FH JOANNEUM UNIVERSITY OF APPLIED SCIENCES

Heat Transfer

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LAV Master Studies, WS 2014/2015

Revision of the basic laws of Thermodynamics

- **0th Law of Thermodynamics:**
  - If two (macroscopic) physical systems, e.g., two reservoirs containing some gas or liquid are in thermal equilibrium with a third system, then they are also in thermal equilibrium with each other; <u>This property of macroscopic systems makes possible the definition of (absolute) temperature</u> by taking one single reference system (e.g. liquid water or, better, some ideal gas) to define a temperature scale. The Celsius scale, for instance, was historically defined via the freezing point (defined as 0°C) and the boiling point (defined as 100°C) of water, and subdividing this temperature interval into 100 equidistant sections.

Revision of the basic laws of Thermodynamics

- Note: By observation, thermal equilibrium of two (otherwise isolated) macroscopic physical systems is achieved when they are in thermal contact for a sufficiently long period of time.

Revision of the basic laws of Thermodynamics

- **1st Law of Thermodynamics:**
  - Recall that Thermodynamics is based on the notions of work, heat and energy.
  - By definition, (mechanical) work is performed by a process if this process can be used to lift some piece of mass in a gravitational field.
  - Energy is the ability of a system to perform work; if work is performed on an otherwise isolated system, then the energy of the system is increased, that is, the ability of the system to perform work (at some later time) increases. It is possible, however, that – in certain configurations – only a fractional amount of the system’s energy can be converted to mechanical work.

Revision of the basic laws of Thermodynamics

- Heat is a special form of energy: Heat is the energy which is exchanged by two systems at different temperature solely by bringing them in thermal contact with each other.

- At the level of atoms, molecules and photons: Heat is the energy stored in the chaotic movement of atoms and molecules (including rotational and vibrational modes) as well as in thermal radiation.

Revision of the basic laws of Thermodynamics

- **2nd Law of Thermodynamics:**
  - So far, it was observed in all experiments that heat is spontaneously transferred from hotter to colder systems but never vice versa
  - Interpretation of this empirical findings by using the notion of entropy: Only those processes take place in nature by which the entropy of the universe (which is a measure for the universe’s disorder) increases.
  - Note: Heat and work are process variables whereas entropy is a state variable (in the same manner as, e.g., temperature, pressure and internal energy)

Revision of the basic laws of Thermodynamics

- **3rd Law of Thermodynamics:**
  - No system can ever reach a temperature of absolute zero.

The mechanisms of heat transfer

- **Conduction:** in a solid, a liquid, or a gas conduction is the transport of thermal energy via the collision of atoms or molecules with each other. Therefore heat conduction is a microscopic (equivalently, molecular) transport mechanism.
- **Convection:** in a gas or liquid convection is the bulk movement of macroscopic amounts of material from one location to another. This material transport is (in a large number of cases) accompanied by the transport of thermal energy (and possibly also other forms of energy, such as chemical energy).

The mechanisms of heat transfer

- **Radiation** is the transfer of heat by photons; this mechanism requires no intermediate material;

- Note: In many practical cases these three mechanisms occur simultaneously.

Heat transfer by conduction

- The thermal energy of a hotter system (or a hotter *location* within a system if conduction is considered within a single system) is transferred by collisions between atoms/molecules to a colder system. Even in a solid, where the molecules are located at fixed positions, they vibrate and rotate and hand over their kinetic energy to atoms or molecules at the neighboring positions.

Heat transfer by conduction

- Heat transfer by conduction typically takes place in <u>solid materials</u> but it also plays an important role in the boundary layers of near-wall flows.
- Fourier's law of heat conduction describes the rate at which heat is transferred from one location in the material to an infinitesimally neighbored position in the material: the heat flux $\vec{q}$ ($\vec{q}$ is a three-dimensional vector field which is meaningful within the material, and $[\vec{q}] = W/m^2$) is proportional to the negative temperature gradient in the material at the location that is considered: $\vec{q} \sim -\vec{\nabla} T$. The proportionality factor depends on the material and describes how well the material conducts heat: $\vec{q} = -k\vec{\nabla} T$, $[k] = W/(m \cdot K)$.

Heat transfer by conduction

The parameter k is called the thermal conductivity of the material. In general, k depends not only on the nature of the material but also on its thermodynamic state (temperature and pressure if the material consists only of a single phase).

Thermal conductivity of various materials

Material	$k\ [W/(m \cdot K)]$	Temperature [K]
Air	$1.810 \cdot 10^{-2}$	200
Air	$2.407 \cdot 10^{-2}$	273.15
Air	$3.951 \cdot 10^{-2}$	500
Air	$1.75 \cdot 10^{-1}$	2500
Aluminum	237	293
Water (lq)	$5.62 \cdot 10^{-1}$	273.15
Water (lq)	$6.77 \cdot 10^{-1}$	373.15

Heat transfer by conduction

- The steady heat equation:
  - Suppose steady heat transfer by conduction is considered, e.g., in a piece of metal. Then the conservation of energy for an arbitrary control volume in the material yields, in conjunction with Fourier's law of heat conduction, the steady state heat equation:

$$-\vec{\nabla} \cdot \left(k \vec{\nabla} T\right) = \text{Heat source terms}$$

  - Heat sources (phys. dimension W/m^3) are, for instance:
    - Heat generated by an electric current
    - Heat released or consumed by phase changes (latent heat) or by chemical reactions

Heat transfer by conduction

- The time-dependent heat equation:
  - Similarly to the steady heat equation, the consideration of conservation of energy for an arbitrary control volume in the material yields, in conjunction with Fourier's law of heat conduction, the unsteady heat equation:

$$\frac{\partial(c\rho T)}{\partial t} - \vec{\nabla} \cdot \left(k\vec{\nabla}T\right) = \text{Heat source terms}$$

$$c = \text{Specific heat capacity of the material } \left([c] = \frac{J}{kg \cdot K}\right)$$

$$\rho = \text{Density of the material } \left([\rho] = \frac{kg}{m^3}\right)$$

Heat transfer by conduction

- Boundary and initial conditions for the heat equation:

  - Initial conditions:

    - The initial temperature distribution within the material must be provided for the unsteady heat equation.

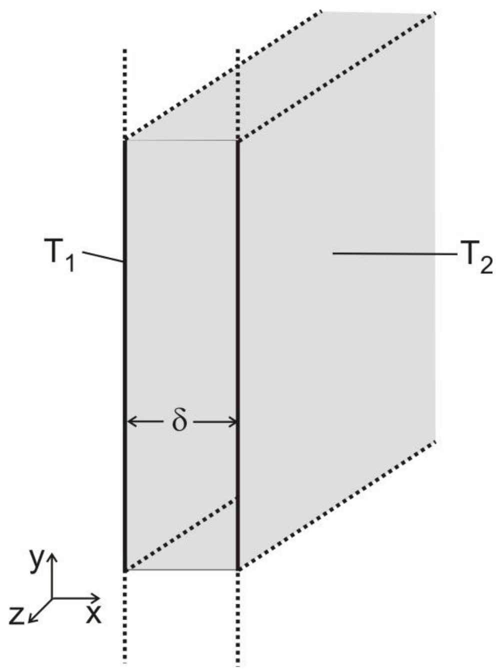
    - No initial conditions need to be specified for the steady heat equation.

Heat transfer by conduction

- Boundary conditions:
  - For the unsteady heat equation either the wall temperature (Dirichlet condition) *or* the wall heat flux (Neumann condition) must be provided at any time.
  - For the steady heat equation time-independent boundary conditions (again Dirichlet *or* Neumann conditions) must be provided at the boundaries.

Heat transfer by conduction

- Example: Solution of the steady heat equation for an infinite homogeneous wall with constant thickness and constant temperature on either side:



Heat transfer by radiation

- Stefan-Boltzmann law for the radiative energy flux M_{rad} ($[M_{rad}] = W/m^2$) of a black body (i.e., a body having perfect absorptivity/emissivity of electromagnetic radiation): $M_{rad} = \sigma T^4$. The constant $\sigma = 5.67 \cdot 10^{-8} \frac{W}{m^2 K^4}$ is called the Stefan-Boltzmann constant.
- For a grey body (i.e., a body having a reduced absorptivity/emissivity relative to a black body) the law for radiative energy loss is $M_{rad} = \varepsilon\sigma T^4$. The emissivity ε ($[\varepsilon] = 1$) is material-dependent and also depends on the structure of the surface. The parameter ε lies in the range $0 < \varepsilon < 1$. Highly polished silver, for instance, has $\varepsilon = 0.02$.

Heat transfer by radiation

- When a hotter body with surface temperature T_h is located in a colder surrounding area with temperature T_l then the radiative energy flux from the body is proportional to the difference of the fourth powers of T_h and T_l: $\dot{E} = C({T_h}^4 - {T_l}^4)$. The constant C depends on various parameters such as the absorptivity, the reflexivity, and the transmission coefficient of the material.

Convective heat transfer

- Convective heat transfer will be the main topic of this course.
- Many (or even most) problems in convective heat transfer, especially when turbulent flows are involved, have very complicated solutions which can be found only empirically by experiments or with the aid of numerical simulations. In a number of cases, however, (e.g. for certain types of heat exchangers) (semi-)empirical, mostly algebraic formulas for the heat transfer coefficients are available and make possible simple heat transfer calculations "by hand" or at least by using "relatively simple" computer programs. These calculations are often of great significance for engineers.

Convective heat transfer

- However, for involved problems in complex geometries numerical solutions of the full Navier-Stokes equations (including turbulence equations and equations describing heat transfer by conduction, convection and radiation) are often the only way of treating heat transfer problems is a satisfactory way.
- Therefore transport equations for mass, momentum and energy as well as thermodynamic equations of state must be considered in order to tackle problems in convective heat transfer in full generality.

Revision of the continuity and the momentum equations in a viscous fluid

- <u>Continuity equation:</u>
  - The continuity equation reads $\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{V}) = 0.$
    - Here ρ is the fluid density, $[\rho] = kg/m^3$
    - and $\vec{V} = (u, v, w)^T$ is the velocity field, $[\vec{V}] = m/s$
  - For steady flow problems the continuity equation reads $\mathrm{div}(\rho \vec{V}) = 0.$
  - For incompressible flows (ρ is independent of time and location) the continuity equation simplifies to $\mathrm{div}(\vec{V}) = 0.$

Revision of the continuity and the momentum equations in a viscous fluid

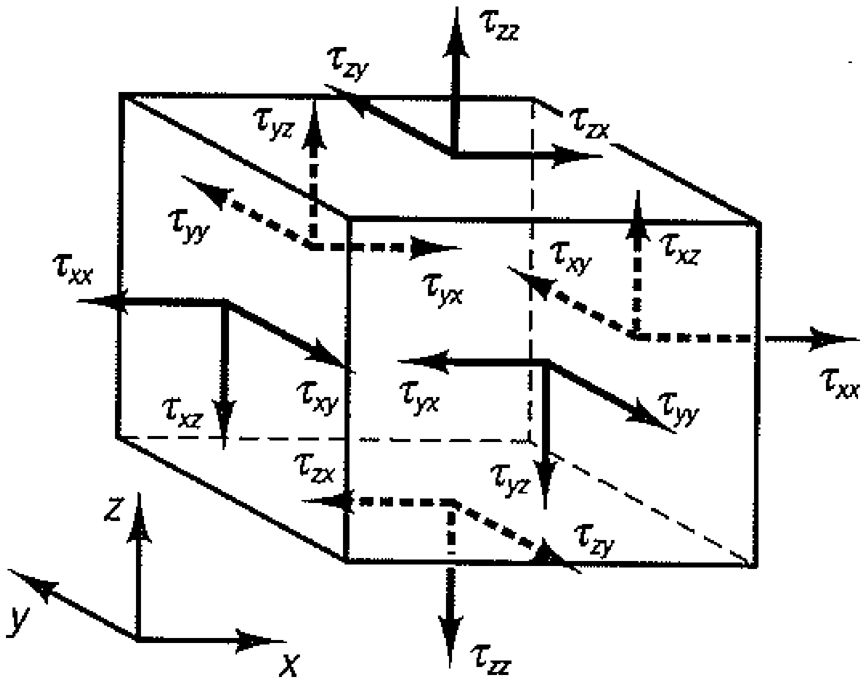
- Momentum equations for a general fluid:

- In x-direction: $\frac{\partial(\rho u)}{\partial t}+\mathrm{div}\left(\rho u\vec{V}\right)=-\frac{\partial p}{\partial x}+\mathrm{div}\begin{pmatrix}\tau_{xx}\\ \tau_{yx}\\ \tau_{zx}\end{pmatrix}$

- In y-direction: $\frac{\partial(\rho v)}{\partial t}+\mathrm{div}\left(\rho v\vec{V}\right)=-\frac{\partial p}{\partial y}+\mathrm{div}\begin{pmatrix}\tau_{xy}\\ \tau_{yy}\\ \tau_{zy}\end{pmatrix}$

- In z-direction: $\frac{\partial(\rho w)}{\partial t}+\mathrm{div}\left(\rho w\vec{V}\right)=-\frac{\partial p}{\partial z}+\mathrm{div}\begin{pmatrix}\tau_{xz}\\ \tau_{yz}\\ \tau_{zz}\end{pmatrix}$

Revision of the continuity and the momentum equations in a viscous fluid



Note: $\tau_{ij} = \tau_{ji}$, i.e., τ is a symmetric tensor

Viscous stress components on the faces of a fluid element; Source: H.K. Versteeg et al., "Computational Fluid Dynamics"

Revision of the continuity and the momentum equations in a viscous fluid

- Stress tensor in a Newtonian fluid:
  - $\tau_{ij} = \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) + \gamma\delta_{ij}\mathrm{div}(\vec{V})$
  - μ = dynamic viscosity, $[\mu] = \frac{kg}{m \cdot s}$
  - γ = volume viscosity, $[\gamma] = \frac{kg}{m \cdot s}$
  - Stokes' hypothesis says that $\gamma = -\frac{2}{3}\mu$

Revision of the continuity and the momentum equations in a viscous fluid

- Momentum equations for an incompressible Newtonian fluid:
  - In x-direction: $\frac{\partial(\rho u)}{\partial t} + \mathrm{div}\left(\rho u \vec{V}\right) = -\frac{\partial p}{\partial x} + \mu \Delta u$
  - In y-direction: $\frac{\partial(\rho v)}{\partial t} + \mathrm{div}\left(\rho v \vec{V}\right) = -\frac{\partial p}{\partial y} + \mu \Delta v$
  - In z-direction: $\frac{\partial(\rho w)}{\partial t} + \mathrm{div}\left(\rho w \vec{V}\right) = -\frac{\partial p}{\partial z} + \mu \Delta w$
  - Here $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplace operator.

The energy equation

- Notation:
  - e = static internal energy of the fluid per unit mass, $[e] = J/kg$
  - $E = e + \frac{1}{2}\left|\vec{V}\right|^2$ = total energy of the fluid per unit mass, $[E] = J/kg$
  - $\dot{q}$ = a heat source (or sink) per unit mass, e.g. heat generated by electrical heating or heat exchanged by radiation, $[\dot{q}] = \frac{W}{kg}$
  - k = thermal conductivity, $[k] = \frac{W}{m \cdot K}$
  - $\vec{f}$ = body acceleration, e.g., gravitational accelerations, $[\vec{f}] = \frac{m}{s^2}$

The energy equation

- Energy equation:
  - $\frac{\partial(\rho E)}{\partial t} + \mathrm{div}\big(\rho E\vec{V}\big) = \mathrm{div}\big(k \cdot \mathrm{grad}(T)\big) - \mathrm{div}\big(p\vec{V}\big) + \mathrm{div}\big(\tau \cdot \vec{V}\big) + \rho\dot{q} + \rho\vec{f} \cdot \vec{V}$
  - $\mathrm{div}\big(k \cdot \mathrm{grad}(T)\big)$ … thermal conduction term
  - $\mathrm{div}\big(p\vec{V}\big)$ … work performed by pressure forces
  - $\mathrm{div}\big(\tau \cdot \vec{V}\big)$ … viscous force term
  - $\rho\dot{q}$ … heat introduced/extracted by the heat source/sink
  - $\rho\vec{f} \cdot \vec{V}$ … work performed by body forces

Equations of state

- If the medium consists of a single phase and is in (local) thermodynamic equilibrium, then the thermodynamic parameters p, T, ρ, e, h ($h = e + \frac{p}{\rho}$ denotes the specific static enthalpy) are related with each other by so-called equations of state, e.g.
- $p = p(\rho, T)$
- $e = e(p, T)$
- $h = h(\rho, T)$

Equations of state

- If the medium is, for example an ideal gas, then:
  - $p = \rho \frac{R}{m_w} T$
  - Here $R = 8.3145 \frac{J}{mol \cdot K}$ denotes the universal gas constant and m_w denotes the molecular weight of the gas, $[m_w] = \frac{kg}{mol}$
  - Further, the specific internal energy e is related to the temperature by $e = c_v T$, where c_v is the specific heat for constant volume, $[c_v] = \frac{J}{kg \cdot K}$

Initial- and boundary conditions for the dynamic equations

- Everywhere in the solution region ρ, $\vec{V}$ and T must be given at time $t = 0$.
- On solid walls, the fluid velocity $\vec{V}$ relative to the wall must be zero (no-slip condition).
- On solid walls, <u>either the temperature</u> T (fixed temperature) <u>or the heat flux</u> $\vec{q} = -k\frac{\partial T}{\partial n}$ in normal direction (fixed heat flux) must be specified as a function of time.
- On inlets ρ, $\vec{V}$ and T must be specified as a function of time.

Initial- and boundary conditions for the dynamic equations

- On outlets, $-p + \mu \frac{\partial u_n}{\partial n} = F_n$ (specification of the normal stresses), $\mu \frac{\partial u_t}{\partial n} = F_t$ (specification of the tangential stresses), and $q_n = -k \frac{\partial T}{\partial n}$ (specification of the heat flux normal to the outlet surface) must be provided as a function of time.
- The most common outlet-bc in the finite volume method are:
  1. Specified pressure on the outlet surface
  2. $\frac{\partial u_n}{\partial n} = \frac{\partial u_t}{\partial n} = 0$
  3. $\frac{\partial T}{\partial n} = 0$

Initial- and boundary conditions for the dynamic equations

- Note: It is common practice in CFD to place the outlet as far downstream as possible from the place of action – possibly by enlarging the flow geometry. This minimizes the influence of the outlet boundary conditions on the flow solution at the locations of interest.

Boundary layers

- At the beginnings of the 20^{th} century Ludwig Prandtl investigated the influence of the viscosity on large Reynolds number flows. He derived equations (namely, the boundary layer equations) for regions near solid walls where the influence of – even a very small viscosity – on the flow field becomes important. In the course of his discoveries the modern boundary layer theory was born.
- The basic idea behind boundary layer theory is that viscous effects play no role in the bulk flow far off from walls but become – due to high velocity gradients in normal direction to a wall – important in near wall regions.

Boundary layers

- The bulk flow field can (often but not always) be described by equations assuming an inviscid fluid; in particular, potential flow solutions often describe the global flow field very well in cases without boundary layer separation.
- The near-wall flow inside the boundary layer, on the other hand, can be described by simplified dynamic equations which make possible a thorough analysis of the structure of the (usually very thin) boundary layer region.

The external laminar boundary layer

- Recall the boundary layer equations (2D, incompressible, steady, flow, and wall parallel to x-direction):
  - Special conditions inside the boundary layer relative to free stream conditions:
    - $u \gg v$ and $\left|\frac{\partial u}{\partial y}\right| \gg \left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial v}{\partial x}\right|$; (From $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ [continuity eq.] it then follows that also $\left|\frac{\partial u}{\partial y}\right| \gg \left|\frac{\partial v}{\partial y}\right|$)
- By studying the full Navier-Stokes equations in non-dimensional form and using these approximations, the boundary layer equations follow:

The external laminar boundary layer

- Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
- x-momentum: $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2}$
- y-momentum: $\frac{\partial p}{\partial y} = 0$ (and therefore $p = p(x)$, which implies $\frac{\partial p}{\partial x} = \frac{dp}{dx}$)

- Boundary conditions:
  - For $y = 0$: $u = 0$ and $v = 0$
  - For $y \to \infty$: $u(x, y) = u_\infty(x)$, where $u_\infty(x)$ denotes the free-stream velocity
  - The free-stream velocity satisfies the equation $\rho u_\infty \frac{du_\infty}{dx} = -\frac{dp}{dx}$ (which essentially is Bernoulli's equation)

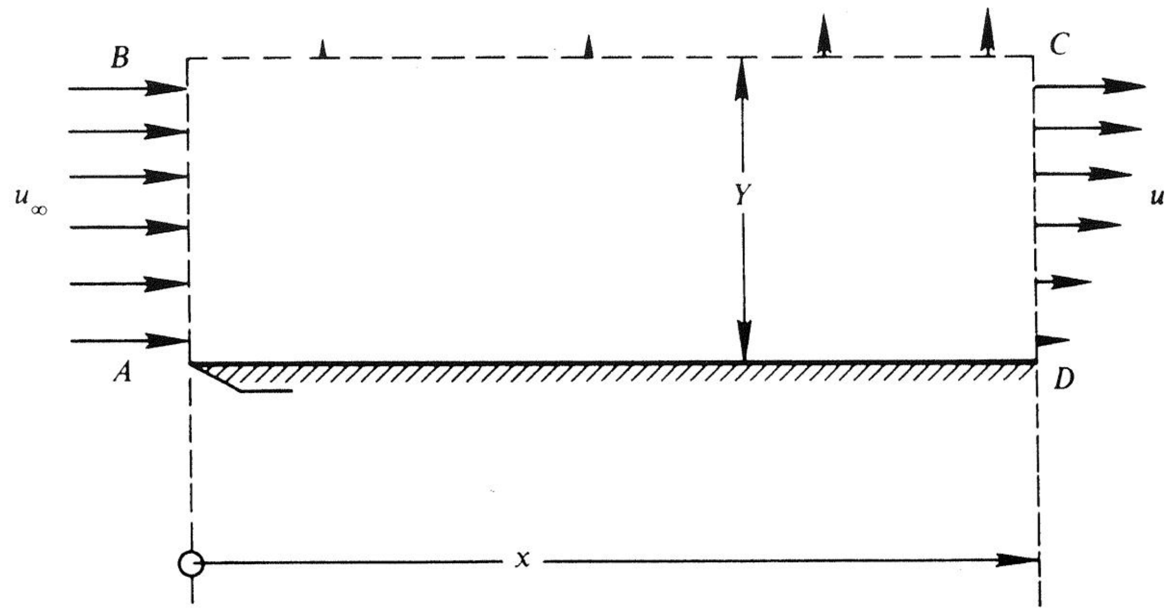
Measures for the boundary layer thickness

- $\delta_{99}(x) \coloneqq y$-distance at which $u(x,y)$ becomes 99% of the free-stream velocity $u_\infty(x)$.

- Displacement thickness: $\delta_1(x) \coloneqq \int_0^\infty \left(1 - \frac{u(x,y)}{u_\infty(x)}\right) dy$

- Momentum thickness: $\delta_2(x) \coloneqq \int_0^\infty \frac{u(x,y)}{u_\infty(x)} \left(1 - \frac{u(x,y)}{u_\infty(x)}\right) dy$

Measures for the boundary layer thickness



Source: W. Kays et al., "Convective Heat and Mass Transfer".

The displacement thickness is defined as the mass flow across boundary BC divided by ρu_∞, and the momentum thickness is defined as the loss of x-momentum through boundary AD divided by ρu_∞^2. In both cases the limit $Y \to \infty$ is applied.

External laminar boundary layers: The flat plate solution (Blasius solution)

- In the case of a semi-infinite plate with constant free-stream velocity $u_\infty(x) = u_\infty$ parallel to the plate the pressure gradient along the plate is zero.

- By using the ansatz $u(x,y) = f(\frac{y}{\sqrt{x}})$ (the quantity $\frac{y}{\sqrt{x}}$ is a so-called similarity parameter), the boundary layer equations (which are non-linear partial differential equations) can be turned to a <u>single non-linear ordinary differential equation</u> – the Blasius equation:
  $\zeta''' + \frac{1}{2}\zeta\zeta'' = 0$; here $\zeta'(\eta) := \frac{u(\eta)}{u_\infty}$, with $\eta = \frac{y}{\sqrt{\nu x/u_\infty}}$

- The boundary conditions for the Blasius equation are $\zeta'(0) = 0$, $\zeta'(\infty) = 1$ and $\zeta'''(0) = 0$ (which follows from the x-momentum equation). The latter implies $\zeta(0) = 0$ since $\zeta''(0) = 0$ would imply zero shear stress at the wall $y = 0$).

External laminar boundary layers: The flat plate solution (Blasius solution)

η	ζ	ζ'	ζ''
0	0	0	0.3321
0.2	0.00664	0.06641	0.3320
0.4	0.02656	0.13277	0.3315
0.6	0.05974	0.19894	
0.8	0.10611	0.26471	
1.0	0.16557	0.32979	
1.2	0.23795	0.39378	
1.4	0.32298	0.45627	
1.6	0.42032	0.51676	
1.8	0.52952	0.57477	
2.0	0.65003	0.62977	
2.2	0.78120	0.68132	
2.4	0.92230	0.72899	
2.6	1.07252	0.77246	
2.8	1.23099	0.81152	
3.0	1.39682	0.84605	
3.2	1.56911	0.87609	
3.4	1.74696	0.90177	
3.6	1.92954	0.92333	
3.8	2.11605	0.94112	
4.0	2.30576	0.95552	
4.2	2.49806	0.96696	
4.4	2.69238	0.97587	
4.6	2.88826	0.98269	
4.8	3.08534	0.98779	
5.0	3.28329	0.99155	

For higher values of η, $\zeta = \eta - 1.72$

Numeric solution to the Blasius equation with $u_\infty = \text{constant}$;
Source: W. Kays et al., “Convective Heat and Mass Transfer”.

External laminar boundary layers: The flat plate solution (Blasius solution)

- Having found the solution to the Blasius solution it is now easy to compute the local wall shear stress and the local friction coefficient for the flat plate laminar boundary layer:

- $\tau_w(x) = \mu \left(\frac{\partial u}{\partial y}\right)_w = \mu \left[\frac{du}{d\eta}\frac{\partial \eta}{\partial y}\right]_w = \mu u_\infty \zeta''(0) \left[\frac{\partial \eta}{\partial y}\right]_w = \mu \sqrt{\frac{u_\infty^3}{\nu x}}\zeta''(0) =$

  $= 0.332 \cdot \mu \sqrt{\frac{u_\infty^3}{\nu x}}$

- $c_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho u_\infty^2} = \frac{0.664}{\sqrt{Re_x}}$, where $Re_x = \frac{u_\infty x}{\nu}$

External laminar boundary layers: The flat plate solution (Blasius solution)

- Exercise: Compute the average friction coefficient along the plate as a function of the running length l.

External laminar boundary layers: The flat plate solution

- Displacement thickness of the laminar boundary layer for the flat plate: $\delta_1(x) = 1.72\frac{x}{\sqrt{Re_x}}$; Exercise: Prove this

- Momentum thickness of the laminar boundary layer for the flat plate: $\delta_2(x) = 0.664\frac{x}{\sqrt{Re_x}}$

- δ_{99} of the flat plate boundary layer: $\delta_{99}(x) = 5.0\frac{x}{\sqrt{Re_x}}$; Exercise: Prove this

Key parameters in heat transfer

- **Heat transfer coefficient**
  - Consider a fluid boundary (e.g. a solid wall) which can absorb or release heat. Then the (local) heat transfer coefficient of the boundary is defined as the quotient of the wall heat flux q and the difference of some reference temperature T_{ref} (e.g. free stream temperature) and the surface temperature T_s:

$$h := \frac{q}{T_s - T_{ref}}$$

  - Note that the physical dimension of h is $\frac{W}{m^2 \cdot K}$.

Key parameters in heat transfer

- **Nusselt number**
  - The Nusselt number is a dimensionless heat transfer coefficient:

$$Nu = \frac{h \cdot L_{ref}}{k}$$

In this equation L_{ref} is a characteristic length scale (e.g. the pipe diameter, running length of the boundary layer or the cylinder diameter), h is the heat transfer coefficient and k is the thermal conductivity of the fluid.

Key parameters in heat transfer

- **Stanton number**
  - The Stanton number is also a dimensionless heat transfer coefficient. It is defined by

$$St = \frac{h}{\rho \cdot V_{ref} \cdot c_p}$$

    In this equation V_{ref} is some reference velocity, e.g. free stream velocity.

  - From the very definition of the Nusselt and Stanton numbers it follows that $Nu = Re \cdot Pr \cdot St$, where $Pr \coloneqq \frac{\mu \cdot c_p}{k}$ is called the Prandtl number of the medium.

External laminar boundary layers: Heat transfer

- In the following discussion we consider a semi-infinite flat plate, and we assume constant free-stream velocity u_∞ and constant free-stream temperature T_∞ of the medium.
- We consider laminar flow along the plate.
- The surface temperature of the plate is assumed to be constant and will be denoted by T_s.
- The local temperature of the medium (which strongly varies inside the boundary layer) is denoted by $T = T(x, y)$.

External laminar boundary layers: Heat transfer

- For a steady, 2-dimensional flow with temperature- and pressure-independent material properties of the medium it can be shown that the energy equation takes on – inside the laminar boundary layer – the following much simpler form:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial x}\right)^2$$

- When heat production by viscous friction within the boundary layer can be neglected (which is reasonable at low speed), then the equation becomes even simpler:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$$

External laminar boundary layers: Heat transfer

- We define the non-dimensional temperature $\theta = \theta(x, y) \coloneqq \frac{T_s - T}{T_s - T_\infty}$
- Then the energy equation $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$ becomes $u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}$, where $\alpha =: \frac{k}{\rho c_p}$ is called the thermal diffusivity of the medium.
- The boundary conditions for the constant-temperature semi-infinite plate are:
  - $\theta = 0$ at $y = 0$
  - $\theta = 1$ as $y \to \infty$
  - $\theta = 1$ as $x \to 0-$

External laminar boundary layers: Heat transfer

- Suppose first that the thermal diffusivity $\alpha = \frac{\mu}{\rho}$, that is, $Pr = 1$. Then the momentum and energy equations for the boundary layer become identical, and also the boundary conditions for $\frac{u}{u_\infty}$ on the one hand and θ on the other hand are identical. Therefore the non-dimensional velocity $\frac{u}{u_\infty}$ and the non-dimensional temperature θ grow along the flat plate at the same rate: $\theta(x, y) = \frac{u(x,y)}{u_\infty}$.

  To find a solution also for the general case $Pr \neq 1$, we assume again that similarity solutions exist – this time for the temperature field: $\theta = \theta(\eta)$, where $\eta = \eta(x, y) = \frac{y}{\sqrt{\nu x / u_\infty}}$

External laminar boundary layers: Heat transfer

- By substituting this into the thermal boundary layer equation we obtain the equation $\theta'' + \frac{Pr}{2}\zeta\theta' = 0$ (with $\zeta'(\eta) \coloneqq \frac{u(\eta)}{u_\infty}$) which resembles the Blasius equation. Since the Blasius equation has been solved (numerically), $\zeta(\eta)$ can be treated as a known function.

  The equation $\theta'' + \frac{Pr}{2}\zeta\theta' = 0$ can be re-written: $\frac{d\theta'}{d\eta} + \frac{Pr}{2}\zeta\theta' = 0$. In this form the equation can be integrated directly. Taking also the boundary conditions into account the solution is

$$\theta(\eta) = C_1 \int_0^\eta \left[\exp\left(-\frac{Pr}{2}\int_0^\eta \zeta d\eta\right)\right] d\eta,$$

where $C_1 = \left(\int_0^\infty \left[\exp\left(-\frac{Pr}{2}\int_0^\eta \zeta d\eta\right)\right] d\eta\right)^{-1}$.

External laminar boundary layers: Heat transfer

- Since ζ is known, these integrals can be solved easily by numeric means.
- To compute the heat transfer from the plate, we recall that

$h = \frac{q}{T_s - T_{ref}}$ and $q = -k\left(\frac{\partial T}{\partial y}\right)_s$; Therefore $h = k\theta'(0)\frac{1}{\sqrt{\frac{\nu x}{u_\infty}}}$, and thus

$Nu_x = \theta'(0) \cdot \sqrt{Re_x}$, where $\theta'(0) = \frac{1}{\int_0^\infty \left[\exp\left(-\frac{Pr}{2}\int_0^\eta \zeta d\eta\right)\right] d\eta}$. Since $\theta'(0)$ is a function of the Prandtl number only, we can write

$Nu_x = F(Pr) \cdot \sqrt{Re_x}$, with $F(Pr) = \frac{1}{\int_0^\infty \left[\exp\left(-\frac{Pr}{2}\int_0^\eta \zeta d\eta\right)\right] d\eta}$.

External laminar boundary layers: Heat transfer

- For the range of Prandtl numbers $0.5 \leq Pr \leq 15$ (this includes air as well as water – the latter in the temperature range 0°C – 100°C), it turns out that $F(Pr)$ can be approximated very accurately by $F(Pr) \approx 0.332 \cdot \sqrt[3]{Pr}$. Therefore we finally obtain

$$Nu_x = 0.332 \cdot \sqrt[3]{Pr} \cdot \sqrt{Re_x}$$

for the local Nusselt number of the flat plate with laminar boundary layer and constant surface temperature.

Insertion: Measures for the thickness of the thermal boundary layer

- **Enthalpy thickness:**
  - Similar to the definition of the displacement and momentum thickness of the hydrodynamic boundary layer one defines the enthalpy thickness of the thermal boundary layer:

$$\delta_{ent} = \int_0^{\infty} \frac{u}{u_\infty} \frac{h_{ref}(T)}{h_{ref}(T_s)} dy$$

- **Conduction thickness:**

$$\delta_{cond} = \frac{k(T_s - T_\infty)}{q} = \frac{k}{h}$$

External laminar boundary layers: Heat transfer

- The local Stanton number for the flat plate at constant temperature (and assuming a callorically perfect gas) can be written, based on the enthalpy thickness, as

$$\mathrm{St} = 0.2204 \cdot Pr^{-4/3} \cdot Re_{\delta_{ent}}^{-1}$$

and for the local Nusselt number one obtains

$$Nu_{\delta_{ent}} = 0.2204 \cdot Pr^{-1/3}$$

External laminar boundary layers: Heat transfer

- In the following we again consider a semi-infinite flat plate; we assume constant free-stream velocity u_∞ and also constant free-stream temperature T_∞ of the medium.
- We again consider laminar flow along the plate.
- This time the surface heat flux of the plate is prescribed; we assume it to be constant along the plate and denote it by q_{plate}.
- The local air temperature is denoted by $T = T(x, y)$, and the surface temperature of the plate (which is a priori unknown in the current case) is denoted by T_s.

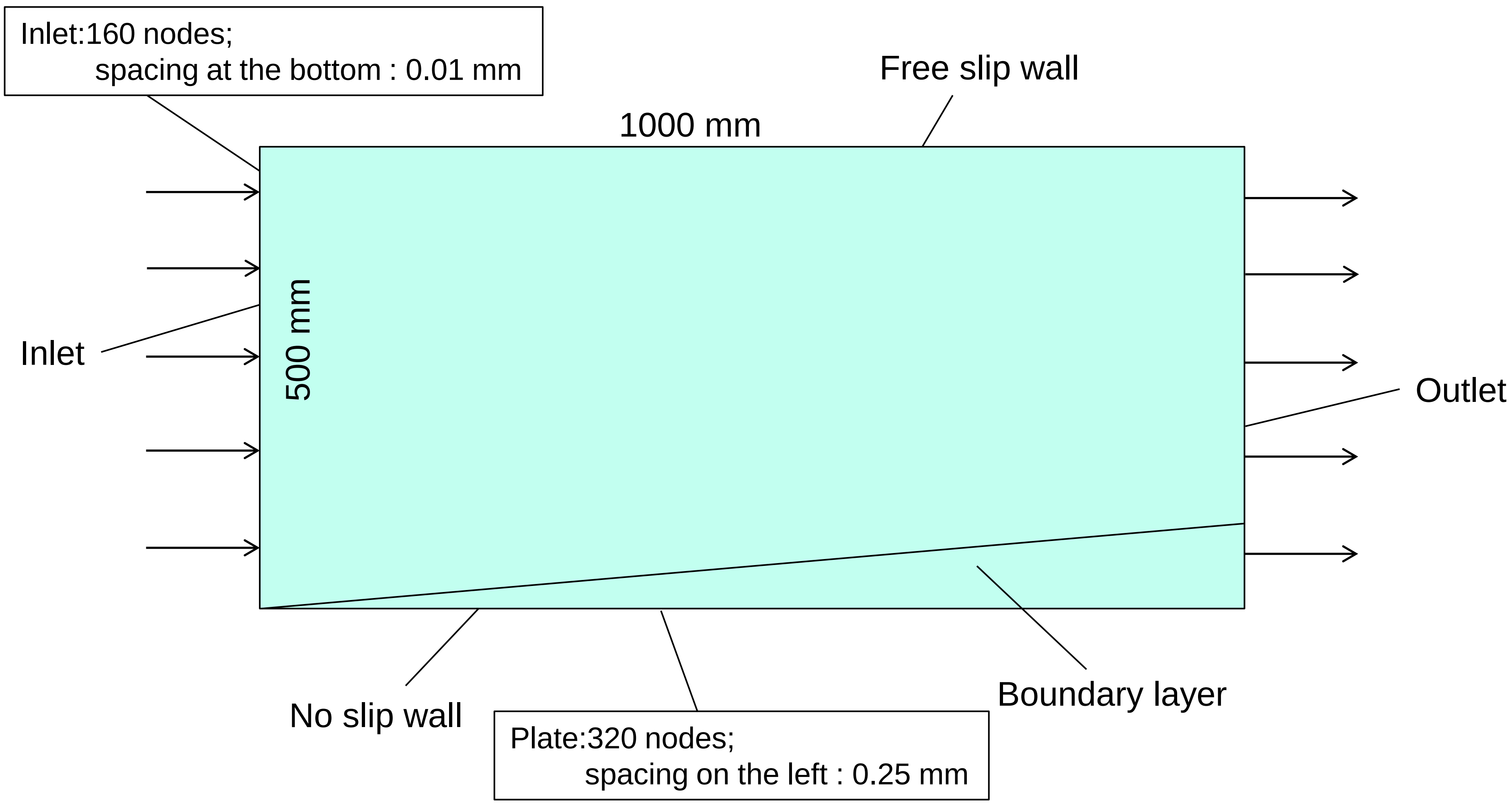
External laminar boundary layers: Heat transfer

- By similar computations as were performed for the constant-temperature plate one obtains for the local Nusselt number in the current case: $Nu_x = 0.453 \cdot \sqrt[3]{Pr} \cdot \sqrt{Re_x}$
- Interpretation: As a general rule it can be stated that the heat transfer coefficient increases when the surface temperature changes from constant to an – in flow direction – increasing temperature profile. Conversely, when the surface temperature changes from constant to a decreasing (again in flow direction) temperature profile, then the local heat transfer coefficient decreases.

External laminar boundary layers: Heat transfer

- **In extreme cases the local heat transfer coefficient loses its significance:** It then becomes by itself a function of the surface temperature (distribution)

External laminar boundary layers: Heat transfer



External laminar boundary layers: Heat transfer

- **First case:**
  - Medium: Air at 25°C
  - $V_{in} = 1.0\,\frac{m}{s}$
  - $T_{in} = 25.0°\mathrm{C}$
  - $T_{plate} = 26.0°\mathrm{C}$

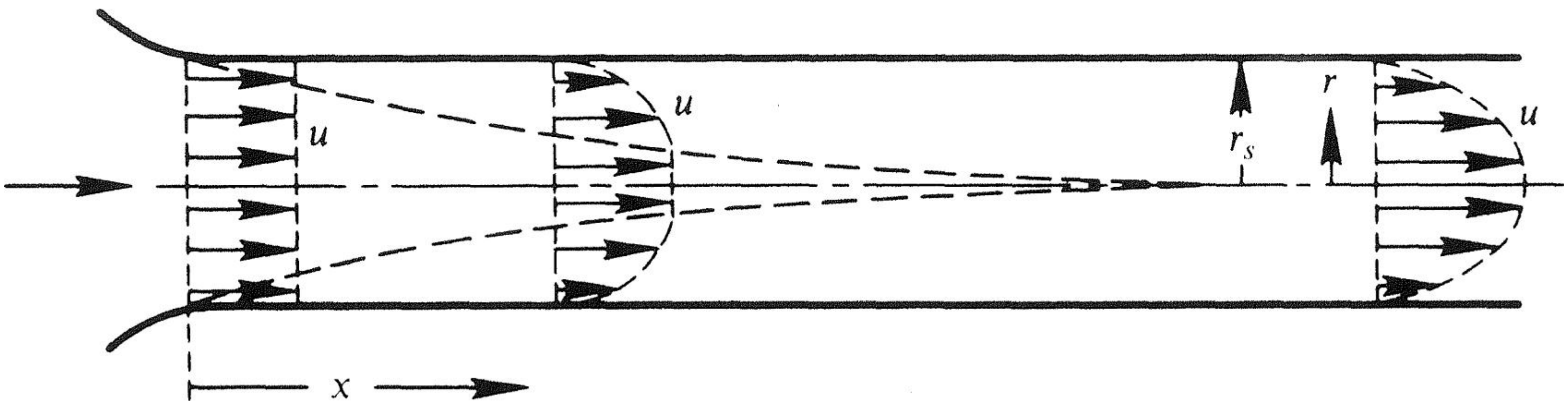
- **Task:** Compute the local wall heat flux with CFX and compare the result with the heat flux predicted by the formula $Nu_x = 0.332 \cdot \sqrt[3]{Pr} \cdot \sqrt{Re_x}$

External laminar boundary layers: Heat transfer

- **Second case:**
  - Medium: Air at 25°C
  - $V_{in} = 1.0 \frac{m}{s}$
  - $T_{in} = 25.0°C$
  - $q_{plate} = 10.0 \frac{W}{m^2}$
- **Task:** Compute the local wall temperature in CFX and compare the result with the wall temperature predicted by the formula $Nu_x = 0.453 \cdot \sqrt[3]{Pr} \cdot \sqrt{Re_x}$ (resp. $Nu_x = 0.464 \cdot \sqrt[3]{Pr} \cdot \sqrt{Re_x}$ [from Schlichting, Boundary Layer Theory])

Internal laminar boundary layers: Momentum transfer

- Consider a steady laminar flow of a viscous fluid inside a circular tube:



Development of the velocity profile in the hydrodynamic entry region of a pipe; Source: W. Kays et al., "Convective Heat and Mass Transfer".

Internal laminar boundary layers: Momentum transfer

- A particular simplifying feature of viscous flow inside tubes is the fact that the boundary layer developing at the entry region eventually meets itself at the tube centerline. Therefore the velocity distribution establishes – far downstream of the entrance region – a fixed pattern being invariant thereafter.
- We refer to the hydrodynamic entry length as that part of the tube in which the momentum boundary layer grows and the velocity distribution changes with length.
- We speak of a fully developed velocity profile as the fixed radial velocity distribution in the fully developed region.

Internal laminar boundary layers: Momentum transfer

- Without taking care (at least for the moment) about how long the hydrodynamic entry length must be in order for a fully developed velocity profile to be obtained, we evaluate the fully developed velocity distribution for a laminar flow in a pipe having circular cross section with radius R.
- In cylindrical coordinates (the centerline of the pipe is assumed to lie on the x-axis) and for axisymmetric flow the boundary layer equation reads: $u\frac{\partial u}{\partial x} + v_r\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$; here v_r is the radial velocity component of the flow field.

Internal laminar boundary layers: Momentum transfer

- Therefore, in the case of a fully developed velocity profile, the boundary layer equation simplifies to $-\frac{dp}{dx} + \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$
- The boundary conditions are:
  - $u(R) = 0$ (no-slip condition)
  - $\frac{du}{dr}(0) = 0$ (this follows from radial symmetry)
- The solution can be found easily:
  - $u(r) = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)$; this is the familiar parabolic law for fully developed pipe flow; in fact, this solution is also an exact solution of the full incompressible Navier Stokes equations.

Internal laminar boundary layers: Momentum transfer

- Further, since the velocity profile is independent of x (fully developed flow, by assumption), $-\frac{dp}{dx} = \text{constant}.$

- Exercise 1:
  - Write $u(r)$ as a function of the mean velocity V in the pipe rather than as a function of the pressure gradient.

Internal laminar boundary layers: Momentum transfer

- Further, since the velocity profile is independent of x (fully developed flow, by assumption), $-\frac{dp}{dx} = \text{constant}$

- Exercise 1:
  - Write $u(r)$ as a function of the mean velocity V in the pipe rather than as a function of the pressure gradient.
- Solution: $u(r) = 2V\left(1 - \frac{r^2}{R^2}\right)$

Internal laminar boundary layers: Momentum transfer

- Exercise 2:
  - Compute the wall shear stress τ_s and the friction coefficient c_f (based on the mean velocity) for fully developed pipe flow. Furthermore, express c_f as a function of the Reynolds number $\mathrm{Re} = \frac{2R\rho V}{\mu}$ (which is based on the mean velocity V and the pipe diameter $2R$)

Internal laminar boundary layers: Momentum transfer

- Exercise 2:
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- Solution: $\tau_s = \frac{4V\mu}{R}$ and $c_f = \frac{16}{\mathrm{Re}}$

Internal laminar boundary layers: Momentum transfer

- Remark: Computing the flow field for fully developed laminar flow in other cross-sectional shape tubes comes down to solving a Poisson problem: $-\frac{dp}{dx} + \mu\, \Delta u = 0$. Such problems can be solved by standard procedures.
- When working with non-circular cross sectional shape tubes, the (natural) length scale defined by the geometry is the hydraulic diameter D_h:

$$D_h = \frac{4\ \times\ \text{cross}-\text{sectional area}}{\text{wetted perimeter}}$$

Internal laminar boundary layers: Momentum transfer

- Remark: The hydraulic radius is defined by

$$r_h = \frac{\text{cross-sectional area}}{\text{wetted perimeter}},$$

so $D_h = 4r_h$; therefore the hydraulic diameter is four times – rather than twice – the hydraulic radius.

- Example: An equilateral triangular tube with side length a of the triangular cross section has hydraulic diameter $D_h = \frac{a}{\sqrt{3}}$; the mean friction coefficient (i.e. the local friction coefficient averaged along the perimeter) turns out to be $c_f = \frac{13.33}{\mathrm{Re}_{D_h}}$, where $\mathrm{Re}_{D_h} = \frac{D_h \rho V}{\mu}$.

Internal laminar boundary layers: Momentum transfer

- Entry length: The preceding discussion has been concerned with the velocity profile at points far downstream of the entrance region, where fully developed conditions prevail. The complete hydrodynamic solution for a tube, however, must include some kind of entry length, as depicted earlier. For a circular tube with axisymmetric constant property flow the differential equation of motion becomes the momentum equation with μ constant:

$$u\frac{\partial u}{\partial x}+v_r\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

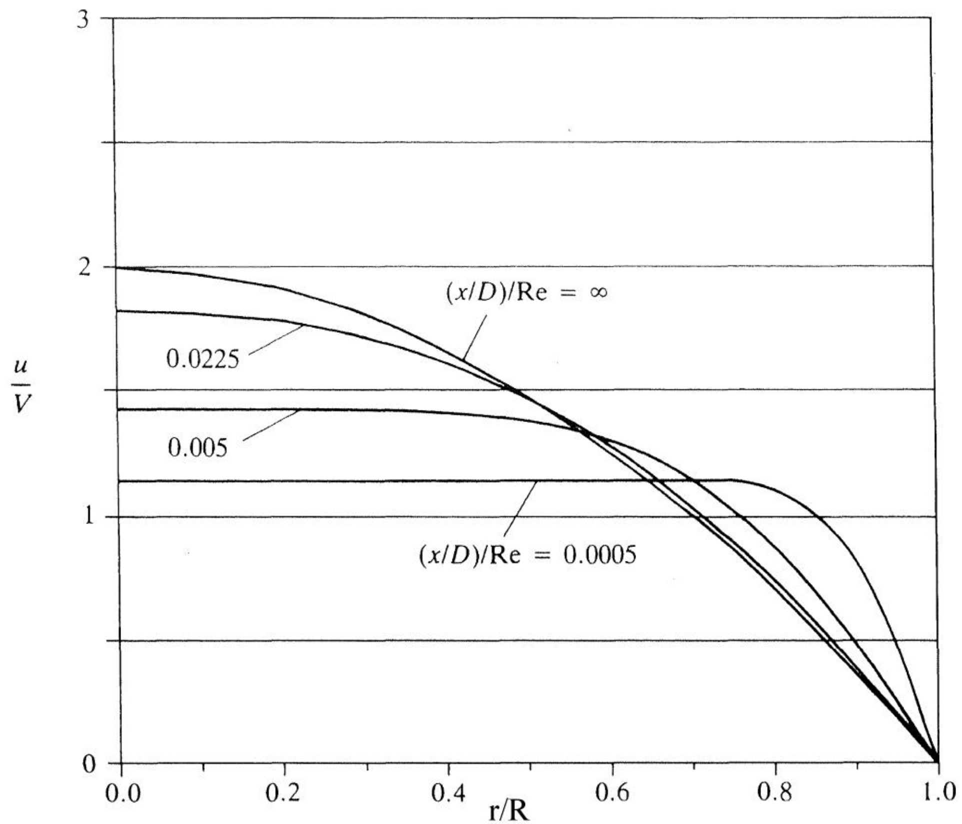
Internal laminar boundary layers: Momentum transfer

- The simplest entry condition would be a uniform velocity profile at $x = 0$, as depicted in the figure shown earlier.
- <u>Note:</u> Experimental and numerical studies show that even for a sharp-cornered or abrupt concentration entrance, the velocity profile develops in much the same manner, although the behavior for the first few diameters from the entrance would be somewhat different.
- Note also, that the boundary layer equation incorporates the boundary layer assumptions which are – due to large x-gradients very near the entrance – not fulfilled in this region.

Internal laminar boundary layers: Momentum transfer

- According to investigations by Shah and London (R.K. Shah and A.L. London, “Laminar Flow Forced Convection in Ducts”, *Advances in Heat Transfer*, Academic Press, NY, 1978), if the full Navier-Stokes equations are employed, it is found that for $Re < 400$ and $((x/D)/Re) < 0.005$ the boundary layer equation will <u>lead to errors in the computed velocity profiles</u>. The solutions depicted in the following figure (which were obtained from solutions of the boundary layer equation) are therefore only valid downstream of this region.

Internal laminar boundary layers: Momentum transfer



Radial velocity distribution at various x-locations in the hydrodynamic entry region of a circular tube; Source: W. Kays et al., "Convective Heat and Mass Transfer".

Internal laminar boundary layers: Momentum transfer

- From this figure it becomes apparent that for $((x/D)/Re) \sim 0.0225$ the velocity profile differs from the fully developed profile by no more than approx. 10%. Thus a good approximate figure for the length of tube necessary for the development of the laminar velocity profile is

$$\frac{x}{D} = \frac{Re}{20}.$$

Using this approximation for the entry length, the deviation from the fully developed profile is (in particular with respect to the local friction coefficient) less than about 2%.

Internal laminar boundary layers: Heat transfer

Heat transfer in internal flows

- We first consider the situation of a circular tube with fully developed velocity and temperature profiles of a constant property fluid.
- Then we look at the case when the velocity profile is fully developed and the fluid temperature upstream of some point is assumed to be uniform and equal to the surface temperature, there being no heat transfer in this region. Following this point, heat transfer takes place. These thermal-entry-length solutions are considered in detail for the circular tube and will be compared with our own CFD-computations.

Internal laminar boundary layers: Heat transfer

- Finally, some results are presented for the combined hydrodynamic- and thermal-entry length, that is, where velocity and fluid temperatures are uniform at the tube entrance.

- The energy differential equation for flow through a circular tube (centerline in x-direction, incompressible constant property fluid, no body and pressure forces, no viscous dissipation, no external heat sources such as radiation, and steady flow assumption) is:

$$u\frac{\partial T}{\partial x} + v_r\frac{\partial T}{\partial r} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial x^2}\right),$$

with the thermal diffusivity $\alpha = k/(\rho c_p)$.

Internal laminar boundary layers: Heat transfer

- We further restrict to the problem of radial symmetric heat transfer. Then:

$$\frac{\partial^2 T}{\partial \phi^2} = 0$$

- For the moment, we also assume hydrodynamically fully developed flow ($v_r = 0$), and we assume that axial heat conduction can be neglected (it will be discussed later when the latter is not the case) relative to radial heat conduction: $\alpha \frac{\partial^2 T}{\partial x^2} = 0$
- Then the thermal equation simplifies to:

$$u \frac{\partial T}{\partial x} = \alpha \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right)$$

Internal laminar boundary layers: Heat transfer

- Under a few possible heating conditions (in particular, constant surface temperature and constant surface heat flux) there exists, far downstream of the entrance region, a generalized non-dimensional radial temperature profile that is invariant with tube length. To define a non-dimensional temperature θ, we need a reference temperature. This could be the temperature at the centerline of the tube or the so-called mixed mean temperature T_m:

$$T_m := \frac{1}{V \cdot A} \int_A uT dA$$

Here A denotes the cross sectional area of the pipe and V denotes the mean velocity.

Internal laminar boundary layers: Heat transfer

- The non-dimensional temperature is then defined by

$$\theta = \frac{T_s - T}{T_s - T_m} \,,$$

where T_s denotes the surface temperature.

- Now we look for such solutions of the energy equation for which θ is only a function of r, i.e., for which θ is invariant along the x-axis. We are particularly interested in the surface heat transfer coefficient, which is defined by $q_s = h(T_s - T_m)$, where q_s denotes the surface heat flux: $q_s = k\left(\frac{\partial T}{\partial r}\right)_{r=R}$

Internal laminar boundary layers: Heat transfer

- Since θ is (assumed to be) only a function of r, the condition

  $\left(\frac{\partial\theta}{\partial r}\right)_{r=R} = -\frac{\left(\frac{\partial T}{\partial r}\right)_{r=R}}{T_s - T_m} = \text{constant}$ follows (note that T_s and T_m do not depend on r). From $q_s = h(T_s - T_m)$, $q_s = k\left(\frac{\partial T}{\partial r}\right)_{r=R}$ and

  $-\frac{\left(\frac{\partial T}{\partial r}\right)_{r=R}}{T_s - T_m} = \text{constant}$ it follows that $\frac{h}{k} = \frac{q_s/k}{q_s/h} = \frac{\left(\frac{\partial T}{\partial r}\right)_{r=R}}{T_s - T_m} = \text{constant}.$

  Therefore the assumption that θ is only a function of r implies that the heat transfer coefficient h is constant along the x-axis.

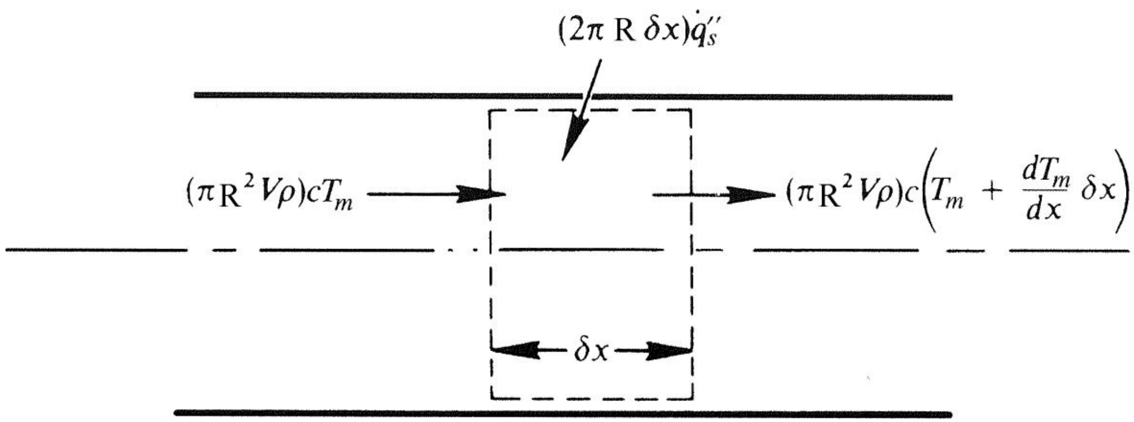
Internal laminar boundary layers: Heat transfer

- We now treat the case when the surface heat flux is fixed and constant along the x-axis. Then, since h is constant, the equation $q_s = h(T_s - T_m)$ implies that $T_s - T_m$ is also constant. From this it follows that $\frac{dT_s}{dx} = \frac{dT_m}{dx}$. Since the non-dimensional temperature profile θ (is assumed to be) x-invariant, it follows that $0 = \frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{T_s - T}{T_s - T_m}\right)$. This implies $\frac{dT_s}{dx} = \frac{\partial T}{\partial x}$. Thus $\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx}$. The energy equation therefore becomes $u\frac{dT_m}{dx} = \alpha \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$ with the boundary conditions $T = T_s$ at $r = R$ and $\frac{\partial T}{\partial r} = 0$ at $r = 0$. The last condition holds since we have assumed a radial symmetric profile.

Internal laminar boundary layers: Heat transfer

- Substituting the fully developed parabolic velocity u yields $\frac{2V}{\alpha}\left(1-\frac{r^2}{R^2}\right)\frac{dT_m}{dx}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$. Integrating this twice with respect to r and incorporating the boundary conditions yields $T=T_s-\frac{2V}{\alpha}\frac{dT_m}{dx}\left(\frac{3R^2}{16}+\frac{r^4}{16R^2}-\frac{r^2}{4}\right)$. Substituting u and T in $T_m=\frac{1}{V\cdot A}\int_A uT\,dA=\frac{1}{VR^2\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{R} uTr\,dr d\phi$ gives, after integration, $T_m=T_s-\frac{11}{96}\frac{2V}{\alpha}\frac{dT_m}{dx}R^2$. Therefore $q_s=h(T_s-T_m)=h\frac{11}{96}\frac{2V}{\alpha}\frac{dT_m}{dx}R^2$. The connection between the mixed mean temperature T_m and the surface heat flux q_s can be found by considering an energy balance:

Internal laminar boundary layers: Heat transfer



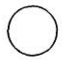
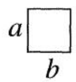
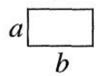
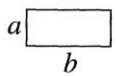
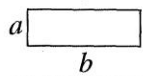
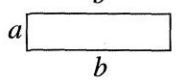
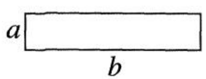
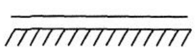
Control volume for the energy flow in a circular pipe; Source: W. Kays et al., "Convective Heat and Mass Transfer".

- From this consideration it follows that $q_s = \frac{RV\rho c_p}{2}\frac{dT_m}{dx}$. Combining this with $q_s = h\frac{11}{96}\frac{2V}{\alpha}\frac{dT_m}{dx}R^2$ and solving for h yields $h = \frac{48}{11}\frac{k}{2R}$. For the Nusselt number based on the pipe diameter $D = 2R$ we therefore obtain $Nu = \frac{48}{11} \approx 4.364$.

Internal laminar boundary layers: Heat transfer

- In a similar manner (but by considering – mathematically more demanding – infinite series expansions of the solution) one can show that the Nusselt number for hydrodynamically and thermally fully developed pipe flow <u>with constant surface temperature</u> is $Nu = 3.6568$.

Internal laminar boundary layers: Heat transfer

Cross-sectional shape	b/a	$Nu_{Ⓗ}$†	$Nu_{Ⓣ}$
		4.364	3.66
	1.0	3.61	2.98
	1.43	3.73	3.08
	2.0	4.12	3.39
	3.0	4.79	3.96
	4.0	5.33	4.44
	8.0	6.49	5.60
	∞	8.235	7.54
		5.385	4.86
		3.11	2.49

†The constant-heat-rate solutions are based on constant *axial* heat rate, but with constant *temperature* around the tube periphery. Nusselt numbers are averages with respect to tube periphery.

Nusselt numbers for fully developed velocity and temperature profiles in tubes of various cross sections; Source: W. Kays et al.,“Convective Heat and Mass Transfer”.

Internal laminar boundary layers: Heat transfer

The effect of axial heat conduction

- In the former discussion the axial heat conduction term $\alpha \frac{\partial^2 T}{\partial x^2}$ was neglected. For the fully developed constant heat rate case it turns out that this term is always 0 (this follows from the fact that $\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx}$ is constant); therefore the Nusselt number is not affected by axial heat conduction in this case.
- However, for constant surface temperature axial heat conduction can be of significance at low values of Re and/or Pr. In fact, the significant parameter is the Péclet number $\mathrm{Pe} \coloneqq \mathrm{Re} \cdot \mathrm{Pr}$.

Internal laminar boundary layers: Heat transfer

- The problem has been investigated by Michelson & Villadsen (M.L. Michelson and J. Villadsen, “The Graetz problem with axial heat conduction”, Int. J. Heat and Mass Transfer 17, 1391-1402, 1974) who recommend the following equations:

$$Nu = \begin{cases} 4.180654 - 0.183460 \cdot Pe & \text{for} \quad Pe < 1.5 \\ 3.656794 + 4.487/Pe^2 & \text{for} \quad Pe > 5 \end{cases}$$

- These expressions imply that axial heat conduction is totally negligible for $Pe > 100$ and fairly small even for $Pe = 10$. Therefore axial heat conduction is – for gases – only important at extremely low Reynolds numbers and for most liquids (except liquid metals which have a very small Prandtl number) it is rarely of importance.

Internal laminar boundary layers: Heat transfer

Circular tube thermal entry length solutions

- We now consider cases where the temperature of the fluid is uniform over the cross section at the point where heat transfer begins (say at $x = 0$). The velocity profile at this point is assumed to be already fully established and invariant. Under these conditions the heat transfer coefficient varies along the length of the tube.
- Before we discuss the solutions, we take a closer look at the meaning of the Prandtl number for heat transfer problems: The Prandtl number $Pr = \frac{\mu \cdot c_p}{k}$ can also be written as $Pr := \frac{\frac{\mu}{\rho}}{\frac{k}{\rho \cdot c_p}} = \frac{\nu}{\alpha}$, that is, as the ratio of the molecular diffusivity ν for momentum and

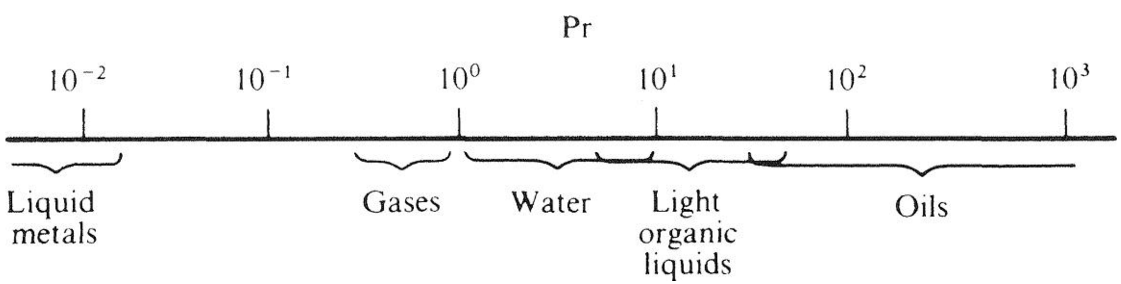
Internal laminar boundary layers: Heat transfer

the thermal diffusivity α.

- If $Pr = 1$, then heat and momentum are diffused through the fluid at the same rates, and the velocity profile and the temperature profile also develop at the same rate. If $Pr > 1$, then the velocity profile develops more rapidly than the temperature profile. If $Pr > 5$ (e.g. for water colder than $\sim 30°C$), then the velocity profile leads the temperature profile sufficiently that a solution based on an already fully developed velocity profile will apply quite accurately – even though there is no hydrodynamic starting length. Of course, the opposite also holds: for a fluid with a Prandtl number less than 1 (e.g. air – see below) the temperature profile develops more rapidly than the velocity profile.

Internal laminar boundary layers: Heat transfer

- The Prandtl number for any particular fluid generally varies somewhat with temperature, but only over a limited range.



Prandtl number spectrum of fluids; Source: W. Kays et al.,"Convective Heat and Mass Transfer".

- For air the Prandtl number varies only weakly with temperature:
  - Pr_{air} at $100\text{K} = 0.787$ and Pr_{air} at $2500\text{K} = 0.673$, and for temperatures between 100K and 2500K the value of the Prandtl number lies between 0.673 and 0.787.

Internal laminar boundary layers: Heat transfer

Fully developed hydrodynamic pipe flow with thermal entry length and constant surface temperature

- The applicable thermal equation is again $u\frac{\partial T}{\partial x} = \alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2}\right)$.
- Again, axial heat conduction will be neglected, i.e., the term $\frac{\partial^2 T}{\partial x^2}$ will be omitted. The equation can be solved (by incorporating the applicable boundary conditions) by transforming it to non-dimensional form and expanding the non-dimensional temperature $\theta = \frac{T_s - T}{T_s - T_e}$ (T_e denotes here the [uniform] fluid temperature at the inlet) and in particular the mixed mean temperature $\theta_m = \frac{T_s - T_m}{T_s - T_e}$ into an infinite series. Then

Internal laminar boundary layers: Heat transfer

$$\theta_m = 8 \sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} e^{-\lambda_n^2 x^+},$$

where $x^+ := \frac{2\left(x/_D\right)}{Re \cdot Pr}$, and the values for λ_n^2 and G_n are given in the following table:

N	λ_n^2	G_n
0	7.313	0.749
1	44.61	0.544
2	113.9	0.463
3	215.2	0.415
4	348.6	0.383

For $n > 2$, $\lambda_n = 4n + \frac{8}{3}$; $G_n = 1.01276\lambda_n^{-1/3}$

Infinite series solution for the thermal entry length in a circular tube; constant surface temperature; Source: W. Kays et al.,“Convective Heat and Mass Transfer”.

Internal laminar boundary layers: Heat transfer

- For the local Nusselt number one obtains

$$(*) \qquad Nu = \frac{\sum_{n=0}^{\infty} G_n \, e^{-\lambda_n^2 x^+}}{2\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} e^{-\lambda_n^2 x^+}}$$

- <u>Exercise:</u> Prove that

$$Nu_m = \frac{1}{2x^+} \ln\left(\frac{1}{8\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} e^{-\lambda_n^2 x^+}}\right)$$

Internal laminar boundary layers: Heat transfer

x^+	Nu_x	Nu_m	θ_m
0	∞	∞	1.000
0.001	12.80	19.29	0.962
0.004	8.03	12.09	0.908
0.01	6.00	8.92	0.837
0.04	4.17	5.81	0.628
0.08	3.77	4.86	0.459
0.10	3.71	4.64	0.396
0.20	3.66	4.15	0.190
∞	3.66	3.66	0.0

Local and average Nusselt numbers and mean temperature for the thermal entry length in the circular tube; constant surface temperature; Source: W. Kays et al., "Convective Heat and Mass Transfer".

- From the table above it can be concluded that the relative error in the local Nusselt number becomes smaller than about 2% if $x^+ > 0.1$. Therefore the thermal entry length is approximately

$$x^+ = 0.1 = \frac{2\left(x/_D\right)}{Re \cdot Pr}, \text{ that is, } \frac{x}{D} = \frac{Re \cdot Pr}{20} = \frac{Pe}{20}.$$

Internal laminar boundary layers: Heat transfer

- Note the analogy to the equation

$$\frac{x}{D} = \frac{Re}{20}$$

  for the hydrodynamic entry length!

- <u>Exercise:</u> Compute numerically in CFX the local heat flux for fully developed hydrodynamic pipe flow (circular cross section) with thermal entry length and constant surface temperature. Then compare the numeric result with the heat flux predicted by formula $(*)$.

Internal laminar boundary layers: Heat transfer

- Geometry:
  - Pipe diameter = 10 cm
  - Length of initial leg = 12.5 m
  - Length of heated leg = 12.5 m
- Fluid: Air at 25°C
- Inlet fluid temperature = 273.15 K (uniform distribution across the inlet)
- Surface temperature of the heated part = 373.15 K
- Reynolds number (based on the mean velocity and the pipe diameter):
  - **Case1:** $\mathrm{Re} = 100$ (uniform velocity profile across the inlet)
  - **Case2:** $\mathrm{Re} = 2000$ (uniform velocity profile across the inlet)

Internal laminar boundary layers: Heat transfer

Fully developed hydrodynamic pipe flow with thermal entry length and constant surface heat flux

- In the case of constant surface heat flux the local Nusselt number is

$$Nu = \left[\frac{1}{Nu_\infty} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{e^{-\gamma_m^2 x^+}}{A_m \gamma_m^4} \right]^{-1}$$

The constants A_m and γ_m are given in the following table:

Internal laminar boundary layers: Heat transfer

m	γ_m^2	A_m
1	25.68	7.630×10^{-3}
2	83.86	2.053×10^{-3}
3	174.2	0.903×10^{-3}
4	296.5	0.491×10^{-3}
5	450.9	0.307×10^{-3}

For larger m, $\gamma_m = 4m + \frac{4}{3}$; $A_m = 0.4165\gamma_m^{-7/3}$.

Infinite series solution for thermal entry length in a circular tube; constant heat rate; Source: W. Kays et al., "Convective Heat and Mass Transfer".

Internal laminar boundary layers: Heat transfer

- **Combined hydrodynamic and thermal entry length**

x^+	Nu_x Pr = 0.7	Nu_x Pr = 2	Nu_x Pr = 5	Nu_m Pr = 0.7	Nu_m Pr = 2	Nu_m Pr = 5
0.001	16.8	14.8	13.5	30.6	25.2	22.1
0.002	12.6	11.4	10.6	22.1	19.1	16.8
0.004	9.6	8.8	8.2	16.7	14.4	12.9
0.006	8.25	7.5	7.1	14.1	12.4	11.0
0.01	6.8	6.2	5.9	11.3	10.2	9.2
0.02	5.3	5.0	4.7	8.7	7.8	7.1
0.05	4.2	4.1	3.9	6.1	5.6	5.1
∞	3.66	3.66	3.66	3.66	3.66	3.66

Nusselt numbers for combined thermal and hydrodynamic entry length in the circular tube; constant surface temperature; Source: W. Kays et al., “Convective Heat and Mass Transfer”.

External turbulent boundary layers: Momentum transfer

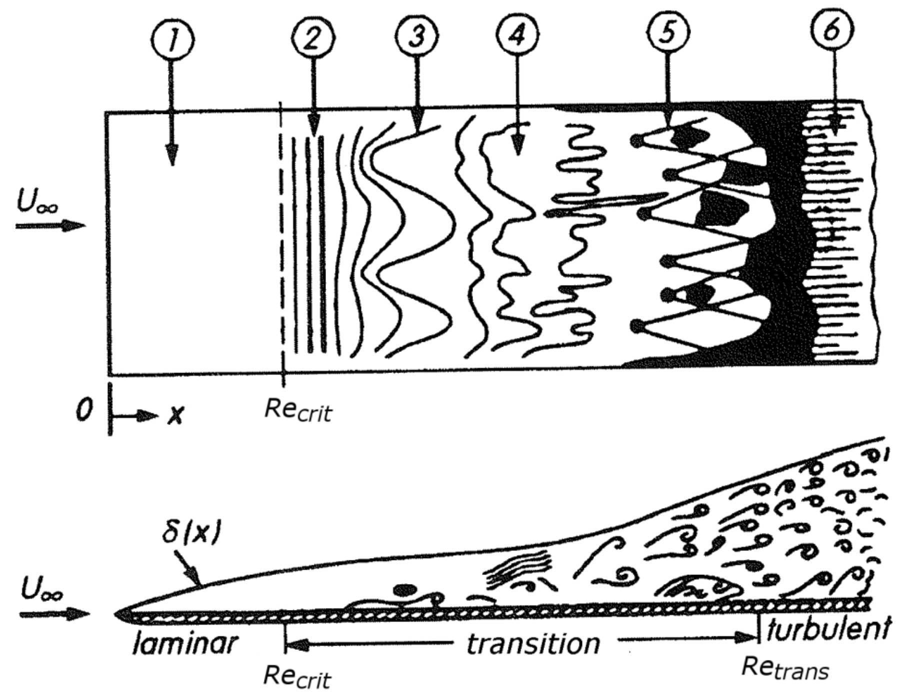
- The transition from a laminar boundary layer to a turbulent boundary layer does not take place abruptly; there is a transition region whose extent may be in the same order of magnitude as that of the preceding laminar boundary layer
- The location where transition sets in depends on the Reynolds number and is very sensitive to influencing factors such as surface smoothness and free stream turbulence
- The following range of transitional Reynolds numbers is encountered in "practical" engineering problems:

$$Re_{x,trans} = 3.0 \cdot 10^5 \text{ through } 1.0 \cdot 10^6$$

($Re_{x,trans}$ is based on the running length x and the free stream velocity)

External turbulent boundary layers: Momentum transfer

(1) Stable laminar flow
(2) Unstable Tollmien-Schlichting waves
(3) 3-dimensional waves and Λ-vortices
(4) Decay of vortices
(5) Turbulent spots
(6) Fully turbulent flow



Source: H. Schlichting and K. Gersten, Boundary Layer Theory (with modifications)

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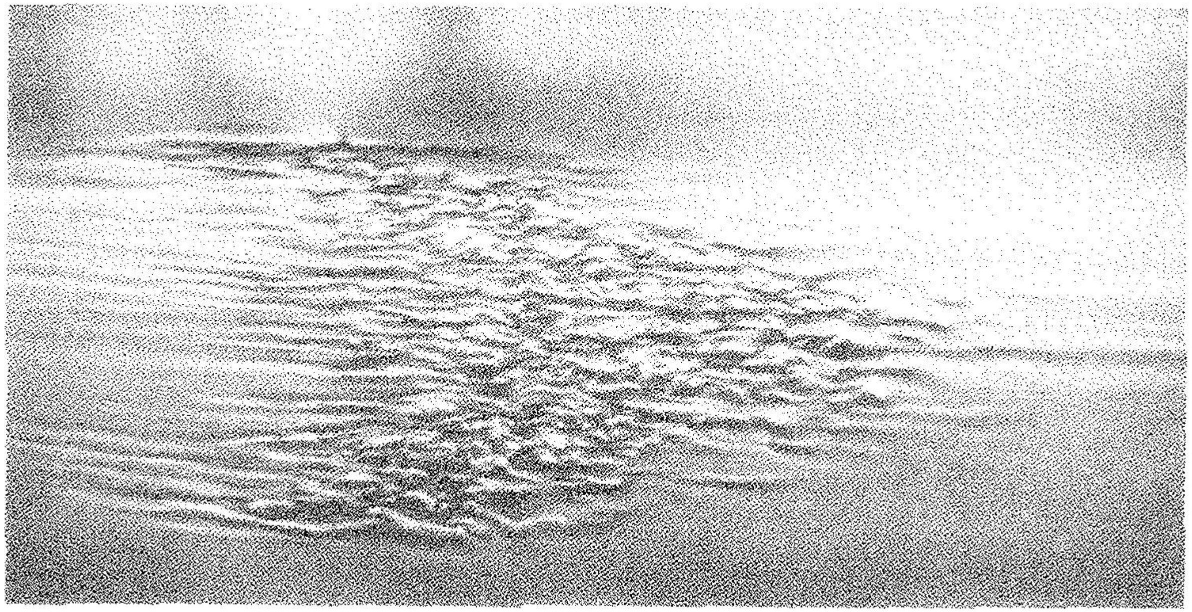
External turbulent boundary layers: Momentum transfer



Turbulent spots in the transition region over a flat plate; Source: H. Schlichting and K. Gersten, Boundary Layer Theory

External turbulent boundary layers: Momentum transfer

- When physical parameters $\vec{V}, p, \ldots$ in a turbulent flow are considered, then, very frequently, their time averages $\langle\vec{V}\rangle, \langle p\rangle, \ldots$ lie in the center of interest. Therefore, in many contexts, <u>time averaged physical quantities are tacitly meant</u> when one talks about flow parameters.

External turbulent boundary layers: Momentum transfer

Recall a number of parameters playing an important role in the description of turbulent boundary layers:

- $u_\tau := \sqrt{\frac{\tau_W}{\rho}}$, where τ_W denotes the local wall shear stress; u_τ is called the friction velocity
- $u^+ := \frac{u}{u_\tau}$ (non-dimensional velocity; well suited to describe the inner region and the near wall region in the turbulent boundary layer)
- $y^+ = \frac{y u_\tau}{\nu}$ (non-dimensional wall distance)

External turbulent boundary layers: Momentum transfer

Logarithmic law of the wall for a fully turbulent flat plate boundary layer:

- $u^+ = y^+$ within the viscous sublayer, that is, for y^+ less than about 5
- $u^+ = \frac{1}{\kappa}\ln(y^+) + 5.0$, for approx. $y^+ > 40$, where $\kappa = 0.41$ is called the von Karman constant.
- In the region $5 < y^+ < 40$ the behavior is inbetween linear and logarithmic

External turbulent boundary layers: Momentum transfer

Boundary layer thickness and friction coefficient

- Recall that for an entirely laminar boundary layer we had for the local parameters describing the boundary layer thickness and wall friction:

  - $\delta_1(x) = 1.72 \frac{x}{\sqrt{Re_x}}$
  - $\delta_2(x) = 0.664 \frac{x}{\sqrt{Re_x}}$
  - $\delta_{99}(x) = 5.0 \frac{x}{\sqrt{Re_x}}$
  - $c_f(x) = \frac{0.664}{\sqrt{Re_x}}$

$$Re_x = \frac{u_\infty x}{\nu}$$

External turbulent boundary layers: Momentum transfer

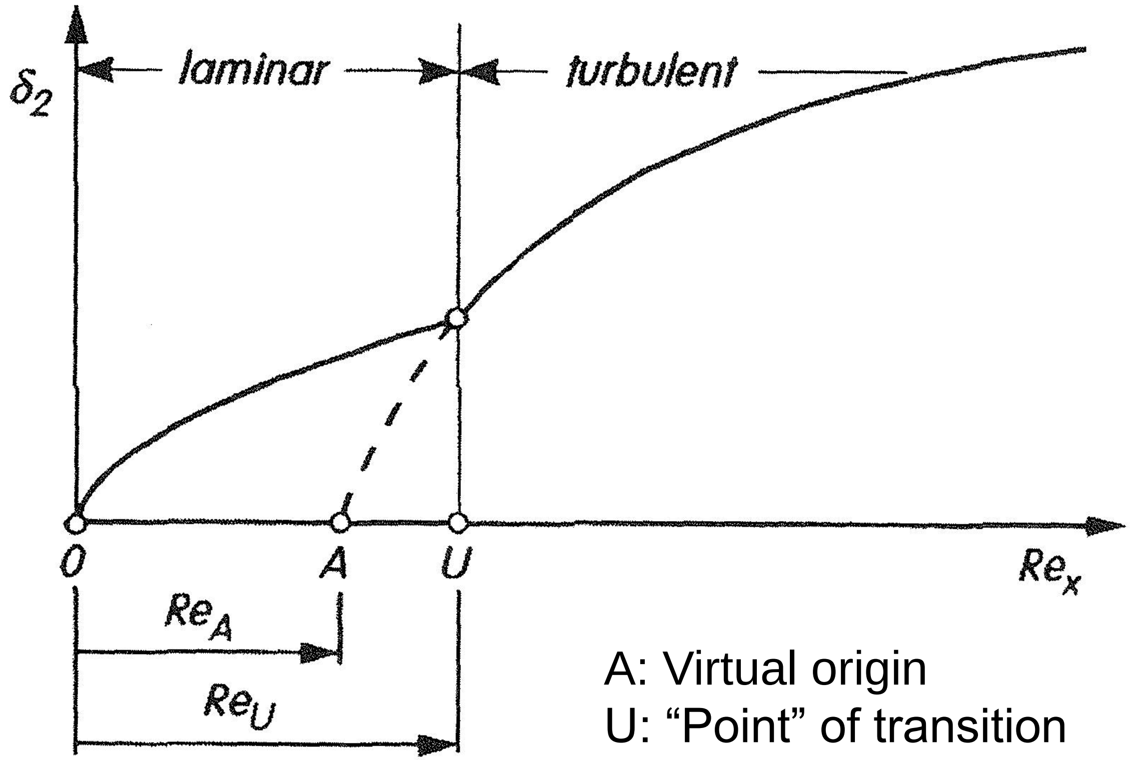
For a fully turbulent boundary layer the following results can be obtained from the turbulent boundary layer equation (which – basically – is the time average of the “ordinary” boundary layer equation in conjunction with an appropriate turbulence model):

- $\delta_2(x) = 0.036 \cdot x \cdot Re_x^{-0.2}$
- $\frac{\delta_1}{\delta_2} = 1.29$
- $\frac{\delta_2}{\delta_{99}} = 0.097$
- $c_f = 0.0574 \cdot Re_x^{-0.2}$

External turbulent boundary layers: Momentum transfer

- For many practical applications it is sufficient to assume that the boundary layer can be split into a fully laminar part and a fully turbulent part, without considering the transitional part explicitly; In other words, one makes the simplification that transition from laminar flow to fully turbulent flow takes place at some specific "point".

External turbulent boundary layers: Momentum transfer



A: Virtual origin
U: “Point” of transition

Development of the momentum thickness when assuming sudden transition from a laminar to a turbulent boundary layer; Source: H. Schlichting and K. Gersten, Boundary Layer Theory

External turbulent boundary layers: Momentum transfer

- Exercise 1: What is the ratio $\frac{\bar{c}_{f,fully\,lam}}{\bar{c}_{f,fully\,turb}}$ of the mean friction coefficient on a plate with length L assuming a fully laminar boundary layer on the one hand and a fully turbulent boundary layer on the other hand?
- Apply your result to the following example:
  - $L = 1\ \mathrm{m}$
  - $u_\infty = 30\ \mathrm{m/s}$
  - $\nu = 1.5 \cdot 10^{-5}\ \mathrm{m^2/s}$

External turbulent boundary layers: Momentum transfer

- Exercise 1: What is the ratio $\frac{\bar{c}_{f,fully\,lam}}{\bar{c}_{f,fully\,turb}}$ of the mean friction coefficient on a plate with length L assuming a fully laminar boundary layer on the one hand and a fully turbulent boundary layer on the other hand? Answer: $\frac{\bar{c}_{f,fully\,lam}}{\bar{c}_{f,fully\,turb}} = 18.51 \cdot Re_L^{-0.3}$
- Apply your result to the following example:
  - $L = 1\ \mathrm{m}$
  - $u_\infty = 30\ \mathrm{m/s}$
  - $\nu = 1.5 \cdot 10^{-5}\ \mathrm{m^2/s}$

Answer: $\frac{\bar{c}_{f,fully\,lam}}{\bar{c}_{f,fully\,turb}} \approx 0.24$

External turbulent boundary layers: Momentum transfer

- Exercise 2: Show that

$$\frac{x_{virt.\,orig}}{x_{trans}} = 1 - \frac{0.664^{5/4}}{0.036^{5/4}} \mathrm{Re}_{x_{trans}}^{-3/8}$$

External turbulent boundary layers: Momentum transfer

- Exercise 3: Denote by $\bar{c}_{f,lam+turb}$ the mean friction coefficient of the combined laminar and turbulent boundary layer. What is the ratio $\frac{\bar{c}_{f,lam+turb}}{\bar{c}_{f,fully\ turb}}$ in terms of Re_L and Re_{trans} (L is the total length of the plate; without loss of generality assume that $L \geq x_{trans}$)?

- Answer: $\frac{\bar{c}_{f,lam+turb}}{\bar{c}_{f,fully\ turb}} = \left[1 - \frac{Re_{trans}}{Re_L}\left(1 - 38.2 \cdot Re_{trans}^{-\frac{3}{8}}\right)\right]^{4/5}$

External turbulent boundary layers: Momentum transfer

- Exercise 4: Re-visit the numerical example from Exercise 1. What error is made relative to the combined laminar-turbulent friction coefficient when a fully turbulent boundary layer is assumed over the full extent of the plate? The transitional Reynolds number is

A) $Re_{trans} = 3.0 \cdot 10^5$; Answer: $\frac{\bar{c}_{f,lam+turb}}{\bar{c}_{f,fully\,turb}} = 0.92$

B) $Re_{trans} = 1.0 \cdot 10^6$; Answer: $\frac{\bar{c}_{f,lam+turb}}{\bar{c}_{f,fully\,turb}} = 0.67$

External turbulent boundary layers: Heat transfer

Logarithmic law of the wall for a turbulent thermal boundary layer on a flat plate:

Definition of a non-dimensional temperature:

- $T^+ = \frac{\rho c_p (T_s - T)\sqrt{\tau_s/\rho}}{q_s}$; Here T_s denotes the surface temperature, τ_s denotes the wall shear stress, and q_s denotes the surface heat flux

- Logarithmic law of the wall for air (and similar gases):
  - $T^+ = \mathrm{Pr} \cdot y^+$ when y^+ is less than about 5
  - $T^+ = 2.075 \ln(y^+) + 13.2 \cdot \mathrm{Pr} - 5.34$ when approx. $y^+ > 40$

External turbulent boundary layers: Heat transfer

- Logarithmic law of the wall for water ($\mathrm{Pr} \sim 6.0$):
  - $T^+ = \mathrm{Pr} \cdot y^+$ for y^+ less than about 5
  - $T^+ = 2.075 \ln(y^+) + 7.55 \cdot \mathrm{Pr} - 3.95$ for approx. $y^+ > 40$

- Universal fit suitable for fluids with Prandtl numbers in the range 0.6 – 6.0:

$$T^+ = 2.2 \ln(y^+) + 13.39 \cdot Pr^{\frac{2}{3}} - 5.66$$

External turbulent boundary layers: Heat transfer

- Note: For materials with very low Prandtl number (most notably, liquid metals) and materials with very high Prandtl number (viscous oils etc.) the logarithmic law of the wall deviates (partly in a significant way) from the law of the wall for fluids with Prandtl number in the range of 1.

External turbulent boundary layers: Heat transfer

- Local Nusselt number of a flat plate with constant surface temperature in the range $0.5 < Pr < 1.0$ and $5 \cdot 10^5 < Re_x < 5 \cdot 10^6$:

$$Nu_x = 0.0287 \cdot Re_x^{\frac{4}{5}} \cdot Pr^{\frac{3}{5}}$$

- Local Nusselt number of a flat plate with constant surface heat rate:

$$Nu_x = 0.030 \cdot Re_x^{\frac{4}{5}} \cdot Pr^{\frac{3}{5}}$$

Remark: The turbulent boundary layer is much less sensitive to variations of the surface temperature than the laminar boundary layer. This is particularly the case for fluids with high Prandtl numbers.

Internal turbulent boundary layers: Momentum transfer

Hydrodynamic entry length

- By analytical as well as numerical means an idealized entry region with a uniform velocity profile at the inlet can be analyzed. A classical result of this type for a tube with circular cross section is that published by Latzko (H. Latzko, *Der Wärmeübergang an einen turbulenten Flussigkeits- oder Gasstrom*, Z. Angew. Math. Mach., vol.1 268-290, 1921):

$$\left(\frac{x}{D}\right)_{entry} = 0.623 \cdot Re_D^{1/4}$$

Internal turbulent boundary layers: Momentum transfer

- As an example, if $Re_D = 50000$, then $x/D = 9.3$. On the other hand, when $Re_D = 2000$, then the entry length for the laminar boundary layer is $x/D = Re_D/20 = 100$. The turbulent hydrodynamic entry length is obviously much shorter than the laminar entry length. In most engineering applications – heat exchangers for example – the tubes are very much longer than the turbulent entry length. For this reason the entry length solution is usually not very important. Therefore we focus on the fully developed hydrodynamic boundary layer throughout.

Internal turbulent boundary layers: Momentum transfer

Velocity profile

- As a good approximation for the radial turbulent velocity profile in a circular pipe a simple power law can be used:

$$u(r) = u_c \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}$$

Here u_c denotes the velocity at the center line and n is an empirical exponent depending (somewhat) on the Reynolds number; typically, n ranges from 6 for $Re = 4 \cdot 10^3$ to 8.6 for $Re = 10^6$; for $Re = 10^5$ a good approximation is obtained when setting $n = 7$.

Internal turbulent boundary layers: Momentum transfer

- Consider a circular pipe with fully developed velocity profile. Close to the wall, where r is near R and τ is near τ_s, the velocity profile outside the viscous sublayer should differ little from the law of the wall for a flat plate. In fact, experimental data over a substantial portion of flow area are quite well represented by a slight modification of the law of the wall, the so-called Nikuradse equation:

$$u^+ = 2.5 \cdot \ln(y^+) + 5.5$$

Here $y := R - r$. Note that, for reasons of symmetry, the gradient $\frac{du^+}{dy^+}$ should be zero at the tube centerline. This is not the case for the Nikuradse equation. A slight modification of the Nikuradse equation is

Internal turbulent boundary layers: Momentum transfer

$$u^{+} = 2.5 \cdot \ln\left(y^{+} \frac{1.5\left(1 + \frac{r}{R}\right)}{1 + 2\left(\frac{r}{R}\right)^{2}} \right) + 5.5$$

which has gradient 0 at the centerline. This equation is in slightly better agreement with experimental data all the way to the centerline. However, the actual difference in the calculated values of u^{+} is small and the Nikuradse equation is often preferred because of its simplicity.

Internal turbulent boundary layers: Momentum transfer

Friction coefficient (circular pipe, fully developed turbulent velocity profile)

- To compute the friction coefficient the “classical” Kármán-Nikuradse equation can be used:

$$\frac{1}{\sqrt{c_f/2}} = 2.46 \cdot \ln\left(Re \cdot \sqrt{c_f/2}\right) + 0.3$$

(This equation can be obtained from the Nikuradse equation when applying also a fit of the numerical constants to better match the experimental data.)

- A simpler expression, valid in the range $3 \cdot 10^4 < Re < 10^6$, is

$$c_f/2 = 0.023 \cdot Re^{-0.2}$$

Internal turbulent boundary layers: Momentum transfer

- Another equation, valid over the range $10^4 < Re < 5 \cdot 10^6$, was proposed by Petukhov (B.S. Petukhov, *Heat Transfer and Friction in Turbulent Pipe Flow with variable physical properties*, Advances in Heat Transfer 6, 503-504, Academic Press, New York, 1970):

$$\frac{c_f}{2} = (2.236 \cdot \ln(Re) - 4.639)^{-2}$$

Internal turbulent boundary layers: Heat transfer

Temperature profile (circular pipe, fully developed turbulent velocity and temperature profile, Prandtl number range 0.6 – 6.0)

- $T^+ = 2.2\ln(y^+) + 13.39 \cdot Pr^{\frac{2}{3}} - 5.66$
- Using the Nikuradse equation, it is not difficult to derive an equation for the constant heat rate Nusselt number:

$$Nu = \frac{Re \cdot Pr \cdot c_f/2}{0.88 + 13.39(Pr^{2/3} - 0.78)\sqrt{c_f/2}}$$

Using the simplified expression for the friction coefficient, this yields

$$Nu = \frac{0.023 \cdot Re^{4/5} \cdot Pr}{0.88 + 2.03(Pr^{2/3} - 0.78)Re^{-1/10}}$$

Internal turbulent boundary layers: Heat transfer

- A much simpler expression for gases (Prandtl number range 0.5 – 1.0) in the Reynolds number range 10000 – 100000 is

$$Nu = 0.022 \cdot Pr^{1/2} Re^{4/5}$$

- Note: Only for very low Prandtl numbers there is a significant difference between the constant heat rate and constant surface temperature solution. Thus, for gases, the expressions developed for the constant heat rate case can also be applied to the constant surface temperature boundary condition.

Internal turbulent boundary layers: Heat transfer

Thermal entry length

- Consider the case of a fully developed velocity profile, but with a uniform fluid temperature at the point where heat transfer begins. Solutions to this problem for both constant surface temperature and constant heat rate have been obtained by Notter et al. (R.H. Notter and C.H. Sleicher, *A solution to the turbulent Graetz problem — III. Fully developed and entry region heat transfer rates*, Chem. Engng. Sci. 27, 2073-2093, 1972). These solutions follow the same procedures as were employed for the laminar flow counterpart to the problem:

Internal turbulent boundary layers: Heat transfer

$$Nu_x = \frac{\sum_{n=0}^{\infty} G_n \, e^{-\lambda_n^2 x^+}}{2 \sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} \, e^{-\lambda_n^2 x^+}}$$

$$Nu_m = \frac{1}{2x^+} \ln\left(\frac{1}{8 \sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} \, e^{-\lambda_n^2 x^+}} \right)$$

The parameters λ_n and G_n can be obtained – for various Prandtl and Reynolds numbers – from the following table:

Internal turbulent boundary layers: Heat transfer

n	Pr	Re	λ_n^2	G_n
0	0.002	50,000	10.61	0.975
1			58.19	0.897
2			144.6	0.862
3			269.7	0.841
4				
0	0.002	100,000	11.23	1.036
1			61.74	0.943
2			153.6	0.903
3			286.7	0.88
4				
0	0.002	500,000	15.91	1.527
1			90.19	1.246
2			227.4	1.137
3			426.8	1.08
4				
0	0.01	50,000	13.39	1.277
1			75.23	1.079
2			188.9	0.996
3			353.9	0.949
4				
0	0.01	100,000	16.75	1.642
1			96.15	1.29
2			243.8	1.154
3			458.5	1.085
4				
0	0.01	500,000	39.31	4.2
1			252.9	2.42
2			665.6	1.93
3			1,271	1.752
4				
0	0.03	50,000	21.28	2.17
1			126.7	1.532
2			325.2	1.32
3			614.8	1.189
4				
0	0.03	100,000	31.06	3.28
1			194.2	2.03
2			506.9	1.649
3			964.8	1.488
4				
0	0.03	500,000	91.5	10.38
1			678.1	4.39
2			1,844	3.22
3			3,562	2.9
4				

n	Pr	Re	λ_n^2	G_n
0	0.72	10,000	64.38	7.596
1			646.8	1.829
2			1,870	1.217
3				
0	0.72	50,000	219	26.6
1			2,350	5.63
2			6,808	3.32
3				
0	0.72	100,000	375.9	45.8
1			4,183	9.25
2			12,130	5.48
3				
0	8	50,000	685.6	85.4
		100,000	1,232	154
		500,000	5,020	625
1	10	50,000		
2				
3				
4				
5				
1	10	100,000		
2				
3				
4				
5				
0	50	100,000	2,570	321
		200,000	4,778	598
		500,000	10,800	1,350
0	100	100,000	3,317	415
		200,000	6,129	766
		500,000	14,040	1,750

Infinite series solution functions for thermal entry length with turbulent flow in a circular pipe; Source: W. Kays et al., "Convective Heat and Mass Transfer".

Internal turbulent boundary layers: Heat transfer

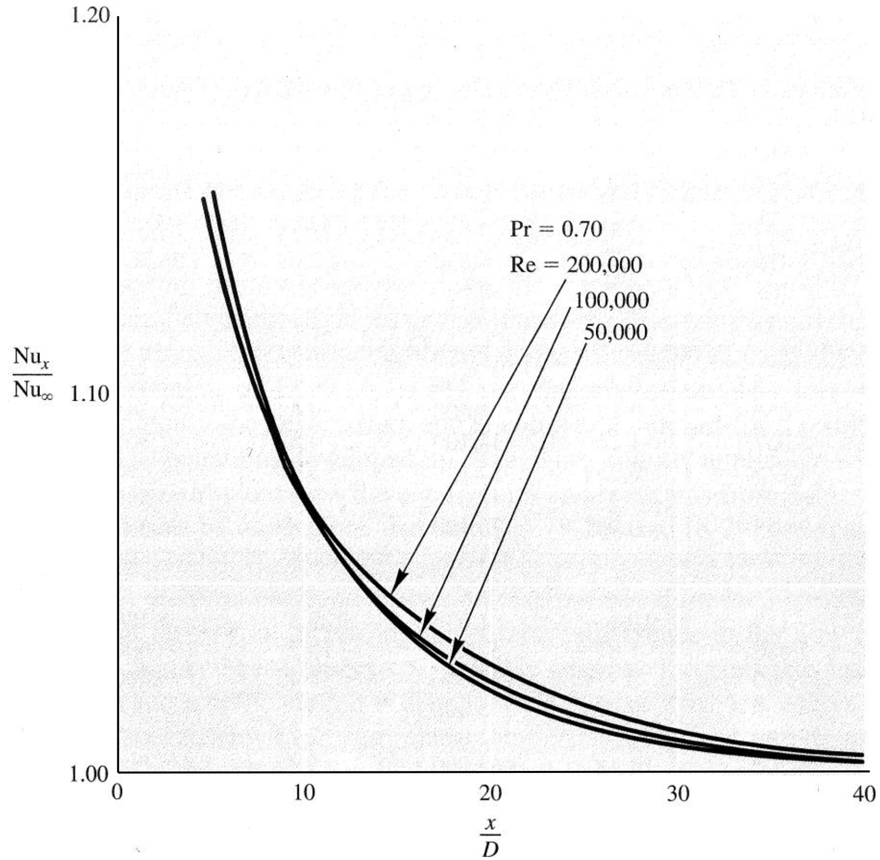
- Exercise: Show, by using the series expansion of the mean Nusselt number, that asymptotically (as $x^+ \to \infty$)

$$Nu_m(x^+) = \frac{\lambda_0^2}{2} + \frac{1}{2x^+}\ln\left(\frac{\lambda_0^2}{8G_0}\right) + O\left(e^{-ax^+}\right) \quad \text{and}$$

$$\frac{Nu_m(x^+)}{Nu_\infty} = 1 + \frac{1}{\lambda_0^2 x^+}\ln\left(\frac{\lambda_0^2}{8G_0}\right) + O\left(e^{-ax^+}\right)$$

  for some $a > 0$.
- Example: Consider $Pr = 0.72$ and $Re = 50000$. From the table, $\lambda_0^2 = 219$, $G_0 = 26.6$; further, $x^+ = \frac{2\left(x/_D\right)}{Re \cdot Pr}$; thus $\frac{Nu_m}{Nu_\infty} = 1 + \frac{2.36}{x/D}$
- Thus, even for a tube 100 diameter in length, there is still a 2% effect of the entry length on the mean heat transfer coefficient, primarily due to the high heat transfer coefficient in the first 10 diameters.

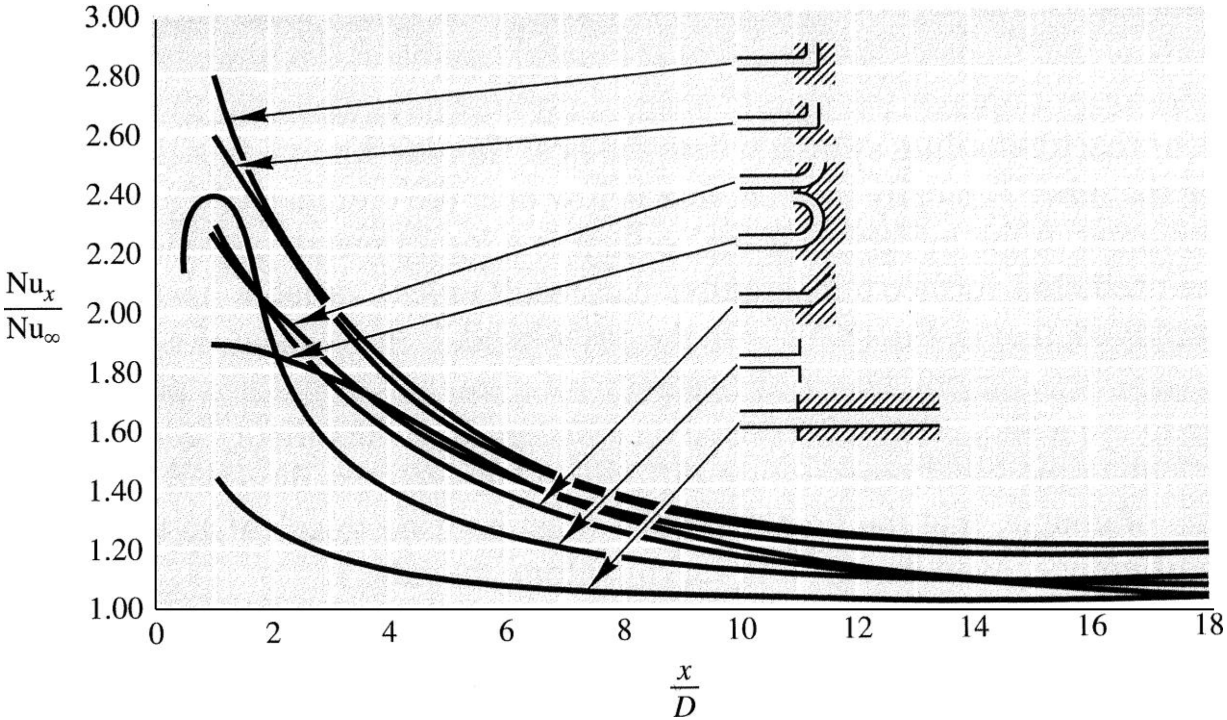
Internal turbulent boundary layers: Heat transfer



Nusselt numbers in the thermal entry region of a circular pipe, for constant heat rate: influence of Re at $Pr = 0.7$; Source: W. Kays et al., “Convective Heat and Mass Transfer”.

Internal turbulent boundary layers: Heat transfer

Combined hydrodynamic and thermal entry length



Measured local Nusselt numbers in the entry region of a circular tube for various entry configurations; air with constant surface temperature; Source: W. Kays et al., "Convective Heat and Mass Transfer".