Frequency Control and Power Sharing in Combined Heat and Power Networks

Xin Qin and Ioannis Lestas

***Abstract*— We consider the problem of using district heating systems as ancillary services for primary frequency control in power networks. We propose a novel power sharing scheme for heating systems based on the average temperature, which enables an optimal power allocation among the diverse heat sources without having a prior knowledge of the disturbances. We then discuss two approaches for heating systems to contribute to frequency regulation in power networks. We show that both approaches ensure stability in the combined heat and power network and facilitate optimal power allocation among the different energy sources.**

I. Introduction

As decarbonization efforts intensify, the shift towards electrified heating underscores the significance of heat pumps. These high-efficiency devices for converting electricity to heat are gaining popularity with increasing government support. Unlike traditional electric loads, heat pumps can provide rapid response capabilities and interlink two different energy sectors, the electric power system and the heating system. The power system requires real-time power balance whereas the temperature dynamics in a heating system have much higher inertia. Thus, if controlled properly, the heating system can provide important support to the frequency control mechanisms in power systems.

However, integrating heat pumps in district heating systems into power system frequency control is a non-trivial problem where various challenges need to be addressed. The dynamics of primary frequency control raise questions about maintaining stability of the combined heat and power systems amid renewable and load fluctuations. Moreover, achieving an appropriate power distribution among the different electric and heat sources is also challenging when heat pumps contribute to frequency regulation. Additionally, the combined heat and power network has diverse coupled dynamics associated with the heat and power system that are more challenging to address.

A number of studies such as the one in [1] have contributed to the modeling of heat pumps and heating networks, focusing on the design of control strategies specifically tailored to frequency regulation with a single heat pump. Despite these efforts, the broader system-wide stability has not been extensively explored. Various control strategies for frequency regulation via heat pumps include those in [2], [3], [4]. However, these studies do not provide a stability analysis in a general power network and its coupling with heating networks. Recent advancements, such as the work in [5], introduce models that incorporate the combined dynamics of heat and power networks. However, the problem of achieving power sharing in heating systems without having a prior knowledge of the disturbances needs to be addressed. In particular, without such power sharing there is a risk that costly heat sources in a heating network may disproportionately compensate for the heat pump contributions to frequency control, discouraging users from actively participating in frequency regulation efforts.

In this paper, we consider heat pumps that are part of a district heating system and provide an ancillary service to the power grid by contributing to primary frequency regulation. Our contributions can be summarized as follows. We introduce a novel optimal power sharing scheme among the different heat sources in the heating network. This is based on the use of the average temperature as a control signal and achieves optimal power sharing without having a prior knowledge of the disturbances applied. We also present two schemes for heating systems to contribute to frequency regulation in power networks. When these schemes are implemented, it is shown that the stability of the combined heat and power network is maintained when a general network topology is considered. Furthermore, an optimal power sharing can be achieved between diverse electric and heat sources that is quantified within the paper.

The paper is structured as follows. In section II we provide various preliminaries associated with the problem formulation. In section III we describe the combined heat and power network that will be considered. In sections IV and V we present the main results associated with stability and power sharing. Extension associated with general passive power and heat generation dynamics are presented in section VI. Finally conclusions are drawn in section VII.

II. Preliminaries

We use $\mathbf{0}_n$ and $\mathbf{1}_n$ to denote the n-dimensional vector of zeros and ones, respectively. For simplicity in the presentation we will omit the subscript in the text. For a vector $x \in \mathbb{R}^n$, we use $diag(x)$ to denote a diagonal matrix with elements x_i in its main diagonal. For vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ we use the notation $col(x, y)$ to denote the vector $[x^T \ y^T]^T$. For a function $f(x)$, we use $f'(x) = \frac{df(x)}{dx}$ to denote its first-order derivative. We use $f^{-1}(y)$ to represent the preimage of the point y under the function f, i.e., $f^{-1}(y) = \{x : \ f(x) = y\}$.

We consider in this paper an electric power network at sub-transmission or transmission level and a local heating

X. Qin and I. Lestas are with the Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ, UK {xq234, icl20}@cam.ac.uk

network coupled together via a heat pump providing ancillary services to the power grid. A graph will be defined for each of these two network systems, respectively, which will then be used to define the underlying physical dynamics in the two networks and the way these are coupled.

In particular, to describe the interconnections in the electric power network we consider a directed graph (N_e, E_e), where $N_e = \{1, ..., n_{N_e}\}$ is the set of electric buses and $E_e \subseteq N_e \times N_e$ is set of all power lines. We use (i, j) to denote the link connecting buses i and j and assume that the directed graph (N_e, E_e) has an arbitrary orientation, so that if $(i, j) \in E$ then $(j, i) \notin E$. We use the set $N_e^G = \{1, ..., n_{N_e^G}\}$ to denote the set of generators. We use $i: i \to j$ and $k: j \to k$ to indicate the predecessors and successors of bus j, respectively. We use the set $H = \{1, ..., n_H\}$ to denote the set of heat pumps that interconnect the electric and heating systems.

Similarly, we use a directed graph $G_h = (N_h, E_h)$ to describe the heating network as detailed in [6], where the direction of the edge is the same as the mass flow. N_h is the set of heat nodes and E_h is the set of heat edges that represent all types of single-input-single-output heat devices including pipelines and heat exchangers. There are three types of heat edges: edges associated with Heat pumps denoted by set H_h, conventional heat sources like gas boilers denoted by set E_h^G, and heat loads, denoted by set E_h^L, which use heat exchangers to exchange heat energy with the heating network. We use the temperature vectors $T^E = col(T^E_{i\in H_h}, T^E_{i\in E_h^G}, T^E_{i\in E_h^L})$ to denote the edge temperatures of the different type of edges. We use vector T^N to denote the temperature of heat nodes. Then we use vector $T = col(T^E, T^N)$ to aggregate the edge and node temperatures in the heating system.

The following simplifying assumptions are made for the combined heat and power network in the models that will be described in the next section. These are assumptions relevant at slower/intermediate timescales as is the case with the temperature dynamics of the heating systems that will be considered. Bus voltage magnitudes in the power network are approximately 1 per-unit (p.u.) for each $j \in N_e$. Lines $(i, j) \in E_e$ are lossless and characterized by their susceptances $B_{ij} = B_{ji} > 0$. Also reactive power flows do not affect bus voltage angles and frequencies.

III. Model Formulation

In this section, we describe the dynamic model of the combined heat and power network.

A. Electric power system

We will be using the swing equation to describe the frequency dynamics at each bus i in the electrical power network.

$$M_j \dot{\omega}_j = -p_j^L - p_j^P + p_j^G - p_j^U + \sum_{i:\ i\to j} p_{ij} - \sum_{k:\ j\to k} p_{jk}, \quad j \in N_e \tag{1a}$$

$$\dot{\eta}_{ij} = \omega_i - \omega_j, \qquad (i, j) \in E_e \tag{1b}$$

$$p_{ij} = B_{ij} \sin(\eta_{ij}) - p_{ij}^{\text{nom}}, \qquad (i, j) \in E_e. \tag{1c}$$

Variable ω_j is the frequency deviation from a nominal value (50 or 60Hz) at bus j. Variable η_{ij} is the angle difference between bus i and j. Variables p_j^G, p_j^U, p_j^P, p_j^L represent deviations from nominal values, with p^G, p^U, p^P, p^L denoting corresponding vectors. In particular, p_j^G represents generation at bus j, and variable $p_j^U = D_j\omega_j$ where $D_j > 0$ is a constant, represents system damping and frequency dependent uncontrollable loads at bus j. Variable p_j^L denotes the deviation from a nominal value of a step change in the demand at bus j. Constant $M_j > 0$ indicates the generator inertia at bus j. Constants $B_{ij} > 0$ are the line susceptance. Variable p_j^P denotes the electric power consumed by the heat pump at bus j. Note that this variable is zero if there is no heat pump connected to bus j.

B. Heating system

In this section, we describe the temperature dynamics of the heating system based on the model in [6]. Equation (2a) quantifies the rate of change of the temperature at an edge which is dependent on the temperature difference between its inlet and outlet as well as the heat power consumed/generated at this edge. Equation (2b) describes the thermal dynamics at a heat node, indicating that the rate of change of the thermal energy at node k (left hand side) equals its net sum thermal energy input flows (right hand side).

$$\rho C_p V_j^E \dot{T}_j^E = \rho C_p q_j^E \left(T_k^N - T_j^E\right) + h_j^G + h_j^P - h_j^L, \quad j \in E_h,\ k \in N_h \tag{2a}$$

$$\rho C_p V_k^N \dot{T}_k^N = \sum_{j\in\mathfrak{T}_k} \rho C_p q_j^E T_j^E - \sum_{j\in\mathfrak{T}_k} \rho C_p q_j^E T_k^N, \quad k \in N_h, \tag{2b}$$

where q_j^E is the mass flow of edge j. Variable T_j^E is the temperature of edge j. Variables V_j^E and V_k^N are the volumes at edge j and node k, respectively. Variables h_j^P, h_j^G, and h_j^L are the powers of the heat pump, the conventional heat source, and the load power at edge j, respectively. If edge j is associated with a heat pump, $h_j^G = h_j^L = 0$; Similarly, if edge j is a non-pump heat source, $h_j^P = h_j^L = 0$, and if edge j is a heat load, $h_j^G = h_j^P = 0$ whereas h_j^L is a given constant. C_p and ρ are the capacity and density of water, respectively. In the normalized per-unit system, $\rho C_p = 1$. Set $\mathfrak{T}_k$ indicates the edges that inject flow to node k.

For convenience in the analysis, we write equations (2a) and (2b) into matrix form:

$$V \begin{bmatrix} \dot{T}^E \\ \dot{T}^N \end{bmatrix} = -A_h \begin{bmatrix} T^E \\ T^N \end{bmatrix} + \begin{bmatrix} h^G + h^P - h^L \\ \mathbf{0} \end{bmatrix}, \tag{3}$$

where h^P, h^G, and h^L are vectors with elements h_j^P , h_j^G, and h_j^L, respectively. $V = diag(V^E, V^N)$ indicates the volume matrix.

Note that

$$A_h = \begin{bmatrix} diag(q^E) & -diag(q^E)B_{sh} \\ -B_{th}diag(q^E) & diag(B_t q^E) \end{bmatrix} \quad (4)$$

is a constant matrix for a given mass flow vector q^E, where $q^E = col(q_j^E)$. $B_{th} = \frac{1}{2}(|B_h| + B_h)$, and $B_{sh} = \frac{1}{2}(|B_h| - B_h)$, where B_h is the incidence matrix of the heating system. It can be shown that $A_h + A_h^T$ is positive semidefinite with a simple eigenvalue at the origin and A_h satisfies $A_h\mathbf{1} = 0, \mathbf{1}^T A_h = 0$ [6].

C. Generation control

Generators contribute to frequency control in the power grid. We consider heat pumps that operate in a mode that provides an ancillary service to the power grid by contributing to frequency control. The heat pumps are also part of a heating network. In particular, we have the following frequency regulation schemes for generators:

$$\dot{p}_j^G = -p_j^G - \tilde{Q}_{e,jj}\omega_j, \quad j \in N_e^G \quad (5a)$$

where $\tilde{Q}_{e,jj} = \frac{1}{Q_{e,jj}}$, with scalar $Q_{e,jj}$ a positive cost coefficient reflecting the electric generation cost. For convenience, we define two diagonal matrices $Q_e := diag(Q_{e,jj})$ and $\tilde{Q}_e := diag(\tilde{Q}_{e,jj})$.

We consider also conventional heat sources in the heating system contributing to temperature control via the scheme:

$$\dot{h}_j^G = -h_j^G - \tilde{Q}_{h,jj}\bar{T}, \quad j \in E_h^G \quad (5b)$$

where $\tilde{Q}_{h,jj} = \frac{1}{Q_{h,jj}}$, with scalar $Q_{h,jj}$ a positive constant. We also denote $Q_h = diag(Q_{h,jj})$. Variable

$$\bar{T} := \frac{\mathbf{1}^T V T}{\mathcal{V}}$$

is the average temperature of the heating system, which is calculated as a weighted mean of temperature deviations. The weights used reflect the energy stored in the nodes and edges of the heating system, where $\mathcal{V} = \mathbf{1}^T V \mathbf{1}$ is a constant equal to the sum of the heating system volumes.

Remark 1. The significance of the use of the average temperature $\bar{T}$ in the control policies is that it leads to appropriate power sharing properties in the heating system in real time, without having an a prior knowledge of the disturbances applied. This is shown in section V. The stability of the combined heat and power network is also shown in section IV when these regulation schemes are used, in conjunction with frequency support mechanisms by heat pumps described in the next section.

Remark 2. First order dynamics in (5) are used here for simplicity in the presentation. In Section VI we will generalize the power generation dynamics in (5) to general classes of input strictly passive systems from ω_j to p_j^G and from $\bar{T}$ to h_j^G.

D. Heat pump

When participating in frequency regulation, we assume that the heat pump operates around an operating point where the coefficient of performance (CoP) is approximately a constant [7]. The heat pump's electric power consumption and heat power generation are

$$h_{j_k}^P = C_o p_{i_k}^P, \quad k \in H \quad (6)$$

where C_o indicates the CoP of the heat pump. Note that for each $k \in H$ there are indices i_k, j_k, where $i_k \in H_e$ is the bus the heat pump connects to in the electrical network and $j_k \in H_h$ is the corresponding edge in the heating network it is associated with. We denote H_e the set of buses in the electrical network heat pumps are connected to and H_h the set of edges in the heat network associated with heat pumps.

We consider two schemes for the heat pump participating in frequency regulation:

- **Mode 1:** Frequency dependent load. In this mode, the heat pump works as a frequency dependent load as shown in (7) to provide frequency support for the power network.

$$p_j^P = a_{1,j}\omega_j, \quad j \in H_e \quad (7)$$

  where $a_{1,j}$ is a positive constant. Equation (7) indicates that we regulate the shaft speed of the heat pump in proportion to the bus frequency.
- **Mode 2:** Converter-linked load. In this mode, we consider the output of the heat pump converter as a separate bus with zero inertia, where the frequency is set by the heat pump, and we have power balance considering the power transfers from neighbouring buses.
  The scheme is given by equation (8). In particular, the pump converter interlinks the electric frequency with the thermal temperature, which allows to achieve appropriate power sharing between the combined heat and power network.

$$0 = -p_j^P + \sum_{i:\ i\to j} p_{ij} - \sum_{k:\ j\to k} p_{jk}, \quad j \in H_e \quad (8a)$$

$$\omega_j = m\bar{T} \quad (8b)$$

  Note that the heat pump power is dependent on the power transfer from other buses as shown in the power balance constraint (8a). Scalar m is a coefficient indicating the relation between electric frequency and heat average temperature.

Remark 3. Mode 2 has two major differences from Mode 1. First, as shown in equation (8b) the frequency of the heat pump bus is set directly by the converter, with this then determining the power consumed by the heat pump via the power transfer from neighboring buses (analogous to matching control schemes in hybrid AC/DC networks). This is in contrast to Mode 1 where the heat pump power is set directly as a function of the grid frequency. Second, Mode 2 treats the heat pump converter as a separate bus, which implies that the underlying graph of the electric power system is different from that in Mode 1.

IV. Stability of Combined Heat and Power Network

In this section, we analyze the stability of the combined heat and power system under the two different participation modes of heat pumps described in the previous section, where the heat network also contributes to frequency control via the control policies of the heat pumps.

Assumption 4. The combined heat and power network (1)-(6), together with (7) or (8), respectively, admits an equilibrium point $x^* = (\eta^*, \omega^*, p^{G*}, T^*, h^{G*})$ with $|\eta^*_{ij}| < \frac{\pi}{2}$, $\forall ij \in E_e$.

This assumption on η^*_{ij} is a security constraint and is ubiquitous in the power network literature.

Theorem 5 (Stability)**.** Consider the combined heat and power network described by (1)-(6), (7) or (1)-(6), (8). Consider an equilibrium point of this system that satisfies Assumption 4. Then, there exists an open neighbourhood of this equilibrium point such that all solutions of the system starting in this region converge to an equilibrium point.

The proof of Theorem 5 is provided in Sections IV-A and IV-B below. In particular, in Section IV-A we prove the theorem when Mode 1 is used, and in Section IV-B we prove the theorem when Mode 2 is used.

A. Proof of Theorem 5 under Mode 1

We consider the combined heat and electrical power network described by (1)-(6), (7). We will first prove convergence to an equilibrium point for the states of the electrical power system, then we will prove convergence of the average temperature $\bar{T}$ in the heating system, and we will finally prove convergence of the temperature vector T in the heating system.

The arguments for the convergence in the power system are analogous to those in [8], though to deduce convergence in the heating system requires more involved arguments due to the non-unique equilibrium points in the latter.

For the dynamics of the electric power system, the state vector is $x_e = (\eta, \omega, p^G)$. We consider storage functions $V_{1,\omega}(\omega) = \frac{1}{2}\sum_{j\in N_e} M_j(\omega_j - \omega^*_j)^2$ and $V_{1,l}(\eta) = \sum_{ij\in E_e} B_{ij}\int_{\eta^*_{ij}}^{\eta_{ij}}(\sin\theta - \sin\eta^*_{ij})d\theta$ and then consider the time derivative of $V_{1,\omega}(\omega) + V_{1,l}(\eta)$ along the system trajectories

$$\begin{aligned}\dot{V}_{1,\omega} + \dot{V}_{1,l} &= \sum_{j\in N_e} -D_j(\omega_j - \omega^*_j)^2 \\ &\quad + \sum_{j\in N_e}(\omega_j - \omega^*_j)(p^G_j - p^{G*}_j) - \sum_{j\in H_e} a_{1,j}(\omega_j - \omega^*_j)^2.\end{aligned} \tag{9}$$

We consider also the storage function

$$V_{1,g}(p^G) = \frac{1}{2}\sum_{j\in N^G_e}\frac{1}{\tilde{Q}_{e,jj}}(p^G_j - p^{G*}_j)^2.$$

Then

$$\dot{V}_{1,g} = -\sum_{j\in N^G_e}(p^G_j - p^{G*}_j)\left[\frac{p^G_j - p^{G*}_j}{\tilde{Q}_{e,jj}} + (\omega_j - \omega^*_j)\right].$$

We define $V_{1,e}(x_e) = V_{1,\omega} + V_{1,l} + V_{1,g}$ and hence have

$$\begin{aligned}\dot{V}_{1,e}(x_e) &= \dot{V}_{1,\omega} + \dot{V}_{1,l} + \dot{V}_{1,g} \\ &= \sum_{j\in N_e} -(D_j + a_j)(\omega_j - \omega^*_j)^2 - \sum_{j\in N^G_e}\frac{1}{\tilde{Q}_{e,jj}}(p^G_j - p^{G*}_j)^2,\end{aligned} \tag{10}$$

where $a_j = 0$ for $j \notin H_e$.

We then consider $V_{1,e}$ as the candidate Lyapunov function for the electric power system which from the analysis above is non-increasing with time. From the condition that $\eta_{ij} < \frac{\pi}{2}$, the function $V_{1,e}(x_e)$ has a strict local minimum at $x^*_e = (\eta^*, \omega^*, p^{G*})$. Hence we can choose a neighborhood $\mathcal{S}_1$ of x^*_e given by $\mathcal{S}_1 = \{(\eta, \omega, p^G) : V_{1,e} \le \epsilon_1\}$ for some sufficiently small $\epsilon_1 > 0$ such that this is a compact and positively-invariant set that contains x^*_e.

We then apply LaSalle's Theorem, with $V_{1,e}$ as the Lyapunov-like function, which states that all trajectories of the system starting from within $\mathcal{S}_1$ converge to the largest invariant set within $\mathcal{S}_1$ that satisfies $\dot{V}_{1,e}(x_e) = 0$. Note that coefficients D_j, a_j, $\tilde{Q}_{e,jj}$ are all positive, indicating $\dot{V}_{1,e}(x_e) = 0$ implies $(\omega_j, p^G_j) = (\omega^*_j, p^{G*}_j)$ and therefore $\dot{\omega} = 0, \dot{p}^G_j = 0$. Hence from the system dynamics, we deduce that the largest invariant set in $\mathcal{S}_1$ for which $\dot{V}_{1,e} = 0$ is the set of equilibrium points in $\mathcal{S}_1$. Therefore, by LaSalle's Theorem we have convergence to the set of equilibrium points in $\mathcal{S}_1$. Convergence to an equilibrium point in $\mathcal{S}_1$ for ϵ_1 sufficiently small, can be deduced using arguments analogous to those in [9][Prop. 4.7, Thm. 4.20], noting that the equilibrium points in $\mathcal{S}_1$ are Lyapunov stable.

We can also deduce that the convergence to an equilibrium point is exponential. This follows by using the fact that[1] $\eta_{ij} = \theta_i - \theta_j$, and by considering the trajectories of x_e where the initial conditions of η satisfy this property. These trajectories lie in an invariant subspace of $\mathbb{R}^{|E_e|+|N_e|+1}$ where this property is satisfied. In this subspace it can be shown that linearization of (1) about x^*_e leads to a linear system where the origin is asymptotically stable, therefore local exponential convergence can be deduced for the nonlinear system in this subspace (using arguments analogous to those in the proof of [10][Theorem 4.7]).

We will now prove that the average temperature $\bar{T}$ converges to a constant value. The average temperature dynamics are given by

$$\dot{\bar{T}} = \frac{1}{\mathcal{V}}\mathbf{1}^T V\dot{T} = \frac{1}{\mathcal{V}}\mathbf{1}^T[-A_h + h] = \frac{1}{\mathcal{V}}\mathbf{1}^T h, \tag{11}$$

where

$$h = \begin{bmatrix} h^G + h^P - h^L \\ \mathbf{0}\end{bmatrix}. \tag{12}$$

[1]It can also be shown that due to this property the equilibrium points x^*_e for a given equilibrium frequency ω^* are unique.

In (11) have used the property $\mathbf{1}^T A_h = 0$. Variable h^G satisfies the Ordinary Differential Equation (ODE) (5b) and h^P equation (7). We will first show that when h^P is constant the corresponding equilibrium point $(\bar{T}^*, h^{G*})$ of the state vector $x_h = (\bar{T}, h^G)$ is asymptotically stable.

We consider the storage function $V_{1,t}(x_h) = \frac{1}{2}\mathcal{V}(\bar{T} - \bar{T}^*)^2$. Using the expression for $\dot{\bar{T}}$ in (11) we obtain

$$\begin{aligned}\dot{V}_{1,t} &= \mathcal{V}(\bar{T} - \bar{T}^*)\frac{\mathbf{1}^T}{\mathcal{V}} V\dot{T} \\ &= \sum_{j\in E_h^G} (\bar{T} - \bar{T}^*)(h_j^G - h_j^{G*}).\end{aligned}$$

We consider also the storage function

$$V_{1,s}(h^G) = \frac{1}{2}\sum_{j\in N_h^G} \frac{1}{\tilde{Q}_{h,jj}}(h_j^G - h_j^{G*})^2.$$

Then

$$\dot{V}_{1,s} = -\sum_{j\in N_h^G}(h_j^G - h_j^{G*})\left[\frac{h_j^G - h_j^{G*}}{\tilde{Q}_{h,jj}} + (\bar{T} - \bar{T}^*)\right].$$

We then define $V_{1,h}(x_h) = V_{1,t} + V_{1,s}$ and hence have

$$\dot{V}_{1,h}(x_h) = \dot{V}_{1,t} + \dot{V}_{1,s} = -\sum_{j\in N_h^G}\frac{1}{\tilde{Q}_{h,jj}}(h_j^G - h_j^{G*})^2. \tag{13}$$

Using $V_{1,h}$ as a Lyapunov like function we can deduce using Lasalle's theorem that $(\bar{T}^*, h^{G,*})$ is asymptotically stable when h^P is constant. Since the system is linear we can deduce that x_h converges exponentially to $(\bar{T}^*, h^{G,*})$ also when h^P tends exponentially to its equilibrium value. The latter is the case for h^P from the convergence properties of the frequency ω_i established in the first part of the proof associated with the electric power system.

We will finally prove that the temperature vector T also converges to a constant value. We consider deviations of T and variable h in (12) from equilibrium values T^*, h^*, respectively. We denote the deviations as $\tilde{T} = T - T^*$, $\tilde{h} = h - h^*$, and these satisfy the ODE

$$\dot{\tilde{T}} = -V^{-1}A_h\tilde{T} + V^{-1}\tilde{h}. \tag{14}$$

We make use of the fact that $V^{-1}A_h$ has the same non-zero eigenvalues as $V^{-1/2}A_hV^{-1/2}$. Due to the fact that $A_h + A_h^T$ is positive semidefinite with a simple eigenvalue at the origin, $V^{-1/2}A_hV^{-1/2}$ also satisfies this property, and we can hence deduce that $V^{-1}A_h$ has all its eigenvalues in the right half-plane with a simple value at the origin.

From our previous analysis associated with the electrical power system and the convergence of $\bar{T}$ we have that $\tilde{h}$ converges exponentially to 0 (when the initial condition of the system is sufficiently close to the equilibrium point). Therefore, from the linearity of the system in (14) and the location of the eigenvalues of $V^{-1}A_h$ we deduce that $\tilde{T}$ converges to a constant value, and hence T also converges to a constant value. This completes the proof.

B. *Proof of Theorem 5 under Mode 2*

The proof under Mode 2 follows a similar logic to the one used under Mode 1.

For the dynamics of the combined heat and power system, we consider the state vector $\bar{x} = (\eta, \omega, p^G, \bar{T}, h^G)$, i.e. the heating system is described by its dynamics for the average temperature $\bar{T}$ with the aggregate system described by (1), (5), (6), (8), (11), (12). Below we construct a Lyapunov function for this aggregate system so as to prove convergence of $\bar{x}$ to an equilibrium point.

We consider storage functions $V_{2,\omega}(\omega) = \frac{1}{2}\sum_{j\in N_e} M_j(\omega_j - \omega_j^*)^2$ and $V_{2,l}(\eta) = \sum_{ij\in E_e} B_{ij}\int_{\eta_{ij}^*}^{\eta_{ij}}(\sin\theta - \sin\eta_{ij}^*)d\theta$ and the time derivative of $V_{2,\omega}(\omega) + V_{2,l}(\eta)$ along the system trajectories is

$$\begin{aligned}\dot{V}_{2,\omega} + \dot{V}_{2,l} &= \sum_{j\in N_e} -D_j(\omega_j - \omega_j^*)^2 \\ &+ \sum_{j\in N_e}(\omega_j - \omega_j^*)(p_j^G - p_j^{G*}) - \sum_{j\in H_e}(\omega_j - \omega_j^*)(p^P - p^{P*}).\end{aligned} \tag{15}$$

We consider also the storage function

$$V_{2,g}(p^G) = \frac{1}{2}\sum_{j\in N_e^G}\frac{1}{\tilde{Q}_{e,jj}}(p_j^G - p_j^{G*})^2,$$

then

$$\dot{V}_{2,g} = -\sum_{j\in N_e^G}(p_j^G - p_j^{G*})\left[\frac{p_j^G - p_j^{G*}}{\tilde{Q}_{e,jj}} + (\omega_j - \omega_j^*)\right].$$

We consider the storage function $V_{2,t}(\bar{T}) = \frac{1}{2}\alpha\mathcal{V}(\bar{T} - \bar{T}^*)^2$, where $\alpha = \frac{m}{C_o}$ is a constant coefficient. Using $\dot{\bar{T}} = \frac{\mathbf{1}^T}{\mathcal{V}}V\dot{T}$ we obtain

$$\begin{aligned}\dot{V}_{2,t} &= \alpha\mathcal{V}\bar{T}\frac{\mathbf{1}^T}{\mathcal{V}}V\dot{T} \\ &= \sum_{j\in H}\alpha(\bar{T} - \bar{T}^*)(h^P - h^{P*}) + \sum_{j\in N_h^G}\alpha(\bar{T} - \bar{T}^*)(h^G - h^{G*}) \\ &= \sum_{j\in H_e}(\omega_j - \omega_j^*)(p^P - p^{P*}) + \sum_{j\in N_h^G}\alpha(\bar{T} - \bar{T}^*)(h^G - h^{G*})\end{aligned}$$

where we have used the property $\mathbf{1}^T A_h(T - T^*) = 0$.

We consider also the storage function

$$V_{2,s}(h^G) = \frac{1}{2}\alpha\sum_{j\in N_h^G}\frac{1}{\tilde{Q}_{h,jj}}(h_j^G - h_j^{G*})^2,$$

then

$$\dot{V}_{2,s} = -\alpha\sum_{j\in N_h^G}(h_j^G - h_j^{G*})\left[\frac{h_j^G - h_j^{G*}}{\tilde{Q}_{h,jj}} + (\bar{T} - \bar{T}^*)\right].$$

Then consider the candidate Lyapunov function for the aggregate system $V_2 = V_{2,\omega} + V_{2,l} + V_{2,g} + V_{2,t} + V_{2,s}$:

$$\begin{aligned}\dot{V}_2(\bar{x}) &= \dot{V}_{2,\omega} + \dot{V}_{2,l} + \dot{V}_{2,g} + \dot{V}_{2,t} + \dot{V}_{2,s} \\ &= -\sum_{j\in N_e} D_j(\omega_j - \omega_j^*)^2 \\ &\quad - \sum_{j\in N_e^G}\frac{1}{\tilde{Q}_{e,jj}}(p_j^G - p_j^{G*})^2 - \sum_{j\in N_h^G}\frac{\alpha}{\tilde{Q}_{h,jj}}(h_j^G - h_j^{G*})^2\end{aligned} \tag{16}$$

where equation (8) is used to derive the final expression.

Hence the candidate Lyapunov function $V_2(\bar{x})$ is non-increasing with time. From the condition that $\eta_{ij} < \frac{\pi}{2}$, the function $V_2(\bar{x})$ has a strict local minimum at $\bar{x}^* = (\eta^*, \omega^*, p^{G*}, \bar{T}^*, h_j^{G*})$. Hence we can choose a neighborhood $\mathcal{S}_2$ of x^* given by $\mathcal{S}_2 = \{(\eta, \omega, p^G, \bar{T}, h^G) : V_2 \leq \epsilon_2\}$ for some sufficiently small $\epsilon_2 > 0$ such that this is a compact and forward-invariant set that contains $\bar{x}^*$.

We then apply LaSalle's Theorem, with V_2 as the Lyapunov-like function, which states that all trajectories of the system starting from within $\mathcal{S}_2$ converge to the largest invariant set within $\mathcal{S}_2$ that satisfies $\dot{V}_2(\bar{x}) = 0$. Note that coefficients D_j, $\tilde{Q}_{e,jj}$, and $\tilde{Q}_{h,jj}$ are all positive, indicating $\dot{V}_2(\bar{x}) = 0$ implies $(\omega_j, p_j^G, h_j^G, \bar{T}) = (\omega_j^*, p_j^{G*}, h_j^{G*}, \bar{T}^*)$ and therefore $\dot{\omega} = 0$, $\dot{p}_j^G = 0$, $\dot{\bar{T}} = 0$ and $\dot{h}_j^G = 0$. Hence from the system dynamics we deduce that the largest invariant set in $\mathcal{S}_2$ for which $\dot{V}_2 = 0$ is the set of equilibrium points in $\mathcal{S}_2$. Therefore, by LaSalle's Theorem we have convergence to the set of equilibrium points in $\mathcal{S}_2$. Convergence to an equilibrium point in $\mathcal{S}_2$ for ϵ_2 sufficiently small, can be deduced using arguments analogous to those in [9][Prop. 4.7, Thm. 4.20], noting that the equilibrium points in $\mathcal{S}_2$ are Lyapunov stable.

Finally, we can prove the convergence of T to a constant equilibrium value from the convergence of $\bar{T}$ to $\bar{T}^*$, following arguments analogous to those used in subsection IV-A. This completes the proof.

V. Power Sharing in the Combined Heat and Power Network

In this section, we show that the use of the average temperature in the control policies in the heating system allows to achieve optimal power sharing. In Mode 1, we show that the power sharing is optimal in the electric and the heating system, separately. In Mode 2, we show that the optimal combined electric and heat power sharing is achieved among all electric and heat sources.

A. Mode 1

Proposition 6 (Optimality under Mode 1)**.** Consider the combined heat and power network described by (1)-(6) and (7). The deviation in the power generation p^G and of the heat pump load p^P at equilibrium after a step change in electrical load p^L is the solution to the optimization problem

$$\min_{p^G, p^P, p^U} C_{1,e} = \frac{1}{2}(p^G)^T Q_e p^G + \frac{1}{2}(p^P)^T Q_p (p^P)^T + \frac{1}{2}(p^U)^T Q_u p^U, \tag{17a}$$

subject to

$$\mathbf{1}^T p^G = \mathbf{1}^T p^L + \mathbf{1}^T p^P + \mathbf{1}^T p^U, \tag{17b}$$

where $Q_p = diag(Q_{p,jj})$ with $Q_{p,jj} = \frac{1}{a_{1,j}}$, $j \in H_e$ and $Q_u = diag(Q_{u,jj})$ with $Q_{u,jj} = \frac{1}{D_j}$, $j \in N_e$.

Given the heat pump power consumption $\bar{p}^P$ in the solution of (17), the deviation in power h^G of the conventional heat sources is the solution to the optimization problem

$$\min_{h^G} C_{1,h} = \frac{1}{2}(h^G)^T Q_h h^G, \tag{18a}$$

subject to

$$\mathbf{1}^T h^G = \mathbf{1}^T (h^L - h^P). \tag{18b}$$

Proof: We will show that at steady state the electrical power system variables are the solution to optimization problem (17). Then we will show that at steady state the heating system variables are the solution to the optimization problem (18).

First, consider the equilibrium of the power network, $\dot{\omega} = 0$ and $\dot{p}^G = 0$, which results in $p^{G*} = -\tilde{Q}_e \mathbf{1}\omega^*$, $p_j^{P*} = a_{1,j}\omega^*$, $\forall j \in H_e$, and $p_j^{U*} = D_j \omega^*$, $\forall j \in N_e$ from the control scheme (5a).

We then consider the optimal solution to the power network optimization problem (17) which we denote as $(\bar{p}^G, \bar{p}^P, \bar{p}^U)$. Using the Karush–Kuhn–Tucker (KKT) conditions for optimality, we obtain

$$\begin{aligned} Q_e \bar{p}^G + \mathbf{1}\lambda_1 &= 0, \\ Q_p \bar{p}^P - \mathbf{1}\lambda_1 &= 0, \\ Q_u \bar{p}^U - \mathbf{1}\lambda_1 &= 0, \end{aligned}$$

where λ_1 is the dual variable associated with constraint (17b). Now we set at the optimal point, $\lambda_1 = \omega^*$. Then $\bar{p}^G = -\tilde{Q}_e \mathbf{1}\omega^* = p^{G*}$, $\bar{p}^P = a_1 \omega^* = p^{P*}$, and $\bar{p}^U = D\omega^* = p^{U*}$, where a_1 and D are vectors of $a_{1,j}$ and D_j, respectively. Thus, $(p^{G*}, p^{P*}, p^{U*}) = (\bar{p}^G, \bar{p}^P, \bar{p}^U)$.

Given the heat pump heat generation $h^{P*} = C_o p^{P*}$, consider the equilibrium of the heating system, where $\dot{\bar{T}} = 0$, $\dot{T} = 0$, and $h^{G*} = -\tilde{Q}_h \mathbf{1}\bar{T}^*$ from the control scheme (5b).

Then consider the optimal solution to the power network optimization problem (18) which is denoted as $\bar{h}^G$. Using the KKT conditions we obtain

$$Q_h \bar{h}^G + \mathbf{1}\mu_1 = 0,$$

where μ_1 is the dual variable associated with constraint (18b). Now we set at the optimal point, $\mu_1 = \bar{T}^*$. Then $\bar{h}^G = -\tilde{Q}_h \mathbf{1}\bar{T}^* = h^{G*}$. This completes the proof.

B. Mode 2

Theorem 7 (Optimality under Mode 2)**.** Consider the combined heat and power network described by (1)-(6) and (8). Then the power contribution of the electric generators p^G and conventional heat sources h^G is the solution to the combined heat and power optimization problem

$$\min_{p^G, h^G, p^U} C_2 = \frac{1}{2}(p^G)^T Q_e p^G + \frac{1}{2}(h^G)^T \frac{m}{C_o} Q_h h^G + \frac{1}{2}(p^U)^T Q_u p^U, \tag{19a}$$

subject to

$$\mathbf{1}^T p^G = \mathbf{1}^T p^L + \mathbf{1}^T p^P + \mathbf{1}^T p^U, \tag{19b}$$

$$\mathbf{1}^T h^G = \mathbf{1}^T h^L - \mathbf{1}^T h^P, \tag{19c}$$

$$h^P_{j_k} = C_o p^P_{i_k}, \quad k \in H. \tag{19d}$$

where Q_u is as defined in Proposition 6.

Proof: We will show that at steady state the system variables are the optimal solution to the combined heat and power optimization problem (19) .

First, consider the equilibrium of the aggregate dynamical system. At equilibrium, $\dot{\omega} = 0$, $\dot{\bar{T}} = 0$, $\dot{T} = 0$. Hence $p^G = -\tilde{Q}_e \mathbf{1}\omega^*$ and $h^G = -\tilde{Q}_h \mathbf{1}\bar{T}^* = -\tilde{Q}_h \mathbf{1}\omega^*/m$ from the control scheme (5) and equation (8) .

Then we consider the optimal solution to the combined heat and power optimization problem (19) which we denote as $(\bar{p}^G, \bar{h}^G, \bar{p}^P, \bar{p}^U)$. Consider the Lagrangian for (19)

$$\begin{aligned} L(p^G, h^G, p^P, \lambda, \mu) &= \frac{1}{2}(p^G)^T Q_e p^G + \frac{1}{2}(h^G)^T \frac{Q_h}{C_o} h^G \\ &+ \frac{1}{2}(p^U)^T Q_u p^U + \mu_2 \left[\mathbf{1}^T h^G - \mathbf{1}^T h^L + \mathbf{1}^T C_o \cdot p^P\right] \\ &+ \lambda_2 \left[\mathbf{1}^T p^G - \mathbf{1}^T p^L - \mathbf{1}^T p^P - \mathbf{1}^T p^U\right], \end{aligned}$$

where λ_2 and μ_2 are dual variables associated with the constraints (19b) and (19c), respectively. Then form the KKT conditions we have

$$\frac{\partial L}{p^G} = Q_e p^G + \mathbf{1}\lambda_2 = 0, \tag{20a}$$

$$\frac{\partial L}{h^G} = \frac{m}{C_o} \cdot Q_h h^G + \mathbf{1}\mu_2 = 0, \tag{20b}$$

$$\frac{\partial L}{p^P} = -\lambda_2 + C_o \cdot \mu_2 = 0, \tag{20c}$$

$$\frac{\partial L}{p^U} = Q_u \bar{p}^U - \mathbf{1}\lambda_2 = 0, \tag{20d}$$

$$\frac{\partial L}{\lambda} = \mathbf{1}^T(p^G - p^L - p^P - p^U) = 0, \tag{20e}$$

$$\frac{\partial L}{\mu} = \mathbf{1}^T(h^G - h^L + C_o \cdot p^P) = 0. \tag{20f}$$

Then we set $\lambda_2 = \omega^*$. Hence $\mu_2 = \frac{1}{C_o}\lambda_2 = \frac{1}{C_o}\omega^*$ from (20c). Then we have

$$\begin{aligned} \bar{p}^G &= -\tilde{Q}_e \mathbf{1}\omega^* &&= p^{G*}, \\ \frac{1}{C_o}\bar{h}^G &= -\frac{1}{C_o}\tilde{Q}_h \mathbf{1}\frac{\omega^*}{m} &&= \frac{1}{C_o} h^{G*}, \quad j \in N_e \\ \bar{p}^U &= \quad D\omega^* &&= p^{U*}. \end{aligned}$$

Thus, $(p^{G*}, h^{G*}, p^{U*}) = (\bar{p}^G, \bar{h}^G, \bar{p}^U)$. Then from the power balance constraint (19b), we know $p^{P*} = \bar{p}^P$, thus completing the proof.

VI. General passive dynamics

In practical combined heat and power networks, the dynamics of generators and conventional heat sources are diverse, extending beyond simple first-order dynamics as in (5). To address this practical issue, we describe in this section how the approach in the paper can accommodate general higher order generation dynamics characterized by input strictly passive systems, relating ω_j to p^G_j and $\bar{T}$ to h^G_j, respectively.

More precisely, in the power network we consider p^G_j now as the output of a nonlinear system with input ω_j, namely

$$\dot{x}^s_{e,j} = -f_{e,j}(x^s_{e,j},\ \omega_j), \quad j \in N^G_e, \tag{21a}$$

$$p^G_j = -g_{e,j}(x^s_{e,j},\ \omega_j), \quad j \in N^G_e. \tag{21b}$$

where $x^s_{e,j}$ is the state of the system, and $f_{e,j}$ and $g_{e,j}$ are locally Lipschitz functions. System (21) is said to be locally input strictly passive around the equilibrium point ω^*_j, $x^{s*}_{e,j}$, if there exist open neighborhoods Ω_j of ω^*_j and $X^s_{e,j}$ of $x^{s*}_{e,j}$ and a continuously differentiable positive semi-definite function $V^s_{e,j}(x^s_{e,j})$, such that for all $\omega_j \in \Omega_j$ and all $x^s_{e,j} \in X^s_{e,j}$, $j \in N^G_e$, one has

$$\dot{V}^g_{e,j}(x^s_{e,j}) \leq (-\omega_j + \omega^*_j)(p^G_j - p^{G*}_j) - \phi_{e,j}(-\omega_j + \omega^*_j),$$

where $\phi_{e,j}$ is a positive definite function. At the equilibrium point, we define

$$p^{G*}_j = K_{p^G_j}(-\omega^*_j), \tag{22}$$

where $K_{p^G_j}(-\omega^*_j)$ is a static input-output characteristic map, which is defined under the assumption that the equilibrium value p^{G*}_j is unique for a given constant input ω^*_j. We also assume that the equilibrium point $x^{s*}_{e,j}$ is asymptotically stable for subsystem (21) for constant input ω^*_j.

Similarly, we consider a general class of generation dynamics of conventional heat sources, where we define h^G_j is the output of a nonlinear system

$$\dot{x}^s_{h,j} = -f_{h,j}(x^s_{h,j},\ \bar{T}), \quad j \in E^G_h, \tag{23a}$$

$$h^G_j = -g_{h,j}(x^s_{h,j},\ \bar{T}), \quad j \in E^G_h. \tag{23b}$$

where $x^s_{h,j}$ is the state of the system, and $f_{h,j}$ and $g_{h,j}$ are locally Lipschitz. System (23) is said to be locally input strictly passive around the equilibrium point $\bar{T}^*$, $x^{s*}_{h,j}$, if there exist open neighborhoods $\mathcal{T}_j$ of $\bar{T}^*$ and $X^s_{h,j}$ of $x^{s*}_{h,j}$ and a continuously differentiable positive semi-definite function $V^s_{h,j}(x^s_{h,j})$, such that for all $\bar{T} \in \mathcal{T}_j$ and all $x^s_{h,j} \in X^s_{h,j}$, $j \in E^G_h$, one has

$$\dot{V}^g_{h,j}(x^s_{h,j}) \leq (-\bar{T} + \bar{T}^*)(h^G_j - h^{G*}_j) - \phi_{h,j}(-\bar{T} + \bar{T}^*), \tag{24}$$

where $\phi_{h,j}$ is a positive definite function. At the equilibrium point, we similarly define

$$h^{G*}_j = K_{h^G_j}(-\bar{T}^*), \tag{25}$$

where $K_{h^G_j}(-\bar{T}^*)$ is a static input-output characteristic map and assume $x^{s*}_{h,j}$ is asymptotically stable for subsystem (23).

Assumption 8. Each of the systems defined in (21) and (23) with given inputs and outputs respectively are locally input strictly passive about equilibrium values associated with an equilibrium point of the combined heat and power network. The corresponding storage functions also have strict local minima at this equilibrium point.

A. Stability of Combined Heat and Power Network under General Passivity

Corollary 9. Consider the combined heat and power network described by (1)-(3),(6),(21),(23),(7) or (1)-(3), (6),(21),(23),(8). Consider an equilibrium point of this system that satisfies Assumptions 4 and 8. Then there exists an open neighbourhood of this equilibrium point such that all solutions of the system starting in this region converge to an equilibrium point.

Proof: In Mode 1, consider a candidate Lyapunov function for the power network, $\hat{V}_{1,e}(x_e) = V_{1,\omega} + V_{1,l} + \sum_{j\in N_e^G} V_{e,j}^g(x_{e,j})$. Hence we have

$$\begin{aligned}\dot{\hat{V}}_{1,e}(x_e) &= \dot{V}_{1,\omega} + \dot{V}_{1,l} + \sum_{j\in N_e^G} \dot{V}_{e,j}^g(x_{e,j}) \\ &\leq \sum_{j\in N_e} -(D_j + a_j)(\omega_j - \omega_j^*)^2 - \sum_{j\in N_e^G} \phi_{e,j}(-\omega_j + \omega_j^*) \leq 0.\end{aligned}$$

Similarly, consider the following sotrage function for the heating network, $\hat{V}_{1,h}(x_h) = V_{1,t} + \sum_{j\in E_h^G} V_{h,j}^g(x_{h,j})$. Hence we have

$$\begin{aligned}\dot{\hat{V}}_{1,h}(x_h) &= \dot{V}_{1,t} + \sum_{j\in E_h^G} \dot{V}_{h,j}^g(x_{h,j}) \\ &= -\sum_{j\in E_h^G} \phi_{h,j}(-\bar{T} + \bar{T}^*) \leq 0.\end{aligned}$$

The proof then proceeds in a way analogous to that followed in subsection IV-A.

In Mode 2, consider the candidate Lyapunov function for the aggregate system $\hat{V}_2 = V_{2,\omega} + V_{2,l} + \sum_{j\in N_e^G} V_{e,j}^g(x_{e,j}) + V_{2,t} + \alpha \sum_{j\in E_h^G} V_{h,j}^s$. Hence we have

$$\begin{aligned}\dot{\hat{V}}_2(\bar{x}) \leq & \sum_{j\in N_e} -D_j(\omega_j - \omega_j^*)^2 \\ & - \alpha \sum_{j\in E_h^G} \phi_{h,j}(-\bar{T} + \bar{T}^*) - \sum_{j\in N_e^G} \phi_{e,j}(-\omega_j + \omega_j^*)\end{aligned}$$

The proof then proceeds in a way analogous to that in subsection IV-B.

B. Power Sharing under General Passive dynamics

It can be shown that analogous power sharing results to those stated in Proposition 6 and Proposition 7 can be obtained when the generalized generation dynamics (21), (23) are considered. In particular, Propositions 6, 7 hold but with the quadratic terms $\frac{1}{2}(p^G)^T Q_e p^G$, $\frac{1}{2}(h^G)^T \frac{m}{C_o} Q_h h^G$ replaced by $\hat{C}_e(p^G)$, $\frac{m}{C_o}\hat{C}_h(h^G)$ respectively, where

$$K_{p_j^G}(\omega^*) = (\hat{C}'_{e,jj})^{-1}(\omega^*),\ j \in N_e^G \tag{26a}$$

$$K_{h_j^G}(\bar{T}^*) = (\hat{C}'_{h,jj})^{-1}(\bar{T}^*),\ j \in E_h^G \tag{26b}$$

and $\hat{C}_e(p^G)$, $\hat{C}_h(h^G)$ are assumed to be strictly convex. This follows from the KKT conditions for optimality as in the derivations in section V.

VII. Conclusion

We have shown that the use of the average temperature as a control signal in a district heating system can lead to appropriate optimal power sharing, while maintaining stability guarantees in general network topologies and general generation dynamics. We have also proposed two schemes through which heat pumps can contribute to frequency control with the combined heat and power network maintaining its stability and also achieving appropriate power sharing.

References

[1] Y.-J. Kim, E. Fuentes, and L. K. Norford, "Experimental study of grid frequency regulation ancillary service of a variable speed heat pump," *IEEE Transactions on Power Systems*, vol. 31, no. 4, pp. 3090–3099, 2015.

[2] X. C. Miow, Y. S. Lim, L. C. Hau, J. Wong, and H. Patsios, "Demand response for frequency regulation with neural network load controller under high intermittency photovoltaic systems," *Energy Reports*, vol. 9, pp. 2869–2880, 2023.

[3] M. T. Muhssin, L. M. Cipcigan, S. S. Sami, and Z. A. Obaid, "Potential of demand side response aggregation for the stabilization of the grids frequency," *Applied energy*, vol. 220, pp. 643–656, 2018.

[4] T. B. H. Rasmussen, Q. Wu, and M. Zhang, "Primary frequency support from local control of large-scale heat pumps," *International Journal of Electrical Power & Energy Systems*, vol. 133, p. 107270, 2021.

[5] A. Krishna and J. Schiffer, "A port-hamiltonian approach to modeling and control of an electro-thermal microgrid," *IFAC-PapersOnLine*, vol. 54, no. 19, pp. 287–293, 2021.

[6] J. E. Machado, M. Cucuzzella, and J. M. Scherpen, "Modeling and passivity properties of multi-producer district heating systems," *Automatica*, vol. 142, p. 110397, 2022.

[7] Y.-J. Kim, L. K. Norford, and J. L. Kirtley, "Modeling and analysis of a variable speed heat pump for frequency regulation through direct load control," *IEEE Transactions on Power Systems*, vol. 30, no. 1, pp. 397–408, 2014.

[8] A. Kasis, E. Devane, C. Spanias, and I. Lestas, "Primary frequency regulation with load-side participation—part i: Stability and optimality," *IEEE Transactions on Power Systems*, vol. 32, no. 5, pp. 3505–3518, 2016.

[9] W. M. Haddad and V. Chellaboina, *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton university press, 2008.

[10] H. Khalil, "Nonlinear systems," *3rd edition*, 2002.