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Point-based methods for estimating the length of a parametric curve

Michael S. Floater, Atgeirr F. Rasmussen*

CMA, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway

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Abstract

This paper studies a general method for estimating the length of a parametric curve using only samples of points. We show that by making a special choice of points, namely the Gauss–Lobatto nodes, we get higher orders of approximation, similar to the behaviour of Gauss quadrature, and we derive some explicit examples.

Keywords: Curve; Arc length; Polynomial interpolation

1. Introduction

Computing the arc length of a parametric curve is a basic problem in geometric modelling and computer graphics, and has been treated in various ways. In [11], Guenter and Parent use numerical integration on the derivative of the curve. In [16], Vincent and Forsey derive a method based entirely on point evaluations. Gravesen has derived a method specifically for Bézier curves [10]. The estimation of arc length is an important issue in [13,17,18], where approximate arc length parametrizations were sought for spline curves. This is necessary, since apart from trivial cases, polynomial curves never have unit speed [6]. The article [2] treats the issue of reparametrizing NURBS curves so that the resulting curve parametrization is close to arc length. The articles [3,4] deal with optimal, i.e., as close to arc length as possible, rational reparametrizations of polynomial curves. In [15], the authors calculate approximate arc length parametrizations for general parametric curves. Recently, results have been obtained on approximating the length of a curve, given only as a sequence of points (without parameter values), using polynomials and splines [7,8].

Suppose $\mathbf{f} : [\alpha, \beta] \to \mathbb{R}^d$, $d \geqslant 2$ is a regular parametric curve, by which we mean a continuously differentiable function such that $\mathbf{f}'(t) \neq \mathbf{0}$ for all $t \in [\alpha, \beta]$, and $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^d$. Then its arc length (see [14, Section 9]) is

$$L(\mathbf{f}) = \int_\alpha^\beta |\mathbf{f}'(t)|\,\mathrm{d}t. \tag{1}$$

Since $L(\mathbf{f})$ is simply the integral of the 'speed' function $|\mathbf{f}'|$, a natural approach is simply to apply to $|\mathbf{f}'|$ some standard composite quadrature rule: we split the parameter interval $[\alpha, \beta]$ into small pieces, apply a quadrature rule to

* Corresponding author. Tel.: +47 977 48 679; fax: +47 22 85 43 49.
E-mail addresses: michaelf@ifi.uio.no (M.S. Floater), atgeirr@math.uio.no (A.F. Rasmussen).

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$|\mathbf{f}'|$ in each piece, and add up the contributions. If $[a, b]$ is one such piece, with $\alpha \leqslant a < b \leqslant \beta$, then a typical rule has the form

$$L(\mathbf{f}|_{[a,b]}) = \int_a^b |\mathbf{f}'(t)|\,\mathrm{d}t \approx \sum_{i=0}^{n} w_i |\mathbf{f}'(q_i)|, \tag{2}$$

for some quadrature nodes

$$a \leqslant q_0 < q_1 < \cdots < q_n \leqslant b, \tag{3}$$

and weights $w_0, w_1, \ldots, w_n$. Guenter and Parent [11] apply such a method adaptively.

This method, however, has the drawback that it involves derivatives of $\mathbf{f}$, which might be more time-consuming to evaluate than points of $\mathbf{f}$, or might simply not be available. One alternative is the 'chord length' rule (16), but it only has second order accuracy (as will be shown in 4.1). This motivated Vincent and Forsey [16] to find a higher order method using only point evaluations (18). In this paper, we investigate the following much more general point-based method, which turns out to include these two methods as special cases.

We can first interpolate $\mathbf{f}$ with a polynomial $\mathbf{p}_n : [a, b] \to \mathbb{R}^d$, of degree $\leqslant n$, at some points

$$a \leqslant t_0 < t_1 < \cdots < t_n \leqslant b,$$

for some $n \geqslant 1$, i.e., $\mathbf{p}_n(t_i) = \mathbf{f}(t_i)$ for $i = 0, 1, \ldots, n$, giving the approximation

$$L(\mathbf{f}|_{[a,b]}) \approx L(\mathbf{p}_n|_{[a,b]}). \tag{4}$$

We can then estimate the length of $\mathbf{p}_n$ by quadrature, giving the estimate

$$L(\mathbf{p}_n|_{[a,b]}) \approx \sum_{j=0}^{m} w_j |\mathbf{p}_n'(q_j)|, \tag{5}$$

and by expressing $\mathbf{p}_n$ in the Lagrange form

$$\mathbf{p}_n(t) = \sum_{i=0}^{n} L_i(t)\mathbf{f}(t_i), \quad L_i(t) = \prod_{j=0, j\neq i}^{n} \frac{t - t_j}{t_i - t_j},$$

we get the point-based rule

$$L(\mathbf{f}|_{[a,b]}) \approx \sum_{j=0}^{m} w_j \left| \sum_{i=0}^{n} L_i'(q_j)\mathbf{f}(t_i) \right|. \tag{6}$$

In view of the definition of the length $L(\mathbf{f}|_{[a,b]})$ in (2), it is reasonable to expect that the error in (4) will be small due to the well-known fact that $\mathbf{p}_n'$ is a good approximation to $\mathbf{f}'$ when

$$h := b - a$$

is small. However, we have not seen this method explicitly referred to in the literature, nor are we aware of any error analysis. The main contribution of this paper is to offer a thorough analysis of the approximation order of the method, in terms of h, which depends on the points t_i, and the quadrature nodes and weights q_j and w_j as well as the smoothness of $\mathbf{f}$. One result of our analysis is that the interpolation points t_i can be chosen to maximize the approximation order, analogously to the use of Gauss–Legendre points for numerical integration.

2. Error of the derivative-based method

For the sake of comparison, we start with a comment about the approximation order of the derivative-based method (2). If the quadrature rule used in (2) has degree of precision r then the error will be of order $\mathrm{O}(h^{r+2})$ provided the $(r + 1)$th derivative of $F := |\mathbf{f}'|$ is bounded [12].

Lemma 1. *If* $\mathbf{f} \in C^{r+2}[\alpha, \beta]$, *and* $\mathbf{f}$ *is regular, then all the derivatives* $F', F'', \ldots, F^{(r+1)}$ *are bounded in* $[\alpha, \beta]$.

Proof. Let $k \in \{1, \ldots, r+1\}$. By Leibniz' rule,

$$2FF^{(k)} + \sum_{i=1}^{k-1} \binom{k}{i} F^{(i)} F^{(k-i)} = (F^2)^{(k)} = (\mathbf{f}' \cdot \mathbf{f}')^{(k)} = \sum_{i=0}^{k} \binom{k}{i} \mathbf{f}^{(i+1)} \cdot \mathbf{f}^{(k-i+1)},$$

and so

$$|F^{(k)}| \leqslant \frac{1}{2F} \left(\sum_{i=0}^{k} \binom{k}{i} |\mathbf{f}^{(i+1)}||\mathbf{f}^{(k-i+1)}| + \sum_{i=1}^{k-1} \binom{k}{i} |F^{(i)}||F^{(k-i)}| \right). \tag{7}$$

Now since $\mathbf{f}$ is regular on the closed interval $[\alpha, \beta]$, F attains a strictly positive minimum $\varepsilon > 0$. Further, by assumption, all the derivatives $\mathbf{f}', \ldots, \mathbf{f}^{k+1}$ are bounded. Therefore, assuming by induction that all the lower derivatives $F', \ldots, F^{(k-1)}$ are bounded, we see that $F^{(k)}$ is also bounded. □

This leads to the approximation order of the derivative-based method.

Theorem 1. *Suppose* $\mathbf{f} \in C^{r+2}[\alpha, \beta]$, $\mathbf{f}$ *is regular, and that rule* (2) *has degree of precision* r. *Then*

$$L(\mathbf{f}|_{[a,b]}) - \sum_{j=0}^{m} w_j |\mathbf{f}'(q_j)| = \mathrm{O}(h^{r+2}) \quad \text{as } h \to 0.$$

For example since the midpoint rule has degree of precision $r = 1$, we get

$$L(\mathbf{f}|_{[a,b]}) - h|\mathbf{f}'(q_0)| = \mathrm{O}(h^3), \tag{8}$$

where $q_0 = (a+b)/2$, provided $\mathbf{f} \in C^3[\alpha, \beta]$. Since Simpson's rule has degree of precision $r = 3$, we find

$$L(\mathbf{f}|_{[a,b]}) - h(|\mathbf{f}'(a)| + 4|\mathbf{f}'(q_1)| + |\mathbf{f}'(b)|)/6 = \mathrm{O}(h^5),$$

where $q_1 = (a+b)/2$, provided $\mathbf{f} \in C^5[\alpha, \beta]$.

If we take the $q_0, \ldots, q_m$ to be the Gauss nodes of order m, then the rule has degree of precision $2m+1$ and so provided $\mathbf{f} \in C^{2m+3}[\alpha, \beta]$, we get

$$L(\mathbf{f}|_{[a,b]}) - \sum_{j=0}^{m} w_j |\mathbf{f}'(q_j)| = \mathrm{O}(h^{2m+3}).$$

3. Error of the point-based method

There are two contributions to the error of the point-based method, namely the errors in the interpolation part (4) and the quadrature part (5). We will treat them both, starting with the quadrature error (5). Letting f_i and $p_{n,i}$ be the d components of the vector-valued $\mathbf{f}$ and $\mathbf{p}_n$, we recall a classical result of polynomial interpolation due to [12, Section 6.5, p. 290]:

$$|f_i^{(k)}(t) - p_{n,i}^{(k)}(t)| \leqslant h^{n+1-k} \frac{\max_{s \in [a,b]} |f_i^{(n+1)}(s)|}{(n+1-k)!}. \tag{9}$$

This equation does not hold for vector-valued functions, but we can still use it to derive some error bounds:

$$|f_i^{(k)}(t) - p_{n,i}^{(k)}(t)| \leqslant h^{n+1-k} \frac{\max_{s \in [a,b]} |\mathbf{f}^{(n+1)}(s)|}{(n+1-k)!}.$$

Using the notation

$$\|\mathbf{f}^{(n+1)}\|_{[a,b]} := \max_{s\in[a,b]} |\mathbf{f}^{(n+1)}(s)|,$$

we exploit the fact that the right-hand side above does not depend on the component i to write

$$\|\mathbf{f}^{(k)} - \mathbf{p}_n^{(k)}\|_{[a,b]} \leqslant C_k h^{n+1-k} \|\mathbf{f}^{(n+1)}\|_{[\alpha,\beta]}, \quad k = 0, 1, \ldots, n, \tag{10}$$

where $C_k = \sqrt{d}/(n+1-k)!$.

Lemma 2. *If* $\mathbf{f} \in C^{n+1}[\alpha, \beta]$ *and* $\mathbf{f}$ *is regular, then all derivatives of the function* $|\mathbf{p}_n'|$ *are bounded independently of* h *for small enough* h.

Proof. We will prove this by showing that $\mathbf{p}_n$ is regular for sufficiently small h, then apply Lemma 1. By the triangle inequality $|\mathbf{p}_n'(t)| \geqslant |\mathbf{f}'(t)| - |\mathbf{f}'(t) - \mathbf{p}_n'(t)|$ for all t. Using Eq. (10) in the case $k = 1$ we then see that

$$|\mathbf{p}_n'(t)| \geqslant |\mathbf{f}'(t)| - \|\mathbf{f}' - \mathbf{p}_n'\|_{[a,b]} \geqslant |\mathbf{f}'(t)| - C_1 h^n \|\mathbf{f}^{(n+1)}\|_{[\alpha,\beta]}.$$

Thus, since $\mathbf{f}'$ is bounded away from zero, so will $\mathbf{p}_n'$ be for sufficiently small h. Then $\mathbf{p}_n$ is regular. Since $\mathbf{p}_n$ is a polynomial, it is in C^{r+2} for all r and we can apply Lemma 1 to show that all derivatives of $|\mathbf{p}_n'|$ are bounded. $\square$

The approximation order of the quadrature part of the point-based method now immediately follows, analogously to Theorem 1. Provided $\mathbf{f} \in C^{n+1}[\alpha, \beta]$, we can make the order of this part of the error as high as we like simply by using a quadrature rule of high enough precision, independently of n.

Lemma 3. *Suppose* $\mathbf{f} \in C^{n+1}[\alpha, \beta]$, $\mathbf{f}$ *is regular, and that the quadrature rule in* (5) *has degree of precision* r *for any* $r \geqslant 0$. *Then*

$$L(\mathbf{p}_n|_{[a,b]}) - \sum_{j=0}^{m} w_j |\mathbf{p}_n'(q_j)| = \mathrm{O}(h^{r+2}).$$

Next we turn to the error in the interpolation part of method (4). The approximation order of this part depends crucially on the smoothness of $\mathbf{f}$. Again we will need to show that derivatives of certain terms are bounded.

Lemma 4. *If* $\mathbf{f} \in C^{n+1}[\alpha, \beta]$ *and* $\mathbf{f}$ *is regular, then all derivatives up to order* n *of the function* $\mathbf{g} := \mathbf{f}'/(|\mathbf{f}'| + |\mathbf{p}_n'|)$ *are bounded independently of* h *for small enough* h.

Proof. Clearly $\mathbf{g}$ itself is bounded independently of h, since $\mathbf{f}$ is regular. Next let $k \in \{1, \ldots, n\}$. Since,

$$((|\mathbf{f}'| + |\mathbf{p}_n'|)\mathbf{g})^{(k)} = \mathbf{f}^{(k+1)},$$

Leibniz' rule gives

$$\mathbf{g}^{(k)} = \frac{1}{|\mathbf{f}'| + |\mathbf{p}_n'|} \left(\mathbf{f}^{(k+1)} - \sum_{i=1}^{k} \binom{k}{i} (|\mathbf{f}'|^{(i)} + |\mathbf{p}_n'|^{(i)}) \mathbf{g}^{(k-i)} \right).$$

By Lemma 2, $|\mathbf{p}_n'|^{(i)}$ is bounded for each $i \geqslant 0$ when h is small enough. By Lemma 1, so is $|\mathbf{f}'|^{(i)}$ for $i = 0, \ldots, n$. Thus, if all derivatives of $\mathbf{g}$ up to order $k-1$ are bounded, so is $\mathbf{g}^{(k)}$. $\square$

This gives us our first result on the approximation order of the point-based method.

Lemma 5. *If* $\mathbf{f} \in C^{n+1}[\alpha, \beta]$ *and regular then, as* $h \to 0$,

$$L(\mathbf{f}|_{[a,b]}) - L(\mathbf{p}_n|_{[a,b]}) = \mathrm{O}(h^{n+1}). \tag{11}$$

If in addition $t_0 = a$ *and* $t_n = b$ *then*

$$L(\mathbf{f}|_{[a,b]}) - L(\mathbf{p}_n|_{[a,b]}) = \mathrm{O}(h^{n+2}). \tag{12}$$

Proof. Letting $\mathbf{e}(t) := \mathbf{f}(t) - \mathbf{p}_n(t)$, we use the identity

$$|\mathbf{f}'| - |\mathbf{p}_n'| = \frac{2\mathbf{e}'\cdot\mathbf{f}' - \mathbf{e}'\cdot\mathbf{e}'}{|\mathbf{f}'| + |\mathbf{p}_n'|}.$$

This gives us

$$\int_a^b |\mathbf{f}'(t)| - |\mathbf{p}_n'(t)|\,\mathrm{d}t = 2I_1 - I_2, \tag{13}$$

where

$$I_1 = \int_a^b \mathbf{e}'(t)\cdot\mathbf{g}(t)\,\mathrm{d}t, \quad I_2 = \int_a^b \frac{|\mathbf{e}'(t)|^2}{|\mathbf{f}'(t)| + |\mathbf{p}_n'(t)|}\,\mathrm{d}t$$

and $\mathbf{g} := \mathbf{f}'/(|\mathbf{f}'| + |\mathbf{p}_n'|)$. Since $\mathbf{e}'$ is of order $\mathrm{O}(h^n)$ by (10), and $|\mathbf{f}'(t)|$ is bounded away from zero, we see that $I_1 = \mathrm{O}(h^{n+1})$ and $I_2 = \mathrm{O}(h^{2n+1})$, and since $2n+1 \geqslant n+1$, this establishes (11).

If in addition $t_0 = a$ and $t_n = b$ then $\mathbf{e}(a) = \mathbf{e}(b) = \mathbf{0}$, and so integration by parts implies

$$I_1 = -\int_a^b \mathbf{e}(t)\cdot\mathbf{g}'(t)\,\mathrm{d}t. \tag{14}$$

Since $\mathbf{e}$ is $\mathrm{O}(h^{n+1})$ by (10), and $\mathbf{g}'(t)$ is bounded as $h \to 0$ by Lemma 4, we now have $I_1 = \mathrm{O}(h^{n+2})$. Since $n \geqslant 1$ we also have $I_2 = \mathrm{O}(h^{n+2})$, and this establishes (12). $\square$

It is interesting to note that *without* needing to raise the smoothness assumption on $\mathbf{f}$, we raise the approximation order by one simply by including the end points of the interval $[a, b]$ in the interpolation points t_i. Similar observations were made in [7,8]. Now the point is that we can continue to raise the order of approximation by further restricting the locations of the t_i. Notice that the order of the integral I_2 in (13) is already very high, namely $2n+1$ which means that we can raise the order of the whole error (13) by manipulating the first integral I_1. To do this we borrow from the idea of Gauss quadrature.

Lemma 6. *Suppose* $\mathbf{f} \in C^{2n}[\alpha, \beta]$ *and regular, and that* $t_0 = a$, $t_n = b$ *and*

$$\int_a^b \psi_n(t) t^k\,\mathrm{d}t = 0, \quad k = 0, 1, \ldots, n-2, \tag{15}$$

where $\psi_n(t) := (t - t_0)\cdots(t - t_n)$. *Then*

$$L(\mathbf{f}|_{[a,b]}) - L(\mathbf{p}_n|_{[a,b]}) = \mathrm{O}(h^{2n+1}).$$

Proof. It is enough to show that I_1 in (13) is of order $\mathrm{O}(h^{2n+1})$. Since

$$\mathbf{e}(t) = \psi_n(t)[t_0, t_1, \ldots, t_n, t]\mathbf{f},$$

where $[t_0, t_1, \ldots, t_n, t]\mathbf{f}$ denotes the divided difference of $\mathbf{f}$ at the points $t_0, t_1, \ldots, t_n, t$, we can write I_1 in (14) as

$$I_1 = -\int_a^b \psi_n(t)\gamma(t)\,\mathrm{d}t, \quad \gamma(t) := ([t_0, t_1, \ldots, t_n, t]\mathbf{f})\cdot\mathbf{g}'(t).$$

Thus if we expand γ in a Taylor series about a,

$$\gamma(t) = \sum_{k=0}^{n-2} \frac{1}{k!}(t-a)^k \gamma^{(k)}(a) + \frac{1}{(n-1)!}(t-a)^{n-1}\gamma^{(n-1)}(\xi_t),$$

with $a \leqslant \xi_t \leqslant t$, the orthogonality conditions (15) imply that

$$I_1 = -\frac{1}{(n-1)!}\int_a^b \psi_n(t)(t-a)^{n-1}\gamma^{(n-1)}(\xi_t)\,\mathrm{d}t.$$

Therefore since

$$|\psi_n(t)(t-a)^{n-1}| \leqslant h^{2n}, \quad a \leqslant t \leqslant b,$$

the lemma will be complete when we have shown that $\gamma^{(n-1)}$ is bounded as $h \to 0$. To see this, observe that Leibniz' rule gives

$$\gamma^{(n-1)}(t) = \sum_{j=0}^{n-1} \frac{(n-1)!}{(n-1-j)!}([t_0, t_1, \ldots, t_n, \underbrace{t, \ldots, t}_{j+1}]\mathbf{f}) \cdot \mathbf{g}^{(n-j)}(t).$$

Since

$$\left[t_0, t_1, \ldots, t_n, \underbrace{t, \ldots, t}_{j+1}\right] f_i = f_i^{(n+1+j)}(\mu_{j,i})/(n+1+j)!$$

for each component f_i of $\mathbf{f}$ and $\mathbf{f} \in C^{2n}[\alpha, \beta]$, and since all the derivatives $\mathbf{g}', \ldots, \mathbf{g}^{(n)}$ are bounded by Lemma 4, this shows that $\gamma^{(n-1)}$ is bounded as claimed. $\square$

Thus in order to increase the approximation order we can choose the t_i so that both $t_0 = a$ and $t_n = b$ and ψ_n is orthogonal to π_{n-2} (the space of polynomials of degree at most $n-2$) on $[a, b]$. This can be done by choosing

$$\psi_n(t) = (t-a)(t-b)P_n'(t),$$

where P_n is the Legendre polynomial of degree n on the interval $[a, b]$. A short calculation yields

$$\int_a^b \psi_n(t)t^k\,\mathrm{d}t = -\int_a^b P_n(t)((t-a)(t-b)kt^{k-1} + (2t-a-b)t^k)\,\mathrm{d}t + [P_n(t)(t-a)(t-b)t^k]_a^b.$$

For $k = 0, \ldots, n-2$ this is zero, since P_n is orthogonal to π_{n-1}.

The interpolation nodes we achieve in this manner are known in numerical integration as Gauss–Lobatto quadrature nodes. A table of nodes can be found in [1].

We are now able to give our main result.

Theorem 2. *Suppose that* $\mathbf{f} \in C^{2n}[\alpha, \beta]$, $\mathbf{f}$ *is regular, and that* $\{t_i\}_{i=0}^n$ *are the Gauss–Lobatto points in* $[a, b]$. *Suppose also that* $\{q_j\}_{j=0}^m$ *and* $\{w_j\}_{j=0}^m$ *are the nodes and weights, respectively, of a quadrature rule with degree of accuracy* $2n-1$ *on* $[a, b]$. *Then*

$$L(\mathbf{f}|_{[a,b]}) - \sum_{j=0}^m w_j |\mathbf{p}_n'(q_j)| = \mathrm{O}(h^{2n+1}).$$

Proof. This follows from the triangle inequality

$$\left| L(\mathbf{f}|_{[a,b]}) - \sum_{j=0}^m w_j |\mathbf{p}_n'(q_j)| \right| \leqslant |L(\mathbf{f}|_{[a,b]}) - L(\mathbf{p}_n|_{[a,b]})| + \left| L(\mathbf{p}_n|_{[a,b]}) - \sum_{j=0}^m w_j |\mathbf{p}_n'(q_j)| \right|.$$

and Lemmas 6 and 3. $\square$

Using our analysis, we now see that the point-based method is more robust than the derivative-based method from the point of view of the smoothness of $\mathbf{f}$. Given a desired local order of approximation, say $2n+1$, the point-based method of Theorem 2 only requires $\mathbf{f} \in C^{2n}[\alpha, \beta]$, while the derivative-based method of Theorem 1 requires $\mathbf{f} \in C^{2n+1}[\alpha, \beta]$.

4. Examples

4.1. Second order method

For $n=1$ the only choice of interpolation points satisfying Lemma 5 is $t_0=a$ and $t_1=b$. Computing the length of a linear curve does not call for quadrature, and we are left with the familiar chord length rule:

$$L(\mathbf{f}|_{[a,b]}) \approx |\mathbf{f}(b)-\mathbf{f}(a)|. \tag{16}$$

By Theorem 2, this rule has a local error of $\mathrm{O}(h^3)$, so when used as a composite rule, it has a global error of $\mathrm{O}(h^2)$. We have thus proved that the chord length rule has order of accuracy 2. According to Theorem 2, the required smoothness is that $\mathbf{f} \in C^2[\alpha,\beta]$. If we compare this to the midpoint method (8), we see that we have the same order of accuracy, but the midpoint rule requires $\mathbf{f} \in C^3[\alpha,\beta]$.

4.2. Fourth order methods

For $n=2$ there is precisely one choice of the points t_0, t_1, t_2 which satisfies the condition of Lemma 6. We must set $t_0=a$ and $t_2=b$. Then we must choose t_1 in order to make ψ_2 orthogonal with π_0, i.e., with the constant function 1. The only way this can be achieved is by the symmetric solution $t_1=(a+b)/2$. With this choice, if $\mathbf{f} \in C^4[\alpha,\beta]$ then

$$L(\mathbf{f}|_{[a,b]}) - L(\mathbf{p}_2) = \mathrm{O}(h^5) \quad \text{as } h \to 0.$$

Now we consider three choices of quadrature rule for $|\mathbf{p}_2'|$ in order to achieve an $\mathrm{O}(h^5)$ rule for $L(\mathbf{f}|_{[a,b]})$. All methods presented in this subsection will thus have local approximation order 5, and global order 4 (when used as a composite method).

4.2.1. Simpson-based rule

Simpson's rule applied to $|\mathbf{p}_2'|$ gives

$$L(\mathbf{f}|_{[a,b]}) \approx \frac{(b-a)}{6}(|\mathbf{p}_2'(q_0)| + 4|\mathbf{p}_2'(q_1)| + |\mathbf{p}_2'(q_2)|),$$

where $t_i = q_i$. Writing out the rule with $\mathbf{f}_i := \mathbf{f}(t_i)$, we get

$$L(\mathbf{f}|_{[a,b]}) \approx \tfrac{1}{6}(|-3\mathbf{f}_0 + 4\mathbf{f}_1 - \mathbf{f}_2| + 4|\mathbf{f}_2 - \mathbf{f}_0| + |\mathbf{f}_0 - 4\mathbf{f}_1 + 3\mathbf{f}_2|).$$

4.2.2. Gauss-based ('$\sqrt{3}$') rule

The two-point Gauss rule gives

$$L(\mathbf{f}|_{[a,b]}) \approx \frac{(b-a)}{2}(|\mathbf{p}_2'(q_0)| + |\mathbf{p}_2'(q_1)|),$$

where q_0, q_1 are $(a+b)/2 \mp ((b-a)/6)\sqrt{3}$. Writing out this rule gives

$$L(\mathbf{f}|_{[a,b]}) \approx |\mathbf{r} - \mathbf{f}_0| + |\mathbf{f}_2 - \mathbf{r}|, \tag{17}$$

where

$$\mathbf{r} = \tfrac{1}{2}(\mathbf{f}_0 + \mathbf{f}_2) + \tfrac{1}{3}\sqrt{3}(-\mathbf{f}_0 + 2\mathbf{f}_1 - \mathbf{f}_2).$$

This rule may be the one best suited for implementation, as it requires the computation of only two Euclidian norms, i.e., square roots. The other fourth order methods require three such computations.

4.2.3. The Vincent–Forsey rule

A third choice gives a very simple rule in terms of the points $\mathbf{f}_i$, $i = 0, 1, 2$. The open Newton–Cotes rule with three nodes has degree of precision 3, and gives

$$L(\mathbf{f}|_{[a,b]}) \approx \frac{(b-a)}{3}(2|\mathbf{p}_2'(q_0)| - |\mathbf{p}_2'(q_1)| + 2|\mathbf{p}_2'(q_2)|),$$

where $q_0 = (3a+b)/4$, $q_1 = (a+b)/2$, $q_2 = (a+3b)/4$. This can be written as

$$L(\mathbf{f}|_{[a,b]}) \approx \tfrac{4}{3}(|\mathbf{f}(q_1) - \mathbf{f}(a)| + |\mathbf{f}(b) - \mathbf{f}(q_1)|) - \tfrac{1}{3}|\mathbf{f}(b) - \mathbf{f}(a)|, \tag{18}$$

which is the method of Vincent and Forsey proposed in [16]. Their reasoning was based on approximating a circular segment, however, and not polynomials. Since the method satisfies the conditions of Theorem 2, we have proved that the Vincent–Forsey method has local error $O(h^5)$, and global error $O(h^4)$. Therefore it has fourth order of accuracy when used as a composite method.

4.3. Sixth order method

We now derive a sixth order method, by taking $n = 3$ and choosing interpolation nodes fulfiling the conditions of Lemma 6. To do this, we must take the interpolation nodes to be the nodes of the four-node Gauss–Lobatto scheme (see for instance [1]):

$$t_0 = a, \quad t_1 = a + \frac{b-a}{2}(1-\alpha), \quad t_2 = a + \frac{b-a}{2}(1+\alpha), \quad t_3 = b,$$

where $\alpha = \frac{1}{5}\sqrt{5}$.

In order to get optimum order, we must pick a quadrature method with local error $O(h^7)$. If we use the three-point Gauss method

$$L(\mathbf{f}|_{[a,b]}) \approx \frac{(b-a)}{18}(5|\mathbf{p}_3'(q_0)| + 8|\mathbf{p}_3'(q_1)| + 5|\mathbf{p}_3'(q_2)|),$$

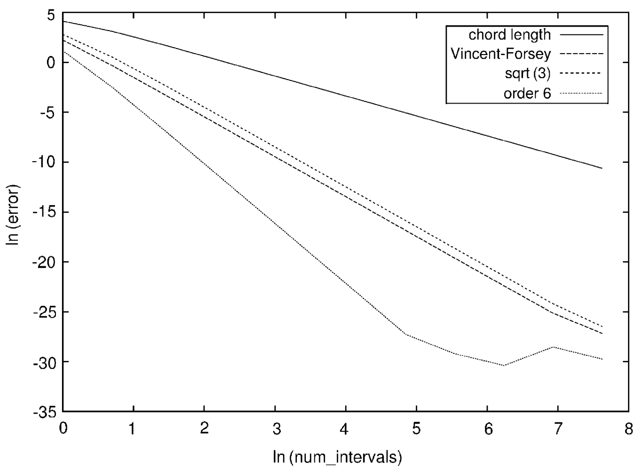

Fig. 1. Method error comparison.

with the nodes

$$q_0 = a + \frac{b-a}{2}(1-\beta), \quad q_1 = a + \frac{b-a}{2}, \quad q_2 = a + \frac{b-a}{2}(1+\beta),$$

where $\beta = \frac{1}{5}\sqrt{15}$, then we get the formula

$$L(\mathbf{f}|_{[a,b]}) \approx |\mathbf{r}_1 - \mathbf{f}_0| + |\mathbf{r}_2 - \mathbf{r}_1| + |\mathbf{f}_3 - \mathbf{r}_2|,$$

$$\mathbf{r}_1 = \sum_{i=0}^{3} \eta_i \mathbf{f}_i, \quad \mathbf{r}_2 = \sum_{i=0}^{3} \eta_{3-i} \mathbf{f}_i, \tag{19}$$

where the coefficients η_i are given by

$$\eta = \tfrac{1}{36}\left(16 - 5\sqrt{15},\ 10\sqrt{5} + 5\sqrt{15},\ -10\sqrt{5} + 5\sqrt{15},\ 20 - 5\sqrt{15}\right).$$

In Fig. 1, we have results from evaluating the length of a sample curve (in this case a circular segment) with composite rules built on various basic rules. We can see that we get the expected slope of -6 for the order 6 method until roundoff error becomes dominant. For the other methods, we also get the expected approximation order.

5. Geometric properties

As we have seen, the approximations of the '$\sqrt{3}$' method (17) and the sixth order method (19) can be written as the lengths of certain polygons. This geometric interpretation of the point-based method turns out to hold under fairly general conditions.

Theorem 3. *Suppose the quadrature weights w_j of rule* (6) *are positive, that the rule has precision of degree $\leqslant n-1$, and that $t_0 = a$ and $t_n = b$. Then the length estimate of* (6) *is equal to the length of a polygon with end points $\mathbf{f}(a)$ and $\mathbf{f}(b)$.*

Proof. We start from (6) and compute

$$\sum_{j=0}^{m} w_j \left| \sum_{i=0}^{n} L_i'(q_j)\mathbf{f}(t_i) \right| = \sum_{j=0}^{m} \left| w_j \sum_{i=0}^{n} L_i'(q_j)\mathbf{f}(t_i) \right| = \sum_{j=0}^{m} |\mathbf{a}_j| = \sum_{j=0}^{m} |\mathbf{r}_{j+1} - \mathbf{r}_j|,$$

where $\mathbf{r}_0 = \mathbf{f}(a)$ and $\mathbf{r}_j = \mathbf{f}(a) + \mathbf{a}_0 + \cdots + \mathbf{a}_{j-1}$. This is the length of the polygon with vertices $\mathbf{r}_0, \ldots, \mathbf{r}_{m+1}$. It remains to show that $\mathbf{r}_{m+1} = \mathbf{f}(b)$. This follows from

$$\begin{aligned}
\mathbf{r}_{m+1} &= \mathbf{f}(a) + \sum_{j=0}^{m} \mathbf{a}_j = \mathbf{f}(a) + \sum_{j=0}^{m} w_j \sum_{i=0}^{n} L_i'(q_j)\mathbf{f}(t_i) \\
&= \mathbf{f}(a) + \sum_{i=0}^{n} \mathbf{f}(t_i) \sum_{j=0}^{m} w_j L_i'(q_j) = \mathbf{f}(a) + \sum_{i=0}^{n} \mathbf{f}(t_i) \int_a^b L_i'(t)\,\mathrm{d}t \\
&= \mathbf{f}(a) + \sum_{i=0}^{n} \mathbf{f}(t_i)(L_i(b) - L_i(a)) = \mathbf{f}(b). \qquad \square
\end{aligned}$$

Now, we know that for any (continuous) curve $\mathbf{f}$,

$$L(\mathbf{f}|_{[a,b]}) \geqslant |\mathbf{f}(b) - \mathbf{f}(a)|.$$

It turns out that the estimated curve length given by the point-based rule (6) has the same property:

Corollary 1. *Under the assumptions of Theorem* 3, *the length estimate of* (6) *has the chord length as a lower bound*:

$$\sum_{j=0}^{m} w_j |\mathbf{p}_n'(q_j)| \geqslant |\mathbf{f}(b) - \mathbf{f}(a)|.$$

Proof. The length of any polygon from $\mathbf{f}(a)$ to $\mathbf{f}(b)$ is greater than or equal to the length of the straight line from $\mathbf{f}(a)$ to $\mathbf{f}(b)$ by the triangle inequality. □

Note that the conditions of the theorem are sufficient, but not necessary. For example, the Vincent–Forsey rule (18) is bounded below by chord length, in spite of not fulfiling the conditions of the theorem.

6. PH curve exactness

For general curves, it is not possible to find an analytic form for the arc length. However, there are classes of curves for which the arc length indeed has an analytic form. Examples of this include the pythagorean hodograph (PH) curves of Farouki [5], and the curve family introduced by Gil and Keren [9]. In this section we show that some of the point-based methods constructed are exact for PH curves.

The PH curves are planar polynomial curves $\mathbf{f} : [\alpha, \beta] \to \mathbb{R}^2$ with the property that $|\mathbf{f}'|$ is also a polynomial. One of the simplest examples is the curve $\mathbf{f}(t) = (x(t), y(t))$ where

$$x(t) = t - t^3/3, \quad y(t) = t^2.$$

Since

$$(x'(t))^2 + (y'(t))^2 = (1 + t^2)^2,$$

it follows that

$$|\mathbf{f}'(t)| = 1 + t^2.$$

Thus $\mathbf{f}$ is a PH cubic. In general a PH curve is any planar polynomial curve of degree $2k+1$ such that $|\mathbf{f}'|$ is a polynomial of degree $2k$.

If we apply the derivative-based method (2) to estimate the length of a curve $\mathbf{f}$ over an interval $[a, b]$, as long as we use a quadrature rule with degree of precision $\geqslant 2k$, the method will clearly be exact when $\mathbf{f}$ is a PH curve of degree $2k + 1$. Thus for example, if we apply Simpson's rule or the two-point Gauss rule to estimate the length of a PH cubic, the error will be zero.

Next consider the point-based method (6). Clearly, if $\mathbf{f}$ is a polynomial of degree $\leqslant n$ then $\mathbf{p}_n = \mathbf{f}$ and so $\mathbf{p}_n' = \mathbf{f}'$. In this case the point-based method reduces to the derivative-based one. Thus, for example, a point-based method with $n \geqslant 3$ (at least four points) will be exact for PH cubics $\mathbf{f}$.

An interesting situation is the case that $\mathbf{f}$ is a polynomial of exact degree $n + 1$, one higher than $\mathbf{p}_n$. This is the case when $\mathbf{f}$ is for example a PH cubic and we use the Gauss-based '$\sqrt{3}$' rule (17). Recall that

$$\mathbf{f}(t) - \mathbf{p}_n(t) = \psi_n(t)[t_0, t_1, \ldots, t_n, t]\mathbf{f}.$$

Therefore if $\mathbf{f}$ is a polynomial of degree $n + 1$,

$$\mathbf{f}'(t) - \mathbf{p}_n'(t) = \psi_n'(t)[t_0, t_1, \ldots, t_n, t]\mathbf{f}.$$

Thus we again find $\mathbf{p}_n'(q_i) = \mathbf{f}'(q_i)$ at certain points q_i, namely those for which $\psi_n'(q_i) = 0$. Now if the points $t_0, t_1, \ldots, t_n$ are the Gauss–Lobatto points then one can show that the points $q_1, \ldots, q_n$ for which $\psi_n'(q_i) = 0$ are precisely the Gauss points. Thus if we use Gauss–Lobatto points in the first part and Gauss points in the second, we get exactness for PH curves $\mathbf{f}$ of degree $n+1$. This is precisely what happens in the '$\sqrt{3}$' rule when applied to a PH cubic. The Vincent–Forsey rule on the other hand does not share this property.

More generally, if $\mathbf{f}$ is *any* cubic polynomial curve then the '$\sqrt{3}$' rule is the same as applying two-point Gauss integration to the speed function $|\mathbf{f}'|$.

7. Concluding remarks

We have made a framework for computing lengths of curves with only point evaluations, and shown that we do not lose accuracy compared to methods based on evaluating derivatives. We have also observed that the methods are robust, requiring one less order of smoothness than derivative-based methods with the same order of accuracy.

We have shown that some previously investigated methods fit in the framework, and thereby been able to give proofs of their approximation order.

In a future article we will investigate evaluating the areas of surfaces with only point evaluations.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, ninth ed., Dover, New York, 1972.
[2] G. Casciola, S. Morigi, Reparametrization of nurbs curves, Internat. J. Shape Modelling 2 (1996) 103–116.
[3] P. Constantini, R.T. Farouki, C. Manni, A. Sestini, Computation of optimal composite re-parametrizations, Comput. Aided Geom. Design 18 (2001) 875–897.
[4] R.T. Farouki, Optimal parametrizations, Comput. Aided Geom. Design 8 (1997) 153–168.
[5] R.T. Farouki, Pythagorean-hodograph curves, in: Handbook of Computer Aided Geometric Design, 2002, pp. 405–427.
[6] R.T. Farouki, T. Sakkalis, Real rational curves are not 'unit speed', Comput. Aided Geom. Design 8 (1991) 151–157.
[7] M.S. Floater, Arc length estimation and the convergence of polynomial curve interpolation, BIT, to appear.
[8] M.S. Floater, Chordal cubic spline interpolation is fourth order accurate, IMA J. Numer. Anal. (2005), to appear.
[9] J. Gil, D. Keren, New approach to the arc length parameterization problem, in: 13th Spring Conference on Computer Graphics, 1997, pp. 27–34.
[10] J. Gravesen, Adaptive subdivision and the length and energy of Bézier curves, Comput. Geom. 8 (1997) 13–31.
[11] B. Guenter, R. Parent, Computing the arc length of parametric curves, IEEE Comp. Graph. Appl. 5 (1990) 72–78.
[12] E. Isaacson, H.B. Keller, Analysis of Numerical Methods, Wiley, New York, 1966.
[13] J. Kearney, H. Wang, K. Atkinson, Arc-length parameterized spline curves for real-time simulation, in: Curve and Surface Design, Saint-Malo, 2002, pp. 387–396.
[14] E. Kreyszig, Differential Geometry, Dover, New York, 1991.
[15] R.J. Sharpe, R.W. Thorne, Numerical method for extracting an arc length parameterization from parametric curves, Comput. Aided Design 14 (2) (1982) 79–81.
[16] S. Vincent, D. Forsey, Fast and accurate parametric curve length computation, J. Graph. Tools 6 (4) (2002) 29–40.
[17] M. Walter, A. Fournier, Approximate arc length parameterization, in: Proceedings of the Ninth Brazilian Symposium on Computer Graphics and Image Processing, 1996, pp. 143–150.
[18] F.-C. Wang, D.C.H. Yang, Nearly arc-length parameterized quintic-spline interpolation for precision machining, Comput. Aided Design 25 (5) (1993) 281–288.