FN6905

Exotic Options and Structured Products

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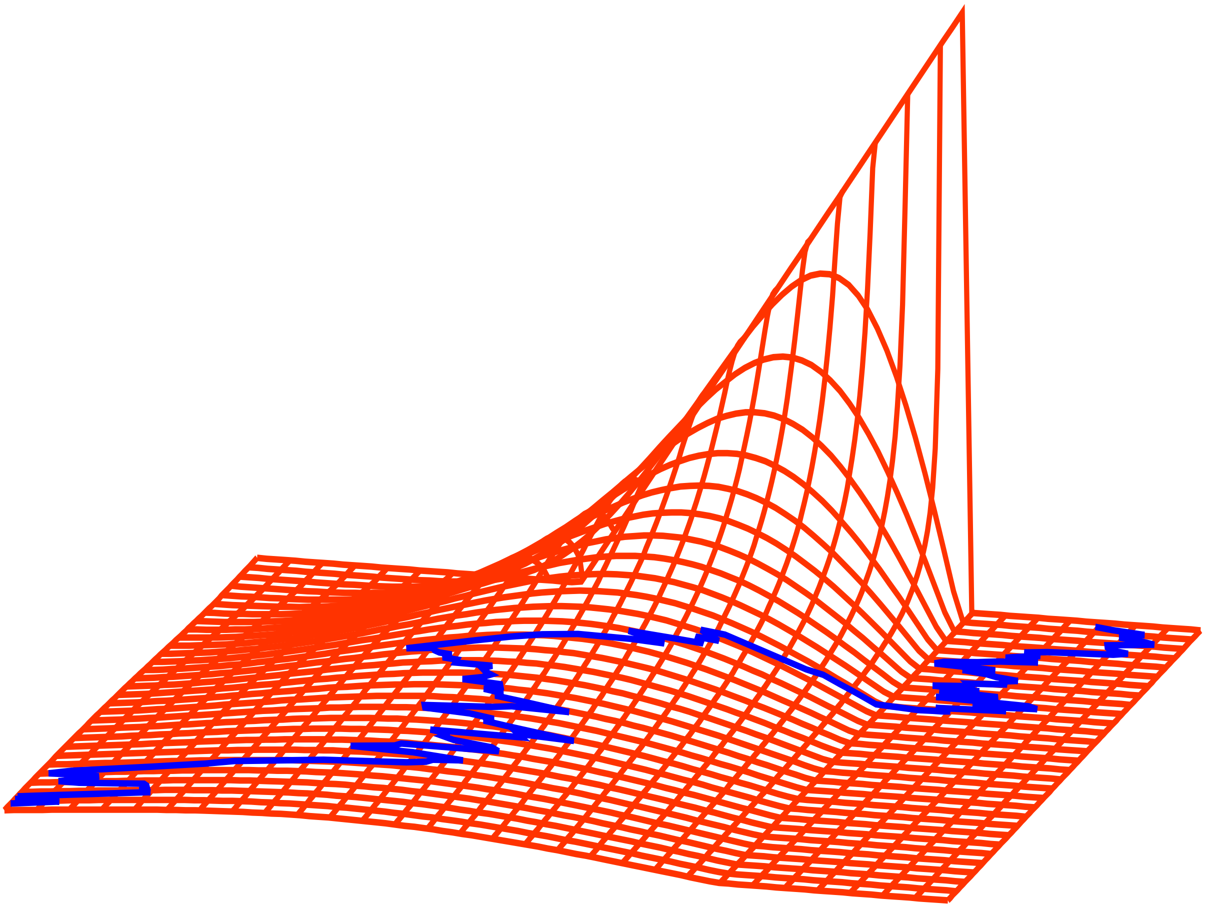

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This course deals with the pricing of exotic options and related financial derivatives, such as volatility and credit derivatives. An introduction to stochastic volatility is given in Chapter 5, followed by a presentation of volatility estimation tools including historical, local, and implied volatilities, in Chapter 6. This chapter also contains a comparison of the prices obtained by the Black-Scholes formula with actual option price market data.

Exotic options such as barrier, lookback, and Asian options are treated in Chapters 8, 9 and 10 respectively, following an introduction to the properties of the maximum of Brownian motion given in Chapter 7.

Credit risk is covered in Chapters 1 and 3 on the reduced-form and structural approaches to credit risk and valuation, which require a basic knowledge of stochastic calculus in continuous time as well as preliminaries on correlation and dependence covered in Chapter 2 Credit default is treated via defaultable bonds and Credit Default Swaps (CDS) and collateralized debt obligations (CDOs) in Chapter 4.

This text contains external links and 140 figures, including 15 animated Figures 7.1, 7.2, 7.3, 7.6, 8.11, 9.1, 9.6, 9.14, and 10.1, 1 embedded video in Figure 6.3 and one interacting 3D graph in Figure 8.1, that may require using Acrobat Reader for viewing on the complete pdf file.

The document also contains 66 exercises with solutions, and includes 3 Python codes on page 87 and 25 R codes e.g. on pages 87, 88, 89, 101, 101, 101, 69, 136, and 144. Supplementary exercises, problems and solutions are available from the textbook **Introduction to Stochastic Finance with Market Examples**, Chapman & Hall Financial Mathematics Series, 2022.

The cover picture represents the price map of an up-and-out barrier call (or down-and-out barrier put) option price, depending on the reading of the price axis.

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*Animated figures (work in Acrobat reader).

1. Reduced-Form Approach to Credit Risk

In this chapter, credit risk is estimated by modeling default probabilities using stochastic failure rate processes. In addition, information on default events is incorporated to the model by the use of exogeneous random variables and enlargement of filtrations. This is in contrast to the structural approach to credit risk of Chapter 3, in which bankruptcy is modeled from a firm's asset value. Applications are given to the pricing of default bonds.

1.1 Survival Probabilities

The reduced-form approach to credit risk relies on the concept of survival probability, defined as the probability $\mathbb{P}(\tau > t)$ that a random system with lifetime τ survives at least over t years, $t > 0$. Assuming that survival probabilities $\mathbb{P}(\tau > t)$ are strictly positive for all $t > 0$, we can compute the conditional probability for that system to survive up to time T, given that it was still functioning at time $t \in [0,T]$, as

$$\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T,$$

with

$$\begin{aligned}
\mathbb{P}(\tau \leqslant T \mid \tau > t) &= 1 - \mathbb{P}(\tau > T \mid \tau > t) \\
&= \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{P}(\tau \leqslant T) - \mathbb{P}(\tau \leqslant t)}{\mathbb{P}(\tau > t)} \\
&= \frac{\mathbb{P}(t < \tau \leqslant T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T. \qquad (1.1.1)
\end{aligned}
$$

Such survival probabilities are typically found in life (or mortality) tables:

Age t	$\mathbb{P}(\tau \leqslant t+1 \mid \tau > t)$
20	0.0894%
30	0.1008%
40	0.2038%
50	0.4458%
60	0.9827%

Table 1.1: Mortality table.

The corresponding conditional survival probability distribution can be computed as follows:

$$
\begin{aligned}
\mathbb{P}(\tau \in dx \mid \tau > t) &= \mathbb{P}(x < \tau \leqslant x + dx \mid \tau > t) \\
&= \mathbb{P}(\tau \leqslant x + dx \mid \tau > t) - \mathbb{P}(\tau \leqslant x \mid \tau > t) \\
&= \frac{\mathbb{P}(\tau \leqslant x + dx) - \mathbb{P}(\tau \leqslant x)}{\mathbb{P}(\tau > t)} \\
&= \frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau \leqslant x) \\
&= -\frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau > x), \qquad x > t.
\end{aligned}
$$

Proposition 1.1 The *failure rate* function, defined as

$$
\lambda(t) := \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt},
$$

satisfies

$$
\mathbb{P}(\tau > t) = \exp\left(- \int_0^t \lambda(u) du \right), \qquad t \geqslant 0. \qquad (1.1.2)
$$

Proof. By (1.1.1), we have

$$
\begin{aligned}
\lambda(t) &:= \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt} \\
&= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(t < \tau \leqslant t + dt)}{dt} \\
&= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > t + dt)}{dt} \\
&= -\frac{d}{dt} \log \mathbb{P}(\tau > t) \\
&= -\frac{1}{\mathbb{P}(\tau > t)} \frac{d}{dt} \mathbb{P}(\tau > t), \qquad t > 0,
\end{aligned}
$$

and the differential equation

$$\frac{d}{dt}\mathbb{P}(\tau > t) = -\lambda(t)\mathbb{P}(\tau > t),$$

which can be solved as in (1.1.2) under the initial condition $\mathbb{P}(\tau > 0) = 1$. □

Proposition 1.1 allows us to rewrite the (conditional) survival probability as

$$\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \exp\left(-\int_t^T \lambda(u)du\right), \qquad 0 \leqslant t \leqslant T,$$

with

$$\mathbb{P}(\tau > t+h \mid \tau > t) = \mathrm{e}^{-\lambda(t)h} \simeq 1 - \lambda(t)h, \qquad [h \searrow 0],$$

and

$$\mathbb{P}(\tau \leqslant t+h \mid \tau > t) = 1 - \mathrm{e}^{-\lambda(t)h} \simeq \lambda(t)h, \qquad [h \searrow 0],$$

as h tends to 0. When the failure rate $\lambda(t) = \lambda > 0$ is a constant function of time, Relation (1.1.2) shows that

$$\mathbb{P}(\tau > T) = \mathrm{e}^{-\lambda T}, \qquad T \geqslant 0,$$

i.e. τ has the exponential distribution with parameter λ. Note that given $(\tau_n)_{n \geqslant 1}$ a sequence of *i.i.d.* exponentially distributed random variables, letting

$$T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \qquad n \geqslant 1,$$

defines the sequence of jump times of a standard Poisson process with intensity $\lambda > 0$.

1.2 Stochastic Default

When the random time τ is a *stopping time* with respect to $(\mathscr{F}_t)_{t \in \mathbb{R}_+}$ we have

$$\{\tau > t\} \in \mathscr{F}_t, \qquad t \geqslant 0,$$

i.e. the knowledge of whether default or bankruptcy has already occurred at time t is contained in $\mathscr{F}_t$, $t \in \mathbb{R}_+$, cf. *e.g.* Section 14.3 of Privault, 2022. As a consequence, we can write

$$\mathbb{P}(\tau > t \mid \mathscr{F}_t) = \mathbb{E}\left[\mathbb{1}_{\{\tau > t\}} \mid \mathscr{F}_t\right] = \mathbb{1}_{\{\tau > t\}}, \qquad t \geqslant 0.$$

In what follows we will not assume that τ is an $\mathscr{F}_t$-stopping time, and by analogy with (1.1.2) we will write $\mathbb{P}(\tau > t \mid \mathscr{F}_t)$ as

$$\mathbb{P}(\tau > t \mid \mathscr{F}_t) = \exp\left(-\int_0^t \lambda_u du\right), \qquad t \geqslant 0, \tag{1.2.1}$$

where the failure rate function $(\lambda_t)_{t \in \mathbb{R}_+}$ is modeled as a random process adapted to a filtration $(\mathscr{F}_t)_{t \in \mathbb{R}_+}$.

The process $(\lambda_t)_{t \in \mathbb{R}_+}$ can also be chosen among the classical mean-reverting diffusion processes, including jump-diffusion processes. In Lando, 1998, the process $(\lambda_t)_{t \in \mathbb{R}_+}$ is constructed as $\lambda_t := h(X_t)$, $t \in \mathbb{R}_+$, where h is a nonnegative function and $(X_t)_{t \in \mathbb{R}_+}$ is a stochastic process generating the filtration $(\mathscr{F}_t)_{t \in \mathbb{R}_+}$.

The default time τ is then *defined* as

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ \int_0^t h(X_u)du \geqslant L\right\},$$

where L is an exponentially distributed random variable with parameter $\mu > 0$ and distribution function $\mathbb{P}(L > x) = \mathrm{e}^{-\mu x}$, $x \geqslant 0$, independent of $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. In this case, as τ is not an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time, we have

$$\begin{aligned}
\mathbb{P}(\tau > t \mid \mathscr{F}_t) &= \mathbb{P}\left(\int_0^t h(X_u)du < L \,\middle|\, \mathscr{F}_t\right) \\
&= \exp\left(-\mu \int_0^t h(X_u)du\right) \\
&= \exp\left(-\mu \int_0^t \lambda_u du\right), \qquad t \geqslant 0.
\end{aligned}$$

Definition 1.2 Let $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$ be the filtration defined by $\mathscr{G}_\infty := \mathscr{F}_\infty \vee \sigma(\tau)$ and

$$\mathscr{G}_t := \big\{B \in \mathscr{G}_\infty \ : \ \exists A \in \mathscr{F}_t \text{ such that } A \cap \{\tau > t\} = B \cap \{\tau > t\}\big\}, \tag{1.2.2}$$

with $\mathscr{F}_t \subset \mathscr{G}_t$, $t \geqslant 0$.

In other words, $\mathscr{G}_t$ contains insider information on whether default at time τ has occurred or not before time t, and τ is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-stopping time. Note that this information on τ may not be available to a generic user who has only access to the smaller filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. The next key Lemma 1.3, see Lando, 1998, Guo, Jarrow, and Menn, 2007, allows us to price a contingent claim given the information in the larger filtration $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$, by only using information in $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ and factoring in the default rate factor $\exp\left(-\int_t^T \lambda_u du\right)$.

Lemma 1.3 (Guo, Jarrow, and Menn, 2007, Theorem 1) For any $\mathscr{F}_T$-measurable integrable random variable F, we have

$$\begin{aligned}
\mathbb{E}\big[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\big] &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\big[F\mathbb{P}(\tau > T \mid \tau > t) \mid \mathscr{F}_t\big] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \,\middle|\, \mathscr{F}_t\right], \quad 0 \leqslant t \leqslant T.
\end{aligned}$$

Proof. By (1.2.1) we have

$$\frac{\mathbb{P}(\tau > T \mid \mathscr{F}_T)}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)} = \frac{\mathrm{e}^{-\int_0^T \lambda_u du}}{\mathrm{e}^{-\int_0^t \lambda_u du}} = \exp\left(-\int_t^T \lambda_u du\right),$$

hence, since F is $\mathscr{F}_T$-measurable,

$$\begin{aligned}
&\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \,\middle|\, \mathscr{F}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\frac{\mathbb{P}(\tau > T \mid \mathscr{F}_T)}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)} \,\middle|\, \mathscr{F}_t\right] \\
&= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)}\mathbb{E}\big[F\,\mathbb{E}[\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_T] \mid \mathscr{F}_t\big] \\
&= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)}\mathbb{E}\big[\mathbb{E}[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_T] \mid \mathscr{F}_t\big] \\
&= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\big[F\mathbb{1}_{\{\tau>T\}} \big| \mathscr{F}_t\big]}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)} \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\big[F\mathbb{1}_{\{\tau>T\}} \big| \mathscr{G}_t\big]
\end{aligned}$$

$$= \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}}\middle|\mathscr{G}_t\right], \qquad 0 \leqslant t \leqslant T.$$

In the last step of the above argument, we used the key relation

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}}\,\middle|\,\mathscr{G}_t\right] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right],$$

cf. Relation (75.2) in § XX-75 page 186 of Dellacherie, Maisonneuve, and Meyer, 1992, Theorem VI-3-14 page 371 of Protter, 2004, and Lemma 3.1 of Elliott, Jeanblanc, and Yor, 2000, under the conditional probability measure $\mathbb{P}_{|\mathscr{F}_t}$, $0 \leqslant t \leqslant T$. Indeed, according to (1.2.2), for any $B \in \mathscr{G}_t$ we have, for some event $A \in \mathscr{F}_t$,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_B\mathbb{1}_{\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] &= \mathbb{E}\left[\mathbb{1}_{B\cap\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{A\cap\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[\mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}F\mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_A\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A\mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_A\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_B\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right],
\end{aligned}$$

hence by a standard characterization of conditional expectations, we have

$$\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}}F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]$$

□

Taking $F = 1$ in Lemma 1.3 allows one to write the survival probability up to time T, given the information known up to time t, as

$$\begin{aligned}
\mathbb{P}(\tau>T \mid \mathscr{G}_t) &= \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right)\,\middle|\,\mathscr{F}_t\right], \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{1.2.3}$$

In particular, applying Lemma 1.3 for $t = T$ and $F = 1$ shows that

$$\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} \mid \mathscr{G}_t\right] = \mathbb{1}_{\{\tau>t\}},$$

which shows that $\{\tau > t\} \in \mathscr{G}_t$ for all $t > 0$, and recovers the fact that τ is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-stopping time, while in general, τ is not $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time.

The computation of $\mathbb{P}(\tau > T \mid \mathscr{G}_t)$ according to (1.2.3) is then similar to that of a bond price, by considering the failure rate $\lambda(t)$ as a "virtual" short-term interest rate. In particular the failure rate $\lambda(t,T)$ can be modeled in the HJM framework, cf. *e.g.* Chapter 18.3 of Privault, 2022, and

$$\mathbb{P}(\tau > T \mid \mathscr{G}_t) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda(t,u)du\right) \Bigm| \mathscr{F}_t\right]$$

can then be computed by applying HJM bond pricing techniques.

The computation of expectations given $\mathscr{G}_t$ as in Lemma 1.3 can be useful for pricing under insider trading, in which the insider has access to the augmented filtration $\mathscr{G}_t$ while the ordinary trader has only access to $\mathscr{F}_t$, therefore generating two different prices $\mathbb{E}^*[F \mid \mathscr{F}_t]$ and $\mathbb{E}^*[F \mid \mathscr{G}_t]$ for the same claim payoff F under the same risk-neutral probability measure $\mathbb{P}^*$. This leads to the issue of computing the dynamics of the underlying asset price by decomposing it using a $\mathscr{F}_t$-martingale *vs.* a $\mathscr{G}_t$-martingale instead of using different forward measures as in *e.g.* § 19.1 of Privault, 2022. This can be obtained by the technique of enlargement of filtration, cf. Jeulin, 1980, Jacod, 1985, Yor, 1985, Elliott and Jeanblanc, 1999.

1.3 Defaultable Bonds

Bond pricing models are generally based on the terminal condition $P(T,T) = \$1$ according to which the bond payoff at maturity is always equal to \$1, and default does not occurs. In this chapter we allow for the possibility of default at a random time τ, in which case the terminal payoff of a bond is allowed to vanish at maturity.

The price $P_d(t,T)$ at time t of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$\begin{aligned} P_d(t,T) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}}\exp\left(-\int_t^T r_u du\right) \Bigm| \mathscr{G}_t\right] \\ &\quad + \mathbb{E}^*\left[\xi \mathbb{1}_{\{\tau\leqslant T\}}\exp\left(-\int_t^T r_u du\right) \Bigm| \mathscr{G}_t\right], \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

Proposition 1.4 The default bond with maturity T and default time τ can be priced at time $t \in [0,T]$ as

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u)du\right) \Bigm| \mathscr{F}_t\right] \\ &\quad + \mathbb{E}^*\left[\xi \mathbb{1}_{\{\tau\leqslant T\}}\exp\left(-\int_t^T r_u du\right) \Bigm| \mathscr{G}_t\right], \quad 0 \leqslant t \leqslant T. \end{aligned}$$

Proof. We take $F = \exp\left(-\int_t^T r_u du\right)$ in Lemma 1.3, which shows that

$$\mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}}\exp\left(-\int_t^T r_u du\right) \Bigm| \mathscr{G}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u)du\right) \Bigm| \mathscr{F}_t\right],$$

cf. *e.g.* Lando, 1998, Duffie and Singleton, 2003, Guo, Jarrow, and Menn, 2007. □

In the case of complete default (zero-recovery), we have $\xi = 0$ and

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right) \Bigm| \mathscr{F}_t\right], \quad 0 \leqslant t \leqslant T. \tag{1.3.1}$$

From the above expression (1.3.1) we note that the effect of the presence of a default time τ is to decrease the bond price, which can be viewed as an increase of the short rate by the amount λ_u. In a simple setting where the interest rate $r > 0$ and failure rate $\lambda > 0$ are constant, the default bond price becomes

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}}\,\mathrm{e}^{-(r+\lambda)(T-t)}, \qquad 0 \leqslant t \leqslant T.$$

In this case, the failure rate λ can be estimated at time $t \in [0,T]$ from a default bond price $P_d(t,T)$ and a non-default bond price $P(t,T) = \mathrm{e}^{-(T-t)r}$ as

$$\lambda = \frac{1}{T-t}\log\frac{P(t,T)}{P_d(t,T)}.$$

Finally, from *e.g.* Proposition 19.1 in Privault, 2022 the bond price (1.3.1) can also be expressed under the forward measure $\widehat{\mathbb{P}}$ with maturity T, as

$$\begin{aligned}
P_d(t,T) &= \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}^*\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)\,\Big|\,\mathcal{F}_t\right]\\
&= \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\,\Big|\,\mathcal{F}_t\right]\widehat{\mathbb{E}}\left[\exp\left(-\int_t^T \lambda_s ds\right)\,\Big|\,\mathcal{F}_t\right]\\
&= \mathbb{1}_{\{\tau>t\}}N_t\widehat{\mathbb{P}}(\tau > T \mid \mathcal{G}_t),
\end{aligned}$$

where $(N_t)_{t\in\mathbb{R}_+}$ is the numéraire process

$$N_t := P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\,\Big|\,\mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T,$$

and by (1.2.3),

$$\widehat{\mathbb{P}}(\tau > T \mid \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\widehat{\mathbb{E}}\left[\exp\left(-\int_t^T \lambda_s ds\right)\,\Big|\,\mathcal{F}_t\right]$$

is the survival probability under the forward measure $\widehat{\mathbb{P}}$ defined as

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} := \frac{N_T}{N_0}\mathrm{e}^{-\int_0^T r_t dt},$$

see Chen and Huang, 2001 and Chen, Cheng, et al., 2008.

Estimating the default rates

Recall that the price of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$\begin{aligned}
P_d(t,T) &= \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u)du\right)\,\Big|\,\mathcal{F}_t\right]\\
&\quad + \mathbb{E}^*\left[\xi\,\mathbb{1}_{\{\tau\leqslant T\}}\exp\left(-\int_t^T r_u du\right)\,\Big|\,\mathcal{G}_t\right], \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

where ξ denotes the recovery rate. We consider a simplified deterministic step function model with zero recovery rate and tenor structure

$$\{t = T_0 < T_1 < \cdots < T_n = T\},$$

where

$$r(t) = \sum_{l=0}^{n-1} r_l\mathbb{1}_{(T_l,T_{l+1}]}(t) \quad \text{and} \quad \lambda(t) = \sum_{l=0}^{n-1}\lambda_l\mathbb{1}_{(T_l,T_{l+1}]}(t), \quad t \geqslant 0. \tag{1.3.2}$$

i) Estimating the default rates from default bond prices.

From Proposition 1.4, we have

$$\begin{aligned} P_d(t,T_k) &= \mathbb{1}_{\{\tau>t\}} \exp\left(-\int_t^{T_k}(r(u)+\lambda(u))du\right) \\ &= \mathbb{1}_{\{\tau>t\}} \exp\left(-\sum_{l=0}^{k-1}(r_l+\lambda_l)(T_{l+1}-T_l)\right), \end{aligned}$$

$k=1,2,\ldots,n$, from which we can infer

$$\lambda_k = -r_k + \frac{1}{T_{k+1}-T_k}\log\frac{P_d(t,T_k)}{P_d(t,T_{k+1})} > 0, \quad k=0,1,\ldots,n-1.$$

ii) Estimating (implied) default probabilities $\mathbb{P}^*(\tau<T \mid \mathscr{G}_t)$ from default rates.

Based on the expression

$$\begin{aligned} \mathbb{P}^*(\tau>T \mid \mathscr{G}_t) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T, \end{aligned} \tag{1.3.3}$$

of the survival probability up to time T, see (1.2.1), and given the information known up to time t, in terms of the hazard rate process $(\lambda_u)_{u\in\mathbb{R}_+}$ adapted to a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, we find

$$\begin{aligned} \mathbb{P}(\tau>T \mid \mathscr{G}_{T_k}) &= \mathbb{1}_{\{\tau>T_k\}} \exp\left(-\int_{T_k}^T \lambda_u du\right) \\ &= \mathbb{1}_{\{\tau>t\}} \exp\left(-\sum_{l=k}^{n-1}\lambda_l(T_{l+1}-T_l)\right), \quad k=0,1,\ldots,n-1, \end{aligned}$$

where

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\{\tau \leqslant u\} : 0 \leqslant u \leqslant t), \quad t \geqslant 0,$$

i.e. $\mathscr{G}_t$ contains the additional information on whether default at time τ has occurred or not before time t.

In Table 1.2, bond ratings are determined according to hazard (or failure) rate thresholds.

Bond Credit	Moody's		S & P	
Ratings	Municipal	Corporate	Municipal	Corporate
Aaa/AAAs	0.00	0.52	0.00	0.60
Aa/AA	0.06	0.52	0.00	1.50
A/A	0.03	1.29	0.23	2.91
Baa/BBB	0.13	4.64	0.32	10.29
Ba/BB	2.65	19.12	1.74	29.93
B/B	11.86	43.34	8.48	53.72
Caa-C/CCC-C	16.58	69.18	44.81	69.19
Investment Grade	0.07	2.09	0.20	4.14
Non-Invest. Grade	4.29	31.37	7.37	42.35
All	0.10	9.70	0.29	12.98

Table 1.2: Cumulative historic default rates (in percentage).*

*Sources: Moody's, S&P.

Exercises

Exercise 1.1 Consider a standard zero-coupon bond with constant yield $r > 0$ and a defaultable (risky) bond with constant yield r_d and default probability $\alpha \in (0,1)$. Find a relation between r, r_d, α and the bond maturity T.

Exercise 1.2 A standard zero-coupon bond with constant yield $r > 0$ and maturity T is priced $P(t,T) = \mathrm{e}^{-(T-t)r}$ at time $t \in [0,T]$. Assume that the company can get bankrupt at a random time $t+\tau$, and default on its final \$1 payment if $\tau < T-t$.

a) Explain why the defaultable bond price $P_d(t,T)$ can be expressed as

$$P_d(t,T) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\mathbb{1}_{\{\tau > T-t\}}\right]. \tag{1.3.4}$$

b) Assuming that the default time τ is exponentially distributed with parameter $\lambda > 0$, compute the default bond price $P_d(t,T)$ using (1.3.4).

c) Find a formula that can estimate the parameter λ from the risk-free rate r and the market data $P_M(t,T)$ of the defaultable bond price at time $t \in [0,T]$.

Exercise 1.3 Consider an interest rate process $(r_t)_{t\in\mathbb{R}_+}$ and a default rate process $(\lambda_t)_{t\in\mathbb{R}_+}$, modeled according to the Vasicek processes

$$\begin{cases} dr_t = -ar_t dt + \sigma dB_t^{(1)}, \\ d\lambda_t = -b\lambda_t dt + \eta dB_t^{(2)}, \end{cases}$$

where $\left(B_t^{(1)}\right)_{t\in\mathbb{R}_+}$ and $\left(B_t^{(2)}\right)_{t\in\mathbb{R}_+}$ are standard $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-Brownian motions with correlation $\rho \in [-1,1]$, and $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$.

a) Taking $r_0 := 0$, show that we have

$$\int_t^T r_s ds = C(a,t,T)r_t + \sigma \int_t^T C(a,s,T)dB_s^{(1)},$$

and

$$\int_t^T \lambda_s ds = C(b,t,T)\lambda_t + \eta \int_t^T C(b,s,T)dB_s^{(2)},$$

where

$$C(a,t,T) = -\frac{1}{a}\left(\mathrm{e}^{-(T-t)a} - 1\right).$$

b) Show that the random variable

$$\int_t^T r_s ds + \int_t^T \lambda_s ds$$

is has a Gaussian distribution, and compute its conditional mean

$$\mathbb{E}^*\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right]$$

and variance

$$\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right],$$

conditionally to $\mathscr{F}_t$.

Exercise 1.4 (Exercise 1.3 continued). Consider a (random) default time τ with cumulative distribution function

$$\mathbb{P}(\tau > t \mid \mathscr{F}_t) = \exp\left(-\int_0^t \lambda_u du\right), \qquad t \geqslant 0,$$

where λ_t is a (random) default rate process which is adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Recall that the probability of survival up to time T, given the information known up to time t, is given by

$$\mathbb{P}(\tau > T \mid \mathscr{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \Big| \mathscr{F}_t\right],$$

where $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\{\tau < u\} : 0 \leqslant u \leqslant t)$, $t \in \mathbb{R}_+$, is the filtration defined by adding the default time information to the history $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. In this framework, the price $P(t,T)$ of defaultable bond with maturity T, short-term interest rate r_t and (random) default time τ is given by

$$\begin{aligned} P(t,T) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \exp\left(-\int_t^T r_u du\right) \Big| \mathscr{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \Big| \mathscr{F}_t\right]. \end{aligned} \tag{1.3.5}$$

a) Give a justification for the fact that

$$\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \Big| \mathscr{F}_t\right]$$

can be written as a function $F(t,r_t,\lambda_t)$ of t, r_t and λ_t, $t \in [0,T]$.

b) Show that

$$t \longmapsto \exp\left(-\int_0^t (r_s+\lambda_s) ds\right) \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \Big| \mathscr{F}_t\right]$$

is an $\mathscr{F}_t$-martingale under $\mathbb{P}$.

c) Use the Itô formula with two variables to derive a PDE on $\mathbb{R}^2$ for the function $F(t,x,y)$.

d) Compute $P(t,T)$ from its expression (1.3.5) as a conditional expectation.

e) Show that the solution $F(t,x,y)$ to the 2-dimensional PDE of Question (c)) is

$$\begin{aligned} F(t,x,y) &= \exp(-C(a,t,T)x - C(b,t,T)y) \\ &\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right) \\ &\quad \times \exp\left(\rho\sigma\eta \int_t^T C(a,s,T)C(b,s,T)ds\right). \end{aligned}$$

f) Show that the defaultable bond price $P(t,T)$ can also be written as

$$P(t,T) = \mathrm{e}^{U(t,T)} \mathbb{P}(\tau > T \mid \mathscr{G}_t) \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathscr{F}_t\right],$$

where

$$U(t,T) = \rho\frac{\sigma\eta}{ab}(T - t - C(a,t,T) - C(b,t,T) + C(a+b,t,T)).$$

g) By partial differentiation of $\log P(t,T)$ with respect to T, compute the corresponding instantaneous short rate

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

h) Show that $\mathbb{P}(\tau > T \mid \mathscr{G}_t)$ can be written using an HJM type default rate as

$$\mathbb{P}(\tau > T \mid \mathscr{G}_t) = \mathbb{1}_{\{\tau > t\}} \exp\left(- \int_t^T f_2(t,u) du \right),$$

where

$$f_2(t,u) = \lambda_t \, \mathrm{e}^{-(u-t)b} - \frac{\eta^2}{2} C^2(b,t,u).$$

i) Show how the result of Question (f)) can be simplified when the processes $\left(B_t^{(1)}\right)_{t\in\mathbb{R}_+}$ and $\left(B_t^{(2)}\right)_{t\in\mathbb{R}_+}$ are independent.

2. Correlation and Dependence

Correlation and dependence play a capital role in risk management, in particular when assessing or preventing any potential "domino effect" arising from interactions between different entities exposed to uncertainties.* This chapter presents several standard models for the statistical interactions that arise in the modeling of correlated risk. For this, we use the concept of copulas, which can model the uncertainty and dependence properties observed between random variables or data samples.

2.1 Joint Bernoulli Distribution

Our study of dependence structures starts with the simplest case of two correlated random variables X and Y, each of them taking only two possible values. For this, let X and Y by two Bernoulli random variables, with

$$p_X = \mathbb{P}(X=1) = \mathbb{E}[\mathbb{1}_{\{X=1\}}] \quad \text{and} \quad p_Y = \mathbb{P}(Y=1) = \mathbb{E}[\mathbb{1}_{\{Y=1\}}]$$

and correlation coefficient

$$\rho := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}}$$

*Correlation does not imply causation. Try "Spurious Correlations".

$$
\begin{aligned}
&= \frac{\mathbb{E}[XY]-\mathbb{E}[X]\,\mathbb{E}[Y]}{\sqrt{\mathrm{Var}\,[X]\mathrm{Var}\,[Y]}} \\
&= \frac{\mathbb{P}(X=1 \text{ and } Y=1)-p_Xp_Y}{\sqrt{p_X(1-p_X)p_Y(1-p_Y)}},
\end{aligned}
$$

with $\rho \in [-1,1]$ from the Cauchy-Schwarz inequality. We note that in this case, the joint distribution $\mathbb{P}(X=i \text{ and } Y=j)$, $i,j=0,1$, is fully determined by the data of $p_X=\mathbb{P}(X=1)$, $p_Y=\mathbb{P}(Y=1)$ and the correlation coefficient $\rho \in [-1,1]$, as

$$
\begin{cases}
\mathbb{P}(X=1 \text{ and } Y=1)=\mathbb{E}[XY] \\
\qquad = p_Xp_Y+\rho\sqrt{p_Xp_Y(1-p_X)(1-p_Y)}, \\
\\
\mathbb{P}(X=0 \text{ and } Y=1)=\mathbb{E}[(1-X)Y]=\mathbb{P}(Y=1)-\mathbb{P}(X=1 \text{ and } Y=1) \\
\qquad = (1-p_X)p_Y-\rho\sqrt{p_Xp_Y(1-p_X)(1-p_Y)}, \\
\\
\mathbb{P}(X=1 \text{ and } Y=0)=\mathbb{E}[X(1-Y)]=\mathbb{P}(X=1)-\mathbb{P}(X=1 \text{ and } Y=1) \\
\qquad = p_X(1-p_Y)-\rho\sqrt{p_Xp_Y(1-p_X)(1-p_Y)}, \\
\\
\mathbb{P}(X=0 \text{ and } Y=0)=\mathbb{E}[(1-X)(1-Y)] \\
\qquad = (1-p_X)(1-p_Y)+\rho\sqrt{p_Xp_Y(1-p_X)(1-p_Y)},
\end{cases}
$$

see Exercise 2.2.

2.2 Joint Gaussian Distribution

Consider now two *centered* Gaussian random variables $X \simeq \mathcal{N}(0,\sigma^2)$ and $Y \simeq \mathcal{N}(0,\eta^2)$ with probability density functions

$$
f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-x^2/(2\sigma^2)} \quad \text{and} \quad f_Y(x)=\frac{1}{\sqrt{2\pi\eta^2}}\mathrm{e}^{-x^2/(2\eta^2)}, \qquad x \in \mathbb{R}.
$$

Let

$$
\rho = \mathrm{corr}\,(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,[X]\mathrm{Var}\,[Y]}} = \frac{\mathbb{E}[XY]-\mathbb{E}[X]\,\mathbb{E}[Y]}{\sqrt{\mathrm{Var}\,[X]\mathrm{Var}\,[Y]}}.
$$

When the covariance matrix

$$
\Sigma := \begin{bmatrix} \mathbb{E}\left[X^2\right] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}\left[Y^2\right] \end{bmatrix} = \begin{bmatrix} \sigma^2 & \rho\sigma\eta \\ \rho\sigma\eta & \eta^2 \end{bmatrix} \tag{2.2.1}
$$

with determinant

$$
\begin{aligned}
\det\Sigma &= \mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]-\left(\mathbb{E}[XY]\right)^2 \\
&= \mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]\left(1-\left(\mathrm{corr}\,(X,Y)\right)^2\right) \\
&\geqslant 0,
\end{aligned}
$$

is invertible, there exists a probability density function

$$
f_\Sigma(x,y) = \frac{1}{\sqrt{2\pi\det\Sigma}}\exp\left(-\frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^\top \Sigma^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right) \tag{2.2.2}
$$

$$= \frac{1}{\sqrt{2\pi \det \Sigma}} \exp \left(-\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} \mathbb{E}[X^2] & \mathbb{E}[XY] \\ \mathbb{E}[XY] & \mathbb{E}[Y^2] \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right),$$

with respective marginals $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(0,\eta^2)$.

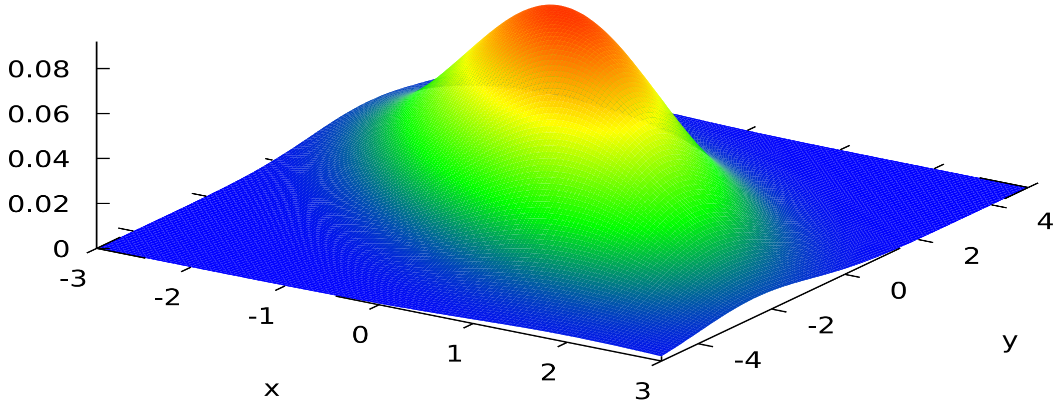


Figure 2.1: Joint Gaussian probability density.

The probability density function (2.2.2) is called the centered joint (bivariate) Gaussian probability density with covariance matrix Σ.

Note that when $\rho = \operatorname{corr}(X,Y) = \pm 1$ we have $\det \Sigma = 0$ and the joint probability density function $f_\Sigma(x,y)$ is *not defined.*

Definition 2.1 A random vector $(X_1,\ldots,X_n)$ is said to have a multivariate centered Gaussian distribution if every linear combination

$$a_1X_1 + \cdots + a_nX_n, \quad a_1,\ldots,a_n \in \mathbb{R}, \ n \geqslant 1,$$

has a centered Gaussian distribution.

Recall that if $(X_1,\ldots,X_n)$ has a multivariate centered Gaussian distribution, then its probability density function takes the form

$$f_\Sigma(x_1,\ldots,x_n) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(x_1,\ldots,x_n)^T \Sigma^{-1}(x_1,\ldots,x_n) \right),$$

$(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$, where Σ is the covariance matrix

$$\Sigma = \begin{bmatrix} \operatorname{Var}[X_1] & \operatorname{Cov}(X_1,X_2) & \cdots & \operatorname{Cov}(X_1,X_{n-1}) & \operatorname{Cov}(X_1,X_n) \\ \operatorname{Cov}(X_2,X_1) & \operatorname{Var}[X_2] & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \operatorname{Var}[X_{n-1}] & \operatorname{Cov}(X_{n-1},X_n) \\ \operatorname{Cov}(X_1,X_n) & \operatorname{Cov}(X_2,X_n) & \cdots & \operatorname{Cov}(X_{n-1},X_n) & \operatorname{Var}[X_n] \end{bmatrix}.$$

The next remark plays an important role in the modeling of joint default probabilities, see **here** for a detailed discussion.

Remark 2.2 There exist couples (X,Y) of random variables with Gaussian marginals $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(0,\eta^2)$, such that

i) (X,Y) does not have the bivariate Gaussian distribution with probability density function

$f_\Sigma(x,y)$, where Σ is the covariance matrix (2.2.1) of (X,Y).

ii) the random variable $X+Y$ is not even Gaussian.

Proof. See Exercise 2.5. □

2.3 Copulas and Dependence Structures

The word copula derives from the Latin noun for a "link" or "tie" that connects two different objects or concepts.

Definition 2.3 A two-dimensional copula is any joint cumulative distribution function

$$\begin{array}{rcl} C: & [0,1]\times[0,1] & \longrightarrow [0,1] \\ & (u,v) & \longmapsto C(u,v) \end{array}$$

with uniform $[0,1]$-valued marginals.

In other words, any copula function $C(u,v)$ can be written as

$$C(u,v) = \mathbb{P}(U \leqslant u \text{ and } V \leqslant v), \qquad 0 \leqslant u,v \leqslant 1,$$

where U and V are uniform $[0,1]$-valued random variables.

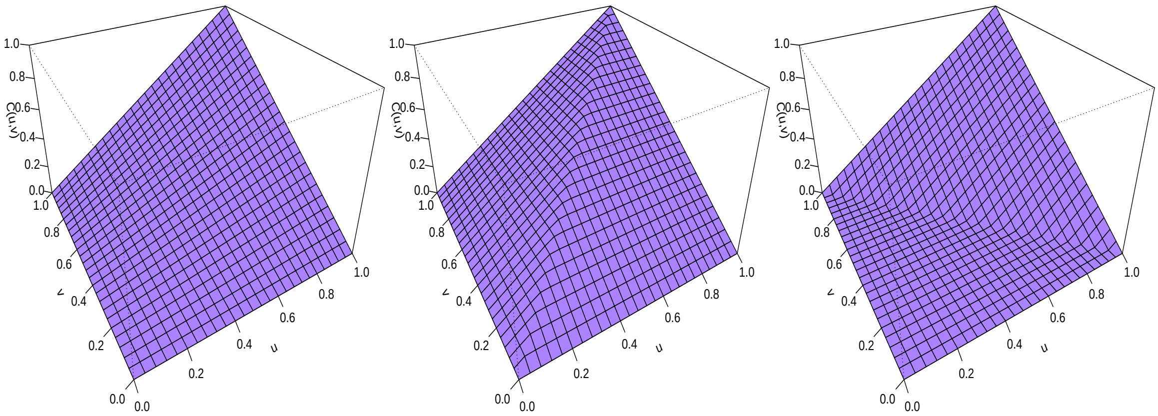
Examples.

i) The copula corresponding to independent uniform random variables (U,V) is given by
$$\begin{aligned} C(u,v) &= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}(U \leqslant u)\mathbb{P}(V \leqslant v) \\ &= uv, \qquad 0 \leqslant u,v \leqslant 1. \end{aligned}$$

ii) The copula corresponding to the fully correlated case $U=V$ is given by
$$\begin{aligned} C(u,v) &= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}(U \leqslant \min(u,v)) \\ &= \min(u,v), \qquad 0 \leqslant u,v \leqslant 1. \end{aligned}$$

iii) The copula corresponding to the fully anticorrelated case $U = 1-V$ is given by
$$\begin{aligned} C(u,v) &:= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}(U \leqslant u \text{ and } 1-U \leqslant v) \\ &= \mathbb{P}(1-v \leqslant U \leqslant u) \\ &= (u+v-1)^+, \qquad 0 \leqslant u,v \leqslant 1. \end{aligned}$$

The above copulas are plotted in Figure 2.3a.



(a) uv. (b) $\min(u,v)$. (c) $(u+v-1)^+$.

Figure 2.2: Copula graphs $C(u,v)=uv$, $C(u,v)=\min(u,v)$, $C(u,v)=(u+v-1)^+$.

In what follows, F_X^{-1} denotes the inverse of the *Cumulative Distribution Function* F_X of X.

Lemma 2.4 Assume that the random variable X has a *continuous and strictly increasing* cumulative distribution function $F_X(x) := \mathbb{P}(X \leqslant x)$, $x \in \mathbb{R}$. Then, $U := F_X(X)$ is uniformly distributed on $[0,1]$.

Proof. We have

$$\begin{aligned} F_U(u) &= \mathbb{P}(U \leqslant u) \\ &= \mathbb{P}(F_X(X) \leqslant u) \\ &= \mathbb{P}(X \leqslant F_X^{-1}(u)) \\ &= F_X(F_X^{-1}(u)) \\ &= u, \qquad 0 \leqslant u \leqslant 1. \end{aligned}$$

□

As in Lemma 2.4, given (X,Y) a couple of random variables with joint cumulative distribution function

$$F_{(X,Y)}(x,y) := \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y), \qquad x,y \in \mathbb{R},$$

and continuous strictly increasing marginal cumulative distribution functions

$$F_X(x) = F_{(X,Y)}(x,\infty) = \mathbb{P}(X \leqslant x) \text{ and } F_Y(y) = F_{(X,Y)}(\infty,y) = \mathbb{P}(Y \leqslant y),$$

we note the following points.

i) The random variables

$$U := F_X(X) \quad \text{and} \quad V := F_Y(Y)$$

are uniformly distributed on $[0,1]$.

ii) The copula function

$$C_{(X,Y)}(u,v) := \mathbb{P}(U \leqslant u \text{ and } V \leqslant v), \qquad 0 \leqslant u,v \leqslant 1,$$

satisfies

$$\begin{aligned} C_{(X,Y)}(u,v) &:= \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \\ &= \mathbb{P}\big(F_X(X) \leqslant u \text{ and } F_Y(Y) \leqslant v\big) \\ &= \mathbb{P}\big(X \leqslant F_X^{-1}(u) \text{ and } Y \leqslant F_Y^{-1}(v)\big) \\ &= F_{(X,Y)}\big(F_X^{-1}(u), F_Y^{-1}(v)\big), \qquad 0 \leqslant u,v \leqslant 1. \end{aligned}$$

iii) The joint cumulative distribution function of (X,Y) can be recovered from the copula $C_{(X,Y)}$ and the marginal cumulative distribution functions F_X, F_Y as

$$\begin{aligned}
F_{(X,Y)}(x,y) &= \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) \\
&= \mathbb{P}\big(F_X(X) \leqslant F_X(x) \text{ and } F_Y(Y) \leqslant F_Y(v)\big) \\
&= \mathbb{P}\big(U \leqslant F_X(x) \text{ and } V \leqslant F_Y(v)\big) \\
&= C_{(X,Y)}\big(F_X(x), F_Y(y)\big), \qquad x,y \in \mathbb{R}.
\end{aligned}$$

Higher dimensional copulas

Definition 2.5 An n-dimensional copula is any joint cumulative distribution function

$$\begin{aligned}
C: \quad [0,1]\times\cdots\times[0,1] &\longrightarrow [0,1] \\
(u_1,\ldots,u_n) &\longmapsto C(u_1,\ldots,u_n)
\end{aligned}$$

of n uniform $[0,1]$-valued random variables.

Consider the joint cumulative distribution function

$$F_{(X_1,\ldots,X_n)}(x_1,\ldots,x_n) := \mathbb{P}(X_1 \leqslant x_1,\ldots,X_n \leqslant x_n)$$

of a family $(X_1,\ldots,X_n)$ of random variables with marginal cumulative distribution functions

$$F_{X_i}(x) = F_{(X_1,\ldots,X_n)}(\infty,\ldots,+infty,x,\infty,\ldots,\infty), \quad x \in \mathbb{R},$$

$i = 1,2,\ldots,n$. The copula defined in the next Sklar's Theorem 2.6 encodes the dependence structure of the vector $(X_1,\ldots,X_n)$.

Theorem 2.6 [Sklar's theorem[a] (Sklar, 1959; Sklar, 2010)] Given a joint cumulative distribution function $F_{(X_1,\ldots,X_n)}$, there exists an n-dimensional copula $C(u_1,\ldots,u_n)$ such that

$$F_{(X_1,\ldots,X_n)}(x_1,x_2,\ldots,x_n) = C\big(F_{X_1}(x_1),F_{X_2}(x_2),\ldots,F_{X_n}(x_n)\big),$$

$x_1,x_2,\ldots,x_n \in \mathbb{R}$.

[a] "The author considers continuous non-decreasing functions C_n on the n-dimensional cube $[0,1]^n$ with $C_n(0,\ldots,0) = 0$, $C_n(1,\ldots,1,\alpha,1,\ldots,1) = \alpha$. Several theorems are stated relating n-dimensional distribution functions and their marginals in terms of functions C_n. No proofs are given." M. Loève, Math. Reviews MR0125600.

The following corollary is a consequence of Sklar's Theorem 2.6.

Corollary 2.7 Assume that the marginal distribution functions F_{X_i} are continuous and strictly increasing. Then the joint cumulative distribution function $F_{(X_1,\ldots,X_n)}$ defines a n-dimensional copula

$$C(u_1,\ldots,u_n) := F_{(X_1,\ldots,X_n)}\big(F_{X_1}^{-1}(u_1),\ldots,F_{X_n}^{-1}(u_n)\big), \tag{2.3.1}$$

$u_1,u_2,\ldots,u_n \in [0,1]$, which encodes the dependence structure of the vector $(X_1,\ldots,X_n)$.

It can be checked as in Lemma 2.4 that $C(u_1,\ldots,u_n)$ defined in (2.3.1) has uniform marginal distributions on $[0,1]$, as

$$\begin{aligned}
&C(1,\ldots,1,u,1,\ldots,1) \\
&\quad = F_{(X_1,\ldots,X_n)}\big(F_{X_1}^{-1}(1),\ldots,F_{X_{i-1}}^{-1}(1),F_{X_i}^{-1}(u),F_{X_{i+1}}^{-1}(1),\ldots,F_{X_n}^{-1}(1)\big) \\
&\quad = F_{(X_1,\ldots,X_n)}\big(\infty,\ldots,\infty,F_{X_i}^{-1}(u),\infty,\ldots,\infty\big)
\end{aligned}$$

$$
\begin{aligned}
&= F_{X_i}\left(F_{X_i}^{-1}(u)\right) \\
&= u, \qquad 0 \leqslant u \leqslant 1.
\end{aligned}
$$

In the following proposition, we construct a vector of random variables from the data of a copula and a family of marginal distributions.

Proposition 2.8 Given a family of continuous strictly increasing marginal cumulative distribution functions $F_1, \ldots, F_n$ and a multidimensional copula $C(u_1, \ldots, u_n)$, the function

$$
F^C_{(X_1,\ldots,X_n)}(x_1, \ldots, x_n) := C(F_1(x_1), \ldots, F_n(x_n)), \quad x_1, x_2, \ldots, x_n \in \mathbb{R}, \tag{2.3.2}
$$

defines a joint cumulative distribution function with marginals $X_1, \ldots, X_n$.

Proof. Given $(U_1, \ldots, U_n)$ a vector of n uniform random variables having the copula $C(u_1, \ldots, u_n)$ for cumulative distribution function, we let

$$
X_1 := F_1^{-1}(U_1), \ldots, X_n := F_n^{-1}(U_n).
$$

Then, we have

$$
\begin{aligned}
\mathbb{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n) &= \mathbb{P}\left(F_1^{-1}(U_1) \leqslant x_1, \ldots, F_n(U_n) \leqslant x_n\right) \\
&= \mathbb{P}(U_1 \leqslant F_1(x_1), \ldots, U_n \leqslant F_n(x_n)) \\
&= C(F_1(x_1), \ldots, F_n(x_n)) \\
&= F^C_{(X_1,\ldots,X_n)}(x_1, \ldots, x_n), \quad x_1, x_2, \ldots, x_n \in \mathbb{R}.
\end{aligned}
$$

We can also check that the marginal distributions generated by $F^C_{(X_1,\ldots,X_n)}$ coincide with the respective marginals of $(X_1, \ldots, X_n)$, as we have

$$
\begin{aligned}
&F^C_{(X_1,\ldots,X_n)}(\infty, \ldots, \infty, u, \infty, \ldots, \infty) \\
&\quad = C\left(F_1(\infty), \ldots, F_{i-1}(\infty), F_i(u), F_{i+1}(\infty), \ldots, F_n(\infty)\right) \\
&\quad = C\left(1, \ldots, 1, F_i(u), 1, \ldots, 1\right) \\
&\quad = F_i(u), \qquad 0 \leqslant u \leqslant 1.
\end{aligned}
$$

□

2.4 Examples of Copulas

Gaussian copulas

The choice of (2.2.2) above as joint probability density function, see Figure 2.1, actually induces a particular dependence structure between the Gaussian random variables X and Y, and corresponding to the joint cumulative distribution function

$$
\begin{aligned}
\Phi_\Sigma(x,y) &:= \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) \\
&= \frac{1}{\sqrt{2\pi \det \Sigma}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{1}{2}\left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \Sigma^{-1} \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle\right) du dv,
\end{aligned}
$$

$x, y \in \mathbb{R}$. In case (X,Y) are normalized centered Gaussian random variables with unit variance, Σ is given by

$$
\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},
$$

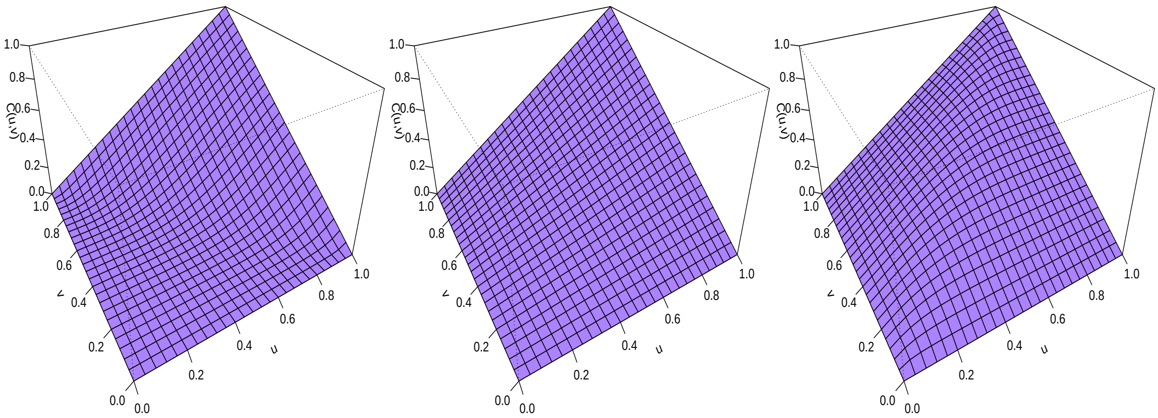
with correlation parameter $\rho \in (-1,1)$. Letting

$$F_X(x) := \mathbb{P}(X \leqslant x) \quad \text{and} \quad F_Y(y) := \mathbb{P}(Y \leqslant y),$$

denote the cumulative distribution functions of X and Y, the random variables $F_X(X)$ and $F_Y(Y)$ are known to be uniformly distributed on $[0,1]$, and $(F_X(X), F_Y(Y))$ is a $[0,1] \times [0,1]$-valued random variable with joint cumulative distribution function

$$\begin{aligned} C_\Sigma(u,v) &:= \mathbb{P}\big(F_X(X) \leqslant u \text{ and } F_Y(Y) \leqslant v\big) \\ &= \mathbb{P}\big(X \leqslant F_X^{-1}(u) \text{ and } Y \leqslant F_Y^{-1}(v)\big) \\ &= \Phi_\Sigma\big(F_X^{-1}(u), F_Y^{-1}(v)\big), \qquad 0 \leqslant u,v \leqslant 1. \end{aligned} \tag{2.4.1}$$

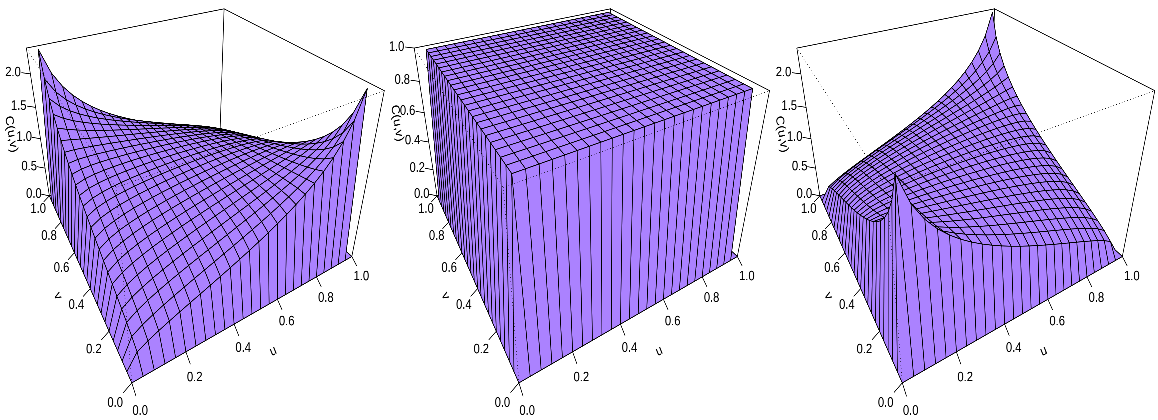
The function $C_\Sigma(u,v)$, which is the joint cumulative distribution function of a couple of uniformly distributed $[0,1]$-valued random variables, is called the *Gaussian copula* generated by the jointly Gaussian distribution of (X,Y) with covariance matrix Σ.



(a) $\rho = -0.9$. (b) $\rho = 0$. (c) $\rho = 0.9$.

Figure 2.3: Gaussian copula graphs for $\rho = -0.9$, $\rho = 0$, and $\rho = 0.9$.

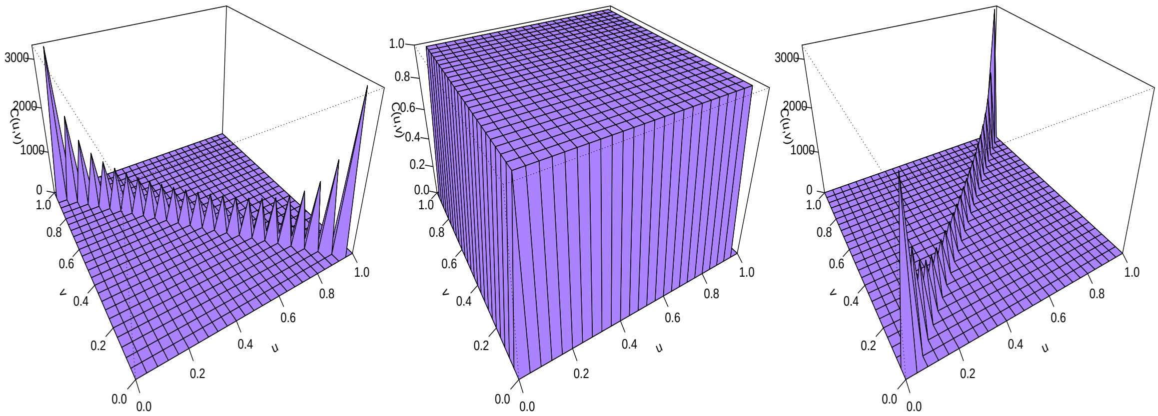
The graphs of Figures 2.3-(
a) and Figures 2.3-(
c) correspond to intermediate dependence levels given by Gaussian copulas, cf. (2.4.1).



(a) $\rho = -0.35$. (b) $\rho = 0$. (c) $\rho = 0.35$.

Figure 2.4: Different Gaussian copula *density* graphs for $\rho = -0.35$, $\rho = 0$ and $\rho = 0.35$.

Figures 2.4 and 2.5 present the corresponding copula density graphs.



(a) $\rho = -0.999$. (b) $\rho = 0$. (c) $\rho = 0.999$.

Figure 2.5: Different Gaussian copula *density* graphs for $\rho = -0.999$, $\rho = 0$ and $\rho = 0.999$.

Figure 2.4-(

a) represents a uniform (product) probability density function on the square $[0,1] \times [0,1]$, which corresponds to two independent uniformly distributed $[0,1]$-valued random variables U, V. Figure 2.4-(

c) shows the probability distribution of the fully correlated couple (U,U), which does not admit a probability density on the square $[0,1] \times [0,1]$.

The Gaussian copula $C_\Sigma(u,u)$ admits a probability density function on $[0,1] \times [0,1]$ given by

$$\begin{aligned}
c_\Sigma(u,v) &= \frac{\partial^2 C_\Sigma}{\partial u \partial v}(u,v) \\
&= \frac{\partial^2}{\partial u \partial v} \Phi_\Sigma\big(F_X^{-1}(u), F_Y^{-1}(v)\big) \\
&= \frac{\partial}{\partial u}\left(\frac{1}{F_Y'(F_Y^{-1}(v))} \frac{\partial \Phi_\Sigma}{\partial y}\big(F_X^{-1}(u), F_Y^{-1}(v)\big)\right) \\
&= \frac{\partial}{\partial u}\left(\frac{1}{f_Y(F_Y^{-1}(v))} \frac{\partial \Phi_\Sigma}{\partial y}\big(F_X^{-1}(u), F_Y^{-1}(v)\big)\right) \\
&= \frac{1}{f_X(F_X^{-1}(u)) f_Y(F_Y^{-1}(v))} \frac{\partial^2 \Phi_\Sigma}{\partial x \partial y}\big(F_X^{-1}(u), F_Y^{-1}(v)\big) \\
&= \frac{f_\Sigma\big(F_X^{-1}(u), F_Y^{-1}(v)\big)}{f_X(F_X^{-1}(u)) f_Y(F_Y^{-1}(v))},
\end{aligned}$$

hence the Gaussian copula $C_\Sigma(u,v)$ can be computed as

$$\begin{aligned}
C_\Sigma(u,v) &= \int_0^u \int_0^v c_\Sigma(a,b)\,dadb \\
&= \int_0^u \int_0^v \frac{f_\Sigma\big(F_X^{-1}(a), F_Y^{-1}(b)\big)}{f_X(F_X^{-1}(a)) f_Y(F_Y^{-1}(b))} dadb, \qquad 0 \leqslant u,v \leqslant 1.
\end{aligned}$$

The joint cumulative distribution function $F_{(X,Y)}(x,y)$ of (X,Y) can be recovered from Corollary 2.7 as

$$F_{(X,Y)}(x,y) = C_\Sigma(F_X(x), F_Y(y)), \qquad x,y \in \mathbb{R}. \tag{2.4.2}$$

from the Gaussian copula $C_\Sigma(x,y)$ and the respective cumulative distribution functions $F_X(x)$, $F_Y(y)$ of X and Y.

In that sense, the Gaussian copula $C_\Sigma(x,y)$ encodes the Gaussian dependence structure of the covariance matrix Σ. Moreover, the Gaussian copula $C_\Sigma(x,y)$ can be used to generate a joint distribution function $F^C_{(X,Y)}(x,y)$ by letting

$$F^C_{(X,Y)}(x,y) := C_\Sigma(F_X(x), F_Y(y)), \qquad x,y \in \mathbb{R}, \tag{2.4.3}$$

based on other, *possibly non-Gaussian* cumulative distribution functions $F_X(x)$, $F_Y(y)$ of two random variables X and Y. In this case we note that the marginals of the joint cumulative distribution function $F^C_{(X,Y)}(x,y)$ are $F_X(x)$ and $F_Y(y)$ because $C_\Sigma(x,y)$ has uniform marginals on $[0,1]$.

Gumbel copula

The Gumbel copula is given by

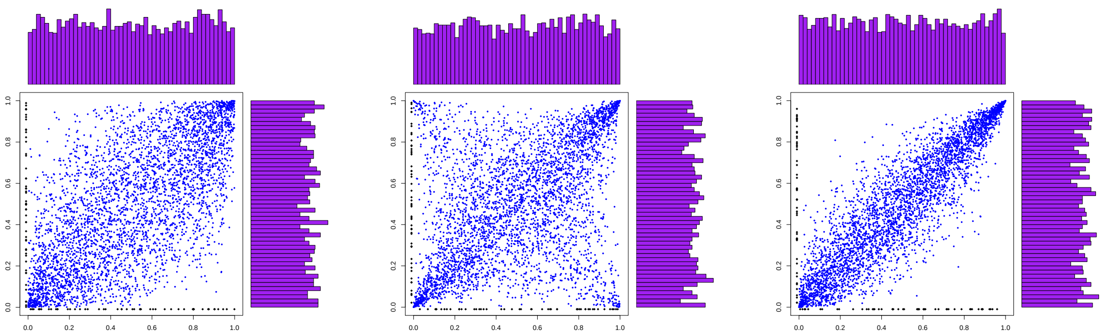
$$C(u,v) = \exp\left(-\left((-\log u)^\theta + (-\log v)^\theta\right)^{1/\theta}\right), \qquad 0 \leqslant u,v \leqslant 1,$$

with $\theta \geqslant 1$, and $C(u,v) = uv$ when $\theta = 1$.

Uniform marginals with given copulas

The following R code generates random samples according to the Gaussian, Student, and Gumbel copulas with uniform marginals, as illustrated in Figure 2.6.

```
install.packages("copula"); install.packages("gumbel")
library(copula);library(gumbel)
norm.cop <- normalCopula(0.35);norm.cop
persp(norm.cop, pCopula, n.grid = 51, xlab="u", ylab="v", zlab="C(u,v)", main="", sub="", col='lightblue')
persp(norm.cop, dCopula, n.grid = 51, xlab="u", ylab="v", zlab="c(u,v)", main="", sub="", col='lightblue')
norm <- rCopula(4000,normalCopula(0.7))
plot(norm[,1],norm[,2],cex=3,pch='.',col='blue')
stud <- rCopula(4000,tCopula(0.5,dim=2,df=1))
points(stud[,1],stud[,2],cex=3,pch='.',col='red')
gumb <- rgumbel(4000,4)
points(gumb[,1],gumb[,2],cex=3,pch='.',col='green')
```



(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

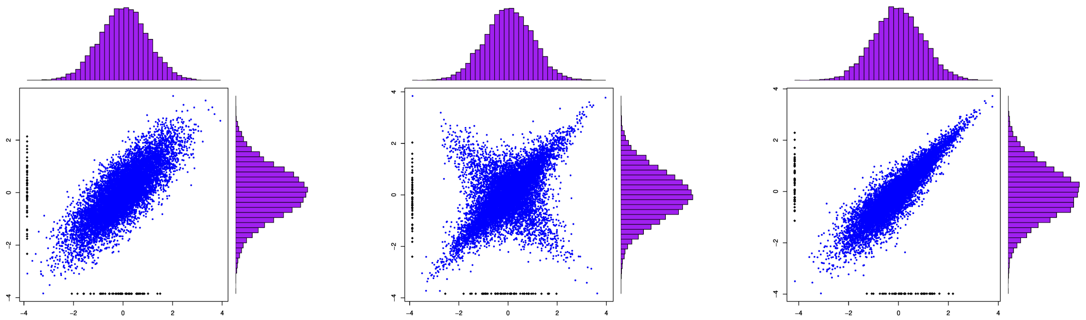
Figure 2.6: Samples with uniform marginals and given copulas.

The following R code is plotting the histograms of Figure 2.6.

```
joint_hist <- function(u) {x <- u[,1]; y <- u[,2]
xhist <- hist(x, breaks=40,plot=FALSE) ; yhist <- hist(y, breaks=40,plot=FALSE)
top <- max(c(xhist$counts, yhist$counts))
```

```
nf <- layout(matrix(c(2,0,1,3),2,2,byrow=TRUE), c(3,1), c(1,3), TRUE)
par(mar=c(3,3,1,1))
plot(x, y, xlab="", ylab="",col="blue",pch=19,cex=0.4)
points(x, -0.01+rep(min(y),length(x)), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(x),length(y)), y, xlab="", ylab="",col="black",pch=18,cex=0.8)
par(mar=c(0,3,1,1))
barplot(xhist$counts, axes=FALSE, ylim=c(0, top), space=0,col="purple")
par(mar=c(3,0,1,1))
barplot(yhist$counts, axes=FALSE, xlim=c(0, top), space=0, horiz=TRUE,col="purple")}
joint_hist(norm);joint_hist(stud);joint_hist(gumb)
```

Gaussian marginals with given copulas

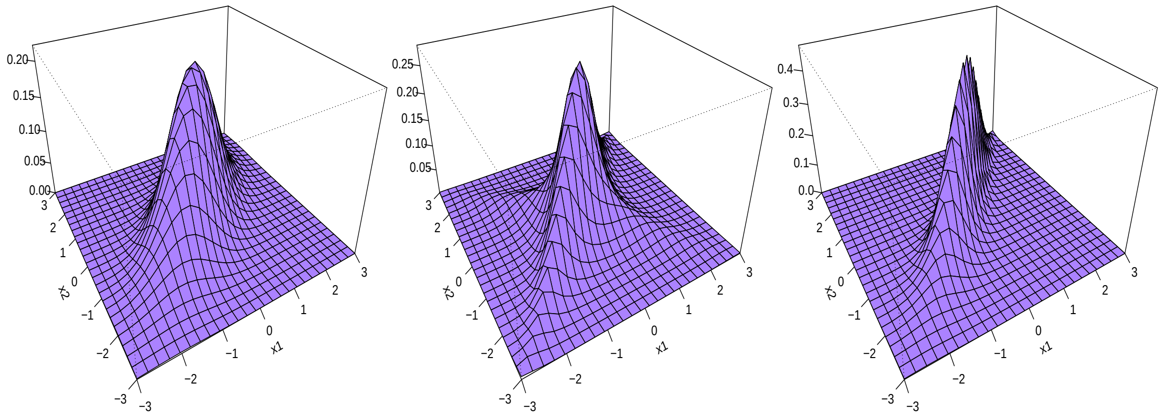


(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Figure 2.7: Samples with Gaussian marginals and given copulas.

The next R code generates random samples according to the Gaussian, Student, and Gumbel copulas with Gaussian marginals, as illustrated in Figure 2.7.

```
set.seed(100);N=10000
gaussMVD<-mvdc(normalCopula(0.8), margins=c("norm","norm"),
      paramMargins=list(list(mean=0,sd=1),list(mean=0,sd=1)))
norm <- rMvdc(N,gaussMVD)
studentMVD<-mvdc(tCopula(0.5,dim=2,df=1), margins=c("norm","norm"),
      paramMargins=list(list(mean=0,sd=1),list(mean=0,sd=1)))
stud <- rMvdc(N,studentMVD)
gumbelMVD<-mvdc(gumbelCopula(param=4, dim=2), margins=c("norm","norm"),
      paramMargins=list(list(mean=0,sd=1),list(mean=0,sd=1)))
gumb <- rMvdc(N,gumbelMVD)
plot(norm[,1],norm[,2],cex=3,pch='.',col='blue')
points(norm[,1], -0.01+rep(min(norm[,2]),N), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(norm[,1]),N), norm[,2], xlab="", ylab="",col="black",pch=18,cex=0.8)
plot(stud[,1],stud[,2],cex=3,pch='.',col='blue')
points(stud[,1], -0.01+rep(min(stud[,2]),N), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(stud[,1]),N), stud[,2], xlab="", ylab="",col="black",pch=18,cex=0.8)
plot(gumb[,1],gumb[,2],cex=3,pch='.',col='blue')
points(gumb[,1], -0.01+rep(min(gumb[,2]),N), xlab="", ylab="",col="black",pch=18,cex=0.8)
points(-0.01+rep(min(gumb[,1]),N), gumb[,2], xlab="", ylab="",col="black",pch=18,cex=0.8)
joint_hist(norm);joint_hist(stud);joint_hist(gumb)
```

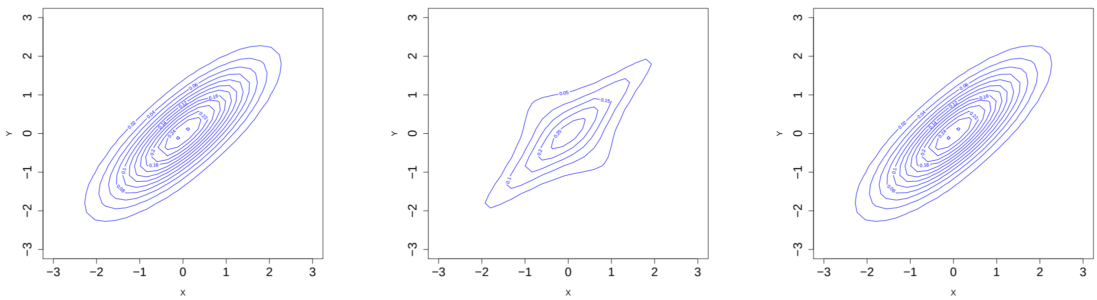


(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Figure 2.8: Joint densities with Gaussian marginals and given copulas.

The following R code is plotting joint densities with Gaussian marginals and given copulas, as illustrated in Figure 2.8.

```
persp(gaussMVD, dMvdc, xlim = c(-3,3), ylim=c(-3,3),col="lightblue")
persp(studentMVD, dMvdc, xlim = c(-3,3), ylim=c(-3,3),col="lightblue")
persp(gumbelMVD, dMvdc, xlim = c(-3,3), ylim=c(-3,3),col="lightblue")
```



(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Figure 2.9: Joint density contour plots with Gaussian marginals and given copulas.

The following R code generates countour plots with Gaussian marginals and given copulas, as illustrated in Figure 2.9.

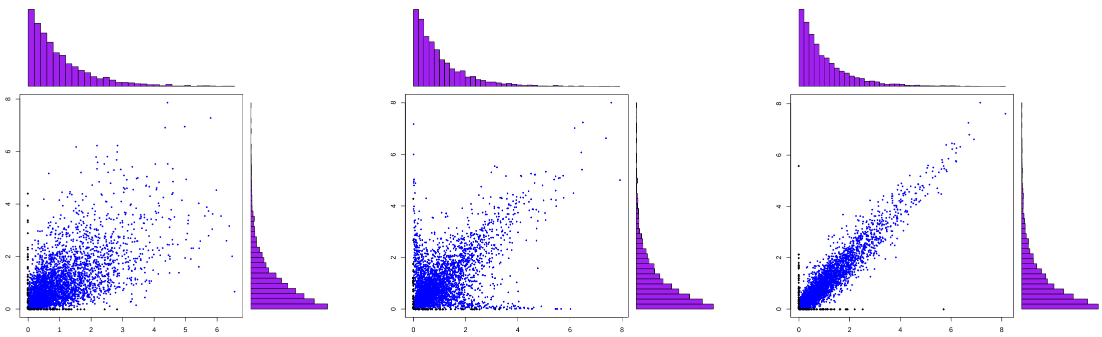
```
contour(gaussMVD,dMvdc,xlim=c(-3,3),ylim=c(-3,3),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(studentMVD,dMvdc,xlim=c(-3,3),ylim=c(-3,3),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(gaussMVD,dMvdc,xlim=c(-3,3),ylim=c(-3,3),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
```

Exponential marginals with given copulas

The following R code generates random samples with exponential marginals according to the Gaussian, Student, and Gumbel copulas as illustrated in Figure 2.10.

```
library(copula);set.seed(100);N=4000
gaussMVD<-mvdc(normalCopula(0.7), margins=c("exp","exp"), paramMargins=list(list(rate=1),list(rate=1)))
norm <- rMvdc(N,gaussMVD)
studentMVD<-mvdc(tCopula(0.5,dim=2,df=1), margins=c("exp","exp"),
     paramMargins=list(list(rate=1),list(rate=1)))
stud <- rMvdc(N,studentMVD)
gumbelMVD<-mvdc(gumbelCopula(param=4, dim=2), margins=c("exp","exp"),
     paramMargins=list(list(rate=1),list(rate=1)))
```

```
gumb <- rMvdc(N,gumbelMVD)
plot(norm[,1],norm[,2],cex=3,pch='.',col='blue')
plot(stud[,1],stud[,2],cex=3,pch='.',col='blue')
plot(gumb[,1],gumb[,2],cex=3,pch='.',col='blue')
persp(gaussMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
persp(studentMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
persp(gumbelMVD, dMvdc, xlim = c(0,1), ylim=c(0,1), col='lightblue')
contour(gaussMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(studentMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
contour(gaussMVD,dMvdc,xlim=c(0,1),ylim=c(0,1),nlevels=10,xlab="X",ylab="Y",cex.axis=1.5)
joint_hist(norm);joint_hist(stud);joint_hist(gumb)
```



(a) Gaussian copula. (b) Student copula. (c) Gumbel copula.

Figure 2.10: Samples with exponential marginals and given copulas.

Exercises

Exercise 2.1 Copulas. In what follows, U denotes a uniformly distributed $[0,1]$-valued random variable.

a) To which couple (U,V) of uniformly distributed $[0,1]$-valued random variables does the copula function

$$C_M(u,v) = \min(u,v), \qquad 0 \leqslant u,v \leqslant 1,$$

correspond?

b) Show that the function

$$C_m(u,v) := (u+v-1)^+, \qquad 0 \leqslant u,v \leqslant 1,$$

is the copula on $[0,1]\times[0,1]$ corresponding to $(U,V) = (U,1-U)$.

c) Show that for any copula function $C(u,v)$ on $[0,1]\times[0,1]$ we have

$$C(u,v) \leqslant C_M(u,v), \qquad 0 \leqslant u,v \leqslant 1. \tag{2.4.4}$$

d) Show that for any copula function $C(u,v)$ on $[0,1]\times[0,1]$ we also have

$$C_m(u,v) \leqslant C(u,v), \qquad 0 \leqslant u,v \leqslant 1. \tag{2.4.5}$$

Hint: For fixed $v \in [0,1]$, let $h(u) := C(u,v) - (u+v-1)$ and show that $h(1) = 0$ and $h'(u) \leqslant 0$.

Exercise 2.2 Consider two Bernoulli random variables X and Y, with $p_X = \mathbb{P}(X=1)$, $p_Y = \mathbb{P}(Y=1)$, correlation coefficient $\rho \in [-1,1]$, and

$$\begin{cases} \mathbb{P}(X=1 \text{ and } Y=1) = p_X p_Y + \rho\sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ \\ \mathbb{P}(X=0 \text{ and } Y=1) = (1-p_X)p_Y - \rho\sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ \\ \mathbb{P}(X=1 \text{ and } Y=0) = p_X(1-p_Y) - \rho\sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ \\ \mathbb{P}(X=0 \text{ and } Y=0) = (1-p_X)(1-p_Y) + \rho\sqrt{p_X p_Y (1-p_X)(1-p_Y)}. \end{cases}$$

a) Is it possible to have $\rho = 1$ *without* having $p_X = p_Y$ and

$$\begin{cases} \mathbb{P}(X=1 \text{ and } Y=1) = p_X = p_Y, \\ \\ \mathbb{P}(X=0 \text{ and } Y=1) = 0, \\ \\ \mathbb{P}(X=1 \text{ and } Y=0) = 0, \\ \\ \mathbb{P}(X=0 \text{ and } Y=0) = 1-p_X = 1-p_Y\ ? \end{cases}$$

b) Similarly, is it possible to have $\rho = 1$ *without* having $p_X = 1 - p_Y$ and

$$\begin{cases} \mathbb{P}(X=1 \text{ and } Y=1) = 0, \\ \\ \mathbb{P}(X=0 \text{ and } Y=1) = p_Y, \\ \\ \mathbb{P}(X=1 \text{ and } Y=0) = p_X, \\ \\ \mathbb{P}(X=0 \text{ and } Y=0) = 0\ ? \end{cases}$$

Exercise 2.3 Exponential copulas. Consider the random vector (X,Y) of nonnegative random variables, whose joint distribution is given by the survival function

$$\mathbb{P}(X \geqslant x \text{ and } Y \geqslant y) := \mathrm{e}^{-\lambda x - \mu y - \nu \operatorname{Max}(x,y)}, \qquad x,y \in \mathbb{R}_+,$$

where $\lambda, \mu, \nu > 0$.

a) Find the marginal distributions of X and Y.
b) Find the joint cumulative distribution function $F(x,y) := \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y)$ of (X,Y).
c) Construct an "exponential copula" based on the joint cumulative distribution function of (X,Y).

Exercise 2.4 Gumbel bivariate logistic distribution. Consider the random vector (X,Y) of non-negative random variables, whose joint distribution is given by the joint cumulative distribution function (CDF)

$$F_{(X,Y)}(x,y) := \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) := \frac{1}{1+\mathrm{e}^{-x}+\mathrm{e}^{-y}}, \qquad x,y \in \mathbb{R}.$$

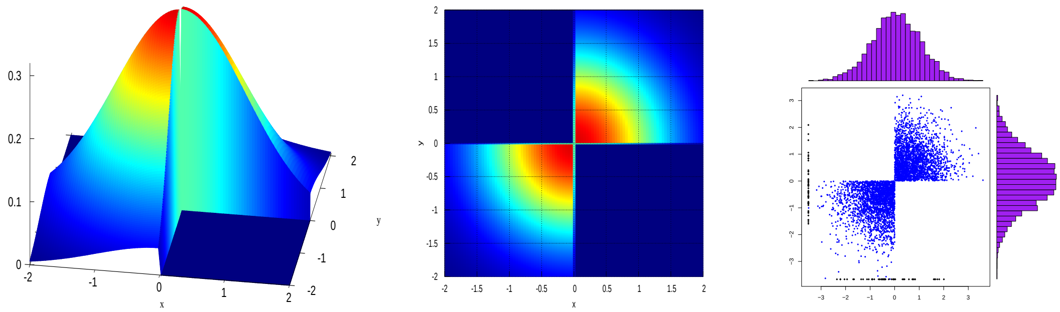
a) Find the marginal distributions of X and Y.

b) Construct the copula based on the joint CDF of (X,Y).

Exercise 2.5 Consider the random vector (X,Y) with the joint probability density function

$$\widetilde{f}(x,y) := \frac{1}{\pi\sigma\eta}\mathbb{1}_{\mathbb{R}^2_-}(x,y)\,\mathrm{e}^{-x^2/(2\sigma^2)-y^2/(2\eta^2)} + \frac{1}{\pi\sigma\eta}\mathbb{1}_{\mathbb{R}^2_+}(x,y)\,\mathrm{e}^{-x^2/(2\sigma^2)-y^2/(2\eta^2)},$$

plotted as a heat map in Figure 2.11b.



(a) 3D view. (b) Heat map. (c) Samples.

Figure 2.11: Truncated two-dimensional Gaussian density.

```
library(MASS)
Sigma <- matrix(c(1,0,0,1),2,2);N=10000
u<-mvrnorm(N,rep(0,2),Sigma);j=1
for (i in 1:N){
if (u[i,1]>0 && u[i,2]>0) {j<-j+1;}
if (u[i,1]<0 && u[i,2]<0) {j<-j+1;}}
v<-matrix(nrow=j-1, ncol=2);j=1
for (i in 1:N){
if (u[i,1]>0 && u[i,2]>0) {v[j,]=u[i,];j<-j+1;}
if (u[i,1]<0 && u[i,2]<0) {v[j,]=u[i,];j<-j+1;}}
joint_hist(v) # Function defined the previous section
```

a) Show that (X,Y) has the Gaussian marginals $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(0,\eta^2)$.
b) Does the couple (X,Y) have the bivariate Gaussian distribution with probability density function $f_\Sigma(x,y)$, where Σ is the covariance matrix (2.2.1) of (X,Y)?
c) Show that the random variable $X+Y$ is not Gaussian (take $\sigma=\eta=1$ for simplicity).
d) Show that under the rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

of angle $\theta\in[0,2\pi]$ the random vector $(X\cos\theta - Y\sin\theta, X\sin\theta + Y\cos\theta)$ can have an arbitrary covariance depending on the value of $\theta\in[0,2\pi]$.

Exercise 2.6 Let τ_1, τ_2 and τ denote three independent exponentially distributed random times with respective parameters $\lambda_1,\lambda_2,\lambda>0$. Consider two firms with respective default times $\tau_1\wedge\tau=\min(\tau_1,\tau)$ and $\tau_2\wedge\tau=\min(\tau_2,\tau)$, where τ represents the time of a macro-economic shock.
a) Find the tail (or survival) distribution functions of $\tau_1\wedge\tau$ and $\tau_2\wedge\tau$.
b) Compute the joint survival probability

$$\mathbb{P}(\tau_1\wedge\tau > s \text{ and } \tau_2\wedge\tau > t), \qquad s,t\geqslant 0.$$

Hint: Use the relation

$$\mathrm{Max}(s,t) = s+t-\min(s,t), \qquad s,t\geqslant 0.$$

c) Compute the joint cumulative distribution function

$$\mathbb{P}(\tau_1 \wedge \tau \leqslant s \text{ and } \tau_2 \wedge \tau \leqslant t), \qquad s,t \geqslant 0.$$

d) Compute the resulting copula

$$C(u,v) := F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)), \qquad 0 \leqslant u,v \leqslant 1.$$

e) Compute the resulting copula density function $\dfrac{\partial^2 C}{\partial u \partial v}(u,v)$, $u,v \in [0,1]$.

3. Structural Approach to Credit Risk

Credit risk can be defined as the risk of default on the payment of a debt. In this chapter, credit risk is modeled using the value of a firm's assets, *i.e.* the default event is said to occur when the value of assets drops below a certain pre-defined level. This is in contrast to the reduced form approach to credit risk of Chapter 1, in which stochastic processes are used to model default probabilities. We also consider the modeling of correlation and dependence between multiple default times.

3.1 Merton Model

The Merton, 1974 credit risk model reframes corporate debt as an option on a firm's underlying value. Precisely the value S_t of a firm's asset is modeled by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

under the historical (or physical) measure $\mathbb{P}$. Recall that, using the standard Brownian motion

$$\widehat{B}_t = \frac{\mu - r}{\sigma} t + B_t, \qquad t \geqslant 0,$$

under the risk-neutral probability measure $\mathbb{P}^*$, the process $(S_t)_{t\in\mathbb{R}_+}$ is modeled as

$$dS_t = rS_t dt + \sigma S_t d\widehat{B}_t.$$

The company's debt is represented by an amount $K > 0$ in bonds to be paid at maturity T, cf. § 4.1 of Grasselli and Hurd, 2010.

Default occurs if $S_T < K$ with probability $\mathbb{P}(S_T < K)$, the bond holder will receive the recovery value S_T. Otherwise, if $S_T \geqslant K$ the bond holder receives K and the equity holder is entitled to receive $S_T - K$, which can be represented as $(S_T - K)^+$ in general.

The discounted expected cash flow (or dividend) $\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathscr{F}_t]$ received by the equity holder can be estimated at time $t \in [0,T]$ as the price of a European call option, from the Black-Scholes formula

$$\begin{aligned}\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathscr{F}_t] &= S_t\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right)\\ &\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right), \quad 0 \leqslant t \leqslant T.\end{aligned}$$

Proposition 3.1 The default probability $\mathbb{P}(S_T < K \mid \mathscr{F}_t)$ can be computed from the lognormal distribution of S_T as

$$\mathbb{P}(S_T < K \mid \mathscr{F}_t) = \Phi\big(-d_-^{\mu}\big),$$

where Φ is the cumulative distribution function of the standard normal distribution, and

$$d_-^{\mu} := \frac{(\mu-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

Proof. The default probability $\mathbb{P}(S_T < K \mid \mathscr{F}_t)$ can be computed from the lognormal distribution of S_T as

$$\begin{aligned}\mathbb{P}(S_T < K \mid \mathscr{F}_t) &= \mathbb{P}\big(S_0\,\mathrm{e}^{\sigma B_T+(\mu-\sigma^2/2)T} < K \mid \mathscr{F}_t\big)\\ &= \mathbb{P}\left(B_T < \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)T+\log\frac{K}{S_0}\right) \,\Big|\, \mathscr{F}_t\right)\\ &= \mathbb{P}\left(B_T - B_t + y < \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)T+\log\frac{K}{S_0}\right)\right)_{|y=B_t}\\ &= \mathbb{P}\left(B_T - B_t + \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)t+\log\frac{K}{x}\right)\right.\\ &\qquad\qquad \left.< \frac{1}{\sigma}\left(-\left(\mu-\frac{\sigma^2}{2}\right)T+\log\frac{K}{S_0}\right)\right)_{|x=S_t}\\ &= \frac{1}{\sqrt{2(T-t)\pi}}\int_{-\infty}^{(-(\mu-\sigma^2/2)(T-t)+\log(K/S_t))/\sigma}\mathrm{e}^{-x^2/(2(T-t))}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(-(\mu-\sigma^2/2)(T-t)+\log(K/S_t))/(\sigma\sqrt{T-t})}\mathrm{e}^{-x^2/2}dx\\ &= 1-\Phi\left(\frac{(\mu-\sigma^2/2))(T-t)+\log(S_t/K))}{\sigma\sqrt{T-t}}\right)\\ &= 1-\Phi(d_-^{\mu})\\ &= \Phi(-d_-^{\mu}).\end{aligned}$$

□

Note that under the risk-neutral probability measure $\mathbb{P}^*$ we have, replacing μ with r,

$$\mathbb{P}^*(S_T < K \mid \mathscr{F}_t) = \Phi(-d_-^{r}),$$

with

$$d_-^r = \frac{(r - \sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}},$$

which implies the relation

$$d_-^r = d_-^\mu - \frac{\mu - r}{\sigma}\sqrt{T-t},$$

or, denoting by Φ^{-1} the inverse function of Φ,

$$\Phi^{-1}(\mathbb{P}(S_T < K \mid \mathscr{F}_t)) = -\frac{\mu - r}{\sigma}\sqrt{T-t} + \Phi^{-1}(\mathbb{P}^*(S_T < K \mid \mathscr{F}_t)).$$

If the level of the firm's assets falls below the level K at time T, default may have occurred at a random time τ such that

$$\mathbb{P}(\tau < T \mid \mathscr{F}_t) = \mathbb{P}(S_T < K \mid \mathscr{F}_t).$$

In this case, the result of Proposition 3.1 can be reinterpreted in the next corollary.

Corollary 3.2 The conditional distribution of the default time τ is given by

$$\mathbb{P}(\tau < T \mid \mathscr{F}_t) = \mathbb{P}(S_T < K \mid \mathscr{F}_t) = \Phi\left(-\frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right), \quad (3.1.1)$$

$0 \leqslant t \leqslant T$.

We also have

$$\begin{aligned}
\mathbb{P}(\tau < T \mid \mathscr{F}_t) &= \mathbb{P}(S_T < K \mid \mathscr{F}_t) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}^*(S_T < K \mid \mathscr{F}_t)) - \frac{\mu - r}{\sigma}\sqrt{T-t}\right) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}^*(\tau < T \mid \mathscr{F}_t)) - \frac{\mu - r}{\sigma}\sqrt{T-t}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}^*(\tau < T \mid \mathscr{F}_t) &= \mathbb{P}^*(S_T < K \mid \mathscr{F}_t) \\
&= \Phi\left(-\frac{(r - \sigma^2/2))(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}(S_T < K \mid \mathscr{F}_t)) + \frac{\mu - r}{\sigma}\sqrt{T-t}\right) \\
&= \Phi\left(\Phi^{-1}(\mathbb{P}(\tau < T \mid \mathscr{F}_t)) + \frac{\mu - r}{\sigma}\sqrt{T-t}\right). \qquad (3.1.2)
\end{aligned}$$

Note that when $\mu < r$, we have

$$\mathbb{P}(\tau < T \mid \mathscr{F}_t) > \mathbb{P}^*(\tau < T \mid \mathscr{F}_t),$$

whereas when $\mu > r$ we get

$$\mathbb{P}(\tau < T \mid \mathscr{F}_t) < \mathbb{P}^*(\tau < T \mid \mathscr{F}_t),$$

as illustrated in the next Figure 3.1.

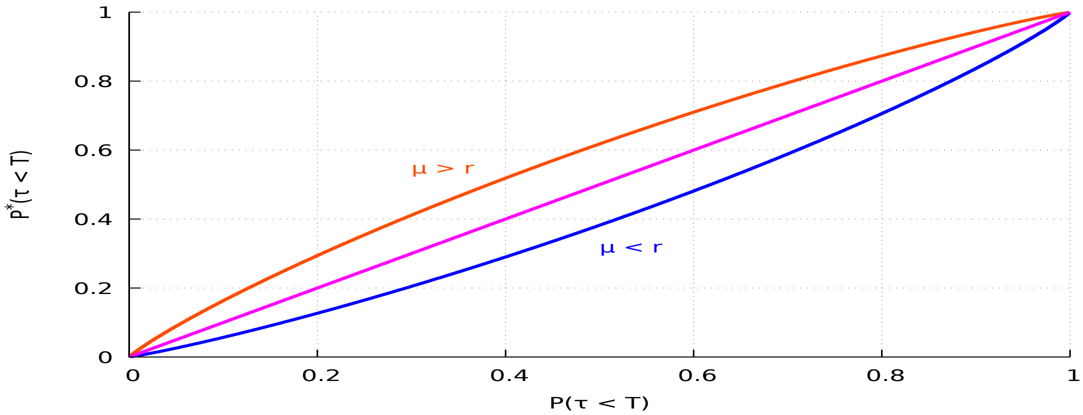


Figure 3.1: Graph of the function $x \longmapsto \Phi\big(\Phi^{-1}(x) - (\mu - r)\sqrt{T}/\sigma\big)$ for $\mu > r$, $\mu = r$, and $\mu < r$.

3.2 Default Bonds

In the following proposition we price at time $t \in [0,T]$ the amount $\min(S_T, K)$ received by the bond holder (or junior creditor) at maturity, based on the recovery value S_T when $S_T < K$. This price can interpreted at the price $P(t,T)$ at time $t \in [0,T]$ of a default bond with face value \$1, maturity T and recovery value $\min(S_T/K, 1)$.

Proposition 3.3 The amount received by the bond holder (or junior creditor) at maturity is priced at time $t \in [0,T]$ as

$$e^{-(T-t)r}\,\mathbb{E}^*\big[\min(S_T,K) \mid \mathscr{F}_t\big] = K\,e^{-(T-t)r}\Phi(d_-^r) - S_t\Phi(-d_+^r), \quad 0 \leqslant t \leqslant T.$$

Proof. Using the Black-Scholes put option pricing formula and the identity

$$\min(x,K) = K - (K-x)^+, \qquad x \in \mathbb{R},$$

we have

$$\begin{aligned} e^{-(T-t)r}\,\mathbb{E}^*\big[\min(S_T,K) \mid \mathscr{F}_t\big] &= e^{-(T-t)r}\,\mathbb{E}^*\big[K - (K-S_T)^+ \mid \mathscr{F}_t\big] \\ &= e^{-(T-t)r}K - e^{-(T-t)r}\,\mathbb{E}^*\big[(K-S_T)^+ \mid \mathscr{F}_t\big] \\ &= e^{-(T-t)r}K - \big(S_t\Phi(-d_+^r) - K\,e^{-(T-t)r}\Phi(-d_-^r)\big) \\ &= K\,e^{-(T-t)r}\Phi(d_-^r) - S_t\Phi(-d_+^r). \end{aligned}$$

□

Writing

$$\begin{aligned} P(t,T) &= e^{-(T-t)y_{t,T}} \\ &= \frac{1}{K}e^{-(T-t)r}\,\mathbb{E}^*[\min(S_T,K) \mid \mathscr{F}_t] \\ &= e^{-(T-t)r}\Phi(d_-^r) - \frac{S_t}{K}\Phi(-d_+^r), \end{aligned}$$

gives the default bond yield

$$y_{t,T} = -\frac{1}{T-t}\log(P(t,T))$$

$$
\begin{aligned}
&= -\frac{1}{T-t}\log\left(\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\min\left(1,\frac{S_T}{K}\right)\Big|\mathscr{F}_t\right]\right)\\
&= r-\frac{1}{T-t}\log\left(\mathbb{E}^*\left[\min\left(1,\frac{S_T}{K}\right)\Big|\mathscr{F}_t\right]\right)\\
&= r-\frac{1}{T-t}\log\left(\frac{1}{K}\mathbb{E}^*\left[\min(K,S_T)\Big|\mathscr{F}_t\right]\right)\\
&= r-\frac{1}{T-t}\log\left(\Phi(d_-^r)-\frac{S_t}{K}\mathrm{e}^{(T-t)r}\Phi(-d_+^r)\right),
\end{aligned}
$$

which is usually higher than the risk-free yield r.

3.3 Black-Cox Model

In the Black and Cox, 1976 model the firm has to maintain an account balance above the level K throughout time, therefore default occurs at the first time the process S_t hits the level K, cf. § 4.2 of Grasselli and Hurd, 2010. The default time τ_K is therefore the first hitting time

$$
\tau_K := \inf\left\{t \geqslant 0 \ : \ S_t := S_0\,\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t} \leqslant K\right\},
$$

of the level K by

$$
(S_t)_{t\in\mathbb{R}_+} = \left(S_0\,\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t}\right)_{t\in\mathbb{R}_+},
$$

after starting from $S_0 > K$.

Proposition 3.4 The probability distribution function of the default time τ_K is given by

$$
\mathbb{P}(\tau_K \leqslant T) = \mathbb{P}(S_T \leqslant K) + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2}\Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right),
$$

with $S_0 \geqslant K$.

Proof. By *e.g.* Corollary 7.2.2 and pages 297-299 of Shreve, 2004, or from Relation (10.13) in Privault, 2022, we have

$$
\begin{aligned}
\mathbb{P}(\tau_K \leqslant T) &= \mathbb{P}\left(\min_{t\in[0,T]} S_t \leqslant K\right)\\
&= \mathbb{P}\left(\min_{t\in[0,T]} \mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t} \leqslant \frac{K}{S_0}\right)\\
&= \mathbb{P}\left(\min_{t\in[0,T]}\left(B_t+\frac{(\mu-\sigma^2/2)t}{\sigma}\right) \leqslant \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)\right)\\
&= \Phi\left(\frac{\log(K/S_0)-(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right)\\
&\quad + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2}\Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \qquad (3.3.1)\\
&= \mathbb{P}(S_T \leqslant K) + \left(\frac{S_0}{K}\right)^{1-2\mu/\sigma^2}\Phi\left(\frac{\log(K/S_0)+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}\right),
\end{aligned}
$$

with $S_0 \geqslant K$. □

The cash flow

$$(S_T - K)^+ \mathbb{1}_{\{\tau_K > T\}} = (S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{t\in[0,T]} S_t > K \right\}}$$

received at maturity T by the equity holder can be priced at time $t \in [0,T]$ as a down-and-out barrier call option with strike price K and barrier level K is priced in the next proposition, in which Bl_c denotes the Black-Scholes call pricing formula.

Proposition 3.5 We have

$$e^{-(T-t)r} \mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0\leqslant t\leqslant T} S_t > K \right\}} \,\middle|\, \mathscr{F}_t \right] = \mathbb{1}_{\left\{ \min\limits_{t\in[0,T]} S_t > B \right\}} g(t, S_t),$$

$t \in [0,T]$, where

$$g(t, S_t) = \mathrm{Bl}_c(S_t, K, r, T-t, \sigma) - S_t \left(\frac{K}{S_t} \right)^{2r/\sigma^2} \mathrm{Bl}_c(K/S_t, 1, r, T-t, \sigma),$$

$0 \leqslant t \leqslant T$.

Proof. By *e.g.* Relation (11.10) and Exercise 11.1 in Privault, 2022, we have

$$\mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0\leqslant t\leqslant T} S_t > K \right\}} \,\middle|\, \mathscr{F}_t \right] = \mathbb{1}_{\left\{ \min\limits_{t\in[0,T]} S_t > B \right\}} g(t, S_t),$$

$t \in [0,T]$, where

$$\begin{aligned}
& g(t, S_t) \\
= \; & S_t \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K} \right) \right) - e^{-(T-t)r} K \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K} \right) \right) \\
& -K \left(\frac{K}{S_t} \right)^{2r/\sigma^2} \Phi\left(\delta_+^{T-t}\left(\frac{K}{S_t} \right) \right) + e^{-(T-t)r} K \left(\frac{S_t}{K} \right)^{1-2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{K}{S_t} \right) \right) \\
= \; & \mathrm{Bl}_c(S_t, K, r, T-t, \sigma) \\
& -K \left(\frac{S_t}{K} \right)^{-2r/\sigma^2} \Phi\left(\delta_+^{T-t}\left(\frac{K}{S_t} \right) \right) + e^{-(T-t)r} S_t \left(\frac{S_t}{K} \right)^{-2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{K}{S_t} \right) \right) \\
= \; & \mathrm{Bl}_c(S_t, K, r, T-t, \sigma) - S_t \left(\frac{K}{S_t} \right)^{2r/\sigma^2} \mathrm{Bl}_c(K/S_t, 1, r, T-t, \sigma),
\end{aligned}$$

$0 \leqslant t \leqslant T$. $\square$

For $t \geqslant 0$, taking now

$$\tau_K := \inf\left\{ u \in [t, \infty) \ : \ S_u := S_0 e^{\sigma B_u + (\mu - \sigma^2/2)u} \leqslant K \right\},$$

the recovery value received by the bond holder at time $\min(\tau_K, T)$ is K, and it can be priced as in the next proposition.

Proposition 3.6 After discounting from time $\min(\tau_K, T)$ to time $t \in [0,T]$, we have

$$\mathbb{E}^*\left[K\mathrm{e}^{-(\min(\tau_K,T)-t)r} \mid \mathscr{F}_t\right]$$
$$= K\mathbb{1}_{\{\tau_K \geqslant t\}} \int_t^T \mathrm{e}^{-(u-t)r} d\mathbb{P}^*(\tau_K \leqslant u \mid \mathscr{F}_t) + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*(\tau_K > T \mid \mathscr{F}_t).$$

Proof. We have

$$\begin{aligned}
&\mathbb{E}^*\left[K\mathrm{e}^{-(\min(\tau_K,T)-t)r} \mid \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[K\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} + K\mathrm{e}^{-(T-t)r}\mathbb{1}_{\{\tau_K>T\}} \mid \mathscr{F}_t\right] \\
&= K\mathbb{E}^*\left[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} \mid \mathscr{F}_t\right] + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*(\tau_K > T \mid \mathscr{F}_t) \\
&= K\mathbb{1}_{\{\tau_K\geqslant t\}}\mathbb{E}^*\left[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{t\leqslant\tau_K\leqslant T\}} \mid \mathscr{F}_t\right] + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*(\tau_K > T \mid \mathscr{F}_t) \\
&= K\mathbb{1}_{\{\tau_K\geqslant t\}} \int_t^T \mathrm{e}^{-(u-t)r} d\mathbb{P}(\tau_K \leqslant u \mid \mathscr{F}_t) + K\mathrm{e}^{-(T-t)r}\mathbb{P}^*(\tau_K > T \mid \mathscr{F}_t),
\end{aligned}$$

$0 \leqslant t \leqslant T$. □

The above probabilities $\mathbb{P}^*(\tau_K \leqslant u \mid \mathscr{F}_t)$ and $\mathbb{P}^*(\tau_K > T \mid \mathscr{F}_t) = 1 - \mathbb{P}^*(\tau_K \leqslant T \mid \mathscr{F}_t)$ can be computed from (3.3.1) as

$$\begin{aligned}
\mathbb{P}^*(\tau_K \leqslant u \mid \mathscr{F}_t) &= \Phi\left(\frac{\log(K/S_t) - (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right) \\
&\quad + \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2} \Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right) \\
&= \mathbb{P}(S_u \leqslant K \mid \mathscr{F}_t) + \left(\frac{S_t}{K}\right)^{1-2r/\sigma^2} \Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right),
\end{aligned}$$

with $S_t \geqslant K$ and $u > t$, from which the probability density function of the hitting time τ_K can be estimated by differentiation with respect to $u > t$. Note also that we have

$$\begin{aligned}
\mathbb{P}^*(\tau_K < \infty \mid \mathscr{F}_t) &= \lim_{u\to\infty} \mathbb{P}^*(\tau_K \leqslant u \mid \mathscr{F}_t) \\
&= \begin{cases} \left(\dfrac{K}{S_t}\right)^{-1+2r/\sigma^2} & \text{if } r > \sigma^2/2 \\ 1 & \text{if } r \leqslant \sigma^2/2. \end{cases}
\end{aligned}$$

3.4 Correlated Default Times

In order to model correlated default and possible "domino effects", one can regard two given default times τ_1 and τ_2 are correlated random variables.

Namely, given τ_1 and τ_2 two default times we can consider the correlation

$$\rho = \frac{\mathrm{Cov}(\tau_1, \tau_2)}{\sqrt{\mathrm{Var}[\tau_1]\,\mathrm{Var}[\tau_2]}} \in [-1, 1].$$

When trying to build a dependence structure for the default times τ_1 and τ_2, the idea of D. Li, 2000 is to use the normalized Gaussian copula $C_\Sigma(x,y)$, with

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

with correlation parameter $\rho \in [-1,1]$, and to model the joint default probability $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$ as

$$\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) := C_\Sigma \left(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T)\right),$$

where C_Σ is given by (2.4.1). Given two default events $A = \{\tau_1 \leqslant T\}$ and $B = \{\tau_2 \leqslant T\}$ with probabilities

$$\mathbb{P}(\tau_1 \leqslant T) = 1 - \exp\left(-\int_0^T \lambda_1(s)ds\right) \text{ and } \mathbb{P}(\tau_2 \leqslant T) = 1 - \exp\left(-\int_0^T \lambda_2(s)ds\right)$$

we can also define the default correlation $\rho^D \in [-1,1]$ as

$$\begin{aligned} \rho^D &= \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)}{\sqrt{\mathbb{P}(A)(1-\mathbb{P}(A))}\sqrt{\mathbb{P}(B)(1-\mathbb{P}(B))}} \\ &= \frac{C_\Sigma(\mathbb{P}(\tau_1 \leqslant T), \mathbb{P}(\tau_2 \leqslant T)) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1-\mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1-\mathbb{P}(\tau_2 \leqslant T))}}. \end{aligned} \tag{3.4.1}$$

When the default probabilities are specified in the Merton model of credit risk as

$$\begin{aligned} \mathbb{P}(\tau_i \leqslant T) &= \mathbb{P}(S_T < K) \\ &= \mathbb{P}\left(\mathrm{e}^{\sigma_i B_T + (\mu_i - \sigma_i^2/2)T} < \frac{K}{S_0}\right) \\ &= \mathbb{P}\left(B_T \leqslant -\frac{(\mu_i - \sigma_i^2/2)T}{\sigma_i} + \frac{1}{\sigma_i}\log\frac{K}{S_0}\right) \\ &= \Phi\left(\frac{\log(K/S_0) - (\mu_i - \sigma_i^2/2)T}{\sigma_i\sqrt{T}}\right), \qquad i = 1,2, \end{aligned}$$

where

$$\left(A_t^i\right)_{t\in\mathbb{R}_+} := \left(S_0\,\mathrm{e}^{\sigma_i B_t + (\mu - \sigma_i^2/2)t}\right)_{t\in\mathbb{R}_+}, \qquad i = 1,2,$$

the default correlation ρ^D becomes

$$\begin{aligned} \rho^D &= \frac{\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1-\mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1-\mathbb{P}(\tau_2 \leqslant T))}} \\ &= \frac{\Phi_\Sigma\left(\frac{\log(S_0/K)+(\mu_1-\sigma_1^2/2)T}{\sigma_1\sqrt{T}}, \frac{\log(S_0/K)+(\mu_2-\sigma_2^2/2)T}{\sigma_2\sqrt{T}}\right) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1-\mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1-\mathbb{P}(\tau_2 \leqslant T))}}. \end{aligned}$$

In D. Li, 2000 it was suggested to use a single *average correlation* estimate, see (8.1) page 82 of the Credit Metrics™ Technical Document Gupton, Finger, and Bhatia, 1997, and also the Appendix F therein.

It is worth noting that the outcomes of this methodology have been discussed in a number of magazine articles in recent years, to name a few:

1. "Recipe for disaster: the formula that killed Wall Street", *Wired Magazine*, by F. Salmon, 2009;

2. "The formula that felled Wall Street", *Financial Times Magazine*, by S. Jones, 2009;

3. "Formula from hell", *Forbes.com*, by S. S. Lee, 2009,

see also **here**.

On the other hand, a more proper definition of the default correlation ρ^D should be

$$\rho^D := \frac{\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T) - \mathbb{P}(\tau_1 \leqslant T)\mathbb{P}(\tau_2 \leqslant T)}{\sqrt{\mathbb{P}(\tau_1 \leqslant T)(1-\mathbb{P}(\tau_1 \leqslant T))}\sqrt{\mathbb{P}(\tau_2 \leqslant T)(1-\mathbb{P}(\tau_2 \leqslant T))}},$$

which requires the actual computation of the joint default probability $\mathbb{P}(\tau_1 \leqslant T \text{ and } \tau_2 \leqslant T)$. An exact expression for this joint default probability in the first passage time Black-Cox model, and the associated correlation, have been recently obtained in W. Li and Krehbiel, 2016.

Multiple default times

Consider now a sequence $(\tau_k)_{k=1,2,\ldots,n}$ of random default times and, for more flexibility, a standardized random variable M with probability density function $\phi(m)$ and variance $\mathrm{Var}[M] = 1$.

As in the Merton, 1974 model, cf. § 3.1, a common practice, see Vašiček, 1987, Gibson, 2004, Hull and White, 2004 is to parametrize the default probability associated to each τ_k by a conditioning of the form

$$\mathbb{P}(\tau_k \leqslant T \mid M = m) = \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k m}{\sqrt{1-a_k^2}}\right), \quad k = 1,2,\ldots,n, \tag{3.4.2}$$

see (3.1.2), where $a_k \in (-1,1)$, $k = 1,2,\ldots,n$. Note that we have

$$\begin{aligned}\mathbb{P}(\tau_k \leqslant T) &= \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leqslant T \mid M = m)\phi(m)dm \\ &= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k m}{\sqrt{1-a_k^2}}\right)\phi(m)dm, \end{aligned}\tag{3.4.3}$$

and $\phi(m)$ can be typically chosen as a standard normal Gaussian probability density function.

Next, we present a dependence structure which implements of the Gaussian copula correlation method of D. Li, 2000 in the case of multiple default times.

Definition 3.7 Given $X_1, X_2, \ldots, X_n$ Gaussian samples defined as

$$X_k := a_k M + \sqrt{1-a_k^2} Z_k, \quad k = 1,2,\ldots,n, \tag{3.4.4}$$

conditionally to M, where $Z_1, Z_2, \ldots, Z_n$ are normal random variables with same cumulative distribution function Φ, independent of M, we let the correlated default times $(\tau_1, \ldots, \tau_n)$ be defined as

$$\tau_k := F_{\tau_k}^{-1}(\Phi_{X_k}(X_k)), \tag{3.4.5}$$

where $F_{\tau_k}^{-1}$ denotes the inverse function of F_{τ_k} and Φ_{X_k} denotes the cumulative distribution function of X_k, $k = 1,2,\ldots,n$.

In the next proposition we compute the joint distribution of the default times $(\tau_1, \ldots, \tau_n)$ according to the above dependence structure.

Proposition 3.8 The default times $(\tau_k)_{k=1,2,\ldots,n}$ have the joint distribution

$$\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = C\big(\mathbb{P}(\tau_1 \leqslant y_1), \ldots, \mathbb{P}(\tau_n \leqslant y_n)\big), \tag{3.4.6}$$

where

$$C(x_1, \ldots, x_n)$$

$$:= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi_{X_k}^{-1}(x_1) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi_{X_k}^{-1}(x_n) - a_n m}{\sqrt{1-a_n^2}}\right) \phi(m) dm,$$

$x_1, x_2, \ldots, x_n \in [0,1]$.

Proof. We start by recovering the conditional distribution (3.4.2), as follows:

$$\begin{aligned}
\mathbb{P}(\tau_k \leqslant T \mid M = m) &= \mathbb{P}\big(F_{\tau_k}^{-1}(\Phi_{X_k}(X_k)) \leqslant T \mid M = m\big) \\
&= \mathbb{P}\big(\Phi_{X_k}(X_k) \leqslant F_{\tau_k}(T) \mid M = m\big) \\
&= \mathbb{P}\big(X_k \leqslant \Phi_{X_k}^{-1}(F_{\tau_k}(T)) \mid M = m\big) \\
&= \mathbb{P}\left(a_k m + \sqrt{1-a_k^2} Z_k \leqslant \Phi_{X_k}^{-1}(F_{\tau_k}(T))\right) \\
&= \mathbb{P}\left(\sqrt{1-a_k^2} Z_k \leqslant \Phi_{X_k}^{-1}(F_{\tau_k}(T)) - a_k m\right) \\
&= \mathbb{P}\left(Z_k \leqslant \frac{\Phi_{X_k}^{-1}(F_{\tau_k}(T)) - a_k m}{\sqrt{1-a_k^2}}\right) \\
&= \Phi\left(\frac{\Phi_{X_k}^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k m}{\sqrt{1-a_k^2}}\right), \qquad k = 1, 2, \ldots, n.
\end{aligned}$$

Note that the above recovers the correct marginal distributions (3.4.3), *i.e.* we have

$$\begin{aligned}
\mathbb{P}(\tau_k \leqslant y_k) &= \mathbb{P}(\tau_1 \leqslant \infty, \ldots, \tau_{k-1} \leqslant \infty, \tau_k \leqslant y_k, \tau_{k+1} \leqslant \infty, \ldots, \tau_n \leqslant \infty) \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi_{X_k}^{-1}(\mathbb{P}(\tau_k \leqslant y_k)) - a_k m}{\sqrt{1-a_k^2}}\right) \phi(m) dm \\
&= \int_{-\infty}^{\infty} \mathbb{P}(\tau_k \leqslant T \mid M = m) \phi(m) dm, \quad k = 1, 2, \ldots, n.
\end{aligned}$$

Knowing that, given the sample $M = m$, the default times τ_k, $k = 1, 2, \ldots, n$, are independent random variables, we can compute the joint distribution

$$\begin{aligned}
&\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n \mid M = m) \\
&\qquad = \mathbb{P}(\tau_1 \leqslant y_1 \mid M = m) \times \cdots \times \mathbb{P}(\tau_n \leqslant y_n \mid M = m),
\end{aligned}$$

conditionally to $M = m$. This yields

$$\begin{aligned}
&\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = \int_{-\infty}^{\infty} \mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n \mid M = m) \phi(m) dm \\
&= \int_{-\infty}^{\infty} \mathbb{P}(\tau_1 \leqslant y_1 \mid M = m) \cdots \mathbb{P}(\tau_n \leqslant y_n \mid M = m) \phi(m) dm \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi_{X_1}^{-1}(\mathbb{P}(\tau_1 \leqslant y_1)) - a_1 m}{\sqrt{1-a_1^2}}\right) \cdots \Phi\left(\frac{\Phi_{X_n}^{-1}(\mathbb{P}(\tau_n \leqslant y_n)) - a_n m}{\sqrt{1-a_n^2}}\right) \phi(m) dm,
\end{aligned}$$

which is (3.4.6). □

The next corollary deals with the case where M is normally distributed.

Corollary 3.9 Assume that M has the standard normal distribution with probability density function ϕ and is independent of $X_1, \ldots, X_n$. Then, the joint distribution of the default times $(\tau_k)_{k=1,2,\ldots,n}$ is given by

$$\mathbb{P}(\tau_1 \leqslant y_1, \ldots, \tau_n \leqslant y_n) = C\big(\mathbb{P}(\tau_1 \leqslant y_1), \ldots, \mathbb{P}(\tau_n \leqslant y_n)\big),$$

where

$$\begin{aligned} & C(x_1,\ldots,x_n) \\ & \quad := \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(x_1)-a_1 m}{\sqrt{1-a_1^2}}\right)\cdots\Phi\left(\frac{\Phi^{-1}(x_n)-a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm, \end{aligned}$$

$x_1,x_2,\ldots,x_n \in [0,1]$, is the Gaussian copula with covariance matrix

$$\Sigma = \begin{bmatrix} 1 & a_1a_2 & \cdots & a_1a_{n-1} & a_1a_n \\ a_2a_1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & a_{n-1}a_n \\ a_na_1 & a_na_2 & \cdots & a_na_{n-1} & 1 \end{bmatrix}. \tag{3.4.7}$$

Proof. When is normally distributed and independent of $X_1,\ldots,X_n$, the random vector $(X_1,\ldots,X_n)$ has the covariance matrix (3.4.7), and the function

$$\begin{aligned} & C(x_1,\ldots,x_n) \\ & \quad := \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(x_1)-a_1 m}{\sqrt{1-a_1^2}}\right)\cdots\Phi\left(\frac{\Phi^{-1}(x_n)-a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm, \end{aligned}$$

$x_1,x_2,\ldots,x_n \in [0,1]$, is a Gaussian copula on $[0,1]^n$, built as

$$C(x_1,\ldots,x_n) = F\left(\Phi^{-1}(x_1),\ldots,\Phi^{-1}(x_n)\right),$$

from the Gaussian cumulative distribution function

$$\begin{aligned} F(x_1,\ldots,x_n) &:= \int_{-\infty}^{\infty} \Phi\left(\frac{x_1-a_1 m}{\sqrt{1-a_1^2}}\right)\cdots\Phi\left(\frac{x_n-a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm \\ &= \int_{-\infty}^{\infty} \mathbb{P}\left(Z_1 \leqslant \frac{x_1-a_1 m}{\sqrt{1-a_1^2}}\right)\cdots\mathbb{P}\left(Z_n \leqslant \frac{x_n-a_n m}{\sqrt{1-a_n^2}}\right)\phi(m)dm \\ &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 \leqslant x_1,\ldots,X_n \leqslant x_n \mid M=m)\phi(m)dm \\ &= \mathbb{P}(X_1 \leqslant x_1,\ldots,X_n \leqslant x_n), \qquad 0 \leqslant x_1,x_2,\ldots,x_n \leqslant 1, \end{aligned}$$

of the vector $(X_1,\ldots,X_n)$, with covariance matrix given by (3.4.7). We conclude by Proposition 3.8. □

Exercises

Exercise 3.1 Compute the conditional probability density function of the default time τ defined in (3.1.1).

Exercise 3.2 Credit Default Contract. The assets of a company are modeled using a geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$ with drift $r > 0$ under the risk-neutral probability measure $\mathbb{P}^*$. A Credit Default Contract pays $1 as soon as the asset S_t hits a level $K > 0$. Price this contract at time $t > 0$ assuming that $S_t > K$.

Exercise 3.3

a) Check that the vector $(X_1, X_2, \ldots, X_n)$ defined in (3.4.4) has the covariance matrix given by (3.4.7).
b) Show that the vector $(X_1, X_2, \ldots, X_n)$, with covariance matrix (3.4.7) has standard Gaussian marginals.
c) By computing explicitly the probability density function of $(X_1, \ldots, X_n)$, recover the fact that it is a jointly Gaussian random vector with covariance matrix (3.4.7).

Exercise 3.4 Compute the inverse Σ^{-1} of the covariance matrix (3.4.7) in case $n = 2$.

4. Credit Valuation

Credit derivatives are option contracts that can be used as a protection against default risk in a creditor/debtor relationship, by transferring risk to a third party. This chapter reviews the construction and properties of several credit derivatives such as Collateralized Debt Obligations (CDOs) and Credit Default Swaps (CDSs). We also address the issue of counterparty default risk via the computation of Credit Valuation Adjustments (CVAs).

4.1 Credit Default Swaps (CDS)

Detailed information on the status of credit default swap (CDS) contracts can be obtained from the Bank for International Settlements. We note in particular that the outstanding notional amount of CDS contracts has decreased from its historical high of \$61.2 trillion at year-end 2007 to \$7.6 trillion at year-end 2019.

In this chapter, we work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$ that represents a sequence of possible payment dates. We also let τ be a default time, and given a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, we consider the enlarged filtration $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$ given by $\mathscr{G}_t := \mathscr{F}_t \vee \sigma(\tau)$, $t \geqslant 0$, which contains the additional information given by τ, see Definition 1.2.

Definition 4.1 A Credit Default Swap (CDS) is a contract consisting in

- ***A premium leg*:** the buyer is purchasing protection at time t against default at time T_k, $k = i+1,\ldots,j$, and has to make a fixed spread payment $S_t^{i,j}$ at times $T_{i+1},\ldots,T_j$ between t

and T in compensation.

- ***A protection leg***: the seller or issuer of the contract makes a compensation payment $1-\xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, $k=i,\ldots,j-1$, where ξ_{k+1} is the *recovery rate*.

In the sequel, we let

$$P(t,T_k) := \mathbb{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^{T_k}(r_s+\lambda_s)ds\right)\,\Big|\,\mathscr{F}_t\right], \quad 0\leqslant t\leqslant T_k,$$

denote the default bond price with maturity T_k, $k=i,\ldots,j-1$, see Lemma 1.3 and Proposition 1.4.

Proposition 4.2 The discounted value at time t of the premium leg is given by

$$V^p(t,T) = S_t^{i,j}P(t,T_i,T_j), \tag{4.1.1}$$

where $\delta_k := T_{k+1}-T_k$, $k=i,\ldots,j-1$, and

$$P(t,T_i,T_j) := \sum_{k=i}^{j-1}\delta_k P(t,T_{k+1})$$

is the (default) annuity numéraire, cf. *e.g.* Relation (19.27) in Privault, 2022.

Proof. We have

$$\begin{aligned}
V^p(t,T) &= \mathbb{E}\left[\sum_{k=i}^{j-1}S_t^{i,j}\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_sds\right)\,\Big|\,\mathscr{G}_t\right]\\
&= \sum_{k=i}^{j-1}S_t^{i,j}\delta_k\,\mathbb{E}\left[\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_sds\right)\,\Big|\,\mathscr{G}_t\right]\\
&= S_t^{i,j}\sum_{k=i}^{j-1}\delta_k P(t,T_{k+1})\\
&= S_t^{i,j}P(t,T_i,T_j).
\end{aligned}$$

□

For simplicity, in the above proof we have ignored a possible accrual interest term over the time interval $[T_k,\tau]$ when $\tau\in[T_k,T_{k+1}]$ in the above value of the premium leg. Similarly, we have the following result.

Proposition 4.3 The value at time t of the protection leg is given by

$$V^d(t,T) := \mathbb{E}\left[\sum_{k=i}^{j-1}\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}}r_sds\right)\,\Big|\,\mathscr{G}_t\right], \tag{4.1.2}$$

where ξ_{k+1} is the recovery rate associated with the maturity T_{k+1}, $k=i,\ldots,j-1$.

In the case of a non-random recovery rate ξ_k, the value of the protection leg becomes

$$\sum_{k=i}^{j-1}(1-\xi_{k+1})\,\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_sds\right)\,\Big|\,\mathscr{G}_t\right].$$

The spread $S_t^{i,j}$ is computed by equating the values of the premium (4.1.1) and protection (4.1.2) legs as $V^p(t,T) = V^d(t,T)$, *i.e.* from the relation

$$\begin{aligned} V^p(t,T) &= S_t^{i,j} P(t,T_i,T_j) \\ &= \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right] \\ &= V^d(t,T), \end{aligned}$$

which yields

$$S_t^{i,j} = \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right]. \tag{4.1.3}$$

The spread $S_t^{i,j}$, which is quoted in basis points per year and paid at regular time intervals, gives protection against defaults on payments of \$1. For a notional amount N the premium payment will become $N \times S_t^{i,j}$.

Figure 4.1: CDS price evolution on Credit Suisse, 2023.

In the case of a constant recovery rate ξ, we find

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right],$$

and if τ is constrained to take values in the tenor structure $\{t = T_i, \ldots, T_j\}$, we get

$$S_t^{i,j} = \frac{1-\xi}{P(t,T_i,T_j)} \mathbb{E}\left[\mathbb{1}_{(t,T]}(\tau) \exp\left(-\int_t^{\tau} r_s ds \right) \Big| \mathcal{G}_t \right].$$

The buyer of a Credit Default Swap (CDS) is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, by making a fixed payment $S_t^{i,j}$ (the premium leg) at times $T_{i+1}, \ldots, T_j$. On the other hand, the issuer of the contract makes a payment $1-\xi_{k+1}$ to the buyer in case default occurs at time T_{k+1}, $k = i, \ldots, j-1$.

The contract is priced in terms of the swap rate $S_t^{i,j}$ (or spread) computed by equating the values $V^d(t,T)$ and $V^p(t,T)$ of the protection and premium legs, and acts as a compensation that makes the deal fair to both parties. Recall that from (4.1.3) and Lemma 1.3, we have

$$S_t^{i,j} = \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathcal{G}_t \right]$$

$$= \frac{1}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(\mathbb{1}_{\{T_k<\tau\}} - \mathbb{1}_{\{T_{k+1}<\tau\}})(1-\xi_{k+1}) \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathscr{G}_t \right]$$

$$= \frac{\mathbb{1}_{\{\tau>t\}}}{P(t,T_i,T_j)} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1}) \left(\exp\left(-\int_t^{T_k} \lambda_s ds \right) - \exp\left(-\int_t^{T_{k+1}} \lambda_s ds \right) \right) \right.$$
$$\left. \times \exp\left(-\int_t^{T_{k+1}} r_s ds \right) \Big| \mathscr{F}_t \right].$$

Estimating a deterministic failure rate

In case the rates $r(s)$, $\lambda(s)$ and the recovery rate ξ_{k+1} are deterministic, the above spread can be written as

$$\begin{aligned} S_t^{i,j} P(t,T_i,T_j) &= \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} (1-\xi_k) \exp\left(-\int_t^{T_{k+1}} r(s) ds \right) \\ &\quad \times \left(\exp\left(-\int_t^{T_k} \lambda_s ds \right) - \exp\left(-\int_t^{T_{k+1}} \lambda_s ds \right) \right). \end{aligned}$$

Given that

$$P(t,T_i,T_j) = \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t,T_{k+1}), \qquad T_i \leqslant t \leqslant T_{i+1},$$

we can write

$$\begin{aligned} & S_t^{i,j} \sum_{k=i}^{j-1} (T_{k+1} - T_k) \exp\left(-\int_t^{T_{k+1}} (r(s) + \lambda(s)) ds \right) \\ &= \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} (1-\xi_k) \exp\left(-\int_t^{T_{k+1}} r(s) ds \right) \left(\exp\left(-\int_t^{T_k} \lambda(s) ds \right) - \exp\left(-\int_t^{T_{k+1}} \lambda(s) ds \right) \right). \end{aligned}$$

In particular, when $r(t)$ and $\lambda(t)$ are written as in (1.3.2) and assuming that $\xi_k = \xi$ is constant, $k = i, \ldots, j$, we get, with $t = T_i$ and writing $\delta_k = T_{k+1} - T_k$, $k = i, \ldots, j-1$,

$$\begin{aligned} & S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(-\sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right) \\ &= \mathbb{1}_{\{\tau>t\}} (1-\xi) \sum_{k=i}^{j-1} \exp\left(-\sum_{p=i}^{k} \delta_p (r_p + \lambda_p) \right) (\mathrm{e}^{\delta_k \lambda_k} - 1). \end{aligned}$$

Assuming further that $\lambda_k = \lambda$ is constant, $k = i, \ldots, j$, we have

$$\begin{aligned} & S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(-\sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right) \\ &= (1-\xi) \sum_{k=i}^{j-1} (\mathrm{e}^{-\lambda \delta_k} - 1) \exp\left(-\sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p \right), \end{aligned} \tag{4.1.4}$$

which can be solved numerically for λ, cf. Sections 4 and 5 of Castellacci, 2008 for the JP Morgan model, and Exercises 4.1 and 4.2.

4.2 Collateralized Debt Obligations (CDO)

Consider a portfolio consisting of $N = j - i$ bonds with default times $\tau_k \in (T_k, T_{k+1}]$, $k = i, \ldots, j-1$, and recovery rates $\xi_k \in [0,1]$, $k = i+1, \ldots, j$.

A synthetic CDO is a structured investment product constructed by splitting the above portfolio into n ordered tranches numbered $i = 1, 2, \ldots, n$, where tranche n°i represents a percentage $p_i\%$ of the total portfolio value. We let

$$\alpha_l := p_1 + p_2 + \cdots + p_l, \qquad l = 1, 2, \ldots, n, \tag{4.2.1}$$

denote the corresponding cumulative percentages, with $\alpha_0 = 0$ and $\alpha_n = p_1 + p_2 + \cdots + p_n = 100\%$.

The tranches are ordered according to increasing default risk, tranche n°1 being the riskiest one ("equity tranche"), and tranche n°n being the safest one ("senior tranche"), while the intermediate tranches are referred to as "mezzanine tranches". In practice, losses occur first to the "equity" tranches, next to the "mezzanine" tranche holders, and finally to "senior" tranches.

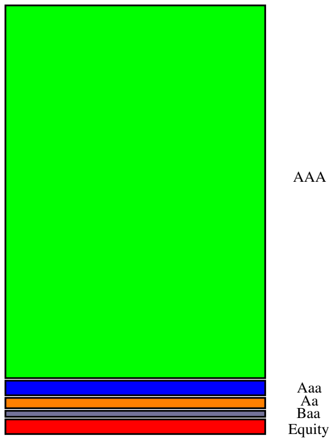


Figure 4.2: A representation of CDO tranches.

CDOs can attract different types of investors.

- Unfunded investors (usually for the higher tranches) are receiving premiums and make payments in case of default.
- Funded investors (usually in the lower tranches) are investing in risky bonds to receive principal payments at maturity, and they are the first in line to incur losses.
- A CDO can also be used as a Credit Default Swap (CDS) for the "short investors" who make premium payments in exchange for credit protection in case of default.

The market for synthetic CDOs has been significantly reduced since the 2006-2008 subprime crisis.

Synthetic CDOs are based on $N = j - i$ bonds that can potentially generate a cumulative loss

$$L_t := \sum_{l=i}^{j-1} (1 - \xi_{l+1}) \mathbb{1}_{\{\tau_l \leqslant t\}} \in [0, N],$$

at time $t \in [T_i, T_j]$, based on the default time τ_l and recovery rate ξ_{l+1} of each involved CDS, $k = i, \ldots, j-1$, with $N = j - i$.

When the first loss occurs, tranche n°1 is the first in line, and it loses the amount

$$L_t^1 = L_t \mathbb{1}_{\{L_t \leqslant p_1 N\}} + N p_1 \mathbb{1}_{\{L_t > p_1 N\}} = N \min(L_t / N, p_1).$$

In case $L_t > p_1 N$, then tranche n°2 takes the remaining loss up to the amount $N p_2$, that means the loss L_t^2 of tranche n°2 is

$$L_t^2 \quad = \quad (L_t - N p_1) \mathbb{1}_{\{p_1 N < L_t \leqslant (p_1 + p_2) N\}} + N p_2 \mathbb{1}_{\{L_t > (p_1 + p_2) N\}}$$

$$
\begin{aligned}
&= (L_t - Np_1)\mathbb{1}_{\{p_1 N < L_t \leqslant \alpha_2 N\}} + Np_2 \mathbb{1}_{\{L_t > \alpha_2 N\}} \\
&= (L_t - Np_1)^+ \mathbb{1}_{\{L_t \leqslant \alpha_2 N\}} + Np_2 \mathbb{1}_{\{L_t > \alpha_2 N\}} \\
&= \min((L_t - Np_1)^+, Np_2) \\
&= \operatorname{Max}(\min(L_t, Np_1 + Np_2) - Np_1, 0) \\
&= \operatorname{Max}(\min(L_t, N\alpha_2) - Np_1, 0).
\end{aligned}
$$

By induction, the potential loss taken by tranche n°i is given by

$$
\begin{aligned}
L_t^i &= (L_t - N\alpha_{i-1})\mathbb{1}_{\{\alpha_{i-1} N < L_t \leqslant \alpha_i N\}} + Np_i \mathbb{1}_{\{L_t > \alpha_i N\}} \\
&= (L_t - N\alpha_{i-1})^+ \mathbb{1}_{\{L_t \leqslant \alpha_i N\}} + Np_i \mathbb{1}_{\{L_t > \alpha_i N\}} \\
&= \min((L_t - N\alpha_{i-1})^+, Np_i) \\
&= \operatorname{Max}(\min(L_t, N\alpha_i) - N\alpha_{i-1}, 0),
\end{aligned}
$$

where $\alpha_i := p_1 + p_2 + \cdots + p_i$, $i = 1, 2, \ldots, n$.

In the end, tranche n°n will take the loss

$$
L_t^n = (L_t - N\alpha_{n-1})\mathbb{1}_{\{\alpha_{n-1} N < L_t\}} = (L_t - N\alpha_{n-1})^+.
$$

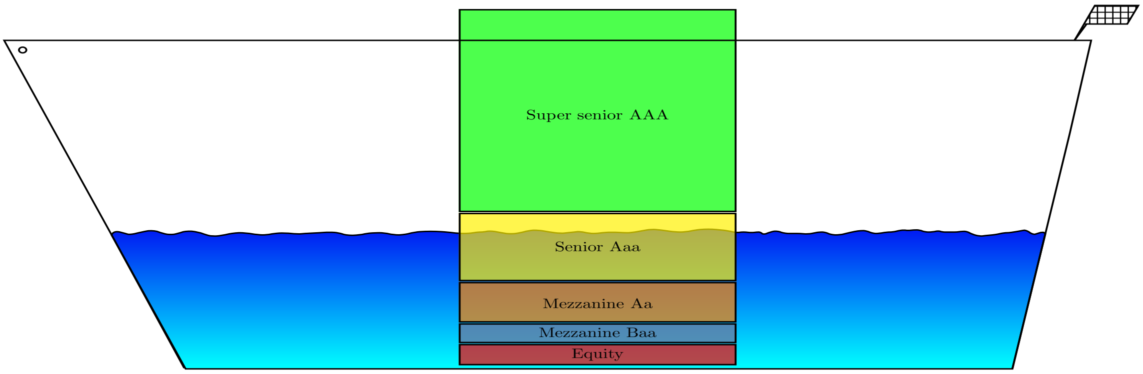


Figure 4.3: A Titanic-style representation of cumulative tranche losses.

The CDO tranche n°l, $l = 1, 2, \ldots, n$, can be decomposed into:

- A premium leg: the short investor in tranche n°l is purchasing protection at time t against default at time T_k, $k = i+1, \ldots, j$, by making fixed payments $S_t^{i,j}$ at times $T_{i+1}, \ldots, T_j$ between t and T in compensation. This premium can also be received by the unfunded investor.

The discounted value at time t of the premium leg for the tranche n°l is

$$
\begin{aligned}
V_l^p(t,T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} S_t^l \delta_k (Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right] \\
&= S_t^l \sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \Bigm| \mathcal{G}_t\right], \qquad (4.2.2)
\end{aligned}
$$

$l = 1, 2, \ldots, N$, where the premium spread S_t^l is quoted as a proportion of the compensation $Np_l - L_{T_{k+1}}^l$ and is paid at each time T_{k+1} until $k = j-1$ or $L_{T_{k+1}} = 100\%$, whichever comes first.

- A protection leg: the short investor receives protection against default from the premium leg, which can also be paid by the unfunded investors. Noting that at each default time $\tau_k \in$

$(T_k, T_{k+1}]$, $k = i, \ldots, j-1$, the loss L_t^l taken by tranche n°l jumps by the amount $\Delta L_{\tau_k}^l = L_{\tau_k}^l - L_{\tau_k^-}^l$, the value at time t of the protection leg for tranche n°l can be written as

$$\begin{aligned}
V_l^d(t,T) &= \mathbb{E}\left[\sum_{k=i}^{j-1} \mathbb{1}_{[T_i,T_j]}(\tau_k)\Delta L_{\tau_k}^l \exp\left(-\int_t^{\tau_k} r_u du\right) \,\Big|\, \mathcal{G}_t\right] \\
&= \mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(-\int_t^s r_u du\right) dL_s^l \,\Big|\, \mathcal{G}_t\right] \qquad (4.2.3)\\
&= \mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L_{T_j}^l - \exp\left(-\int_t^{T_i} r_u du\right) L_{T_i}^l \,\Big|\, \mathcal{G}_t\right] \\
&\quad + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L_s^l ds \,\Big|\, \mathcal{G}_t\right] \\
&= \mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L_{T_j}^l \,\Big|\, \mathcal{G}_t\right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L_s^l ds \,\Big|\, \mathcal{G}_t\right],
\end{aligned}$$

$l = 1, 2, \ldots, n$, where we applied integration by parts on $[T_i, T_j]$ and used the fact that $L_{T_i} = 0$. The spread S_t^l paid by tranche n°l is computed by equating the values $V_l^p(t,T) = V_l^d(t,T)$ of the protection and premium legs in (4.2.2) and (4.2.3), which yields

$$\begin{aligned}
S_t^l &= \frac{\mathbb{E}\left[\int_{T_i}^{T_j} \exp\left(-\int_t^s r_u du\right) dL_s^l \,\Big|\, \mathcal{G}_t\right]}{\sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\Big|\, \mathcal{G}_t\right]} \\
&= \frac{\mathbb{E}\left[\exp\left(-\int_t^{T_j} r_u du\right) L_{T_j}^l \,\Big|\, \mathcal{G}_t\right] + \mathbb{E}\left[\int_{T_i}^{T_j} r_s \exp\left(-\int_t^s r_u du\right) L_s^l ds \,\Big|\, \mathcal{G}_t\right]}{\sum_{k=i}^{j-1} \delta_k \mathbb{E}\left[(Np_l - L_{T_{k+1}}^l) \exp\left(-\int_t^{T_{k+1}} r_s ds\right) \,\Big|\, \mathcal{G}_t\right]} \\
&\geqslant 0,
\end{aligned}$$

$l = 1, 2, \ldots, n$.

Expected tranche loss

The expected cumulative loss given the parameter M can be computed by linearity in the multiple default time model (3.4.2) of Chapter 3 as

$$\begin{aligned}
\mathbb{E}[L_t \mid M = m] &= \sum_{l=i}^{j-1} \mathbb{E}\left[(1-\xi_{l+1})\mathbb{1}_{\{\tau_l \leqslant t\}} \mid M = m\right] \\
&= \sum_{l=i}^{j-1} (1-\xi_{l+1})\mathbb{P}(\tau_l \leqslant t \mid M = m) \\
&= \sum_{l=i}^{j-1} (1-\xi_{l+1})\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_l \leqslant t)) + a_k m}{\sqrt{1-a_k^2}}\right),
\end{aligned}$$

by (3.4.2), and the expected cumulative loss can be written as

$$\mathbb{E}[L_t] = \int_{-\infty}^{\infty} \mathbb{E}[L_t \mid M = m]\, \phi(m) dm = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{E}[L_t \mid M = m]\, e^{-m^2/2} dm.$$

The situation is different for the expected loss of tranche n°k is written as the expected value

$$\mathbb{E}[L_t^k] = \mathbb{E}\left[\min((L_t - N\alpha_{k-1})^+, Np_k)\right], \qquad k = 1, 2, \ldots, n,$$

of the *nonlinear* function $f_k(x) := \min((x - N\alpha_{k-1})^+, Np_k)$ of L_t, where α_{k-1} is defined in (4.2.1).

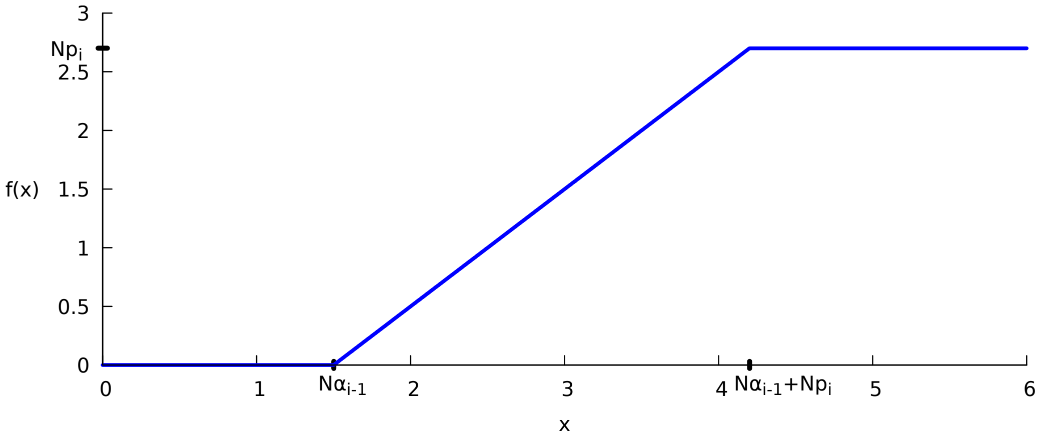


Figure 4.4: Function $f_k(x) = \min((x - N\alpha_{k-1})^+, Np_k)$.

The expected tranche loss $\mathbb{E}[L_t^k]$ n°k can be estimated by the Monte Carlo method when the default times are generated according to (3.4.5).

In order to compute expected tranche losses we can use the fact that the cumulative loss L_t is a discrete random variable, with for example

$$\mathbb{P}\left(L_t = N - \sum_{k=i}^{j-1} \xi_{k+1}\right) = \mathbb{P}(\tau_i \leqslant t, \ldots, \tau_{j-1} \leqslant t),$$

and

$$\mathbb{P}(L_t = 0) = \mathbb{P}(\tau_i > t, \ldots, \tau_{j-1} > t),$$

which require the knowledge of the joint distribution of the default times $\tau_i, \ldots, \tau_{j-1}$.

If the $\tau_k' s$ are independent and identically distributed with common cumulative distribution function F_τ and $a_k = a$, $\xi_k = \xi$, $k = i+1, \ldots, j$, then the cumulative loss L_t has a binomial distribution given M, given by

$$\begin{aligned}\mathbb{P}(L_t = (1-\xi)k \mid M) &= \binom{N}{k}(1 - \mathbb{P}(\tau \leqslant T \mid M))^{N-k}(\mathbb{P}(\tau \leqslant T \mid M))^k \\ &= \binom{N}{k}\left(1 - \Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}}\right)\right)^{N-k}\left(\Phi\left(\frac{\Phi^{-1}(F_\tau(T)) - aM}{\sqrt{1-a^2}}\right)\right)^k,\end{aligned}$$

$k = 0, 1, \ldots, N$. The expected loss of tranche n°k can then be expressed as

$$\begin{aligned}\mathbb{E}[L_t^k] &= \int_{-\infty}^{\infty} \mathbb{E}[f_k(L_t) \mid M = m]\,\phi(m)dm \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathbb{E}[f_k(L_t) \mid M = m]\,\mathrm{e}^{-m^2/2}dm,\end{aligned}$$

$k = 1, 2, \ldots, n$, where $\mathbb{E}[f_k(L_t) \mid M = m]$ is computed either by the Monte Carlo method, from the distribution of L_t.

In Vašiček, 2002, the tranche loss has been approximated by a Gaussian random variable for very large portfolios with $N \to \infty$.

The α-percentile loss of the portfolio can be estimated as

$$\mathbb{E}[L_t \mid M = m] = \sum_{k=i}^{j-1}(1 - \xi_{k+1})\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right),$$

where $m = \Phi^{-1}(\alpha)$.

Such (Gaussian) Merton, 1974 and Vašiček, 2002 type models have been implemented in the Basel II recommendations Banking Supervision, 2005. Namely in Basel II, banks are expected to hold capital in prevision of unexpected losses in a worst case scenario, according to the Internal Ratings-Based (IRB) formula

$$\sum_{k=i}^{j-1}(1-\xi_{k+1})\left(\Phi\left(\frac{\Phi^{-1}(\mathbb{P}(\tau_k \leqslant T)) - a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right) - \mathbb{P}(\tau_k \leqslant T)\right),$$

with confidence level set at $\alpha = 0.999$ *i.e.* $m = \Phi^{-1}(0.999) = 3.09$, cf. Relation (2.4) page 10 of Aas, 2005. Recall that the function

$$x \longmapsto \Phi\left(\frac{\Phi^{-1}(x) + a_k\Phi^{-1}(\alpha)}{\sqrt{1-a_k^2}}\right)$$

always lies above the graph of x when $a_k < 0$, as in the next figure.

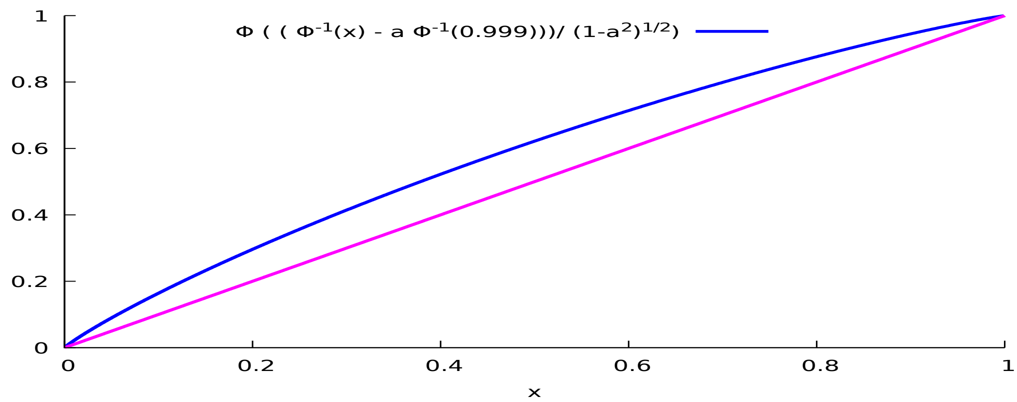


Figure 4.5: Internal Ratings-Based (IRB) formula.

4.3 Credit Valuation Adjustment (CVA)

Credit Valuation Adjustments (CVA) aim at estimating the amount of capital required in the event of counterparty default, and are specially relevant to the Basel III regulatory framework. Other credit value adjustments (XVA) include the Funding Valuation Adjustments (FVA), Debit Valuation Adjustments (DVA), Capital Valuation Adjustments (KVA), and Margin Valuation Adjustments (MVA). The purpose of XVAs is also to take into account the future value of trades and their associated risks. The real-time estimation of XVA measures is generally highly demanding from a computational point of view.

Net Present Value (NPV) of a CDS

As above, we work with a tenor structure $\{t = T_i < \cdots < T_j = T\}$. Let

$$\begin{aligned}\Pi(T_l, T_j) &:= \text{protection\_leg} - \text{premium\_leg} \\ &= \sum_{k=l}^{j-1} \mathbb{1}_{(T_k, T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r_s ds\right) \\ &\quad - \sum_{k=l}^{j-1} S_t^{i,j}\delta_k \mathbb{1}_{\{\tau > T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}} r_s ds\right)\end{aligned}$$

$$
\begin{aligned}
&= (1-\xi)\sum_{k=l}^{j-1}\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\\
&\quad -\sum_{k=l}^{j-1}S_t^{i,j}\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\\
&= \sum_{k=l}^{j-1}\left((1-\xi)\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\right.\\
&\quad \left. -\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\right)
\end{aligned}
$$

denote the difference between the remaining protection and premium legs from time T_l until time T_j. Note that by definition of the spread $S_t^{i,j}$ we have $\Pi(t,T_j)=0$, $0\leqslant t\leqslant T_i$.

Definition 4.4 The Net Present Value (NPV) at time T_l of the CDS is the conditional expected value

$$\mathrm{NPV}(T_l,T_j) := \mathbb{E}\left[\Pi(T_l,T_j)\mid\mathcal{G}_{T_l}\right]$$

of the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j, where $(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ is the filtration (1.2.2) enlarged as with the additional information on the default time τ.

The Net Present Value (NPV) at time T_l of the CDS satisfies

$$
\begin{aligned}
\mathrm{NPV}(T_l,T_j) &:= \mathbb{E}\left[\Pi(T_l,T_j)\mid\mathcal{G}_{T_l}\right]\\
&= \mathbb{E}\left[\sum_{k=l}^{j-1}\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\,\Bigg|\,\mathcal{G}_{T_l}\right] \qquad (4.3.1)\\
&\quad -\mathbb{E}\left[\sum_{k=l}^{j-1}S_t^{i,j}\delta_k\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\,\Bigg|\,\mathcal{G}_{T_l}\right]\\
&= (1-\xi)\sum_{k=l}^{j-1}\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\,\Bigg|\,\mathcal{G}_t\right]-S_t^{i,j}\sum_{k=l}^{j-1}\delta_k P(t,T_{k+1})\\
&= \sum_{k=l}^{j-1}\left((1-\xi)\mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)\exp\left(-\int_t^{T_{k+1}}r_s ds\right)\,\Bigg|\,\mathcal{G}_t\right]-S_t^{i,j}\delta_k P(t,T_{k+1})\right)
\end{aligned}
$$

of the difference between the values at time T_l of the remaining protection and premium legs from time T_l until time T_j.

In addition to the credit default time τ we introduce a second stopping time $\nu\in[T_l,T_j]$ representing the possible default time of the party providing the protection leg.

The Net Present Value $\mathrm{NPV}(\nu,T_j)$ is estimated when default occurs at time ν.

i) If $\mathrm{NPV}(\nu,T_j)>0$ then a payment is due from the party providing the protection leg, and only a fraction $\eta\mathrm{NPV}(\nu,T_j)$ of this payment may be recovered, where $\eta\in[0,1]$ is the recovery rate of the party providing protection in the CDS.

ii) On the other hand, if $\mathrm{NPV}(\nu,T_j)<0$ then the original fee payment $-\mathrm{NPV}(\nu,T_j)$ is still due.

As a consequence, in the event of default at time $\nu\in[T_l,T_j]$, the net present value of the CDS at time ν is

$$
\begin{aligned}
&\eta\mathrm{NPV}(\nu,T_j)\mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)>0\}}+\mathrm{NPV}(\nu,T_j)\mathbb{1}_{\{\mathrm{NPV}(\nu,T_j)<0\}}\\
&\quad = \eta\left(\mathrm{NPV}(\nu,T_j)\right)^+-\left(\mathrm{NPV}(\nu,T_j)\right)^-
\end{aligned}
$$

$$
\begin{aligned}
&= \eta\left(\mathrm{NPV}(v,T_j)\right)^+ - \left(-\mathrm{NPV}(v,T_j)\right)^+ \\
&= \eta\left(\mathrm{NPV}(v,T_j)\right)^+ + \left(\mathrm{NPV}(v,T_j) - \left(\mathrm{NPV}(v,T_j)\right)^+\right) \\
&= \mathrm{NPV}(v,T_j) - (1-\eta)\left(\mathrm{NPV}(v,T_j)\right)^+. \qquad (4.3.2)
\end{aligned}
$$

Credit Valuation Adjustment (CVA)

Under the event of counterparty default at a time $v \in [T_l, T_j]$, the discounted payment estimated at time T_l becomes

$$
\begin{aligned}
&\Pi(T_l,v) + \exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\eta\left(\mathrm{NPV}(v,T_j)\right)^+ - \left(-\mathrm{NPV}(v,T_j)\right)^+\right) \\
&= \Pi(T_l,v) + \exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\mathrm{NPV}(v,T_j) - (1-\eta)\left(\mathrm{NPV}(v,T_j)\right)^+\right) \\
&= \Pi(T_l,T_j) - (1-\eta)\exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\mathrm{NPV}(v,T_j)\right)^+,
\end{aligned}
$$

since

$$
\Pi(T_l,T_j) = \Pi(T_l,v) + \exp\left(-\int_{T_l}^{v} r_s ds\right)\mathrm{NPV}(v,T_j).
$$

More generally, the total discounted payment due at time T_l under counterparty risk rewrites as

$$
\begin{aligned}
&\Pi^D(T_l,T_j) = \mathbb{1}_{\{T_j<v\}}\Pi(T_l,T_j) \\
&+ \mathbb{1}_{\{T_l<v\leqslant T_j\}}\left(\Pi(T_l,v) + \exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\eta\left(\mathrm{NPV}(v,T_j)\right)^+ - \left(-\mathrm{NPV}(v,T_j)\right)^+\right)\right) \\
&= \mathbb{1}_{\{T_j<v\}}\Pi(T_l,T_j) \\
&\quad + \mathbb{1}_{\{T_l<v\leqslant T_j\}}\left(\Pi(T_l,T_j) - (1-\eta)\exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\mathrm{NPV}(v,T_j)\right)^+\right) \\
&= \Pi(T_l,T_j) - \mathbb{1}_{\{T_l<v\leqslant T_j\}}(1-\eta)\exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\mathrm{NPV}(v,T_j)\right)^+, \qquad (4.3.3)
\end{aligned}
$$

see Brigo and Chourdakis, 2009; Brigo and Masetti, 2006. As a consequence of (4.3.3), we derive the following result.

Proposition 4.5 The price at time T_l of the payoff $\Pi^D(T_l,T_j)$ under counterparty risk is given by

$$
\begin{aligned}
&\mathbb{E}\left[\Pi^D(T_l,T_j) \mid \mathcal{F}_{T_l}\right] = \mathbb{E}\left[\Pi(T_l,T_j) \mid \mathcal{F}_{T_l}\right] \\
&\qquad -(1-\eta)\,\mathbb{E}\left[\mathbb{1}_{\{T_l<v\leqslant T_j\}}\exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\mathrm{NPV}(v,T_j)\right)^+ \,\Big|\, \mathcal{F}_{T_l}\right].
\end{aligned}
$$

The quantity

$$
(1-\eta)\,\mathbb{E}\left[\mathbb{1}_{\{T_l<v\leqslant T_j\}}\exp\left(-\int_{T_l}^{v} r_s ds\right)\left(\mathrm{NPV}(v,T_j)\right)^+ \,\Big|\, \mathcal{F}_{T_l}\right]
$$

is called the (positive) Counterparty Risk (CR) Credit Valuation Adjustment (CVA).

Exercises

Exercise 4.1 Show that the equation (4.1.4) admits a numerical solution $\lambda > 0$.

Exercise 4.2 Credit default swaps. From the CDS market data of Figure 4.6 on McDonald's Corp, estimate the first default rate λ_1 and the associated default probability in the framework of (4.1.4), cf. also Castellacci, 2008.

Figure 4.6: Cashflow data.

Exercise 4.3 Consider a tenor structure $\{t = T_i < \cdots < T_j = T\}$, a sequence

$$P(t,T_k) = \exp\left(-\int_t^{T_k} r(s)ds\right) = \mathrm{e}^{-(T_k-t)r_k}, \qquad k = i,\ldots,j,$$

of *deterministic* discount factors, and a family

$$Q(t,T_k) = \mathbb{E}\left[\exp\left(-\int_t^{T_k} \lambda_s ds\right) \,\Big|\, \mathscr{F}_t\right]$$

of survival probabilities.

a) Show that the discounted value at time t of the protection leg equals

$$\sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \,\Big|\, \mathscr{G}_t\right]$$
$$= \mathbb{1}_{\{\tau>t\}}(1-\xi)\sum_{k=i}^{j-1} P(t,T_{k+1})\left(Q(t,T_k)-Q(t,T_{k+1})\right).$$

b) Letting $\delta_k := T_{k+1} - T_k$, $k = i,\ldots,j-1$, show that the discounted value at time t of the premium leg, equals

$$V^p(t,T) = \mathbb{1}_{\{\tau>t\}} S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1}) Q(t,T_{k+1}).$$

c) By equating the protection and premium legs, find the value of $Q(t,T_{i+1})$ with $Q(t,T_i) = 1$, and derive a recurrence relation between $Q(t,T_{j+1})$ and $Q(t,T_i),\ldots,Q(t,T_j)$.

Exercise 4.4 (Exercise 4.3 continued). From the spread data and survival probabilities data of Figure 4.7 on the Coca-Cola Company, retrieve the corresponding CDS spreads $S_t^{i,j}$ and discount factors $P(t,T_i),\ldots,P(t,T_n)$, and estimate the corresponding survival probabilities $Q(t,T_i),\ldots,Q(t,T_n)$.

Figure 4.7: CDS Market data.

5. Stochastic Volatility

Stochastic volatility refers to the modeling of volatility using time-dependent stochastic processes, in contrast to the constant volatility assumption made in the standard Black-Scholes model. In this setting, we consider the pricing of realized variance swaps and options using moment matching approximations. We also cover the pricing of vanilla options by PDE arguments in the Heston model, and by perturbation analysis approximations in more general stochastic volatility models.

5.1 Stochastic Volatility Models

Time-dependent stochastic volatility

The next Figure 5.1 refers to the EURO/SGD exchange rate, and shows some spikes that cannot be generated by Gaussian returns with constant variance.

Figure 5.1: Euro / SGD exchange rate.

This type data shows that, in addition to jump models that are commonly used to take into account the slow decrease of probability tails observed in market data, other tools should be implemented in order to model a possibly random and time-varying volatility.

We consider an asset price driven by the stochastic differential equation

$$dS_t = rS_t dt + S_t \sqrt{v_t} dB_t \tag{5.1.1}$$

under the risk-neutral probability measure $\mathbb{P}^*$, with solution

$$S_T = S_t \exp\left((T-t)r + \int_t^T \sqrt{v_s} dB_s - \frac{1}{2} \int_t^T v_s ds \right) \tag{5.1.2}$$

where $(v_t)_{t\in\mathbb{R}_+}$ is a (possibly random) squared volatility (or variance) process adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$.

Time-dependent deterministic volatility

When the variance process $(v(t))_{t\in\mathbb{R}_+}$ is a deterministic function of time, the solution (5.1.2) of (5.1.1) is a lognormal random variable at time T with conditional log-variance

$$\int_t^T v(s) ds$$

given $\mathscr{F}_t$. In particular, the European call option on S_T can be priced by the Black-Scholes formula as

$$\mathrm{e}^{-(T-t)r} \mathbb{E}^*[(S_T - K)^+ \mid \mathscr{F}_t] = \mathrm{Bl}\big(S_t, K, r, T-t, \sqrt{\widehat{v}(t)}\big),$$

with integrated squared volatility parameter

$$\widehat{v}(t) := \frac{\int_t^T v(s) ds}{T-t}, \qquad t \in [0, T).$$

Independent (stochastic) volatility

When the volatility $(v_t)_{t\in\mathbb{R}_+}$ is a random process generating a filtration $\big(\mathscr{F}_t^{(2)}\big)_{t\in\mathbb{R}_+}$, independent of the filtration $\big(\mathscr{F}_t^{(1)}\big)_{t\in\mathbb{R}_+}$ generated by the driving Brownian motion $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$, the equation (5.1.1) can still be solved as

$$S_T = S_t \exp\left((T-t)r + \int_t^T \sqrt{v_s} dB_s^{(1)} - \frac{1}{2} \int_t^T v_s ds \right),$$

and, given $\mathscr{F}_T^{(2)}$, the asset price S_T is a lognormal random variable with random variance

$$\int_t^T v_s ds.$$

In this case, taking

$$\mathscr{F}_t := \mathscr{F}_t^{(1)} \vee \mathscr{F}_t^{(2)}, \qquad 0 \leqslant t \leqslant T,$$

where $\big(\mathscr{F}_t^{(1)}\big)_{t\in\mathbb{R}_+}$ is the filtration generated by $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$, we can still price an option with payoff $\phi(S_T)$ on the underlying asset price S_T using the tower property

$$\mathbb{E}^*[\phi(S_T) \mid \mathscr{F}_t] = \mathbb{E}^*\left[\mathbb{E}^*\left[\phi(S_T) \,\middle|\, \mathscr{F}_t^{(1)} \vee \mathscr{F}_T^{(2)}\right] \,\middle|\, \mathscr{F}_t^{(1)} \vee \mathscr{F}_t^{(2)}\right].$$

As an example, the European call option on S_T can be priced by averaging the Black-Scholes formula as follows:

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathscr{F}_t] \\
&\quad = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\mathbb{E}^*\left[(S_T-K)^+ \,\middle|\, \mathscr{F}_t^{(1)} \vee \mathscr{F}_T^{(2)}\right] \,\middle|\, \mathscr{F}_t^{(1)} \vee \mathscr{F}_t^{(2)}\right]. \\
&\quad = \mathbb{E}^*\left[\mathrm{Bl}\left(S_t, K, r, T-t, \sqrt{\frac{\int_t^T v_s ds}{T-t}}\right) \,\middle|\, \mathscr{F}_t\right] \\
&\quad = \mathbb{E}^*\left[\mathrm{Bl}\big(x, K, r, T-t, \sqrt{\widehat{v}(t,T)}\big) \mid \mathscr{F}_t^{(2)}\right]_{|x=S_t},
\end{aligned}$$

which represents an averaged version of Black-Scholes prices, with the random integrated volatility

$$\widehat{v}(t,T) := \frac{1}{T-t}\int_t^T v_s ds, \qquad 0 \leqslant t \leqslant T.$$

On the other hand, the probability distribution of the time integral $\int_t^T v_s ds$ given $\mathscr{F}_t^{(2)}$ can be computed using integral expressions, see Yor, 1992 and Proposition 10.5 when $(v_t)_{t\in\mathbb{R})_+}$ is a geometric Brownian motion, and Lemma 9 in Feller, 1951 or Corollary 24 in Albanese and Lawi, 2005 when $(v_t)_{t\in\mathbb{R})_+}$ is the CIR process.

Two-factor stochastic volatility model

Evidence based on financial market data, see Figure 6.16, Figure 1 of A. Papanicolaou and Sircar, 2014 or § 2.3.1 in Fouque, G. Papanicolaou, Sircar, and Sølna, 2011, shows that the variations in volatility tend to be negatively correlated with the variations of underlying asset prices. For this reason we need to consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ and a stochastic volatility process $(v_t)_{t\in\mathbb{R}_+}$ driven by

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dB_t^{(1)} \\ \\ dv_t = \mu(t, v_t)dt + \beta(t, v_t)dB_t^{(2)}, \end{cases}$$

Here, $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are possibly correlated Brownian motions, with

$$\mathrm{Cov}\left(B_t^{(1)}, B_t^{(2)}\right) = \rho t \quad \text{and} \quad dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt,$$

where the correlation parameter ρ satisfies $-1 \leqslant \rho \leqslant 1$, and the coefficients $\mu(t,x)$ and $\beta(t,x)$ can be chosen *e.g.* from mean-reverting models (CIR) or geometric Brownian models, as follows. Note that the observed correlation coefficient ρ is usually negative, cf. *e.g.* § 2.1 in A. Papanicolaou and Sircar, 2014 and Figures 6.16 and 6.17.

The Heston model

In the Heston, 1993 model, the stochastic volatility $(v_t)_{t\in\mathbb{R}_+}$ is chosen to be a Cox, Ingersoll, and Ross, 1985 (CIR) process, *i.e.* we have

$$\begin{cases} dS_t = rS_t dt + S_t\sqrt{v_t}dB_t^{(1)} \\ \\ dv_t = -\lambda(v_t - m)dt + \eta\sqrt{v_t}dB_t^{(2)}, \end{cases}$$

and $\mu(t,v) = -\lambda(v-m)$, $\beta(t,v) = \eta\sqrt{v}$, where λ, m, $\eta > 0$.

Option pricing formulas can be derived in the Heston model using Fourier inversion and complex integrals, cf. (5.4.5) below.

The SABR model

In the Sigma-Alpha-Beta-Rho (σ-α-β-ρ-SABR) model Hagan et al., 2002, based on the parameters (α,β,ρ), the stochastic volatility process $(\sigma_t)_{t\in\mathbb{R}_+}$ is modeled as a geometric Brownian motion with

$$\begin{cases} dF_t = \sigma_t F_t^{\beta} dB_t^{(1)} \\ \\ d\sigma_t = \alpha\sigma_t dB_t^{(2)}, \end{cases}$$

where $(F_t)_{t\in\mathbb{R}_+}$ typically models a forward interest rate. Here, we have $\alpha > 0$ and $\beta \in (0,1]$, and $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$, $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are standard Brownian motions with the correlation

$$dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt.$$

This setting is typically used for the modeling of LIBOR rates and is *not* mean-reverting, hence it is preferably used with a short time horizon. It allows in particular for short time asymptotics of Black implied volatilities that can be used for pricing by inputting them into the Black pricing formula, cf. § 3.3 in Rebonato, 2009.

5.2 Realized Variance Swaps

Another look at historical volatility

In this section, given $T > 0$ and $N \geqslant 1$, we let

$$t_k := k\frac{T}{N}, \qquad k = 0,1,\ldots,N.$$

a natural estimator for the trend parameter μ can be written in terms of actual returns as

$$\widehat{\mu}_N := \frac{1}{N}\sum_{k=1}^{N}\frac{1}{t_k - t_{k-1}}\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}},$$

or in terms of log-returns as

$$\begin{aligned} \widehat{\mu}_N &:= \frac{1}{N}\sum_{k=1}^{N}\frac{1}{t_k - t_{k-1}}\log\frac{S_{t_k}}{S_{t_{k-1}}} \\ &= \frac{1}{T}\sum_{k=1}^{N}\big(\log(S_{t_k}) - \log(S_{t_{k-1}})\big) \\ &= \frac{1}{T}\log\frac{S_T}{S_0}. \end{aligned}$$

Similarly, one can use the squared volatility (or realized variance) estimator

$$\begin{aligned}\widehat{\sigma}_N^2 &:= \frac{1}{N-1}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}-(t_{k+1}-t_k)\widehat{\mu}_N\right)^2\\ &= \frac{1}{N-1}\sum_{k=1}^{N}\frac{1}{t_k-t_{k-1}}\left(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2-\frac{T}{N-1}(\widehat{\mu}_N)^2\end{aligned}$$

using actual returns, or, using log-returns,*

$$\begin{aligned}\widehat{\sigma}_N^2 &:= \frac{1}{N-1}\sum_{k=1}^{N}\frac{1}{t_k-t_{k-1}}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}-(t_k-t_{k-1})\widehat{\mu}_N\right)^2\\ &= \frac{1}{N-1}\sum_{k=1}^{N}\frac{1}{t_k-t_{k-1}}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2-\frac{T}{N-1}(\widehat{\mu}_N)^2.\end{aligned} \tag{5.2.1}$$

Realized variance swaps

Realized variance swaps are forward contracts that allow for the exchange of the estimated volatility (5.2.1) against a fixed value κ_σ. They can be priced using log-returns and expected value as

$$\mathbb{E}\left[\widehat{\sigma}_N^2\right]=\frac{1}{T}\mathbb{E}\left[\sum_{k=1}^{N}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2-\frac{1}{N-1}\left(\log\frac{S_T}{S_0}\right)^2\right]-\kappa_\sigma^2$$

of their payoff

$$\frac{1}{T}\left(\sum_{k=1}^{N}\left(\log\frac{S_{t_k}}{S_{t_{k-1}}}\right)^2-\frac{1}{N-1}\left(\log\frac{S_T}{S_0}\right)^2\right)-\kappa_\sigma^2,$$

where κ_σ is the volatility level. Note that the above payoff has to be multiplied by the *Vega notional*, which is part of the contract, in order to convert it into currency units.

Heston model

Consider the Heston, 1993 model driven by the stochastic differential equation

$$dv_t=(a-bv_t)dt+\sigma\sqrt{v_t}dW_t,$$

where $a,b,\sigma>0$. We have

$$\mathbb{E}[v_T]=v_0e^{-bT}+\frac{a}{b}\left(1-\mathrm{e}^{-bT}\right),$$

see Exercise 5.2-(a)), from which it follows that the realized variance $R_{0,T}^2:=\int_0^T v_t dt$ can be averaged as

$$\begin{aligned}\mathbb{E}\left[R_{0,T}^2\right] &= \mathbb{E}\left[\int_0^T v_t dt\right]\\ &= \int_0^T \mathbb{E}[v_t]dt\\ &= v_0\frac{1-\mathrm{e}^{-bT}}{b}+a\frac{\mathrm{e}^{-bT}+bT-1}{b^2},\end{aligned} \tag{5.2.2}$$

and the variance swap with strike level κ_σ^2 and payoff $R_{0,T}^2-\kappa_\sigma^2$ can be priced as

$$\mathbb{E}\left[R_{0,T}^2-\kappa_\sigma^2\right]=v_0\frac{1-\mathrm{e}^{-bT}}{b}+a\frac{\mathrm{e}^{-bT}+bT-1}{b^2}-\kappa_\sigma^2.$$

*We apply the identity $\sum_{k=1}^n\left(a_k-\sum_{l=1}^n a_l\right)^2=\sum_{k=1}^n a_k^2-\left(\sum_{l=1}^n a_l\right)^2$.

We can also express the variances

$$\operatorname{Var}[v_T] = v_0 \frac{\sigma^2}{b}\left(\mathrm{e}^{-bT} - \mathrm{e}^{-2bT}\right) + \frac{a\sigma^2}{2b^2}\left(1 - \mathrm{e}^{-bT}\right)^2,$$

and

$$\begin{aligned}\operatorname{Var}\left[R_{0,T}^2\right] &= v_0\sigma^2 \frac{1 - 2bTe^{-bT} - \mathrm{e}^{-2bT}}{b^3} \\ &\quad + a\sigma^2 \frac{\mathrm{e}^{-2bT} + 2bT + 4(bT+1)\mathrm{e}^{-bT} - 5}{2b^4},\end{aligned} \tag{5.2.3}$$

see *e.g.* Relation (3.3) in Prayoga and Privault, 2017.

Stochastic volatility

In what follows, we assume that the risky asset price process is given by

$$\frac{dS_t}{S_t} = rdt + \sigma_t dB_t, \tag{5.2.4}$$

under the risk-neutral probability measure $\mathbb{P}^*$, *i.e.*

$$S_t = S_0 \exp\left(rt + \int_0^t \sigma_s dB_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right), \quad t \geqslant 0, \tag{5.2.5}$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$ is a stochastic volatility process. In this setting, we have the following proposition.

Proposition 5.1 Denoting by $F_0 := \mathrm{e}^{rT} S_0$ the futures contract price on S_T, we have the relation

$$\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2\,\mathbb{E}^*\left[\log \frac{F_0}{S_T}\right]. \tag{5.2.6}$$

Proof. From (5.2.5), we have

$$\begin{aligned}\mathbb{E}^*\left[\log \frac{S_T}{F_0}\right] &= \mathbb{E}^*\left[\log \frac{S_T}{S_0}\right] - rT \\ &= \mathbb{E}^*\left[\int_0^T \sigma_t dB_t - \frac{1}{2}\int_0^T \sigma_t^2 dt\right] \\ &= -\frac{1}{2}\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right].\end{aligned}$$

□

Independent stochastic volatility

In this subsection, we assume that the stochastic volatility process $(\sigma_t)_{t\in\mathbb{R}_+}$ in (5.2.4) is *independent* of the Brownian motion $(B_t)_{t\in\mathbb{R}_+}$.

Lemma 5.2 (Carr and R. Lee, 2008, Proposition 5.1) Assume that $(\sigma_t)_{t\in\mathbb{R}_+}$ is *independent* of $(B_t)_{t\in\mathbb{R}_+}$. Then, for every $\lambda > 0$ we have

$$\mathbb{E}^*\left[\exp\left(\lambda \int_0^T \sigma_t^2 dt\right)\right] = \mathrm{e}^{-rp_\lambda^\pm T}\,\mathbb{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda^\pm}\right], \tag{5.2.7}$$

where $p_\lambda^\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$.

Proof. Letting $(\mathscr{F}_t^\sigma)_{t\in\mathbb{R}_+}$ denote the filtration generated by the process $(\sigma_t)_{t\in\mathbb{R}_+}$, we have

$$\begin{aligned}
\mathrm{e}^{-rp_\lambda T}\,\mathbf{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda}\,\middle|\,\mathscr{F}_T^\sigma\right] &= \mathbf{E}^*\left[\exp\left(p_\lambda\int_0^T\sigma_t dB_t - \frac{p_\lambda}{2}\int_0^T\sigma_t^2 dt\right)\,\middle|\,\mathscr{F}_T^\sigma\right]\\
&= \exp\left(-\frac{p_\lambda}{2}\int_0^T\sigma_t^2 dt\right)\mathbf{E}^*\left[\exp\left(p_\lambda\int_0^T\sigma_t dB_t\right)\,\middle|\,\mathscr{F}_T^\sigma\right]\\
&= \exp\left(-\frac{p_\lambda}{2}\int_0^T\sigma_t^2 dt\right)\exp\left(\frac{p_\lambda^2}{2}\int_0^T\sigma_t^2 dt\right)\\
&= \exp\left(\frac{p_\lambda}{2}(p_\lambda-1)\int_0^T\sigma_t^2 dt\right)\\
&= \exp\left(\lambda\int_0^T\sigma_t^2 dt\right),
\end{aligned}$$

provided that $\lambda = p_\lambda(p_\lambda-1)/2$, and in this case we have

$$\begin{aligned}
\mathrm{e}^{-rp_\lambda T}\,\mathbf{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda}\right] &= \mathrm{e}^{-rp_\lambda T}\,\mathbf{E}^*\left[\mathbf{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda}\,\middle|\,\mathscr{F}_T^\sigma\right]\right]\\
&= \mathbf{E}^*\left[\exp\left(\lambda\int_0^T\sigma_t^2 dt\right)\right].
\end{aligned}$$

It remains to note that the equation $\lambda = p_\lambda(p_\lambda-1)/2$, *i.e.* $p_\lambda^2 - p_\lambda - 2\lambda = 0$, has for solutions

$$p_\lambda^\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4}+2\lambda},$$

with $p_\lambda^- < 0 < p_\lambda^+$ when $\lambda > 0$. □

By differentiating the moment generating function computed in Lemma 5.2 with respect to $\lambda > 0$, we can compute the first moment of the realized variance $R_{0,T}^2 = \int_0^T \sigma_t^2 dt$ in the following corollary.

Corollary 5.3 Assume that $(\sigma_t)_{t\in\mathbb{R}_+}$ is *independent* of $(B_t)_{t\in\mathbb{R}_+}$. Denoting by $F_0 := \mathrm{e}^{rT}S_0$ the futures contract price on S_T, we have

$$\mathbf{E}^*\left[\int_0^T\sigma_t^2 dt\right] = 2\,\mathbf{E}^*\left[\frac{S_T}{F_0}\log\frac{S_T}{F_0}\right].$$

Proof. Rewriting (5.2.7) as

$$\mathbf{E}^*\left[\exp\left(\lambda\int_0^T\sigma_t^2 dt\right)\right] = \mathbf{E}^*\left[\exp\left(-rp_\lambda^\pm T + p_\lambda^\pm\log\frac{S_T}{S_0}\right)\right]$$

and differentiating this relation with respect to λ, we get

$$\begin{aligned}
\mathbf{E}^*\left[\int_0^T\sigma_t^2 dt\exp\left(\lambda\int_0^T\sigma_t^2 dt\right)\right] &= -rp_\lambda' T\,\mathbf{E}^*\left[\exp\left(-rp_\lambda^\pm T + p_\lambda^\pm\log\frac{S_T}{S_0}\right)\right]\\
&\quad + p_\lambda'\,\mathbf{E}^*\left[\exp\left(-rp_\lambda^\pm T + p_\lambda^\pm\log\frac{S_T}{S_0}\right)\log\frac{S_T}{S_0}\right]\\
&= \mp\frac{rT}{\sqrt{2\lambda+1/4}}\mathbf{E}^*\left[\exp\left(-rp_\lambda^\pm T\right)\left(\frac{S_T}{S_0}\right)^{p_\lambda^\pm}\right]
\end{aligned}$$

$$\pm\frac{1}{\sqrt{2\lambda+1/4}}\mathbb{E}^*\left[\exp\left(-rp_\lambda^\pm T+p_\lambda^\pm\log\frac{S_T}{S_0}\right)\log\frac{S_T}{S_0}\right],$$

which, when $\lambda = 0$, recovers (5.2.6) in Proposition 5.1 as

$$\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2rT - 2\,\mathbb{E}^*\left[\log\frac{S_T}{S_0}\right] = -2\,\mathbb{E}^*\left[\int_0^T \sigma_t dB_t - \frac{1}{2}\int_0^T \sigma_t^2 dt\right]$$

if $p_0^- = 0$, and yields

$$\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt\right] = 2\,\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\frac{S_T}{S_0}\log\frac{\mathrm{e}^{-rT}S_T}{S_0}\right]$$

for $p_0^+ = 1$. □

5.3 Realized Variance Options

In this section, we consider the realized variance call option with payoff

$$\left(\int_0^T \sigma_t^2 dt - \kappa_\sigma^2\right)^+.$$

Proposition 5.4 Under the condition $\int_0^t \sigma_u^2 du \geqslant \kappa_\sigma^2$, the price of the realized variance call option in the money is given by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\int_0^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \middle| \mathcal{F}_t\right]\\
&\quad= \mathrm{e}^{-(T-t)r}\int_0^t \sigma_u^2 du - \mathrm{e}^{-(T-t)r}\kappa_\sigma^2 + \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right].
\end{aligned}$$

Proof. In case $\int_0^t \sigma_u^2 du \geqslant \kappa_\sigma^2$, we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\int_0^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \middle| \mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(x+\int_t^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \middle| \mathcal{F}_t\right]_{x=\int_0^t \sigma_u^2 du}\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[x+\int_t^T \sigma_u^2 du - \kappa_\sigma^2 \middle| \mathcal{F}_t\right]_{x=\int_0^t \sigma_u^2 du}\\
&= \mathrm{e}^{-(T-t)r}\int_0^t \sigma_u^2 du - \mathrm{e}^{-(T-t)r}\kappa_\sigma^2 + \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right].
\end{aligned}$$

□

In Proposition 5.4, the futures contract price $\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \middle| \mathcal{F}_t\right]$ can be computed from Proposition 5.1.

Lognormal approximation

When $R_{0,t}^2 := \int_0^t \sigma_u^2 du < \kappa_\sigma^2$, in order to estimate the price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(x+\int_t^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \Bigg| \mathscr{F}_t\right]_{x=\int_0^t \sigma_u^2 du}, \tag{5.3.1}$$

of the realized variance call option out of the money, we can approximate $R_{t,T} := \sqrt{\int_t^T \sigma_u^2 du}$ by a lognormal random variable

$$R_{t,T} = \sqrt{\int_t^T \sigma_u^2 du} \simeq \mathrm{e}^{\tilde{\mu}_{t,T}+\tilde{\sigma}_{t,T}X}$$

with mean $\tilde{\mu}_{t,T}$ and variance $\eta_{t,T}^2$, where $X \simeq \mathcal{N}(0, T-t)$ is a centered Gaussian random variable with variance $T-t$.

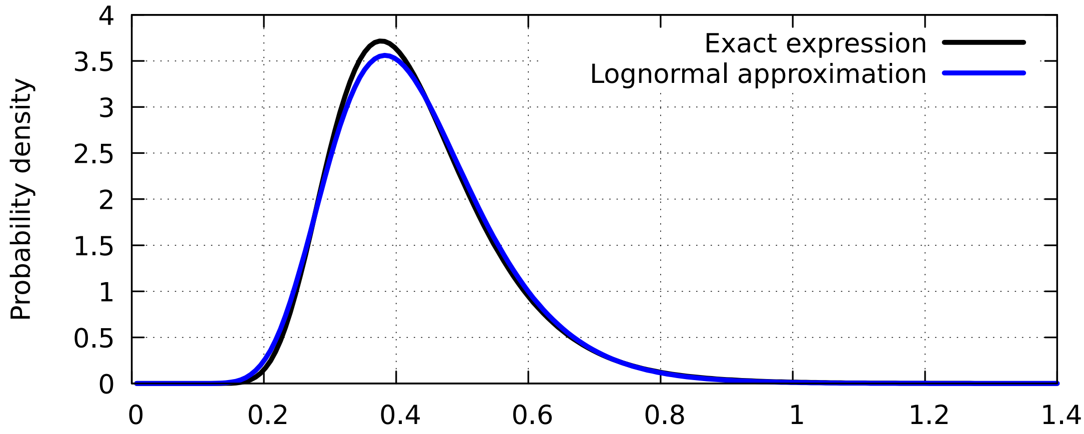


Figure 5.2: Fitting of a lognormal probability density function (example).

Proposition 5.5 (Lognormal approximation by volatility swap moment matching). The probability density function $\varphi_{R_{t,T}}$ of $R_{t,T} := \sqrt{\int_t^T \sigma_u^2 du}$ can be approximated as

$$\varphi_{R_{t,T}}(x) \approx \frac{1}{x\tilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}} \exp\left(-\frac{(\tilde{\mu}_{t,T}-\log x)^2}{2(T-t)\tilde{\sigma}_{t,T}^2}\right), \quad x>0, \tag{5.3.2}$$

where

$$\tilde{\mu}_{t,T} := \log\left(\frac{(\mathbb{E}[R_{t,T}])^2}{\sqrt{\mathbb{E}\left[R_{t,T}^2\right]}}\right) \quad \text{and} \quad \tilde{\sigma}_{t,T}^2 := \frac{2}{T-t}\log\left(\frac{\sqrt{\mathbb{E}\left[R_{t,T}^2\right]}}{\mathbb{E}[R_{t,T}]}\right), \tag{5.3.3}$$

and $\mathbb{E}\left[R_{t,T}^2\right]$, $\mathbb{E}[R_{t,T}]$ can be estimated from realized variance and volatility swap prices.

Proof. The parameters $\tilde{\mu}_{t,T}$ and $\tilde{\sigma}_{t,T}$ are estimated by matching the first and second moments $\mathbb{E}\left[R_{t,T}\right]$ and $\mathbb{E}\left[R_{t,T}^2\right]$ of $R_{t,T}$ to those of the lognormal distribution with mean $\tilde{\mu}_{t,T}$ and variance $(T-t)\tilde{\sigma}_{t,T}^2$, which yields

$$\mathbb{E}[R_{t,T}] = \mathrm{e}^{\tilde{\mu}_{t,T}+(T-t)\tilde{\sigma}_{t,T}^2/2} \quad \text{and} \quad \mathbb{E}\left[R_{t,T}^2\right] = \mathrm{e}^{2(\tilde{\mu}_{t,T}+(T-t)\tilde{\sigma}_{t,T}^2)},$$

and (5.3.3). □

By (5.3.3), the parameters $\tilde{\mu}_{t,T}$ and $\tilde{\sigma}^2_{t,T}$ can be estimated from the realized volatility swap price

$$e^{-(T-t)r}\mathbb{E}^*\left[R_{t,T} \mid \mathcal{F}_t\right] = e^{-(T-t)r}\mathbb{E}^*\left[\sqrt{\int_t^T \sigma_u^2 du}\,\middle|\,\mathcal{F}_t\right],$$

and from the realized variance swap price

$$e^{-(T-t)r}\mathbb{E}^*\left[R^2_{t,T} \mid \mathcal{F}_t\right] = e^{-(T-t)r}\mathbb{E}^*\left[\int_t^T \sigma_u^2 du\,\middle|\,\mathcal{F}_t\right].$$

By Proposition 5.7, we can estimate the price (5.3.1) of the realized variance call option by approximating $R^2_{t,T} = \int_t^T \sigma_u^2 du$ by a lognormal random variable. We refer to § 8.4 in Friz and Gatheral, 2005 or to Relation (11.15) page 152 of Gatheral, 2006 for the following result.

Proposition 5.6 Under the lognormal approximation (5.3.2), the price

$$\mathrm{VC}_{t,T}(\kappa_\sigma) = e^{-(T-t)r}\mathbb{E}\left[\left(x + R^2_{t,T} - \kappa_\sigma^2\right)^+\right]_{x=R^2_{0,t}}$$

of the realized variance call option can be approximated as

$$\mathrm{VC}_{t,T}(\kappa_\sigma) \approx e^{-(T-t)r}\mathbb{E}\left[R^2_{t,T}\right]\Phi(d_+) - e^{-(T-t)r}\left(\kappa_\sigma^2 - R^2_{0,t}\right)\Phi(d_-), \qquad (5.3.4)$$

where

$$\begin{aligned} d_+ &:= \frac{\log\left((\mathbb{E}[R_{t,T}])^2/\left(\kappa_\sigma^2 - R^2_{0,t}\right)\right)}{2\tilde{\sigma}_{t,T}\sqrt{T-t}} + 2\tilde{\sigma}_{t,T}\sqrt{T-t} \\ &= \frac{-\log\left(\kappa_\sigma^2 - R^2_{0,t}\right) + 2\tilde{\mu}_{t,T} + 4(T-t)\tilde{\sigma}^2_{t,T}}{2\tilde{\sigma}_{t,T}\sqrt{T-t}}, \end{aligned}$$

and

$$d_- := d_+ - 2\tilde{\sigma}_{t,T}\sqrt{T-t} = \frac{2\tilde{\mu}_{t,T} - \log\left(\kappa_\sigma^2 - R^2_{0,t}\right)}{2\tilde{\sigma}_{t,T}\sqrt{T-t}},$$

and Φ denotes the standard Gaussian cumulative distribution function.

Proof. The lognormal approximation (5.3.4) by realized variance moment matching states that

$$\varphi_{R_{t,T}}(x) \approx \frac{1}{x\tilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}} e^{-(-\tilde{\mu}_{t,T}+\log x)^2/(2(T-t)\tilde{\sigma}^2_{t,T})}, \quad x > 0,$$

or equivalently

$$\begin{aligned} \varphi_{R^2_{t,T}}(x) &= \frac{1}{2\sqrt{x}}\varphi_{R_{t,T}}(\sqrt{x}) \\ &\approx \frac{1}{2x\tilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}} e^{-(-2\tilde{\mu}_{t,T}+\log x)^2/(2(T-t)(2\tilde{\sigma}_{t,T})^2)}, \quad x > 0. \end{aligned}$$

In other words, the distribution of $R^2_{t,T}$ is approximately that of $e^{2\tilde{\mu}_{t,T}+2\tilde{\sigma}_{t,T}X}$ where $X \simeq \mathcal{N}(0, T-t)$, hence

$$\begin{aligned} \mathrm{VC}_{t,T}(\kappa_\sigma) &= e^{-(T-t)r}\mathbb{E}\left[\left(x + R^2_{t,T} - \kappa_\sigma^2\right)^+\right]_{x=R^2_{0,t}} \\ &= e^{-(T-t)r}\int_{\kappa_\sigma}^\infty \left(y - \left(\kappa_\sigma^2 - R^2_{0,t}\right)\right)^+ \varphi_{R^2_{t,T}}(y)dy \\ &\approx e^{-(T-t)r}\mathbb{E}\left[\left(e^{2\tilde{\mu}_{t,T}+2\tilde{\sigma}_{t,T}X} - \left(\kappa_\sigma^2 - x\right)\right)^+\right]_{x=R^2_{0,t}} \end{aligned}$$

$$= \mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{2\tilde{\mu}_{t,T}+2(T-t)\tilde{\sigma}_{t,T}^2}\Phi(d_+) - \left(\kappa_\sigma^2 - R_{0,t}^2\right)\Phi(d_-)\right)$$
$$= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[R_{t,T}^2\right]\Phi(d_+) - \mathrm{e}^{-(T-t)r}\left(\kappa_\sigma^2 - R_{0,t}^2\right)\Phi(d_-). \tag{5.3.5}$$

□

In order to estimate the price

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2\right)^+ \,\middle|\, \mathscr{F}_t\right]_{x=\int_0^t \sigma_u^2 du},$$

of the realized variance call option when $R_{0,t} := \sqrt{\int_0^t \sigma_u^2 du} < \kappa_\sigma$, we can also approximate $R_{t,T}^2 := \int_t^T \sigma_u^2 du$ by a lognormal random variable

$$R_{t,T}^2 = \int_t^T \sigma_u^2 du \simeq \mathrm{e}^{\tilde{\mu}_{t,T}+\tilde{\sigma}_{t,T}X}$$

with mean $\tilde{\mu}_{t,T}$ and variance $\sigma_{t,T}^2$, where $X \simeq \mathcal{N}(0,1)$ is a standard normal random variable.

Proposition 5.7 (Lognormal approximation by realized variance moment matching). Under the lognormal approximation, the probability density function $\varphi_{R_{t,T}^2}$ of $R_{t,T}^2 := \int_t^T \sigma_u^2 du$ can be approximated as

$$\varphi_{R_{t,T}^2}(x) \approx \frac{1}{x\tilde{\sigma}_{t,T}\sqrt{2(T-t)\pi}} \exp\left(-\frac{(\tilde{\mu}_{t,T} - \log x)^2}{2(T-t)\tilde{\sigma}_{t,T}^2}\right), \quad x > 0, \tag{5.3.6}$$

where

$$\tilde{\mu}_{t,T} := -(T-t)\frac{\tilde{\sigma}_{t,T}^2}{2} + \log \mathbb{E}[R_{t,T}^2], \tag{5.3.7}$$

and

$$\tilde{\sigma}_{t,T}^2 = \frac{1}{T-t}\log\left(1 + \frac{\mathrm{Var}[R_{t,T}^2]}{(\mathbb{E}[R_{t,T}^2])^2}\right). \tag{5.3.8}$$

Proof. The parameters $\tilde{\mu}_{t,T}$ and $\tilde{\sigma}_{t,T}$ are estimated by matching the first and second moments $\mathbb{E}\left[R_{t,T}^2\right]$ and $\mathbb{E}\left[R_{t,T}^4\right]$ of $R_{t,T}^4$ to those of the lognormal distribution with mean $\tilde{\mu}_{t,T}$ and variance $(T-t)\tilde{\sigma}_{t,T}^2$, which yields

$$\mathbb{E}\left[R_{t,T}^2\right] = \mathrm{e}^{\tilde{\mu}_{t,T}+(T-t)\tilde{\sigma}_{t,T}^2/2}, \qquad \mathbb{E}\left[R_{t,T}^4\right] = \mathrm{e}^{2(\tilde{\mu}_{t,T}+(T-t)\tilde{\sigma}_{t,T}^2)},$$

and

$$\tilde{\mu}_{t,T} = -(T-t)\frac{\tilde{\sigma}_{t,T}^2}{2} + \log \mathbb{E}\left[R_{t,T}^2\right] \text{ and } \tilde{\sigma}_{t,T}^2 := \frac{1}{T-t}\log\left(\frac{\mathbb{E}\left[R_{t,T}^4\right]}{\left(\mathbb{E}\left[R_{t,T}^2\right]\right)^2}\right).$$

□

By (5.3.7)-(5.3.8), the parameters $\tilde{\mu}_{t,T}$ and $\tilde{\sigma}_{t,T}^2$ can be estimated from the realized variance swap price

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[R_{t,T}^2 \mid \mathscr{F}_t\right] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\int_t^T \sigma_u^2 du \,\middle|\, \mathscr{F}_t\right],$$

and from the realized variance power option price

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[R_{t,T}^4 \mid \mathscr{F}_t\right] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\int_t^T \sigma_u^2 du\right)^2 \middle| \mathscr{F}_t\right].$$

The next proposition is obtained by the same argument as in the proof of Proposition 5.6.

Proposition 5.8 Under the lognormal approximation (5.3.6), the price

$$\mathrm{VC}_{t,T}(\kappa_\sigma) = \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\left(x + R_{t,T}^2 - \kappa_\sigma^2\right)^+\right]_{x=R_{0,t}}$$

of the realized variance call option can be approximated as

$$\mathrm{VC}_{t,T}(\kappa_\sigma) \approx \mathrm{e}^{-(T-t)r}\mathbb{E}\left[R_{t,T}^2\right]\Phi(d_+) - \mathrm{e}^{-(T-t)r}\left(\kappa_\sigma^2 - R_{0,t}^2\right)\Phi(d_-), \tag{5.3.9}$$

where

$$\begin{aligned} d_+ \quad &:= \quad \frac{\log\left(\mathbb{E}\left[R_{t,T}^2\right]/\left(\kappa_\sigma^2 - R_{0,t}^2\right)\right)}{\widetilde{\sigma}_{t,T}\sqrt{T-t}} + \widetilde{\sigma}_{t,T}\frac{\sqrt{T-t}}{2} \\ &= \quad \frac{-\log\left(\kappa_\sigma^2 - R_{0,t}^2\right) + \widetilde{\mu}_{t,T} + (T-t)\widetilde{\sigma}_{t,T}^2}{\widetilde{\sigma}_{t,T}\sqrt{T-t}}, \end{aligned}$$

and

$$d_- := d_+ - \widetilde{\sigma}_{t,T}\sqrt{T-t} = \frac{\widetilde{\mu}_{t,T} - \log\left(\kappa_\sigma^2 - R_{0,t}^2\right)}{\widetilde{\sigma}_{t,T}\sqrt{T-t}},$$

and Φ denotes the standard Gaussian cumulative distribution function.

Note that, using the integral identity

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - \mathrm{e}^{-\lambda x}\right)\frac{d\lambda}{\lambda^{3/2}},$$

see *e.g.* Relation 3.434.1 in Gradshteyn and Ryzhik, 2007 and Exercise 6.9-(a)), the realized volatility swap price $\mathbb{E}[R_{t,T}]$ can be expressed as

$$\mathbb{E}[R_{t,T}] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - \mathbb{E}\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]\right)\frac{d\lambda}{\lambda^{3/2}}, \tag{5.3.10}$$

see § 3.1 in Friz and Gatheral, 2005, where $\mathbb{E}\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]$ can be expressed from Lemma 5.2. In particular, by *e.g.* Relation (3.25) in Brigo and Mercurio, 2006, in the Cox, Ingersoll, and Ross, 1985 (CIR)

$$dv_t = (a - bv_t)dt + \eta\sqrt{v_t}dW_t$$

variance model with $v_t = \sigma_t^2$, we have

$$\begin{aligned} &\mathbb{E}\left[\mathrm{e}^{-\lambda R_{0,T}^2}\right] \\ &= \quad \exp\left(-\frac{2v_0\lambda\left(1 - e^{-\bar{b}T}\right)}{\bar{b} + b + (\bar{b} - b)\,\mathrm{e}^{-\bar{b}T}} - \frac{a}{\eta^2}(\bar{b} - b)T - \frac{2a}{\eta^2}\log\frac{\bar{b} + b + (\bar{b} - b)e^{-\bar{b}T}}{2\bar{b}}\right), \end{aligned}$$

where $\bar{b} := \sqrt{b^2 + 2\lambda\eta^2}$.

Gamma approximation

In case $R^2_{0,t} = \int_0^t \sigma_u^2 du < \kappa_\sigma^2$, the realized variance call option price

$$\mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(x + \int_t^T \sigma_u^2 du - \kappa_\sigma^2 \right)^+ \Big| \mathcal{F}_t \right]_{x=\int_0^t \sigma_u^2 du}$$

can be estimated by approximating $R^2_{t,T} = \int_t^T \sigma_u^2 du$ by a gamma random variable as in the probability density graph of Figure 5.3.

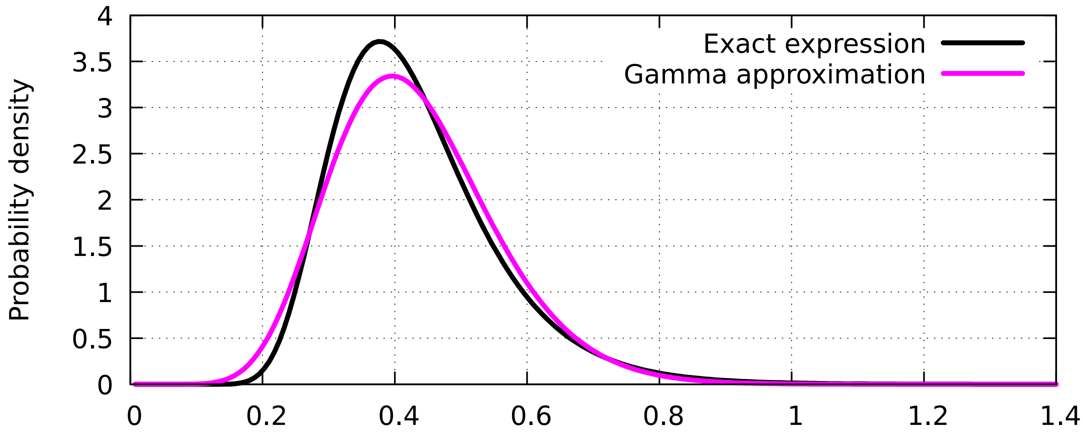


Figure 5.3: Fitting of a gamma probability density function (example).

Proposition 5.9 (Gamma approximation). Under the gamma approximation the probability density function $\varphi_{R^2_{t,T}}$ of $R^2_{t,T} := \int_t^T \sigma_u^2 du$ can be approximated as

$$\varphi_{R^2_{t,T}}(x) \approx \frac{(x/\theta_{t,T})^{-1+\nu_{t,T}}}{\theta_{t,T}\Gamma(\nu_{t,T})} \mathrm{e}^{-x/\theta_{t,T}}, \quad x > 0, \tag{5.3.11}$$

where

$$\theta_{t,T} = \frac{\mathrm{Var}\left[R^2_{t,T}\right]}{\mathbb{E}\left[R^2_{t,T}\right]} \quad \text{and} \quad \nu_{t,T} = \frac{\mathbb{E}\left[R^2_{t,T}\right]}{\theta_{t,T}} = \frac{\left(\mathbb{E}\left[R^2_{t,T}\right]\right)^2}{\mathrm{Var}\left[R^2_{t,T}\right]}. \tag{5.3.12}$$

Proof. The parameters $\theta_{t,T}$, $\nu_{t,T}$ are estimated by matching the first and second moments of $R^2_{t,T}$ to those of the gamma distribution with scale and shape parameters $\theta_{t,T}$ and $\nu_{t,T}$, which yields

$$\mathbb{E}\left[R^2_{t,T}\right] = \nu_{t,T}\theta_{t,T} \quad \text{and} \quad \mathrm{Var}\left[R^2_{t,T}\right] = \nu_{t,T}\theta^2_{t,T},$$

and (5.3.12). □

Proposition 5.10 Under the gamma approximation (5.3.11), the price

$$\mathrm{EA}(\kappa_\sigma, T) = \mathrm{e}^{-(T-t)r} \mathbb{E}\left[\left(x + R^2_{t,T} - \kappa_\sigma^2\right)^+\right]_{x=R^2_{0,t}}$$

of the realized variance call option can be approximated as

$$\mathrm{EA}(\kappa_\sigma, T) = \mathrm{e}^{-(T-t)r} \left(\mathbb{E}\left[R^2_{t,T}\right] Q\left(1 + \nu_{t,T}, \frac{\kappa_\sigma^2}{\theta_{t,T}}\right) - \kappa_\sigma^2 Q\left(\nu_{t,T}, \frac{\kappa_\sigma^2}{\theta_{t,T}}\right)\right), \tag{5.3.13}$$

where

$$Q(\lambda,z) := \frac{1}{\Gamma(\lambda)}\int_z^\infty t^{\lambda-1}\mathrm{e}^{-t}dt, \qquad z>0,$$

is the (normalized) upper incomplete gamma function.

Proof. Using the gamma approximation

$$\varphi_{R^2_{t,T}}(x) \approx \frac{\mathrm{e}^{-x/\theta_{t,T}}}{\Gamma(\nu_{t,T})}\frac{x^{-1+\nu_{t,T}}}{(\theta_{t,T})^{\nu_{t,T}}}, \tag{5.3.14}$$

where $\theta_{t,T}$ and $\nu_{t,T}$ are given by (5.3.12), we have

$$\begin{aligned}
\mathbb{E}\left[\left(R^2_{t,T}-\kappa^2_\sigma\right)^+\right] &= \int_{\kappa^2_\sigma}^\infty (x-\kappa^2_\sigma)^+\varphi_{R^2_{t,T}}(x)dx \\
&\approx \frac{1}{\Gamma(\nu_{t,T})}\int_{\kappa^2_\sigma}^\infty (x-\kappa^2_\sigma)\frac{x^{-1+\nu_{t,T}}}{(\theta_{t,T})^{\nu_{t,T}}}\mathrm{e}^{-x/\theta_{t,T}}dx \\
&= \frac{1}{\Gamma(\nu_{t,T})}\int_{\kappa^2_\sigma}^\infty (x/\theta_{t,T})^{\nu_{t,T}}\mathrm{e}^{-x/\theta_{t,T}}dx - \frac{\kappa^2_\sigma}{\Gamma(\nu_{t,T})}\int_{\kappa^2_\sigma}^\infty \frac{x^{-1+\nu_{t,T}}}{(\theta_{t,T})^{\nu_{t,T}}}\mathrm{e}^{-x/\theta_{t,T}}dx \\
&= \frac{\theta_{t,T}}{\Gamma(\nu_{t,T})}\int_{\kappa^2_\sigma/\theta_{t,T}}^\infty x^{\nu_{t,T}}\mathrm{e}^{-x}dx - \frac{\kappa^2_\sigma}{\Gamma(\nu_{t,T})}\int_{\kappa^2_\sigma/\theta_{t,T}}^\infty x^{-1+\nu_{t,T}}\mathrm{e}^{-x}dx \\
&= \theta_{t,T}\nu_{t,T}Q\left(1+\nu_{t,T},\frac{\kappa^2_\sigma}{\theta_{t,T}}\right) - \kappa^2_\sigma Q\left(\nu_{t,T},\frac{\kappa^2_\sigma}{\theta_{t,T}}\right),
\end{aligned}$$

where

$$Q(\lambda,z) := \frac{1}{\Gamma(\lambda)}\int_z^\infty t^{\lambda-1}\mathrm{e}^{-t}dt, \qquad z>0,$$

is the (normalized) upper incomplete gamma function, which yields

$$\begin{aligned}
\mathrm{EA}(\kappa_\sigma,T) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(x+R^2_{t,T}-\kappa^2_\sigma\right)^+\right] \\
&\approx \mathrm{e}^{-(T-t)r}\left(\nu_{t,T}\theta_{t,T}Q\left(1+\nu_{t,T},\frac{\kappa^2_\sigma}{\theta_{t,T}}\right) - \kappa^2_\sigma Q\left(\nu_{t,T},\frac{\kappa^2_\sigma}{\theta_{t,T}}\right)\right) \\
&= \mathrm{e}^{-(T-t)r}\left(\mathbb{E}\left[R^2_{t,T}\right]Q\left(1+\nu_{t,T},\frac{\kappa^2_\sigma}{\theta_{t,T}}\right) - \kappa^2_\sigma Q\left(\nu_{t,T},\frac{\kappa^2_\sigma}{\theta_{t,T}}\right)\right).
\end{aligned} \tag{5.3.15}$$

□

Realized variance options in the Heston model

Taking $r=0$, $t=0$ and $R_{0,0}=0$, and using the parameters

$$\sigma = 0.39,\ b=1.15,\ a=0.04\times b,\ v_0=0.04,\ T=1$$

in the Heston stochastic differential equation

$$dv_t = (a-bv_t)dt + \sigma\sqrt{v_t}dW_t,$$

in Figures 5.4-5.5 we plot the graphs of the lognormal volatility swap and realized variance moment matching approximations (5.3.4), (5.3.9), and of the gamma approximation (5.3.13) for realized variance call option prices with $\kappa^2_\sigma \in [0,0.2]$, based on the expressions (5.2.2)-(5.2.3) of $\mathbb{E}\left[R^2_{0,T}\right]$ and $\mathrm{Var}\left[R^2_{0,T}\right]$.

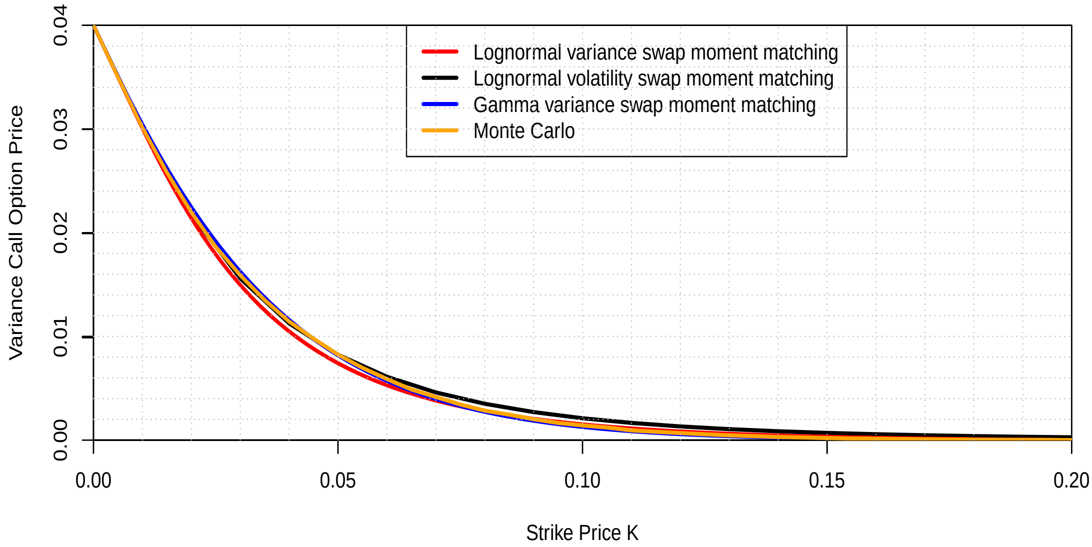


Figure 5.4: One-year variance call option prices with $b = 0.15$.

The graphs of Figures 5.4-5.5 are obtained using this **code** and **data file**.

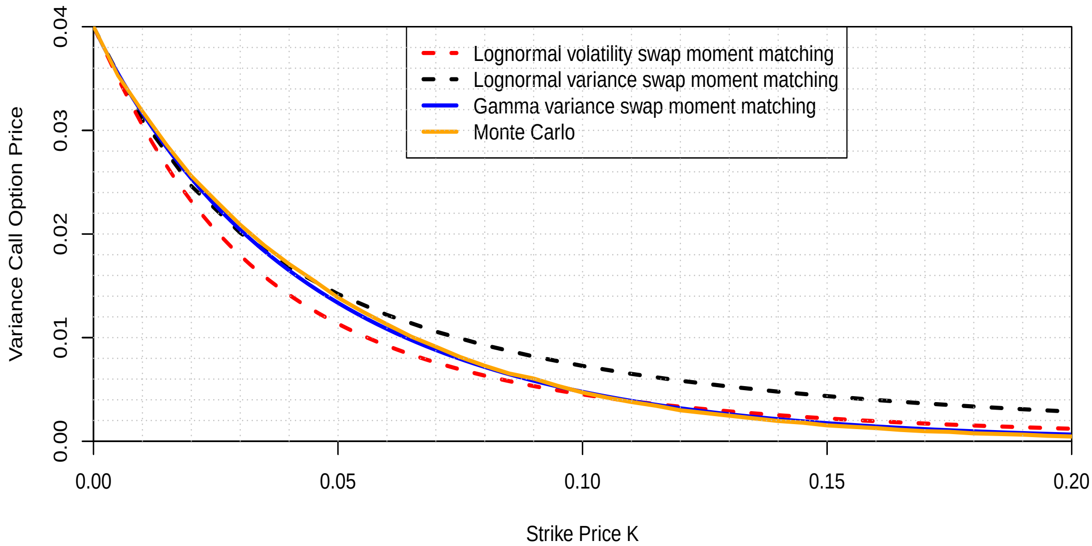


Figure 5.5: One-year variance call option prices with $b = -0.05$.

As can be checked from in Figure 5.5 with

$$\sigma = 0.39, \ b = 1.15, \ a = 0.04 \times b, \ v_0 = 0.04, \ T = 1,$$

the gamma approximation (5.3.13) appears to be more accurate than the lognormal approximations for large values of κ_σ^2, which can be consistent with the fact that the long run distribution of the CIR-Heston process has the gamma probability density function

$$f(x) = \frac{1}{\Gamma(2a/\sigma^2)} \left(\frac{2b}{\sigma^2}\right)^{2a/\sigma^2} x^{-1+2a/\sigma^2} \mathrm{e}^{-2bx/\sigma^2}, \quad x > 0.$$

with shape parameter $2a/\sigma^2$ and scale parameter $\sigma^2/(2b)$, which is also the *invariant distribution* of v_t.

5.4 European Options - PDE Method

In what follows we consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ in the stochastic volatility model

$$dS_t = rS_t dt + S_t\sqrt{v_t}dB_t^{(1)}$$

under the risk-neutral probability measure $\mathbb{P}^*$, where $(v_t)_{t\in\mathbb{R}_+}$ is a squared volatility (or variance) process satisfying a stochastic differential equation of the form

$$dv_t = \mu(t,v_t)dt + \beta(t,v_t)dB_t^{(2)}.$$

Here, $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are correlated standard Brownian motions started at 0 with correlation $\mathrm{Corr}\big(B_s^{(1)},B_t^{(2)}\big) = \rho\min(s,t)$ under the risk-neutral probability measure $\mathbb{P}^*$, *i.e.* $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$.

Proposition 5.11 Assume that $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ is also a standard Brownian motion under the risk-neutral probability measure[a] $\mathbb{P}^*$. Consider a vanilla option with payoff $h(S_T)$ priced as

$$V_t = f(t,v_t,S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[h(S_T) \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T.$$

The function $f(t,y,x)$ satisfies the PDE

$$\begin{aligned}
&\frac{\partial f}{\partial t}(t,v,x) + rx\frac{\partial f}{\partial x}(t,v,x) + \frac{1}{2}vx^2\frac{\partial^2 f}{\partial x^2}(t,v,x) \\
&+\mu(t,v)\frac{\partial f}{\partial v}(t,v,x) + \frac{1}{2}\beta^2(t,v)\frac{\partial^2 f}{\partial v^2}(t,v,x) + \rho\beta(t,v)x\sqrt{v}\frac{\partial^2 f}{\partial v\partial x}(t,v,x) \\
&= rf(t,v,x),
\end{aligned} \tag{5.4.1}$$

[a]When this condition is not satisfied, we need to introduce a drift that will yield a market price of volatility.

under the terminal condition $f(T,v,x) = h(x)$.

Proof. By Itô calculus with respect to the correlated Brownian motions $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$, the portfolio value $f(t,v_t,S_t)$ can be differentiated as follows:

$$\begin{aligned}
&df(t,v_t,S_t) \\
&= \frac{\partial f}{\partial t}(t,v_t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,v_t,S_t)dt + \sqrt{v_t}S_t\frac{\partial f}{\partial x}(t,v_t,S_t)dB_t^{(1)} \\
&\quad +\frac{1}{2}v_tS_t^2\frac{\partial^2 f}{\partial x^2}(t,v_t,S_t)dt + \mu(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dt \\
&\quad +\beta(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dB_t^{(2)} + \frac{1}{2}\beta^2(t,v_t)\frac{\partial^2 f}{\partial v^2}(t,v_t,S_t)dt \\
&\quad +\beta(t,v_t)\sqrt{v_t}S_t\frac{\partial^2 f}{\partial v\partial x}(t,v_t,S_t)dB_t^{(1)} \bullet dB_t^{(2)} \\
&= \frac{\partial f}{\partial t}(t,v_t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,v_t,S_t)dt + \sqrt{v_t}S_t\frac{\partial f}{\partial x}(t,v_t,S_t)dB_t^{(1)}
\end{aligned} \tag{5.4.2}$$

$$
\begin{aligned}
&+\frac{1}{2}v_tS_t^2\frac{\partial^2 f}{\partial x^2}(t,v_t,S_t)dt+\mu(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dt\\
&+\beta(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dB_t^{(2)}+\frac{1}{2}\beta^2(t,v_t)\frac{\partial^2 f}{\partial v^2}(t,v_t,S_t)dt\\
&+\rho\beta(t,v_t)\sqrt{v_t}S_t\frac{\partial^2 f}{\partial v\partial x}(t,v_t,S_t)dt.
\end{aligned}
$$

Knowing that the discounted portfolio value process $(\mathrm{e}^{-rt}f(t,v_t,S_t))_{t\in\mathbb{R}_+}$ is also a martingale under $\mathbb{P}^*$, from the relation

$$
d(\mathrm{e}^{-rt}f(t,v_t,S_t))=-r\mathrm{e}^{-rt}f(t,v_t,S_t)dt+\mathrm{e}^{-rt}df(t,v_t,S_t),
$$

we obtain

$$
\begin{aligned}
&-rf(t,v_t,S_t)dt+\frac{\partial f}{\partial t}(t,v_t,S_t)dt+rS_t\frac{\partial f}{\partial x}(t,v_t,S_t)dt+\frac{1}{2}v_tS_t^2\frac{\partial^2 f}{\partial x^2}(t,v_t,S_t)dt\\
&+\mu(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dt+\frac{1}{2}\beta^2(t,v_t)\frac{\partial^2 f}{\partial v^2}(t,v_t,S_t)dt\\
&+\rho\beta(t,v_t)S_t\sqrt{v_t}\frac{\partial^2 f}{\partial v\partial x}(t,v_t,S_t)dt\\
&=0,
\end{aligned}
$$

and the pricing PDE (5.4.1). □

Heston model

In the Heston model with $\mu(t,v)=-\lambda(v-m)$ and $\beta(t,v)=\eta\sqrt{v}$, from (5.4.1) we find the Heston PDE

$$
\begin{aligned}
&\frac{\partial f}{\partial t}(t,v,x)+rx\frac{\partial f}{\partial x}(t,v,x)+\frac{1}{2}vx^2\frac{\partial^2 f}{\partial x^2}(t,v,x)\\
&-\lambda(v-m)\frac{\partial f}{\partial v}(t,v,x)+\frac{1}{2}\eta^2v\frac{\partial^2 f}{\partial v^2}(t,v,x)+\rho\eta xv\frac{\partial^2 f}{\partial v\partial x}(t,v,x)=rf(t,v,x).
\end{aligned}
\tag{5.4.3}
$$

The solution of this PDE has been expressed in Heston, 1993 as a complex integral by inversion of a characteristic function.

Using the change of variable $y=\log x$ where with $g(t,v,y)=f(t,v,e^y)$, the PDE (5.4.3) is transformed into

$$
\begin{aligned}
&\frac{\partial g}{\partial t}(t,v,y)+\frac{1}{2}v\frac{\partial^2 g}{\partial y^2}(t,v,y)+\left(r-\frac{v}{2}\right)\frac{\partial g}{\partial y}(t,v,x)\\
&+\lambda(m-v)\frac{\partial g}{\partial v}(t,v,y)+v\frac{\eta^2}{2}\frac{\partial^2 g}{\partial v^2}(t,v,y)+\rho\eta v\frac{\partial^2 g}{\partial v\partial y}(t,v,y)=rg(t,v,y).
\end{aligned}
$$

The following proposition shows that the Fourier transform of $g(t,v,y)$ satisfies an affine PDE with respect to the variable v, when z is regarded as a constant parameter.

Proposition 5.12 Assume that $\rho=0$. The Fourier transform

$$
\widehat{g}(t,v,z):=\int_{-\infty}^{\infty}\mathrm{e}^{-iyz}g(t,v,y)dy
$$

satisfies the partial differential equation

$$\frac{\partial \widehat{g}}{\partial t}(t,v,z) + \left(irz - \frac{1}{2}vz^2\right)\widehat{g}(t,v,z) - iz\frac{1}{2}v\widehat{g}(t,v,z) \tag{5.4.4}$$
$$+ (\lambda(m-v) + i\rho\eta zv)\frac{\partial \widehat{g}}{\partial v}(t,v,z) + v\frac{\eta^2}{2}\frac{\partial^2 \widehat{g}}{\partial v^2}(t,v,z) = r\widehat{g}(t,v,z).$$

Proof. We apply the relations $i^2 = -1$ and

$$iz\widehat{g}(t,v,z) = \int_{-\infty}^{\infty} \mathrm{e}^{-iyz}\frac{\partial g}{\partial y}(t,v,y)dy.$$

□

The equation (5.4.4) can be solved in closed form, and the final solution $g(t,v,y)$ can then be obtained by the Fourier inversion relation

$$g(t,v,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{izy}\widehat{g}(t,v,z)dz, \tag{5.4.5}$$

see Heston, 1993, Attari, 2004, Albrecher et al., 2007, and Rouah, 2013 for details.

Delta hedging in the Heston model

Consider a portfolio of the form

$$V_t = \eta_t \mathrm{e}^{rt} + \xi_t S_t$$

based on the riskless asset $A_t = \mathrm{e}^{rt}$ and on the risky asset S_t. When this portfolio is self-financing we have

$$\begin{aligned} dV_t &= df(t,v_t,S_t) \\ &= r\eta_t \mathrm{e}^{rt}dt + \xi_t dS_t \\ &= r\eta_t \mathrm{e}^{rt}dt + \xi_t\left(rS_t dt + S_t\sqrt{v_t}dB_t^{(1)}\right) \\ &= rV_t dt + \xi_t S_t\sqrt{v_t}dB_t^{(1)} \\ &= rf(t,v_t,S_t)dt + \xi_t S_t\sqrt{v_t}dB_t^{(1)}. \end{aligned} \tag{5.4.6}$$

However, trying to match (5.4.2) to (5.4.6) yields

$$\sqrt{v_t}S_t\frac{\partial f}{\partial x}(t,v_t,S_t)dB_t^{(1)} + \beta(t,v_t)\frac{\partial f}{\partial v}(t,v_t,S_t)dB_t^{(2)} = \xi_t S_t\sqrt{v_t}dB_t^{(1)}, \tag{5.4.7}$$

which admits no solution unless $\beta(t,v) = 0$, *i.e.* when volatility is deterministic. A solution to that problem is to consider instead a portfolio

$$V_t = f(t,v_t,S_t) = \eta_t \mathrm{e}^{rt} + \xi_t S_t + \zeta_t P(t,v_t,S_t)$$

that includes an additional asset with price $P(t,v_t,S_t)$, which can be an option depending on the volatility v_t.

Proposition 5.13 Assume that $\rho = 0$. The self-financing portfolio allocation $(\xi_t,\zeta_t)_{t\in[0,T]}$ in the assets $\left(\mathrm{e}^{rt}, S_t, P(t,v_t,S_t)\right)_{t\in[0,T]}$ with portfolio value

$$V_t = f(t,v_t,S_t) = \eta_t \mathrm{e}^{rt} + \xi_t S_t + \zeta_t P(t,v_t,S_t) \tag{5.4.8}$$

is given by

$$\zeta_t = \frac{\dfrac{\partial f}{\partial v}(t, v_t, S_t)}{\dfrac{\partial P}{\partial v}(t, v_t, S_t)}, \tag{5.4.9}$$

and

$$\xi_t = \frac{\partial f}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\dfrac{\partial P}{\partial x}(t, v_t, S_t)}{\dfrac{\partial P}{\partial v}(t, v_t, S_t)}. \tag{5.4.10}$$

Proof. Using (5.4.8), we replace (5.4.6) with the self-financing condition

$$\begin{aligned}
dV_t &= df(t, v_t, S_t) \\
&= r\eta_t \, \mathrm{e}^{rt} dt + \xi_t dS_t + \zeta_t dP(t, v_t, S_t) \\
&= r\eta_t \, \mathrm{e}^{rt} dt + \xi_t (r S_t dt + S_t \sqrt{v_t} dB_t^{(1)}) + r\zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt \\
&\quad + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt \\
&\quad + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt \\
&\quad + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}, \\
&= (V_t - \zeta_t P(t, v_t, S_t)) r dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r\zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt \\
&\quad + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt \\
&\quad + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt \\
&\quad + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)} \\
&= r f(t, v_t, S_t) dt + \xi_t S_t \sqrt{v_t} dB_t^{(1)} + r\zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) dt \\
&\quad + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dt + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) dt + \frac{1}{2} \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) dt \\
&\quad + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) dt + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) dt \\
&\quad + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)},
\end{aligned} \tag{5.4.11}$$

and by matching (5.4.11) to (5.4.2), the equation (5.4.7) now becomes

$$\begin{aligned}
&\sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \beta(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) dB_t^{(2)} \\
&= \xi_t S_t \sqrt{v_t} dB_t^{(1)} + \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t) dB_t^{(1)} + \zeta_t \beta(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) dB_t^{(2)}.
\end{aligned}$$

This leads to the equations

$$\begin{cases} \sqrt{v_t} S_t \dfrac{\partial f}{\partial x}(t, v_t, S_t) = \xi_t S_t \sqrt{v_t} + \zeta_t S_t \sqrt{v_t} \dfrac{\partial P}{\partial x}(t, v_t, S_t), \\ \\ \beta(t, v_t) \dfrac{\partial f}{\partial v}(t, v_t, S_t) = \zeta_t \beta(t, v_t) \dfrac{\partial P}{\partial v}(t, v_t, S_t), \end{cases}$$

which show that

$$\zeta_t = \frac{\dfrac{\partial f}{\partial v}(t, v_t, S_t)}{\dfrac{\partial P}{\partial v}(t, v_t, S_t)},$$

and

$$\begin{aligned} \xi_t &= \frac{1}{S_t \sqrt{v_t}} \left(\sqrt{v_t} S_t \frac{\partial f}{\partial x}(t, v_t, S_t) - \zeta_t S_t \sqrt{v_t} \frac{\partial P}{\partial x}(t, v_t, S_t), \right) \\ &= \frac{\partial f}{\partial x}(t, v_t, S_t) - \zeta_t \frac{\partial P}{\partial x}(t, v_t, S_t) \\ &= \frac{\partial f}{\partial x}(t, v_t, S_t) - \frac{\partial f}{\partial v}(t, v_t, S_t) \frac{\dfrac{\partial P}{\partial x}(t, v_t, S_t)}{\dfrac{\partial P}{\partial v}(t, v_t, S_t)}. \end{aligned}$$

□

We note in addition that identifying the "dt" terms when equating (5.4.11) to (5.4.2) would now lead to the more complicated PDE

$$\begin{aligned} & (f(t, v_t, S_t) - \zeta_t P(t, v_t, S_t)) r + r \zeta_t S_t \frac{\partial P}{\partial x}(t, v_t, S_t) + \zeta_t \mu(t, v_t) \frac{\partial P}{\partial v}(t, v_t, S_t) \\ & \quad + \zeta_t \frac{\partial P}{\partial t}(t, v_t, S_t) + \frac{1}{2} \zeta_t S_t^2 v_t \frac{\partial^2 P}{\partial x^2}(t, v_t, S_t) + \frac{1}{2} \zeta_t \beta^2(t, v_t) \frac{\partial^2 P}{\partial v^2}(t, v_t, S_t) \\ & \quad + \rho \zeta_t \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 P}{\partial x \partial v}(t, v_t, S_t) \\ = & \ \frac{\partial f}{\partial t}(t, v_t, S_t) + r S_t \frac{\partial f}{\partial x}(t, v_t, S_t) + \frac{1}{2} v_t S_t^2 \frac{\partial^2 f}{\partial x^2}(t, v_t, S_t) + \mu(t, v_t) \frac{\partial f}{\partial v}(t, v_t, S_t) \\ & \quad + \frac{1}{2} \beta^2(t, v_t) \frac{\partial^2 f}{\partial v^2}(t, v_t, S_t) + \rho \beta(t, v_t) S_t \sqrt{v_t} \frac{\partial^2 f}{\partial v \partial x}(t, v_t, S_t), \end{aligned}$$

which can be rewritten using (5.4.9) as

$$\begin{aligned} & \frac{\partial f}{\partial v}(t, v, x) \left(-r P(t, v, x) + \frac{\partial P}{\partial t}(t, v, x) + r x \frac{\partial P}{\partial x}(t, v, x) + \mu(t, v) \frac{\partial P}{\partial v}(t, v, x) \right) \\ & + \frac{\partial f}{\partial v}(t, v, x) \left(\frac{x^2 v}{2} \frac{\partial^2 P}{\partial x^2}(t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 P}{\partial v^2}(t, v, x) + \rho \beta(t, v) x \sqrt{v} \frac{\partial^2 P}{\partial x \partial v}(t, v, x) \right) \\ & = \frac{\partial P}{\partial v}(t, v, x) \left(-r f(t, v, x) + \frac{\partial f}{\partial t}(t, v, x) + r x \frac{\partial f}{\partial x}(t, v, x) + \frac{v x^2}{2} \frac{\partial^2 f}{\partial x^2}(t, v, x) \right) \\ & + \frac{\partial P}{\partial v}(t, v, x) \left(\mu(t, v) \frac{\partial f}{\partial v}(t, v, x) + \frac{1}{2} \beta^2(t, v) \frac{\partial^2 f}{\partial v^2}(t, v, x) + \rho \beta(t, v) x \sqrt{v} \frac{\partial^2 f}{\partial v \partial x}(t, v, x) \right). \end{aligned}$$

Therefore, dividing both sides by $\frac{\partial P}{\partial v}(t,v,x)$ and letting

$$\begin{aligned}
&\lambda(t,v,x) && (5.4.12)\\
&:= \frac{1}{\frac{\partial P}{\partial v}(t,v,x)}\left(-rP(t,v,x)+rx\frac{\partial P}{\partial x}(t,v,x)+\frac{\partial P}{\partial t}(t,v,x)\right)\\
&+\frac{1}{\frac{\partial P}{\partial v}(t,v,x)}\left(\frac{x^2v}{2}\frac{\partial^2 P}{\partial x^2}(t,v,x)+\frac{1}{2}\beta^2(t,v)\frac{\partial^2 P}{\partial v^2}(t,v,x)+\rho\beta(t,v)x\sqrt{v}\frac{\partial^2 P}{\partial x\partial v}(t,v,x)\right)\\
&= \frac{1}{\frac{\partial f}{\partial v}(t,v,x)}\left(-rf(t,v,x)+\frac{\partial f}{\partial t}(t,v,x)+rx\frac{\partial f}{\partial x}(t,v,x)+\frac{vx^2}{2}\frac{\partial^2 f}{\partial x^2}(t,v,x)\right) && (5.4.13)\\
&+\frac{1}{\frac{\partial f}{\partial v}(t,v,x)}\left(\frac{1}{2}\beta^2(t,v)\frac{\partial^2 f}{\partial v^2}(t,v,x)+\rho\beta(t,v)x\sqrt{v}\frac{\partial^2 f}{\partial v\partial x}(t,v,x)\right) && (5.4.14)
\end{aligned}$$

defines a function $\lambda(t,v,x)$ that depends only on the parameters (t,v,x) and not on P, without requiring $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ to be a standard Brownian motion under $\mathbb{P}$. The function $\lambda(t,v,x)$ is linked to the market price of volatility risk, cf. Chapter 1 of Gatheral, 2006 § 2.4.1 in Fouque, G. Papanicolaou, and Sircar, 2000; Fouque, G. Papanicolaou, Sircar, and Sølna, 2011 for details.

Combining (5.4.12)-(5.4.14) allows us to rewrite the pricing PDE as

$$\begin{aligned}
&\frac{\partial f}{\partial t}(t,v,x)+rx\frac{\partial f}{\partial x}(t,v,x)+\frac{vx^2}{2}\frac{\partial^2 f}{\partial x^2}(t,v,x)+\frac{1}{2}\beta^2(t,v)\frac{\partial^2 f}{\partial v^2}(t,v,x)\\
&\qquad +\rho\beta(t,v)x\sqrt{v}\frac{\partial^2 f}{\partial v\partial x}(t,v,x)=rf(t,v,x)+\lambda(t,v,x)\frac{\partial f}{\partial v}(t,v,x),
\end{aligned}$$

and (5.4.1) corresponds to the choice $\lambda(t,v,x)=-\mu(t,v)$, which corresponds to a vanishing "market price of volatility risk".

5.5 Perturbation Analysis

We refer to Chapter 4 of Fouque, G. Papanicolaou, Sircar, and Sølna, 2011 for the contents of this section. Consider the time-rescaled model

$$\begin{cases} dS_t = rS_t dt + S_t\sqrt{v_{t/\varepsilon}}\,dB_t^{(1)} \\ \\ dv_t = \mu(v_t)dt+\beta(v_t)dB_t^{(2)}. \end{cases} \tag{5.5.1}$$

We note that $v_t^{(\varepsilon)} := v_{t/\varepsilon}$ satisfies the SDE

$$\begin{aligned}
dv_t^{(\varepsilon)} &= dv_{t/\varepsilon}\\
&\simeq v_{(t+dt)/\varepsilon}-v_{t/\varepsilon}\\
&= v_{t/\varepsilon+dt/\varepsilon}-v_{t/\varepsilon}\\
&= \frac{1}{\varepsilon}\mu(v_{t/\varepsilon})dt+\beta(v_{t/\varepsilon})dB_{t/\varepsilon}^{(2)},
\end{aligned}$$

with

$$\big(dB_{t/\varepsilon}^{(2)}\big)^2\simeq\frac{dt}{\varepsilon}\simeq\frac{1}{\varepsilon}\big(dB_t^{(2)}\big)^2\simeq\left(\frac{1}{\sqrt{\varepsilon}}dB_t^{(2)}\right)^2,$$

hence the SDE for $v_t^{(\varepsilon)}$ can be rewritten as the slow-fast system

$$dv_t^{(\varepsilon)}=\frac{1}{\varepsilon}\mu(v_t^{(\varepsilon)})dt+\frac{1}{\sqrt{\varepsilon}}\beta(v_t^{(\varepsilon)})dB_t^{(2)}.$$

In other words, $\varepsilon \to 0$ corresponds to fast mean-reversion and (5.5.1) can be rewritten as

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t^{(\varepsilon)}} S_t dB_t^{(1)} \\ dv_t^{(\varepsilon)} = \dfrac{1}{\varepsilon}\mu\big(v_t^{(\varepsilon)}\big)dt + \dfrac{1}{\sqrt{\varepsilon}}\beta\big(v_t^{(\varepsilon)}\big)dB_t^{(2)}, \qquad \varepsilon > 0. \end{cases}$$

The perturbed PDE

$$\frac{\partial f_\varepsilon}{\partial t}(t,v,x) + rx\frac{\partial f_\varepsilon}{\partial x}(t,v,x) + \frac{vx^2}{2}\frac{\partial^2 f_\varepsilon}{\partial x^2}(t,v,x) + \frac{1}{\varepsilon}\mu(v)\frac{\partial f_\varepsilon}{\partial v}(t,v,x)$$
$$+\frac{1}{2\varepsilon}\beta^2(v)\frac{\partial^2 f_\varepsilon}{\partial v^2}(t,v,x) + \frac{\rho}{\sqrt{\varepsilon}}\beta(v)x\sqrt{v}\frac{\partial^2 f_\varepsilon}{\partial v \partial x}(t,v,x) = rf_\varepsilon(t,v,x)$$

with terminal condition $f_\varepsilon(T,v,x) = (x-K)^+$ rewrites as

$$\frac{1}{\varepsilon}\mathscr{L}_0 f_\varepsilon(t,v,x) + \frac{1}{\sqrt{\varepsilon}}\mathscr{L}_1 f_\varepsilon(t,v,x) + \mathscr{L}_2 f_\varepsilon(t,v,x) = rf_\varepsilon(t,v,x), \tag{5.5.2}$$

where

$$\begin{cases} \mathscr{L}_0 f_\varepsilon(t,v,x) := \dfrac{1}{2}\beta^2(v)\dfrac{\partial^2 f_\varepsilon}{\partial v^2}(t,v,x) + \mu(v)\dfrac{\partial f_\varepsilon}{\partial v}(t,v,x), \\ \mathscr{L}_1 f_\varepsilon(t,v,x) := \rho x \beta(v)\sqrt{v}\dfrac{\partial^2 f_\varepsilon}{\partial v \partial x}(t,v,x), \\ \mathscr{L}_2 f_\varepsilon(t,v,x) := \dfrac{\partial f_\varepsilon}{\partial t}(t,v,x) + rx\dfrac{\partial f_\varepsilon}{\partial x}(t,v,x) + \dfrac{vx^2}{2}\dfrac{\partial^2 f_\varepsilon}{\partial x^2}(t,v,x). \end{cases}$$

Note that

- $\mathscr{L}_0$ is the infinitesimal generator of the process $(v_s^1)_{s\in\mathbb{R}_+}$, see (5.5.6) below,

  and

- $\mathscr{L}_2$ is the Black-Scholes operator, *i.e.* $\mathscr{L}_2 f = rf$ is the Black-Scholes PDE.

The solution $f_\varepsilon(t,v,x)$ will be expanded as

$$f_\varepsilon(t,v,x) = f^{(0)}(t,v,x) + \sqrt{\varepsilon} f^{(1)}(t,v,x) + \varepsilon f^{(2)}(t,v,x) + \cdots \tag{5.5.3}$$

with $f(T,v,x) = (x-K)^+$, $f^{(1)}(T,v,x) = 0$, and $f^{(2)}(T,v,x) = 0$. Since $\mathscr{L}_0$ contains only differentials with respect to v, we will choose $f^{(0)}(t,v,x)$ of the form

$$f^{(0)}(t,v,x) = f^{(0)}(t,x),$$

cf. § 4.2.1 of Fouque, G. Papanicolaou, Sircar, and Sølna, 2011 for details, with

$$\mathscr{L}_0 f^{(0)}(t,x) = \mathscr{L}_1 f^{(0)}(t,x) = 0. \tag{5.5.4}$$

Proposition 5.14 (Fouque, G. Papanicolaou, Sircar, and Sølna, 2011, § 3.2). The first-order term $f_0(t,v)$ in (5.5.3) satisfies the Black-Scholes PDE

$$rf^{(0)}(t,x) = \frac{\partial f^{(0)}}{\partial t}(t,x) + rx\frac{\partial f^{(0)}}{\partial x}(t,x) + \frac{\eta^2}{2}\int_0^\infty v\phi(v)dv\frac{\partial^2 f^{(0)}}{\partial x^2}(t,x)$$

with the terminal condition $f^{(0)}(T,x) = (x-K)^+$, where $\phi(v)$ is the stationary (or invariant) probability density function of the process $\left(v_t^{(1)}\right)_{t\in\mathbb{R}_+}$.

Proof. By identifying the terms of order $1/\sqrt{\varepsilon}$ when plugging (5.5.3) in (5.5.2), we have

$$\mathscr{L}_0 f^{(1)}(t,v,x) + \mathscr{L}_1 f^{(0)}(t,x) = 0,$$

hence $\mathscr{L}_0 f^{(1)}(t,v,x) = 0$. Similarly, by identifying the terms that do not depend on ε in (5.5.2) and taking $f^{(1)}(t,v,x) = f^{(1)}(t,x)$, we have $\mathscr{L}_1 f^{(1)} = 0$ and

$$\mathscr{L}_0 f^{(2)}(t,v,x) + \mathscr{L}_2 f^{(0)}(t,x) = 0. \tag{5.5.5}$$

Using the Itô formula, we have

$$\begin{aligned}
\mathbb{E}\left[f^{(2)}(t,v_s^1,x)\right] &= f^{(2)}(t,v_0^1,x) + \mathbb{E}\left[\int_0^s \frac{\partial f^{(2)}}{\partial x}(t,v_\tau^1,x)\,dB_\tau^{(2)}\right] \\
&\quad + \mathbb{E}\left[\int_0^s \left(\mu(v_\tau^1)\frac{\partial f^{(2)}}{\partial v}(t,v_\tau^1,x) + \frac{1}{2}\beta^2(v_\tau^1)\frac{\partial^2 f^{(2)}}{\partial v^2}(t,v_\tau^1,x)\right)d\tau\right] \\
&= f^{(2)}(t,v_0^1,x) + \int_0^s \mathbb{E}\left[\mathscr{L}_0 f^{(2)}(t,v_\tau^1,x)\right]d\tau.
\end{aligned} \tag{5.5.6}$$

When the process $\left(v_t^{(1)}\right)_{t\in\mathbb{R}_+}$ is started under its stationary (or invariant) probability distribution with probability density function $\phi(v)$, we have

$$\mathbb{E}\left[f^{(2)}(t,v_\tau^1,x)\right] = \int_0^\infty f^{(2)}(t,v,x)\phi(v)dv, \qquad \tau \geqslant 0,$$

hence (5.5.6) rewrites as

$$\int_0^\infty f^{(2)}(t,v,x)\phi(v)dv = \int_0^\infty f^{(2)}(t,v,x)\phi(v)dv + \int_0^s \int_0^\infty \mathscr{L}_0 f^{(2)}(t,v,x)\phi(v)dvd\tau.$$

By differentiation with respect to $s > 0$ this yields

$$\int_0^\infty \mathscr{L}_0 f^{(2)}(t,v,x)\phi(v)dv = 0,$$

hence by (5.5.5) we find

$$\int_0^\infty \mathscr{L}_2 f^{(0)}(t,x)\phi(v)dv = 0,$$

cf. § 3.2 of Fouque, G. Papanicolaou, Sircar, and Sølna, 2011, *i.e.* we find

$$\frac{\partial f^{(0)}}{\partial t}(t,x) + rx\frac{\partial f^{(0)}}{\partial x}(t,x) + \frac{\eta^2}{2}\int_0^\infty v\phi(v)dv\frac{\partial^2 f^{(0)}}{\partial x^2}(t,x) = rf^{(0)}(t,x),$$

with the terminal condition $f^{(0)}(T,x) = (x-K)^+$. □

As a consequence of Proposition 5.14, the first-order term $f^{(0)}(t,x)$ in the expansion (5.5.3) is the Black-Scholes function

$$f^{(0)}(t,x) = \mathrm{Bl}\left(S_t, K, r, T-t, \sqrt{\int_0^\infty v\phi(v)dv}\right),$$

with the averaged squared volatility

$$\int_0^\infty v\phi(v)dv = \mathbb{E}\left[v_\tau^1\right], \qquad \tau \geqslant 0, \tag{5.5.7}$$

under the stationary distribution of the process with infinitesimal generator $\mathscr{L}_0$, *i.e.* the stationary distribution of the solution to

$$dv_t^{(1)} = \mu(v_t^{(1)})dt + \beta(v_t^{(1)})dB_t^{(2)}.$$

Perturbation analysis in the Heston model

We have

$$\begin{cases} dS_t = rS_t dt + S_t\sqrt{v_t^{(\varepsilon)}}dB_t^{(1)} \\ \\ dv_t^{(\varepsilon)} = -\dfrac{\lambda}{\varepsilon}\big(v_t^{(\varepsilon)} - m\big)dt + \eta\sqrt{\dfrac{v_t^{(\varepsilon)}}{\varepsilon}}dB_t^{(2)}, \end{cases}$$

under the modified short mean-reversion time scale, and the SDE can be rewritten as

$$dv_t^{(\varepsilon)} = -\frac{\lambda}{\varepsilon}\big(v_t^{(\varepsilon)} - m\big)dt + \eta\sqrt{\frac{v_t^{(\varepsilon)}}{\varepsilon}}dB_t^{(2)}.$$

In other words, $\varepsilon \to 0$ corresponds to fast mean reversion, in which $v_t^{(\varepsilon)}$ becomes close to its mean (5.5.7).

Recall that the CIR process $\big(v_t^{(1)}\big)_{t\in\mathbb{R}_+}$ has a gamma invariant (or stationary) distribution with shape parameter $2\lambda m/\eta^2$, scale parameter $\eta^2/(2\lambda)$, and probability density function ϕ given by

$$\phi(v) = \frac{1}{\Gamma(2\lambda m/\eta^2)(\eta^2/(2\lambda))^{2\lambda m/\eta^2}} v^{-1+2\lambda m/\eta^2}\,\mathrm{e}^{-2v\lambda/\eta^2}\mathbb{1}_{[0,\infty)}(v), \quad v > 0,$$

and mean

$$m = \int_0^\infty v\phi(v)dv.$$

Hence the first-order term $f^{(0)}(t,x)$ in the expansion (5.5.3) reads

$$f^{(0)}(t,x) = \mathrm{Bl}\left(S_t, K, r, T-t, \sqrt{m}\right),$$

with the averaged squared volatility

$$m = \int_0^\infty v\phi(v)dv = \mathbb{E}\left[v_\tau^1\right], \qquad \tau \geqslant 0,$$

under the stationary distribution of the process with infinitesimal generator $\mathscr{L}_0$, *i.e.* the stationary distribution of the solution to

$$dv_t^{(1)} = \mu\big(v_t^{(1)}\big)dt + \beta\big(v_t^{(1)}\big)dB_t^{(2)}.$$

In Figure 5.6, cf. Privault and She, 2016, related approximations of put option prices are plotted against the value of v with correlation $\rho = -0.5$ and $\varepsilon = 0.01$ in the α-hypergeometric stochastic volatility model of Fonseca and Martini, 2016, based on the series expansion of Han et al., 2013, and compared to a Monte Carlo curve requiring 300,000 samples and 30,000 time steps.

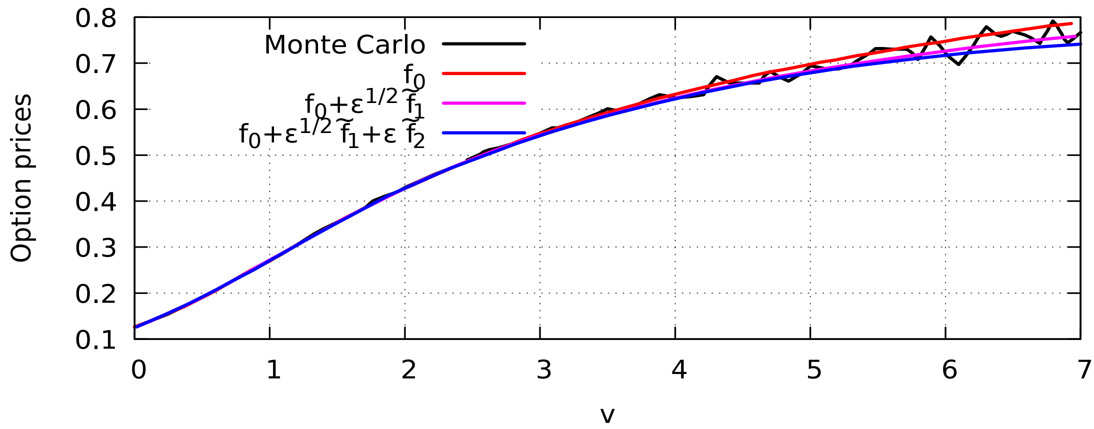


Figure 5.6: Option price approximations plotted against v with $\rho = -0.5$.

Exercises

Exercise 5.1 (Gatheral, 2006, Chapter 11). Compute the expected realized variance on the time interval $[0,T]$ in the Heston model, with

$$dv_t = -\lambda(v_t - m)dt + \eta\sqrt{v_t}dB_t, \qquad 0 \leqslant t \leqslant T.$$

Exercise 5.2 Compute the variance swap rate

$$\mathrm{VS}_T := \frac{1}{T}\mathbb{E}\left[\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{S_{kT/n} - S_{(k-1)T/n}}{S_{(k-1)T/n}}\right)^2\right] = \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}(dS_t)^2\right]$$

on the index whose level S_t is given in the following two models.

a) Heston, 1993 model. Here, $(S_t)_{t\in\mathbb{R}_+}$ is given by the system of stochastic differential equations

$$\begin{cases} dS_t = (r - \alpha v_t)S_t dt + S_t\sqrt{\beta + v_t}dB_t^{(1)} \\ \\ dv_t = -\lambda(v_t - m)dt + \gamma\sqrt{v_t}dB_t^{(2)}, \end{cases}$$

where $\left(B_t^{(1)}\right)_{t\in\mathbb{R}_+}$ and $\left(B_t^{(2)}\right)_{t\in\mathbb{R}_+}$ are standard Brownian motions with correlation $\rho \in [-1,1]$ and $\alpha \geqslant 0$, $\beta \geqslant 0$, $\lambda > 0$, $m > 0$, $r > 0$, $\gamma > 0$.

b) SABR model with $\beta = 1$. The index level S_t is given by the system of stochastic differential equations

$$\begin{cases} dS_t = \sigma_t S_t dB_t^{(1)} \\ \\ d\sigma_t = \alpha\sigma_t dB_t^{(2)}, \end{cases}$$

where $\alpha > 0$ and $\left(B_t^{(1)}\right)_{t\in\mathbb{R}_+}$ and $\left(B_t^{(2)}\right)_{t\in\mathbb{R}_+}$ are standard Brownian motions with correlation $\rho \in [-1,1]$.

Exercise 5.3 Convexity adjustment (§ 2.3 of Broadie and Jain, 2008).

a) Using Taylor's formula

$$\sqrt{x} = \sqrt{x_0} + \frac{x - x_0}{2\sqrt{x_0}} - \frac{(x - x_0)^2}{8x_0^{3/2}} + o((x - x_0)^2),$$

find an approximation of $R_{0,T} = \sqrt{R_{0,T}^2}$ using $\sqrt{\mathbb{E}\left[R_{0,T}^2\right]}$ and correction terms.

b) Find an (approximate) relation between the variance swap price $\mathbb{E}^*\left[R_{0,T}^2\right]$ and the volatility swap price $\mathbb{E}^*[R_{0,T}]$ up to a correction term.

Exercise 5.4 Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ with the log-return dynamics

$$d\log S_t = \mu dt + Z_{N_{t^-}} dN_t, \qquad t \geqslant 0,$$

i.e. $S_t := S_0 \mathrm{e}^{\mu t + Y_t}$ in a pure jump Merton model, where $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process with intensity $\lambda > 0$ and $(Z_k)_{k\geqslant 0}$ is a family of independent identically distributed Gaussian $\mathcal{N}(\delta, \eta^2)$ random variables. Compute the price of the log-return variance swap

$$\begin{aligned}
\mathbb{E}\left[\int_0^T (d\log S_t)^2 dN_t\right] &= \mathbb{E}\left[\int_0^T (\mu dt + Z_{N_{t^-}} dN_t)^2 dN_t\right] \\
&= \mathbb{E}\left[\int_0^T (Z_{N_{t^-}} dN_t)^2 dN_t\right] \\
&= \mathbb{E}\left[\int_0^T \left(\log \frac{S_t}{S_{t^-}}\right)^2 dN_t\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{N_T} \left(\log \frac{S_{T_k}}{S_{T_{k-1}}}\right)^2\right]
\end{aligned}$$

using the "smoothing lemma".

Exercise 5.5 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{5.5.8}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r \in \mathbb{R}$ and $\sigma > 0$.

a) Write down the solution $(S_t)_{t\in\mathbb{R}_+}$ of Equation (5.5.8) in explicit form.

b) Show by a direct calculation that Corollary 5.3 is satisfied by $(S_t)_{t\in\mathbb{R}_+}$.

Exercise 5.6 (Carr and R. Lee, 2008) Consider an underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ given by $dS_t = rS_t dt + \sigma_t S_t dB_t$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $(\sigma_t)_{t\in\mathbb{R}_+}$ is an (adapted) stochastic volatility process. The riskless asset is priced $A_t := \mathrm{e}^{rt}$, $t \in [0,T]$. We consider a realized variance swap with payoff $R_{0,T}^2 = \int_0^T \sigma_t^2 dt$.

a) Show that the payoff $\int_0^T \sigma_t^2 dt$ of the realized variance swap satisfies

$$\int_0^T \sigma_t^2 dt = 2\int_0^T \frac{dS_t}{S_t} - 2\log\frac{S_T}{S_0}. \tag{5.5.9}$$

b) Show that the price $V_t := \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt \,\middle|\, \mathcal{F}_t\right]$ of the variance swap at time $t \in [0,T]$ satisfies

$$V_t = L_t + 2(T-t)r\mathrm{e}^{-(T-t)r} + 2\mathrm{e}^{-(T-t)r}\int_0^t \frac{dS_u}{S_u}, \tag{5.5.10}$$

where

$$L_t := -2\,\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\log \frac{S_T}{S_0}\,\middle|\,\mathscr{F}_t\right]$$

is the price at time t of the log contract (see Neuberger, 1994, Demeterfi et al., 1999) with payoff $-2\log(S_T/S_0)$.

c) Show that the portfolio made at time $t \in [0,T]$ of:
   - one log contract priced L_t,
   - $2\,\mathrm{e}^{-(T-t)r}/S_t$ in shares priced S_t,
   - $2\,\mathrm{e}^{-rT}\left(\int_0^t \frac{dS_u}{S_u} + (T-t)r - 1\right)$ in the riskless asset $A_t = \mathrm{e}^{rt}$,

   hedges the realized variance swap.

d) Show that the above portfolio is self-financing.

Exercise 5.7 Compute the moment $\mathbb{E}^*\left[R_{0,T}^4\right]$ from Lemma 5.2.

6. Volatility Estimation

Volatility estimation methods include historical, implied and local volatility, and the VIX® volatility index. This chapter presents such estimation methods, together with examples of how the Black-Scholes formula can be fitted to market data. While the market parameters r, t, S_t, T, and K used in Black-Scholes option pricing can be easily obtained from market terms and data, the estimation of volatility parameters can be a more complex task.

6.1 Historical Volatility

We consider the problem of estimating the parameters μ and σ from market data in the stock price model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{6.1.1}$$

Historical trend estimation

By discretization of (6.1.1) along a family $t_0, t_1, \ldots, t_N$ of observation times as

$$\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} = (t_{k+1} - t_k)\mu + (B_{t_{k+1}} - B_{t_k})\sigma, \quad k = 0, 1, \ldots, N-1, \tag{6.1.2}$$

a natural estimator for the trend parameter μ can be constructed as

$$\widehat{\mu}_N := \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S^M_{t_{k+1}}-S^M_{t_k}}{S^M_{t_k}}\right), \tag{6.1.3}$$

where $(S^M_{t_{k+1}}-S^M_{t_k})/S^M_{t_k}$, $k=0,1,\ldots,N-1$ denotes market returns observed at discrete times $t_0,t_1,\ldots,t_N$.

Historical log-return estimation

Alternatively, observe that, replacing* (6.1.3) by the log-returns

$$\begin{aligned}\log\frac{S_{t_{k+1}}}{S_{t_k}} &= \log S_{t_{k+1}} - \log S_{t_k}\\ &= \log\left(1+\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}\right)\\ &\simeq \frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}},\end{aligned}$$

with $t_{k+1}-t_k = T/N$, $k=0,1,\ldots,N-1$, one can replace (6.1.3) with the simpler telescoping estimate

$$\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}(\log S_{t_{k+1}}-\log S_{t_k}) = \frac{1}{T}\log\frac{S_T}{S_0}.$$

Historical volatility estimation

The volatility parameter σ can be estimated by writing, from (6.1.2),

$$\sigma^2\sum_{k=0}^{N-1}\frac{(B_{t_{k+1}}-B_{t_k})^2}{t_{k+1}-t_k} = \sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}-(t_{k+1}-t_k)\mu\right)^2,$$

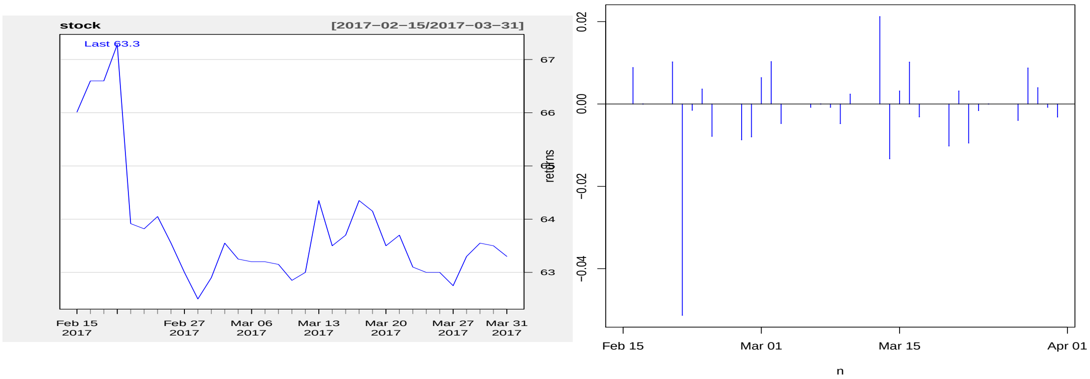
which yields the (unbiased) realized variance estimator

$$\widehat{\sigma}^2_N := \frac{1}{N-1}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}-(t_{k+1}-t_k)\widehat{\mu}_N\right)^2.$$

```
library(quantmod)
getSymbols("0005.HK",from="2017-02-15",to=Sys.Date(),src="yahoo")
stock=Ad(`0005.HK`)
chartSeries(stock,up.col="blue",theme="white")
```

```
stock=Ad(`0005.HK`);logreturns=diff(log(stock));returns=(stock-lag(stock))/lag(stock)
times=index(returns);returns <- as.vector(returns)
n = sum(is.na(returns))+sum(!is.na(returns))
plot(times,returns,pch=19,cex=0.05,col="blue", ylab="returns", xlab="n", main = '')
segments(x0 = times, x1 = times, cex=0.05,y0 = 0, y1 = returns,col="blue")
abline(seq(1,n),0,FALSE);dt=1.0/365;mu=mean(returns,na.rm=TRUE)/dt
sigma=sd(returns,na.rm=TRUE)/sqrt(dt);mu;sigma
```

*This approximation does not include the correction term $(dS_t)^2/(2S_t^2)$ in the Itô formula $d\log S_t = dS_t/S_t - (dS_t)^2/(2S_t^2)$.



(a) Underlying asset price. (b) Log-returns.

Figure 6.1: Graph of underlying asset price *vs.* log-returns.

```
library(PerformanceAnalytics);
returns <- exp(CalculateReturns(stock,method="compound")) - 1; returns[1,] <- 0
histvol <- rollapply(returns, width = 30, FUN=sd.annualized)
myPars <- chart_pars();myPars$cex<-1.4
myTheme <- chart_theme(); myTheme$col$line.col <- "blue"
dev.new(width=16,height=7)
chart_Series(stock,name="0005.HK",pars=myPars,theme=myTheme)
add_TA(histvol, name="Historical Volatility")
```

Figure 6.2 presents a historical volatility graph with a 30 days rolling window.

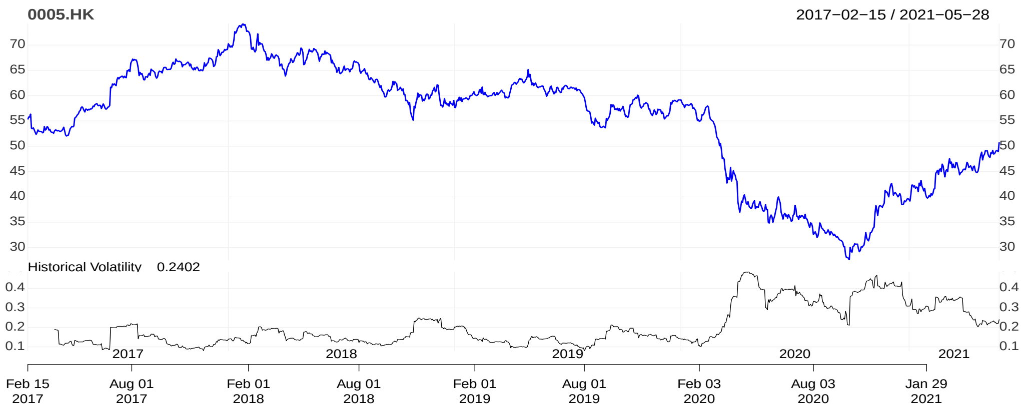


Figure 6.2: Historical volatility graph.

Parameter estimation based on historical data usually requires a lot of samples and it can only be valid on a given time interval, or as a moving average. Moreover, it can only rely on past data, which may not reflect future data.

Figure 6.3: "The fugazi: it's a wazy, it's a woozie. It's fairy dust."*

6.2 Implied Volatility

Recall that when $h(x) = (x-K)^+$, the solution of the Black-Scholes PDE is given by

$$\mathrm{Bl}(t,x,K,\sigma,r,T) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

and

$$\begin{cases} d_+(T-t) = \dfrac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \\ d_-(T-t) = \dfrac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \end{cases}$$

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data. Equating the Black-Scholes formula

$$\mathrm{Bl}(t,S_t,K,\sigma,r,T) = M \tag{6.2.1}$$

to the observed value M of a given market price allows one to infer a value of σ when t, S_t, r, T are known, as in *e.g.* Figure 6.4.

*Scorsese, 2013 Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).

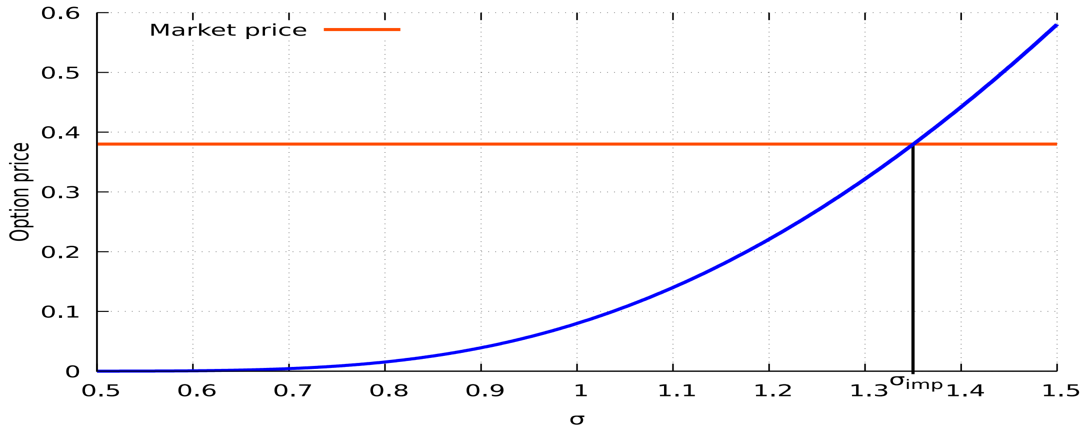


Figure 6.4: Option price as a function of the volatility σ.

This value of σ is called the implied volatility, and it is denoted here by $\sigma_{\text{imp}}(K,T)$. Various algorithms can be implemented to solve (6.2.1) numerically for $\sigma_{\text{imp}}(K,T)$, such as the bisection method and the Newton-Raphson method.*

```
BS <- function(S, K, T, r, sig){d1 <- (log(S/K) + (r + sig^2/2)*T) / (sig*sqrt(T))
d2 <- d1 - sig*sqrt(T);return(S*pnorm(d1) - K*exp(-r*T)*pnorm(d2))}
implied.vol <- function(S, K, T, r, market){
sig <- 0.20;sig.up <- 10;sig.down <- 0.0001;count <- 0;err <- BS(S, K, T, r, sig) - market
while(abs(err) > 0.00001 && count<1000){
if(err < 0){sig.down <- sig;sig <- (sig.up + sig)/2} else{sig.up <- sig;sig <- (sig.down + sig)/2}
err <- BS(S, K, T, r, sig) - market;count <- count + 1};if(count==1000){return(NA)}else{return(sig)}}
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02; implied.vol(S, K, T, r, market)
BS(S, K, T, r, implied.vol(S, K, T, r, market))
```

The implied volatility value can be used as an alternative way to quote the option price, based on the knowledge of the remaining parameters (such as underlying asset price, time to maturity, interest rate, and strike price). For example, market option price data provided by the Hong Kong stock exchange includes implied volatility computed by inverting the Black-Scholes formula.

```
library(devtools); install_github("https://github.com/cran/fOptions")
library(fOptions)
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02
sig=GBSVolatility(market,"c",S,K,T,r,r,1e-4,maxiter = 10000)
BS(S, K, T, r, sig)
```

*Download the corresponding R **code** or the **IPython notebook** that can be run **here** or **here**.

Option chain data in R

```r
install.packages("quantmod");install.packages("jsonlite");
library(quantmod);library(jsonlite)
getSymbols("^GSPC",src="yahoo",from=as.Date("2018-01-01"), to = as.Date("2018-03-01"))
head(GSPC)
# Only the front-month expiry
SPX.OPT <- getOptionChain("^SPX")
AAPL.OPT <- getOptionChain("AAPL")
# All expiries
SPX.OPTS <- getOptionChain("^SPX", NULL)
AAPL.OPTS <- getOptionChain("AAPL", NULL)
# All 2021 to 2023 expiries
SPX.OPTS <- getOptionChain("^SPX", "2021/2023")
AAPL.OPTS <- getOptionChain("AAPL", "2021/2023")
```

Exporting option price data

```r
write.table(AAPL.OPT$puts, file = "AAPLputs")
write.csv(AAPL.OPT$puts, file = "AAPLputs.csv")
install.packages("xlsx")
library(xlsx)
write.xlsx(AAPL.OPTS$Jun.19.2020$puts, file = "AAPL.OPTS$Jun.19.2020$puts.xlsx")
```

Volatility smiles

Given two European call options with strike prices K_1, resp. K_2, maturities T_1, resp. T_2, and prices C_1, resp. C_2, on the same stock S, this procedure should yield two estimates $\sigma_{\mathrm{imp}}(K_1,T_1)$ and $\sigma_{\mathrm{imp}}(K_2,T_2)$ of implied volatilities according to the following equations.

$$\begin{cases} \mathrm{Bl}(t,S_t,K_1,\sigma_{\mathrm{imp}}(K_1,T_1),r,T_1) = M_1, & (6.2.2a) \\ \\ \mathrm{Bl}(t,S_t,K_2,\sigma_{\mathrm{imp}}(K_2,T_2),r,T_2) = M_2, & (6.2.2b) \end{cases}$$

Clearly, there is no reason a priori for the implied volatilities $\sigma_{\mathrm{imp}}(K_1,T_1)$, $\sigma_{\mathrm{imp}}(K_2,T_2)$ solutions of (6.2.2a)-(6.2.2b) to coincide across different strike prices and different maturities. However, in the standard Black-Scholes model the value of the parameter σ should be unique for a given stock S. This contradiction between a model and market data is motivating the development of more sophisticated stochastic volatility models.

Figure 6.5 presents an estimation of implied volatility surface for Asian options on light sweet crude oil futures traded on the New York Mercantile Exchange (NYMEX), based on contract specifications and market data obtained from the Chicago Mercantile Exchange (CME).

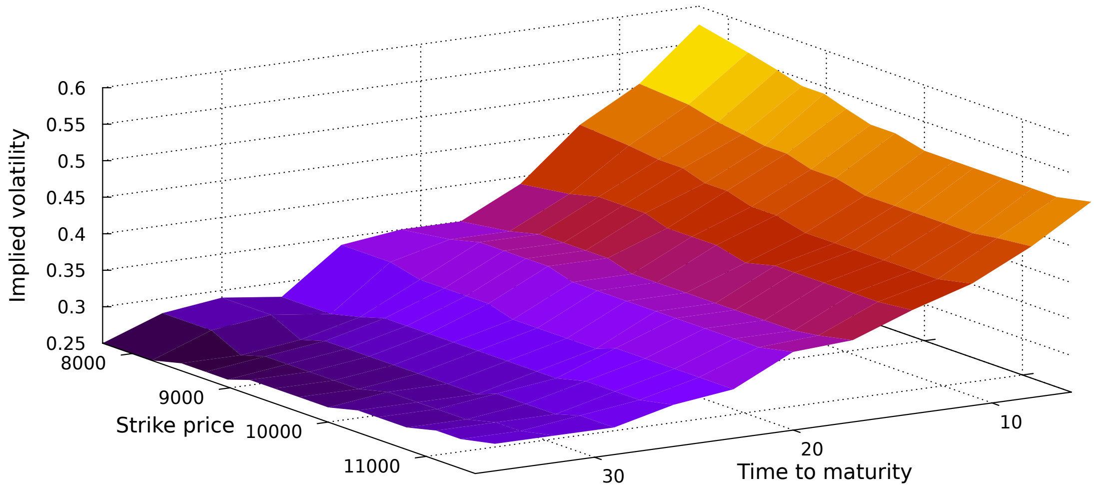


Figure 6.5: Implied volatility surface of Asian options on light sweet crude oil futures.*

As observed in Figure 6.5, the volatility surface can exhibit a *smile* phenomenon, in which implied volatility is higher at a given end (or at both ends) of the range of strike price values.

```
install.packages("jsonlite");install.packages("lubridate")
library(jsonlite);library(lubridate);library(quantmod)
# Maturity to be updated as needed
maturity <- as.Date("2021-08-20", format="%Y-%m-%d")
CHAIN <- getOptionChain("GOOG",maturity)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
getSymbols("GOOG", src = "yahoo")
lastBusDay=last(row.names(as.data.frame(Ad(GOOG))))
T <- as.numeric(difftime(maturity, lastBusDay, units = "days")/365);r = 0.02;ImpVol<-1:1;
      S=as.vector(tail(Ad(GOOG),1))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S,CHAIN$calls$Strike[i],T,r,
      CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab = "Implied
      volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4, data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])),
      col="red",lwd=2)
```

```
currentyear<-format(Sys.Date(), "%Y")
# Maturity to be updated as needed
maturity <- as.Date("2021-12-17", format="%Y-%m-%d")
CHAIN <- getOptionChain("^SPX",maturity)
# Last trading day (may require update)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
getSymbols("^SPX", src = "yahoo")
lastBusDay=last(row.names(as.data.frame(Ad(SPX))))
T <- as.numeric(difftime(maturity, lastBusDay, units = "days")/365);r = 0.02;ImpVol<-1:1;
      S=as.vector(tail(Ad(SPX),1))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S, CHAIN$calls$Strike[i], T, r,
      CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab = "Implied
      volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4, data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])),
      col="red",lwd=3)
```

*© Tan Yu Jia.

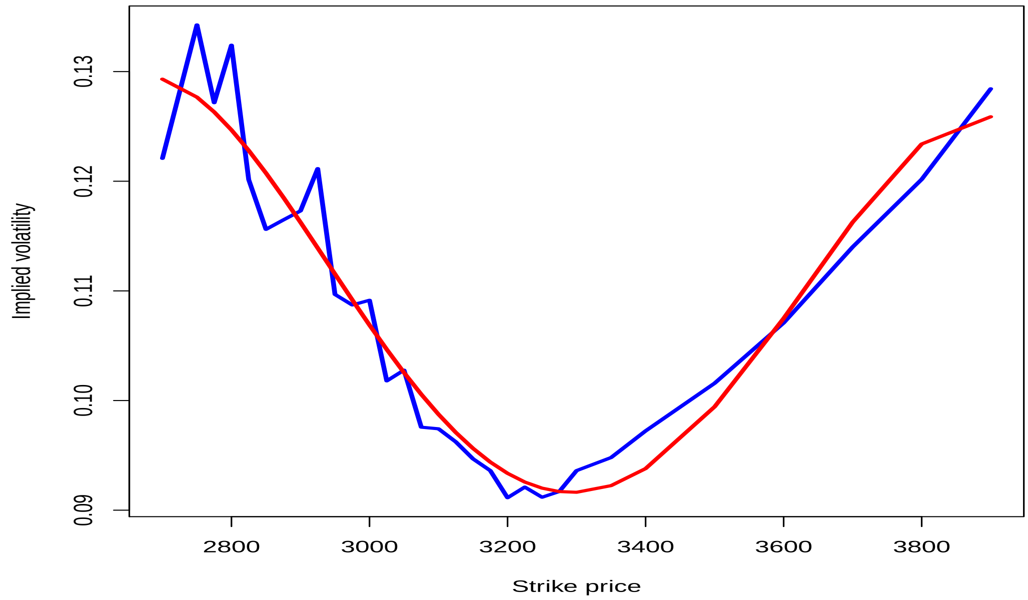


Figure 6.6: S&P500 option prices plotted against strike prices.

When reading option prices on the volatility scale, the smile phenomenon shows that the Black-Scholes formula tends to underprice extreme events for which the underlying asset price S_T is far away from the strike price K. In that sense, the Black-Scholes formula, which is modeling asset returns using Gaussian distribution tails, tends to underestimate the probability of extreme events.

Plotting the different values of the implied volatility σ as a function of K and T will yield a three-dimensional plot called the volatility surface.*

Black-Scholes Formula *vs.* Market Data

On July 28, 2009 a call warrant has been issued by Merrill Lynch on the stock price S of Cheung Kong Holdings (0001.HK) with strike price K=$109.99, Maturity T = December 13, 2010, and entitlement ratio 100.

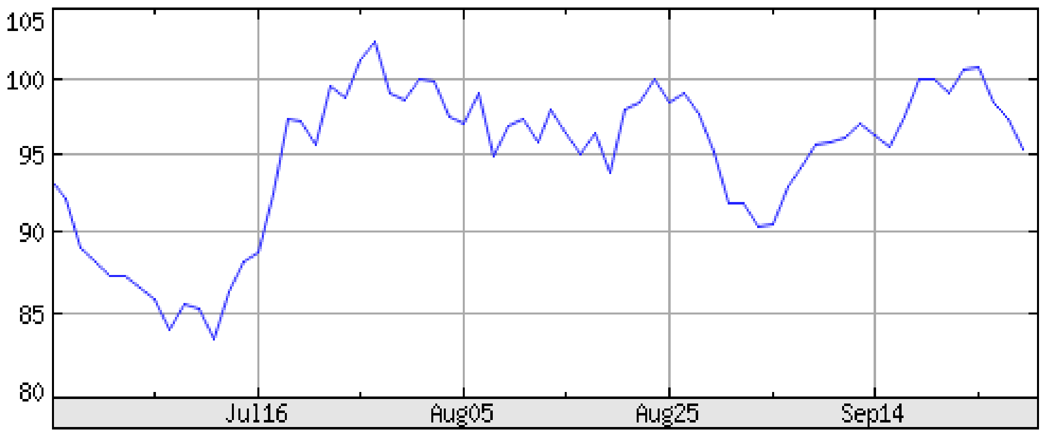


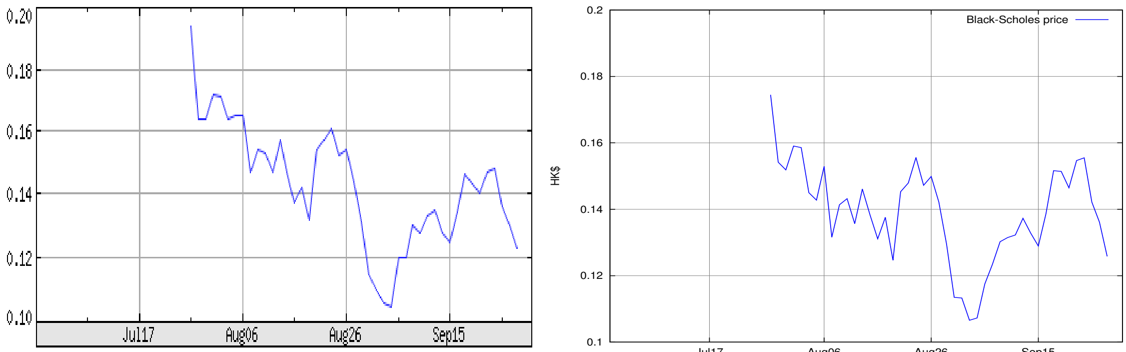
Figure 6.7: Graph of the (market) stock price of Cheung Kong Holdings.

The market price of the option (17838.HK) on September 28 was $12.30, as obtained from

*Download the corresponding **IPython notebook** that can be run **here** or **here** (© Qu Mengyuan).

https://www.hkex.com.hk/eng/dwrc/search/listsearch.asp.

The next graph in Figure 6.8a shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying asset price.



(a) Graph of (market) option prices. (b) Graph of Black-Scholes prices.

Figure 6.8: Comparison of market option prices *vs.* calibrated Black-Scholes prices.

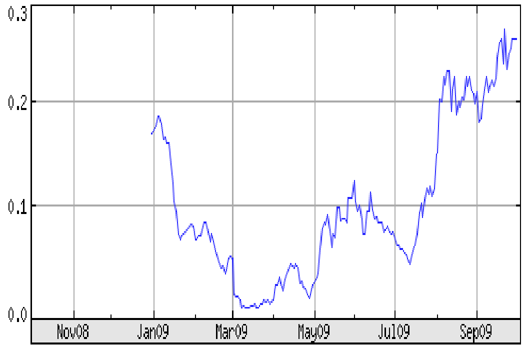
In Figure 6.8b we have fitted the time evolution $t \mapsto g_c(t, S_t)$ of Black-Scholes prices to the data of Figure 6.8a using the market stock price data of Figure 6.7, by varying the values of the volatility σ.

Another example

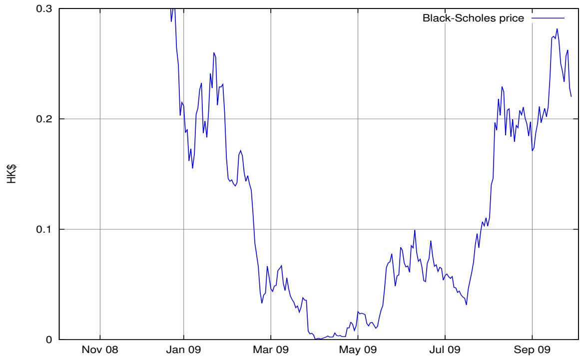
Let us consider the stock price of HSBC Holdings (0005.HK) over one year:

Figure 6.9: Graph of the (market) stock price of HSBC Holdings.

Next, we consider the graph of the price of the call option issued by Societe Generale on 31 December 2008 with strike price K=$63.704, maturity T = October 05, 2009, and entitlement ratio 100.



(a) Graph of (market) option prices.

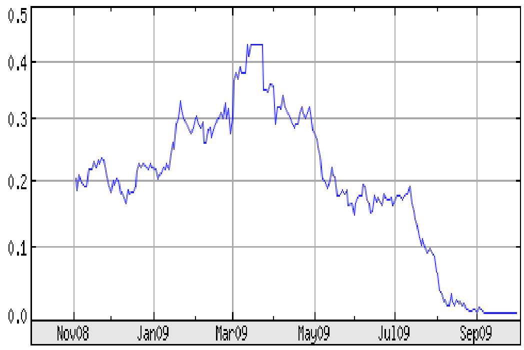


(b) Graph of Black-Scholes prices.

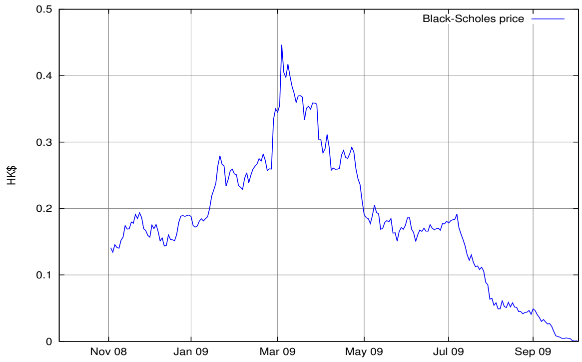
Figure 6.10: Comparison of market option prices *vs.* calibrated Black-Scholes prices.

As above, in Figure 6.10b we have fitted the path $t \longmapsto g_c(t, S_t)$ of the Black-Scholes option price to the data of Figure 6.10a using the stock price data of Figure 6.9.
In this case the option is *in the money* at maturity. We can also check that the option is worth $100 \times 0.2650 = \$26.650$ at that time, which, according to absence of arbitrage, is quite close to the actual value \$90 - \$63.703 = \$26.296 of its payoff.

For one more example, consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 on the underlying asset HSBC, with strike price K=\$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593.



(a) Graph of (market) option prices.



(b) Graph of Black-Scholes prices.

Figure 6.11: Comparison of market option prices *vs.* calibrated Black-Scholes prices.

One checks easily that at maturity, the price of the put option is worth \$0.01 (a market price cannot be lower), which almost equals the option payoff \$0, by absence of arbitrage opportunities. Figure 6.11b is a fit of the Black-Scholes put price graph

$$t \longmapsto g_p(t, S_t)$$

to Figure 6.11a as a function of the stock price data of Figure 6.10b. Note that the Black-Scholes price at maturity is strictly equal to 0 while the corresponding market price cannot be lower than one cent.

The normalized market data graph in Figure 6.12 shows how the option price can track the values of the underlying asset price. Note that the range of values $[26.55, 26.90]$ for the underlying asset price corresponds to $[0.675, 0.715]$ for the option price, meaning 1.36% *vs.* 5.9% in percentage. This is a European call option on the ALSTOM underlying asset with strike price $K = €20$, maturity March 20, 2015, and entitlement ratio 10.

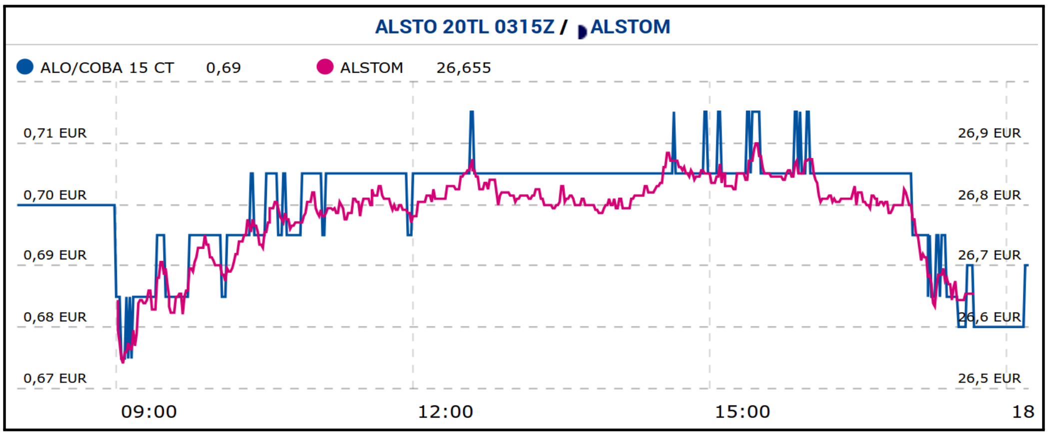


Figure 6.12: Call option price *vs.* underlying asset price.

6.3 Local Volatility

As the constant volatility assumption in the Black-Scholes model does not appear to be satisfactory due to the existence of volatility smiles, it can make more sense to consider models of the form

$$\frac{dS_t}{S_t} = rdt + \sigma_t dB_t$$

where σ_t is a random process. Such models are called stochastic volatility models.

A particular class of stochastic volatility models can be written as

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dB_t \tag{6.3.1}$$

where $\sigma(t,x) \geqslant 0$ is a deterministic function of time t and of the underlying asset price x. Such models are called local volatility models.

As an example, consider the stochastic differential equation with local volatility

$$dY_t = rdt + \sigma Y_t^2 dB_t, \tag{6.3.2}$$

where $\sigma > 0$.

```
dev.new(width=16,height=7)
N=10000; t <- 0:(N-1); dt <- 1.0/N;r=0.5;sigma=1.2;
Z <- rnorm(N,mean=0,sd=sqrt(dt));Y <- rep(0,N);Y[1]=1
for (j in 2:N){ Y[j]=max(0,Y[j-1]+r*Y[j-1]*dt+sigma*Y[j-1]**2*Z[j])}
plot(t*dt, Y, xlab = "t", ylab = "", type = "l", col = "blue", xaxs='i', yaxs='i', cex.lab=2,
     cex.axis=1.6,las=1)
abline(h=0)
```

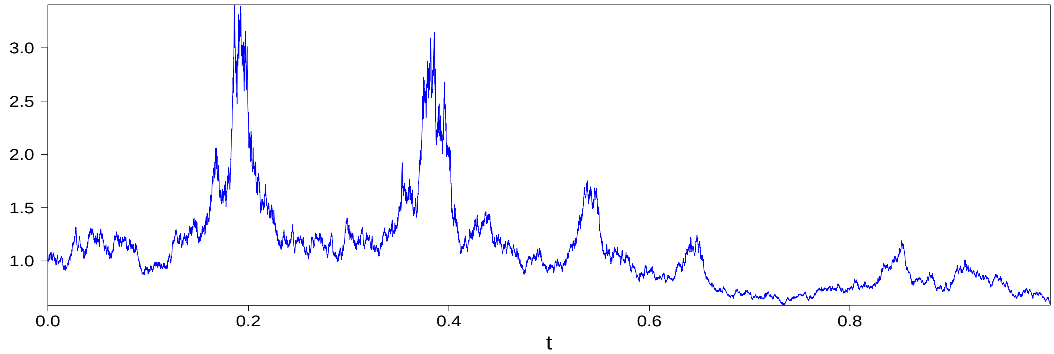


Figure 6.13: Simulated path of (6.3.2) with $r = 0.5$ and $\sigma = 1.2$.

In the general case, the corresponding Black-Scholes PDE for the option prices

$$g(t,x,K) := \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[(S_T-K)^+ \,\middle|\, S_t = x\right], \tag{6.3.3}$$

where $(S_t)_{t\in\mathbb{R}_+}$ is defined by (6.3.1), can be written as

$$\begin{cases} rg(t,x,K) = \dfrac{\partial g}{\partial t}(t,x,K) + rx\dfrac{\partial g}{\partial x}(t,x,K) + \dfrac{1}{2}x^2\sigma^2(t,x)\dfrac{\partial^2 g}{\partial x^2}(t,x,K), \\ \\ g(T,x,K) = (x-K)^+, \end{cases} \tag{6.3.4}$$

with terminal condition $g(T,x,K) = (x-K)^+$, *i.e.* we consider European call options.

Lemma 6.1 (Relation (1) in Breeden and Litzenberger, 1978). Consider a family $(C^M(T,K))_{T,K>0}$ of market call option prices with maturities T and strike prices K given at time 0. Then, the probability density function $\varphi_T(y)$ of S_T is given by

$$\varphi_T(K) = \mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial K^2}(T,K), \qquad K,T > 0. \tag{6.3.5}$$

Proof. Assume that the market option prices $C^M(T,K)$ match the Black-Scholes prices $\mathrm{e}^{-rT}\,\mathbb{E}[(S_T - K)^+]$, $K > 0$. Letting $\varphi_T(y)$ denote the probability density function of S_T, Condition (6.3.8) can be written at time $t = 0$ as

$$\begin{aligned} C^M(T,K) &= \mathrm{e}^{-rT}\,\mathbb{E}[(S_T-K)^+] \\ &= \mathrm{e}^{-rT}\int_0^\infty (y-K)^+\varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty (y-K)\varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty y\varphi_T(y)dy - K\,\mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty y\varphi_T(y)dy - K\,\mathrm{e}^{-rT}\,\mathbb{P}(S_T \geqslant K). \end{aligned} \tag{6.3.6}$$

By differentiation of (6.3.6) with respect to K, one gets

$$\begin{aligned} \frac{\partial C^M}{\partial K}(T,K) &= -\mathrm{e}^{-rT}K\varphi_T(K) - \mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy + \mathrm{e}^{-rT}K\varphi_T(K) \\ &= -\mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy, \end{aligned}$$

which yields (6.3.5) by twice differentiation of $C^M(T,K)$ with respect to K. □

In order to implement a stochastic volatility model such as (6.3.1), it is important to first calibrate the local volatility function $\sigma(t,x)$ to market data.

In principle, the Black-Scholes PDE (6.3.4) could allow one to recover the value of $\sigma(t,x)$ as a function of the option price $g(t,x,K)$, as

$$\sigma(t,x) = \sqrt{\frac{2rg(t,x,K) - 2\frac{\partial g}{\partial t}(t,x,K) - 2rx\frac{\partial g}{\partial x}(t,x,K)}{x^2\frac{\partial^2 g}{\partial x^2}(t,x,K)}}, \qquad x,t > 0,$$

however, this formula requires the knowledge of the option price for different values of the underlying asset price x, in addition to the knowledge of the strike price K.

The Dupire, 1994 formula brings a solution to the local volatility calibration problem by providing an estimator of $\sigma(t,x)$ as a function $\sigma(t,K)$ based on the values of the strike price K.

Proposition 6.2 (Dupire, 1994, Derman and Kani, 1994) Consider a family $(C^M(T,K))_{T,K>0}$ of market call option prices at time 0 with maturity T and strike price K, and define the volatility function $\sigma(t,y)$ by

$$\boxed{\sigma(t,y) := \sqrt{\frac{2\frac{\partial C^M}{\partial t}(t,y) + 2ry\frac{\partial C^M}{\partial y}(t,y)}{y^2\frac{\partial^2 C^M}{\partial y^2}(t,y)}} = \frac{\sqrt{\frac{\partial C^M}{\partial t}(t,y) + ry\frac{\partial C^M}{\partial y}(t,y)}}{y\mathrm{e}^{-rT/2}\sqrt{\varphi_t(y)/2}},} \qquad (6.3.7)$$

where $\varphi_t(y)$ denotes the probability density function of S_t, $t \in [0,T]$. Then, the prices generated from the Black-Scholes PDE (6.3.4) will be compatible with the market option prices $C^M(T,K)$ in the sense that

$$C^M(T,K) = \mathrm{e}^{-rT}\mathbb{E}[(S_T - K)^+], \qquad K > 0. \qquad (6.3.8)$$

Proof. For any sufficiently smooth function $f \in \mathscr{C}_0^\infty(\mathbb{R})$, with $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$, using the Itô formula, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(y)\varphi_T(y)dy = \mathbb{E}[f(S_T)]\\
&= \mathbb{E}\left[f(S_0) + r\int_0^T S_t f'(S_t)dt + \int_0^T S_t f'(S_t)\sigma(t,S_t)dB_t \right.\\
&\qquad \left. + \frac{1}{2}\int_0^T S_t^2 f''(S_t)\sigma^2(t,S_t)dt\right]\\
&= f(S_0) + \mathbb{E}\left[r\int_0^T S_t f'(S_t)dt + \frac{1}{2}\int_0^T S_t^2 f''(S_t)\sigma^2(t,S_t)dt\right]\\
&= f(S_0) + r\int_0^T \mathbb{E}[S_t f'(S_t)]dt + \frac{1}{2}\int_0^T \mathbb{E}[S_t^2 f''(S_t)\sigma^2(t,S_t)]dt\\
&= f(S_0) + r\int_{-\infty}^{\infty} yf'(y)\int_0^T \varphi_t(y)dtdy\\
&\qquad + \frac{1}{2}\int_{-\infty}^{\infty} y^2 f''(y)\int_0^T \sigma^2(t,y)\varphi_t(y)dtdy,
\end{aligned}$$

hence, after differentiating both sides of the equality with respect to T,

$$\int_{-\infty}^{\infty} f(y)\frac{\partial \varphi_T}{\partial T}(y)dy = r\int_{-\infty}^{\infty} yf'(y)\varphi_T(y)dy + \frac{1}{2}\int_{-\infty}^{\infty} y^2 f''(y)\sigma^2(T,y)\varphi_T(y)dy.$$

Integrating by parts in the above relation yields

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\partial \varphi_T}{\partial T}(y)f(y)dy\\ &= \quad -r\int_{-\infty}^{\infty} f(y)\frac{\partial}{\partial y}(y\varphi_T(y))dy + \frac{1}{2}\int_{-\infty}^{\infty} f(y)\frac{\partial^2}{\partial y^2}\big(y^2\sigma^2(T,y)\varphi_T(y)\big)dy,\end{aligned}$$

for all smooth functions $f(y)$ with compact support in $\mathbb{R}$, hence

$$\frac{\partial \varphi_T}{\partial T}(y) = -r\frac{\partial}{\partial y}(y\varphi_T(y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(y^2\sigma^2(T,y)\varphi_T(y)\big), \qquad y\in\mathbb{R}.$$

From Relation (6.3.5) in Lemma 6.1, we have

$$\frac{\partial \varphi_T}{\partial T}(K) = r\,\mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial K^2}(T,K) + \mathrm{e}^{rT}\frac{\partial^3 C^M}{\partial T\partial K^2}(T,K),$$

hence we get

$$\begin{aligned}&-r\frac{\partial^2 C^M}{\partial y^2}(T,y) - \frac{\partial^3 C^M}{\partial T\partial y^2}(T,y)\\ &= r\frac{\partial}{\partial y}\left(y\frac{\partial^2 C^M}{\partial y^2}(T,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right), \qquad y\in\mathbb{R}.\end{aligned}$$

After a first integration with respect to y under the boundary condition $\lim_{y\to+\infty} C^M(T,y) = 0$, we obtain

$$\begin{aligned}&-r\frac{\partial C^M}{\partial y}(T,y) - \frac{\partial}{\partial T}\frac{\partial C^M}{\partial y}(T,y)\\ &= \quad ry\frac{\partial^2 C^M}{\partial y^2}(T,y) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right),\end{aligned}$$

i.e.

$$\begin{aligned}&-r\frac{\partial C^M}{\partial y}(T,y) - \frac{\partial}{\partial T}\frac{\partial C^M}{\partial y}(T,y)\\ &= \quad r\frac{\partial}{\partial y}\left(y\frac{\partial C^M}{\partial y}(T,y)\right) - r\frac{\partial C^M}{\partial y}(T,y) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right),\end{aligned}$$

or

$$-\frac{\partial}{\partial y}\frac{\partial C^M}{\partial T}(T,y) = r\frac{\partial}{\partial y}\left(y\frac{\partial C^M}{\partial y}(T,y)\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right).$$

Integrating one more time with respect to y yields

$$-\frac{\partial C^M}{\partial T}(T,y) = ry\frac{\partial C^M}{\partial y}(T,y) - \frac{1}{2}y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y), \qquad y\in\mathbb{R}, \tag{6.3.9}$$

which conducts to (6.3.7) and is called the Dupire, 1994 PDE. □

Partial derivatives in time can be approximated using *forward* finite difference approximations as

$$\frac{\partial C}{\partial t}(t_i, y_j) \simeq \frac{C(t_{i+1}, y_j) - C(t_i, y_j)}{\Delta t}, \tag{6.3.10}$$

or, using *backward* finite difference approximations, as

$$\frac{\partial C}{\partial t}(t_i, y) \simeq \frac{C(t_i, y_j) - C(t_{i-1}, y_j)}{\Delta t}. \tag{6.3.11}$$

First order spatial derivatives can be approximated as

$$\frac{\partial C}{\partial y}(t, y_j) \simeq \frac{C(t_i, y_j) - C(t_i, y_{j-1})}{\Delta y}, \quad \frac{\partial C}{\partial y}(t, y_{j+1}) \simeq \frac{C(t_i, y_{j+1}) - C(t_i, y_j)}{\Delta y}. \tag{6.3.12}$$

Reusing (6.3.12), second order spatial derivatives can be similarly approximated as

$$\begin{aligned} \frac{\partial^2 C}{\partial y^2}(t_i, y_j) &\simeq \frac{1}{\Delta y}\left(\frac{\partial C}{\partial y}(t_i, y_{j+1}) - \frac{\partial C}{\partial y}(t_i, y_j)\right) \\ &\simeq \frac{C(t_i, y_{j+1}) - 2C(t_i, y_j) + C(t_i, y_{j-1})}{(\Delta y)^2}. \end{aligned} \tag{6.3.13}$$

Figure 6.14* presents an estimation of local volatility by the finite differences (6.3.10)-(6.3.13), based on Boeing (NYSE:BA) option price data.

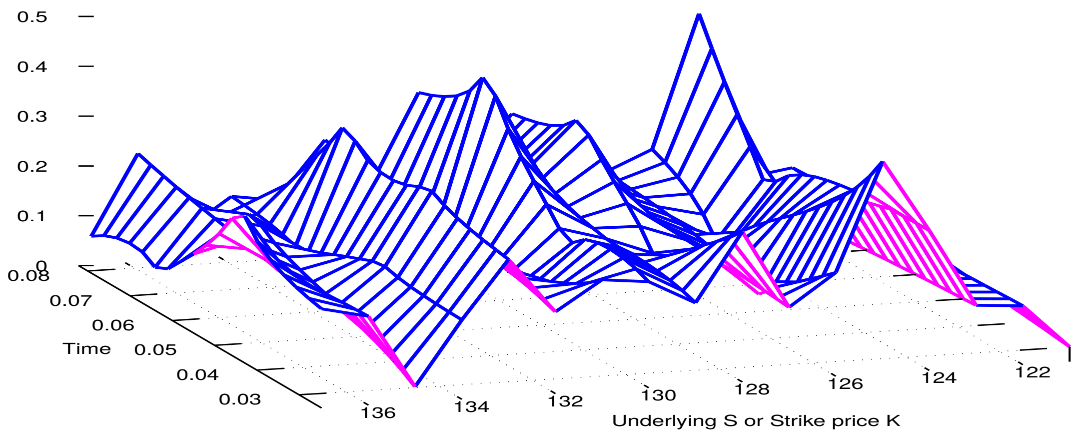


Figure 6.14: Local volatility estimated from Boeing Co. option price data.

See Achdou and Pironneau, 2005 and in particular Figure 8.1 therein for numerical methods applied to local volatility estimation using spline functions instead of the discretization (6.3.10)-(6.3.13). See also Ackerer, Tagasovska, and Vatter, 2020, Chataigner et al., 2021 for deep learning approaches to the estimation of local volatility.

The attached R **code**† plots a local volatility estimate for a given stock.

Based on (6.3.7), the local volatility $\sigma(t, y)$ can also be estimated by computing $C^M(T, y)$ from the Black-Scholes formula, from a value of the implied volatility σ.

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Local volatility from put option prices

Note that by the call-put parity relation

$$C^M(T,y) = P^M(T,y) + x - y\,\mathrm{e}^{-rT}, \qquad y,T>0,$$

where $S_0 = y$, we have

$$\begin{cases} \dfrac{\partial C^M}{\partial T}(T,y) = ry\,\mathrm{e}^{-rT} + \dfrac{\partial P^M}{\partial T}(T,y), \\ \\ \dfrac{\partial P^M}{\partial y}(T,y) = \mathrm{e}^{-rT} + \dfrac{\partial C^M}{\partial y}(T,y), \end{cases}$$

and

$$\frac{\partial C^M}{\partial T}(T,y) + ry\frac{\partial C^M}{\partial y}(T,y) = \frac{\partial P^M}{\partial T}(T,y) + ry\frac{\partial P^M}{\partial y}(T,y).$$

Consequently, the local volatility in Proposition 6.2 can be rewritten in terms of market put option prices as

$$\sigma(t,y) := \sqrt{\frac{2\dfrac{\partial P^M}{\partial t}(t,y) + 2ry\dfrac{\partial P^M}{\partial y}(t,y)}{y^2\dfrac{\partial^2 P^M}{\partial y^2}(t,y)}} = \frac{\sqrt{\dfrac{\partial P^M}{\partial t}(t,y) + ry\dfrac{\partial P^M}{\partial y}(t,y)}}{y\,\mathrm{e}^{-rT/2}\sqrt{\varphi_t(y)/2}},$$

which is formally identical to (6.3.7) after replacing market call option prices $C^M(T,K)$ with market put option prices $P^M(T,K)$. In addition, we have the relation

$$\varphi_T(K) = \mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial y^2}(T,y) = \mathrm{e}^{rT}\frac{\partial^2 P^M}{\partial y^2}(T,y) \tag{6.3.14}$$

between the probability density function φ_T of S_T and the call/put option pricing functions $C^M(T,y)$, $P^M(T,y)$.

6.4 The VIX® Index

Other ways to estimate market volatility include the CBOE Volatility Index® (VIX) for the S&P 500 Index (SPX). Let the asset price process $(S_t)_{t\in\mathbb{R}_+}$ satisfy

$$dS_t = rS_t dt + \sigma_t S_t dB_t,$$

i.e.

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dB_s + rt - \frac{1}{2}\int_0^t \sigma_s^2 ds\right), \qquad t \geqslant 0,$$

where, as in Section 5.2, $(\sigma_t)_{t\in\mathbb{R}_+}$ denotes a stochastic volatility process.

Lemma 6.3 Let $\phi \in \mathscr{C}^2((0,\infty))$. For all $y>0$, we have the following version of the Taylor formula:

$$\phi(x) = \phi(y) + (x-y)\phi'(y) + \int_0^y (z-x)^+\phi''(z)dz + \int_y^\infty (x-z)^+\phi''(z)dz,$$

$x>0$.

Proof. We use the Taylor formula with integral remainder:

$$\phi(x) = \phi(y) + (x-y)\phi'(y) + |x-y|^2 \int_0^1 (1-\tau)\phi''(\tau x + (1-\tau)y)d\tau, \quad x,y \in \mathbb{R}.$$

Letting $z = \tau x + (1-\tau)y = y + \tau(x-y)$, if $x \leqslant y$ we have

$$\begin{aligned} |x-y|^2 \int_0^1 (1-\tau)\phi''(\tau x + (1-\tau)y)d\tau &= |x-y| \int_y^x \left(1 - \frac{z-y}{x-y}\right)\phi''(z)dz \\ &= \int_y^x (x-z)\phi''(z)dz \\ &= \int_y^\infty (x-z)^+\phi''(z)dz. \end{aligned}$$

If $x \geqslant y$, we have

$$\begin{aligned} |x-y|^2 \int_0^1 (1-\tau)\phi''(\tau x + (1-\tau)y)d\tau &= |y-x| \int_x^y \left(1 - \frac{y-z}{y-x}\right)\phi''(z)dz \\ &= \int_x^y (z-x)\phi''(z)dz \\ &= \int_0^y (z-x)^+\phi''(z)dz. \end{aligned}$$

□

The next Proposition 6.4, cf. Remark 5 in Friz and Gatheral, 2005 and page 4 of the CBOE white paper, shows that the VIX® Volatility Index defined as

$$\mathrm{VIX}_t := \sqrt{\frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\int_0^{F_{t,t+\tau}} \frac{P(t,t+\tau,K)}{K^2}dK + \int_{F_{t,t+\tau}}^\infty \frac{C(t,t+\tau,K)}{K^2}dK\right)} \tag{6.4.1}$$

at time $t > 0$ can be interpreted as an average of future volatility values, see also § 3.1.1 of A. Papanicolaou and Sircar, 2014. Here, τ =30 days,

$$F_{t,t+\tau} := \mathbb{E}^*[S_{t+\tau} \mid \mathscr{F}_t] = \mathrm{e}^{r\tau}S_t$$

represents the future price on $S_{t+\tau}$, and $P(t,t+\tau,K)$, $C(t,t+\tau,K)$ are OTM (Out-Of-the-Money) call and put option prices with respect to $F_{t,t+\tau}$, with strike price K and maturity $t+\tau$.

Proposition 6.4 The value of the VIX® Volatility Index at time $t \geqslant 0$ is given from the averaged realized variance swap price as

$$\mathrm{VIX}_t := \sqrt{\frac{1}{\tau}\mathbb{E}^*\left[\int_t^{t+\tau} \sigma_u^2 du \,\middle|\, \mathscr{F}_t\right]}.$$

Proof. We take $t = 0$ for simplicity. Applying Lemma 6.3 to the function

$$\phi(x) = \frac{x}{y} - 1 - \log\frac{x}{y}$$

with $\phi'(x) = 1/y - 1/x$ and $\phi''(x) = 1/x^2$ shows that

$$\frac{x}{y} - 1 - \log\frac{x}{y} = \int_0^y (z-x)^+ \frac{1}{z^2}dz + \int_y^\infty (x-z)^+ \frac{1}{z^2}dz, \quad x,y > 0.$$

Alternatively, we can use the following relationships which are obtained by integration by parts:

$$
\begin{aligned}
\int_0^y (z-x)^+ \frac{dz}{z^2} &= \mathbb{1}_{\{x\leqslant y\}} \int_x^y (z-x) \frac{dz}{z^2} \\
&= \mathbb{1}_{\{x\leqslant y\}} \left(\int_x^y \frac{dz}{z} - x \int_x^y \frac{dz}{z^2} \right) \\
&= \mathbb{1}_{\{x\leqslant y\}} \left(\frac{x}{y} - 1 + \log \frac{y}{x} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_y^\infty (x-z)^+ \frac{dz}{z^2} &= \mathbb{1}_{\{x\geqslant y\}} \int_y^x (x-z) \frac{dz}{z^2} \\
&= \mathbb{1}_{\{x\geqslant y\}} \left(x \int_y^x \frac{dz}{z^2} - \int_y^x \frac{dz}{z} \right) \\
&= \mathbb{1}_{\{x\geqslant y\}} \left(\frac{x}{y} - 1 + \log \frac{y}{x} \right).
\end{aligned}
$$

Hence, taking $y := F_{0,\tau} = \mathrm{e}^{r\tau} S_0$ and $x := S_\tau$, we have

$$
\frac{S_\tau}{F_{0,\tau}} - 1 + \log \frac{F_{0,\tau}}{S_\tau} = \int_0^{F_{0,\tau}} (K - S_\tau)^+ \frac{dK}{K^2} + \int_{F_{0,\tau}}^\infty (S_\tau - K)^+ \frac{dK}{K^2}. \tag{6.4.2}
$$

Next, taking expectations under $\mathbb{P}^*$ on both sides of (6.4.2), we find

$$
\begin{aligned}
\mathrm{VIX}_0^2 &= \frac{2\,\mathrm{e}^{r\tau}}{\tau} \left(\int_0^{F_{0,\tau}} \frac{P(0,\tau,K)}{K^2} dK + \int_{F_{0,\tau}}^\infty \frac{C(0,\tau,K)}{K^2} dK \right) \\
&= \frac{2}{\tau} \int_0^{F_{0,\tau}} \mathbb{E}^*[(K - S_\tau)^+] \frac{dK}{K^2} + \frac{2}{\tau} \int_{F_{0,\tau}}^\infty \mathbb{E}^*[(S_\tau - K)^+] \frac{dK}{K^2} \\
&= \frac{2}{\tau} \mathbb{E}^* \left[\int_0^{F_{0,\tau}} (K - S_\tau)^+ \frac{dK}{K^2} + \int_{F_{0,\tau}}^\infty (S_\tau - K)^+ \frac{dK}{K^2} \right] \\
&= \frac{2}{\tau} \mathbb{E}^* \left[\frac{S_\tau}{F_{0,\tau}} - 1 + \log \frac{F_{0,\tau}}{S_\tau} \right] \\
&= \frac{2}{\tau} \left(\frac{\mathbb{E}^*[S_\tau]}{F_{0,\tau}} - 1 \right) + \frac{2}{\tau} \mathbb{E}^* \left[\log \frac{F_{0,\tau}}{S_\tau} \right] \\
&= \frac{2}{\tau} \mathbb{E}^* \left[\log \frac{F_{0,\tau}}{S_\tau} \right] \\
&= \frac{1}{\tau} \mathbb{E}^* \left[\int_0^\tau \sigma_t^2 dt \right],
\end{aligned}
$$

where we applied Proposition 5.1. □

The following R code allows us to estimate the VIX® index based on the discretization of (6.4.1) and market option prices on the S&P 500 Index (SPX). Here, the OTM put strike prices and call strike prices are listed as

$$
K_1^{(p)} < \cdots < K_{n_p-1}^{(p)} < K_{n_p}^{(p)} := F_{t,t+\tau} =: K_0^{(c)} < K_1^{(c)} < \cdots < K_{n_c}^{(c)},
$$

and (6.4.1) may for example be discretized as

$$
\mathrm{VIX}_t^2 = \frac{2\,\mathrm{e}^{r\tau}}{\tau} \left(\int_0^{F_{t,t+\tau}} \frac{P(t,t+\tau,K)}{K^2} dK + \int_{F_{t,t+\tau}}^\infty \frac{C(t,t+\tau,K)}{K^2} dK \right)
$$

$$
\begin{aligned}
&= \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\sum_{i=1}^{n_p-1}\int_{K_i^{(p)}}^{K_{i+1}^{(p)}}\frac{P(t,t+\tau,K)}{K^2}dK+\sum_{i=1}^{n_c}\int_{K_{i-1}^{(c)}}^{K_i^{(c)}}\frac{C(t,t+\tau,K)}{K^2}dK\right)\\
&\simeq \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\sum_{i=1}^{n_p-1}\int_{K_i^{(p)}}^{K_{i+1}^{(p)}}\frac{P(t,t+\tau,K_i^{(p)})}{K^2}dK+\sum_{i=1}^{n_c}\int_{K_{i-1}^{(c)}}^{K_i^{(c)}}\frac{C(t,t+\tau,K_i^{(c)})}{K^2}dK\right)\\
&= \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\sum_{i=1}^{n_p-1}P(t,t+\tau,K_i^{(p)})\left(\frac{1}{K_i^{(p)}}-\frac{1}{K_{i+1}^{(p)}}\right)\right.\\
&\qquad\left.+\sum_{i=1}^{n_c}\int_{K_{i-1}^{(c)}}^{K_i^{(c)}}C(t,t+\tau,K_i^{(c)})\left(\frac{1}{K_{i-1}^{(c)}}-\frac{1}{K_i^{(c)}}\right)\right),
\end{aligned}
$$

see page 158 of Gatheral, 2006 for the implementation of the discretization of the CBOE white paper.

```r
library(quantmod)
today <- as.Date(Sys.Date(), format="%Y-%m-%d"); getSymbols("^SPX", src = "yahoo")
lastBusDay=last(row.names(as.data.frame(Ad(SPX))))
S0 = as.vector(tail(Ad(SPX),1)); T = 30/365;r=0.02;F0 = S0*exp(r*T)
maturity <- as.Date("2021-07-07", format="%Y-%m-%d") # Choose a maturity in 30 days
SPX.OPTS <- getOptionChain("^SPX", maturity)
Call <- as.data.frame(SPX.OPTS$calls);Put <- as.data.frame(SPX.OPTS$puts)
Call_OTM <- Call[Call$Strike>F0,];Put_OTM <-Put[Put$Strike<F0,];
Call_OTM$dif = c(1/F0-1/min(Call_OTM$Strike),-diff(1/Call_OTM$Strike))
Put_OTM$dif = c(-diff(1/Put_OTM$Strike),1/max(Put_OTM$Strike)-1/F0)
VIX_imp = 100*sqrt((2*exp(r*T)/T)*(sum(Put_OTM$Last*Put_OTM$dif)
      +sum(Call_OTM$Last*Call_OTM$dif)))
getSymbols("^VIX", src = "yahoo", from = lastBusDay);VIX_market = as.vector(Ad(VIX)[1])
c("Estimated VIX"= VIX_imp, "CBOE VIX"=VIX_market)
VIX.OPTS <- getOptionChain("^VIX")
```

The following R code is fetching VIX® index data using the quantmod R package.

```r
library(quantmod)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo")
dev.new(width=16,height=7); myPars <- chart_pars();myPars$cex<-1.4
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(Ad(`GSPC`),name="S&P500",pars=myPars,theme=myTheme)
add_TA(Ad(`VIX`))
```

The impact of various world events can be identified on the VIX® index in Figure 6.15.

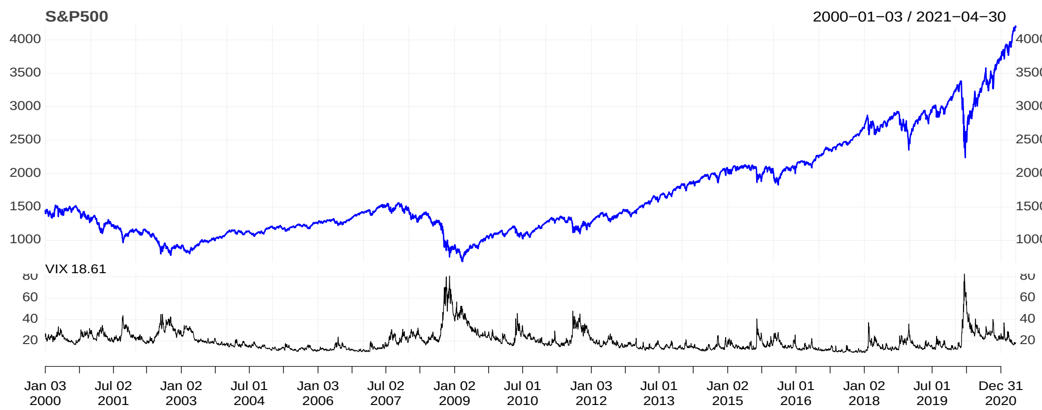


Figure 6.15: VIX® Index *vs.* S&P 500.

```
library(quantmod);library(PerformanceAnalytics)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo");SP500=Ad(`GSPC`)
SP500.rtn <- exp(CalculateReturns(SP500,method="compound")) - 1;SP500.rtn[1,] <- 0
histvol <- rollapply(SP500.rtn, width = 30, FUN=sd.annualized)
dev.new(width=16,height=7)
myPars <- chart_pars();myPars$cex<-1.4
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(SP500,name="SP500",theme=myTheme,pars=myPars)
add_TA(histvol, name="Historical Volatility");add_TA(Ad(`VIX`), name="VIX")
```

Figure 6.16 compares the VIX® index estimate to the historical volatility of Section 6.1.

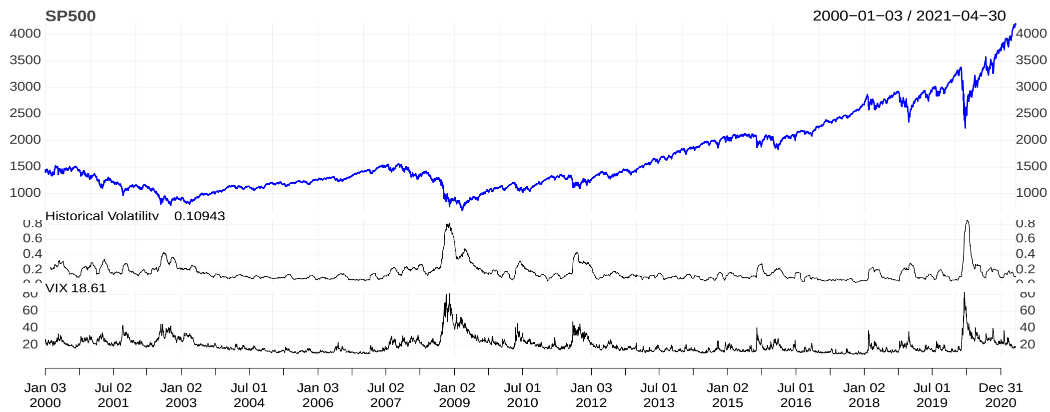
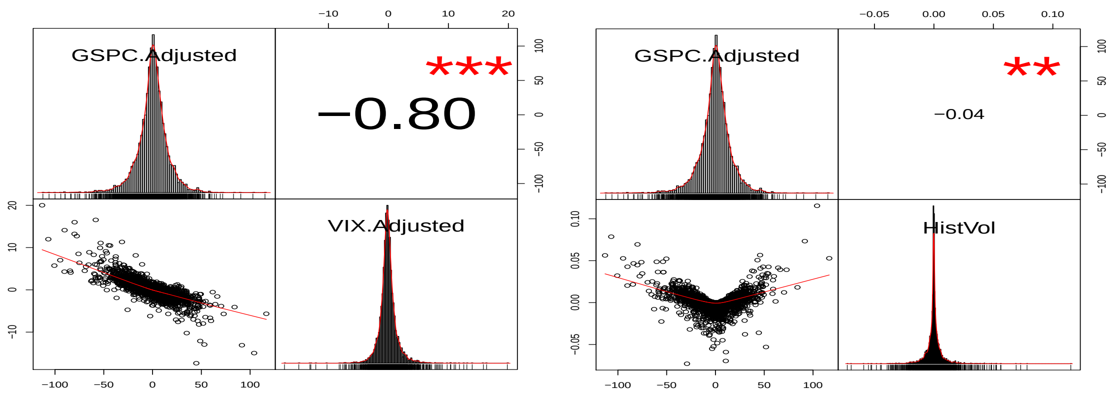


Figure 6.16: VIX® Index *vs.* historical volatility for the year 2011.

We note that the variations of the stock index are negatively correlated to the variations of the VIX® index, however the same cannot be said of the correlation to the variations of historical volatility.



(a) GSPC returns *vs.* VIX®.

(b) GSPC returns *vs.* hist. vol.

Figure 6.17: Correlation estimates between GSPC and the VIX®.

```
chart.Correlation(cbind(Ad(`GSPC`)-lag(Ad(`GSPC`)),Ad(`VIX`)-lag(Ad(`VIX`))), histogram=TRUE,
      pch="+")
colnames(histvol) <- "HistVol"
chart.Correlation(cbind(Ad(`GSPC`)-lag(Ad(`GSPC`)),histvol-lag(histvol)), histogram=TRUE, pch="+")
```

Figure 6.18 shortens the time range to year 2011 and shows the increased reactivity of the VIX® index to volatility spikes, in comparison with the moving average of historical volatility.

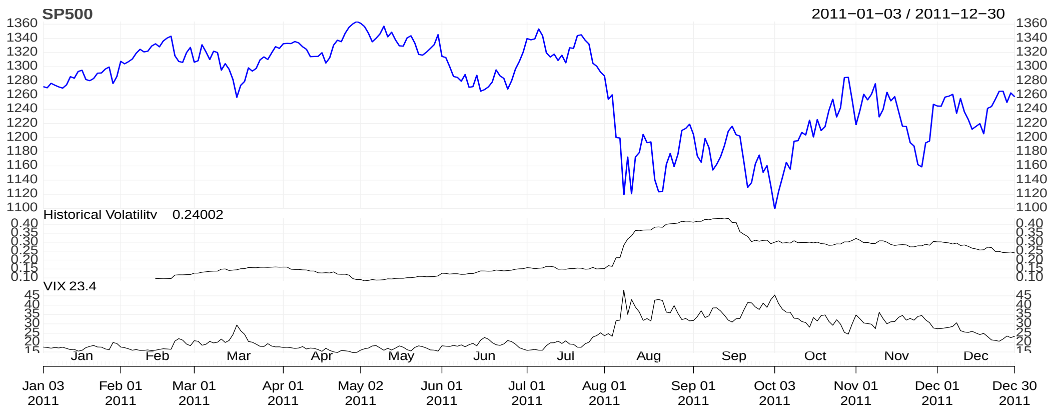


Figure 6.18: VIX® Index *vs.* 30 day historical volatility for the S&P 500.

Exercises

Exercise 6.1 We consider an entropy swap with discrete-time payoff

$$\frac{1}{T}\sum_{k=1}^{N} S_{t_k}\log\frac{S_{t_k}}{S_{t_{k-1}}} - \kappa_\sigma^2 = \frac{1}{T}\sum_{k=1}^{N} S_{t_k}(\log S_{t_k} - \log S_{t_{k-1}}) - \kappa_\sigma^2,$$

approximated in continous time as

$$\frac{1}{T}\int_0^T S_t d\log S_t - \kappa_\sigma^2,$$

where κ_σ is the volatility level.

a) Show that for any $K^* > 0$ we have

$$\int_0^T S_t d\log S_t = f_{K^*}(S_T) - f_{K^*}(S_0) + \int_0^T dS_t + \int_0^T \log\frac{S_t}{K^*} dS_t, \tag{6.4.3}$$

where $f_{K^*}(x) := x - K^* - x\log(x/K^*)$.

Hint: Use $d\log S_t$ as well.

b) Show that the payoff $f_{K^*}(S_T)$ can be hedged using a portfolio of call and put options.

Hint: Use Lemma 6.3 with $y := K^*$.

Exercise 6.2 Strike arbitrage.

a) Given a set of three strike prices $K_1 < K_2 < K_3$ with $K_3 - K_2 = K_2 - K_1 = \Delta K > 0$, write down a discretized expression of the second partial derivative

$$\frac{\partial^2 C}{\partial K^2}(T,K)_{|K=K_2}.$$

b) Show that if $\frac{\partial^2 C}{\partial K^2}(T,K)_{|K=K_2} < 0$, one can construct a portfolio leading to an arbitrage opportunity.

Hint: Choose your own values of K_1, K_2, K_3, use https://optioncreator.com/, and Save your portfolio design.

Exercise 6.3 Consider the Black-Scholes call pricing formula

$$C(T-t,x,K) = Kf\left(T-t,\frac{x}{K}\right)$$

written using the function

$$f(\tau,z) := z\Phi\left(\frac{(r+\sigma^2/2)\tau+\log z}{|\sigma|\sqrt{\tau}}\right) - \mathrm{e}^{-r\tau}\Phi\left(\frac{(r-\sigma^2/2)\tau+\log z}{|\sigma|\sqrt{\tau}}\right).$$

a) Compute $\dfrac{\partial C}{\partial x}$ and $\dfrac{\partial C}{\partial K}$ using the function f, and find the relation between $\dfrac{\partial C}{\partial K}(T-t,x,K)$ and $\dfrac{\partial C}{\partial x}(T-t,x,K)$.

b) Compute $\dfrac{\partial^2 C}{\partial x^2}$ and $\dfrac{\partial^2 C}{\partial K^2}$ using the function f, and find the relation between $\dfrac{\partial C^2}{\partial K^2}(T-t,x,K)$ and $\dfrac{\partial C^2}{\partial x^2}(T-t,x,K)$.

c) From the Black-Scholes PDE

$$\begin{aligned} rC(T-t,x,K) &= \frac{\partial C}{\partial t}(T-t,x,K) + rx\frac{\partial C}{\partial x}(T-t,x,K) \\ &\quad + \frac{\sigma^2 x^2}{2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K), \end{aligned}$$

recover the Dupire, 1994 PDE (6.3.9) for the constant volatility σ.

Exercise 6.4 The prices of call options in a certain local volatility model of the form $dS_t = S_t\sigma(t,S_t)dB_t$ with risk-free rate $\underline{r=0}$ are given by

$$C(S_0,K,T) = \sqrt{\frac{T}{2\pi}}\mathrm{e}^{-(K-S_0)^2/(2T)} - (K-S_0)\Phi\left(-\frac{K-S_0}{\sqrt{T}}\right), \quad K,T>0.$$

Recover the local volatility function $\sigma(t,x)$ of this model by applying the Dupire formula.

Exercise 6.5 Let $\sigma_{\mathrm{imp}}(K)$ denote the implied volatility of a call option with strike price K, defined from the relation

$$M_C(K,S,r,\tau) = C(K,S,\sigma_{\mathrm{imp}}(K),r,\tau),$$

where M_C is the market price of the call option, $C(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)$ is the Black-Scholes call pricing function, S is the underlying asset price, τ is the time remaining until maturity, and r is the risk-free interest rate.

a) Compute the partial derivative

$$\frac{\partial M_C}{\partial K}(K,S,r,\tau).$$

using the functions C and σ_{imp}.

b) Knowing that market call option prices $M_C(K,S,r,\tau)$ are *decreasing* in the strike prices K, find an upper bound for the slope $\sigma'_{\mathrm{imp}}(K)$ of the implied volatility curve.

c) Similarly, knowing that the market *put* option prices $M_P(K,S,r,\tau)$ are *increasing* in the strike prices K, find a lower bound for the slope $\sigma'_{\mathrm{imp}}(K)$ of the implied volatility curve.

Exercise 6.6 (Hagan et al., 2002) Consider the European option priced as $e^{-rT}\,\mathbb{E}^*[(S_T-K)^+]$ in a local volatility model $dS_t=\sigma_{\rm loc}(S_t)S_t dB_t$. The implied volatility $\sigma_{\rm imp}(K,S_0)$, computed from the equation

$$\mathrm{Bl}\big(S_0,K,T,\sigma_{\rm imp}(K,S_0),r\big)=\mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T-K)^+],$$

is known to admit the approximation

$$\sigma_{\rm imp}(K,S_0)\simeq\sigma_{\rm loc}\left(\frac{K+S_0}{2}\right).$$

a) Taking a local volatility of the form $\sigma_{\rm loc}(x):=\sigma_0+\beta(x-S_0)^2$, estimate the implied volatility $\sigma_{\rm imp}(K,S)$ when the underlying asset price is at the level S.
b) Express the Delta of the Black Scholes call option price given by

$$\mathrm{Bl}\big(S,K,T,\sigma_{\rm imp}(K,S),r\big),$$

using the standard Black-Scholes Delta and the Black-Scholes Vega.

Exercise 6.7 Show that the result of Proposition 6.4 can be recovered from Lemma 5.2 and Relation (6.3.14).

Exercise 6.8 (Exercise 5.7 continued). Find an expression for $\mathbb{E}^*\left[R^4_{0,T}\right]$ using call and put pricing functions.

Exercise 6.9 (Henry-Labordère, 2009, § 3.5).
a) Using the gamma probability density function and integration by parts or Laplace transform inversion, prove the formula

$$\int_0^\infty\frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-\mu x}}{x^{\rho+1}}dx=\frac{\mu^\rho-\nu^\rho}{\rho}\Gamma(1-\rho)$$

for all $\rho\in(0,1)$ and $\mu,\nu>0$, see Relation 3.434.1 in Gradshteyn and Ryzhik, 2007.
b) By the result of Question (a)), generalize the volatility swap pricing expression (5.3.10).
c) By Lemma 5.2 and the result of Question (b)), find an expression of the volatility swap price using call and put functions.

7. Maximum of Brownian motion

The probability distribution of the maximum of Brownian motion on a given interval can be computed in closed form using the reflection principle. As a consequence, the expected value of the running maximum of Brownian motion can also be computed explicitly. Those properties will be applied in the next Chapters 8 and 9 to the pricing of barrier and lookback options, whose payoffs may depend on extrema of the underlying asset price process $(S_t)_{t\in[0,T]}$, as well as on its terminal value S_T.

7.1 Running Maximum of Brownian Motion

Figure 7.1 represents the running maximum process

$$X_0^t := \operatorname*{Max}_{s\in[0,t]} W_s, \qquad t \geqslant 0,$$

of Brownian motion $(W_t)_{t\in\mathbb{R}_+}$.

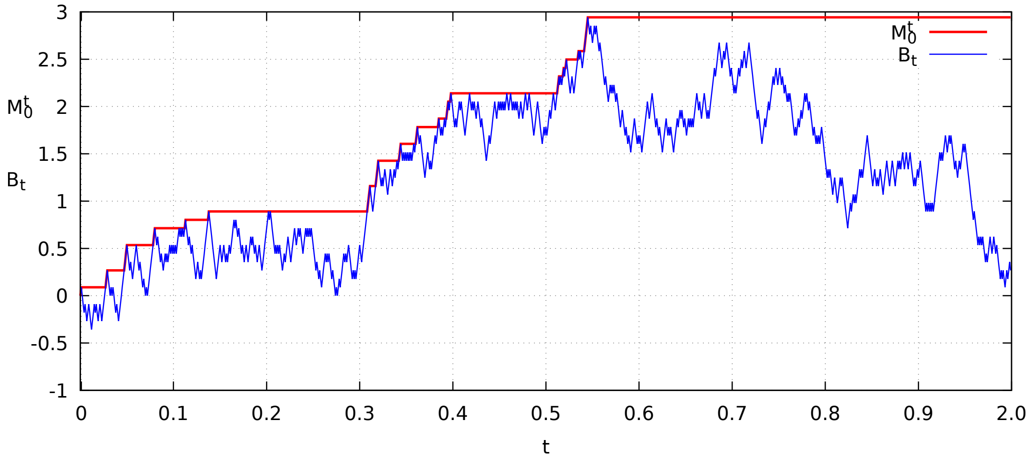


Figure 7.1: Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ and its running maximum $(X_0^t)_{t\in\mathbb{R}_+}$.*

Note that Brownian motion admits (almost surely) no "point of increase". More precisely, there does not exist $t > 0$ and $\varepsilon > 0$ such that

$$\operatorname*{Max}_{s\in(t-\varepsilon,t)} W_s \leqslant W_t \leqslant \min_{s\in(t,t+\varepsilon)} W_s,$$

see, *e.g.*, Dvoretzky, Erdős, and Kakutani, 1961 and Burdzy, 1990. This property is illustrated in Figure 7.2, see also (7.2.4)-(7.2.5) below.

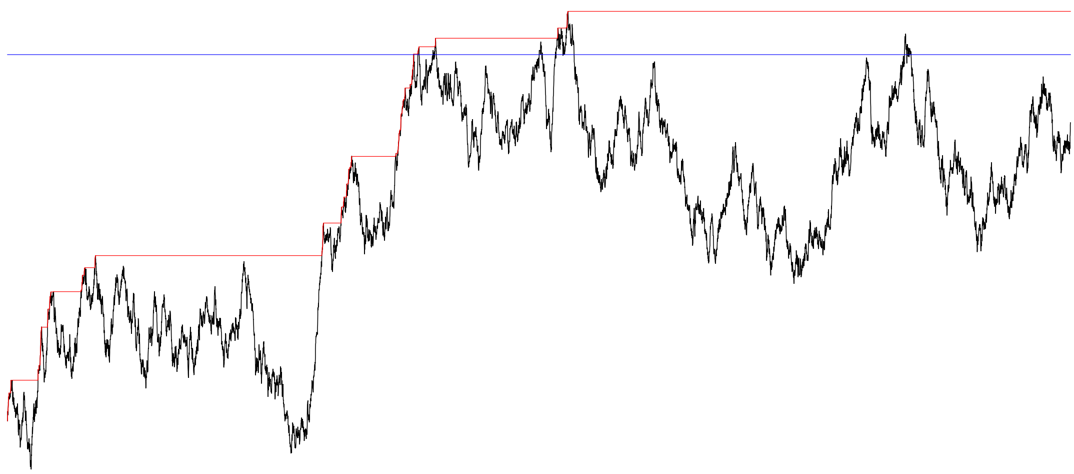


Figure 7.2: Running maximum of Brownian motion.*

Related properties can be observed with the zeroes of Brownian motion which form an *uncountable* set (see *e.g.* Theorem 2.28 page 48 of Mörters and Peres, 2010) which has *zero measure* $\mathbb{P}$-almost surely, as we have

$$\mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{W_t=0\}}dt\right] = \int_0^\infty \mathbb{E}\left[\mathbb{1}_{\{W_t=0\}}\right]dt = \int_0^\infty \mathbb{P}(W_t=0)dt = 0,$$

*The animation works in Acrobat Reader on the entire pdf file.

see Figure 7.3.

Figure 7.3: Zeroes of Brownian motion.*

See also the Cantor function presented in the next Figure 7.4, which is continuous on $[0,1]$ and flat (with a vanishing derivative) everywhere except on the *Cantor set*, which is an *uncountable* set of *zero measure* in $[0,1]$.

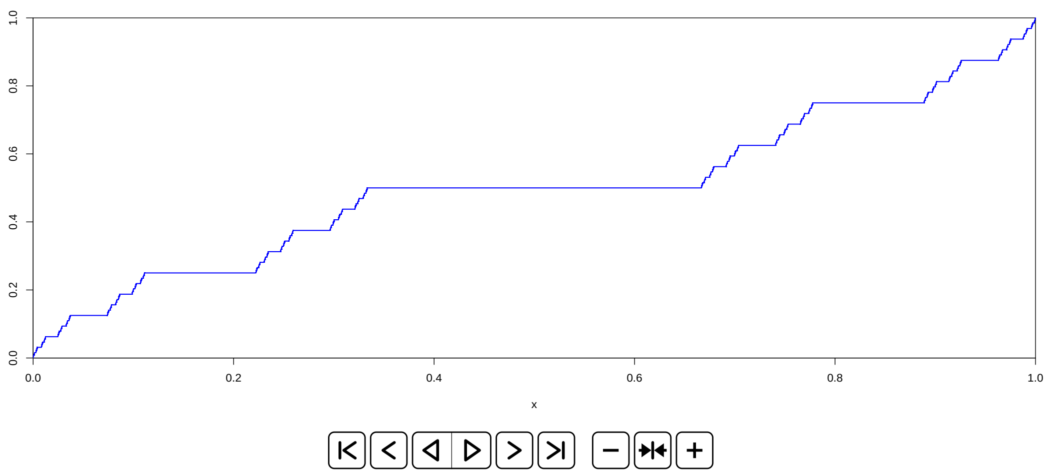


Figure 7.4: Graph of the Cantor function.†

Examples of deterministic functions having no "last point of increase" can be built for some $\varepsilon \in (0,1)$ as

$$f(t) := (1-\varepsilon)\sum_{n\geqslant 1}\varepsilon^{n-1}\mathbb{1}_{[1-\varepsilon^n,1)}(t) + \mathbb{1}_{[1,\infty)}(t), \qquad t \geqslant 0,$$

which admits no "last" point of increase before $t=1$, as illustrated in Figure 7.5 with $\varepsilon = 3/4$.

*The animation works in Acrobat Reader on the entire pdf file.
†The animation works in Acrobat Reader on the entire pdf file.

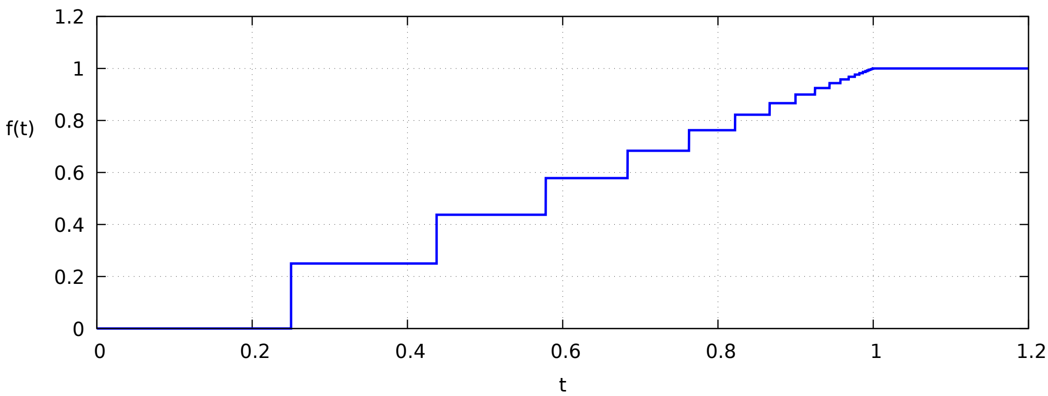


Figure 7.5: A function with no last point of increase before $t = 1$.

7.2 The Reflection Principle

Let $(W_t)_{t\in\mathbb{R}_+}$ denote the standard Brownian motion started at $W_0 = 0$. While it is well-known that $W_T \simeq \mathcal{N}(0,T)$, computing the distribution of the maximum

$$X_0^T := \operatorname*{Max}_{t\in[0,T]} W_t$$

might seem a difficult problem. However, this is not the case, due to the *reflection principle*.

Note that since $W_0 = 0$, we have

$$X_0^T = \operatorname*{Max}_{t\in[0,T]} W_t \geqslant 0,$$

almost surely, *i.e.* with probability one. Given $a > W_0 = 0$, let

$$\tau_a = \inf\{t \geqslant 0 \ : \ W_t = a\}$$

denote the first time $(W_t)_{t\in\mathbb{R}_+}$ hits the level $a > 0$. Due to the spatial symmetry of Brownian motion we note the identity

$$\mathbb{P}(W_T \geqslant a \mid \tau_a \leqslant T) = \mathbb{P}(W_T > a \mid \tau_a \leqslant T) = \mathbb{P}(W_T \leqslant a \mid \tau_a \leqslant T) = \frac{1}{2}.$$

In addition, due to the relation

$$\{X_0^T \geqslant a\} = \{\tau_a \leqslant T\}, \tag{7.2.1}$$

we have

$$\begin{aligned}
\mathbb{P}(\tau_a \leqslant T) &= \mathbb{P}(\tau_a \leqslant T \text{ and } W_T > a) + \mathbb{P}(\tau_a \leqslant T \text{ and } W_T \leqslant a) \\
&= 2\mathbb{P}(\tau_a \leqslant T \text{ and } W_T \geqslant a) \\
&= 2\mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T \geqslant a\big) \\
&= 2\mathbb{P}(W_T \geqslant a) \\
&= \mathbb{P}(W_T \geqslant a) + \mathbb{P}(W_T \leqslant -a) \\
&= \mathbb{P}(|W_T| \geqslant a),
\end{aligned}$$

where we used the fact that

$$\{W_T \geqslant a\} \subset \big\{X_0^T \geqslant a \text{ and } W_T \geqslant a\big\} \subset \{W_T \geqslant a\}.$$

Figure 7.6 shows a graph of Brownian motion and its reflected path, with $0 < b < a < 2a - b$.

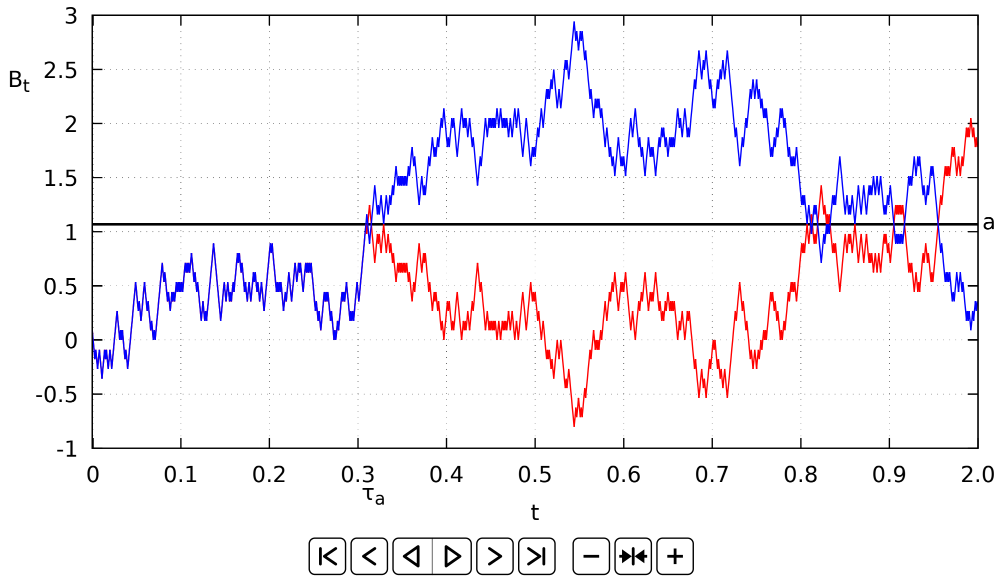


Figure 7.6: Reflected Brownian motion with $a = 1.07$.*

As a consequence of the equality

$$\mathbb{P}(\tau_a \leqslant T) = \mathbb{P}(|W_T| \geqslant a), \qquad a > 0, \tag{7.2.2}$$

the maximum X_0^T of Brownian motion has *same distribution* as the absolute value $|W_T|$ of W_T. Precisely, X_0^T is a nonnegative random variable with cumulative distribution function given by

$$\begin{aligned}\mathbb{P}\big(X_0^T < a\big) &= \mathbb{P}(\tau_a > T)\\ &= \mathbb{P}(|W_T| < a)\\ &= \frac{1}{\sqrt{2\pi T}} \int_{-a}^{a} \mathrm{e}^{-x^2/(2T)} dx\\ &= \frac{2}{\sqrt{2\pi T}} \int_0^a \mathrm{e}^{-x^2/(2T)} dx, \qquad a \geqslant 0,\end{aligned}$$

i.e.

$$\mathbb{P}\big(X_0^T \leqslant a\big) = \frac{2}{\sqrt{2\pi T}} \int_0^a \mathrm{e}^{-x^2/(2T)} dx, \qquad a \geqslant 0,$$

and probability density function

$$\varphi_{X_0^T}(a) = \frac{d\mathbb{P}\big(X_0^T \leqslant a\big)}{da} = \sqrt{\frac{2}{\pi T}}\, \mathrm{e}^{-a^2/(2T)} \mathbb{1}_{[0,\infty)}(a), \quad a \in \mathbb{R}, \tag{7.2.3}$$

which vanishes over $a \in (-\infty, 0]$ because $X_0^T \geqslant 0$ almost surely.

*The animation works in Acrobat Reader on the entire pdf file.

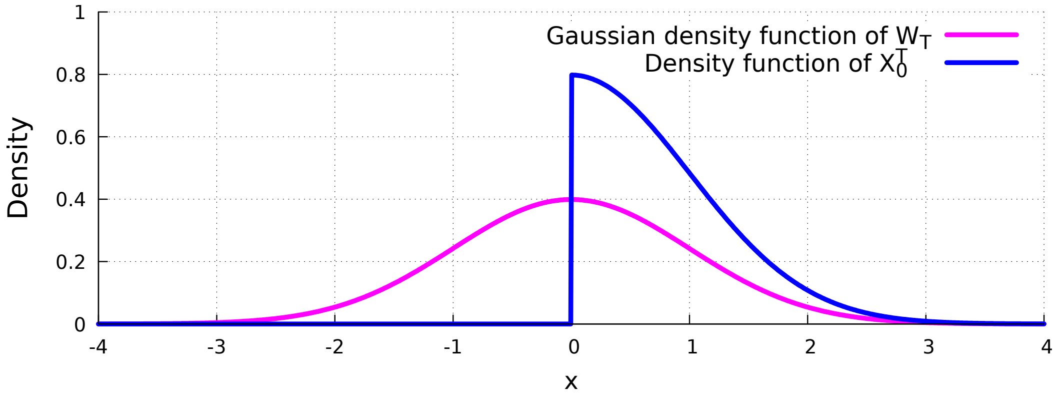


Figure 7.7: Probability density of the maximum X_0^1 of Brownian motion over [0,1].

We note that, as a consequence of the existence of the probability density function (7.2.3), we have

$$\mathbb{P}(W_t \leqslant 0, \ \forall t \in [0, \varepsilon]) = \mathbb{P}(X_0^\varepsilon = 0) = \int_0^0 \varphi_{X_0^\varepsilon}(a) ds = 0, \tag{7.2.4}$$

for all $\varepsilon > 0$. Similarly, by a symmetry argument, for all $\varepsilon > 0$ we find

$$\mathbb{P}(W_t \geqslant 0, \ \forall t \in [0, \varepsilon]) = 0, \tag{7.2.5}$$

and similarly

$$\mathbb{P}(W_t \leqslant 0, \ \forall t \in [0, \varepsilon]) = 0.$$

Using the probability density function of X_0^T, we can price an option with payoff $\phi(X_0^T)$, as

$$\begin{aligned} \mathrm{e}^{-rT} \mathbb{E}^* [\phi(X_0^T)] &= \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \phi(x) d\mathbb{P}(X_0^T \leqslant x) \\ &= \mathrm{e}^{-rT} \sqrt{\frac{2}{\pi T}} \int_0^{\infty} \phi(x) \, \mathrm{e}^{-x^2/(2T)} dx. \end{aligned}$$

Proposition 7.1 Let $\sigma > 0$ and $(S_t)_{t \in [0,T]} := \left(S_0 \, \mathrm{e}^{\sigma W_t}\right)_{t \in [0,T]}$. The probability density function of the maximum

$$M_0^T := \underset{t \in [0,T]}{\mathrm{Max}} \, S_t$$

of $(S_t)_{t \in [0,T]}$ over the time interval $[0, T]$ is given by the truncated lognormal probability density function

$$\varphi_{M_0^T}(y) = \mathbb{1}_{[S_0, \infty)}(y) \frac{1}{\sigma y} \sqrt{\frac{2}{\pi T}} \exp\left(-\frac{1}{2\sigma^2 T} (\log(y/S_0))^2\right), \quad y > 0,$$

see Figure 7.8.

Proof. Since $\sigma > 0$, we have

$$\begin{aligned} M_0^T &= \underset{t \in [0,T]}{\mathrm{Max}} \, S_t \\ &= S_0 \underset{t \in [0,T]}{\mathrm{Max}} \, \mathrm{e}^{\sigma W_t} \end{aligned}$$

$$
\begin{aligned}
&= S_0 \, \mathrm{e}^{\sigma \operatorname{Max}_{t\in[0,T]} W_t} \\
&= S_0 \, \mathrm{e}^{\sigma X_0^T}.
\end{aligned}
$$

Hence $M_0^T = h(X_0^T)$ with $h(x) = S_0 \, \mathrm{e}^{\sigma x}$, and

$$
h'(x) = \sigma S_0 \, \mathrm{e}^{\sigma x}, \quad x \in \mathbb{R}, \text{ and } h^{-1}(y) = \frac{1}{\sigma} \log\left(\frac{y}{S_0}\right), \quad y > 0,
$$

hence

$$
\begin{aligned}
\varphi_{M_0^T}(y) &= \frac{1}{|h'(h^{-1}(y))|} \varphi_{X_0^T}(h^{-1}(y)) \\
&= \mathbb{1}_{[0,\infty)}(h^{-1}(y)) \frac{\sqrt{2}}{|h'(h^{-1}(y))|\sqrt{\pi T}} \mathrm{e}^{-(h^{-1}(y))^2/(2T)} \\
&= \mathbb{1}_{[S_0,\infty)}(y) \frac{1}{\sigma y} \sqrt{\frac{2}{\pi T}} \exp\left(-\frac{1}{2\sigma^2 T}(\log(y/S_0))^2\right), \quad y > 0.
\end{aligned}
$$

□

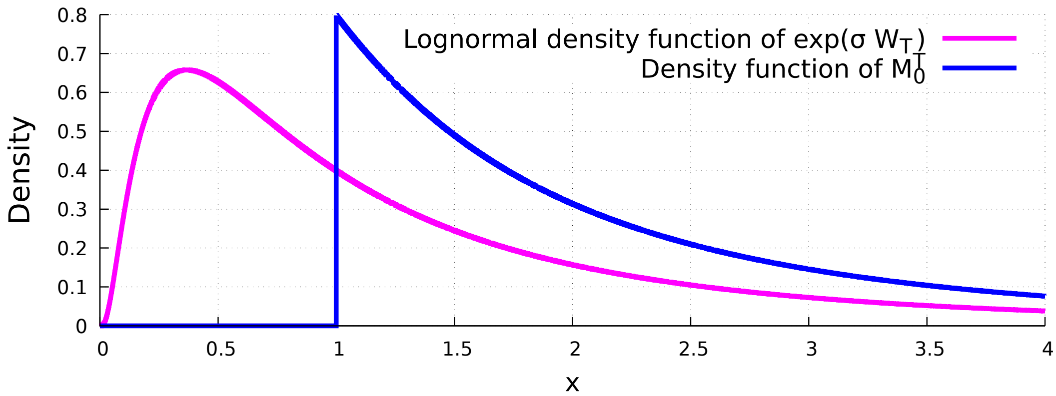


Figure 7.8: Density of the maximum $M_0^T = \operatorname{Max}_{t\in[0,T]} S_t$ of geometric Brownian motion with $S_0 = 1$.

When the claim payoff takes the form $C = \phi(M_0^T)$, where $S_T = S_0 \, \mathrm{e}^{\sigma W_T}$, we have

$$
C = \phi(M_0^T) = \phi(S_0 \, \mathrm{e}^{\sigma X_0^T}),
$$

hence

$$
\begin{aligned}
\mathrm{e}^{-rT} \mathbb{E}^*[C] &= \mathrm{e}^{-rT} \mathbb{E}^*\left[\phi(S_0 \, \mathrm{e}^{\sigma X_0^T})\right] \\
&= \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \phi(S_0 \, \mathrm{e}^{\sigma x}) d\mathbb{P}(X_0^T \leqslant x) \\
&= \sqrt{\frac{2}{\pi T}} \mathrm{e}^{-rT} \int_0^{\infty} \phi(S_0 \, \mathrm{e}^{\sigma x}) \, \mathrm{e}^{-x^2/(2T)} dx \\
&= \sqrt{\frac{2}{\pi \sigma^2 T}} \mathrm{e}^{-rT} \int_1^{\infty} \phi(y) \exp\left(-\frac{1}{2\sigma^2 T}(\log(y/S_0))^2\right) \frac{dy}{y},
\end{aligned}
$$

after the change of variable $y = S_0 \, \mathrm{e}^{\sigma x}$ with $dx = dy/(\sigma y)$.

The above computation is however not sufficient for practical applications as it imposes the condition $r = \sigma^2/2$. In order to do away with this condition we need to consider the maximum of *drifted* Brownian motion, and for this we have to compute the *joint* probability density function of X_0^T and W_T.

7.3 Density of the Maximum of Brownian Motion

The reflection principle also allows us to compute the *joint* probability density function of Brownian motion W_T and its maximum $X_0^T = \underset{t\in[0,T]}{\operatorname{Max}} W_t$. Recall that the probability density function $\varphi_{X_0^T,W_T}$ can be recovered from the joint cumulative distribution function

$$\begin{aligned}(x,y) \longmapsto F_{X_0^T,W_T}(x,y) &:= \mathbb{P}\big(X_0^T \leqslant x \text{ and } W_T \leqslant y\big)\\ &= \int_{-\infty}^{x}\int_{-\infty}^{y} \varphi_{X_0^T,W_T}(s,t)\,ds\,dt,\end{aligned}$$

and

$$(x,y) \longmapsto \mathbb{P}\big(X_0^T \geqslant x \text{ and } W_T \geqslant y\big) = \int_x^\infty \int_y^\infty \varphi_{X_0^T,W_T}(s,t)\,ds\,dt,$$

as

$$\begin{aligned}\varphi_{X_0^T,W_T}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{X_0^T,W_T}(x,y) &(7.3.1)\\ &= \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{x}\int_{-\infty}^{y} \varphi_{X_0^T,W_T}(s,t)\,ds\,dt &(7.3.2)\\ &= \frac{\partial^2}{\partial x \partial y} \int_x^\infty \int_y^\infty \varphi_{X_0^T,W_T}(s,t)\,ds\,dt, \qquad x,y\in\mathbb{R}.\end{aligned}$$

The probability densities $\varphi_{X_0^T} : \mathbb{R} \longrightarrow \mathbb{R}_+$ and $\varphi_{W_T} : \mathbb{R} \longrightarrow \mathbb{R}_+$ of X_0^T and W_T are called the marginal densities of $\big(X_0^T, W_T\big)$, and are given by

$$\varphi_{X_0^T}(x) = \int_{-\infty}^{\infty} \varphi_{X_0^T,W_T}(x,y)\,dy, \qquad x\in\mathbb{R},$$

and

$$\varphi_{W_T}(y) = \int_{-\infty}^{\infty} \varphi_{X_0^T,W_T}(x,y)\,dx, \qquad y\in\mathbb{R}.$$

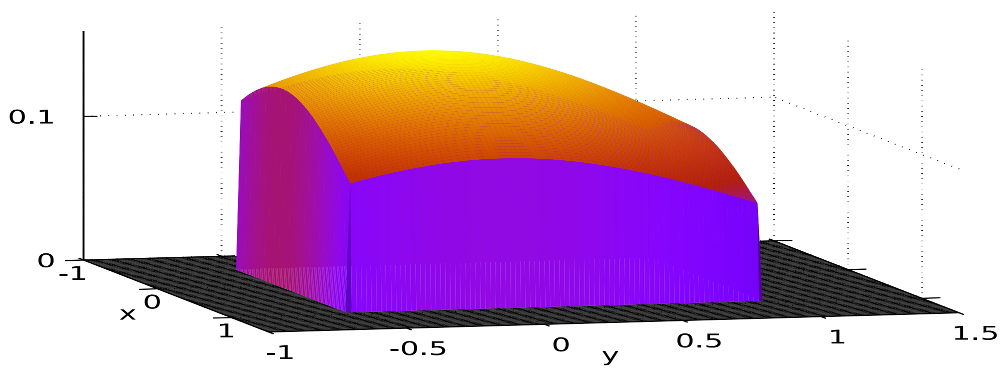


Figure 7.9: Probability $\mathbb{P}((X,Y)\in[-0.5,1]\times[-0.5,1])$ computed as a volume integral.

In order to compute the *joint* probability density function of Brownian motion W_T and its maximum $X_0^T = \underset{t\in[0,T]}{\operatorname{Max}} W_t$ by the reflection principle, we note that for any $b \leqslant a$ we have

$$\mathbb{P}(W_T < b \mid \tau_a < T) = \mathbb{P}(W_T > a + (a-b) \mid \tau_a < T)$$

as shown in Figure 7.10, *i.e.*

$$\mathbb{P}(W_T < b \text{ and } \tau_a < T) = \mathbb{P}(W_T > 2a - b \text{ and } \tau_a < T),$$

or, by (7.2.1),

$$\mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T < b\big) \quad = \quad \mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T > 2a-b\big).$$

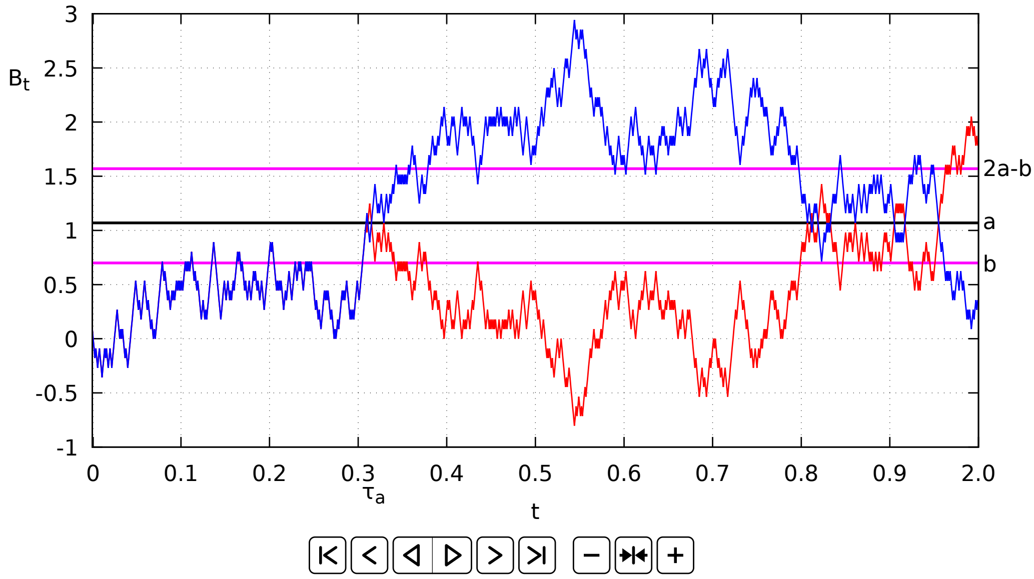


Figure 7.10: Reflected Brownian motion with $a = 1.07$.*

Hence, since $2a - b \geqslant a$ we have

$$\mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T < b\big) = \mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T > 2a-b\big) = \mathbb{P}(W_T \geqslant 2a-b), \tag{7.3.3}$$

where we used the fact that

$$\begin{aligned}\{W_T \geqslant 2a-b\} \quad &\subset \quad \big\{X_0^T \geqslant 2a-b \text{ and } W_T > 2a-b\big\} \\ &\subset \quad \big\{X_0^T \geqslant a \text{ and } W_T > 2a-b\big\} \subset \{W_T > 2a-b\},\end{aligned}$$

which shows that

$$\{W_T \geqslant 2a-b\} = \big\{X_0^T \geqslant a \text{ and } \ W_T > 2a-b\big\}.$$

Consequently, by (7.3.3) we find

$$\begin{aligned}\mathbb{P}\big(X_0^T > a \text{ and } W_T \leqslant b\big) \quad &= \quad \mathbb{P}\big(X_0^T \geqslant a \text{ and } W_T < b\big) \\ &= \quad \mathbb{P}\big(W_T \geqslant 2a-b\big) \\ &= \quad \frac{1}{\sqrt{2\pi T}} \int_{2a-b}^{\infty} \mathrm{e}^{-x^2/(2T)} dx,\end{aligned} \tag{7.3.4}$$

$0 \leqslant b \leqslant a$, which yields the joint probability density function

$$\begin{aligned}\varphi_{X_0^T, W_T}(a,b) \quad &= \quad \frac{\partial^2}{\partial a \partial b} \mathbb{P}\big(X_0^T \leqslant a \text{ and } W_T \leqslant b\big) \\ &= \quad \frac{\partial^2}{\partial a \partial b} \big(\mathbb{P}\big(W_T \leqslant b\big) - \mathbb{P}\big(X_0^T > a \text{ and } W_T \leqslant b\big)\big)\end{aligned}$$

*The animation works in Acrobat Reader on the entire pdf file.

$$= -\frac{d\mathbb{P}\left(X_0^T > a \text{ and } W_T \leqslant b\right)}{dadb}, \qquad a,b \in \mathbb{R}.$$

By (7.3.4), we obtain the following proposition.

Proposition 7.2 The joint probability density function $\varphi_{X_0^T, W_T}$ of Brownian motion W_T and its maximum $X_0^T = \underset{t\in[0,T]}{\operatorname{Max}} W_t$ is given by

$$\boxed{\begin{aligned}\varphi_{X_0^T, W_T}(a,b) &= \sqrt{\frac{2}{\pi}}\frac{(2a-b)}{T^{3/2}}\mathrm{e}^{-(2a-b)^2/(2T)}\mathbb{1}_{\{a\geqslant \operatorname{Max}(b,0)\}} \\ &= \begin{cases} \sqrt{\frac{2}{\pi}}\frac{(2a-b)}{T^{3/2}}\mathrm{e}^{-(2a-b)^2/(2T)}, & a > \operatorname{Max}(b,0), \\ 0, & a < \operatorname{Max}(b,0).\end{cases}\end{aligned}} \quad (7.3.5)$$

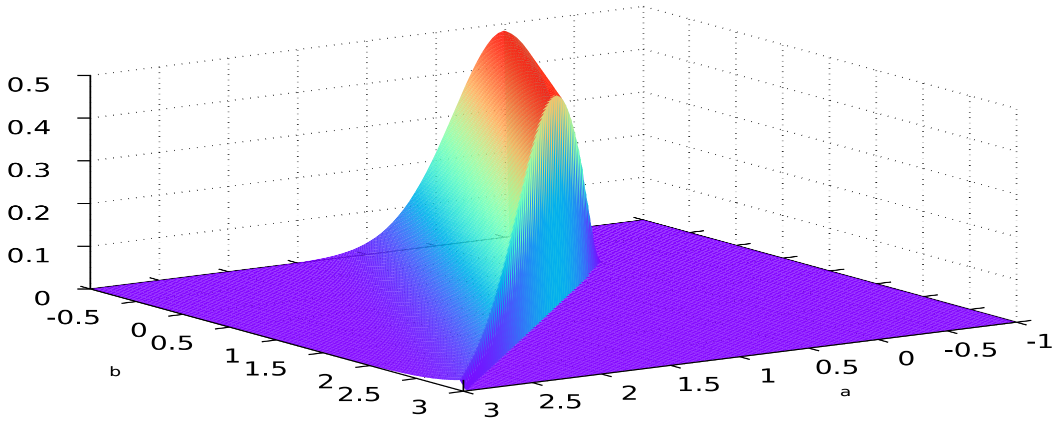


Figure 7.11: Joint probability density of W_1 and the maximum X_0^1 over $[0,1]$.

Figure 7.12 presents the *heat map* of Figure 7.11, as viewed from above.

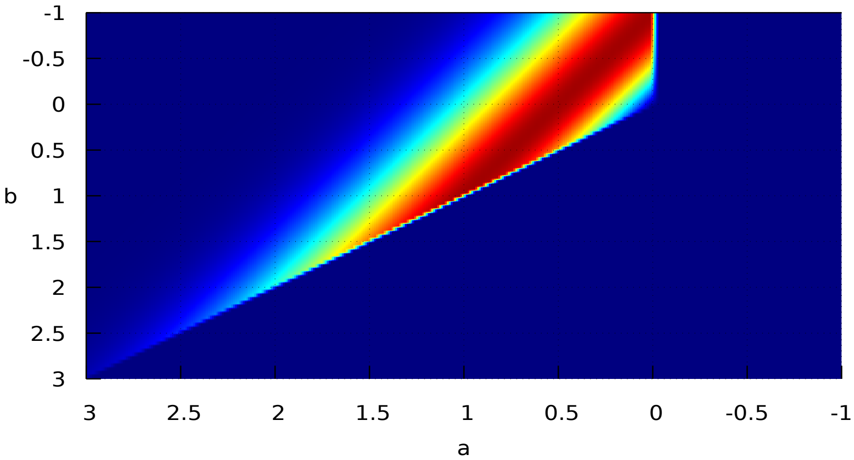


Figure 7.12: Heat map of the joint density of W_1 and its maximum $\widehat{X}_0^1$ over [0,1].

See Relation (4.44) in Borodin, 2017 for the joint distribution of the minimum $\min_{t\in[0,T]} W_t$, the maximum $\operatorname{Max}_{t\in[0,T]} W_t$ and the endpoint W_t of Brownian motion.

Maximum of drifted Brownian motion

Using the Girsanov Theorem, it is even possible to compute the probability density function of the maximum

$$\widehat{X}_0^T := \operatorname*{Max}_{t\in[0,T]} \widetilde{W}_t = \operatorname*{Max}_{t\in[0,T]} (W_t + \mu t)$$

of drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t \in [0,T]$, for any $\mu \in \mathbb{R}$.

Proposition 7.3 The joint probability density function $\varphi_{\widehat{X}_0^T,\widetilde{W}_T}$ of the drifted Brownian motion $\widetilde{W}_T := W_T + \mu T$ and its maximum $\widehat{X}_0^T = \operatorname*{Max}_{t\in[0,T]} \widetilde{W}_t$ is given by

$$\varphi_{\widehat{X}_0^T,\widetilde{W}_T}(a,b) = \mathbb{1}_{\{a\geqslant \operatorname{Max}(b,0)\}} \frac{1}{T}\sqrt{\frac{2}{\pi T}}(2a-b)\,\mathrm{e}^{\mu b-(2a-b)^2/(2T)-\mu^2 T/2}$$
$$= \begin{cases} \dfrac{1}{T}\sqrt{\dfrac{2}{\pi T}}(2a-b)\,\mathrm{e}^{-\mu^2 T/2+\mu b-(2a-b)^2/(2T)}, & a > \operatorname{Max}(b,0), \\ 0, & a < \operatorname{Max}(b,0). \end{cases} \tag{7.3.6}$$

Proof. The arguments previously applied to the standard Brownian motion $(W_t)_{t\in[0,T]}$ cannot be directly applied to $(\widetilde{W}_t)_{t\in[0,T]}$ because drifted Brownian motion is no longer symmetric in space when $\mu \neq 0$. On the other hand, the drifted process $(\widetilde{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the probability measure $\widetilde{\mathbb{P}}$ defined from the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} := e^{-\mu W_T - \mu^2 T/2}, \tag{7.3.7}$$

and the joint probability density function of $\left(\widehat{X}_0^T, \widetilde{W}_T\right)$ under $\widetilde{\mathbb{P}}$ is given by (7.3.5). Now, using the probability density function (7.3.7) and the relation $\widetilde{W}_t := W_t + \mu t$, we get

$$\begin{aligned}
&\mathbb{P}\big(\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\big) = \mathbb{E}\left[\mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \int_\Omega \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}} d\mathbb{P} \\
&= \int_\Omega \frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}} d\widetilde{\mathbb{P}} \\
&= \widetilde{\mathbb{E}}\left[\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{\mu W_T + \mu^2 T/2} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{\mu \widetilde{W}_T - \mu^2 T/2} \mathbb{1}_{\{\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\}}\right] \\
&= \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^b \mathbb{1}_{(-\infty,x]}(y)\, e^{\mu y - \mu^2 T/2} \frac{(2x-y)}{T} e^{-(2x-y)^2/(2T)} dy dx,
\end{aligned}$$

$0 \leqslant b \leqslant a$, which yields the joint probability density function (7.3.6) from the differentiation

$$\varphi_{\widehat{X}_0^T, \widetilde{W}_T}(a,b) = \frac{d\mathbb{P}\big(\widehat{X}_0^T \leqslant a \text{ and } \widetilde{W}_T \leqslant b\big)}{da db}.$$

□

The following proposition is consistent with (7.2.3) in case $\mu = 0$.

Proposition 7.4 The cumulative distribution function of the maximum

$$\widehat{X}_0^T := \operatorname*{Max}_{t \in [0,T]} \widetilde{W}_t = \operatorname*{Max}_{t \in [0,T]} (W_t + \mu t)$$

of drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t \in [0,T]$ is given by

$$\mathbb{P}\big(\widehat{X}_0^T \leqslant a\big) = \Phi\left(\frac{a - \mu T}{\sqrt{T}}\right) - e^{2\mu a} \Phi\left(\frac{-a - \mu T}{\sqrt{T}}\right), \quad a \geqslant 0, \tag{7.3.8}$$

and the probability density function $\varphi_{\widehat{X}_0^T}$ of $\widehat{X}_0^T$ satisfies

$$\varphi_{\widehat{X}_0^T}(a) = \sqrt{\frac{2}{\pi T}} e^{-(a-\mu T)^2/(2T)} - 2\mu e^{2\mu a} \Phi\left(\frac{-a - \mu T}{\sqrt{T}}\right), \quad a \geqslant 0. \tag{7.3.9}$$

Proof. Letting $\underline{a \vee b := \mathrm{Max}(a,b)}$, $a, b \in \mathbb{R}$, since the condition ($y \leqslant x$ and $0 \leqslant x \leqslant a$) is equivalent to the condition $(y \vee 0 \leqslant x \leqslant a)$, we have

$$\begin{aligned}
\mathbb{P}\big(\widehat{X}_0^T \leqslant a\big) &= \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^\infty \mathbb{1}_{(-\infty,x]}(y)\, e^{\mu y - \mu^2 T/2} \frac{(2x-y)}{T} e^{-(2x-y)^2/(2T)} dy dx \\
&= \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^x e^{\mu y - \mu^2 T/2} \frac{(2x-y)}{T} e^{-(2x-y)^2/(2T)} dy dx
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi T}} \mathrm{e}^{-\mu^2 T/2} \int_{-\infty}^{a} \mathrm{e}^{\mu y} \int_{y\vee 0}^{a} \frac{(2x-y)}{T} \mathrm{e}^{-(2x-y)^2/(2T)} dx dy.$$

Next, since

$$2(y\vee 0)^2 - y = \begin{cases} 2\times 0 - y = -y, & y \leqslant 0, \\ \\ 2y - y = y, & y \geqslant 0, \end{cases}$$

and using the "completion of the square" identity

$$\mu y - \frac{(2a-y)^2}{2T} - \frac{\mu^2 T}{2} = 2a\mu - \frac{1}{2T}(y - (\mu T + 2a))^2$$

and a standard changes of variables, we have

$$\begin{aligned}
\mathbb{P}(\widehat{X}_0^T \leqslant a) &= \sqrt{\frac{2}{\pi T}} \int_0^a \int_{-\infty}^{\infty} \mathbb{1}_{(-\infty,x]}(y) \mathrm{e}^{\mu y - \mu^2 T/2} \frac{(2x-y)}{T} \mathrm{e}^{-(2x-y)^2/(2T)} dy dx \\
&= \frac{1}{\sqrt{2\pi T}} \mathrm{e}^{-\mu^2 T/2} \int_{-\infty}^{a} \left(\mathrm{e}^{\mu y - (2(y\vee 0)-y)^2/(2T)} - \mathrm{e}^{\mu y - (2a-y)^2/(2T)} \right) dy \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{a} \left(\mathrm{e}^{\mu y - y^2/(2T) - \mu^2 T/2} - \mathrm{e}^{\mu y - (2a-y)^2/(2T) - \mu^2 T/2} \right) dy \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{a} \left(\mathrm{e}^{-(y-\mu T)^2/(2T)} - \mathrm{e}^{-(y-(\mu T+2a))^2/(2T) + 2a\mu} \right) dy \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{a} \mathrm{e}^{-(y-\mu T)^2/(2T)} dy - \mathrm{e}^{2a\mu} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{a} \mathrm{e}^{-(y-(\mu T+2a))^2/(2T)} dy \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{a-\mu T} \mathrm{e}^{-y^2/(2T)} dy - \mathrm{e}^{2a\mu} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-a-\mu T} \mathrm{e}^{-y^2/(2T)} dy \\
&= \Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) - \mathrm{e}^{2\mu a} \Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right), \qquad a \geqslant 0,
\end{aligned}$$

cf. Corollary 7.2.2 and pages 297-299 of Shreve, 2004 for another derivation. □

See Profeta, Roynette, and Yor, 2010 for interpretations of (7.3.8) and (7.3.10) in terms of the Black-Scholes formula.

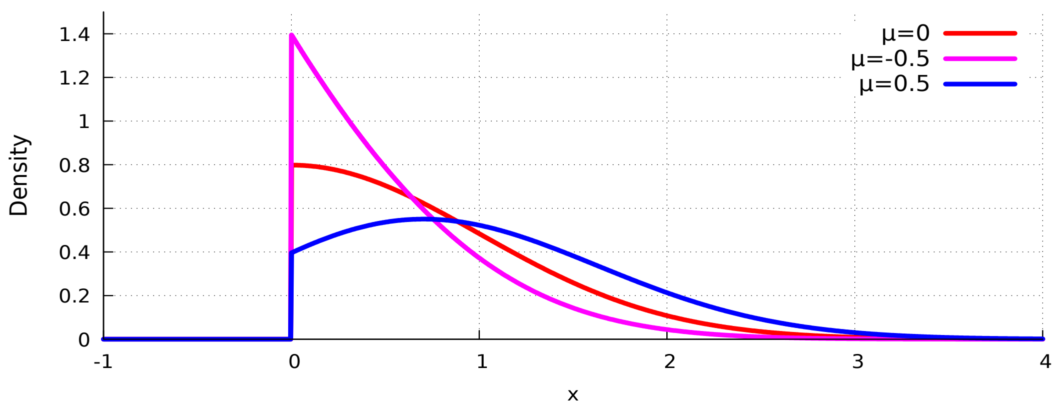


Figure 7.13: Probability density of the maximum $\widehat{X}_0^T$ of drifted Brownian motion.

We note from Figure 7.13 that small values of the maximum are more likely to occur when μ takes large negative values. As T tends to infinity, Proposition 7.4 also shows that when $\mu < 0$, the maximum of drifted Brownian motion $\big(\widetilde{W}_t\big)_{t\in\mathbb{R}_+} = (W_t+\mu t)_{t\in\mathbb{R}_+}$ over all time has an exponential distribution with parameter $2|\mu|$, *i.e.*

$$\varphi_{\widehat{X}_0^T}(a) = 2\mu\,\mathrm{e}^{2\mu a}, \qquad a \geqslant 0.$$

Relation (7.3.8), resp. Relation (7.3.11) below, will be used for the pricing of lookback call, resp. put options in Section 7.4. See also Exercise 7.8 for the joint probability density function of geometric Brownian motion $S_T := S_0\,\mathrm{e}^{\sigma W_T+(r-\sigma^2/2)T}$ and its maximum $M_0^T := \operatorname*{Max}_{t\in[0,T]} S_t$.

Corollary 7.5 The cumulative distribution function of the maximum

$$M_0^T := \operatorname*{Max}_{t\in[0,T]} S_t = S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t+(r-\sigma^2/2)t}$$

of geometric Brownian motion over $t\in[0,T]$ is given by

$$\begin{aligned} \mathbb{P}\big(M_0^T \leqslant x\big) &= \Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right) \\ &\quad -\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\Phi\left(\frac{-(r-\sigma^2/2)T-\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad x\geqslant S_0, \end{aligned} \tag{7.3.10}$$

and the probability density function $\varphi_{M_0^T}$ of M_0^T satisfies

$$\begin{aligned} \varphi_{M_0^T}(x) &= \frac{1}{\sigma x\sqrt{2\pi T}}\exp\left(-\frac{(-(r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right) \\ &+\frac{1}{\sigma x\sqrt{2\pi T}}\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\exp\left(-\frac{((r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right) \\ &+\frac{1}{x}\left(1-\frac{2r}{\sigma^2}\right)\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\Phi\left(\frac{-(r-\sigma^2/2)T-\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad x\geqslant S_0. \end{aligned}$$

Proof. Taking

$$\widetilde{W}_t := W_t+\mu t = W_t+\frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)t$$

with $\mu := r/\sigma-\sigma/2$, by (7.3.8) we find

$$\begin{aligned} \mathbb{P}\big(M_0^T \leqslant x\big) &= \mathbb{P}\left(\mathrm{e}^{\sigma\widehat{X}_0^T}\leqslant\frac{x}{S_0}\right) \\ &= \mathbb{P}\left(\widehat{X}_0^T\leqslant\frac{1}{\sigma}\log\frac{x}{S_0}\right) \\ &= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)-\mathrm{e}^{2\mu\sigma^{-1}\log(x/S_0)}\Phi\left(\frac{-\mu T-\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) \\ &= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)-\left(\frac{x}{S_0}\right)^{2\mu/\sigma}\Phi\left(\frac{-\mu T-\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right) \\ &= \Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right) \\ &\quad -\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\Phi\left(\frac{-(r-\sigma^2/2)T-\log(x/S_0)}{\sigma\sqrt{T}}\right). \end{aligned}$$

□

Minimum of drifted Brownian motion

Proposition 7.6 The joint probability density function $\varphi_{\breve{X}_0^T,\widetilde{W}_T}$ of the minimum of the drifted Brownian motion $\widetilde{W}_t := W_t + \mu t$ and its value $\widetilde{W}_T$ at time T is given by

$$\varphi_{\breve{X}_0^T,\widetilde{W}_T}(a,b) = \mathbb{1}_{\{a\leqslant \min(b,0)\}} \frac{1}{T}\sqrt{\frac{2}{\pi T}}(b-2a)\,\mathrm{e}^{\mu b-(2a-b)^2/(2T)-\mu^2T/2}$$

$$= \begin{cases} \dfrac{1}{T}\sqrt{\dfrac{2}{\pi T}}(b-2a)\,\mathrm{e}^{-\mu^2T/2+\mu b-(2a-b)^2/(2T)}, & a<\min(b,0), \\ 0, & a>\min(b,0). \end{cases}$$

Proof. We use the relations

$$\min_{t\in[0,T]} \widetilde{W}_t = -\operatorname*{Max}_{t\in[0,T]}\big(-\widetilde{W}_t\big),$$

and

$$\begin{aligned} \breve{X}_0^T &:= \min_{t\in[0,T]} \widetilde{W}_t \\ &= \min_{t\in[0,T]}(W_t+\mu t) \\ &= -\operatorname*{Max}_{t\in[0,T]}\big(-\widetilde{W}_t\big) \\ &= -\operatorname*{Max}_{t\in[0,T]}(-W_t-\mu t) \\ &\simeq -\operatorname*{Max}_{t\in[0,T]}(W_t-\mu t), \end{aligned}$$

where the last equality "$\simeq$" follows from the identity in distribution of $(W_t)_{t\in\mathbb{R}_+}$ and $(-W_t)_{t\in\mathbb{R}_+}$, and we conclude by applying the change of variables $(a,b,\mu)\mapsto(-a,-b,-\mu)$ to (7.3.6). □

Similarly to the above, the following proposition holds for the minimum drifted Brownian motion, and Relation (7.3.12) below can be obtained by changing the signs of both a and μ in Proposition 7.4.

Proposition 7.7 The cumulative distribution function and probability density function of the minimum

$$\breve{X}_0^T := \min_{t\in[0,T]} \widetilde{W}_t = \min_{t\in[0,T]}(W_t+\mu t)$$

of the drifted Brownian motion $\widetilde{W}_t = W_t+\mu t$ over $t\in[0,T]$ are given by

$$\mathbb{P}\big(\breve{X}_0^T \leqslant a\big) = \Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right), \qquad a\leqslant 0, \tag{7.3.11}$$

and

$$\varphi_{\breve{X}_0^T}(a) = \sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(a-\mu T)^2/(2T)} + 2\mu\,\mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right), \quad a \leqslant 0. \tag{7.3.12}$$

Proof. From (7.3.8), the cumulative distribution function of the minimum of drifted Brownian motion can be expressed as

$$\begin{aligned}
\mathbb{P}\big(\breve{X}_0^T \leqslant a\big) &= \mathbb{P}\left(\min_{t\in[0,T]} \widetilde{W}_t \leqslant a\right) \\
&= \mathbb{P}\left(\min_{t\in[0,T]} (W_t + \mu t) \leqslant a\right) \\
&= \mathbb{P}\left(-\operatorname*{Max}_{t\in[0,T]} (-W_t - \mu t) \leqslant a\right) \\
&= \mathbb{P}\left(-\operatorname*{Max}_{t\in[0,T]} (W_t - \mu t) \leqslant a\right) \\
&= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} (W_t - \mu t) \geqslant -a\right) \\
&= 1 - \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} (W_t - \mu t) \leqslant -a\right) \\
&= 1 - \Phi\left(\frac{-a+\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right) \\
&= \Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\Phi\left(\frac{a+\mu T}{\sqrt{T}}\right), \qquad a \leqslant 0,
\end{aligned}$$

where we used the identity in distribution of $(W_t)_{t\in\mathbb{R}_+}$ and $(-W_t)_{t\in\mathbb{R}_+}$, hence the probability density function of the minimum of drifted Brownian motion is given by (7.3.12). □

Similarly, we have

$$\mathbb{P}\big(\breve{X}_0^T > a\big) = \Phi\left(\frac{\mu T - a}{\sqrt{T}}\right) - \mathrm{e}^{2a\mu}\Phi\left(\frac{\mu T + a}{\sqrt{T}}\right), \quad a \leqslant 0,$$

and, if $\mu > 0$, the minimum of the positively drifted Brownian motion $\big(\widetilde{W}_t\big)_{t\in\mathbb{R}_+} = (W_t + \mu t)_{t\in\mathbb{R}_+}$ over all time has an exponential distribution with parameter 2μ on $\mathbb{R}_-$, *i.e.*

$$\varphi_{\breve{X}_0^T}(a) = 2\mu\,\mathrm{e}^{2\mu a}, \qquad a \leqslant 0.$$

In addition, as in Corollary 7.5, we have the following result.

Corollary 7.8 The cumulative distribution function of the minimum

$$m_0^T := \min_{t\in[0,T]} S_t = S_0 \min_{t\in[0,T]} \mathrm{e}^{\sigma W_t + (r-\sigma^2/2)t}$$

of geometric Brownian motion over $t \in [0,T]$ is given by

$$\begin{aligned}
\mathbb{P}\big(m_0^T \leqslant x\big) &= \Phi\left(\frac{-(r-\sigma^2/2)T + \log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{S_0}{x}\right)^{1-2r/\sigma^2} \Phi\left(\frac{(r-\sigma^2/2)T + \log(x/S_0)}{\sigma\sqrt{T}}\right), \quad 0 < x \leqslant S_0,
\end{aligned} \tag{7.3.13}$$

and the probability density function $\varphi_{m_0^T}$ of m_0^T satisfies

$$\begin{aligned}\varphi_{m_0^T}(x) &= \frac{1}{\sigma x\sqrt{2\pi T}} \exp\left(-\frac{(-(r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right)\\ &+\frac{1}{\sigma x\sqrt{2\pi T}}\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2} \exp\left(-\frac{((r-\sigma^2/2)T+\log(x/S_0))^2}{2\sigma^2 T}\right)\\ &+\frac{1}{x}\left(\frac{2r}{\sigma^2}-1\right)\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\Phi\left(\frac{(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad 0<x\leqslant S_0.\end{aligned}$$

Proof. From (7.3.11) we have

$$\begin{aligned}&\mathbb{P}(m_0^T \leqslant x)\\ &= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)+\mathrm{e}^{2\mu\sigma^{-1}\log(x/S_0)}\Phi\left(\frac{\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)\\ &= \Phi\left(\frac{-\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)+\left(\frac{x}{S_0}\right)^{2\mu/\sigma}\Phi\left(\frac{\mu T+\sigma^{-1}\log(x/S_0)}{\sqrt{T}}\right)\\ &= \Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right)\\ &\quad+\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\Phi\left(\frac{(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right), \quad 0<x\leqslant S_0,\end{aligned}$$

with $\mu := r/\sigma-\sigma/2$. The probability density function $\varphi_{m_0^T}$ is computed from

$$\varphi_{m_0^T}(x) = \frac{\partial}{\partial x}\mathbb{P}(m_0^T \leqslant x), \qquad 0<x\leqslant S_0.$$

□

7.4 Average of Geometric Brownian Extrema

Let

$$m_s^t = \min_{u\in[s,t]} S_u \quad \text{and} \quad M_s^t = \operatorname{Max}_{u\in[s,t]} S_u,$$

$0\leqslant s\leqslant t\leqslant T$, and let $\mathscr{M}_s^t$ be either m_s^t or M_s^t. In the lookback option case the payoff $\phi(S_T,\mathscr{M}_0^T)$ depends not only on the price of the underlying asset at maturity but it also depends on all price values of the underlying asset over the period which starts from the initial time and ends at maturity.

The payoff of such of an option is of the form $\phi(S_T,\mathscr{M}_0^T)$ with $\phi(x,y)=x-y$ in the case of lookback call options, and $\phi(x,y)=y-x$ in the case of lookback put options. We let

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi(S_T,\mathscr{M}_0^T)\,|\,\mathscr{F}_t\right]$$

denote the price at time $t\in[0,T]$ of such an option.

Maximum selling price over $[0,T]$

In the next proposition we start by computing the average of the maximum selling price $M_0^T := \operatorname{Max}_{t\in[0,T]} S_t$ of $(S_t)_{t\in[0,T]}$ over the time interval $[0,T]$. We denote

$$\delta_\pm^\tau(s) := \frac{1}{\sigma\sqrt{\tau}}\left(\log s+\left(r\pm\frac{1}{2}\sigma^2\right)\tau\right), \qquad s>0. \tag{7.4.1}$$

Proposition 7.9 The average maximum value of $(S_t)_{t\in[0,T]}$ over $[0,T]$ is given by

$$\begin{aligned}&\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right] \\ &\quad = M_0^t\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)+S_t\,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\ &\qquad -S_t\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right),\end{aligned} \tag{7.4.2}$$

where $\delta_\pm^{T-t}$ is defined in (7.4.1).

When $t=0$ we have $S_0=M_0^0$, and given that

$$\delta_\pm^T(1)=\frac{r\pm\sigma^2/2}{\sigma}\sqrt{T}, \tag{7.4.3}$$

the formula (7.4.2) simplifies to

$$\begin{aligned}&\mathbb{E}^*\left[M_0^T\right] \\ &\quad = S_0\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(\frac{\sigma^2/2-r}{\sigma}\sqrt{T}\right)+S_0\,\mathrm{e}^{rT}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\frac{\sigma^2/2+r}{\sigma}\sqrt{T}\right),\end{aligned}$$

with

$$\mathbb{E}^*\left[M_0^T\right]=2S_0\left(1+\frac{\sigma^2T}{4}\Phi\left(\sigma\frac{\sqrt{T}}{2}\right)\right)+\sigma S_0\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2T/8}$$

when $r=0$, cf. Exercise 9.2.

In general, when T tends to infinity we find that

$$\lim_{T\to\infty}\frac{\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right]}{\mathbb{E}^*[S_T \mid \mathcal{F}_t]}=\begin{cases}1+\dfrac{\sigma^2}{2r} & \text{if } r>0,\\[2ex] \infty & \text{if } r=0,\end{cases}$$

see Exercise 7.3-(d)) in the case $r=\sigma^2/2$.

Proof of Proposition 7.9. We have

$$\begin{aligned}\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right] &= \mathbb{E}^*\left[\operatorname{Max}\left(M_0^t,M_t^T\right) \mid \mathcal{F}_t\right]\\ &= \mathbb{E}^*\left[M_0^t\mathbb{1}_{\{M_0^t>M_t^T\}} \mid \mathcal{F}_t\right]+\mathbb{E}^*\left[M_t^T\mathbb{1}_{\{M_t^T>M_0^t\}} \mid \mathcal{F}_t\right]\\ &= M_0^t\,\mathbb{E}^*\left[\mathbb{1}_{\{M_0^t>M_t^T\}} \mid \mathcal{F}_t\right]+\mathbb{E}^*\left[M_t^T\mathbb{1}_{\{M_t^T>M_0^t\}} \mid \mathcal{F}_t\right]\\ &= M_0^t\mathbb{P}\left(M_0^t>M_t^T \mid \mathcal{F}_t\right)+\mathbb{E}^*\left[M_t^T\mathbb{1}_{\{M_t^T>M_0^t\}} \mid \mathcal{F}_t\right].\end{aligned}$$

Next, we have

$$\begin{aligned}\mathbb{P}\left(M_0^t>M_t^T \mid \mathcal{F}_t\right) &= \mathbb{P}\left(\frac{M_0^t}{S_t}>\frac{M_t^T}{S_t}\ \middle|\ \mathcal{F}_t\right)\\ &= \mathbb{P}\left(x>\frac{M_t^T}{S_t}\ \middle|\ \mathcal{F}_t\right)_{x=M_0^t/S_t}\end{aligned}$$

$$= \quad \mathbb{P}\left(\frac{M_0^{T-t}}{S_0} < x\right)_{x=M_0^t/S_t}.$$

On the other hand, letting $\mu := r/\sigma - \sigma/2$, from (7.3.8) or (7.3.10) in Corollary 7.5 we have

$$\begin{aligned}
\mathbb{P}\left(\frac{M_0^T}{S_0} < x\right) &= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t + rt - \sigma^2 t/2} < x\right) \\
&= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{(W_t+\mu t)\sigma} < x\right) \\
&= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma \widetilde{W}_t} < x\right) \\
&= \mathbb{P}\left(\mathrm{e}^{\sigma \widehat{X}_0^T} < x\right) \\
&= \mathbb{P}\left(\widehat{X}_T < \frac{1}{\sigma}\log x\right) \\
&= \Phi\left(\frac{-\mu T + \sigma^{-1}\log x}{\sqrt{T}}\right) - \mathrm{e}^{2\mu\sigma^{-1}\log x}\Phi\left(\frac{-\mu T - \sigma^{-1}\log x}{\sqrt{T}}\right) \\
&= \Phi\left(-\delta_-^T\left(\frac{1}{x}\right)\right) - x^{-1+2r/\sigma^2}\Phi\left(-\delta_-^T(x)\right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathbb{P}\left(M_0^t > M_t^T \mid \mathscr{F}_t\right) &= \mathbb{P}\left(\frac{M_0^{T-t}}{S_0} < x\right)_{x=M_0^t/S_t} \\
&= \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \left(\frac{M_0^t}{S_t}\right)^{-1+2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\mathbf{E}^*\left[M_t^T \mathbb{1}_{\{M_t^T > M_0^t\}} \,\Big|\, \mathscr{F}_t\right] = S_t\,\mathbf{E}^*\left[\frac{M_t^T}{S_t}\mathbb{1}_{\{M_t^T/S_t > M_0^t/S_t\}} \,\Big|\, \mathscr{F}_t\right] \\
&= S_t\,\mathbf{E}^*\left[\mathbb{1}_{\{\operatorname{Max}_{u\in[t,T]} S_u/S_t > x\}} \operatorname*{Max}_{u\in[t,T]}\frac{S_u}{S_t} \,\Big|\, \mathscr{F}_t\right]_{x=M_0^t/S_t} \\
&= S_t\,\mathbf{E}^*\left[\mathbb{1}_{\{\operatorname{Max}_{u\in[0,T-t]} S_u/S_0 > x\}} \operatorname*{Max}_{u\in[0,T-t]}\frac{S_u}{S_0}\right]_{x=M_0^t/S_t},
\end{aligned}$$

and by Proposition 7.4 we have

$$\begin{aligned}
&\mathbf{E}^*\left[\mathbb{1}_{\{\operatorname{Max}_{u\in[0,T]} S_u/S_0 > x\}} \operatorname*{Max}_{u\in[0,T]}\frac{S_u}{S_0}\right] \qquad (7.4.4)\\
&= \mathbf{E}^*\left[\mathbb{1}_{\{\operatorname{Max}_{u\in[0,T]} \mathrm{e}^{\sigma\widehat{W}_u} > x\}} \operatorname*{Max}_{u\in[0,T]} \mathrm{e}^{\sigma\widehat{W}_u}\right] \\
&= \mathbf{E}^*\left[\mathrm{e}^{\sigma \operatorname{Max}_{u\in[0,T]}\widehat{W}_u}\mathbb{1}_{\{\operatorname{Max}_{u\in[0,T]}\widehat{W}_u > \sigma^{-1}\log x\}}\right] \\
&= \mathbf{E}^*\left[\mathrm{e}^{\sigma\widehat{X}_T}\mathbb{1}_{\{\widehat{X}_T > \sigma^{-1}\log x\}}\right] \\
&= \int_{\sigma^{-1}\log x}^{\infty} \mathrm{e}^{\sigma z} f_{\widehat{X}_T}(z)\,dz \\
&= \int_{\sigma^{-1}\log x}^{\infty} \mathrm{e}^{\sigma z}\left(\sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(z-\mu T)^2/(2T)} - 2\mu\,\mathrm{e}^{2\mu z}\Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right)\right)dz
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi T}} \int_{\sigma^{-1}\log x}^{\infty} e^{\sigma z-(z-\mu T)^2/(2T)} dz - 2\mu \int_{\sigma^{-1}\log x}^{\infty} e^{(\sigma+2\mu)z} \Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right) dz.$$

By a standard "completion of the square" argument, we find

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log x}^{\infty} e^{\sigma z-(z-\mu T)^2/(2T)} dz \\
&= \frac{1}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log x}^{\infty} e^{-(z^2+\mu^2T^2-2(\mu+\sigma)Tz)/(2T)} dz \\
&= \frac{1}{\sqrt{2\pi T}} e^{\sigma^2 T/2+\mu\sigma T} \int_{\sigma^{-1}\log x}^{\infty} e^{-(z-(\mu+\sigma)T)^2/(2T)} dz \\
&= \frac{1}{\sqrt{2\pi T}} e^{rT} \int_{-(\mu+\sigma)T+\sigma^{-1}\log x}^{\infty} e^{-z^2/(2T)} dz \\
&= e^{rT} \Phi\left(\delta_+^T\left(\frac{1}{x}\right)\right),
\end{aligned}$$

since $\mu\sigma + \sigma^2/2 = r$. The second integral

$$\int_{\sigma^{-1}\log x}^{\infty} e^{(\sigma+2\mu)z} \Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right) dz$$

can be computed by integration by parts using the identity

$$\int_a^\infty v'(z)u(z)dz = u(+\infty)v(+\infty) - u(a)v(a) - \int_a^\infty v(z)u'(z)dz,$$

with $a := \sigma^{-1}\log x$. We let

$$u(z) = \Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right) \quad \text{and} \quad v'(z) = e^{(\sigma+2\mu)z}$$

which satisfy

$$u'(z) = -\frac{1}{\sqrt{2\pi T}} e^{-(z+\mu T)^2/(2T)} \quad \text{and} \quad v(z) = \frac{1}{\sigma+2\mu} e^{(\sigma+2\mu)z},$$

and using the completion of square identity

$$\frac{1}{\sqrt{2\pi T}} \int_c^b e^{\gamma y-y^2/(2T)} dy = e^{\gamma^2 T/2} \left(\Phi\left(\frac{-c+\gamma T}{\sqrt{T}}\right) - \Phi\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right) \tag{7.4.5}$$

for $b = +\infty$, we find

$$\begin{aligned}
&\int_a^\infty e^{(\sigma+2\mu)z} \Phi\left(\frac{-z-\mu T}{\sqrt{T}}\right) dz = \int_a^\infty v'(z)u(z)dz \\
&= u(+\infty)v(+\infty) - u(a)v(a) - \int_a^\infty v(z)u'(z)dz \\
&= -\frac{1}{\sigma+2\mu} e^{a(\sigma+2\mu)} \Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) \\
&\quad + \frac{1}{(\sigma+2\mu)\sqrt{2\pi T}} \int_a^\infty e^{(\sigma+2\mu)z} e^{-(z+\mu T)^2/(2T)} dz \\
&= -\frac{1}{\sigma+2\mu} e^{a(\sigma+2\mu)} \Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) \\
&\quad + \frac{1}{(\sigma+\mu)\sqrt{2\pi T}} e^{(T(\sigma+\mu)^2-\mu^2 T)/2} \int_a^\infty e^{-(z-T(\sigma+\mu))^2/(2T)} dz
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\sigma+2\mu}\,\mathrm{e}^{a(\sigma+2\mu)}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) \\
&\quad +\frac{1}{(\sigma+2\mu)\sqrt{2\pi}}\,\mathrm{e}^{(T(\sigma+\mu)^2-\mu^2T)/2}\int_{(a-T(\sigma+\mu))/\sqrt{T}}^{\infty}\mathrm{e}^{-z^2/2}dz \\
&= -\frac{1}{\sigma+2\mu}\,\mathrm{e}^{a(\sigma+2\mu)}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) \\
&\quad +\frac{1}{\sigma+2\mu}\,\mathrm{e}^{(T(\sigma+\mu)^2-\mu^2T)/2}\Phi\left(\frac{-a+T(\sigma+\mu)}{\sqrt{T}}\right) \\
&= -\frac{2r}{\sigma}\,(x)^{2r/\sigma^2}\,\Phi\left(\frac{-(r/\sigma-\sigma/2)T-\sigma^{-1}\log x}{\sqrt{T}}\right) \\
&\quad +\frac{2r}{\sigma}\,\mathrm{e}^{\sigma T(\sigma+2\mu)/2}\Phi\left(\frac{T(r/\sigma+\sigma/2)-\sigma^{-1}\log x}{\sqrt{T}}\right) \\
&= \frac{\sigma}{2r}\,\mathrm{e}^{rT}\Phi\left(\delta_+^T\left(\frac{1}{x}\right)\right)-\frac{\sigma}{2r}x^{2r/\sigma^2}\Phi\left(-\delta_-^T(x)\right),
\end{aligned}
$$

cf. pages 317-319 of Shreve, 2004 for a different derivation using double integrals. Hence we have

$$
\begin{aligned}
&\mathbb{E}^*\left[M_t^T\mathbb{1}_{\{M_t^T>M_0^t\}}\,\middle|\,\mathscr{F}_t\right] = S_t\,\mathbb{E}^*\left[\mathbb{1}_{\{\mathrm{Max}_{u\in[0,T-t]}S_u/S_0>x\}}\operatorname*{Max}_{u\in[0,T-t]}\frac{S_u}{S_0}\right]_{x=M_0^t/S_t} \\
&= 2S_t\,\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)-S_t\frac{\mu\sigma}{r}\,\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\
&\quad +S_t\frac{\mu\sigma}{r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right),
\end{aligned}
$$

and consequently this yields, since $\mu\sigma/r = 1-\sigma^2/(2r)$,

$$
\begin{aligned}
&\mathbb{E}^*\left[M_0^T\mid\mathscr{F}_t\right] = \mathbb{E}^*\left[M_0^T\mid M_0^t\right] \\
&= M_0^t\mathbb{P}\left(M_0^t>M_t^T\mid M_0^t\right)+\mathbb{E}^*\left[M_t^T\mathbb{1}_{\{M_t^T>M_0^t\}}\mid M_0^t\right] \\
&= M_0^t\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)-S_t\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right) \\
&\quad +2S_t\,\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\
&\quad -S_t\left(1-\frac{\sigma^2}{2r}\right)\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\
&\quad +S_t\left(1-\frac{\sigma^2}{2r}\right)\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right) \\
&= M_0^t\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)+S_t\,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\
&\quad -S_t\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).
\end{aligned}
$$

This concludes the proof of Proposition 7.9. □

See Exercise 7.6-(a)) for a computation of the average minimum $\mathbb{E}^*\left[m_0^T\right]=\mathbb{E}^*\left[\min_{t\in[0,T]}S_t\right]$.

Minimum buying price over $[0,T]$

In the next proposition we compute the average of the minimum buying price $m_0^T := \min_{t\in[0,T]} S_t$ of $(S_t)_{t\in[0,T]}$ over the time interval $[0,T]$.

Proposition 7.10 The average minimum value of $(S_t)_{t\in[0,T]}$ over $[0,T]$ is given by

$$\begin{aligned} & \mathbb{E}^*\left[m_0^T \mid \mathscr{F}_t\right] \qquad (7.4.6) \\ &= m_0^t \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - S_t \frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) \\ & \quad + S_t \,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right), \end{aligned}$$

where $\delta_\pm^{T-t}$ is defined in (7.4.1).

We note a certain symmetry between the expressions (7.4.2) and (7.4.6).

When $t=0$ we have $S_0 = m_0^0$, and given (7.4.3) the formula (7.4.6) simplifies to

$$\begin{aligned} \mathbb{E}^*\left[m_0^T\right] &= S_0 \Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right) - S_0\frac{\sigma^2}{2r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right) \\ & \quad + S_0 \,\mathrm{e}^{rT}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{\sigma^2/2+r}{\sigma}\sqrt{T}\right), \end{aligned}$$

with

$$\mathbb{E}^*\left[m_0^T\right] = 2S_0\left(1+\frac{\sigma^2 T}{4}\right)\Phi\left(-\frac{\sigma^2 T/2}{\sigma\sqrt{T}}\right) - \sigma S_0\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2 T/8}.$$

when $r=0$, cf. Exercise 9.1.

In general, when T tends to infinity we find that

$$\lim_{T\to\infty} \frac{\mathbb{E}^*\left[m_0^T \mid \mathscr{F}_t\right]}{\mathbb{E}^*[S_T \mid \mathscr{F}_t]} = 0, \qquad r \geqslant 0,$$

see Exercise 7.3-(f)) in the case $r = \sigma^2/2$.

Proof of Proposition 7.10. We have

$$\begin{aligned} \mathbb{E}^*\left[m_0^T \mid \mathscr{F}_t\right] &= \mathbb{E}^*\left[\min\left(m_0^t, m_t^T\right) \mid \mathscr{F}_t\right] \\ &= \mathbb{E}^*\left[m_0^t \mathbb{1}_{\{m_0^t < m_t^T\}} \,\middle|\, \mathscr{F}_t\right] + \mathbb{E}^*\left[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\middle|\, \mathscr{F}_t\right] \\ &= m_0^t\,\mathbb{E}^*\left[\mathbb{1}_{\{m_0^t < m_t^T\}} \,\middle|\, \mathscr{F}_t\right] + \mathbb{E}^*\left[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\middle|\, \mathscr{F}_t\right] \\ &= m_0^t \mathbb{P}\left(m_0^t < m_t^T \mid \mathscr{F}_t\right) + \mathbb{E}^*\left[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\middle|\, \mathscr{F}_t\right]. \end{aligned}$$

By (7.3.12) we find the cumulative distribution function

$$\mathbb{P}\left(\frac{m_0^{T-t}}{S_0} > x\right)_{x=m_0^t/S_t} = \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \left(\frac{m_0^t}{S_t}\right)^{-1+2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right),$$

of the minimum m_0^{T-t} of $(S_t)_{t\in\mathbb{R}_+}$ over the time interval $[0,T-t]$, hence

$$\mathbb{P}\left(m_0^t < m_t^T \mid \mathscr{F}_t\right) = \mathbb{P}\left(\frac{m_0^t}{S_t} < \frac{m_t^T}{S_t} \,\middle|\, \mathscr{F}_t\right)$$

$$
\begin{aligned}
&= \mathbb{P}\left(x < \frac{m_t^T}{S_t} \,\Big|\, \mathscr{F}_t \right)_{x=m_0^t/S_t} \\
&= \mathbb{P}\left(\frac{m_0^{T-t}}{S_0} > x \right)_{x=m_0^t/S_t} \\
&= \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) - \left(\frac{m_0^t}{S_t} \right)^{-1+2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t} \right) \right).
\end{aligned}
$$

Next, by integration with respect to the probability density function (7.3.11) as in (7.4.4) in the proof of Proposition 7.9, we find

$$
\begin{aligned}
&\mathbb{E}^*\left[m_t^T \mathbb{1}_{\{m_0^t > m_t^T\}} \,\big|\, \mathscr{F}_t \right] = S_t\, \mathbb{E}^*\left[\mathbb{1}_{\{m_0^t/S_t > m_t^T/S_t\}} \min_{u \in [t,T]} \frac{S_u}{S_t} \right]_{x=m_0^t, y=S_t} \\
&= S_t\, \mathbb{E}^*\left[\mathbb{1}_{\{\min_{u\in[t,T]} S_u/S_t < x\}} \min_{u \in [t,T]} \frac{S_u}{S_t} \right]_{x=m_0^t/S_t} \\
&= S_t\, \mathbb{E}^*\left[\mathbb{1}_{\{\min_{u\in[0,T-t]} S_u/S_0 < x\}} \min_{u \in [0,T-t]} \frac{S_u}{S_0} \right]_{x=m_0^t/S_t} \\
&= 2S_t\, \mathrm{e}^{(T-t)r} \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) - S_t \frac{\mu\sigma}{r} \mathrm{e}^{(T-t)r} \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) \\
&\quad + S_t \frac{\mu\sigma}{r} \left(\frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t} \right) \right).
\end{aligned}
$$

Given the relation $\mu\sigma/r = 1 - \sigma^2/(2r)$, this yields

$$
\begin{aligned}
&\mathbb{E}^*\left[m_0^T \,\big|\, \mathscr{F}_t \right] = m_0^t \mathbb{P}\left(\frac{m_0^{T-t}}{S_0} > x \right)_{x=m_0^t/S_t} \\
&\quad + S_t\, \mathbb{E}^*\left[\mathbb{1}_{\{\min_{u\in[0,T-t]} S_u/S_0 < x\}} \min_{u \in [0,T-t]} \frac{S_u}{S_0} \right]_{x=m_0^t/S_t} \\
&= m_0^t \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) - m_0^t \left(\frac{m_0^t}{S_t} \right)^{-1+2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t} \right) \right) \\
&\quad + 2S_t\, \mathrm{e}^{(T-t)r} \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) - S_t\, \mathrm{e}^{(T-t)r} \frac{\mu\sigma}{r} \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) \\
&\quad + S_t \frac{\mu\sigma}{r} \left(\frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t} \right) \right) \\
&= m_0^t \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) + S_t\, \mathrm{e}^{(T-t)r} \left(1 + \frac{\sigma^2}{2r} \right) \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t} \right) \right) \\
&\quad - S_t \frac{\sigma^2}{2r} \left(\frac{m_0^t}{S_t} \right)^{2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t} \right) \right).
\end{aligned}
$$

□

Exercises

Exercise 7.1 Let $(W_t)_{t\in\mathbb{R}_+}$ be standard Brownian motion, and let $a > W_0 = 0$.

a) Using the equality (7.2.2), find the probability density function φ_{τ_a} of the first time

$$
\tau_a := \inf\{t \geqslant 0 \ :\ W_t = a\}
$$

that $(W_t)_{t\in\mathbb{R}_+}$ hits the level $a>0$.

b) Let $\mu\in\mathbb{R}$. By Proposition 7.4, find the probability density function φ_{τ_a} of the first time

$$\tilde{\tau}_a := \inf\{t \geqslant 0 \ : \ \widetilde{W}_t = a\}$$

that the drifted Brownian motion $\big(\widetilde{W}_t\big)_{t\in\mathbb{R}_+} := (W_t+\mu t)_{t\in\mathbb{R}_+}$ hits the level $a>0$.

c) Let $\sigma>0$ and $r\in\mathbb{R}$. By Corollary 7.5, find the probability density function φ_{τ_a} of the first time

$$\hat{\tau}_x := \inf\{t \geqslant 0 \ : \ S_t = x\}$$

that the geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+} := \big(\mathrm{e}^{\sigma W_t + rt - \sigma^2 t/2}\big)_{t\in\mathbb{R}_+}$ hits the level $x>0$.

Exercise 7.2

a) Compute the mean value

$$\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} \widetilde{W}_t\right] = \mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} (\sigma W_t + \mu t)\right]$$

of the maximum of drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t\in[0,T]$, for $\sigma>0$ and $\mu\in\mathbb{R}$. The probability density function of the maximum is given in (7.3.9).

b) Compute the mean value $\mathbb{E}\left[\min_{t\in[0,T]} \widetilde{W}_t\right] = \mathbb{E}\left[\min_{t\in[0,T]} (\sigma W_t + \mu t)\right]$ of the *minimum* of drifted Brownian motion $\widetilde{W}_t = \sigma W_t + \mu t$ over $t\in[0,T]$, for $\sigma>0$ and $\mu\in\mathbb{R}$. The probability density function of the minimum is given in (7.3.12).

Exercise 7.3 Consider a risky asset whose price S_t is given by

$$dS_t = \sigma S_t dW_t + \frac{\sigma^2}{2} S_t dt, \tag{7.4.7}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Solve the stochastic differential equation (7.4.7).

b) Compute the expected stock price value $\mathbb{E}^*[S_T]$ at time T.

c) What is the probability distribution of the maximum $\operatorname*{Max}_{t\in[0,T]} W_t$ over the interval $[0,T]$?

d) Compute the expected value $\mathbb{E}^*\left[M_0^T\right]$ of the maximum

$$M_0^T := \operatorname*{Max}_{t\in[0,T]} S_t = S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t} = S_0 \exp\left(\sigma \operatorname*{Max}_{t\in[0,T]} W_t\right).$$

of the stock price over the interval $[0,T]$.

e) What is the probability distribution of the *minimum* $\min_{t\in[0,T]} W_t$ over the interval $[0,T]$?

f) Compute the expected value $\mathbb{E}^*\left[m_0^T\right]$ of the *minimum*

$$m_0^T := \min_{t\in[0,T]} S_t = S_0 \min_{t\in[0,T]} \mathrm{e}^{\sigma W_t} = S_0 \exp\left(\sigma \min_{t\in[0,T]} W_t\right).$$

of the stock price over the interval $[0,T]$.

Exercise 7.4 Arcsine law. Let τ denote the first time a standard Brownian motion $(B_t)_{t\in[0,T]}$ reaches its maximum over $[0,T]$.

a) Write down $\mathbb{P}(\tau\leqslant t)$ using two independent Gaussian random variables $Z_1\sim\mathcal{N}(0,t)$ and $Z_2\sim\mathcal{N}(0,T-t)$.

Hint: By (7.2.3), $\operatorname{Max}_{s\in[0,t]} B_s$ has same distribution as $|Z_1|$.

b) Write down $\mathbb{P}(\tau \leqslant t)$ as an integral.

Hint: Use Answer 2 on https://math.stackexchange.com/questions/3534598/let-x-y-be-independent-normally-distributed-random-variables-find-the-density.

Exercise 7.5 (Exercise 7.3 continued).

a) Compute the "optimal call option" prices $\mathbb{E}\left[\left(M_0^T - K\right)^+\right]$ estimated by optimally exercising at the maximum value M_0^T of $(S_t)_{t\in[0,T]}$ before maturity T.

b) Compute the "optimal put option" prices $\mathbb{E}\left[\left(K - m_0^T\right)^+\right]$ estimated by optimally exercising at the minimum value m_0^T of $(S_t)_{t\in[0,T]}$ before maturity T.

Exercise 7.6 (Exercise 7.5 continued). Consider an asset price S_t given by $S_t = S_0\,\mathrm{e}^{rt+\sigma B_t-\sigma^2 t/2}$, $t \geqslant 0$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r \geqslant 0$ and $\sigma > 0$.

a) Compute the average $\mathbb{E}^*\left[m_0^T\right]$ of the minimum $m_0^T := \min_{t\in[0,T]} S_t$ of $(S_t)_{t\in[0,T]}$ over $[0,T]$.

b) Compute the expected payoff $\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right]$ for $r > 0$. Using a finite expiration American put option pricer, compare the American put option price to the above expected payoff.

c) Compute the expected payoff $\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right]$ for $r = 0$.

Exercise 7.7 Recall that the maximum $X_0^t := \mathrm{Max}_{s\in[0,t]} W_s$ over $[0,t]$ of standard Brownian motion $(W_s)_{s\in[0,t]}$ has the probability density function

$$\varphi_{X_0^t}(x) = \sqrt{\frac{2}{\pi t}}\,\mathrm{e}^{-x^2/(2t)}, \qquad x \geqslant 0.$$

a) Let $\tau_a = \inf\{s \geqslant 0 \ : \ W_s = a\}$ denote the first hitting time of $a > 0$ by $(W_s)_{s\in\mathbb{R}_+}$. Using the relation between $\{\tau_a \leqslant t\}$ and $\left\{X_0^t \geqslant a\right\}$, write down the probability $\mathbb{P}(\tau_a \leqslant t)$ as an integral from a to ∞.

b) Using integration by parts on $[a,\infty)$, compute the probability density function of τ_a.

Hint: the derivative of $\mathrm{e}^{-x^2/(2t)}$ with respect to x is $-x\,\mathrm{e}^{-x^2/(2t)}/t$.

c) Compute the mean value $\mathbb{E}^*[(\tau_a)^{-2}]$ of $1/\tau_a^2$.

Exercise 7.8 From Relation (7.3.6) in Proposition 7.3 and the Jacobian change of variable formula, see *e.g.* `https://online.stat.psu.edu/stat414/lesson/23/23.1`, and assuming $S_0 > 0$, compute the joint probability density function of geometric Brownian motion $S_T := S_0\,\mathrm{e}^{\sigma W_T+(r-\sigma^2/2)T}$ and its maximum

$$M_0^T := \mathop{\mathrm{Max}}_{t\in[0,T]} S_t = S_0 \mathop{\mathrm{Max}}_{t\in[0,T]} \mathrm{e}^{\sigma W_t+(r-\sigma^2/2)t}.$$

8. Barrier Options

Barrier options are financial derivatives whose payoffs depend on the crossing of a certain predefined barrier level by the underlying asset price process $(S_t)_{t\in[0,T]}$. In this chapter, we consider barrier options whose payoffs depend on an extremum of $(S_t)_{t\in[0,T]}$, in addition to the terminal value S_T. Barrier options are then priced by computing the discounted expected values of their claim payoffs, or by PDE arguments.

8.1 Options on Extrema

Vanilla options with payoff $C = \phi(S_T)$ can be priced as

$$\mathrm{e}^{-rT}\,\mathbb{E}^*[\phi(S_T)] = \mathrm{e}^{-rT}\int_0^\infty \phi(y)\varphi_{S_T}(y)dy$$

where $\varphi_{S_T}(y)$ is the (one parameter) *probability density* function of S_T, which satisfies

$$\mathbb{P}(S_T \leqslant y) = \int_0^y \varphi_{S_T}(v)dv, \qquad y > 0.$$

Recall that typically we have

$$\phi(x) = (x-K)^+ = \begin{cases} x-K & \text{if } x \geqslant K, \\ 0 & \text{if } x < K, \end{cases}$$

for the European call option with strike price K, and

$$\phi(x) = \mathbb{1}_{[K,\infty)}(x) = \begin{cases} \$1 & \text{if } x \geqslant K, \\ \\ 0 & \text{if } x < K, \end{cases}$$

for the binary call option with strike price K. On the other hand, exotic options, also called path-dependent options, are options whose payoff C may depend on the whole path

$$\{S_t \ : \ 0 \leqslant t \leqslant T\}$$

of the underlying asset price process via a "complex" operation such as averaging or computing a maximum. They are opposed to vanilla options whose payoff

$$C = \phi(S_T),$$

depends only on the terminal value S_T of the price process via a payoff function ϕ, and can be priced by the computation of path integrals.

For example, the payoff of an option on extrema may take the form

$$C := \phi\big(M_0^T, S_T\big),$$

where

$$M_0^T = \operatorname*{Max}_{t\in[0,T]} S_t$$

is the maximum of $(S_t)_{t\in\mathbb{R}_+}$ over the time interval $[0,T]$. In such situations the option price at time $t=0$ can be expressed as

$$\mathrm{e}^{-rT}\,\mathbb{E}^*\big[\phi\big(M_0^T, S_T\big)\big] = \mathrm{e}^{-rT}\int_0^\infty\int_0^\infty \phi(x,y)\varphi_{M_0^T,S_T}(x,y)dxdy$$

where $\varphi_{M_0^T,S_T}$ is the *joint probability density* function of (M_0^T, S_T), which satisfies

$$\mathbb{P}(M_0^T \leqslant x \text{ and } \ S_T \leqslant y) = \int_0^x\int_0^y \varphi_{M_0^T,S_T}(u,v)dudv, \qquad x,y \geqslant 0.$$

General case

Using the joint probability density function of $\widetilde{W}_T = W_T + \mu T$ and

$$\widehat{X}_0^T = \operatorname*{Max}_{t\in[0,T]} \widetilde{W} = \operatorname*{Max}_{t\in[0,T]} (W_t + \mu t),$$

see Proposition 7.2, we are able to price any exotic option with payoff $\phi(\widetilde{W}_T, \widehat{X}_0^T)$, as

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\big[\phi\big(\widehat{X}_0^T, \widetilde{W}_T\big)\,\big|\,\mathscr{F}_t\big],$$

with in particular, letting $a \vee b := \mathrm{Max}(a,b)$,

$$\mathrm{e}^{-rT}\,\mathbb{E}^*\big[\phi\big(\widehat{X}_0^T, \widetilde{W}_T\big)\big] = \mathrm{e}^{-rT}\int_{-\infty}^\infty\int_{y\vee 0}^\infty \phi(x,y)d\mathbb{P}^*\big(\widehat{X}_0^T \leqslant x, \widetilde{W}_T \leqslant y\big).$$

In this chapter, we work in a (continuous) geometric Brownian model, in which the asset price $(S_t)_{t\in[0,T]}$ has the dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad t \geqslant 0,$$

where $\sigma > 0$ and $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. In particular, the value V_t of a self-financing portfolio satisfies

$$V_T\,\mathrm{e}^{-rT} = V_0 + \sigma \int_0^T \xi_t S_t\,\mathrm{e}^{-rt} dW_t, \quad t \in [0,T].$$

In order to price barrier* options by the above probabilistic method, we will use the probability density function of the maximum

$$M_0^T = \operatorname*{Max}_{t\in[0,T]} S_t$$

of geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$ over a given time interval $[0,T]$ and the joint probability density function $\varphi_{M_0^T,S_T}(u,v)$ derived in Chapter 7 by the *reflection principle*.

Proposition 8.1 An exotic option with integrable claim payoff of the form

$$C = \phi\left(M_0^T, S_T\right) = \phi\left(\operatorname*{Max}_{t\in[0,T]} S_t, S_T\right)$$

can be priced at time $t = 0$ as

$$\begin{aligned}
&\mathrm{e}^{-rT}\,\mathbb{E}^*[C] \\
&= \frac{\mathrm{e}^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_0^\infty \int_y^\infty \phi\left(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\right)(2x-y)\,\mathrm{e}^{-\mu^2 T/2+\mu y-(2x-y)^2/(2T)} dxdy \\
&\quad + \frac{\mathrm{e}^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^0 \int_0^\infty \phi\left(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\right)(2x-y)\,\mathrm{e}^{-\mu^2 T/2+\mu y-(2x-y)^2/(2T)} dxdy.
\end{aligned}$$

Proof. We have

$$S_T = S_0\,\mathrm{e}^{\sigma W_T - \sigma^2 T/2 + rT} = S_0\,\mathrm{e}^{(W_T+\mu T)\sigma} = S_0\,\mathrm{e}^{\sigma \widetilde{W}_T},$$

with

$$\mu := -\frac{\sigma}{2} + \frac{r}{\sigma} \quad \text{and} \quad \widetilde{W}_T = W_T + \mu T,$$

and

$$\begin{aligned}
M_0^T &= \operatorname*{Max}_{t\in[0,T]} S_t = S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma W_t - \sigma^2 t/2 + rt} \\
&= S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma \widetilde{W}_t} = S_0\,\mathrm{e}^{\sigma \operatorname{Max}_{t\in[0,T]} \widetilde{W}_t} \\
&= S_0\,\mathrm{e}^{\sigma \widehat{X}_0^T},
\end{aligned}$$

since $\sigma > 0$. Hence,

$$C = \phi\left(S_T, M_0^T\right) = \phi\left(S_0\,\mathrm{e}^{\sigma W_T - \sigma^2 T/2 + rT}, M_0^T\right) = \phi\left(S_0\,\mathrm{e}^{\sigma \widetilde{W}_T}, S_0\,\mathrm{e}^{\sigma \widehat{X}_0^T}\right),$$

and

$$\begin{aligned}
&\mathrm{e}^{-rT}\,\mathbb{E}^*[C] = \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi\left(S_0\,\mathrm{e}^{\sigma \widetilde{W}_T}, S_0\,\mathrm{e}^{\sigma \widehat{X}_0^T}\right)\right] \\
&= \mathrm{e}^{-rT}\int_{-\infty}^\infty \int_{y\vee 0}^\infty \phi\left(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\right) d\mathbb{P}\left(\widehat{X}_0^T \leqslant x, \widetilde{W}_T \leqslant y\right) \\
&= \frac{\mathrm{e}^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty \int_{y\vee 0}^\infty \phi\left(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\right)(2x-y)\,\mathrm{e}^{-\mu^2 T/2+\mu y-(2x-y)^2/(2T)} dxdy
\end{aligned}$$

*"A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank" Lyuu, 2021.

$$
\begin{aligned}
&= \frac{\mathrm{e}^{-rT}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_0^\infty\int_y^\infty \phi\left(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\right)(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy \\
&+ \frac{\mathrm{e}^{-rT}1}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^0\int_0^\infty \phi\left(S_0\,\mathrm{e}^{\sigma y}, S_0\,\mathrm{e}^{\sigma x}\right)(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy.
\end{aligned}
$$

□

Pricing barrier options

The payoff of an up-and-out barrier put option on the underlying asset price S_t with exercise date T, strike price K and barrier level (or call level) B is

$$
C = (K-S_T)^+ \mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\operatorname{Max}} S_t < B\right\}} = \begin{cases} (K-S_T)^+ & \text{if } \underset{0\leqslant t\leqslant T}{\operatorname{Max}} S_t < B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\operatorname{Max}} S_t \geqslant B. \end{cases}
$$

This option is also called a *Callable Bear Contract*, or a Bear CBBC with no residual value, or a turbo warrant with no rebate, in which the call level B usually satisfies $B \leqslant K$.

The payoff of a down-and-out barrier call option on the underlying asset price S_t with exercise date T, strike price K and barrier level B is

$$
C = (S_T-K)^+ \mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\min} S_t > B\right\}} = \begin{cases} (S_T-K)^+ & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t > B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t \leqslant B. \end{cases}
$$

This option is also called a *Callable Bull Contract*, or a Bull CBBC with no residual value, or a turbo warrant with no rebate, in which the call level B usually satisfies $B \geqslant K$. [*]

Category 'R' Callable Bull/Bear Contracts, or CBBCs, also called turbo warrants, involve a rebate or residual value computed as the payoff of a down-and-in lookback option. Category 'N' Callable Bull/Bear Contracts do not involve a residual value or rebate, and they usually satisfy $B = K$. See J. Eriksson and Persson, 2006, Wong and Chan, 2008 and Exercise 8.2 for the pricing of Category 'R' CBBCs with rebate.

[*] Download this R **code** for the pricing of Bull CBBCs (down-and-out barrier call options) with $B \geqslant K$ (right-click to save as attachment).

Option type	CBBC	Behavior	Payoff		Price	Figure
Barrier call	Bull	down-and-out (knock-out)	$(S_T-K)^+ \mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t > B\right\}}$	$B \leqslant K$	(8.2.6)	8.4a
				$B \geqslant K$	(8.2.7)	8.4b
		down-and-in (knock-in)	$(S_T-K)^+ \mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t < B\right\}}$	$B \leqslant K$	(8.3.1)	8.7a
				$B \geqslant K$	(8.3.2)	8.7b
		up-and-out (knock-out)	$(S_T-K)^+ \mathbb{1}_{\left\{\mathrm{Max}\limits_{0\leqslant t\leqslant T} S_t < B\right\}}$	$B \leqslant K$	0	N.A.
				$B \geqslant K$	(8.2.1)	8.1
		up-and-in (knock-in)	$(S_T-K)^+ \mathbb{1}_{\left\{\mathrm{Max}\limits_{0\leqslant t\leqslant T} S_t > B\right\}}$	$B \leqslant K$	BSCall	
				$B \geqslant K$	(8.3.3)	8.8
Barrier put		down-and-out (knock-out)	$(K-S_T)^+ \mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t > B\right\}}$	$B \leqslant K$	(8.2.8)	8.6
				$B \geqslant K$	0	N.A.
		down-and-in (knock-in)	$(K-S_T)^+ \mathbb{1}_{\left\{\min\limits_{0\leqslant t\leqslant T} S_t < B\right\}}$	$B \leqslant K$	(8.3.4)	8.9
				$B \geqslant K$	BSPut	
	Bear	up-and-out (knock-out)	$(K-S_T)^+ \mathbb{1}_{\left\{\mathrm{Max}\limits_{0\leqslant t\leqslant T} S_t < B\right\}}$	$B \leqslant K$	(8.2.4)	8.2a
				$B \geqslant K$	(8.2.5)	8.2b
		up-and-in (knock-in)	$(K-S_T)^+ \mathbb{1}_{\left\{\mathrm{Max}\limits_{0\leqslant t\leqslant T} S_t > B\right\}}$	$B \leqslant K$	(8.3.5)	8.10a
				$B \geqslant K$	(8.3.6)	8.10b

Table 8.1: Barrier option types.

We can distinguish between eight different variations on barrier options, according to Table 8.1.

In-out parity

We have the following parity relations between the prices of barrier options and vanilla call and put options:

$$\begin{cases} C_{\text{up-in}}(t) + C_{\text{up-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t], & (8.1.1) \\ \\ C_{\text{down-in}}(t) + C_{\text{down-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t], & (8.1.2) \\ \\ P_{\text{up-in}}(t) + P_{\text{up-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid \mathcal{F}_t], & (8.1.3) \\ \\ P_{\text{down-in}}(t) + P_{\text{down-out}}(t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid \mathcal{F}_t], & (8.1.4) \end{cases}$$

where the price of the European call, resp. put option with strike price K are obtained from the Black-Scholes formula. Consequently, in what follows we will only compute the prices of the up-and-out barrier call and put options and of the down-and-out barrier call and put options.

Note that all knock-out barrier option prices vanish when $M_0^t > B$ or $m_0^t < B$, while the barrier up-and-out call, resp. the down-and-out barrier put option prices require $B > K$, resp. $B < K$, in order not to vanish.

8.2 Knock-Out Barrier

Up-and-out barrier call option

Let us consider an up-and-out barrier call option with maturity T, strike price K, barrier (or call level) B, and payoff

$$C = (S_T - K)^+ \mathbb{1}_{\left\{\operatorname*{Max}_{0\leqslant t\leqslant T} S_t < B\right\}} = \begin{cases} S_T - K & \text{if } \operatorname*{Max}_{0\leqslant t\leqslant T} S_t \leqslant B, \\ 0 & \text{if } \operatorname*{Max}_{0\leqslant t\leqslant T} S_t > B, \end{cases}$$

with $B \geqslant K$.

Proposition 8.2 When $K \leqslant B$, the price

$$e^{-(T-t)r} \mathbb{1}_{\{M_0^t < B\}} \mathbb{E}^* \left[\left(x \frac{S_{T-t}}{S_0} - K \right)^+ \mathbb{1}_{\left\{ x \operatorname*{Max}_{0\leqslant u\leqslant T-t} \frac{S_u}{S_0} < B \right\}} \right]_{x=S_t}$$

of the up-and-out barrier call option with maturity T, strike price K and barrier level B is given by

$$\begin{aligned} & e^{-(T-t)r} \mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} \,\middle|\, \mathcal{F}_t \right] \qquad (8.2.1) \\ &= S_t \mathbb{1}_{\{M_0^t < B\}} \left\{ \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \right. \\ &\quad \left. - \left(\frac{B}{S_t}\right)^{1+2r/\sigma^2} \left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{K S_t}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right) \right) \right\} \\ &\quad - e^{-(T-t)r} K \mathbb{1}_{\{M_0^t < B\}} \left\{ \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right) \right. \\ &\quad \left. - \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{K S_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right) \right) \right\}, \end{aligned}$$

where

$$\delta_\pm^\tau(z) = \frac{1}{\sigma\sqrt{\tau}} \left(\log z + \left(r \pm \frac{\sigma^2}{2} \right) \tau \right), \qquad z > 0. \qquad (8.2.2)$$

The price of the up-and-out barrier call option vanishes when $B \leqslant K$.

We also have

$$\begin{aligned} & e^{-(T-t)r} \mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} \,\middle|\, \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{M_0^t < B\}} \mathrm{Bl}(S_t, K, r, T-t, \sigma) - S_t \mathbb{1}_{\{M_0^t < B\}} \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&- B\left(\frac{B}{S_t}\right)^{2r/\sigma^2} \mathbb{1}_{\{M_0^t<B\}}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\\
&+ \mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{M_0^t<B\}}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\
&+ \mathrm{e}^{-(T-t)r}K\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \mathbb{1}_{\{M_0^t<B\}}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right).
\end{aligned}
$$

The following R code implements the up-and-out pricing formula (8.2.1).

```
dp <- function( T , r, v, z ) { ( log(z) + ( r + v*v/2.0)*T )/v/sqrt(T) }
dm <- function( T , r , v, z ) { ( log(z) + ( r - v*v/2.0)*T )/v/sqrt(T) }
ind<-function(condition) ifelse(condition,1,0)
CBBC <- function(S,K,B,T,r,sig){ S*ind(S<B)*(pnorm(dp(T,r,sig,S/K)) -pnorm(dp(T,r,sig,S/B))
     -(B/S)**(1+2*r/sig**2)*(pnorm(dp(T,r,sig,B**2/K/S)) -pnorm(dp(T,r,sig,B/S))))
     -K*exp(-r*T)*ind(S<B)*((pnorm(dm(T,r,sig,S/K)) -pnorm(dm(T,r,sig,S/B)))
     -(S/B)**(1-2*r/sig**2)*(pnorm(dm(T,r,sig,B**2/K/S)) -pnorm(dm(T,r,sig,B/S))))}
CBBC(S=90,K=100,B=120,T=1,r=0.01,sig=0.1)
library(devtools);
install_github("https://github.com/cran/fOptions")
install_github("https://github.com/cran/fExoticOptions")
library(fExoticOptions);StandardBarrierOption("cuo",90,100,120,0,1,0.01,0.01,0.1)
```

Note that taking $B = +\infty$ in the above identity (8.2.1) recovers the Black-Scholes formula

$$
\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathscr{F}_t] = S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - \mathrm{e}^{-(T-t)r}K\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)
$$

for the price of European call options.

The graph of Figure 8.1 represents the up-and-out barrier call option price given the value S_t of the underlying asset and the time $t \in [0,T]$ with $T = 220$ days.

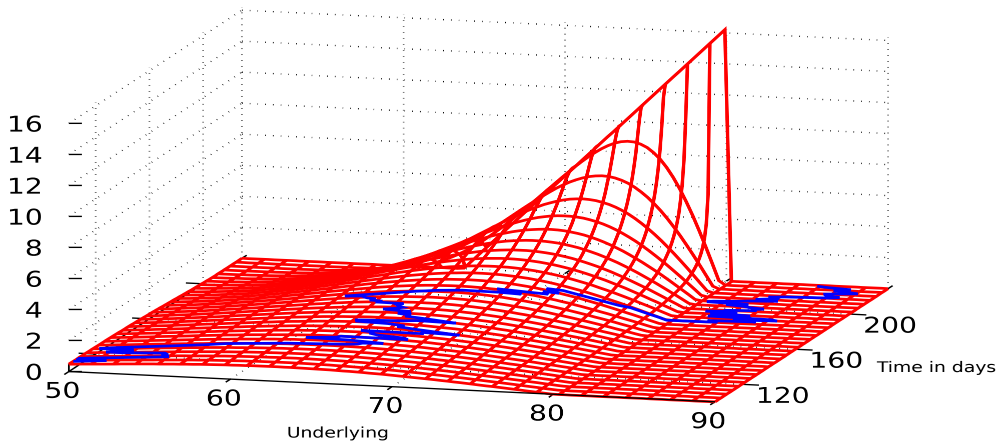


Figure 8.1: Graph of the up-and-out call option price with $B = 80 > K = 65$.*

Proof of Proposition 8.2. We have $C = \phi\left(S_T, M_0^T\right)$ with

$$
\phi(x,y) = (x-K)^+\mathbb{1}_{\{y<B\}} = \begin{cases} (x-K)^+ & \text{if } y < B,\\ 0 & \text{if } y \geqslant B, \end{cases}
$$

*Right-click on the figure for interaction and "Full Screen Multimedia" view.

hence

$$\begin{aligned}
& \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\,\middle|\,\mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{1}_{\left\{M_t^T<B\right\}}\,\middle|\,\mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\mathrm{Max}}S_r<B\right\}}\,\middle|\,\mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[\left(x\frac{S_T}{S_t}-K\right)^+\mathbb{1}_{\left\{x\underset{t\leqslant r\leqslant T}{\mathrm{Max}}\frac{S_r}{S_t}>B\right\}}\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[\left(x\frac{S_{T-t}}{S_0}-K\right)^+\mathbb{1}_{\left\{x\underset{0\leqslant r\leqslant T-t}{\mathrm{Max}}\frac{S_r}{S_0}<B\right\}}\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\mathbb{E}^*\left[\left(x\,\mathrm{e}^{\sigma\widetilde{W}_{T-t}}-K\right)^+\mathbb{1}_{\left\{x\underset{0\leqslant r\leqslant T-t}{\mathrm{Max}}\mathrm{e}^{\sigma\widetilde{W}_r}<B\right\}}\right]_{x=S_t}.
\end{aligned}$$

It then suffices to compute, using (7.3.6),

$$\begin{aligned}
& \mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\right] \\
&= \mathbb{E}^*\left[\left(S_0\,\mathrm{e}^{\sigma\widetilde{W}_T}-K\right)\mathbb{1}_{\left\{S_0\,\mathrm{e}^{\sigma\widetilde{W}_T}>K\right\}}\mathbb{1}_{\left\{S_0\,\mathrm{e}^{\sigma\widehat{X}_0^T}<B\right\}}\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma y}>K\}}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma x}<B\}}d\mathbb{P}\left(\widehat{X}_0^T\leqslant x,\widetilde{W}_T\leqslant y\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{\sigma y>\log(K/S_0)\}}\mathbb{1}_{\{\sigma x<\log(B/S_0)\}}\varphi_{\widehat{X}_T,\widetilde{W}_T}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\mathbb{1}_{\{\sigma y>\log(K/S_0)\}}\mathbb{1}_{\{\sigma x<\log(B/S_0)\}}\mathbb{1}_{\{y\vee 0<x\}}\varphi_{\widehat{X}_T,\widetilde{W}_T}(x,y)dxdy \\
&= \frac{1}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)} \\
&\quad \int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy \\
&= \frac{\mathrm{e}^{-\mu^2T/2}}{T^{3/2}}\sqrt{\frac{2}{\pi}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)}\left(S_0\,\mathrm{e}^{\sigma y}-K\right)\mathrm{e}^{\mu y-y^2/(2T)} \\
&\qquad \times\int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}(2x-y)\,\mathrm{e}^{2x(y-x)/T}dxdy,
\end{aligned}$$

if $B\geqslant K$ and $B\geqslant S_0$ (otherwise the option price is 0), with $\mu:=r/\sigma-\sigma/2$ and $y\vee 0=\mathrm{Max}(y,0)$. Letting $a:=y\vee 0$ and $b:=\sigma^{-1}\log(B/S_0)$, we have

$$\begin{aligned}
\int_a^b(2x-y)\,\mathrm{e}^{2x(y-x)/T}dx &= \int_a^b(2x-y)\,\mathrm{e}^{2x(y-x)/T}dx \\
&= -\frac{T}{2}\left[\mathrm{e}^{2x(y-x)/T}\right]_{x=a}^{x=b} \\
&= \frac{T}{2}\left(\mathrm{e}^{2a(y-a)/T}-\mathrm{e}^{2b(y-b)/T}\right) \\
&= \frac{T}{2}\left(\mathrm{e}^{2(y\vee 0)(y-y\vee 0)/T}-\mathrm{e}^{2b(y-b)/T}\right) \\
&= \frac{T}{2}\left(1-\mathrm{e}^{2b(y-b)/T}\right),
\end{aligned}$$

hence, letting $c := \sigma^{-1}\log(K/S_0)$, we obtain

$$\begin{aligned}
&\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{M_0^T<B\}}\right]\\
&= \frac{e^{-\mu^2T/2}}{\sqrt{2\pi T}}\int_c^b (S_0 e^{\sigma y}-K)\, e^{\mu y-y^2/(2T)}(1-e^{2b(y-b)/T})dy\\
&= S_0 e^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b e^{(\sigma+\mu)y-y^2/(2T)}(1-e^{2b(y-b)/T})dy\\
&\quad -K e^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b e^{\mu y-y^2/(2T)}(1-e^{2b(y-b)/T})dy\\
&= S_0 e^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b e^{(\sigma+\mu)y-y^2/(2T)}dy\\
&\quad -S_0 e^{-\mu^2T/2-2b^2/T}\frac{1}{\sqrt{2\pi T}}\int_c^b e^{(\sigma+\mu+2b/T)y-y^2/(2T)}dy\\
&\quad -K e^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b e^{\mu y-y^2/(2T)}dy\\
&\quad +K e^{-\mu^2T/2-2b^2/T}\frac{1}{\sqrt{2\pi T}}\int_c^b e^{(\mu+2b/T)y-y^2/(2T)}dy.
\end{aligned}$$

Using Relation (7.4.5), we find

$$\begin{aligned}
&e^{-rT}\,\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{M_0^T<B\}}\right]\\
&= S_0 e^{-(r+\mu^2/2)T+(\sigma+\mu)^2T/2}\left(\Phi\left(\frac{-c+(\sigma+\mu)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\sigma+\mu)T}{\sqrt{T}}\right)\right)\\
&\quad -S_0 e^{-(r+\mu^2/2)T-2b^2/T+(\sigma+\mu+2b/T)^2T/2}\\
&\qquad \times\left(\Phi\left(\frac{-c+(\sigma+\mu+2b/T)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\sigma+\mu+2b/T)T}{\sqrt{T}}\right)\right)\\
&\quad -K e^{-rT}\left(\Phi\left(\frac{-c+\mu T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+\mu T}{\sqrt{T}}\right)\right)\\
&\quad +K e^{-(r+\mu^2/2)T-2b^2/T+(\mu+2b/T)^2T/2}\\
&\qquad \times\left(\Phi\left(\frac{-c+(\mu+2b/T)T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+(\mu+2b/T)T}{\sqrt{T}}\right)\right)\\
&= S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -S_0 e^{-(r+\mu^2/2)T-2b^2/T+(\sigma+\mu+2b/T)^2T/2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&\quad -K e^{-rT}\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad +K e^{-(r+\mu^2/2)T-2b^2T+(\mu+2b/T)^2T/2}\left(\Phi\left(\delta_-\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-\left(\frac{B}{S_0}\right)\right)\right),
\end{aligned}$$

$0\leqslant x\leqslant B$, where $\delta_\pm^T(z)$ is defined in (8.2.2). Given the relations

$$-T\left(r+\frac{\mu^2}{2}\right)-2\frac{b^2}{T}+\frac{T}{2}\left(\sigma+\mu+\frac{2b}{T}\right)^2=2b\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)=\left(1+\frac{2r}{\sigma^2}\right)\log\frac{B}{S_0},$$

and

$$-T\left(r+\frac{\mu^2}{2}\right)-2\frac{b^2}{T}+\frac{T}{2}\left(\mu+\frac{2b}{T}\right)^2=-rT+2\mu b=-rT+\left(-1+\frac{2r}{\sigma^2}\right)\log\frac{B}{S_0},$$

this yields

$$
\begin{aligned}
& \mathrm{e}^{-rT}\,\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\right] \qquad (8.2.3)\\
&= S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -\mathrm{e}^{-rT}K\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -B\left(\frac{B}{S_0}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&\quad +\mathrm{e}^{-rT}K\left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-^T\left(\frac{B}{S_0}\right)\right)\right)\\
&= S_0\left(\Phi\left(\delta_+^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_+^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad -S_0\left(\frac{B}{S_0}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_+^T\left(\frac{B}{S_0}\right)\right)\right)\\
&\quad -\mathrm{e}^{-rT}K\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right)-\Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right)\\
&\quad +\mathrm{e}^{-rT}K\left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0}\right)\right)-\Phi\left(\delta_-^T\left(\frac{B}{S_0}\right)\right)\right),
\end{aligned}
$$

and this yields the result of Proposition 8.2, cf. § 7.3.3 pages 304-307 of Shreve, 2004 for a different approach to this calculation. This concludes the proof of Proposition 8.2. □

Up-and-out barrier put option

This option is also called a *Callable Bear Contract*, or a Bear CBBC with no residual value, or a turbo warrant with no rebate, in which B denotes the call level*. The price

$$
\mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{M_0^t<B\right\}}\,\mathbb{E}^*\left[\left(K-x\frac{S_{T-t}}{S_0}\right)^+\mathbb{1}_{\left\{x\underset{0\leqslant r\leqslant T-t}{\mathrm{Max}}\frac{S_r}{S_0}<B\right\}}\right]_{x=S_t}
$$

of the up-and-out barrier put option with maturity T, strike price K and barrier level B is given, if $B\leqslant K$, by

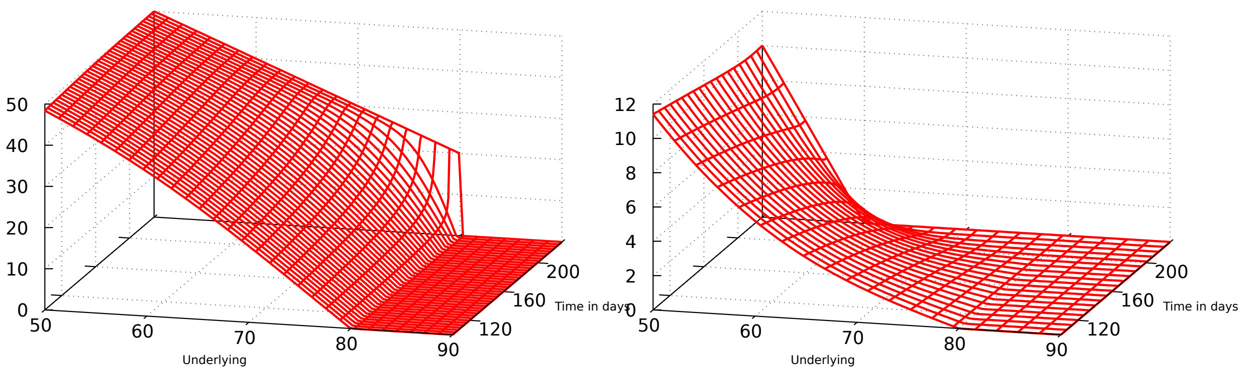
$$
\begin{aligned}
& \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\left\{M_0^T<B\right\}}\,\middle|\,\mathscr{F}_t\right]\\
&= S_t\mathbb{1}_{\left\{M_0^t<B\right\}}\left(\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)-1-\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)-1\right)\right)\\
&\quad -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{M_0^t<B\right\}}\left(\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)-1-\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)-1\right)\right)\\
&= S_t\mathbb{1}_{\left\{M_0^t<B\right\}}\left(-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)+\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\\
&\quad -K\,\mathrm{e}^{-(T-t)r}
\end{aligned}
$$

*Download this R **code** for the pricing of Bear CBBCs (up-and-out barrier put options) with $B\leqslant K$ (right-click to save as attachment).

$$\times \mathbb{1}_{\{M_0^t < B\}} \left(-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right) + \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(-\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right). \tag{8.2.4}$$

and, if $B \geqslant K$, by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K - S_T)^+ \mathbb{1}_{\{M_0^T < B\}} \,\Big|\, \mathscr{F}_t \right] \\
&= S_t \mathbb{1}_{\{M_0^t < B\}} \left(\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - 1 - \left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - 1\right)\right) \\
&- \mathrm{e}^{-(T-t)r} K \\
&\quad \times \mathbb{1}_{\{M_0^t < B\}} \left(\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right) - 1 - \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - 1\right)\right) \\
&= S_t \mathbb{1}_{\{M_0^t < B\}} \left(-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) + \left(\frac{B}{S_t}\right)^{1+2r/\sigma^2} \Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\right) \\
&- K\,\mathrm{e}^{-(T-t)r} \\
&\quad \times \mathbb{1}_{\{M_0^t < B\}} \left(-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right) + \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\right), \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(K - x\frac{S_{T-t}}{S_0}\right)^+ \mathbb{1}_{\left\{x \underset{0\leqslant r \leqslant T-t}{\mathrm{Max}} \frac{S_r}{S_0} < B\right\}} \right]_{x=S_t} \\
&= -S_t \mathbb{1}_{\{M_0^t < B\}} \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) + S_t \mathbb{1}_{\{M_0^t < B\}} \left(\frac{B}{S_t}\right)^{1+2r/\sigma^2} \Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) \\
&+ K \mathbb{1}_{\{M_0^t < B\}} \mathrm{e}^{-(T-t)r} \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right) - K\,\mathrm{e}^{-(T-t)r}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right) \\
&= \mathbb{1}_{\{M_0^t < B\}} \mathrm{Bl}_{\mathrm{put}}(S_t, K, r, T-t, \sigma) + S_t \mathbb{1}_{\{M_0^t < B\}} \left(\frac{B}{S_t}\right)^{1+2r/\sigma^2} \Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) \\
&- K \mathbb{1}_{\{M_0^t < B\}} \mathrm{e}^{-(T-t)r} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right).
\end{aligned} \tag{8.2.5}$$



(a) Case $K = 100 > B = 80$. (b) Case $B = 80 > K = 60$.

Figure 8.2: Graphs of the up-and-out put option prices (8.2.4)-(8.2.5).

The following Figure 8.3 shows the market pricing data of an up-and-out barrier put option on BHP Billiton Limited ASX:BHP with $B = K = \$28$ for half a share, priced at 1.79.

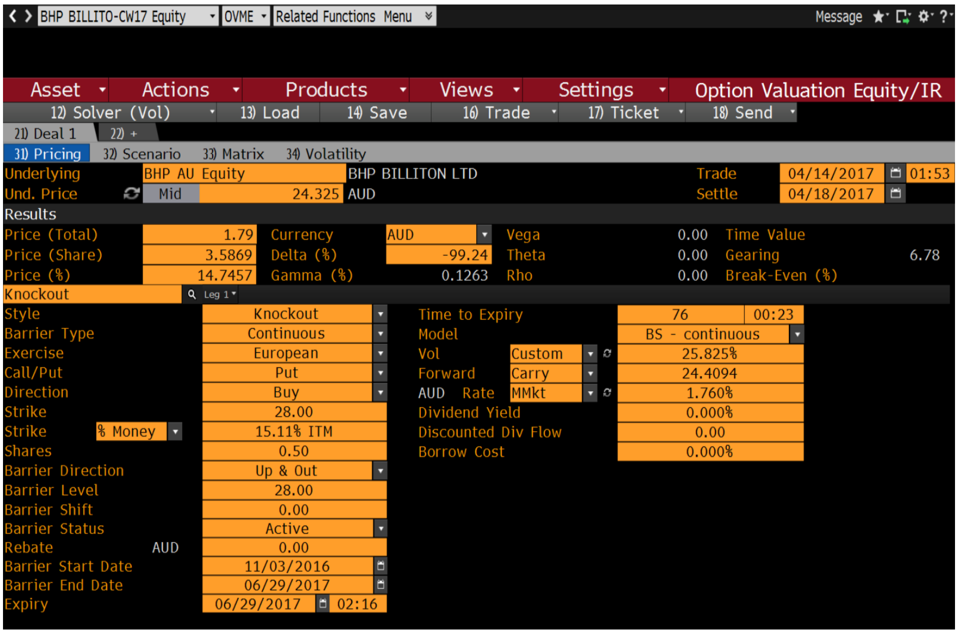


Figure 8.3: Pricing data for an up-and-out put option with $K = B = \$28$.

The attached **code** performs an implied volatility calculation for up-and-out barrier put option (or Bear CBBC) prices with $B < K$, based on this **market data** set.

Down-and-out barrier call option

Let us now consider a down-and-out barrier call option on the underlying asset price S_t with exercise date T, strike price K, barrier level B, and payoff

$$C = (S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}} = \begin{cases} S_T - K & \text{if } \min\limits_{0 \leqslant t \leqslant T} S_t > B, \\ \\ 0 & \text{if } \min\limits_{0 \leqslant t \leqslant T} S_t \leqslant B, \end{cases}$$

with $0 \leqslant B \leqslant K$. The down-and-out barrier call option is also called a *Callable Bull Contract*, or a Bull CBBC with no residual value, or a turbo warrant with no rebate, in which B denotes the call level.* When $B \leqslant K$, we have

$$\begin{aligned} & e^{-(T-t)r} \mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}} \,\middle|\, \mathcal{F}_t \right] \qquad (8.2.6) \\ &= S_t \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K} \right) \right) - e^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K} \right) \right) \\ &\quad - B \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t} \right)^{2r/\sigma^2} \Phi\left(\delta_+^{T-t}\left(\frac{B^2}{K S_t} \right) \right) \\ &\quad + e^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{B^2}{K S_t} \right) \right) \end{aligned}$$

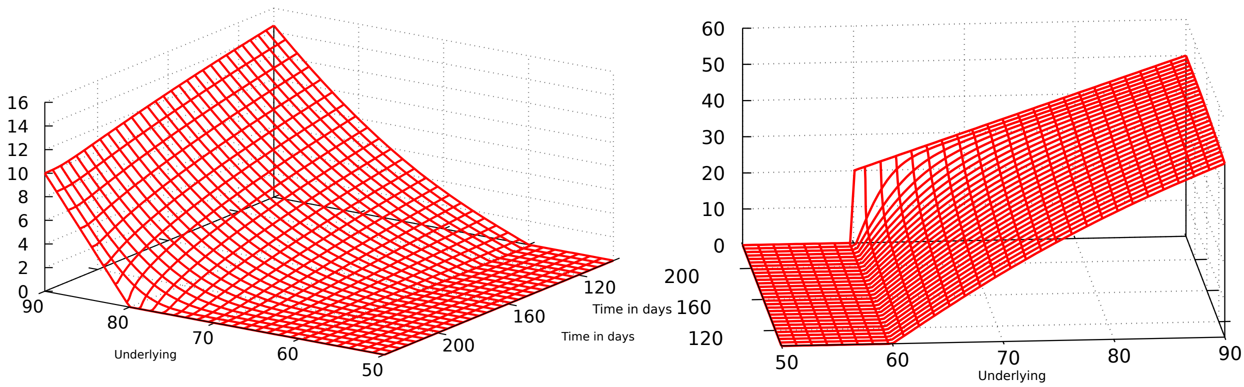
*Download this **code** for the pricing of Bull CBBC (down-and-out barrier call options) with $B \geqslant K$ (right-click to save as attachment).

$$
\begin{aligned}
&= \mathbb{1}_{\{m_0^t > B\}} \mathrm{Bl}(S_t, K, r, T-t, \sigma) \\
&\quad - B \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t} \right)^{2r/\sigma^2} \Phi\left(\delta_+^{T-t}\left(\frac{B^2}{K S_t} \right) \right) \\
&\quad + \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{B^2}{K S_t} \right) \right) \\
&= \mathbb{1}_{\{m_0^t > B\}} \mathrm{Bl}(S_t, K, r, T-t, \sigma) \\
&\quad - S_t \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t} \right)^{2r/\sigma^2} \mathrm{Bl}\left(\frac{B}{S_t}, \frac{K}{B}, r, T-t, \sigma \right),
\end{aligned}
$$

$0 \leqslant t \leqslant T$. When $B \geqslant K$, we find

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0 \leqslant t \leqslant T} S_t > B \right\}} \,\middle|\, \mathcal{F}_t \right] \qquad (8.2.7) \\
&= S_t \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B} \right) \right) - \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B} \right) \right) \\
&\quad - B \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t} \right)^{2r/\sigma^2} \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t} \right) \right) \\
&\quad + \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B} \right)^{1-2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t} \right) \right),
\end{aligned}
$$

$S_t > B$, $0 \leqslant t \leqslant T$, see Exercise 8.1 below.



(a) Case $B = 60 < K = 80$. (b) Case $K = 40 < B = 60$.

Figure 8.4: Graphs of the down-and-out call option price (8.2.6)-(8.2.7).

In the next Figure 8.5 we plot* the down-and-out barrier call option price (8.2.7) as a function of volatility with $B = 349.2 > K = 346.4$, $r = 0.03$, $T = 99/365$, and $S_0 = 360$.

*Download this **code** for the pricing of down-and-out barrier call options (right-click to save as attachment).

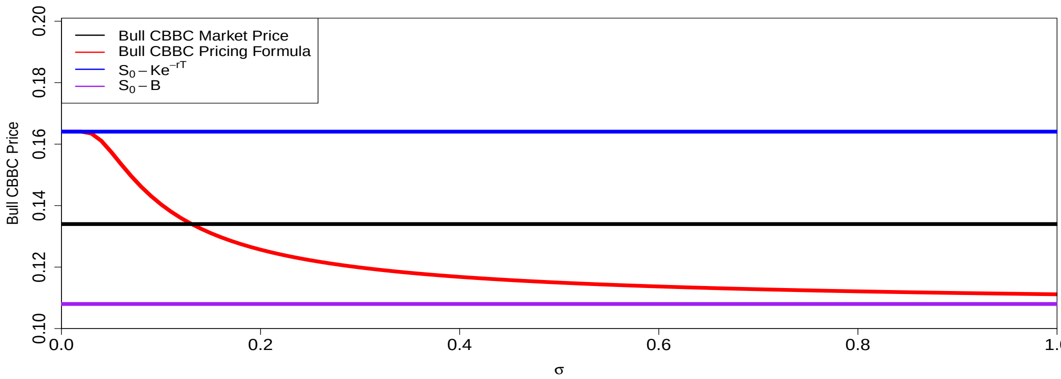


Figure 8.5: Down-and-out call option price as a function of σ.

We note that with such parameters, the down-and-out barrier call option price (8.2.7) is upper bounded by the forward contract price $S_0 - K\mathrm{e}^{-rT}$ in the limit as σ tends to zero, and that it decreases to $S_0 - B$ in the limit as σ tends to infinity.

Down-and-out barrier put option

When $K \geqslant B$, the price

$$\mathrm{e}^{-(T-t)r}\mathbb{1}_{\{m_0^t>B\}}\mathbb{E}^*\left[\left(K - x\frac{S_{T-t}}{S_0}\right)^+ \mathbb{1}_{\left\{x \min\limits_{0\leqslant r\leqslant T-t} S_r/S_0 > B\right\}}\right]_{x=S_t}$$

of the down-and-out barrier put option with maturity T, strike price K and barrier level B is given by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{m_0^T>B\}}\,\Big|\,\mathscr{F}_t\right] \\
&= S_t\mathbb{1}_{\{m_0^t>B\}}\left\{\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\right. \\
&\qquad\left.-\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\right\} \\
&\quad -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\left\{\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\right. \\
&\qquad\left.-\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\right\} \\
&= S_t\mathbb{1}_{\{m_0^t>B\}}\left\{\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)\right. \\
&\qquad\left.-\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\right\} \\
&\quad -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\{m_0^t>B\}}\left\{\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{K}\right)\right)\right. \\
&\qquad\left.-\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \mathbb{1}_{\{m_0^t > B\}} \mathrm{Bl}_{\mathrm{put}}(S_t, K, r, T-t, \sigma) + S_t \mathbb{1}_{\{m_0^t > B\}} \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \\
& - B \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t}\right)^{2r/\sigma^2} \left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right) \\
& - \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right) \\
& + \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right),
\end{aligned}
\tag{8.2.8}
$$

while the corresponding price vanishes when $K \leqslant B$.

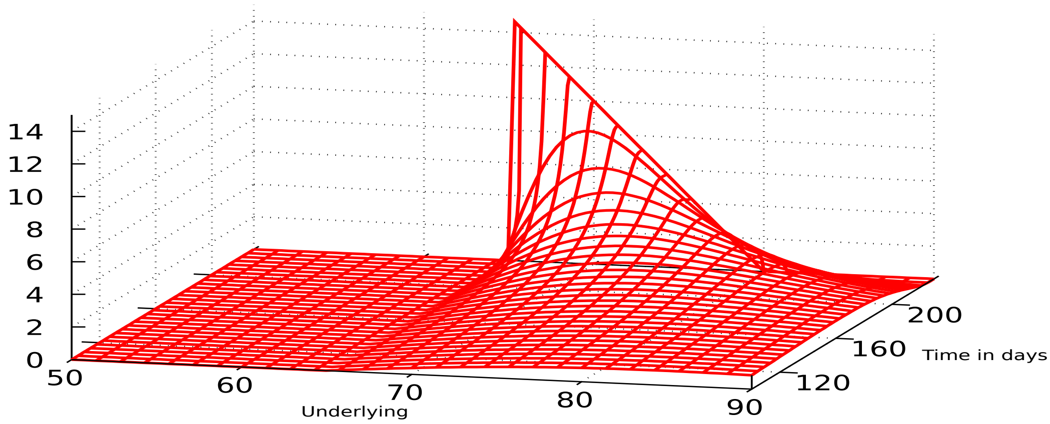


Figure 8.6: Graph of the down-and-out put option price (8.2.8) with $K = 80 > B = 65$.

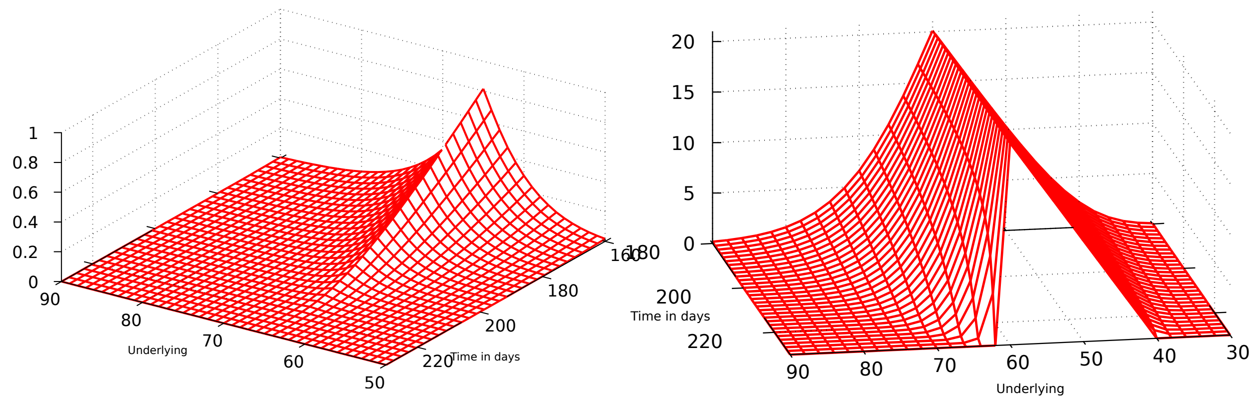
Note that although Figures 8.2b and 8.4a, resp. 8.2a and 8.4b, appear to share some symmetry property, the functions themselves are not exactly symmetric. Regarding Figures 8.1 and 8.6, the pricing function is actually the same, but the conditions $B < K$ and $B > K$ play opposite roles.

8.3 Knock-In Barrier

Down-and-in barrier call option

When $B \leqslant K$, the price of the down-and-in barrier call option is given from the down-and-out barrier call option price (8.2.6) and the down-in-out call parity relation (8.1.2) as

$$
\begin{aligned}
& \mathrm{e}^{-(T-t)r} \mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{m_0^T < B\}} \,\middle|\, \mathcal{F}_t \right] \\
= \quad & \mathbb{1}_{\{m_0^t \leqslant B\}} \mathrm{Bl}(S_t, K, r, T-t, \sigma) \\
& + S_t \mathbb{1}_{\{m_0^t > B\}} \left(\frac{B}{S_t}\right)^{1+2r/\sigma^2} \Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) \\
& - \mathrm{e}^{-(T-t)r} K \mathbb{1}_{\{m_0^t > B\}} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right).
\end{aligned}
\tag{8.3.1}
$$



(a) Case $K = 80 > B = 65$.

(b) Case $K = 40 < B = 60$.

Figure 8.7: Graphs of the down-and-in call option price (8.3.1)-(8.3.2).

When $B \geqslant K$, the price of the down-and-in barrier call option is given from the down-and-out barrier call option price (8.2.7) and the down-in-out call parity relation (8.1.2) as

$$\begin{aligned} & \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{m_t^T<B\right\}}\,\middle|\,\mathscr{F}_t\right] \qquad (8.3.2)\\ = \;& \mathrm{Bl}(S_t,K,r,T-t,\sigma)\\ & -S_t\mathbb{1}_{\left\{m_0^t>B\right\}}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)+\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{m_0^t>B\right\}}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\ & +\mathbb{1}_{\left\{m_0^t>B\right\}}S_t\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\\ & -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{m_0^t>B\right\}}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right), \qquad 0\leqslant t\leqslant T. \end{aligned}$$

Up-and-in barrier call option

When $B \geqslant K$, the price of the up-and-in barrier call option is given from (8.2.1) and the up-in-out call parity relation (8.1.1) as

$$\begin{aligned} & \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{M_0^T>B\right\}}\,\middle|\,\mathscr{F}_t\right] \qquad (8.3.3)\\ = \;& \mathbb{1}_{\left\{M_0^t\geqslant B\right\}}\mathrm{Bl}(S_t,K,r,T-t,\sigma)+S_t\mathbb{1}_{\left\{M_0^t<B\right\}}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\\ & +B\mathbb{1}_{\left\{M_0^t<B\right\}}\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\\ & -\mathrm{e}^{-(T-t)r}K\mathbb{1}_{\left\{M_0^t<B\right\}}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\\ & -\mathrm{e}^{-(T-t)r}K\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right). \end{aligned}$$

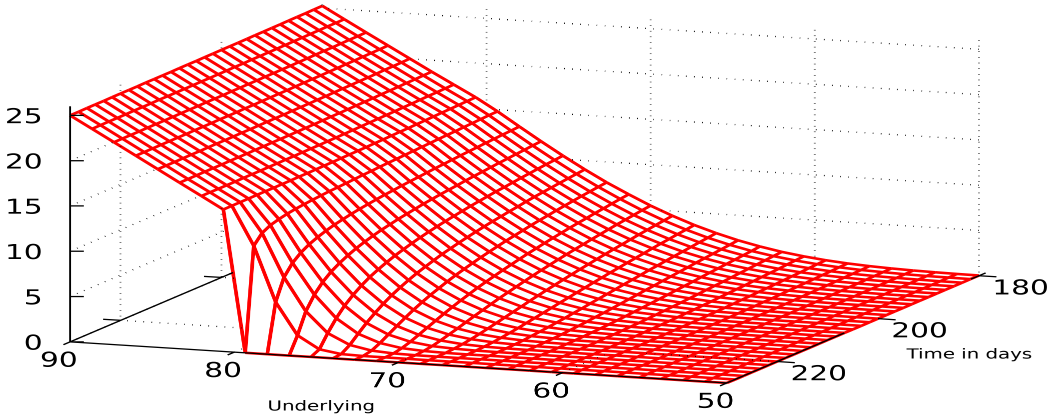


Figure 8.8: Graph of the up-and-in call option price (8.3.3) with $B = 80 > K = 65$.

When $B \leqslant K$, the price of the up-and-in barrier call option is given from the Black-Scholes formula and the up-in-out call parity relation (8.1.1) as

$$e^{-(T-t)r} \mathbb{E}^* \left[(S_T - K)^+ \mathbb{1}_{\left\{M_0^T > B\right\}} \Big| \mathcal{F}_t \right] = \mathrm{Bl}(S_t, K, r, T-t, \sigma).$$

Down-and-in barrier put option

When $B \leqslant K$, the price of the down-and-in barrier put option is given from (8.2.8) and the down-in-out put parity relation (8.1.4) as

$$\begin{aligned} & e^{-(T-t)r} \mathbb{E}^* \left[(K - S_T)^+ \mathbb{1}_{\left\{m_t^T < B\right\}} \Big| \mathcal{F}_t \right] \qquad (8.3.4) \\ & = \mathbb{1}_{\left\{m_0^t \leqslant B\right\}} \mathrm{Bl}_{\mathrm{put}}(S_t, K, r, T-t, \sigma) - S_t \mathbb{1}_{\left\{m_0^t > B\right\}} \Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \\ & + B \mathbb{1}_{\left\{m_0^t > B\right\}} \left(\frac{B}{S_t}\right)^{2r/\sigma^2} \left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right) \\ & + e^{-(T-t)r} K \mathbb{1}_{\left\{m_0^t > B\right\}} \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right) \\ & - e^{-(T-t)r} K \mathbb{1}_{\left\{m_0^t > B\right\}} \left(\frac{S_t}{B}\right)^{1-2r/\sigma^2} \left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right), \end{aligned}$$

$0 \leqslant t \leqslant T$.

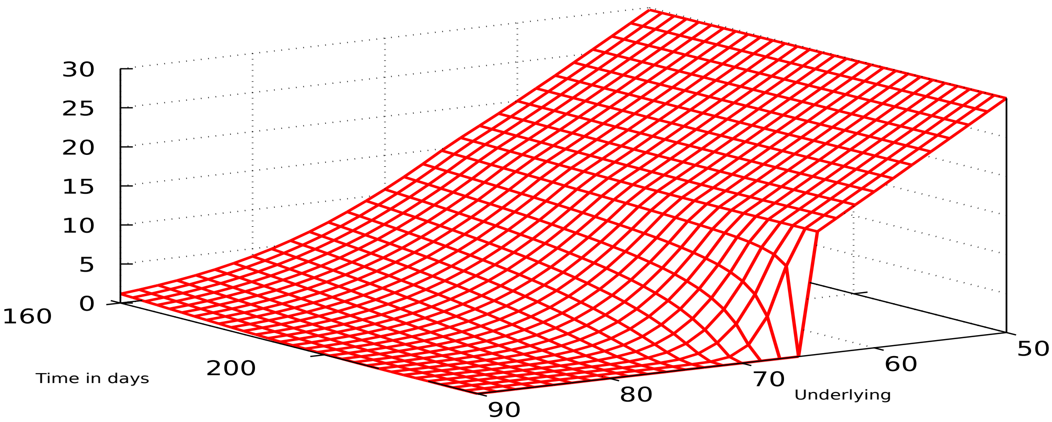


Figure 8.9: Graph of the down-and-in put option price (8.3.4) with $K = 80 > B = 65$.

When $B \geqslant K$, the price of the down-and-in barrier put option is given from the Black-Scholes put function and the down-in-out put parity relation (8.1.4) as

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{m_t^T<B\}}\,\middle|\,\mathscr{F}_t\right]=\mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma),$$

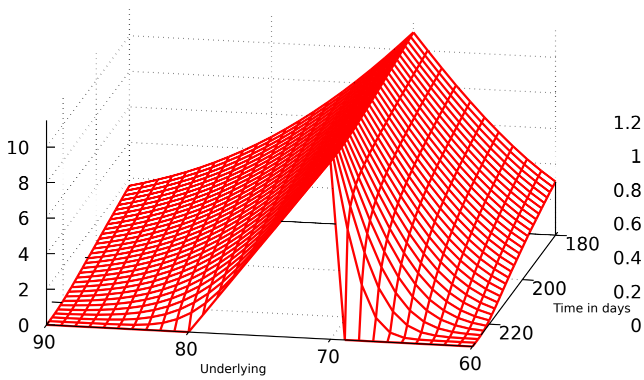
$0 \leqslant t \leqslant T$.

Up-and-in barrier put option

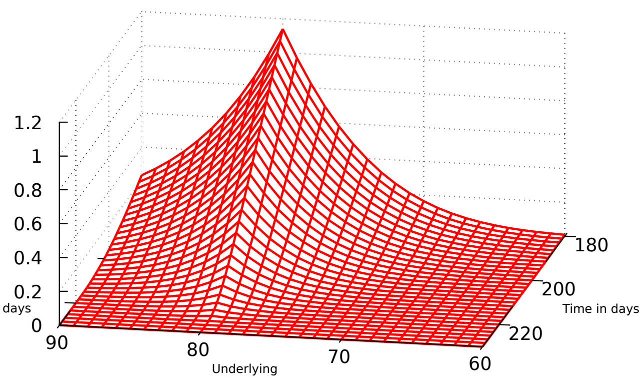
When $B \leqslant K$, the price of the down-and-in barrier put option is given from (8.2.4) and the up-in-out put parity relation (8.1.3) as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{M_0^T>B\}}\,\middle|\,\mathscr{F}_t\right] \qquad (8.3.5)\\
&= \mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma)\\
&\quad -S_t\mathbb{1}_{\{M_0^t<B\}}\left(\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)-\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)\right)\\
&\quad +K\,\mathrm{e}^{-(T-t)r}\\
&\qquad \times\mathbb{1}_{\{M_0^t<B\}}\left(\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)-\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{B}\right)\right)\right).
\end{aligned}$$

$0 \leqslant t \leqslant T$.



(a) $K = 80 > B = 70$.



(b) Case $K = 70 < B = 80$.

Figure 8.10: Graphs of the up-and-in put option price (8.3.5)-(8.3.6).

By (8.2.5) and the up-in-out put parity relation (8.1.3), the price of the up-and-in barrier put option is given when $B \geqslant K$ by

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{M_0^T>B\}}\,\middle|\,\mathscr{F}_t\right] \qquad (8.3.6)\\
&= \mathbb{1}_{\{M_0^t\geqslant B\}}\mathrm{Bl}_{\mathrm{put}}(S_t,K,r,T-t,\sigma)\\
&\quad -S_t\mathbb{1}_{\{M_0^t<B\}}\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\\
&\quad +K\mathbb{1}_{\{M_0^t<B\}}\,\mathrm{e}^{-(T-t)r}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right).
\end{aligned}$$

8.4 PDE Method

The up-and-out barrier call option price has been evaluated by probabilistic arguments in the previous sections. In this section we complement this approach with the derivation of a Partial Differential Equation (PDE) for this option price function.

The up-and-out barrier call option price can be written as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{M_0^T<B\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{0\leqslant r\leqslant t}{\operatorname{Max}} S_r<B\right\}}\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\operatorname{Max}} S_r<B\right\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{1}_{\left\{\underset{0\leqslant r\leqslant t}{\operatorname{Max}} S_r<B\right\}}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\operatorname{Max}} S_r<B\right\}}\,\Big|\,\mathcal{F}_t\right]\\
&= \mathbb{1}_{\{M_0^t<B\}}g(t,S_t),
\end{aligned}$$

where the function $g(t,x)$ of t and S_t is given by

$$g(t,x)=\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\operatorname{Max}} S_r<B\right\}}\,\Big|\,S_t=x\right]. \tag{8.4.1}$$

Next, we derive the Black-Scholes partial differential equation (PDE) satisfied by $g(t,x)$, and written for the value of a self-financing portfolio.

Proposition 8.3 Let $(\eta_t,\xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy such that

(i) $(\eta_t,\xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the portfolio value $V_t:=\eta_t A_t+\xi_t S_t$, $t\geqslant 0$, is given as in (8.4.1) by

$$V_t=\mathbb{1}_{\{M_0^t<B\}}g(t,S_t), \qquad t\geqslant 0.$$

Then, the function $g(t,x)$ pricing the up-and-out barrier call option satisfies the Black-Scholes PDE

$$rg(t,x)=\frac{\partial g}{\partial t}(t,x)+rx\frac{\partial g}{\partial x}(t,x)+\frac{1}{2}x^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,x), \tag{8.4.2}$$

$t>0$, $0<x<B$, and ξ_t is given by

$$\xi_t=\frac{\partial g}{\partial x}(t,S_t), \qquad 0\leqslant t\leqslant T, \tag{8.4.3}$$

provided that $M_0^t<B$.

Proof. By (8.4.1) the price at time t of the up-and-out barrier call option discounted to time 0 is given by

$$\begin{aligned}
&\mathrm{e}^{-rt}\mathbb{1}_{\{M_0^t<B\}}g(t,S_t)\\
&= \mathrm{e}^{-rT}\mathbb{1}_{\{M_0^t<B\}}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\operatorname{Max}} S_r<B\right\}}\,\Big|\,\mathcal{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-rT}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{M_0^t<B\}}\mathbb{1}_{\left\{\underset{t\leqslant r\leqslant T}{\operatorname{Max}} S_r<B\right\}}\,\Big|\,\mathcal{F}_t\right] \\
&= \mathrm{e}^{-rT}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\left\{\underset{0\leqslant r\leqslant T}{\operatorname{Max}} S_r<B\right\}}\,\Big|\,S_t\right],
\end{aligned}
$$

which is a martingale indexed by $t \geqslant 0$. Next, applying the Itô formula to $t \longmapsto \mathrm{e}^{-rt}g(t,S_t)$ "on $\{M_0^t \leqslant B,\ 0 \leqslant t \leqslant T\}$", we have

$$
\begin{aligned}
d(\mathrm{e}^{-rt}g(t,S_t)) &= -r\,\mathrm{e}^{-rt}g(t,S_t)dt + \mathrm{e}^{-rt}dg(t,S_t) \\
&= -r\,\mathrm{e}^{-rt}g(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial g}{\partial t}(t,S_t)dt \\
&\quad + r\,\mathrm{e}^{-rt}S_t\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}\mathrm{e}^{-rt}\sigma^2S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt \\
&\quad + \mathrm{e}^{-rt}\sigma S_t\frac{\partial g}{\partial x}(t,S_t)dW_t. \qquad (8.4.4)
\end{aligned}
$$

In order to derive (8.4.3) we note that the self-financing condition implies

$$
\begin{aligned}
d(\mathrm{e}^{-rt}V_t) &= -r\,\mathrm{e}^{-rt}V_t dt + \mathrm{e}^{-rt}dV_t \\
&= -r\,\mathrm{e}^{-rt}V_t dt + \eta_t\,\mathrm{e}^{-rt}dA_t + \xi_t\,\mathrm{e}^{-rt}dS_t \\
&= -r(\eta_tA_t+\xi_tS_t)\,\mathrm{e}^{-rt}dt + r\eta_tA_t\,\mathrm{e}^{-rt}dt + r\xi_tS_t\,\mathrm{e}^{-rt}dt + \sigma\xi_tS_t\,\mathrm{e}^{-rt}dW_t \\
&= \sigma\xi_tS_t\,\mathrm{e}^{-rt}dW_t, \qquad t \geqslant 0, \qquad (8.4.5)
\end{aligned}
$$

and (8.4.3) follows by identification of (8.4.4) with (8.4.5) which shows that the sum of components in factor of dt have to vanish, hence

$$
-rg(t,S_t) + \frac{\partial g}{\partial t}(t,S_t) + rS_t\frac{\partial g}{\partial x}(t,S_t) + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t) = 0.
$$

□

In the next proposition we add a boundary condition to the Black-Scholes PDE (8.4.2) in order to hedge the up-and-out barrier call option with maturity T, strike price K, barrier (or call level) B, and payoff

$$
C = (S_T-K)^+\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\operatorname{Max}} S_t<B\right\}} = \begin{cases} S_T-K & \text{if } \underset{0\leqslant t\leqslant T}{\operatorname{Max}} S_t \leqslant B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\operatorname{Max}} S_t > B, \end{cases}
$$

with $B \geqslant K$.

Proposition 8.4 The value $V_t = \mathbb{1}_{\{M_0^t<B\}}g(t,S_t)$ of the self-financing portfolio hedging the up-and-out barrier call option satisfies the Black-Scholes PDE

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x), & (8.4.6a) \\ g(t,x) = 0, \qquad x \geqslant B, \quad t \in [0,T], & (8.4.6b) \\ g(T,x) = (x-K)^+ \mathbb{1}_{\{x<B\}}, & (8.4.6c) \end{cases}$$

on the time-space domain $[0,T] \times [0,B]$ with terminal condition

$$g(T,x) = (x-K)^+ \mathbb{1}_{\{x<B\}}$$

and additional boundary condition

$$g(t,x) = 0, \qquad x \geqslant B. \tag{8.4.7}$$

Condition (8.4.7) holds since the price of the claim at time t is 0 whenever $S_t = B$. When $K \leqslant B$, the closed-form solution of the PDE (8.4.6a) under the boundary conditions (8.4.6b)-(8.4.6c) is given from (8.2.1) in Proposition 8.2 as

$$\begin{aligned} g(t,x) = {} & x\left(\Phi\left(\delta_+^{T-t}\left(\frac{x}{K}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right)\right) \\ & -x\left(\frac{x}{B}\right)^{-1-2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{x}\right)\right)\right) \\ & -K\,\mathrm{e}^{-(T-t)r}\left(\Phi\left(\delta_-^{T-t}\left(\frac{x}{K}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{x}{B}\right)\right)\right) \\ & +K\,\mathrm{e}^{-(T-t)r}\left(\frac{x}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right)\right), \end{aligned} \tag{8.4.8}$$

$0 < x \leqslant B$, $0 \leqslant t \leqslant T$,

see Figure 8.1. We note that the expression (8.4.8) can be rewritten using the standard Black-Scholes formula

$$\mathrm{Bl}(S,K,r,T,\sigma) = S\Phi\left(\delta_+^T\left(\frac{S}{K}\right)\right) - K\,\mathrm{e}^{-rT}\Phi\left(\delta_-^T\left(\frac{S}{K}\right)\right)$$

for the price of the European call option, as

$$\begin{aligned} g(t,x) &= \mathrm{Bl}(x,K,r,T-t,\sigma) - x\Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right) + \mathrm{e}^{-(T-t)r}K\Phi\left(\delta_-^{T-t}\left(\frac{x}{B}\right)\right) \\ & \quad -B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{x}\right)\right)\right) \\ & \quad +\mathrm{e}^{-(T-t)r}K\left(\frac{x}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{Kx}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right)\right), \end{aligned}$$

$0 < x \leqslant B$, $0 \leqslant t \leqslant T$.

Table 8.2 summarizes the boundary conditions satisfied for barrier option pricing in the Black-Scholes PDE.

Option type	CBBC	Behavior		Boundary conditions	
				Maturity T	Barrier B
Barrier call	Bull	down-and-out	$B \leqslant K$	$(x-K)^+$	0
		(knock-out)	$B \geqslant K$	$(x-K)^+ \mathbb{1}_{\{x>B\}}$	0
		down-and-in	$B \leqslant K$	0	$\mathrm{Bl}(B,K,r,T-t,\sigma)$
		(knock-in)	$B \geqslant K$	$(x-K)^+ \mathbb{1}_{\{x<B\}}$	$\mathrm{Bl}(B,K,r,T-t,\sigma)$
		up-and-out	$B \leqslant K$	0	0
		(knock-out)	$B \geqslant K$	$(x-K)^+ \mathbb{1}_{\{x<B\}}$	0
		up-and-in	$B \leqslant K$	$(x-K)^+$	0
		(knock-in)	$B \geqslant K$	$(x-K)^+ \mathbb{1}_{\{x>B\}}$	$\mathrm{Bl}(B,K,r,T-t,\sigma)$
Barrier put		down-and-out	$B \leqslant K$	$(K-x)^+ \mathbb{1}_{\{x>B\}}$	0
		(knock-out)	$B \geqslant K$	0	0
		down-and-in	$B \leqslant K$	$(K-x)^+ \mathbb{1}_{\{x<B\}}$	$\mathrm{Bl}_{\mathrm{p}}(B,K,r,T-t,\sigma)$
		(knock-in)	$B \geqslant K$	$(K-x)^+$	0
	Bear	up-and-out	$B \leqslant K$	$(K-x)^+ \mathbb{1}_{\{x<B\}}$	0
		(knock-out)	$B \geqslant K$	$(K-x)^+$	0
		up-and-in	$B \leqslant K$	$(K-x)^+ \mathbb{1}_{\{x>B\}}$	$\mathrm{Bl}_{\mathrm{p}}(B,K,r,T-t,\sigma)$
		(knock-in)	$B \geqslant K$	0	$\mathrm{Bl}_{\mathrm{p}}(B,K,r,T-t,\sigma)$

Table 8.2: Boundary conditions for barrier option prices.

8.5 Hedging Barrier Options

Figure 8.11 represents the value of Delta obtained from (8.4.3) for the up-and-out barrier call option in Exercise 8.1-(a)).

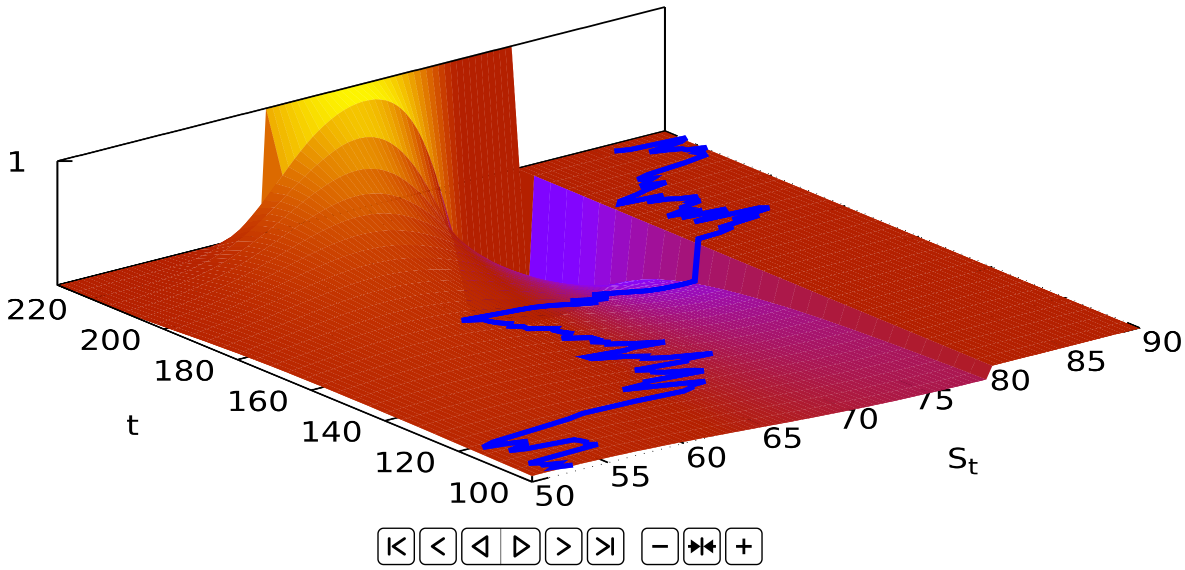


Figure 8.11: Delta of the up-and-out barrier call with $B = 80 > K = 55$.*

Down-and-out barrier call option

Similarly, the price $g(t, S_t)$ at time t of the down-and-out barrier call option satisfies the Black-Scholes PDE

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(t,B) = 0, \quad t \in [0,T], \\ \\ g(T,x) = (x-K)^+ \mathbb{1}_{\{x>B\}}, \end{cases}$$

on the time-space domain $[0,T] \times [0,B]$ with terminal condition $g(T,x) = (x-K)^+ \mathbb{1}_{\{x>B\}}$ and the additional boundary condition

$$g(t,x) = 0, \qquad x \leqslant B,$$

since the price of the claim at time t is 0 whenever $S_t \leqslant B$, see (8.2.6) and Figure 8.4a when $B \leqslant K$, and (8.2.7) and Figure 8.4b when $B \geqslant K$.

Exercises

Exercise 8.1 Barrier options.

a) Compute the delta hedging strategies of the up-and-out barrier call and put options on the underlying asset price S_t with exercise date T, strike price K and barrier level B, with $B \geqslant K$.
b) Compute the joint probability density function

$$\varphi_{Y_T,W_T}(a,b) = \frac{d\mathbb{P}(Y_T \leqslant a \text{ and } W_T \leqslant b)}{dadb}, \qquad a,b \in \mathbb{R},$$

of standard Brownian motion W_T and its *minimum*

$$Y_T = \min_{t\in[0,T]} W_t.$$

*The animation works in Acrobat Reader on the entire pdf file.

c) Compute the joint probability density function

$$\varphi_{\breve{Y}_T,\widetilde{W}_T}(a,b) = \frac{d\mathbb{P}(\breve{Y}_T \leqslant a \text{ and } \widetilde{W}_T \leqslant b)}{dadb}, \qquad a,b \in \mathbb{R},$$

of *drifted* Brownian motion $\widetilde{W}_T = W_T + \mu T$ and its *minimum*

$$\breve{Y}_T = \min_{t\in[0,T]} \widetilde{W}_t = \min_{t\in[0,T]} (W_t + \mu t).$$

d) Compute the price at time $t \in [0,T]$ of the down-and-out barrier call option on the underlying asset price S_t with exercise date T, strike price K, barrier level B, and payoff

$$C = (S_T - K)^+ \mathbb{1}_{\left\{ \min\limits_{0\leqslant t\leqslant T} S_t > B \right\}} = \begin{cases} S_T - K & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t > B, \\ 0 & \text{if } \min\limits_{0\leqslant t\leqslant T} S_t \leqslant B, \end{cases}$$

in cases $0 < B < K$ and $B \geqslant K$.

Exercise 8.2 Pricing Category 'R' CBBC rebates. Given $\tau > 0$, consider an asset price $(S_t)_{t\in[\tau,\infty)}$, given by

$$S_{\tau+t} = S_\tau \mathrm{e}^{rt+\sigma W_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r \geqslant 0$ and $\sigma > 0$. In what follows, $\Delta\tau$ is the *deterministic* length of the Mandatory Call Event (MCE) valuation period which commences from the time upon which a MCE occurs up to the end of the following trading session.

a) Compute the expected rebate (or residual) $\mathbb{E}\left[\left(\min\limits_{s\in[0,\Delta\tau]} S_{\tau+s} - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right]$ of a Category 'R' Bull CBBC Contract (down-and-out barrier call option) having expired at a given time $\tau < T$, knowing that $S_\tau = B > K > 0$, with $r > 0$.

b) Compute the expected rebate $\mathbb{E}\left[\left(\min\limits_{s\in[0,\Delta\tau]} S_{\tau+s} - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right]$ of a Category 'R' Bull CBBC Contract having expired at a given time $\tau < T$, knowing that $S_\tau = B > K > 0$, with $r = 0$.

c) Find the expression of the probability density function of the first hitting time

$$\tau_B = \inf\{t \geqslant 0 \ : \ S_t = B\}$$

of the level $B > 0$ by the process $(S_t)_{t\in\mathbb{R}_+}$.

d) Price the CBBC rebate

$$\begin{aligned} &\mathrm{e}^{-r\Delta\tau}\mathbb{E}\left[\mathrm{e}^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\left(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\right)^+\right] \\ &= \mathrm{e}^{-r\Delta\tau}\mathbb{E}\left[\mathrm{e}^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\,\mathbb{E}\left[\left(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right]\right]. \end{aligned}$$

Exercise 8.3 Barrier forward contracts. Compute the price at time t of the following barrier forward contracts on the underlying asset price S_t with exercise date T, strike price K, barrier level B, and the following payoffs. In addition, compute the corresponding hedging strategies.

a) Up-and-in barrier long forward contract. Take

$$C = (S_T - K)\mathbb{1}_{\left\{ \operatorname{Max}\limits_{0\leqslant t\leqslant T} S_t > B \right\}} = \begin{cases} S_T - K & \text{if } \operatorname{Max}\limits_{0\leqslant t\leqslant T} S_t > B, \\ 0 & \text{if } \operatorname{Max}\limits_{0\leqslant t\leqslant T} S_t \leqslant B. \end{cases}$$

b) Up-and-out barrier long forward contract. Take

$$C = (S_T - K)\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t < B\right\}} = \begin{cases} S_T - K & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t < B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t \geqslant B. \end{cases}$$

c) Down-and-in barrier long forward contract. Take

$$C = (S_T - K)\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\min} S_t < B\right\}} = \begin{cases} S_T - K & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t < B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t \geqslant B. \end{cases}$$

d) Down-and-out barrier long forward contract. Take

$$C = (S_T - K)\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\min} S_t > B\right\}} = \begin{cases} S_T - K & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t > B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t \leqslant B. \end{cases}$$

e) Up-and-in barrier short forward contract. Take

$$C = (K - S_T)\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t > B\right\}} = \begin{cases} K - S_T & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t > B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t \leqslant B. \end{cases}$$

f) Up-and-out barrier short forward contract. Take

$$C = (K - S_T)\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t < B\right\}} = \begin{cases} K - S_T & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t < B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\mathrm{Max}} S_t \geqslant B. \end{cases}$$

g) Down-and-in barrier short forward contract. Take

$$C = (K - S_T)\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\min} S_t < B\right\}} = \begin{cases} K - S_T & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t < B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t \geqslant B. \end{cases}$$

h) Down-and-out barrier short forward contract. Take

$$C = (K - S_T)\mathbb{1}_{\left\{\underset{0\leqslant t\leqslant T}{\min} S_t > B\right\}} = \begin{cases} K - S_T & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t > B, \\ 0 & \text{if } \underset{0\leqslant t\leqslant T}{\min} S_t \leqslant B. \end{cases}$$

Exercise 8.4 Compute the Vega of the down-and-out and down-and-in barrier call option prices, *i.e.* compute the sensitivity of down-and-out and down-and-in barrier option prices with respect to the volatility parameter σ.

Exercise 8.5 Stability warrants. Price the up-and-out binary barrier option with payoff

$$C := \mathbb{1}_{\{S_T > K\}} \mathbb{1}_{\{M_0^T < B\}} = \mathbb{1}_{\{S_T > K \text{ and } M_0^T \leqslant B\}}$$

at time $t = 0$, with $K \leqslant B$.

Exercise 8.6 Check that the function $g(t,x)$ in (8.4.8) satisfies the boundary conditions

$$\begin{cases} g(t,B) = 0, & t \in [0,T], \\ \\ g(T,x) = 0, & x \leqslant K < B, \\ \\ g(T,x) = x - K, & K \leqslant x < B, \\ \\ g(T,x) = 0, & x > B. \end{cases}$$

Exercise 8.7 European knock-in/knock-out barrier options. Price the following vanilla options by computing their conditional discounted expected payoffs:

a) European knock-out barrier call option with payoff $(S_T - K)^+ \mathbb{1}_{\{S_T \leqslant B\}}$,
b) European knock-in barrier put option with payoff $(K - S_T)^+ \mathbb{1}_{\{S_T \leqslant B\}}$,
c) European knock-in barrier call option with payoff $(S_T - K)^+ \mathbb{1}_{\{S_T \geqslant B\}}$,
d) European knock-out barrier put option with payoff $(K - S_T)^+ \mathbb{1}_{\{S_T \geqslant B\}}$,

9. Lookback Options

Lookback call (resp. put) options are financial derivatives that allow their holders to exercise the option by setting the strike price at the minimum (resp. maximum) of the underlying asset price process $(S_t)_{t\in[0,T]}$ over the time interval $[0,T]$. Lookback options can be priced by PDE arguments or by computing the discounted expected values of their claim payoff C, namely $C = S_T - \min_{0\leqslant t\leqslant T} S_t$ in the case of call options, and $C = \mathop{\mathrm{Max}}_{0\leqslant t\leqslant T} S_t - S_T$ in the case of put options.

9.1 The Lookback Put Option

The standard lookback put option gives its holder the right to sell the underlying asset at its historically highest price. In this case, the floating strike price is M_0^T and the payoff is given by the terminal value

$$C = M_0^T - S_T$$

of the drawdown process $(M_0^t - S_t)_{t\in[0,T]}$. The following pricing formula for lookback put options is a direct consequence of Proposition 7.9.

Proposition 9.1 The price at time $t \in [0,T]$ of the lookback put option with payoff $M_0^T - S_T$ is given by

$$\begin{aligned} & e^{-(T-t)r}\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, \mathcal{F}_t\right] \\ = \quad & M_0^t e^{-(T-t)r}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_t\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\ & -S_t e^{-(T-t)r}\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right) - S_t, \end{aligned}$$

where $\delta_\pm^T(s)$ is defined in (8.2.2).

Proof. We have

$$\begin{aligned} \mathbb{E}^*\left[M_0^T - S_T \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}^*\left[M_0^T \,\middle|\, \mathcal{F}_t\right] - \mathbb{E}^*[S_T \mid \mathcal{F}_t] \\ &= \mathbb{E}^*\left[M_0^T \,\middle|\, \mathcal{F}_t\right] - e^{(T-t)r}S_t, \end{aligned}$$

hence Proposition 7.9 shows that

$$\begin{aligned} & e^{-(T-t)r}\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, \mathcal{F}_t\right] \\ & = e^{-(T-t)r}\mathbb{E}^*\left[M_0^T \,\middle|\, \mathcal{F}_t\right] - e^{-(T-t)r}\mathbb{E}^*[S_T \mid \mathcal{F}_t] \\ & = e^{-(T-t)r}\mathbb{E}^*\left[M_0^T \,\middle|\, M_0^t\right] - S_t \\ & = M_0^t e^{-(T-t)r}\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - S_t\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\ & + S_t\frac{\sigma^2}{2r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - S_t\frac{\sigma^2}{2r}e^{-(T-t)r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right). \end{aligned}$$

□

Figure 9.1 represents the lookback put option price as a function of S_t and M_0^t, for different values of the time to maturity $T-t$.

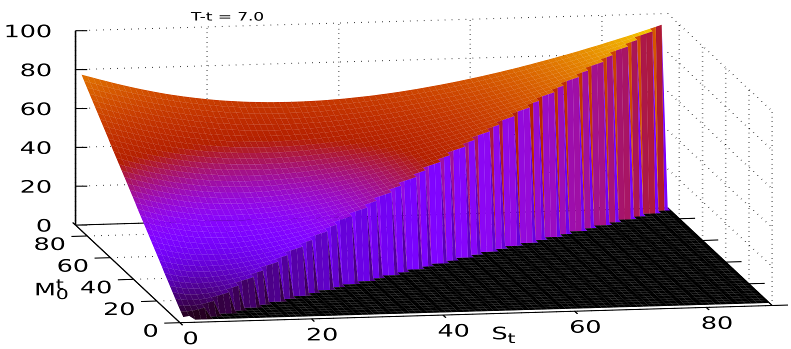


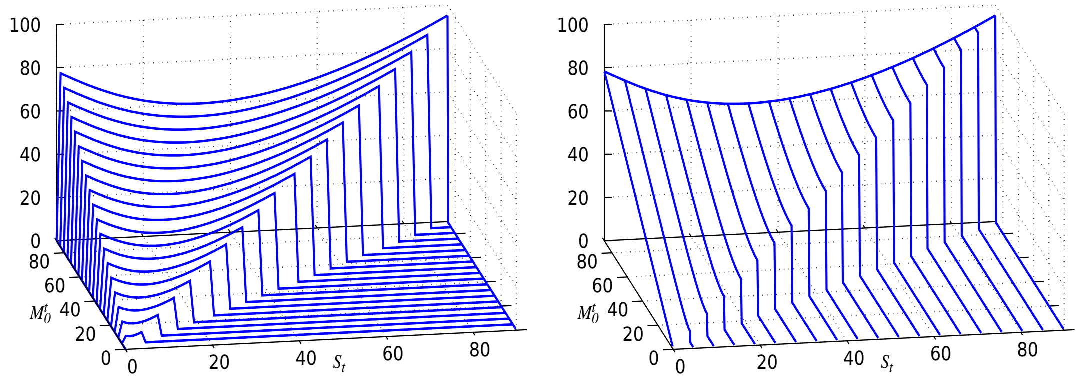
Figure 9.1: Graph of the lookback put option price (3D).*

From Figures 9.1 and 9.2, we make the following observations.

*The animation works in Acrobat Reader on the entire pdf file.

i) Close to maturity, if the underlying asset price S_t is close to M_0^t then an increase in the value S_t can result into a higher put option price, as in this case a variation of S_t may increase the value of M_0^t.

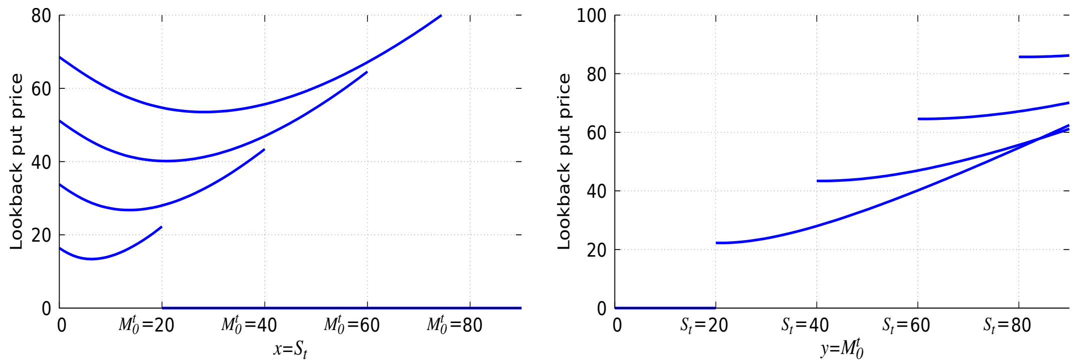
ii) When the underlying asset price S_t is far from M_0^t, an increase in S_t is less likely to affect the value of M_0^t when time t is close to maturity T, and this results into a lower option price.



(a) Put prices as functions of S. (b) Put prices as functions of M.

Figure 9.2: Graph of lookback put option prices.

Figures 9.2 and 9.3 show accordingly that, from the Delta hedging strategy for lookback put options, see Proposition 9.2 below, one should short the underlying asset when S_t is far from M_0^t, and long this asset when S_t becomes closer to M_0^t.



(a) Lookback put prices for fixed M_0^t. (b) Lookback put prices for fixed S_t.

Figure 9.3: Graph of lookback put option prices (2D).

9.2 PDE Method

Since the couple (S_t, M_0^t) is a Markov process, the price of the lookback put option at time $t \in [0,T]$ can be written as a function

$$
\begin{aligned}
f(t, S_t, M_0^t) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(S_T, M_0^T\right) \,\middle|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(S_T, M_0^T\right) \,\middle|\, S_t,\ M_0^t\right]
\end{aligned}
\tag{9.2.1}
$$

of S_t and M_0^t, $0 \leqslant t \leqslant T$.

Black-Scholes PDE for lookback put option prices

In the next proposition we derive the partial differential equation (PDE) for the pricing function $f(t,x,y)$ of a self-financing portfolio hedging a lookback put option. See Exercise 9.5 for the verification of the boundary conditions (9.2.3a)-(9.2.3c).

Proposition 9.2 The function $f(t,x,y)$ defined by

$$f(t,x,y) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_t = x,\ M_0^T = y\right], \quad t \in [0,T],\ x,y > 0,$$

is $\mathscr{C}^2((0,T)\times(0,\infty)^2)$ and satisfies the Black-Scholes PDE

$$rf(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + rx\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y), \tag{9.2.2}$$

$0 \leqslant t \leqslant T$, $x,y > 0$, subject to the boundary conditions

$$\begin{cases} f(t,0^+,y) = \mathrm{e}^{-(T-t)r}y, \qquad 0 \leqslant t \leqslant T, \quad y \geqslant 0, & (9.2.3a)\\[2ex] \dfrac{\partial f}{\partial y}(t,x,y)_{|y=x} = 0, \qquad 0 \leqslant t \leqslant T, \quad y > 0, & (9.2.3b)\\[2ex] f(T,x,y) = y - x, \qquad 0 \leqslant x \leqslant y. & (9.2.3c)\end{cases}$$

The replicating portfolio of the lookback put option is given by

$$\xi_t = \frac{\partial f}{\partial x}(t,S_t,M_0^t), \qquad t \in [0,T]. \tag{9.2.4}$$

Proof. The existence of $f(t,x,y)$ follows from the Markov property, more precisely, from the time homogeneity of the asset price process $(S_t)_{t\in\mathbb{R}_+}$ the function $f(t,x,y)$ satisfies

$$\begin{aligned} f(t,x,y) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(S_T,M_0^T\right) \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(x\frac{S_T}{S_t}, \mathrm{Max}\left(y, M_t^T\right)\right)\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(x\frac{S_{T-t}}{S_0}, \mathrm{Max}\left(y, M_0^{T-t}\right)\right)\right], \quad t \in [0,T]. \end{aligned}$$

Applying the change of variable formula to the discounted portfolio value

$$\widetilde{f}(t,x,y) := \mathrm{e}^{-rt}f(t,x,y) = \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi(S_T,M_0^T) \,\middle|\, S_t = x,\ M_0^t = y\right]$$

which is a martingale indexed by $t \in [0,T]$, we have

$$\begin{aligned} d\widetilde{f}(t,S_t,M_0^t) &= -r\mathrm{e}^{-rt}f(t,S_t,M_0^t)dt + \mathrm{e}^{-rt}df(t,S_t,M_0^t) \\ &= -r\mathrm{e}^{-rt}f(t,S_t,M_0^t)dt + \mathrm{e}^{-rt}\frac{\partial f}{\partial t}(t,S_t,M_0^t)dt + r\mathrm{e}^{-rt}S_t\frac{\partial f}{\partial x}(t,S_t,M_0^t)dt \\ &\quad + \mathrm{e}^{-rt}\sigma S_t\frac{\partial f}{\partial x}(t,S_t,M_0^t)dB_t + \frac{1}{2}\mathrm{e}^{-rt}\sigma^2S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t,M_0^t)dt \\ &\quad + \mathrm{e}^{-rt}\frac{\partial f}{\partial y}(t,S_t,M_0^t)dM_0^t, \end{aligned} \tag{9.2.5}$$

according to the following extension of the Itô multiplication table 9.1.

$\cdot$	dt	dB_t	dM_0^t
dt	0	0	0
dB_t	0	dt	0
dM_0^t	0	0	0

Table 9.1: Extended Itô multiplication table.

Since $\big(\widetilde{f}(t,S_t,M_0^t)\big)_{t\in[0,T]} = \big(\mathrm{e}^{-rT}\,\mathbb{E}^*\big[\phi(S_T,M_0^T)\,\big|\,\mathscr{F}_t\big]\big)_{t\in[0,T]}$ is a martingale under $\mathbb{P}$ and $\big(M_0^t\big)_{t\in[0,T]}$ has finite variation (it is in fact a non-decreasing process), (9.2.5) yields:

$$d\widetilde{f}(t,S_t,M_0^t) = \sigma S_t \frac{\partial \widetilde{f}}{\partial x}(t,S_t,M_0^t)dB_t, \qquad t\in[0,T], \tag{9.2.6}$$

and the function $f(t,x,y)$ satisfies the equation

$$\begin{aligned}\frac{\partial f}{\partial t}(t,S_t,M_0^t)dt + rS_t\frac{\partial f}{\partial x}f(t,S_t,M_0^t)dt \\ +\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t,M_0^t)dt + \frac{\partial f}{\partial y}(t,S_t,M_0^t)dM_0^t = rf(t,S_t,M_0^t)dt,\end{aligned} \tag{9.2.7}$$

which implies

$$\frac{\partial f}{\partial t}(t,S_t,M_0^t) + rS_t\frac{\partial f}{\partial x}(t,S_t,M_0^t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t,M_0^t) = rf(t,S_t,M_0^t),$$

which is (9.2.2), and

$$\frac{\partial f}{\partial y}(t,S_t,M_0^t)dM_0^t = 0.$$

Indeed, M_0^t increases only on a set of zero Lebesgue measure (which has no isolated points), therefore the Lebesgue measure dt and the measure dM_0^t are mutually *singular*, hence by the Lebesgue decomposition theorem, both components in dt and dM_0^t should vanish in (9.2.7) if the sum vanishes, see also the Cantor function. This implies

$$\frac{\partial f}{\partial y}(t,S_t,M_0^t) = 0,$$

when $dM_0^t > 0$, hence since

$$\{S_t = M_0^t\} \quad \Longleftrightarrow dM_0^t > 0$$

and

$$\{S_t < M_0^t\} \quad \Longleftrightarrow dM_0^t = 0,$$

we have

$$\frac{\partial f}{\partial y}(t,S_t,S_t) = \frac{\partial f}{\partial y}(t,x,y)_{x=S_t,\ y=S_t} = 0,$$

since M_0^t hits S_t, *i.e.* $M_0^t = S_t$, only when M_0^t increases at time t, and this shows the boundary condition (9.2.3b).

On the other hand, (9.2.6) shows that

$$\phi(S_T,M_0^T) = \mathbb{E}^*[\phi(S_T,M_0^T)] + \sigma\int_0^T S_t\frac{\partial f}{\partial x}(t,x,M_0^t)_{|x=S_t}dB_t,$$

$0\leqslant t\leqslant T$, which implies (9.2.4) as in the proof of Proposition 8.3. □

In other words, the price of the lookback put option takes the form

$$f(t,S_t,M_0^t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, \mathcal{F}_t\right],$$

where the function $f(t,x,y)$ is given from Proposition 9.1 as

$$\begin{aligned} f(t,x,y) &= y\,\mathrm{e}^{-(T-t)r}\Phi\big(-\delta_-^{T-t}(x/y)\big) + x\left(1+\frac{\sigma^2}{2r}\right)\Phi\big(\delta_+^{T-t}(x/y)\big) \\ &\quad - x\frac{\sigma^2}{2r}\mathrm{e}^{-(T-t)r}\left(\frac{y}{x}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(y/x)\right) - x. \end{aligned} \tag{9.2.8}$$

Remark 9.3 We have

$$f(t,x,x) = xC(T-t),$$

with

$$\begin{aligned} C(\tau) &= \mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau(1)\right) + \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^\tau(1)\right) - \frac{\sigma^2}{2r}\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau(1)\right) - 1 \\ &= \mathrm{e}^{-r\tau}\Phi\left(-\frac{r-\sigma^2/2}{\sigma}\sqrt{\tau}\right) + \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{\tau}\right) \\ &\quad - \frac{\sigma^2}{2r}\mathrm{e}^{-r\tau}\Phi\left(-\frac{r-\sigma^2/2}{\sigma}\sqrt{\tau}\right) - 1, \qquad \tau > 0, \end{aligned}$$

hence

$$\frac{\partial f}{\partial x}(t,x,x) = C(T-t), \qquad t \in [0,T].$$

Scaling property of lookback put option prices

From (9.2.8) and the following argument we note the scaling property

$$\begin{aligned} f(t,x,y) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\mathrm{Max}\left(M_0^t, M_t^T\right) - S_T \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}x\,\mathbb{E}^*\left[\mathrm{Max}\left(\frac{M_0^t}{S_t}, \frac{M_t^T}{S_t}\right) - \frac{S_T}{S_t} \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}x\,\mathbb{E}^*\left[\mathrm{Max}\left(\frac{y}{x}, \frac{M_t^T}{x}\right) - \frac{S_T}{x} \,\middle|\, S_t = x,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}x\,\mathbb{E}^*\left[\mathrm{Max}\left(M_0^t, M_t^T\right) - S_T \,\middle|\, S_t = 1,\ M_0^t = \frac{y}{x}\right] \\ &= \mathrm{e}^{-(T-t)r}x\,\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_t = 1,\ M_0^t = \frac{y}{x}\right] \\ &= xf(t,1,y/x) \\ &= xg(T-t,x/y), \end{aligned}$$

where we let

$$g(\tau,z) :=$$

$$\frac{1}{z}\,\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau(z)\right)+\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^\tau(z)\right)-\frac{\sigma^2}{2r}\,\mathrm{e}^{-r\tau}\left(\frac{1}{z}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right)-1,$$

with the boundary condition

$$\begin{cases}\dfrac{\partial g}{\partial z}(\tau,1)=0, & \tau>0, \qquad (9.2.9a)\\[2ex] g(0,z)=\dfrac{1}{z}-1, & z\in(0,1]. \qquad (9.2.9b)\end{cases}$$

The next Figure 9.4 shows a graph of the function $g(\tau,z)$.

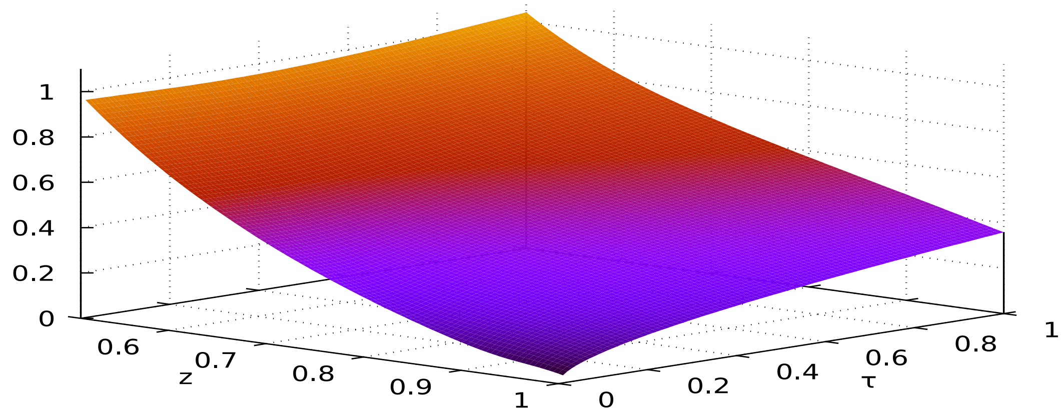


Figure 9.4: Graph of the normalized lookback put option price.

Black-Scholes approximation of lookback put option prices

Letting

$$\mathrm{Bl_p}(x,K,r,\sigma,\tau):=K\,\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau\left(\frac{x}{K}\right)\right)-x\Phi\left(-\delta_+^\tau\left(\frac{x}{K}\right)\right)$$

denote the standard Black-Scholes formula for the price of the European put option.

Proposition 9.4 The lookback put option price can be rewritten as

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^T-S_T\,\middle|\,\mathcal{F}_t\right]=\mathrm{Bl_p}(S_t,M_0^t,r,\sigma,T-t) \qquad (9.2.10)$$
$$+S_t\frac{\sigma^2}{2r}\left(\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)-\mathrm{e}^{-(T-t)r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right).$$

In other words, we have

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^T-S_T\,\middle|\,\mathcal{F}_t\right]=\mathrm{Bl_p}(S_t,M_0^t,r,\sigma,T-t)+S_t h_p\left(T-t,\frac{S_t}{M_0^t}\right)$$

where the function

$$h_p(\tau,z)=\frac{\sigma^2}{2r}\Phi\left(\delta_+^\tau(z)\right)-\frac{\sigma^2}{2r}\,\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right), \qquad (9.2.11)$$

depends only on time τ and $z=S_t/M_0^t$. In other words, due to the relation

$$\mathrm{Bl_p}(x,y,r,\sigma,\tau) \;=\; y\,\mathrm{e}^{-r\tau}\Phi\left(-\delta_-^\tau\left(\frac{x}{y}\right)\right)-x\Phi\left(-\delta_+^\tau\left(\frac{x}{y}\right)\right)$$

$$= \quad x\mathrm{Bl_p}(1, y/x, r, \sigma, \tau)$$

for the standard Black-Scholes put option price formula, we observe that $f(t,x,y)$ satisfies

$$f(t,x,y) = x\mathrm{Bl_p}\left(1, \frac{y}{x}, r, \sigma, T-t\right) + xh\left(T-t, \frac{x}{y}\right),$$

i.e.

$$f(t,x,y) = xg\left(T-t, \frac{x}{y}\right),$$

with

$$g(\tau, z) = \mathrm{Bl_p}\left(1, \frac{1}{z}, r, \sigma, \tau\right) + h_p(\tau, z), \tag{9.2.12}$$

where the function $h_p(\tau,z)$ is a correction term given by (9.2.11) which is small when $z = x/y$ or τ become small.

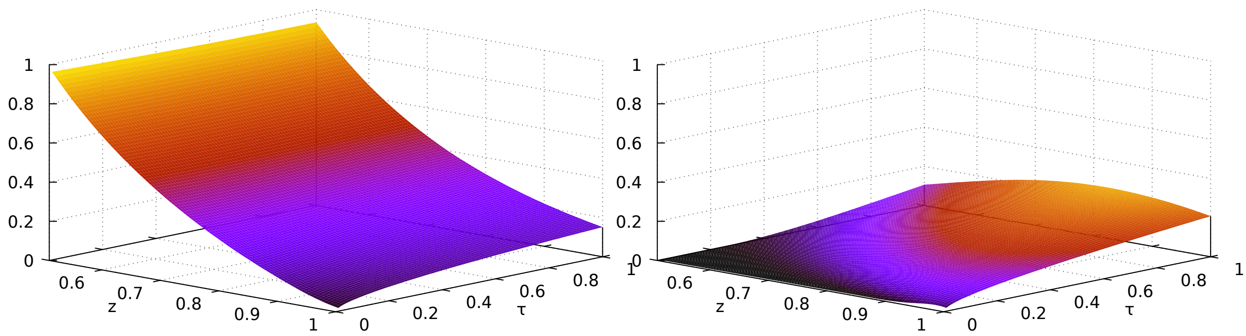
Note that $(x,y) \longmapsto xh_p(T-t, x/y)$ also satisfies the Black-Scholes PDE (9.2.2), in particular $(\tau,z) \longmapsto \mathrm{Bl_p}(1, 1/z, r, \sigma, \tau)$ and $h_p(\tau,z)$ both satisfy the PDE

$$\frac{\partial h_p}{\partial \tau}(\tau,z) = z\left(r+\sigma^2\right)\frac{\partial h_p}{\partial z}(\tau,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 h_p}{\partial z^2}(\tau,z), \tag{9.2.13}$$

$\tau \in [0,T]$, $z \in [0,1]$, subject to the boundary condition

$$h_p(0,z) = 0, \qquad 0 \leqslant z \leqslant 1.$$

The next Figure 9.5b illustrates the decomposition (9.2.12) of the normalized lookback put option price $g(\tau,z)$ in Figure 9.4 into the Black-Scholes put price function $\mathrm{Bl_p}(1, 1/z, r, \sigma, \tau)$ and $h_p(\tau,z)$.



(a) Black-Scholes put price. (b) Correction term $h_p(\tau,z)$.

Figure 9.5: Normalized Black-Scholes put price and correction term in (9.2.12).

Note that in Figure 9.5b the condition $h_p(0,z) = 0$ is not fully respected as z tends to 1, due to numerical instabilities in the approximation of the function Φ.

9.3 The Lookback Call Option

The standard Lookback call option gives the right to buy the underlying asset at its historically lowest price. In this case, the floating strike price is m_0^T and the payoff is

$$C = S_T - m_0^T.$$

The following result gives the price of the lookback call option, cf. *e.g.* Proposition 9.5.1, page 270 of Dana and Jeanblanc, 2007.

Proposition 9.5 The price at time $t \in [0,T]$ of the lookback call option with payoff $S_T - m_0^T$ is given by

$$\begin{aligned} & \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[S_T - m_0^T \mid \mathscr{F}_t\right] \\ & = S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - m_0^t\,\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\ & \quad + \mathrm{e}^{-(T-t)r}S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - S_t\frac{\sigma^2}{2r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right). \end{aligned}$$

Proof. By Proposition 7.10 we have

$$\begin{aligned} & \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[S_T - m_0^T \mid \mathscr{F}_t\right] = S_t - \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[m_0^T \mid \mathscr{F}_t\right] \\ & = S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \mathrm{e}^{-(T-t)r}m_0^t\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\ & \quad + \mathrm{e}^{-(T-t)r}\frac{S_t\sigma^2}{2r}\left(\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - \mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\right). \end{aligned}$$

□

Figure 9.6 represents the price of the lookback call option as a function of m_0^t and S_t for different values of the time to maturity $T-t$.

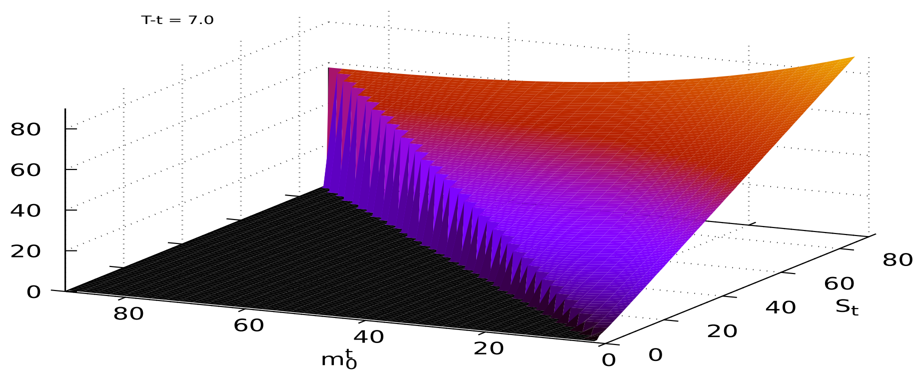


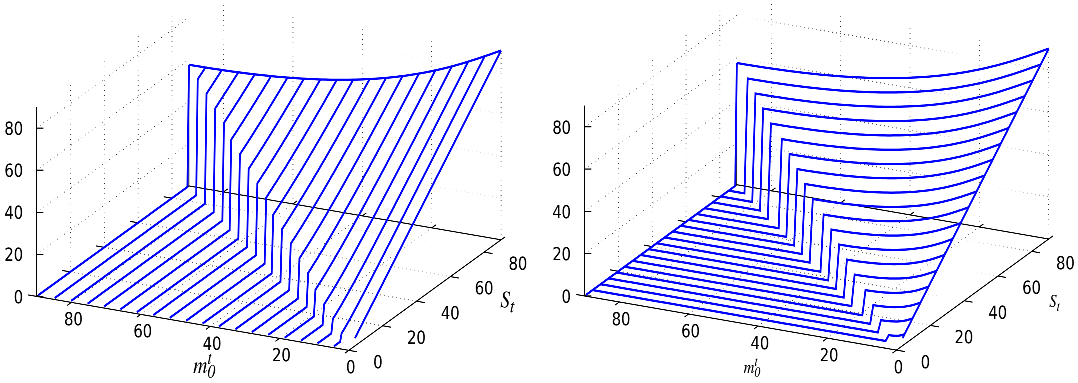
Figure 9.6: Graph of the lookback call option price.*

From Figures 9.6 and 9.7, we note the following.

i) When the underlying asset price S_t is far from m_0^t, an increase in the value S_t clearly results into a higher call option price.

ii) When the underlying asset price S_t is close to m_0^t, a decrease in S_t could lead to a decrease in the value of m_0^t, however on average this appears insufficient to increase the average option payoff.

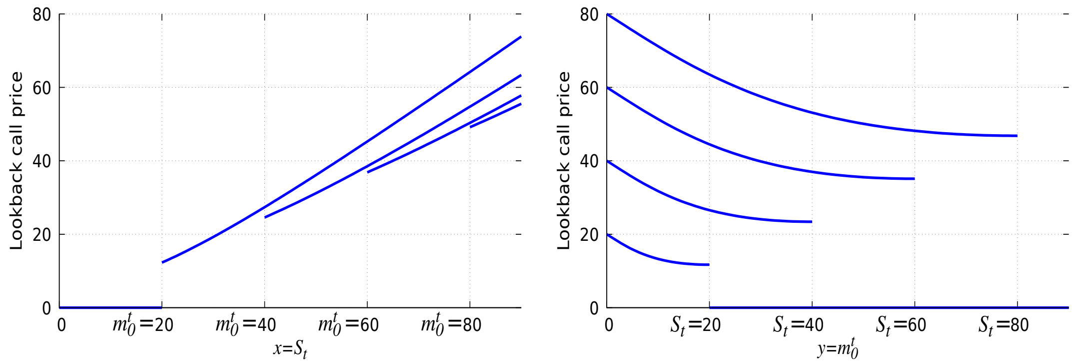
*The animation works in Acrobat Reader on the entire pdf file.



(a) Call prices as functions of S. (b) Call prices as functions of m.

Figure 9.7: Graph of lookback call option prices.

Figures 9.7 and 9.8 show accordingly that, from the Delta hedging strategy for lookback call options, see Propositions 9.6 and 9.8, one should long the underlying asset in order to hedge a lookback call option.



(a) Lookback call prices for fixed m_0^t. (b) Lookback call prices for fixed S_t.

Figure 9.8: Graphs of lookback call option prices (2D).

Black-Scholes PDE for lookback call option prices

Since the couple (S_t, m_0^t) is also a Markov process, the price of the lookback call option at time $t \in [0,T]$ can be written as a function

$$\begin{aligned} f(t, S_t, m_0^t) &= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[\phi\left(S_T, m_0^T\right) \mid \mathscr{F}_t\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[\phi\left(S_T, m_0^T\right) \mid S_t,\ m_0^t\right] \end{aligned}$$

of S_t and m_0^t, $0 \leqslant t \leqslant T$. By the same argument as in the proof of Proposition 9.2, we obtain the following result.

Proposition 9.6 The function $f(t,x,y)$ defined by

$$f(t,x,y) = \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[S_T - m_0^T \mid S_t = x,\ m_0^t = y\right], \quad t \in [0,T],\ x,y > 0,$$

is $\mathscr{C}^2((0,T) \times (0,\infty)^2)$ and satisfies the Black-Scholes PDE

$$rf(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + rx\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y),$$

$0 \leqslant t \leqslant T$, $x > 0$, subject to the boundary conditions

$$\begin{cases} \lim_{y \searrow 0} f(t,x,y) = x, \qquad 0 \leqslant t \leqslant T, \quad x > 0, & (9.3.1a) \\ \\ \frac{\partial f}{\partial y}(t,x,y)_{y=x} = 0, \qquad 0 \leqslant t \leqslant T, \quad y > 0, & (9.3.1b) \\ \\ f(T,x,y) = x - y, \qquad 0 < y \leqslant x, & (9.3.1c) \end{cases}$$

and the corresponding self-financing hedging strategy is given by

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, m_0^t), \qquad t \in [0,T], \tag{9.3.2}$$

which represents the quantity of the risky asset S_t to be held at time t in the hedging portfolio.

In other words, the price of the lookback call option takes the form

$$f(t, S_t, m_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[S_T - m_0^T \mid \mathscr{F}_t \right],$$

where the function $f(t,x,y)$ is given by

$$\begin{aligned} f(t,x,y) &= x\Phi\left(\delta_+^{T-t}\left(\frac{x}{y}\right)\right) - \mathrm{e}^{-(T-t)r} y \Phi\left(\delta_-^{T-t}\left(\frac{x}{y}\right)\right) \\ &\quad + \mathrm{e}^{-(T-t)r} x \frac{\sigma^2}{2r}\left(\left(\frac{y}{x}\right)^{2r/\sigma^2} \Phi\left(\delta_-^{T-t}\left(\frac{y}{x}\right)\right) - \mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right)\right) \\ &= x - y\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{x}{y}\right)\right) - x\left(1 + \frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right) \\ &\quad + x\mathrm{e}^{-(T-t)r}\frac{\sigma^2}{2r}\left(\frac{y}{x}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{y}{x}\right)\right). \end{aligned} \tag{9.3.3}$$

Scaling property of lookback call option prices

We note the scaling property

$$\begin{aligned} f(t,x,y) &= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[S_T - m_0^T \mid S_t = x, \ m_0^t = y \right] \\ &= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[S_T - \min\left(m_0^t, m_t^T\right) \mid S_t = x, \ m_0^t = y \right] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^* \left[\frac{S_T}{S_t} - \min\left(\frac{m_0^t}{S_t}, \frac{m_t^T}{S_t}\right) \middle| S_t = x, \ m_0^t = y \right] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^* \left[\frac{S_T}{x} - \min\left(\frac{y}{x}, \frac{m_t^T}{x}\right) \middle| S_t = x, \ m_0^t = y \right] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^* \left[S_T - \min\left(m_0^t, m_t^T\right) \middle| S_t = 1, \ m_0^t = \frac{y}{x} \right] \\ &= \mathrm{e}^{-(T-t)r} x \mathbb{E}^* \left[S_T - m_0^T \middle| S_t = 1, \ m_0^t = \frac{y}{x} \right] \\ &= x f(t, 1, y/x) \end{aligned}$$

$$= xg\left(T-t,\frac{1}{z}\right),$$

where

$$g(\tau,z) :=$$
$$1-\frac{1}{z}\,\mathrm{e}^{-r\tau}\Phi\left(\delta_-^\tau(z)\right)-\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^\tau(z)\right)+\frac{\sigma^2}{2r}\,\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right),$$

with $g(\tau,1) = C(T-t)$, and

$$f(t,x,y) = xg\left(T-t,\frac{x}{y}\right)$$

and the boundary condition

$$\begin{cases} \dfrac{\partial g}{\partial z}(\tau,1) = 0, & \tau > 0, \quad (9.3.4a) \\ \\ g(0,z) = 1-\dfrac{1}{z}, & z \geqslant 1. \quad (9.3.4b) \end{cases}$$

The next Figure 9.9 shows a graph of the function $g(\tau,z)$.

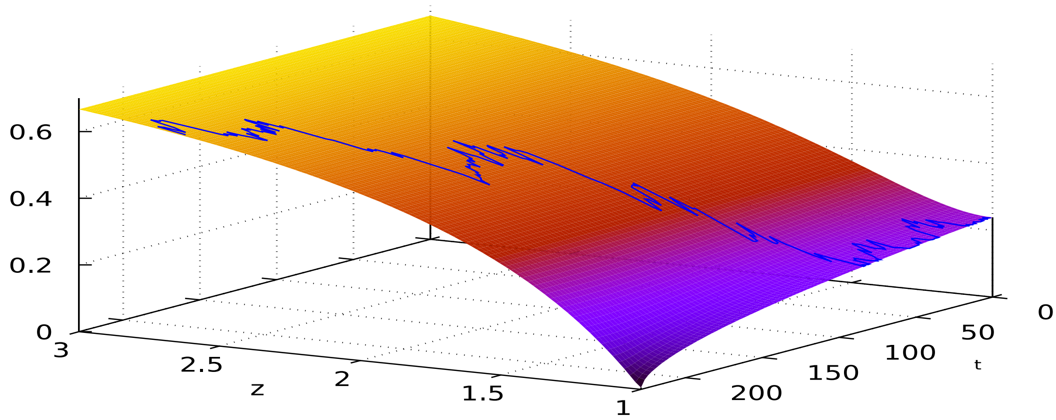


Figure 9.9: Normalized lookback call option price.

The next Figure 9.10 represents the path of the underlying asset price used in Figure 9.9.

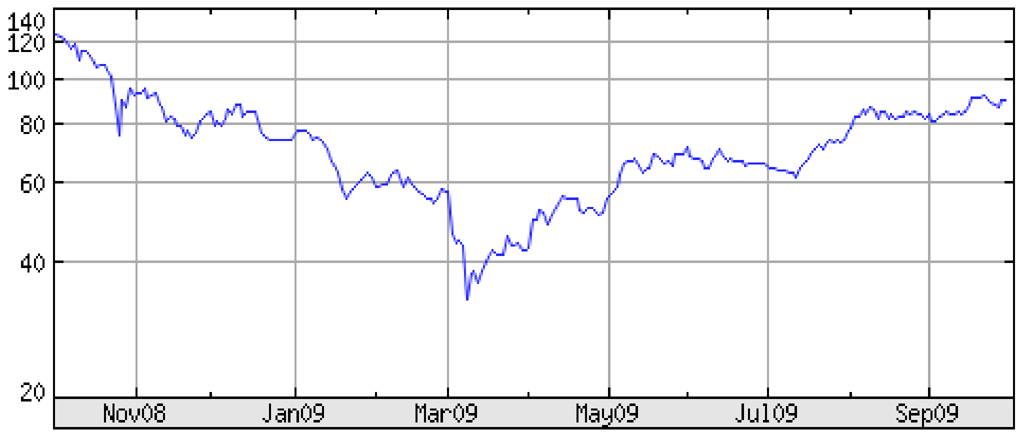


Figure 9.10: Graph of underlying asset prices.

The next Figure 9.11 represents the corresponding underlying asset price and its running minimum.

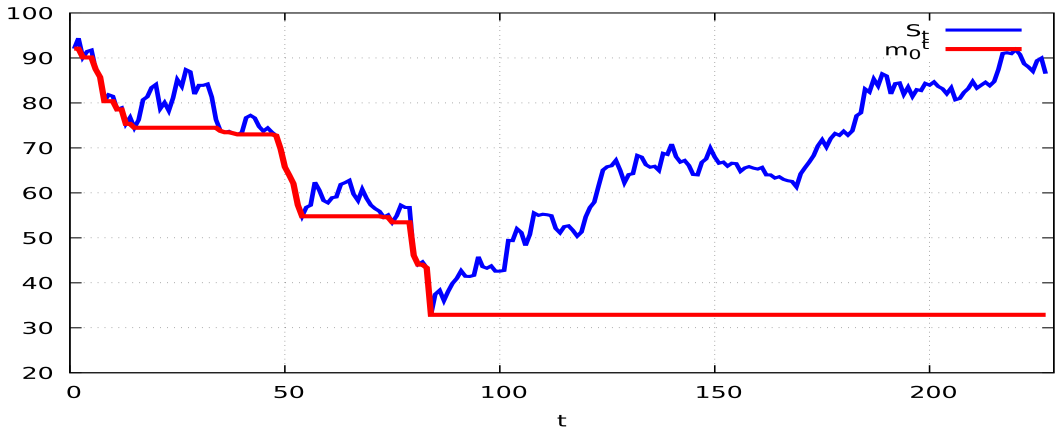


Figure 9.11: Running minimum of the underlying asset price.

Next, we represent the option price as a function of time, together with the process $(S_t - m_0^t)_{t\in\mathbb{R}_+}$.

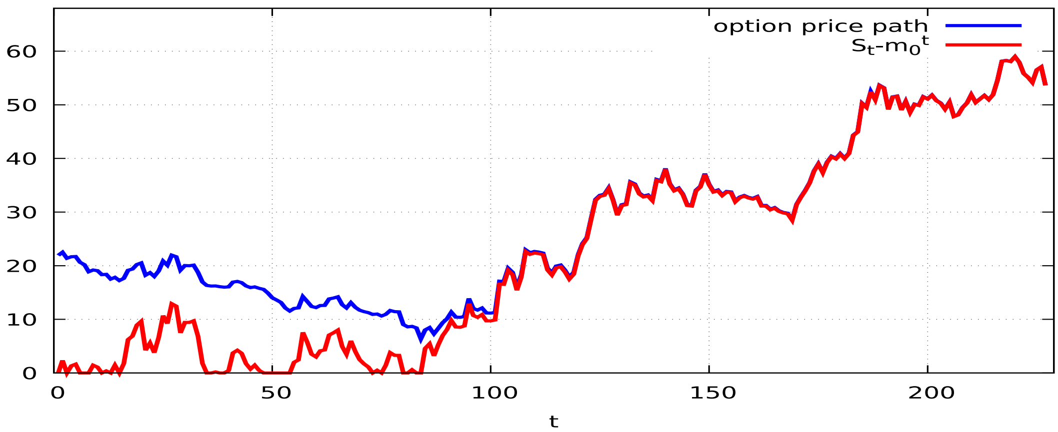


Figure 9.12: Graph of the lookback call option price.

Black-Scholes approximation of lookback call option prices

Let

$$\mathrm{Bl}_{\mathrm{c}}(S, K, r, \sigma, \tau) = S\Phi\left(\delta_+^\tau\left(\frac{S}{K}\right)\right) - K\,\mathrm{e}^{-r\tau}\Phi\left(\delta_-^\tau\left(\frac{S}{K}\right)\right)$$

denote the standard Black-Scholes formula for the price of the European call option.

Proposition 9.7 The lookback call option price can be rewritten as

$$\begin{aligned} \mathrm{e}^{-(T-t)r}\,\mathbf{E}^*\left[S_T - m_0^T \mid \mathcal{F}_t\right] &= \mathrm{Bl}_{\mathrm{c}}(S_t, m_0^t, r, \sigma, T-t) \\ &-S_t\frac{\sigma^2}{2r}\left(\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \mathrm{e}^{-(T-t)r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)\right). \end{aligned} \tag{9.3.5}$$

In other words, we have

$$\mathrm{e}^{-(T-t)r}\,\mathbf{E}^*\left[S_T - m_0^T \mid \mathcal{F}_t\right] := \mathrm{Bl}_{\mathrm{c}}(S_t, m_0^t, r, \sigma, T-t) + S_t h_c\left(T-t, \frac{S_t}{m_0^t}\right)$$

where the correction term

$$h_c(\tau,z) = -\frac{\sigma^2}{2r}\left(\Phi\left(-\delta_+^\tau(z)\right) - \mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right), \tag{9.3.6}$$

is small when $z = S_t/m_0^t$ becomes large or τ becomes small. In addition, $h_p(\tau,z)$ is linked to $h_c(\tau,z)$ by the relation

$$h_c(\tau,z) = h_p(\tau,z) - \frac{\sigma^2}{2r}\left(1 - \mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\right), \qquad \tau \geqslant 0, \quad z \geqslant 0,$$

where $(z,\tau) \longmapsto \mathrm{e}^{-r\tau}z^{-2r/\sigma^2}$ also solves the PDE (9.2.13). Due to the relation

$$\begin{aligned}\mathrm{Bl_c}(x,y,r,\sigma,\tau) &= x\Phi\left(\delta_+^\tau\left(\frac{x}{y}\right)\right) - y\mathrm{e}^{-r\tau}\Phi\left(\delta_-^\tau\left(\frac{x}{y}\right)\right)\\ &= x\mathrm{Bl_c}\left(1,\frac{y}{x},r,\sigma,\tau\right)\end{aligned}$$

for the standard Black-Scholes call price formula, recall that from Proposition 9.7, $f(t,x,y)$ can be decomposed as

$$f(t,x,y) = x\mathrm{Bl_c}\left(1,\frac{y}{x},r,\sigma,T-t\right) + xh_c\left(T-t,\frac{x}{y}\right),$$

where $h_c(\tau,z)$ is the function given by (9.3.6), *i.e.*

$$f(t,x,y) = xg\left(T-t,\frac{x}{y}\right),$$

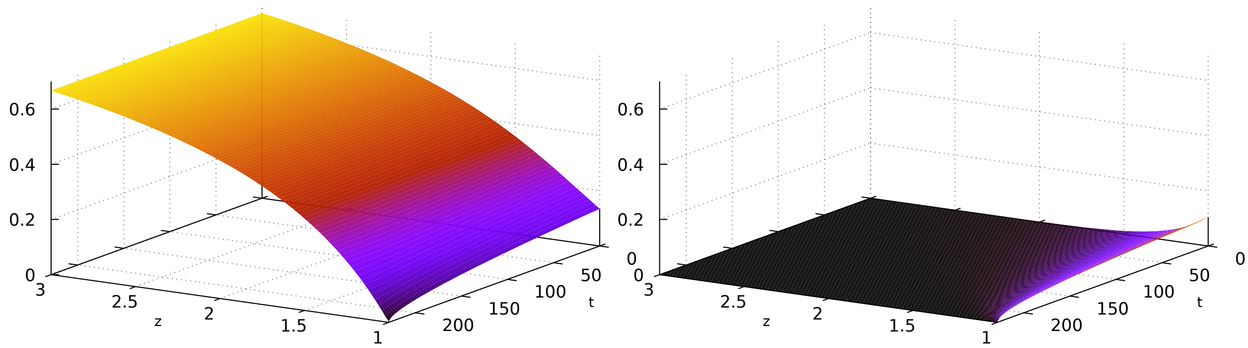
with

$$g(\tau,z) = \mathrm{Bl_c}\left(1,\frac{1}{z},r,\sigma,\tau\right) + h_c(\tau,z), \tag{9.3.7}$$

where $(x,y) \longmapsto xh_c(T-t,x/y)$ also satisfies the Black-Scholes PDE (9.2.2), *i.e.* $(\tau,z) \longmapsto \mathrm{Bl_c}(1,1/z,r,\sigma,\tau)$ and $h_c(\tau,z)$ both satisfy the PDE (9.2.13) subject to the boundary condition

$$h_c(0,z) = 0, \qquad z \geqslant 1.$$

The next Figures 9.13a and 9.13b show the decomposition of $g(t,z)$ in (9.3.7) and Figures 9.9-9.10 into the sum of the Black-Scholes call price function $\mathrm{Bl_c}(1,1/z,r,\sigma,\tau)$ and $h(t,z)$.



(a) Black-Scholes call price $g(\tau,z)$. (b) Correction term $h_c(\tau,z)$.

Figure 9.13: Normalized Black-Scholes call price and correction term in (9.3.7).

We also note that

$$\begin{aligned}
\mathbb{E}^*\left[M_0^T - m_0^T \mid S_0 = x\right] &= x - x\mathrm{e}^{-(T-t)r}\Phi\big(\delta_-^{T-t}(1)\big) \\
&\quad - x\left(1+\frac{\sigma^2}{2r}\right)\Phi\big(-\delta_+^{T-t}(1)\big) + x\mathrm{e}^{-(T-t)r}\frac{\sigma^2}{2r}\Phi\big(\delta_-^{T-t}(1)\big) \\
&\quad + x\mathrm{e}^{-(T-t)r}\Phi\big(-\delta_-^{T-t}(1)\big) + x\left(1+\frac{\sigma^2}{2r}\right)\Phi\big(\delta_+^{T-t}(1)\big) \\
&\quad - x\frac{\sigma^2}{2r}\mathrm{e}^{-(T-t)r}\Phi\big(-\delta_-^{T-t}(1)\big) - x \\
&= x\left(1+\frac{\sigma^2}{2r}\right)\Big(\Phi\big(\delta_+^{T-t}(1)\big) - \Phi\big(-\delta_+^{T-t}(1)\big)\Big) \\
&\quad + x\mathrm{e}^{-(T-t)r}\left(\frac{\sigma^2}{2r}-1\right)\big(\Phi\big(\delta_-^{T-t}(1)\big) - \Phi\big(-\delta_-^{T-t}(1)\big)\big).
\end{aligned}$$

9.4 Delta Hedging for Lookback Options

In this section we compute hedging strategies for lookback call and put options by application of the Delta hedging formula (9.3.2). See Bermin, 1998, § 2.6.1, page 29, for another approach to the following result using the Clark-Ocone formula. Here we use (9.3.2) instead, cf. Proposition 4.6 of El Khatib and Privault, 2003.

Proposition 9.8 The Delta hedging strategy of the lookback call option is given by

$$\begin{aligned}
\xi_t &= 1 - \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\
&\quad + \mathrm{e}^{-(T-t)r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\left(\frac{\sigma^2}{2r}-1\right)\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right), \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{9.4.1}$$

Proof. By (9.3.2) and (9.3.5), we need to differentiate

$$f(t,x,y) = \mathrm{Bl}_\mathrm{c}(x,y,r,\sigma,T-t) + xh_c\left(T-t,\frac{x}{y}\right)$$

with respect to the variable x, where

$$h_c(\tau,z) = -\frac{\sigma^2}{2r}\left(\Phi\left(-\delta_+^\tau(z)\right) - \mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right)$$

is given by (9.3.6) First, we note that the relation

$$\frac{\partial}{\partial x}\mathrm{Bl}_c(x,y,r,\sigma,\tau) = \Phi\left(\delta_+^\tau\left(\frac{x}{y}\right)\right)$$

is known. Next, we have

$$\frac{\partial}{\partial x}\left(xh_c\left(\tau,\frac{x}{y}\right)\right) = h_c\left(\tau,\frac{x}{y}\right) + \frac{x}{y}\frac{\partial h_c}{\partial z}\left(\tau,\frac{x}{y}\right),$$

and

$$\frac{\partial h_c}{\partial z}(\tau,z) = -\frac{\sigma^2}{2r}\left(\frac{\partial}{\partial z}\left(\Phi\left(-\delta_+^\tau(z)\right)\right) - \mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\frac{\partial}{\partial z}\left(\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right)\right)$$

$$
\begin{aligned}
&-\frac{\sigma^2}{2r}\left(\frac{2r}{\sigma^2}\,\mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right)\\
&=\frac{\sigma}{2rz\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}(\delta_+^\tau(z))^2\right)\\
&-\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\frac{\sigma}{2rz\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)^2\right)-\frac{2r}{\sigma^2}\,\mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right).
\end{aligned}
$$

Next, we note that

$$
\begin{aligned}
\mathrm{e}^{-(\delta_-^\tau(1/z))^2/2} &= \exp\left(-\frac{1}{2}(\delta_+^\tau(z))^2-\frac{1}{2}\left(\frac{4r^2}{\sigma^2}\tau-\frac{4r}{\sigma}\delta_+^\tau(z)\sqrt{\tau}\right)\right)\\
&= \mathrm{e}^{-\left(\delta_+^\tau(z)\right)^2/2}\exp\left(-\frac{1}{2}\left(\frac{4r^2}{\sigma^2}\tau-\frac{4r}{\sigma^2}\left(\log z+\left(r+\frac{1}{2}\sigma^2\right)\tau\right)\right)\right)\\
&= \mathrm{e}^{-\left(\delta_+^\tau(z)\right)^2/2}\exp\left(\frac{-2r^2}{\sigma^2}\tau+\frac{2r}{\sigma^2}\log z+\frac{2r^2}{\sigma^2}\tau+r\tau\right)\\
&= \mathrm{e}^{r\tau}z^{2r/\sigma^2}\,\mathrm{e}^{-(\delta_+^\tau(z))^2/2},
\end{aligned}
\tag{9.4.2}
$$

hence

$$
\frac{\partial h_c}{\partial z}\left(\tau,\frac{x}{y}\right)=-\,\mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right),
$$

and

$$
\frac{\partial}{\partial x}\left(xh_c\left(\tau,\frac{x}{y}\right)\right)=h_c\left(\tau,\frac{x}{y}\right)-\mathrm{e}^{-r\tau}\left(\frac{y}{x}\right)^{2r/\sigma^2}\Phi\left(\delta_-^\tau\left(\frac{y}{x}\right)\right),
$$

which concludes the proof. □

We note that $\xi_t = 1 > 0$ as T tends to infinity, and that at maturity $t = T$, the delta hedging strategy satisfies

$$
\xi_T=\begin{cases}
1 & \text{if } m_0^T < S_T,\\[2ex]
1-\dfrac{1}{2}\left(1+\dfrac{\sigma^2}{2r}\right)+\dfrac{1}{2}\left(\dfrac{\sigma^2}{2r}-1\right)=0 & \text{if } m_0^T = S_T.
\end{cases}
$$

In Figure 9.14 we represent the Delta of the lookback call option, as given by (9.4.1).

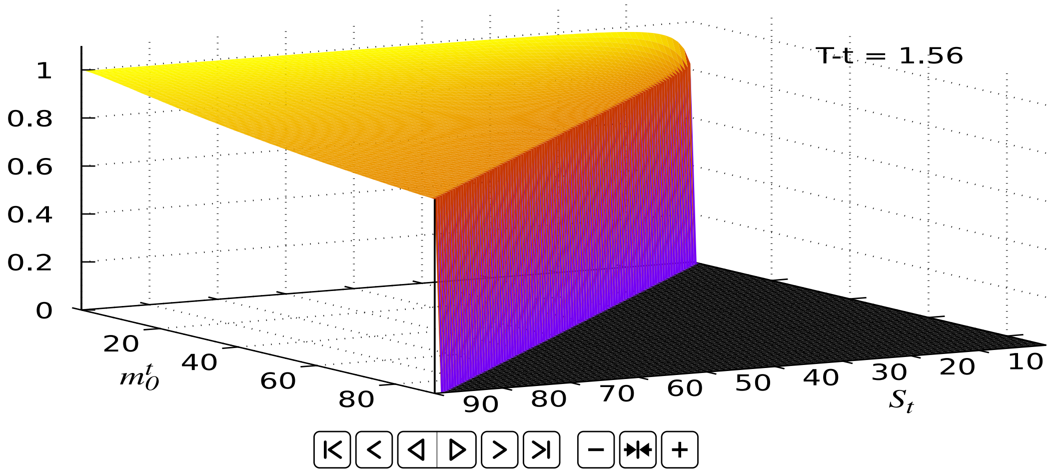


Figure 9.14: Delta of the lookback call option with $r = 2\%$ and $\sigma = 0.41$.*

*The animation works in Acrobat Reader on the entire pdf file.

The above scaling procedure can be applied to the Delta of lookback call options by noting that ξ_t can be written as

$$\xi_t = \zeta\left(t, \frac{S_t}{m_0^t}\right),$$

where the function $\zeta(t,z)$ is given by

$$\begin{aligned}\zeta(t,z) &= \Phi\left(\delta_+^{T-t}(z)\right) - \frac{\sigma^2}{2r}\Phi\left(-\delta_+^{T-t}(z)\right) \\ &\quad + \mathrm{e}^{-(T-t)r} z^{-2r/\sigma^2}\left(\frac{\sigma^2}{2r} - 1\right)\Phi\left(\delta_-^{T-t}\left(\frac{1}{z}\right)\right),\end{aligned} \tag{9.4.3}$$

$t \in [0,T]$, $z \in [0,1]$. The graph of the function $(t,z) \mapsto \zeta(t,z)$ is given in Figure 9.15.

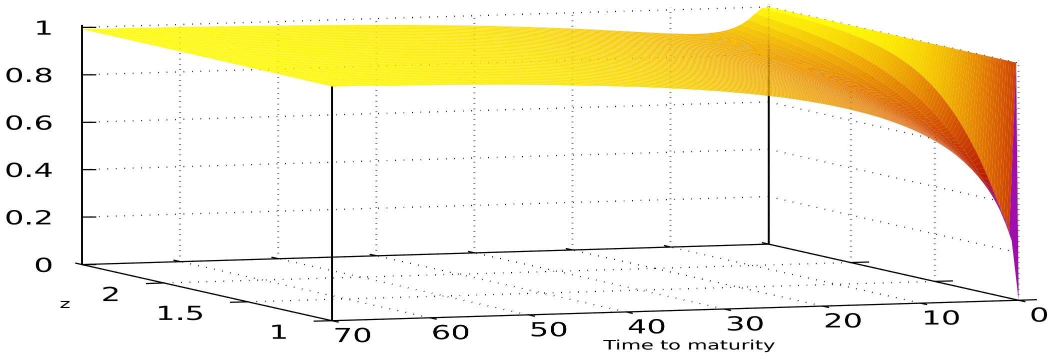


Figure 9.15: Rescaled portfolio strategy for the lookback call option.

Similar calculations using (9.2.4) can be carried out for other types of lookback options, such as options on extrema and partial lookback options, cf. El Khatib, 2003. As a consequence of Propositions 9.5 and 9.8, we have

$$\begin{aligned}&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T - m_0^T \mid \mathscr{F}_t\right] \\ &= S_t\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - m_0^t\,\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\ &\quad + \mathrm{e}^{-(T-t)r}S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - S_t\frac{\sigma^2}{2r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) \\ &= \xi_t S_t + m_0^t\,\mathrm{e}^{-(T-t)r}\left(\left(\frac{S_t}{m_0^t}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\right),\end{aligned}$$

and the quantity of the riskless asset e^{rt} in the portfolio is given by

$$\eta_t = m_0^t\,\mathrm{e}^{-rT}\left(\left(\frac{S_t}{m_0^t}\right)^{1-2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right)\right),$$

so that the portfolio value V_t at time t satisfies

$$V_t = \xi_t S_t + \eta_t\,\mathrm{e}^{rt}, \qquad t \geqslant 0.$$

Proposition 9.9 The Delta hedging strategy of the lookback put option is given by

$$\begin{aligned}\xi_t &= \left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\ &\quad + \mathrm{e}^{-(T-t)r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)-1, \quad 0\leqslant t\leqslant T.\end{aligned} \tag{9.4.4}$$

Proof. By (9.3.2) and (9.2.10), we need to differentiate

$$f(t,x,y) = \mathrm{Bl_p}(x,y,r,\sigma,T-t) + xh_p\left(T-t,\frac{x}{y}\right)$$

where

$$h_p(\tau,z) = \frac{\sigma^2}{2r}\Phi\left(\delta_+^\tau(z)\right) - \mathrm{e}^{-r\tau}\frac{\sigma^2}{2r}z^{-2r/\sigma^2}\Phi\left(-\delta_-^\tau(1/z)\right),$$

and

$$\delta_\pm^\tau(z) := \frac{1}{\sigma\sqrt{\tau}}\left(\log z + \left(r\pm\frac{1}{2}\sigma^2\right)\tau\right), \qquad z>0.$$

We have

$$\begin{aligned}\frac{\partial h_p}{\partial z}(\tau,z) &= \frac{\sigma^2}{2r}\delta_+^{\prime\tau}(z)\,\varphi\left(\delta_+^\tau(z)\right) + \mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right) \\ &\quad + \frac{\sigma^2}{2rz^2}\delta_-^{\prime\tau}\left(\frac{1}{z}\right)\mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\varphi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right) \\ &= \mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right) \\ &\quad + \frac{\sigma}{2rz\sqrt{\tau}}\left(\varphi\left(\delta_+^\tau(z)\right) - \mathrm{e}^{-r\tau}z^{-2r/\sigma^2}\varphi\left(\delta_-^\tau\left(\frac{1}{z}\right)\right)\right).\end{aligned}$$

From the relation

$$\begin{aligned}\left(\delta_+^{T-t}(z)\right)^2 - \left(\delta_-^{T-t}\left(\frac{1}{z}\right)\right)^2 &= \left(\delta_+^{T-t}(z)+\delta_-^{T-t}\left(\frac{1}{z}\right)\right)\left(\delta_+^{T-t}(z)-\delta_-^{T-t}\left(\frac{1}{z}\right)\right) \\ &= \frac{2r}{\sigma^2}\log z + 2r(T-t),\end{aligned}$$

we have

$$\varphi\left(\delta_+^{T-t}(z)\right) = z^{-2r/\sigma^2}\mathrm{e}^{-r(T-t)}\varphi\left(\delta_-^{T-t}\left(\frac{1}{z}\right)\right),$$

hence

$$\frac{\partial h_p}{\partial z}(\tau,z) = \mathrm{e}^{-r\tau}z^{-1-2r/\sigma^2}\Phi\left(-\delta_-^\tau\left(\frac{1}{z}\right)\right).$$

Therefore, knowing that the Black-Scholes put Delta is

$$-\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right) = -1+\Phi\left(\delta_+^{T-t}\left(\frac{x}{y}\right)\right),$$

we have

$$\begin{aligned}\frac{\partial f}{\partial x}(t,x,y) &= -\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right) + h_p\left(T-t,\frac{x}{y}\right) + \frac{x}{y}\frac{\partial h_p}{\partial z}\left(T-t,\frac{x}{y}\right) \\ &= -\Phi\left(-\delta_+^{T-t}\left(\frac{x}{y}\right)\right) + \frac{\sigma^2}{2r}\Phi\left(\delta_+^{T-t}\left(\frac{x}{y}\right)\right) \\ &\quad + \mathrm{e}^{-(T-t)r}\left(\frac{y}{x}\right)^{2r/\sigma^2}\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_-^{T-t}\left(\frac{y}{x}\right)\right),\end{aligned}$$

which yields (9.4.4). □

Note that we have $\xi_t = \sigma^2/(2r) > 0$ as T tends to infinity. At maturity $t = T$, the delta hedging strategy satisfies

$$\xi_T = \begin{cases} -1 & \text{if } M_0^T > S_T, \\ \\ \dfrac{1}{2} + \dfrac{\sigma^2}{4r} + \dfrac{1}{2}\left(1 - \dfrac{\sigma^2}{2r}\right) - 1 = 0 & \text{if } M_0^T = S_T. \end{cases}$$

In Figure 9.16 we represent the Delta of the lookback put option, as given by (9.4.4).

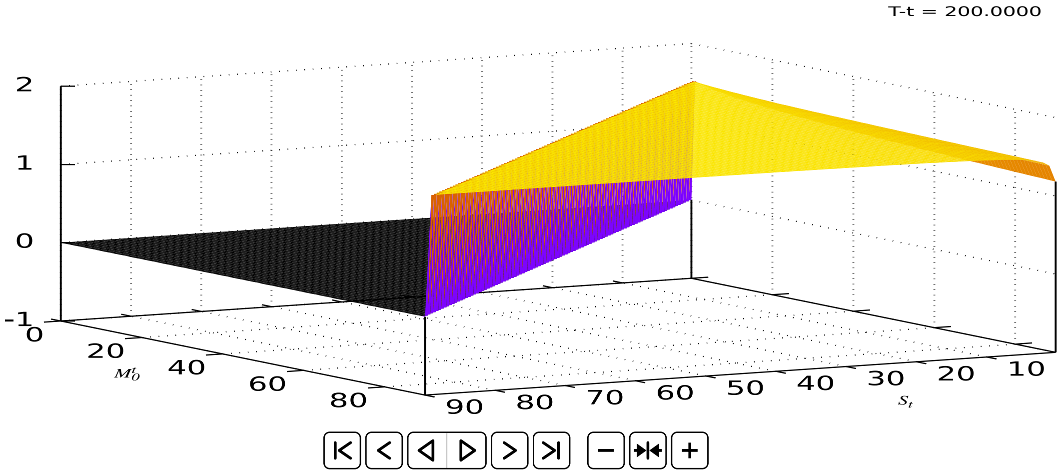


Figure 9.16: Delta of the lookback put option with $r = 2\%$ and $\sigma = 0.25$.*

As a consequence of Propositions 9.1 and 9.9, we have

$$\begin{aligned} & e^{-(T-t)r}\, \mathbb{E}^*\left[M_0^T - S_T \,\middle|\, \mathcal{F}_t\right] \\ = \ & M_0^t\, e^{-(T-t)r} \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_t\left(1 + \frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\ & - S_t\, e^{-(T-t)r}\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right) - S_t \\ = \ & \xi_t S_t + M_0^t\, e^{-(T-t)r}\left(\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \left(\frac{S_t}{M_0^t}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right), \end{aligned}$$

and the quantity of the riskless asset e^{rt} in the portfolio is given by

$$\eta_t = M_0^t\, e^{-rT}\left(\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \left(\frac{S_t}{M_0^t}\right)^{1-2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right)$$

so that the portfolio value V_t at time t satisfies

$$V_t = \xi_t S_t + \eta_t\, e^{rt}, \qquad t \geqslant 0.$$

Exercises

Exercise 9.1

*The animation works in Acrobat Reader on the entire pdf file.

a) Give the probability density function of the maximum of drifted Brownian motion $\operatorname*{Max}_{t\in[0,1]}(B_t + \sigma t/2)$.

b) Taking $S_t := \mathrm{e}^{\sigma B_t - \sigma^2 t/2}$, compute the expected value

$$\begin{aligned}\mathbb{E}\left[\min_{t\in[0,1]} S_t\right] &= \mathbb{E}\left[\min_{t\in[0,1]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{-\sigma \operatorname{Max}_{t\in[0,1]}(B_t+\sigma t/2)}\right].\end{aligned}$$

c) Compute the "optimal exercise" price $\mathbb{E}\left[\left(K - S_0 \min_{t\in[0,1]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2}\right)^+\right]$ of a finite expiration American put option with $S_0 \leqslant K$.

Exercise 9.2 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion.

a) Compute the expected value

$$\mathbb{E}\left[\operatorname*{Max}_{t\in[0,1]} S_t\right] = \mathbb{E}\left[\mathrm{e}^{\sigma \operatorname{Max}_{t\in[0,1]}(B_t - \sigma t/2)}\right].$$

b) Compute the "optimal exercise" price $\mathbb{E}\left[\left(S_0 \operatorname*{Max}_{t\in[0,1]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} - K\right)^+\right]$ of a finite expiration American call option with $S_0 \geqslant K$.

Exercise 9.3 Consider a risky asset whose price S_t is given by

$$dS_t = \sigma S_t dB_t + \sigma^2 S_t dt/2,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Compute the cumulative distribution function and the probability density function of the minimum $\min_{t\in[0,T]} B_t$ over the interval $[0,T]$?

b) Compute the price value

$$\mathrm{e}^{-\sigma^2 T/2}\,\mathbb{E}^*\left[S_T - \min_{t\in[0,T]} S_t\right]$$

of a lookback call option on S_T with maturity T.

Exercise 9.4 (Dassios and Lim, 2019) The digital drawdown call option with qualifying period pays a unit amount when the drawdown period reaches one unit of time, if this happens before fixed maturity T, but only if the size of drawdown at this stopping time is larger than a prespecified K. This provides an insurance against a prolonged drawdown, if the drawdown amount is large. Specifically, the digital drawdown call option is priced as

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau}\mathbb{1}_{\{\tau\leqslant T\}}\mathbb{1}_{\{M_0^\tau - S_\tau \geqslant K\}}\right],$$

where $M_0^t := \operatorname{Max}_{u\in[0,t]} S_u$, $U_t := t - \operatorname{Sup}\{0 \leqslant u \leqslant t \ : \ M_0^t = S_u\}$, and $\tau := \inf\{t\in\mathbb{R}_+ \ : \ U_t = 1\}$. Write the price of the drawdown option as a triple integral using the joint probability density function $f_{(\tau,S_\tau,M_\tau)}(t,x,y)$ of (τ, S_τ, M_τ) under the risk-neutral probability measure $\mathbb{P}^*$.

Exercise 9.5

a) Check explicitly that the boundary conditions (9.2.3a)-(9.2.3c) are satisfied.

b) Check explicitly that the boundary conditions (9.3.1a)-(9.3.1b) are satisfied.

10. Asian Options

Asian options are special cases of average value options, whose claim payoffs are determined by the difference between the average underlying asset price over a certain time interval and a strike price K. This chapter covers several probabilistic and PDE techniques for the pricing and hedging of Asian options. Due to their dependence on averaged asset prices. Asian options are less volatile than plain vanilla options whose claim payoffs depend only on the terminal value of the underlying asset.

10.1 Bounds on Asian Option Prices

Asian options were first traded in Tokyo in 1987, and have become particularly popular in commodities trading.

Arithmetic Asian options

Given an underlying asset price process $(S_t)_{t\in[0,T]}$, the payoff of the Asian call option on $(S_t)_{t\in[0,T]}$ with exercise date T and strike price K is given by

$$C = \left(\frac{1}{T}\int_0^T S_t dt - K\right)^+.$$

Similarly, the payoff of the Asian put option on $(S_t)_{t\in[0,T]}$ with exercise date T and strike price K is

$$C = \left(K - \frac{1}{T}\int_0^T S_t dt\right)^+.$$

Due to their dependence on averaged asset prices, Asian options are less volatile than plain vanilla options whose payoffs depend only on the terminal value of the underlying asset. Examples of market price data for Asian option prices on Light Sweet Crude Oil Futures can be found **here**.

As an example, Figure 10.1 presents a graph of Brownian motion and its moving average process

$$X_t := \frac{1}{t}\int_0^t B_s ds, \qquad t > 0.$$

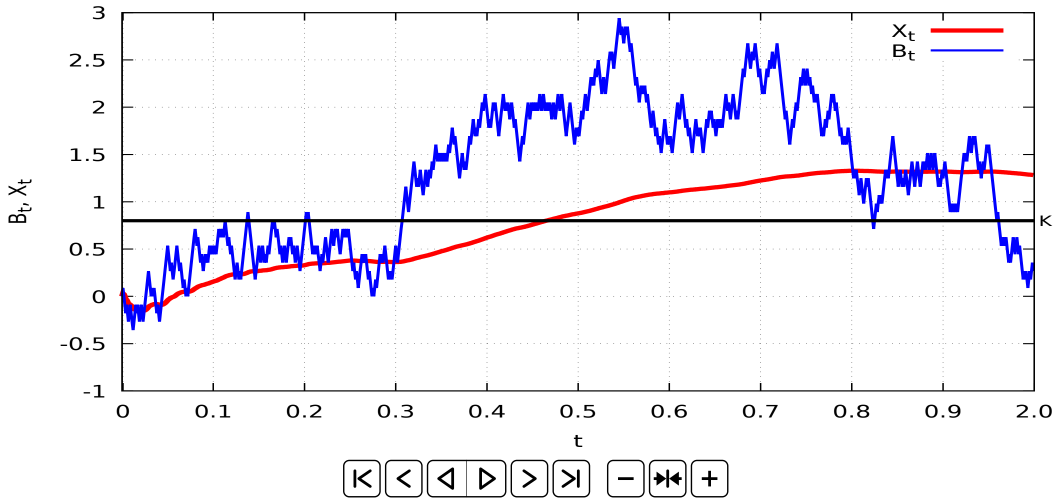


Figure 10.1: Brownian motion B_t and its moving average X_t.*

Related exotic options include the Asian-American options, or Hawaiian options, that combine an Asian claim payoff with American style exercise, and can be priced by variational PDEs, cf. § 8.6.3.2 of Crépey, 2013.

An option on average is an option whose payoff has the form

$$C = \phi(\Lambda_T, S_T),$$

where

$$\Lambda_T = S_0 \int_0^T e^{\sigma B_u + ru - \sigma^2 u/2} du = \int_0^T S_u du, \qquad T \geqslant 0.$$

- For example when $\phi(y,x) = (y/T - K)^+$ this yields the Asian call option with payoff

$$\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ = \left(\frac{\Lambda_T}{T} - K\right)^+, \tag{10.1.1}$$

which is a path-dependent option whose price at time $t \in [0,T]$ is given by

$$e^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \middle| \mathscr{F}_t\right]. \tag{10.1.2}$$

*The animation works in Acrobat Reader on the entire pdf file.

- As another example, when $\phi(y,x) := \mathrm{e}^{-y}$ this yields the price

$$P(0,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_0^T S_u du}\right] = \mathbb{E}^*\left[\mathrm{e}^{-\Lambda_T}\right]$$

at time 0 of a bond with underlying short-term rate process S_t.

In the sequel, we assume that the underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ is a geometric Brownian motion satisfying the equation

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$.

Using the time homogeneity of the process $(S_t)_{t\in\mathbb{R}_+}$, the option with payoff $C = \phi(\Lambda_T, S_T)$ can be priced as

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(\Lambda_T,S_T) \mid \mathscr{F}_t] &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\left(\Lambda_t + \int_t^T S_u du, S_T\right) \middle| \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\left(y + x\int_t^T \frac{S_u}{S_t}du, x\frac{S_T}{S_t}\right)\right]_{y=\Lambda_t, x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\left(y + x\int_0^{T-t} \frac{S_u}{S_0}du, x\frac{S_{T-t}}{S_0}\right)\right]_{y=\Lambda_t, x=S_t}. \qquad (10.1.3)
\end{aligned}$$

Using the Markov property of the process $(S_t,\Lambda_t)_{t\in\mathbb{R}_+}$, we can write down the option price as a function

$$\begin{aligned}
f(t,S_t,\Lambda_t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(\Lambda_T,S_T) \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[\phi(\Lambda_T,S_T) \mid S_t,\ \Lambda_t]
\end{aligned}$$

of (t,S_t,Λ_t), where the function $f(t,x,y)$ is given by

$$f(t,x,y) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\phi\left(y + x\int_0^{T-t}\frac{S_u}{S_0}du, x\frac{S_{T-t}}{S_0}\right)\right].$$

As we will see below there exists no easily tractable closed-form solution for the price of an arithmetically averaged Asian option.

Geometric Asian options

On the other hand, replacing the arithmetic average

$$\frac{1}{T}\sum_{k=1}^n S_{t_k}(t_k - t_{k-1}) \simeq \frac{1}{T}\int_0^T S_u du$$

with the geometric average

$$\begin{aligned}
\prod_{k=1}^n S_{t_k}^{(t_k-t_{k-1})/T} &= \exp\left(\log\prod_{k=1}^n S_{t_k}^{(t_k-t_{k-1})/T}\right) \\
&= \exp\left(\frac{1}{T}\sum_{k=1}^n \log S_{t_k}^{t_k-t_{k-1}}\right) \\
&= \exp\left(\frac{1}{T}\sum_{k=1}^n (t_k-t_{k-1})\log S_{t_k}\right) \\
&\simeq \exp\left(\frac{1}{T}\int_0^T \log S_u du\right)
\end{aligned}$$

leads to closed-form solutions using the Black Scholes formula, see Exercise 10.5.

Pricing by probability density functions

We note that the prices of option on averages can be estimated numerically using the joint probability density function $\psi_{\Lambda_{T-t},B_{T-t}}$ of (Λ_{T-t},B_{T-t}), as follows:

$$\begin{aligned}
f(t,x,y) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(y+x\int_0^{T-t}\frac{S_u}{S_0}du, x\frac{S_{T-t}}{S_0}\right)\right]\\
&= \mathrm{e}^{-(T-t)r}\int_0^\infty\int_{-\infty}^\infty \phi\left(y+xz, x\mathrm{e}^{\sigma u+(T-t)r-(T-t)\sigma^2/2}\right)\psi_{\Lambda_{T-t},B_{T-t}}(z,u)dzdu,
\end{aligned}$$

see Section 10.2 for details.

Bounds on Asian option prices

As noted in the next proposition, arithmetic Asian call option prices can be lower bounded by geometric Asian call prices, as a consequence of the Jensen, 1906 inequality. See Exercise 10.5 for the expression of the geometric Asian option call price.

Proposition 10.1 Let ϕ be a non-decreasing payoff function. We have the bound

$$\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi\left(\exp\left(\frac{1}{T}\int_0^T\log S_u du\right)-K\right)\right]\leqslant \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi\left(\frac{1}{T}\int_0^T S_u du-K\right)\right].$$

Proof. By Jensen's inequality applied to the (concave) log function and the uniform measure with probability density function $(1/T)\mathbb{1}_{[0,T]}$ on $[0,T]$, we have

$$\exp\left(\frac{1}{T}\int_0^T\log S_t dt\right)\leqslant\exp\left(\log\left(\frac{1}{T}\int_0^T S_t dt\right)\right)=\frac{1}{T}\int_0^T S_t dt. \tag{10.1.4}$$

□

We also note (see Lemma 1 of Kemna and Vorst, 1990 and Exercise 10.7 below for the discrete-time version of that result), that the Asian call option price can be upper bounded by the corresponding European call option price using convexity arguments.

Proposition 10.2 Assume that $r\geqslant 0$, and let ϕ be a convex and non-decreasing payoff function. We have the bound

$$\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi\left(\frac{1}{T}\int_0^T S_u du-K\right)\right]\leqslant \mathrm{e}^{-rT}\,\mathbb{E}^*[\phi(S_T-K)].$$

Proof. By Jensen's inequality for the uniform measure with probability density function $(1/T)\mathbb{1}_{[0,T]}$ on $[0,T]$ and for the probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi\left(\int_0^T S_u\frac{du}{T}-K\right)\right] &= \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi\left(\int_0^T (S_u-K)\frac{du}{T}\right)\right]\\
&\leqslant \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\int_0^T\phi(S_u-K)\frac{du}{T}\right]\\
&= \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\int_0^T\phi\left(\mathrm{e}^{-(T-u)r}\,\mathbb{E}^*[S_T\mid\mathscr{F}_u]-K\right)\frac{du}{T}\right]\\
&= \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\int_0^T\phi\left(\mathbb{E}^*[\mathrm{e}^{-(T-u)r}S_T-K\mid\mathscr{F}_u]\right)\frac{du}{T}\right]\\
&\leqslant \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\int_0^T\mathbb{E}^*\left[\phi\left(\mathrm{e}^{-(T-u)r}S_T-K\right)\mid\mathscr{F}_u\right]\frac{du}{T}\right]
\end{aligned} \tag{10.1.5}$$

$$
\begin{aligned}
&\leqslant e^{-rT}\int_0^T \mathbb{E}^*\left[\mathbb{E}^*\left[\phi(S_T-K)\,\middle|\,\mathscr{F}_u\right]\right]\frac{du}{T} && (10.1.6)\\
&= e^{-rT}\int_0^T \mathbb{E}^*[\phi(S_T-K)]\frac{du}{T}\\
&= e^{-rT}\,\mathbb{E}^*[\phi(S_T-K)],
\end{aligned}
$$

where from (10.1.5) to (10.1.6) we used the facts that $r \geqslant 0$ and ϕ is non-decreasing. □

In particular, taking $\phi(x) : (x-K)^+$, Proposition 10.2 shows that Asian option prices are upper bounded by European call option prices due to the lower volatility of arithmetic averages, as

$$
e^{-rT}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+\right] \leqslant e^{-rT}\,\mathbb{E}^*[(S_T-K)^+],
$$

see Figure 10.2 for an illustration, as the averaging feature of Asian options reduces their underlying volatility.

```
nSim=99999;N=1000; t <- 1:N; dt <- 1.0/N; sigma=2;r=0.5; european=0;asian=0;K=1.5
dev.new(width=16,height=7); par(oma=c(0,5,0,0))
for (j in 1:nSim){S<-exp(sigma*cumsum(rnorm( N, 0,sqrt(dt)))+r*t/N-sigma**2*t/2/N);color="blue"
A<-sum(c(1,S))/(N+1);if (S[N]>=K) {european=european+S[N]-K}
if (A>=K) {asian=asian+A-K};if (S[N]>A) {color="darkred"} else {color="darkgreen"}
plot(c(0,t/N),c(1,S), xlab = "Time", type='l', lwd = 3, ylab = "", ylim = c(0,exp(4*r)), col =
      color,main=paste("Asian Price=",format(round(asian,2)),"/", j,"=",format(round(asian/j,2)),"European
      Price=",format(round(european,2)), "/",j,"=",format(round(european/j,2))), xaxs='i',xaxt='n',yaxt='n',
      yaxs='i', yaxp = c(0,10,10), cex.lab=2, cex.main=2)
text(0.3,6,paste("A-Payoff=",format(round(max(A-K,0),2)),", E-Payoff=", format(round(max(S[N]-K,0),2))),
      col=color,cex=2)
axis(1, las=1, cex.axis=2)
axis(2, at=c(0,K,A,1,2,3,4,5,6,7,8,9,10), labels=c(0,"K","Average",1,2,3,4,5,6,7,8,9,10), las=2, cex.axis=2)
lines(c(0,t/N),rep(K,N+1),col = "red",lty = 1, lwd = 4);
lines(c(0,t/N),rep(A,N+1),col = "darkgreen",lty = 2, lwd = 4); Sys.sleep(0.1)
if (S[N]>K || A>K) {readline(prompt = "Pause. Press <Enter> to continue...")}}
```

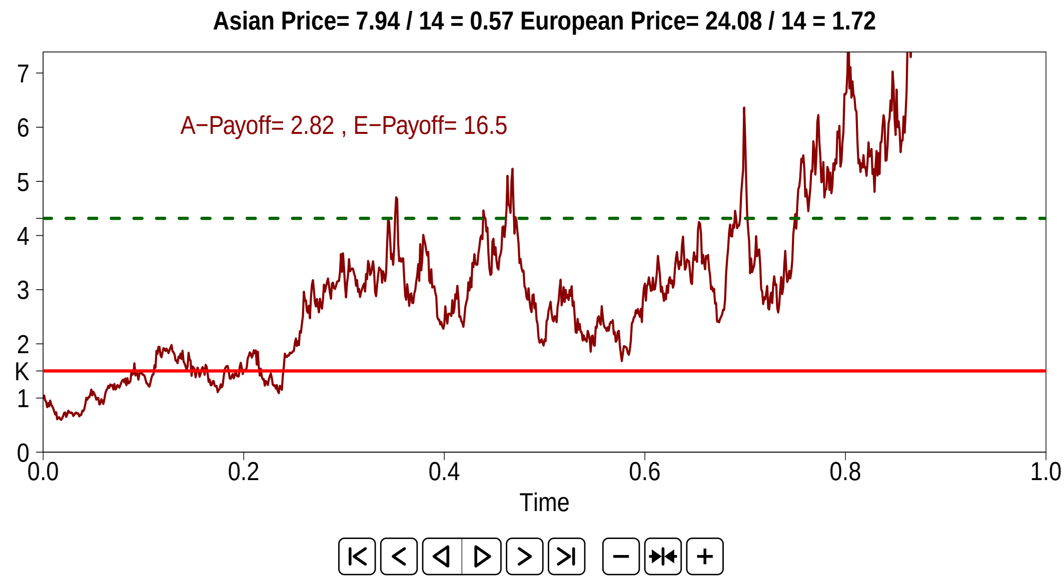


Figure 10.2: Asian option price *vs.* European option price.*

In the case of Asian call options we have the following result.

*The animation works in Acrobat Reader on the entire pdf file.

Proposition 10.3 Assume that $r \geqslant 0$. We have the conditional bound

$$\begin{aligned} & \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| \mathscr{F}_t\right] \\ & \leqslant \ \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T}\int_0^t S_u du + \frac{T-t}{T}S_T - K\right)^+ \Big| \mathscr{F}_t\right] \end{aligned} \tag{10.1.7}$$

on Asian option prices, $t \in [0,T]$.

Proof. Let the function $f(t,x,y)$ be defined as

$$f(t,S_t,\Lambda_t) = \mathbf{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| \mathscr{F}_t\right],$$

i.e., from Proposition 10.2,

$$\begin{aligned} f(t,x,y) &= \mathbf{E}^*\left[\left(\frac{1}{T}\left(y + x\int_0^{T-t}\frac{S_u}{S_0}du\right) - K\right)^+\right] \\ &= \mathbf{E}^*\left[\left(\frac{1}{T}\left(y + \frac{x}{S_0}\Lambda_{T-t}\right) - K\right)^+\right] \\ &= \mathbf{E}^*\left[\left(\frac{y}{T} - K + \frac{x}{TS_0}\Lambda_{T-t}\right)^+\right] \\ &= \frac{(T-t)x}{TS_0}\mathbf{E}^*\left[\left(\frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + \frac{\Lambda_{T-t}}{T-t}\right)^+\right] \\ &\leqslant \frac{(T-t)x}{TS_0}\mathbf{E}^*\left[\left(\frac{yS_0}{(T-t)x} - \frac{KTS_0}{(T-t)x} + S_{T-t}\right)^+\right] \\ &= \mathbf{E}^*\left[\left(\frac{y}{T} - K + \frac{(T-t)xS_{T-t}}{TS_0}\right)^+\right], \qquad x,y > 0, \end{aligned}$$

which yields (10.1.7). □

The right-hand side of the bound (10.1.7) can be computed from the Black-Scholes formula as

$$\begin{aligned} & \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T}\int_0^t S_u du + \frac{T-t}{T}S_T - K\right)^+ \Big| \mathscr{F}_t\right] \\ &= \frac{T-t}{T}\mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[\left(S_T + \frac{1}{T-t}\int_0^t S_u du - \frac{KT}{T-t}\right)^+ \Big| \mathscr{F}_t\right] \\ &= \frac{T-t}{T}\mathrm{Bl}\left(S_t, \frac{KT}{T-t} - \frac{1}{T-t}\int_0^t S_u du, \sigma, r, T-t\right), \quad 0 \leqslant t < T. \end{aligned}$$

See also Proposition 3.2-(ii) of Geman and Yor, 1993 for lower bounds when r takes negative values. We also have the following bound which yields the behavior of Asian call option prices in large time.

Proposition 10.4 Assume that $r \geqslant 0$. The Asian call option price satisfies the bound

$$e^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| \mathscr{F}_t\right] \leqslant \frac{e^{-(T-t)r}}{T}\int_0^t S_u du + S_t \frac{1-e^{-(T-t)r}}{rT},$$

$t \in [0,T]$, and tends to zero (almost surely) as time to maturity T tends to infinity:

$$\lim_{T\to\infty}\left(e^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| \mathscr{F}_t\right]\right) = 0, \quad t \geqslant 0.$$

Proof. Using the inequality $(x-K)^+ \leqslant x$ for $x \geqslant 0$, we have the bound

$$\begin{aligned}
0 &\leqslant e^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| \mathscr{F}_t\right] \\
&\leqslant e^{-(T-t)r}\mathbf{E}^*\left[\frac{1}{T}\int_0^T S_u du \Big| \mathscr{F}_t\right] \\
&= e^{-(T-t)r}\mathbf{E}^*\left[\frac{1}{T}\int_0^t S_u du \Big| \mathscr{F}_t\right] + e^{-(T-t)r}\mathbf{E}^*\left[\frac{1}{T}\int_t^T S_u du \Big| \mathscr{F}_t\right] \\
&= e^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \frac{1}{T}e^{-(T-t)r}\int_t^T \mathbf{E}^*[S_u \mid \mathscr{F}_t]du \\
&= e^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \frac{1}{T}e^{-(T-t)r}\int_t^T e^{(u-t)r}S_t du \\
&= \frac{1}{T}e^{-(T-t)r}\int_0^t S_u du + \frac{S_t}{T}\int_t^T e^{-(T-u)r}du \\
&= \frac{1}{T}e^{-(T-t)r}\int_0^t S_u du + S_t\frac{1-e^{-(T-t)r}}{rT}.
\end{aligned}$$

□

Note that as T tends to infinity the Black-Scholes European call price tends to S_t, *i.e.*, we have

$$\lim_{T\to\infty}\left(e^{-(T-t)r}\mathbf{E}^*[(S_T-K)^+ \mid \mathscr{F}_t]\right) = S_t, \quad t \geqslant 0.$$

10.2 Hartman-Watson Distribution

First, we note that the numerical computation of Asian call option prices can be done using the probability density function of

$$\Lambda_T = \int_0^T S_t dt.$$

In Yor, 1992, Proposition 2, the joint probability density function of

$$(\Lambda_t, B_t) = \left(\int_0^t e^{\sigma B_s - p\sigma^2 s/2} ds, B_t - p\sigma t/2\right), \qquad t > 0,$$

has been computed in the case $\sigma = 2$, cf. also Dufresne, 2001 and Matsumoto and Yor, 2005. In the next proposition, we restate this result for an arbitrary variance parameter σ after rescaling. Let $\theta(v,\tau)$ denote the function defined as

$$\theta(v,\tau) = \frac{v e^{\pi^2/(2\tau)}}{\sqrt{2\pi^3\tau}}\int_0^\infty e^{-\xi^2/(2\tau)} e^{-v\cosh\xi}\sinh(\xi)\sin(\pi\xi/\tau)d\xi, \quad v,\tau > 0. \qquad (10.2.1)$$

Proposition 10.5 For all $t > 0$ we have

$$\mathbb{P}\left(\int_0^t \mathrm{e}^{\sigma B_s - p\sigma^2 s/2} ds \in dy,\ B_t - p\frac{\sigma t}{2} \in dz\right)$$
$$= \frac{\sigma}{2}\mathrm{e}^{-p\sigma z/2 - p^2\sigma^2 t/8}\exp\left(-2\frac{1+\mathrm{e}^{\sigma z}}{\sigma^2 y}\right)\theta\left(\frac{4\,\mathrm{e}^{\sigma z/2}}{\sigma^2 y}, \frac{\sigma^2 t}{4}\right)\frac{dy}{y}dz,$$

$y > 0$, $z \in \mathbb{R}$.

The expression of this probability density function can then be used for the pricing of options on average such as (10.1.3), as

$$\begin{aligned}
f(t,x,y) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(y + x\int_0^{T-t}\frac{S_v}{S_0}dv, x\frac{S_{T-t}}{S_0}\right)\right] \\
&= \mathrm{e}^{-(T-t)r} \\
&\quad \times \int_0^\infty \phi\left(y + xz, x\,\mathrm{e}^{\sigma u + (T-t)r - (T-t)\sigma^2/2}\right)\mathbb{P}\left(\int_0^{T-t}\frac{S_v}{S_0}dv \in dz, B_{T-t} \in du\right) \\
&= \frac{\sigma}{2}\mathrm{e}^{-(T-t)r + (T-t)p^2\sigma^2/8}\int_0^\infty\int_{-\infty}^\infty \phi\left(y + xz, x\,\mathrm{e}^{\sigma u + (T-t)r - (T-t)(1+p)\sigma^2/2}\right) \\
&\quad \times \exp\left(-2\frac{1+\mathrm{e}^{\sigma u - (T-t)p\sigma^2/2}}{\sigma^2 z} - \frac{p}{2}\sigma u\right)\theta\left(\frac{4\,\mathrm{e}^{\sigma u/2 - (T-t)p\sigma^2/4}}{\sigma^2 z}, \frac{(T-t)\sigma^2}{4}\right)du\frac{dz}{z} \\
&= \mathrm{e}^{-(T-t)r - (T-t)p^2\sigma^2/8}\int_0^\infty\int_0^\infty \phi\left(y + x/z, xv^2\,\mathrm{e}^{(T-t)r - (T-t)\sigma^2/2}\right) \\
&\quad \times v^{-1-p}\exp\left(-2z\frac{1+v^2}{\sigma^2}\right)\theta\left(\frac{4vz}{\sigma^2}, \frac{(T-t)\sigma^2}{4}\right)dv\frac{dz}{z},
\end{aligned}$$

which actually stands as a triple integral due to the definition (10.2.1) of $\theta(v,\tau)$. Note that here the order of integration between du and dz cannot be exchanged without particular precautions, at the risk of wrong computations.

By repeating the argument of (10.1.3) for $\phi(x,y) := (x-K)^+$, the Asian call option can be priced as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Bigg| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\left(\Lambda_t + \int_t^T S_u du\right) - K\right)^+ \Bigg| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\left(y + x\int_t^T \frac{S_u}{S_t}du\right) - K\right)^+ \Bigg| \mathcal{F}_t\right]_{x=S_t,\ y=\Lambda_t} \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\left(y + x\int_0^{T-t}\frac{S_u}{S_0}du\right) - K\right)^+\right]_{x=S_t,\ y=\Lambda_t}.
\end{aligned}$$

Hence the Asian call option can be priced as

$$f(t,S_t,\Lambda_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Bigg| \mathcal{F}_t\right],$$

where the function $f(t,x,y)$ is given by

$$f(t,x,y) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\left(y + x\int_0^{T-t}\frac{S_u}{S_0}du\right) - K\right)^+\right]$$

$$= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\left(y+\frac{x}{S_0}\Lambda_{T-t}\right)-K\right)^+\right], \quad x,y>0. \tag{10.2.2}$$

From Proposition 10.5, we deduce the marginal probability density function of Λ_T, also called the Hartman-Watson distribution see *e.g.* Barrieu, Rouault, and Yor, 2004.

Proposition 10.6 The probability density function of

$$\Lambda_T := \int_0^T \mathrm{e}^{\sigma B_t - p\sigma^2 t/2} dt,$$

is given by

$$\begin{aligned}
&\mathbb{P}\left(\int_0^T \mathrm{e}^{\sigma B_t - p\sigma^2 t/2} dt \in du\right) \\
&= \frac{\sigma}{2u}\mathrm{e}^{p^2\sigma^2 T/8}\int_{-\infty}^{\infty}\exp\left(-2\frac{1+\mathrm{e}^{\sigma v - p\sigma^2 T/2}}{\sigma^2 u}-\frac{p}{2}\sigma v\right)\theta\left(\frac{4\,\mathrm{e}^{\sigma v/2 - p\sigma^2 T/4}}{\sigma^2 u},\frac{\sigma^2 T}{4}\right)dvdu \\
&= \mathrm{e}^{-p^2\sigma^2 T/8}\int_0^{\infty} v^{-1-p}\exp\left(-2\frac{1+v^2}{\sigma^2 u}\right)\theta\left(\frac{4v}{\sigma^2 u},\frac{\sigma^2 T}{4}\right)dv\frac{du}{u},
\end{aligned}$$

$u>0$.

From Proposition 10.6, we get

$$\begin{aligned}
\mathbb{P}(\Lambda_T/S_0 \in du) &= \mathbb{P}\left(\int_0^T S_t dt \in du\right) \\
&= \mathrm{e}^{-p^2\sigma^2 T/8}\int_0^{\infty} v^{-1-p}\exp\left(-2\frac{1+v^2}{\sigma^2 u}\right)\theta\left(\frac{4v}{\sigma^2 u},\frac{\sigma^2 T}{4}\right)dv\frac{du}{u},
\end{aligned} \tag{10.2.3}$$

where $S_t = S_0\,\mathrm{e}^{\sigma B_t - p\sigma^2 t/2}$ and $p = 1 - 2r/\sigma^2$. By (10.2.2), this probability density function can then be used for the pricing of Asian options, as

$$\begin{aligned}
f(t,x,y) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\left(y+\frac{x}{S_0}\Lambda_{T-t}\right)-K\right)^+\right] \\
&= \mathrm{e}^{-(T-t)r}\int_0^{\infty}\left(\frac{y+xz}{T}-K\right)^+\mathbb{P}(\Lambda_{T-t}/S_0 \in dz) \\
&= \mathrm{e}^{-(T-t)r}\frac{\sigma}{2}\mathrm{e}^{-(T-t)p^2\sigma^2/8}\int_0^{\infty}\int_0^{\infty}\left(\frac{y+xz}{T}-K\right)^+ \\
&\quad\times v^{-1-p}\exp\left(-2\frac{1+v^2}{\sigma^2 z}\right)\theta\left(\frac{4v}{\sigma^2 z},(T-t)\frac{\sigma^2}{4}\right)dv\frac{dz}{z} \\
&= \frac{1}{T}\mathrm{e}^{-(T-t)r-(T-t)p^2\sigma^2/8}\int_{0\vee(KT-y)/x}^{\infty}\int_0^{\infty}(xz+y-KT) \\
&\quad\times\exp\left(-2\frac{1+v^2}{\sigma^2 z}\right)\theta\left(\frac{4v}{\sigma^2 z},(T-t)\frac{\sigma^2}{4}\right)dv\frac{dz}{z} \\
&= \frac{4x}{\sigma^2 T}\mathrm{e}^{-(T-t)r-(T-t)p^2\sigma^2/8}\int_0^{\infty}\int_0^{\infty}\left(\frac{1}{z}-\frac{(KT-y)\sigma^2}{4x}\right)^+ \\
&\quad\times v^{-1-p}\exp\left(-z\frac{1+v^2}{2}\right)\theta\left(vz,(T-t)\frac{\sigma^2}{4}\right)dv\frac{dz}{z},
\end{aligned} \tag{10.2.4}$$

cf. Theorem in § 5 of Carr and Schröder, 2004, which is actually a triple integral due to the definition (10.2.1) of $\theta(v,t)$. Note that since the integrals are not absolutely convergent, here the order of integration between dv and dz cannot be exchanged without particular precautions, at the risk of wrong computations.

10.3 Laplace Transform Method

The time Laplace transform of the rescaled option price

$$C(t) := \mathbf{E}^*\left[\left(\frac{1}{t}\int_0^t S_u du - K\right)^+\right], \qquad t > 0,$$

as

$$\int_0^\infty \mathrm{e}^{-\lambda t} C(t)dt = \frac{\int_0^{K/2} \mathrm{e}^{-x} x^{-2+(p+\sqrt{2\lambda+p^2})/2}(1-2Kx)^{2+(\sqrt{2\lambda+p^2}-p)/2}dx}{\lambda(\lambda-2+2p)\Gamma\left(-1+\left(p+\sqrt{2\lambda+p^2}\right)/2\right)},$$

with here $\sigma := 2$, and $\Gamma(z)$ denotes the gamma function, see Relation (3.10) in Geman and Yor, 1993. This expression can be used for pricing by numerical inversion of the Laplace transform using *e.g.* the Widder method, the Gaver-Stehfest method, the Durbin-Crump method, or the Papoulis method. The following Figure 10.3 represents Asian call option prices computed by the Geman and Yor, 1993 method.

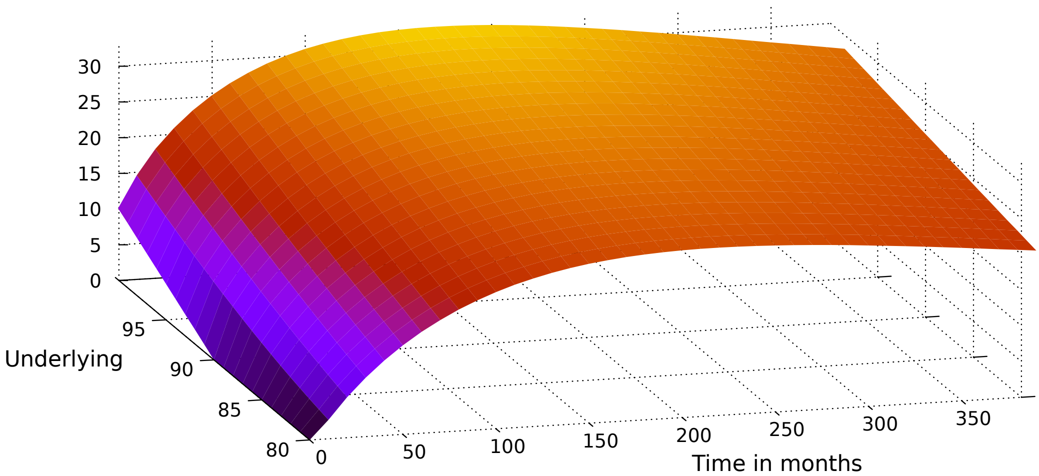


Figure 10.3: Graph of Asian call option prices with $\sigma = 1$, $r = 0.1$ and $K = 90$.

We refer to *e.g.* Carr and Schröder, 2004, Dufresne, 2000, and references therein for more results on Asian option pricing using the probability density function of the averaged geometric Brownian motion.

Figure 6.5 presents a graph of implied volatility surface for Asian options on light sweet crude oil futures.

10.4 Moment Matching Approximations

Lognormal approximation

Other numerical approaches to the pricing of Asian options include Levy, 1992, Turnbull and Wakeman, 1992 which rely on approximations of the average price distribution based on the lognormal distribution. The lognormal distribution has the probability density function

$$g(x) = \frac{1}{\eta\sqrt{2\pi}}\mathrm{e}^{-(\mu-\log x)^2/(2\eta^2)}\frac{dx}{x}, \qquad x > 0,$$

where $\mu \in \mathbb{R}$, $\eta > 0$, with moments

$$\mathbb{E}[X] = \mathrm{e}^{\mu+\eta^2/2} \quad \text{and} \quad \mathbb{E}\left[X^2\right] = \mathrm{e}^{2\mu+2\eta^2}. \tag{10.4.1}$$

The approximation is implemented by matching the above first two moments to those of time integral

$$\Lambda_T := \int_0^T S_t dt$$

of geometric Brownian motion

$$S_t = S_0 \,\mathrm{e}^{\sigma B_t + (r-\sigma^2/2)t}, \qquad 0 \leqslant t \leqslant T,$$

as computed in the next proposition, cf. also (7) and (8) page 480 of Levy, 1992, and Exercise 10.1.

Proposition 10.7 The first and second moments of Λ_T are given by

$$\mathbb{E}^*[\Lambda_T] = S_0 \frac{\mathrm{e}^{rT}-1}{r},$$

and

$$\mathbb{E}^*\left[(\Lambda_T)^2\right] = 2S_0^2 \frac{r\,\mathrm{e}^{(\sigma^2+2r)T} - (\sigma^2+2r)\,\mathrm{e}^{rT} + (\sigma^2+r)}{(\sigma^2+r)(\sigma^2+2r)r}.$$

Proof. The computation of the first moment is straightforward. We have

$$\begin{aligned}
\mathbb{E}^*\left[\Lambda_T\right] &= \mathbb{E}^*\left[\int_0^T S_u du\right] \\
&= \int_0^T \mathbb{E}^*[S_u] du \\
&= S_0 \int_0^T \mathrm{e}^{ru} du \\
&= S_0 \frac{\mathrm{e}^{rT}-1}{r}.
\end{aligned}$$

For the second moment we have, letting $p := 1 - 2r/\sigma^2$,

$$\begin{aligned}
\mathbb{E}^*\left[(\Lambda_T)^2\right] &= S_0^2 \int_0^T \int_0^T \mathrm{e}^{-p\sigma^2 a/2 - p\sigma^2 b/2} \mathbb{E}^*\left[\mathrm{e}^{\sigma B_a} \mathrm{e}^{\sigma B_b}\right] dbda \\
&= 2S_0^2 \int_0^T \int_0^a \mathrm{e}^{-p\sigma^2 a/2 - p\sigma^2 b/2} \mathrm{e}^{(a+b)\sigma^2/2} \mathrm{e}^{b\sigma^2} dbda \\
&= 2S_0^2 \int_0^T \mathrm{e}^{-(p-1)\sigma^2 a/2} \int_0^a \mathrm{e}^{-(p-3)\sigma^2 b/2} dbda \\
&= \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T \mathrm{e}^{-(p-1)\sigma^2 a/2} (1 - \mathrm{e}^{-(p-3)\sigma^2 a/2}) da \\
&= \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T \mathrm{e}^{-(p-1)\sigma^2 a/2} da - \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T \mathrm{e}^{-(p-1)\sigma^2 a/2} \mathrm{e}^{-(p-3)\sigma^2 a/2} da \\
&= \frac{8S_0^2}{(p-3)(p-1)\sigma^4} (1 - \mathrm{e}^{-(p-1)\sigma^2 T/2}) - \frac{4S_0^2}{(p-3)\sigma^2} \int_0^T \mathrm{e}^{-(2p-4)\sigma^2 a/2} da \\
&= \frac{8S_0^2}{(p-3)(p-1)\sigma^4} (1 - \mathrm{e}^{-(p-1)\sigma^2 T/2}) - \frac{4S_0^2}{(p-3)(p-2)\sigma^4} (1 - \mathrm{e}^{-(p-2)\sigma^2 T})
\end{aligned}$$

$$= \quad 2S_0^2 \frac{r\,\mathrm{e}^{(\sigma^2+2r)T} - (\sigma^2+2r)\,\mathrm{e}^{rT} + (\sigma^2+r)}{(\sigma^2+r)(\sigma^2+2r)r},$$

since $r - \sigma^2/2 = -p\sigma^2/2$. $\square$

By matching the first and second moments

$$\mathbb{E}[\Lambda_T] \simeq \mathrm{e}^{\hat{\mu}_T + \hat{\eta}_T^2 T/2} \quad \text{and} \quad \mathbb{E}\left[\Lambda_T^2\right] \simeq \mathrm{e}^{2(\hat{\mu}_T + \hat{\eta}_T^2 T)}$$

of the lognormal distribution with the moments of Proposition 10.7 we estimate $\widehat{\mu}_T$ and $\widehat{\eta}_T$ as

$$\widehat{\eta}_T^2 = \frac{1}{T}\log\left(\frac{E\left[\Lambda_T^2\right]}{(\mathbb{E}^*[\Lambda_T])^2}\right) \quad \text{and} \quad \widehat{\mu}_T = \frac{1}{T}\log \mathbb{E}^*[\Lambda_T] - \frac{1}{2}\widehat{\eta}_T^2.$$

Under this approximation, the probability density function φ_{Λ_T} of $\Lambda_T = \int_0^T S_t dt$ is approximated by the lognormal probability density function

$$\varphi_{\Lambda_T}(x) \approx \frac{1}{x\sigma_{t,T}\sqrt{2(T-t)\pi}} \exp\left(-\frac{(\widehat{\mu}_T - \log x)^2}{2(T-t)\widehat{\eta}_T^2}\right), \quad x > 0.$$

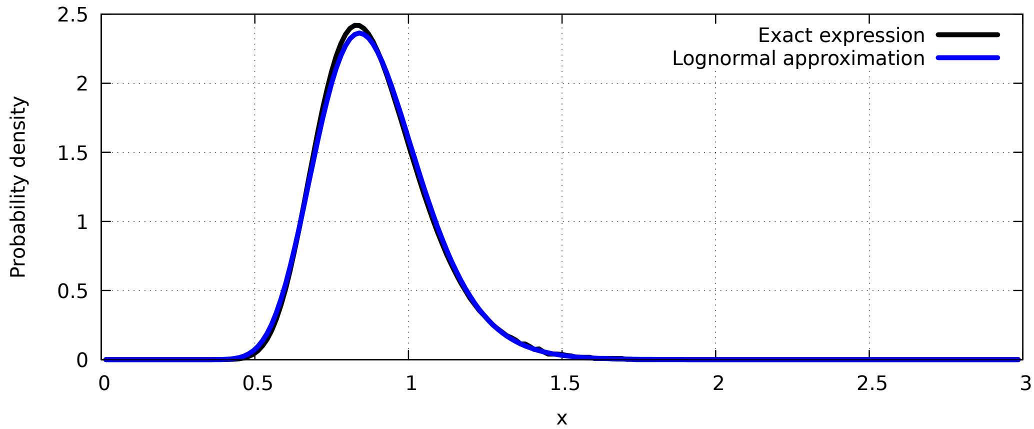


Figure 10.4: Lognormal approximation for the probability density function of Λ_T.

We have the approximation

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] &= \mathrm{e}^{-rT}\int_0^\infty \left(\frac{x}{T} - K\right)^+ \varphi_{\Lambda_T}(x)dx \\
&\simeq \frac{\mathrm{e}^{-rT}}{\sigma_{t,T}\sqrt{2(T-t)\pi}}\int_0^\infty \left(\frac{x}{T} - K\right)^+ \exp\left(-\frac{(\widehat{\mu}_T - \log x)^2}{2(T-t)\widehat{\eta}_T^2}\right)\frac{dx}{x} \\
&= \frac{1}{T}\,\mathrm{e}^{\hat{\mu}_T + \hat{\eta}_T^2 T/2}\Phi(d_1) - K\Phi(d_2) \qquad (10.4.2) \\
&= \frac{\mathbb{E}[\Lambda_T]}{T}\Phi(d_1) - K\Phi(d_2) \\
&= \mathrm{Bl}\left(\frac{\mathbb{E}[\Lambda_T]}{T}, K, 0, \widehat{\eta}_T, T-t\right),
\end{aligned}$$

where

$$d_1 = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\widehat{\eta}_T\sqrt{T}} + \widehat{\eta}_T\frac{\sqrt{T}}{2} = \frac{\widehat{\mu}_T T + \widehat{\eta}_T^2 T - \log(KT)}{\widehat{\eta}_T\sqrt{T}}$$

and

$$d_2 = d_1 - \widehat{\eta}_T\sqrt{T} = \frac{\log(\mathbb{E}^*[\Lambda_T]/(KT))}{\widehat{\eta}_T\sqrt{T}} - \widehat{\eta}_T\frac{\sqrt{T}}{2}.$$

The next Figure 10.5 compares the lognormal approximation to a Monte Carlo estimate of Asian call option prices with $\sigma = 1$, $r = 0.05$, and $S_t = 1.5$, and $K = 1.65$.

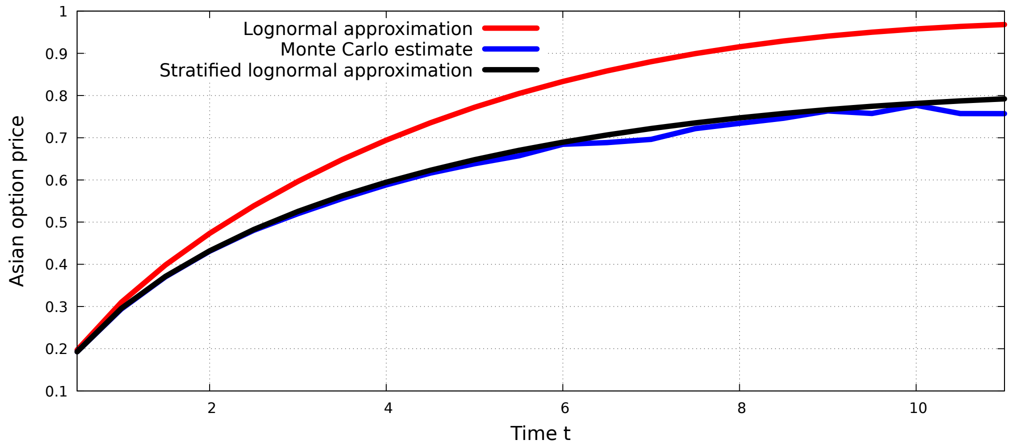


Figure 10.5: Lognormal approximation to the Asian call option price.

```r
library(devtools);
install_github("https://github.com/cran/fOptions")
install_github("https://github.com/cran/fExoticOptions")
library(fExoticOptions);
LevyAsianApproxOption(TypeFlag = "c", S = 1.5, SA = 1.5, X = 1.65, Time = 4, time = 4, r = 0.05, b =
    0.05, sigma = 1.0)
```

Figure 10.5 also includes the stratified approximation

$$\begin{aligned} & e^{-rT}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \qquad (10.4.3) \\ &= e^{-rT}\int_0^\infty \mathbb{E}\left[\left(\frac{x}{T} - K\right)^+ \,\middle|\, S_T = y\right]\varphi_{\Lambda_T|S_T=y}(x)d\mathbb{P}(S_T \leqslant y)dx \\ &\simeq \frac{e^{-rT}}{T}\int_0^\infty \left(e^{-p(y/x)\sigma^2(y/x)T/2+\sigma^2(y/x)T/2}\Phi(d_+(K,y,x)) - KT\Phi(d_-(K,y,x))\right) \\ &\quad \times d\mathbb{P}(S_T \leqslant y)dx, \end{aligned}$$

cf. Privault and Yu, 2016, see the attached codes,* where

$$d_\pm(K,y,x) := \frac{1}{2\sigma(y/x)\sqrt{T}}\log\left(\frac{2x(b_T(y/x) - (1+y/x)a_T(y/x))}{\sigma^2K^2T^2}\right) \pm \frac{\sigma(y/x)\sqrt{T}}{2}$$

*C code - Matlab code - R code (right-click to save as attachment).

and

$$\begin{cases} \sigma^2(z) := \dfrac{1}{T}\log\left(\dfrac{2}{\sigma^2 a_T(z)}\left(\dfrac{b_T(z)}{a_T(z)} - 1 - z\right)\right), \\[2ex] a_T(z) := \dfrac{1}{\sigma^2 p(z)}\left(\Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} + \dfrac{1}{2}\sqrt{\sigma^2 T}\right) - \Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} - \dfrac{1}{2}\sqrt{\sigma^2 T}\right)\right), \\[2ex] b_T(z) := \dfrac{1}{\sigma^2 q(z)}\left(\Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} + \sqrt{\sigma^2 T}\right) - \Phi\left(\dfrac{\log z}{\sqrt{\sigma^2 T}} - \sqrt{\sigma^2 T}\right)\right), \end{cases}$$

and

$$p(z) := \frac{1}{\sqrt{2\pi\sigma^2 T}} \mathrm{e}^{-(\sigma^2 T/2 + \log z)^2/(2\sigma^2 T)}, \quad q(z) := \frac{1}{\sqrt{2\pi\sigma^2 T}} \mathrm{e}^{-(\sigma^2 T + \log z)^2/(2\sigma^2 T)}.$$

Conditioning on the geometric mean price

Asian options on the arithmetic average

$$\frac{1}{T}\int_0^T S_t dt$$

have been priced by conditioning on the geometric mean underlying price

$$G := \exp\left(\frac{1}{T}\int_0^T \log S_t dt\right) \leqslant \exp\left(\log\left(\frac{1}{T}\int_0^T S_t dt\right)\right) = \frac{1}{T}\int_0^T S_t dt$$

in Curran, 1994, as

$$\begin{aligned} & \mathrm{e}^{-rT}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+\right] \\ &= \mathrm{e}^{-rT}\int_0^\infty \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, G = x\right] d\mathbb{P}(G \leqslant x) \\ &= \mathrm{e}^{-rT}\int_0^K \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, G = x\right] d\mathbb{P}(G \leqslant x) \\ &\quad + \mathrm{e}^{-rT}\int_K^\infty \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, G = x\right] d\mathbb{P}(G \leqslant x) \\ &= C_1 + C_2, \end{aligned}$$

where

$$C_1 := \mathrm{e}^{-rT}\int_0^K \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, G = x\right] d\mathbb{P}(G \leqslant x),$$

and

$$\begin{aligned} C_2 &:= \mathrm{e}^{-rT}\int_K^\infty \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, G = x\right] d\mathbb{P}(G \leqslant x) \\ &= \mathrm{e}^{-rT}\int_K^\infty \mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du - K \,\middle|\, G = x\right] d\mathbb{P}(G \leqslant x) \\ &= \frac{\mathrm{e}^{-rT}}{T}\int_K^\infty \mathbb{E}^*\left[\int_0^T S_u du \,\middle|\, G = x\right] d\mathbb{P}(G \leqslant x) - K\mathrm{e}^{-rT}\int_K^\infty d\mathbb{P}(G \leqslant x) \\ &= \frac{\mathrm{e}^{-rT}}{T}\mathbb{E}^*\left[\int_0^T S_u du \mathbb{1}_{\{G \geqslant K\}}\right] - K\mathrm{e}^{-rT}\mathbb{P}(G \geqslant K). \end{aligned}$$

The term C_1 can be estimated by a lognormal approximation given that $G = x$. As for C_2, we note that

$$\begin{aligned} G &= \exp\left(\frac{1}{T}\int_0^T \log S_t dt\right) \\ &= \exp\left(\frac{1}{T}\int_0^T \left(\mu t + \sigma B_t - \frac{\sigma^2 t}{2}\right) dt\right) \\ &= \exp\left(\frac{T}{2}\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma}{T}\int_0^T B_t dt\right), \end{aligned}$$

hence

$$\log G = \frac{T}{2}(\mu - \sigma^2/2) + \frac{\sigma}{T}\int_0^T B_t dt$$

has the Gaussian distribution $\mathcal{N}\left((\mu - \sigma^2/2)T/2, \sigma^2 T/3\right)$ with mean $(\mu - \sigma^2/2)T/2$, and variance

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T B_t dt\right)^2\right] &= \mathbb{E}\left[\int_0^T \int_0^T B_s B_t ds dt\right] \\ &= \int_0^T \int_0^T \mathbb{E}[B_s B_t] ds dt \\ &= 2\int_0^T \int_0^t s ds dt \\ &= \int_0^T t^2 dt \\ &= \frac{T^3}{3}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathbb{P}(G \geqslant K) &= \mathbb{P}(\log G \geqslant \log K) \\ &= \mathbb{P}\left(\frac{T}{2}\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma}{T}\int_0^T B_t dt \geqslant \log K\right) \\ &= \mathbb{P}\left(\int_0^T B_t dt \geqslant \frac{T}{\sigma}\left(-\frac{T}{2}\left(\mu - \frac{\sigma^2}{2}\right) + \log K\right)\right) \\ &= \Phi\left(\frac{\sqrt{3}}{\sigma\sqrt{T}}\left(\frac{T}{2}\left(\mu - \frac{\sigma^2}{2}\right) - \log K\right)\right). \end{aligned}$$

Basket options

Basket options on the portfolio

$$A_T := \sum_{k=1}^{N} \alpha_k S_T^{(k)}$$

have also been priced in Milevsky, 1998 by approximating A_T by a lognormal or a reciprocal gamma random variable, see also Deelstra, Liinev, and Vanmaele, 2004 for additional conditioning on the geometric average of asset prices.

Asian basket options

Moment matching techniques combined with conditioning have been applied to Asian basket options in Deelstra, Diallo, and Vanmaele, 2010. See also Dahl and Benth, 2002 for the pricing of Asian basket options using quasi Monte Carlo simulation.

10.5 PDE Method

Two variables

The price at time t of the Asian call option with payoff (10.1.1) can be written as

$$f(t, S_t, \Lambda_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \Big| \mathscr{F}_t \right], \quad 0 \leqslant t \leqslant T. \tag{10.5.1}$$

Next, we derive the Black-Scholes partial differential equation (PDE) for the value of a self-financing portfolio. Until the end of this chapter we model the asset price $(S_t)_{t\in[0,T]}$ as

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad t \geqslant 0, \tag{10.5.2}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the historical probability measure $\mathbb{P}$.

Proposition 10.8 Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a self-financing portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, $t \geqslant 0$, takes the form

$$V_t = f(t, S_t, \Lambda_t), \qquad t \geqslant 0,$$

where $f \in \mathscr{C}^{1,2,1}((0,T) \times (0,\infty)^2)$ is given by (10.5.1). Then, the function $f(t,x,y)$ satisfies the PDE

$$rf(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + x\frac{\partial f}{\partial y}(t,x,y) + rx\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y), \tag{10.5.3}$$

$0 \leqslant t \leqslant T$, $x > 0$, under the boundary conditions

$$\begin{cases} f(t,0^+,y) = \lim\limits_{x\searrow 0} f(t,x,y) = \mathrm{e}^{-(T-t)r}\left(\dfrac{y}{T} - K\right)^+, & \text{(10.5.4a)} \\ f(t,x,0^+) = \lim\limits_{y\searrow 0} f(t,x,y) = 0, & \text{(10.5.4b)} \\ f(T,x,y) = \left(\dfrac{y}{T} - K\right)^+, & \text{(10.5.4c)} \end{cases}$$

$0 \leqslant t \leqslant T$, $x > 0$, $y \geqslant 0$,

and ξ_t is given by

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, \Lambda_t), \qquad 0 \leqslant t \leqslant T. \tag{10.5.5}$$

Proof. We note that the self-financing condition implies

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \qquad (10.5.6) \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \qquad t \geqslant 0. \end{aligned}$$

Since $d\Lambda_t = S_t dt$, an application of Itô's formula to $f(t,x,y)$ leads to

$$\begin{aligned} dV_t &= f(t,S_t,\Lambda_t) = \frac{\partial f}{\partial t}(t,S_t,\Lambda_t)dt + \frac{\partial f}{\partial y}(t,S_t,\Lambda_t)d\Lambda_t \\ &\quad +\mu S_t \frac{\partial f}{\partial x}(t,S_t,\Lambda_t)dt + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,S_t,\Lambda_t)dt + \sigma S_t \frac{\partial f}{\partial x}(t,S_t,\Lambda_t)dB_t \\ &= \frac{\partial f}{\partial t}(t,S_t,\Lambda_t)dt + S_t\frac{\partial f}{\partial y}(t,S_t,\Lambda_t)dt \\ &\quad +\mu S_t \frac{\partial f}{\partial x}(t,S_t,\Lambda_t)dt + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,S_t,\Lambda_t)dt + \sigma S_t \frac{\partial f}{\partial x}(t,S_t,\Lambda_t)dB_t. \end{aligned} \tag{10.5.7}$$

By respective identification of components in dB_t and dt in (10.5.6) and (10.5.7), we get

$$\begin{cases} r\eta_t A_t dt + \mu\xi_t S_t dt = \displaystyle\frac{\partial f}{\partial t}(t,S_t,\Lambda_t)dt + S_t\frac{\partial f}{\partial y}(t,S_t,\Lambda_t)dt + \mu S_t\frac{\partial f}{\partial x}(t,S_t,\Lambda_t)dt \\ \qquad\qquad\qquad\qquad \displaystyle +\frac{1}{2}S_t^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,S_t,\Lambda_t)dt, \\ \\ \qquad \xi_t S_t \sigma dB_t = S_t\sigma\displaystyle\frac{\partial f}{\partial x}(t,S_t,\Lambda_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rV_t - r\xi_t S_t = \displaystyle\frac{\partial f}{\partial t}(t,S_t,\Lambda_t) + S_t\frac{\partial f}{\partial y}(t,S_t,\Lambda_t)dt + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,S_t,\Lambda_t), \\ \\ \xi_t = \displaystyle\frac{\partial f}{\partial x}(t,S_t,\Lambda_t), \end{cases}$$

i.e.

$$\begin{cases} rf(t,S_t,\Lambda_t) = \displaystyle\frac{\partial f}{\partial t}(t,S_t,\Lambda_t) + S_t\frac{\partial f}{\partial y}(t,S_t,\Lambda_t) + rS_t\frac{\partial f}{\partial x}(t,S_t,\Lambda_t) \\ \qquad\qquad\qquad \displaystyle +\frac{1}{2}S_t^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,S_t,\Lambda_t), \\ \\ \qquad \xi_t = \displaystyle\frac{\partial f}{\partial x}(t,S_t,\Lambda_t). \end{cases}$$

□

Remarks.

i) We have $\xi_T = 0$ at maturity from (10.5.4c) and (10.5.5), which is consistent with the fact that the Asian option is cash-settled at maturity and, close to maturity, its payoff $(\Lambda_T/T - K)^+$ becomes less dependent on the underlying asset price S_T.

ii) If $\Lambda_t/T \geqslant K$, by Exercise 10.8 we have

$$\begin{aligned} f(t,S_t,\Lambda_t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, \mathcal{F}_t\right] \\ &= \mathrm{e}^{-(T-t)r}\left(\frac{\Lambda_t}{T} - K\right) + S_t\frac{1-\mathrm{e}^{-(T-t)r}}{rT}, \end{aligned} \tag{10.5.8}$$

$0 \leqslant t \leqslant T$. In particular, the function

$$f(t,x,y) = \mathrm{e}^{-(T-t)r}\left(\frac{y}{T} - K\right) + x\frac{1-\mathrm{e}^{-(T-t)r}}{rT},$$

$0 \leqslant t \leqslant T$, $x > 0$, $y \geqslant KT$, solves the PDE (10.5.3).

iii) When $\Lambda_t/T \geqslant K$, the Delta ξ_t is given by

$$\xi_t = \frac{\partial f}{\partial x}(t,S_t,\Lambda_t) = \frac{1-\mathrm{e}^{-(T-t)r}}{rT}, \qquad 0 \leqslant t \leqslant T. \tag{10.5.9}$$

Next, we examine two methods which allow one to reduce the Asian option pricing PDE from three variables (t,x,y) to two variables (t,z). Reduction of dimensionality can be of crucial importance when applying discretization scheme whose complexity are of the form N^d where N is the number of discretization steps and d is the dimension of the problem (curse of dimensionality).

(1) One variable with time-independent coefficients

Following Lamberton and Lapeyre, 1996, page 91, we define the auxiliary process

$$Z_t := \frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right) = \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right), \qquad 0 \leqslant t \leqslant T.$$

With this notation, the price of the Asian call option at time t becomes

$$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Big| \mathscr{F}_t\right] = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T(Z_T)^+ \,\big|\, \mathscr{F}_t\right].$$

Lemma 10.9 The price (10.1.2) at time t of the Asian call option with payoff (10.1.1) can be written as

$$\begin{aligned} f(t,S_t,\Lambda_t) &= S_t g(t,Z_t) \qquad (10.5.10)\\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Bigg| \mathscr{F}_t\right], \qquad t \in [0,T], \end{aligned}$$

with the relation

$$f(t,x,y) = xg\left(t, \frac{1}{x}\left(\frac{y}{T} - K\right)\right), \qquad x > 0,\ y \geqslant 0, \quad 0 \leqslant t \leqslant T,$$

where

$$\begin{aligned} g(t,z) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(z + \frac{1}{T}\int_0^{T-t}\frac{S_u}{S_0}du\right)^+\right] \qquad (10.5.11)\\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(z + \frac{\Lambda_{T-t}}{S_0 T}\right)^+\right], \end{aligned}$$

with the boundary condition

$$g(T,z) = z^+, \qquad z \in \mathbb{R}.$$

Proof. For $0 \leqslant s \leqslant t \leqslant T$, we have

$$d(S_t Z_t) = \frac{1}{T} d\left(\int_0^t S_u du - K\right) = \frac{S_t}{T} dt,$$

hence

$$S_t Z_t = S_s Z_s + \int_s^t d(S_u Z_u) = S_s Z_s + \int_s^t \frac{S_u}{T} du,$$

and therefore

$$\frac{S_t Z_t}{S_s} = Z_s + \frac{1}{T}\int_s^t \frac{S_u}{S_s} du, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Since for any $t \in [0,T]$, S_t is positive and $\mathscr{F}_t$-measurable, and S_u/S_t is independent of $\mathscr{F}_t$, $u \geqslant t$, we have:

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\, \mathbf{E}^*\left[S_T (Z_T)^+ \,\middle|\, \mathscr{F}_t\right] &= \mathrm{e}^{-(T-t)r} S_t\, \mathbf{E}^*\left[\left(\frac{S_T}{S_t} Z_T\right)^+ \,\middle|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r} S_t\, \mathbf{E}^*\left[\left(Z_t + \frac{1}{T}\int_t^T \frac{S_u}{S_t} du\right)^+ \,\middle|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r} S_t\, \mathbf{E}^*\left[\left(z + \frac{1}{T}\int_t^T \frac{S_u}{S_t} du\right)^+\right]_{z=Z_t} \\
&= \mathrm{e}^{-(T-t)r} S_t\, \mathbf{E}^*\left[\left(z + \frac{1}{T}\int_0^{T-t} \frac{S_u}{S_0} du\right)^+\right]_{z=Z_t} \\
&= \mathrm{e}^{-(T-t)r} S_t\, \mathbf{E}^*\left[\left(z + \frac{\Lambda_{T-t}}{S_0 T}\right)^+\right]_{z=Z_t} \\
&= S_t g(t, Z_t),
\end{aligned}$$

which proves (10.5.11). □

When $\Lambda_t/T \geqslant K$ we have $Z_t \geqslant 0$, hence in this case by (10.5.8) and (10.5.10) we find

$$g(t, Z_t) = \mathrm{e}^{-(T-t)r} Z_t + \frac{1 - \mathrm{e}^{-(T-t)r}}{rT}, \qquad 0 \leqslant t \leqslant T. \tag{10.5.12}$$

Note that as in (10.2.4), $g(t,z)$ can be computed from the probability density function (10.2.3) of Λ_{T-t}, as

$$\begin{aligned}
g(t,z) &= \mathbf{E}^*\left[\left(z + \frac{\Lambda_{T-t}}{S_0 T}\right)^+\right] \\
&= \int_0^\infty \left(z + \frac{u}{T}\right)^+ d\mathbb{P}\left(\frac{\Lambda_t}{S_0} \leqslant u\right) \\
&= \mathrm{e}^{-p^2\sigma^2 t/8} \\
&\quad \times \int_0^\infty \left(z + \frac{u}{T}\right)^+ \int_0^\infty v^{-1-p} \exp\left(-2\frac{1+v^2}{\sigma^2}\right) \theta\left(\frac{4v}{\sigma^2 u}, (T-t)\frac{\sigma^2}{4}\right) dv \frac{du}{u} \\
&= \mathrm{e}^{-p^2\sigma^2 t/8} \\
&\quad \times \int_{(-zT)\vee 0}^\infty \left(z + \frac{u}{T}\right) \int_0^\infty v^{-1-p} \exp\left(-2\frac{1+v^2}{\sigma^2}\right) \theta\left(\frac{4v}{\sigma^2 u}, (T-t)\frac{\sigma^2}{4}\right) dv \frac{du}{u} \\
&= z\mathrm{e}^{-p^2\sigma^2 t/8} \int_{(-zT)\vee 0}^\infty \int_0^\infty v^{-1-p} \exp\left(-2\frac{1+v^2}{\sigma^2}\right) \theta\left(\frac{4v}{\sigma^2 u}, (T-t)\frac{\sigma^2}{4}\right) dv \frac{du}{u} \\
&\quad + \frac{1}{T}\mathrm{e}^{-p^2\sigma^2 t/8} \int_{(-zT)\vee 0}^\infty \int_0^\infty v^{-1-p} \exp\left(-2\frac{1+v^2}{\sigma^2}\right) \theta\left(\frac{4v}{\sigma^2 u}, (T-t)\frac{\sigma^2}{4}\right) dv du.
\end{aligned}$$

The next proposition gives a replicating hedging strategy for Asian options.

Proposition 10.10 (Rogers and Shi, 1995). Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a self-financing portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, $t \in [0,T]$, is given by

$$V_t = S_t g(t, Z_t) = S_t g\left(t, \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right), \qquad 0 \leqslant t \leqslant T,$$

where $g \in \mathscr{C}^{1,2}((0,T)\times(0,\infty))$ is given by (10.5.11). Then, the function $g(t,z)$ satisfies the PDE

$$\frac{\partial g}{\partial t}(t,z) + \left(\frac{1}{T} - rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) = 0, \tag{10.5.13}$$

$0 \leqslant t \leqslant T$, under the terminal condition

$$g(T,z) = z^+, \qquad z \in \mathbb{R}, \tag{10.5.14}$$

and the corresponding replicating portfolio Delta is given by

$$\xi_t = g(t, Z_t) - Z_t \frac{\partial g}{\partial z}(t, Z_t), \qquad 0 \leqslant t \leqslant T. \tag{10.5.15}$$

Proof. By (10.5.2) and the Itô formula applied to $1/S_t$, we have

$$\begin{aligned} d\left(\frac{1}{S_t}\right) &= -\frac{dS_t}{(S_t)^2} + \frac{2}{2}\frac{(dS_t)^2}{(S_t)^3} \\ &= \frac{1}{S_t}\left((-\mu + \sigma^2)\,dt - \sigma dB_t\right), \end{aligned}$$

hence

$$\begin{aligned} dZ_t &= d\left(\frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right) \\ &= d\left(\frac{\Lambda_t}{TS_t} - \frac{K}{S_t}\right) \\ &= \frac{1}{T} d\left(\frac{\Lambda_t}{S_t}\right) - K d\left(\frac{1}{S_t}\right) \\ &= \frac{1}{T}\frac{d\Lambda_t}{S_t} + \left(\frac{\Lambda_t}{T} - K\right) d\left(\frac{1}{S_t}\right) \\ &= \frac{dt}{T} + S_t Z_t d\left(\frac{1}{S_t}\right) \\ &= \frac{dt}{T} + Z_t\left(-\mu + \sigma^2\right) dt - Z_t \sigma dB_t. \end{aligned}$$

By the self-financing condition, we have

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \qquad t \geqslant 0. \end{aligned} \tag{10.5.16}$$

Another application of Itô's formula to $f(t, S_t, Z_t) = S_t g(t, Z_t)$ leads to

$$d(S_t g(t, Z_t)) = g(t, Z_t) dS_t + S_t dg(t, Z_t) + dS_t \bullet dg(t, Z_t)$$

$$
\begin{aligned}
&= g(t,Z_t)dS_t + S_t\frac{\partial g}{\partial t}(t,Z_t)dt + S_t\frac{\partial g}{\partial z}(t,Z_t)dZ_t \\
&\quad +\frac{1}{2}S_t\frac{\partial^2 g}{\partial z^2}(t,Z_t)(dZ_t)^2 + dS_t \bullet dg(t,Z_t) \\
&= \mu S_t g(t,Z_t)dt + \sigma S_t g(t,Z_t)dB_t + S_t\frac{\partial g}{\partial t}(t,Z_t)dt \\
&\quad +S_tZ_t\left(-\mu+\sigma^2\right)\frac{\partial g}{\partial z}(t,Z_t)dt + \frac{1}{T}S_t\frac{\partial g}{\partial z}(t,Z_t)dt - \sigma S_tZ_t\frac{\partial g}{\partial z}(t,Z_t)dB_t \\
&\quad +\frac{1}{2}\sigma^2Z_t^2S_t\frac{\partial^2 g}{\partial z^2}(t,Z_t)dt - \sigma^2S_tZ_t\frac{\partial g}{\partial z}(t,Z_t)dt \\
&= \mu S_t g(t,Z_t)dt + S_t\frac{\partial g}{\partial t}(t,Z_t)dt + S_tZ_t\left(-\mu+\sigma^2\right)\frac{\partial g}{\partial z}(t,Z_t)dt + \frac{1}{T}S_t\frac{\partial g}{\partial z}(t,Z_t)dt \\
&\quad +\frac{1}{2}\sigma^2Z_t^2S_t\frac{\partial^2 g}{\partial z^2}(t,Z_t)dt - \sigma^2S_tZ_t\frac{\partial g}{\partial z}(t,Z_t)dt \\
&\quad +\sigma S_t g(t,Z_t)dB_t - \sigma S_tZ_t\frac{\partial g}{\partial z}(t,Z_t)dB_t. \qquad (10.5.17)
\end{aligned}
$$

By respective identification of components in dB_t and dt in (10.5.16) and (10.5.17), we get

$$
\begin{cases}
r\eta_tA_t + \mu\xi_tS_t = \mu S_t g(t,Z_t) + S_t\dfrac{\partial g}{\partial t}(t,Z_t) - \mu S_tZ_t\dfrac{\partial g}{\partial z}(t,Z_t) \\
\qquad\qquad + \dfrac{1}{T}S_t\dfrac{\partial g}{\partial z}(t,Z_t) + \dfrac{1}{2}\sigma^2Z_t^2S_t\dfrac{\partial^2 g}{\partial z^2}(t,Z_t), \\
\\
\xi_tS_t\sigma = \sigma S_t g(t,Z_t) - \sigma S_tZ_t\dfrac{\partial g}{\partial z}(t,Z_t),
\end{cases}
$$

hence

$$
\begin{cases}
rV_t - r\xi_tS_t = S_t\dfrac{\partial g}{\partial t}(t,Z_t) + \dfrac{1}{T}S_t\dfrac{\partial g}{\partial z}(t,Z_t) + \dfrac{1}{2}\sigma^2Z_t^2S_t\dfrac{\partial^2 g}{\partial z^2}(t,Z_t), \\
\\
\xi_t = g(t,Z_t) - Z_t\dfrac{\partial g}{\partial z}(t,Z_t),
\end{cases}
$$

i.e.

$$
\begin{cases}
\dfrac{\partial g}{\partial t}(t,z) + \left(\dfrac{1}{T} - rz\right)\dfrac{\partial g}{\partial z}(t,z) + \dfrac{1}{2}\sigma^2z^2\dfrac{\partial^2 g}{\partial z^2}(t,z) = 0, \\
\\
\xi_t = g(t,Z_t) - Z_t\dfrac{\partial g}{\partial z}(t,Z_t),
\end{cases}
$$

under the terminal condition $g(T,z) = z^+$, $z \in \mathbb{R}$, which follows from (10.5.11). □

When $\Lambda_t/T \geqslant K$ we have $Z_t \geqslant 0$ and (10.5.12) and (10.5.15) show that

$$
\begin{aligned}
\xi_t &= g(t,Z_t) - Z_t\frac{\partial g}{\partial z}(t,Z_t) \\
&= \mathrm{e}^{-(T-t)r}Z_t + \frac{1-\mathrm{e}^{-(T-t)r}}{rT} - \mathrm{e}^{-(T-t)r}Z_t \\
&= \frac{1-\mathrm{e}^{-(T-t)r}}{rT}, \qquad 0 \leqslant t \leqslant T,
\end{aligned}
$$

which recovers (10.5.9). Similarly, from (10.5.14) we recover

$$
\xi_T = g(T,Z_T) - Z_T\frac{\partial g}{\partial z}(T,Z_T) = Z_T\mathbb{1}_{\{Z_T\geqslant 0\}} - Z_T\mathbb{1}_{\{Z_T\geqslant 0\}} = 0
$$

at maturity.

We also check that

$$\begin{aligned}
\xi_t &= \mathrm{e}^{-(T-t)r}\sigma S_t\frac{\partial f}{\partial x}f(t,S_t,Z_t)-\sigma Z_t\frac{\partial f}{\partial z}f(t,S_t,Z_t)\\
&= \mathrm{e}^{-(T-t)r}\left(-Z_t\frac{\partial g}{\partial z}(t,Z_t)+g(t,Z_t)\right)\\
&= \mathrm{e}^{-(T-t)r}\left(S_t\frac{\partial g}{\partial x}\left(t,\frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du-K\right)\right)_{|x=S_t}+g(t,Z_t)\right)\\
&= \frac{\partial}{\partial x}\left(x\mathrm{e}^{-(T-t)r}g\left(t,\frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du-K\right)\right)\right)_{|x=S_t}, \quad 0\leqslant t\leqslant T.
\end{aligned}$$

We also find that the amount invested on the riskless asset is given by

$$\eta_t A_t = Z_t S_t\frac{\partial g}{\partial z}(t,Z_t).$$

Next we note that a PDE with no first-order derivative term can be obtained using time-dependent coefficients.

(2) One variable with time-dependent coefficients

Define now the auxiliary process

$$\begin{aligned}
U_t &:= \frac{1-\mathrm{e}^{-(T-t)r}}{rT}+\mathrm{e}^{-(T-t)r}\frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du-K\right)\\
&= \frac{1}{rT}\left(1-\mathrm{e}^{-(T-t)r}\right)+\mathrm{e}^{-(T-t)r}Z_t, \quad 0\leqslant t\leqslant T,
\end{aligned}$$

i.e.

$$Z_t=\mathrm{e}^{(T-t)r}U_t+\frac{\mathrm{e}^{(T-t)r}-1}{rT}, \qquad 0\leqslant t\leqslant T.$$

We have

$$\begin{aligned}
dU_t &= -\frac{1}{T}\mathrm{e}^{-(T-t)r}dt+r\mathrm{e}^{-(T-t)r}Z_t dt+\mathrm{e}^{-(T-t)r}dZ_t\\
&= \mathrm{e}^{-(T-t)r}\sigma^2 Z_t dt-\mathrm{e}^{-(T-t)r}\sigma Z_t dB_t-(\mu-r)\mathrm{e}^{-(T-t)r}Z_t dt\\
&= -\mathrm{e}^{-(T-t)r}\sigma Z_t d\widehat{B}_t, \qquad t\geqslant 0,
\end{aligned}$$

where

$$d\widehat{B}_t=dB_t-\sigma dt+\frac{\mu-r}{\sigma}dt=d\widetilde{B}_t-\sigma dt$$

is a standard Brownian motion under

$$d\widehat{\mathbb{P}}=\mathrm{e}^{\sigma B_T-\sigma^2 t/2}d\mathbb{P}^*=\mathrm{e}^{-rT}\frac{S_T}{S_0}d\mathbb{P}^*.$$

Lemma 10.11 The Asian call option price can be written as

$$S_t h(t,U_t)=\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du-K\right)^+\,\Big|\,\mathscr{F}_t\right],$$

where the function $h(t,y)$ is given by

$$h(t,y)=\widehat{\mathbb{E}}\big[(U_T)^+\,\big|\,U_t=y\big], \qquad 0\leqslant t\leqslant T. \tag{10.5.18}$$

Proof. We have

$$U_T = \frac{1}{S_T}\left(\frac{1}{T}\int_0^T S_u du - K\right) = Z_T,$$

and

$$\frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} = e^{(B_T-B_t)\sigma-(T-t)\sigma^2/2} = \frac{e^{-rT}S_T}{e^{-rt}S_t},$$

hence the price of the Asian call option is

$$\begin{aligned}
e^{-(T-t)r}\mathbb{E}^*[S_T(Z_T)^+ \mid \mathscr{F}_t] &= e^{-(T-t)r}\mathbb{E}^*[S_T(U_T)^+ \mid \mathscr{F}_t] \\
&= S_t\mathbb{E}^*\left[\frac{e^{-rT}S_T}{e^{-rt}S_t}(U_T)^+ \,\middle|\, \mathscr{F}_t\right] \\
&= S_t\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}}(U_T)^+ \,\middle|\, \mathscr{F}_t\right] \\
&= S_t\widehat{\mathbb{E}}[(U_T)^+ \mid \mathscr{F}_t].
\end{aligned}$$

□

The next proposition gives a replicating hedging strategy for Asian options. See § 7.5.3 of Shreve, 2004 and references therein for a different derivation of the PDE (10.5.19).

Proposition 10.12 (Večeř, 2001). Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a self-financing portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, $t \geqslant 0$, is given by

$$V_t = S_t h(t, U_t), \qquad t \geqslant 0,$$

where $h \in \mathscr{C}^{1,2}((0,T)\times(0,\infty))$ is given by (10.5.18). Then, the function $h(t,z)$ satisfies the PDE

$$\frac{\partial h}{\partial t}(t,y) + \frac{\sigma^2}{2}\left(\frac{1-e^{-(T-t)r}}{rT} - y\right)^2 \frac{\partial^2 h}{\partial y^2}(t,y) = 0, \tag{10.5.19}$$

under the terminal condition

$$h(T,z) = z^+,$$

and the corresponding replicating portfolio is given by

$$\xi_t = h(t,U_t) - Z_t\frac{\partial h}{\partial y}(t,U_t), \qquad 0 \leqslant t \leqslant T.$$

Proof. By the self-financing condition (10.5.6) we have

$$\begin{aligned}
dV_t &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\
&= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t,
\end{aligned} \tag{10.5.20}$$

$t \geqslant 0$. By Itô's formula we get

$$\begin{aligned}
d(S_t h(t,U_t)) &= h(t,U_t)dS_t + S_t dh(t,U_t) + dS_t \bullet dh(t,U_t) \\
&= \mu S_t h(t,U_t)dt + \sigma S_t h(t,U_t)dB_t
\end{aligned}$$

$$
\begin{aligned}
&+S_t\left(\frac{\partial h}{\partial t}(t,U_t)dt+\frac{\partial h}{\partial y}(t,U_t)dU_t+\frac{1}{2}\frac{\partial^2 h}{\partial y^2}(t,U_t)(dU_t)^2\right)\\
&+\frac{\partial h}{\partial y}(t,U_t)dS_t\bullet dU_t\\
= \ &\mu S_t h(t,U_t)dt+\sigma S_t h(t,U_t)dB_t-S_t(\mu-r)\frac{\partial h}{\partial y}(t,U_t)Z_t dt\\
&+S_t\left(\frac{\partial h}{\partial t}(t,U_t)dt-\sigma\frac{\partial h}{\partial y}(t,U_t)Z_t d\widetilde{B}_t+\frac{\sigma^2}{2}Z_t^2\frac{\partial^2 h}{\partial y^2}(t,U_t)dt\right)\\
&-\sigma^2 S_t\frac{\partial h}{\partial y}(t,U_t)Z_t dt\\
= \ &\mu S_t h(t,U_t)dt+\sigma S_t h(t,U_t)dB_t-S_t(\mu-r)\frac{\partial h}{\partial y}(t,U_t)Z_t dt\\
&+S_t\left(\frac{\partial h}{\partial t}(t,U_t)dt-\sigma\frac{\partial h}{\partial y}(t,U_t)Z_t(dB_t-\sigma dt)+\frac{\sigma^2}{2}Z_t^2\frac{\partial^2 h}{\partial y^2}(t,U_t)dt\right)\\
&-\sigma^2 S_t\frac{\partial h}{\partial y}(t,U_t)Z_t dt. \qquad (10.5.21)
\end{aligned}
$$

By respective identification of components in dB_t and dt in (10.5.20) and (10.5.21), we get

$$
\begin{cases}
r\eta_t A_t+\mu\xi_t S_t = \mu S_t h(t,U_t)-(\mu-r)S_t Z_t\frac{\partial h}{\partial y}(t,U_t)dt+S_t\frac{\partial h}{\partial t}(t,U_t)\\
\qquad\qquad +\frac{\sigma^2}{2}S_t Z_t^2\frac{\partial^2 h}{\partial y^2}(t,U_t),\\
\\
\xi_t = h(t,U_t)-Z_t\frac{\partial h}{\partial y}(t,U_t),
\end{cases}
$$

hence

$$
\begin{cases}
r\eta_t A_t=-rS_t(\xi_t-h(t,U_t))+S_t\frac{\partial h}{\partial t}(t,U_t)+\frac{\sigma^2}{2}S_t Z_t^2\frac{\partial^2 h}{\partial y^2}(t,U_t),\\
\\
\xi_t=h(t,U_t)-Z_t\frac{\partial h}{\partial y}(t,U_t),
\end{cases}
$$

and

$$
\begin{cases}
\frac{\partial h}{\partial t}(t,y)+\frac{\sigma^2}{2}\left(\frac{1-\mathrm{e}^{-(T-t)r}}{rT}-y\right)^2\frac{\partial^2 h}{\partial y^2}(t,y)=0,\\
\\
\xi_t=h(t,U_t)+\left(\frac{1-\mathrm{e}^{-(T-t)r}}{rT}-U_t\right)\frac{\partial h}{\partial y}(t,U_t),
\end{cases}
$$

under the terminal condition

$$h(T,z)=z^+.$$

□

We also find the riskless portfolio allocation

$$\eta_t A_t=\mathrm{e}^{(T-t)r}S_t\left(U_t-\frac{1-\mathrm{e}^{-(T-t)r}}{rT}\right)\frac{\partial h}{\partial y}(t,U_t)=S_t Z_t\frac{\partial h}{\partial y}(t,U_t).$$

Various implementations of Asian pricing methods can be found in this **IPython notebook** (right-click to save as attachment). See also the **Premia** website.

Exercises

Exercise 10.1 Compute the first and second moments of the time integral $\int_\tau^T S_t dt$ for $\tau \in [0,T)$, where $(S_t)_{t\in\mathbb{R}_+}$ is the geometric Brownian motion $S_t := S_0 \mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \geqslant 0$.

Exercise 10.2 Consider the short rate process $r_t = \sigma B_t$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Find the probability distribution of the time integral $\int_0^T r_s ds$.
b) Compute the price

$$\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\left(\int_0^T r_u du - K\right)^+\right]$$

of a caplet on the forward rate $\int_0^T r_s ds$.

Exercise 10.3 Asian call option with a *negative* strike price. Consider the asset price process

$$S_t = S_0\,\mathrm{e}^{rt+\sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Assuming that $K \leqslant 0$, compute the price

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, \mathscr{F}_t\right]$$

of the Asian option at time $t \in [0,T]$.

Exercise 10.4 Consider the Asian forward contract with payoff

$$\frac{1}{T}\int_0^T S_u du - K, \tag{10.5.22}$$

where $S_u = S_0\,\mathrm{e}^{\sigma B_u + ru - \sigma^2 u/2}$, $u \geqslant 0$, and $(B_u)_{u\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{P}^*$.

a) Price the long forward Asian contract at any time $t \in [0,T]$.
b) Derive a call-put parity relation between the prices

$$C(t,K) := \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, \mathscr{F}_t\right]$$

and

$$P(t,K) := \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(K - \frac{1}{T}\int_0^T S_u du\right)^+ \,\middle|\, \mathscr{F}_t\right]$$

of Asian call and put options.
c) Find the self-financing portfolio strategy $(\xi_t)_{t\in[0,T]}$ hedging the Asian forward contract with payoff (10.5.22), where ξ_t denotes the quantity invested at time $t \in [0,T]$ in the risky asset S_t.
d) Compute the numerical value of the price

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du - K \,\middle|\, \mathscr{F}_t\right]$$

of the long forward Asian contract on Light Sweet Crude Oil Futures (CLZ22.NYM) using the following market data:

Issue date: 2022-01-01,
Maturity T =2022-12-31,
Strike price K =\$80,
Interest rate r =2% per year,
t =2022-04-01,
Number of business days per month: 21, per year: 252.

```
library(quantmod)
getSymbols("CLZ22.NYM",from="2022-01-01",to="2022-04-01",src="yahoo")
futures=Cl(`CLZ22.NYM`)
chartSeries(futures,up.col="blue",theme="white")
n = length(!is.na(futures))
```

Exercise 10.5 Compute the price

$$e^{-(T-t)r}\mathbb{E}^*\left[\left(\exp\left(\frac{1}{T}\int_0^T \log S_u du\right)-K\right)^+ \Big| \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T.$$

at time t of the geometric Asian option with maturity T, where $S_t = S_0 e^{rt+\sigma B_t - \sigma^2 t/2}$, $t \in [0,T]$.

Hint: When $X \simeq \mathscr{N}(0,v^2)$ is a centered Gaussian random variable with variance $v^2 > 0$, we have

$$\mathbb{E}^*[(e^{m+X}-K)^+] = e^{m+v^2/2}\Phi(v+(m-\log K)/v) - K\Phi((m-\log K)/v).$$

Exercise 10.6 Consider a CIR process $(r_t)_{t\in\mathbb{R}_+}$ given by

$$dr_t = -\lambda(r_t - m)dt + \sigma\sqrt{r_t}dB_t, \qquad (10.5.23)$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, and let

$$\Lambda_t := \frac{1}{T-\tau}\int_\tau^t r_s ds, \qquad t \in [\tau, T].$$

Compute the price at time $t \in [\tau, T]$ of the Asian option with payoff $(\Lambda_T - K)^+$, under the condition $\Lambda_t \geqslant K$.

Exercise 10.7 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ which is a *sub*martingale under the risk-neutral probability measure $\mathbb{P}^*$, in a market with risk-free interest rate $r > 0$, and let $\phi(x) = (x-K)^+$ be the (convex) payoff function of the European call option.

Show that, for any sequence $0 < T_1 < \cdots < T_n$, the price of the option on average with payoff

$$\phi\left(\frac{S_{T_1}+\cdots+S_{T_n}}{n}\right)$$

can be upper bounded by the price of the European call option with maturity T_n, *i.e.* show that

$$\mathbb{E}^*\left[\phi\left(\frac{S_{T_1}+\cdots+S_{T_n}}{n}\right)\right] \leqslant \mathbb{E}^*[\phi(S_{T_n})].$$

Exercise 10.8 Let $(S_t)_{t\in\mathbb{R}_+}$ denote a risky asset whose price S_t is given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. Compute the price at time $t \in [\tau, T]$ of the Asian option with payoff

$$\left(\frac{1}{T-\tau}\int_\tau^T S_u du - K\right)^+,$$

under the condition that

$$A_t := \frac{1}{T-\tau}\int_\tau^t S_u du \geqslant K.$$

Exercise 10.9 Pricing Asian options by PDEs. Show that the functions $g(t,z)$ and $h(t,y)$ are linked by the relation

$$g(t,z) = h\left(t, \frac{1-\mathrm{e}^{-(T-t)r}}{rT} + \mathrm{e}^{-(T-t)r}z\right), \qquad 0 \leqslant t \leqslant T, \quad z > 0,$$

and that the PDE (1.35) for $h(t,y)$ can be derived from the PDE (1.33) for $g(t,z)$ and the above relation.

Exercise 10.10 (Brown et al., 2016) Given $S_t := S_0 \mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$ a geometric Brownian motion and letting

$$\widetilde{Z}_t := \frac{\mathrm{e}^{-(T-t)r}}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right) = \frac{\mathrm{e}^{-(T-t)r}}{S_t}\left(\frac{\Lambda_t}{T} - K\right), \qquad 0 \leqslant t \leqslant T,$$

find the PDE satisfied by the pricing function $\widetilde{g}(t,z)$ such that

$$S_t \widetilde{g}(t, \widetilde{Z}_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \middle| \mathcal{F}_t\right].$$

Exercise 10.11 Hedging Asian options (Yang, Ewald, and Menkens, 2011).

a) Compute the Asian option price $f(t, S_t, \Lambda_t)$ when $\Lambda_t/T \geqslant K$.
b) Compute the hedging portfolio allocation (ξ_t, η_t) when $\Lambda_t/T \geqslant K$.
c) At maturity we have $f(T, S_T, \Lambda_T) = (\Lambda_T/T - K)^+$, hence $\xi_T = 0$ and

$$\eta_T A_T = A_T \frac{\mathrm{e}^{-rT}}{A_0}\left(\frac{\Lambda_T}{T} - K\right)\mathbb{1}_{\{\Lambda_T > KT\}} = \left(\frac{\Lambda_T}{T} - K\right)^+.$$

d) Show that the Asian option with payoff $(\Lambda_T - K)^+$ can be hedged by the self-financing portfolio

$$\xi_t = \frac{1}{S_t}\left(f(t, S_t, \Lambda_t) - \mathrm{e}^{-(T-t)r}\left(\frac{\Lambda_t}{T} - K\right) h\left(t, \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right)\right)$$

in the asset S_t and

$$\eta_t = \frac{\mathrm{e}^{-rT}}{A_0}\left(\frac{\Lambda_t}{T} - K\right) h\left(t, \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right), \qquad 0 \leqslant t \leqslant T,$$

in the riskless asset $A_t = A_0 \mathrm{e}^{rt}$, where $h(t,z)$ is solution to a partial differential equation to be written explicitly.

Exercise 10.12 Asian options with dividends. Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled as $dS_t = (\mu-\delta)S_t dt + \sigma S_t dB_t$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $\delta > 0$ is a continuous-time dividend rate.

a) Write down the self-financing condition for the portfolio value $V_t = \xi_t S_t + \eta_t A_t$ with $A_t = A_0\,\mathrm{e}^{rt}$, assuming that all dividends are reinvested.

b) Derive the Black-Scholes PDE for the function $g_\delta(t,x,y)$ such that $V_t = g_\delta(t,S_t,\Lambda_t)$ at time $t\in[0,T]$.

```
install.packages("quantmod")
library(quantmod)
getDividends("Z74.SI",from="2018-01-01",to="2018-12-31",src="yahoo")
getSymbols("Z74.SI",from="2018-11-16",to="2018-12-19",src="yahoo")
T <- chart_theme(); T$col$line.col <- "black"
chart_Series(Op(`Z74.SI`),name="Opening prices (black) - Closing prices (blue)",lty=4,theme=T)
add_TA(Cl(`Z74.SI`),lwd=2,lty=5,legend='Difference',col="blue",on = 1)
```

	Z74.SI.div
2018-07-26	0.107
2018-12-17	0.068
2018-12-18	0.068

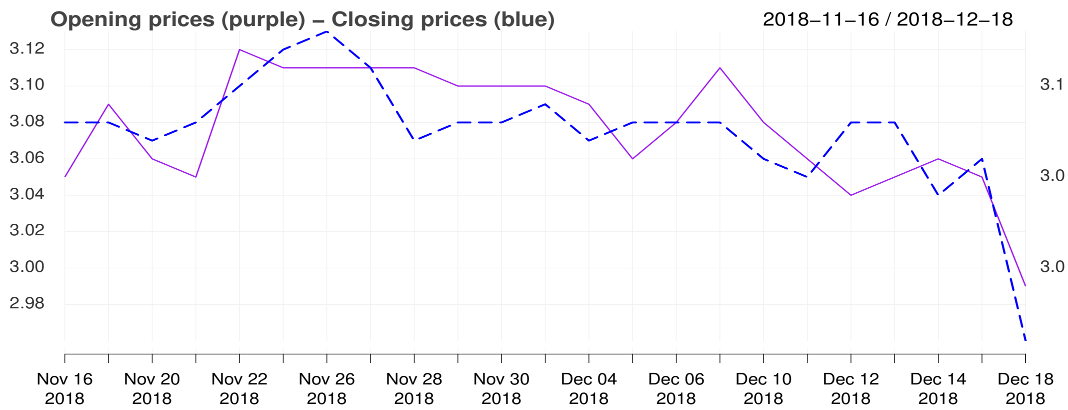


Figure 10.6: SGD0.068 dividend detached on 18 Dec 2018 on Z74.SI.

The difference between the closing price on Dec 17 (\$3.06) and the opening price on Dec 18 (\$2.99) is $\$3.06 - \$2.99 = \$0.07$. The adjusted price on Dec 17 (\$2.992) is the closing price (\$3.06) minus the dividend (\$0.068).

Z74.SI	Open	High	Low	Close	Volume	Adjusted (ex-dividend)
2018-12-17	3.05	3.08	3.05	**3.06**	17441000	**2.992**
2018-12-18	**2.99**	2.99	2.96	2.96	28456400	2.960

The dividend rate α is given by $\alpha = 0.068/3.06 = 2.22\%$.

Exercise Solutions

Chapter 1

Exercise 1.1 By absence of arbitrage we have $(1-\alpha)\,\mathrm{e}^{r_d T} = \mathrm{e}^{rT}$, hence $\alpha = 1 - \mathrm{e}^{(r-r_d)T}$.

Exercise 1.2

a) The bond payoff $\mathbb{1}_{\{\tau > T-t\}}$ is discounted according to the risk-free rate, before taking expectation.

b) We have $\mathbb{E}\left[\mathbb{1}_{\{\tau>T-t\}}\right] = \mathrm{e}^{-\lambda(T-t)}$, hence $P_d(t,T) = \mathrm{e}^{-(\lambda+r)(T-t)}$.

c) We have $P_M(t,T) = \mathrm{e}^{-(\lambda+r)(T-t)}$, hence $\lambda = -r - \frac{1}{T-t}\log P_M(t,T)$.

Exercise 1.3

a) We have

$$r_t = -a\int_0^t r_s ds + \sigma B_t^{(1)}, \quad t \geqslant 0,$$

hence

$$\begin{aligned}
\int_0^t r_s ds &= \frac{1}{a}\left(\sigma B_t^{(1)} - r_t\right)\\
&= \frac{\sigma}{a}\left(B_t^{(1)} - \int_0^t \mathrm{e}^{-(t-s)a} dB_s^{(1)}\right)\\
&= \frac{\sigma}{a}\int_0^t \left(1 - \mathrm{e}^{-(t-s)a}\right) dB_s^{(1)},
\end{aligned}$$

and

$$\begin{aligned}
\int_t^T r_s ds &= \int_0^T r_s ds - \int_0^t r_s ds\\
&= \frac{\sigma}{a}\int_0^T \left(1-\mathrm{e}^{-(T-s)a}\right)dB_s^{(1)} - \frac{\sigma}{a}\int_0^t \left(1-\mathrm{e}^{-(t-s)a}\right)dB_s^{(1)}\\
&= -\frac{\sigma}{a}\left(\int_0^t \left(\mathrm{e}^{-(T-s)a} - \mathrm{e}^{-(t-s)a}\right)dB_s^{(1)} + \int_t^T \left(\mathrm{e}^{-(T-s)a}-1\right)dB_s^{(1)}\right)\\
&= -\frac{\sigma}{a}\left(\mathrm{e}^{-(T-t)a}-1\right)\int_0^t \mathrm{e}^{-(t-s)a} dB_s^{(1)} - \frac{\sigma}{a}\int_t^T \left(\mathrm{e}^{-(T-s)a}-1\right)dB_s^{(1)}\\
&= -\frac{1}{a}\left(\mathrm{e}^{-(T-t)a}-1\right)r_t - \frac{\sigma}{a}\int_t^T \left(\mathrm{e}^{-(T-s)a}-1\right)dB_s^{(1)}.
\end{aligned}$$

The answer for λ_t is similar.

b) As a consequence of the answer to the previous question, we have

$$\mathbb{E}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right] = C(a,t,T)r_t + C(b,t,T)\lambda_t,$$

and

$$\begin{aligned}
&\operatorname{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right] \\
&= \operatorname{Var}\left[\int_t^T r_s ds \,\Big|\, \mathscr{F}_t\right] + \operatorname{Var}\left[\int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right] \\
&\quad +2\operatorname{Cov}\left(\int_t^T X_s ds, \int_t^T Y_s ds \,\Big|\, \mathscr{F}_t\right) \\
&= \frac{\sigma^2}{a^2}\int_t^T (\mathrm{e}^{-(T-s)a} - 1)^2 ds \\
&\quad +2\rho\frac{\sigma\eta}{ab}\int_t^T (\mathrm{e}^{-(T-s)a} - 1)(\mathrm{e}^{-(T-s)b} - 1) ds \\
&\quad +\frac{\eta^2}{b^2}\int_t^T (\mathrm{e}^{-(T-s)b} - 1)^2 ds \\
&= \sigma^2\int_t^T C^2(a,s,T)ds + 2\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds \\
&\quad +\eta^2\int_t^T C^2(b,sT)ds,
\end{aligned}$$

from the Itô isometry.

Exercise 1.4 (Exercise 1.3 continued).

a) We use the fact that $(r_t, \lambda_t)_{t\in[0,T]}$ is a Markov process.

b) We use the tower property of the conditional expectation given $\mathscr{F}_t$.

c) Writing $F(t, r_t, \lambda_t) = P(t,T)$, we have

$$\begin{aligned}
&d\left(\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)\right) \\
&= -(r_t+\lambda_t)\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} dP(t,T) \\
&= -(r_t+\lambda_t)\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} dF(t,r_t,\lambda_t) \\
&= -(r_t+\lambda_t)\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\frac{\partial F}{\partial x}(t,r_t,\lambda_t)dr_t \\
&+ \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\frac{\partial F}{\partial y}(t,r_t,\lambda_t)d\lambda_t + \frac{1}{2}\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t)dt \\
&+ \frac{1}{2}\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t)dt \\
&+ \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\rho\frac{\partial^2 F}{\partial x\partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\frac{\partial F}{\partial t}(t,r_t,\lambda_t)dt \\
&= \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\frac{\partial F}{\partial x}(t,r_t,\lambda_t)\sigma_1(t,r_t)dB_t^{(1)} + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\frac{\partial F}{\partial y}(t,r_t,\lambda_t)\sigma_2(t,\lambda_t)dB_t^{(2)} \\
&+ \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds}\left(-(r_t+\lambda_t)P(t,T) + \frac{\partial F}{\partial x}(t,r_t,\lambda_t)\mu_1(t,r_t)\right. \\
&+\frac{\partial F}{\partial y}(t,r_t,\lambda_t)\mu_2(t,\lambda_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t) \\
&\left.+\rho\frac{\partial^2 F}{\partial x\partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t) + \frac{\partial F}{\partial t}(t,r_t,\lambda_t)\right)dt,
\end{aligned}$$

hence the bond pricing PDE is

$$\begin{aligned}
&-(x+y)F(t,x,y)+\mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y)\\
&+\mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y)+\frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y)\\
&+\frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y)+\rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x\partial y}(t,x,y)+\frac{\partial F}{\partial t}(t,r_t,\lambda_t)=0.
\end{aligned}$$

d) We have

$$\begin{aligned}
P(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T r_s ds-\int_t^T \lambda_s ds\right)\,\Big|\,\mathscr{F}_t\right]\\
&= \mathbb{1}_{\{\tau>t\}}\exp\left(-\mathbb{E}\left[\int_t^T r_s ds\,\Big|\,\mathscr{F}_t\right]-\mathbb{E}\left[\int_t^T \lambda_s ds\,\Big|\,\mathscr{F}_t\right]\right)\\
&\quad\times\exp\left(\frac{1}{2}\mathrm{Var}\left[\int_t^T r_s ds+\int_t^T \lambda_s ds\,\Big|\,\mathscr{F}_t\right]\right)\\
&= \mathbb{1}_{\{\tau>t\}}\exp\left(-C(a,t,T)r_t-C(b,t,T)\lambda_t\right)\\
&\quad\times\exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds+\frac{\eta^2}{2}\int_t^T C^2(b,s,T)\,\mathrm{e}^{-(T-s)b}ds\right)\\
&\quad\times\exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right).
\end{aligned}$$

e) This is a direct consequence of the answers to Questions (c)) and (d)).

f) The above analysis shows that

$$\begin{aligned}
\mathbb{P}(\tau>T\mid\mathscr{G}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right)\,\Big|\,\mathscr{F}_t\right]\\
&= \mathbb{1}_{\{\tau>t\}}\exp\left(-C(b,t,T)\lambda_t+\frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right),
\end{aligned}$$

for $a=0$ and

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\,\Big|\,\mathscr{F}_t\right] = \exp\left(-C(a,t,T)r_t+\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds\right),$$

for $b=0$, and this implies

$$\begin{aligned}
U_\rho(t,T) &= \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right)\\
&= \exp\left(\rho\frac{\sigma\eta}{ab}\left(T-t-C(a,t,T)-C(b,t,T)+C(a+b,t,T)\right)\right).
\end{aligned}$$

g) We have

$$\begin{aligned}
f(t,T) &= -\mathbb{1}_{\{\tau>t\}}\frac{\partial}{\partial T}\log P(t,T)\\
&= \mathbb{1}_{\{\tau>t\}}\left(r_t\,\mathrm{e}^{-(T-t)a}-\frac{\sigma^2}{2}C^2(a,t,T)+\lambda_t\,\mathrm{e}^{-(T-t)b}-\frac{\eta^2}{2}C^2(b,t,T)\right)\\
&\quad-\mathbb{1}_{\{\tau>t\}}\rho\sigma\eta C(a,t,T)C(b,t,T).
\end{aligned}$$

h) We use the relation

$$\begin{aligned}
\mathbb{P}(\tau>T\mid\mathscr{G}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right)\,\Big|\,\mathscr{F}_t\right]\\
&= \mathbb{1}_{\{\tau>t\}}\exp\left(-C(b,t,T)\lambda_t+\frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right)\\
&= \mathbb{1}_{\{\tau>t\}}\,\mathrm{e}^{-\int_t^T f_2(t,u)du},
\end{aligned}$$

where $f_2(t,T)$ is the Vasicek forward rate corresponding to λ_t, *i.e.*

$$f_2(t,u) = \lambda_t \, \mathrm{e}^{-(u-t)b} - \frac{\eta^2}{2} C^2(b,t,u).$$

i) In this case we have $\rho = 0$ and

$$P(t,T) = \mathbb{P}(\tau > T \mid \mathscr{G}_t) \, \mathbb{E}\left[\exp\left(- \int_t^T r_s ds \right) \Bigm| \mathscr{F}_t \right],$$

since $U_\rho(t,T) = 0$.

Chapter 2

Exercise 2.1

a) Taking $(U,V) = (U,U)$, we have

$$\begin{aligned}
\mathbb{P}(U \leqslant u \text{ and } V \leqslant v) &= \mathbb{P}(U \leqslant u \text{ and } U \leqslant v) \\
&= \mathbb{P}(U \leqslant \min(u,v)) \\
&= \min(u,v) \\
&= C_M(u,v), \qquad u,v \in [0,1].
\end{aligned}$$

b) Taking $(U,V) = (U,1-U)$, we have

$$\begin{aligned}
\mathbb{P}(U \leqslant u \text{ and } V \leqslant v) &= \mathbb{P}(U \leqslant u \text{ and } 1-U \leqslant v) \\
&= \mathbb{P}(U \leqslant u \text{ and } U \geqslant 1-v) \\
&= \mathbb{P}(1-v \leqslant U \leqslant u) \\
&= \mathbb{1}_{\{0 \leqslant 1-v \leqslant u \leqslant 1\}} \mathbb{P}(1-v \leqslant U \leqslant u) \\
&= \mathbb{1}_{\{0 \leqslant u+v-1 \leqslant 1\}} (u-(1-v)) \\
&= (u+v-1)^+,
\end{aligned}$$

$u,v \in [0,1]$.

c) We have

$$C(u,v) = \mathbb{P}(U \leqslant u \text{ and } V \leqslant v) \leqslant \mathbb{P}(U \leqslant u \text{ and } V \geqslant 1) \leqslant \mathbb{P}(U \leqslant u) = u,$$

$u,v \in [0,1]$, and similarly we find $C(u,v) \leqslant \mathbb{P}(U \leqslant v) = v$ for all $u,v \in [0,1]$, which yields (2.4.4).

d) For fixed $v \in [0,1]$ we have

$$\begin{aligned}
\frac{\partial C}{\partial u}(u,v) &= \lim_{\varepsilon \to 0} \frac{C(u+\varepsilon,v) - C(u,v)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\mathbb{P}(U \leqslant u+\varepsilon \text{ and } V \leqslant v) - \mathbb{P}(U \leqslant u \text{ and } V \leqslant v)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\mathbb{P}(u \leqslant U \leqslant u+\varepsilon \text{ and } V \leqslant v)}{P(u \leqslant U \leqslant u+\varepsilon)} \\
&= \lim_{\varepsilon \to 0} \mathbb{P}(V \leqslant v \mid u \leqslant U \leqslant u+\varepsilon) \\
&= \mathbb{P}(V \leqslant v \mid U = u) \\
&\leqslant 1,
\end{aligned}$$

$u,v \in [0,1]$, hence

$$h'(u) = \frac{\partial C}{\partial u}(u,v) - 1 = \mathbb{P}(V \leqslant v \mid U = u) - 1 \leqslant 0,$$

$u,v \in [0,1]$, and since $h(1) = C(1,v) - v = \mathbb{P}(V \leqslant v) - v = 0$, $v \in [0,1]$ we conclude that $h(u) \geqslant 0$, $u \in [0,1]$, which shows (2.4.5).

Exercise 2.2

a) When $\rho = 1$, we have

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X p_Y + \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = (1-p_X)p_Y - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = p_X(1-p_Y) - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = (1-p_X)(1-p_Y) + \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \end{cases}$$

hence

$$\begin{cases} (1-p_X)p_Y \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ \\ p_X(1-p_Y) \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \end{cases}$$

hence

$$(1-p_X)p_Y \geqslant p_X(1-p_Y) \quad \text{and} \quad p_X(1-p_Y) \geqslant p_Y(1-p_X),$$

showing that $(1-p_X)p_Y = p_X(1-p_Y)$, which implies $p_X = p_Y$, and

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X^2 + p_X(1-p_X) = p_X = p_Y, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = 0, \\ \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = 0, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = 1 - p_X = 1 - p_Y. \end{cases}$$

b) When $\rho = -1$, we have

$$\begin{cases} \mathbb{P}(X = 1 \text{ and } Y = 1) = p_X p_Y - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 1) = (1-p_X)p_Y + \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \\ \mathbb{P}(X = 1 \text{ and } Y = 0) = p_X(1-p_Y) + \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \\ \\ \mathbb{P}(X = 0 \text{ and } Y = 0) = (1-p_X)(1-p_Y) - \sqrt{p_X p_Y (1-p_X)(1-p_Y)} \geqslant 0, \end{cases}$$

hence

$$\begin{cases} p_X p_Y \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \\ \\ p_X p_Y \geqslant \sqrt{p_X p_Y (1-p_X)(1-p_Y)}, \end{cases}$$

hence

$$p_X p_Y \geqslant (1-p_X)(1-p_Y) \quad \text{and} \quad p_X p_Y \geqslant (1-p_X)(1-p_Y),$$

showing that $p_X p_Y = (1-p_X)(1-p_Y)$, which implies $p_X = 1-p_Y$, and

$$\begin{cases} \mathbb{P}(X=1 \text{ and } Y=1)=0, \\ \mathbb{P}(X=0 \text{ and } Y=1)=1, \\ \mathbb{P}(X=1 \text{ and } Y=0)=1, \\ \mathbb{P}(X=0 \text{ and } Y=0)=0. \end{cases}$$

Exercise 2.3

a) We have

$$\mathbb{P}(X \geqslant x) = \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0) = \mathrm{e}^{-(\lambda+\nu)x},$$

and

$$\mathbb{P}(Y \geqslant y) = \mathbb{P}(X \geqslant 0 \text{ and } Y \geqslant y) := \mathrm{e}^{-(\mu+\nu)y},$$

$x,y \geqslant 0$, *i.e.* X and Y are exponentially distributed with respective parameters $\lambda+\nu$ and $\mu+\nu$.

b) We have

$$\begin{aligned} &\mathbb{P}(X \leqslant x \text{ and } Y \leqslant 0) \\ &\quad = \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0) - (\mathbb{P}(X \geqslant x) - \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0)) \\ &\qquad -(\mathbb{P}(Y \geqslant x) - \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0)) \\ &\quad = \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0) - \mathbb{P}(X \geqslant x) - \mathbb{P}(Y \geqslant x) + \mathbb{P}(X \geqslant x \text{ and } Y \geqslant 0)), \end{aligned}$$

$x,y \geqslant 0$, *i.e.* X and Y are exponentially distributed with respective parameters $\lambda+\nu$ and $\mu+\nu$.

c) Since $\mathrm{e}^{-(\lambda+\nu)X}$ and $\mathrm{e}^{-(\mu+\nu)Y}$ are uniformly distributed on $[0,1]$, a copula function $C(u,v)$ can be defined by

$$\begin{aligned} C(u,v) &:= \mathbb{P}(\mathrm{e}^{-(\lambda+\nu)X} \leqslant u \text{ and } \mathrm{e}^{-(\mu+\nu)Y} \leqslant v) \\ &= \mathbb{P}(X \leqslant -(\lambda+\nu)^{-1}\log u \text{ and } Y \leqslant -(\lambda+\nu)^{-1}\log v) \\ &= \mathrm{e}^{\lambda(\lambda+\nu)^{-1}\log u + \mu(\lambda+\nu)^{-1}\log v)y - \nu \operatorname{Max}(-(\lambda+\nu)^{-1}\log u, -(\lambda+\nu)^{-1}\log v))} \\ &= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)}\, \mathrm{e}^{-\nu \operatorname{Max}(-(\lambda+\nu)^{-1}\log u, -(\lambda+\nu)^{-1}\log v))} \\ &= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)}\, \mathrm{e}^{\nu \min(\log u^{(\lambda+\nu)^{-1}}, \log v^{(\lambda+\nu)^{-1}}))} \\ &= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)}\, \mathrm{e}^{\log \min(u^{\nu/(\lambda+\nu)}, v^{\nu/(\lambda+\nu)}))} \\ &= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)} \min(u^{\nu/(\lambda+\nu)}, v^{\nu/(\lambda+\nu)})) \\ &= u^{\lambda/(\lambda+\nu)} v^{\mu/(\lambda+\nu)} (\min(u,v))^{\nu/(\lambda+\nu)}, \qquad x,y \geqslant 0. \end{aligned}$$

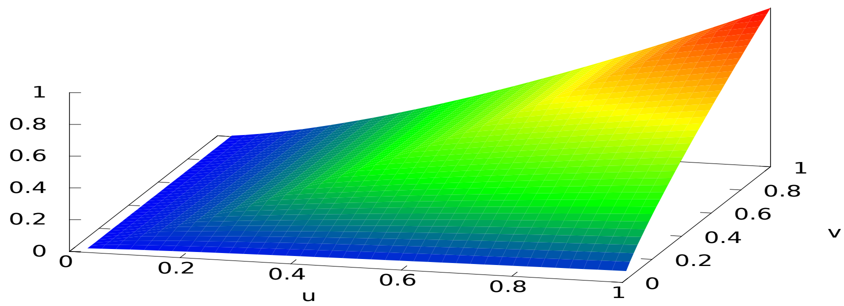


Figure S.1: Exponential copula function $u,v \mapsto C(u,b)$ with $\lambda=1$, $\mu=2$, $\nu=4$.

Exercise 2.4

a) We have

$$F_X(x) = \mathbb{P}(X \leqslant x) = \mathbb{P}(X \leqslant x \text{ and } Y \leqslant \infty) = \frac{1}{1+\mathrm{e}^{-x}}$$

and

$$F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(X \leqslant \infty \text{ and } Y \leqslant y) = \frac{1}{1+\mathrm{e}^{-y}}, \qquad x, y \in \mathbb{R}.$$

The probability densities are given by

$$f_X(x) = f_Y(x) = F_X'(x) = F_Y'(x) = \frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}, \qquad x \in \mathbb{R}.$$

b) We have

$$F_X^{-1}(u) = F_Y^{-1}(u) = -\log \frac{1-u}{u}, \qquad u \in (0,1),$$

and the corresponding copula is given by

$$\begin{aligned}
C(u,v) &= F_{(X,Y)}(F_X^{-1}(u), F_Y^{-1}(v)) \\
&= F_{(X,Y)}\left(-\log \frac{1-u}{u}, -\log \frac{1-v}{v}\right) \\
&= \frac{1}{1+(1-u)/u+(1-v)/v} \\
&= \frac{1}{1+(1-u)/u+(1-v)/v} \\
&= \frac{uv}{u+v-uv}, \qquad u,v \in [0,1],
\end{aligned}$$

which is a particular case of the Ali-Mikhail-Haq copula.

Exercise 2.5

a) We show that (X,Y) have Gaussian marginals $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(0,\eta^2)$, according to the following computation:

$$\begin{aligned}
\int_{-\infty}^{\infty} \widetilde{f}(x,y)dy &= \frac{1}{\pi\sigma\eta} \int_{-\infty}^{\infty} \mathbb{1}_{\mathbb{R}_-^2 \cup \mathbb{R}_+^2}(x,y)\, \mathrm{e}^{-x^2/(2\sigma^2)-y^2/(2\eta^2)} dy \\
&= \frac{1}{\pi\sigma\eta} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_+}(x) \int_0^{\infty} \mathrm{e}^{-y^2/(2\eta^2)} dy + \\
&\quad \frac{1}{\pi\sigma\eta} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_-}(x) \int_{-\infty}^{0} \mathrm{e}^{-y^2/(2\eta^2)} dy \\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_+}(x) + \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-x^2/(2\sigma^2)} \mathbb{1}_{\mathbb{R}_-}(x) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-x^2/(2\sigma^2)}, \qquad x \in \mathbb{R}.
\end{aligned}$$

b) The couple (X,Y) does *not* have a joint Gaussian distribution, and its joint probability density function does *not* coincide with $f_\Sigma(x,y)$.

c) When $\sigma = \eta = 1$, the random variable $X+Y$ has the probability density function

$$\begin{aligned}
\frac{\partial}{\partial a}\mathbb{P}(X+Y \leqslant a) &= \frac{\partial}{\partial a} \int_{-\infty}^{\infty} \int_{-\infty}^{a-x} \widetilde{f}(x,y) dy dx \\
&= \frac{1}{\pi} \frac{\partial}{\partial a} \int_0^a \int_0^{a-x} \mathrm{e}^{-x^2/2-y^2/2} dy dx \\
&= \frac{1}{\pi} \frac{\partial}{\partial a} \int_0^a \mathrm{e}^{-(a-z)^2/2} \int_0^z \mathrm{e}^{-y^2/2} dy dz \\
&= \frac{1}{\pi} \int_0^a \mathrm{e}^{-y^2/2} dy - \frac{1}{\pi} \int_0^a (a-z)\, \mathrm{e}^{-(a-z)^2/2} \int_0^z \mathrm{e}^{-y^2/2} dy dz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^a e^{-y^2/2}dy - \frac{1}{\pi}\int_0^a (a-z)\,e^{-(a-z)^2/2}dz\int_0^a e^{-y^2/2}dy \\
&\quad + \frac{1}{\pi}\int_0^a e^{-y^2/2}\int_0^y e^{-(a-z)^2/2}dzdy \\
&= \frac{1}{\pi}e^{-a^2/2}\int_0^a e^{-y^2/2}dy + \frac{1}{\pi}\int_0^a e^{-y^2/2}\big(e^{-(a-y)^2/2} - e^{-a^2/2}\big)dy \\
&= \frac{1}{\pi}e^{-a^2/2}\int_0^a e^{-y^2/2-(a-y)^2/2}dy \\
&= \frac{1}{\pi}\int_0^a e^{-((\sqrt{2}y-a/\sqrt{2})^2-a^2/2)/2}dy \\
&= \frac{e^{-a^2/4}}{\pi\sqrt{2}}\int_0^{a\sqrt{2}} e^{-((y-a/\sqrt{2})^2)/2}dy \\
&= \frac{e^{-a^2/4}}{\pi\sqrt{2}}\int_{-a/\sqrt{2}}^{a(\sqrt{2}-1/\sqrt{2})} e^{-y^2/2}dy \\
&= \frac{e^{-a^2/4}}{\sqrt{\pi}\sqrt{2\pi}}\int_{-a/\sqrt{2}}^{a/\sqrt{2}} e^{-y^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}}e^{-a^2/4}\frac{1}{\sqrt{\pi}}\big(2\Phi(a/\sqrt{2})-1\big), \quad a \geqslant 0,
\end{aligned}
$$

which vanishes at $a = 0$.

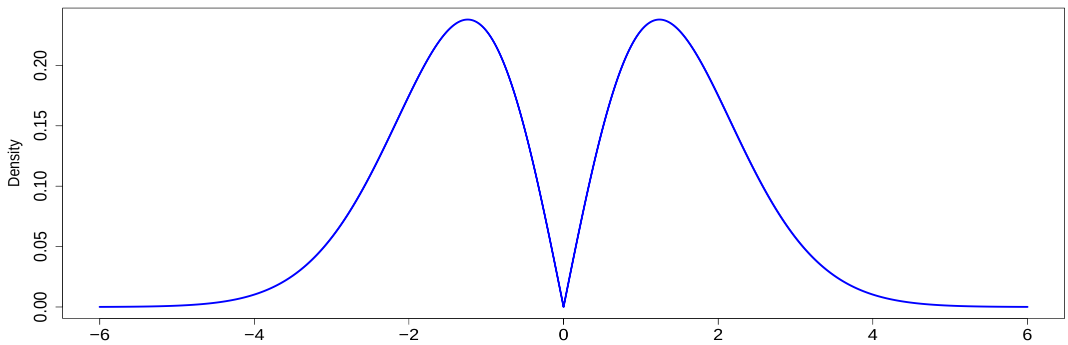


Figure S.2: Density of $X+Y$.

d) The random variables X and Y are positively correlated, as

$$
\begin{aligned}
\int_{-\infty}^{\infty} y f_\Sigma(x,y)dy &= \frac{1}{\pi\sigma\eta}\int_{-\infty}^{\infty}\mathbb{1}_{\mathbb{R}_-^2\cup\mathbb{R}_+^2}(x,y)\,y\,e^{-x^2/(2\sigma^2)-y^2/(2\eta^2)}dy \\
&= \frac{1}{\pi\sigma\eta}e^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_+}(x)\int_0^{\infty} y\,e^{-y^2/(2\eta^2)}dy \\
&\quad + \frac{1}{\pi\sigma\eta}e^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_-}(x)\int_{-\infty}^0 y\,e^{-y^2/(2\eta^2)}dy \\
&= \frac{\eta}{\pi\sigma}e^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_+}(x) - \frac{\eta}{\pi\sigma}e^{-x^2/(2\sigma^2)}\mathbb{1}_{\mathbb{R}_-}(x),
\end{aligned}
$$

hence

$$
\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_\Sigma(x,y)dydx \\
&= \frac{\eta}{\pi\sigma}\int_0^{\infty} x\,e^{-x^2/(2\sigma^2)}dx - \frac{\eta}{\pi\sigma}\int_{-\infty}^0 x\,e^{-x^2/(2\sigma^2)}dx \\
&= \frac{2\sigma\eta}{\pi},
\end{aligned}
$$

and

$$\rho = \frac{\mathbb{E}[XY]}{\sigma\eta} = \frac{2}{\pi}.$$

Under a rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

of angle $\theta \in [0, 2\pi]$ we would find

$$\begin{aligned} & \mathbb{E}[(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)] \\ & = \sin\theta\cos\theta\,\mathbb{E}\left[X^2\right] + (\cos^2\theta - \sin^2\theta)\,\mathbb{E}[XY] - \sin\theta\cos\theta\,\mathbb{E}\left[Y^2\right] \\ & = \sigma^2\sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\frac{2\sigma\eta}{\pi} - \eta^2\sin\theta\cos\theta \\ & = \frac{\sigma^2}{2}\sin(2\theta) + \cos(2\theta)\frac{2\sigma\eta}{\pi} - \frac{\eta^2}{2}\sin(2\theta), \end{aligned}$$

and

$$\rho = \frac{\sigma}{2\eta}\sin(2\theta) + \cos(2\theta)\frac{2}{\pi} - \frac{\eta}{2\sigma}\sin(2\theta),$$

i.e. $\theta = \pi/4$ and $\sigma = \eta$ would lead to uncorrelated random variables.

Exercise 2.6

a) We have

$$\begin{aligned} \mathbb{P}(\tau_i \wedge \tau \geqslant s) & = \mathbb{P}(\tau_i \geqslant s \text{ and } \tau \geqslant s) \\ & = \mathbb{P}(\tau_i \geqslant s)\mathbb{P}(\tau \geqslant s) \\ & = \mathrm{e}^{-\lambda_i s}\mathrm{e}^{-\lambda s} \\ & = \mathrm{e}^{-(\lambda_i+\lambda)s}, \qquad s \geqslant 0, \end{aligned}$$

hence $\tau_i \wedge \tau$ is an exponentially distributed random variable with parameter $\lambda_i + \lambda$, $i = 1, 2$.

b) Next, we have

$$\begin{aligned} \mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t) & = \mathbb{P}(\tau_1 > s \text{ and } \tau > s \text{ and } \tau_2 > t \text{ and } \tau > t) \\ & = \mathbb{P}(\tau_1 > s \text{ and } \tau_2 > t \text{ and } \tau > \mathrm{Max}(s,t)) \\ & = \mathbb{P}(\tau_1 > s)\mathbb{P}(\tau_2 > t)\mathbb{P}(\tau > \mathrm{Max}(s,t)) \\ & = \mathrm{e}^{-\lambda_1 s}\mathrm{e}^{-\lambda_2 t}\mathrm{e}^{-\lambda\,\mathrm{Max}(s,t)} \\ & = \mathrm{e}^{-\lambda_1 s - \lambda_2 t - \lambda\,\mathrm{Max}(s,t)} \\ & = \mathrm{e}^{-(\lambda_1+\lambda)s-(\lambda_2+\lambda)t+\lambda\min(s,t)} \\ & = (1 - F_X(s))(1 - F_Y(t))\min(\mathrm{e}^{\lambda s}, \mathrm{e}^{\lambda t}), \end{aligned}$$

$s, t \geqslant 0$.

c) We have

$$\begin{aligned} F_{X,Y}(s,t) & = \mathbb{P}(\tau_1 \wedge \tau \leqslant s \text{ and } \tau_2 \wedge \tau \leqslant t) \\ & = \mathbb{P}(\tau_1 \wedge \tau \leqslant s) - \mathbb{P}(\tau_1 \wedge \tau \leqslant s \text{ and } \tau_2 \wedge \tau > t) \\ & = \mathbb{P}(\tau_1 \wedge \tau \leqslant s) - \big(\mathbb{P}(\tau_2 \wedge \tau > t) - \mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t)\big) \\ & = \mathbb{P}(\tau_1 \wedge \tau \leqslant s) + \mathbb{P}(\tau_2 \wedge \tau \leqslant t) + \mathbb{P}(\tau_1 \wedge \tau > s \text{ and } \tau_2 \wedge \tau > t) - 1 \\ & = F_X(s) + F_Y(t) + (1 - F_X(s))(1 - F_Y(t))\min(\mathrm{e}^{\lambda s}, \mathrm{e}^{\lambda t}) - 1. \end{aligned}$$

d) We find

$$\begin{aligned} C(u,v) & = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \\ & = F_X(F_X^{-1}(u)) + F_Y(F_X^{-1}(v)) \\ & \quad + (1 - F_X(F_X^{-1}(u)))(1 - F_Y(F_X^{-1}(v)))\min\left(\mathrm{e}^{\lambda F_X^{-1}(u)}, \mathrm{e}^{\lambda F_X^{-1}(v)}\right) - 1 \\ & = u + v - 1 + (1-u)(1-v)\min\left(\mathrm{e}^{\lambda F_X^{-1}(u)}, \mathrm{e}^{\lambda F_Y^{-1}(v)}\right) \end{aligned}$$

$$
\begin{aligned}
&= u+v-1+(1-u)(1-v)\min\left(\mathrm{e}^{-\lambda\log(1-u)/(\lambda_1+\lambda)}, \mathrm{e}^{-\lambda\log(1-v)/(\lambda_2+\lambda)}\right) \\
&= u+v-1+\min\left((1-v)(1-u)^{1-\lambda/(\lambda_1+\lambda)}, (1-u)(1-v)^{1-\lambda/(\lambda_2+\lambda)}\right) \\
&= u+v-1+\min\left((1-v)(1-u)^{1-\theta_1}, (1-u)(1-v)^{1-\theta_2}\right), \quad u,v\in[0,1],
\end{aligned}
$$

with

$$
\theta_1 = \frac{\lambda}{\lambda_1+\lambda} \quad \text{and} \quad \theta_2 = \frac{\lambda}{\lambda_2+\lambda}.
$$

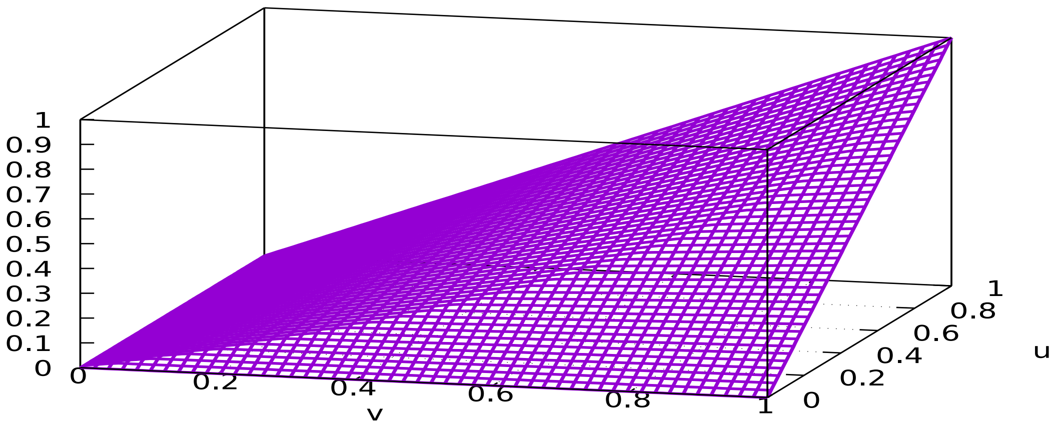


Figure S.3: Survival copula graph with $\theta_1 = 0.3$ and $\theta_2 = 0.7$.

e) We have

$$
\begin{aligned}
C(u,v) &= u+v-1+(1-u)(1-v)^{1-\theta_2}\mathbb{1}_{\{(1-u)^{\theta_1}<(1-v)^{\theta_2}\}} \\
&\quad +(1-v)(1-u)^{1-\theta_1}\mathbb{1}_{\{(1-u)^{\theta_1}>(1-v)^{\theta_2}\}}, \quad u,v\in[0,1],
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{\partial C}{\partial u}(u,v) &= -(1-v)^{1-\theta_2}\mathbb{1}_{\{(1-u)^{\theta_1}<(1-v)^{\theta_2}\}} \\
&\quad -(1-\theta_1)(1-v)(1-u)^{-\theta_1}\mathbb{1}_{\{(1-u)^{\theta_1}>(1-v)^{\theta_2}\}}
\end{aligned}
$$

and the survival copula density is given by

$$
\begin{aligned}
\frac{\partial^2 C}{\partial u\partial v}(u,v) &= (1-\theta_2)(1-v)^{-\theta_2}\mathbb{1}_{\{(1-u)^{\theta_1}<(1-v)^{\theta_2}\}} \\
&\quad +(1-\theta_1)(1-u)^{-\theta_1}\mathbb{1}_{\{(1-u)^{\theta_1}>(1-v)^{\theta_2}\}}, \quad u,v\in[0,1],
\end{aligned}
$$

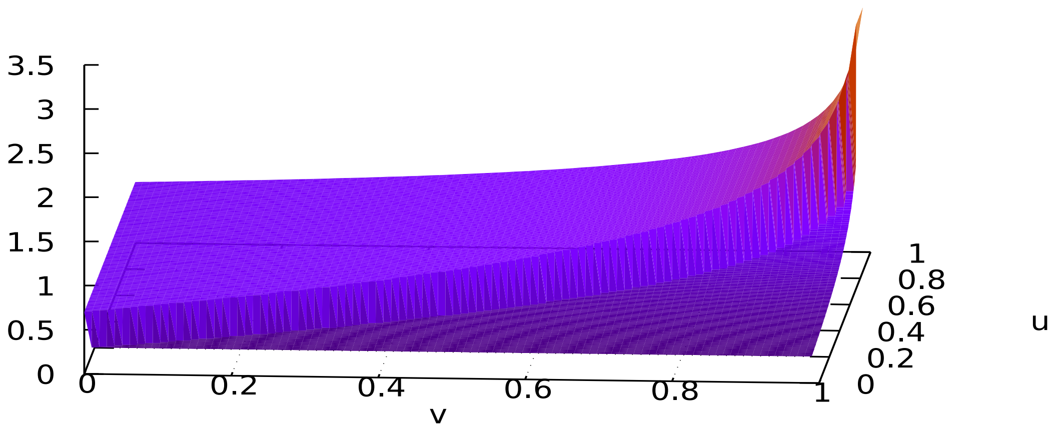


Figure S.4: Survival copula density graph with $\theta_1 = 0.3$ and $\theta_2 = 0.7$.

Remark: When $\lambda = 0$ we have $\theta_1 = \theta_2 = 0$ and $\tau = +\infty$ *a.s.*, therefore we have

$$\min(\tau_1, \tau) = \tau_1 \quad \text{and} \quad \min(\tau_2, \tau) = \tau_2,$$

hence the copula $C(u,v)$ is given by

$$C(u,v) = u + v - 1 + (1-v)(1-u) = uv, \qquad u,v \in [0,1],$$

which coincides with the copula of independence.

Chapter 3

Exercise 3.1 By differentiation of (3.1.1), *i.e.*

$$\begin{aligned} \mathbb{P}(\tau < T \mid \mathscr{F}_t) &:= \mathbb{P}(S_T < K \mid \mathscr{F}_t) \\ &= \Phi\left(-\frac{(\mu - \sigma^2/2)(T-t) + \log(S_t/K))}{\sigma\sqrt{T-t}}\right), \qquad T \geqslant t, \end{aligned}$$

with respect to T, we find

$$\begin{aligned} d\mathbb{P}(\tau \leqslant T \mid \mathscr{F}_t) &= \frac{dT}{2\sigma\sqrt{2\pi(T-t)}}\left(\frac{\sigma^2}{2} - \mu + \frac{\log(S_t/K)}{T-t}\right) \\ &\quad \times \exp\left(-\frac{((\mu - \sigma^2/2))(T-t) + \log(S_t/K)))^2}{2(T-t)\sigma^2}\right), \end{aligned}$$

provided that $\mu < \sigma^2/2$.

Exercise 3.2 Consider the first hitting time

$$\tau_K := \inf\{u \geqslant t \ : \ S_u \leqslant K\}$$

of the level $K > 0$ starting from $S_t > K$. By Lemma 15.1 in Privault, 2022, we have

$$\mathbb{E}^*\left[\mathrm{e}^{-(\tau_K - t)r} \mid \mathscr{F}_t\right] = \left(\frac{K}{S_t}\right)^{2r/\sigma^2},$$

provided that $S_t \geqslant K$.

Exercise 3.3

a) We have

$$\begin{aligned} \mathbb{E}[X_k X_l] &= \mathbb{E}\left[(a_k M + \sqrt{1-a_k^2} Z_k)(a_l M + \sqrt{1-a_l^2} Z_l)\right] \\ &= \mathbb{E}\left[a_k a_l M^2 + a_k M \sqrt{1-a_l^2} Z_l + a_l M \sqrt{1-a_k^2} Z_k + \sqrt{1-a_k^2}\sqrt{1-a_l^2} Z_k Z_l\right] \\ &= a_k a_l \mathbb{E}\left[M^2\right] + a_k \sqrt{1-a_l^2}\, \mathbb{E}[Z_l M] + a_l \sqrt{1-a_k^2}\, \mathbb{E}[Z_k M] \\ &\quad + \sqrt{1-a_k^2}\sqrt{1-a_l^2}\, \mathbb{E}[Z_k Z_l] \\ &= a_k a_l \mathbb{E}\left[M^2\right] + a_k \sqrt{1-a_l^2}\, \mathbb{E}[Z_l]\, \mathbb{E}[M] + a_l \sqrt{1-a_k^2}\, \mathbb{E}[Z_k]\, \mathbb{E}[M] \end{aligned}$$

$$+\sqrt{1-a_k^2}\sqrt{1-a_l^2}\mathbb{1}_{\{k=l\}}$$
$$= a_k a_l + (1-a_k^2)\mathbb{1}_{\{k=l\}}$$
$$= \mathbb{1}_{\{k=l\}} + a_k a_l \mathbb{1}_{\{k\neq l\}}, \qquad k,l = 1,2,\ldots,n,$$

b) We check that the vector $(X_1,\ldots,X_n)$, with covariance matrix (3.4.7) has the probability density function

$$\varphi(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n/2}}\prod_{k=1}^n (1-a_k^2)^{-1/2}\int_{-\infty}^{\infty} e^{-\frac{(x_1-a_1m)^2}{2(1-a_1^2)}}\cdots e^{-\frac{(x_n-a_nm)^2}{2(1-a_n^2)}}\frac{e^{-m^2/2}}{\sqrt{2\pi}}dm$$

which is jointly Gaussian, with marginals given by

$$\begin{aligned}
x_k \longmapsto & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(x_1,\ldots,x_n)dx_1\cdots dx_{k-1}dx_{k+1}\cdots dx_n \\
= & \frac{1}{\sqrt{2\pi(1-a_k^2)}}\int_{-\infty}^{\infty} e^{-\frac{(x_k-a_km)^2}{2(1-a_k^2)}}\frac{e^{-m^2/2}}{\sqrt{2\pi}}dm \\
= & \frac{1}{2\pi\sqrt{1-a_k^2}}\int_{-\infty}^{\infty} e^{-\frac{(x_k-a_km)^2}{2(1-a_k^2)}-m^2/2}dm \\
= & \frac{1}{2\pi\sqrt{1-a_k^2}}\int_{-\infty}^{\infty} e^{-\frac{x_k^2-2a_kx_km+m^2}{2(1-a_k^2)}}dm \\
= & \frac{e^{-x_k^2/2}}{2\pi\sqrt{1-a_k^2}}\int_{-\infty}^{\infty} e^{-\frac{(m-a_kx_k)^2}{2(1-a_k^2)}}dm \\
= & \frac{1}{\sqrt{2\pi}}e^{-x_k^2/2}, \qquad x_k\in\mathbb{R}.
\end{aligned}$$

c) We have

$$\begin{aligned}
\varphi(x_1,\ldots,x_n) = & \frac{1}{(2\pi)^{n/2}}\prod_{k=1}^n(1-a_k^2)^{-1/2}\int_{-\infty}^{\infty} e^{-\frac{(x_1-a_1m)^2}{2(1-a_1^2)}}\cdots e^{-\frac{(x_n-a_nm)^2}{2(1-a_n^2)}}\frac{e^{-m^2/2}}{\sqrt{2\pi}}dm \\
= & \frac{1}{(2\pi)^{n/2}}\prod_{k=1}^n(1-a_k^2)^{-1/2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x_1^2+a_1^2m^2-2x_1a_1m}{1-a_1^2}+\cdots+\frac{x_n^2+a_n^2m^2-2x_na_nm}{1-a_n^2}+m^2\right)}\frac{dm}{\sqrt{2\pi}} \\
= & \frac{1}{(2\pi)^{n/2}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2}+\cdots+\frac{x_n^2}{1-a_n^2}\right)}\prod_{k=1}^n(1-a_k^2)^{-1/2} \\
& \int_{-\infty}^{\infty} e^{-\frac{m^2}{2}\left(1+\frac{a_1^2}{1-a_1^2}+\cdots+\frac{a_n^2}{1-a_n^2}\right)+2m\left(\frac{x_1a_1}{2(1-a_1^2)}+\cdots+\frac{x_na_n}{2(1-a_n^2)}\right)}dm \\
= & \frac{e^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2}+\cdots+\frac{x_n^2}{1-a_n^2}\right)}}{\sqrt{(2\pi)^n(1-a_1^2)\cdots(1-a_n^2)}}\exp\left(\frac{\frac{1}{2}\left(\frac{x_1a_1}{1-a_1^2}+\cdots+\frac{x_na_n}{1-a_n^2}\right)^2}{1+\frac{a_1^2}{1-a_1^2}+\cdots+\frac{a_n^2}{1-a_n^2}}\right) \\
& \times\left(1+\frac{a_1^2}{1-a_1^2}+\cdots+\frac{a_n^2}{1-a_n^2}\right)^{-1/2} \\
= & \frac{e^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2}+\cdots+\frac{x_n^2}{1-a_n^2}\right)}}{\sqrt{(2\pi)^n\alpha^2(1-a_1^2)\cdots(1-a_n^2)}}\exp\left(\frac{1}{2\alpha^2}\left(\frac{x_1a_1}{1-a_1^2}+\cdots+\frac{x_na_n}{1-a_n^2}\right)^2\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2}+\cdots+\frac{x_n^2}{1-a_n^2}\right)}}{\sqrt{(2\pi)^n\alpha^2(1-a_1^2)\cdots(1-a_n^2)}}\exp\left(\frac{1}{2\alpha^2}\left(\frac{x_1a_1}{1-a_1^2}+\cdots+\frac{x_na_n}{1-a_n^2}\right)^2\right)\\
&= \frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x_1^2}{1-a_1^2}\left(1-\frac{a_1^2}{\alpha^2(1-a_1^2)}\right)+\cdots+\frac{x_n^2}{1-a_n^2}\left(1-\frac{a_n^2}{\alpha^2(1-a_n^2)}\right)\right)}}{\sqrt{(2\pi)^n\alpha^2(1-a_1^2)\cdots(1-a_n^2)}}\exp\left(\frac{1}{2\alpha^2}\sum_{1\leqslant p\neq l\leqslant n}\frac{x_px_la_pa_l}{(1-a_p^2)(1-a_l^2)}\right)\\
&= \frac{1}{\sqrt{(2\pi)^n\det\Sigma}}\mathrm{e}^{-\frac{1}{2}\langle x,\Sigma^{-1}x\rangle},
\end{aligned}$$

where

$$\alpha^2 := 1+\frac{a_1^2}{1-a_1^2}+\cdots+\frac{a_n^2}{1-a_n^2},$$

and

$$\Sigma^{-1}=\frac{1}{\alpha^2}\begin{bmatrix}
\frac{\alpha^2(1-a_1^2)-a_1^2}{(1-a_1^2)^2} & \frac{-a_1a_2}{(1-a_1^2)(1-a_2^2)} & \cdots & \frac{-a_1a_n}{(1-a_1^2)(1-a_n^2)}\\
\frac{-a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2(1-a_2^2)-a_2^2}{(1-a_2^2)^2} & \ddots & \vdots\\
\vdots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \frac{\alpha^2(1-a_{n-1}^2)-a_{n-1}^2}{(1-a_{n-1}^4)} & \frac{-a_{n-1}a_n}{(1-a_{n-1}^2)(1-a_n^2)}\\
\frac{-a_na_1}{(1-a_n^2)(1-a_1^2)} & \ddots & \frac{-a_na_{n-1}}{(1-a_n^2)(1-a_{n-1}^2)} & \frac{\alpha^2(1-a_n^2)-a_n^2}{(1-a_n^2)^2}
\end{bmatrix}.$$

Exercise 3.4 We have

$$\Sigma=\begin{bmatrix}1 & a_1a_2\\ a_2a_1 & 1\end{bmatrix},$$

and letting

$$\begin{aligned}
\alpha^2 &:= 1+\frac{a_1^2}{1-a_1^2}+\frac{a_2^2}{1-a_2^2}\\
&= \frac{(1-a_1^2)(1-a_2^2)+a_1^2(1-a_2^2)+a_2^2(1-a_1^2)}{(1-a_1^2)(1-a_2^2)}\\
&= \frac{1-a_2^2a_1^2}{(1-a_1^2)(1-a_2^2)},
\end{aligned}$$

we find

$$\begin{aligned}
\Sigma^{-1} &= \frac{1}{\alpha^2}\begin{bmatrix}\frac{\alpha^2(1-a_1^2)-a_1^2}{(1-a_1^2)^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)}\\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2(1-a_2^2)-a_2^2}{(1-a_2^2)^2}\end{bmatrix}\\
&= \frac{1}{\alpha^2}\begin{bmatrix}\frac{\alpha^2}{1-a_1^2}\left(1-\frac{(1-a_2^2)a_1^2}{1-a_2^2a_1^2}\right) & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)}\\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2}\left(1-\frac{(1-a_1^2)a_2^2}{1-a_2^2a_1^2}\right)\end{bmatrix}\\
&= \frac{1}{\alpha^2}\begin{bmatrix}\frac{\alpha^2}{1-a_2^2a_1^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)}\\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2a_1^2}\end{bmatrix}\\
&= \frac{(1-a_1^2)(1-a_2^2)}{1-a_2^2a_1^2}\begin{bmatrix}\frac{\alpha^2}{1-a_2^2a_1^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)}\\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2a_1^2}\end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha^2} \begin{bmatrix} \frac{\alpha^2}{1-a_2^2a_1^2} & -\frac{a_1a_2}{(1-a_1^2)(1-a_2^2)} \\ -\frac{a_2a_1}{(1-a_2^2)(1-a_1^2)} & \frac{\alpha^2}{1-a_2^2a_1^2} \end{bmatrix} \\
&= \frac{1}{1-a_2^2a_1^2} \begin{bmatrix} 1 & -a_1a_2 \\ -a_1a_2 & 1 \end{bmatrix}.
\end{aligned}
$$

In particular, the case $n = 2$ is able to recover all two-dimensional copulas by setting the correlation coefficient $\rho = a_1a_2$. In the general case, Σ is parametrized by n numbers, which offers less degrees of freedom compared with the joint Gaussian copula correlation method which relies on $n(n-1)/2$ coefficients, see also Exercise 3.3.

Chapter 4

Exercise 4.1 It suffices to check that as λ tends to ∞, the ratio

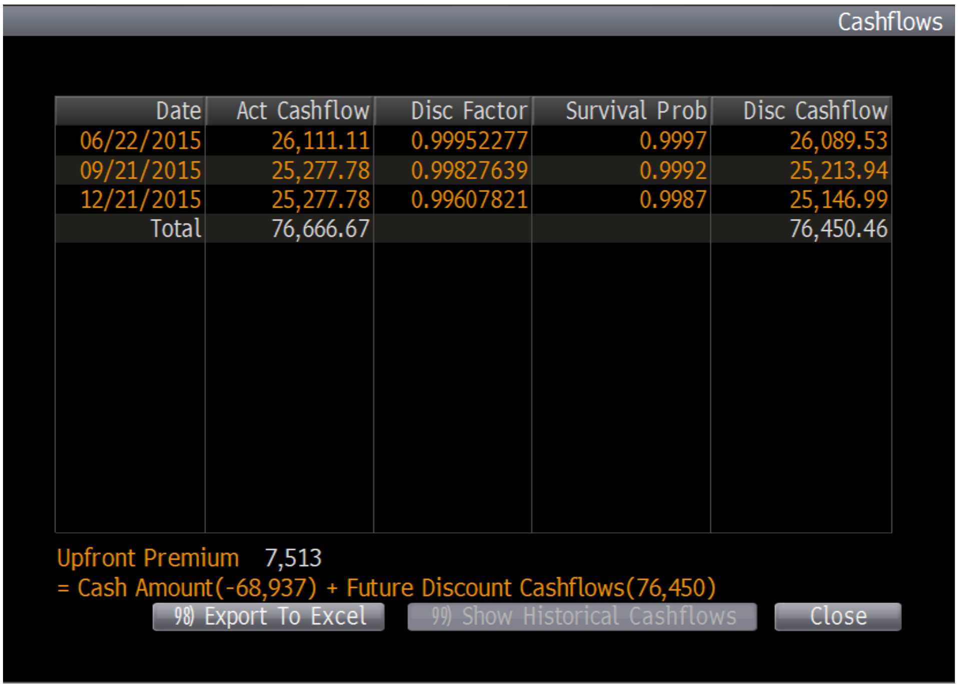
$$
\frac{S_{T_i}^{i,j} \sum_{k=i}^{j-1} \delta_k \exp\left(-\sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p\right)}{(1-\xi) \sum_{k=i}^{j-1} \left(1 - \mathrm{e}^{-\lambda\delta_k}\right) \exp\left(-\sum_{p=i}^{k} \delta_p r_p - \lambda \sum_{p=i}^{k} \delta_p\right)}
$$

converges to 0, while it tends to $+\infty$ as λ goes to 0. Therefore, the equation (4.1.4) admits a numerical solution.

Exercise 4.2 From the terminal data of Figure 4.6 we infer $S_{T_i} = 0.10790\%$, $t =$ Apr 12, 2015, $T_i =$ Mar 20, 2015, $\rho = 40\%$. Next, from the discount factors of Figure S.5 we solve the Equation (4.1.4) numerically in Table 11.1 below to find the default rate $\lambda_1 = 0.0017987468$ and default probability 0.0012460256, which is consistent with the value of 0.0013 in Figure 4.6, see also Castellacci, 2008.

Date	Delta	Discount Factor	Premium Leg	Protection Leg
Jun 22, 2015	0.2611111	0.99952277	0.0002814722	0.0002814708
Sep 21, 2015	0.2527778	0.99827639	0.0002721533	0.000272154
Dec 21, 2015	0.2527778	0.99607821	0.0002715541	0.0002715548
		Sum	0.0008251796	0.0008251796

Table 11.1: CDS Market data.



Cashflows

Date	Act Cashflow	Disc Factor	Survival Prob	Disc Cashflow
06/22/2015	26,111.11	0.99952277	0.9997	26,089.53
09/21/2015	25,277.78	0.99827639	0.9992	25,213.94
12/21/2015	25,277.78	0.99607821	0.9987	25,146.99
Total	76,666.67			76,450.46

Upfront Premium 7,513
= Cash Amount(-68,937) + Future Discount Cashflows(76,450)
98) Export To Excel 99) Show Historical Cashflows Close

Figure S.5: CDS Price data.

Exercise 4.3

a) We have

$$
\begin{aligned}
&\sum_{k=i}^{j-1} \mathbb{E}\left[\mathbb{1}_{(T_k,T_{k+1}]}(\tau)(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \Big| \mathscr{G}_t\right] \\
&= \sum_{k=i}^{j-1} \mathbb{E}\left[(\mathbb{1}_{\{T_k<\tau\}} - \mathbb{1}_{\{T_{k+1}<\tau\}})(1-\xi_{k+1})\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \Big| \mathscr{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} \sum_{k=i}^{j-1} \mathbb{E}\left[(1-\xi_{k+1})\left(\mathrm{e}^{-\int_t^{T_k}\lambda_s ds} - \mathrm{e}^{-\int_t^{T_{k+1}}\lambda_s ds}\right)\mathrm{e}^{-\int_t^{T_{k+1}} r(s)ds} \Big| \mathscr{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}(1-\xi)\sum_{k=i}^{j-1} \mathrm{e}^{-\int_t^{T_{k+1}} r(s)ds}\,\mathbb{E}\left[\mathrm{e}^{-\int_t^{T_k}\lambda_s ds} - \mathrm{e}^{-\int_t^{T_{k+1}}\lambda_s ds} \Big| \mathscr{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}(1-\xi)\sum_{k=i}^{j-1} P(t,T_{k+1})\,(Q(t,T_k)-Q(t,T_{k+1})).
\end{aligned}
$$

b) We have

$$
\begin{aligned}
V^p(t,T) &= S_t^{i,j}\sum_{k=i}^{j-1}\delta_k\,\mathbb{E}\left[\mathbb{1}_{\{\tau>T_{k+1}\}}\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \Big| \mathscr{G}_t\right] \\
&= S_t^{i,j}\sum_{k=i}^{j-1}\delta_k\,\mathbb{E}\left[\mathbb{1}_{\{T_{k+1}<\tau\}}\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \Big| \mathscr{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} S_t^{i,j}\sum_{k=i}^{j-1}\delta_k\,\mathbb{E}\left[\exp\left(-\int_t^{T_{k+1}}\lambda_s ds\right)\exp\left(-\int_t^{T_{k+1}} r(s)ds\right) \Big| \mathscr{F}_t\right] \\
&= S_t^{i,j}\mathbb{1}_{\{\tau>t\}}\sum_{k=i}^{j-1}\delta_k\exp\left(-\int_t^{T_{k+1}} r(s)ds\right)\mathbb{E}\left[\exp\left(-\int_t^{T_{k+1}}\lambda_s ds\right) \Big| \mathscr{F}_t\right]
\end{aligned}
$$

$$= \mathbb{1}_{\{\tau>t\}} S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1}) Q(t,T_{k+1}).$$

c) By equating the protection and premium legs, we find

$$(1-\xi) \sum_{k=i}^{j-1} P(t,T_{k+1}) \left(Q(t,T_k) - Q(t,T_{k+1}) \right)$$
$$= S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1}) Q(t,T_{k+1}).$$

For $j = i+1$, this yields

$$(1-\xi) P(t,T_{i+1}) \left(Q(t,T_i) - Q(t,T_{i+1}) \right) = S_t^{i,i+1} \delta_i P(t,T_{i+1}) Q(t,T_{i+1}),$$

hence

$$Q(t,T_{i+1}) = \frac{1-\xi}{S_t^{i,i+1}\delta_i + 1 - \xi},$$

with $Q(t,T_i) = 1$, and the recurrence relation

$$(1-\xi) P(t,T_{j+1}) \left(Q(t,T_j) - Q(t,T_{j+1}) \right)$$
$$+ (1-\xi) \sum_{k=i}^{j-1} P(t,T_{k+1}) \left(Q(t,T_k) - Q(t,T_{k+1}) \right)$$
$$= S_t^{i,j} \delta_j P(t,T_{j+1}) Q(t,T_{j+1}) + S_t^{i,j} \sum_{k=i}^{j-1} \delta_k P(t,T_{k+1}) Q(t,T_{k+1}),$$

i.e.

$$Q(t,T_{j+1}) = \frac{(1-\xi) Q(t,T_j)}{1-\xi+S_t^{i,j}\delta_j}$$
$$+ \sum_{k=i}^{j-1} \frac{P(t,T_{k+1}) \big((1-\xi) Q(t,T_k) - Q(t,T_{k+1}) \big((1-\xi) + \delta_k S_t^{i,j} \big) \big)}{P(t,T_{j+1})(1-\xi+S_t^{i,j}\delta_j)}.$$

Exercise 4.4 (Exercise 4.3 continued). From the terminal data of Figure 4.7, we find the following spread data and survival probabilities:

Chapter 5

Exercise 5.1 We need to compute the average

$$\frac{1}{T} \mathbb{E}\left[\int_0^T v_t dt \right] = \frac{1}{T} \int_0^T \mathbb{E}[v_t] dt = \frac{1}{T} \int_0^T u(t) dt,$$

where $u(t) := \mathbb{E}[v_t]$. Taking expectation on both sides of the equation

$$v_t = v_0 - \lambda \int_0^t (v_s - m) ds + \eta \int_0^t \sqrt{v_s} dB_s,$$

we find

$$u(t) = \mathbb{E}[v_t]$$

k	Maturity	T_k	$S_t^{1,k}$ (bp)	$Q(t,T_k)$
1	6M	0.5	10.97	0.999087
2	1Y	1	12.25	0.997961
3	2Y	2	14.32	0.995235
4	3Y	3	19.91	0.990037
5	4Y	4	26.48	0.982293
6	5Y	5	33.29	0.972122
7	7Y	7	52.91	0.937632
8	10Y	10	71.91	0.880602

Table 11.2: Spread and survival probabilities.

$$\begin{aligned}
&= \mathbb{E}\left[v_0 - \lambda \int_0^t (v_s - m)ds + \eta \int_0^t \sqrt{v_s} dB_s\right] \\
&= v_0 - \lambda\, \mathbb{E}\left[\int_0^t (v_s - m)ds\right] \\
&= v_0 - \lambda \int_0^t (\mathbb{E}[v_s] - m)ds \\
&= v_0 - \lambda \int_0^t (u(s) - m)ds, \qquad t \geqslant 0,
\end{aligned}$$

hence by differentiation with respect to $t \in \mathbb{R}$ we find the ordinary differential equation

$$u'(t) = \lambda m - \lambda u(t).$$

This equation can be rewritten as

$$(\mathrm{e}^{\lambda t} u(t))' = \lambda\, \mathrm{e}^{\lambda t} u(t) + \mathrm{e}^{\lambda t} u'(t) = \lambda m\, \mathrm{e}^{\lambda t},$$

which can be integrated as

$$\begin{aligned}
\mathrm{e}^{\lambda t} u(t) &= \left(u(0) + \lambda m \int_0^t \mathrm{e}^{\lambda s} ds\right) \\
&= \mathbb{E}[v_0] + m(\mathrm{e}^{\lambda t} - 1) \\
&= m\, \mathrm{e}^{\lambda t} + \mathbb{E}[v_0] - m \qquad t \in \mathbb{R}_+,
\end{aligned}$$

from which we conclude that

$$u(t) = m + (\mathbb{E}[v_0] - m)\, \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+,$$

and

$$\begin{aligned}
\frac{1}{T} \mathbb{E}\left[\int_0^T v_t dt\right] &= \frac{1}{T} \int_0^T u(t) dt \\
&= \frac{1}{T} \int_0^T \left(m + (\mathbb{E}[v_0] - m)\, \mathrm{e}^{-\lambda t}\right) dt \\
&= m + \frac{\mathbb{E}[v_0] - m}{T} \int_0^T \mathrm{e}^{-\lambda t} dt \\
&= m + \left(\mathbb{E}[v_0] - m\right) \frac{1 - \mathrm{e}^{-\lambda T}}{\lambda T}.
\end{aligned}$$

Exercise 5.2

a) We have

$$\mathbb{E}[v_t] = \mathbb{E}[v_0]\,\mathrm{e}^{-\lambda t} + m(1-\mathrm{e}^{-\lambda t}), \qquad t \in \mathbb{R}_+,$$

hence

$$\begin{aligned}
\mathrm{VS}_T &= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}(dS_t)^2\right] \\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}\left((r-\alpha v_t)S_t dt + S_t\sqrt{\beta+v_t}\,dB_t^{(1)}\right)^2\right] \\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T (\beta+v_t)dt\right] \\
&= \beta + \frac{1}{T}\int_0^T \mathbb{E}[v_t]dt,
\end{aligned}$$

which yields

$$\begin{aligned}
\mathrm{VS}_T &= \beta + \frac{1}{T}\int_0^T \left(\mathbb{E}[v_0]\,\mathrm{e}^{-\lambda t} + m(1-\mathrm{e}^{-\lambda t})\right)dt \\
&= \beta + \frac{1}{T}\int_0^T \left(\mathbb{E}[v_0]\,\mathrm{e}^{-\lambda t} + m(1-\mathrm{e}^{-\lambda t})\right)dt \\
&= \beta + m + \frac{1}{T}(\mathbb{E}[v_0]-m)\int_0^T \mathrm{e}^{-\lambda t}dt \\
&= \beta + m + (\mathbb{E}[v_0]-m)\frac{\mathrm{e}^{\lambda T}-1}{\lambda T}.
\end{aligned}$$

Note that if the process $(v_t)_{t\in\mathbb{R}_+}$ is started in the gamma stationary distribution then we have $\mathbb{E}[v_0] = \mathbb{E}[v_t] = m$, $t \in \mathbb{R}_+$, and the variance swap rate $\mathrm{VS}_T = \beta + m$ becomes independent of the time T.

b) The stochastic differential equation $d\sigma_t = \alpha\sigma_t dB_t^{(2)}$ is solved as

$$\sigma_t = \sigma_0\,\mathrm{e}^{\alpha B_t^{(2)} - \alpha^2 t/2}, \qquad t \in \mathbb{R}_+,$$

hence we have

$$\begin{aligned}
\mathrm{VS}_T &= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}(dS_t)^2\right] \\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}\left(\sigma_t S_t dB_t^{(1)}\right)^2\right] \\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] \\
&= \frac{\sigma_0^2}{T}\int_0^T \mathbb{E}\left[\mathrm{e}^{2\alpha B_t^{(2)} - \alpha^2 t}\right]dt \\
&= \frac{\sigma_0^2}{T}\int_0^T \mathrm{e}^{-\alpha^2 t}\,\mathbb{E}\left[\mathrm{e}^{2\alpha B_t^{(2)}}\right]dt \\
&= \frac{\sigma_0^2}{T}\int_0^T \mathrm{e}^{-\alpha^2 t + 2\alpha^2 t}dt \\
&= \frac{\sigma_0^2}{T}\int_0^T \mathrm{e}^{\alpha^2 t}dt \\
&= \frac{\sigma_0^2}{\alpha^2 T}(\mathrm{e}^{\alpha^2 T}-1).
\end{aligned}$$

Exercise 5.3

a) Taking $x = R_{0,T}^2$ and $x_0 = \mathbb{E}\left[R_{0,T}^2\right]$, we have

$$R_{0,T} \approx \sqrt{\mathbb{E}\left[R_{0,T}^2\right]} + \frac{R_{0,T}^2 - \mathbb{E}\left[R_{0,T}^2\right]}{2\sqrt{\mathbb{E}\left[R_{0,T}^2\right]}} - \frac{\left(R_{0,T}^2 - \mathbb{E}\left[R_{0,T}^2\right]\right)^2}{8\left(\mathbb{E}\left[R_{0,T}^2\right]\right)^{3/2}}, \tag{A.1}$$

provided that $R_{0,T}^2$ is sufficiently close to $\mathbb{E}\left[R_{0,T}^2\right]$.

b) Taking expectations on both sides of (A.1), we find

$$\begin{aligned}\mathbb{E}^*[R_{0,T}] &\approx \sqrt{\mathbb{E}\left[R_{0,T}^2\right]}+\frac{\mathbb{E}\left[R_{0,T}^2\right]-\mathbb{E}\left[R_{0,T}^2\right]}{2\sqrt{\mathbb{E}\left[R_{0,T}^2\right]}}-\frac{\mathbb{E}\left[\left(R_{0,T}^2-\mathbb{E}[R_{0,T}^2]\right)^2\right]}{8\left(\mathbb{E}\left[R_{0,T}^2\right]\right)^{3/2}}\\ &= \sqrt{\mathbb{E}\left[R_{0,T}^2\right]}-\frac{\mathbb{E}\left[\left(R_{0,T}^2-\mathbb{E}\left[R_{0,T}^2\right]\right)^2\right]}{8\left(\mathbb{E}\left[R_{0,T}^2\right]\right)^{3/2}}\\ &= \sqrt{\mathbb{E}\left[R_{0,T}^2\right]}-\frac{\mathrm{Var}\left[R_{0,T}^2\right]}{8\left(\mathbb{E}\left[R_{0,T}^2\right]\right)^{3/2}},\end{aligned}$$

provided that $R_{0,T}^2$ is sufficiently close to $\mathbb{E}\left[R_{0,T}^2\right]$.

Exercise 5.4 We have

$$\begin{aligned}\mathbb{E}\left[\sum_{n=1}^{N_T}\left(\log\frac{S_{T_k}}{S_{T_{k-1}}}\right)^2\right] &= \mathbb{E}\left[\int_0^T\left(\log\frac{S_t}{S_{t^-}}\right)^2 dN_t\right]\\ &= \mathbb{E}\left[\int_0^T (Z_{N_{t^-}})^2 dN_t\right]\\ &= \lambda\int_0^T \mathbb{E}\left[(Z_{N_{t^-}})^2\right]dt\\ &= \lambda\int_0^T(\eta^2+\delta^2)dt\\ &= \lambda(\eta^2+\delta^2)T.\end{aligned}$$

Exercise 5.5

a) We have $S_t = S_0\,\mathrm{e}^{\sigma B_t-\sigma^2 t/2+rt}$, $t\in\mathbb{R}_+$.

b) Letting $\widetilde{S}_t := \mathrm{e}^{-rt}S_t$, $t\in\mathbb{R}_+$, we have $\widetilde{S}_T = S_0\,\mathrm{e}^{\sigma B_T-\sigma^2T/2}$ and $d\widetilde{S}_t = \sigma\widetilde{S}_t dB_t$, hence

$$\widetilde{S}_T = S_0+\sigma\int_0^T\widetilde{S}_t dB_t,$$

and

$$\begin{aligned}&2\,\mathbb{E}^*\left[\frac{\mathrm{e}^{-rT}S_T}{S_0}\log\frac{\mathrm{e}^{-rT}S_T}{S_0}\right] = 2\,\mathbb{E}^*\left[\frac{\widetilde{S}_T}{S_0}\log\frac{\widetilde{S}_T}{S_0}\right]\\ &= 2\,\mathbb{E}^*\left[\left(1+\sigma\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right)\left(\sigma B_T-\frac{\sigma^2T}{2}\right)\right]\\ &= 2\,\mathbb{E}^*\left[\sigma B_T-\frac{\sigma^2T}{2}\right]+2\sigma^2\,\mathbb{E}^*\left[B_T\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right]-\sigma^2T\,\mathbb{E}^*\left[\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right]\\ &= -\sigma^2T+2\sigma^2\,\mathbb{E}^*\left[\int_0^T dB_t\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right]\\ &= -\sigma^2T+2\sigma^2\,\mathbb{E}^*\left[\int_0^T\frac{\widetilde{S}_t}{S_0}dt\right]\\ &= -\sigma^2T+2\sigma^2\int_0^T\mathbb{E}^*\left[\frac{\widetilde{S}_t}{S_0}\right]dt\\ &= -\sigma^2T+2\sigma^2\int_0^T dt\end{aligned}$$

$$
\begin{aligned}
&= -\sigma^2 T + 2\sigma^2 T \\
&= \sigma^2 T.
\end{aligned}
$$

Alternatively, we could also write

$$
\begin{aligned}
2\,\mathbb{E}^*\left[\frac{\mathrm{e}^{-rT}S_T}{S_0}\log\frac{\mathrm{e}^{-rT}S_T}{S_0}\right] &= 2\,\mathbb{E}^*\left[\frac{\widetilde{S}_T}{S_0}\log\frac{\widetilde{S}_T}{S_0}\right] \\
&= 2\,\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T-\sigma^2T/2}\log\mathrm{e}^{\sigma B_T-\sigma^2T/2}\right] \\
&= 2\,\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T-\sigma^2T/2}\left(\sigma B_T-\frac{\sigma^2T}{2}\right)\right] \\
&= 2\sigma\,\mathrm{e}^{-\sigma^2T/2}\,\mathbb{E}^*\left[B_T\,\mathrm{e}^{\sigma B_T}\right]-\sigma^2T\,\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T-\sigma^2T/2}\right] \\
&= 2\sigma\,\mathrm{e}^{-\sigma^2T/2}\frac{\partial}{\partial\sigma}\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T}\right]-\sigma^2T \\
&= 2\sigma\,\mathrm{e}^{-\sigma^2T/2}\frac{\partial}{\partial\sigma}\mathrm{e}^{\sigma^2T/2}-\sigma^2T \\
&= 2\sigma^2T\,\mathrm{e}^{-\sigma^2T/2}\mathrm{e}^{\sigma^2T/2}-\sigma^2T \\
&= \sigma^2T.
\end{aligned}
$$

Exercise 5.6

a) By the Itô formula, we have

$$
\log\frac{S_T}{S_0} = \log S_T - \log S_0 = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2}\int_0^T \frac{\sigma_t^2}{S_t^2}dt.
$$

b) By (5.5.9) we have

$$
\begin{aligned}
\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt \,\Big|\, \mathscr{F}_t\right] &= 2\,\mathbb{E}^*\left[\int_0^T \frac{dS_t}{S_t}\,\Big|\,\mathscr{F}_t\right] - 2\,\mathbb{E}^*\left[\log\frac{S_T}{S_0}\,\Big|\,\mathscr{F}_t\right] \\
&= 2\int_0^t \frac{dS_u}{S_u} + 2r(T-t) - 2\,\mathbb{E}^*\left[\log\frac{S_T}{S_0}\,\Big|\,\mathscr{F}_t\right].
\end{aligned}
$$

c) At time $t\in[0,T]$ we check that

$$
\begin{aligned}
&L_t + \mathrm{e}^{-(T-t)r}\frac{2}{S_t}S_t + 2\,\mathrm{e}^{-rT}\left(\int_0^t \frac{dS_u}{S_u} + (T-t)r - 1\right)A_t \\
&= L_t + 2r(T-t)\,\mathrm{e}^{-(T-t)r} + 2\,\mathrm{e}^{-(T-t)r}\int_0^t\frac{dS_u}{S_u} \\
&= V_t.
\end{aligned}
$$

d) By (5.5.10) we have

$$
\begin{aligned}
dV_t &= d\left(L_t + 2r(T-t)\,\mathrm{e}^{-(T-t)r} + 2\,\mathrm{e}^{-(T-t)r}\int_0^t \frac{dS_u}{S_u}\right) \\
&= dL_t - 2r\,\mathrm{e}^{-(T-t)r}dt + 2r^2(T-t)\,\mathrm{e}^{-(T-t)r}dt \\
&\quad +2r\,\mathrm{e}^{-(T-t)r}\int_0^t\frac{dS_u}{S_u}dt + 2\,\mathrm{e}^{-(T-t)r}\frac{dS_t}{S_t} \\
&= dL_t + \mathrm{e}^{-(T-t)r}\frac{2}{S_t}dS_t + 2\,\mathrm{e}^{-rT}\left(\int_0^t\frac{dS_u}{S_u} + (T-t)r - 1\right)dA_t,
\end{aligned}
$$

with $dA_t = r\,\mathrm{e}^{rt}dt$, hence the portfolio is self-financing.

Exercise 5.7 By second differentiation of the moment generating function (5.2.7), we find the two expressions

$$\mathbb{E}^*\left[R_{0,T}^4\right]=4\,\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2+2\log\frac{S_T}{F_0}\right]=4\,\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2\right]-4\,\mathbb{E}^*\left[R_{0,T}^2\right],$$

and

$$\begin{aligned}\mathbb{E}^*\left[R_{0,T}^4\right] &= 4\,\mathbb{E}^*\left[\frac{S_T}{F_0}\left(\left(\log\frac{S_T}{F_0}\right)^2-2\log\frac{S_T}{F_0}\right)\right]\\ &= 4\,\mathbb{E}^*\left[\frac{S_T}{F_0}\left(\log\frac{S_T}{F_0}\right)^2\right]-4\,\mathbb{E}^*\left[R_{0,T}^2\right].\end{aligned}$$

Chapter 6

Exercise 6.1

a) We have $f'_{K^*}(x)=-\log(x/K^*)$ and $f''_{K^*}(x):=-1/x$, hence

$$df_{K^*}(S_t)=f'_{K^*}(S_t)dS_t+\frac{1}{2}f''_{K^*}(S_t)(dS_t)^2=-\log\frac{S_t}{K^*}dS_t-\frac{(dS_t)^2}{2S_t}.$$

On the other hand, we have

$$d\log S_t=\frac{dS_t}{S_t}-\frac{(dS_t)^2}{2S_t^2},$$

hence

$$-\frac{(dS_t)^2}{2S_t}=S_t d\log S_t-dS_t,$$

which yields

$$S_t d\log S_t=df_{K^*}(S_t)+\log\frac{S_t}{K^*}dS_t,$$

and leads to (6.4.3) by integration over $[0,T]$.

b) We note that $f_{K^*}(K^*)=f'_{K^*}(K^*)=0$, hence by Lemma 2.3 we have

$$\begin{aligned}f_{K^*}(x) &= f_{K^*}(K^*)+(x-K^*)f'_{K^*}(K^*)+\int_0^{K^*}(z-x)^+f''_{K^*}(z)dz\\ &\quad+\int_{K^*}^\infty(x-z)^+f''_{K^*}(z)dz\\ &= -\int_0^{K^*}(K-x)^+\frac{dK}{K}-\int_{K^*}^\infty(x-K)^+\frac{dK}{K}.\end{aligned}$$

As a consequence, we have

$$\begin{aligned}\int_0^T S_t d\log S_t &= \int_0^{K^*}\left((K-S_0)^+-(K-S_T)^+\right)\frac{dK}{K}\\ &\quad+\int_{K^*}^\infty\left((S_0-K)^+-(S_T-K)^+\right)\frac{dK}{K}+\int_0^T\left(1+\log\frac{S_t}{K^*}\right)dS_t,\end{aligned}$$

and the entropy swap payoff can be hedged in a self-financing way by:

- holding $(1+\log(S_t/K^*)$ units of S_t,
- holding $\displaystyle\int_0^{K^*}(K-S_0)^+\frac{dK}{K}+\int_{K^*}^\infty(S_0-K)^+\frac{dK}{K}$ in cash,
- shorting a static portfolio of call options with strike prices $K>K^*$ and a static portfolio of put options with strike prices $K<K^*$.

Exercise 6.2

a) We have

$$\frac{\partial C}{\partial K}(T,K_2) \simeq \frac{C(T,K_2)-C(T,K_1)}{\Delta K}, \quad \frac{\partial C}{\partial K}(T,K_3) \simeq \frac{C(T,K_3)-C(T,K_2)}{\Delta K}. \tag{A.2}$$

Reusing (A.2), second order spatial derivatives can be similarly approximated as

$$\begin{aligned}\frac{\partial^2 C}{\partial K^2}(T,K_2) &\simeq \frac{1}{\Delta K}\left(\frac{\partial C}{\partial K}(T,K_3)-\frac{\partial C}{\partial K}(T,K_2)\right)\\ &\simeq \frac{C(T,K_3)+C(T,K_1)-2C(T,K_2)}{(\Delta K)^2}.\end{aligned}$$

b) Under the condition $\frac{\partial^2 C}{\partial K^2}(T,K_2) < 0$, the portfolio with terminal payoff

$$(S_T-K_3)^+ + (S_T-K_1)^+ - 2(S_T-K_2)^+$$

has the negative initial price

$$C(T,K_3)+C(T,K_1)-2C(T,K_2) < 0,$$

hence it constitutes an arbitrage opportunity, see https://optioncreator.com/stzrvij with $K_1 = \$80$, $K_2 = \$100$, $K_3 = \$120$.

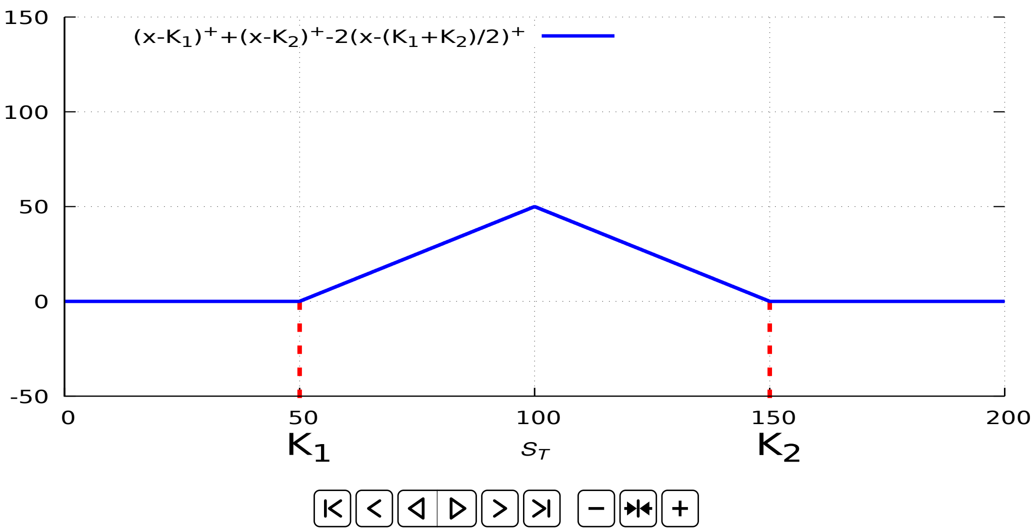


Figure S.6: Butterfly option payoff as a combination of call and put options.*

See page 30 of Bergomi, 2016 for the construction of arbitrage opportunities based on the negativity of the numerator in (6.3.7) (maturity arbitrage).

Exercise 6.3

a) We have $\frac{\partial C}{\partial x}(T-t,x,K) = \frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right)$ and

$$\begin{aligned}\frac{\partial C}{\partial K}(T-t,x,K) &= \frac{\partial}{\partial K}\left(Kf\left(T-t,\frac{x}{K}\right)\right)\\ &= f\left(T-t,\frac{x}{K}\right) - \frac{x}{K}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right)\\ &= \frac{1}{K}C(T-t,x,K) - \frac{x}{K}\frac{\partial C}{\partial x}(T-t,x,K),\end{aligned}$$

*The animation works in Acrobat Reader on the entire pdf file.

hence

$$\frac{\partial C}{\partial x}(T-t,x,K)=\frac{1}{x}C(T-t,x,K)-\frac{K}{x}\frac{\partial C}{\partial K}(T-t,x,K).$$

b) We have $\frac{\partial^2 C}{\partial x^2}(T-t,x,K)=\frac{1}{K}\frac{\partial^2 f}{\partial z^2}\left(T-t,\frac{x}{K}\right)$ and

$$\begin{aligned}\frac{\partial^2 C}{\partial K^2}&(T-t,x,K)\\ &= -\frac{x}{K^2}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right)+\frac{x}{K^2}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right)+\frac{x^2}{K^3}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right)\\ &= \frac{x^2}{K^3}\frac{\partial^2 f}{\partial z^2}\left(T-t,\frac{x}{K}\right)\\ &= \frac{x^2}{K^2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K),\end{aligned}$$

hence

$$\frac{\partial^2 C}{\partial x^2}(T-t,x,K)=\frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K).$$

c) Noting that

$$\frac{\partial C}{\partial t}(T-t,x,K)=-\frac{\partial C}{\partial T}(T-t,x,K),$$

we can rewrite the Black-Scholes PDE as

$$\begin{aligned}rC(T-t,x,K) &= -\frac{\partial C}{\partial T}(T-t,x,K)\\ &\quad +rx\left(\frac{1}{x}C(T-t,x,K)-\frac{K}{x}\frac{\partial C}{\partial K}(T-t,x,K)\right)\\ &\quad +\frac{\sigma^2x^2}{2}\frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K),\end{aligned}$$

i.e.

$$\frac{\partial C}{\partial T}(T-t,x,K)=-rK\frac{\partial C}{\partial K}(T-t,x,K)+\frac{\sigma^2x^2}{2}\frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K).$$

Remarks:

1. Using the Black-Scholes Greek Gamma expression

$$\begin{aligned}\frac{\partial^2 C}{\partial x^2}(T-t,x,K) &= \frac{1}{\sigma x\sqrt{T-t}}\Phi'(d_+(T-t))\\ &= \frac{1}{\sigma x\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_+(T-t))^2/2},\end{aligned}$$

we can recover the lognormal probability density function $\varphi_T(y)$ of geometric Brownian motion S_T as follows:

$$\begin{aligned}\varphi_T(K) &= \mathrm{e}^{(T-t)r}\frac{\partial^2 C}{\partial K^2}(T-t,x,K)\\ &= \mathrm{e}^{(T-t)r}\frac{x^2}{K^2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K)\\ &= \frac{\mathrm{e}^{(T-t)r}x}{\sigma K^2\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_+(T-t))^2/2}\\ &= \frac{1}{\sigma K\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_-(T-t))^2/2}\\ &= \frac{1}{\sigma K\sqrt{2\pi(T-t)}}\exp\left(-\frac{\left((r-\sigma^2/2)(T-t)+\log(x/K)\right)^2}{2(T-t)\sigma^2}\right),\end{aligned}$$

knowing that

$$\begin{aligned}
-\frac{1}{2}\big(d_-(T-t)\big)^2 &= -\frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\\
&= -\frac{1}{2}\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2+(T-t)r+\log\frac{x}{K}\\
&= -\frac{1}{2}\big(d_+(T-t)\big)^2+(T-t)r+\log\frac{x}{K},
\end{aligned}$$

which can be obtained from the relation

$$\begin{aligned}
&\big(d_+(T-t)\big)^2-\big(d_-(T-t)\big)^2\\
&= \big((d_+(T-t)+d_-(T-t)\big)\big((d_+(T-t)-d_-(T-t)\big)\\
&= 2r(T-t)+2\log\frac{x}{K}.
\end{aligned}$$

2. Using the expressions of the Black-Scholes Greeks Delta and Theta we can also recover

$$\begin{aligned}
&2\frac{\dfrac{\partial C}{\partial T}(T-t,x,K)+rK\dfrac{\partial C}{\partial K}(T-t,x,K)}{K^2\dfrac{\partial^2 C}{\partial K^2}(T-t,x,K)}\\
&= 2\frac{-\dfrac{\partial C}{\partial t}(T-t,x,K)+rK\left(\dfrac{1}{K}C(T-t,x,K)-\dfrac{x}{K}\dfrac{\partial C}{\partial x}(T-t,x,K)\right)}{x^2\dfrac{\partial^2 C}{\partial x^2}(T-t,x,K)}\\
&= 2\frac{x\sigma\Phi'(d_+(T-t))/(2\sqrt{T-t})+rK\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))}{x^2\Phi'(d_+(T-t))/(x\sigma\sqrt{T-t})}\\
&\quad +2\frac{rC(T-t,x,K)-rx\Phi(d_+(T-t))}{x^2\Phi'(d_+(T-t))/(x\sigma\sqrt{T-t})}\\
&= \sigma^2.
\end{aligned}$$

Exercise 6.4 We have

$$\begin{aligned}
\frac{\partial C}{\partial K}(S_0,K,T) &= -(K-S_0)\frac{\mathrm{e}^{-(K-S_0)^2/(2T)}}{\sqrt{2\pi T}}\\
&\quad -\Phi\left(-\frac{K-S_0}{\sqrt{T}}\right)+\frac{K-S_0}{\sqrt{T}}\varphi\left(-\frac{K-S_0}{\sqrt{T}}\right)\\
&= -\Phi\left(-\frac{K-S_0}{\sqrt{T}}\right),
\end{aligned}$$

and

$$\frac{\partial C^2}{\partial K^2}(S_0,K,T) = \frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-(K-S_0)^2/(2T)},$$

which is the Gaussian probability density function of $S_T = S_0+B_T$. We also have

$$\begin{aligned}
\frac{\partial C}{\partial T}(S_0,K,T) &= \frac{1}{2\sqrt{2\pi T}}\mathrm{e}^{-(K-S_0)^2/(2T)}-\frac{(K-S_0)^2}{2T^2}\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-(K-S_0)^2/(2T)}\\
&\quad +\frac{(K-S_0)^2}{2T^{3/2}}\varphi\left(-\frac{K-S_0}{\sqrt{T}}\right).\\
&= \frac{1}{2\sqrt{2\pi T}}\mathrm{e}^{-(K-S_0)^2/(2T)}\\
&= \frac{1}{2}\frac{\partial C^2}{\partial K^2}(S_0,K,T),
\end{aligned}$$

hence

$$|\sigma(t,y)| = \sqrt{\frac{2\frac{\partial C^M}{\partial t}(t,y) + 2ry\frac{\partial C^M}{\partial y}(t,y)}{y^2\frac{\partial^2 C^M}{\partial y^2}(t,y)}} = \sqrt{\frac{2\frac{\partial C^M}{\partial t}(t,y)}{y^2\frac{\partial^2 C^M}{\partial y^2}(t,y)}} = \sqrt{\frac{1}{y^2}} = \frac{1}{|y|},$$

and the equation satisfied by $(S_t)_{t\in\mathbb{R}_+}$ is

$$dS_t = S_t\sigma(t,S_t)dB_t = \frac{S_t}{|S_t|}dB_t = \operatorname{sign}(S_t)dB_t = dW_t,$$

where $dW_t := \operatorname{sign}(S_t)dB_t$ is also a standard Brownian motion by the Lévy characterization theorem, $\sigma(t,y) = 1/y$, and $S_t = S_0 + B_t$. Indeed, as in Quiz 2 of FE8815, the price of the call option in the Bachelier model is given by

$$\begin{aligned}
C(S_0,K,T) &= \mathbb{E}[(S_T-K)^+] \\
&= \mathbb{E}[(S_0+B_T-K)^+] \\
&= \int_{K-S_0}^{\infty}(x+S_0-K)\,\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}} \\
&= \int_{K-S_0}^{\infty}x\,\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}} - (K-S_0)\int_{K-S_0}^{\infty}\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}} \\
&= \sqrt{\frac{T}{2\pi}}\left[-\mathrm{e}^{-x^2/(2T)}\right]_{K-S_0}^{\infty} - (K-S_0)\int_{(K-S_0)/\sqrt{T}}^{\infty}\mathrm{e}^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\
&= \sqrt{\frac{T}{2\pi}}\mathrm{e}^{-(K-S_0)^2/(2T)} - (K-S_0)\Phi\left(-\frac{K-S_0}{\sqrt{T}}\right).
\end{aligned}$$

Exercise 6.5

a) We have
$$\frac{\partial M_C}{\partial K}(K,S,r,\tau) = \frac{\partial C}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau) + \sigma'_{\mathrm{imp}}(K)\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau).$$

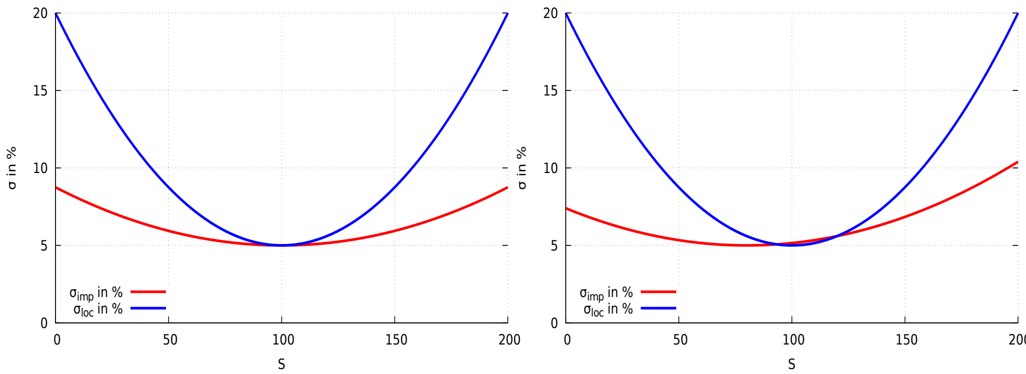
b) We have
$$\frac{\partial C}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau) + \sigma'_{\mathrm{imp}}(K)\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau) \leqslant 0,$$
which shows that
$$\sigma'_{\mathrm{imp}}(K) \leqslant -\frac{\frac{\partial C}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}{\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}$$

c) We have
$$\frac{\partial P}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau) + \sigma'_{\mathrm{imp}}(K)\frac{\partial P}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau) \geqslant 0,$$
which shows that
$$\sigma'_{\mathrm{imp}}(K) \geqslant -\frac{\frac{\partial P}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}{\frac{\partial P}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}$$

Exercise 6.6

a) We have

$$\begin{aligned}\sigma_{\mathrm{imp}}(K,S) &\simeq \sigma_{\mathrm{loc}}\left(\frac{K+S}{2}\right)\\ &= \sigma_0+\beta\left(\frac{K+S}{2}-S_0\right)^2\\ &= \sigma_0+\frac{\beta}{4}(K-(2S_0-S))^2.\end{aligned}$$



(a) At the money $K=S_0$. (b) Out of the money $K>S_0$.

Figure S.7: Implied *vs.* local volatility.

b) We find

$$\begin{aligned}\frac{\partial}{\partial S}\left(\mathrm{Bl}(S,K,T,\sigma_{\mathrm{imp}}(K,S),r)\right) &= \frac{\partial \mathrm{Bl}}{\partial x}(x,K,T,\sigma_{\mathrm{imp}}(K,S),r)_{x=S}\\ &\quad +\frac{\partial \sigma_{\mathrm{imp}}}{\partial S}\frac{\partial \mathrm{Bl}}{\partial \sigma}(x,K,T,\sigma,r)_{\sigma=\sigma_{\mathrm{imp}}(K,S)}\\ &= \Delta+v\frac{\beta}{2}(K-(2S_0-S)),\end{aligned}$$

where

$$\Delta=\frac{\partial \mathrm{Bl}}{\partial x}(x,K,T,\sigma_{\mathrm{imp}}(K,S),r)_{x=S}$$

is the Black-Scholes Delta and

$$v=\frac{\partial \mathrm{Bl}}{\partial \sigma}(S,K,T,\sigma,r)_{\sigma=\sigma_{\mathrm{imp}}(K,S)}$$

is the Black-Scholes Vega, cf. §2.2 of Hagan et al., 2002.

Exercise 6.7 We take $t=0$ for simplicity. We start by showing that for every $\lambda>0$, we have

$$\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau \sigma_t^2dt\right)-1\right]=\frac{2\,\mathrm{e}^{r\tau}}{S_0^{p_\lambda}}\left(\int_0^{F_0}P(\tau,K)\frac{dK}{K^{2-p_\lambda}}+\int_{F_0}^\infty C(\tau,K)\frac{dK}{K^{2-p_\lambda}}\right).$$

By Lemma 5.2, we have

$$\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau \sigma_t^2dt\right)-1\right]=\frac{1}{\lambda}\mathbb{E}\left[\left(\frac{S_\tau}{F_0}\right)^{p_\lambda}-1\right],\quad \lambda>0.$$

Using Relation (6.3.14), *i.e.*

$$\varphi_\tau(K)=\mathrm{e}^{r\tau}\frac{\partial^2C^M}{\partial y^2}(\tau,y)=\mathrm{e}^{r\tau}\frac{\partial^2P^M}{\partial y^2}(\tau,y),$$

we have

$$\begin{aligned}
\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau\sigma_t^2dt\right)-1\right] &= \frac{1}{\lambda}\mathbb{E}\left[\left(\frac{S_\tau}{F_0}\right)^{p_\lambda}-1\right] \\
&= \frac{1}{\lambda F_0^{p_\lambda}}\mathbb{E}\left[S_\tau^{p_\lambda}-F_0^{p_\lambda}\right] \\
&= \frac{1}{\lambda F_0^{p_\lambda}}\left(\int_0^\infty K^{p_\lambda}\varphi_\tau(K)dK-F_0^{p_\lambda}\right) \\
&= \frac{1}{\lambda F_0^{p_\lambda}}\left(\int_0^{F_0} K^{p_\lambda}\varphi_\tau(K)dK+\int_{F_0}^\infty K^{p_\lambda}\varphi_\tau(K)dK-F_0^{p_\lambda}\right) \\
&= \frac{1}{\lambda S_0^{p_\lambda}}\left(\mathrm{e}^{r\tau}\int_0^{F_0} K^{p_\lambda}\frac{\partial^2 P}{\partial K^2}(\tau,K)dK+\mathrm{e}^{r\tau}\int_{F_0}^\infty K^{p_\lambda}\frac{\partial^2 C}{\partial K^2}(\tau,K)dK-S_0^{p_\lambda}\right).
\end{aligned}$$

Next, integrating by parts over the intervals $[0,F_0]$ and $[F_0,\infty)$ and using the boundary conditions

$$P(\tau,0)=C(\tau,\infty)=0, \qquad \frac{\partial P}{\partial K}(\tau,0)=\frac{\partial C}{\partial K}(\tau,\infty)=0,$$

with the relation

$$\frac{\partial P}{\partial K}(\tau,K)-\frac{\partial C}{\partial K}(\tau,K)-\mathrm{e}^{-r\tau}=0$$

and the call-put parity

$$P(\tau,F_0)-C(\tau,F_0)=S_0-F_0\,\mathrm{e}^{r\tau}=0$$

as boundary conditions, we find

$$\begin{aligned}
&\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau\sigma_t^2dt\right)-1\right] \\
&= \frac{1}{\lambda S_0^{p_\lambda}}\left(\mathrm{e}^{r\tau}S_0^{p_\lambda}\frac{\partial P}{\partial K}(\tau,F_0)-p_\lambda\,\mathrm{e}^{r\tau}\int_0^{F_0}K^{p_\lambda-1}\frac{\partial P}{\partial K}(\tau,K)dK\right. \\
&\qquad \left.-\,\mathrm{e}^{r\tau}S_0^{p_\lambda}\frac{\partial C}{\partial K}(\tau,F_0)-p_\lambda\,\mathrm{e}^{r\tau}\int_{F_0}^\infty K^{p_\lambda-1}\frac{\partial C}{\partial K}(\tau,K)dK-S_0^{p_\lambda}\right) \\
&= -p_\lambda\frac{\mathrm{e}^{r\tau}}{\lambda S_0^{p_\lambda}}\left(\int_0^{F_0}K^{p_\lambda-1}\frac{\partial P}{\partial K}(\tau,K)dK+\int_{F_0}^\infty K^{p_\lambda-1}\frac{\partial C}{\partial K}(\tau,K)dK\right) \\
&= \frac{p_\lambda\,\mathrm{e}^{r\tau}}{\lambda S_0^{p_\lambda}}\left(S_0^{p_\lambda-1}P(\tau,F_0)+(p_\lambda-1)\int_0^{F_0}K^{p_\lambda-2}P(\tau,K)dK\right. \\
&\qquad \left.-S_0^{p_\lambda-1}C(\tau,F_0)+(p_\lambda-1)\int_{F_0}^\infty K^{p_\lambda-2}C(\tau,K)dK\right) \\
&= \frac{p_\lambda(p_\lambda-1)}{\lambda S_0^{p_\lambda}}\mathrm{e}^{r\tau}\left(\int_0^{F_0}K^{p_\lambda-2}P(\tau,K)dK+\int_{F_0}^\infty K^{p_\lambda-2}C(\tau,K)dK\right) \\
&= \frac{2\,\mathrm{e}^{r\tau}}{S_0^{p_\lambda}}\left(\int_0^{F_0}P(\tau,K)\frac{dK}{K^{2-p_\lambda}}+\int_{F_0}^\infty C(\tau,K)\frac{dK}{K^{2-p_\lambda}}\right) \\
&= \frac{2\,\mathrm{e}^{r\tau}}{S_0^{p_\lambda}}\left(\int_0^{F_0}P(\tau,K)\frac{dK}{K^{2-p_\lambda}}+\int_{F_0}^\infty C(\tau,K)\frac{dK}{K^{2-p_\lambda}}\right).
\end{aligned}$$

Finally, taking

$$p_\lambda := p_\lambda^- = 1/2-\sqrt{1/4+2\lambda}$$

and letting λ tend to zero, we find

$$\mathbb{E}\left[\int_0^\tau\sigma_t^2dt\right] = \lim_{\lambda\to 0}\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau\sigma_t^2dt\right)-1\right]$$

$$
\begin{aligned}
&= \lim_{\lambda\to 0} \frac{2\,\mathrm{e}^{r\tau}}{S_0^{p_\lambda}} \left(\int_0^{F_0} P(\tau,K)\frac{dK}{K^{2-p_\lambda}} + \int_{F_0}^{\infty} C(\tau,K)\frac{dK}{K^{2-p_\lambda}} \right) \\
&= 2\,\mathrm{e}^{r\tau} \left(\int_0^{F_0} P(\tau,K)\frac{dK}{K^2} + \int_{F_0}^{\infty} C(\tau,K)\frac{dK}{K^2} \right).
\end{aligned}
$$

Exercise 6.8 (Exercise 5.7 continued). Taking $\phi(x) = (\log(x/F_0))^2$ with $y = F_0$, we have

$$
\phi'(x) = \frac{2}{x}\log\frac{x}{F_0} \quad \text{and} \quad \phi''(x) = \frac{2}{x^2}\left(1 - \log\frac{x}{F_0}\right),
$$

hence

$$
\begin{aligned}
\left(\log\frac{S_T}{F_0}\right)^2 &= \phi(F_0) + (S_T - F_0)\phi'(F_0) \\
&\quad + \int_0^{F_0} (z - S_T)^+ \phi''(z)dz + \int_{F_0}^{\infty} (S_T - z)^+ \phi''(z)dz \\
&= 2\int_0^{F_0} (K - S_T)^+ \left(1 - \log\frac{K}{F_0}\right)\frac{dK}{K^2} \\
&\quad + 2\int_{F_0}^{\infty} (S_T - K)^+ \left(1 - \log\frac{K}{F_0}\right)\frac{dK}{K^2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2\right] &= 2\int_0^{F_0} \mathbb{E}^*[(K - S_T)^+]\frac{dK}{K^2} + 2\int_{F_0}^{\infty} \mathbb{E}^*[(S_T - K)^+]\frac{dK}{K^2} \\
&\quad -2\int_0^{F_0} \mathbb{E}^*[(K - S_T)^+]\log\frac{K}{F_0}\frac{dK}{K^2} - 2\int_{F_0}^{\infty} \mathbb{E}^*[(S_T - K)^+]\log\frac{K}{F_0}\frac{dK}{K^2} \\
&= \mathbb{E}^*\left[R_{0,T}^2\right] - 2\,\mathrm{e}^{rT}\int_0^{F_0} P(T,K)\log\frac{K}{F_0}\frac{dK}{K^2} - 2\,\mathrm{e}^{rT}\int_{F_0}^{\infty} C(T,K)\log\frac{K}{F_0}\frac{dK}{K^2},
\end{aligned}
$$

and

$$
\mathbb{E}^*\left[R_{0,T}^4\right] = 8\,\mathrm{e}^{rT}\int_0^{F_0} P(T,K)\left(\log\frac{F_0}{K}\right)\frac{dK}{K^2} - 8\,\mathrm{e}^{rT}\int_{F_0}^{\infty} C(T,K)\left(\log\frac{K}{F_0}\right)\frac{dK}{K^2}. \tag{A.3}
$$

Alternatively, taking $\phi(x) = (x/F_0)(\log(x/F_0))^2$ with $y = F_0$, we have

$$
\phi'(x) = \frac{1}{F_0}\left(\log\frac{x}{F_0}\right)^2 + \frac{2}{F_0}\log\frac{x}{F_0}
$$

and

$$
\phi''(x) = \frac{2}{xF_0}\log\frac{x}{F_0} + \frac{2}{xF_0} = \frac{2}{xF_0}\left(1 + \log\frac{x}{F_0}\right),
$$

hence

$$
\begin{aligned}
\left(\log\frac{S_T}{F_0}\right)^2 &= \int_0^{F_0} (K - S_T)^+ \frac{2}{KF_0}\left(1 + \log\frac{K}{F_0}\right)dK \\
&\quad + \int_{F_0}^{\infty} (S_T - K)^+ \frac{2}{KF_0}\left(1 + \log\frac{K}{F_0}\right)dK.
\end{aligned}
$$

Therefore, we have

$$
\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2\right] = \int_0^{F_0} \mathbb{E}[(K - S_T)^+]\frac{2}{KF_0}\left(1 + \log\frac{K}{F_0}\right)dK
$$

$$+\int_{F_0}^{\infty}\mathbb{E}[(S_T-K)^+]\frac{2}{KF_0}\left(1+\log\frac{K}{F_0}\right)dK.,$$

and

$$\begin{aligned}\mathbb{E}^*\left[R_{0,T}^4\right] &= \frac{8}{F_0}\mathrm{e}^{rT}\int_0^{F_0}P(T,K)\left(1+\log\frac{K}{F_0}\right)\frac{dK}{K}\\ &+\frac{8}{F_0}\mathrm{e}^{rT}\int_{F_0}^{\infty}P(T,K)\left(1+\log\frac{K}{F_0}\right)\frac{dK}{K}-4\,\mathbb{E}^*\left[R_{0,T}^2\right].\end{aligned}$$

Exercise 6.9

a) We have

$$\begin{aligned}\int_0^\infty\frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-\mu x}}{x^{\rho+1}}dx &= \int_0^\infty\frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-\mu x}}{x^{\rho+1}}dx\\ &= -\frac{1}{\rho}\left[\frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-\mu x}}{x^{\rho}}\right]_0^\infty+\frac{1}{\rho}\int_0^\infty\frac{-\nu\,\mathrm{e}^{-\nu x}+\mu\,\mathrm{e}^{-\mu x}}{x^{\rho}}dx\\ &= -\frac{\nu}{\rho}\int_0^\infty\mathrm{e}^{-\nu x}x^{-\rho}dx+\frac{\mu}{\rho}\int_0^\infty\mathrm{e}^{-\mu x}x^{-\rho}dx\\ &= \frac{\mu^\rho-\nu^\rho}{\rho}\Gamma(1-\rho).\end{aligned}$$

b) Taking $\nu=0$ and $\mu=R_{t,T}$, we find

$$\begin{aligned}\mathbb{E}^*[R_{t,T}] &= \frac{\rho}{\Gamma(1-\rho)}\mathbb{E}^*\left[\int_0^\infty\left(1-\mathrm{e}^{-\lambda R_{t,T}^2}\right)\frac{d\lambda}{\lambda^{\rho+1}}\right]\\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty\left(1-\mathbb{E}^*\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]\right)\frac{d\lambda}{\lambda^{\rho+1}},\end{aligned}$$

see § 3.1 in Friz and Gatheral, 2005 with $\rho=1/2$.

c) Letting $p_\lambda^\pm:=1/2\pm\sqrt{1/4-2\lambda}$, we have

$$\begin{aligned}\mathbb{E}^*[R_{t,T}] &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty\left(1-\mathbb{E}^*\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]\right)\frac{d\lambda}{\lambda^{\rho+1}}\\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty\left(1-\mathrm{e}^{-rp_\lambda^\pm T}\,\mathbb{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda^\pm}\right]\right)\frac{d\lambda}{\lambda^{\rho+1}}\\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty\mathbb{E}^*\left[1-\left(\frac{S_T}{F_0}\right)^{p_\lambda^\pm}\right]\frac{d\lambda}{\lambda^{\rho+1}}\\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^{1/8}\mathbb{E}^*\left[1-\left(\frac{S_T}{F_0}\right)^{p_\lambda^\pm}\right]\frac{d\lambda}{\lambda^{\rho+1}}\\ &\quad+\frac{\rho}{\Gamma(1-\rho)}\int_{1/8}^\infty\mathbb{E}^*\left[1-\sqrt{\frac{S_T}{F_0}}\exp\left(\pm\frac{i}{2}\sqrt{8\lambda-1}\log\frac{S_T}{F_0}\right)\right]\frac{d\lambda}{\lambda^{\rho+1}}\\ &= \frac{8^\rho}{\rho+1}+\frac{\rho}{\Gamma(1-\rho)}\int_0^{1/8}\mathbb{E}^*\left[1-\left(\frac{S_T}{F_0}\right)^{p_\lambda^\pm}\right]\frac{d\lambda}{\lambda^{\rho+1}}\\ &\quad-\frac{\rho}{\Gamma(1-\rho)}\int_{1/8}^\infty\mathbb{E}^*\left[\sqrt{\frac{S_T}{F_0}}\cos\left(\frac{1}{2}\sqrt{8\lambda-1}\log\frac{S_T}{F_0}\right)\right]\frac{d\lambda}{\lambda^{\rho+1}}\\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^{1/8}\mathbb{E}^*[\phi_\lambda(S_T)]\frac{d\lambda}{\lambda^{\rho+1}}+\frac{\rho}{\Gamma(1-\rho)}\mathbb{E}^*\left[\psi(S_T)\right],\end{aligned}$$

where

$$\phi_\lambda(x)=1-\left(\frac{x}{F_0}\right)^{p_\lambda^\pm},$$

we have

$$\phi_\lambda'(x) = -p_\lambda^\pm \frac{x^{p_\lambda^\pm - 1}}{F_0^{p_\lambda^\pm}} \quad \text{and} \quad \phi_\lambda''(x) = -p_\lambda^\pm (p_\lambda^\pm - 1)\frac{x^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}} = 2\lambda \frac{x^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}},$$

hence with $y := F_0$ we have $\phi_\lambda(y) = 0$ and

$$\begin{aligned}
&\mathbb{E}^*[\phi_\lambda(S_T)] \\
&= \mathbb{E}^*\left[(S_T - F_0)\frac{p_\lambda^\pm}{F_0} - 2\lambda\int_0^{F_0}(K - S_T)^+\frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}}dK - 2\lambda\int_{F_0}^\infty (S_T - K)^+\frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}}dK\right] \\
&= 2\lambda\int_0^{F_0}\mathbb{E}^*[(K - S_T)^+]\frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}}dK + \int_{F_0}^\infty \mathbb{E}^*[(S_T - K)^+]\frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}}dK \\
&= 2\lambda\frac{\mathrm{e}^{rT}}{F_0^{p_\lambda^\pm}}\left(\int_0^{F_0} P(T,K)K^{p_\lambda^\pm - 2}dK + \int_{F_0}^\infty C(T,K)K^{p_\lambda^\pm - 2}dK\right).
\end{aligned}$$

Taking now

$$\psi(x) := \int_{1/8}^\infty \left(1 - \sqrt{\frac{x}{F_0}}\cos\left(\frac{1}{2}\sqrt{8\lambda - 1}\log\frac{x}{F_0}\right)\right)\frac{d\lambda}{\lambda^{\rho+1}},$$

we have

$$\begin{aligned}
\psi'(x) &= -\frac{1}{2\sqrt{F_0 x}}\int_{1/8}^\infty \cos\left(\frac{1}{2}\sqrt{8\lambda - 1}\log\frac{x}{F_0}\right)\frac{d\lambda}{\lambda^{\rho+1}} \\
&\quad + \frac{1}{2\sqrt{F_0 x}}\int_{1/8}^\infty \sin\left(\frac{1}{2}\sqrt{8\lambda - 1}\log\frac{x}{F_0}\right)\sqrt{8\lambda - 1}\frac{d\lambda}{\lambda^{\rho+1}},
\end{aligned}$$

which converges provided that $\rho > 1/2$, while $\psi''(x)$ cannot be written as a converging integral but can be estimated numerically from $\psi'(x)$. Hence, we have

$$\begin{aligned}
&\mathbb{E}^*[R_{t,T}] \\
&= \frac{\rho\,\mathrm{e}^{rT}}{\Gamma(1-\rho)} \\
&\times \left(\int_0^{1/8}\frac{2}{F_0^{\pm\sqrt{1/4 - 2\lambda}}}\left(\int_0^{F_0} P(T,K)K^{p_\lambda^\pm - 2}dK + \int_{F_0}^\infty C(T,K)K^{p_\lambda^\pm - 2}dK\right)\frac{d\lambda}{\lambda^\rho}\right. \\
&\left. + \int_0^{F_0} P(T,K)\psi''(K)dK + \int_{F_0}^\infty C(T,K)\psi''(K)dK\right).
\end{aligned}$$

```
library(quantmod)
today <- as.Date(Sys.Date(), format="%Y-%m-%d"); getSymbols("^SPX", src = "yahoo")
lastBusDay=last(row.names(as.data.frame(Ad(SPX))))
S0 = as.vector(tail(Ad(SPX),1)); T = 30/365;r=0.02;F0 = S0*exp(r*T)
maturity<- as.Date("2021-07-07", format="%Y-%m-%d") # Choose a maturity in 30 days
SPX.OPTS <- getOptionChain("^SPX", maturity)
Call <- as.data.frame(SPX.OPTS$calls); Put <- as.data.frame(SPX.OPTS$puts)
Call_OTM <- Call[Call$Strike>F0,];Call_OTM$dif = c(min(Call_OTM$Strike)-F0,
    diff(Call_OTM$Strike))
Put_OTM <-Put[Put$Strike<F0,];Put_OTM$dif = c(diff(Put_OTM$Strike),
    F0-max(Put_OTM$Strike))
```

```
pl <- function(lambda){return( 1/2+sqrt(1/4-2*lambda ))}; rho=0.9
g1 <- function(x){ f1 <- function(lambda){ - cos ( 0.5*sqrt(lambda*8-1)*log
      (x/F0))/lambda^(rho+1)/sqrt(x*F0)/2}; return(f1)}
g2 <- function(x){ f2 <- function(lambda){ sin ( 0.5*sqrt(lambda*8-1)*log
      (x/F0))/lambda^(rho+1)/sqrt(lambda*8-1)/sqrt(x*F0)/2}; return(f2)}
g3 <- function (x) { integrate(g1(x), lower=0.125, upper=Inf,stop.on.error = FALSE)$value}
g4 <- function (x) { if (x>F0) {integrate(g2(x), lower=0.125, upper=1000000,stop.on.error =
      FALSE)$value} else {integrate(g2(x), lower=0.125, upper=100000,stop.on.error =
      FALSE)$value}}
eps=1;psi2nd <- function(x){(g3(x+eps)+g4(x+eps)-g3(x)-g4(x))/eps}
f <- function(lambda){ return (2*(sum(Put_OTM$Last*Put_OTM$Strike**(pl(lambda)-2)
      *Put_OTM$dif)) +sum(Call_OTM$Last *Call_OTM$Strike**(pl(lambda)-2)
      *Call_OTM$dif)/F0**pl(lambda)/lambda**rho)}
(sum(Put_OTM$Last*as.numeric(lapply(Put_OTM$Strike,psi2nd)) *Put_OTM$dif)
      +sum(Call_OTM$Last*as.numeric(lapply(Call_OTM$Strike,psi2nd))
      *Call_OTM$dif)+integrate(Vectorize(f), lower=0,
      upper=0.125)$value)*rho*exp(r*T)/gamma(1-rho)
```

Chapter 7

Exercise 7.1

a) By differentiating (7.2.2) with respect to T, we find

$$\begin{aligned}
\varphi_{\tau_a}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\tau_a < T) \\
&= 2\frac{\partial}{\partial T}\mathbb{P}(W_T > a) \\
&= \frac{2}{\sqrt{2\pi T}}\frac{\partial}{\partial T}\int_a^\infty \mathrm{e}^{-x^2/(2T)}dx \\
&= \frac{2}{\sqrt{2\pi}}\frac{\partial}{\partial T}\int_{a/\sqrt{T}}^\infty \mathrm{e}^{-y^2/2}dy \\
&= \frac{a}{\sqrt{2\pi T^3}}\mathrm{e}^{-a^2/(2T)}, \qquad T > 0.
\end{aligned} \tag{A.4}$$

b) By differentiating (7.3.8) with respect to T, we find

$$\begin{aligned}
\varphi_{\tilde{\tau}_a}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\tilde{\tau}_a < T) \\
&= -\frac{\partial}{\partial T}\mathbb{P}(\tilde{\tau}_a \geqslant T) \\
&= -\frac{\partial}{\partial T}\mathbb{P}\big(\widehat{X}_0^T \leqslant a\big) \\
&= -\frac{\partial}{\partial T}\Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\frac{\partial}{\partial T}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) \\
&= \left(\frac{a}{2\sqrt{2\pi T^3}} + \frac{\mu}{\sqrt{2\pi T}}\right)\mathrm{e}^{-(a-\mu T)^2/(2T)} \\
&\quad + \left(\frac{a}{2\sqrt{2\pi T^3}} - \frac{\mu}{\sqrt{2\pi T}}\right)\mathrm{e}^{2\mu a-(a+\mu T)^2/(2T)} \\
&= \frac{a}{2\sqrt{2\pi T^3}}\mathrm{e}^{-(a-\mu T)^2/(2T)}, \qquad T > 0.
\end{aligned}$$

c) By differentiating (7.3.10) with respect to T, for $x > S_0$ we find

$$\begin{aligned}
\varphi_{\hat{\tau}_x}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\hat{\tau}_a < T) \\
&= -\frac{\partial}{\partial T}\mathbb{P}(\hat{\tau}_x \geqslant T)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\partial}{\partial T}\mathbb{P}\big(M_0^T \leqslant x\big) \\
&= -\frac{\partial}{\partial T}\Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad +\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\frac{\partial}{\partial T}\Phi\left(\frac{-(r-\sigma^2/2)T-\log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&= \frac{\log(x/S_0)}{\sigma\sqrt{2\pi T^3}}\exp\left(-\frac{1}{2\sigma^2 T}((r-\sigma^2/2)T-\log(x/S_0))^2\right), \quad T>0,
\end{aligned}
$$

which can also be recovered from (A.4) by taking $a := \log(S_0/x)/\sigma$ and $\mu := r/\sigma - \sigma/2$. Similarly, when $0 < x < S_0$ we can differentiate (7.3.13) in Corollary 7.8 to find

$$
\begin{aligned}
\varphi_{\hat{\tau}_x}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\hat{\tau}_x < T) \\
&= \frac{\partial}{\partial T}\mathbb{P}\big(m_0^T \leqslant x\big) \\
&= \frac{\partial}{\partial T}\Phi\left(\frac{-(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad +\left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\frac{\partial}{\partial T}\Phi\left(\frac{(r-\sigma^2/2)T+\log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&= \frac{\log(S_0/x)}{\sigma\sqrt{2\pi T^3}}\exp\left(-\frac{1}{2\sigma^2 T}((r-\sigma^2/2)T-\log(x/S_0))^2\right), \quad T>0,
\end{aligned}
$$

which yields

$$
\varphi_{\hat{\tau}_x}(T) = \frac{|\log(S_0/x)|}{\sigma\sqrt{2\pi T^3}}\exp\left(-\frac{1}{2\sigma^2 T}((r-\sigma^2/2)T-\log(x/S_0))^2\right), \quad T>0,
$$

for all $x > 0$.

Exercise 7.2

a) We use Relation (7.3.9) and the integration by parts identity

$$
\int_0^\infty v'(z)u(z)dz = u(+\infty)v(+\infty) - u(0)v(0) - \int_0^\infty v(z)u'(z)dz
$$

with

$$
u(y) = \Phi\left(\frac{-y-\mu T/\sigma}{\sqrt{T}}\right) \quad \text{and} \quad v'(y) = \frac{2\mu}{\sigma}y\,\mathrm{e}^{2\mu y/\sigma}
$$

which satisfy

$$
u'(y) = -\frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-(y+\mu T/\sigma)^2/(2T)} \quad \text{and} \quad v(y) = y\,\mathrm{e}^{2\mu y/\sigma} - \frac{\sigma}{2\mu}\mathrm{e}^{2\mu y/\sigma},
$$

we have

$$
\begin{aligned}
&\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]}\widetilde{W}_t\right] = \sigma\,\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]}(W_t+\mu t/\sigma)\right] \\
&= \sigma\sqrt{\frac{2}{\pi T}}\int_0^\infty y\,\mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy - 2\mu\int_0^\infty y\,\mathrm{e}^{2\mu y/\sigma}\Phi\left(\frac{-y-\mu T/\sigma}{\sqrt{T}}\right)dy \\
&= \sigma\sqrt{\frac{2}{\pi T}}\int_0^\infty y\,\mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy - \sigma\int_0^\infty v'(y)u(y)dy \\
&= \sigma\sqrt{\frac{2}{\pi T}}\int_0^\infty y\,\mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy - \sigma u(+\infty)v(+\infty) + \sigma u(0)v(0) + \sigma\int_0^\infty u'(y)v(y)dy
\end{aligned}
$$

$$\begin{aligned}
&= \sigma\sqrt{\frac{2}{\pi T}}\int_0^\infty y\,\mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right)\\
&\quad - \frac{\sigma}{\sqrt{2\pi T}}\int_0^\infty y\,\mathrm{e}^{2\mu y/\sigma-(y+\mu T/\sigma)^2/(2T)}dy + \frac{\sigma^2}{2\mu\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{2\mu y/\sigma-(y+\mu T/\sigma)^2/(2T)}dy\\
&= \sigma\sqrt{\frac{2}{\pi T}}\int_0^\infty y\,\mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy - \frac{\sigma}{\sqrt{2\pi T}}\int_0^\infty y\,\mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy\\
&\quad - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy\\
&= \frac{\sigma}{\sqrt{2\pi T}}\int_0^\infty y\,\mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right)\\
&\quad + \frac{\sigma^2}{2\mu\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)}dy\\
&= \frac{\sigma}{\sqrt{2\pi T}}\int_{-\mu T/\sigma}^\infty \left(y+\frac{\mu T}{\sigma}\right)\mathrm{e}^{-y^2/(2T)}dy - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right)\\
&\quad + \frac{\sigma^2}{2\mu\sqrt{2\pi T}}\int_{-\mu T/\sigma}^\infty \mathrm{e}^{-y^2/(2T)}dy\\
&= \frac{\sigma}{\sqrt{2\pi T}}\int_{-\mu T/\sigma}^\infty y\,\mathrm{e}^{-y^2/(2T)}dy + \frac{\mu T+\sigma^2/(2\mu)}{\sqrt{2\pi T}}\int_{-\mu T/\sigma}^\infty \mathrm{e}^{-y^2/(2T)}dy - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right)\\
&= \frac{\sigma}{\sqrt{2\pi T}}\left[-T\,\mathrm{e}^{-y^2/(2T)}\right]_{-\mu T/\sigma}^\infty + \mu T\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu}\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right)\\
&= \sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\mu^2T/2} + \left(\mu T+\frac{\sigma^2}{2\mu}\right)\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right).
\end{aligned}$$

As σ tends to zero, we find

$$\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]}\widetilde{W}_t\right] = \begin{cases} \mu T\Phi(+\infty) = \mu T & \text{if } \mu \geqslant 0,\\ \mu T\Phi(-\infty) = 0 & \text{if } \mu \leqslant 0.\end{cases}$$

We also have

$$\begin{aligned}
\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]}\widetilde{W}_t\right] &= \sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\mu^2T/2} + \mu T\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu}\left(\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) - \Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right)\right)\\
&= \sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\mu^2T/2} + \mu T\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu\sqrt{2\pi}}\int_{-\mu\sqrt{T}/\sigma}^{\mu\sqrt{T}/\sigma}\mathrm{e}^{-y^2/2}dy.
\end{aligned}$$

Hence, as μ tends to zero we find

$$\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]}\widetilde{W}_t\right] = \sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\mu^2T/2} + \mu T\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \sigma\sqrt{\frac{T}{2\pi}} + o(\mu), \qquad [\mu\to 0],$$

and for $\mu = 0$ and $\sigma = 1$ we recover the average maximum of standard Brownian motion

$$\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]}W_t\right] = \sqrt{\frac{2T}{\pi}},$$

which represents two times the expected maximum

$$\mathbb{E}[\mathrm{Max}(W_T,0)] = \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^\infty \mathrm{Max}(y,0)\,\mathrm{e}^{-y^2/(2T)}dy$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi T}} \int_0^\infty y \,\mathrm{e}^{-y^2/(2T)} dy \\
&= \frac{1}{\sqrt{2\pi T}} \left[-T \,\mathrm{e}^{-y^2/(2T)} \right]_0^\infty \\
&= \sqrt{\frac{T}{2\pi}}.
\end{aligned}$$

b) By part (a)) the identity in distribution $(-W_t)_{t\in\mathbb{R}_+} \approx (W_t)_{t\in\mathbb{R}_+}$, we have

$$\begin{aligned}
\mathbb{E}\left[\min_{t\in[0,T]} \widetilde{W}_t\right] &= \sigma\,\mathbb{E}\left[\min_{t\in[0,T]} (W_t + \mu t/\sigma)\right] \\
&= -\sigma\,\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} (-W_t - \mu t/\sigma)\right] \\
&= -\sigma\,\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} (W_t - \mu t/\sigma)\right] \\
&= -\sigma\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\mu^2 T/2} + \left(\mu T + \frac{\sigma^2}{2\mu}\right)\Phi\left(\frac{-\mu\sqrt{T}}{\sigma}\right) - \frac{\sigma^2}{2\mu}\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right).
\end{aligned}$$

In particular, as σ tends to zero, we find

$$\mathbb{E}\left[\min_{t\in[0,T]} \widetilde{W}_t\right] = \begin{cases} \mu T \Phi(+\infty) = \mu T & \text{if } \mu \leqslant 0, \\[2ex] \mu T \Phi(-\infty) = 0 & \text{if } \mu \geqslant 0. \end{cases}$$

Exercise 7.3

a) We have $S_t = S_0 \,\mathrm{e}^{\sigma W_t}$, $t \in \mathbb{R}_+$.

b) We have

$$\mathbb{E}[S_T] = S_0\,\mathbb{E}\left[\mathrm{e}^{\sigma W_T}\right] = S_0\,\mathrm{e}^{\sigma^2 T/2}.$$

c) We have

$$\mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} W_t \geqslant a\right) = 2\int_a^\infty \mathrm{e}^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \qquad a \geqslant 0,$$

and

$$P\left(\operatorname*{Max}_{t\in[0,T]} W_t \leqslant a\right) = 2\int_0^a \mathrm{e}^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \qquad a \geqslant 0,$$

hence the probability density function φ of $\operatorname*{Max}_{t\in[0,T]} W_t$ is given by

$$\varphi(a) = \sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-a^2/(2T)} \mathbb{1}_{[0,\infty)}(a), \qquad a \in \mathbb{R}.$$

d) We have

$$\begin{aligned}
\mathbb{E}\left[M_0^T\right] &= S_0\,\mathbb{E}\left[\exp\left(\sigma \operatorname*{Max}_{t\in[0,T]} W_t\right)\right] = S_0 \int_0^\infty \mathrm{e}^{\sigma x} \varphi(x) dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_0^\infty \mathrm{e}^{\sigma x - x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi\sigma^2 T}} \int_0^\infty \mathrm{e}^{-(x-\sigma T)^2/(2T) + \sigma^2 T/2} dx \\
&= \frac{2S_0}{\sqrt{2\pi T}}\,\mathrm{e}^{\sigma^2 T/2} \int_{-\sigma T}^\infty \mathrm{e}^{-x^2/(2T)} dx = \frac{2S_0}{\sqrt{2\pi}}\,\mathrm{e}^{\sigma^2 T/2} \int_{-\sigma\sqrt{T}}^\infty \mathrm{e}^{-x^2/2} dx \\
&= 2S_0\,\mathrm{e}^{\sigma^2 T/2} \int_{-\infty}^{\sigma\sqrt{T}} \mathrm{e}^{-x^2/2} dx \\
&= 2S_0\,\mathrm{e}^{\sigma^2 T/2} \Phi(\sigma\sqrt{T}) \\
&= 2\,\mathbb{E}[S_T]\Phi(\sigma\sqrt{T}).
\end{aligned}$$

Remarks:

(i) From the inequality

$$\begin{aligned}
0 &\leqslant \mathbb{E}[(W_T-\sigma T)^+] \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}(x-\sigma T)^+\,\mathrm{e}^{-x^2/(2T)}dx \\
&= -\frac{1}{\sqrt{2\pi T}}\int_{\sigma T}^{\infty}(x-\sigma T)\,\mathrm{e}^{-x^2/(2T)}dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_{\sigma T}^{\infty}x\,\mathrm{e}^{-x^2/(2T)}dx-\frac{\sigma T}{\sqrt{2\pi T}}\int_{\sigma T}^{\infty}\mathrm{e}^{-x^2/(2T)}dx \\
&= \sqrt{\frac{T}{2\pi}}\int_{\sigma\sqrt{T}}^{\infty}x\,\mathrm{e}^{-x^2/2}dx-\frac{\sigma T}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}}^{\infty}\mathrm{e}^{-x^2/2}dx \\
&= \sqrt{\frac{T}{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_{\sigma\sqrt{T}}^{\infty}-\sigma T\Phi(-\sigma\sqrt{T}) \\
&= \sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2T/2}-\sigma T(1-\Phi(\sigma\sqrt{T})),
\end{aligned}$$

we get

$$\Phi(\sigma\sqrt{T})\geqslant 1-\frac{\mathrm{e}^{-\sigma^2T/2}}{\sigma\sqrt{2\pi T}},$$

hence

$$\begin{aligned}
\mathbb{E}\left[M_0^T\right] &= 2S_0\,\mathrm{e}^{\sigma^2T/2}\Phi(\sigma\sqrt{T}) \\
&\geqslant 2S_0\,\mathrm{e}^{\sigma^2T/2}\left(1-\frac{\mathrm{e}^{-\sigma^2T/2}}{\sigma\sqrt{2\pi T}}\right) \\
&= 2\,\mathbb{E}[S_T]\left(1-\frac{\mathrm{e}^{-\sigma^2T/2}}{\sigma\sqrt{2\pi T}}\right) \\
&= 2S_0\left(\mathrm{e}^{\sigma^2T/2}-\frac{1}{\sigma\sqrt{2\pi T}}\right).
\end{aligned}$$

(ii) We observe that the ratio between the expected gains by selling at the maximum and selling at time T is given by $2\Phi(\sigma\sqrt{T})$, which cannot be greater than 2.

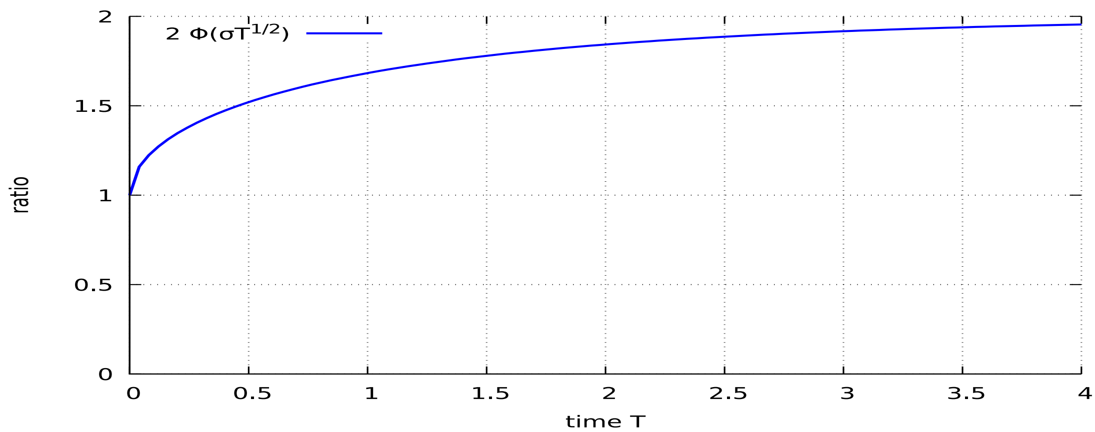


Figure S.8: Average return by selling at the maximum *vs.* selling at maturity.

e) By a symmetry argument, we have

$$\mathbb{P}\left(\min_{t\in[0,T]}W_t\leqslant a\right)=\mathbb{P}\left(-\operatorname*{Max}_{t\in[0,T]}(-W_t)\leqslant a\right)$$

$$
\begin{aligned}
&= \mathbb{P}\left(-\operatorname*{Max}_{t\in[0,T]} W_t \leqslant a\right)\\
&= \mathbb{P}\left(\operatorname*{Max}_{t\in[0,T]} W_t \geqslant -a\right)\\
&= 2\int_{-a}^{\infty} e^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}}, \qquad a<0,
\end{aligned}
$$

i.e. the probability density function φ of $\min_{t\in[0,T]} W_t$ is given by

$$
\varphi(a)=\sqrt{\frac{2}{\pi T}}\, e^{-a^2/(2T)}\mathbb{1}_{(-\infty,0]}(a), \qquad a\in\mathbb{R}.
$$

f) We have

$$
\begin{aligned}
\mathbb{E}\left[m_0^T\right] &= S_0\,\mathbb{E}\left[\exp\left(\sigma \min_{t\in[0,T]} W_t\right)\right]\\
&= S_0\int_{-\infty}^{0} e^{\sigma x}\varphi(x)dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_{-\infty}^{0} e^{\sigma x - x^2/(2T)}dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_{-\infty}^{0} e^{-(x-\sigma T)^2/(2T)+\sigma^2T/2}dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}e^{\sigma^2T/2}\int_{-\infty}^{-\sigma T} e^{-x^2/(2T)}dx\\
&= \frac{2S_0}{\sqrt{2\pi}}e^{\sigma^2T/2}\int_{-\infty}^{-\sigma\sqrt{T}} e^{-x^2/2}dx\\
&= 2S_0\, e^{\sigma^2T/2}\Phi\big(-\sigma\sqrt{T}\big)\\
&= 2\,\mathbb{E}[S_T]\Phi\big(-\sigma\sqrt{T}\big).
\end{aligned}
$$

Remarks:

(i) From the inequality

$$
\begin{aligned}
0 &\leqslant \mathbb{E}[(-\sigma T - W_T)^+]\\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}(-\sigma T - x)^+ e^{-x^2/(2T)}dx\\
&= -\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{-\sigma T}(\sigma T + x)e^{-x^2/(2T)}dx\\
&= -\frac{\sigma T}{\sqrt{2\pi T}}\int_{-\infty}^{-\sigma T} e^{-x^2/(2T)}dx - \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{-\sigma T} x\, e^{-x^2/(2T)}dx\\
&= -\frac{\sigma T}{\sqrt{2\pi}}\int_{-\infty}^{-\sigma\sqrt{T}} e^{-x^2/2}dx - \sqrt{\frac{T}{2\pi}}\int_{-\infty}^{-\sigma\sqrt{T}} x\, e^{-x^2/2}dx\\
&= -\sigma T\Phi\big(-\sigma\sqrt{T}\big) + \sqrt{\frac{T}{2\pi}}\left[e^{-x^2/2}\right]_{-\infty}^{-\sigma\sqrt{T}}\\
&= -\sigma T\Phi\big(-\sigma\sqrt{T}\big) + \sqrt{\frac{T}{2\pi}}\, e^{-\sigma^2T/2},
\end{aligned}
$$

we get

$$
e^{\sigma^2T/2}\Phi\big(-\sigma\sqrt{T}\big) \leqslant \frac{1}{\sigma\sqrt{2\pi T}}, \quad \text{hence} \quad \mathbb{E}\left[m_0^T\right] \leqslant \frac{2S_0}{\sigma\sqrt{2\pi T}}.
$$

(ii) The ratio between the expected gains by maturity T *vs.* selling at the minimum is given by $2\Phi\big(-\sigma\sqrt{T}\big)$, which is at most 1 and tends to 0 as σ and T tend to infinity.

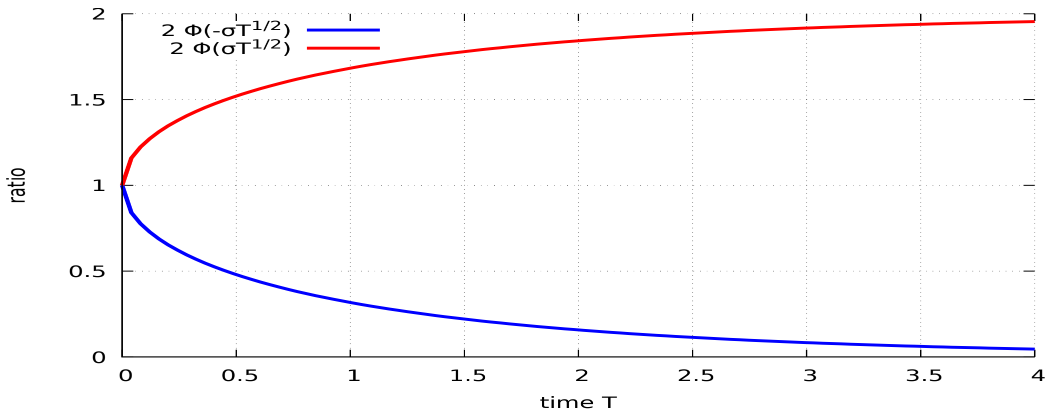


Figure S.9: Average returns by selling at the minimum *vs.* selling at maturity.

(iii) Given that $\mathbb{E}\left[M_0^T\right] = 2\,\mathbb{E}[S_T]\Phi\left(\sigma\sqrt{T}\right)$, we find the bound

$$2\,\mathbb{E}[S_T]\Phi\left(-\sigma\sqrt{T}\right) \leqslant \mathbb{E}[S_T] \leqslant 2\,\mathbb{E}[S_T]\Phi\left(\sigma\sqrt{T}\right),$$

with equality if $\sigma = 0$ or $T = 0$. We also have

$$\begin{aligned} 2\,\mathbb{E}[S_T] - \mathbb{E}\left[M_0^T\right] &= 2\,\mathrm{e}^{\sigma^2 T/2}\left(1 - \Phi\left(\sigma\sqrt{T}\right)\right) \\ &= 2\,\mathrm{e}^{\sigma^2 T/2}\Phi\left(-\sigma\sqrt{T}\right) \\ &= \mathbb{E}\left[m_0^T\right], \end{aligned}$$

hence we have

$$\mathbb{E}\left[m_0^T\right] + \mathbb{E}\left[M_0^T\right] = 2\,\mathbb{E}[S_T], \quad \text{or} \quad \mathbb{E}[S_T] - \mathbb{E}\left[m_0^T\right] = \mathbb{E}\left[M_0^T\right] - \mathbb{E}[S_T],$$

and

$$2\,\mathbb{E}[S_T] - \frac{2S_0}{\sigma\sqrt{2\pi T}} \leqslant \mathbb{E}\left[M_0^T\right] \leqslant 2\,\mathbb{E}[S_T].$$

Exercise 7.4

a) Letting $M_0^t := \operatorname*{Max}_{s\in[0,t]} B_s$ and $M_t^T := \operatorname*{Max}_{s\in[t,T]} B_s$, we have

$$\begin{aligned} \mathbb{P}(\tau \leqslant t) &= \mathbb{P}(M_0^t \geqslant M_t^T) \\ &= \mathbb{P}\left(\operatorname*{Max}_{s\in[0,t]} B_s \geqslant \operatorname*{Max}_{s\in[t,T]} B_s\right) \\ &= \mathbb{P}\left(\operatorname*{Max}_{s\in[0,t]} (B_s - B_t) \geqslant \operatorname*{Max}_{s\in[t,T]} (B_s - B_t)\right) \\ &= \mathbb{P}\left(\operatorname*{Max}_{s\in[0,t]} B_s \geqslant \operatorname*{Max}_{s\in[t,T]} (B_s - B_t)\right) \\ &= \mathbb{P}(|Z_1| \geqslant |Z_2|). \end{aligned}$$

b) Given $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$ two independent standard normal random variables, we have

$$\begin{aligned} \mathbb{P}(\tau \leqslant t) &= \mathbb{P}(|Z_1| \geqslant |Z_2|) \\ &= \mathbb{P}(\sqrt{t}|X| \geqslant \sqrt{T-t}|Y|) \\ &= \mathbb{P}(tX^2 \geqslant (T-t)Y^2) \\ &= \mathbb{P}(t(X^2+Y^2) \geqslant TY^2) \\ &= \mathbb{P}\left(\frac{Y^2}{X^2+Y^2} \leqslant \frac{t}{T}\right) \end{aligned}$$

$$= \int_0^{t/T} \frac{1}{\pi\sqrt{(1-z)z}} dz$$
$$= \frac{2}{\pi} \arcsin\sqrt{\frac{t}{T}},$$

see **here**. For example, we find

$$\mathbb{P}\left(\tau \leqslant \frac{T}{2}\right) = \frac{2}{\pi} \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}.$$

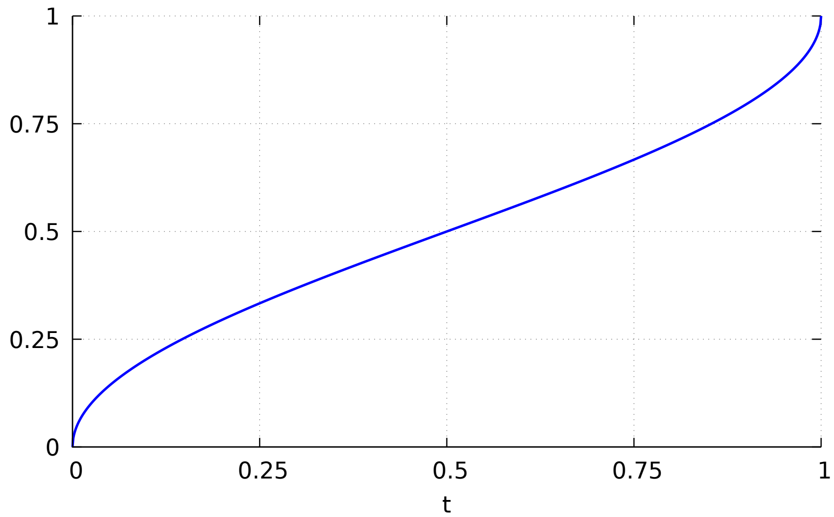


Figure S.10: Cumulative distribution function $\mathbb{P}(\tau \leqslant t)$ of the time of maximum of Brownian motion with $T = 1$.

Exercise 7.5 (Exercise 7.3 continued).

a) Regarding call option prices we have, assuming $K \geqslant S_0$,

$$\mathbb{E}\left[\left(M_0^T - K\right)^+\right] = S_0\,\mathbb{E}\left[\left(\exp\left(\sigma \operatorname*{Max}_{t\in[0,T]} W_t\right) - K\right)^+\right]$$
$$= \int_0^\infty (S_0\,\mathrm{e}^{\sigma x} - K)^+ \varphi(x) dx$$
$$= \frac{2}{\sqrt{2\pi T}} \int_0^\infty \left(S_0\,\mathrm{e}^{\sigma x} - K\right)^+ \mathrm{e}^{-x^2/(2T)} dx$$
$$= \frac{2}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log(K/S_0)}^\infty \left(S_0\,\mathrm{e}^{\sigma x} - K\right) \mathrm{e}^{-x^2/(2T)} dx$$
$$= \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{\sigma x - x^2/(2T)} dx$$
$$- \frac{2K}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-x^2/(2T)} dx$$
$$= \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2} dx$$
$$- \frac{2K}{\sqrt{2\pi T}} \int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-x^2/(2T)} dx$$
$$= \frac{2S_0}{\sqrt{2\pi T}} \mathrm{e}^{\sigma^2 T/2} \int_{-\sigma T+\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-x^2/(2T)} dx$$

$$
\begin{aligned}
& -\frac{2K}{\sqrt{2\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^{\infty} \mathrm{e}^{-x^2/(2T)}dx \\
= \quad & 2S_0\,\mathrm{e}^{\sigma^2T/2}\Phi\big(\sigma\sqrt{T}+\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) \\
& -2K\Phi\big(\sigma^{-1}\log(S_0/K)/\sqrt{T}\big).
\end{aligned}
$$

When $K \leqslant S_0$, by "completion of the square" and use of the Gaussian cumulative distribution function $\Phi(\cdot)$, we find

$$
\begin{aligned}
\mathbb{E}\left[\left(\operatorname*{Max}_{t\in[0,T]} S_t - K\right)^+\right] &= \mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} S_t - K\right] \\
&= \mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} S_t\right] - \mathbb{E}[K] \\
&= \mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} S_t\right] - K \\
&= S_0\,\mathbb{E}\left[\exp\left(\sigma \operatorname*{Max}_{t\in[0,T]} W_t\right)\right] - K \\
&= S_0\int_0^\infty \mathrm{e}^{\sigma x}\varphi(x)dx - K \\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{\sigma x - x^2/(2T)}dx \\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2T/2}dx - K \\
&= \frac{2S_0}{\sqrt{2\pi T}}\mathrm{e}^{\sigma^2T/2}\int_{-\sigma T}^\infty \mathrm{e}^{-x^2/(2T)}dx - K \\
&= 2S_0\,\mathrm{e}^{\sigma^2T/2}\Phi\big(\sigma\sqrt{T}\big) - K \\
&= 2S_0\,\mathrm{e}^{\sigma^2T/2}\Big(1-\Phi\big(-\sigma\sqrt{T}\big)\Big) - K \\
&= 2\,\mathbb{E}[S_T]\Phi\big(\sigma\sqrt{T}\big) - K,
\end{aligned}
$$

hence

$$
\mathrm{e}^{-\sigma^2T/2}\,\mathbb{E}\left[\big(M_0^T-K\big)^+\right] = 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\,\mathrm{e}^{-\sigma^2T/2}.
$$

Recall that when $r=\sigma^2/2$ the price of the finite expiration American call option price is the Black-Scholes price with maturity T, with

$$
\begin{aligned}
& \mathrm{Bl}_{\mathrm{Call}}(S_0,K,\sigma,r,T) \\
= \quad & S_0\Phi\big(\sigma\sqrt{T}+\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) - K\,\mathrm{e}^{-\sigma^2T/2}\Phi\big(\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) \\
\leqslant \quad & \begin{cases} 2S_0\Phi\big(\sigma\sqrt{T}+\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) - 2K\,\mathrm{e}^{-\sigma^2T/2}\Phi\big(\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) \\ \qquad \text{if } K \geqslant S_0, \\ \\ 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\,\mathrm{e}^{-\sigma^2T/2} \\ \qquad \text{if } K \leqslant S_0. \end{cases} \\
= \quad & \begin{cases} 2\times \mathrm{Bl}_{\mathrm{Call}}(S_0,K,\sigma,r,T) & \text{if } K \geqslant S_0, \\ \\ 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\,\mathrm{e}^{-\sigma^2T/2} & \text{if } K \leqslant S_0, \end{cases} \\
= \quad & \operatorname{Max}\Big(2\times \mathrm{Bl}_{\mathrm{Call}}(S_0,K,\sigma,r,T), 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\,\mathrm{e}^{-\sigma^2T/2}\Big).
\end{aligned}
$$

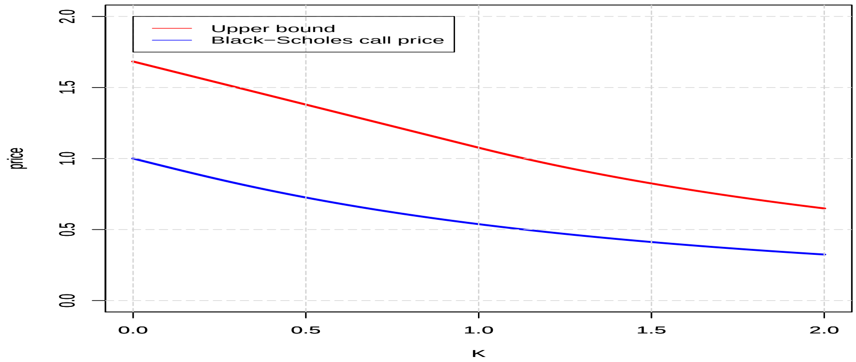


Figure S.11: Black-Scholes call price upper bound with $S_0 = 1$.

b) Regarding put option prices we have, assuming $S_0 \geqslant K$,

$$
\begin{aligned}
\mathbb{E}\left[\left(K-m_0^T\right)^+\right] &= S_0\,\mathbb{E}\left[\left(K-\exp\left(\sigma \min_{t\in[0,T]} W_t\right)\right)^+\right] \\
&= \int_0^\infty (K-S_0\,\mathrm{e}^{\sigma x})^+\varphi(x)dx \\
&= \frac{2}{\sqrt{2\pi T}}\int_{-\infty}^0 \left(K-S_0\,\mathrm{e}^{\sigma x}\right)^+ \mathrm{e}^{-x^2/(2T)}dx \\
&= \frac{2}{\sqrt{2\pi T}}\int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \left(K-S_0\,\mathrm{e}^{\sigma x}\right) \mathrm{e}^{-x^2/(2T)}dx \\
&= \frac{2K}{\sqrt{2\pi T}}\int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)}dx \\
&\quad -\frac{2S_0}{\sqrt{2\pi T}}\int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{\sigma x-x^2/(2T)}dx \\
&= \frac{2K}{\sqrt{2\pi T}}\int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)}dx \\
&\quad -\frac{2S_0}{\sqrt{2\pi T}}\int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2}dx \\
&= \frac{2K}{\sqrt{2\pi T}}\int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)}dx \\
&\quad -\frac{2S_0}{\sqrt{2\pi T}}\,\mathrm{e}^{\sigma^2 T/2}\int_{-\infty}^{-\sigma T+\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)}dx \\
&= 2K\Phi(-\sigma^{-1}\log\left(S_0/K\right)/\sqrt{T}) \\
&\quad -2S_0\,\mathrm{e}^{\sigma^2 T/2}\Phi\left(-\sigma\sqrt{T}-\sigma^{-1}\log(S_0/K)/\sqrt{T}\right),
\end{aligned}
$$

with

$$
\mathrm{e}^{-\sigma^2 T/2}\,\mathbb{E}\left[\left(K-m_0^T\right)^+\right] = K\,\mathrm{e}^{-\sigma^2 T/2} - 2S_0\Phi\left(-\sigma\sqrt{T}\right)
$$

if $S_0 \leqslant K$. Therefore we deduce the bounds

$$
\begin{aligned}
&\mathrm{Bl}_{\mathrm{Put}}(S_0,K,\sigma,r,T) \\
&= K\,\mathrm{e}^{-\sigma^2 T/2}\Phi\left(-\sigma^{-1}\log(S_0/K)/\sqrt{T}\right) - S_0\Phi\left(-\sigma\sqrt{T}-\sigma^{-1}\log(S_0/K)/\sqrt{T}\right) \\
&\leqslant \text{American put option price}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \begin{cases} 2K\,\mathrm{e}^{-\sigma^2T/2}\Phi\big(-\sigma^{-1}\log(S_0/K)/\sqrt{T}\big)-2S_0\Phi\big(-\sigma\sqrt{T}-\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) \\ \quad \text{if } S_0 \geqslant K, \\ \\ K\,\mathrm{e}^{-\sigma^2T/2}-2S_0\Phi\big(-\sigma\sqrt{T}\big) \qquad \text{if } S_0 \leqslant K, \end{cases} \\
&= \begin{cases} 2\times \mathrm{Bl}_{\mathrm{Put}}(S_0,K,\sigma,r,T) & \text{if } S_0 \geqslant K, \\ \\ K\,\mathrm{e}^{-\sigma^2T/2}-2S_0\Phi\big(-\sigma\sqrt{T}\big) & \text{if } S_0 \leqslant K, \end{cases} \\
&= \ \mathrm{Max}\Big(2\times \mathrm{Bl}_{\mathrm{Put}}(S_0,K,\sigma,r,T), K\,\mathrm{e}^{-\sigma^2T/2}-2S_0\Phi\big(-\sigma\sqrt{T}\big)\Big)
\end{aligned}
$$

for the finite expiration American put option price when $r=\sigma^2/2$.

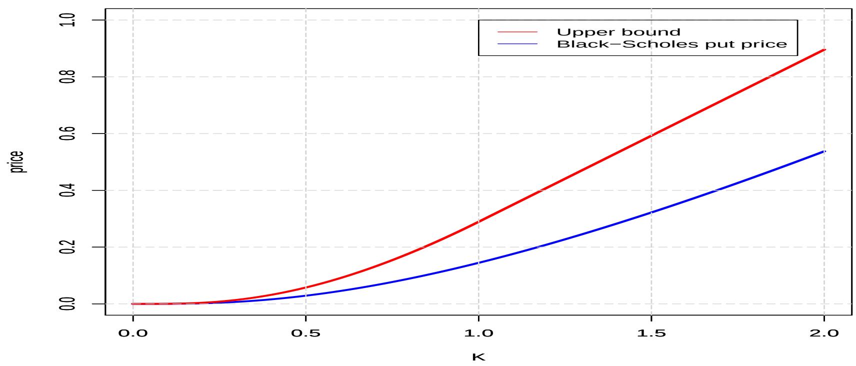


Figure S.12: Black-Scholes put price upper bound with $S_0=1$.

Exercise 7.6 (Exercise 7.5 continued).

a) Using the expression

$$\varphi_{\check{X}_0^T}(x)=\sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(x-\mu T)^2/(2T)}+2\mu\,\mathrm{e}^{2\mu x}\Phi\left(\frac{x+\mu T}{\sqrt{T}}\right), \quad x\leqslant 0.$$

of the probability density function of the minimum

$$\check{X}_0^T := \min_{t\in[0,T]} \widetilde{W}_t = \min_{t\in[0,T]}(W_t+\mu t)$$

of drifted Brownian motion $\widetilde{W}_t = W_t+\mu t$ over $t\in[0,T]$ given in Proposition 7.7, we find

$$
\begin{aligned}
\mathbb{E}\Big[\min_{t\in[0,T]} S_t\Big] &= S_0\int_{-\infty}^0 \mathrm{e}^{\sigma x}\varphi_{\check{X}_0^T}(x)dx \\
&= S_0\int_{-\infty}^0 \mathrm{e}^{\sigma x}\sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(x-\mu T)^2/(2T)}dx \\
&\quad +2\mu S_0\int_{-\infty}^0 \mathrm{e}^{\sigma x}\,\mathrm{e}^{2\mu x}\Phi\left(\frac{x+\mu T}{\sqrt{T}}\right)dx \\
&= 2S_0\,\mathrm{e}^{\sigma^2T/2-\mu\sigma T}\Phi\big((\mu-\sigma)\sqrt{T}\big)+\frac{2\mu S_0}{2\mu-\sigma}\Phi\big(-\mu\sqrt{T}\big) \\
&\quad -\frac{2\mu S_0}{2\mu-\sigma}\,\mathrm{e}^{\sigma^2T/2-\mu\sigma T}\Phi\big((\mu-\sigma)\sqrt{T}\big),
\end{aligned}
$$

with $\mu:=r/\sigma-\sigma/2$, which yields

$$\mathbb{E}\Big[\min_{t\in[0,T]} S_t\Big] = S_0\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right)$$

$$+S_0\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{r+\sigma^2/2}{\sigma}\sqrt{T}\right).$$

See Exercise 9.1-(b)) for the computation of $\mathbb{E}\left[\min_{t\in[0,1]} S_t\right]$ when $r=0$.

b) When $S_0 \leqslant K$, we have

$$\begin{aligned}\mathbb{E}\left[\left(K-\min_{t\in[0,T]} S_t\right)^+\right] &= \mathbb{E}\left[K-\min_{t\in[0,T]} S_t\right]\\ &= K-\mathbb{E}\left[\min_{t\in[0,T]} S_t\right]\\ &= K-S_0\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right)\\ &\quad -S_0\,\mathrm{e}^{rT}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{r+\sigma^2/2}{\sigma}\sqrt{T}\right).\end{aligned}$$

Next, when $S_0 \geqslant K$ we have, using the probability density function $\varphi_{\check{X}_0^T}(x)$,

$$\begin{aligned}&\mathbb{E}\left[\left(K-\min_{t\in[0,T]} S_t\right)^+\right] = \mathbb{E}\left[\left(K-S_0\min_{t\in[0,T]}\mathrm{e}^{\sigma\check{X}_0^T}\right)^+\right]\\ &= \int_{-\infty}^0 \left(K-S_0\,\mathrm{e}^{\sigma x}\right)^+\varphi_{\check{X}_0^T}(x)dx\\ &= S_0\sqrt{\frac{2}{\pi T}}\int_{-\infty}^0 \left(K-S_0\,\mathrm{e}^{\sigma x}\right)^+\mathrm{e}^{-(x-\mu T)^2/(2T)}dx\\ &\quad +2\mu S_0\int_{-\infty}^0 \left(K-S_0\,\mathrm{e}^{\sigma x}\right)^+\mathrm{e}^{2\mu x}\Phi\left(\frac{x+\mu T}{\sqrt{T}}\right)dx\\ &= K\Phi\left(-\frac{(r-\sigma^2/2)T+\log(S_0/K)}{\sigma\sqrt{T}}\right)\\ &\quad +K\left(\frac{S_0}{K}\right)^{1-2r/\sigma^2}\Phi\left(\frac{(r-\sigma^2/2)T+\log(K/S_0)}{\sigma\sqrt{T}}\right)\\ &\quad -S_0\left(1-\frac{\sigma^2}{2r}\right)\left(\frac{S_0}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r-\sigma^2/2)T+\log(K/S_0)}{\sigma\sqrt{T}}\right)\\ &\quad -S_0\,\mathrm{e}^{rT}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{(r+\sigma^2/2)T+\log(S_0/K)}{\sigma\sqrt{T}}\right).\end{aligned}$$

In Figure S.13, using a finite expiration American put option pricer from the R fOptions package, we plot the graph of American put option price *vs.* (A.5)-(A.6), together with the European put option price, according to the following R code.

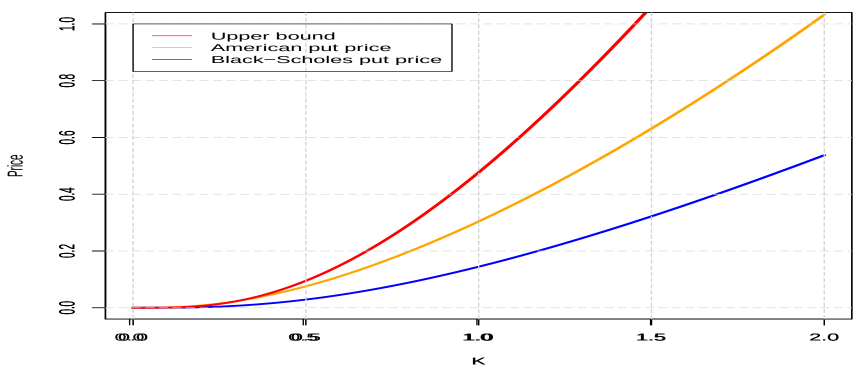


Figure S.13: "Optimal exercise" put price upper bound with $S_0 = 1$.

```
d1 <- function(S,K,r,T,sig) {return((log(S/K)+(r+sig^2/2)*T)/(sig*sqrt(T)))}
d2 <- function(S,K,r,T,sig) {return(d1(S, K, r, T, sig) - sig * sqrt(T))}
BSPut <- function(S, K, r, T, sig){return(K*exp(-r*T) * pnorm(-d2(S, K, r, T, sig)) - S*pnorm(-d1(S, K,
    r, T, sig)))}
Optimal_Put_Option <- function(S,K,r,T,sig){
if (r==0) {if (S>=K) {return(K*pnorm(d1(K,S,0,T,sig))-S*(1+sig*sig*T/2
    +log(S/K))*pnorm(-d1(S,K,0,T,sig))
    +S*sig*sqrt(T/(2*pi))*exp(-d1(S,K,0,T,sig)*d1(S,K,0,T,sig)/(2*sig*sig*T)))}
else {return(K-2*S*(1+sig*sig*T/4)*pnorm(-sig*sqrt(T)/2) +S*sig*sqrt(T/(2*pi))*exp(-sig*sig*T/8))}}
else {if (S>=K) {return(K*pnorm(-d2(S,K,r,T,sig)) +K*(S/K)**(1-2*r/sig/sig)*pnorm(d2(K,S,r,T,sig))
    -2*S*exp(r*T)*pnorm(-d1(S,K,r,T,sig))
    -S*(1-sig*sig/2/r)*(S/K)**(-2*r/sig/sig)*pnorm(d1(K,S,r,T,sig))
    +S*exp(r*T)*(1-sig*sig/2/r)*pnorm(-d1(S,K,r,T,sig)))}
else {return(K-S*(1-sig*sig/2/r)*pnorm((r-sig*sig/2)*sqrt(T)/sig)
    -S*(1+sig*sig/2/r)*pnorm(-(r+sig*sig/2)*sqrt(T)/sig))}}}
r=0.5;sig=1;S=1;T=1
library(fOptions)
curve(BAWAmericanApproxOption("p",S,x,T,r,b=0,sig,title = NULL, description = NULL)@price,
    from=0.01, to=2 , xlab="K", lwd = 3, ylim=c(0,1),ylab="",col="orange")
par(new=TRUE)
curve(BSPut(S,x,r,T,sig), from=0, to=2 , xlab="K", lwd = 3, ylim=c(0,1), ylab="Price",col="blue")
par(new=TRUE)
curve(Optimal_Put_Option(S,x,r,T,sig), from=0, to=2 , xlab="K", lwd = 3,
    ylim=c(0,1),ylab="",col="red")
grid (lty = 5)
legend(.0,1.0,legend=c("Upper bound","American put price","Black-Scholes put
    price"),col=c("red","orange", "blue"), lty=1:1, cex=1.)
```

c) When $r = 0$ and $S_0 \leqslant K$, we find

$$\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right] = K - 2S_0\left(1+\frac{\sigma^2 T}{4}\right)\Phi\left(-\frac{\sigma\sqrt{T}}{2}\right) + \sigma S_0\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2 T/8}. \quad (A.5)$$

Next, when $r = 0$ and $S_0 \geqslant K$, we find

$$\begin{aligned}\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right] &= K\Phi\left(\frac{\sigma^2 T/2 + \log(K/S_0)}{\sigma\sqrt{T}}\right) \quad (A.6)\\ &-S_0\left(1+\log\frac{S_0}{K}+\frac{\sigma^2 T}{2}\right)\Phi\left(-\frac{\sigma^2 T/2+\log(S_0/K)}{\sigma\sqrt{T}}\right)\\ &+S_0\sigma\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-(\sigma^2 T/2+\log(S_0/K))^2/(2\sigma^2 T)}.\end{aligned}$$

From the above R code we can check that when $r \simeq 0$ the price of the finite expiration American put option coincides with the price of the standard European put option.

Exercise 7.7

a) We have

$$\begin{aligned} P(\tau_a \geqslant t) &= P(X_t > a)\\ &= \int_a^\infty \varphi_{X_t}(x)dx\\ &= \sqrt{\frac{2}{\pi t}}\int_y^\infty \mathrm{e}^{-x^2/(2t)}dx, \qquad y > 0. \end{aligned}$$

b) We have

$$\begin{aligned} \varphi_{\tau_a}(t) &= \frac{d}{dt}P(\tau_a \leqslant t)\\ &= \frac{d}{dt}\int_a^\infty \varphi_{X_t}(x)dx\\ &= -\frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\int_a^\infty \mathrm{e}^{-x^2/(2t)}dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\int_a^\infty \frac{x^2}{t}\mathrm{e}^{-x^2/(2t)}dx \end{aligned}$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} t^{-3/2}\left(-\int_a^\infty e^{-x^2/(2t)}dx + a\,e^{-a^2/(2t)} + \int_a^\infty e^{-x^2/(2t)}dx\right)$$
$$= \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/(2t)}, \qquad t > 0.$$

c) We have

$$\begin{aligned}
\mathbb{E}\left[(\tau_a)^{-2}\right] &= \int_0^\infty t^{-2}\varphi_{\tau_a}(t)dt \\
&= \frac{a}{\sqrt{2\pi}}\int_0^\infty t^{-7/2} e^{-a^2/(2t)}dt \\
&= \frac{2a}{\sqrt{2\pi}}\int_0^\infty x^4 e^{-a^2x^2/2}dx \\
&= \frac{3}{a^4},
\end{aligned}$$

by the change of variable $x = t^{-1/2}$, *i.e.* $x^2 = 1/t$, $t = x^{-2}$, and $dt = -2x^{-3}dx$.

Remark: We have

$$\mathbb{E}[\tau_a] = \frac{a}{\sqrt{2\pi}}\int_0^\infty t^{-1/2} e^{-a^2/(2t)}dt = +\infty.$$

Exercise 7.8 Starting from the probability density function

$$\varphi_{\widehat{X}_0^T,\widetilde{W}_T}(a,b) = \mathbb{1}_{\{a\geqslant \mathrm{Max}(b,0)\}}\frac{1}{T}\sqrt{\frac{2}{\pi T}}(2a-b)\,e^{\mu b-(2a-b)^2/(2T)-\mu^2T/2}$$

of the drifted Brownian motion $\widetilde{W}_T := W_T + \mu T$ and its maximum $\widehat{X}_0^T = \underset{t\in[0,T]}{\mathrm{Max}}\,\widetilde{W}_t$, we take $\mu := r/\sigma - \sigma/2$ and let the functions f and g be defined as

$$f(x,y) := \frac{1}{\sigma}\log\frac{x}{S_0} \quad \text{and} \quad g(x,y) := \frac{1}{\sigma}\log\frac{y}{S_0},$$

with the Jacobian

$$|J(x,y)| = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y}\end{vmatrix} = \begin{vmatrix} \dfrac{1}{\sigma x} & 0 \\ 0 & \dfrac{1}{\sigma y}\end{vmatrix} = \frac{1}{\sigma^2 xy},$$

$x,y > 0$, which yields the joint density

$$\begin{aligned}
\varphi_{M_0^T,S_T}(x,y) &= |J(x,y)|\varphi_{\widehat{X}_0^T,\widetilde{W}_T}(f(x,y),g(x,y)) \\
&= \frac{1}{\sigma^3 Txy}\mathbb{1}_{\{x\geqslant \mathrm{Max}(y,S_0)\}}\sqrt{\frac{2}{\pi T}}\left(\log\frac{x^2}{S_0 y}\right)\exp\left(\frac{\mu}{\sigma}\log\frac{y}{S_0} - \frac{1}{2\sigma^2 T}\left(\log\frac{x^2}{S_0 y}\right)^2 - \mu^2\frac{T}{2}\right)
\end{aligned}$$

of S_T and its maximum $M_0^T = \mathrm{Max}_{t\in[0,T]} S_t$ over $t\in[0,T]$, $x,y>0$.

Chapter 8

Exercise 8.1 Barrier options.

a) By (8.4.8) and (9.3.2), for the up-and-out barrier call option we find

$$\xi_t = \frac{\partial g}{\partial y}(t,S_t) = \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)$$

$$+\frac{K}{B}\mathrm{e}^{-(T-t)r}\left(1-\frac{2r}{\sigma^2}\right)\left(\frac{S_t}{B}\right)^{-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)$$
$$+\frac{2r}{\sigma^2}\left(\frac{S_t}{B}\right)^{-1-2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)$$
$$-\frac{2}{\sigma\sqrt{2\pi(T-t)}}\left(1-\frac{K}{B}\right)\exp\left(-\frac{1}{2}\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)^2\right),$$

$0 < S_t \leqslant B$, $0 \leqslant t \leqslant T$, see also Exercise 7.1-$(ix)$ of Shreve, 2004 and Figure 8.11 above. At maturity for $t = T$, we find $\xi_T = \mathbb{1}_{[K,B]}(S_T)$.

Regarding the up-and-out barrier put option, we find

$$\xi_t = \frac{\partial g}{\partial y}(t,S_t) = -\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right)-\frac{2r}{\sigma^2}\left(\frac{B}{S_t}\right)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)$$
$$-\frac{K}{B}\mathrm{e}^{-(T-t)r}\left(1-\frac{2r}{\sigma^2}\right)\left(\frac{B}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right),$$

$0 \leqslant S_t \leqslant B$, $0 \leqslant t \leqslant T$. At maturity for $t = T$, we find $\xi_T = -\mathbb{1}_{[0,K]}(S_T)$.

b) We have

$$\mathbb{P}(Y_T \leqslant a \text{ and } W_T \geqslant b) = \mathbb{P}(W_T \leqslant 2a-b), \qquad a < b < 0,$$

hence

$$f_{Y_T,W_T}(a,b) = \frac{d\mathbb{P}(Y_T \leqslant a \text{ and } W_T \leqslant b)}{dadb} = -\frac{d\mathbb{P}(Y_T \leqslant a \text{ and } W_T \geqslant b)}{dadb},$$

$a,b \in \mathbb{R}$, satisfies

$$\begin{aligned} f_{Y_T,W_T}(a,b) &= \sqrt{\frac{2}{\pi T}}\mathbb{1}_{(-\infty,\min(0,b)]}(a)\frac{(b-2a)}{T}\mathrm{e}^{-(2a-b)^2/(2T)} \\ &= \begin{cases} \sqrt{\dfrac{2}{\pi T}}\dfrac{(b-2a)}{T}\mathrm{e}^{-(2a-b)^2/(2T)}, & a < \min(0,b), \\ 0, & a > \min(0,b). \end{cases} \end{aligned}$$

c) We find

$$\begin{aligned} f_{\breve{Y}_T,\widehat{W}_T}(a,b) &= \mathbb{1}_{(-\infty,\min(0,b)]}(a)\frac{1}{T}\sqrt{\frac{2}{\pi T}}(b-2a)\,\mathrm{e}^{-\mu^2T/2+\mu b-(2a-b)^2/(2T)} \\ &= \begin{cases} \dfrac{1}{T}\sqrt{\dfrac{2}{\pi T}}(b-2a)\,\mathrm{e}^{-\mu^2T/2+\mu b-(2a-b)^2/(2T)}, & a < \min(0,b), \\ 0, & a > \min(0,b). \end{cases} \end{aligned}$$

d) The function $g(t,x)$ is given by the Relations (8.2.6) and (8.2.7) above.

Exercise 8.2

a) By Corollary 7.8, the probability density function of the minimum

$$m_0^{\Delta\tau} = \min_{s\in[0,\Delta\tau]} S_{\tau+s}$$

with $S_\tau = B$ is given by

$$\varphi_{m_0^{\Delta\tau}}(x) = \frac{1}{\sigma x\sqrt{2\pi\Delta\tau}} \exp\left(-\frac{(-(r-\sigma^2/2)\Delta\tau + \log(x/B))^2}{2\sigma^2\Delta\tau}\right)$$
$$+\frac{1}{\sigma x\sqrt{2\pi\Delta\tau}}\left(\frac{B}{x}\right)^{1-2r/\sigma^2} \exp\left(-\frac{((r-\sigma^2/2)\Delta\tau + \log(x/B))^2}{2\sigma^2\Delta\tau}\right)$$
$$+\frac{1}{x}\left(\frac{2r}{\sigma^2}-1\right)\left(\frac{B}{x}\right)^{1-2r/\sigma^2} \Phi\left(\frac{(r-\sigma^2/2)\Delta\tau + \log(x/B)}{\sigma\sqrt{\Delta\tau}}\right),$$

$0 < x \leqslant B$, see also Proposition 7.7 for the probability density function of the minimum of the drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t \in [0,T]$. Hence, we have

$$\mathbb{E}\left[\left(\min_{s\in[0,\Delta\tau]} S_{\tau+s} - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right] = \int_0^B (x-K)^+ \varphi_{m_0^{\Delta\tau}}(x)dx$$
$$= B\left(1+\frac{\sigma^2}{2r}\right) e^{r\Delta\tau}\Phi\left(-\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\sqrt{\Delta\tau}\right) + K\Phi\left(-\frac{\log(B/K)+(r-\sigma^2/2)\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right)$$
$$-B\left(1+\frac{\sigma^2}{2r}\right) e^{r\Delta\tau}\Phi\left(-\frac{\log(B/K)+(r+\sigma^2/2)\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right)$$
$$+B\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\Delta\tau}\right)$$
$$+B\frac{\sigma^2}{2r}\left(\frac{K}{B}\right)^{2r/\sigma^2}\Phi\left(-\frac{\log(B/K)-(r-\sigma^2/2)\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right) - K,$$

with $r > 0$.

b) When $r = 0$, we find

$$\mathbb{E}\left[\left(\min_{s\in[0,\Delta\tau]} S_{\tau+s} - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right] = B\left(2+\frac{\sigma^2}{2}\Delta\tau\right)\Phi\left(-\frac{\sigma}{2}\sqrt{\Delta\tau}\right)$$
$$-B\left(1+\frac{\sigma^2}{2}\Delta\tau + \log\frac{B}{K}\right)\Phi\left(-\frac{\log(B/K)+\frac{\sigma^2}{2}\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right)$$
$$+K\Phi\left(-\frac{\log(B/K)-\frac{\sigma^2}{2}\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right) - \frac{1}{\sqrt{2\pi}}\sigma B\sqrt{\Delta\tau}\left(e^{-\sigma^2\Delta\tau/8} - e^{-d_+^2/2}\right) - K.$$

c) By the solution of Exercise 7.1-(c)), the probability density function of τ_B is given by

$$\varphi_{\tau_B}(x) = \frac{|\log(S_0/B)|}{\sigma\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2\sigma^2 x}((r-\sigma^2/2)x - \log(B/S_0))^2\right),$$

$x > 0$.

d) We have

$$e^{-r\Delta\tau}\mathbb{E}\left[e^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\left(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\right)^+\right]$$
$$= e^{-r\Delta\tau}\mathbb{E}\left[e^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\,\mathbb{E}\left[\left(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right]\right]$$
$$= e^{-r\Delta\tau}\mathbb{E}\left[e^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\right]\mathbb{E}\left[\left(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right], \tag{A.7}$$

where $\mathbb{E}\left[\left(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\right)^+ \,\Big|\, \mathscr{F}_\tau\right]$ is given by Questions (a))-(b)), and

$$\mathbb{E}\left[e^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\right] = \int_0^T e^{-rx}\varphi_{\tau_B}(x)dx$$
$$= \left|\log\frac{S_0}{B}\right|\int_0^T e^{-rx\tau}\frac{1}{\sigma\sqrt{2\pi x^3}}\exp\left(-\frac{1}{2\sigma^2 x}((r-\sigma^2/2)x - \log(B/S_0))^2\right)dx$$

$$
\begin{aligned}
&= \left|\log\frac{S_0}{B}\right| \exp\left(\frac{\log(B/S_0)\left(r-\sigma^2/2-\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\right)}{\sigma^2}\right) \\
&\quad \times \int_0^T \frac{1}{\sigma\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2\sigma^2 x}\left(x\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}-\log(B/S_0)\right)^2\right) dx \\
&= \exp\left(\frac{\log(B/S_0)\left(r-\sigma^2/2-\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\right)}{\sigma^2}\right) \\
&\quad \times \int_0^T \frac{1}{\sigma\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2\sigma^2 x}\left(x\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}-\log(B/S_0)\right)^2\right) dx \\
&= \left(\frac{B}{S_0}\right)^{\left(r-\sigma^2/2-\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\right)/\sigma^2} \Phi\left(\frac{\log(B/S_0)-T\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{B}{S_0}\right)^{\left(r-\sigma^2/2+\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\right)/\sigma^2} \Phi\left(\frac{\log(B/S_0)+T\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}}{\sigma\sqrt{T}}\right),
\end{aligned}
$$

where the last identity follows from Proposition 7.4 and the relation

$$
\begin{aligned}
\Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) - \mathrm{e}^{2\mu a}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) &= \mathbb{P}\left(\widehat{X}_0^T \leqslant a\right) \\
&= \mathbb{P}(\tilde{\tau}_a \leqslant T) \\
&= \int_0^T \varphi_{\tilde{\tau}_a}(x)dx \\
&= \frac{1}{2}\int_0^T \frac{a}{\sqrt{2\pi x^3}} \mathrm{e}^{-(a-\mu x)^2/(2x)} dx, \quad T > 0.
\end{aligned}
$$

with $a := \log(B/S_0)/\sigma$, for a Brownian motion with drift

$$
\mu = \frac{\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}}{\sigma},
$$

see Exercise 7.1-(b)).

In Figure S.14 we plot* the down-and-out barrier call warrant price (A.7) with rebate as a function of volatility, with $B = 18,800 > K = 18,700$, $r = 0.02627$, $T = 110/365$, and $S_0 = 20,331.20$, and entitlement ratio $10,000$.

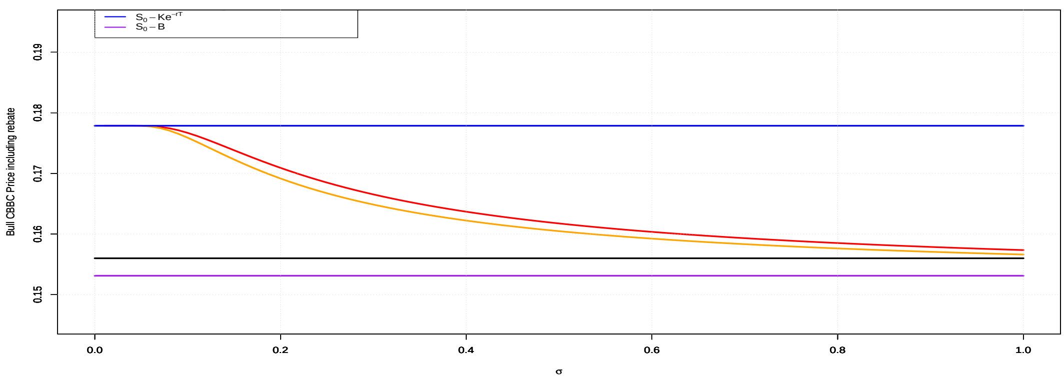


Figure S.14: Down-and-out call option price with rebate as a function of σ.

Exercise 8.3 Barrier forward contracts.

*Download this R **code** for the pricing of down-and-out barrier call options with and without rebate (right-click to save as attachment).

a) Up-and-in barrier long forward contract. We have

$$
\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}[C \mid \mathscr{F}_t] &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T-K)\mathbb{1}_{\left\{\underset{0\leqslant u\leqslant T}{\mathrm{Max}} S_u > B\right\}} \,\middle|\, \mathscr{F}_t\right] \\
&= \mathbb{1}_{\left\{\underset{0\leqslant u\leqslant t}{\mathrm{Max}} S_u > B\right\}}\left(S_t - K\mathrm{e}^{-(T-t)r}\right) + \mathbb{1}_{\left\{\underset{0\leqslant u\leqslant t}{\mathrm{Max}} S_u \leqslant B\right\}}\phi(t,S_t),
\end{aligned}
\tag{A.8}
$$

where the function

$$
\begin{aligned}
\phi(t,x) := {} & x\Phi\left(\delta_+^{T-t}(x/B)\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}(x/B)\right) \\
& + B(B/x)^{2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
& - K\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right)
\end{aligned}
$$

solves the Black-Scholes PDE with the terminal condition

$$
\phi(T,x) = \left(x - K + \left(\frac{B}{x}\right)^{2r/\sigma^2}\left(B - x\frac{K}{B}\right)\right)\mathbb{1}_{[B,\infty)}(x),
$$

as in the proof of Proposition 8.3. Note that only the values of $\phi(t,x)$ with $x \in [0,B]$ are used for pricing.

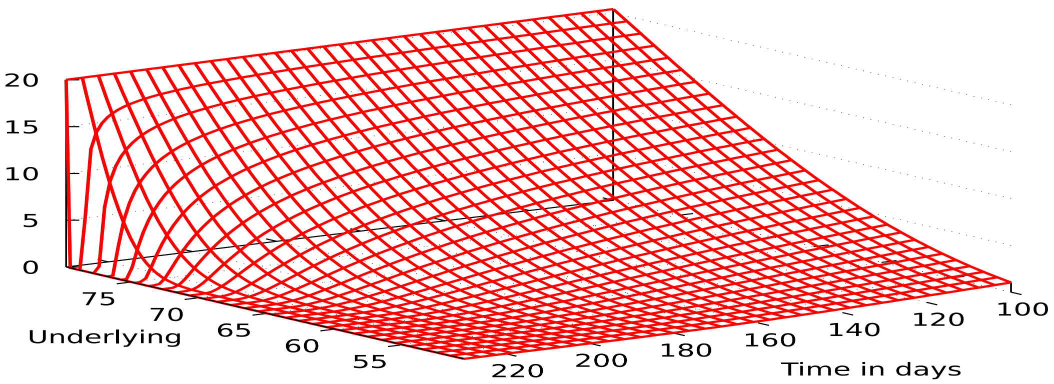


Figure S.15: Price of the up-and-in long forward contract with $K = 60 < B = 80$.

As for the delta hedging strategy, we find

$$
\begin{aligned}
\xi_t &= \frac{\partial\phi}{\partial x}(t,S_t) = \Phi\left(\delta_+^{T-t}(x/B)\right) + \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} \\
&\quad - \frac{1}{x\sqrt{2\pi}}K\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} - \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad + \frac{1}{\sqrt{2\pi}}(B/x)^{1+2r/\sigma^2}\mathrm{e}^{-(\delta_+^{T-t}(B/x))^2/2} \\
&\quad - \frac{K(1-2r/\sigma^2)}{B}\mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right) \\
&\quad - \frac{K}{B\sqrt{2\pi}}(B/x)^{2r/\sigma^2}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(B/x))^2/2} \\
&= \Phi\left(\delta_+^{T-t}(x/B)\right) - \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad + \frac{1}{\sqrt{2\pi}}(1-K/B)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right)
\end{aligned}
$$

$$-\frac{K}{B}(1-2r/\sigma^2)\,\mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right),$$

since by (9.4.2) we have

$$\mathrm{e}^{-(\delta_-^{T-t}(B/x))^2/2} = \mathrm{e}^{r(T-t)}(x/B)^{2r/\sigma^2}\,\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2}$$

and

$$\mathrm{e}^{-(\delta_-^{T-t}(x/B))^2/2} = \mathrm{e}^{r(T-t)}(B/x)^{2r/\sigma^2}\,\mathrm{e}^{-(\delta_+^{T-t}(B/x))^2/2}.$$

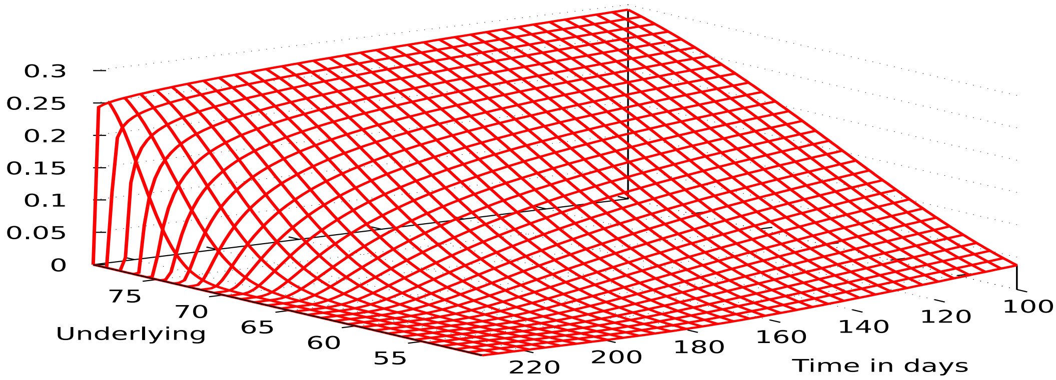


Figure S.16: Delta of the up-and-in long forward contract with $K = 60 < B = 80$.

b) Up-and-out barrier long forward contract. We have

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\,\mathbb{E}[C \mid \mathcal{F}_t] &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[(S_T-K)\,\mathbb{1}_{\left\{\underset{0\leqslant u\leqslant T}{\mathrm{Max}}\, S_u < B\right\}}\,\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{1}_{\left\{\underset{0\leqslant u\leqslant t}{\mathrm{Max}}\, S_u \leqslant B\right\}}\phi(t,S_t),
\end{aligned} \tag{A.9}$$

where the function

$$\begin{aligned}
\phi(t,x) := \; & x\Phi\left(-\delta_+^{T-t}(x/B)\right) - K\,\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}(x/B)\right) \\
& -B(B/x)^{2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
& +K\,\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right)
\end{aligned}$$

solves the Black-Scholes PDE with the terminal condition

$$\phi(T,x) = (x-K)\mathbb{1}_{[0,B]}(x) - \left(\frac{B}{x}\right)^{2r/\sigma^2}\left(B - x\frac{K}{B}\right)\mathbb{1}_{[B,\infty)}(x).$$

Note that only the values of $\phi(t,x)$ with $x \in [B,\infty)$ are used for pricing.

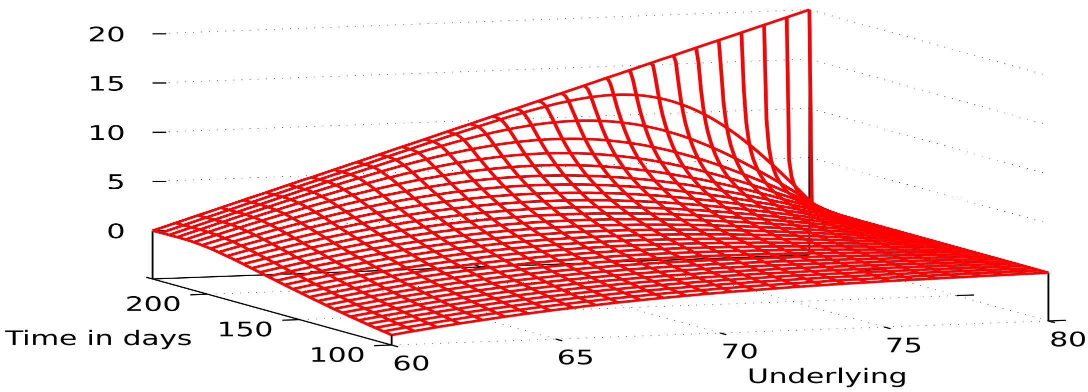


Figure S.17: Price of the up-and-out long forward contract with $K = 60 < B = 80$.

As for the delta hedging strategy, we find

$$
\begin{aligned}
\xi_t &= \frac{\partial \phi}{\partial x}(t, S_t) = \Phi\left(-\delta_+^{T-t}(x/B)\right) - \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} \\
&\quad + \frac{1}{x\sqrt{2\pi}} K \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad - \frac{1}{\sqrt{2\pi}}(B/x)^{1+2r/\sigma^2} \mathrm{e}^{-(\delta_+^{T-t}(B/x))^2/2} \\
&\quad + \frac{K(1-2r/\sigma^2)}{B} \mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right) \\
&\quad + \frac{K}{B\sqrt{2\pi}}(B/x)^{2r/\sigma^2} \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(B/x))^2/2} \\
&= \Phi\left(-\delta_+^{T-t}(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad - \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} - \frac{1}{\sqrt{2\pi}}\frac{B}{x} \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} \\
&\quad + \frac{K}{B\sqrt{2\pi}} \mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{1}{\sqrt{2\pi}}\frac{K}{x} \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} \\
&\quad + \frac{K}{B}(1-2r/\sigma^2) \mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right) \\
&= \Phi\left(-\delta_+^{T-t}(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad - \frac{1}{\sqrt{2\pi}}(1-K/B)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right) \\
&\quad + \frac{K}{B}\left(1-\frac{2r}{\sigma^2}\right)\mathrm{e}^{-(T-t)r}\left(\frac{B}{x}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B}{x}\right)\right),
\end{aligned}
$$

by (9.4.2).

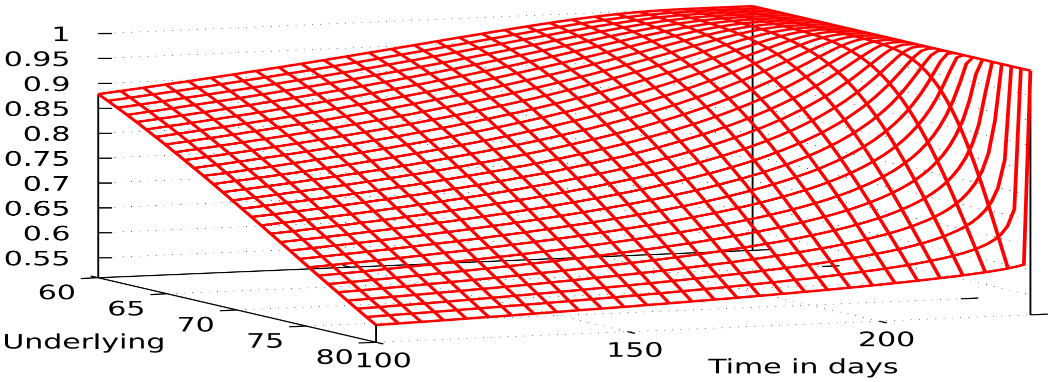


Figure S.18: Delta of the up-and-out long forward contract price with $K = 60 < B = 80$.

c) Down-and-in barrier long forward contract. We have

$$
\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}[C \mid \mathscr{F}_t] &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T-K)\mathbb{1}_{\left\{\min\limits_{0\leqslant u\leqslant T} S_u < B\right\}} \,\middle|\, \mathscr{F}_t\right] \\
&= \mathbb{1}_{\left\{\min\limits_{0\leqslant u\leqslant t} S_u < B\right\}}(S_t - K\mathrm{e}^{-(T-t)r}) + \mathbb{1}_{\left\{\min\limits_{0\leqslant u\leqslant t} S_u \geqslant B\right\}}\phi(t,S_t)
\end{aligned} \tag{A.10}
$$

where the function

$$
\begin{aligned}
\phi(t,x) &:= x\Phi\left(-\delta_+^{T-t}(x/B)\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}(x/B)\right) \\
&\quad + B(B/x)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}(B/x)\right) \\
&\quad - K\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(\delta_-^{T-t}(B/x)\right)
\end{aligned}
$$

solves the Black-Scholes PDE with the terminal condition

$$
\phi(T,x) = \left(x - K + \left(\frac{B}{x}\right)^{2r/\sigma^2}\left(B - x\frac{K}{B}\right)\right)\mathbb{1}_{[0,B]}(x).
$$

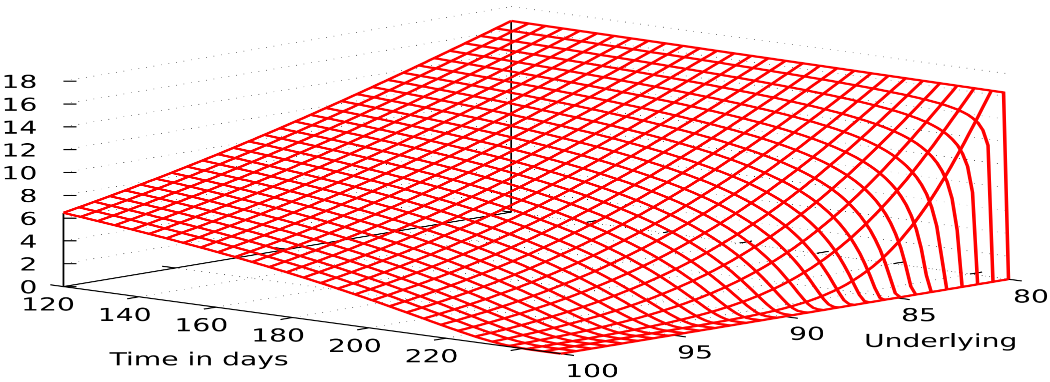


Figure S.19: Price of the down-and-in long forward contract with $K = 60 < B = 80$.

As for the delta hedging strategy, we find

$$
\begin{aligned}
\xi_t &= \frac{\partial \phi}{\partial x}(t,S_t) \\
&= \Phi\left(-\delta_+^{T-t}(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(\delta_+^{T-t}(B/x)\right)
\end{aligned}
$$

$$-\frac{1}{\sqrt{2\pi}}(1-K/B)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2}+\frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right)$$
$$+\frac{K}{B}(1-2r/\sigma^2)\,\mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}(B/x)\right).$$

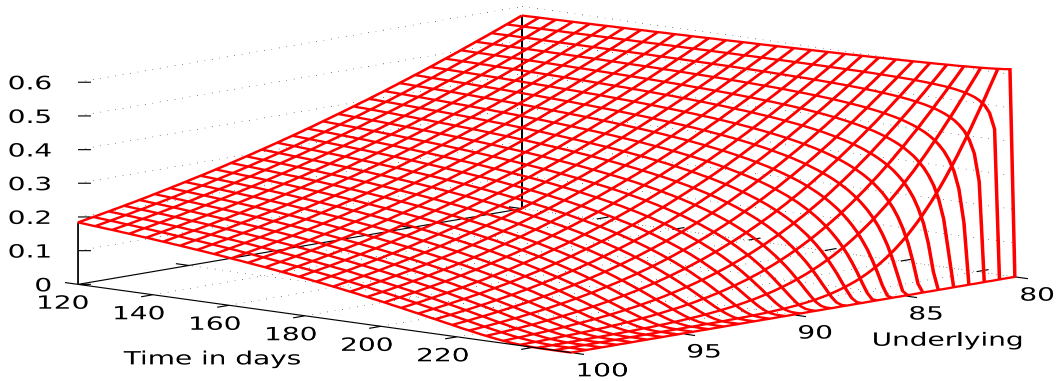


Figure S.20: Delta of the down-and-in long forward contract with $K=60<B=80$.

d) Down-and-out barrier long forward contract. We have

$$\begin{aligned}\mathrm{e}^{-(T-t)r}\mathbb{E}[C\mid\mathscr{F}_t] &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[(S_T-K)\mathbb{1}_{\left\{\min\limits_{0\leqslant u\leqslant T}S_u>B\right\}}\ \Big|\ \mathscr{F}_t\right]\\ &= \mathbb{1}_{\left\{\min\limits_{0\leqslant u\leqslant t}S_u\geqslant B\right\}}\phi(t,S_t)\end{aligned} \tag{A.11}$$

where the function

$$\begin{aligned}\phi(t,x) &:= x\Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right)-K\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{x}{B}\right)\right)\\ &\quad -B(B/x)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B}{x}\right)\right)\\ &\quad +K\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right)\Bigg)\end{aligned}$$

solves the Black-Scholes PDE with the terminal condition

$$\phi(T,x)=(x-K)\mathbb{1}_{[B,\infty)}(x)-\left(B-x\frac{K}{B}\right)\left(\frac{B}{x}\right)^{2r/\sigma^2}\mathbb{1}_{[0,B]}(x).$$

Note that $\phi(t,x)$ above coincides with the price of (8.2.7) of the standard down-and-out barrier call option in the case $K<B$, cf. Exercise 8.1-(d)).

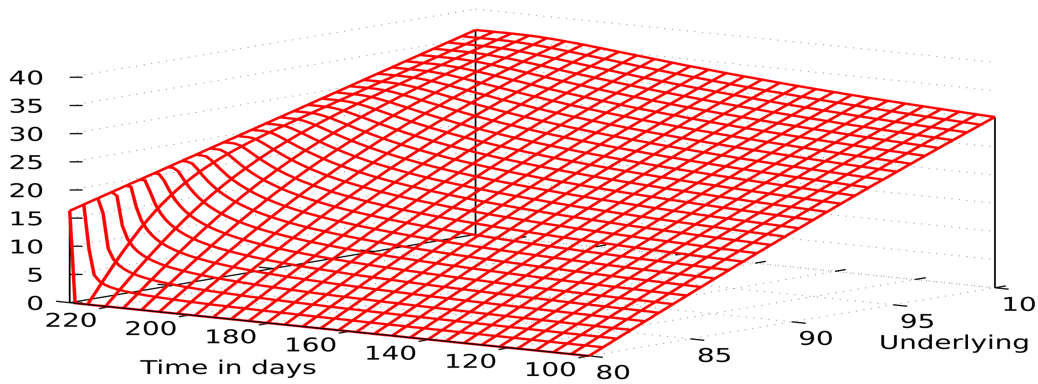


Figure S.21: Price of the down-and-out long forward contract with $K = 60 < B = 80$.

As for the delta hedging strategy, we find

$$\begin{aligned}
\xi_t &= \frac{\partial \phi}{\partial x}(t, S_t) \\
&= \Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right) - \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(\delta_+^{T-t}(B/x)\right) \\
&\quad + \frac{1}{\sqrt{2\pi}}\left(1 - \frac{K}{B}\right)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right) \\
&\quad - \frac{K}{B}\left(1 - \frac{2r}{\sigma^2}\right)\mathrm{e}^{-(T-t)r}\left(\frac{B}{x}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right).
\end{aligned}$$

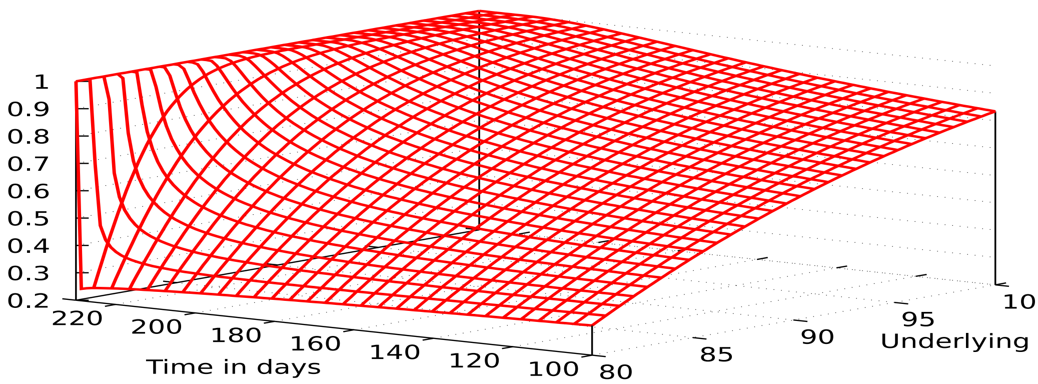


Figure S.22: Delta of the down-and-out long forward contract with $K = 60 < B = 80$.

e) Up-and-in barrier short forward contract. The price of the up-and-in barrier short forward contract is identical to (A.8) with a negative sign.

f) Up-and-out barrier short forward contract. The price of the up-and-out barrier short forward contract is identical to (A.9) with a negative sign. Note that $\phi(t,x)$ coincides with the price of (8.2.4) of the standard up-and-out barrier put option in the case $B < K$.

g) Down-and-in barrier short forward contract. The price of the down-and-in barrier short forward contract is identical to (A.10) with a negative sign.

h) Down-and-out barrier short forward contract. The price of the down-and-out barrier short forward contract is identical to (A.11) with a negative sign.

Exercise 8.4 When $B < K$, we find

$$\begin{aligned}
&\mathrm{Vega}_{\text{down-and-out-call}}\\
&= S_t\sqrt{\frac{T-t}{2\pi}}\,\mathrm{e}^{-(\delta_+^{T-t}(S_t/K))^2/2}\\
&-\frac{4r}{\sigma^3}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\frac{B^2}{S_t}\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-K\,\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\right)\log\frac{S_t}{B}\\
&-\sqrt{\frac{T-t}{2\pi}}\frac{B^2}{S_t}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\mathrm{e}^{-(\delta_+^{T-t}(B^2/K/S_t))^2/2}.
\end{aligned}$$

When $B > K$, we find

$$\begin{aligned}
&\mathrm{Vega}_{\text{down-and-out-call}}\\
&= \frac{S_t}{\sqrt{2\pi}}\,\mathrm{e}^{-(\delta_+^{T-t}(S_t/K))^2/2}\left(\left(\frac{K}{B}-1\right)\left(\frac{\delta_-^{T-t}(S_t/B)}{\sigma}+\sqrt{T-t}\right)+\sqrt{T-t}\right)\\
&-\frac{4r}{\sigma^3}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\frac{B^2}{S_t}\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)-K\,\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\log\frac{S_t}{B}\\
&-\frac{1}{\sqrt{2\pi}}\frac{B^2}{S_t}\,\mathrm{e}^{-(\delta_+^{T-t}(S_t/B))^2/2}\left(\left(\frac{K}{B}-1\right)\left(\frac{\delta_-^{T-t}(B/S_t)}{\sigma}+\sqrt{T-t}\right)+\sqrt{T-t}\right).
\end{aligned}$$

The corresponding formulas for the down-and-in call option can be obtained from the parity relation (8.1.2) and the value $S_t\sqrt{\frac{T-t}{2\pi}}\,\mathrm{e}^{-(\delta_+^{T-t}(S_t/K))^2/2}$ of the Black-Scholes Vega.

Exercise 8.5 We have

$$\begin{aligned}
\mathbb{E}^*[C] &= \mathbb{E}^*\left[\mathbb{1}_{\{S_T\geqslant K\}}\mathbb{1}_{\{M_0^T\leqslant B\}}\right]\\
&= \mathbb{E}^*\left[\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma\widehat{W}_T}\geqslant K\}}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma\widehat{X}_0^T}\leqslant B\}}\right]\\
&= \int_{-\infty}^{\infty}\int_{y\vee 0}^{\infty}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma y}\geqslant K\}}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma x}\leqslant B\}}d\mathbb{P}(\widehat{X}_0^T\leqslant x,\widehat{W}_T\leqslant y)\\
&= \int_{-\infty}^{\infty}\int_{y\vee 0}^{\infty}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma y}\geqslant K\}}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma x}\leqslant B\}}f_{\widehat{X}_T,\widehat{W}_T}(x,y)dxdy\\
&= \frac{1}{T}\sqrt{\frac{2}{\pi T}}\int_{-\infty}^{\sigma^{-1}\log(B/S_0)}\\
&\quad\int_{y\vee 0}^{\infty}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma y}\geqslant K\}}\mathbb{1}_{\{S_0\,\mathrm{e}^{\sigma x}\leqslant B\}}(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy\\
&= \frac{1}{T}\sqrt{\frac{2}{\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)}\int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}(2x-y)\,\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy,
\end{aligned}$$

if $B\geqslant S_0$ (otherwise the option price is 0), with $\mu = r/\sigma-\sigma/2$ and $y\vee 0=\mathrm{Max}(y,0)$. Next, letting $a = y\vee 0$ and $b=\sigma^{-1}\log(B/S_0)$, we have

$$\int_a^b(2x-y)\,\mathrm{e}^{2x(y-x)/T}dx = \frac{T}{2}(1-\mathrm{e}^{2b(y-b)/T}),$$

hence, letting $c=\sigma^{-1}\log(K/S_0)$, we have

$$\mathbb{E}^*[C] = \mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b \mathrm{e}^{\mu y-y^2/(2T)}(1-\mathrm{e}^{2b(y-b)/T})dy$$

$$
\begin{aligned}
&= \mathrm{e}^{-\mu^2 T/2} \frac{1}{\sqrt{2\pi T}} \int_c^b \mathrm{e}^{\mu y - y^2/(2T)} dy \\
&\quad - \mathrm{e}^{-\mu^2 T/2 - 2b^2/T} \frac{1}{\sqrt{2\pi T}} \int_c^b \mathrm{e}^{y(\mu + 2b/T) - y^2/(2T)} dy.
\end{aligned}
$$

Using the relation

$$
\frac{1}{\sqrt{2\pi T}} \int_c^b \mathrm{e}^{\gamma y - y^2/(2T)} dy = \mathrm{e}^{\gamma^2 T/2} \left(\Phi\left(\frac{-c + \gamma T}{\sqrt{T}} \right) - \Phi\left(\frac{-b + \gamma T}{\sqrt{T}} \right) \right),
$$

we find

$$
\begin{aligned}
&\mathbb{E}^*[C] = \mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{M_0^T \leqslant B\}} \right] \\
&= \Phi\left(\frac{-c + \mu T}{\sqrt{T}} \right) - \Phi\left(\frac{-b + \mu T}{\sqrt{T}} \right) \\
&- \mathrm{e}^{-\mu^2 T/2 - 2b^2/T + (\mu + 2b/T)^2 T/2} \left(\Phi\left(\frac{-c + (\mu + 2b/T)T}{\sqrt{T}} \right) - \Phi\left(\frac{-b + (\mu + 2b/T)T}{\sqrt{T}} \right) \right) \\
&= \Phi\left(\delta_-^T\left(\frac{S_0}{K} \right) \right) - \Phi\left(\delta_-^T\left(\frac{S_0}{B} \right) \right) \\
&\quad - \mathrm{e}^{-\mu^2 T/2 - 2b^2 T + (\mu + 2b/T)^2 T/2} \left(\Phi\left(\delta_-\left(\frac{B^2}{KS_0} \right) \right) - \Phi\left(\delta_-\left(\frac{B}{S_0} \right) \right) \right),
\end{aligned}
$$

$0 \leqslant x \leqslant B$. Given the relation

$$
-\frac{\mu^2 T}{2} - 2\frac{b^2}{T} + \frac{T}{2}\left(\mu + \frac{2b}{T} \right)^2 = \left(-1 + \frac{2r}{\sigma^2} \right) \log \frac{B}{S_0},
$$

we get

$$
\begin{aligned}
&\mathrm{e}^{-rT}\, \mathbb{E}^*[C] = \mathrm{e}^{-rT}\, \mathbb{E}^*\left[\mathbb{1}_{\{S_T \geqslant K\}} \mathbb{1}_{\{M_0^T \leqslant B\}} \right] \\
&= \mathrm{e}^{-rT} \left(\Phi\left(\delta_-^T\left(\frac{S_0}{K} \right) \right) - \Phi\left(\delta_-^T\left(\frac{S_0}{B} \right) \right) \right. \\
&\quad \left. - \left(\frac{S_0}{B} \right)^{1 - 2r/\sigma^2} \left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0} \right) \right) - \Phi\left(\delta_-^T\left(\frac{B}{S_0} \right) \right) \right) \right).
\end{aligned}
$$

Exercise 8.6

a) For $x = B$ and $t \in [0, T]$ we check that

$$
\begin{aligned}
g(t, B) &= B\left(\Phi\left(\delta_+^{T-t}\left(\frac{B}{K} \right) \right) - \Phi\left(\delta_+^{T-t}(1) \right) \right) \\
&\quad - \mathrm{e}^{-(T-t)r} K \left(\Phi\left(\delta_-^{T-t}\left(\frac{B}{K} \right) \right) - \Phi\left(\delta_-^{T-t}(1) \right) \right) \\
&\quad - B\left(\Phi\left(\delta_+^{T-t}\left(\frac{B}{K} \right) \right) - \Phi\left(\delta_+^{T-t}(1) \right) \right) \\
&\quad + \mathrm{e}^{-(T-t)r} K \left(\Phi\left(\delta_-^{T-t}\left(\frac{B}{K} \right) \right) - \Phi\left(\delta_-^{T-t}(1) \right) \right) \\
&= 0,
\end{aligned}
$$

and the function $g(t, x)$ is extended to $x > B$ by letting

$$
g(t, x) = 0, \qquad x > B.
$$

b) For $x = K$ and $t = T$, we find

$$\delta_\pm^0(s) = -\infty \times \mathbb{1}_{\{s<1\}} + \infty \times \mathbb{1}_{\{s>1\}} = \begin{cases} +\infty & \text{if } s > 1, \\ 0 & \text{if } s = 1, \\ -\infty & \text{if } s < 1, \end{cases}$$

hence when $x < K < B$ we have

$$\begin{aligned} g(T,x) = {} & x\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ & -K\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ & -B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ & +K\left(\frac{B}{K}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ = {} & 0, \end{aligned}$$

c) when $K < x < B$, we get

$$\begin{aligned} g(T,x) & = x\left(\Phi(+\infty) - \Phi(-\infty)\right) \\ & \quad -K\left(\Phi(+\infty) - \Phi(-\infty)\right) \\ & \quad -B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ & \quad +K\left(\frac{B}{K}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ & = x - K. \end{aligned}$$

Finally, for $x > B$ we obtain

$$\begin{aligned} g(T,K) & = x\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ & \quad -K\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ & \quad -B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ & \quad +K\left(\frac{B}{K}\right)^{2r/\sigma^2}\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ & = 0. \end{aligned}$$

Exercise 8.7

a) The price at time $t \in [0,T]$ of the European knock-out call option is given by

$$\mathrm{EKOC}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{S_T \leqslant B\}} \,\middle|\, \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T.$$

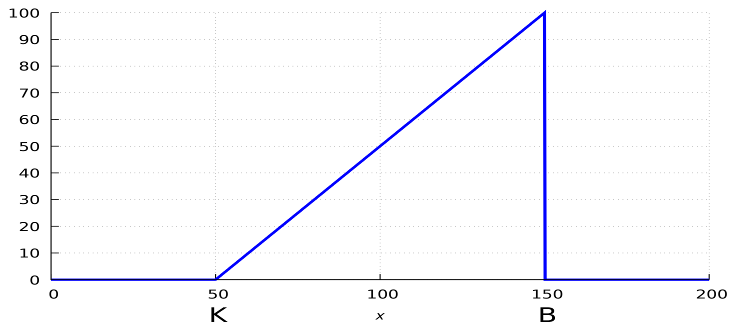


Figure S.23: Payoff function of the European knock-out call option.

Using the relation

$$S_T = S_t \,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}, \qquad 0 \leqslant t \leqslant T,$$

we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{S_T\leqslant B\}}\,\big|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\Big[\big(x\,\mathrm{e}^{(T-t)r+\left(\widehat{B}_T-\widehat{B}_t\right)\sigma-(T-t)\sigma^2/2}-K\big)^+\\
&\quad \times \mathbb{1}_{\{x\,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}\leqslant B\}}\Big]_{x=S_t}\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\mathrm{e}^{m(x)+X}-K\right)^+\mathbb{1}_{\{\mathrm{e}^{m(x)+X}\leqslant B\}}\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2}(T-t) + \log x$$

and

$$X := \left(\widehat{B}_T - \widehat{B}_t\right)\sigma \simeq \mathcal{N}(0,(T-t)\sigma^2)$$

under $\mathbb{P}^*$. Next, we note that if X is a centered Gaussian random variable with variance $v^2 > 0$ and $B \geqslant K$, for any $m \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathrm{e}^{m+X}-K\right)^+\mathbb{1}_{\{\mathrm{e}^{m+X}\leqslant B\}}\right]\\
&= \frac{1}{\sqrt{2\pi v^2}}\int_{-\infty}^{\infty}\left(\mathrm{e}^{m+x}-K\right)^+\mathbb{1}_{\{\mathrm{e}^{m+x}\leqslant B\}}\,\mathrm{e}^{-x^2/(2v^2)}dx\\
&= \frac{1}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{-m+\log B}\left(\mathrm{e}^{m+x}-K\right)\mathrm{e}^{-x^2/(2v^2)}dx\\
&= \frac{\mathrm{e}^m}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{-m+\log B}\mathrm{e}^{x-x^2/(2v^2)}dx - \frac{K}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{-m+\log B}\mathrm{e}^{-x^2/(2v^2)}dx\\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{-m+\log B}\mathrm{e}^{-(v^2-x)^2/(2v^2)}dx - \frac{K}{\sqrt{2\pi}}\int_{(-m+\log K)/v}^{(-m+\log B)/v}\mathrm{e}^{-x^2/2}dx\\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}}\int_{-v^2-m+\log K}^{-v^2-m+\log B}\mathrm{e}^{-y^2/(2v^2)}dy\\
&\quad -K\big(\Phi((m-\log K)/v)-\Phi((m-\log B)/v)\big)\\
&= \mathrm{e}^{m+v^2/2}\big(\Phi(v+(m-\log K)/v)-\Phi(v+(m-\log B)/v)\big)\\
&\quad -K\big(\Phi((m-\log K)/v)-\Phi((m-\log B)/v)\big).
\end{aligned}$$

Hence, the price of the European knock-out call option is given, if $B \geqslant K$, by

$$\mathrm{EKOC}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{S_T\leqslant B\}}\,\big|\,\mathcal{F}_t\right]$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)r}\,\mathrm{e}^{m(S_t)+\sigma^2(T-t)/2}\left(\Phi\left(v+\frac{m(S_t)-\log K}{v}\right)-\Phi\left(v+\frac{m(S_t)-\log K}{v}\right)\right)\\
&\quad -K\,\mathrm{e}^{-(T-t)r}\left(\Phi\left(\frac{m(S_t)-\log K}{v}\right)-\Phi\left(\frac{m(S_t)-\log B}{v}\right)\right)\\
&= S_t\Phi\left(\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right)\\
&\quad -S_t\Phi\left(\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right)\\
&\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right)\\
&\quad +K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

$0 \leqslant t \leqslant T$, with $\mathrm{EKOC}_t = 0$ if $B \leqslant K$.

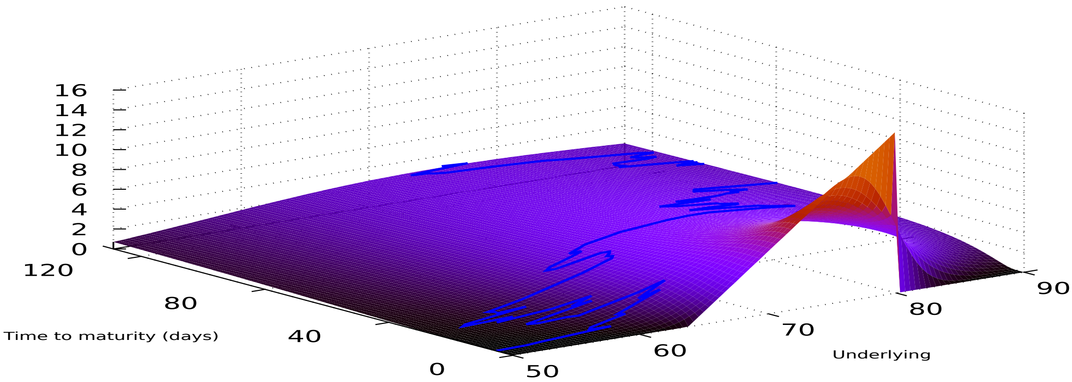


Figure S.24: Price map of the European knock-out call option.

b) By computations similar to part (a)) we find that, if $B \leqslant K$,

$$
\begin{aligned}
\mathrm{EKIP}_t &= K\,\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right)\\
&\quad -S_t\Phi\left(-\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

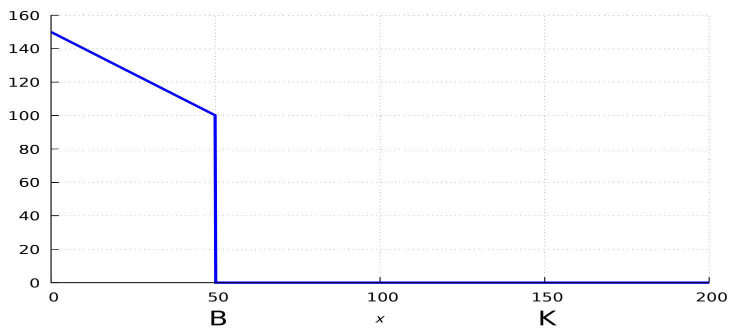


Figure S.25: Payoff function of the European knock-in put option.

When $B \geqslant K$, we find the Black-Scholes put option price

$$
\mathrm{EKIP}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{S_T\leqslant B\}}\,\middle|\,\mathcal{F}_t\right]
$$

$$
\begin{aligned}
&= e^{-(T-t)r}\mathbb{E}^*\left[(K-S_T)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= K e^{-(T-t)r}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad - S_t\Phi\left(-\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

$0 \leqslant t \leqslant T$.

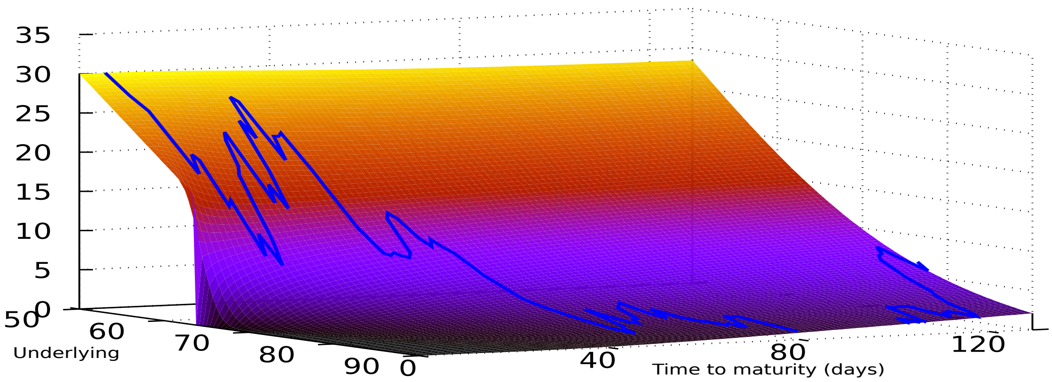


Figure S.26: Price map of the European knock-in put option.

c) Using the in-out parity relation

$$
\begin{aligned}
\mathrm{EKOC}_t + \mathrm{EKIC}_t &= e^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t] \\
&= S_t\Phi\left(\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K e^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

which is the price of the European call put option with strike price K, the price at time $t \in [0,T]$ of the European knock-in call option is given, if $B \geqslant K$, as

$$
\begin{aligned}
\mathrm{EKIC}_t &= e^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{S_T \geqslant B\}} \,\middle|\, \mathcal{F}_t\right] \\
&= S_t\Phi\left(\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K e^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

$0 \leqslant t \leqslant T$.

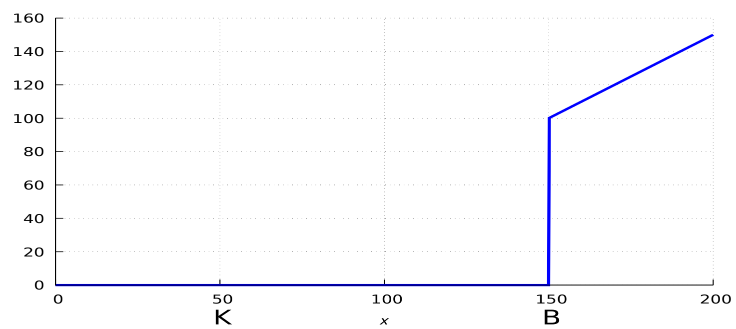


Figure S.27: Payoff function of the European knock-in call option.

When $B \leqslant K$, we find the Black-Scholes call option price

$$\begin{aligned} \mathrm{EKIC}_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{S_T\geqslant B\}}\,\middle|\,\mathscr{F}_t\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+\,\middle|\,\mathscr{F}_t\right] \\ &= S_t\Phi\left(\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\ &\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

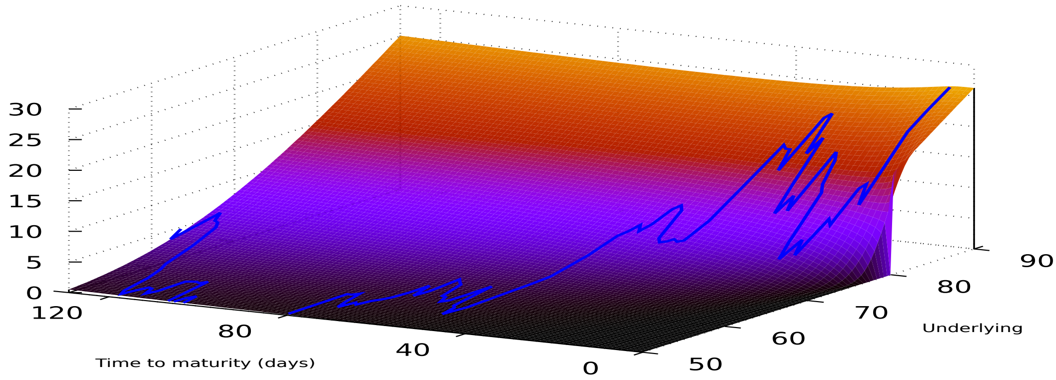


Figure S.28: Price map of the European knock-in call option.

d) Using the in-out parity relation

$$\mathrm{EKOP}_t+\mathrm{EKIP}_t=\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(K-S_T)^+\mid\mathscr{F}_t],$$

which is the price of the European put option with strike price K, we find that the price at time $t\in[0,T]$ of the European knock-in put option is given, if $B\leqslant K$, as

$$\begin{aligned} \mathrm{EKOP}_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{S_T\geqslant B\}}\,\middle|\,\mathscr{F}_t\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(K-S_T)^+\mid\mathscr{F}_t]-\mathrm{EKIP}_t \\ &= K\,\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\ &\quad -S_t\Phi\left(-\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\ &\quad +S_t\Phi\left(-\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right) \\ &\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

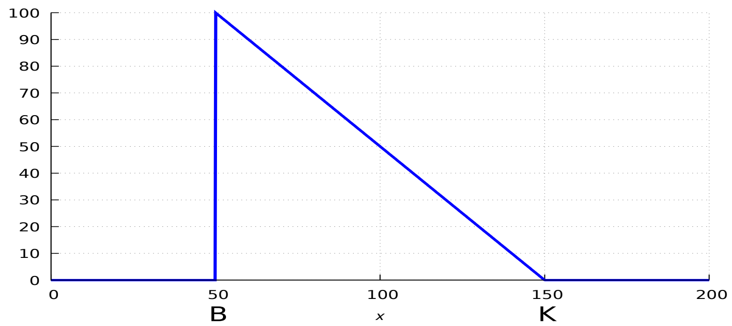


Figure S.29: Payoff function of the European knock-out put option.

When $B \geqslant K$, we have

$$\text{EKIP}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{S_T \geqslant B\}}\,\middle|\,\mathcal{F}_t\right] = 0.$$

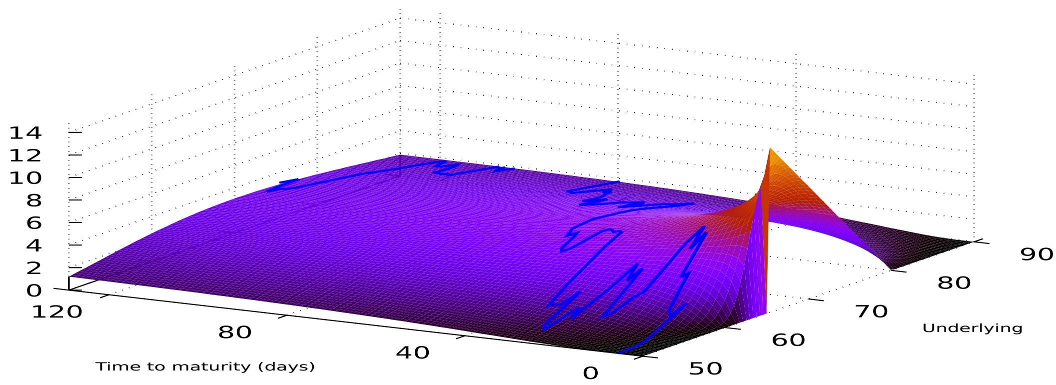


Figure S.30: Price map of the European knock-out put option.

In addition, by the results of Questions (d)) and (c)) we can verify the call-put parity relation

$$\begin{aligned}
\text{EKIC}_t - \text{EKIP}_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)\mathbb{1}_{\{S_T \geqslant B\}}\,\middle|\,\mathcal{F}_t\right] \\
&= S_t\Phi\left(\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

Chapter 9

Exercise 9.1

a) This probability density function is given by

$$x \longmapsto \sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(x-\sigma T/2)^2/(2T)} - \sigma\,\mathrm{e}^{\sigma x}\Phi\left(\frac{-x-\sigma T/2}{\sqrt{T}}\right), \qquad x \geqslant 0.$$

b) We have

$$\begin{aligned}
\mathbb{E}\left[\min_{t\in[0,T]} S_t\right] &= S_0\,\mathbb{E}\left[\min_{t\in[0,T]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2}\right] \\
&= S_0\,\mathbb{E}\left[\mathrm{e}^{-\sigma \operatorname{Max}_{t\in[0,T]}(B_t+\sigma t/2)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= S_0 \int_0^\infty \mathrm{e}^{-\sigma x}\left(\sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-(x-\sigma T/2)^2/(2T)} - \sigma\,\mathrm{e}^{\sigma x}\Phi\left(\frac{-x-\sigma T/2}{\sqrt{T}}\right)\right)dx \\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{-(x+\sigma T/2)^2/(2T)}dx - S_0\sigma\int_0^\infty \Phi\left(\frac{-x-\sigma T/2}{\sqrt{T}}\right)dx \\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_{\sigma T/2}^\infty \mathrm{e}^{-x^2/(2T)}dx - \frac{\sigma S_0}{\sqrt{2\pi T}}\int_0^\infty x\,\mathrm{e}^{-(x+\sigma T/2)^2/(2T)}dx \\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_{\sigma T/2}^\infty \mathrm{e}^{-x^2/(2T)}dx - \frac{\sigma S_0}{\sqrt{2\pi T}}\int_{\sigma T/2}^\infty (x-\sigma T/2)\,\mathrm{e}^{-x^2/(2T)}dx \\
&= 2S_0(1+\sigma^2 T/4)\Phi(-\sigma\sqrt{T}/2) - S_0\sigma\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2 T/8}. \qquad (A.12)
\end{aligned}
$$

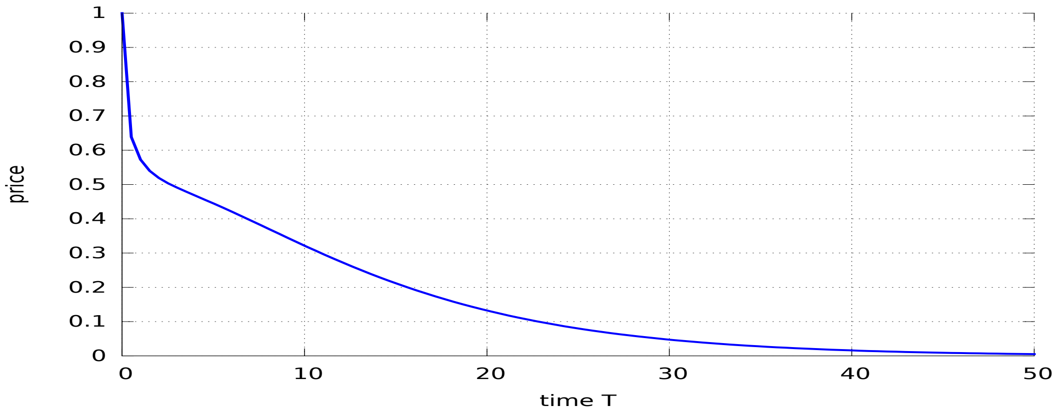


Figure S.31: Expected minimum of geometric Brownian motion over $[0,T]$.

c) We have

$$
\begin{aligned}
\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right] &= \mathbb{E}\left[K - \min_{t\in[0,T]} S_t\right] \\
&= K - S_0\left(2\left(1+\frac{\sigma^2 T}{4}\right)\Phi\left(-\frac{\sigma\sqrt{T}}{2}\right) - \sigma\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2 T/8}\right).
\end{aligned}
$$

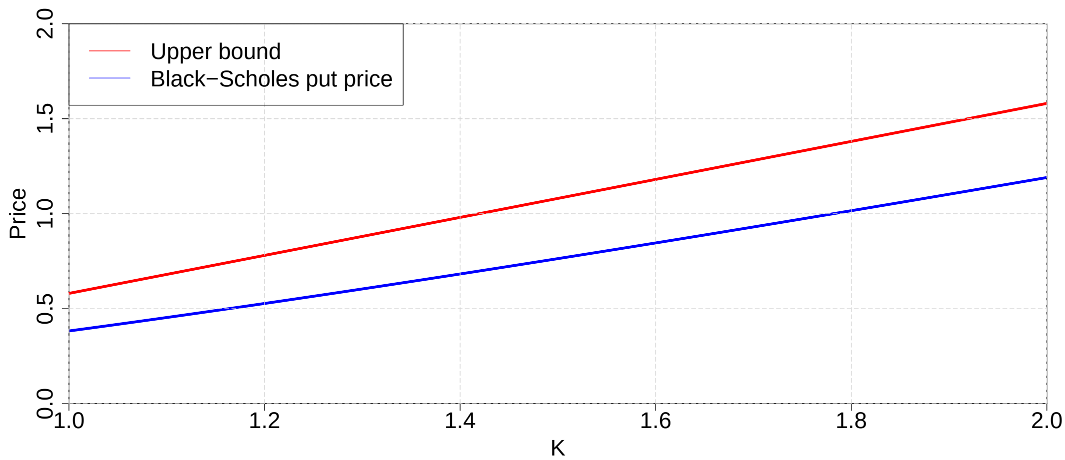


Figure S.32: Black-Scholes put price upper bound with $S_0 = 1$.

The derivative with respect to time is given by

$$\begin{aligned}
\frac{\partial}{\partial T}\mathbb{E}\left[\min_{t\in[0,T]} S_t\right] &= S_0\frac{\sigma^2}{2}\Phi\big(-\sigma\sqrt{T}/2\big) - 2S_0\left(1+\frac{\sigma^2 T}{4}\right)\frac{\sigma}{4\sqrt{2\pi T}}\,\mathrm{e}^{-\sigma^2 T/8} \\
&\quad -\frac{\sigma S_0}{\sqrt{8\pi T}}\,\mathrm{e}^{-\sigma^2 T/8} + \frac{S_0\sigma^3}{8}\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2 T/8} \\
&= \frac{S_0\sigma^2}{2}\Phi\left(-\sigma\frac{\sqrt{T}}{2}\right) - \frac{S_0\sigma}{\sqrt{2\pi T}}\,\mathrm{e}^{-\sigma^2 T/8}\left(1+\frac{3\sigma^2 T}{4}\right).
\end{aligned}$$

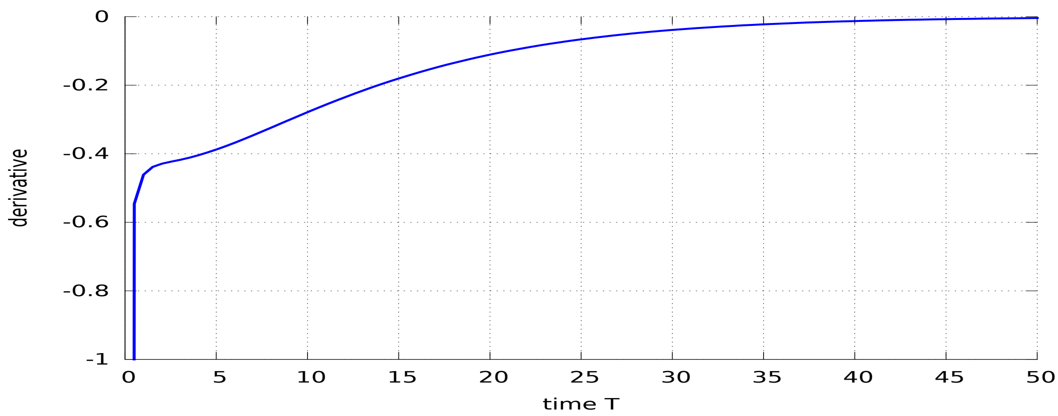


Figure S.33: Time derivative of the expected minimum of geometric Brownian motion.

On the other hand, when $r > 0$ we have

$$\begin{aligned}
\mathbb{E}^*\left[m_0^T \mid \mathscr{F}_t\right] &= m_0^t\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right) \\
&\quad + S_t\,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right).
\end{aligned}$$

When r tends to 0, this minimum tends to

$$\begin{aligned}
& m_0^t\Phi\left(\frac{\log(S_t/m_0^t)-\sigma^2 T/2}{\sigma\sqrt{T}}\right) + S_t\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2 T/2}{\sigma\sqrt{T}}\right) \\
&\quad + \sigma^2 S_t\lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
& \lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)\right) \\
&= \lim_{r\to 0}\frac{1}{2r}\left((1+(T-t)r)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2 T/2+rT}{\sigma\sqrt{T}}\right)\right. \\
&\quad \left. -\left(1+\frac{2r}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(\frac{\log(m_0^t/S_t)-\sigma^2 T/2+rT}{\sigma\sqrt{T}}\right)\right) \\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2 T/2}{\sigma\sqrt{T}}\right) \\
&\quad + \lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\left(\int_{-\infty}^{-(\log(S_t/m_0^t)+\sigma^2 T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\right.
\end{aligned}$$

$$
\begin{aligned}
& -\int_{-\infty}^{(-\log(S_t/m_0^t)-\sigma^2 T/2+rT)/(\sigma\sqrt{T})} \mathrm{e}^{-y^2/2} dy\Bigg) \\
= \; & \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2 T/2}{\sigma\sqrt{T}}\right) \\
& -\lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\int_{(-\log(S_t/m_0^t)-\sigma^2 T/2-rT)/(\sigma\sqrt{T})}^{(-\log(S_t/m_0^t)-\sigma^2 T/2+rT)/(\sigma\sqrt{T})} \mathrm{e}^{-y^2/2} dy \\
= \; & \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2 T/2}{\sigma\sqrt{T}}\right) \\
& -\frac{\sqrt{T}}{\sigma\sqrt{2\pi}}\mathrm{e}^{-((\log(S_t/m_0^t)+\sigma^2 T/2)/(\sigma\sqrt{T}))^2/2},
\end{aligned}
$$

hence

$$
\begin{aligned}
\mathbb{E}^*\left[m_0^T \mid \mathscr{F}_t\right] \; = \; & m_0^t \Phi\left(\frac{\log(S_t/m_0^t)-\sigma^2 T/2}{\sigma\sqrt{T}}\right)+S_t\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2 T/2}{\sigma\sqrt{T}}\right) \\
& +\frac{S_t}{2}\left((T-t)\sigma^2+2\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2 T/2}{\sigma\sqrt{T}}\right) \\
& -\sigma S_t\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-((\log(S_t/m_0^t)+\sigma^2 T/2)/(\sigma\sqrt{T}))^2/2}.
\end{aligned}
$$

In particular, when T tends to infinity we find that

$$
\lim_{T\to\infty}\frac{\mathbb{E}^*\left[m_0^T \mid \mathscr{F}_t\right]}{\mathbb{E}^*[S_T \mid \mathscr{F}_t]}=0, \qquad r\geqslant 0.
$$

When $t=0$ we have $S_0=m_0^0$, and we recover

$$
\mathbb{E}^*\left[m_0^T\right]=2\left(S_0+\frac{\sigma^2 T}{4}\right)\Phi\left(-\sigma\frac{\sqrt{T}}{2}\right)-\sigma S_0\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2 T/8}.
$$

Exercise 9.2

a) By (A.12), we have

$$
\begin{aligned}
\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} S_t\right] \; & = \; \mathbb{E}\left[\mathrm{e}^{\sigma \operatorname{Max}_{t\in[0,T]}(B_t-\sigma t/2)}\right] \\
& = \; S_0\,\mathbb{E}\left[\mathrm{e}^{-(-\sigma)\operatorname{Max}_{t\in[0,T]}(B_t-(-\sigma)t/2)}\right] \\
& = \; 2S_0(1+\sigma^2 T/4)\Phi(\sigma\sqrt{T}/2)+S_0\sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2 T/8}.
\end{aligned}
$$

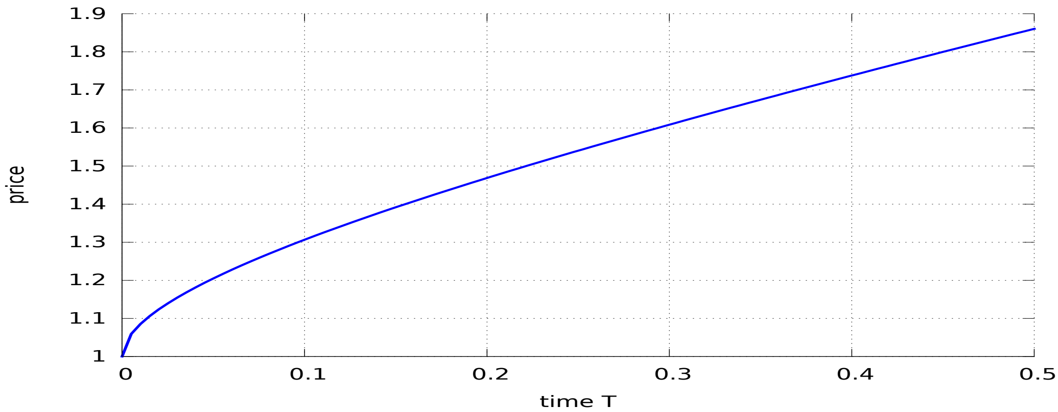


Figure S.34: Expected maximum of geometric Brownian motion over $[0,T]$.

b) We have

$$\mathbb{E}\left[\left(S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma B_t-\sigma^2t/2}-K\right)^+\right] = \mathbb{E}\left[S_0 \operatorname*{Max}_{t\in[0,T]} \mathrm{e}^{\sigma B_t-\sigma^2t/2}\right]-K$$

$$= 2S_0\left(1+\frac{\sigma^2T}{4}\right)\Phi\left(\sigma\frac{\sqrt{T}}{2}\right)+S_0\sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2T/8}-K.$$

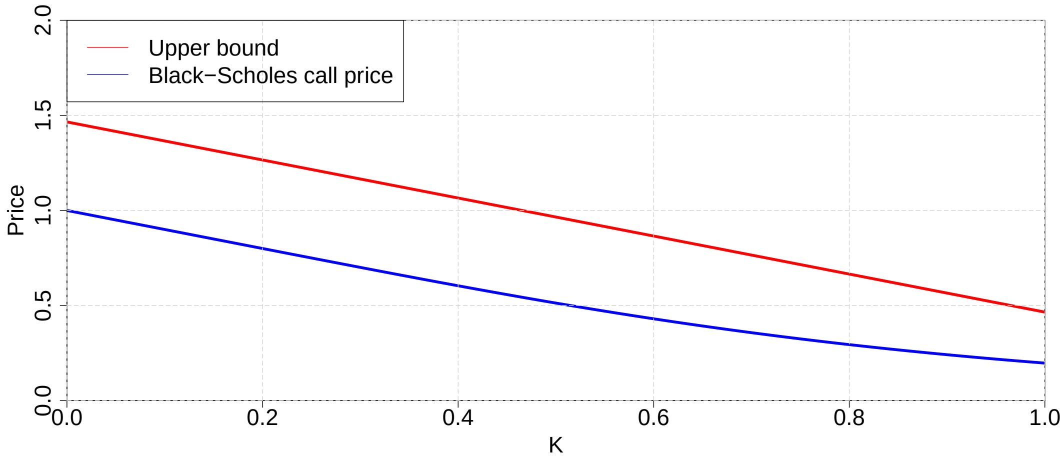


Figure S.35: Black-Scholes call price upper bound with $S_0 = 1$.

The derivative with respect to time is given by

$$\frac{\partial}{\partial T}\mathbb{E}\left[\operatorname*{Max}_{t\in[0,T]} S_t\right] = \frac{S_0\sigma^2}{2}\Phi\left(\sigma\frac{\sqrt{T}}{2}\right)+\frac{S_0\sigma}{\sqrt{2\pi T}}\mathrm{e}^{-\sigma^2T/8}\left(1+\frac{3\sigma^2T}{4}\right).$$

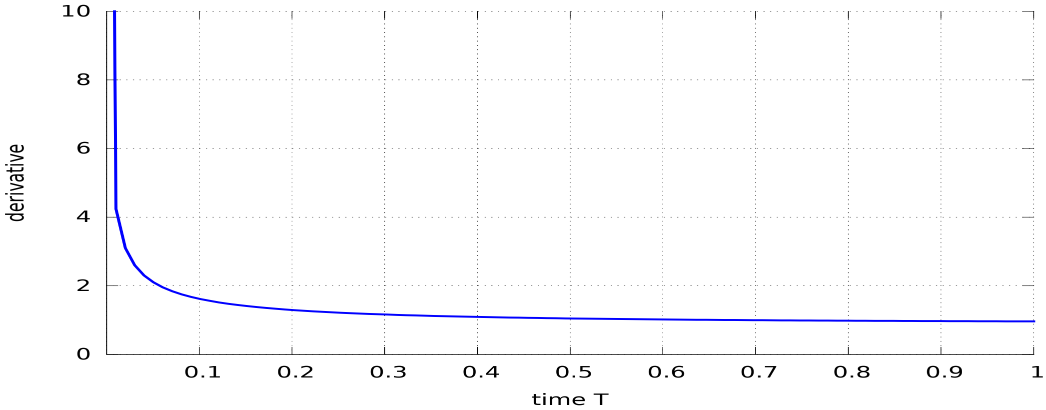


Figure S.36: Time derivative of the expected maximum of geometric Brownian motion.

Note that when $r > 0$ we have

$$\begin{aligned}\mathbb{E}^*\left[M_0^T \mid \mathscr{F}_t\right] &= M_0^t\Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_t\,\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)\\ &\quad -S_t\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).\end{aligned}$$

When r tends to 0, this maximum tends to

$$\begin{aligned}& M_0^t\Phi\left(-\frac{\log(S_t/M_0^t)-\sigma^2T/2}{\sigma\sqrt{T}}\right) + S_t\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\ &\quad +\sigma^2 S_t\lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)-\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right),\end{aligned}$$

where

$$\begin{aligned}&\lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)-\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right)\\
&= \lim_{r\to 0}\frac{1}{2r}\left((1+(T-t)r)\Phi\left(\frac{\log\left(\frac{S_t}{M_0^t}\right)+\frac{\sigma^2}{2}T+rT}{\sigma\sqrt{T}}\right)\right.\\
&\qquad \left.-\left(1+\frac{2r}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2-rT}{\sigma\sqrt{T}}\right)\right)\\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\qquad +\lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\left(\int_{-\infty}^{(\log(S_t/M_0^t)+\sigma^2T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\right.\\
&\qquad \left.-\int_{-\infty}^{(\log(S_t/M_0^t)+\sigma^2T/2-rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\right)\\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\qquad +\lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\int_{(\log(S_t/M_0^t)+\sigma^2T/2-rT)/(\sigma\sqrt{T})}^{(\log(S_t/M_0^t)+\sigma^2T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\end{aligned}$$

$$+\frac{\sqrt{T}}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-((\log(S_t/M_0^t)+\sigma^2T/2)/(\sigma\sqrt{T}))^2/2},$$

hence

$$\begin{aligned}\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right] &= M_0^t\Phi\left(-\frac{\log(S_t/M_0^t)-\sigma^2T/2}{\sigma\sqrt{T}}\right)+S_t\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\ &\quad +\frac{S_t}{2}\left((T-t)\sigma^2+2\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\ &\quad +\sigma S_t\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-((\log(S_t/M_0^t)+\sigma^2T/2)/(\sigma\sqrt{T}))^2/2}.\end{aligned}$$

In particular, when T tends to infinity we find that

$$\lim_{T\to\infty}\frac{\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right]}{\mathbb{E}^*[S_T \mid \mathcal{F}_t]}=\begin{cases}1+\dfrac{\sigma^2}{2r} & \text{if } r>0,\\[2ex] +\infty & \text{if } r=0.\end{cases}$$

When $t=0$ we have $S_0=M_0^0$, and we recover

$$\mathbb{E}^*\left[M_0^T\right]=2\left(S_0+\frac{\sigma^2T}{4}\right)\Phi\left(\sigma\frac{\sqrt{T}}{2}\right)+\sigma S_0\sqrt{\frac{T}{2\pi}}\,\mathrm{e}^{-\sigma^2T/8}.$$

Exercise 9.3

a) We have

$$P\left(\min_{t\in[0,T]}B_t\leqslant a\right)=2\int_{-\infty}^{a}\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}},\qquad a<0,$$

i.e. the probability density function φ of $\operatorname*{Sup}_{t\in[0,T]}B_t$ is given by

$$\varphi(a)=\sqrt{\frac{2}{\pi T}}\,\mathrm{e}^{-a^2/(2T)}\mathbb{1}_{(-\infty,0]}(a),\qquad a\in\mathbb{R}.$$

b) We have

$$\begin{aligned}\mathbb{E}\left[\min_{t\in[0,T]}S_t\right]&=S_0\,\mathbb{E}\left[\exp\left(\sigma\min_{t\in[0,T]}B_t\right)\right]\\ &=\frac{2S_0}{\sqrt{2\pi T}}\int_{-\infty}^{0}\mathrm{e}^{\sigma x-x^2/(2T)}dx=\frac{2S_0}{\sqrt{2\pi\sigma^2T}}\int_{-\infty}^{0}\mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2T/2}dx\\ &=\frac{2S_0}{\sqrt{2\pi T}}\,\mathrm{e}^{\sigma^2T/2}\int_{-\infty}^{-\sigma T}\mathrm{e}^{-x^2/(2T)}dx=\frac{2S_0}{\sqrt{2\pi}}\,\mathrm{e}^{\sigma^2T/2}\int_{-\infty}^{-\sigma\sqrt{T}}\mathrm{e}^{-x^2/2}dx\\ &=2S_0\,\mathrm{e}^{\sigma^2T/2}\Phi(-\sigma\sqrt{T})=2\,\mathbb{E}[S_T](1-\Phi(\sigma\sqrt{T})),\end{aligned}$$

hence

$$\begin{aligned}\mathbb{E}\left[S_T-\min_{t\in[0,T]}S_t\right]&=\mathbb{E}[S_T]-\mathbb{E}\left[\min_{t\in[0,T]}S_t\right]=\mathbb{E}[S_T]-2\,\mathbb{E}[S_T]\left(1-\Phi\left(\sigma\sqrt{T}\right)\right)\\ &=\mathbb{E}[S_T]\left(2\Phi\left(\sigma\sqrt{T}\right)-1\right)=2S_0\,\mathrm{e}^{\sigma^2T/2}\left(\Phi\left(\sigma\sqrt{T}\right)-\frac{1}{2}\right),\end{aligned}$$

and

$$\mathrm{e}^{-\sigma^2T/2}\,\mathbb{E}\left[S_T-\min_{t\in[0,T]}S_t\right]=S_0\left(2\Phi\left(\sigma\sqrt{T}\right)-1\right)=S_0\left(1-2\Phi\left(-\sigma\sqrt{T}\right)\right).$$

Remark: We note that the price of the lookback option converges to S_0 as T goes to infinity.

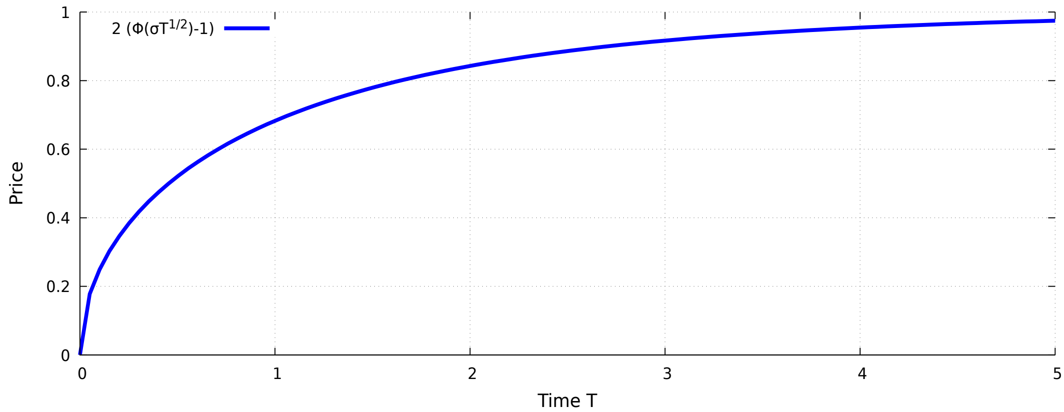


Figure S.37: Lookback call option price as a function of T with $S_0 = 1$.

Exercise 9.4 We have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-r\tau}\mathbb{1}_{\{\tau\leqslant T\}}\mathbb{1}_{\{M_0^\tau - S_\tau \geqslant K\}}\right]\\
&= \int_1^T\int_0^\infty\int_{K+x}^\infty \mathrm{e}^{-rt} f_{(\tau,S_\tau,M_\tau)}(t,x,y)dydxdt\\
&= \int_1^T\int_K^\infty\int_0^{y-K} \mathrm{e}^{-rt} f_{(\tau,S_\tau,M_\tau)}(t,x,y)dxdydt
\end{aligned}$$

for $T \geqslant 1$, and $\mathbb{E}^*\left[\mathrm{e}^{-r\tau}\mathbb{1}_{\{\tau\leqslant T\}}\mathbb{1}_{\{M_0^\tau - S_\tau \geqslant K\}}\right] = 0$ if $T \in [0,1]$.

Exercise 9.5

a) i) The boundary condition (9.2.3a) is explained by the fact that

$$\begin{aligned}
f(t,0,y) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_t = 0,\ M_0^t = y\right]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^t - S_T \,\middle|\, S_t = 0,\ M_0^t = y\right]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^t \,\middle|\, M_0^t = y\right] - \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid S_t = 0]\\
&= y\,\mathrm{e}^{-(T-t)r},
\end{aligned}$$

since $\mathbb{E}^*[S_T \mid S_t = 0] = 0$ as $S_t = 0$ implies $S_T = 0$ from the relation

$$S_T = S_t\,\mathrm{e}^{\sigma(B_T - B_t) + (\mu - \sigma^2/2)(T-t)}, \qquad 0 \leqslant t \leqslant T.$$

ii) The boundary condition (9.2.3b), *i.e.*

$$\frac{\partial f}{\partial y}(t,x,y)_{y=x} = 0, \qquad 0 \leqslant x \leqslant y,$$

is illustrated in the following Figure S.38, see also Figure 9.3.

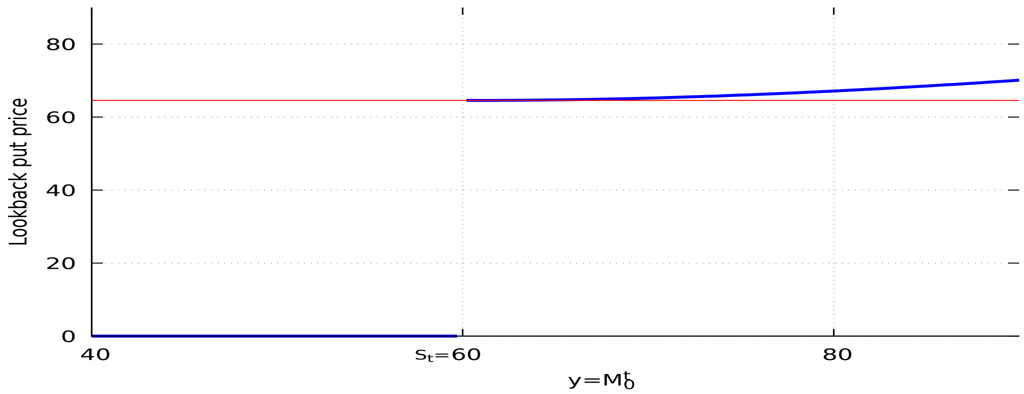


Figure S.38: Graph of the lookback put option price (2D) with $S_t = 60$.

iii) Condition (9.2.3c) follows from the fact that

$$f(T,x,y) = \mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_T = x,\ M_0^T = y\right] = y - x.$$

b) i) The boundary condition (9.3.1a) is explained by the fact that

$$\begin{aligned} f(t,x,0) &= e^{-(T-t)r}\,\mathbb{E}^*\left[S_T - m_0^T \,\middle|\, S_t = x,\ m_0^t = 0\right] \\ &= e^{-(T-t)r}\,\mathbb{E}^*\left[S_T \,\middle|\, S_t = x,\ m_0^t = 0\right] \\ &= e^{-(T-t)r}\,\mathbb{E}^*[S_T \mid S_t = x] \\ &= e^{-(T-t)r}x, \qquad x > 0. \end{aligned}$$

ii) Condition (9.3.1b) follows from the fact that

$$f(T,x,y) = \mathbb{E}^*\left[S_T - m_0^T \,\middle|\, S_T = x,\ m_0^T = y\right] = x - y.$$

We have

$$f(t,x,x) = xC(T-t),$$

with

$$\begin{aligned} C(\tau) &= 1 - e^{-r\tau}\Phi\big(\delta_-^\tau(1)\big) \\ &\quad - \left(1 + \frac{\sigma^2}{2r}\right)\Phi\big(-\delta_+^\tau(1)\big) + e^{-r\tau}\frac{\sigma^2}{2r}\Phi\big(\delta_-^\tau(1)\big), \quad \tau > 0, \end{aligned}$$

hence

$$\frac{\partial f}{\partial x}(t,x,x) = C(T-t), \qquad 0 \leqslant t \leqslant T,$$

while we also have

$$\frac{\partial f}{\partial y}(t,x,y)_{y=x} = 0, \qquad 0 \leqslant x \leqslant y,$$

see also Figure 9.8.

Chapter 10

Exercise 10.1 We have

$$\begin{aligned} \mathbb{E}\left[\int_\tau^T S_t dt\right] &= \int_\tau^T \mathbb{E}[S_t]dt \\ &= S_0 \int_\tau^T \mathbb{E}[e^{\sigma B_t + rt - \sigma^2 t/2}]dt \end{aligned}$$

$$
\begin{aligned}
&= S_0 \int_\tau^T \mathrm{e}^{rt-\sigma^2 t/2} \mathbb{E}[\mathrm{e}^{\sigma B_t}] dt \\
&= S_0 \int_\tau^T \mathrm{e}^{rt} dt \\
&= S_0 \frac{\mathrm{e}^{rT} - \mathrm{e}^{r\tau}}{r}, \qquad 0 \leqslant \tau \leqslant T,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_\tau^T S_t dt\right)^2\right] &= \mathbb{E}\left[\int_\tau^T S_t dt \int_\tau^T S_u du\right] \\
&= \mathbb{E}\left[\int_\tau^T \int_\tau^T S_u S_t dt du\right] \\
&= 2\int_\tau^T \int_\tau^u \mathbb{E}[S_u S_t] dt du \\
&= 2S_0^2 \int_\tau^T \int_\tau^u \mathbb{E}[\mathrm{e}^{\sigma B_u + ru - \sigma^2 u/2} \mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}] dt du \\
&= 2S_0^2 \int_\tau^T \int_\tau^u \mathrm{e}^{ru - \sigma^2 u/2 + rt - \sigma^2 t/2} \mathbb{E}[\mathrm{e}^{\sigma B_u + \sigma B_t}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t} \mathbb{E}[\mathrm{e}^{\sigma B_u + \sigma B_t}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t} \mathbb{E}[\mathrm{e}^{2\sigma B_t + \sigma(B_u - B_t)}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t} \mathbb{E}[\mathrm{e}^{2\sigma B_t}] \mathbb{E}[\mathrm{e}^{\sigma(B_u - B_t)}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t} \mathrm{e}^{2\sigma^2 t} \mathrm{e}^{\sigma^2(u-t)/2} dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{ru} \int_\tau^u \mathrm{e}^{rt+\sigma^2 t} dt du \\
&= \frac{2S_0^2}{\sigma^2 + r} \int_\tau^T \left(\mathrm{e}^{(2r+\sigma^2)u} - \mathrm{e}^{ru} \mathrm{e}^{(r+\sigma^2)\tau}\right) du \\
&= 2S_0^2 \frac{r\,\mathrm{e}^{(\sigma^2+2r)T} - (\sigma^2 + 2r)\,\mathrm{e}^{rT + (\sigma^2+r)\tau} + (\sigma^2 + r)\,\mathrm{e}^{(\sigma^2+2r)\tau}}{(\sigma^2 + r)(\sigma^2 + 2r) r}, \quad 0 \leqslant \tau \leqslant T.
\end{aligned}
$$

Exercise 10.2

a) The integral $\int_0^T r_s ds$ has a centered Gaussian distribution with variance

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^T r_s ds\right)^2\right] &= \sigma^2 \mathbb{E}\left[\int_0^T \int_0^T B_s B_t ds dt\right] \\
&= \sigma^2 \int_0^T \int_0^T \mathbb{E}[B_s B_t] ds dt \\
&= \sigma^2 \int_0^T \int_0^T \min(s,t) ds dt \\
&= 2\sigma^2 \int_0^T \int_0^t s ds dt \\
&= \sigma^2 \int_0^T t^2 dt \\
&= T^3 \frac{\sigma^2}{3}.
\end{aligned}
$$

b) Since the integral $\int_0^T r_s ds$ is a random variable with probability density

$$
\varphi(x) = \frac{1}{\sqrt{2\pi T^3/3}} \mathrm{e}^{-3x^2/(2\pi T^3)},
$$

we have

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}\left[\left(\int_0^T r_u du-\kappa\right)^+\right] &= \mathrm{e}^{-rT}\int_{-\infty}^{\infty}(x-\kappa)^+\varphi(x)dx \\
&= \frac{\mathrm{e}^{-rT}}{\sqrt{2\pi\sigma^2T^3/3}}\int_{\kappa}^{\infty}(x-\kappa)\,\mathrm{e}^{-3x^2/(2\sigma^2T^3)}dx \\
&= \frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}\int_{\kappa/\sqrt{\sigma^2T^3/3}}^{\infty}(x\sqrt{\sigma^2T^3/3}-\kappa)\,\mathrm{e}^{-x^2/2}dx \\
&= \frac{\mathrm{e}^{-rT}\sqrt{\sigma^2T^3/3}}{\sqrt{2\pi}}\int_{\kappa/\sqrt{\sigma^2T^3/3}}^{\infty}x\,\mathrm{e}^{-x^2/2}dx-\kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}\int_{\kappa/\sqrt{\sigma^2T^3/3}}^{\infty}\mathrm{e}^{-x^2/2}dx \\
&= -\frac{\mathrm{e}^{-rT}\sqrt{\sigma^2T^3/3}}{\sqrt{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_{\kappa/\sqrt{\sigma^2T^3/3}}^{\infty}-\kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}(1-\Phi(\kappa/\sqrt{\sigma^2T^3/3})) \\
&= \frac{\mathrm{e}^{-rT}\sqrt{\sigma^2T^3/3}}{\sqrt{2\pi}}\,\mathrm{e}^{-3\kappa^2/(2\sigma^2T^3)}-\kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}(1-\Phi(\kappa/\sqrt{\sigma^2T^3/3})) \\
&= \mathrm{e}^{-rT}\sqrt{\frac{\sigma^2T^3}{6\pi}}\,\mathrm{e}^{-3\kappa^2/(2\sigma^2T^3)}-\kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}\Phi\left(-\kappa\sqrt{\frac{3}{\sigma^2T^3}}\right).
\end{aligned}$$

Exercise 10.3 We have

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u du-\kappa\right)^+ \,\Big|\, \mathscr{F}_t\right] &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du-\kappa \,\Big|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du \,\Big|\, \mathscr{F}_t\right]-\kappa\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\,\mathbb{E}\left[\int_0^t S_u du \,\Big|\, \mathscr{F}_t\right]+\mathrm{e}^{-(T-t)r}\frac{1}{T}\,\mathbb{E}\left[\int_t^T S_u du \,\Big|\, \mathscr{F}_t\right]-\kappa\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du+\mathrm{e}^{-(T-t)r}\frac{1}{T}\,\mathbb{E}\left[\int_t^T S_u du \,\Big|\, \mathscr{F}_t\right]-\kappa\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du+\mathrm{e}^{-(T-t)r}\frac{1}{T}\int_t^T \mathbb{E}[S_u \mid \mathscr{F}_t]du-\kappa\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du+\mathrm{e}^{-(T-t)r}\frac{1}{T}\int_t^T S_t\,\mathrm{e}^{(u-t)r}du-\kappa\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du+\mathrm{e}^{-(T-t)r}\frac{S_t}{T}\int_0^{T-t}\mathrm{e}^{ru}du-\kappa\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du+\mathrm{e}^{-(T-t)r}\frac{S_t}{rT}(\mathrm{e}^{(T-t)r}-1)-\kappa\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du+S_t\frac{1-\mathrm{e}^{-(T-t)r}}{rT}-\kappa\,\mathrm{e}^{-(T-t)r},
\end{aligned}$$

$t\in[0,T]$, cf. Geman and Yor, 1993 page 361. We check that the function $f(t,x,y)=\mathrm{e}^{-(T-t)r}(y/T-\kappa)+x(1-\mathrm{e}^{-(T-t)r})/(rT)$ satisfies the PDE

$$rf(t,x,y)=\frac{\partial f}{\partial t}(t,x,y)+x\frac{\partial f}{\partial y}(t,x,y)+rx\frac{\partial f}{\partial x}(t,x,y)+\frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y),$$

$t,x>0$, and the boundary conditions $f(t,0,y)=\mathrm{e}^{-(T-t)r}(y/T-\kappa)$, $0\leqslant t\leqslant T$, $y\in\mathbb{R}_+$, and $f(T,x,y)=y/T-\kappa$, $x,y\in\mathbb{R}_+$. However, the condition $\lim_{y\to-\infty}f(t,x,y)=0$ is not satisfied because we need to take $y>0$ in the above calculation.

Exercise 10.4

a) We have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du - K \,\Big|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du \,\Big|\, \mathscr{F}_t\right] - K\,\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\mathbb{E}^*\left[\int_0^t S_u du \,\Big|\, \mathscr{F}_t\right] + \frac{\mathrm{e}^{-(T-t)r}}{T}\mathbb{E}^*\left[\int_t^T S_u du \,\Big|\, \mathscr{F}_t\right] - K\,\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + \frac{\mathrm{e}^{-(T-t)r}}{T}\int_t^T \mathbb{E}^*[S_u \mid \mathscr{F}_t] du - K\,\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + \frac{\mathrm{e}^{-(T-t)r}}{T}\int_t^T \mathrm{e}^{(u-t)r} S_t du - K\,\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + S_t\frac{\mathrm{e}^{-rT}}{T}\int_t^T \mathrm{e}^{ru} du - K\,\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + \frac{S_t}{rT}\big(1-\mathrm{e}^{-(T-t)r}\big) - K\,\mathrm{e}^{-(T-t)r}.
\end{aligned}$$

b) Using the relation

$$(x-K)^+ - (K-x)^+ = x-K, \qquad K,x\in\mathbb{R},$$

we have

$$\begin{aligned}
C(t,K)-P(t,K) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\Bigg|\, \mathscr{F}_t\right] \\
&\quad - \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(K-\frac{1}{T}\int_0^T S_u du\right)^+ \,\Bigg|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ - \left(K-\frac{1}{T}\int_0^T S_u du\right)^+ \,\Bigg|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du - K \,\Big|\, \mathscr{F}_t\right] \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + \frac{S_t}{rT}\big(1-\mathrm{e}^{-(T-t)r}\big) - K\,\mathrm{e}^{-(T-t)r}. \qquad \text{(A.13)}
\end{aligned}$$

c) Any self-financing portfolio strategy $(\xi_t)_{t\in\mathbb{R}_+}$ with price process $(V_t)_{t\in\mathbb{R}_+}$ has to satisfy the equation

$$\begin{aligned}
dV_t &= \eta_t dA_t + \xi_t dS_t \\
&= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\
&= rV_t dt + (\mu-r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \qquad t\in\mathbb{R}_+.
\end{aligned}$$

On the other hand, by part (a)) we have

$$\begin{aligned}
dV_t &= d\left(\frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_s ds + \frac{S_t}{rT}(1-\mathrm{e}^{-(T-t)r}) - K\,\mathrm{e}^{-(T-t)r}\right) \\
&= \frac{r}{T}\mathrm{e}^{-(T-t)r}\int_0^t S_s ds dt + \frac{\mathrm{e}^{-(T-t)r}}{T} S_t dt - \frac{S_t}{T}\mathrm{e}^{-(T-t)r} dt \\
&\quad + \frac{1-\mathrm{e}^{-(T-t)r}}{rT} dS_t - rK\,\mathrm{e}^{-(T-t)r} dt \\
&= \frac{r}{T}\mathrm{e}^{-(T-t)r}\int_0^t S_s ds dt + \frac{1-\mathrm{e}^{-(T-t)r}}{rT} dS_t - rK\,\mathrm{e}^{-(T-t)r} dt \\
&= rV_t dt + \frac{1-\mathrm{e}^{-(T-t)r}}{rT} dS_t - S_t(1-\mathrm{e}^{-(T-t)r}) dt
\end{aligned}$$

$$= rV_t dt + \frac{1-\mathrm{e}^{-(T-t)r}}{rT}((\mu - r)S_t dt + \sigma S_t dB_t),$$

hence

$$\xi_t = \frac{1-\mathrm{e}^{-(T-t)r}}{rT}, \qquad t \in [0,T],$$

which can be recovered by differentiating the pricing function

$$\frac{\mathrm{e}^{-(T-t)r}}{T} y + \frac{x}{rT}\left(1-\mathrm{e}^{-(T-t)r}\right) - K\,\mathrm{e}^{-(T-t)r}$$

in (A.13) with respect to $x = S_t$, with $y = \int_0^t S_u du$. We also have

$$V_t = \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_s ds + \xi_t S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad t \in [0,T].$$

d) The following code yields \$7.906436 for the price of the long forward contract.

```
T=1;t=63/252;r=0.0209;K=80;dt=1/252;S=as.numeric(last(futures));
exp(-(T-t)*r)*sum(futures)*dt/T+S*(1-exp(-(T-t)*r))/(r*T)-K*exp(-(T-t)*r)
```

Exercise 10.5 The geometric mean price G satisfies

$$\begin{aligned}
G &= \exp\left(\frac{1}{T}\int_0^T \log S_u du\right) = \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{1}{T}\int_t^T \log S_u du\right) \\
&= \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t + \frac{1}{T}\int_t^T \log\frac{S_u}{S_t} du\right) \\
&= \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t\right. \\
&\quad \left. + \frac{1}{T}\int_t^T (r(u-t) + (B_u - B_t)\sigma - (u-t)\sigma^2/2)du\right) \\
&= \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t\right. \\
&\quad \left. + \frac{1}{T}\int_0^{T-t} (ru - \sigma^2 u/2)du + \frac{\sigma}{T}\int_t^T (B_u - B_t)du\right) \\
&= (S_t)^{(T-t)/T}\exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{(T-t)^2}{2T}(r-\sigma^2/2) + \frac{\sigma}{T}\int_t^T (B_u - B_t)du\right)
\end{aligned}$$

where $\int_t^T B_u du$ is centered Gaussian with conditional variance

$$\begin{aligned}
\mathbb{E}\left[\left(\int_t^T B_u du\right)^2 \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\left(\int_t^T (B_u - B_t)du\right)^2 \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\left(\int_t^T (B_u - B_t)du\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^{T-t} (B_u - B_t)du\right)^2\right] = \int_0^{T-t}\int_0^{T-t}\mathbb{E}[B_s B_u]\,dsdu \\
&= 2\int_0^{T-t}\int_0^u sdsdu = \int_0^{T-t} u^2 du = \frac{(T-t)^3}{3}.
\end{aligned}$$

Hence, letting

$$m := \frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t + \frac{(T-t)^2}{2T}(r-\sigma^2/2), \quad X := \frac{\sigma}{T}\int_t^T B_u du,$$

and $v^2 = (T-t)\sigma^2/3$, we find

$$\begin{aligned}
& e^{-(T-t)r}\mathbf{E}^*\left[\left(\exp\left(\frac{1}{T}\int_0^T \log S_u du\right) - K\right)^+ \Big| \mathscr{F}_t\right] \\
= \ & (S_t)^{(T-t)/T} e^{-(T-t)r}\exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{(T-t)^2}{4T}(2r-\sigma^2) + \frac{\sigma^2}{6}(T-t)\right) \\
& \times\Phi\left(\frac{(T-t)\sigma^2/3 + \frac{1}{T}\int_0^t \log S_u du + \log\frac{S_t^{(T-t)/T}}{K} + \frac{(T-t)^2}{2T}(r-\sigma^2/2)}{\sigma\sqrt{(T-t)/3}}\right) \\
& -K e^{-(T-t)r}\Phi\left(\frac{\frac{1}{T}\int_0^t \log S_u du + \log\frac{S_t^{(T-t)/T}}{K} + \frac{(T-t)^2}{2T}(r-\sigma^2/2)}{\sigma\sqrt{(T-t)/3}}\right),
\end{aligned}$$

$0 \leqslant t \leqslant T$. In case $t = 0$, we get

$$\begin{aligned}
& e^{-rT}\mathbf{E}^*\left[\left(\exp\left(\frac{1}{T}\int_0^T \log S_u du\right) - K\right)^+\right] \\
= \ & S_0 e^{-T(r+\sigma^2/6)/2}\Phi\left(\frac{\log(S_0/K) + T(r+\sigma^2/6)/2}{\sigma\sqrt{T/3}}\right) \\
& -K e^{-rT}\Phi\left(\frac{\log(S_0/K) + T(r-\sigma^2/2)/2}{\sigma\sqrt{T/3}}\right).
\end{aligned}$$

Exercise 10.6 Under the above condition we have, taking $t \in [\tau, T]$,

$$\begin{aligned}
& e^{-(T-t)r}\mathbf{E}^*\left[\left(\frac{1}{T-\tau}\int_\tau^T r_s ds - K\right)^+ \Big| \mathscr{F}_t\right] \\
= \ & e^{-(T-t)r}\mathbf{E}^*\left[\left(\Lambda_t + \frac{1}{T-\tau}\int_t^T r_s ds - K\right)^+ \Big| \mathscr{F}_t\right] \\
= \ & e^{-(T-t)r}\mathbf{E}^*\left[\Lambda_t + \frac{1}{T-\tau}\int_t^T r_s ds - K \Big| \mathscr{F}_t\right] \\
= \ & e^{-(T-t)r}(\Lambda_t - K) + \frac{e^{-(T-t)r}}{T-\tau}\mathbf{E}^*\left[\int_t^T r_s ds \Big| \mathscr{F}_t\right] \\
= \ & e^{-(T-t)r}(\Lambda_t - K) + \frac{e^{-(T-t)r}}{T-\tau}\int_t^T \mathbf{E}^*[r_s \mid \mathscr{F}_t] ds, \qquad t \in [\tau, T],
\end{aligned}$$

where

$$\mathbf{E}^*[r_s \mid \mathscr{F}_t] = v_t e^{-(s-t)\lambda} + m(1 - e^{-(s-t)\lambda}), \qquad 0 \leqslant s \leqslant t,$$

hence

$$\begin{aligned}
& \mathbf{E}^*\left[\left(\frac{1}{T-\tau}\int_\tau^T r_s ds - K\right)^+ \Big| \mathscr{F}_t\right] = \mathbf{E}^*\left[\left(\Lambda_t + \frac{1}{T-\tau}\int_t^T r_s ds - K\right)^+ \Big| \mathscr{F}_t\right] \\
= \ & \Lambda_t - K + \frac{1}{T-\tau}\int_t^T \mathbf{E}^*[r_s \mid \mathscr{F}_t] ds
\end{aligned}$$

$$\begin{aligned}
&= \Lambda_t - K + \frac{1}{T-\tau}\int_t^T \left(r_t\, \mathrm{e}^{-(s-t)\lambda} + m(1-\mathrm{e}^{-(s-t)\lambda})\right) ds \\
&= \Lambda_t - K + \frac{1}{T-\tau}(r_t - m)\int_0^{T-t} \mathrm{e}^{-\lambda s} ds + m(T-t)\frac{\mathrm{e}^{-(T-t)r}}{T-\tau} \\
&= \Lambda_t - K + (r_t - m)\frac{1}{T-\tau}\int_0^{T-t} \mathrm{e}^{-\lambda s} ds + m\frac{T-t}{T-\tau} \\
&= \Lambda_t - K + \frac{1-\mathrm{e}^{-(T-t)\lambda}}{(T-\tau)\lambda}(r_t - m) + m\frac{T-t}{T-\tau}.
\end{aligned}$$

Exercise 10.7 If $(S_t)_{t\in\mathbb{R}_+}$ is a martingale then for any convex payoff function ϕ we can write

$$\begin{aligned}
&\mathbf{E}^*\left[\phi\left(\frac{S_{T_1}+\cdots+S_{T_n}}{n}\right)\right] \leqslant \mathbf{E}^*\left[\frac{\phi(S_{T_1})+\cdots+\phi(S_{T_n})}{n}\right] && \text{since } \phi \text{ is convex,} \\
&= \frac{\mathbf{E}^*[\phi(S_{T_1})]+\cdots+\mathbf{E}^*[\phi(S_{T_n})]}{n} \\
&= \frac{\mathbf{E}^*[\phi(\mathbf{E}^*[S_{T_n} \mid \mathcal{F}_{T_1}])]+\cdots+\mathbf{E}^*[\phi(\mathbf{E}^*[S_{T_n} \mid \mathcal{F}_{T_n}])]}{n} && \text{because } (S_t)_{t\in\mathbb{R}_+} \text{ is a martingale,} \\
&\leqslant \frac{\mathbf{E}^*[\mathbf{E}^*[\phi(S_{T_n}) \mid \mathcal{F}_{T_1}]]+\cdots+\mathbf{E}^*[\mathbf{E}^*[\phi(S_{T_n}) \mid \mathcal{F}_{T_n}]]}{n} && \text{by Jensen's inequality,} \\
&= \frac{\mathbf{E}^*[\phi(S_{T_n})]+\cdots+\mathbf{E}^*[\phi(S_{T_n})]}{n} && \text{by the tower property,} \\
&= \mathbf{E}^*[\phi(S_{T_n})].
\end{aligned}$$

On the other hand, if $(S_t)_{t\in\mathbb{R}_+}$ is only a *sub*martingale then the above argument still applies to a convex non-decreasing payoff function ϕ such as $\phi(x) = (x-K)^+$.

Exercise 10.8 Taking $t \in [\tau, T]$, under the condition

$$\Lambda_t := \frac{1}{T-\tau}\int_\tau^t S_s ds \geqslant K,$$

we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbf{E}^*\left[\left(\frac{1}{T-\tau}\int_\tau^T S_s ds - K\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbf{E}^*\left[\left(\Lambda_t + \frac{1}{T-\tau}\int_t^T S_s ds - K\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbf{E}^*\left[\Lambda_t + \frac{1}{T-\tau}\int_t^T S_s ds - K \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + \frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\mathbf{E}^*\left[\int_t^T S_s ds \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + \frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\int_t^T \mathbf{E}^*[S_s \mid \mathcal{F}_t] ds \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t\frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\int_t^T \mathrm{e}^{(s-t)r} ds \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t\frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\int_0^{T-t} \mathrm{e}^{rs} ds \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t\frac{\mathrm{e}^{-(T-t)r}}{(T-\tau)r}(\mathrm{e}^{(T-t)r} - 1)
\end{aligned}$$

$$= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t \frac{1-\mathrm{e}^{-(T-t)r}}{(T-\tau)r}, \qquad t \in [\tau, T].$$

Exercise 10.9 The Asian option price can be written as

$$\begin{aligned} \mathrm{e}^{-r(T-t)} \mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, \mathscr{F}_t\right] &= S_t \widehat{\mathbb{E}}\left[(U_T)^+ \mid U_t\right] \\ &= S_t h(t, U_t) = S_t g(t, Z_t), \end{aligned}$$

which shows that

$$g(t, Z_t) = h(t, U_t),$$

and it remains to use the relation

$$U_t = \frac{1-\mathrm{e}^{-(T-t)r}}{rT} + \mathrm{e}^{-(T-t)r} Z_t, \qquad t \in [0, T].$$

Exercise 10.10

i) By change of variable. We note that $\widetilde{Z}_t = \mathrm{e}^{-(T-t)r} Z_t$, where

$$Z_t := \frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right) = \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right), \qquad 0 \leqslant t \leqslant T,$$

and the pricing function $g(t, Z_t)$ satisfies the Rogers-Shi PDE

$$\frac{\partial g}{\partial t}(t,z) + \left(\frac{1}{T} - rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) = 0.$$

Letting $\widetilde{z} := \mathrm{e}^{-(T-t)r} z$ and $\widetilde{g}(t, \widetilde{z}) := g\big(t, \mathrm{e}^{(T-t)r}\widetilde{z}\big) = g(t,z) = \widetilde{g}\big(t, \mathrm{e}^{-(T-t)r} z\big)$, we note that

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,z) &= \dfrac{\partial}{\partial t}\widetilde{g}\big(t, \mathrm{e}^{-(T-t)r} z\big) \\ &= \dfrac{\partial \widetilde{g}}{\partial t}\big(t, \mathrm{e}^{-(T-t)r} z\big) + r\,\mathrm{e}^{-(T-t)r} z \dfrac{\partial \widetilde{g}}{\partial x}\big(t, \mathrm{e}^{-(T-t)r} z\big) \\ &= \dfrac{\partial \widetilde{g}}{\partial t}(t, \widetilde{z}) + r\widetilde{z}\dfrac{\partial \widetilde{g}}{\partial x}(t, \widetilde{z}), \\ \dfrac{\partial g}{\partial z}(t,z) &= \mathrm{e}^{-(T-t)r}\dfrac{\partial \widetilde{g}}{\partial \widetilde{z}}\big(t, \mathrm{e}^{-(T-t)r} z\big) = \mathrm{e}^{-(T-t)r}\dfrac{\partial \widetilde{g}}{\partial \widetilde{z}}(t, \widetilde{z}), \\ \dfrac{\partial^2 g}{\partial z^2}(t,z) &= \mathrm{e}^{-2(T-t)r}\dfrac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}\big(t, \mathrm{e}^{-(T-t)r} z\big) = \mathrm{e}^{-2(T-t)r}\dfrac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t, \widetilde{z}), \end{cases}$$

hence

$$\begin{aligned} 0 &= \frac{\partial g}{\partial t}(t,z) + \left(\frac{1}{T} - rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) \\ &= \frac{\partial \widetilde{g}}{\partial t}(t, \widetilde{z}) + r\widetilde{z}\frac{\partial \widetilde{g}}{\partial x}(t, \widetilde{z}) + \left(\frac{1}{T} - rz\right)\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial \widetilde{z}}(t, \widetilde{z}) \\ &\quad + \frac{1}{2}\sigma^2 z^2\,\mathrm{e}^{-2(T-t)r}\frac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t, \widetilde{z}) \end{aligned}$$

$$= \frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{z}) + \frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial \widetilde{z}}(t,\widetilde{z}) + \frac{1}{2}\sigma^2\widetilde{z}^2\frac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t,\widetilde{z}),$$

and the (simpler) PDE

$$\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{z}) + \frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial \widetilde{z}}(t,\widetilde{z}) + \frac{1}{2}\sigma^2\widetilde{z}^2\frac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t,\widetilde{z}) = 0.$$

ii) Using the Itô formula. Given that

$$\begin{aligned} d\widetilde{Z}_t &= d(\mathrm{e}^{-(T-t)r}Z_t) \\ &= r\mathrm{e}^{-(T-t)r}Z_t dt + \mathrm{e}^{-(T-t)r}dZ_t \\ &= r\widetilde{Z}_t dt + \mathrm{e}^{-(T-t)r}dZ_t, \end{aligned}$$

and

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

under the risk-neutral probability measure $\mathbb{P}^*$, an application of Itô's formula to the discounted portfolio price leads to

$$\begin{aligned} & d\big(\mathrm{e}^{-rt}S_t\widetilde{g}(t,\widetilde{Z}_t)\big) \\ &= \mathrm{e}^{-rt}\big(-r\widetilde{g}(t,\widetilde{Z}_t)dt + \widetilde{g}(t,\widetilde{Z}_t)dS_t + S_t d\widetilde{g}(t,\widetilde{Z}_t) + dS_t \bullet d\widetilde{g}(t,\widetilde{Z}_t)\big) \\ &= \mathrm{e}^{-rt}\left(-rS_t\widetilde{g}(t,\widetilde{Z}_t)dt + \widetilde{g}(t,\widetilde{Z}_t)dS_t + S_t\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{Z}_t)dt + S_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)d\widetilde{Z}_t\right) \\ &\quad + \frac{1}{2}\mathrm{e}^{-rt}\left(S_t\frac{\partial^2 \widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)\big(d\widetilde{Z}_t\big)^2 + dS_t \bullet d\widetilde{g}(t,\widetilde{Z}_t)\right) \\ &= \mathrm{e}^{-rt}\left(-rS_t\widetilde{g}(t,\widetilde{Z}_t)dt + \widetilde{g}(t,\widetilde{Z}_t)dS_t + S_t\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{Z}_t)dt\right. \\ &\quad \left. + r\widetilde{Z}_t S_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt + S_t\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dZ_t\right) \\ &\quad + \frac{1}{2}\mathrm{e}^{-rt}\left(S_t\frac{\partial^2 \widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)\big(d\widetilde{Z}_t\big) + dS_t \bullet d\widetilde{g}(t,\widetilde{Z}_t)\right) \\ &= \mathrm{e}^{-rt}\left(-rS_t\widetilde{g}(t,\widetilde{Z}_t)dt + rS_t\widetilde{g}(t,\widetilde{Z}_t)dt + \sigma S_t\widetilde{g}(t,\widetilde{Z}_t)dB_t + r\widetilde{Z}_t S_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt\right) \\ &\quad + \mathrm{e}^{-rt}\left(\mathrm{e}^{-(T-t)r}S_t Z_t\left(-r+\sigma^2\right)\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt\right. \\ &\quad \left. + \frac{1}{T}\mathrm{e}^{-(T-t)r}S_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt - \sigma\mathrm{e}^{-(T-t)r}S_t Z_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dB_t\right) \\ &\quad + \mathrm{e}^{-rt}\left(\frac{1}{2}\sigma^2\widetilde{Z}_t^2 S_t\frac{\partial^2 \widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)dt - \sigma^2 S_t\widetilde{Z}_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt\right) \\ &= \mathrm{e}^{-rt}S_t\left(\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{Z}_t) + \frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t) + \frac{1}{2}\sigma^2\widetilde{Z}_t^2\frac{\partial^2 \widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)\right)dt \\ &\quad + S_t\mathrm{e}^{-rt}\left(\sigma\widetilde{g}(t,\widetilde{Z}_t) - \sigma\widetilde{Z}_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)\right)dB_t. \end{aligned}$$

Since the discounted portfolio price process is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, the sum of components in dt should vanish in the above expression, which yields

$$\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{Z}_t) + \frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t) + \frac{1}{2}\sigma^2\widetilde{Z}_t^2\frac{\partial^2 \widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t) = 0,$$

and the PDE

$$\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{z}) + \frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial \widetilde{z}}(t,\widetilde{z}) + \frac{1}{2}\sigma^2\widetilde{z}^2\frac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t,\widetilde{z}) = 0,$$

under the terminal condition $\widetilde{g}(T,\widetilde{z}) = \widetilde{z}^+$, $\widetilde{z} \in \mathbb{R}$.

Exercise 10.11

a) When $\Lambda_t/T \geqslant K$ we have

$$f(t,S_t,\Lambda_t) = \mathrm{e}^{-(T-t)r}\left(\frac{\Lambda_t}{T}-K\right) + S_t\frac{1-\mathrm{e}^{-(T-t)r}}{rT},$$

see Exercise 10.8.

b) When $\Lambda_t/T \geqslant K$ we have

$$\xi_t = \frac{1-\mathrm{e}^{-(T-t)r}}{rT} \quad \text{and} \quad \eta_t A_t = \mathrm{e}^{(T-t)r}\left(\frac{\Lambda_t}{T}-K\right), \qquad 0 \leqslant t \leqslant T.$$

c) At maturity we have $f(T,S_T,\Lambda_T) = (\Lambda_T/T-K)^+$, hence $\xi_T = 0$ and

$$\eta_T A_T = A_T\frac{\mathrm{e}^{-rT}}{A_0}\left(\frac{\Lambda_T}{T}-K\right)\mathbb{1}_{\{\Lambda_T>KT\}} = \left(\frac{\Lambda_T}{T}-K\right)^+.$$

d) By Proposition 10.12 we have

$$\xi_t = \frac{1}{S_t}\left(f(t,S_t,\Lambda_t) - \left(\frac{\Lambda_t}{T}-K\right)\frac{\partial g}{\partial z}\left(t,\frac{1}{S_t}\left(\frac{\Lambda_t}{T}-K\right)\right)\right)$$

where the function $g(t,z)$ satisfies $f(t,x,y) = xg(t,(y/T-K)/x))$ and

$$g(t,z) = z\mathrm{e}^{-(T-t)r} + \frac{1-\mathrm{e}^{-(T-t)r}}{rT}, \qquad z > 0,$$

and solves the PDE

$$\frac{\partial g}{\partial t}(t,z) + \left(\frac{1}{T}-rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2\frac{\partial^2 g}{\partial z^2}(t,z) = 0,$$

under the terminal condition $g(T,z) = z^+$, hence letting

$$h(t,z) := \mathrm{e}^{(T-t)r}\frac{\partial g}{\partial z}(t,z),$$

we have

$$\mathrm{e}^{(T-t)r}\frac{\partial g}{\partial t}(t,z) + \mathrm{e}^{(T-t)r}\left(\frac{1}{T}-rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2\frac{\partial^2 g}{\partial z^2}(t,z) = 0,$$

with $h(t,z) = 1$, $z > 0$, hence

$$\begin{aligned}\mathrm{e}^{(T-t)r}\frac{\partial^2 g}{\partial t\partial z}(t,z) - r\,\mathrm{e}^{(T-t)r}\frac{\partial g}{\partial z}(t,z) + \mathrm{e}^{(T-t)r}\left(\frac{1}{T}-rz\right)\frac{\partial^2 g}{\partial z^2}(t,z)\\ +\sigma^2 z\mathrm{e}^{(T-t)r}\frac{\partial^2 g}{\partial z^2}(t,z) + \frac{1}{2}\mathrm{e}^{(T-t)r}\sigma^2 z^2\frac{\partial^3 g}{\partial z^3}(t,z) = 0,\end{aligned}$$

or

$$\frac{\partial h}{\partial t}(t,z) + \left(\frac{1}{T}+(\sigma^2-r)z\right)\frac{\partial h}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2\frac{\partial^2 h}{\partial z^2}(t,z) = 0,$$

with the terminal condition $h(T,z) = \mathbb{1}_{\{z>0\}}$. On the other hand, we have

$$\eta_t = \frac{1}{A_t}\left(f(t,S_t,\Lambda_t) - \xi_t S_t\right)$$

$$
\begin{aligned}
&= \frac{1}{A_t}\left(\frac{\Lambda_t}{T}-K\right)\frac{\partial g}{\partial z}\left(t,\frac{1}{S_t}\left(\frac{\Lambda_t}{T}-K\right)\right)\\
&= \frac{\mathrm{e}^{-(T-t)r}}{A_t}\left(\frac{\Lambda_t}{T}-K\right)h\left(t,\frac{1}{S_t}\left(\frac{\Lambda_t}{T}-K\right)\right).
\end{aligned}
$$

Exercise 10.12 Asian options with dividends. When reinvesting dividends, the portfolio self-financing condition reads

$$
\begin{aligned}
dV_t &= \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{Trading profit and loss}} + \underbrace{\delta\xi_t S_t dt}_{\text{Dividend payout}}\\
&= r\eta_t A_t dt + \xi_t((\mu-\delta)S_t dt + \sigma S_t dB_t) + \delta\xi_t S_t dt\\
&= r\eta_t A_t dt + \xi_t(\mu S_t dt + \sigma S_t dB_t)\\
&= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \qquad t\in\mathbb{R}_+.
\end{aligned}
$$

On the other hand, by Itô's formula we have

$$
\begin{aligned}
&dg_\delta(t,S_t,\Lambda_t)\\
&= \frac{\partial g_\delta}{\partial t}(t,S_t,\Lambda_t)dt + \frac{\partial g_\delta}{\partial y}(t,S_t,\Lambda_t)d\Lambda_t + (\mu-\delta)S_t\frac{\partial g_\delta}{\partial x}(t,S_t,\Lambda_t)dt\\
&\quad +\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 g_\delta}{\partial x^2}(t,S_t,\Lambda_t)dt + \sigma S_t\frac{\partial g_\delta}{\partial x}(t,S_t,\Lambda_t)dB_t\\
&= \frac{\partial g_\delta}{\partial t}(t,S_t,\Lambda_t)dt + S_t\frac{\partial g_\delta}{\partial y}(t,S_t,\Lambda_t)dt + (\mu-\delta)S_t\frac{\partial g_\delta}{\partial x}(t,S_t,\Lambda_t)dt\\
&\quad +\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 g_\delta}{\partial x^2}(t,S_t,\Lambda_t)dt + \sigma S_t\frac{\partial g_\delta}{\partial x}(t,S_t,\Lambda_t)dB_t,
\end{aligned}
$$

hence by identification of the terms in dB_t and dt in the expressions of dV_t and $dg_\delta(t,S_t)$, we get

$$
\xi_t = \frac{\partial g_\delta}{\partial x}(t,S_t,\Lambda_t),
$$

and we derive the Black-Scholes PDE with dividend

$$
\begin{aligned}
rg_\delta(t,x,y) &= \frac{\partial g_\delta}{\partial t}(t,x,y) + y\frac{\partial g_\delta}{\partial y}(t,x,y)\\
&\quad +(r-\delta)x\frac{\partial g_\delta}{\partial x}(t,x,y) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 g_\delta}{\partial x^2}(t,x,y).
\end{aligned}
\tag{A.14}
$$

Defining $f(t,x,y) := \mathrm{e}^{(T-t)\delta}g_\delta(t,x,y)$ and substituting

$$
g_\delta(t,x,y) = \mathrm{e}^{(T-t)\delta}f(t,x,y)
$$

in (A.14) yields the equation

$$
\begin{aligned}
rf(t,x,y) &= \delta f(t,x,y) + y\frac{\partial f}{\partial y}(t,x,y) + \frac{\partial f}{\partial t}(t,x,y)\\
&\quad +(r-\delta)x\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}(t,x,y),
\end{aligned}
$$

i.e.

$$
(r-\delta)f(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + y\frac{\partial f}{\partial y}(t,x,y)
$$

$$+(r-\delta)x\frac{\partial f}{\partial x}(t,x,y)+\frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}(t,x,y),$$

whose solution $f(t,x,y)$ is the Asian option pricing function with modified interest rate $r-\delta$ and no dividends, under the terminal condition

$$f(T,x,y)=g_\delta(T,x,y)=\left(\frac{y}{T}-K\right)^+.$$

Therefore the Asian option price $g_\delta(t,S_t,\Lambda_t)$ with dividend rate δ can be recovered from the relation

$$g_\delta(t,x,y)=\mathrm{e}^{(T-t)\delta}f(t,x,y), \qquad t\in[0,T],\ x,y>0.$$

Note that we can also define

$$h(t,x,y):=g_\delta\big(t,x\mathrm{e}^{-\delta(T-t)},y\big)$$

and substituting

$$g_\delta(t,x,y)=h\big(t,x\mathrm{e}^{\delta(T-t)},y\big)$$

in (A.14) yields the equation

$$\begin{aligned} rh(t,x,y) &= y\frac{\partial h}{\partial y}(t,x,y)+\frac{\partial h}{\partial t}(t,x,y)\\ &\quad +rx\frac{\partial h}{\partial x}(t,x,y)+\frac{1}{2}\sigma^2x^2\frac{\partial^2 h}{\partial x^2}(t,x,y), \end{aligned}$$

whose solution $h(t,x,y)$ is the Asian option pricing function with interest rate r and no dividends, under the terminal condition

$$h(T,x,y)=g_\delta(T,x,y)=\left(\frac{y}{T}-K\right)^+.$$

Finally, the Asian option price $g_\delta(t,S_t,\Lambda_t)$ with dividend rate δ can be also recovered from the relation

$$g_\delta(t,x,y)=h\big(t,x\mathrm{e}^{-(T-t)\delta},y\big), \qquad t\in[0,T],\ x,y>0.$$

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