

Machine Learning 1 HW 3

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1 The l_2 Support Vector Machine [10pts]

We are given

$$\arg \min_{\mathbf{w}, b, \xi} \frac{1}{2} \|\mathbf{w}\|^2 + \frac{C}{2} \sum_{i=1}^n \xi_i^2 \quad \text{s.t.} \quad y_i(\mathbf{w}^T \mathbf{x}_i + b) \geq 1 - \xi_i \quad \forall i \in [n]$$

The Lagrangian is therefore given by

$$L(\mathbf{w}, b, \xi, \alpha) = \frac{1}{2} \|\mathbf{w}\|^2 + \frac{C}{2} \sum_{i=1}^n \xi_i^2 - \sum_{i=1}^n \alpha_i (y_i(\mathbf{w}^T \mathbf{x}_i + b) - 1 + \xi_i)$$

Where $\alpha_i \geq 0$. Differentiating with respect \mathbf{w} , we have:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{w}} &= \mathbf{w} - \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i = \mathbf{0} \\ \Rightarrow \mathbf{w} &= \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i \end{aligned}$$

Differentiating w.r.t. ξ_i we have

$$\begin{aligned} \frac{\partial L}{\partial \xi_i} &= C\xi_i - \alpha_i = 0 \\ \Rightarrow C\xi_i &= \alpha_i \end{aligned}$$

Differentiating w.r.r. b :

$$\frac{\partial L}{\partial b} = - \sum_{i=1}^n \alpha_i y_i = 0 \Rightarrow \sum_{i=1}^n \alpha_i y_i = 0$$

Plugging the derived expression for \mathbf{w} into the original Lagrangian eliminates the explicit dependence on \mathbf{w} , and recalling that $\alpha_i y_i (\mathbf{w}^T \mathbf{x}_i + b) = 0$:

$$\begin{aligned}
L(\alpha) &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + \frac{C}{2} \sum_{i=1}^n \xi_i^2 - \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - \sum_{i=1}^n \alpha_i (y_i b - 1 + \xi_i) \\
&= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + \frac{C}{2} \sum_{i=1}^n \xi_i^2 + \sum_{i=1}^n \alpha_i (1 - \xi_i) \\
&= \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + \frac{1}{2C} \sum_{i=1}^n (C \xi_i)^2 - \frac{1}{C} \sum_{i=1}^n \alpha_i (C \xi_i) \\
&= \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + \frac{1}{2C} \sum_{i=1}^n \alpha_i^2 - \frac{1}{C} \sum_{i=1}^n \alpha_i^2 \\
&= \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - \frac{1}{2C} \sum_{i=1}^n \alpha_i^2
\end{aligned}$$

Thus, the dual form is therefore given by:

$$\arg \min_{\alpha} \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - \frac{1}{2C} \sum_{i=1}^n \alpha_i^2$$

2 Domain Adaptation Support Vector Machines [10pts]

We are given the primal form

$$\arg \min_{\mathbf{w}_T, \xi} \frac{1}{2} \|\mathbf{w}_T\|^2 + C \sum_{i=1}^n \xi_i - B \mathbf{w}_T^T \mathbf{w}_S \quad \text{s.t.} \quad y_i (\mathbf{w}_T^T \mathbf{x}_i + b) \geq 1 - \xi_i, \quad \xi_i \geq 0 \quad \forall i \in [n]$$

The Lagrangian is therefore:

$$L(\mathbf{w}_T, \xi, \mu, \alpha) = \frac{1}{2} \|\mathbf{w}_T\|^2 + C \sum_{i=1}^n \xi_i - B \mathbf{w}_T^T \mathbf{w}_S - \sum_{i=1}^n \mu_i \xi_i - \sum_{i=1}^n \alpha_i [y_i (\mathbf{w}_T^T \mathbf{x}_i + b) - 1 + \xi_i]$$

Deriving the dual form, we differentiate L w.r.t. \mathbf{w}_T :

$$\begin{aligned}
\frac{\partial L}{\partial \mathbf{w}_T} &= \mathbf{w}_T - B \mathbf{w}_S - \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i = 0 \\
\Rightarrow \mathbf{w}_T &= B \mathbf{w}_S + \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i
\end{aligned}$$

Differentiating w.r.t. ξ_i we have

$$\begin{aligned}
\frac{\partial L}{\partial \xi_i} &= C - \mu_i - \alpha_i = 0 \\
\Rightarrow C &= \mu_i + \alpha_i
\end{aligned}$$

Differentiating w.r.t. b

$$\frac{\partial L}{\partial b} = \sum_{i=1}^n \alpha_i y_i = 0$$

Let's take a separate look at each of the components of the Lagrangian and simplify using the results from the derivatives. First note that

$$\begin{aligned} \frac{1}{2} \|\mathbf{w}_T\|^2 &= \frac{1}{2} \left(B\mathbf{w}_S + \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i \right)^T \left(B\mathbf{w}_S + \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i \right) \\ &= \frac{1}{2} B^2 \|\mathbf{w}_S\|^2 + B\mathbf{w}_S^T \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i \mathbf{x}_j \end{aligned}$$

Next, let's look at

$$\begin{aligned} -\sum_{i=1}^n \alpha_i [y_i(\mathbf{w}_T^T \mathbf{x}_i + b) - 1 + \xi_i] &= -\sum_{i=1}^n \alpha_i y_i \mathbf{w}_T^T \mathbf{x}_i + \sum_{i=1}^n \alpha_i - \sum_{i=1}^n \alpha_i \xi_i \\ &= -\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i \mathbf{x}_j - B \sum_{i=1}^n \alpha_i y_i \mathbf{w}_S^T \mathbf{x}_i + \sum_{i=1}^n \alpha_i - \sum_{i=1}^n \alpha_i \xi_i \end{aligned}$$

And finally,

$$\begin{aligned} -B\mathbf{w}_T^T \mathbf{w}_S &= -B(B\mathbf{w}_S + \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i)^T \mathbf{w}_S \\ &= -B^2 \|\mathbf{w}_S\|^2 - B\mathbf{w}_S^T \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i \end{aligned}$$

Where I have colored like terms the same. The Lagrangian is therefore

$$\begin{aligned} L(\alpha) &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - B \sum_{i=1}^n \alpha_i y_i \mathbf{w}_S^T \mathbf{x}_i + C \sum_{i=1}^n \xi_i - \sum_{i=1}^n \mu_i \xi_i + \sum_{i=1}^n \alpha_i - \sum_{i=1}^n \alpha_i \xi_i \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - B \sum_{i=1}^n \alpha_i y_i \mathbf{w}_S^T \mathbf{x}_i + C \sum_{i=1}^n \xi_i + \sum_{i=1}^n \alpha_i - \sum_{i=1}^n (\alpha_i + \mu_i) \xi_i \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - B \sum_{i=1}^n \alpha_i y_i \mathbf{w}_S^T \mathbf{x}_i + C \sum_{i=1}^n \xi_i + \sum_{i=1}^n \alpha_i - C \sum_{i=1}^n \xi_i \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j - B \sum_{i=1}^n \alpha_i y_i \mathbf{w}_S^T \mathbf{x}_i + \sum_{i=1}^n \alpha_i \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + \sum_{i=1}^n \alpha_i (1 - B y_i \mathbf{w}_S^T \mathbf{x}_i) \end{aligned}$$

The dual problem is therefore,

$$\arg \max_{\alpha} -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \mathbf{x}_i^T \mathbf{x}_j + \sum_{i=1}^n \alpha_i (1 - B y_i \mathbf{w}_S^T \mathbf{x}_i) \quad \text{s.t.} \quad 0 \leq \alpha \leq C, \quad \sum_{i=1}^n \alpha_i y_i = 0$$