Deriving the BSSN equations from the ADM equations

The following Cadabra[1] codes verifies, for the particular case of vacuum spacetimes, the main equations (with zero-shift) in the Phys Rev D papers Phys.Rev.D. (62) 044034 (2000) and Phys.Rev.D. (67) 084023 (2003) by Miguel Alcubierre, Bernd Brugmann etal. .

The two papers are abbreviated as pdr62 and prd67 respectively.

All of the essential BSSN equations are covered, in particlular equations 9, 10, 11, 12 , 15, 17, 18, 19 and 20 from prd62 and equations 19, 20 and 27 from prd67.

The first few pages provides a summary of the calculations. For each equation the summary is of the form

$$\texttt{eq18.lcb} := \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}$$

$$\texttt{eq18.prd} := -\frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right)$$

$$\texttt{eq18.chk} := 0$$

The first line records the output from the Cadabra code, the second line is the corresponding equation in prd62 (or prd67) while the third line shows the difference. The good news is that for each equation the difference is zero (as it should be).

The summary is then followed by the full details of the calculations for each equation. The output includes all of the steps. The odd looking equation numbers are tags that match the line in the corresponding Cadabra source. Thus the following output (from prd62 equation 11)

$$\partial_t \mathrm{tr}K = \partial_t\left(g^{ij}K_{ij}\right) \tag{eq11.102}$$

$$= \partial_t g^{ij}K_{ij} + g^{ij}\partial_t K_{ij} \tag{eq11.103}$$

$$= 2NK^{ij}K_{ij} + g^{ij}\partial_t K_{ij} \tag{eq11.104}$$

$$= 2NK^{ij}K_{ij} + g^{ij}\left(-D_{ij}N + N\left(R_{ij} + \mathrm{tr}K K_{ij} - 2K_{ic}K_{jd}g^{cd}\right)\right) \tag{eq11.105}$$

can be matched line-by-line with the Cadabra code

```
substitute      (dotK,trK)          # cdb (eq11.102,dotK)
product_rule    (dotK)              # cdb (eq11.103,dotK)
substitute      (dotK,DhijDt)       # cdb (eq11.104,dotK)
substitute      (dotK,DKijDt)       # cdb (eq11.105,dotK)
```

[1]Based on Cadabra 2.3.7 (build 2778.b0ba2dbb80 dated 2021-09-27)

Running the codes

To run any of the codes you will need to install Cadabra. Compiling and installing Cadabra is straightforward with full details provided on the https://github.com/kpeeters/cadabra2.git repository on GitHub.

You will also need to install the hybrid-latex tools. These are simple scripts and LaTeX macros that allow the Cadabra output to be caught and integrated back into the LaTeX source. See the `INSTALL.txt` file in `hybrid-latex` for full details. The documentation for hybrid-latex tools is included in the `hybrid-latex` directory.

To compile all of the files in one go just use

```
cd source
make
```

You can compile any one of the files (e.g., `eqtn11.tex`) using any of the following (from within the `source` directory)

```
cdblatex.sh -i eqtn11
make  eqtn11
make .eqtn11
```

The first two commands will force a recompile of `eqtn11` while the third command will use the magic of makefiles to determine if it (and its dependencies) needs to be recompiled.

The `cdblatex.sh` is the main script in the hybrid-latex codes. It does the job of extracting rhe embedded Cadabra code, running that code through Cadabra, collecting the output and finally making that output available to the LaTeX source.

You may want to look inside the `Makefile` to see which targets are available.

Note that there are dependencies amongst the files and these dependencies are encoded in the makefile. See the file `SEQUENCE.txt` for the details.

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$$\texttt{eq09.lcb} := -2N\bar{A}_{ij}$$

$$\texttt{eq09.prd} := -2N\bar{A}_{ij}$$

$$\texttt{eq09.chk} := 0$$

$$\texttt{eq10.lcb} := -\frac{1}{6}\mathrm{tr}KN$$

$$\texttt{eq10.prd} := -\frac{1}{6}N\mathrm{tr}K$$

$$\texttt{eq10.chk} := 0$$

$$\texttt{eq11.lcb} := -g^{ab}D_{ab}N + N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}\mathrm{tr}K^2N$$

$$\texttt{eq11.prd} := -g^{ij}D_{ij}N + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\mathrm{tr}K^2\right)$$

$$\texttt{eq11.chk} := 0$$

$$\texttt{eq12.lcb} := \mathrm{tr}KN\bar{A}_{ij} - 2N\bar{A}_i{}^b\bar{A}_{jb} + \exp\left(-4\phi\right)\left(-D_{ij}N + NR_{ij} - \frac{1}{3}Ng_{ij}g^{ab}R_{ab} + \frac{1}{3}g_{ij}g^{ab}D_{ab}N\right)$$

$$\texttt{eq12.prd} := N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - D_{ij}N - \frac{1}{3}g_{ij}\left(NR_{ab} - D_{ab}N\right)g^{ab}\right)$$

$$\texttt{eq12.chk} := 0$$

$$\texttt{eq15.lcb} := -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_{a}\phi\bar{D}_{b}\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{c}\phi\bar{D}_{d}\phi$$

$$\texttt{eq15.prd} := -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_{a}\phi\bar{D}_{b}\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{c}\phi\bar{D}_{d}\phi$$

$$\texttt{eq15.chk} := 0$$

$$\texttt{eq17.lcb} := -\partial_{b}\bar{g}^{ib}$$

$$\texttt{eq17.prd} := -\partial_{j}\bar{g}^{ij}$$

$$\texttt{eq17.chk} := 0$$

$$\texttt{eq18.lcb} := \frac{1}{2}\bar{g}_{bc}\partial_{a}\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}_{ac}\partial_{b}\bar{\Gamma}^{c} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^{c}\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^{c}\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}$$

$$\texttt{eq18.prd} := -\frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_{b}\bar{\Gamma}^{k} + \frac{1}{2}\bar{g}_{kb}\partial_{a}\bar{\Gamma}^{k} + \frac{1}{2}\bar{\Gamma}^{k}\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^{k}\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right)$$

$$\texttt{eq18.chk} := 0$$

$$\texttt{eq19.lcb} := -2\bar{A}^{ia}\partial_{a}N - 2N\partial_{a}\bar{A}^{ia}$$

$$\texttt{eq19.prd} := -2\partial_{j}\left(N\bar{A}^{ij}\right)$$

$$\texttt{eq19.chk} := 0$$

$$\texttt{eq20.lcb} := -2\bar{A}^{ia}\partial_{a}N - 2N\left(-6\bar{A}^{ia}\partial_{a}\phi - \bar{A}^{ab}\bar{\Gamma}^{i}{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_{a}\mathrm{tr}K\right)$$

$$\texttt{eq20.prd} := -2\bar{A}^{ij}\partial_{j}N + 2N\left(\bar{\Gamma}^{i}{}_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{g}^{ij}\partial_{j}\mathrm{tr}K + 6\bar{A}^{ij}\partial_{j}\phi\right)$$

$$\texttt{eq20.chk} := 0$$

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$$\texttt{prd67.eq19.lcb} := R + \frac{2}{3}\mathrm{tr}K^2 - \bar{A}^{ab}\bar{A}_{ab}$$

$$\texttt{prd67.eq19.prd} := R - \bar{A}_{ab}\bar{A}^{ab} + \frac{2}{3}\mathrm{tr}K^2$$

$$\texttt{prd67.eq19.chk} := 0$$

$$\texttt{prd67.eq20.lcb} := 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K$$

$$\texttt{prd67.eq20.prd} := \partial_a\bar{A}^{ia} + 6\bar{A}^{ia}\partial_a\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K$$

$$\texttt{prd67.eq20.chk} := 0$$

$$\texttt{prd67.eq27.lcb} := 2\bar{g}^a{}_c\partial_b\phi + 2\bar{g}^a{}_b\partial_c\phi - 2\bar{g}^{ae}\partial_e\phi\bar{g}_{bc} + \bar{\Gamma}^a{}_{bc}$$

$$\texttt{prd67.eq27.prd} := \bar{\Gamma}^a{}_{bc} + 2\bar{g}^a{}_c\partial_b\phi + 2\bar{g}^a{}_b\partial_c\phi - 2\bar{g}_{bc}\bar{g}^{ae}\partial_e\phi$$

$$\texttt{prd67.eq27.chk} := 0$$

PhysRevD.62.044034 equation (9)

```
from shared import *
import cdblib

jsonfile = 'eqtn09.json'
cdblib.create (jsonfile)

DgijDt = cdblib.get ('adm.DgijDt','adm.json')
DhijDt = cdblib.get ('adm.DhijDt','adm.json')

DphiDt = cdblib.get ('DphiDt','eqtn10.json')

# -----------------------------------------------------------------------------

gBarij := gBar_{i j} -> \exp(-4\phi) g_{i j}.         # prd62 eqn 05

Kij    := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.     # prd62 eqn 07

A2ABar := \exp(-4\phi) A_{i j} -> ABar_{i j}.         # prd62 eqn 08

# -----------------------------------------------------------------------------
# dgBar_{ij}/dt

dotgBarij := \partial_{t}{gBar_{i j}}.          # cdb (eq09.101,dotgBarij)

substitute   (dotgBarij, gBarij)                # cdb (eq09.102,dotgBarij)
product_rule (dotgBarij)                        # cdb (eq09.103,dotgBarij)
substitute   (dotgBarij, dexp)                  # cdb (eq09.104,dotgBarij)
substitute   (dotgBarij, DgijDt)                # cdb (eq09.105,dotgBarij)
substitute   (dotgBarij, DphiDt)                # cdb (eq09.106,dotgBarij)
substitute   (dotgBarij, Kij)                   # cdb (eq09.107,dotgBarij)
distribute   (dotgBarij)                        # cdb (eq09.108,dotgBarij)
map_sympy    (dotgBarij, "simplify")            # cdb (eq09.109,dotgBarij)
substitute   (dotgBarij, A2ABar)                # cdb (eq09.110,dotgBarij)

DgBarijDt := \partial_{t}{gBar_{i j}} -> @(dotgBarij).
```

```
cdblib.put ('dotgBarij',dotgBarij,jsonfile)
```

$$\partial_t \bar{g}_{ij} = \partial_t \left(\exp\left(-4\phi\right) g_{ij}\right) \tag{eq09.102}$$

$$= \partial_t \left(\exp\left(-4\phi\right)\right) g_{ij} + \exp\left(-4\phi\right) \partial_t g_{ij} \tag{eq09.103}$$

$$= -4 \exp\left(-4\phi\right) \partial_t \phi g_{ij} + \exp\left(-4\phi\right) \partial_t g_{ij} \tag{eq09.104}$$

$$= -4 \exp\left(-4\phi\right) \partial_t \phi g_{ij} - 2 \exp\left(-4\phi\right) N K_{ij} \tag{eq09.105}$$

$$= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} - 2 \exp\left(-4\phi\right) N K_{ij} \tag{eq09.106}$$

$$= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} - 2 \exp\left(-4\phi\right) N \left(A_{ij} + \frac{1}{3} g_{ij} \mathrm{tr} K\right) \tag{eq09.107}$$

$$= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} - 2 \exp\left(-4\phi\right) N A_{ij} - \frac{2}{3} \exp\left(-4\phi\right) N g_{ij} \mathrm{tr} K \tag{eq09.108}$$

$$= -2N \exp\left(-4\phi\right) A_{ij} \tag{eq09.109}$$

$$= -2N \bar{A}_{ij} \tag{eq09.110}$$

```
# ---------------------------------------------------------------------------
# Check against prd62.

foo := @(dotgBarij).                                # cdb(eq09.lcb,foo)
bah  = cdblib.get('prd62.eq09.rhs','prd62.json')    # cdb(eq09.prd,bah)

diff := @(foo) - @(bah).

diff = product_sort (diff)
rename_dummies (diff)
canonicalise   (diff)                               # cdb(eq09.chk,diff)
```

$$\texttt{eq09.lcb} := -2N\bar{A}_{ij}$$

$$\texttt{eq09.prd} := -2N\bar{A}_{ij}$$

$$\texttt{eq09.chk} := 0$$

PhysRevD.62.044034 equation (10)

```
from shared import *
import cdblib

jsonfile = 'eqtn10.json'
cdblib.create (jsonfile)

DgijDt  = cdblib.get ('adm.DgijDt','adm.json')
DdetgDt = cdblib.get ('adm.DdetgDt','adm.json')

# ----------------------------------------------------------------------------

phi   := \phi -> (1/12) \log(detg).
gdotK := g^{i j} K_{i j} -> trK.

# ----------------------------------------------------------------------------
# d\phi/dt

dotphi := \partial_{t}{\phi}.     # cdb (eq10.101,dotphi)

substitute (dotphi, phi)          # cdb (eq10.102,dotphi)
substitute (dotphi, dlog)         # cdb (eq10.103,dotphi)
substitute (dotphi, DdetgDt)      # cdb (eq10.104,dotphi)
substitute (dotphi, DgijDt)       # cdb (eq10.105,dotphi)
substitute (dotphi, gdotK)        # cdb (eq10.106,dotphi)
map_sympy  (dotphi, "simplify")   # cdb (eq10.107,dotphi)

DphiDt := \partial_{t}{\phi} -> @(dotphi).

cdblib.put ('DphiDt',DphiDt,jsonfile)
```

$$
\begin{aligned}
\partial_t \phi &= \frac{1}{12} \partial_t \left(\log \left(g \right) \right) && \texttt{(eq10.102)} \\
&= \frac{1}{12} g^{-1} \partial_t g && \texttt{(eq10.103)} \\
&= \frac{1}{12} g^{-1} g g^{ij} \partial_t g_{ij} && \texttt{(eq10.104)} \\
&= -\frac{1}{6} g^{-1} g g^{ij} N K_{ij} && \texttt{(eq10.105)} \\
&= -\frac{1}{6} g^{-1} g \operatorname{tr} K N && \texttt{(eq10.106)} \\
&= -\frac{1}{6} \operatorname{tr} K N && \texttt{(eq10.107)}
\end{aligned}
$$

```
# ---------------------------------------------------------------------------
# Check against prd62.

foo := @(dotphi).                                  # cdb(eq10.lcb,foo)
bah  = cdblib.get('prd62.eq10.rhs','prd62.json')   # cdb(eq10.prd,bah)

diff := @(foo) - @(bah).

diff = product_sort (diff)
rename_dummies (diff)
canonicalise   (diff)                              # cdb(eq10.chk,diff)
```

$$\texttt{eq10.lcb} := -\frac{1}{6}\mathrm{tr}KN$$

$$\texttt{eq10.prd} := -\frac{1}{6}N\mathrm{tr}K$$

$$\texttt{eq10.chk} := 0$$

PhysRevD.62.044034 equation (11)

```
from shared import *
import cdblib

jsonfile = 'eqtn11.json'
cdblib.create (jsonfile)

DhijDt = cdblib.get ('adm.DhijDt','adm.json')
DKijDt = cdblib.get ('adm.DKijDt','adm.json')

# ----------------------------------------------------------------------------

trK    := trK -> g^{i j} K_{i j}.
gdotK  := g^{i j} K_{i j} -> trK.

Kup := g^{i a} g^{j b} K_{i j} -> K^{a b}.

Ham := g^{i j} R_{i j} -> K_{i j} K^{i j} - trK trK.

Kij := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.   # prd62 eqn 07
Lij := K^{i j} -> A^{i j} + (1/3) g^{i j} trK.   # prd62 eqn 07

trA1 := A_{i j} g^{i j} -> 0.                     # Aij is trace free
trA2 := A^{i j} g_{i j} -> 0.

Asq := A_{i j} A^{i j} -> ABar_{i j} ABar^{i j}.

gdotg := g_{i j} g^{i j} -> 3.

# ----------------------------------------------------------------------------
# dK/dt

dotK := \partial_{t}{trK}.          # cdb (eq11.101,dotK)

substitute     (dotK,trK)           # cdb (eq11.102,dotK)
product_rule   (dotK)               # cdb (eq11.103,dotK)
substitute     (dotK,DhijDt)        # cdb (eq11.104,dotK)
```

```
substitute      (dotK,DKijDt)         # cdb (eq11.105,dotK)
distribute      (dotK)                # cdb (eq11.106,dotK)
substitute      (dotK,gdotK)          # cdb (eq11.107,dotK)
substitute      (dotK,Kup)            # cdb (eq11.108,dotK)
dotK = product_sort (dotK)            # cdb (eq11.109,dotK)
substitute      (dotK,Ham)            # cdb (eq11.110,dotK)
distribute      (dotK)                # cdb (eq11.111,dotK)
substitute      (dotK,Kij)            # cdb (eq11.112,dotK)
substitute      (dotK,Lij)            # cdb (eq11.113,dotK)
distribute      (dotK)                # cdb (eq11.114,dotK)
substitute      (dotK,trA1)           # cdb (eq11.115,dotK)
substitute      (dotK,trA2)           # cdb (eq11.116,dotK)
substitute      (dotK,Asq)            # cdb (eq11.117,dotK)
substitute      (dotK,gdotg)          # cdb (eq11.118,dotK)
map_sympy       (dotK, "simplify")    # cdb (eq11.119,dotK)

DKDt := \partial_{t}{trK} -> @(dotK).

cdblib.put ('DKDt',DKDt,jsonfile)
```

$$
\begin{aligned}
\partial_t \mathrm{tr}K &= \partial_t \left(g^{ij} K_{ij}\right) && \texttt{(eq11.102)}\\
&= \partial_t g^{ij} K_{ij} + g^{ij}\partial_t K_{ij} && \texttt{(eq11.103)}\\
&= 2NK^{ij}K_{ij} + g^{ij}\partial_t K_{ij} && \texttt{(eq11.104)}\\
&= 2NK^{ij}K_{ij} + g^{ij}\left(-D_{ij}N + N\left(R_{ij} + \mathrm{tr}KK_{ij} - 2K_{ic}K_{jd}g^{cd}\right)\right) && \texttt{(eq11.105)}\\
&= 2NK^{ij}K_{ij} - g^{ij}D_{ij}N + g^{ij}NR_{ij} + g^{ij}N\mathrm{tr}KK_{ij} - 2g^{ij}NK_{ic}K_{jd}g^{cd} && \texttt{(eq11.106)}\\
&= 2NK^{ij}K_{ij} - g^{ij}D_{ij}N + g^{ij}NR_{ij} + \mathrm{tr}KN\mathrm{tr}K - 2g^{ij}NK_{ic}K_{jd}g^{cd} && \texttt{(eq11.107)}\\
&= 2NK^{ij}K_{ij} - g^{ij}D_{ij}N + g^{ij}NR_{ij} + \mathrm{tr}KN\mathrm{tr}K - 2K^{jd}NK_{jd} && \texttt{(eq11.108)}\\
&= -g^{ab}D_{ab}N + Ng^{ab}R_{ab} + N\mathrm{tr}K\mathrm{tr}K && \texttt{(eq11.109)}\\
&= -g^{ab}D_{ab}N + N\left(K_{ab}K^{ab} - \mathrm{tr}K\mathrm{tr}K\right) + N\mathrm{tr}K\mathrm{tr}K && \texttt{(eq11.110)}\\
&= -g^{ab}D_{ab}N + NK_{ab}K^{ab} && \texttt{(eq11.111)}\\
&= -g^{ab}D_{ab}N + N\left(A_{ab} + \frac{1}{3}g_{ab}\mathrm{tr}K\right)K^{ab} && \texttt{(eq11.112)}\\
&= -g^{ab}D_{ab}N + N\left(A_{ab} + \frac{1}{3}g_{ab}\mathrm{tr}K\right)\left(A^{ab} + \frac{1}{3}g^{ab}\mathrm{tr}K\right) && \texttt{(eq11.113)}\\
&= -g^{ab}D_{ab}N + NA_{ab}A^{ab} + \frac{1}{3}NA_{ab}g^{ab}\mathrm{tr}K + \frac{1}{3}Ng_{ab}\mathrm{tr}KA^{ab} + \frac{1}{9}Ng_{ab}\mathrm{tr}Kg^{ab}\mathrm{tr}K && \texttt{(eq11.114)}\\
&= -g^{ab}D_{ab}N + NA_{ab}A^{ab} + \frac{1}{3}Ng_{ab}\mathrm{tr}KA^{ab} + \frac{1}{9}Ng_{ab}\mathrm{tr}Kg^{ab}\mathrm{tr}K && \texttt{(eq11.115)}\\
&= -g^{ab}D_{ab}N + NA_{ab}A^{ab} + \frac{1}{9}Ng_{ab}\mathrm{tr}Kg^{ab}\mathrm{tr}K && \texttt{(eq11.116)}\\
&= -g^{ab}D_{ab}N + N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{9}Ng_{ab}\mathrm{tr}Kg^{ab}\mathrm{tr}K && \texttt{(eq11.117)}\\
&= -g^{ab}D_{ab}N + N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K\mathrm{tr}K && \texttt{(eq11.118)}\\
&= -g^{ab}D_{ab}N + N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}\mathrm{tr}K^2N && \texttt{(eq11.119)}
\end{aligned}
$$

```
# ------------------------------------------------------------------------
# Check against prd62.

foo := @(dotK).                                    # cdb(eq11.lcb,foo)
bah  = cdblib.get('prd62.eq11.rhs','prd62.json')     # cdb(eq11.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                              # cdb(eq11.chk,diff)
```

$$\texttt{eq11.lcb} := -g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{3} \mathrm{tr} K^2 N$$

$$\texttt{eq11.prd} := -g^{ij} D_{ij} N + N \left(\bar{A}_{ij} \bar{A}^{ij} + \frac{1}{3} \mathrm{tr} K^2 \right)$$

$$\texttt{eq11.chk} := 0$$

PhysRevD.62.044034 equation (12)

```
from shared import *
import cdblib

jsonfile = 'eqtn12.json'
cdblib.create (jsonfile)

DgijDt = cdblib.get ('adm.DgijDt','adm.json')
DKijDt = cdblib.get ('adm.DKijDt','adm.json')

DphiDt = cdblib.get ('DphiDt','eqtn10.json')
DKDt = cdblib.get ('DKDt','eqtn11.json')

# ---------------------------------------------------------------------------
ABar2A := ABar_{i j} -> \exp(-4\phi) A_{i j}.         # prd62 eqn 08
A2ABar := A_{i j} -> \exp(4\phi) ABar_{i j}.          # prd62 eqn 08

Aij    := A_{i j} -> K_{i j} - (1/3) g_{i j} trK.     # prd62 eqn 07
Kij    := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.     # prd62 eqn 07

gginv := {g_{i a} g^{a j} -> g_{i}^{j},
          g_{i a} g^{j a} -> g_{i}^{j}}.

ABarUp := ABar_{i j} g^{j k} -> \exp(-4\phi) ABar_{i}^{k}.

ABardotABar := ABar_{i j} ABar^{i j} ->
               (K_{i j}-(1/3)g_{i j} trK) (K^{i j}-(1/3)g^{i j} trK).

trg := g_{i j} g^{i j} -> 3.

trK := {K_{i j} g^{i j} -> trK,
        K^{i j} g_{i j} -> trK}.

Ham := trK**2 -> K_{i j} K^{i j} - g^{i j} R_{i j}.

# ---------------------------------------------------------------------------
# dABarij/dt
```

```
dotABarij := \partial_{t}{ABar_{i j}}.          # cdb (eq12.101,dotABarij)

substitute     (dotABarij, ABar2A)              # cdb (eq12.102,dotABarij)
product_rule   (dotABarij)                      # cdb (eq12.103,dotABarij)
map_sympy      (dotABarij, "simplify")          # cdb (eq12.104,dotABarij)
substitute     (dotABarij, DphiDt)              # cdb (eq12.105,dotABarij)
substitute     (dotABarij, Aij)                 # cdb (eq12.106,dotABarij)
distribute     (dotABarij)                      # cdb (eq12.107,dotABarij)
substitute     (dotABarij, DKijDt)              # cdb (eq12.108,dotABarij)
product_rule   (dotABarij)                      # cdb (eq12.109,dotABarij)
distribute     (dotABarij)                      # cdb (eq12.110,dotABarij)
substitute     (dotABarij, DKDt)                # cdb (eq12.111,dotABarij)
substitute     (dotABarij, DgijDt)              # cdb (eq12.112,dotABarij)
distribute     (dotABarij)                      # cdb (eq12.113,dotABarij)
substitute     (dotABarij, Kij)                 # cdb (eq12.114,dotABarij)
distribute     (dotABarij)                      # cdb (eq12.115,dotABarij)
substitute     (dotABarij, gginv)               # cdb (eq12.116,dotABarij)
eliminate_kronecker (dotABarij)                 # cdb (eq12.117,dotABarij)
substitute     (dotABarij, A2ABar)              # cdb (eq12.118,dotABarij)
canonicalise   (dotABarij)                      # cdb (eq12.119,dotABarij)
substitute     (dotABarij, ABardotABar)         # cdb (eq12.120,dotABarij)
distribute     (dotABarij)                      # cdb (eq12.121,dotABarij)
substitute     (dotABarij, trg)                 # cdb (eq12.122,dotABarij)
substitute     (dotABarij, trK)                 # cdb (eq12.123,dotABarij)
map_sympy      (dotABarij, "simplify")          # cdb (eq12.124,dotABarij)
substitute     (dotABarij, Ham)                 # cdb (eq12.125,dotABarij)
distribute     (dotABarij)                      # cdb (eq12.126,dotABarij)
dotABarij = product_sort (dotABarij)            # cdb (eq12.127,dotABarij)
substitute     (dotABarij, ABarUp)              # cdb (eq12.128,dotABarij)
map_sympy      (dotABarij, "simplify")          # cdb (eq12.129,dotABarij)
factor_out     (dotABarij,$\exp(-4\phi)$)       # cdb (eq12.130,dotABarij)

DABarijDt := \partial_{t}{ABar_{ij}} -> @(dotABarij).

cdblib.put ('DABarijDt',DABarijDt,jsonfile)
```

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= \partial_t \left(\exp\left(-4\phi\right) A_{ij} \right) && \texttt{(eq12.102)} \\
&= \partial_t \left(\exp\left(-4\phi\right) \right) A_{ij} + \exp\left(-4\phi\right) \partial_t A_{ij} && \texttt{(eq12.103)} \\
&= -4 \exp\left(-4\phi\right) \partial_t \phi A_{ij} + \exp\left(-4\phi\right) \partial_t A_{ij} && \texttt{(eq12.104)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N A_{ij} + \exp\left(-4\phi\right) \partial_t A_{ij} && \texttt{(eq12.105)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N \left(K_{ij} - \frac{1}{3} g_{ij} \mathrm{tr} K \right) + \exp\left(-4\phi\right) \partial_t \left(K_{ij} - \frac{1}{3} g_{ij} \mathrm{tr} K \right) && \texttt{(eq12.106)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N K_{ij} - \frac{2}{9} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} \mathrm{tr} K + \exp\left(-4\phi\right) \partial_t K_{ij} - \frac{1}{3} \exp\left(-4\phi\right) \partial_t \left(g_{ij} \mathrm{tr} K \right) && \texttt{(eq12.107)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N K_{ij} - \frac{2}{9} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} \mathrm{tr} K + \exp\left(-4\phi\right) \left(-D_{ij} N + N \left(R_{ij} + \mathrm{tr} K K_{ij} - 2 K_{ic} K_{jd} g^{cd} \right) \right) \\
&\quad - \frac{1}{3} \exp\left(-4\phi\right) \partial_t \left(g_{ij} \mathrm{tr} K \right) && \texttt{(eq12.108)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N K_{ij} - \frac{2}{9} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} \mathrm{tr} K + \exp\left(-4\phi\right) \left(-D_{ij} N + N \left(R_{ij} + \mathrm{tr} K K_{ij} - 2 K_{ic} K_{jd} g^{cd} \right) \right) \\
&\quad - \frac{1}{3} \exp\left(-4\phi\right) \left(\partial_t g_{ij} \mathrm{tr} K + g_{ij} \partial_t \mathrm{tr} K \right) && \texttt{(eq12.109)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N K_{ij} - \frac{2}{9} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} \mathrm{tr} K - \exp\left(-4\phi\right) D_{ij} N + \exp\left(-4\phi\right) N R_{ij} + \exp\left(-4\phi\right) N \mathrm{tr} K K_{ij} \\
&\quad - 2 \exp\left(-4\phi\right) N K_{ic} K_{jd} g^{cd} - \frac{1}{3} \exp\left(-4\phi\right) \partial_t g_{ij} \mathrm{tr} K - \frac{1}{3} \exp\left(-4\phi\right) g_{ij} \partial_t \mathrm{tr} K && \texttt{(eq12.110)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N K_{ij} - \frac{2}{9} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} \mathrm{tr} K - \exp\left(-4\phi\right) D_{ij} N + \exp\left(-4\phi\right) N R_{ij} + \exp\left(-4\phi\right) N \mathrm{tr} K K_{ij} \\
&\quad - 2 \exp\left(-4\phi\right) N K_{ic} K_{jd} g^{cd} - \frac{1}{3} \exp\left(-4\phi\right) \partial_t g_{ij} \mathrm{tr} K - \frac{1}{3} \exp\left(-4\phi\right) g_{ij} \left(-g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{3} \mathrm{tr} K^2 N \right) && \texttt{(eq12.111)} \\
&= \frac{2}{3} \exp\left(-4\phi\right) \mathrm{tr} K N K_{ij} - \frac{2}{9} \exp\left(-4\phi\right) \mathrm{tr} K N g_{ij} \mathrm{tr} K - \exp\left(-4\phi\right) D_{ij} N + \exp\left(-4\phi\right) N R_{ij} + \exp\left(-4\phi\right) N \mathrm{tr} K K_{ij} \\
&\quad - 2 \exp\left(-4\phi\right) N K_{ic} K_{jd} g^{cd} + \frac{2}{3} \exp\left(-4\phi\right) N K_{ij} \mathrm{tr} K - \frac{1}{3} \exp\left(-4\phi\right) g_{ij} \left(-g^{ab} D_{ab} N + N \bar{A}_{ab} \bar{A}^{ab} + \frac{1}{3} \mathrm{tr} K^2 N \right) && \texttt{(eq12.112)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}KNK_{ij} - \frac{2}{9}\exp\left(-4\phi\right)\mathrm{tr}KNg_{ij}\mathrm{tr}K - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \exp\left(-4\phi\right)N\mathrm{tr}KK_{ij} - 2\exp\left(-4\phi\right)NK_{ic}K_{jd}g^{cd} \\
&\quad + \frac{2}{3}\exp\left(-4\phi\right)NK_{ij}\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\bar{A}_{ab}\bar{A}^{ab} - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.113)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}KN\left(A_{ij} + \frac{1}{3}g_{ij}\mathrm{tr}K\right) - \frac{2}{9}\exp\left(-4\phi\right)\mathrm{tr}KNg_{ij}\mathrm{tr}K - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} \\
&\quad + \exp\left(-4\phi\right)N\mathrm{tr}K\left(A_{ij} + \frac{1}{3}g_{ij}\mathrm{tr}K\right) - 2\exp\left(-4\phi\right)N\left(A_{ic} + \frac{1}{3}g_{ic}\mathrm{tr}K\right)\left(A_{jd} + \frac{1}{3}g_{jd}\mathrm{tr}K\right)g^{cd} + \frac{2}{3}\exp\left(-4\phi\right)N\left(A_{ij} + \frac{1}{3}g_{ij}\mathrm{tr}K\right)\mathrm{tr}K \\
&\quad + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\bar{A}_{ab}\bar{A}^{ab} - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.114)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}KNA_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \exp\left(-4\phi\right)N\mathrm{tr}KA_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}Kg_{ij}\mathrm{tr}K - 2\exp\left(-4\phi\right)NA_{ic}A_{jd}g^{cd} \\
&\quad - \frac{2}{3}\exp\left(-4\phi\right)NA_{ic}g_{jd}\mathrm{tr}Kg^{cd} - \frac{2}{3}\exp\left(-4\phi\right)Ng_{ic}\mathrm{tr}KA_{jd}g^{cd} - \frac{2}{9}\exp\left(-4\phi\right)Ng_{ic}\mathrm{tr}Kg_{jd}\mathrm{tr}Kg^{cd} + \frac{2}{3}\exp\left(-4\phi\right)NA_{ij}\mathrm{tr}K \\
&\quad + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\bar{A}_{ab}\bar{A}^{ab} - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.115)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}KNA_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \exp\left(-4\phi\right)N\mathrm{tr}KA_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}Kg_{ij}\mathrm{tr}K - 2\exp\left(-4\phi\right)NA_{ic}A_{jd}g^{cd} \\
&\quad - \frac{2}{3}\exp\left(-4\phi\right)NA_{ic}{g_j}^{c}\mathrm{tr}K - \frac{2}{3}\exp\left(-4\phi\right)N{g_i}^{d}\mathrm{tr}KA_{jd} - \frac{2}{9}\exp\left(-4\phi\right)N{g_i}^{d}\mathrm{tr}Kg_{jd}\mathrm{tr}K + \frac{2}{3}\exp\left(-4\phi\right)NA_{ij}\mathrm{tr}K \\
&\quad + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\bar{A}_{ab}\bar{A}^{ab} - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.116)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}KNA_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \exp\left(-4\phi\right)N\mathrm{tr}KA_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}Kg_{ij}\mathrm{tr}K - 2\exp\left(-4\phi\right)NA_{ic}A_{jd}g^{cd} \\
&\quad - \frac{2}{3}\exp\left(-4\phi\right)N\mathrm{tr}KA_{ji} - \frac{2}{9}\exp\left(-4\phi\right)N\mathrm{tr}Kg_{ji}\mathrm{tr}K + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\bar{A}_{ab}\bar{A}^{ab} \\
&\quad - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.117)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}K N\exp\left(4\phi\right)\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \exp\left(-4\phi\right)N\mathrm{tr}K\exp\left(4\phi\right)\bar{A}_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}K g_{ij}\mathrm{tr}K \\
&\quad - 2\exp\left(-4\phi\right)N\exp\left(4\phi\right)\bar{A}_{ic}\exp\left(4\phi\right)\bar{A}_{jd}g^{cd} - \frac{2}{3}\exp\left(-4\phi\right)N\mathrm{tr}K\exp\left(4\phi\right)\bar{A}_{ji} - \frac{2}{9}\exp\left(-4\phi\right)N\mathrm{tr}K g_{ji}\mathrm{tr}K + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K \\
&\quad + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\bar{A}_{ab}\bar{A}^{ab} - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.118)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}K N\exp\left(4\phi\right)\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}K\exp\left(4\phi\right)\bar{A}_{ij} + \frac{1}{9}\exp\left(-4\phi\right)N\mathrm{tr}K g_{ij}\mathrm{tr}K \\
&\quad - 2\exp\left(-4\phi\right)N\exp\left(4\phi\right)\bar{A}_{ic}\exp\left(4\phi\right)\bar{A}_{jd}g^{cd} + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\bar{A}_{ab}\bar{A}^{ab} \\
&\quad - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.119)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}K N\exp\left(4\phi\right)\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}K\exp\left(4\phi\right)\bar{A}_{ij} + \frac{1}{9}\exp\left(-4\phi\right)N\mathrm{tr}K g_{ij}\mathrm{tr}K \\
&\quad - 2\exp\left(-4\phi\right)N\exp\left(4\phi\right)\bar{A}_{ic}\exp\left(4\phi\right)\bar{A}_{jd}g^{cd} + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N \\
&\quad - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}N\left(K_{ab} - \frac{1}{3}g_{ab}\mathrm{tr}K\right)\left(K^{ab} - \frac{1}{3}g^{ab}\mathrm{tr}K\right) - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.120)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}K N\exp\left(4\phi\right)\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}K\exp\left(4\phi\right)\bar{A}_{ij} + \frac{1}{9}\exp\left(-4\phi\right)N\mathrm{tr}K g_{ij}\mathrm{tr}K \\
&\quad - 2\exp\left(-4\phi\right)N\exp\left(4\phi\right)\bar{A}_{ic}\exp\left(4\phi\right)\bar{A}_{jd}g^{cd} + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}NK_{ab}K^{ab} \\
&\quad + \frac{1}{9}\exp\left(-4\phi\right)g_{ij}NK_{ab}g^{ab}\mathrm{tr}K + \frac{1}{9}\exp\left(-4\phi\right)g_{ij}Ng_{ab}\mathrm{tr}KK^{ab} - \frac{1}{27}\exp\left(-4\phi\right)g_{ij}Ng_{ab}\mathrm{tr}Kg^{ab}\mathrm{tr}K - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N \qquad \texttt{(eq12.121)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}K N\exp\left(4\phi\right)\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}K\exp\left(4\phi\right)\bar{A}_{ij} + \frac{1}{9}\exp\left(-4\phi\right)N\mathrm{tr}K g_{ij}\mathrm{tr}K \\
&\quad - 2\exp\left(-4\phi\right)N\exp\left(4\phi\right)\bar{A}_{ic}\exp\left(4\phi\right)\bar{A}_{jd}g^{cd} + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}NK_{ab}K^{ab} \\
&\quad + \frac{1}{9}\exp\left(-4\phi\right)g_{ij}NK_{ab}g^{ab}\mathrm{tr}K + \frac{1}{9}\exp\left(-4\phi\right)g_{ij}Ng_{ab}\mathrm{tr}KK^{ab} - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}N\mathrm{tr}K\mathrm{tr}K - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N && \texttt{(eq12.122)} \\
&= \frac{2}{3}\exp\left(-4\phi\right)\mathrm{tr}K N\exp\left(4\phi\right)\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + \exp\left(-4\phi\right)NR_{ij} + \frac{1}{3}\exp\left(-4\phi\right)N\mathrm{tr}K\exp\left(4\phi\right)\bar{A}_{ij} + \frac{1}{9}\exp\left(-4\phi\right)N\mathrm{tr}K g_{ij}\mathrm{tr}K \\
&\quad - 2\exp\left(-4\phi\right)N\exp\left(4\phi\right)\bar{A}_{ic}\exp\left(4\phi\right)\bar{A}_{jd}g^{cd} + \frac{2}{9}\exp\left(-4\phi\right)Ng_{ij}\mathrm{tr}K\mathrm{tr}K + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}\exp\left(-4\phi\right)g_{ij}NK_{ab}K^{ab} \\
&\quad + \frac{1}{9}\exp\left(-4\phi\right)g_{ij}N\mathrm{tr}K\mathrm{tr}K - \frac{1}{9}\exp\left(-4\phi\right)g_{ij}\mathrm{tr}K^2N && \texttt{(eq12.123)} \\
&= \mathrm{tr}KN\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + N\exp\left(-4\phi\right)R_{ij} + \frac{1}{3}\mathrm{tr}K^2N\exp\left(-4\phi\right)g_{ij} - 2N\exp\left(4\phi\right)\bar{A}_{ic}\bar{A}_{jd}g^{cd} + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N \\
&\quad - \frac{1}{3}N\exp\left(-4\phi\right)g_{ij}K_{ab}K^{ab} && \texttt{(eq12.124)} \\
&= \mathrm{tr}KN\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + N\exp\left(-4\phi\right)R_{ij} + \frac{1}{3}\left(K_{ab}K^{ab} - g^{ab}R_{ab}\right)N\exp\left(-4\phi\right)g_{ij} - 2N\exp\left(4\phi\right)\bar{A}_{ic}\bar{A}_{jd}g^{cd} \\
&\quad + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}N\exp\left(-4\phi\right)g_{ij}K_{ab}K^{ab} && \texttt{(eq12.125)} \\
&= \mathrm{tr}KN\bar{A}_{ij} - \exp\left(-4\phi\right)D_{ij}N + N\exp\left(-4\phi\right)R_{ij} + \frac{1}{3}K_{ab}K^{ab}N\exp\left(-4\phi\right)g_{ij} - \frac{1}{3}g^{ab}R_{ab}N\exp\left(-4\phi\right)g_{ij} - 2N\exp\left(4\phi\right)\bar{A}_{ic}\bar{A}_{jd}g^{cd} \\
&\quad + \frac{1}{3}\exp\left(-4\phi\right)g_{ij}g^{ab}D_{ab}N - \frac{1}{3}N\exp\left(-4\phi\right)g_{ij}K_{ab}K^{ab} && \texttt{(eq12.126)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - D_{ij}N\exp\left(-4\phi\right) + NR_{ij}\exp\left(-4\phi\right) - \frac{1}{3}Ng_{ij}g^{ab}R_{ab}\exp\left(-4\phi\right) - 2N\bar{A}_{ia}\bar{A}_{jb}g^{ab}\exp\left(4\phi\right) + \frac{1}{3}g_{ij}g^{ab}D_{ab}N\exp\left(-4\phi\right) && \texttt{(eq12.127)}
\end{aligned}
$$

$$\partial_t \bar{A}_{ij} = N \mathrm{tr} K \bar{A}_{ij} - D_{ij} N \exp(-4\phi) + N R_{ij} \exp(-4\phi) - \frac{1}{3} N g_{ij} g^{ab} R_{ab} \exp(-4\phi) - 2N \exp(-4\phi) \bar{A}_i{}^b \bar{A}_{jb} \exp(4\phi) + \frac{1}{3} g_{ij} g^{ab} D_{ab} N \exp(-4\phi) \quad (\texttt{eq12.128})$$

$$= \mathrm{tr} K N \bar{A}_{ij} - D_{ij} N \exp(-4\phi) + N \exp(-4\phi) R_{ij} - \frac{1}{3} N \exp(-4\phi) g_{ij} g^{ab} R_{ab} - 2N \bar{A}_i{}^b \bar{A}_{jb} + \frac{1}{3} g_{ij} g^{ab} D_{ab} N \exp(-4\phi) \quad (\texttt{eq12.129})$$

$$= \mathrm{tr} K N \bar{A}_{ij} - 2N \bar{A}_i{}^b \bar{A}_{jb} + \exp(-4\phi) \left(-D_{ij} N + N R_{ij} - \frac{1}{3} N g_{ij} g^{ab} R_{ab} + \frac{1}{3} g_{ij} g^{ab} D_{ab} N \right) \quad (\texttt{eq12.130})$$

```
# ---------------------------------------------------------------------------
# Check against prd62.

foo := @(dotABarij).                                  # cdb(eq12.lcb,foo)
bah  = cdblib.get('prd62.eq12.rhs','prd62.json')     # cdb(eq12.prd,bah)

diff := @(foo) - @(bah).

foo := ABar_{a}^{b} -> gBar^{b c} ABar_{a c}.
bah := ABar^{a}_{b} -> gBar^{a c} ABar_{c b}.

substitute     (diff, foo)
substitute     (diff, bah)
distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                                 # cdb(eq12.chk,diff)
```

$$\texttt{eq12.lcb} := \mathrm{tr}K N\bar{A}_{ij} - 2N\bar{A}_i{}^b\bar{A}_{jb} + \exp\left(-4\phi\right)\left(-D_{ij}N + NR_{ij} - \frac{1}{3}Ng_{ij}g^{ab}R_{ab} + \frac{1}{3}g_{ij}g^{ab}D_{ab}N\right)$$

$$\texttt{eq12.prd} := N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - D_{ij}N - \frac{1}{3}g_{ij}\left(NR_{ab} - D_{ab}N\right)g^{ab}\right)$$

$$\texttt{eq12.chk} := 0$$

PhysRevD.62.044034 equation (15)

```
from shared import *
import cdblib

jsonfile = 'eqtn15.json'
cdblib.create (jsonfile)

defG2GBar = cdblib.get ('defG2GBar','gamma.json')

# --------------------------------------------------------------------------
# Rphi = the part of Rab from the conformal factor

Rab := R_{a b}.                                                    # cdb (eq15.101,Rab)

substitute     (Rab, defRab)                                       # cdb (eq15.102,Rab)
substitute     (Rab, defRiem)                                      # cdb (eq15.103,Rab)
substitute     (Rab, defG2GBar)                                    # cdb (eq15.104,Rab)
distribute     (Rab)                                               # cdb (eq15.105,Rab)
product_rule   (Rab)                                               # cdb (eq15.106,Rab)
Rab = product_sort (Rab)                                           # cdb (eq15.107,Rab)
rename_dummies (Rab)                                               # cdb (eq15.108,Rab)
canonicalise   (Rab)                                               # cdb (eq15.109,Rab)
substitute     (Rab, $gBar_{b c} gBar^{c a} -> gBar^{a}_{b}$)
substitute     (Rab, $\partial_{a}{gBar^{a}_{b}} -> 0$)
substitute     (Rab, $\partial_{a}{gBar_{b}^{c}} -> 0$)
substitute     (Rab, $gBar^{a}_{a} -> 3$)
eliminate_kronecker (Rab)                                          # cdb (eq15.110,Rab)
Rab = product_sort (Rab)                                           # cdb (eq15.111,Rab)
rename_dummies (Rab)                                               # cdb (eq15.112,Rab)
canonicalise   (Rab)                                               # cdb (eq15.113,Rab)
substitute     (Rab, $gBar_{b c} gBar^{c a} -> gBar^{a}_{b}$)      # cdb (eq15.114,Rab)
substitute     (Rab, $gBar^{a}_{a} -> 3$)                          # cdb (eq15.115,Rab)
eliminate_kronecker (Rab)                                          # cdb (eq15.116,Rab)

#
```

```
# isolate Rphi from Rab by switching to local RNC

Rphi := @(Rab).

substitute (Rphi, $GammaBar^{a}_{b c}->0$)                         # cdb (eq15.117,Rphi)
substitute (Rphi, $\partial_{a}{gBar_{b c}}->0$)                   # cdb (eq15.118,Rphi)
substitute (Rphi, $\partial_{a}{gBar^{b c}}->0$)                   # cdb (eq15.119,Rphi)

substitute (Rphi, $\partial_{a b}{\phi} -> DBar_{a b}{\phi}$)      # cdb (eq15.120,Rphi)
substitute (Rphi, $\partial_{a}{\phi} -> DBar_{a}{\phi}$)          # cdb (eq15.121,Rphi)

defRphi := Rphi_{a b} -> @(Rphi).

cdblib.put ('defRphi',defRphi,jsonfile)
```

$$
\begin{aligned}
R_{ab} &= R^{c}{}_{acb} && \text{(eq15.102)}\\
&= \partial_c \Gamma^{c}{}_{ab} + \Gamma^{c}{}_{ec}\Gamma^{e}{}_{ab} - \partial_b \Gamma^{c}{}_{ac} - \Gamma^{c}{}_{eb}\Gamma^{e}{}_{ac} && \text{(eq15.103)}\\
&= \partial_c \left(2\bar{g}^{c}{}_{b}\partial_a\phi + 2\bar{g}^{c}{}_{a}\partial_b\phi - 2\bar{g}^{ce}\partial_e\phi\bar{g}_{ab} + \bar{\Gamma}^{c}{}_{ab}\right) + \left(2\bar{g}^{c}{}_{c}\partial_e\phi + 2\bar{g}^{c}{}_{e}\partial_c\phi - 2\bar{g}^{cd}\partial_d\phi\bar{g}_{ec} + \bar{\Gamma}^{c}{}_{ec}\right)\left(2\bar{g}^{e}{}_{b}\partial_a\phi + 2\bar{g}^{e}{}_{a}\partial_b\phi - 2\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} + \bar{\Gamma}^{e}{}_{ab}\right)\\
&\quad -\partial_b \left(2\bar{g}^{c}{}_{c}\partial_a\phi + 2\bar{g}^{c}{}_{a}\partial_c\phi - 2\bar{g}^{ce}\partial_e\phi\bar{g}_{ac} + \bar{\Gamma}^{c}{}_{ac}\right) - \left(2\bar{g}^{c}{}_{b}\partial_e\phi + 2\bar{g}^{c}{}_{e}\partial_b\phi - 2\bar{g}^{cd}\partial_d\phi\bar{g}_{eb} + \bar{\Gamma}^{c}{}_{eb}\right)\left(2\bar{g}^{e}{}_{c}\partial_a\phi + 2\bar{g}^{e}{}_{a}\partial_c\phi - 2\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} + \bar{\Gamma}^{e}{}_{ac}\right) && \text{(eq15.104)}\\
&= 2\partial_c\left(\bar{g}^{c}{}_{b}\partial_a\phi\right) + 2\partial_c\left(\bar{g}^{c}{}_{a}\partial_b\phi\right) - 2\partial_c\left(\bar{g}^{ce}\partial_e\phi\bar{g}_{ab}\right) + \partial_c\bar{\Gamma}^{c}{}_{ab} + 4\bar{g}^{c}{}_{c}\partial_e\phi\bar{g}^{e}{}_{b}\partial_a\phi + 4\bar{g}^{c}{}_{c}\partial_e\phi\bar{g}^{e}{}_{a}\partial_b\phi - 4\bar{g}^{c}{}_{c}\partial_e\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} + 2\bar{g}^{c}{}_{c}\partial_e\phi\bar{\Gamma}^{e}{}_{ab} + 4\bar{g}^{c}{}_{e}\partial_c\phi\bar{g}^{e}{}_{b}\partial_a\phi\\
&\quad + 4\bar{g}^{c}{}_{e}\partial_c\phi\bar{g}^{e}{}_{a}\partial_b\phi - 4\bar{g}^{c}{}_{e}\partial_c\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} + 2\bar{g}^{c}{}_{e}\partial_c\phi\bar{\Gamma}^{e}{}_{ab} - 4\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{g}^{e}{}_{b}\partial_a\phi - 4\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{g}^{e}{}_{a}\partial_b\phi + 4\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} - 2\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{\Gamma}^{e}{}_{ab}\\
&\quad + 2\bar{\Gamma}^{c}{}_{ec}\bar{g}^{e}{}_{b}\partial_a\phi + 2\bar{\Gamma}^{c}{}_{ec}\bar{g}^{e}{}_{a}\partial_b\phi - 2\bar{\Gamma}^{c}{}_{ec}\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} + \bar{\Gamma}^{c}{}_{ec}\bar{\Gamma}^{e}{}_{ab} - 2\partial_b\left(\bar{g}^{c}{}_{c}\partial_a\phi\right) - 2\partial_b\left(\bar{g}^{c}{}_{a}\partial_c\phi\right) + 2\partial_b\left(\bar{g}^{ce}\partial_e\phi\bar{g}_{ac}\right) - \partial_b\bar{\Gamma}^{c}{}_{ac} - 4\bar{g}^{c}{}_{b}\partial_e\phi\bar{g}^{e}{}_{c}\partial_a\phi\\
&\quad - 4\bar{g}^{c}{}_{b}\partial_e\phi\bar{g}^{e}{}_{a}\partial_c\phi + 4\bar{g}^{c}{}_{b}\partial_e\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} - 2\bar{g}^{c}{}_{b}\partial_e\phi\bar{\Gamma}^{e}{}_{ac} - 4\bar{g}^{c}{}_{e}\partial_b\phi\bar{g}^{e}{}_{c}\partial_a\phi - 4\bar{g}^{c}{}_{e}\partial_b\phi\bar{g}^{e}{}_{a}\partial_c\phi + 4\bar{g}^{c}{}_{e}\partial_b\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} - 2\bar{g}^{c}{}_{e}\partial_b\phi\bar{\Gamma}^{e}{}_{ac} + 4\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{g}^{e}{}_{c}\partial_a\phi\\
&\quad + 4\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{g}^{e}{}_{a}\partial_c\phi - 4\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} + 2\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{\Gamma}^{e}{}_{ac} - 2\bar{\Gamma}^{c}{}_{eb}\bar{g}^{e}{}_{c}\partial_a\phi - 2\bar{\Gamma}^{c}{}_{eb}\bar{g}^{e}{}_{a}\partial_c\phi + 2\bar{\Gamma}^{c}{}_{eb}\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} - \bar{\Gamma}^{c}{}_{eb}\bar{\Gamma}^{e}{}_{ac} && \text{(eq15.105)}\\
&= 2\partial_c\bar{g}^{c}{}_{b}\partial_a\phi + 2\bar{g}^{c}{}_{b}\partial_{ca}\phi + 2\partial_c\bar{g}^{c}{}_{a}\partial_b\phi + 2\bar{g}^{c}{}_{a}\partial_{cb}\phi - 2\partial_c\bar{g}^{ce}\partial_e\phi\bar{g}_{ab} - 2\bar{g}^{ce}\partial_{ce}\phi\bar{g}_{ab} - 2\bar{g}^{ce}\partial_e\phi\partial_c\bar{g}_{ab} + \partial_c\bar{\Gamma}^{c}{}_{ab} + 4\bar{g}^{c}{}_{c}\partial_e\phi\bar{g}^{e}{}_{b}\partial_a\phi + 4\bar{g}^{c}{}_{c}\partial_e\phi\bar{g}^{e}{}_{a}\partial_b\phi\\
&\quad - 4\bar{g}^{c}{}_{c}\partial_e\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} + 2\bar{g}^{c}{}_{c}\partial_e\phi\bar{\Gamma}^{e}{}_{ab} + 4\bar{g}^{c}{}_{e}\partial_c\phi\bar{g}^{e}{}_{b}\partial_a\phi + 4\bar{g}^{c}{}_{e}\partial_c\phi\bar{g}^{e}{}_{a}\partial_b\phi - 4\bar{g}^{c}{}_{e}\partial_c\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} + 2\bar{g}^{c}{}_{e}\partial_c\phi\bar{\Gamma}^{e}{}_{ab} - 4\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{g}^{e}{}_{b}\partial_a\phi\\
&\quad - 4\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{g}^{e}{}_{a}\partial_b\phi + 4\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} - 2\bar{g}^{cd}\partial_d\phi\bar{g}_{ec}\bar{\Gamma}^{e}{}_{ab} + 2\bar{\Gamma}^{c}{}_{ec}\bar{g}^{e}{}_{b}\partial_a\phi + 2\bar{\Gamma}^{c}{}_{ec}\bar{g}^{e}{}_{a}\partial_b\phi - 2\bar{\Gamma}^{c}{}_{ec}\bar{g}^{ef}\partial_f\phi\bar{g}_{ab} + \bar{\Gamma}^{c}{}_{ec}\bar{\Gamma}^{e}{}_{ab} - 2\partial_b\bar{g}^{c}{}_{c}\partial_a\phi\\
&\quad - 2\bar{g}^{c}{}_{c}\partial_{ba}\phi - 2\partial_b\bar{g}^{c}{}_{a}\partial_c\phi - 2\bar{g}^{c}{}_{a}\partial_{bc}\phi + 2\partial_b\bar{g}^{ce}\partial_e\phi\bar{g}_{ac} + 2\bar{g}^{ce}\partial_{be}\phi\bar{g}_{ac} + 2\bar{g}^{ce}\partial_e\phi\partial_b\bar{g}_{ac} - \partial_b\bar{\Gamma}^{c}{}_{ac} - 4\bar{g}^{c}{}_{b}\partial_e\phi\bar{g}^{e}{}_{c}\partial_a\phi - 4\bar{g}^{c}{}_{b}\partial_e\phi\bar{g}^{e}{}_{a}\partial_c\phi\\
&\quad + 4\bar{g}^{c}{}_{b}\partial_e\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} - 2\bar{g}^{c}{}_{b}\partial_e\phi\bar{\Gamma}^{e}{}_{ac} - 4\bar{g}^{c}{}_{e}\partial_b\phi\bar{g}^{e}{}_{c}\partial_a\phi - 4\bar{g}^{c}{}_{e}\partial_b\phi\bar{g}^{e}{}_{a}\partial_c\phi + 4\bar{g}^{c}{}_{e}\partial_b\phi\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} - 2\bar{g}^{c}{}_{e}\partial_b\phi\bar{\Gamma}^{e}{}_{ac} + 4\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{g}^{e}{}_{c}\partial_a\phi\\
&\quad + 4\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{g}^{e}{}_{a}\partial_c\phi - 4\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} + 2\bar{g}^{cd}\partial_d\phi\bar{g}_{eb}\bar{\Gamma}^{e}{}_{ac} - 2\bar{\Gamma}^{c}{}_{eb}\bar{g}^{e}{}_{c}\partial_a\phi - 2\bar{\Gamma}^{c}{}_{eb}\bar{g}^{e}{}_{a}\partial_c\phi + 2\bar{\Gamma}^{c}{}_{eb}\bar{g}^{ef}\partial_f\phi\bar{g}_{ac} - \bar{\Gamma}^{c}{}_{eb}\bar{\Gamma}^{e}{}_{ac} && \text{(eq15.106)}\\
&= 2\partial_a\phi\partial_c\bar{g}^{c}{}_{b} + 2\partial_{ca}\phi\bar{g}^{c}{}_{b} + 2\partial_b\phi\partial_c\bar{g}^{c}{}_{a} + 2\partial_{cb}\phi\bar{g}^{c}{}_{a} - 2\bar{g}_{ab}\partial_d\phi\partial_c\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_d\phi\partial_c\bar{g}_{ab} + \partial_c\bar{\Gamma}^{c}{}_{ab} + 4\partial_a\phi\partial_c\phi\bar{g}^{c}{}_{b}\bar{g}^{d}{}_{d} + 4\partial_b\phi\partial_c\phi\bar{g}^{c}{}_{a}\bar{g}^{d}{}_{d}\\
&\quad - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^{e}{}_{e} + 2\bar{\Gamma}^{c}{}_{ab}\partial_c\phi\bar{g}^{d}{}_{d} + 4\partial_a\phi\partial_c\phi\bar{g}^{c}{}_{d}\bar{g}^{d}{}_{b} - 4\bar{g}_{ab}\bar{g}^{cd}\partial_e\phi\partial_d\phi\bar{g}^{e}{}_{c} + 2\bar{\Gamma}^{c}{}_{ab}\partial_d\phi\bar{g}^{d}{}_{c} - 4\bar{g}_{cd}\bar{g}^{de}\partial_a\phi\partial_e\phi\bar{g}^{c}{}_{b} - 4\bar{g}_{cd}\bar{g}^{de}\partial_b\phi\partial_e\phi\bar{g}^{c}{}_{a}\\
&\quad + 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{de}\bar{g}^{cf}\partial_e\phi\partial_f\phi - 2\bar{g}_{cd}\bar{g}^{de}\bar{\Gamma}^{c}{}_{ab}\partial_e\phi + 2\bar{\Gamma}^{c}{}_{dc}\partial_a\phi\bar{g}^{d}{}_{b} + 2\bar{\Gamma}^{c}{}_{dc}\partial_b\phi\bar{g}^{d}{}_{a} - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^{e}{}_{ce}\partial_d\phi + \bar{\Gamma}^{c}{}_{ab}\bar{\Gamma}^{d}{}_{cd} - 2\partial_a\phi\partial_b\bar{g}^{c}{}_{c} - 2\partial_{ba}\phi\bar{g}^{c}{}_{c} - 2\partial_c\phi\partial_b\bar{g}^{c}{}_{a}\\
&\quad - 2\partial_{bc}\phi\bar{g}^{c}{}_{a} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}_{ac}\bar{g}^{cd}\partial_{bd}\phi + 2\bar{g}^{cd}\partial_d\phi\partial_b\bar{g}_{ac} - \partial_b\bar{\Gamma}^{c}{}_{ac} - 4\partial_a\phi\partial_c\phi\bar{g}^{d}{}_{b}\bar{g}^{c}{}_{d} - 4\partial_c\phi\partial_d\phi\bar{g}^{c}{}_{b}\bar{g}^{d}{}_{a} + 4\bar{g}_{ac}\bar{g}^{de}\partial_d\phi\partial_e\phi\bar{g}^{c}{}_{b} - 2\bar{\Gamma}^{c}{}_{ad}\partial_c\phi\bar{g}^{d}{}_{b}\\
&\quad - 4\partial_a\phi\partial_b\phi\bar{g}^{c}{}_{d}\bar{g}^{d}{}_{c} + 4\bar{g}_{ac}\bar{g}^{de}\partial_b\phi\partial_e\phi\bar{g}^{c}{}_{d} - 2\bar{\Gamma}^{c}{}_{ad}\partial_b\phi\bar{g}^{d}{}_{c} + 4\bar{g}_{cb}\bar{g}^{de}\partial_a\phi\partial_e\phi\bar{g}^{c}{}_{d} + 4\bar{g}_{cb}\bar{g}^{de}\partial_d\phi\partial_e\phi\bar{g}^{c}{}_{a} - 4\bar{g}_{ac}\bar{g}_{db}\bar{g}^{ce}\bar{g}^{df}\partial_e\phi\partial_f\phi + 2\bar{g}_{cb}\bar{g}^{de}\bar{\Gamma}^{c}{}_{ad}\partial_e\phi\\
&\quad - 2\bar{\Gamma}^{c}{}_{db}\partial_a\phi\bar{g}^{d}{}_{c} - 2\bar{\Gamma}^{c}{}_{db}\partial_c\phi\bar{g}^{d}{}_{a} + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^{c}{}_{db}\partial_e\phi - \bar{\Gamma}^{c}{}_{db}\bar{\Gamma}^{d}{}_{ac} && \text{(eq15.107)}
\end{aligned}
$$

$$
\begin{aligned}
R_{ab} = {}& 2\partial_a\phi\partial_c\bar{g}^{c}{}_{b} + 2\partial_{ca}\phi\bar{g}^{c}{}_{b} + 2\partial_b\phi\partial_c\bar{g}^{c}{}_{a} + 2\partial_{cb}\phi\bar{g}^{c}{}_{a} - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{dc} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{dc}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^{c}{}_{ab} + 4\partial_a\phi\partial_c\phi\bar{g}^{c}{}_{b}\bar{g}^{d}{}_{d} + 4\partial_b\phi\partial_c\phi\bar{g}^{c}{}_{a}\bar{g}^{d}{}_{d} \\
& - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^{e}{}_{e} + 2\bar{\Gamma}^{c}{}_{ab}\partial_c\phi\bar{g}^{d}{}_{d} + 4\partial_a\phi\partial_c\phi\bar{g}^{c}{}_{d}\bar{g}^{d}{}_{b} - 4\bar{g}_{ab}\bar{g}^{ed}\partial_c\phi\partial_d\phi\bar{g}^{c}{}_{e} + 2\bar{\Gamma}^{c}{}_{ab}\partial_d\phi\bar{g}^{d}{}_{c} - 4\bar{g}_{de}\bar{g}^{ec}\partial_a\phi\partial_c\phi\bar{g}^{d}{}_{b} - 4\bar{g}_{de}\bar{g}^{ec}\partial_b\phi\partial_c\phi\bar{g}^{d}{}_{a} \\
& + 4\bar{g}_{ab}\bar{g}_{ef}\bar{g}^{fc}\bar{g}^{ed}\partial_c\phi\partial_d\phi - 2\bar{g}_{ce}\bar{g}^{ed}\bar{\Gamma}^{c}{}_{ab}\partial_d\phi + 2\bar{\Gamma}^{c}{}_{dc}\partial_a\phi\bar{g}^{d}{}_{b} + 2\bar{\Gamma}^{c}{}_{dc}\partial_b\phi\bar{g}^{d}{}_{a} - 2\bar{g}_{ab}\bar{g}^{de}\bar{\Gamma}^{c}{}_{dc}\partial_e\phi + \bar{\Gamma}^{c}{}_{ab}\bar{\Gamma}^{d}{}_{cd} - 2\partial_a\phi\partial_b\bar{g}^{c}{}_{c} - 2\partial_{ba}\phi\bar{g}^{c}{}_{c} - 2\partial_c\phi\partial_b\bar{g}^{c}{}_{a} \\
& - 2\partial_{bc}\phi\bar{g}^{c}{}_{a} + 2\bar{g}_{ad}\partial_c\phi\partial_b\bar{g}^{dc} + 2\bar{g}_{ad}\bar{g}^{dc}\partial_{bc}\phi + 2\bar{g}^{dc}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^{c}{}_{ac} - 4\partial_a\phi\partial_c\phi\bar{g}^{d}{}_{b}\bar{g}^{c}{}_{d} - 4\partial_c\phi\partial_d\phi\bar{g}^{c}{}_{b}\bar{g}^{d}{}_{a} + 4\bar{g}_{ae}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^{e}{}_{b} - 2\bar{\Gamma}^{c}{}_{ad}\partial_c\phi\bar{g}^{d}{}_{b} \\
& - 4\partial_a\phi\partial_b\phi\bar{g}^{c}{}_{d}\bar{g}^{d}{}_{c} + 4\bar{g}_{ad}\bar{g}^{ec}\partial_b\phi\partial_c\phi\bar{g}^{d}{}_{e} - 2\bar{\Gamma}^{c}{}_{ad}\partial_b\phi\bar{g}^{d}{}_{c} + 4\bar{g}_{db}\bar{g}^{ec}\partial_a\phi\partial_c\phi\bar{g}^{d}{}_{e} + 4\bar{g}_{eb}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^{e}{}_{a} - 4\bar{g}_{ae}\bar{g}_{fb}\bar{g}^{ec}\bar{g}^{fd}\partial_c\phi\partial_d\phi + 2\bar{g}_{cb}\bar{g}^{de}\bar{\Gamma}^{c}{}_{ad}\partial_e\phi \\
& - 2\bar{\Gamma}^{c}{}_{db}\partial_a\phi\bar{g}^{d}{}_{c} - 2\bar{\Gamma}^{c}{}_{db}\partial_c\phi\bar{g}^{d}{}_{a} + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^{c}{}_{db}\partial_e\phi - \bar{\Gamma}^{c}{}_{db}\bar{\Gamma}^{d}{}_{ac}
\end{aligned}
\tag{eq15.108}
$$

$$
\begin{aligned}
= {}& 2\partial_a\phi\partial_c\bar{g}_{b}{}^{c} + 2\partial_{ac}\phi\bar{g}_{b}{}^{c} + 2\partial_b\phi\partial_c\bar{g}_{a}{}^{c} - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^{c}{}_{ab} + 4\partial_a\phi\partial_c\phi\bar{g}_{b}{}^{c}\bar{g}^{d}{}_{d} + 4\partial_b\phi\partial_c\phi\bar{g}_{a}{}^{c}\bar{g}^{d}{}_{d} - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi\bar{g}^{e}{}_{e} \\
& + 2\bar{\Gamma}^{c}{}_{ab}\partial_c\phi\bar{g}^{d}{}_{d} - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_e\phi\bar{g}_{d}{}^{e} + 2\bar{\Gamma}^{c}{}_{ab}\partial_d\phi\bar{g}_{c}{}^{d} - 4\bar{g}_{cd}\bar{g}^{ce}\partial_a\phi\partial_e\phi\bar{g}_{b}{}^{d} - 4\bar{g}_{cd}\bar{g}^{ce}\partial_b\phi\partial_e\phi\bar{g}_{a}{}^{d} + 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ce}\bar{g}^{df}\partial_e\phi\partial_f\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^{d}{}_{ab}\partial_e\phi \\
& + 2\bar{\Gamma}^{c}{}_{cd}\partial_a\phi\bar{g}_{b}{}^{d} + 2\bar{\Gamma}^{c}{}_{cd}\partial_b\phi\bar{g}_{a}{}^{d} - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^{e}{}_{ce}\partial_d\phi + \bar{\Gamma}^{c}{}_{ab}\bar{\Gamma}^{d}{}_{cd} - 2\partial_a\phi\partial_b\bar{g}^{c}{}_{c} - 2\partial_{ab}\phi\bar{g}^{c}{}_{c} - 2\partial_c\phi\partial_b\bar{g}_{a}{}^{c} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}_{ac}\bar{g}^{cd}\partial_{bd}\phi + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} \\
& - \partial_b\bar{\Gamma}^{c}{}_{ac} - 4\partial_c\phi\partial_d\phi\bar{g}_{a}{}^{c}\bar{g}_{b}{}^{d} + 4\bar{g}_{ac}\bar{g}^{de}\partial_d\phi\partial_e\phi\bar{g}_{b}{}^{c} - 2\bar{\Gamma}^{c}{}_{ad}\partial_c\phi\bar{g}_{b}{}^{d} - 4\partial_a\phi\partial_b\phi\bar{g}^{c}{}_{d}\bar{g}_{c}{}^{d} + 4\bar{g}_{ac}\bar{g}^{de}\partial_b\phi\partial_d\phi\bar{g}^{c}{}_{e} - 2\bar{\Gamma}^{c}{}_{ad}\partial_b\phi\bar{g}_{c}{}^{d} + 4\bar{g}_{bc}\bar{g}^{de}\partial_a\phi\partial_d\phi\bar{g}^{c}{}_{e} \\
& + 4\bar{g}_{bc}\bar{g}^{de}\partial_d\phi\partial_e\phi\bar{g}_{a}{}^{c} - 4\bar{g}_{ac}\bar{g}_{bd}\bar{g}^{ce}\bar{g}^{df}\partial_e\phi\partial_f\phi + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^{c}{}_{ad}\partial_e\phi - 2\bar{\Gamma}^{c}{}_{bd}\partial_a\phi\bar{g}_{c}{}^{d} - 2\bar{\Gamma}^{c}{}_{bd}\partial_c\phi\bar{g}_{a}{}^{d} + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^{c}{}_{bd}\partial_e\phi - \bar{\Gamma}^{c}{}_{ad}\bar{\Gamma}^{d}{}_{bc}
\end{aligned}
\tag{eq15.109}
$$

$$
\begin{aligned}
= {}& -4\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^{c}{}_{ab} + 8\partial_a\phi\partial_b\phi + 12\partial_b\phi\partial_a\phi - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^{c}{}_{ab}\partial_c\phi - 4\bar{g}_{ab}\bar{g}^{ce}\partial_c\phi\partial_e\phi \\
& + 2\bar{\Gamma}^{d}{}_{ab}\partial_d\phi - 4\bar{g}_{cb}\bar{g}^{ce}\partial_a\phi\partial_e\phi - 4\bar{g}_{ca}\bar{g}^{ce}\partial_b\phi\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{fe}\partial_e\phi\partial_f\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^{d}{}_{ab}\partial_e\phi + 2\bar{\Gamma}^{c}{}_{cb}\partial_a\phi + 2\bar{\Gamma}^{c}{}_{ca}\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^{e}{}_{ce}\partial_d\phi + \bar{\Gamma}^{c}{}_{ab}\bar{\Gamma}^{d}{}_{cd} \\
& + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\partial_{ba}\phi + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^{c}{}_{ac} + 4\bar{g}_{ab}\bar{g}^{de}\partial_d\phi\partial_e\phi - 4\partial_a\phi\partial_b\phi\bar{g}_{d}{}^{d} + 4\bar{g}_{ae}\bar{g}^{de}\partial_b\phi\partial_d\phi - 2\bar{\Gamma}^{d}{}_{ad}\partial_b\phi + 4\bar{g}_{be}\bar{g}^{de}\partial_a\phi\partial_d\phi \\
& + 4\bar{g}_{ba}\bar{g}^{de}\partial_d\phi\partial_e\phi - 4\bar{g}_{bd}\bar{g}^{df}\partial_a\phi\partial_f\phi + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^{c}{}_{ad}\partial_e\phi - 2\bar{\Gamma}^{d}{}_{bd}\partial_a\phi - 2\bar{\Gamma}^{c}{}_{ba}\partial_c\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^{c}{}_{bd}\partial_e\phi - \bar{\Gamma}^{c}{}_{ad}\bar{\Gamma}^{d}{}_{bc}
\end{aligned}
\tag{eq15.110}
$$

$$
\begin{aligned}
= {}& -4\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^{c}{}_{ab} + 20\partial_a\phi\partial_b\phi - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 6\bar{\Gamma}^{c}{}_{ab}\partial_c\phi - 4\bar{g}_{cb}\bar{g}^{cd}\partial_a\phi\partial_d\phi - 4\bar{g}_{ca}\bar{g}^{cd}\partial_b\phi\partial_d\phi \\
& + 4\bar{g}_{ab}\bar{g}^{cd}\partial_d\phi\partial_c\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^{d}{}_{ab}\partial_e\phi + 2\bar{\Gamma}^{c}{}_{cb}\partial_a\phi + 2\bar{\Gamma}^{c}{}_{ca}\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^{e}{}_{ce}\partial_d\phi + \bar{\Gamma}^{c}{}_{ab}\bar{\Gamma}^{d}{}_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\partial_{ba}\phi + 2\bar{g}^{dc}\partial_d\phi\partial_b\bar{g}_{ac} - \partial_b\bar{\Gamma}^{c}{}_{ac} \\
& - 4\partial_a\phi\partial_b\phi\bar{g}_{c}{}^{c} + 4\bar{g}_{ac}\bar{g}^{dc}\partial_b\phi\partial_d\phi - 2\bar{\Gamma}^{c}{}_{ac}\partial_b\phi + 4\bar{g}_{bc}\bar{g}^{dc}\partial_a\phi\partial_d\phi + 4\bar{g}_{ba}\bar{g}^{cd}\partial_c\phi\partial_d\phi - 4\bar{g}_{bc}\bar{g}^{cd}\partial_a\phi\partial_d\phi + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^{c}{}_{ad}\partial_e\phi - 2\bar{\Gamma}^{c}{}_{bc}\partial_a\phi - 2\bar{\Gamma}^{c}{}_{ba}\partial_c\phi \\
& + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^{c}{}_{bd}\partial_e\phi - \bar{\Gamma}^{c}{}_{ad}\bar{\Gamma}^{d}{}_{bc}
\end{aligned}
\tag{eq15.111}
$$

$$
\begin{aligned}
= {}& -4\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^{c}{}_{ab} + 20\partial_a\phi\partial_b\phi - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 6\bar{\Gamma}^{c}{}_{ab}\partial_c\phi - 4\bar{g}_{db}\bar{g}^{dc}\partial_a\phi\partial_c\phi - 4\bar{g}_{da}\bar{g}^{dc}\partial_b\phi\partial_c\phi \\
& + 4\bar{g}_{ab}\bar{g}^{dc}\partial_c\phi\partial_d\phi - 2\bar{g}_{ec}\bar{g}^{ed}\bar{\Gamma}^{c}{}_{ab}\partial_d\phi + 2\bar{\Gamma}^{c}{}_{cb}\partial_a\phi + 2\bar{\Gamma}^{c}{}_{ca}\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{de}\bar{\Gamma}^{c}{}_{dc}\partial_e\phi + \bar{\Gamma}^{c}{}_{ab}\bar{\Gamma}^{d}{}_{cd} + 2\bar{g}_{ad}\partial_c\phi\partial_b\bar{g}^{dc} + 2\partial_{ba}\phi + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^{c}{}_{ac} \\
& - 4\partial_a\phi\partial_b\phi\bar{g}_{c}{}^{c} + 4\bar{g}_{ad}\bar{g}^{cd}\partial_b\phi\partial_c\phi - 2\bar{\Gamma}^{c}{}_{ac}\partial_b\phi + 4\bar{g}_{bd}\bar{g}^{cd}\partial_a\phi\partial_c\phi + 4\bar{g}_{ba}\bar{g}^{cd}\partial_c\phi\partial_d\phi - 4\bar{g}_{bd}\bar{g}^{dc}\partial_a\phi\partial_c\phi + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^{c}{}_{ad}\partial_e\phi - 2\bar{\Gamma}^{c}{}_{bc}\partial_a\phi - 2\bar{\Gamma}^{c}{}_{ba}\partial_c\phi \\
& + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^{c}{}_{bd}\partial_e\phi - \bar{\Gamma}^{c}{}_{ad}\bar{\Gamma}^{d}{}_{bc}
\end{aligned}
\tag{eq15.112}
$$

$$\begin{aligned}
R_{ab} &= -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 20\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c{}_{ab}\partial_c\phi - 4\bar{g}_{bc}\bar{g}^{cd}\partial_a\phi\partial_d\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d{}_{ab}\partial_e\phi\\
&\quad - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{ce}\partial_d\phi + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c{}_{ac} - 4\partial_a\phi\partial_b\phi\bar{g}^c{}_c + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{bd}\partial_e\phi - \bar{\Gamma}^c{}_{ad}\bar{\Gamma}^d{}_{bc} && \texttt{(eq15.113)}\\
&= -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 20\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c{}_{ab}\partial_c\phi - 4\bar{g}^d{}_b\partial_a\phi\partial_d\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d{}_{ab}\partial_e\phi\\
&\quad - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{ce}\partial_d\phi + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c{}_{ac} - 4\partial_a\phi\partial_b\phi\bar{g}^c{}_c + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{bd}\partial_e\phi - \bar{\Gamma}^c{}_{ad}\bar{\Gamma}^d{}_{bc} && \texttt{(eq15.114)}\\
&= -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 8\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c{}_{ab}\partial_c\phi - 4\bar{g}^d{}_b\partial_a\phi\partial_d\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d{}_{ab}\partial_e\phi\\
&\quad - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{ce}\partial_d\phi + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c{}_{ac} + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{bd}\partial_e\phi - \bar{\Gamma}^c{}_{ad}\bar{\Gamma}^d{}_{bc} && \texttt{(eq15.115)}\\
&= -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + \partial_c\bar{\Gamma}^c{}_{ab} + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 4\bar{\Gamma}^c{}_{ab}\partial_c\phi - 2\bar{g}_{cd}\bar{g}^{ce}\bar{\Gamma}^d{}_{ab}\partial_e\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{ce}\partial_d\phi\\
&\quad + \bar{\Gamma}^c{}_{ab}\bar{\Gamma}^d{}_{cd} + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} - \partial_b\bar{\Gamma}^c{}_{ac} + 2\bar{g}_{bc}\bar{g}^{de}\bar{\Gamma}^c{}_{ad}\partial_e\phi + 2\bar{g}_{ac}\bar{g}^{de}\bar{\Gamma}^c{}_{bd}\partial_e\phi - \bar{\Gamma}^c{}_{ad}\bar{\Gamma}^d{}_{bc} && \texttt{(eq15.116)}
\end{aligned}$$

The above doesn't look much like equation (15). So, what do we do? First note that `(eq15.116)` represents the full R_{ab}, that is, equation (14). To isolate the contributions from ϕ we can first set $\bar{\Gamma}$ and its derivatives to zero (which in turn requires setting $\partial_a\bar{g}_{bc} = 0$). The result is equation `(eq15.119)` below. Having set $\bar{\Gamma}$ to zero means that we can replace ∂ with $\bar{D}$ leading to equation `(eq15.121)`. But that is clearly a tensor equation and so by the usual arguments it must be true in all frames (not just this frame with $\bar{\Gamma} = 0$). It's a standard argument and I've probably overdone the discussion. Anyway, equation `(eq15.121)` is exactly equation (15) from the paper. Yeah.

$$\begin{aligned}
R^\phi_{ab} &= -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}^{cd}\partial_c\phi\partial_d\bar{g}_{ab} + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} + 2\bar{g}^{cd}\partial_c\phi\partial_b\bar{g}_{ad} && \texttt{(eq15.117)}\\
&= -2\partial_{ab}\phi - 2\bar{g}_{ab}\partial_c\phi\partial_d\bar{g}^{cd} - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi + 2\bar{g}_{ac}\partial_d\phi\partial_b\bar{g}^{cd} && \texttt{(eq15.118)}\\
&= -2\partial_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi && \texttt{(eq15.119)}\\
&= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi && \texttt{(eq15.120)}\\
&= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi && \texttt{(eq15.121)}
\end{aligned}$$

```
# ------------------------------------------------------------------------
# Check against prd62.

foo := @(Rphi).                                         # cdb(eq15.lcb,foo)
bah  = cdblib.get('prd62.eq15.rhs','prd62.json')        # cdb(eq15.prd,bah)

diff := @(foo) - @(bah).

distribute      (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy       (diff, "simplify")
canonicalise    (diff)                                  # cdb(eq15.chk,diff)
```

$$\texttt{eq15.lcb} := -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_{a}\phi\bar{D}_{b}\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{c}\phi\bar{D}_{d}\phi$$

$$\texttt{eq15.prd} := -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_{a}\phi\bar{D}_{b}\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{c}\phi\bar{D}_{d}\phi$$

$$\texttt{eq15.chk} := 0$$

PhysRevD.62.044034 equation (17)

```
from shared import *
import cdblib

jsonfile = 'eqtn17.json'
cdblib.create (jsonfile)

# -------------------------------------------------------------------------
defGammaBar := GammaBar^{a}_{b c} ->
               (1/2) gBar^{a e} (   \partial_{b}{gBar_{e c}}
                                  + \partial_{c}{gBar_{b e}}
                                  - \partial_{e}{gBar_{b c}}).

foo := \partial_{a}{gBar_{b c}} gBar^{i b} gBar^{j c} -> - \partial_{a}{gBar^{i j}}.
bah := \partial_{a}{gBar_{b c}} gBar^{b c} -> 0.   # follows from det gBar = 1

# -------------------------------------------------------------------------
# GiBar

GiBar := gBar^{j k} GammaBar^{i}_{j k}.            # cdb (eq17.101,GiBar)

substitute     (GiBar, defGammaBar)                # cdb (eq17.102,GiBar)
distribute     (GiBar)                             # cdb (eq17.103,GiBar)
GiBar = product_sort (GiBar)                       # cdb (eq17.104,GiBar)
rename_dummies (GiBar)                             # cdb (eq17.105,GiBar)
canonicalise   (GiBar)                             # cdb (eq17.106,GiBar)
substitute     (GiBar, foo)                        # cdb (eq17.107,GiBar)
substitute     (GiBar, bah)                        # cdb (eq17.108,GiBar)

defGiBar := GammaBar^{i} -> @(GiBar).

cdblib.put ('defGiBar',defGiBar,jsonfile)
```

$$
\begin{aligned}
\bar{g}^{jk}\bar{\Gamma}^{i}{}_{jk} &= \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}\left(\partial_j \bar{g}_{ek} + \partial_k \bar{g}_{je} - \partial_e \bar{g}_{jk}\right) && \texttt{(eq17.102)} \\
&= \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}\partial_j \bar{g}_{ek} + \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}\partial_k \bar{g}_{je} - \frac{1}{2}\bar{g}^{jk}\bar{g}^{ie}\partial_e \bar{g}_{jk} && \texttt{(eq17.103)} \\
&= \frac{1}{2}\bar{g}^{ia}\bar{g}^{cb}\partial_c \bar{g}_{ab} + \frac{1}{2}\bar{g}^{ib}\bar{g}^{ac}\partial_c \bar{g}_{ab} - \frac{1}{2}\bar{g}^{ic}\bar{g}^{ab}\partial_c \bar{g}_{ab} && \texttt{(eq17.104)} \\
&= \frac{1}{2}\bar{g}^{ib}\bar{g}^{ac}\partial_a \bar{g}_{bc} + \frac{1}{2}\bar{g}^{ib}\bar{g}^{ca}\partial_a \bar{g}_{cb} - \frac{1}{2}\bar{g}^{ia}\bar{g}^{bc}\partial_a \bar{g}_{bc} && \texttt{(eq17.105)} \\
&= \bar{g}^{ia}\bar{g}^{bc}\partial_b \bar{g}_{ac} - \frac{1}{2}\bar{g}^{ia}\bar{g}^{bc}\partial_a \bar{g}_{bc} && \texttt{(eq17.106)} \\
&= -\partial_b \bar{g}^{ib} - \frac{1}{2}\bar{g}^{ia}\bar{g}^{bc}\partial_a \bar{g}_{bc} && \texttt{(eq17.107)} \\
&= -\partial_b \bar{g}^{ib} && \texttt{(eq17.108)}
\end{aligned}
$$

```
# ----------------------------------------------------------------------------
# Check against prd62.

foo := @(GiBar).                                   # cdb(eq17.lcb,foo)
bah  = cdblib.get('prd62.eq17.rhs','prd62.json')   # cdb(eq17.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                              # cdb(eq17.chk,diff)
```

$$\texttt{eq17.lcb} := -\partial_b \bar{g}^{ib}$$

$$\texttt{eq17.prd} := -\partial_j \bar{g}^{ij}$$

$$\texttt{eq17.chk} := 0$$

PhysRevD.62.044034 equation (18)

```
from shared import *
import cdblib

jsonfile = 'eqtn18.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------
# RBar pt.1 = split into two terms

defGammaBar := GammaBar^{a}_{b c} ->
               (1/2) gBar^{a e} (   \partial_{b}{gBar_{e c}}
                                  + \partial_{c}{gBar_{b e}}
                                  - \partial_{e}{gBar_{b c}}).

defRiemBar := RBar^{a}_{b c d} ->
              \partial_{c}{GammaBar^{a}_{b d}} + GammaBar^{a}_{e c} GammaBar^{e}_{b d}
            - \partial_{d}{GammaBar^{a}_{b c}} - GammaBar^{a}_{e d} GammaBar^{e}_{b c}.

defRBar := RBar_{a b} -> RBar^{c}_{a c b}.

RBar := RBar_{a b}.                                                   # cdb(eq18.000,RBar)

substitute   (RBar, defRBar)                                          # cdb(eq18.001,RBar)
substitute   (RBar, defRiemBar)                                       # cdb(eq18.002,RBar)
substitute   (RBar, $GammaBar^{a}_{b a} -> 0$)                        # cdb(eq18.003,RBar)  # follows from det g = 1
canonicalise (RBar)
```

$$
\begin{aligned}
\bar{R}_{ab} &= \bar{R}^{c}{}_{acb} && \texttt{(eq18.001)}\\
&= \partial_c \bar{\Gamma}^{c}{}_{ab} + \bar{\Gamma}^{c}{}_{ec}\bar{\Gamma}^{e}{}_{ab} - \partial_b \bar{\Gamma}^{c}{}_{ac} - \bar{\Gamma}^{c}{}_{eb}\bar{\Gamma}^{e}{}_{ac} && \texttt{(eq18.002)}\\
&= \partial_c \bar{\Gamma}^{c}{}_{ab} - \bar{\Gamma}^{c}{}_{eb}\bar{\Gamma}^{e}{}_{ac} && \texttt{(eq18.003)}
\end{aligned}
$$

From here the computations will be splt into two threads, one for each of the two terms in the above result.

```
# -------------------------------------------------------------------------
# get tmpA & tmpB from RBar

GammaBar^{a}_{b c}::Weight(label=numG).
\partial{#}::WeightInherit(label=all, type=multiplicative).

tmpA := @(RBar).                                        # cdb(tmp18.101,tmpA)
keep_weight (tmpA, $numG=1$)                            # cdb(tmp18.102,tmpA)  # the derivative terms

tmpB := @(RBar).
keep_weight (tmpB, $numG=2$)                            # cdb(tmp18.103,tmpB)  # the quadrtaic terms
```

$$\texttt{tmp18.101} := \partial_{c}\bar{\Gamma}^{c}{}_{ab} - \bar{\Gamma}^{c}{}_{ae}\bar{\Gamma}^{e}{}_{bc}$$

$$\texttt{tmp18.102} := \partial_{c}\bar{\Gamma}^{c}{}_{ab}$$

$$\texttt{tmp18.103} := -\bar{\Gamma}^{c}{}_{ae}\bar{\Gamma}^{e}{}_{bc}$$

```
# ------------------------------------------------------------------------
# tmpA pt. 1

substitute   (tmpA, defGammaBar)                                     # cdb(tmp18.201,tmpA)
distribute   (tmpA)                                                  # cdb(tmp18.202,tmpA)
product_rule (tmpA)                                                  # cdb(tmp18.203,tmpA)
substitute   (tmpA, $\partial_{a}{gBar^{a b}} -> - GammaBar^{b}$)    # cdb(tmp18.204,tmpA)
```

$$\partial_c \bar{\Gamma}^c{}_{ab} = \frac{1}{2}\partial_c \left(\bar{g}^{ce}\left(\partial_a \bar{g}_{eb} + \partial_b \bar{g}_{ae} - \partial_e \bar{g}_{ab}\right)\right) \tag{tmp18.201}$$

$$= \frac{1}{2}\partial_c \left(\bar{g}^{ce}\partial_a \bar{g}_{eb}\right) + \frac{1}{2}\partial_c \left(\bar{g}^{ce}\partial_b \bar{g}_{ae}\right) - \frac{1}{2}\partial_c \left(\bar{g}^{ce}\partial_e \bar{g}_{ab}\right) \tag{tmp18.202}$$

$$= \frac{1}{2}\partial_c \bar{g}^{ce}\partial_a \bar{g}_{eb} + \frac{1}{2}\bar{g}^{ce}\partial_{ca}\bar{g}_{eb} + \frac{1}{2}\partial_c \bar{g}^{ce}\partial_b \bar{g}_{ae} + \frac{1}{2}\bar{g}^{ce}\partial_{cb}\bar{g}_{ae} - \frac{1}{2}\partial_c \bar{g}^{ce}\partial_e \bar{g}_{ab} - \frac{1}{2}\bar{g}^{ce}\partial_{ce}\bar{g}_{ab} \tag{tmp18.203}$$

$$= -\frac{1}{2}\bar{\Gamma}^e\partial_a \bar{g}_{eb} + \frac{1}{2}\bar{g}^{ce}\partial_{ca}\bar{g}_{eb} - \frac{1}{2}\bar{\Gamma}^e\partial_b \bar{g}_{ae} + \frac{1}{2}\bar{g}^{ce}\partial_{cb}\bar{g}_{ae} + \frac{1}{2}\bar{\Gamma}^e\partial_e \bar{g}_{ab} - \frac{1}{2}\bar{g}^{ce}\partial_{ce}\bar{g}_{ab} \tag{tmp18.204}$$

Notice that this result contains two terms containg second derivatives of $\bar{g}_{ij}$. This pair of terms will now be replaced with an expression built from the first derivatives of Γ^i.

```
# ------------------------------------------------------------------------
# tmpC

defGi := GammaBar^{i} -> - \partial_{j}{gBar^{i j}}.

# lower the indices on gBar^{b c}

defLowerIndices := \partial_{a}{gBar^{b c}} -> - gBar^{i b} gBar^{j c} \partial_{a}{gBar_{i j}}.

substitute (defGi, defLowerIndices)

tmpC := gBar_{a i} \partial_{b}{GammaBar^{i}}
      + gBar_{b i} \partial_{a}{GammaBar^{i}}.                        # cdb(tmp18.301,tmpC)

saveC := @(tmpC).

substitute   (tmpC, defGi)                                            # cdb(tmp18.302,tmpC)
product_rule (tmpC)                                                   # cdb(tmp18.303,tmpC)
distribute   (tmpC)                                                   # cdb(tmp18.304,tmpC)
canonicalise (tmpC)                                                   # cdb(tmp18.305,tmpC)
substitute   (tmpC, $gBar_{a b} gBar^{b c} -> gBar_{a}^{c}$)          # cdb(tmp18.306,tmpC)
eliminate_kronecker (tmpC)                                            # cdb(tmp18.307,tmpC)

# foo is the target expression to be moved to the lhs

foo := gBar^{i j} \partial_{a i}{gBar_{b j}}
     + gBar^{i j} \partial_{b i}{gBar_{a j}} -> X_{a b}.

# bah helps when rebuilding the equation

bah := X_{a b} ->
       gBar^{i j} \partial_{a i}{gBar_{b j}}
     + gBar^{i j} \partial_{b i}{gBar_{a j}}.

substitute (tmpC, foo)                                                # cdb(tmp18.308,tmpC)

#
```

```
# rearrange to move the target to the lhs

tmpE := @(tmpC).
tmpF := @(tmpC).

X_{a b}::Weight(label=numX).

# get the two pieces of the equation
keep_weight (tmpE, $numX=0$)                                          # cdb(tmp18.309,tmpE)
keep_weight (tmpF, $numX=1$)                                          # cdb(tmp18.310,tmpF)

substitute  (tmpF, bah)                                               # cdb(tmp18.311,tmpF)

# now rebuild with terms reorderd
tmpG := @(saveC) - @(tmpE).                                           # cdb(tmp18.312,tmpG)

defTmpSub := @(tmpF) -> @(tmpG).                                      # cdb(tmp18.313,defTmpSub)
```

$$
\begin{aligned}
\bar{g}_{ai}\partial_b\bar{\Gamma}^i + \bar{g}_{bi}\partial_a\bar{\Gamma}^i &= \bar{g}_{ai}\partial_b\left(\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd}\right) + \bar{g}_{bi}\partial_a\left(\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd}\right) && \texttt{(tmp18.302)}\\
&= \bar{g}_{ai}\left(\partial_b\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\partial_b\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\bar{g}^{dj}\partial_{bj}\bar{g}_{cd}\right) + \bar{g}_{bi}\left(\partial_a\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\partial_a\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}^{ci}\bar{g}^{dj}\partial_{aj}\bar{g}_{cd}\right) && \texttt{(tmp18.303)}\\
&= \bar{g}_{ai}\partial_b\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{ai}\bar{g}^{ci}\partial_b\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{ai}\bar{g}^{ci}\bar{g}^{dj}\partial_{bj}\bar{g}_{cd} + \bar{g}_{bi}\partial_a\bar{g}^{ci}\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{bi}\bar{g}^{ci}\partial_a\bar{g}^{dj}\partial_j\bar{g}_{cd} + \bar{g}_{bi}\bar{g}^{ci}\bar{g}^{dj}\partial_{aj}\bar{g}_{cd} && \texttt{(tmp18.304)}\\
&= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{ac}\bar{g}^{cd}\partial_b\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{ac}\bar{g}^{cd}\bar{g}^{ij}\partial_{bi}\bar{g}_{dj} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{bc}\bar{g}^{cd}\partial_a\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_{bc}\bar{g}^{cd}\bar{g}^{ij}\partial_{ai}\bar{g}_{dj} && \texttt{(tmp18.305)}\\
&= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_a{}^d\partial_b\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_a{}^d\bar{g}^{ij}\partial_{bi}\bar{g}_{dj} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_b{}^d\partial_a\bar{g}^{ij}\partial_i\bar{g}_{dj} + \bar{g}_b{}^d\bar{g}^{ij}\partial_{ai}\bar{g}_{dj} && \texttt{(tmp18.306)}\\
&= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_b\bar{g}^{ij}\partial_i\bar{g}_{aj} + \bar{g}^{ij}\partial_{bi}\bar{g}_{aj} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_a\bar{g}^{ij}\partial_i\bar{g}_{bj} + \bar{g}^{ij}\partial_{ai}\bar{g}_{bj} && \texttt{(tmp18.307)}\\
&= \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_b\bar{g}^{ij}\partial_i\bar{g}_{aj} + X_{ba} + \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} + \partial_a\bar{g}^{ij}\partial_i\bar{g}_{bj} && \texttt{(tmp18.308)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{g}^{ij}\partial_{bi}\bar{g}_{aj} + \bar{g}^{ij}\partial_{ai}\bar{g}_{bj} &= X_{ba} && \texttt{(tmp18.310)}\\
&= \bar{g}_{ai}\partial_b\bar{\Gamma}^i + \bar{g}_{bi}\partial_a\bar{\Gamma}^i - \bar{g}_{ac}\partial_b\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} - \partial_b\bar{g}^{ij}\partial_i\bar{g}_{aj} - \bar{g}_{bc}\partial_a\bar{g}^{cd}\bar{g}^{ij}\partial_i\bar{g}_{dj} - \partial_a\bar{g}^{ij}\partial_i\bar{g}_{bj} && \texttt{(tmp18.312)}
\end{aligned}
$$

This result will now be applied to the earlier equation `(tmp18.204)`.

```
# ------------------------------------------------------------------------
# tmpA pt.2 eliminate second partial derivatives of gBar

canonicalise    (tmpA)                                             # cdb(tmp18.401,tmpA)

substitute      (tmpA, defTmpSub)                                  # cdb(tmp18.402,tmpA)
tmpA = product_sort (tmpA)
rename_dummies (tmpA)
canonicalise    (tmpA)                                             # cdb(tmp18.403,tmpA)

foo := gBar^{d e} \partial_{c}{gBar_{e f}} -> - gBar_{e f} \partial_{c}{gBar^{d e}}.
bah := \partial_{d}{gBar^{d f}} -> - GammaBar^{f}.

substitute      (tmpA, foo)                                        # cdb(tmp18.404,tmpA)
substitute      (tmpA, bah)                                        # cdb(tmp18.405,tmpA)

foo := gBar_{e f} \partial_{a}{gBar^{c f}}  -> - \partial_{a}{gBar_{e f}} gBar^{c f}.

substitute      (tmpA, foo)                                        # cdb(tmp18.406,tmpA)

foo := gBar_{b d} gBar^{d e}  -> gBar_{b}^{e}.

substitute      (tmpA, foo)                                        # cdb(tmp18.407,tmpA)
eliminate_kronecker (tmpA)                                         # cdb(tmp18.408,tmpA)
tmpA = product_sort (tmpA)
rename_dummies (tmpA)
canonicalise    (tmpA)                                             # cdb(tmp18.409,tmpA)
```

$$
\begin{aligned}
\partial_c\bar{\Gamma}^c{}_{ab} &= -\frac{1}{2}\bar{\Gamma}^e\partial_a\bar{g}_{be} + \frac{1}{2}\bar{g}^{ce}\partial_{ac}\bar{g}_{be} - \frac{1}{2}\bar{\Gamma}^e\partial_b\bar{g}_{ae} + \frac{1}{2}\bar{g}^{ce}\partial_{bc}\bar{g}_{ae} + \frac{1}{2}\bar{\Gamma}^e\partial_e\bar{g}_{ab} - \frac{1}{2}\bar{g}^{ce}\partial_{ce}\bar{g}_{ab} && (\texttt{tmp18.401})\\
&= -\frac{1}{2}\bar{\Gamma}^e\partial_a\bar{g}_{be} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\bar{g}_{bf}\partial_a\bar{g}^{fd}\bar{g}^{ce}\partial_c\bar{g}_{de} - \frac{1}{2}\partial_a\bar{g}^{ce}\partial_c\bar{g}_{be} - \frac{1}{2}\bar{g}_{af}\partial_b\bar{g}^{fd}\bar{g}^{ce}\partial_c\bar{g}_{de} - \frac{1}{2}\partial_b\bar{g}^{ce}\partial_c\bar{g}_{ae} - \frac{1}{2}\bar{\Gamma}^e\partial_b\bar{g}_{ae} + \frac{1}{2}\bar{\Gamma}^e\partial_e\bar{g}_{ab}\\
&\quad - \frac{1}{2}\bar{g}^{ce}\partial_{ce}\bar{g}_{ab} && (\texttt{tmp18.402})\\
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\bar{g}_{bc}\bar{g}^{de}\partial_d\bar{g}_{ef}\partial_a\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} - \frac{1}{2}\bar{g}_{ac}\bar{g}^{de}\partial_d\bar{g}_{ef}\partial_b\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab}\\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && (\texttt{tmp18.403})\\
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\bar{g}_{ef}\partial_d\bar{g}^{de}\partial_a\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\bar{g}_{ac}\bar{g}_{ef}\partial_d\bar{g}^{de}\partial_b\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab}\\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && (\texttt{tmp18.404})\\
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\bar{g}_{bc}\bar{g}_{ef}\bar{\Gamma}^e\partial_a\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} - \frac{1}{2}\bar{g}_{ac}\bar{g}_{ef}\bar{\Gamma}^e\partial_b\bar{g}^{cf} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab}\\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && (\texttt{tmp18.405})\\
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{g}_{ef}\bar{g}^{cf}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{g}_{ef}\bar{g}^{cf}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab}\\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && (\texttt{tmp18.406})\\
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_b{}^f\partial_a\bar{g}_{ef}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\bar{g}_a{}^f\partial_b\bar{g}_{ef}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab}\\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && (\texttt{tmp18.407})\\
&= -\frac{1}{2}\bar{\Gamma}^c\partial_a\bar{g}_{bc} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\partial_a\bar{g}_{eb}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} + \frac{1}{2}\partial_b\bar{g}_{ea}\bar{\Gamma}^e - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{2}\bar{\Gamma}^c\partial_b\bar{g}_{ac} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && (\texttt{tmp18.408})\\
&= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && (\texttt{tmp18.409})
\end{aligned}
$$

```
# ------------------------------------------------------------------------
# tmpB

substitute     (tmpB, defGammaBar)                                  # cdb(tmp18.501,tmpB)
distribute     (tmpB)                                               # cdb(tmp18.502,tmpB)
tmpB = product_sort (tmpB)                                          # cdb(tmp18.503,tmpB)
rename_dummies (tmpB)                                               # cdb(tmp18.504,tmpB)
canonicalise   (tmpB)                                               # cdb(tmp18.505,tmpB)
```

$$-\bar{\Gamma}^{c}{}_{ae}\bar{\Gamma}^{e}{}_{bc} = -\frac{1}{4}\bar{g}^{cd}\left(\partial_{a}\bar{g}_{de} + \partial_{e}\bar{g}_{ad} - \partial_{d}\bar{g}_{ae}\right)\bar{g}^{ef}\left(\partial_{b}\bar{g}_{fc} + \partial_{c}\bar{g}_{bf} - \partial_{f}\bar{g}_{bc}\right) \tag{tmp18.501}$$

$$\begin{aligned} &= -\frac{1}{4}\bar{g}^{cd}\partial_{a}\bar{g}_{de}\bar{g}^{ef}\partial_{b}\bar{g}_{fc} - \frac{1}{4}\bar{g}^{cd}\partial_{a}\bar{g}_{de}\bar{g}^{ef}\partial_{c}\bar{g}_{bf} + \frac{1}{4}\bar{g}^{cd}\partial_{a}\bar{g}_{de}\bar{g}^{ef}\partial_{f}\bar{g}_{bc} - \frac{1}{4}\bar{g}^{cd}\partial_{e}\bar{g}_{ad}\bar{g}^{ef}\partial_{b}\bar{g}_{fc} - \frac{1}{4}\bar{g}^{cd}\partial_{e}\bar{g}_{ad}\bar{g}^{ef}\partial_{c}\bar{g}_{bf} + \frac{1}{4}\bar{g}^{cd}\partial_{e}\bar{g}_{ad}\bar{g}^{ef}\partial_{f}\bar{g}_{bc} \\ &\quad + \frac{1}{4}\bar{g}^{cd}\partial_{d}\bar{g}_{ae}\bar{g}^{ef}\partial_{b}\bar{g}_{fc} + \frac{1}{4}\bar{g}^{cd}\partial_{d}\bar{g}_{ae}\bar{g}^{ef}\partial_{c}\bar{g}_{bf} - \frac{1}{4}\bar{g}^{cd}\partial_{d}\bar{g}_{ae}\bar{g}^{ef}\partial_{f}\bar{g}_{bc} \end{aligned} \tag{tmp18.502}$$

$$\begin{aligned} &= -\frac{1}{4}\bar{g}^{fc}\bar{g}^{de}\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}_{ef} - \frac{1}{4}\bar{g}^{fc}\bar{g}^{de}\partial_{a}\bar{g}_{cd}\partial_{f}\bar{g}_{be} + \frac{1}{4}\bar{g}^{ec}\bar{g}^{df}\partial_{a}\bar{g}_{cd}\partial_{f}\bar{g}_{be} - \frac{1}{4}\bar{g}^{de}\bar{g}^{fc}\partial_{b}\bar{g}_{cd}\partial_{f}\bar{g}_{ae} - \frac{1}{4}\bar{g}^{ed}\bar{g}^{fc}\partial_{e}\bar{g}_{bc}\partial_{f}\bar{g}_{ad} + \frac{1}{4}\bar{g}^{dc}\bar{g}^{ef}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} \\ &\quad + \frac{1}{4}\bar{g}^{df}\bar{g}^{ec}\partial_{b}\bar{g}_{cd}\partial_{f}\bar{g}_{ae} + \frac{1}{4}\bar{g}^{ef}\bar{g}^{dc}\partial_{e}\bar{g}_{bc}\partial_{f}\bar{g}_{ad} - \frac{1}{4}\bar{g}^{de}\bar{g}^{cf}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} \end{aligned} \tag{tmp18.503}$$

$$\begin{aligned} &= -\frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_{a}\bar{g}_{de}\partial_{b}\bar{g}_{fc} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_{a}\bar{g}_{de}\partial_{c}\bar{g}_{bf} + \frac{1}{4}\bar{g}^{de}\bar{g}^{fc}\partial_{a}\bar{g}_{ef}\partial_{c}\bar{g}_{bd} - \frac{1}{4}\bar{g}^{de}\bar{g}^{cf}\partial_{b}\bar{g}_{fd}\partial_{c}\bar{g}_{ae} - \frac{1}{4}\bar{g}^{ce}\bar{g}^{df}\partial_{c}\bar{g}_{bf}\partial_{d}\bar{g}_{ae} + \frac{1}{4}\bar{g}^{ef}\bar{g}^{cd}\partial_{c}\bar{g}_{af}\partial_{d}\bar{g}_{be} \\ &\quad + \frac{1}{4}\bar{g}^{dc}\bar{g}^{ef}\partial_{b}\bar{g}_{fd}\partial_{c}\bar{g}_{ae} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_{c}\bar{g}_{bf}\partial_{d}\bar{g}_{ae} - \frac{1}{4}\bar{g}^{ec}\bar{g}^{fd}\partial_{c}\bar{g}_{af}\partial_{d}\bar{g}_{be} \end{aligned} \tag{tmp18.504}$$

$$= -\frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_{a}\bar{g}_{ce}\partial_{b}\bar{g}_{df} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_{c}\bar{g}_{ae}\partial_{f}\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_{c}\bar{g}_{ae}\partial_{d}\bar{g}_{bf} \tag{tmp18.505}$$

```
# ---------------------------------------------------------------------------
# RBar pt.2 = Rebuild Rab from tmpA and tmpB

RBar := @(tmpA) + @(tmpB).

canonicalise   (RBar)                                                  # cdb(eq18.601,RBar)

foo := \partial_{a}{gBar^{c d}} -> - gBar^{c i} gBar^{d j} \partial_{a}{gBar_{i j}}.

substitute     (RBar, foo)
distribute     (RBar)
RBar = product_sort (RBar)
rename_dummies (RBar)
canonicalise   (RBar)                                                 # cdb(eq18.602,RBar)

foo := \partial_{a}{gBar_{b c}} -> GammaBar_{b c a} + GammaBar_{c b a}.

substitute     (RBar, foo)                                            # cdb(eq18.603,RBar)
distribute     (RBar)
RBar = product_sort (RBar)
rename_dummies (RBar)
canonicalise   (RBar)                                                 # cdb(eq18.604,RBar)

foo := GammaBar_{d e f} gBar^{d e} -> 0.

substitute     (RBar, foo)                                            # cdb(eq18.605,RBar)

defRab := RBar_{a b} -> @(RBar).
```

$$
\begin{aligned}
\bar{R}_{ab} &= \partial_c \bar{\Gamma}^c{}_{ab} - \bar{\Gamma}^c{}_{eb} \bar{\Gamma}^e{}_{ac} && \texttt{(eq18.003)}\\
&= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c - \frac{1}{2}\partial_c\bar{g}_{bd}\partial_a\bar{g}^{cd} - \frac{1}{2}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_a\bar{g}_{ce}\partial_b\bar{g}_{df} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd}\\
&\quad + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} && \texttt{(eq18.601)}\\
&= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_a\bar{g}_{ce}\partial_d\bar{g}_{bf} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_d\bar{g}_{af} + \frac{1}{2}\bar{\Gamma}^c\partial_c\bar{g}_{ab} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_a\bar{g}_{ce}\partial_b\bar{g}_{df} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd}\\
&\quad + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} && \texttt{(eq18.602)}\\
&= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\left(\bar{\Gamma}_{cea} + \bar{\Gamma}_{eca}\right)\left(\bar{\Gamma}_{bfd} + \bar{\Gamma}_{fbd}\right) + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\left(\bar{\Gamma}_{ceb} + \bar{\Gamma}_{ecb}\right)\left(\bar{\Gamma}_{afd} + \bar{\Gamma}_{fad}\right) + \frac{1}{2}\bar{\Gamma}^c\left(\bar{\Gamma}_{abc} + \bar{\Gamma}_{bac}\right) - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}\\
&\quad - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\left(\bar{\Gamma}_{cea} + \bar{\Gamma}_{eca}\right)\left(\bar{\Gamma}_{dfb} + \bar{\Gamma}_{fdb}\right) - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\left(\bar{\Gamma}_{aec} + \bar{\Gamma}_{eac}\right)\left(\bar{\Gamma}_{bdf} + \bar{\Gamma}_{dbf}\right) + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\left(\bar{\Gamma}_{aec} + \bar{\Gamma}_{eac}\right)\left(\bar{\Gamma}_{bfd} + \bar{\Gamma}_{fbd}\right) && \texttt{(eq18.603)}\\
&= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && \texttt{(eq18.604)}\\
&= \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} && \texttt{(eq18.605)}
\end{aligned}
$$

```
# -----------------------------------------------------------------------------
# Check against prd62.

foo := @(RBar).                                         # cdb(eq18.lcb,foo)
bah  = cdblib.get('prd62.eq18.rhs','prd62.json')      # cdb(eq18.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                                   # cdb(eq18.chk,diff)
```

$$\texttt{eq18.lcb} := \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}$$

$$\texttt{eq18.prd} := -\frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right)$$

$$\texttt{eq18.chk} := 0$$

PhysRevD.62.044034 equation (19)

```
from shared import *
import cdblib

jsonfile = 'eqtn19.json'
cdblib.create (jsonfile)

defGiBar = cdblib.get ('defGiBar','eqtn17.json')

# ----------------------------------------------------------------------------
# DGiBarDt pt.1

dotgBar_{a b}::Symmetric.
dotgBar^{a b}::Symmetric.
dotgBar{#}::LaTeXForm("{\bar{dg}}").

dotGiBar := \partial_{t}{GammaBar^{i}}.             # cdb (eq19.101,dotGiBar)

substitute   (dotGiBar, defGiBar)                   # cdb (eq19.102,dotGiBar)
substitute   (dotGiBar, $\partial_{t a}{gBar^{i a}} -> \partial_{a}{dotgBar^{i a}}$)
                                                    # cdb (eq19.103,dotGiBar)

defdotgBarD := dotgBar_{i j} -> -2 N ABar_{i j}.
defdotgBarU := dotgBar^{i j} ->  2 N ABar^{i j}.
# defABarD2ABarU := ABar_{i j} -> ABar^{a b} gBar_{a i} gBar_{b j}.

substitute   (dotGiBar, defdotgBarU )               # cdb (eq19.104,dotGiBar)
product_rule (dotGiBar)                             # cdb (eq19.105,dotGiBar)

dotGiBar = product_sort (dotGiBar)                  # cdb (eq19.106,dotGiBar)

cdblib.put ('dotGiBar',dotGiBar,jsonfile)
```

$$
\begin{aligned}
\partial_t \bar{\Gamma}^i &= -\partial_{tb} \bar{g}^{ib} && \text{(eq19.102)} \\
&= -\partial_b \bar{d} g^{ib} && \text{(eq19.103)} \\
&= -2\partial_b \left(N \bar{A}^{ib} \right) && \text{(eq19.104)} \\
&= -2\partial_b N \bar{A}^{ib} - 2N \partial_b \bar{A}^{ib} && \text{(eq19.105)} \\
&= -2\bar{A}^{ia} \partial_a N - 2N \partial_a \bar{A}^{ia} && \text{(eq19.106)}
\end{aligned}
$$

```
# ------------------------------------------------------------------------
# Check against prd62.

foo := @(dotGiBar).                                      # cdb (eq19.lcb,foo)
bah  = cdblib.get('prd62.eq19.rhs','prd62.json')     # cdb (eq19.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
product_rule   (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                                    # cdb (eq19.chk,diff)
```

$$\texttt{eq19.lcb} := -2\bar{A}^{ia}\partial_a N - 2N\partial_a \bar{A}^{ia}$$

$$\texttt{eq19.prd} := -2\partial_j \left(N\bar{A}^{ij}\right)$$

$$\texttt{eq19.chk} := 0$$

PhysRevD.62.044034 equation (20)

```python
from shared import *
import cdblib

jsonfile = 'eqtn20.json'
cdblib.create (jsonfile)

dotGiBar = cdblib.get ('dotGiBar','eqtn19.json')
defMomSub = cdblib.get ('defMomSub','momentum.json')

# ------------------------------------------------------------------------
# DGiBarDt pt.2

substitute (dotGiBar, defMomSub)      # cdb(eq20.101,dotGiBar)
```

$$\begin{aligned}\partial_t \bar{\Gamma}^i &= -2\bar{A}^{ia}\partial_a N - 2N\partial_a \bar{A}^{ia} && \text{(eq19.106)}\\ &= -2\bar{A}^{ia}\partial_a N - 2N\left(-6\bar{A}^{ia}\partial_a \phi - \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_a \mathrm{tr}K\right) && \text{(eq20.101)}\end{aligned}$$

```
# ----------------------------------------------------------------------------
# Check against prd62.

foo := @(dotGiBar).                                     # cdb (eq20.lcb,foo)
bah  = cdblib.get('prd62.eq20.rhs','prd62.json')        # cdb (eq20.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                                   # cdb (eq20.chk,diff)
```

$$\texttt{eq20.lcb} := -2\bar{A}^{ia}\partial_{a}N - 2N\left(-6\bar{A}^{ia}\partial_{a}\phi - \bar{A}^{ab}\bar{\Gamma}^{i}{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_{a}\mathrm{tr}K\right)$$

$$\texttt{eq20.prd} := -2\bar{A}^{ij}\partial_{j}N + 2N\left(\bar{\Gamma}^{i}{}_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{g}^{ij}\partial_{j}\mathrm{tr}K + 6\bar{A}^{ij}\partial_{j}\phi\right)$$

$$\texttt{eq20.chk} := 0$$

PhysRevD.67.084023 equation (27)

```
from shared import *
import cdblib

jsonfile = 'gamma.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------
# Gamma in terms of GammaBar and phi, see prd67 eqn 27

Gamma := \Gamma^{a}_{b c}.                                          # cdb (eq27.101,Gamma)

substitute   (Gamma, defGamma)                                      # cdb (eq27.102,Gamma)
substitute   (Gamma, defG2GBarD)                                    # cdb (eq27.103,Gamma)
substitute   (Gamma, defG2GBarU)                                    # cdb (eq27.104,Gamma)
distribute   (Gamma)                                                # cdb (eq27.105,Gamma)
product_rule (Gamma)                                                # cdb (eq27.106,Gamma)
substitute   (Gamma, dexp)                                          # cdb (eq27.107,Gamma)
distribute   (Gamma)                                                # cdb (eq27.108,Gamma)
map_sympy    (Gamma, "simplify")                                    # cdb (eq27.109,Gamma)

foo := gBar^{a e} \partial_{e}{gBar_{b c}} ->
       - 2 GammaBar^{a}_{b c}
       + gBar^{a e} \partial_{b}{gBar_{e c}}
       + gBar^{a e} \partial_{c}{gBar_{b e}}.

substitute (Gamma, foo)                                             # cdb (eq27.110,Gamma)
substitute (Gamma, $gBar^{a i} gBar_{i b} -> gBar^{a}_{b}$)       # cdb (eq27.111,Gamma)
substitute (Gamma, $gBar^{a i} gBar_{b i} -> gBar^{a}_{b}$)       # cdb (eq27.112,Gamma)

defG2GBar :=  \Gamma^{a}_{b c} -> @(Gamma).

cdblib.put ('defG2GBar',defG2GBar,jsonfile)
```

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ae} \left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right) \tag{eq27.102}$$

$$= \frac{1}{2} g^{ae} \left(\partial_b \left(\exp\left(4\phi\right) \bar{g}_{ec}\right) + \partial_c \left(\exp\left(4\phi\right) \bar{g}_{be}\right) - \partial_e \left(\exp\left(4\phi\right) \bar{g}_{bc}\right)\right) \tag{eq27.103}$$

$$= \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \left(\partial_b \left(\exp\left(4\phi\right) \bar{g}_{ec}\right) + \partial_c \left(\exp\left(4\phi\right) \bar{g}_{be}\right) - \partial_e \left(\exp\left(4\phi\right) \bar{g}_{bc}\right)\right) \tag{eq27.104}$$

$$= \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \partial_b \left(\exp\left(4\phi\right) \bar{g}_{ec}\right) + \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \partial_c \left(\exp\left(4\phi\right) \bar{g}_{be}\right) - \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \partial_e \left(\exp\left(4\phi\right) \bar{g}_{bc}\right) \tag{eq27.105}$$

$$\begin{aligned}= &\frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \left(\partial_b \left(\exp\left(4\phi\right)\right) \bar{g}_{ec} + \exp\left(4\phi\right) \partial_b \bar{g}_{ec}\right) + \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \left(\partial_c \left(\exp\left(4\phi\right)\right) \bar{g}_{be} + \exp\left(4\phi\right) \partial_c \bar{g}_{be}\right) \\ &- \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \left(\partial_e \left(\exp\left(4\phi\right)\right) \bar{g}_{bc} + \exp\left(4\phi\right) \partial_e \bar{g}_{bc}\right)\end{aligned} \tag{eq27.106}$$

$$\begin{aligned}= &\frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \left(4 \exp\left(4\phi\right) \partial_b \phi \bar{g}_{ec} + \exp\left(4\phi\right) \partial_b \bar{g}_{ec}\right) + \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \left(4 \exp\left(4\phi\right) \partial_c \phi \bar{g}_{be} + \exp\left(4\phi\right) \partial_c \bar{g}_{be}\right) \\ &- \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \left(4 \exp\left(4\phi\right) \partial_e \phi \bar{g}_{bc} + \exp\left(4\phi\right) \partial_e \bar{g}_{bc}\right)\end{aligned} \tag{eq27.107}$$

$$\begin{aligned}= &2 \exp\left(-4\phi\right) \bar{g}^{ae} \exp\left(4\phi\right) \partial_b \phi \bar{g}_{ec} + \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \exp\left(4\phi\right) \partial_b \bar{g}_{ec} + 2 \exp\left(-4\phi\right) \bar{g}^{ae} \exp\left(4\phi\right) \partial_c \phi \bar{g}_{be} + \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \exp\left(4\phi\right) \partial_c \bar{g}_{be} \\ &- 2 \exp\left(-4\phi\right) \bar{g}^{ae} \exp\left(4\phi\right) \partial_e \phi \bar{g}_{bc} - \frac{1}{2} \exp\left(-4\phi\right) \bar{g}^{ae} \exp\left(4\phi\right) \partial_e \bar{g}_{bc}\end{aligned} \tag{eq27.108}$$

$$= 2\bar{g}^{ae} \partial_b \phi \bar{g}_{ec} + \frac{1}{2} \bar{g}^{ae} \partial_b \bar{g}_{ec} + 2\bar{g}^{ae} \partial_c \phi \bar{g}_{be} + \frac{1}{2} \bar{g}^{ae} \partial_c \bar{g}_{be} - 2\bar{g}^{ae} \partial_e \phi \bar{g}_{bc} - \frac{1}{2} \bar{g}^{ae} \partial_e \bar{g}_{bc} \tag{eq27.109}$$

$$= 2\bar{g}^{ae} \partial_b \phi \bar{g}_{ec} + 2\bar{g}^{ae} \partial_c \phi \bar{g}_{be} - 2\bar{g}^{ae} \partial_e \phi \bar{g}_{bc} + \bar{\Gamma}^a{}_{bc} \tag{eq27.110}$$

$$= 2\bar{g}^a{}_c \partial_b \phi + 2\bar{g}^{ae} \partial_c \phi \bar{g}_{be} - 2\bar{g}^{ae} \partial_e \phi \bar{g}_{bc} + \bar{\Gamma}^a{}_{bc} \tag{eq27.111}$$

$$= 2\bar{g}^a{}_c \partial_b \phi + 2\bar{g}^a{}_b \partial_c \phi - 2\bar{g}^{ae} \partial_e \phi \bar{g}_{bc} + \bar{\Gamma}^a{}_{bc} \tag{eq27.112}$$

```
# ------------------------------------------------------------------------------
# Check against prd67.

foo := @(Gamma).                                      # cdb(prd67.eq27.lcb,foo)
bah  = cdblib.get('prd67.eq27.rhs','prd67.json')      # cdb(prd67.eq27.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                                 # cdb(prd67.eq27.chk,diff)
```

$$\texttt{prd67.eq27.lcb} := 2\bar{g}^{a}{}_{c}\partial_{b}\phi + 2\bar{g}^{a}{}_{b}\partial_{c}\phi - 2\bar{g}^{ae}\partial_{e}\phi\bar{g}_{bc} + \bar{\Gamma}^{a}{}_{bc}$$

$$\texttt{prd67.eq27.prd} := \bar{\Gamma}^{a}{}_{bc} + 2\bar{g}^{a}{}_{c}\partial_{b}\phi + 2\bar{g}^{a}{}_{b}\partial_{c}\phi - 2\bar{g}_{bc}\bar{g}^{ae}\partial_{e}\phi$$

$$\texttt{prd67.eq27.chk} := 0$$

PhysRevD.67.084023 equation (19)

```
from shared import *
import cdblib

jsonfile = 'hamiltonian.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------
# Hamiltonian constraint

Ham := R + K_{a b} g^{a b} K_{c d} g^{c d} - K_{a b} K_{c d} g^{a c} g^{b d}.  # cdb (Ham.101,Ham)

defK2ABarD := K_{i j} -> \exp(4\phi) ABar_{i j} + (1/3) g_{i j} trK.
defG2GBarD := g_{a b} -> \exp(4\phi) gBar_{a b}.
defG2GBarU := g^{a b} -> \exp(-4\phi) gBar^{a b}.

substitute     (Ham, defK2ABarD)   # cdb (Ham.102,Ham)
substitute     (Ham, defG2GBarD)   # cdb (Ham.103,Ham)
substitute     (Ham, defG2GBarU)   # cdb (Ham.104,Ham)
distribute     (Ham)               # cdb (Ham.105,Ham)
Ham = product_sort (Ham)           # cdb (Ham.106,Ham)
rename_dummies (Ham)               # cdb (Ham.107,Ham)
canonicalise   (Ham)               # cdb (Ham.108,Ham)
map_sympy      (Ham, "simplify")   # cdb (Ham.109,Ham)

foo := gBar_{a b} gBar^{a b} -> 3.
bah := gBar_{a c} gBar^{b c} -> gBar_{a}^{b}.

substitute (Ham, foo)              # cdb (Ham.110,Ham)
substitute (Ham, bah)              # cdb (Ham.111,Ham)
eliminate_kronecker (Ham)          # cdb (Ham.112,Ham)

foo := gBar_{a b} gBar^{a b} -> 3.
bah := gBar_{a}^{a} -> 3.
moo := ABar_{a b} gBar^{a b} -> 0.

substitute     (Ham, foo)          # cdb (Ham.113,Ham)
```

```
substitute     (Ham, bah)           # cdb (Ham.114,Ham)
substitute     (Ham, moo)           # cdb (Ham.115,Ham)

foo := ABar_{c d} gBar^{c a} gBar^{d b} -> ABar^{a b}.

substitute     (Ham, foo)           # cdb (Ham.116,Ham)
rename_dummies (Ham)                # cdb (Ham.117,Ham)

cdblib.put ('Ham',Ham,jsonfile)
```

$$\mathcal{H} = R + K_{ab} g^{ab} K_{cd} g^{cd} - K_{ab} K_{cd} g^{ac} g^{bd} \tag{Ham.101}$$

$$= R + \left(\exp\left(4\phi\right) \bar{A}_{ab} + \frac{1}{3} g_{ab} \mathrm{tr} K\right) g^{ab} \left(\exp\left(4\phi\right) \bar{A}_{cd} + \frac{1}{3} g_{cd} \mathrm{tr} K\right) g^{cd} - \left(\exp\left(4\phi\right) \bar{A}_{ab} + \frac{1}{3} g_{ab} \mathrm{tr} K\right) \left(\exp\left(4\phi\right) \bar{A}_{cd} + \frac{1}{3} g_{cd} \mathrm{tr} K\right) g^{ac} g^{bd} \tag{Ham.102}$$

$$\begin{aligned}
&= R + \left(\exp\left(4\phi\right) \bar{A}_{ab} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K\right) g^{ab} \left(\exp\left(4\phi\right) \bar{A}_{cd} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K\right) g^{cd} \\
&\quad - \left(\exp\left(4\phi\right) \bar{A}_{ab} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K\right) \left(\exp\left(4\phi\right) \bar{A}_{cd} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K\right) g^{ac} g^{bd}
\end{aligned} \tag{Ham.103}$$

$$\begin{aligned}
&= R + \left(\exp\left(4\phi\right) \bar{A}_{ab} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K\right) \exp\left(-4\phi\right) \bar{g}^{ab} \left(\exp\left(4\phi\right) \bar{A}_{cd} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K\right) \exp\left(-4\phi\right) \bar{g}^{cd} \\
&\quad - \left(\exp\left(4\phi\right) \bar{A}_{ab} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K\right) \left(\exp\left(4\phi\right) \bar{A}_{cd} + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K\right) \exp\left(-4\phi\right) \bar{g}^{ac} \exp\left(-4\phi\right) \bar{g}^{bd}
\end{aligned} \tag{Ham.104}$$

$$\begin{aligned}
&= R + \exp\left(4\phi\right) \bar{A}_{ab} \exp\left(-4\phi\right) \bar{g}^{ab} \exp\left(4\phi\right) \bar{A}_{cd} \exp\left(-4\phi\right) \bar{g}^{cd} + \frac{1}{3} \exp\left(4\phi\right) \bar{A}_{ab} \exp\left(-4\phi\right) \bar{g}^{ab} \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K \exp\left(-4\phi\right) \bar{g}^{cd} \\
&\quad + \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K \exp\left(-4\phi\right) \bar{g}^{ab} \exp\left(4\phi\right) \bar{A}_{cd} \exp\left(-4\phi\right) \bar{g}^{cd} + \frac{1}{9} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K \exp\left(-4\phi\right) \bar{g}^{ab} \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K \exp\left(-4\phi\right) \bar{g}^{cd} \\
&\quad - \exp\left(4\phi\right) \bar{A}_{ab} \exp\left(4\phi\right) \bar{A}_{cd} \exp\left(-4\phi\right) \bar{g}^{ac} \exp\left(-4\phi\right) \bar{g}^{bd} - \frac{1}{3} \exp\left(4\phi\right) \bar{A}_{ab} \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K \exp\left(-4\phi\right) \bar{g}^{ac} \exp\left(-4\phi\right) \bar{g}^{bd} \\
&\quad - \frac{1}{3} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K \exp\left(4\phi\right) \bar{A}_{cd} \exp\left(-4\phi\right) \bar{g}^{ac} \exp\left(-4\phi\right) \bar{g}^{bd} - \frac{1}{9} \exp\left(4\phi\right) \bar{g}_{ab} \mathrm{tr} K \exp\left(4\phi\right) \bar{g}_{cd} \mathrm{tr} K \exp\left(-4\phi\right) \bar{g}^{ac} \exp\left(-4\phi\right) \bar{g}^{bd}
\end{aligned} \tag{Ham.105}$$

$$\begin{aligned}
&= R + \bar{A}_{ab} \bar{A}_{cd} \bar{g}^{ab} \bar{g}^{cd} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right) + \frac{1}{3} \mathrm{tr} K \bar{A}_{ab} \bar{g}_{cd} \bar{g}^{ab} \bar{g}^{cd} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right) \\
&\quad + \frac{1}{3} \mathrm{tr} K \bar{A}_{ab} \bar{g}_{cd} \bar{g}^{cd} \bar{g}^{ab} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right) + \frac{1}{9} \mathrm{tr} K \mathrm{tr} K \bar{g}_{ab} \bar{g}_{cd} \bar{g}^{ab} \bar{g}^{cd} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right) \\
&\quad - \bar{A}_{ab} \bar{A}_{cd} \bar{g}^{ac} \bar{g}^{bd} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right) - \frac{1}{3} \mathrm{tr} K \bar{A}_{ab} \bar{g}_{cd} \bar{g}^{ac} \bar{g}^{bd} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right) \\
&\quad - \frac{1}{3} \mathrm{tr} K \bar{A}_{ab} \bar{g}_{cd} \bar{g}^{ca} \bar{g}^{db} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right) - \frac{1}{9} \mathrm{tr} K \mathrm{tr} K \bar{g}_{ab} \bar{g}_{cd} \bar{g}^{ac} \bar{g}^{bd} \exp\left(-4\phi\right) \exp\left(-4\phi\right) \exp\left(4\phi\right) \exp\left(4\phi\right)
\end{aligned} \tag{Ham.106}$$

$$
\begin{aligned}
\mathcal{H} &= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) + \frac{1}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) \\
&\quad + \frac{1}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) + \frac{1}{9}\mathrm{tr}K\mathrm{tr}K\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) \\
&\quad - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) - \frac{1}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) \\
&\quad - \frac{1}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ca}\bar{g}^{db}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) - \frac{1}{9}\mathrm{tr}K\mathrm{tr}K\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) && (\texttt{Ham.107}) \\
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) + \frac{2}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) \\
&\quad + \frac{1}{9}\mathrm{tr}K\mathrm{tr}K\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) \\
&\quad - \frac{2}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) - \frac{1}{9}\mathrm{tr}K\mathrm{tr}K\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\exp(4\phi) && (\texttt{Ham.108}) \\
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + \frac{2}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} + \frac{1}{9}\mathrm{tr}K^2\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ab}\bar{g}^{cd} - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{1}{9}\mathrm{tr}K^2\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} && (\texttt{Ham.109}) \\
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\mathrm{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\mathrm{tr}K^2\bar{g}_{cd}\bar{g}^{cd} - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{1}{9}\mathrm{tr}K^2\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{ac}\bar{g}^{bd} && (\texttt{Ham.110}) \\
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\mathrm{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\mathrm{tr}K^2\bar{g}_c{}^c - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\mathrm{tr}K\bar{A}_{ab}\bar{g}_c{}^b\bar{g}^{ac} - \frac{1}{9}\mathrm{tr}K^2\bar{g}_{ab}\bar{g}_c{}^b\bar{g}^{ac} && (\texttt{Ham.111}) \\
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\mathrm{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\mathrm{tr}K^2\bar{g}_c{}^c - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\mathrm{tr}K\bar{A}_{ac}\bar{g}^{ac} - \frac{1}{9}\mathrm{tr}K^2\bar{g}_{ac}\bar{g}^{ac} && (\texttt{Ham.112}) \\
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\mathrm{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{1}{3}\mathrm{tr}K^2\bar{g}_c{}^c - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\mathrm{tr}K\bar{A}_{ac}\bar{g}^{ac} - \frac{1}{3}\mathrm{tr}K^2 && (\texttt{Ham.113}) \\
&= R + \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ab}\bar{g}^{cd} + 2\mathrm{tr}K\bar{A}_{ab}\bar{g}^{ab} + \frac{2}{3}\mathrm{tr}K^2 - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} - \frac{2}{3}\mathrm{tr}K\bar{A}_{ac}\bar{g}^{ac} && (\texttt{Ham.114}) \\
&= R + \frac{2}{3}\mathrm{tr}K^2 - \bar{A}_{ab}\bar{A}_{cd}\bar{g}^{ac}\bar{g}^{bd} && (\texttt{Ham.115}) \\
&= R + \frac{2}{3}\mathrm{tr}K^2 - \bar{A}^{cd}\bar{A}_{cd} && (\texttt{Ham.116}) \\
&= R + \frac{2}{3}\mathrm{tr}K^2 - \bar{A}^{ab}\bar{A}_{ab} && (\texttt{Ham.117})
\end{aligned}
$$

```
# ----------------------------------------------------------------------------
# Check against prd67.

foo := @(Ham).                                        # cdb(prd67.eq19.lcb,foo)
bah  = cdblib.get('prd67.eq19.rhs','prd67.json')      # cdb(prd67.eq19.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                                 # cdb(prd67.eq19.chk,diff)
```

$$\texttt{prd67.eq19.lcb} := R + \frac{2}{3}\mathrm{tr}K^2 - \bar{A}^{ab}\bar{A}_{ab}$$

$$\texttt{prd67.eq19.prd} := R - \bar{A}_{ab}\bar{A}^{ab} + \frac{2}{3}\mathrm{tr}K^2$$

$$\texttt{prd67.eq19.chk} := 0$$

PhysRevD.67.084023 equation (20)

```
from shared import *
import cdblib

jsonfile = 'momentum.json'
cdblib.create (jsonfile)

defG2GBar = cdblib.get ('defG2GBar','gamma.json')

# ----------------------------------------------------------------------------
# Momentum constraint pt.1

Mom := D_{j}{K^{i j} - g^{i j} trK}.                                        # cdb(Mom.101,Mom)

defDgD := D_{a}{g_{b c}} -> 0.
defDgU := D_{a}{g^{b c}} -> 0.

defDtrK   := D_{a}{trK} -> \partial_{a}{trK}.
defDexp   := D_{a}{\exp(-4\phi)} -> -4\exp(-4\phi) \partial_{a}{\phi}.

distribute   (Mom)                                                          # cdb(Mom.102,Mom)
product_rule (Mom)                                                          # cdb(Mom.103,Mom)
substitute   (Mom, defDgU)                                                  # cdb(Mom.104,Mom)

defK2ABarU := K^{i j} -> \exp(-4\phi) ABar^{i j} + (1/3) g^{i j} trK.

substitute   (Mom, defK2ABarU)                                              # cdb(Mom.105,Mom)
distribute   (Mom)                                                          # cdb(Mom.106,Mom)
product_rule (Mom)                                                          # cdb(Mom.107,Mom)
substitute   (Mom, defDtrK)                                                 # cdb(Mom.108,Mom)
substitute   (Mom, defDgU)                                                  # cdb(Mom.109,Mom)
substitute   (Mom, defDexp)                                                 # cdb(Mom.110,Mom)
```

$$
\begin{aligned}
\mathcal{D}^j &= D_j\left(K^{ij} - g^{ij}\mathrm{tr}K\right) && \texttt{(Mom.101)}\\
&= D_j K^{ij} - D_j\left(g^{ij}\mathrm{tr}K\right) && \texttt{(Mom.102)}\\
&= D_j K^{ij} - D_j g^{ij}\mathrm{tr}K - g^{ij} D_j \mathrm{tr}K && \texttt{(Mom.103)}\\
&= D_j K^{ij} - g^{ij} D_j \mathrm{tr}K && \texttt{(Mom.104)}\\
&= D_j\left(\exp\left(-4\phi\right)\bar{A}^{ij} + \frac{1}{3}g^{ij}\mathrm{tr}K\right) - g^{ij} D_j \mathrm{tr}K && \texttt{(Mom.105)}\\
&= D_j\left(\exp\left(-4\phi\right)\bar{A}^{ij}\right) + \frac{1}{3}D_j\left(g^{ij}\mathrm{tr}K\right) - g^{ij} D_j \mathrm{tr}K && \texttt{(Mom.106)}\\
&= D_j\left(\exp\left(-4\phi\right)\right)\bar{A}^{ij} + \exp\left(-4\phi\right)D_j\bar{A}^{ij} + \frac{1}{3}D_j g^{ij}\mathrm{tr}K - \frac{2}{3}g^{ij} D_j \mathrm{tr}K && \texttt{(Mom.107)}\\
&= D_j\left(\exp\left(-4\phi\right)\right)\bar{A}^{ij} + \exp\left(-4\phi\right)D_j\bar{A}^{ij} + \frac{1}{3}D_j g^{ij}\mathrm{tr}K - \frac{2}{3}g^{ij} \partial_j \mathrm{tr}K && \texttt{(Mom.108)}\\
&= D_j\left(\exp\left(-4\phi\right)\right)\bar{A}^{ij} + \exp\left(-4\phi\right)D_j\bar{A}^{ij} - \frac{2}{3}g^{ij} \partial_j \mathrm{tr}K && \texttt{(Mom.109)}\\
&= -4\exp\left(-4\phi\right)\partial_j\phi\bar{A}^{ij} + \exp\left(-4\phi\right)D_j\bar{A}^{ij} - \frac{2}{3}g^{ij} \partial_j \mathrm{tr}K && \texttt{(Mom.110)}
\end{aligned}
$$

```
# ------------------------------------------------------------------------
# Momentum constraint pt.2

confMom := \exp(4\phi) @(Mom).

defG2GBarU := g^{i j} -> \exp(-4\phi) gBar^{i j}.

distribute   (confMom)                                              # cdb(confMom.101,confMom)
substitute   (confMom, defG2GBarU)                                  # cdb(confMom.102,confMom)
map_sympy    (confMom, "simplify")                                  # cdb(confMom.103,confMom)

defDAabU  := D_{a}{ABar^{b c}} ->  \partial_{a}{ABar^{b c}}
                                 + \Gamma^{b}_{i a} ABar^{i c}
                                 + \Gamma^{c}_{i a} ABar^{b i}.

substitute     (confMom, defDAabU)                                  # cdb(confMom.104,confMom)
substitute     (confMom, defG2GBar)                                 # cdb(confMom.105,confMom)
distribute     (confMom)                                            # cdb(confMom.106,confMom)
confMom = product_sort (confMom)                                    # cdb(confMom.107,confMom)
rename_dummies (confMom)                                            # cdb(confMom.108,confMom)
canonicalise   (confMom)                                            # cdb(confMom.109,confMom)
substitute     (confMom, $gBar^{i}_{i} -> 3$)                       # cdb(confMom.110,confMom)
substitute     (confMom, $gBar_{i j} ABar^{i j} -> 0$)              # cdb(confMom.111,confMom)
substitute     (confMom, $gBar_{a i} gBar^{i b} -> gBar_{a}^{b}$)   # cdb(confMom.112,confMom)
substitute     (confMom, $GammaBar^{b}_{a b} -> 0$)                 # cdb(confMom.113,confMom) # follows from det gBar = 1
eliminate_kronecker (confMom)                                       # cdb(confMom.114,confMom)
rename_dummies (confMom)
canonicalise   (confMom)                                            # cdb(confMom.115,confMom)

cdblib.put ('confMom',confMom,jsonfile)
```

$$\exp(4\phi)\mathcal{D}^j = -4\exp\left(4\phi\right)\exp\left(-4\phi\right)\partial_j\phi\bar{A}^{ij} + \exp\left(4\phi\right)\exp\left(-4\phi\right)D_j\bar{A}^{ij} - \frac{2}{3}\exp\left(4\phi\right)g^{ij}\partial_j\mathrm{tr}K \tag{confMom.101}$$

$$= -4\exp\left(4\phi\right)\exp\left(-4\phi\right)\partial_j\phi\bar{A}^{ij} + \exp\left(4\phi\right)\exp\left(-4\phi\right)D_j\bar{A}^{ij} - \frac{2}{3}\exp\left(4\phi\right)\exp\left(-4\phi\right)\bar{g}^{ij}\partial_j\mathrm{tr}K \tag{confMom.102}$$

$$= -4\partial_j\phi\bar{A}^{ij} + D_j\bar{A}^{ij} - \frac{2}{3}\bar{g}^{ij}\partial_j\mathrm{tr}K \tag{confMom.103}$$

$$= -4\partial_j\phi\bar{A}^{ij} + \partial_j\bar{A}^{ij} + \Gamma^i{}_{aj}\bar{A}^{aj} + \Gamma^j{}_{aj}\bar{A}^{ia} - \frac{2}{3}\bar{g}^{ij}\partial_j\mathrm{tr}K \tag{confMom.104}$$

$$= -4\partial_j\phi\bar{A}^{ij} + \partial_j\bar{A}^{ij} + \left(2\bar{g}^i{}_j\partial_a\phi + 2\bar{g}^i{}_a\partial_j\phi - 2\bar{g}^{ie}\partial_e\phi\bar{g}_{aj} + \bar{\Gamma}^i{}_{aj}\right)\bar{A}^{aj} + \left(2\bar{g}^j{}_j\partial_a\phi + 2\bar{g}^j{}_a\partial_j\phi - 2\bar{g}^{je}\partial_e\phi\bar{g}_{aj} + \bar{\Gamma}^j{}_{aj}\right)\bar{A}^{ia} - \frac{2}{3}\bar{g}^{ij}\partial_j\mathrm{tr}K \tag{confMom.105}$$

$$= -4\partial_j\phi\bar{A}^{ij} + \partial_j\bar{A}^{ij} + 2\bar{g}^i{}_j\partial_a\phi\bar{A}^{aj} + 2\bar{g}^i{}_a\partial_j\phi\bar{A}^{aj} - 2\bar{g}^{ie}\partial_e\phi\bar{g}_{aj}\bar{A}^{aj} + \bar{\Gamma}^i{}_{aj}\bar{A}^{aj} + 2\bar{g}^j{}_j\partial_a\phi\bar{A}^{ia} + 2\bar{g}^j{}_a\partial_j\phi\bar{A}^{ia} - 2\bar{g}^{je}\partial_e\phi\bar{g}_{aj}\bar{A}^{ia} + \bar{\Gamma}^j{}_{aj}\bar{A}^{ia} - \frac{2}{3}\bar{g}^{ij}\partial_j\mathrm{tr}K \tag{confMom.106}$$

$$= -4\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 2\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b + 2\bar{A}^{ab}\partial_b\phi\bar{g}^i{}_a - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_a\phi\bar{g}^b{}_b + 2\bar{A}^{ia}\partial_b\phi\bar{g}^b{}_a - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.107}$$

$$= -4\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 2\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b + 2\bar{A}^{ab}\partial_b\phi\bar{g}^i{}_a - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_a\phi\bar{g}^b{}_b + 2\bar{A}^{ia}\partial_b\phi\bar{g}^b{}_a - 2\bar{A}^{ia}\bar{g}_{ac}\bar{g}^{cb}\partial_b\phi + \bar{A}^{ia}\bar{\Gamma}^b{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.108}$$

$$= -4\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_a\phi\bar{g}^b{}_b + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a{}^b - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.109}$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b - 2\bar{A}^{ab}\bar{g}_{ab}\bar{g}^{ic}\partial_c\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a{}^b - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.110}$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a{}^b - 2\bar{A}^{ia}\bar{g}_{ab}\bar{g}^{bc}\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.111}$$

$$\exp(4\phi)\mathcal{D}^j = 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a{}^b - 2\bar{A}^{ia}\bar{g}_a{}^c\partial_c\phi + \bar{A}^{ia}\bar{\Gamma}^b{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \qquad (\texttt{confMom.112})$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ab}\partial_a\phi\bar{g}^i{}_b + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ia}\partial_b\phi\bar{g}_a{}^b - 2\bar{A}^{ia}\bar{g}_a{}^c\partial_c\phi - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \qquad (\texttt{confMom.113})$$

$$= 2\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + 4\bar{A}^{ai}\partial_a\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + 2\bar{A}^{ib}\partial_b\phi - 2\bar{A}^{ic}\partial_c\phi - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \qquad (\texttt{confMom.114})$$

$$= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \qquad (\texttt{confMom.115})$$

```
tmpA := @(confMom).                                       # cdb(confMom.201,tmpA)
tmpB := @(confMom).

X^{b c}_{a}::Weight(label=numX).

Xbca := \partial_{a}{ABar^{b c}}.                         # cdb(confMom.202,Xbca)

foo := \partial_{a}{ABar^{b c}} -> X^{b c}_{a}.
bah := X^{b c}_{a} -> \partial_{a}{ABar^{b c}}.

substitute  (tmpA, foo)                                   # cdb(confMom.203,tmpA)
substitute  (tmpB, foo)                                   # cdb(confMom.204,tmpB)
drop_weight (tmpA, $numX=1$)                              # cdb(confMom.205,tmpA)
keep_weight (tmpB, $numX=1$)                              # cdb(confMom.206,tmpB)
substitute  (tmpB, bah)                                   # cdb(confMom.207,tmpB)

tmpC := - @(tmpA).                                        # cdb(confMom.208,tmpC)

defMomSub := @(tmpB) -> @(tmpC).                          # cdb(confMom.209,defMomSub)

cdblib.put ('defMomSub',defMomSub,jsonfile)
```

$$0 = 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.201}$$

$$0 = 6\bar{A}^{ia}\partial_a\phi + X^{ia}{}_a + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.203}$$

$$\begin{aligned}\partial_a\bar{A}^{bc} &= X^{ia}{}_a && \text{(confMom.206)}\\ &= -6\bar{A}^{ia}\partial_a\phi - \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && \text{(confMom.208)}\end{aligned}$$

$$\partial_a\bar{A}^{ia} \to -6\bar{A}^{ia}\partial_a\phi - \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} + \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K \tag{confMom.209}$$

```
# ------------------------------------------------------------------------
# Check against prd67.

foo := @(confMom).                                     # cdb(prd67.eq20.lcb,foo)
bah  = cdblib.get('prd67.eq20.rhs','prd67.json')     # cdb(prd67.eq20.prd,bah)

diff := @(foo) - @(bah).

distribute     (diff)
diff = product_sort (diff)
rename_dummies (diff)
map_sympy      (diff, "simplify")
canonicalise   (diff)                                  # cdb(prd67.eq20.chk,diff)
```

$$\texttt{prd67.eq20.lcb} := 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K$$

$$\texttt{prd67.eq20.prd} := \partial_a\bar{A}^{ia} + 6\bar{A}^{ia}\partial_a\phi + \bar{A}^{ab}\bar{\Gamma}^i{}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K$$

$$\texttt{prd67.eq20.chk} := 0$$