

Deriving the BSSN equations from the ADM equations

The following Cadabra¹ codes verifies, for the particular case of vacuum spacetimes, the main equations (with zero-shift) in the Phys Rev D papers Phys.Rev.D. (62) 044034 (2000) and Phys.Rev.D. (67) 084023 (2003) by Miguel Alcubierre, Bernd Bruggmann et al. .

The two papers are abbreviated as pdr62 and prd67 respectively.

All of the essential BSSN equations are covered, in particular equations 9, 10, 11, 12 , 15, 17, 18, 19 and 20 from prd62 and equations 19, 20 and 27 from prd67.

The first few pages provides a summary of the calculations. For each equation the summary is of the form

$$\begin{aligned} \text{eq18.1cb} &:= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\ \text{eq18.prd} &:= -\frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}) \\ \text{eq18.chk} &:= 0 \end{aligned}$$

The first line records the output from the Cadabra code, the second line is the corresponding equation in prd62 (or prd67) while the third line shows the difference. The good news is that for each equation the difference is zero (as it should be).

The summary is then followed by the full details of the calculations for each equation. The output includes all of the steps. The odd looking equation numbers are tags that match the line in the corresponding Cadabra source. Thus the following output (from prd62 equation 11)

$$\partial_t \text{tr} K = \partial_t (g^{ij} K_{ij}) \quad (\text{eq11.102})$$

$$= \partial_t g^{ij} K_{ij} + g^{ij} \partial_t K_{ij} \quad (\text{eq11.103})$$

$$= 2N K^{ij} K_{ij} + g^{ij} \partial_t K_{ij} \quad (\text{eq11.104})$$

$$= 2N K^{ij} K_{ij} + g^{ij} (-D_{ij} N + N (R_{ij} + \text{tr} K K_{ij} - 2K_{ic} K_{jd} g^{cd})) \quad (\text{eq11.105})$$

can be matched line-by-line with the Cadabra code

substitute	(dotK,trK)	# cdb (eq11.102,dotK)
product_rule	(dotK)	# cdb (eq11.103,dotK)
substitute	(dotK,DhijDt)	# cdb (eq11.104,dotK)
substitute	(dotK,DKijDt)	# cdb (eq11.105,dotK)

¹Based on Cadabra 2.5.8 (build private dated 2024-10-25)

Running the codes

To run any of the codes you will need to install Cadabra. Compiling and installing Cadabra is straightforward with full details provided on the <https://github.com/kpeeters/cadabra2.git> repository on GitHub.

You will also need to install the hybrid-latex tools. These are simple scripts and LaTeX macros that allow the Cadabra output to be caught and integrated back into the LaTeX source. See the `INSTALL.txt` file in `hybrid-latex` for full details. The documentation for hybrid-latex tools is included in the `hybrid-latex` directory.

To compile all of the files in one go just use

```
cd source
make
```

You can compile any one of the files (e.g., `eqtn11.tex`) using any of the following (from within the `source` directory)

```
cdbllatex.sh -i eqtn11
make eqtn11
make .eqtn11
```

The first two commands will force a recompile of `eqtn11` while the third command will use the magic of makefiles to determine if it (and its dependencies) needs to be recompiled.

The `cdbllatex.sh` is the main script in the hybrid-latex codes. It does the job of extracting the embedded Cadabra code, running that code through Cadabra, collecting the output and finally making that output available to the LaTeX source.

You may want to look inside the `Makefile` to see which targets are available.

Note that there are dependencies amongst the files and these dependencies are encoded in the makefile. See the file `SEQUENCE.txt` for the details.

$$\text{eq09.lcb} := -2N\bar{A}_{ij}$$

$$\text{eq09.prd} := -2N\bar{A}_{ij}$$

$$\text{eq09.chk} := 0$$

$$\text{eq10.lcb} := -\frac{1}{6}\text{tr}KN$$

$$\text{eq10.prd} := -\frac{1}{6}N\text{tr}K$$

$$\text{eq10.chk} := 0$$

$$\text{eq11.lcb} := -g^{ab}D_{ab}N + N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}\text{tr}K^2N$$

$$\text{eq11.prd} := -g^{ij}D_{ij}N + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\text{tr}K^2\right)$$

$$\text{eq11.chk} := 0$$

$$\text{eq12.lcb} := \text{tr}KN\bar{A}_{ij} - 2N\bar{A}_i{}^b\bar{A}_{jb} + \exp(-4\phi)\left(-D_{ij}N + NR_{ij} - \frac{1}{3}Ng_{ij}g^{ab}R_{ab} + \frac{1}{3}g_{ij}g^{ab}D_{ab}N\right)$$

$$\text{eq12.prd} := N\left(\text{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp(-4\phi)\left(NR_{ij} - D_{ij}N - \frac{1}{3}g_{ij}(NR_{ab} - D_{ab}N)g^{ab}\right)$$

$$\text{eq12.chk} := 0$$

$$\begin{aligned}
\text{eq15.lcb} &:= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi \\
\text{eq15.prd} &:= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi \\
\text{eq15.chk} &:= 0
\end{aligned}$$

$$\begin{aligned}
\text{eq17.lcb} &:= -\partial_b\bar{g}^{ib} \\
\text{eq17.prd} &:= -\partial_j\bar{g}^{ij} \\
\text{eq17.chk} &:= 0
\end{aligned}$$

$$\begin{aligned}
\text{eq18.lcb} &:= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^e\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^e\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
\text{eq18.prd} &:= -\frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}) \\
\text{eq18.chk} &:= 0
\end{aligned}$$

$$\begin{aligned}
\text{eq19.lcb} &:= -2\bar{A}^{ia}\partial_a N - 2N\partial_a\bar{A}^{ia} \\
\text{eq19.prd} &:= -2\partial_j(N\bar{A}^{ij}) \\
\text{eq19.chk} &:= 0
\end{aligned}$$

$$\begin{aligned}
\text{eq20.lcb} &:= -2\bar{A}^{ia}\partial_a N - 2N\left(-6\bar{A}^{ia}\partial_a\phi - \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K\right) \\
\text{eq20.prd} &:= -2\bar{A}^{ij}\partial_j N + 2N\left(\bar{\Gamma}^i{}_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{g}^{ij}\partial_j\text{tr}K + 6\bar{A}^{ij}\partial_j\phi\right) \\
\text{eq20.chk} &:= 0
\end{aligned}$$

$$\text{prd67.eq19.lcb} := R + \frac{2}{3}\text{tr}K^2 - \bar{A}^{ab}\bar{A}_{ab}$$

$$\text{prd67.eq19.prd} := R - \bar{A}_{ab}\bar{A}^{ab} + \frac{2}{3}\text{tr}K^2$$

$$\text{prd67.eq19.chk} := 0$$

$$\text{prd67.eq20.lcb} := 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K$$

$$\text{prd67.eq20.prd} := \partial_a\bar{A}^{ia} + 6\bar{A}^{ia}\partial_a\phi + \bar{A}^{ab}\bar{\Gamma}^i_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K$$

$$\text{prd67.eq20.chk} := 0$$

$$\text{prd67.eq27.lcb} := 2\bar{g}^a_c\partial_b\phi + 2\bar{g}^a_b\partial_c\phi - 2\bar{g}^{ae}\partial_e\phi\bar{g}_{bc} + \bar{\Gamma}^a_{bc}$$

$$\text{prd67.eq27.prd} := \bar{\Gamma}^a_{bc} + 2\bar{g}^a_c\partial_b\phi + 2\bar{g}^a_b\partial_c\phi - 2\bar{g}_{bc}\bar{g}^{ae}\partial_e\phi$$

$$\text{prd67.eq27.chk} := 0$$

PhysRevD.62.044034 equation (9)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn09.json'
5  cdblib.create (jsonfile)
6
7  DgijDt = cdblib.get ('adm.DgijDt','adm.json')
8  DhijDt = cdblib.get ('adm.DhijDt','adm.json')
9
10 DphiDt = cdblib.get ('DphiDt','eqtn10.json')
11
12 # -----
13
14 gBarij := gBar_{i j} -> \exp(-4\phi) g_{i j}.          # prd62 eqn 05
15
16 Kij     := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.      # prd62 eqn 07
17
18 A2ABar := \exp(-4\phi) A_{i j} -> ABar_{i j}.          # prd62 eqn 08
19
20 # -----
21 # dgBar_{ij}/dt
22
23 dotgBarij := \partial_{t}{gBar_{i j}}.                  # cdb (eq09.101,dotgBarij)
24
25 substitute (dotgBarij, gBarij)                         # cdb (eq09.102,dotgBarij)
26 product_rule (dotgBarij)                               # cdb (eq09.103,dotgBarij)
27 substitute (dotgBarij, dexp)                           # cdb (eq09.104,dotgBarij)
28 substitute (dotgBarij, DgijDt)                         # cdb (eq09.105,dotgBarij)
29 substitute (dotgBarij, DphiDt)                         # cdb (eq09.106,dotgBarij)
30 substitute (dotgBarij, Kij)                             # cdb (eq09.107,dotgBarij)
31 distribute (dotgBarij)                                 # cdb (eq09.108,dotgBarij)
32 map_sympy (dotgBarij, "simplify")                      # cdb (eq09.109,dotgBarij)
33 substitute (dotgBarij, A2ABar)                         # cdb (eq09.110,dotgBarij)
34
35 DgBarijDt := \partial_{t}{gBar_{i j}} -> @(dotgBarij).
36

```

37

```
cdblib.put ('dotgBarij',dotgBarij,jsonfile)
```

$$\partial_t \bar{g}_{ij} = \partial_t (\exp(-4\phi) g_{ij}) \quad (\text{eq09.102})$$

$$= \partial_t (\exp(-4\phi)) g_{ij} + \exp(-4\phi) \partial_t g_{ij} \quad (\text{eq09.103})$$

$$= -4 \exp(-4\phi) \partial_t \phi g_{ij} + \exp(-4\phi) \partial_t g_{ij} \quad (\text{eq09.104})$$

$$= -4 \exp(-4\phi) \partial_t \phi g_{ij} - 2 \exp(-4\phi) N K_{ij} \quad (\text{eq09.105})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N g_{ij} - 2 \exp(-4\phi) N K_{ij} \quad (\text{eq09.106})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N g_{ij} - 2 \exp(-4\phi) N \left(A_{ij} + \frac{1}{3} g_{ij} \text{tr} K \right) \quad (\text{eq09.107})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N g_{ij} - 2 \exp(-4\phi) N A_{ij} - \frac{2}{3} \exp(-4\phi) N g_{ij} \text{tr} K \quad (\text{eq09.108})$$

$$= -2N \exp(-4\phi) A_{ij} \quad (\text{eq09.109})$$

$$= -2N \bar{A}_{ij} \quad (\text{eq09.110})$$


```

1  # -----
2  # Check against prd62.
3
4  foo := @(dotgBarij).           # cdb(eq09.lcb,foo)
5  bah  = cdblib.get('prd62.eq09.rhs','prd62.json') # cdb(eq09.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  diff = product_sort (diff)
10 rename_dummies (diff)
11 canonicalise    (diff)         # cdb(eq09.chk,diff)

```

$$\text{eq09.lcb} := -2N\bar{A}_{ij}$$

$$\text{eq09.prd} := -2N\bar{A}_{ij}$$

$$\text{eq09.chk} := 0$$

PhysRevD.62.044034 equation (10)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn10.json'
5  cdblib.create (jsonfile)
6
7  DgijDt = cdblib.get ('adm.DgijDt','adm.json')
8  DdetgDt = cdblib.get ('adm.DdetgDt','adm.json')
9
10 # -----
11
12 phi := \phi -> (1/12) \log(detg).
13 gdotK := g^{i j} K_{i j} -> trK.
14
15 # -----
16 # d\phi/dt
17
18 dotphi := \partial_t{\phi}.      # cdb (eq10.101,dotphi)
19
20 substitute (dotphi, phi)      # cdb (eq10.102,dotphi)
21 substitute (dotphi, dlog)     # cdb (eq10.103,dotphi)
22 substitute (dotphi, DdetgDt)  # cdb (eq10.104,dotphi)
23 substitute (dotphi, DgijDt)   # cdb (eq10.105,dotphi)
24 substitute (dotphi, gdotK)    # cdb (eq10.106,dotphi)
25 map_sympy (dotphi, "simplify") # cdb (eq10.107,dotphi)
26
27 DphiDt := \partial_t{\phi} -> @(dotphi).
28
29 cdblib.put ('DphiDt',DphiDt,jsonfile)

```

$$\partial_t \phi = \frac{1}{12} \partial_t (\log (g)) \tag{eq10.102}$$

$$= \frac{1}{12} g^{-1} \partial_t g \tag{eq10.103}$$

$$= \frac{1}{12} g^{-1} g g^{ij} \partial_t g_{ij} \tag{eq10.104}$$

$$= -\frac{1}{6} g^{-1} g g^{ij} N K_{ij} \tag{eq10.105}$$

$$= -\frac{1}{6} g^{-1} g \text{tr} K N \tag{eq10.106}$$

$$= -\frac{1}{6} \text{tr} K N \tag{eq10.107}$$

```

1  # -----
2  # Check against prd62.
3
4  foo := @(dotphi).                # cdb(eq10.lcb,foo)
5  bah  = cdblib.get('prd62.eq10.rhs','prd62.json') # cdb(eq10.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  diff = product_sort (diff)
10 rename_dummies (diff)
11 canonicalise    (diff)          # cdb(eq10.chk,diff)

```

$$\text{eq10.lcb} := -\frac{1}{6}\text{tr}KN$$

$$\text{eq10.prd} := -\frac{1}{6}N\text{tr}K$$

$$\text{eq10.chk} := 0$$

PhysRevD.62.044034 equation (11)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn11.json'
5  cdblib.create (jsonfile)
6
7  DhijDt = cdblib.get ('adm.DhijDt','adm.json')
8  DKijDt = cdblib.get ('adm.DKijDt','adm.json')
9
10 # -----
11
12 trK      := trK -> g^{i j} K_{i j}.
13 gdotK    := g^{i j} K_{i j} -> trK.
14
15 Kup := g^{i a} g^{j b} K_{i j} -> K^{a b}.
16
17 Ham := g^{i j} R_{i j} -> K_{i j} K^{i j} - trK trK.
18
19 Kij := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.    # prd62 eqn 07
20 Lij := K^{i j} -> A^{i j} + (1/3) g^{i j} trK.    # prd62 eqn 07
21
22 trA1 := A_{i j} g^{i j} -> 0.                    # Aij is trace free
23 trA2 := A^{i j} g_{i j} -> 0.
24
25 Asq := A_{i j} A^{i j} -> ABar_{i j} ABar^{i j}.
26
27 gdotg := g_{i j} g^{i j} -> 3.
28
29 # -----
30 # dK/dt
31
32 dotK := \partial_t{trK}.                          # cdb (eq11.101,dotK)
33
34 substitute      (dotK,trK)                        # cdb (eq11.102,dotK)
35 product_rule    (dotK)                            # cdb (eq11.103,dotK)
36 substitute      (dotK,DhijDt)                     # cdb (eq11.104,dotK)

```

```

37 substitute      (dotK,DKijDt)      # cdb (eq11.105,dotK)
38 distribute      (dotK)              # cdb (eq11.106,dotK)
39 substitute      (dotK,gdotK)        # cdb (eq11.107,dotK)
40 substitute      (dotK,Kup)          # cdb (eq11.108,dotK)
41 dotK = product_sort (dotK)          # cdb (eq11.109,dotK)
42 substitute      (dotK,Ham)          # cdb (eq11.110,dotK)
43 distribute      (dotK)              # cdb (eq11.111,dotK)
44 substitute      (dotK,Kij)          # cdb (eq11.112,dotK)
45 substitute      (dotK,Lij)          # cdb (eq11.113,dotK)
46 distribute      (dotK)              # cdb (eq11.114,dotK)
47 substitute      (dotK,trA1)         # cdb (eq11.115,dotK)
48 substitute      (dotK,trA2)         # cdb (eq11.116,dotK)
49 substitute      (dotK,Asq)          # cdb (eq11.117,dotK)
50 substitute      (dotK,gdotg)        # cdb (eq11.118,dotK)
51 map_sympy       (dotK, "simplify")  # cdb (eq11.119,dotK)
52
53 DKDt := \partial_{t}{trK} -> @(dotK).
54
55 cdblib.put ('DKDt',DKDt,jsonfile)

```

$$\partial_t \text{tr} K = \partial_t (g^{ij} K_{ij}) \quad (\text{eq11.102})$$

$$= \partial_t g^{ij} K_{ij} + g^{ij} \partial_t K_{ij} \quad (\text{eq11.103})$$

$$= 2N K^{ij} K_{ij} + g^{ij} \partial_t K_{ij} \quad (\text{eq11.104})$$

$$= 2N K^{ij} K_{ij} + g^{ij} (-D_{ij} N + N (R_{ij} + \text{tr} K K_{ij} - 2K_{ic} K_{jd} g^{cd})) \quad (\text{eq11.105})$$

$$= 2N K^{ij} K_{ij} - g^{ij} D_{ij} N + g^{ij} N R_{ij} + g^{ij} N \text{tr} K K_{ij} - 2g^{ij} N K_{ic} K_{jd} g^{cd} \quad (\text{eq11.106})$$

$$= 2N K^{ij} K_{ij} - g^{ij} D_{ij} N + g^{ij} N R_{ij} + \text{tr} K N \text{tr} K - 2g^{ij} N K_{ic} K_{jd} g^{cd} \quad (\text{eq11.107})$$

$$= 2N K^{ij} K_{ij} - g^{ij} D_{ij} N + g^{ij} N R_{ij} + \text{tr} K N \text{tr} K - 2K^{jd} N K_{jd} \quad (\text{eq11.108})$$

$$= -g^{ab} D_{ab} N + N g^{ab} R_{ab} + N \text{tr} K \text{tr} K \quad (\text{eq11.109})$$

$$= -g^{ab} D_{ab} N + N (K_{ab} K^{ab} - \text{tr} K \text{tr} K) + N \text{tr} K \text{tr} K \quad (\text{eq11.110})$$

$$= -g^{ab} D_{ab} N + N K_{ab} K^{ab} \quad (\text{eq11.111})$$

$$= -g^{ab} D_{ab} N + N \left(A_{ab} + \frac{1}{3} g_{ab} \text{tr} K \right) K^{ab} \quad (\text{eq11.112})$$

$$= -g^{ab} D_{ab} N + N \left(A_{ab} + \frac{1}{3} g_{ab} \text{tr} K \right) \left(A^{ab} + \frac{1}{3} g^{ab} \text{tr} K \right) \quad (\text{eq11.113})$$

$$= -g^{ab} D_{ab} N + N A_{ab} A^{ab} + \frac{1}{3} N A_{ab} g^{ab} \text{tr} K + \frac{1}{3} N g_{ab} \text{tr} K A^{ab} + \frac{1}{9} N g_{ab} \text{tr} K g^{ab} \text{tr} K \quad (\text{eq11.114})$$

$$= -g^{ab} D_{ab} N + N A_{ab} A^{ab} + \frac{1}{3} N g_{ab} \text{tr} K A^{ab} + \frac{1}{9} N g_{ab} \text{tr} K g^{ab} \text{tr} K \quad (\text{eq11.115})$$

$$= -g^{ab} D_{ab} N + N A_{ab} A^{ab} + \frac{1}{9} N g_{ab} \text{tr} K g^{ab} \text{tr} K \quad (\text{eq11.116})$$

$$= -g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{9} N g_{ab} \text{tr} K g^{ab} \text{tr} K \quad (\text{eq11.117})$$

$$= -g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{3} N \text{tr} K \text{tr} K \quad (\text{eq11.118})$$

$$= -g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{3} \text{tr} K^2 N \quad (\text{eq11.119})$$

```

1  # -----
2  # Check against prd62.
3
4  foo := @(dotK).                # cdb(eq11.lcb,foo)
5  bah  = cdblib.get('prd62.eq11.rhs','prd62.json')  # cdb(eq11.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy      (diff, "simplify")
13 canonicalise   (diff)          # cdb(eq11.chk,diff)

```

$$\text{eq11.lcb} := -g^{ab}D_{ab}N + N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}\text{tr}K^2N$$

$$\text{eq11.prd} := -g^{ij}D_{ij}N + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\text{tr}K^2\right)$$

$$\text{eq11.chk} := 0$$

PhysRevD.62.044034 equation (12)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn12.json'
5  cdblib.create (jsonfile)
6
7  DgijDt = cdblib.get ('adm.DgijDt','adm.json')
8  DKijDt = cdblib.get ('adm.DKijDt','adm.json')
9
10 DphiDt = cdblib.get ('DphiDt','eqtn10.json')
11 DKDt = cdblib.get ('DKDt','eqtn11.json')
12
13 # -----
14 ABar2A := ABar_{i j} -> \exp(-4\phi) A_{i j}.          # prd62 eqn 08
15 A2ABar := A_{i j} -> \exp(4\phi) ABar_{i j}.          # prd62 eqn 08
16
17 Aij     := A_{i j} -> K_{i j} - (1/3) g_{i j} trK.    # prd62 eqn 07
18 Kij     := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.    # prd62 eqn 07
19
20 gginv := {g_{i a} g^{a j} -> g_{i}^{j},
21           g_{i a} g^{j a} -> g_{i}^{j}}.
22
23 ABarUp := ABar_{i j} g^{j k} -> \exp(-4\phi) ABar_{i}^{k}.
24
25 ABardotABar := ABar_{i j} ABar^{i j} ->
26               (K_{i j}-(1/3)g_{i j} trK) (K^{i j}-(1/3)g^{i j} trK).
27
28 trg := g_{i j} g^{i j} -> 3.
29
30 trK := {K_{i j} g^{i j} -> trK,
31         K^{i j} g_{i j} -> trK}.
32
33 Ham := trK**2 -> K_{i j} K^{i j} - g^{i j} R_{i j}.
34
35 # -----
36 # dABar_{ij}/dt

```

```

37
38 dotABarij := \partial_{t}{ABar_{i j}}.          # cdb (eq12.101,dotABarij)
39
40 substitute      (dotABarij, ABar2A)           # cdb (eq12.102,dotABarij)
41 product_rule    (dotABarij)                   # cdb (eq12.103,dotABarij)
42 map_sympy       (dotABarij, "simplify")        # cdb (eq12.104,dotABarij)
43 substitute      (dotABarij, DphiDt)           # cdb (eq12.105,dotABarij)
44 substitute      (dotABarij, Aij)              # cdb (eq12.106,dotABarij)
45 distribute      (dotABarij)                   # cdb (eq12.107,dotABarij)
46 substitute      (dotABarij, DKijDt)           # cdb (eq12.108,dotABarij)
47 product_rule    (dotABarij)                   # cdb (eq12.109,dotABarij)
48 distribute      (dotABarij)                   # cdb (eq12.110,dotABarij)
49 substitute      (dotABarij, DKDt)             # cdb (eq12.111,dotABarij)
50 substitute      (dotABarij, DgijDt)           # cdb (eq12.112,dotABarij)
51 distribute      (dotABarij)                   # cdb (eq12.113,dotABarij)
52 substitute      (dotABarij, Kij)              # cdb (eq12.114,dotABarij)
53 distribute      (dotABarij)                   # cdb (eq12.115,dotABarij)
54 substitute      (dotABarij, gginv)            # cdb (eq12.116,dotABarij)
55 eliminate_kronecker (dotABarij)             # cdb (eq12.117,dotABarij)
56 substitute      (dotABarij, A2ABar)           # cdb (eq12.118,dotABarij)
57 canonicalise    (dotABarij)                   # cdb (eq12.119,dotABarij)
58 substitute      (dotABarij, ABardotABar)      # cdb (eq12.120,dotABarij)
59 distribute      (dotABarij)                   # cdb (eq12.121,dotABarij)
60 substitute      (dotABarij, trg)              # cdb (eq12.122,dotABarij)
61 substitute      (dotABarij, trK)              # cdb (eq12.123,dotABarij)
62 map_sympy       (dotABarij, "simplify")        # cdb (eq12.124,dotABarij)
63 substitute      (dotABarij, Ham)              # cdb (eq12.125,dotABarij)
64 distribute      (dotABarij)                   # cdb (eq12.126,dotABarij)
65 dotABarij = product_sort (dotABarij)          # cdb (eq12.127,dotABarij)
66 substitute      (dotABarij, ABarUp)           # cdb (eq12.128,dotABarij)
67 map_sympy       (dotABarij, "simplify")        # cdb (eq12.129,dotABarij)
68 factor_out      (dotABarij,$\exp(-4\phi)$)    # cdb (eq12.130,dotABarij)
69
70 DABarijDt := \partial_{t}{ABar_{ij}} -> @(dotABarij).
71
72 cdblib.put ('DABarijDt',DABarijDt,jsonfile)

```

$$\partial_t \bar{A}_{ij} = \partial_t (\exp(-4\phi) A_{ij}) \quad (\text{eq12.102})$$

$$= \partial_t (\exp(-4\phi)) A_{ij} + \exp(-4\phi) \partial_t A_{ij} \quad (\text{eq12.103})$$

$$= -4 \exp(-4\phi) \partial_t \phi A_{ij} + \exp(-4\phi) \partial_t A_{ij} \quad (\text{eq12.104})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N A_{ij} + \exp(-4\phi) \partial_t A_{ij} \quad (\text{eq12.105})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N \left(K_{ij} - \frac{1}{3} g_{ij} \text{tr} K \right) + \exp(-4\phi) \partial_t \left(K_{ij} - \frac{1}{3} g_{ij} \text{tr} K \right) \quad (\text{eq12.106})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N K_{ij} - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K + \exp(-4\phi) \partial_t K_{ij} - \frac{1}{3} \exp(-4\phi) \partial_t (g_{ij} \text{tr} K) \quad (\text{eq12.107})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N K_{ij} - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K + \exp(-4\phi) (-D_{ij} N + N (R_{ij} + \text{tr} K K_{ij} - 2K_{ic} K_{jd} g^{cd})) - \frac{1}{3} \exp(-4\phi) \partial_t (g_{ij} \text{tr} K) \quad (\text{eq12.108})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N K_{ij} - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K + \exp(-4\phi) (-D_{ij} N + N (R_{ij} + \text{tr} K K_{ij} - 2K_{ic} K_{jd} g^{cd})) - \frac{1}{3} \exp(-4\phi) (\partial_t g_{ij} \text{tr} K + g_{ij} \partial_t \text{tr} K) \quad (\text{eq12.109})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N K_{ij} - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \text{tr} K K_{ij} - 2 \exp(-4\phi) N K_{ic} K_{jd} g^{cd} - \frac{1}{3} \exp(-4\phi) \partial_t g_{ij} \text{tr} K - \frac{1}{3} \exp(-4\phi) g_{ij} \partial_t \text{tr} K \quad (\text{eq12.110})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N K_{ij} - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \text{tr} K K_{ij} - 2 \exp(-4\phi) N K_{ic} K_{jd} g^{cd} - \frac{1}{3} \exp(-4\phi) \partial_t g_{ij} \text{tr} K - \frac{1}{3} \exp(-4\phi) g_{ij} \left(-g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{3} \text{tr} K^2 N \right) \quad (\text{eq12.111})$$

$$= \frac{2}{3} \exp(-4\phi) \text{tr} K N K_{ij} - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \text{tr} K K_{ij} - 2 \exp(-4\phi) N K_{ic} K_{jd} g^{cd} + \frac{2}{3} \exp(-4\phi) N K_{ij} \text{tr} K - \frac{1}{3} \exp(-4\phi) g_{ij} \left(-g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{3} \text{tr} K^2 N \right) \quad (\text{eq12.112})$$

$$\begin{aligned}
\partial_t \bar{A}_{ij} &= \frac{2}{3} \exp(-4\phi) \text{tr} K N K_{ij} - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \text{tr} K K_{ij} - 2 \exp(-4\phi) N K_{ic} K_{jd} g^{cd} \\
&\quad + \frac{2}{3} \exp(-4\phi) N K_{ij} \text{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N \bar{A}_{ab} \bar{A}^{ab} - \frac{1}{9} \exp(-4\phi) g_{ij} \text{tr} K^2 N
\end{aligned} \tag{eq12.113}$$

$$\begin{aligned}
&= \frac{2}{3} \exp(-4\phi) \text{tr} K N \left(A_{ij} + \frac{1}{3} g_{ij} \text{tr} K \right) - \frac{2}{9} \exp(-4\phi) \text{tr} K N g_{ij} \text{tr} K - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} \\
&\quad + \exp(-4\phi) N \text{tr} K \left(A_{ij} + \frac{1}{3} g_{ij} \text{tr} K \right) - 2 \exp(-4\phi) N \left(A_{ic} + \frac{1}{3} g_{ic} \text{tr} K \right) \left(A_{jd} + \frac{1}{3} g_{jd} \text{tr} K \right) g^{cd} + \frac{2}{3} \exp(-4\phi) N \left(A_{ij} + \frac{1}{3} g_{ij} \text{tr} K \right) \text{tr} K \\
&\quad + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N \bar{A}_{ab} \bar{A}^{ab} - \frac{1}{9} \exp(-4\phi) g_{ij} \text{tr} K^2 N
\end{aligned} \tag{eq12.114}$$

$$\begin{aligned}
&= \frac{2}{3} \exp(-4\phi) \text{tr} K N A_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \text{tr} K A_{ij} + \frac{1}{3} \exp(-4\phi) N \text{tr} K g_{ij} \text{tr} K - 2 \exp(-4\phi) N A_{ic} A_{jd} g^{cd} \\
&\quad - \frac{2}{3} \exp(-4\phi) N A_{ic} g_{jd} \text{tr} K g^{cd} - \frac{2}{3} \exp(-4\phi) N g_{ic} \text{tr} K A_{jd} g^{cd} - \frac{2}{9} \exp(-4\phi) N g_{ic} \text{tr} K g_{jd} \text{tr} K g^{cd} + \frac{2}{3} \exp(-4\phi) N A_{ij} \text{tr} K \\
&\quad + \frac{2}{9} \exp(-4\phi) N g_{ij} \text{tr} K \text{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N \bar{A}_{ab} \bar{A}^{ab} - \frac{1}{9} \exp(-4\phi) g_{ij} \text{tr} K^2 N
\end{aligned} \tag{eq12.115}$$

$$\begin{aligned}
&= \frac{2}{3} \exp(-4\phi) \text{tr} K N A_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \text{tr} K A_{ij} + \frac{1}{3} \exp(-4\phi) N \text{tr} K g_{ij} \text{tr} K - 2 \exp(-4\phi) N A_{ic} A_{jd} g^{cd} \\
&\quad - \frac{2}{3} \exp(-4\phi) N A_{ic} g_j^c \text{tr} K - \frac{2}{3} \exp(-4\phi) N g_i^d \text{tr} K A_{jd} - \frac{2}{9} \exp(-4\phi) N g_i^d \text{tr} K g_{jd} \text{tr} K + \frac{2}{3} \exp(-4\phi) N A_{ij} \text{tr} K \\
&\quad + \frac{2}{9} \exp(-4\phi) N g_{ij} \text{tr} K \text{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N \bar{A}_{ab} \bar{A}^{ab} - \frac{1}{9} \exp(-4\phi) g_{ij} \text{tr} K^2 N
\end{aligned} \tag{eq12.116}$$

$$\begin{aligned}
&= \frac{2}{3} \exp(-4\phi) \text{tr} K N A_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \text{tr} K A_{ij} + \frac{1}{3} \exp(-4\phi) N \text{tr} K g_{ij} \text{tr} K - 2 \exp(-4\phi) N A_{ic} A_{jd} g^{cd} \\
&\quad - \frac{2}{3} \exp(-4\phi) N \text{tr} K A_{ji} - \frac{2}{9} \exp(-4\phi) N \text{tr} K g_{ji} \text{tr} K + \frac{2}{9} \exp(-4\phi) N g_{ij} \text{tr} K \text{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N \bar{A}_{ab} \bar{A}^{ab} \\
&\quad - \frac{1}{9} \exp(-4\phi) g_{ij} \text{tr} K^2 N
\end{aligned} \tag{eq12.117}$$

$$\begin{aligned}
\partial_t \bar{A}_{ij} = & \frac{2}{3} \exp(-4\phi) \operatorname{tr} K N \exp(4\phi) \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \exp(-4\phi) N \operatorname{tr} K \exp(4\phi) \bar{A}_{ij} + \frac{1}{3} \exp(-4\phi) N \operatorname{tr} K g_{ij} \operatorname{tr} K \\
& - 2 \exp(-4\phi) N \exp(4\phi) \bar{A}_{ic} \exp(4\phi) \bar{A}_{jd} g^{cd} - \frac{2}{3} \exp(-4\phi) N \operatorname{tr} K \exp(4\phi) \bar{A}_{ji} - \frac{2}{9} \exp(-4\phi) N \operatorname{tr} K g_{ji} \operatorname{tr} K + \frac{2}{9} \exp(-4\phi) N g_{ij} \operatorname{tr} K \operatorname{tr} K \\
& + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N \bar{A}_{ab} \bar{A}^{ab} - \frac{1}{9} \exp(-4\phi) g_{ij} \operatorname{tr} K^2 N \quad (\text{eq12.118})
\end{aligned}$$

$$\begin{aligned}
= & \frac{2}{3} \exp(-4\phi) \operatorname{tr} K N \exp(4\phi) \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \frac{1}{3} \exp(-4\phi) N \operatorname{tr} K \exp(4\phi) \bar{A}_{ij} + \frac{1}{9} \exp(-4\phi) N \operatorname{tr} K g_{ij} \operatorname{tr} K \\
& - 2 \exp(-4\phi) N \exp(4\phi) \bar{A}_{ic} \exp(4\phi) \bar{A}_{jd} g^{cd} + \frac{2}{9} \exp(-4\phi) N g_{ij} \operatorname{tr} K \operatorname{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N \bar{A}_{ab} \bar{A}^{ab} \\
& - \frac{1}{9} \exp(-4\phi) g_{ij} \operatorname{tr} K^2 N \quad (\text{eq12.119})
\end{aligned}$$

$$\begin{aligned}
= & \frac{2}{3} \exp(-4\phi) \operatorname{tr} K N \exp(4\phi) \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \frac{1}{3} \exp(-4\phi) N \operatorname{tr} K \exp(4\phi) \bar{A}_{ij} + \frac{1}{9} \exp(-4\phi) N \operatorname{tr} K g_{ij} \operatorname{tr} K \\
& - 2 \exp(-4\phi) N \exp(4\phi) \bar{A}_{ic} \exp(4\phi) \bar{A}_{jd} g^{cd} + \frac{2}{9} \exp(-4\phi) N g_{ij} \operatorname{tr} K \operatorname{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N \\
& - \frac{1}{3} \exp(-4\phi) g_{ij} N \left(K_{ab} - \frac{1}{3} g_{ab} \operatorname{tr} K \right) \left(K^{ab} - \frac{1}{3} g^{ab} \operatorname{tr} K \right) - \frac{1}{9} \exp(-4\phi) g_{ij} \operatorname{tr} K^2 N \quad (\text{eq12.120})
\end{aligned}$$

$$\begin{aligned}
= & \frac{2}{3} \exp(-4\phi) \operatorname{tr} K N \exp(4\phi) \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \frac{1}{3} \exp(-4\phi) N \operatorname{tr} K \exp(4\phi) \bar{A}_{ij} + \frac{1}{9} \exp(-4\phi) N \operatorname{tr} K g_{ij} \operatorname{tr} K \\
& - 2 \exp(-4\phi) N \exp(4\phi) \bar{A}_{ic} \exp(4\phi) \bar{A}_{jd} g^{cd} + \frac{2}{9} \exp(-4\phi) N g_{ij} \operatorname{tr} K \operatorname{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N K_{ab} K^{ab} \\
& + \frac{1}{9} \exp(-4\phi) g_{ij} N K_{ab} g^{ab} \operatorname{tr} K + \frac{1}{9} \exp(-4\phi) g_{ij} N g_{ab} \operatorname{tr} K K^{ab} - \frac{1}{27} \exp(-4\phi) g_{ij} N g_{ab} \operatorname{tr} K g^{ab} \operatorname{tr} K - \frac{1}{9} \exp(-4\phi) g_{ij} \operatorname{tr} K^2 N \quad (\text{eq12.121})
\end{aligned}$$

$$\begin{aligned}
\partial_t \bar{A}_{ij} = & \frac{2}{3} \exp(-4\phi) \text{tr} K N \exp(4\phi) \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \frac{1}{3} \exp(-4\phi) N \text{tr} K \exp(4\phi) \bar{A}_{ij} + \frac{1}{9} \exp(-4\phi) N \text{tr} K g_{ij} \text{tr} K \\
& - 2 \exp(-4\phi) N \exp(4\phi) \bar{A}_{ic} \exp(4\phi) \bar{A}_{jd} g^{cd} + \frac{2}{9} \exp(-4\phi) N g_{ij} \text{tr} K \text{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N K_{ab} K^{ab} \\
& + \frac{1}{9} \exp(-4\phi) g_{ij} N K_{ab} g^{ab} \text{tr} K + \frac{1}{9} \exp(-4\phi) g_{ij} N g_{ab} \text{tr} K K^{ab} - \frac{1}{9} \exp(-4\phi) g_{ij} N \text{tr} K \text{tr} K - \frac{1}{9} \exp(-4\phi) g_{ij} \text{tr} K^2 N \quad (\text{eq12.122})
\end{aligned}$$

$$\begin{aligned}
= & \frac{2}{3} \exp(-4\phi) \text{tr} K N \exp(4\phi) \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + \exp(-4\phi) N R_{ij} + \frac{1}{3} \exp(-4\phi) N \text{tr} K \exp(4\phi) \bar{A}_{ij} + \frac{1}{9} \exp(-4\phi) N \text{tr} K g_{ij} \text{tr} K \\
& - 2 \exp(-4\phi) N \exp(4\phi) \bar{A}_{ic} \exp(4\phi) \bar{A}_{jd} g^{cd} + \frac{2}{9} \exp(-4\phi) N g_{ij} \text{tr} K \text{tr} K + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} \exp(-4\phi) g_{ij} N K_{ab} K^{ab} \\
& + \frac{1}{9} \exp(-4\phi) g_{ij} N \text{tr} K \text{tr} K - \frac{1}{9} \exp(-4\phi) g_{ij} \text{tr} K^2 N \quad (\text{eq12.123})
\end{aligned}$$

$$\begin{aligned}
= & \text{tr} K N \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + N \exp(-4\phi) R_{ij} + \frac{1}{3} \text{tr} K^2 N \exp(-4\phi) g_{ij} - 2N \exp(4\phi) \bar{A}_{ic} \bar{A}_{jd} g^{cd} + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N \\
& - \frac{1}{3} N \exp(-4\phi) g_{ij} K_{ab} K^{ab} \quad (\text{eq12.124})
\end{aligned}$$

$$\begin{aligned}
= & \text{tr} K N \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + N \exp(-4\phi) R_{ij} + \frac{1}{3} (K_{ab} K^{ab} - g^{ab} R_{ab}) N \exp(-4\phi) g_{ij} - 2N \exp(4\phi) \bar{A}_{ic} \bar{A}_{jd} g^{cd} \\
& + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} N \exp(-4\phi) g_{ij} K_{ab} K^{ab} \quad (\text{eq12.125})
\end{aligned}$$

$$\begin{aligned}
= & \text{tr} K N \bar{A}_{ij} - \exp(-4\phi) D_{ij} N + N \exp(-4\phi) R_{ij} + \frac{1}{3} K_{ab} K^{ab} N \exp(-4\phi) g_{ij} - \frac{1}{3} g^{ab} R_{ab} N \exp(-4\phi) g_{ij} - 2N \exp(4\phi) \bar{A}_{ic} \bar{A}_{jd} g^{cd} \\
& + \frac{1}{3} \exp(-4\phi) g_{ij} g^{ab} D_{ab} N - \frac{1}{3} N \exp(-4\phi) g_{ij} K_{ab} K^{ab} \quad (\text{eq12.126})
\end{aligned}$$

$$= N \text{tr} K \bar{A}_{ij} - D_{ij} N \exp(-4\phi) + N R_{ij} \exp(-4\phi) - \frac{1}{3} N g_{ij} g^{ab} R_{ab} \exp(-4\phi) - 2N \bar{A}_{ia} \bar{A}_{jb} g^{ab} \exp(4\phi) + \frac{1}{3} g_{ij} g^{ab} D_{ab} N \exp(-4\phi) \quad (\text{eq12.127})$$

$$\begin{aligned}
\partial_t \bar{A}_{ij} &= N \text{tr} K \bar{A}_{ij} - D_{ij} N \exp(-4\phi) + N R_{ij} \exp(-4\phi) - \frac{1}{3} N g_{ij} g^{ab} R_{ab} \exp(-4\phi) - 2N \exp(-4\phi) \bar{A}_i{}^b \bar{A}_{jb} \exp(4\phi) \\
&\quad + \frac{1}{3} g_{ij} g^{ab} D_{ab} N \exp(-4\phi)
\end{aligned} \tag{eq12.128}$$

$$= \text{tr} K N \bar{A}_{ij} - D_{ij} N \exp(-4\phi) + N \exp(-4\phi) R_{ij} - \frac{1}{3} N \exp(-4\phi) g_{ij} g^{ab} R_{ab} - 2N \bar{A}_i{}^b \bar{A}_{jb} + \frac{1}{3} g_{ij} g^{ab} D_{ab} N \exp(-4\phi) \tag{eq12.129}$$

$$= \text{tr} K N \bar{A}_{ij} - 2N \bar{A}_i{}^b \bar{A}_{jb} + \exp(-4\phi) \left(-D_{ij} N + N R_{ij} - \frac{1}{3} N g_{ij} g^{ab} R_{ab} + \frac{1}{3} g_{ij} g^{ab} D_{ab} N \right) \tag{eq12.130}$$

```

1  # -----
2  # Check against prd62.
3
4  foo := @(dotABarij).                # cdb(eq12.lcb,foo)
5  bah  = cdblib.get('prd62.eq12.rhs','prd62.json') # cdb(eq12.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  foo := ABar_{a}^{b} -> gBar^{b c} ABar_{a c}.
10 bah := ABar^{a}_{a}_{b} -> gBar^{a c} ABar_{c b}.
11
12 substitute (diff, foo)
13 substitute (diff, bah)
14 distribute (diff)
15 diff = product_sort (diff)
16 rename_dummies (diff)
17 map_sympy (diff, "simplify")
18 canonicalise (diff)                # cdb(eq12.chk,diff)

```

$$\text{eq12.lcb} := \text{tr} K N \bar{A}_{ij} - 2N \bar{A}_i^b \bar{A}_{jb} + \exp(-4\phi) \left(-D_{ij}N + N R_{ij} - \frac{1}{3} N g_{ij} g^{ab} R_{ab} + \frac{1}{3} g_{ij} g^{ab} D_{ab}N \right)$$

$$\text{eq12.prd} := N \left(\text{tr} K \bar{A}_{ij} - 2\bar{A}_{ia} \bar{A}^a_j \right) + \exp(-4\phi) \left(N R_{ij} - D_{ij}N - \frac{1}{3} g_{ij} (N R_{ab} - D_{ab}N) g^{ab} \right)$$

$$\text{eq12.chk} := 0$$

PhysRevD.62.044034 equation (15)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn15.json'
5  cdblib.create (jsonfile)
6
7  defG2GBar = cdblib.get ('defG2GBar','gamma.json')
8
9  # -----
10 # Rphi = the part of Rab from the conformal factor
11
12 Rab := R_{a b}. # cdb (eq15.101,Rab)
13
14 substitute      (Rab, defRab) # cdb (eq15.102,Rab)
15 substitute      (Rab, defRiem) # cdb (eq15.103,Rab)
16 substitute      (Rab, defG2GBar) # cdb (eq15.104,Rab)
17 distribute      (Rab) # cdb (eq15.105,Rab)
18 product_rule    (Rab) # cdb (eq15.106,Rab)
19 Rab = product_sort (Rab) # cdb (eq15.107,Rab)
20 rename_dummies  (Rab) # cdb (eq15.108,Rab)
21 canonicalise    (Rab) # cdb (eq15.109,Rab)
22 substitute      (Rab, $gBar_{b c} gBar^{c a} -> gBar^{a}_{b}$)
23 substitute      (Rab, $\partial_{a}\{gBar^{a}_{b}\} -> 0$)
24 substitute      (Rab, $\partial_{a}\{gBar_{b}^{c}\} -> 0$)
25 substitute      (Rab, $gBar^{a}_{a} -> 3$)
26 eliminate_kronecker (Rab) # cdb (eq15.110,Rab)
27 Rab = product_sort (Rab) # cdb (eq15.111,Rab)
28 rename_dummies  (Rab) # cdb (eq15.112,Rab)
29 canonicalise    (Rab) # cdb (eq15.113,Rab)
30 substitute      (Rab, $gBar_{b c} gBar^{c a} -> gBar^{a}_{b}$) # cdb (eq15.114,Rab)
31 substitute      (Rab, $gBar^{a}_{a} -> 3$) # cdb (eq15.115,Rab)
32 eliminate_kronecker (Rab) # cdb (eq15.116,Rab)
33
34 #

```

```

35  # isolate Rphi from Rab by switching to local RNC
36
37  Rphi := @(Rab).
38
39  substitute (Rphi, $GammaBar^{a}_{b c}->0$)           # cdb (eq15.117,Rphi)
40  substitute (Rphi, $\partial_{a}\{gBar_{b c}\}->0$)      # cdb (eq15.118,Rphi)
41  substitute (Rphi, $\partial_{a}\{gBar^{b c}\}->0$)        # cdb (eq15.119,Rphi)
42
43  substitute (Rphi, $\partial_{a b}\{\phi\} \rightarrow DBar_{a b}\{\phi\}$) # cdb (eq15.120,Rphi)
44  substitute (Rphi, $\partial_{a}\{\phi\} \rightarrow DBar_{a}\{\phi\}$)      # cdb (eq15.121,Rphi)
45
46  defRphi := Rphi_{a b} -> @(Rphi).
47
48  cdblib.put ('defRphi',defRphi,jsonfile)

```

$$R_{ab} = R^c_{acb} \quad (\text{eq15.102})$$

$$= \partial_c \Gamma^c_{ab} + \Gamma^c_{ec} \Gamma^e_{ab} - \partial_b \Gamma^c_{ac} - \Gamma^c_{eb} \Gamma^e_{ac} \quad (\text{eq15.103})$$

$$= \partial_c (2\bar{g}^c_b \partial_a \phi + 2\bar{g}^c_a \partial_b \phi - 2\bar{g}^{ce} \partial_e \phi \bar{g}_{ab} + \bar{\Gamma}^c_{ab}) + (2\bar{g}^c_c \partial_e \phi + 2\bar{g}^e_c \partial_c \phi - 2\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} + \bar{\Gamma}^c_{ec}) (2\bar{g}^e_b \partial_a \phi + 2\bar{g}^e_a \partial_b \phi - 2\bar{g}^{ef} \partial_f \phi \bar{g}_{ab} + \bar{\Gamma}^e_{ab}) \\ - \partial_b (2\bar{g}^c_c \partial_a \phi + 2\bar{g}^c_a \partial_c \phi - 2\bar{g}^{ce} \partial_e \phi \bar{g}_{ac} + \bar{\Gamma}^c_{ac}) - (2\bar{g}^c_b \partial_e \phi + 2\bar{g}^e_b \partial_c \phi - 2\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} + \bar{\Gamma}^c_{eb}) (2\bar{g}^e_c \partial_a \phi + 2\bar{g}^e_a \partial_c \phi - 2\bar{g}^{ef} \partial_f \phi \bar{g}_{ac} + \bar{\Gamma}^e_{ac}) \quad (\text{eq15.104})$$

$$= 2\partial_c (\bar{g}^c_b \partial_a \phi) + 2\partial_c (\bar{g}^c_a \partial_b \phi) - 2\partial_c (\bar{g}^{ce} \partial_e \phi \bar{g}_{ab}) + \partial_c \bar{\Gamma}^c_{ab} + 4\bar{g}^c_c \partial_e \phi \bar{g}^e_b \partial_a \phi + 4\bar{g}^c_c \partial_e \phi \bar{g}^e_a \partial_b \phi - 4\bar{g}^c_c \partial_e \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} + 2\bar{g}^c_c \partial_e \phi \bar{\Gamma}^e_{ab} + 4\bar{g}^e_c \partial_c \phi \bar{g}^e_b \partial_a \phi \\ + 4\bar{g}^e_c \partial_c \phi \bar{g}^e_a \partial_b \phi - 4\bar{g}^e_c \partial_c \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} + 2\bar{g}^e_c \partial_c \phi \bar{\Gamma}^e_{ab} - 4\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{g}^e_b \partial_a \phi - 4\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{g}^e_a \partial_b \phi + 4\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} - 2\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{\Gamma}^e_{ab} \\ + 2\bar{\Gamma}^c_{ec} \bar{g}^e_b \partial_a \phi + 2\bar{\Gamma}^c_{ec} \bar{g}^e_a \partial_b \phi - 2\bar{\Gamma}^c_{ec} \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} + \bar{\Gamma}^c_{ec} \bar{\Gamma}^e_{ab} - 2\partial_b (\bar{g}^c_c \partial_a \phi) - 2\partial_b (\bar{g}^c_a \partial_c \phi) + 2\partial_b (\bar{g}^{ce} \partial_e \phi \bar{g}_{ac}) - \partial_b \bar{\Gamma}^c_{ac} - 4\bar{g}^c_b \partial_e \phi \bar{g}^e_c \partial_a \phi \\ - 4\bar{g}^c_b \partial_e \phi \bar{g}^e_a \partial_c \phi + 4\bar{g}^c_b \partial_e \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} - 2\bar{g}^c_b \partial_e \phi \bar{\Gamma}^e_{ac} - 4\bar{g}^e_c \partial_b \phi \bar{g}^e_c \partial_a \phi - 4\bar{g}^e_c \partial_b \phi \bar{g}^e_a \partial_c \phi + 4\bar{g}^e_c \partial_b \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} - 2\bar{g}^e_c \partial_b \phi \bar{\Gamma}^e_{ac} + 4\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{g}^e_c \partial_a \phi \\ + 4\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{g}^e_a \partial_c \phi - 4\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} + 2\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{\Gamma}^e_{ac} - 2\bar{\Gamma}^c_{eb} \bar{g}^e_c \partial_a \phi - 2\bar{\Gamma}^c_{eb} \bar{g}^e_a \partial_c \phi + 2\bar{\Gamma}^c_{eb} \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} - \bar{\Gamma}^c_{eb} \bar{\Gamma}^e_{ac} \quad (\text{eq15.105})$$

$$= 2\partial_c \bar{g}^c_b \partial_a \phi + 2\bar{g}^c_b \partial_{ca} \phi + 2\partial_c \bar{g}^c_a \partial_b \phi + 2\bar{g}^c_a \partial_{cb} \phi - 2\partial_c \bar{g}^{ce} \partial_e \phi \bar{g}_{ab} - 2\bar{g}^{ce} \partial_{ce} \phi \bar{g}_{ab} - 2\bar{g}^{ce} \partial_e \phi \partial_c \bar{g}_{ab} + \partial_c \bar{\Gamma}^c_{ab} + 4\bar{g}^c_c \partial_e \phi \bar{g}^e_b \partial_a \phi + 4\bar{g}^c_c \partial_e \phi \bar{g}^e_a \partial_b \phi \\ - 4\bar{g}^c_c \partial_e \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} + 2\bar{g}^c_c \partial_e \phi \bar{\Gamma}^e_{ab} + 4\bar{g}^e_c \partial_c \phi \bar{g}^e_b \partial_a \phi + 4\bar{g}^e_c \partial_c \phi \bar{g}^e_a \partial_b \phi - 4\bar{g}^e_c \partial_c \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} + 2\bar{g}^e_c \partial_c \phi \bar{\Gamma}^e_{ab} - 4\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{g}^e_b \partial_a \phi \\ - 4\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{g}^e_a \partial_b \phi + 4\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} - 2\bar{g}^{cd} \partial_d \phi \bar{g}_{ec} \bar{\Gamma}^e_{ab} + 2\bar{\Gamma}^c_{ec} \bar{g}^e_b \partial_a \phi + 2\bar{\Gamma}^c_{ec} \bar{g}^e_a \partial_b \phi - 2\bar{\Gamma}^c_{ec} \bar{g}^{ef} \partial_f \phi \bar{g}_{ab} + \bar{\Gamma}^c_{ec} \bar{\Gamma}^e_{ab} - 2\partial_b \bar{g}^c_c \partial_a \phi \\ - 2\bar{g}^c_c \partial_{ba} \phi - 2\partial_b \bar{g}^c_a \partial_c \phi - 2\bar{g}^c_a \partial_{bc} \phi + 2\partial_b \bar{g}^{ce} \partial_e \phi \bar{g}_{ac} + 2\bar{g}^{ce} \partial_{be} \phi \bar{g}_{ac} + 2\bar{g}^{ce} \partial_e \phi \partial_b \bar{g}_{ac} - \partial_b \bar{\Gamma}^c_{ac} - 4\bar{g}^c_b \partial_e \phi \bar{g}^e_c \partial_a \phi - 4\bar{g}^c_b \partial_e \phi \bar{g}^e_a \partial_c \phi \\ + 4\bar{g}^c_b \partial_e \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} - 2\bar{g}^c_b \partial_e \phi \bar{\Gamma}^e_{ac} - 4\bar{g}^e_c \partial_b \phi \bar{g}^e_c \partial_a \phi - 4\bar{g}^e_c \partial_b \phi \bar{g}^e_a \partial_c \phi + 4\bar{g}^e_c \partial_b \phi \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} - 2\bar{g}^e_c \partial_b \phi \bar{\Gamma}^e_{ac} + 4\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{g}^e_c \partial_a \phi \\ + 4\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{g}^e_a \partial_c \phi - 4\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} + 2\bar{g}^{cd} \partial_d \phi \bar{g}_{eb} \bar{\Gamma}^e_{ac} - 2\bar{\Gamma}^c_{eb} \bar{g}^e_c \partial_a \phi - 2\bar{\Gamma}^c_{eb} \bar{g}^e_a \partial_c \phi + 2\bar{\Gamma}^c_{eb} \bar{g}^{ef} \partial_f \phi \bar{g}_{ac} - \bar{\Gamma}^c_{eb} \bar{\Gamma}^e_{ac} \quad (\text{eq15.106})$$

$$= 2\partial_a \phi \partial_c \bar{g}^c_b + 2\partial_{ca} \phi \bar{g}^c_b + 2\partial_b \phi \partial_c \bar{g}^c_a + 2\partial_{cb} \phi \bar{g}^c_a - 2\bar{g}_{ab} \partial_d \phi \partial_c \bar{g}^{cd} - 2\bar{g}_{ab} \bar{g}^{cd} \partial_{cd} \phi - 2\bar{g}^{cd} \partial_d \phi \partial_c \bar{g}_{ab} + \partial_c \bar{\Gamma}^c_{ab} + 4\partial_a \phi \partial_c \phi \bar{g}^c_b \bar{g}^d_d + 4\partial_b \phi \partial_c \phi \bar{g}^c_a \bar{g}^d_d \\ - 4\bar{g}_{ab} \bar{g}^{cd} \partial_c \phi \partial_d \phi \bar{g}^e_e + 2\bar{\Gamma}^c_{ab} \partial_c \phi \bar{g}^d_d + 4\partial_a \phi \partial_c \phi \bar{g}^c_d \bar{g}^d_b - 4\bar{g}_{ab} \bar{g}^{cd} \partial_e \phi \partial_d \phi \bar{g}^e_c + 2\bar{\Gamma}^c_{ab} \partial_d \phi \bar{g}^d_c - 4\bar{g}_{cd} \bar{g}^{de} \partial_a \phi \partial_e \phi \bar{g}^c_b - 4\bar{g}_{cd} \bar{g}^{de} \partial_b \phi \partial_e \phi \bar{g}^c_a \\ + 4\bar{g}_{ab} \bar{g}_{cd} \bar{g}^{de} \bar{g}^{cf} \partial_e \phi \partial_f \phi - 2\bar{g}_{cd} \bar{g}^{de} \bar{\Gamma}^c_{ab} \partial_e \phi + 2\bar{\Gamma}^c_{dc} \partial_a \phi \bar{g}^d_b + 2\bar{\Gamma}^c_{dc} \partial_b \phi \bar{g}^d_a - 2\bar{g}_{ab} \bar{g}^{cd} \bar{\Gamma}^e_{ce} \partial_d \phi + \bar{\Gamma}^c_{ab} \bar{\Gamma}^d_{cd} - 2\partial_a \phi \partial_b \bar{g}^c_c - 2\partial_{ba} \phi \bar{g}^c_c - 2\partial_c \phi \partial_b \bar{g}^c_a \\ - 2\partial_{bc} \phi \bar{g}^c_a + 2\bar{g}_{ac} \partial_d \phi \partial_b \bar{g}^{cd} + 2\bar{g}_{ac} \bar{g}^{cd} \partial_{bd} \phi + 2\bar{g}^{cd} \partial_d \phi \partial_b \bar{g}_{ac} - \partial_b \bar{\Gamma}^c_{ac} - 4\partial_a \phi \partial_c \phi \bar{g}^d_b \bar{g}^d_c - 4\partial_c \phi \partial_d \phi \bar{g}^c_b \bar{g}^d_a + 4\bar{g}_{ac} \bar{g}^{de} \partial_d \phi \partial_e \phi \bar{g}^c_b - 2\bar{\Gamma}^c_{ad} \partial_c \phi \bar{g}^d_b \\ - 4\partial_a \phi \partial_b \phi \bar{g}^c_a \bar{g}^d_c + 4\bar{g}_{ac} \bar{g}^{de} \partial_b \phi \partial_e \phi \bar{g}^c_d - 2\bar{\Gamma}^c_{ad} \partial_b \phi \bar{g}^d_c + 4\bar{g}_{cb} \bar{g}^{de} \partial_a \phi \partial_e \phi \bar{g}^c_d + 4\bar{g}_{cb} \bar{g}^{de} \partial_d \phi \partial_e \phi \bar{g}^c_a - 4\bar{g}_{ac} \bar{g}_{db} \bar{g}^{ce} \bar{g}^{df} \partial_e \phi \partial_f \phi + 2\bar{g}_{cb} \bar{g}^{de} \bar{\Gamma}^c_{ad} \partial_e \phi \\ - 2\bar{\Gamma}^c_{db} \partial_a \phi \bar{g}^d_c - 2\bar{\Gamma}^c_{db} \partial_c \phi \bar{g}^d_a + 2\bar{g}_{ac} \bar{g}^{de} \bar{\Gamma}^c_{db} \partial_e \phi - \bar{\Gamma}^c_{db} \bar{\Gamma}^d_{ac} \quad (\text{eq15.107})$$

$$\begin{aligned}
R_{ab} = & 2\partial_a\phi\partial_c\bar{g}^c{}_b + 2\partial_{ca}\phi\bar{g}^c{}_b + 2\partial_b\phi\partial_c\bar{g}^c{}_a + 2\partial_{cb}\phi\bar{g}^c{}_a - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{dc} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{dc}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 4\partial_a\phi\partial_c\phi\bar{g}^c{}_b\bar{g}^d{}_d + 4\partial_b\phi\partial_c\phi\bar{g}^c{}_a\bar{g}^d{}_d \\
& - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^e{}_e + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi\bar{g}^d{}_d + 4\partial_a\phi\partial_c\phi\bar{g}^c{}_d\bar{g}^d{}_b - 4\bar{g}_{ab}\bar{g}^{ed}\partial_c\phi\partial_d\phi\bar{g}^c{}_e + 2\bar{\Gamma}^c{}_{ab}\partial_d\phi\bar{g}^d{}_c - 4\bar{g}_{de}\bar{g}^{ec}\partial_a\phi\partial_c\phi\bar{g}^d{}_b - 4\bar{g}_{de}\bar{g}^{ec}\partial_b\phi\partial_c\phi\bar{g}^d{}_a \\
& + 4\bar{g}_{ab}\bar{g}_{ef}\bar{g}^{fc}\bar{g}^{ed}\partial_c\phi\partial_d\phi - 2\bar{g}_{ce}\bar{g}^{ed}\bar{\Gamma}^c{}_{ab}\partial_d\phi + 2\bar{\Gamma}^c{}_{dc}\partial_a\phi\bar{g}^d{}_b + 2\bar{\Gamma}^c{}_{dc}\partial_b\phi\bar{g}^d{}_a - 2\bar{g}_{ab}\bar{g}^{de}\bar{\Gamma}^c{}_{dc}\partial_e\phi + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} - 2\partial_a\phi\partial_b\bar{g}^c{}_c - 2\partial_{ba}\phi\bar{g}^c{}_c - 2\partial_c\phi\partial_b\bar{g}^c{}_a \\
& - 2\partial_{bc}\phi\bar{g}^c{}_a + 2\bar{g}_{ad}\partial_c\phi\partial_b\bar{g}^{dc} + 2\bar{g}_{ad}\bar{g}^{dc}\partial_{bc}\phi + 2\bar{g}^{dc}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c{}_{ac} - 4\partial_a\phi\partial_c\phi\bar{g}^d{}_b\bar{g}^c{}_d - 4\partial_c\phi\partial_d\phi\bar{g}^c{}_b\bar{g}^d{}_a + 4\bar{g}_{ae}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^e{}_b - 2\bar{\Gamma}^c{}_{ad}\partial_c\phi\bar{g}^d{}_b \\
& - 4\partial_a\phi\partial_b\phi\bar{g}^c{}_d\bar{g}^d{}_c + 4\bar{g}_{ad}\bar{g}^{ec}\partial_b\phi\partial_c\phi\bar{g}^d{}_e - 2\bar{\Gamma}^c{}_{ad}\partial_b\phi\bar{g}^d{}_c + 4\bar{g}_{db}\bar{g}^{ec}\partial_a\phi\partial_c\phi\bar{g}^d{}_e + 4\bar{g}_{eb}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^e{}_a - 4\bar{g}_{ae}\bar{g}_{fb}\bar{g}^{ec}\bar{g}^{fd}\partial_c\phi\partial_d\phi + 2\bar{g}_{cb}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi \\
& - 2\bar{\Gamma}^c{}_{db}\partial_a\phi\bar{g}^d{}_c - 2\bar{\Gamma}^c{}_{db}\partial_c\phi\bar{g}^d{}_a + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{db}\partial_e\phi - \bar{\Gamma}^c{}_{db}\bar{\Gamma}^d{}_{ac}
\end{aligned} \tag{eq15.108}$$

$$\begin{aligned}
&= 2\partial_a\phi\partial_c\bar{g}_b{}^c + 2\partial_{ac}\phi\bar{g}_b{}^c + 2\partial_b\phi\partial_c\bar{g}_a{}^c - 2\bar{g}_{ab}\partial_c\phi\partial_a\bar{g}{}^{cd} - 2\bar{g}_{ab}\bar{g}{}^{cd}\partial_{cd}\phi - 2\bar{g}{}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}{}^c{}_{ab} + 4\partial_a\phi\partial_c\phi\bar{g}_b{}^c\bar{g}{}^d{}_d + 4\partial_b\phi\partial_c\phi\bar{g}_a{}^c\bar{g}{}^d{}_d - 4\bar{g}_{ab}\bar{g}{}^{cd}\partial_c\phi\partial_d\phi\bar{g}{}^e{}_e \\
&\quad + 2\bar{\Gamma}{}^c{}_{ab}\partial_c\phi\bar{g}{}^d{}_d - 4\bar{g}_{ab}\bar{g}{}^{cd}\partial_c\phi\partial_e\phi\bar{g}{}^d{}_e + 2\bar{\Gamma}{}^c{}_{ab}\partial_d\phi\bar{g}_c{}^d - 4\bar{g}_{cd}\bar{g}{}^{ce}\partial_a\phi\partial_e\phi\bar{g}_b{}^d - 4\bar{g}_{cd}\bar{g}{}^{ce}\partial_b\phi\partial_e\phi\bar{g}_a{}^d + 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}{}^{ce}\bar{g}{}^{df}\partial_e\phi\partial_f\phi - 2\bar{g}_{cd}\bar{g}{}^{ce}\bar{\Gamma}{}^d{}_{ab}\partial_e\phi \\
&\quad + 2\bar{\Gamma}{}^c{}_{cd}\partial_a\phi\bar{g}_b{}^d + 2\bar{\Gamma}{}^c{}_{cd}\partial_b\phi\bar{g}_a{}^d - 2\bar{g}_{ab}\bar{g}{}^{cd}\bar{\Gamma}{}^e{}_{ce}\partial_d\phi + \bar{\Gamma}{}^c{}_{ab}\bar{\Gamma}{}^d{}_{cd} - 2\partial_a\phi\partial_b\bar{g}{}^c{}_c - 2\partial_{ab}\phi\bar{g}{}^c{}_c - 2\partial_c\phi\partial_b\bar{g}_a{}^c + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}{}^{cd} + 2\bar{g}_{ac}\bar{g}{}^{cd}\partial_{bd}\phi + 2\bar{g}{}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} \\
&\quad - \partial_b\bar{\Gamma}{}^c{}_{ac} - 4\partial_c\phi\partial_d\phi\bar{g}_a{}^c\bar{g}_b{}^d + 4\bar{g}_{ac}\bar{g}{}^{de}\partial_d\phi\partial_e\phi\bar{g}_b{}^c - 2\bar{\Gamma}{}^c{}_{ad}\partial_c\phi\bar{g}_b{}^d - 4\partial_a\phi\partial_b\phi\bar{g}{}^c{}_d\bar{g}_c{}^d + 4\bar{g}_{ac}\bar{g}{}^{de}\partial_b\phi\partial_d\phi\bar{g}{}^c{}_e - 2\bar{\Gamma}{}^c{}_{ad}\partial_b\phi\bar{g}_c{}^d + 4\bar{g}_{bc}\bar{g}{}^{de}\partial_a\phi\partial_d\phi\bar{g}{}^c{}_e \\
&\quad + 4\bar{g}_{bc}\bar{g}{}^{de}\partial_d\phi\partial_e\phi\bar{g}_a{}^c - 4\bar{g}_{ac}\bar{g}_{bd}\bar{g}{}^{ce}\bar{g}{}^{df}\partial_e\phi\partial_f\phi + 2\bar{g}_{bc}\bar{g}{}^{de}\bar{\Gamma}{}^c{}_{ad}\partial_e\phi - 2\bar{\Gamma}{}^c{}_{bd}\partial_a\phi\bar{g}_c{}^d - 2\bar{\Gamma}{}^c{}_{bd}\partial_c\phi\bar{g}_a{}^d + 2\bar{g}_{ac}\bar{g}{}^{de}\bar{\Gamma}{}^c{}_{bd}\partial_e\phi - \bar{\Gamma}{}^c{}_{ad}\bar{\Gamma}{}^d{}_{bc} \quad (\text{eq15.109})
\end{aligned}$$

$$\begin{aligned}
= & -4\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_a\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 8\partial_a\phi\partial_b\phi + 12\partial_b\phi\partial_a\phi - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c{}_{ab}\partial_c\phi - 4\bar{g}_{ab}\bar{g}^{ce}\partial_c\phi\partial_e\phi \\
& + 2\bar{\Gamma}^d{}_{ab}\partial_d\phi - 4\bar{g}_{cb}\bar{g}^{ce}\partial_a\phi\partial_e\phi - 4\bar{g}_{ca}\bar{g}^{ce}\partial_b\phi\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{fe}\partial_e\phi\partial_f\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d{}_{ab}\partial_e\phi + 2\bar{\Gamma}^c{}_{cb}\partial_a\phi + 2\bar{\Gamma}^c{}_{ca}\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{ce}\partial_d\phi + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} \\
& + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\partial_{ba}\phi + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c{}_{ac} + 4\bar{g}_{ab}\bar{g}^{de}\partial_d\phi\partial_e\phi - 4\partial_a\phi\partial_b\phi\bar{g}^d{}_d + 4\bar{g}_{ae}\bar{g}^{de}\partial_b\phi\partial_d\phi - 2\bar{\Gamma}^d{}_{ad}\partial_b\phi + 4\bar{g}_{be}\bar{g}^{de}\partial_a\phi\partial_d\phi \\
& + 4\bar{g}_{ba}\bar{g}^{de}\partial_d\phi\partial_e\phi - 4\bar{g}_{bd}\bar{g}^{df}\partial_a\phi\partial_f\phi + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi - 2\bar{\Gamma}^d{}_{bd}\partial_a\phi - 2\bar{\Gamma}^c{}_{ba}\partial_c\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{bd}\partial_e\phi - \bar{\Gamma}^c{}_{ad}\bar{\Gamma}^d{}_{bc} \quad (\text{eq15.110})
\end{aligned}$$

$$\begin{aligned}
= & -4\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 20\partial_a\phi\partial_b\phi - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 6\bar{\Gamma}^c{}_{ab}\partial_c\phi - 4\bar{g}_{cb}\bar{g}^{cd}\partial_a\phi\partial_d\phi - 4\bar{g}_{ca}\bar{g}^{cd}\partial_b\phi\partial_d\phi \\
& + 4\bar{g}_{ab}\bar{g}^{cd}\partial_d\phi\partial_c\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d{}_{ab}\partial_e\phi + 2\bar{\Gamma}^c{}_{cb}\partial_a\phi + 2\bar{\Gamma}^c{}_{ca}\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{ce}\partial_d\phi + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\partial_{ba}\phi + 2\bar{g}^{dc}\partial_d\phi\partial_b\bar{g}_{ac} - \partial_b\bar{\Gamma}^c{}_{ac} \\
& - 4\partial_a\phi\partial_b\phi\bar{g}^c{}_c + 4\bar{g}_{ac}\bar{g}^{dc}\partial_b\phi\partial_d\phi - 2\bar{\Gamma}^c{}_{ac}\partial_b\phi + 4\bar{g}_{bc}\bar{g}^{dc}\partial_a\phi\partial_d\phi + 4\bar{g}_{ba}\bar{g}^{cd}\partial_c\phi\partial_d\phi - 4\bar{g}_{bc}\bar{g}^{cd}\partial_a\phi\partial_d\phi + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi - 2\bar{\Gamma}^c{}_{bc}\partial_a\phi - 2\bar{\Gamma}^c{}_{ba}\partial_c\phi \\
& + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{bd}\partial_e\phi - \bar{\Gamma}^c{}_{ad}\bar{\Gamma}^d{}_{bc}
\end{aligned} \tag{eq15.111}$$

$$\begin{aligned}
= & -4\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 20\partial_a\phi\partial_b\phi - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 6\bar{\Gamma}^c{}_{ab}\partial_c\phi - 4\bar{g}_{ab}\bar{g}^{dc}\partial_a\phi\partial_c\phi - 4\bar{g}_{da}\bar{g}^{dc}\partial_b\phi\partial_c\phi \\
& + 4\bar{g}_{ab}\bar{g}^{dc}\partial_c\phi\partial_d\phi - 2\bar{g}_{ec}\bar{g}^{ed}\bar{\Gamma}^c{}_{ab}\partial_d\phi + 2\bar{\Gamma}^c{}_{cb}\partial_a\phi + 2\bar{\Gamma}^c{}_{ca}\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{de}\bar{\Gamma}^c{}_{dc}\partial_e\phi + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} + 2\bar{g}_{ad}\partial_c\phi\partial_b\bar{g}^{dc} + 2\partial_{ba}\phi + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c{}_{ac} \\
& - 4\partial_a\phi\partial_b\phi\bar{g}^c{}_c + 4\bar{g}_{ad}\bar{g}^{cd}\partial_b\phi\partial_c\phi - 2\bar{\Gamma}^c{}_{ac}\partial_b\phi + 4\bar{g}_{bd}\bar{g}^{cd}\partial_a\phi\partial_c\phi + 4\bar{g}_{ba}\bar{g}^{cd}\partial_c\phi\partial_d\phi - 4\bar{g}_{bd}\bar{g}^{dc}\partial_a\phi\partial_c\phi + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi - 2\bar{\Gamma}^c{}_{bc}\partial_a\phi - 2\bar{\Gamma}^c{}_{ba}\partial_c\phi \\
& + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{bd}\partial_e\phi - \bar{\Gamma}^c{}_{ad}\bar{\Gamma}^d{}_{bc}
\end{aligned} \tag{eq15.112}$$

$$\begin{aligned}
R_{ab} = & -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c_{ab} + 20\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c_{ab}\partial_c\phi - 4\bar{g}_{bc}\bar{g}^{cd}\partial_a\phi\partial_d\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d_{ab}\partial_e\phi \\
& - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e_{ce}\partial_d\phi + \bar{\Gamma}^c_{ab}\bar{\Gamma}^d_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c_{ac} - 4\partial_a\phi\partial_b\phi\bar{g}^c_c + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c_{bd}\partial_e\phi - \bar{\Gamma}^c_{ad}\bar{\Gamma}^d_{bc} \quad (\text{eq15.113})
\end{aligned}$$

$$\begin{aligned}
= & -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c_{ab} + 20\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c_{ab}\partial_c\phi - 4\bar{g}^d_b\partial_a\phi\partial_d\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d_{ab}\partial_e\phi \\
& - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e_{ce}\partial_d\phi + \bar{\Gamma}^c_{ab}\bar{\Gamma}^d_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c_{ac} - 4\partial_a\phi\partial_b\phi\bar{g}^c_c + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c_{bd}\partial_e\phi - \bar{\Gamma}^c_{ad}\bar{\Gamma}^d_{bc} \quad (\text{eq15.114})
\end{aligned}$$

$$\begin{aligned}
= & -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c_{ab} + 8\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c_{ab}\partial_c\phi - 4\bar{g}^d_b\partial_a\phi\partial_d\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d_{ab}\partial_e\phi \\
& - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e_{ce}\partial_d\phi + \bar{\Gamma}^c_{ab}\bar{\Gamma}^d_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c_{ac} + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c_{bd}\partial_e\phi - \bar{\Gamma}^c_{ad}\bar{\Gamma}^d_{bc} \quad (\text{eq15.115})
\end{aligned}$$

$$\begin{aligned}
= & -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c_{ab} + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c_{ab}\partial_c\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d_{ab}\partial_e\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e_{ce}\partial_d\phi \\
& + \bar{\Gamma}^c_{ab}\bar{\Gamma}^d_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c_{ac} + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c_{bd}\partial_e\phi - \bar{\Gamma}^c_{ad}\bar{\Gamma}^d_{bc} \quad (\text{eq15.116})
\end{aligned}$$

The above doesn't look much like equation (15). So, what do we do? First note that (eq15.116) represents the full R_{ab} , that is, equation (14). To isolate the contributions from ϕ we can first set $\bar{\Gamma}$ and its derivatives to zero (which in turn requires setting $\partial_a\bar{g}_{bc} = 0$). The result is equation (eq15.119) below. Having set $\bar{\Gamma}$ to zero means that we can replace ∂ with \bar{D} leading to equation (eq15.121). But that is clearly a tensor equation and so by the usual arguments it must be true in all frames (not just this frame with $\bar{\Gamma} = 0$). It's a standard argument and I've probably overdone the discussion. Anyway, equation (eq15.121) is exactly equation (15) from the paper. Yeah.

$$R_{ab}^\phi = -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} \quad (\text{eq15.117})$$

$$= -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} \quad (\text{eq15.118})$$

$$= -2\partial_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi \quad (\text{eq15.119})$$

$$= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi \quad (\text{eq15.120})$$

$$= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi \quad (\text{eq15.121})$$

```

1  # -----
2  # Check against prd62.
3
4  foo := @(Rphi).                                # cdb(eq15.lcb,foo)
5  bah  = cdblib.get('prd62.eq15.rhs','prd62.json') # cdb(eq15.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy      (diff, "simplify")
13 canonicalise   (diff)                        # cdb(eq15.chk,diff)

```

$$\text{eq15.lcb} := -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi$$

$$\text{eq15.prd} := -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi$$

$$\text{eq15.chk} := 0$$

PhysRevD.62.044034 equation (17)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn17.json'
5  cdblib.create (jsonfile)
6
7  # -----
8  defGammaBar := GammaBar^{a}_{b c} ->
9              (1/2) gBar^{a e} ( \partial_{b}{gBar_{e c}}
10                             + \partial_{c}{gBar_{b e}}
11                             - \partial_{e}{gBar_{b c}}).
12
13  foo := \partial_{a}{gBar_{b c}} gBar^{i b} gBar^{j c} -> - \partial_{a}{gBar^{i j}}.
14  bah := \partial_{a}{gBar_{b c}} gBar^{b c} -> 0.  # follows from det gBar = 1
15
16  # -----
17  # GiBar
18
19  GiBar := gBar^{j k} GammaBar^{i}_{j k}.          # cdb (eq17.101,GiBar)
20
21  substitute      (GiBar, defGammaBar)             # cdb (eq17.102,GiBar)
22  distribute      (GiBar)                          # cdb (eq17.103,GiBar)
23  GiBar = product_sort (GiBar)                     # cdb (eq17.104,GiBar)
24  rename_dummies  (GiBar)                          # cdb (eq17.105,GiBar)
25  canonicalise    (GiBar)                          # cdb (eq17.106,GiBar)
26  substitute      (GiBar, foo)                     # cdb (eq17.107,GiBar)
27  substitute      (GiBar, bah)                     # cdb (eq17.108,GiBar)
28
29  defGiBar := GammaBar^{i} -> @(GiBar).
30
31  cdblib.put ('defGiBar',defGiBar,jsonfile)

```

$$\bar{g}^{jk}\bar{\Gamma}^i_{jk} = \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}(\partial_j\bar{g}_{ek} + \partial_k\bar{g}_{je} - \partial_e\bar{g}_{jk}) \quad (\text{eq17.102})$$

$$= \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}\partial_j\bar{g}_{ek} + \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}\partial_k\bar{g}_{je} - \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}\partial_e\bar{g}_{jk} \quad (\text{eq17.103})$$

$$= \frac{1}{2}\bar{g}^{ia}\bar{g}^{cb}\partial_c\bar{g}_{ab} + \frac{1}{2}\bar{g}^{ib}\bar{g}^{ac}\partial_c\bar{g}_{ab} - \frac{1}{2}\bar{g}^{ic}\bar{g}^{ab}\partial_c\bar{g}_{ab} \quad (\text{eq17.104})$$

$$= \frac{1}{2}\bar{g}^{ib}\bar{g}^{ac}\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}^{ib}\bar{g}^{ca}\partial_a\bar{g}_{cb} - \frac{1}{2}\bar{g}^{ia}\bar{g}^{bc}\partial_a\bar{g}_{bc} \quad (\text{eq17.105})$$

$$= \bar{g}^{ia}\bar{g}^{bc}\partial_b\bar{g}_{ac} - \frac{1}{2}\bar{g}^{ia}\bar{g}^{bc}\partial_a\bar{g}_{bc} \quad (\text{eq17.106})$$

$$= -\partial_b\bar{g}^{ib} - \frac{1}{2}\bar{g}^{ia}\bar{g}^{bc}\partial_a\bar{g}_{bc} \quad (\text{eq17.107})$$

$$= -\partial_b\bar{g}^{ib} \quad (\text{eq17.108})$$


```

1  # -----
2  # Check against prd62.
3
4  foo := @(GiBar).                # cdb(eq17.lcb,foo)
5  bah  = cdblib.get('prd62.eq17.rhs','prd62.json') # cdb(eq17.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy      (diff, "simplify")
13 canonicalise   (diff)          # cdb(eq17.chk,diff)

```

$$\text{eq17.lcb} := -\partial_b \bar{g}^{ib}$$

$$\text{eq17.prd} := -\partial_j \bar{g}^{ij}$$

$$\text{eq17.chk} := 0$$

PhysRevD.62.044034 equation (18)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn18.json'
5  cdblib.create (jsonfile)
6
7  # -----
8  # RBar pt.1 = split into two terms
9
10 defGammaBar := GammaBar^{a}_{b c} ->
11     (1/2) gBar^{a e} ( \partial_{b}{gBar_{e c}}
12                       + \partial_{c}{gBar_{b e}}
13                       - \partial_{e}{gBar_{b c}}).
14
15 defRiemBar := RBar^{a}_{b c d} ->
16     \partial_{c}{GammaBar^{a}_{b d}} + GammaBar^{a}_{e c} GammaBar^{e}_{b d}
17     - \partial_{d}{GammaBar^{a}_{b c}} - GammaBar^{a}_{e d} GammaBar^{e}_{b c}.
18
19 defRBar := RBar_{a b} -> RBar^{c}_{a c b}.
20
21 RBar := RBar_{a b}.                                # cdb(eq18.000,RBar)
22
23 substitute (RBar, defRBar)                          # cdb(eq18.001,RBar)
24 substitute (RBar, defRiemBar)                       # cdb(eq18.002,RBar)
25 substitute (RBar, $GammaBar^{a}_{b a} -> 0$)        # cdb(eq18.003,RBar) # follows from det g = 1
26 canonicalise (RBar)

```

$$\bar{R}_{ab} = \bar{R}^c{}_{acb} \tag{eq18.001}$$

$$= \partial_c \bar{\Gamma}^c{}_{ab} + \bar{\Gamma}^c{}_{ec} \bar{\Gamma}^e{}_{ab} - \partial_b \bar{\Gamma}^c{}_{ac} - \bar{\Gamma}^c{}_{eb} \bar{\Gamma}^e{}_{ac} \tag{eq18.002}$$

$$= \partial_c \bar{\Gamma}^c{}_{ab} - \bar{\Gamma}^c{}_{eb} \bar{\Gamma}^e{}_{ac} \tag{eq18.003}$$

From here the computations will be spltd into two threads, one for each of the two terms in the above result.

```

1  # -----
2  # get tmpA & tmpB from RBar
3
4  GammaBar^{a}_{b c}::Weight(label=numG).
5  \partial{#}::WeightInherit(label=all, type=multiplicative).
6
7  tmpA := @(RBar).                                # cdb(tmp18.101,tmpA)
8  keep_weight (tmpA, $numG=1$)                     # cdb(tmp18.102,tmpA) # the derivative terms
9
10 tmpB := @(RBar).
11 keep_weight (tmpB, $numG=2$)                     # cdb(tmp18.103,tmpB) # the quadrtaic terms

```

$$\text{tmp18.101} := \partial_c \bar{\Gamma}^c_{ab} - \bar{\Gamma}^c_{ae} \bar{\Gamma}^e_{bc}$$

$$\text{tmp18.102} := \partial_c \bar{\Gamma}^c_{ab}$$

$$\text{tmp18.103} := -\bar{\Gamma}^c_{ae} \bar{\Gamma}^e_{bc}$$

```

1 # -----
2 # tmpA pt. 1
3
4 substitute (tmpA, defGammaBar) # cdb(tmp18.201,tmpA)
5 distribute (tmpA) # cdb(tmp18.202,tmpA)
6 product_rule (tmpA) # cdb(tmp18.203,tmpA)
7 substitute (tmpA, $\partial_{a}\{\bar{g}^{b}\} \rightarrow -\Gamma^{b}_{a}$) # cdb(tmp18.204,tmpA)

```

$$\partial_c \bar{\Gamma}^c_{ab} = \frac{1}{2} \partial_c (\bar{g}^{ce} (\partial_a \bar{g}_{eb} + \partial_b \bar{g}_{ae} - \partial_e \bar{g}_{ab})) \quad (\text{tmp18.201})$$

$$= \frac{1}{2} \partial_c (\bar{g}^{ce} \partial_a \bar{g}_{eb}) + \frac{1}{2} \partial_c (\bar{g}^{ce} \partial_b \bar{g}_{ae}) - \frac{1}{2} \partial_c (\bar{g}^{ce} \partial_e \bar{g}_{ab}) \quad (\text{tmp18.202})$$

$$= \frac{1}{2} \partial_c \bar{g}^{ce} \partial_a \bar{g}_{eb} + \frac{1}{2} \bar{g}^{ce} \partial_{ca} \bar{g}_{eb} + \frac{1}{2} \partial_c \bar{g}^{ce} \partial_b \bar{g}_{ae} + \frac{1}{2} \bar{g}^{ce} \partial_{cb} \bar{g}_{ae} - \frac{1}{2} \partial_c \bar{g}^{ce} \partial_e \bar{g}_{ab} - \frac{1}{2} \bar{g}^{ce} \partial_{ce} \bar{g}_{ab} \quad (\text{tmp18.203})$$

$$= -\frac{1}{2} \bar{\Gamma}^e \partial_a \bar{g}_{eb} + \frac{1}{2} \bar{g}^{ce} \partial_{ca} \bar{g}_{eb} - \frac{1}{2} \bar{\Gamma}^e \partial_b \bar{g}_{ae} + \frac{1}{2} \bar{g}^{ce} \partial_{cb} \bar{g}_{ae} + \frac{1}{2} \bar{\Gamma}^e \partial_e \bar{g}_{ab} - \frac{1}{2} \bar{g}^{ce} \partial_{ce} \bar{g}_{ab} \quad (\text{tmp18.204})$$

Notice that this result contains two terms containing second derivatives of \bar{g}_{ij} . This pair of terms will now be replaced with an expression built from the first derivatives of Γ^i .

```

1  # -----
2  # tmpC
3
4  defGi := GammaBar^{i} -> - \partial_{j}{gBar^{i j}}.
5
6  # lower the indices on gBar^{b c}
7
8  defLowerIndices := \partial_{a}{gBar^{b c}} -> - gBar^{i b} gBar^{j c} \partial_{a}{gBar_{i j}}.
9
10 substitute (defGi, defLowerIndices)
11
12 tmpC := gBar_{a i} \partial_{b}{GammaBar^{i}}
13        + gBar_{b i} \partial_{a}{GammaBar^{i}}. # cdb(tmp18.301,tmpC)
14
15 saveC := @(tmpC).
16
17 substitute (tmpC, defGi) # cdb(tmp18.302,tmpC)
18 product_rule (tmpC) # cdb(tmp18.303,tmpC)
19 distribute (tmpC) # cdb(tmp18.304,tmpC)
20 canonicalise (tmpC) # cdb(tmp18.305,tmpC)
21 substitute (tmpC, $gBar_{a b} gBar^{b c} -> gBar_{a}^{c}$) # cdb(tmp18.306,tmpC)
22 eliminate_kronecker (tmpC) # cdb(tmp18.307,tmpC)
23
24 # foo is the target expression to be moved to the lhs
25
26 foo := gBar^{i j} \partial_{a i}{gBar_{b j}}
27        + gBar^{i j} \partial_{b i}{gBar_{a j}} -> X_{a b}.
28
29 # bah helps when rebuilding the equation
30
31 bah := X_{a b} ->
32        gBar^{i j} \partial_{a i}{gBar_{b j}}
33        + gBar^{i j} \partial_{b i}{gBar_{a j}}.
34
35 substitute (tmpC, foo) # cdb(tmp18.308,tmpC)
36
37 #

```

```

38  # rearrange to move the target to the lhs
39
40  tmpE := @(tmpC).
41  tmpF := @(tmpC).
42
43  X_{a b}::Weight(label=numX).
44
45  # get the two pieces of the equation
46  keep_weight (tmpE, $numX=0$)           # cdb(tmp18.309,tmpE)
47  keep_weight (tmpF, $numX=1$)           # cdb(tmp18.310,tmpF)
48
49  substitute (tmpF, bah)                  # cdb(tmp18.311,tmpF)
50
51  # now rebuild with terms reordered
52  tmpG := @(saveC) - @(tmpE).             # cdb(tmp18.312,tmpG)
53
54  defTmpSub := @(tmpF) -> @(tmpG).         # cdb(tmp18.313,defTmpSub)

```

$$\bar{g}_{ai}\partial_b\bar{\Gamma}^i + \bar{g}_{bi}\partial_a\bar{\Gamma}^i = \bar{g}_{ai}\partial_b(\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd}) + \bar{g}_{bi}\partial_a(\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd}) \quad (\text{tmp18.302})$$

$$= \bar{g}_{ai}(\partial_b\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\partial_b\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\bar{g}^{dj}\partial_{bj}\bar{g}_{cd}) + \bar{g}_{bi}(\partial_a\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\partial_a\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\bar{g}^{dj}\partial_{aj}\bar{g}_{cd}) \quad (\text{tmp18.303})$$

$$= \bar{g}_{ai}\partial_b\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{ai}\bar{g}^{ci}\partial_b\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{ai}\bar{g}^{ci}\bar{g}^{dj}\partial_{bj}\bar{g}_{cd} + \bar{g}_{bi}\partial_a\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{bi}\bar{g}^{ci}\partial_a\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{bi}\bar{g}^{ci}\bar{g}^{dj}\partial_{aj}\bar{g}_{cd} \quad (\text{tmp18.304})$$

$$= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{ac}\bar{g}^{cd}\partial_b\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{ac}\bar{g}^{cd}\bar{g}^{ij}\partial_{bi}\bar{g}_{dj} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{bc}\bar{g}^{cd}\partial_a\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{bc}\bar{g}^{cd}\bar{g}^{ij}\partial_{ai}\bar{g}_{dj} \quad (\text{tmp18.305})$$

$$= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_a{}^d\partial_b\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_a{}^d\bar{g}^{ij}\partial_{bi}\bar{g}_{dj} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_b{}^d\partial_a\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_b{}^d\bar{g}^{ij}\partial_{ai}\bar{g}_{dj} \quad (\text{tmp18.306})$$

$$= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_b\bar{g}^{ij}\partial_i\bar{g}_{aj} + \bar{g}^{ij}\partial_{bi}\bar{g}_{aj} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_a\bar{g}^{ij}\partial_i\bar{g}_{bj} + \bar{g}^{ij}\partial_{ai}\bar{g}_{bj} \quad (\text{tmp18.307})$$

$$= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_b\bar{g}^{ij}\partial_i\bar{g}_{aj} + X_{ba} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_a\bar{g}^{ij}\partial_i\bar{g}_{bj} \quad (\text{tmp18.308})$$

$$\bar{g}^{ij}\partial_{bi}\bar{g}_{aj} + \bar{g}^{ij}\partial_{ai}\bar{g}_{bj} = X_{ba} \quad (\text{tmp18.310})$$

$$= \bar{g}_{ai}\partial_b\bar{\Gamma}^i + \bar{g}_{bi}\partial_a\bar{\Gamma}^i - \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} - \partial_b\bar{g}^{ij}\partial_i\bar{g}_{aj} - \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} - \partial_a\bar{g}^{ij}\partial_i\bar{g}_{bj} \quad (\text{tmp18.312})$$

This result will now be applied to the earlier equation (tmp18.204).


```

1  # -----
2  # tmpA pt.2 eliminate second partial derivatives of gBar
3
4  canonicalise      (tmpA)                                # cdb(tmp18.401,tmpA)
5
6  substitute        (tmpA, defTmpSub)                      # cdb(tmp18.402,tmpA)
7  tmpA = product_sort (tmpA)
8  rename_dummies    (tmpA)
9  canonicalise      (tmpA)                                # cdb(tmp18.403,tmpA)
10
11  foo := gBar^{d e} \partial_{c}{gBar_{e f}} -> - gBar_{e f} \partial_{c}{gBar^{d e}}.
12  bah := \partial_{d}{gBar^{d f}} -> - GammaBar^{f}.
13
14  substitute        (tmpA, foo)                            # cdb(tmp18.404,tmpA)
15  substitute        (tmpA, bah)                            # cdb(tmp18.405,tmpA)
16
17  foo := gBar_{e f} \partial_{a}{gBar^{c f}} -> - \partial_{a}{gBar_{e f}} gBar^{c f}.
18
19  substitute        (tmpA, foo)                            # cdb(tmp18.406,tmpA)
20
21  foo := gBar_{b d} gBar^{d e} -> gBar_{b}^{e}.
22
23  substitute        (tmpA, foo)                            # cdb(tmp18.407,tmpA)
24  eliminate_kronecker (tmpA)                              # cdb(tmp18.408,tmpA)
25  tmpA = product_sort (tmpA)
26  rename_dummies    (tmpA)
27  canonicalise      (tmpA)                                # cdb(tmp18.409,tmpA)

```

$$\partial_c \bar{\Gamma}^c_{ab} = -\frac{1}{2} \bar{\Gamma}^e_a \partial_a \bar{g}_{be} + \frac{1}{2} \bar{g}^{ce} \partial_{ac} \bar{g}_{be} - \frac{1}{2} \bar{\Gamma}^e_b \partial_b \bar{g}_{ae} + \frac{1}{2} \bar{g}^{ce} \partial_{bc} \bar{g}_{ae} + \frac{1}{2} \bar{\Gamma}^e_e \partial_e \bar{g}_{ab} - \frac{1}{2} \bar{g}^{ce} \partial_{ce} \bar{g}_{ab} \quad (\text{tmp18.401})$$

$$\begin{aligned}
&= -\frac{1}{2}\bar{\Gamma}^e\partial_a\bar{g}_{be} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\bar{g}_{bf}\partial_a\bar{g}^{fd}\bar{g}^{ce}\partial_c\bar{g}_{de} - \frac{1}{2}\partial_a\bar{g}^{ce}\partial_c\bar{g}_{be} - \frac{1}{2}\bar{g}_{af}\partial_b\bar{g}^{fd}\bar{g}^{ce}\partial_c\bar{g}_{de} - \frac{1}{2}\partial_b\bar{g}^{ce}\partial_c\bar{g}_{ae} - \frac{1}{2}\bar{\Gamma}^e\partial_b\bar{g}_{ae} + \frac{1}{2}\bar{\Gamma}^e\partial_e\bar{g}_{ab} \\
&\quad - \frac{1}{2}\bar{g}^{ce}\partial_{ce}\bar{g}_{ab} \tag{tmp18.402}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\bar{g}_{bc}\bar{g}^{de}\partial_d\bar{g}_{ef}\partial_a\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} - \frac{1}{2}\bar{g}_{ac}\bar{g}^{de}\partial_d\bar{g}_{ef}\partial_b\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} \\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}
\end{aligned}
\tag{tmp18.403}$$

$$\begin{aligned}
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\bar{g}_{ef}\partial_d\bar{g}^{de}\partial_a\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\bar{g}_{ac}\bar{g}_{ef}\partial_d\bar{g}^{de}\partial_b\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} \\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \tag{tmp18.404}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\bar{g}_{bc}\bar{g}_{ef}\bar{\Gamma}^e\partial_a\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} - \frac{1}{2}\bar{g}_{ac}\bar{g}_{ef}\bar{\Gamma}^e\partial_b\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} \\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}
\end{aligned}
\tag{tmp18.405}$$

$$\begin{aligned}
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{g}_{ef}\bar{g}^{cf}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{g}_{ef}\bar{g}^{cf}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} \\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}
\end{aligned}
\tag{tmp18.406}$$

$$\begin{aligned}
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_b{}^f\partial_a\bar{g}_{ef}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\bar{g}_a{}^f\partial_b\bar{g}_{ef}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} \\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}
\end{aligned}
\tag{tmp18.407}$$

$$= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\partial_a\bar{g}_{eb}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\partial_b\bar{g}_{ea}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \quad (\text{tmp18.408})$$

$$= \frac{1}{2} \bar{g}_{bc} \partial_a \bar{\Gamma}^c + \frac{1}{2} \bar{g}_{ac} \partial_b \bar{\Gamma}^c - \frac{1}{2} \partial_c \bar{g}_{bd} \partial_a \bar{g}^{cd} - \frac{1}{2} \partial_c \bar{g}_{ad} \partial_b \bar{g}^{cd} + \frac{1}{2} \bar{\Gamma}^c \partial_c \bar{g}_{ab} - \frac{1}{2} \bar{g}^{cd} \partial_{cd} \bar{g}_{ab} \quad (\text{tmp18.409})$$

```

1 # -----
2 # tmpB
3
4 substitute      (tmpB, defGammaBar)          # cdb(tmp18.501,tmpB)
5 distribute      (tmpB)                        # cdb(tmp18.502,tmpB)
6 tmpB = product_sort (tmpB)                    # cdb(tmp18.503,tmpB)
7 rename_dummies  (tmpB)                        # cdb(tmp18.504,tmpB)
8 canonicalise    (tmpB)                        # cdb(tmp18.505,tmpB)

```

$$-\bar{\Gamma}_{ae}^c \bar{\Gamma}_{bc}^e = -\frac{1}{4} \bar{g}^{cd} (\partial_a \bar{g}_{de} + \partial_e \bar{g}_{ad} - \partial_d \bar{g}_{ae}) \bar{g}^{ef} (\partial_b \bar{g}_{fc} + \partial_c \bar{g}_{bf} - \partial_f \bar{g}_{bc}) \quad (\text{tmp18.501})$$

$$\begin{aligned}
&= -\frac{1}{4} \bar{g}^{cd} \partial_a \bar{g}_{de} \bar{g}^{ef} \partial_b \bar{g}_{fc} - \frac{1}{4} \bar{g}^{cd} \partial_a \bar{g}_{de} \bar{g}^{ef} \partial_c \bar{g}_{bf} + \frac{1}{4} \bar{g}^{cd} \partial_a \bar{g}_{de} \bar{g}^{ef} \partial_f \bar{g}_{bc} - \frac{1}{4} \bar{g}^{cd} \partial_e \bar{g}_{ad} \bar{g}^{ef} \partial_b \bar{g}_{fc} - \frac{1}{4} \bar{g}^{cd} \partial_e \bar{g}_{ad} \bar{g}^{ef} \partial_c \bar{g}_{bf} + \frac{1}{4} \bar{g}^{cd} \partial_e \bar{g}_{ad} \bar{g}^{ef} \partial_f \bar{g}_{bc} \\
&\quad + \frac{1}{4} \bar{g}^{cd} \partial_d \bar{g}_{ae} \bar{g}^{ef} \partial_b \bar{g}_{fc} + \frac{1}{4} \bar{g}^{cd} \partial_d \bar{g}_{ae} \bar{g}^{ef} \partial_c \bar{g}_{bf} - \frac{1}{4} \bar{g}^{cd} \partial_d \bar{g}_{ae} \bar{g}^{ef} \partial_f \bar{g}_{bc}
\end{aligned} \quad (\text{tmp18.502})$$

$$\begin{aligned}
&= -\frac{1}{4} \bar{g}^{fc} \bar{g}^{de} \partial_a \bar{g}_{cd} \partial_b \bar{g}_{ef} - \frac{1}{4} \bar{g}^{fc} \bar{g}^{de} \partial_a \bar{g}_{cd} \partial_f \bar{g}_{be} + \frac{1}{4} \bar{g}^{ec} \bar{g}^{df} \partial_a \bar{g}_{cd} \partial_f \bar{g}_{be} - \frac{1}{4} \bar{g}^{de} \bar{g}^{fc} \partial_b \bar{g}_{cd} \partial_f \bar{g}_{ae} - \frac{1}{4} \bar{g}^{ed} \bar{g}^{fc} \partial_e \bar{g}_{bc} \partial_f \bar{g}_{ad} + \frac{1}{4} \bar{g}^{dc} \bar{g}^{ef} \partial_e \bar{g}_{ac} \partial_f \bar{g}_{bd} \\
&\quad + \frac{1}{4} \bar{g}^{df} \bar{g}^{ec} \partial_b \bar{g}_{cd} \partial_f \bar{g}_{ae} + \frac{1}{4} \bar{g}^{ef} \bar{g}^{dc} \partial_e \bar{g}_{bc} \partial_f \bar{g}_{ad} - \frac{1}{4} \bar{g}^{de} \bar{g}^{cf} \partial_e \bar{g}_{ac} \partial_f \bar{g}_{bd}
\end{aligned} \quad (\text{tmp18.503})$$

$$\begin{aligned}
&= -\frac{1}{4} \bar{g}^{cd} \bar{g}^{ef} \partial_a \bar{g}_{de} \partial_b \bar{g}_{fc} - \frac{1}{4} \bar{g}^{cd} \bar{g}^{ef} \partial_a \bar{g}_{de} \partial_c \bar{g}_{bf} + \frac{1}{4} \bar{g}^{de} \bar{g}^{fc} \partial_a \bar{g}_{ef} \partial_c \bar{g}_{bd} - \frac{1}{4} \bar{g}^{de} \bar{g}^{cf} \partial_b \bar{g}_{fd} \partial_c \bar{g}_{ae} - \frac{1}{4} \bar{g}^{ce} \bar{g}^{df} \partial_c \bar{g}_{bf} \partial_d \bar{g}_{ae} + \frac{1}{4} \bar{g}^{ef} \bar{g}^{cd} \partial_c \bar{g}_{af} \partial_d \bar{g}_{be} \\
&\quad + \frac{1}{4} \bar{g}^{dc} \bar{g}^{ef} \partial_b \bar{g}_{fd} \partial_c \bar{g}_{ae} + \frac{1}{4} \bar{g}^{cd} \bar{g}^{ef} \partial_c \bar{g}_{bf} \partial_d \bar{g}_{ae} - \frac{1}{4} \bar{g}^{ec} \bar{g}^{fd} \partial_c \bar{g}_{af} \partial_d \bar{g}_{be}
\end{aligned} \quad (\text{tmp18.504})$$

$$= -\frac{1}{4} \bar{g}^{cd} \bar{g}^{ef} \partial_a \bar{g}_{ce} \partial_b \bar{g}_{df} - \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_c \bar{g}_{ae} \partial_f \bar{g}_{bd} + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_c \bar{g}_{ae} \partial_d \bar{g}_{bf} \quad (\text{tmp18.505})$$

```

1  # -----
2  # RBar pt.2 = Rebuild Rab from tmpA and tmpB
3
4  RBar := @(tmpA) + @(tmpB).
5
6  canonicalise    (RBar)                                # cdb(eq18.601,RBar)
7
8  foo := \partial_{a}{gBar^{c d}} -> - gBar^{c i} gBar^{d j} \partial_{a}{gBar_{i j}}.
9
10 substitute      (RBar, foo)
11 distribute      (RBar)
12 RBar = product_sort (RBar)
13 rename_dummies  (RBar)
14 canonicalise    (RBar)                                # cdb(eq18.602,RBar)
15
16 foo := \partial_{a}{gBar_{b c}} -> GammaBar_{b c a} + GammaBar_{c b a}.
17
18 substitute      (RBar, foo)                            # cdb(eq18.603,RBar)
19 distribute      (RBar)
20 RBar = product_sort (RBar)
21 rename_dummies  (RBar)
22 canonicalise    (RBar)                                # cdb(eq18.604,RBar)
23
24 foo := GammaBar_{d e f} gBar^{d e} -> 0.
25
26 substitute      (RBar, foo)                            # cdb(eq18.605,RBar)
27
28 defRab := RBar_{a b} -> @(RBar).

```

$$\bar{R}_{ab} = \partial_c \bar{\Gamma}^c_{ab} - \bar{\Gamma}^c_{eb} \bar{\Gamma}^e_{ac} \quad (\text{eq18.003})$$

$$\begin{aligned} &= \frac{1}{2} \bar{g}_{bc} \partial_a \bar{\Gamma}^c + \frac{1}{2} \bar{g}_{ac} \partial_b \bar{\Gamma}^c - \frac{1}{2} \partial_c \bar{g}_{bd} \partial_a \bar{g}^{cd} - \frac{1}{2} \partial_c \bar{g}_{ad} \partial_b \bar{g}^{cd} + \frac{1}{2} \bar{\Gamma}^c \partial_c \bar{g}_{ab} - \frac{1}{2} \bar{g}^{cd} \partial_{cd} \bar{g}_{ab} - \frac{1}{4} \bar{g}^{cd} \bar{g}^{ef} \partial_a \bar{g}_{ce} \partial_b \bar{g}_{df} - \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_c \bar{g}_{ae} \partial_f \bar{g}_{bd} \\ &\quad + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_c \bar{g}_{ae} \partial_d \bar{g}_{bf} \end{aligned} \quad (\text{eq18.601})$$

$$\begin{aligned} &= \frac{1}{2} \bar{g}_{bc} \partial_a \bar{\Gamma}^c + \frac{1}{2} \bar{g}_{ac} \partial_b \bar{\Gamma}^c + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_a \bar{g}_{ce} \partial_d \bar{g}_{bf} + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_b \bar{g}_{ce} \partial_d \bar{g}_{af} + \frac{1}{2} \bar{\Gamma}^c \partial_c \bar{g}_{ab} - \frac{1}{2} \bar{g}^{cd} \partial_{cd} \bar{g}_{ab} - \frac{1}{4} \bar{g}^{cd} \bar{g}^{ef} \partial_a \bar{g}_{ce} \partial_b \bar{g}_{df} - \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_c \bar{g}_{ae} \partial_f \bar{g}_{bd} \\ &\quad + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} \partial_c \bar{g}_{ae} \partial_d \bar{g}_{bf} \end{aligned} \quad (\text{eq18.602})$$

$$\begin{aligned} &= \frac{1}{2} \bar{g}_{bc} \partial_a \bar{\Gamma}^c + \frac{1}{2} \bar{g}_{ac} \partial_b \bar{\Gamma}^c + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} (\bar{\Gamma}_{cea} + \bar{\Gamma}_{eca}) (\bar{\Gamma}_{bfd} + \bar{\Gamma}_{fbd}) + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} (\bar{\Gamma}_{ceb} + \bar{\Gamma}_{ecb}) (\bar{\Gamma}_{afd} + \bar{\Gamma}_{fad}) + \frac{1}{2} \bar{\Gamma}^c (\bar{\Gamma}_{abc} + \bar{\Gamma}_{bac}) - \frac{1}{2} \bar{g}^{cd} \partial_{cd} \bar{g}_{ab} \\ &\quad - \frac{1}{4} \bar{g}^{cd} \bar{g}^{ef} (\bar{\Gamma}_{cea} + \bar{\Gamma}_{eca}) (\bar{\Gamma}_{dfb} + \bar{\Gamma}_{fdb}) - \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} (\bar{\Gamma}_{aec} + \bar{\Gamma}_{eac}) (\bar{\Gamma}_{bdf} + \bar{\Gamma}_{dbf}) + \frac{1}{2} \bar{g}^{cd} \bar{g}^{ef} (\bar{\Gamma}_{aec} + \bar{\Gamma}_{eac}) (\bar{\Gamma}_{bfd} + \bar{\Gamma}_{fbd}) \end{aligned} \quad (\text{eq18.603})$$

$$= \frac{1}{2} \bar{g}_{bc} \partial_a \bar{\Gamma}^c + \frac{1}{2} \bar{g}_{ac} \partial_b \bar{\Gamma}^c + \bar{g}^{cd} \bar{g}^{ef} \bar{\Gamma}_{bce} \bar{\Gamma}_{daf} + \bar{g}^{cd} \bar{g}^{ef} \bar{\Gamma}_{cae} \bar{\Gamma}_{dbf} + \bar{g}^{cd} \bar{g}^{ef} \bar{\Gamma}_{ace} \bar{\Gamma}_{dbf} + \frac{1}{2} \bar{\Gamma}^c \bar{\Gamma}_{abc} + \frac{1}{2} \bar{\Gamma}^c \bar{\Gamma}_{bac} - \frac{1}{2} \bar{g}^{cd} \partial_{cd} \bar{g}_{ab} \quad (\text{eq18.604})$$

$$= \frac{1}{2} \bar{g}_{bc} \partial_a \bar{\Gamma}^c + \frac{1}{2} \bar{g}_{ac} \partial_b \bar{\Gamma}^c + \bar{g}^{cd} \bar{g}^{ef} \bar{\Gamma}_{bce} \bar{\Gamma}_{daf} + \bar{g}^{cd} \bar{g}^{ef} \bar{\Gamma}_{cae} \bar{\Gamma}_{dbf} + \bar{g}^{cd} \bar{g}^{ef} \bar{\Gamma}_{ace} \bar{\Gamma}_{dbf} + \frac{1}{2} \bar{\Gamma}^c \bar{\Gamma}_{abc} + \frac{1}{2} \bar{\Gamma}^c \bar{\Gamma}_{bac} - \frac{1}{2} \bar{g}^{cd} \partial_{cd} \bar{g}_{ab} \quad (\text{eq18.605})$$

```

1  # -----
2  # Check against prd62.
3
4  foo := @(RBar).                                # cdb(eq18.lcb,foo)
5  bah  = cdblib.get('prd62.eq18.rhs','prd62.json') # cdb(eq18.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy       (diff, "simplify")
13 canonicalise    (diff)                        # cdb(eq18.chk,diff)

```

$$\text{eq18.lcb} := \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}$$

$$\text{eq18.prd} := -\frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb})$$

$$\text{eq18.chk} := 0$$

PhysRevD.62.044034 equation (19)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn19.json'
5  cdblib.create (jsonfile)
6
7  defGiBar = cdblib.get ('defGiBar','eqtn17.json')
8
9  # -----
10 # DGiBarDt pt.1
11
12 dotgBar_{a b}::Symmetric.
13 dotgBar^{a b}::Symmetric.
14 dotgBar{#}::LaTeXForm("{\bar{dg}}").
15
16 dotGiBar := \partial_{t}{GammaBar^{i}}.          # cdb (eq19.101,dotGiBar)
17
18 substitute (dotGiBar, defGiBar)                # cdb (eq19.102,dotGiBar)
19 substitute (dotGiBar, $\partial_{t a}{gBar^{i a}} \rightarrow \partial_{a}{dotgBar^{i a}}$)
20                                                # cdb (eq19.103,dotGiBar)
21
22 defdotgBarD := dotgBar_{i j} -> -2 N ABar_{i j}.
23 defdotgBarU := dotgBar^{i j} -> 2 N ABar^{i j}.
24 # defABarD2ABarU := ABar_{i j} -> ABar^{a b} gBar_{a i} gBar_{b j}.
25
26 substitute (dotGiBar, defdotgBarU )            # cdb (eq19.104,dotGiBar)
27 product_rule (dotGiBar)                        # cdb (eq19.105,dotGiBar)
28
29 dotGiBar = product_sort (dotGiBar)              # cdb (eq19.106,dotGiBar)
30
31 cdblib.put ('dotGiBar',dotGiBar,jsonfile)

```

$$\partial_t \bar{\Gamma}^i = -\partial_{tb} \bar{g}^{ib} \tag{eq19.102}$$

$$= -\partial_b \bar{d}g^{ib} \tag{eq19.103}$$

$$= -2\partial_b (N\bar{A}^{ib}) \tag{eq19.104}$$

$$= -2\partial_b N\bar{A}^{ib} - 2N\partial_b \bar{A}^{ib} \tag{eq19.105}$$

$$= -2\bar{A}^{ia}\partial_a N - 2N\partial_a \bar{A}^{ia} \tag{eq19.106}$$


```

1  # -----
2  # Check against prd62.
3
4  foo := @(dotGiBar).                # cdb (eq19.lcb,foo)
5  bah  = cdblib.get('prd62.eq19.rhs','prd62.json')  # cdb (eq19.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 product_rule    (diff)
11 diff = product_sort (diff)
12 rename_dummies  (diff)
13 map_sympy       (diff, "simplify")
14 canonicalise    (diff)              # cdb (eq19.chk,diff)

```

$$\text{eq19.lcb} := -2\bar{A}^{ia}\partial_a N - 2N\partial_a \bar{A}^{ia}$$

$$\text{eq19.prd} := -2\partial_j (N\bar{A}^{ij})$$

$$\text{eq19.chk} := 0$$

PhysRevD.62.044034 equation (20)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'eqtn20.json'
5  cdblib.create (jsonfile)
6
7  dotGiBar = cdblib.get ('dotGiBar','eqtn19.json')
8  defMomSub = cdblib.get ('defMomSub','momentum.json')
9
10 # -----
11 # DGiBarDt pt.2
12
13 substitute (dotGiBar, defMomSub)      # cdb(eq20.101,dotGiBar)

```

$$\partial_t \bar{\Gamma}^i = -2\bar{A}^{ia} \partial_a N - 2N \partial_a \bar{A}^{ia} \quad (\text{eq19.106})$$

$$= -2\bar{A}^{ia} \partial_a N - 2N \left(-6\bar{A}^{ia} \partial_a \phi - \bar{A}^{ab} \bar{\Gamma}^i_{ab} + \frac{2}{3} \bar{g}^{ia} \partial_a \text{tr} K \right) \quad (\text{eq20.101})$$

```

1  # -----
2  # Check against prd62.
3
4  foo := @(dotGiBar).                # cdb (eq20.lcb,foo)
5  bah  = cdblib.get('prd62.eq20.rhs','prd62.json')  # cdb (eq20.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy      (diff, "simplify")
13 canonicalise   (diff)                # cdb (eq20.chk,diff)

```

$$\text{eq20.lcb} := -2\bar{A}^{ia}\partial_a N - 2N \left(-6\bar{A}^{ia}\partial_a \phi - \bar{A}^{ab}\bar{\Gamma}^i_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_a \text{tr} K \right)$$

$$\text{eq20.prd} := -2\bar{A}^{ij}\partial_j N + 2N \left(\bar{\Gamma}^i_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{g}^{ij}\partial_j \text{tr} K + 6\bar{A}^{ij}\partial_j \phi \right)$$

$$\text{eq20.chk} := 0$$

PhysRevD.67.084023 equation (27)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'gamma.json'
5  cdblib.create (jsonfile)
6
7  # -----
8  # Gamma in terms of GammaBar and phi, see prd67 eqn 27
9
10 Gamma := \Gamma^{a}_{b c}. # cdb (eq27.101,Gamma)
11
12 substitute (Gamma, defGamma) # cdb (eq27.102,Gamma)
13 substitute (Gamma, defG2GBarD) # cdb (eq27.103,Gamma)
14 substitute (Gamma, defG2GBarU) # cdb (eq27.104,Gamma)
15 distribute (Gamma) # cdb (eq27.105,Gamma)
16 product_rule (Gamma) # cdb (eq27.106,Gamma)
17 substitute (Gamma, dexp) # cdb (eq27.107,Gamma)
18 distribute (Gamma) # cdb (eq27.108,Gamma)
19 map_sympy (Gamma, "simplify") # cdb (eq27.109,Gamma)
20
21 foo := gBar^{a e} \partial_{e}\{gBar_{b c}\} ->
22     - 2 GammaBar^{a}_{b c}
23     + gBar^{a e} \partial_{b}\{gBar_{e c}\}
24     + gBar^{a e} \partial_{c}\{gBar_{b e}\}.
25
26 substitute (Gamma, foo) # cdb (eq27.110,Gamma)
27 substitute (Gamma, $gBar^{a i} gBar_{i b} -> gBar^{a}_{b}$) # cdb (eq27.111,Gamma)
28 substitute (Gamma, $gBar^{a i} gBar_{b i} -> gBar^{a}_{b}$) # cdb (eq27.112,Gamma)
29
30 defG2GBar := \Gamma^{a}_{b c} -> @(Gamma).
31
32 cdblib.put ('defG2GBar',defG2GBar,jsonfile)

```

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ae} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) \quad (\text{eq27.102})$$

$$= \frac{1}{2} g^{ae} (\partial_b (\exp(4\phi) \bar{g}_{ec}) + \partial_c (\exp(4\phi) \bar{g}_{be}) - \partial_e (\exp(4\phi) \bar{g}_{bc})) \quad (\text{eq27.103})$$

$$= \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} (\partial_b (\exp(4\phi) \bar{g}_{ec}) + \partial_c (\exp(4\phi) \bar{g}_{be}) - \partial_e (\exp(4\phi) \bar{g}_{bc})) \quad (\text{eq27.104})$$

$$= \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} \partial_b (\exp(4\phi) \bar{g}_{ec}) + \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} \partial_c (\exp(4\phi) \bar{g}_{be}) - \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} \partial_e (\exp(4\phi) \bar{g}_{bc}) \quad (\text{eq27.105})$$

$$= \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} (\partial_b (\exp(4\phi)) \bar{g}_{ec} + \exp(4\phi) \partial_b \bar{g}_{ec}) + \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} (\partial_c (\exp(4\phi)) \bar{g}_{be} + \exp(4\phi) \partial_c \bar{g}_{be}) \\ - \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} (\partial_e (\exp(4\phi)) \bar{g}_{bc} + \exp(4\phi) \partial_e \bar{g}_{bc}) \quad (\text{eq27.106})$$

$$= \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} (4 \exp(4\phi) \partial_b \phi \bar{g}_{ec} + \exp(4\phi) \partial_b \bar{g}_{ec}) + \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} (4 \exp(4\phi) \partial_c \phi \bar{g}_{be} + \exp(4\phi) \partial_c \bar{g}_{be}) \\ - \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} (4 \exp(4\phi) \partial_e \phi \bar{g}_{bc} + \exp(4\phi) \partial_e \bar{g}_{bc}) \quad (\text{eq27.107})$$

$$= 2 \exp(-4\phi) \bar{g}^{ae} \exp(4\phi) \partial_b \phi \bar{g}_{ec} + \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} \exp(4\phi) \partial_b \bar{g}_{ec} + 2 \exp(-4\phi) \bar{g}^{ae} \exp(4\phi) \partial_c \phi \bar{g}_{be} + \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} \exp(4\phi) \partial_c \bar{g}_{be} \\ - 2 \exp(-4\phi) \bar{g}^{ae} \exp(4\phi) \partial_e \phi \bar{g}_{bc} - \frac{1}{2} \exp(-4\phi) \bar{g}^{ae} \exp(4\phi) \partial_e \bar{g}_{bc} \quad (\text{eq27.108})$$

$$= 2 \bar{g}^{ae} \partial_b \phi \bar{g}_{ec} + \frac{1}{2} \bar{g}^{ae} \partial_b \bar{g}_{ec} + 2 \bar{g}^{ae} \partial_c \phi \bar{g}_{be} + \frac{1}{2} \bar{g}^{ae} \partial_c \bar{g}_{be} - 2 \bar{g}^{ae} \partial_e \phi \bar{g}_{bc} - \frac{1}{2} \bar{g}^{ae} \partial_e \bar{g}_{bc} \quad (\text{eq27.109})$$

$$= 2 \bar{g}^{ae} \partial_b \phi \bar{g}_{ec} + 2 \bar{g}^{ae} \partial_c \phi \bar{g}_{be} - 2 \bar{g}^{ae} \partial_e \phi \bar{g}_{bc} + \bar{\Gamma}^a{}_{bc} \quad (\text{eq27.110})$$

$$= 2 \bar{g}^a{}_c \partial_b \phi + 2 \bar{g}^{ae} \partial_c \phi \bar{g}_{be} - 2 \bar{g}^{ae} \partial_e \phi \bar{g}_{bc} + \bar{\Gamma}^a{}_{bc} \quad (\text{eq27.111})$$

$$= 2 \bar{g}^a{}_c \partial_b \phi + 2 \bar{g}^a{}_b \partial_c \phi - 2 \bar{g}^{ae} \partial_e \phi \bar{g}_{bc} + \bar{\Gamma}^a{}_{bc} \quad (\text{eq27.112})$$

```

1  # -----
2  # Check against prd67.
3
4  foo := @(Gamma).                # cdb(prd67.eq27.lcb,foo)
5  bah  = cdblib.get('prd67.eq27.rhs','prd67.json')  # cdb(prd67.eq27.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy      (diff, "simplify")
13 canonicalise   (diff)           # cdb(prd67.eq27.chk,diff)

```

$$\text{prd67.eq27.lcb} := 2\bar{g}^a{}_c \partial_b \phi + 2\bar{g}^a{}_b \partial_c \phi - 2\bar{g}^{ae} \partial_e \phi \bar{g}_{bc} + \bar{\Gamma}^a{}_{bc}$$

$$\text{prd67.eq27.prd} := \bar{\Gamma}^a{}_{bc} + 2\bar{g}^a{}_c \partial_b \phi + 2\bar{g}^a{}_b \partial_c \phi - 2\bar{g}_{bc} \bar{g}^{ae} \partial_e \phi$$

$$\text{prd67.eq27.chk} := 0$$

PhysRevD.67.084023 equation (19)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'hamiltonian.json'
5  cdblib.create (jsonfile)
6
7  # -----
8  # Hamiltonian constraint
9
10 Ham := R + K_{a b} g^{a b} K_{c d} g^{c d} - K_{a b} K_{c d} g^{a c} g^{b d}. # cdb (Ham.101,Ham)
11
12 defK2ABarD := K_{i j} -> \exp(4\phi) ABar_{i j} + (1/3) g_{i j} trK.
13 defG2GBarD := g_{a b} -> \exp(4\phi) gBar_{a b}.
14 defG2GBarU := g^{a b} -> \exp(-4\phi) gBar^{a b}.
15
16 substitute (Ham, defK2ABarD) # cdb (Ham.102,Ham)
17 substitute (Ham, defG2GBarD) # cdb (Ham.103,Ham)
18 substitute (Ham, defG2GBarU) # cdb (Ham.104,Ham)
19 distribute (Ham) # cdb (Ham.105,Ham)
20 Ham = product_sort (Ham) # cdb (Ham.106,Ham)
21 rename_dummies (Ham) # cdb (Ham.107,Ham)
22 canonicalise (Ham) # cdb (Ham.108,Ham)
23 map_sympy (Ham, "simplify") # cdb (Ham.109,Ham)
24
25 foo := gBar_{a b} gBar^{a b} -> 3.
26 bah := gBar_{a c} gBar^{b c} -> gBar_{a}^{b}.
27
28 substitute (Ham, foo) # cdb (Ham.110,Ham)
29 substitute (Ham, bah) # cdb (Ham.111,Ham)
30 eliminate_kronecker (Ham) # cdb (Ham.112,Ham)
31
32 foo := gBar_{a b} gBar^{a b} -> 3.
33 bah := gBar_{a}^{a} -> 3.
34 moo := ABar_{a b} gBar^{a b} -> 0.
35
36 substitute (Ham, foo) # cdb (Ham.113,Ham)

```

```
37 substitute      (Ham, bah)          # cdb (Ham.114,Ham)
38 substitute      (Ham, moo)          # cdb (Ham.115,Ham)
39
40 foo := ABar_{c d} gBar^{c a} gBar^{d b} -> ABar^{a b}.
41
42 substitute      (Ham, foo)          # cdb (Ham.116,Ham)
43 rename_dummies  (Ham)              # cdb (Ham.117,Ham)
44
45 cdblib.put ('Ham',Ham,jsonfile)
```


$$\mathcal{H} = R + K_{ab}g^{ab}K_{cd}g^{cd} - K_{ab}K_{cd}g^{ac}g^{bd} \quad (\text{Ham.101})$$

$$= R + \left(\exp(4\phi) \bar{A}_{ab} + \frac{1}{3}g_{ab}\text{tr}K \right) g^{ab} \left(\exp(4\phi) \bar{A}_{cd} + \frac{1}{3}g_{cd}\text{tr}K \right) g^{cd} - \left(\exp(4\phi) \bar{A}_{ab} + \frac{1}{3}g_{ab}\text{tr}K \right) \left(\exp(4\phi) \bar{A}_{cd} + \frac{1}{3}g_{cd}\text{tr}K \right) g^{ac}g^{bd} \quad (\text{Ham.102})$$

$$= R + \left(\exp(4\phi) \bar{A}_{ab} + \frac{1}{3}\exp(4\phi) \bar{g}_{ab}\text{tr}K \right) g^{ab} \left(\exp(4\phi) \bar{A}_{cd} + \frac{1}{3}\exp(4\phi) \bar{g}_{cd}\text{tr}K \right) g^{cd} \\ - \left(\exp(4\phi) \bar{A}_{ab} + \frac{1}{3}\exp(4\phi) \bar{g}_{ab}\text{tr}K \right) \left(\exp(4\phi) \bar{A}_{cd} + \frac{1}{3}\exp(4\phi) \bar{g}_{cd}\text{tr}K \right) g^{ac}g^{bd} \quad (\text{Ham.103})$$

$$= R + \left(\exp(4\phi) \bar{A}_{ab} + \frac{1}{3}\exp(4\phi) \bar{g}_{ab}\text{tr}K \right) \exp(-4\phi) \bar{g}^{ab} \left(\exp(4\phi) \bar{A}_{cd} + \frac{1}{3}\exp(4\phi) \bar{g}_{cd}\text{tr}K \right) \exp(-4\phi) \bar{g}^{cd} \\ - \left(\exp(4\phi) \bar{A}_{ab} + \frac{1}{3}\exp(4\phi) \bar{g}_{ab}\text{tr}K \right) \left(\exp(4\phi) \bar{A}_{cd} + \frac{1}{3}\exp(4\phi) \bar{g}_{cd}\text{tr}K \right) \exp(-4\phi) \bar{g}^{ac} \exp(-4\phi) \bar{g}^{bd} \quad (\text{Ham.104})$$

$$= R + \exp(4\phi) \bar{A}_{ab} \exp(-4\phi) \bar{g}^{ab} \exp(4\phi) \bar{A}_{cd} \exp(-4\phi) \bar{g}^{cd} + \frac{1}{3}\exp(4\phi) \bar{A}_{ab} \exp(-4\phi) \bar{g}^{ab} \exp(4\phi) \bar{g}_{cd}\text{tr}K \exp(-4\phi) \bar{g}^{cd} \\ + \frac{1}{3}\exp(4\phi) \bar{g}_{ab}\text{tr}K \exp(-4\phi) \bar{g}^{ab} \exp(4\phi) \bar{A}_{cd} \exp(-4\phi) \bar{g}^{cd} + \frac{1}{9}\exp(4\phi) \bar{g}_{ab}\text{tr}K \exp(-4\phi) \bar{g}^{ab} \exp(4\phi) \bar{g}_{cd}\text{tr}K \exp(-4\phi) \bar{g}^{cd} \\ - \exp(4\phi) \bar{A}_{ab} \exp(4\phi) \bar{A}_{cd} \exp(-4\phi) \bar{g}^{ac} \exp(-4\phi) \bar{g}^{bd} - \frac{1}{3}\exp(4\phi) \bar{A}_{ab} \exp(4\phi) \bar{g}_{cd}\text{tr}K \exp(-4\phi) \bar{g}^{ac} \exp(-4\phi) \bar{g}^{bd} \\ - \frac{1}{3}\exp(4\phi) \bar{g}_{ab}\text{tr}K \exp(4\phi) \bar{A}_{cd} \exp(-4\phi) \bar{g}^{ac} \exp(-4\phi) \bar{g}^{bd} - \frac{1}{9}\exp(4\phi) \bar{g}_{ab}\text{tr}K \exp(4\phi) \bar{g}_{cd}\text{tr}K \exp(-4\phi) \bar{g}^{ac} \exp(-4\phi) \bar{g}^{bd} \quad (\text{Ham.105})$$

$$= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) + \frac{1}{3}\text{tr}K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\ + \frac{1}{3}\text{tr}K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ab} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) + \frac{1}{9}\text{tr}K \text{tr}K \bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\ - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) - \frac{1}{3}\text{tr}K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\ - \frac{1}{3}\text{tr}K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ca}\bar{g}^{db} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) - \frac{1}{9}\text{tr}K \text{tr}K \bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \quad (\text{Ham.106})$$

$$\begin{aligned} \mathcal{H} = & R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) + \frac{1}{3} \text{tr} K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\ & + \frac{1}{3} \text{tr} K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ab} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) + \frac{1}{9} \text{tr} K \text{tr} K \bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\ & - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) - \frac{1}{3} \text{tr} K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\ & - \frac{1}{3} \text{tr} K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ca}\bar{g}^{db} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) - \frac{1}{9} \text{tr} K \text{tr} K \bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \quad (\text{Ham.107}) \end{aligned}$$

$$\begin{aligned}
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) + \frac{2}{3} \text{tr} K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\
&\quad + \frac{1}{9} \text{tr} K \text{tr} K \bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \\
&\quad - \frac{2}{3} \text{tr} K \bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) - \frac{1}{9} \text{tr} K \text{tr} K \bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \exp(-4\phi) \exp(-4\phi) \exp(4\phi) \exp(4\phi) \quad (\text{Ham.108})
\end{aligned}$$

$$= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + \frac{2}{3}\text{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} + \frac{1}{9}\text{tr}K^2\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\text{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{1}{9}\text{tr}K^2\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \quad (\text{Ham.109})$$

$$= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\text{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\text{tr}K^2\bar{g}_{cd}\bar{g}^{cd} - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\text{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{1}{9}\text{tr}K^2\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} \quad (\text{Ham.110})$$

$$= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\text{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\text{tr}K^2\bar{g}^c_c - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\text{tr}K\bar{A}_{ab}\bar{g}_c^b\bar{g}^{ac} - \frac{1}{9}\text{tr}K^2\bar{g}_{ab}\bar{g}_c^b\bar{g}^{ac} \quad (\text{Ham.111})$$

$$= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\text{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\text{tr}K^2\bar{g}^c_c - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\text{tr}K\bar{A}_{ac}\bar{g}^{ac} - \frac{1}{9}\text{tr}K^2\bar{g}_{ac}\bar{g}^{ac} \quad (\text{Ham.112})$$

$$= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\text{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\text{tr}K^2\bar{g}^c_c - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\text{tr}K\bar{A}_{ac}\bar{g}^{ac} - \frac{1}{3}\text{tr}K^2 \quad (\text{Ham.113})$$

$$= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\text{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{2}{3}\text{tr}K^2 - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\text{tr}K\bar{A}_{ac}\bar{g}^{ac} \quad (\text{Ham.114})$$

$$= R + \frac{2}{3} \text{tr} K^2 - \bar{A}_{ab} \bar{A}_{cd} \bar{g}^{ac} \bar{g}^{bd} \quad (\text{Ham.115})$$

$$= R + \frac{2}{3} \text{tr} K^2 - \bar{A}^{cd} \bar{A}_{cd} \quad (\text{Ham.116})$$

$$= R + \frac{2}{3} \text{tr} K^2 - \bar{A}^{ab} \bar{A}_{ab} \quad (\text{Ham.117})$$

```

1  # -----
2  # Check against prd67.
3
4  foo := @(Ham).                                # cdb(prd67.eq19.lcb,foo)
5  bah  = cdblib.get('prd67.eq19.rhs','prd67.json') # cdb(prd67.eq19.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy      (diff, "simplify")
13 canonicalise   (diff)                        # cdb(prd67.eq19.chk,diff)

```

$$\text{prd67.eq19.lcb} := R + \frac{2}{3}\text{tr}K^2 - \bar{A}^{ab}\bar{A}_{ab}$$

$$\text{prd67.eq19.prd} := R - \bar{A}_{ab}\bar{A}^{ab} + \frac{2}{3}\text{tr}K^2$$

$$\text{prd67.eq19.chk} := 0$$

PhysRevD.67.084023 equation (20)

```

1  from shared import *
2  import cdblib
3
4  jsonfile = 'momentum.json'
5  cdblib.create (jsonfile)
6
7  defG2GBar = cdblib.get ('defG2GBar','gamma.json')
8
9  # -----
10 # Momentum constraint pt.1
11
12 Mom := D_{j}{K^{i j} - g^{i j} trK}. # cdb(Mom.101,Mom)
13
14 defDgD := D_{a}{g_{b c}} -> 0.
15 defDgU := D_{a}{g^{b c}} -> 0.
16
17 defDtrK := D_{a}{trK} -> \partial_{a}{trK}.
18 defDexp := D_{a}{\exp(-4\phi)} -> -4\exp(-4\phi) \partial_{a}{\phi}.
19
20 distribute (Mom) # cdb(Mom.102,Mom)
21 product_rule (Mom) # cdb(Mom.103,Mom)
22 substitute (Mom, defDgU) # cdb(Mom.104,Mom)
23
24 defK2ABarU := K^{i j} -> \exp(-4\phi) ABar^{i j} + (1/3) g^{i j} trK.
25
26 substitute (Mom, defK2ABarU) # cdb(Mom.105,Mom)
27 distribute (Mom) # cdb(Mom.106,Mom)
28 product_rule (Mom) # cdb(Mom.107,Mom)
29 substitute (Mom, defDtrK) # cdb(Mom.108,Mom)
30 substitute (Mom, defDgU) # cdb(Mom.109,Mom)
31 substitute (Mom, defDexp) # cdb(Mom.110,Mom)

```

$$\mathcal{D}^j = D_j (K^{ij} - g^{ij} \text{tr} K) \quad (\text{Mom. 101})$$

$$= D_j K^{ij} - D_j (g^{ij} \text{tr} K) \quad (\text{Mom. 102})$$

$$= D_j K^{ij} - D_j g^{ij} \text{tr} K - g^{ij} D_j \text{tr} K \quad (\text{Mom. 103})$$

$$= D_j K^{ij} - g^{ij} D_j \text{tr} K \quad (\text{Mom. 104})$$

$$= D_j \left(\exp(-4\phi) \bar{A}^{ij} + \frac{1}{3} g^{ij} \text{tr} K \right) - g^{ij} D_j \text{tr} K \quad (\text{Mom. 105})$$

$$= D_j (\exp(-4\phi) \bar{A}^{ij}) + \frac{1}{3} D_j (g^{ij} \text{tr} K) - g^{ij} D_j \text{tr} K \quad (\text{Mom. 106})$$

$$= D_j (\exp(-4\phi)) \bar{A}^{ij} + \exp(-4\phi) D_j \bar{A}^{ij} + \frac{1}{3} D_j g^{ij} \text{tr} K - \frac{2}{3} g^{ij} D_j \text{tr} K \quad (\text{Mom. 107})$$

$$= D_j (\exp(-4\phi)) \bar{A}^{ij} + \exp(-4\phi) D_j \bar{A}^{ij} + \frac{1}{3} D_j g^{ij} \text{tr} K - \frac{2}{3} g^{ij} \partial_j \text{tr} K \quad (\text{Mom. 108})$$

$$= D_j (\exp(-4\phi)) \bar{A}^{ij} + \exp(-4\phi) D_j \bar{A}^{ij} - \frac{2}{3} g^{ij} \partial_j \text{tr} K \quad (\text{Mom. 109})$$

$$= -4 \exp(-4\phi) \partial_j \phi \bar{A}^{ij} + \exp(-4\phi) D_j \bar{A}^{ij} - \frac{2}{3} g^{ij} \partial_j \text{tr} K \quad (\text{Mom. 110})$$

```

1  # -----
2  # Momentum constraint pt.2
3
4  confMom := \exp(4\phi) @ (Mom).
5
6  defG2GBarU := g^{i j} -> \exp(-4\phi) gBar^{i j}.
7
8  distribute      (confMom)                                # cdb(confMom.101,confMom)
9  substitute      (confMom, defG2GBarU)                    # cdb(confMom.102,confMom)
10 map_sympy      (confMom, "simplify")                      # cdb(confMom.103,confMom)
11
12 defDAabU := D_{a}{ABar^{b c}} -> \partial_{a}{ABar^{b c}}
13                                     + \Gamma^{b}_{i a} ABar^{i c}
14                                     + \Gamma^{c}_{i a} ABar^{b i}.
15
16 substitute      (confMom, defDAabU)                      # cdb(confMom.104,confMom)
17 substitute      (confMom, defG2GBarU)                    # cdb(confMom.105,confMom)
18 distribute      (confMom)                                # cdb(confMom.106,confMom)
19 confMom = product_sort (confMom)                          # cdb(confMom.107,confMom)
20 rename_dummies  (confMom)                                # cdb(confMom.108,confMom)
21 canonicalise    (confMom)                                # cdb(confMom.109,confMom)
22 substitute      (confMom, $gBar^{i}_{i} -> 3$)           # cdb(confMom.110,confMom)
23 substitute      (confMom, $gBar_{i j} ABar^{i j} -> 0$)  # cdb(confMom.111,confMom)
24 substitute      (confMom, $gBar_{a i} gBar^{i b} -> gBar_{a}^{b}$) # cdb(confMom.112,confMom)
25 substitute      (confMom, $GammaBar^{b}_{a b} -> 0$)      # cdb(confMom.113,confMom) # follows from det gBar = 1
26 eliminate_kronecker (confMom)                            # cdb(confMom.114,confMom)
27 rename_dummies  (confMom)
28 canonicalise    (confMom)                                # cdb(confMom.115,confMom)
29
30 cdblib.put ('confMom',confMom,jsonfile)

```

$$\exp(4\phi)\mathcal{D}^j = -4\exp(4\phi)\exp(-4\phi)\partial_j\phi\bar{A}^{ij} + \exp(4\phi)\exp(-4\phi)D_j\bar{A}^{ij} - \frac{2}{3}\exp(4\phi)g^{ij}\partial_j\text{tr}K \quad (\text{confMom.101})$$

$$= -4\exp(4\phi)\exp(-4\phi)\partial_j\phi\bar{A}^{ij} + \exp(4\phi)\exp(-4\phi)D_j\bar{A}^{ij} - \frac{2}{3}\exp(4\phi)\exp(-4\phi)\bar{g}^{ij}\partial_j\text{tr}K \quad (\text{confMom.102})$$

$$= -4\partial_j\phi\bar{A}^{ij} + D_j\bar{A}^{ij} - \frac{2}{3}\bar{g}^{ij}\partial_j\text{tr}K \quad (\text{confMom.103})$$

$$= -4\partial_j\phi\bar{A}^{ij} + \partial_j\bar{A}^{ij} + \Gamma^i_{aj}\bar{A}^{aj} + \Gamma^j_{aj}\bar{A}^{ia} - \frac{2}{3}\bar{g}^{ij}\partial_j\text{tr}K \quad (\text{confMom.104})$$

$$= -4\partial_j\phi\bar{A}^{ij} + \partial_j\bar{A}^{ij} + (2\bar{g}^i_j\partial_a\phi + 2\bar{g}^i_a\partial_j\phi - 2\bar{g}^{ie}\partial_e\phi\bar{g}_{aj} + \bar{\Gamma}^i_{aj})\bar{A}^{aj} + (2\bar{g}^j_j\partial_a\phi + 2\bar{g}^j_a\partial_j\phi - 2\bar{g}^{je}\partial_e\phi\bar{g}_{aj} + \bar{\Gamma}^j_{aj})\bar{A}^{ia} - \frac{2}{3}\bar{g}^{ij}\partial_j\text{tr}K \quad (\text{confMom.105})$$

$$= -4\partial_j\phi\bar{A}^{ij} + \partial_j\bar{A}^{ij} + 2\bar{g}^i_j\partial_a\phi\bar{A}^{aj} + 2\bar{g}^i_a\partial_j\phi\bar{A}^{aj} - 2\bar{g}^{ie}\partial_e\phi\bar{g}_{aj}\bar{A}^{aj} + \bar{\Gamma}^i_{aj}\bar{A}^{aj} + 2\bar{g}^j_j\partial_a\phi\bar{A}^{ia} + 2\bar{g}^j_a\partial_j\phi\bar{A}^{ia} - 2\bar{g}^{je}\partial_e\phi\bar{g}_{aj}\bar{A}^{ia} + \bar{\Gamma}^j_{aj}\bar{A}^{ia} - \frac{2}{3}\bar{g}^{ij}\partial_j\text{tr}K \quad (\text{confMom.106})$$

$$= -4\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 2\bar{A}^{ab}\partial_a\phi\bar{g}^i_b + 2\bar{A}^{ab}\partial_b\phi\bar{g}^i_a - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i_{ab} + 2\bar{A}^{ia}\partial_a\phi\bar{g}^b_b + 2\bar{A}^{ia}\partial_b\phi\bar{g}^b_a - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.107})$$

$$= -4\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 2\bar{A}^{ab}\partial_a\phi\bar{g}^i_b + 2\bar{A}^{ab}\partial_b\phi\bar{g}^i_a - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i_{ab} + 2\bar{A}^{ia}\partial_a\phi\bar{g}^b_b + 2\bar{A}^{ia}\partial_b\phi\bar{g}^b_a - 2\bar{A}^{ia}\bar{g}_{ac}\bar{g}^{cb}\partial_b\phi + \bar{A}^{ia}\bar{\Gamma}^b_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.108})$$

$$= -4\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i_b - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i_{ab} + 2\bar{A}^{ia}\partial_a\phi\bar{g}^b_b + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a^b - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.109})$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i_b - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a^b - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.110})$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i_b + \bar{A}^{ab}\bar{\Gamma}^i_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a^b - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.111})$$

$$\exp(4\phi)\mathcal{D}^j = 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a{}^b - 2\bar{A}^{ia}\bar{g}_a{}^c\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.112})$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a{}^b - 2\bar{A}^{ia}\bar{g}_a{}^c\partial_c\phi - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.113})$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ai}\partial_a\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ib}\partial_b\phi - 2\bar{A}^{ic}\partial_c\phi - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.114})$$

$$= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.115})$$


```

1  tmpA := @(confMom).                                # cdb(confMom.201,tmpA)
2  tmpB := @(confMom).
3
4  X^{b c}_{a}::Weight(label=numX).
5
6  Xbca := \partial_{a}{A\bar{b c}}.                    # cdb(confMom.202,Xbca)
7
8  foo := \partial_{a}{A\bar{b c}} -> X^{b c}_{a}.
9  bah := X^{b c}_{a} -> \partial_{a}{A\bar{b c}}.
10
11 substitute (tmpA, foo)                               # cdb(confMom.203,tmpA)
12 substitute (tmpB, foo)                               # cdb(confMom.204,tmpB)
13 drop_weight (tmpA, $numX=1$)                         # cdb(confMom.205,tmpA)
14 keep_weight (tmpB, $numX=1$)                         # cdb(confMom.206,tmpB)
15 substitute (tmpB, bah)                               # cdb(confMom.207,tmpB)
16
17 tmpC := - @(tmpA).                                    # cdb(confMom.208,tmpC)
18
19 defMomSub := @(tmpB) -> @(tmpC).                     # cdb(confMom.209,defMomSub)
20
21 cdblib.put ('defMomSub',defMomSub,jsonfile)

```

$$0 = 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.201})$$

$$0 = 6\bar{A}^{ia}\partial_a\phi + X^{ia}{}_a + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.203})$$

$$\partial_a\bar{A}^{bc} = X^{ia}{}_a \quad (\text{confMom.206})$$

$$= -6\bar{A}^{ia}\partial_a\phi - \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.208})$$

$$\partial_a\bar{A}^{ia} \rightarrow -6\bar{A}^{ia}\partial_a\phi - \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K \quad (\text{confMom.209})$$

```

1  # -----
2  # Check against prd67.
3
4  foo := @(confMom).                # cdb(prd67.eq20.lcb,foo)
5  bah  = cdblib.get('prd67.eq20.rhs','prd67.json')  # cdb(prd67.eq20.prd,bah)
6
7  diff := @(foo) - @(bah).
8
9  distribute      (diff)
10 diff = product_sort (diff)
11 rename_dummies (diff)
12 map_sympy      (diff, "simplify")
13 canonicalise   (diff)                # cdb(prd67.eq20.chk,diff)

```

$$\text{prd67.eq20.lcb} := 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K$$

$$\text{prd67.eq20.prd} := \partial_a\bar{A}^{ia} + 6\bar{A}^{ia}\partial_a\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\text{tr}K$$

$$\text{prd67.eq20.chk} := 0$$