PhysRevD.62.044034 equation (12)

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-12.json'
cdblib.create (jsonfile)

# ------------------------------------------------------------------------

DABarDt := \partial_{t}{ABar_{i j}}.          # cdb(eq12.00,DABarDt)
DABarDt := N (trK ABar_{i j} - 2 ABar_{i a} ABar^{a}_{j})
           + \exp(-4\phi) (N R_{i j} - D_{i j}{N}
                           - (1/3) g_{i j} (N R_{a b} - D_{a b}{N}) g^{a b}).

                                          # cdb(eq12.01,DABarDt)

# ------------------------------------------------------------------------

substitute (DABarDt, defD2)                   # cdb(eq12.02,DABarDt)
substitute (DABarDt, defGamma2GammaBar)       # cdb(eq12.03,DABarDt)

foo := g_{a b} -> \exp(4\phi) gBar_{a b}.
bah := g^{a b} -> \exp(-4\phi) gBar^{a b}.

substitute (DABarDt, foo)                     # cdb(eq12.04,DABarDt)
substitute (DABarDt, bah)                     # cdb(eq12.05,DABarDt)
distribute (DABarDt)                          # cdb(eq12.06,DABarDt)
eliminate_kronecker (DABarDt)                 # cdb(eq12.07,DABarDt)
substitute (DABarDt,defGBarSq)                # cdb(eq12.08,DABarDt)

DABarDt = product_sort (DABarDt)              # cdb(eq12.09,DABarDt)

rename_dummies (DABarDt)                      # cdb(eq12.10,DABarDt)
canonicalise   (DABarDt)                      # cdb(eq12.11,DABarDt)

map_sympy  (DABarDt, "simplify")              # cdb(eq12.12,DABarDt)
factor_out (DABarDt,$\exp(-4\phi)$)           # cdb(eq12.13,DABarDt)
```

```
foo := ABar^{a}_{b} -> gBar^{a c} ABar_{c b}.

substitute (DABarDt, foo)

DABarDt = product_sort (DABarDt)              # cdb(eq12.14,DABarDt)

substitute (DABarDt,defGammaBarU)             # cdb(eq12.15,DABarDt)
distribute (DABarDt)

DABarDt = product_sort (DABarDt)              # cdb(eq12.16,DABarDt)

canonicalise (DABarDt)                        # cdb(eq12.17,DABarDt)

foo := gBar^{b c} \partial_{a}{gBar_{b c}} -> 0.   # follows from det(g) = 1
bah := gBar^{e b} gBar^{f c} \partial_{a}{gBar_{b c}} -> - \partial_{a}{gBar^{e f}}.

substitute (DABarDt,foo)                      # cdb(eq12.18,DABarDt)
substitute (DABarDt,bah)                      # cdb(eq12.19,DABarDt)

DABarDt = product_sort (DABarDt)

canonicalise (DABarDt)                        # cdb(eq12.20,DABarDt)
factor_out   (DABarDt, $\exp(-4\phi)$)        # cdb(eq12.21,DABarDt)

                                              # cdb(eq12.99,DABarDt)

cdblib.put ('DABarDt',DABarDt,jsonfile)
```

$$\partial_t \bar{A}_{ij} = N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - D_{ij}N - \frac{1}{3}g_{ij}\left(NR_{ab} - D_{ab}N\right)g^{ab}\right) \tag{eq12.01}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \Gamma^c{}_{ij}\partial_c N - \frac{1}{3}g_{ij}\left(NR_{ab} - \partial_{ab}N + \Gamma^c{}_{ab}\partial_c N\right)g^{ab}\right) \tag{eq12.02}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \left(\bar{\Gamma}^c{}_{ij} + 2\bar{g}^c{}_j\partial_i\phi + 2\bar{g}^c{}_i\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N\right.$$
$$\left. - \frac{1}{3}g_{ij}\left(NR_{ab} - \partial_{ab}N + \left(\bar{\Gamma}^c{}_{ab} + 2\bar{g}^c{}_b\partial_a\phi + 2\bar{g}^c{}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c N\right)g^{ab}\right) \tag{eq12.03}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \left(\bar{\Gamma}^c{}_{ij} + 2\bar{g}^c{}_j\partial_i\phi + 2\bar{g}^c{}_i\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N\right.$$
$$\left. - \frac{1}{3}\exp\left(4\phi\right)\bar{g}_{ij}\left(NR_{ab} - \partial_{ab}N + \left(\bar{\Gamma}^c{}_{ab} + 2\bar{g}^c{}_b\partial_a\phi + 2\bar{g}^c{}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c N\right)g^{ab}\right) \tag{eq12.04}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \left(\bar{\Gamma}^c{}_{ij} + 2\bar{g}^c{}_j\partial_i\phi + 2\bar{g}^c{}_i\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N\right.$$
$$\left. - \frac{1}{3}\exp\left(4\phi\right)\bar{g}_{ij}\left(NR_{ab} - \partial_{ab}N + \left(\bar{\Gamma}^c{}_{ab} + 2\bar{g}^c{}_b\partial_a\phi + 2\bar{g}^c{}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c N\right)\exp\left(-4\phi\right)\bar{g}^{ab}\right) \tag{eq12.05}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + \exp\left(-4\phi\right)NR_{ij} - \exp\left(-4\phi\right)\partial_{ij}N + \exp\left(-4\phi\right)\bar{\Gamma}^c{}_{ij}\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^c{}_j\partial_i\phi\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^c{}_i\partial_j\phi\partial_c N$$
$$- 2\exp\left(-4\phi\right)\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N - \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}NR_{ab}\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_{ab}N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{\Gamma}^c{}_{ab}\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} - \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^c{}_b\partial_a\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^c{}_a\partial_b\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} \tag{eq12.06}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + \exp\left(-4\phi\right)NR_{ij} - \exp\left(-4\phi\right)\partial_{ij}N + \exp\left(-4\phi\right)\bar{\Gamma}^c{}_{ij}\partial_c N + 2\exp\left(-4\phi\right)\partial_i\phi\partial_j N + 2\exp\left(-4\phi\right)\partial_j\phi\partial_i N$$
$$- 2\exp\left(-4\phi\right)\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N - \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}NR_{ab}\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_{ab}N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{\Gamma}^c{}_{ab}\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} - \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_a\phi\partial_b N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_b\phi\partial_a N\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} \tag{eq12.07}$$

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + \exp(-4\phi)\, NR_{ij} - \exp(-4\phi)\,\partial_{ij}N + \exp(-4\phi)\,\bar{\Gamma}^c{}_{ij}\partial_c N + 2\exp(-4\phi)\,\partial_i\phi\partial_j N + 2\exp(-4\phi)\,\partial_j\phi\partial_i N \\
&\quad - 2\exp(-4\phi)\,\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N - \frac{1}{3}\exp(-4\phi)\exp(4\phi)\,\bar{g}_{ij}NR_{ab}\exp(-4\phi)\,\bar{g}^{ab} + \frac{1}{3}\exp(-4\phi)\exp(4\phi)\,\bar{g}_{ij}\partial_{ab}N\exp(-4\phi)\,\bar{g}^{ab} \\
&\quad - \frac{1}{3}\exp(-4\phi)\exp(4\phi)\,\bar{g}_{ij}\bar{\Gamma}^c{}_{ab}\partial_c N\exp(-4\phi)\,\bar{g}^{ab} - \frac{2}{3}\exp(-4\phi)\exp(4\phi)\,\bar{g}_{ij}\partial_a\phi\partial_b N\exp(-4\phi)\,\bar{g}^{ab} \\
&\quad - \frac{2}{3}\exp(-4\phi)\exp(4\phi)\,\bar{g}_{ij}\partial_b\phi\partial_a N\exp(-4\phi)\,\bar{g}^{ab} + 2\exp(-4\phi)\exp(4\phi)\,\bar{g}_{ij}\bar{g}^{ce}\partial_e\phi\partial_c N\exp(-4\phi) \qquad \texttt{(eq12.08)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + NR_{ij}\exp(-4\phi) - \exp(-4\phi)\,\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp(-4\phi)\,\partial_a N + 2\partial_i\phi\exp(-4\phi)\,\partial_j N + 2\partial_j\phi\exp(-4\phi)\,\partial_i N \\
&\quad - 2\bar{g}_{ij}\bar{g}^{ab}\partial_b\phi\exp(-4\phi)\,\partial_a N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi) + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_{ab}N \\
&\quad - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_c N - \frac{2}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_b N \\
&\quad + \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_b\phi\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_a N \qquad \texttt{(eq12.09)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + NR_{ij}\exp(-4\phi) - \exp(-4\phi)\,\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp(-4\phi)\,\partial_a N + 2\partial_i\phi\exp(-4\phi)\,\partial_j N + 2\partial_j\phi\exp(-4\phi)\,\partial_i N \\
&\quad - 2\bar{g}_{ij}\bar{g}^{ba}\partial_a\phi\exp(-4\phi)\,\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi) + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_{ab}N \\
&\quad - \frac{1}{3}\bar{g}_{ij}\bar{g}^{bc}\bar{\Gamma}^a{}_{bc}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_a N - \frac{2}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_b N \\
&\quad + \frac{4}{3}\bar{g}_{ij}\bar{g}^{ba}\partial_a\phi\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_b N \qquad \texttt{(eq12.10)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + NR_{ij}\exp(-4\phi) - \exp(-4\phi)\,\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp(-4\phi)\,\partial_a N + 2\partial_i\phi\exp(-4\phi)\,\partial_j N + 2\partial_j\phi\exp(-4\phi)\,\partial_i N \\
&\quad - 2\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp(-4\phi)\,\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi) + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_{ab}N \\
&\quad - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_c N + \frac{2}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp(-4\phi)\exp(-4\phi)\exp(4\phi)\,\partial_b N \qquad \texttt{(eq12.11)} \\
&= \mathrm{tr}KN\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + N\exp(-4\phi)\,R_{ij} - \exp(-4\phi)\,\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp(-4\phi)\,\partial_a N + 2\partial_i\phi\exp(-4\phi)\,\partial_j N + 2\partial_j\phi\exp(-4\phi)\,\partial_i N \\
&\quad - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp(-4\phi)\,\partial_b N - \frac{1}{3}N\exp(-4\phi)\,\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp(-4\phi)\,\partial_{ab}N - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\exp(-4\phi)\,\partial_c N \qquad \texttt{(eq12.12)} \\
&= \mathrm{tr}KN\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j \\
&\quad + \exp(-4\phi)\left(NR_{ij} - \partial_{ij}N + \bar{\Gamma}^a{}_{ij}\partial_a N + 2\partial_i\phi\partial_j N + 2\partial_j\phi\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_{ab}N - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\partial_c N\right) \qquad \texttt{(eq12.13)}
\end{aligned}
$$

$$\partial_t \bar{A}_{ij} = N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{aj}\bar{A}_{ib}\bar{g}^{ba}$$
$$+\exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \bar{\Gamma}^{a}{}_{ij}\partial_a N + 2\partial_i\phi\partial_j N + 2\partial_j\phi\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_{ab}N - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^{c}{}_{ab}\partial_c N\right) \quad \texttt{(eq12.14)}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{aj}\bar{A}_{ib}\bar{g}^{ba} + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \frac{1}{2}\bar{g}^{ae}\left(\partial_i\bar{g}_{ej} + \partial_j\bar{g}_{ie} - \partial_e\bar{g}_{ij}\right)\partial_a N + 2\partial_i\phi\partial_j N + 2\partial_j\phi\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\partial_b N\right.$$
$$\left. - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_{ab}N - \frac{1}{6}\bar{g}_{ij}\bar{g}^{ab}\bar{g}^{ce}\left(\partial_a\bar{g}_{eb} + \partial_b\bar{g}_{ae} - \partial_e\bar{g}_{ab}\right)\partial_c N\right) \quad \texttt{(eq12.15)}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{aj}\bar{A}_{ib}\bar{g}^{ba} + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \frac{1}{2}\bar{g}^{ba}\exp\left(-4\phi\right)\partial_b N\partial_i\bar{g}_{aj} + \frac{1}{2}\bar{g}^{ba}\exp\left(-4\phi\right)\partial_b N\partial_j\bar{g}_{ia}$$
$$- \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)$$
$$+ \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N - \frac{1}{6}\bar{g}_{ij}\bar{g}^{cb}\bar{g}^{da}\exp\left(-4\phi\right)\partial_d N\partial_c\bar{g}_{ab} - \frac{1}{6}\bar{g}_{ij}\bar{g}^{ac}\bar{g}^{db}\exp\left(-4\phi\right)\partial_d N\partial_c\bar{g}_{ab} + \frac{1}{6}\bar{g}_{ij}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_c N\partial_d\bar{g}_{ab} \quad \texttt{(eq12.16)}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_j\bar{g}_{ib}$$
$$- \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)$$
$$+ \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_a N\partial_c\bar{g}_{bd} + \frac{1}{6}\bar{g}_{ij}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{cd} \quad \texttt{(eq12.17)}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_j\bar{g}_{ib}$$
$$- \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)$$
$$+ \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_a N\partial_c\bar{g}_{bd} \quad \texttt{(eq12.18)}$$

$$\partial_t \bar{A}_{ij} = N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_j\bar{g}_{ib}$$
$$- \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)$$
$$+ \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N + \frac{1}{3}\bar{g}_{ij}\partial_c\bar{g}^{ac}\exp\left(-4\phi\right)\partial_a N \qquad \texttt{(eq12.19)}$$
$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_j\bar{g}_{ib}$$
$$- \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)$$
$$+ \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N + \frac{1}{3}\bar{g}_{ij}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}^{ab} \qquad \texttt{(eq12.20)}$$
$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\partial_a N\partial_j\bar{g}_{ib} - \frac{1}{2}\bar{g}^{ab}\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\partial_j N + 2\partial_j\phi\partial_i N\right.$$
$$\left. - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_{ab}N + \frac{1}{3}\bar{g}_{ij}\partial_a N\partial_b\bar{g}^{ab}\right) \qquad \texttt{(eq12.21)}$$