The ADM evolution equations.

The vacuum ADM equations, exactly as written in the following Cadabra code, are as follows.

$$\partial_t g_{ab} = -2NK_{ab} \quad \texttt{(dotgab.101)}$$

$$\partial_t K_{ab} = -N_{|ab} + N\left(R_{ab} + \mathrm{tr}K K_{ab} - 2K_{ac}K_{bd}g^{cd}\right) \quad \texttt{(dotKab.101)}$$

$$\partial_t N = 0 \quad \texttt{(dotN.101)}$$

$$\mathcal{H} = R + K_{ab}g^{ab}K_{cd}g^{cd} - K_{ab}K_{cd}g^{ac}g^{bd} \quad \texttt{(Ham.101)}$$

$$\mathcal{D}_c = g^{ab}K_{ac|b} - \partial_c\left(g^{ab}K_{ab}\right) \quad \texttt{(Mom.101)}$$

Cadabra's job was to express R_{ab}, R, N_{ab} and D_c in terms of the ADM variables and their partial derivatives. It's all plain sailing from here, so cutting to the chase, here are the results.

$$\begin{aligned} R_{ab} = {} & \frac{1}{2}\partial_a g_{bc}\partial_d g^{cd} + \frac{1}{2}\partial_b g_{ac}\partial_d g^{cd} - \frac{1}{2}\partial_c g_{ab}\partial_d g^{cd} + \frac{1}{2}g^{cd}\partial_{ac}g_{bd} + \frac{1}{2}g^{cd}\partial_{bc}g_{ad} - \frac{1}{2}g^{cd}\partial_{cd}g_{ab} + \frac{1}{4}g^{cd}g^{ef}\partial_a g_{bc}\partial_d g_{ef} + \frac{1}{4}g^{cd}g^{ef}\partial_b g_{ac}\partial_d g_{ef} \\ & - \frac{1}{4}g^{cd}g^{ef}\partial_c g_{ab}\partial_d g_{ef} - \frac{1}{4}\partial_a g_{cd}\partial_b g^{cd} - \frac{1}{2}g^{cd}\partial_{ab}g_{cd} - \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_f g_{bd} + \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_d g_{bf} \end{aligned} \quad \texttt{(Rab.112)}$$

$$R = g^{ab}\partial_a g_{bc}\partial_d g^{cd} - g^{ab}\partial_c g_{ab}\partial_d g^{cd} + g^{ab}g^{cd}\partial_{ac}g_{bd} - g^{ab}g^{cd}\partial_{ab}g_{cd} - \frac{1}{4}g^{ab}g^{cd}g^{ef}\partial_a g_{cd}\partial_b g_{ef} - \frac{3}{4}g^{ab}\partial_a g_{cd}\partial_b g^{cd} + \frac{1}{2}g^{ab}\partial_c g_{ad}\partial_b g^{cd} \quad \texttt{(R.110)}$$

$$N_{|ab} = \partial_{ab}N - \frac{1}{2}g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\partial_c N \quad \texttt{(Nab.102)}$$

$$\mathcal{D}_c = g^{ab}\partial_a K_{cb} + K_{ca}\partial_b g^{ab} + \frac{1}{2}g^{ab}g^{de}K_{ca}\partial_b g_{de} - \frac{1}{2}K_{ab}\partial_c g^{ab} - g^{ab}\partial_c K_{ab} \quad \texttt{(Mom.110)}$$

The ADM evolution equations. The big picture.

```
from shared import *
import cdblib

jsonfile = 'adm-eqtns.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------------
# generic rules for covariant derivs

deriv1 := A?_{; m}   -> \partial_{m}{A?}.                                    # cdb(deriv1.lhs,deriv1)
deriv2 := A?_{; m n} -> \partial_{m}{A?_{; n}} - \Gamma^{c}_{m n} A?_{; c}.  # cdb(deriv2.lhs,deriv2)

substitute (deriv2, deriv1)                                                  # cdb (deriv2.101,deriv2)

deriv3 := A?_{m n ; p} -> \partial_{p}{A?_{m n}} - \Gamma^{c}_{m p} A?_{c n}
                                                 - \Gamma^{c}_{n p} A?_{m c}.  # cdb(deriv3.lhs,deriv3)

# ----------------------------------------------------------------------------------
# partial derivs of g_{ab} in terms of partial of g^{ab}

defDgab := {g^{a e} g^{b f} \partial_{c}{g_{e f}} -> - \partial_{c}{g^{a b}},
            g^{e a} g^{b f} \partial_{c}{g_{e f}} -> - \partial_{c}{g^{a b}},
            g^{a e} g^{f b} \partial_{c}{g_{e f}} -> - \partial_{c}{g^{a b}},
            g^{e a} g^{f b} \partial_{c}{g_{e f}} -> - \partial_{c}{g^{a b}}}.  # cdb (defDgab.lhs,defDgab)

# ----------------------------------------------------------------------------------
# standard defintions

defGamma := \Gamma^{a}_{b c} ->
            (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                            + \partial_{c}{g_{b e}}
                            - \partial_{e}{g_{b c}}).                        # cdb (defGamma.lhs,defGamma)

defRabcd := R^{a}_{b c d} ->
            \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
          - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.  # cdb (defRabcd.lhs,defRabcd)
```

```
defRab := R_{a b} -> R^{c}_{a c b}.                                       # cdb (defRab.lhs,defRab)

# ---------------------------------------------------------------------------------
# Ricci tensor

Rab := R_{a b}.                                     # cdb (Rab.lhs,Rab)

substitute     (Rab, defRab)                        # cdb (Rab.101,Rab)
substitute     (Rab, defRabcd)                      # cdb (Rab.102,Rab)
substitute     (Rab, defGamma)                      # cdb (Rab.103,Rab)
product_rule   (Rab)                                # cdb (Rab.104,Rab)
distribute     (Rab)                                # cdb (Rab.105,Rab)

Rab = product_sort (Rab)                            # cdb (Rab.106,Rab)

rename_dummies (Rab)                                # cdb (Rab.107,Rab)
canonicalise   (Rab)                                # cdb (Rab.108,Rab)
substitute     (Rab, defDgab)                       # cdb (Rab.109,Rab)

Rab = product_sort (Rab)                            # cdb (Rab.110,Rab)

rename_dummies (Rab)                                # cdb (Rab.111,Rab)
canonicalise   (Rab)                                # cdb (Rab.112,Rab)

defRab := R_{a b} -> @(Rab).

# ---------------------------------------------------------------------------------
# Ricci scalar

Rscalar := R.                                       # cdb (R.lhs,Rscalar)
Rscalar := g^{a b} R_{a b}.                         # cdb (R.101,Rscalar)

substitute     (Rscalar, defRab)                    # cdb (R.102,Rscalar)
distribute     (Rscalar)                            # cdb (R.103,Rscalar)

Rscalar = product_sort (Rscalar)                    # cdb (R.104,Rscalar)
```

```
rename_dummies (Rscalar)                               # cdb (R.105,Rscalar)
canonicalise   (Rscalar)                               # cdb (R.106,Rscalar)
substitute     (Rscalar, defDgab)                      # cdb (R.107,Rscalar)

Rscalar = product_sort (Rscalar)                       # cdb (R.108,Rscalar)

rename_dummies (Rscalar)                               # cdb (R.109,Rscalar)
canonicalise   (Rscalar)                               # cdb (R.110,Rscalar)

defRscalar := R -> @(Rscalar).

# ------------------------------------------------------------------------------------
# Hessian

Nab := N_{; a b}.                                      # cdb (Nab.lhs,Nab)

substitute (Nab, deriv2)                               # cdb (Nab.101,Nab)
substitute (Nab, defGamma)                             # cdb (Nab.102,Nab)

defHess := N_{; a b} -> @(Nab).                        # cdb (Hess.lhs,defHess)

# ------------------------------------------------------------------------------------
# ADM evolution equations

DgabDt := \partial_{t}{g_{a b}}.                       # cdb (dotgab.lhs,DgabDt)
DKabDt := \partial_{t}{K_{a b}}.                       # cdb (dotKab.lhs,DKabDt)
DNDt   := \partial_{t}{N}.                             # cdb (dotN.lhs,DNDt)

DgabDt := -2 N K_{a b}.                                                  # cdb (dotgab.101,DgabDt)
DKabDt := -N_{; a b} + N (R_{a b} + trK K_{a b} - 2 K_{a c} K_{b d} g^{c d}).  # cdb (dotKab.101,DKabDt)
# DNDt := -2 N trK.     # 1+log
# DNDt := -N*N trK.     # Harmonic
DNDt := 0.                                             # cdb (dotN.101,DNDt)  # Static

substitute (DKabDt,defHess)                            # cdb (dotKab.102,DKabDt)
distribute (DKabDt)                                    # cdb (dotKab.103,DKabDt)

# ------------------------------------------------------------------------------------
```

```
# The Hamiltonian contsraint

defHam := Ham     -> R + K_{a b} g^{a b} K_{c d} g^{c d} - K_{a b} K_{c d} g^{a c} g^{b d}.
Ham    := Ham.                                      # cdb (Ham.lhs,Ham)
substitute     (Ham, defHam)                        # cdb (Ham.101,Ham)

canonicalise   (Ham)                                # cdb (Ham.102,Ham)

# ----------------------------------------------------------------------------------
# The momentum contsraint

defMom := Mom_{c} -> g^{a b} K_{a c ; b} - \partial_{c}{g^{a b} K_{a b}}.
Mom    := Mom_{c}.                                  # cdb (Mom.lhs,Mom)
substitute     (Mom, defMom)                        # cdb (Mom.101,Mom)

substitute     (Mom, deriv3)                        # cdb (Mom.102,Mom)
product_rule   (Mom)                                # cdb (Mom.103,Mom)
distribute     (Mom)                                # cdb (Mom.104,Mom)
substitute     (Mom, defGamma)                      # cdb (Mom.105,Mom)
distribute     (Mom)                                # cdb (Mom.106,Mom)
substitute     (Mom, defDgab)                       # cdb (Mom.107,Mom)

Mom = product_sort (Mom)                            # cdb (Mom.108,Mom)

rename_dummies (Mom)                                # cdb (Mom.109,Mom)
canonicalise   (Mom)                                # cdb (Mom.110,Mom)

cdblib.put ('Rscalar', Rscalar, jsonfile)
cdblib.put ('Rab',     Rab,     jsonfile)
cdblib.put ('Nab',     Nab,     jsonfile)
cdblib.put ('DgabDt',  DgabDt,  jsonfile)
cdblib.put ('DKabDt',  DKabDt,  jsonfile)
cdblib.put ('DNDt',    DNDt,    jsonfile)
cdblib.put ('Ham',     Ham,     jsonfile)
cdblib.put ('Mom',     Mom,     jsonfile)
```

The Hessian of the lapse.

$$
\begin{aligned}
N_{|ab} &= \partial_{ab}N - \Gamma^{c}{}_{ab}\partial_c N && (\texttt{Nab.101})\\
&= \partial_{ab}N - \frac{1}{2}g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\partial_c N && (\texttt{Nab.102})
\end{aligned}
$$

The Ricci curvature.

$$
\begin{aligned}
R_{ab} &= R^{c}{}_{acb} && \texttt{(Rab.101)}\\
&= \partial_c \Gamma^{c}{}_{ab} + \Gamma^{c}{}_{ec}\Gamma^{e}{}_{ab} - \partial_b \Gamma^{c}{}_{ac} - \Gamma^{c}{}_{eb}\Gamma^{e}{}_{ac} && \texttt{(Rab.102)}\\
&= \frac{1}{2}\partial_c \left(g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\right) + \frac{1}{4}g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right) - \frac{1}{2}\partial_b \left(g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right)\right)\\
&\quad - \frac{1}{4}g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right) g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) && \texttt{(Rab.103)}\\
&= \frac{1}{2}\partial_c g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right) + \frac{1}{2}g^{ce}\partial_c\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right) + \frac{1}{4}g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right)\\
&\quad - \frac{1}{2}\partial_b g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - \frac{1}{2}g^{ce}\partial_b\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - \frac{1}{4}g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right) g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) && \texttt{(Rab.104)}\\
&= \frac{1}{2}\partial_c g^{ce}\partial_a g_{eb} + \frac{1}{2}\partial_c g^{ce}\partial_b g_{ae} - \frac{1}{2}\partial_c g^{ce}\partial_e g_{ab} + \frac{1}{2}g^{ce}\partial_{ca} g_{eb} + \frac{1}{2}g^{ce}\partial_{cb} g_{ae} - \frac{1}{2}g^{ce}\partial_{ce} g_{ab} + \frac{1}{4}g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{fb} + \frac{1}{4}g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{af} - \frac{1}{4}g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ab}\\
&\quad + \frac{1}{4}g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{fb} + \frac{1}{4}g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{af} - \frac{1}{4}g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ab} - \frac{1}{4}g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{fb} - \frac{1}{4}g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{af} + \frac{1}{4}g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ab} - \frac{1}{2}\partial_b g^{ce}\partial_a g_{ec}\\
&\quad - \frac{1}{2}\partial_b g^{ce}\partial_c g_{ae} + \frac{1}{2}\partial_b g^{ce}\partial_e g_{ac} - \frac{1}{2}g^{ce}\partial_{ba} g_{ec} - \frac{1}{2}g^{ce}\partial_{bc} g_{ae} + \frac{1}{2}g^{ce}\partial_{be} g_{ac} - \frac{1}{4}g^{cd}\partial_e g_{db} g^{ef}\partial_a g_{fc} - \frac{1}{4}g^{cd}\partial_e g_{db} g^{ef}\partial_c g_{af} + \frac{1}{4}g^{cd}\partial_e g_{db} g^{ef}\partial_f g_{ac}\\
&\quad - \frac{1}{4}g^{cd}\partial_b g_{ed} g^{ef}\partial_a g_{fc} - \frac{1}{4}g^{cd}\partial_b g_{ed} g^{ef}\partial_c g_{af} + \frac{1}{4}g^{cd}\partial_b g_{ed} g^{ef}\partial_f g_{ac} + \frac{1}{4}g^{cd}\partial_d g_{eb} g^{ef}\partial_a g_{fc} + \frac{1}{4}g^{cd}\partial_d g_{eb} g^{ef}\partial_c g_{af} - \frac{1}{4}g^{cd}\partial_d g_{eb} g^{ef}\partial_f g_{ac} && \texttt{(Rab.105)}\\
&= \frac{1}{2}\partial_a g_{cb}\partial_d g^{dc} + \frac{1}{2}\partial_b g_{ac}\partial_d g^{dc} - \frac{1}{2}\partial_d g_{ab}\partial_c g^{cd} + \frac{1}{2}g^{dc}\partial_{da} g_{cb} + \frac{1}{2}g^{dc}\partial_{db} g_{ac} - \frac{1}{2}g^{cd}\partial_{cd} g_{ab} + \frac{1}{4}g^{ed}g^{fc}\partial_a g_{cb}\partial_f g_{de} + \frac{1}{4}g^{ed}g^{fc}\partial_b g_{ac}\partial_f g_{de} - \frac{1}{4}g^{dc}g^{ef}\partial_e g_{cd}\partial_f g_{ab}\\
&\quad + \frac{1}{4}g^{fe}g^{dc}\partial_a g_{cb}\partial_f g_{de} + \frac{1}{4}g^{fe}g^{dc}\partial_b g_{ac}\partial_f g_{de} - \frac{1}{4}g^{ed}g^{cf}\partial_e g_{cd}\partial_f g_{ab} - \frac{1}{4}g^{ef}g^{dc}\partial_a g_{cb}\partial_f g_{de} - \frac{1}{4}g^{ef}g^{dc}\partial_b g_{ac}\partial_f g_{de} + \frac{1}{4}g^{de}g^{cf}\partial_e g_{cd}\partial_f g_{ab} - \frac{1}{2}\partial_a g_{cd}\partial_b g^{dc}\\
&\quad - \frac{1}{2}\partial_d g_{ac}\partial_b g^{dc} + \frac{1}{2}\partial_d g_{ac}\partial_b g^{cd} - \frac{1}{2}g^{dc}\partial_{ba} g_{cd} - \frac{1}{2}g^{dc}\partial_{bd} g_{ac} + \frac{1}{2}g^{cd}\partial_{bd} g_{ac} - \frac{1}{4}g^{de}g^{fc}\partial_a g_{cd}\partial_f g_{eb} - \frac{1}{4}g^{ed}g^{fc}\partial_e g_{ac}\partial_f g_{db} + \frac{1}{4}g^{dc}g^{ef}\partial_e g_{cb}\partial_f g_{ad}\\
&\quad - \frac{1}{4}g^{df}g^{ec}\partial_a g_{cd}\partial_b g_{ef} - \frac{1}{4}g^{fd}g^{ce}\partial_b g_{cd}\partial_f g_{ae} + \frac{1}{4}g^{ed}g^{cf}\partial_b g_{cd}\partial_f g_{ae} + \frac{1}{4}g^{df}g^{ec}\partial_a g_{cd}\partial_f g_{eb} + \frac{1}{4}g^{ef}g^{dc}\partial_e g_{ac}\partial_f g_{db} - \frac{1}{4}g^{de}g^{cf}\partial_e g_{cb}\partial_f g_{ad} && \texttt{(Rab.106)}
\end{aligned}
$$

$$
\begin{aligned}
R_{ab} = {}& \frac{1}{2}\partial_a g_{db}\partial_c g^{cd} + \frac{1}{2}\partial_b g_{ad}\partial_c g^{cd} - \frac{1}{2}\partial_c g_{ab}\partial_d g^{dc} + \frac{1}{2}g^{cd}\partial_{ca}g_{db} + \frac{1}{2}g^{cd}\partial_{cb}g_{ad} - \frac{1}{2}g^{cd}\partial_{cd}g_{ab} + \frac{1}{4}g^{de}g^{cf}\partial_a g_{fb}\partial_c g_{ed} + \frac{1}{4}g^{de}g^{cf}\partial_b g_{af}\partial_c g_{ed} - \frac{1}{4}g^{ef}g^{cd}\partial_c g_{fe}\partial_d g_{ab} \\
& + \frac{1}{4}g^{cd}g^{ef}\partial_a g_{fb}\partial_c g_{ed} + \frac{1}{4}g^{cd}g^{ef}\partial_b g_{af}\partial_c g_{ed} - \frac{1}{4}g^{ce}g^{fd}\partial_c g_{fe}\partial_d g_{ab} - \frac{1}{4}g^{dc}g^{ef}\partial_a g_{fb}\partial_c g_{ed} - \frac{1}{4}g^{dc}g^{ef}\partial_b g_{af}\partial_c g_{ed} + \frac{1}{4}g^{ec}g^{fd}\partial_c g_{fe}\partial_d g_{ab} - \frac{1}{2}\partial_a g_{cd}\partial_b g^{dc} \\
& - \frac{1}{2}\partial_c g_{ad}\partial_b g^{cd} + \frac{1}{2}\partial_c g_{ad}\partial_b g^{dc} - \frac{1}{2}g^{cd}\partial_{ba}g_{dc} - \frac{1}{2}g^{cd}\partial_{bc}g_{ad} + \frac{1}{2}g^{dc}\partial_{bc}g_{ad} - \frac{1}{4}g^{de}g^{cf}\partial_a g_{fd}\partial_c g_{eb} - \frac{1}{4}g^{ce}g^{df}\partial_c g_{af}\partial_d g_{eb} + \frac{1}{4}g^{ef}g^{cd}\partial_c g_{fb}\partial_d g_{ae} \\
& - \frac{1}{4}g^{cd}g^{ef}\partial_a g_{fc}\partial_b g_{ed} - \frac{1}{4}g^{cd}g^{ef}\partial_b g_{ed}\partial_c g_{af} + \frac{1}{4}g^{de}g^{fc}\partial_b g_{fe}\partial_c g_{ad} + \frac{1}{4}g^{dc}g^{ef}\partial_a g_{fd}\partial_c g_{eb} + \frac{1}{4}g^{cd}g^{ef}\partial_c g_{af}\partial_d g_{eb} - \frac{1}{4}g^{ec}g^{fd}\partial_c g_{fb}\partial_d g_{ae} \qquad (\texttt{Rab.107}) \\
= {}& \frac{1}{2}\partial_a g_{bc}\partial_d g^{cd} + \frac{1}{2}\partial_b g_{ac}\partial_d g^{cd} - \frac{1}{2}\partial_c g_{ab}\partial_d g^{cd} + \frac{1}{2}g^{cd}\partial_{ac}g_{bd} + \frac{1}{2}g^{cd}\partial_{bc}g_{ad} - \frac{1}{2}g^{cd}\partial_{cd}g_{ab} + \frac{1}{4}g^{cd}g^{ef}\partial_a g_{bc}\partial_d g_{ef} + \frac{1}{4}g^{cd}g^{ef}\partial_b g_{ac}\partial_d g_{ef} \\
& - \frac{1}{4}g^{cd}g^{ef}\partial_c g_{ab}\partial_d g_{ef} - \frac{1}{2}\partial_a g_{cd}\partial_b g^{cd} - \frac{1}{2}g^{cd}\partial_{ab}g_{cd} - \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_f g_{bd} + \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_d g_{bf} - \frac{1}{4}g^{cd}g^{ef}\partial_a g_{ce}\partial_b g_{df} \qquad (\texttt{Rab.108}) \\
= {}& \frac{1}{2}\partial_a g_{bc}\partial_d g^{cd} + \frac{1}{2}\partial_b g_{ac}\partial_d g^{cd} - \frac{1}{2}\partial_c g_{ab}\partial_d g^{cd} + \frac{1}{2}g^{cd}\partial_{ac}g_{bd} + \frac{1}{2}g^{cd}\partial_{bc}g_{ad} - \frac{1}{2}g^{cd}\partial_{cd}g_{ab} + \frac{1}{4}g^{cd}g^{ef}\partial_a g_{bc}\partial_d g_{ef} + \frac{1}{4}g^{cd}g^{ef}\partial_b g_{ac}\partial_d g_{ef} \\
& - \frac{1}{4}g^{cd}g^{ef}\partial_c g_{ab}\partial_d g_{ef} - \frac{1}{2}\partial_a g_{cd}\partial_b g^{cd} - \frac{1}{2}g^{cd}\partial_{ab}g_{cd} - \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_f g_{bd} + \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_d g_{bf} + \frac{1}{4}\partial_b g^{ce}\partial_a g_{ce} \qquad (\texttt{Rab.109}) \\
= {}& \frac{1}{2}\partial_a g_{bc}\partial_d g^{cd} + \frac{1}{2}\partial_b g_{ac}\partial_d g^{cd} - \frac{1}{2}\partial_c g_{ab}\partial_d g^{cd} + \frac{1}{2}g^{dc}\partial_{ad}g_{bc} + \frac{1}{2}g^{dc}\partial_{bd}g_{ac} - \frac{1}{2}g^{cd}\partial_{cd}g_{ab} + \frac{1}{4}g^{cf}g^{de}\partial_a g_{bc}\partial_f g_{de} + \frac{1}{4}g^{cf}g^{de}\partial_b g_{ac}\partial_f g_{de} \\
& - \frac{1}{4}g^{ef}g^{cd}\partial_e g_{ab}\partial_f g_{cd} - \frac{1}{4}\partial_a g_{cd}\partial_b g^{cd} - \frac{1}{2}g^{cd}\partial_{ab}g_{cd} - \frac{1}{2}g^{ed}g^{cf}\partial_e g_{ac}\partial_f g_{bd} + \frac{1}{2}g^{ef}g^{cd}\partial_e g_{ac}\partial_f g_{bd} \qquad (\texttt{Rab.110}) \\
= {}& \frac{1}{2}\partial_a g_{bd}\partial_c g^{dc} + \frac{1}{2}\partial_b g_{ad}\partial_c g^{dc} - \frac{1}{2}\partial_c g_{ab}\partial_d g^{cd} + \frac{1}{2}g^{cd}\partial_{ac}g_{bd} + \frac{1}{2}g^{cd}\partial_{bc}g_{ad} - \frac{1}{2}g^{cd}\partial_{cd}g_{ab} + \frac{1}{4}g^{dc}g^{ef}\partial_a g_{bd}\partial_c g_{ef} + \frac{1}{4}g^{dc}g^{ef}\partial_b g_{ad}\partial_c g_{ef} \\
& - \frac{1}{4}g^{cd}g^{ef}\partial_c g_{ab}\partial_d g_{ef} - \frac{1}{4}\partial_a g_{cd}\partial_b g^{cd} - \frac{1}{2}g^{cd}\partial_{ab}g_{cd} - \frac{1}{2}g^{ce}g^{fd}\partial_c g_{af}\partial_d g_{be} + \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_d g_{bf} \qquad (\texttt{Rab.111}) \\
= {}& \frac{1}{2}\partial_a g_{bc}\partial_d g^{cd} + \frac{1}{2}\partial_b g_{ac}\partial_d g^{cd} - \frac{1}{2}\partial_c g_{ab}\partial_d g^{cd} + \frac{1}{2}g^{cd}\partial_{ac}g_{bd} + \frac{1}{2}g^{cd}\partial_{bc}g_{ad} - \frac{1}{2}g^{cd}\partial_{cd}g_{ab} + \frac{1}{4}g^{cd}g^{ef}\partial_a g_{bc}\partial_d g_{ef} + \frac{1}{4}g^{cd}g^{ef}\partial_b g_{ac}\partial_d g_{ef} \\
& - \frac{1}{4}g^{cd}g^{ef}\partial_c g_{ab}\partial_d g_{ef} - \frac{1}{4}\partial_a g_{cd}\partial_b g^{cd} - \frac{1}{2}g^{cd}\partial_{ab}g_{cd} - \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_f g_{bd} + \frac{1}{2}g^{cd}g^{ef}\partial_c g_{ae}\partial_d g_{bf} \qquad (\texttt{Rab.112})
\end{aligned}
$$

The Ricci scalar.

$$R = g^{ab} R_{ab} \tag{R.101}$$

$$\begin{aligned}
&= g^{ab} \Big(\frac{1}{2} \partial_a g_{bc} \partial_d g^{cd} + \frac{1}{2} \partial_b g_{ac} \partial_d g^{cd} - \frac{1}{2} \partial_c g_{ab} \partial_d g^{cd} + \frac{1}{2} g^{cd} \partial_{ac} g_{bd} + \frac{1}{2} g^{cd} \partial_{bc} g_{ad} - \frac{1}{2} g^{cd} \partial_{cd} g_{ab} + \frac{1}{4} g^{cd} g^{ef} \partial_a g_{bc} \partial_d g_{ef} + \frac{1}{4} g^{cd} g^{ef} \partial_b g_{ac} \partial_d g_{ef} \\
&\qquad - \frac{1}{4} g^{cd} g^{ef} \partial_c g_{ab} \partial_d g_{ef} - \frac{1}{4} \partial_a g_{cd} \partial_b g^{cd} - \frac{1}{2} g^{cd} \partial_{ab} g_{cd} - \frac{1}{2} g^{cd} g^{ef} \partial_c g_{ae} \partial_f g_{bd} + \frac{1}{2} g^{cd} g^{ef} \partial_c g_{ae} \partial_d g_{bf} \Big)
\end{aligned} \tag{R.102}$$

$$\begin{aligned}
&= \frac{1}{2} g^{ab} \partial_a g_{bc} \partial_d g^{cd} + \frac{1}{2} g^{ab} \partial_b g_{ac} \partial_d g^{cd} - \frac{1}{2} g^{ab} \partial_c g_{ab} \partial_d g^{cd} + \frac{1}{2} g^{ab} g^{cd} \partial_{ac} g_{bd} + \frac{1}{2} g^{ab} g^{cd} \partial_{bc} g_{ad} - \frac{1}{2} g^{ab} g^{cd} \partial_{cd} g_{ab} + \frac{1}{4} g^{ab} g^{cd} g^{ef} \partial_a g_{bc} \partial_d g_{ef} \\
&\quad + \frac{1}{4} g^{ab} g^{cd} g^{ef} \partial_b g_{ac} \partial_d g_{ef} - \frac{1}{4} g^{ab} g^{cd} g^{ef} \partial_c g_{ab} \partial_d g_{ef} - \frac{1}{4} g^{ab} \partial_a g_{cd} \partial_b g^{cd} - \frac{1}{2} g^{ab} g^{cd} \partial_{ab} g_{cd} - \frac{1}{2} g^{ab} g^{cd} g^{ef} \partial_c g_{ae} \partial_f g_{bd} + \frac{1}{2} g^{ab} g^{cd} g^{ef} \partial_c g_{ae} \partial_d g_{bf}
\end{aligned} \tag{R.103}$$

$$\begin{aligned}
&= \frac{1}{2} g^{ca} \partial_c g_{ab} \partial_d g^{bd} + \frac{1}{2} g^{ac} \partial_c g_{ab} \partial_d g^{bd} - \frac{1}{2} g^{ab} \partial_c g_{ab} \partial_d g^{cd} + \frac{1}{2} g^{ca} g^{db} \partial_{cd} g_{ab} + \frac{1}{2} g^{ac} g^{db} \partial_{cd} g_{ab} - \frac{1}{2} g^{ab} g^{cd} \partial_{cd} g_{ab} + \frac{1}{4} g^{ea} g^{bf} g^{cd} \partial_e g_{ab} \partial_f g_{cd} \\
&\quad + \frac{1}{4} g^{ae} g^{bf} g^{cd} \partial_e g_{ab} \partial_f g_{cd} - \frac{1}{4} g^{ab} g^{ef} g^{cd} \partial_e g_{ab} \partial_f g_{cd} - \frac{1}{4} g^{cd} \partial_c g_{ab} \partial_d g^{ab} - \frac{1}{2} g^{cd} g^{ab} \partial_{cd} g_{ab} - \frac{1}{2} g^{ac} g^{ed} g^{bf} \partial_e g_{ab} \partial_f g_{cd} + \frac{1}{2} g^{ac} g^{ef} g^{bd} \partial_e g_{ab} \partial_f g_{cd}
\end{aligned} \tag{R.104}$$

$$\begin{aligned}
&= \frac{1}{2} g^{ac} \partial_a g_{cd} \partial_b g^{db} + \frac{1}{2} g^{ca} \partial_a g_{cd} \partial_b g^{db} - \frac{1}{2} g^{cd} \partial_a g_{cd} \partial_b g^{ab} + \frac{1}{2} g^{ac} g^{bd} \partial_{ab} g_{cd} + \frac{1}{2} g^{ca} g^{bd} \partial_{ab} g_{cd} - \frac{1}{2} g^{cd} g^{ab} \partial_{ab} g_{cd} + \frac{1}{4} g^{ac} g^{db} g^{ef} \partial_a g_{cd} \partial_b g_{ef} \\
&\quad + \frac{1}{4} g^{ca} g^{db} g^{ef} \partial_a g_{cd} \partial_b g_{ef} - \frac{1}{4} g^{cd} g^{ab} g^{ef} \partial_a g_{cd} \partial_b g_{ef} - \frac{1}{4} g^{ab} \partial_a g_{cd} \partial_b g^{cd} - \frac{1}{2} g^{ab} g^{cd} \partial_{ab} g_{cd} - \frac{1}{2} g^{cd} g^{ae} g^{fb} \partial_a g_{cf} \partial_b g_{de} + \frac{1}{2} g^{cd} g^{ab} g^{ef} \partial_a g_{ce} \partial_b g_{df}
\end{aligned} \tag{R.105}$$

$$\begin{aligned}
&= g^{ab} \partial_a g_{bc} \partial_d g^{cd} - \frac{1}{2} g^{ab} \partial_c g_{ab} \partial_d g^{cd} + g^{ab} g^{cd} \partial_{ac} g_{bd} - g^{ab} g^{cd} \partial_{ab} g_{cd} + \frac{1}{2} g^{ab} g^{cd} g^{ef} \partial_a g_{bc} \partial_d g_{ef} - \frac{1}{4} g^{ab} g^{cd} g^{ef} \partial_a g_{cd} \partial_b g_{ef} - \frac{1}{4} g^{ab} \partial_a g_{cd} \partial_b g^{cd} \\
&\quad - \frac{1}{2} g^{ab} g^{cd} g^{ef} \partial_a g_{ce} \partial_d g_{bf} + \frac{1}{2} g^{ab} g^{cd} g^{ef} \partial_a g_{ce} \partial_b g_{df}
\end{aligned} \tag{R.106}$$

$$\begin{aligned}
&= g^{ab} \partial_a g_{bc} \partial_d g^{cd} - \frac{1}{2} g^{ab} \partial_c g_{ab} \partial_d g^{cd} + g^{ab} g^{cd} \partial_{ac} g_{bd} - g^{ab} g^{cd} \partial_{ab} g_{cd} - \frac{1}{2} \partial_a g^{ad} g^{ef} \partial_d g_{ef} - \frac{1}{4} g^{ab} g^{cd} g^{ef} \partial_a g_{cd} \partial_b g_{ef} - \frac{1}{4} g^{ab} \partial_a g_{cd} \partial_b g^{cd} + \frac{1}{2} \partial_d g^{ae} g^{cd} \partial_a g_{ce} \\
&\quad - \frac{1}{2} g^{ab} \partial_b g^{ce} \partial_a g_{ce}
\end{aligned} \tag{R.107}$$

$$R = g^{ca}\partial_c g_{ab}\partial_d g^{bd} - \frac{1}{2}g^{ab}\partial_c g_{ab}\partial_d g^{cd} + g^{ca}g^{db}\partial_{cd}g_{ab} - g^{cd}g^{ab}\partial_{cd}g_{ab} - \frac{1}{2}g^{ab}\partial_d g_{ab}\partial_c g^{cd} - \frac{1}{4}g^{ef}g^{ab}g^{cd}\partial_e g_{ab}\partial_f g_{cd} - \frac{3}{4}g^{cd}\partial_c g_{ab}\partial_d g^{ab} + \frac{1}{2}g^{ac}\partial_d g_{ab}\partial_c g^{db} \tag{R.108}$$

$$= g^{ac}\partial_a g_{cd}\partial_b g^{db} - \frac{1}{2}g^{cd}\partial_a g_{cd}\partial_b g^{ab} + g^{ac}g^{bd}\partial_{ab}g_{cd} - g^{ab}g^{cd}\partial_{ab}g_{cd} - \frac{1}{2}g^{cd}\partial_a g_{cd}\partial_b g^{ba} - \frac{1}{4}g^{ab}g^{cd}g^{ef}\partial_a g_{cd}\partial_b g_{ef} - \frac{3}{4}g^{ab}\partial_a g_{cd}\partial_b g^{cd} + \frac{1}{2}g^{cb}\partial_a g_{cd}\partial_b g^{ad} \tag{R.109}$$

$$= g^{ab}\partial_a g_{bc}\partial_d g^{cd} - g^{ab}\partial_c g_{ab}\partial_d g^{cd} + g^{ab}g^{cd}\partial_{ac}g_{bd} - g^{ab}g^{cd}\partial_{ab}g_{cd} - \frac{1}{4}g^{ab}g^{cd}g^{ef}\partial_a g_{cd}\partial_b g_{ef} - \frac{3}{4}g^{ab}\partial_a g_{cd}\partial_b g^{cd} + \frac{1}{2}g^{ab}\partial_c g_{ad}\partial_b g^{cd} \tag{R.110}$$

The ADM constraints.

$$\begin{aligned}\mathcal{H} &= R + K_{ab}g^{ab}K_{cd}g^{cd} - K_{ab}K_{cd}g^{ac}g^{bd} && \texttt{(Ham.101)}\\ &= R + K_{ab}g^{ab}K_{cd}g^{cd} - K_{ab}K_{cd}g^{ac}g^{bd} && \texttt{(Ham.102)}\end{aligned}$$

$$\begin{aligned}\mathcal{D}_c &= g^{ab}K_{ac|b} - \partial_c\left(g^{ab}K_{ab}\right) && \texttt{(Mom.101)}\\ &= g^{ab}\left(\partial_b K_{ac} - \Gamma^d{}_{ab}K_{dc} - \Gamma^d{}_{cb}K_{ad}\right) - \partial_c\left(g^{ab}K_{ab}\right) && \texttt{(Mom.102)}\\ &= g^{ab}\left(\partial_b K_{ac} - \Gamma^d{}_{ab}K_{dc} - \Gamma^d{}_{cb}K_{ad}\right) - \partial_c g^{ab}K_{ab} - g^{ab}\partial_c K_{ab} && \texttt{(Mom.103)}\\ &= g^{ab}\partial_b K_{ac} - g^{ab}\Gamma^d{}_{ab}K_{dc} - g^{ab}\Gamma^d{}_{cb}K_{ad} - \partial_c g^{ab}K_{ab} - g^{ab}\partial_c K_{ab} && \texttt{(Mom.104)}\\ &= g^{ab}\partial_b K_{ac} - \frac{1}{2}g^{ab}g^{de}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)K_{dc} - \frac{1}{2}g^{ab}g^{de}\left(\partial_c g_{eb} + \partial_b g_{ce} - \partial_e g_{cb}\right)K_{ad} - \partial_c g^{ab}K_{ab} - g^{ab}\partial_c K_{ab} && \texttt{(Mom.105)}\\ &= g^{ab}\partial_b K_{ac} - \frac{1}{2}g^{ab}g^{de}\partial_a g_{eb}K_{dc} - \frac{1}{2}g^{ab}g^{de}\partial_b g_{ae}K_{dc} + \frac{1}{2}g^{ab}g^{de}\partial_e g_{ab}K_{dc} - \frac{1}{2}g^{ab}g^{de}\partial_c g_{eb}K_{ad} - \frac{1}{2}g^{ab}g^{de}\partial_b g_{ce}K_{ad} + \frac{1}{2}g^{ab}g^{de}\partial_e g_{cb}K_{ad} - \partial_c g^{ab}K_{ab}\\ &\quad - g^{ab}\partial_c K_{ab} && \texttt{(Mom.106)}\\ &= g^{ab}\partial_b K_{ac} + \frac{1}{2}\partial_a g^{da}K_{dc} + \frac{1}{2}\partial_b g^{bd}K_{dc} + \frac{1}{2}g^{ab}g^{de}\partial_e g_{ab}K_{dc} + \frac{1}{2}\partial_c g^{da}K_{ad} - \frac{1}{2}g^{ab}g^{de}\partial_b g_{ce}K_{ad} + \frac{1}{2}g^{ab}g^{de}\partial_e g_{cb}K_{ad} - \partial_c g^{ab}K_{ab} - g^{ab}\partial_c K_{ab} && \texttt{(Mom.107)}\\ &= g^{ab}\partial_b K_{ac} + \frac{1}{2}K_{ac}\partial_b g^{ab} + \frac{1}{2}K_{bc}\partial_a g^{ab} + \frac{1}{2}g^{ab}g^{de}K_{dc}\partial_e g_{ab} + \frac{1}{2}K_{ba}\partial_c g^{ab} - \frac{1}{2}g^{bd}g^{ea}K_{be}\partial_d g_{ca} + \frac{1}{2}g^{ba}g^{de}K_{bd}\partial_e g_{ca} - K_{ab}\partial_c g^{ab} - g^{ab}\partial_c K_{ab} && \texttt{(Mom.108)}\\ &= g^{ab}\partial_b K_{ac} + \frac{1}{2}K_{ac}\partial_b g^{ab} + \frac{1}{2}K_{ac}\partial_b g^{ba} + \frac{1}{2}g^{de}g^{ab}K_{ac}\partial_b g_{de} + \frac{1}{2}K_{ab}\partial_c g^{ba} - \frac{1}{2}g^{ad}g^{be}K_{ab}\partial_d g_{ce} + \frac{1}{2}g^{ae}g^{bd}K_{ab}\partial_d g_{ce} - K_{ab}\partial_c g^{ab} - g^{ab}\partial_c K_{ab} && \texttt{(Mom.109)}\\ &= g^{ab}\partial_a K_{cb} + K_{ca}\partial_b g^{ab} + \frac{1}{2}g^{ab}g^{de}K_{ca}\partial_b g_{de} - \frac{1}{2}K_{ab}\partial_c g^{ab} - g^{ab}\partial_c K_{ab} && \texttt{(Mom.110)}\end{aligned}$$