PhysRevD.62.044034 equation (14)

The advice given by Miguel Alcubierre, Bernd Brugmann etal (Phys Rev D (67) 084023, 2nd-3rd paragraph on pg. 084023-4)

> ... if one wants to achieve numerical stability. In the computer code we do not use the numerically evolved $\bar{\Gamma}^i$ in all places, but we follow this rule:
>
> Partial derivatives $\partial_j \bar{\Gamma}^i$ are computed as finite differences of the independent variables $\bar{\Gamma}^i$ that are evolved using ...

The Einstein Toolkit code uses the same rule – the only place where the *evolved* $\bar{\Gamma}^i$ are used is in computing the $\partial_j \bar{\Gamma}^i$ terms in the equation for $\bar{R}_{ij}$, that is equation (18) of the Phys Rev D (62) 044034 paper.

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-14.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------

Rphi := -2 DBar_{a b}{\phi} - 2 gBar_{a b} gBar^{c d} DBar_{c d}{\phi}
        +4 DBar_{a}{\phi} DBar_{b}{\phi} - 4 gBar_{a b} gBar^{c d} DBar_{c}{\phi} DBar_{d}{\phi}.

                                                        # cdb(eq15.prd,Rphi)

RBar := - (1/2) gBar^{l m} \partial_{l m}{gBar_{a b}}
        + (1/2) gBar_{k a} \partial_{b}{GammaBar^{k}}
        + (1/2) gBar_{k b} \partial_{a}{GammaBar^{k}}
        + (1/2) GammaBar^{k} GammaBar_{a b k}
        + (1/2) GammaBar^{k} GammaBar_{b a k}
        + gBar^{l m} gBar^{k e} (  GammaBar_{e l a} GammaBar_{b k m}
                                 + GammaBar_{e l b} GammaBar_{a k m}
                                 + GammaBar_{k a m} GammaBar_{e l b}).

                                                        # cdb(eq18.prd,RBar)

defRab := R_{a b} -> @(Rphi) + @(RBar).

Rab := RBar_{a b} + Rphi_{a b}.                         # cdb(eq14.01,Rab)
Rab := R_{a b}.                                         # cdb(eq14.00,Rab)

substitute (Rab, defRab)                                # cdb(eq14.02,Rab)
substitute (Rab, defDBar1)                              # cdb(eq14.03,Rab)
substitute (Rab, defDBar2)                              # cdb(eq14.04,Rab)
substitute (Rab, defGamma2GammaBar)                     # cdb(eq14.05,Rab)
distribute (Rab)                                        # cdb(eq14.06,Rab)
eliminate_kronecker (Rab)                               # cdb(eq14.07,Rab)

Rab = product_sort (Rab)                                # cdb(eq14.08,Rab)
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rename_dummies (Rab)                                  # cdb(eq14.09,Rab)
canonicalise   (Rab)                                  # cdb(eq14.10,Rab)

foo := GammaBar^{a} GammaBar_{b c a} -> gBar^{d e} GammaBar^{a}_{d e} GammaBar_{b c a}.

substitute (Rab, foo)                                 # cdb(eq14.11,Rab)
substitute (Rab, defGBarSq)                           # cdb(eq14.12,Rab)
substitute (Rab, defGammaBarD)                        # cdb(eq14.13,Rab)
substitute (Rab, defGammaBarU)                        # cdb(eq14.14,Rab)
distribute (Rab)                                      # cdb(eq14.15,Rab)

foo := \partial_{a}{gBar_{b c}} gBar^{b c} -> 0.   # follows from det(g) = 1

substitute   (Rab,foo)                                # cdb(eq14.16,Rab)
canonicalise (Rab)                                    # cdb(eq14.17,Rab)

foo := gBar^{b e} gBar^{c f} \partial_{a}{gBar_{b c}}  -> - \partial_{a}{gBar^{e f}}.
bah := gBar^{e b} gBar^{f c} \partial_{a}{gBar_{b c}}  -> - \partial_{a}{gBar^{e f}}.
moo := gBar^{e b} gBar^{c f} \partial_{a}{gBar_{b c}}  -> - \partial_{a}{gBar^{e f}}.

substitute (Rab,foo)                                  # cdb(eq14.18,Rab)
substitute (Rab,bah)                                  # cdb(eq14.19,Rab)
substitute (Rab,moo)                                  # cdb(eq14.20,Rab)

Rab = product_sort (Rab)                              # cdb(eq14.21,Rab)
                                                      # cdb(eq14.99,Rab)

defRab := R_{a b} -> @(Rab).   # used later in bssn-ricci-scalar.tex

cdblib.put ('Rab',Rab,jsonfile)
cdblib.put ('defRab',defRab,jsonfile)
```

$$R_{ab} = \bar{R}_{ab} + {R^{\phi}}_{ab} \tag{eq14.01}$$

$$\begin{aligned} &= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} \\ &\quad + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) \end{aligned} \tag{eq14.02}$$

$$\begin{aligned} &= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} \\ &\quad + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) \end{aligned} \tag{eq14.03}$$

$$\begin{aligned} &= -2\partial_{ab}\phi + 2{\Gamma^c}_{ab}\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\left(\partial_{cd}\phi - {\Gamma^e}_{cd}\partial_e\phi\right) + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} \\ &\quad + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) \end{aligned} \tag{eq14.04}$$

$$\begin{aligned} &= -2\partial_{ab}\phi + 2\left({\bar{\Gamma}^c}_{ab} + 2{\bar{g}^c}_b\partial_a\phi + 2{\bar{g}^c}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\left(\partial_{cd}\phi - \left({\bar{\Gamma}^e}_{cd} + 2{\bar{g}^e}_d\partial_c\phi + 2{\bar{g}^e}_c\partial_d\phi - 2\bar{g}^{ef}\bar{g}_{cd}\partial_f\phi\right)\partial_e\phi\right) + 4\partial_a\phi\partial_b\phi \\ &\quad - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) \end{aligned} \tag{eq14.05}$$

$$\begin{aligned} &= -2\partial_{ab}\phi + 2{\bar{\Gamma}^c}_{ab}\partial_c\phi + 4{\bar{g}^c}_b\partial_a\phi\partial_c\phi + 4{\bar{g}^c}_a\partial_b\phi\partial_c\phi - 4\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}{\bar{\Gamma}^e}_{cd}\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{cd}{\bar{g}^e}_d\partial_c\phi\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{cd}{\bar{g}^e}_c\partial_d\phi\partial_e\phi \\ &\quad - 4\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\bar{g}_{cd}\partial_f\phi\partial_e\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} \\ &\quad + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{kam}\bar{\Gamma}_{elb} \end{aligned} \tag{eq14.06}$$

$$\begin{aligned} &= -2\partial_{ab}\phi + 2{\bar{\Gamma}^c}_{ab}\partial_c\phi + 8\partial_a\phi\partial_b\phi + 4\partial_b\phi\partial_a\phi - 4\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}{\bar{\Gamma}^e}_{cd}\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{ce}\partial_c\phi\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{ed}\partial_d\phi\partial_e\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\bar{g}_{cd}\partial_f\phi\partial_e\phi \\ &\quad - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{kam}\bar{\Gamma}_{elb} \end{aligned} \tag{eq14.07}$$

$$\begin{aligned} &= -2\partial_{ab}\phi + 2{\bar{\Gamma}^c}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}{\bar{\Gamma}^e}_{cd}\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{cd}\partial_d\phi\partial_c\phi - 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ef}\partial_e\phi\partial_f\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\ &\quad + \frac{1}{2}\bar{g}_{ca}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{cb}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bcf}\bar{\Gamma}_{dea} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{acf}\bar{\Gamma}_{deb} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{deb}\bar{\Gamma}_{caf} \end{aligned} \tag{eq14.08}$$

$$\begin{aligned} &= -2\partial_{ab}\phi + 2{\bar{\Gamma}^c}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{de}{\bar{\Gamma}^c}_{de}\partial_c\phi + 4\bar{g}_{ab}\bar{g}^{dc}\partial_c\phi\partial_d\phi - 4\bar{g}_{ab}\bar{g}_{ef}\bar{g}^{ef}\bar{g}^{cd}\partial_c\phi\partial_d\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\ &\quad + \frac{1}{2}\bar{g}_{ca}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{cb}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} + \bar{g}^{ce}\bar{g}^{fd}\bar{\Gamma}_{bcd}\bar{\Gamma}_{efa} + \bar{g}^{ce}\bar{g}^{fd}\bar{\Gamma}_{acd}\bar{\Gamma}_{efb} + \bar{g}^{ec}\bar{g}^{df}\bar{\Gamma}_{cdb}\bar{\Gamma}_{eaf} \end{aligned} \tag{eq14.09}$$

$$\begin{aligned} &= -2\partial_{ab}\phi + 2{\bar{\Gamma}^c}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}{\bar{\Gamma}^e}_{cd}\partial_e\phi - 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} \\ &\quad + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} \end{aligned} \tag{eq14.10}$$

$$
\begin{aligned}
R_{ab} &= -2\partial_{ab}\phi + 2\bar{\Gamma}^{c}{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^{e}{}_{cd}\partial_e\phi - 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^{c}{}_{de}\bar{\Gamma}_{abc} \\
&\quad + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^{c}{}_{de}\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} \qquad \texttt{(eq14.11)} \\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^{c}{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^{e}{}_{cd}\partial_e\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^{c}{}_{de}\bar{\Gamma}_{abc} \\
&\quad + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^{c}{}_{de}\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} \qquad \texttt{(eq14.12)} \\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^{c}{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^{e}{}_{cd}\partial_e\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^{c}{}_{de}\left(\frac{1}{2}\partial_b\bar{g}_{ac} + \frac{1}{2}\partial_c\bar{g}_{ba} - \frac{1}{2}\partial_a\bar{g}_{bc}\right) + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^{c}{}_{de}\left(\frac{1}{2}\partial_a\bar{g}_{bc} + \frac{1}{2}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_b\bar{g}_{ac}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{be} + \frac{1}{2}\partial_e\bar{g}_{cb} - \frac{1}{2}\partial_b\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_a\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{ad} - \frac{1}{2}\partial_d\bar{g}_{af}\right) + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{ae} + \frac{1}{2}\partial_e\bar{g}_{ca} - \frac{1}{2}\partial_a\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_a\bar{g}_{ce} + \frac{1}{2}\partial_e\bar{g}_{ac} - \frac{1}{2}\partial_c\bar{g}_{ae}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) \qquad \texttt{(eq14.13)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\left(\partial_a\bar{g}_{eb} + \partial_b\bar{g}_{ae} - \partial_e\bar{g}_{ab}\right)\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\left(\partial_c\bar{g}_{fd} + \partial_d\bar{g}_{cf} - \partial_f\bar{g}_{cd}\right)\partial_e\phi \\
&\quad - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^{c} + \frac{1}{4}\bar{g}^{de}\bar{g}^{cf}\left(\partial_d\bar{g}_{fe} + \partial_e\bar{g}_{df} - \partial_f\bar{g}_{de}\right)\left(\frac{1}{2}\partial_b\bar{g}_{ac} + \frac{1}{2}\partial_c\bar{g}_{ba} - \frac{1}{2}\partial_a\bar{g}_{bc}\right) \\
&\quad + \frac{1}{4}\bar{g}^{de}\bar{g}^{cf}\left(\partial_d\bar{g}_{fe} + \partial_e\bar{g}_{df} - \partial_f\bar{g}_{de}\right)\left(\frac{1}{2}\partial_a\bar{g}_{bc} + \frac{1}{2}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_b\bar{g}_{ac}\right) + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{be} + \frac{1}{2}\partial_e\bar{g}_{cb} - \frac{1}{2}\partial_b\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_a\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{ad} - \frac{1}{2}\partial_d\bar{g}_{af}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{ae} + \frac{1}{2}\partial_e\bar{g}_{ca} - \frac{1}{2}\partial_a\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_a\bar{g}_{ce} + \frac{1}{2}\partial_e\bar{g}_{ac} - \frac{1}{2}\partial_c\bar{g}_{ae}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) \qquad \texttt{(eq14.14)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{eb}\partial_c\phi + \bar{g}^{ce}\partial_b\bar{g}_{ae}\partial_c\phi - \bar{g}^{ce}\partial_e\bar{g}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{fd}\partial_e\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_d\bar{g}_{cf}\partial_e\phi - \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_f\bar{g}_{cd}\partial_e\phi \\
&\quad - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^{c} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ba} - \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_f\bar{g}_{de}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ab} \\
&\quad + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ab} - \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_f\bar{g}_{de}\partial_c\bar{g}_{ab} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_a\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_a\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_d\bar{g}_{af} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_a\bar{g}_{df} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_f\bar{g}_{ad} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_b\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_d\bar{g}_{bf} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_b\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_d\bar{g}_{bf} \qquad \texttt{(eq14.15)}
\end{aligned}
$$

$$
\begin{aligned}
R_{ab} &= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{eb}\partial_c\phi + \bar{g}^{ce}\partial_b\bar{g}_{ae}\partial_c\phi - \bar{g}^{ce}\partial_e\bar{g}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{fd}\partial_e\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_d\bar{g}_{cf}\partial_e\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi \\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ab} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ab} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_a\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_a\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_d\bar{g}_{af} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_a\bar{g}_{df} \\
&\quad - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_f\bar{g}_{ad} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_b\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_d\bar{g}_{bf} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_b\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_d\bar{g}_{bf} \qquad \texttt{(eq14.16)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{de}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ab}\partial_e\bar{g}_{df} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_a\bar{g}_{ce}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_a\bar{g}_{ce}\partial_b\bar{g}_{df} \\
&\quad + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_d\bar{g}_{af} \qquad \texttt{(eq14.17)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{de}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ab}\partial_e\bar{g}_{df} - \frac{1}{2}\partial_a\bar{g}^{df}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} + \frac{1}{4}\partial_a\bar{g}^{df}\partial_b\bar{g}_{df} - \frac{1}{2}\partial_b\bar{g}^{df}\partial_d\bar{g}_{af} \qquad \texttt{(eq14.18)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{de}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c - \frac{1}{2}\partial_e\bar{g}^{ce}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_a\bar{g}^{df}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} + \frac{1}{4}\partial_a\bar{g}^{df}\partial_b\bar{g}_{df} - \frac{1}{2}\partial_b\bar{g}^{df}\partial_d\bar{g}_{af} \qquad \texttt{(eq14.19)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}_{ab}\partial_c\bar{g}^{cf}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c \\
&\quad + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c - \frac{1}{2}\partial_e\bar{g}^{ce}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_a\bar{g}^{df}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} + \frac{1}{4}\partial_a\bar{g}^{df}\partial_b\bar{g}_{df} - \frac{1}{2}\partial_b\bar{g}^{df}\partial_d\bar{g}_{af} \qquad \texttt{(eq14.20)} \\
&= -2\partial_{ab}\phi + \bar{g}^{cd}\partial_d\phi\partial_a\bar{g}_{bc} + \bar{g}^{cd}\partial_d\phi\partial_b\bar{g}_{ac} - \bar{g}^{cd}\partial_d\phi\partial_c\bar{g}_{ab} + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}_{ab}\partial_d\phi\partial_c\bar{g}^{cd} - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c \\
&\quad + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c - \frac{1}{2}\partial_c\bar{g}_{ab}\partial_d\bar{g}^{cd} - \frac{1}{2}\partial_d\bar{g}_{bc}\partial_a\bar{g}^{dc} - \frac{1}{2}\bar{g}^{ed}\bar{g}^{cf}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{ef}\bar{g}^{cd}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} + \frac{1}{4}\partial_b\bar{g}_{cd}\partial_a\bar{g}^{cd} - \frac{1}{2}\partial_d\bar{g}_{ac}\partial_b\bar{g}^{dc} \qquad \texttt{(eq14.21)}
\end{aligned}
$$

There is a single term in this final expression that appears to be neither symmetric in ab nor part of a symmetric pair, namely

$$\partial_b \bar{g}_{cd} \partial_a \bar{g}^{cd}$$

It is, however, easy to show that this term is symmetric in ab. Start by noting that, for any $\bar{g}_{ab}$,

$$\partial_a \bar{g}^{cd} = -\bar{g}^{ce} \bar{g}^{df} \partial_a \bar{g}_{ef}$$

Now contract both sides with $\partial_b \bar{g}_{cd}$ to obtain

$$\partial_a \bar{g}^{cd} \partial_b \bar{g}_{cd} = -\bar{g}^{ce} \bar{g}^{df} \partial_a \bar{g}_{ef} \partial_b \bar{g}_{cd}$$

The right hand side is clearly symmetric in ab and thus the left hand must also be symmtric in ab.