The BSSN evolution equations. Pt.1

These are the equations as given in the Phys.Rev.D (62) 044034 paper.

$$\partial_t \bar{g}_{ij} = -2N\bar{A}_{ij} \tag{eq09.01}$$

$$\partial_t \phi = -\frac{1}{6}N\mathrm{tr}K \tag{eq10.01}$$

$$\partial_t \mathrm{tr}K = -g^{ij}D_{ij}N + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\mathrm{tr}K^2\right) \tag{eq11.01}$$

$$\partial_t \bar{A}_{ij} = N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - D_{ij}N - \frac{1}{3}g_{ij}\left(NR_{ab} - D_{ab}N\right)g^{ab}\right) \tag{eq12.01}$$

$$\partial_t \bar{\Gamma}^i = -2\bar{A}^{ij}\partial_j N + 2N\left(\bar{\Gamma}^i{}_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{g}^{ij}\partial_j\mathrm{tr}K + 6\bar{A}^{ij}\partial_j\phi\right) \tag{eq20.01}$$

$$R_{ab} = \bar{R}_{ab} + R^\phi{}_{ab} \tag{eq14.01}$$

$$= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak}$$
$$+ \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) \tag{eq14.02}$$

The next task is to rewrite the right hand sides of this set of equations in terms of the BSSN variables and their partial derivatives.

The results are summarised on the following page.

The BSSN evolution equations. Pt.2

Notice that equation (14) contains $\partial\bar\Gamma^i$ but not $\bar\Gamma^i$. This is a consequence of the advice given by Alcubierre, Brugmann etal. (Phys Rev D (67) 084023, 2nd-3rd paragraph on pg. 084023-4)

> ... if one wants to achieve numerical stability. In the computer code we do not use the numerically evolved $\bar\Gamma^i$ in all places, but we follow this rule:
>
> Partial derivatives $\partial_j\bar\Gamma^i$ are computed as finite differences of the independent variables $\bar\Gamma^i$ that are evolved using ...

Compare equation (14) on the previous page and that on this page.

$$\partial_t \bar g_{ij} = -2N\bar A_{ij} \tag{eq09.99}$$

$$\partial_t \phi = -\frac{1}{6}N\operatorname{tr}K \tag{eq10.99}$$

$$\partial_t \operatorname{tr}K = N\bar A_{ab}\bar A^{ab} + \frac{1}{3}N\operatorname{tr}K^2 + \exp\left(-4\phi\right)\left(-\bar g^{ab}\partial_{ab}N - \partial_a N\partial_b\bar g^{ab} - 2\bar g^{ab}\partial_a\phi\partial_b N\right) \tag{eq11.99}$$

$$\begin{aligned}\partial_t \bar A_{ij} = N\operatorname{tr}K\bar A_{ij} - 2N\bar A_{ia}\bar A_{jb}\bar g^{ab} + \exp\left(-4\phi\right)\Big(&NR_{ij} - \partial_{ij}N + \frac{1}{2}\bar g^{ab}\partial_a N\partial_i\bar g_{jb} + \frac{1}{2}\bar g^{ab}\partial_a N\partial_j\bar g_{ib} - \frac{1}{2}\bar g^{ab}\partial_a N\partial_b\bar g_{ij} + 2\partial_i\phi\partial_j N + 2\partial_j\phi\partial_i N\\ &- \frac{4}{3}\bar g_{ij}\bar g^{ab}\partial_a\phi\partial_b N - \frac{1}{3}N\bar g_{ij}\bar g^{ab}R_{ab} + \frac{1}{3}\bar g_{ij}\bar g^{ab}\partial_{ab}N + \frac{1}{3}\bar g_{ij}\partial_a N\partial_b\bar g^{ab}\Big)\end{aligned} \tag{eq12.99}$$

$$\partial_t \bar\Gamma^i = -2\bar A^{ia}\partial_a N + 2N\bar A^{ab}\bar g^{ic}\partial_a\bar g_{bc} - N\bar A^{ab}\bar g^{ic}\partial_c\bar g_{ab} - \frac{4}{3}N\bar g^{ia}\partial_a\operatorname{tr}K + 12N\bar A^{ia}\partial_a\phi \tag{eq20.99}$$

$$\begin{aligned}R_{ab} = &-2\partial_{ab}\phi + \bar g^{cd}\partial_d\phi\partial_a\bar g_{bc} + \bar g^{cd}\partial_d\phi\partial_b\bar g_{ac} - \bar g^{cd}\partial_d\phi\partial_c\bar g_{ab} + 12\partial_a\phi\partial_b\phi - 2\bar g_{ab}\bar g^{cd}\partial_{cd}\phi - 2\bar g_{ab}\partial_d\phi\partial_c\bar g^{cd} - 12\bar g_{ab}\bar g^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar g^{cd}\partial_{cd}\bar g_{ab} + \frac{1}{2}\bar g_{ac}\partial_b\bar\Gamma^c\\ &+ \frac{1}{2}\bar g_{bc}\partial_a\bar\Gamma^c - \frac{1}{2}\partial_c\bar g_{ab}\partial_d\bar g^{cd} - \frac{1}{2}\partial_d\bar g_{bc}\partial_a\bar g^{dc} - \frac{1}{2}\bar g^{ed}\bar g^{cf}\partial_e\bar g_{ac}\partial_f\bar g_{bd} + \frac{1}{2}\bar g^{ef}\bar g^{cd}\partial_e\bar g_{ac}\partial_f\bar g_{bd} + \frac{1}{4}\partial_b\bar g_{cd}\partial_a\bar g^{cd} - \frac{1}{2}\partial_d\bar g_{ac}\partial_b\bar g^{dc}\end{aligned} \tag{eq14.99}$$

The full details of the above computations can be found on the following pages.

PhysRevD.62.044034 equation (9)

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-09.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------

DgBarDt := \partial_{t}{gBar_{ij}}.            # cdb(eq09.00,DgBarDt)
DgBarDt := -2 N ABar_{i j}.                    # cdb(eq09.01,DgBarDt)

canonicalise (DgBarDt)                         # cdb(eq09.02,DgBarDt)  # no change
                                               # cdb(eq09.99,DgBarDt)  # no change

cdblib.put ('DgBarDt',DgBarDt,jsonfile)
```

$$\partial_t \bar{g}_{ij} = -2N\bar{A}_{ij} \qquad \text{(eq09.02)}$$

PhysRevD.62.044034 equation (10)

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-10.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------

DphiDt  := \partial_{t}{\phi}.                    # cdb(eq10.00,DphiDt)
DphiDt  := -(1/6) N trK.                          # cdb(eq10.01,DphiDt)

canonicalise (DphiDt)                             # cdb(eq10.02,DphiDt)  # no change
                                                  # cdb(eq10.99,DphiDt)  # no change

cdblib.put ('DphiDt',DphiDt,jsonfile)
```

$$\partial_t\phi = -\frac{1}{6}N\mathrm{tr}K \qquad (\mathtt{eq10.02})$$

PhysRevD.62.044034 equation (11)

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-11.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------

DtrKDt  := \partial_{t}{trK}.                 # cdb(eq11.00,DtrKDt)
DtrKDt  := - g^{i j} D_{i j}{N}
           + N (ABar_{i j} ABar^{i j} + (1/3) trK**2).

                                              # cdb(eq11.01,DtrKDt)

substitute (DtrKDt, defD2)                    # cdb(eq11.02,DtrKDt)
substitute (DtrKDt, defGamma2GammaBar)        # cdb(eq11.03,DtrKDt)

foo := g^{a b} -> \exp(-4\phi) gBar^{a b}.

substitute (DtrKDt, foo)                      # cdb(eq11.04,DtrKDt)
distribute (DtrKDt)                           # cdb(eq11.05,DtrKDt)
eliminate_kronecker (DtrKDt)                  # cdb(eq11.06,DtrKDt)
canonicalise (DtrKDt)                         # cdb(eq11.07,DtrKDt)
substitute   (DtrKDt, defGBarSq)              # cdb(eq11.08,DtrKDt)

DtrKDt = product_sort (DtrKDt)

factor_out (DtrKDt, $\exp(-4\phi)$)           # cdb(eq11.08,DtrKDt)
substitute (DtrKDt, defGammaBarU)             # cdb(eq11.09,DtrKDt)
distribute (DtrKDt)                           # cdb(eq11.10,DtrKDt)

DtrKDt = product_sort (DtrKDt)                # cdb(eq11.11,DtrKDt)

canonicalise (DtrKDt)                         # cdb(eq11.12,DtrKDt)

foo := gBar^{b c} \partial_{a}{gBar_{b c}} -> 0.   # follows from det(g) = 1
```

```
bah := gBar^{e b} gBar^{f c} \partial_{a}{gBar_{b c}} -> - \partial_{a}{gBar^{e f}}.

substitute (DtrKDt, foo)                              # cdb(eq11.13,DtrKDt)
substitute (DtrKDt, bah)                              # cdb(eq11.14,DtrKDt)

DtrKDt = product_sort (DtrKDt)

canonicalise (DtrKDt)                                 # cdb(eq11.15,DtrKDt)
factor_out   (DtrKDt, $\exp(-4\phi)$)                 # cdb(eq11.16,DtrKDt)
                                                      # cdb(eq11.99,DtrKDt)

cdblib.put ('DtrKDt',DtrKDt,jsonfile)
```

$$
\begin{aligned}
\partial_t \mathrm{tr}K &= -g^{ij} D_{ij} N + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\mathrm{tr}K^2\right) && \texttt{(eq11.01)}\\
&= -g^{ij}\left(\partial_{ij}N - \Gamma^{c}{}_{ij}\partial_c N\right) + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\mathrm{tr}K^2\right) && \texttt{(eq11.02)}\\
&= -g^{ij}\left(\partial_{ij}N - \left(\bar{\Gamma}^{c}{}_{ij} + 2\bar{g}^{c}{}_{j}\partial_i\phi + 2\bar{g}^{c}{}_{i}\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N\right) + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\mathrm{tr}K^2\right) && \texttt{(eq11.03)}\\
&= -\exp\left(-4\phi\right)\bar{g}^{ij}\left(\partial_{ij}N - \left(\bar{\Gamma}^{c}{}_{ij} + 2\bar{g}^{c}{}_{j}\partial_i\phi + 2\bar{g}^{c}{}_{i}\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N\right) + N\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}\mathrm{tr}K^2\right) && \texttt{(eq11.04)}\\
&= -\exp\left(-4\phi\right)\bar{g}^{ij}\partial_{ij}N + \exp\left(-4\phi\right)\bar{g}^{ij}\bar{\Gamma}^{c}{}_{ij}\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^{ij}\bar{g}^{c}{}_{j}\partial_i\phi\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^{ij}\bar{g}^{c}{}_{i}\partial_j\phi\partial_c N - 2\exp\left(-4\phi\right)\bar{g}^{ij}\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N\\
&\quad + N\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}N\mathrm{tr}K^2 && \texttt{(eq11.05)}\\
&= -\exp\left(-4\phi\right)\bar{g}^{ij}\partial_{ij}N + \exp\left(-4\phi\right)\bar{g}^{ij}\bar{\Gamma}^{c}{}_{ij}\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^{ic}\partial_i\phi\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^{cj}\partial_j\phi\partial_c N - 2\exp\left(-4\phi\right)\bar{g}^{ij}\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N\\
&\quad + N\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}N\mathrm{tr}K^2 && \texttt{(eq11.06)}\\
&= -\exp\left(-4\phi\right)\bar{g}^{ij}\partial_{ij}N + \exp\left(-4\phi\right)\bar{g}^{ci}\bar{\Gamma}^{j}{}_{ci}\partial_j N + 2\exp\left(-4\phi\right)\bar{g}^{ci}\partial_c\phi\partial_i N + 2\exp\left(-4\phi\right)\bar{g}^{cj}\partial_c\phi\partial_j N - 2\exp\left(-4\phi\right)\bar{g}^{ce}\bar{g}^{ij}\bar{g}_{ce}\partial_i\phi\partial_j N\\
&\quad + N\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}N\mathrm{tr}K^2 && \texttt{(eq11.07)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 + \exp\left(-4\phi\right)\left(-\bar{g}^{ab}\partial_{ab}N + \bar{g}^{ab}\bar{\Gamma}^{c}{}_{ab}\partial_c N - 2\bar{g}^{ab}\partial_a\phi\partial_b N\right) && \texttt{(eq11.08)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 + \exp\left(-4\phi\right)\left(-\bar{g}^{ab}\partial_{ab}N + \frac{1}{2}\bar{g}^{ab}\bar{g}^{ce}\left(\partial_a\bar{g}_{eb} + \partial_b\bar{g}_{ae} - \partial_e\bar{g}_{ab}\right)\partial_c N - 2\bar{g}^{ab}\partial_a\phi\partial_b N\right) && \texttt{(eq11.09)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 - \exp\left(-4\phi\right)\bar{g}^{ab}\partial_{ab}N + \frac{1}{2}\exp\left(-4\phi\right)\bar{g}^{ab}\bar{g}^{ce}\partial_a\bar{g}_{eb}\partial_c N + \frac{1}{2}\exp\left(-4\phi\right)\bar{g}^{ab}\bar{g}^{ce}\partial_b\bar{g}_{ae}\partial_c N - \frac{1}{2}\exp\left(-4\phi\right)\bar{g}^{ab}\bar{g}^{ce}\partial_e\bar{g}_{ab}\partial_c N\\
&\quad - 2\exp\left(-4\phi\right)\bar{g}^{ab}\partial_a\phi\partial_b N && \texttt{(eq11.10)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N + \frac{1}{2}\bar{g}^{cb}\bar{g}^{da}\exp\left(-4\phi\right)\partial_d N\partial_c\bar{g}_{ab} + \frac{1}{2}\bar{g}^{ac}\bar{g}^{db}\exp\left(-4\phi\right)\partial_d N\partial_c\bar{g}_{ab} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_c N\partial_d\bar{g}_{ab}\\
&\quad - 2\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N && \texttt{(eq11.11)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \mathrm{tr} K &= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N + \bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_a N\partial_c\bar{g}_{bd} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{cd} - 2\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N && \texttt{(eq11.12)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N + \bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_a N\partial_c\bar{g}_{bd} - 2\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N && \texttt{(eq11.13)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N - \partial_c\bar{g}^{ac}\exp\left(-4\phi\right)\partial_a N - 2\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N && \texttt{(eq11.14)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N - \exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}^{ab} - 2\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N && \texttt{(eq11.15)}\\
&= N\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}N\mathrm{tr}K^2 + \exp\left(-4\phi\right)\left(-\bar{g}^{ab}\partial_{ab}N - \partial_a N\partial_b\bar{g}^{ab} - 2\bar{g}^{ab}\partial_a\phi\partial_b N\right) && \texttt{(eq11.16)}
\end{aligned}
$$

PhysRevD.62.044034 equation (12)

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-12.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------

DABarDt := \partial_{t}{ABar_{i j}}.          # cdb(eq12.00,DABarDt)
DABarDt := N (trK ABar_{i j} - 2 ABar_{i a} ABar^{a}_{j})
           + \exp(-4\phi) (N R_{i j} - D_{i j}{N}
                           - (1/3) g_{i j} (N R_{a b} - D_{a b}{N}) g^{a b}).

                                              # cdb(eq12.01,DABarDt)

# ----------------------------------------------------------------------

substitute (DABarDt, defD2)                   # cdb(eq12.02,DABarDt)
substitute (DABarDt, defGamma2GammaBar)       # cdb(eq12.03,DABarDt)

foo := g_{a b} -> \exp(4\phi) gBar_{a b}.
bah := g^{a b} -> \exp(-4\phi) gBar^{a b}.

substitute (DABarDt, foo)                     # cdb(eq12.04,DABarDt)
substitute (DABarDt, bah)                     # cdb(eq12.05,DABarDt)
distribute (DABarDt)                          # cdb(eq12.06,DABarDt)
eliminate_kronecker (DABarDt)                 # cdb(eq12.07,DABarDt)
substitute (DABarDt,defGBarSq)                # cdb(eq12.08,DABarDt)

DABarDt = product_sort (DABarDt)              # cdb(eq12.09,DABarDt)

rename_dummies (DABarDt)                      # cdb(eq12.10,DABarDt)
canonicalise   (DABarDt)                      # cdb(eq12.11,DABarDt)

map_sympy  (DABarDt, "simplify")              # cdb(eq12.12,DABarDt)
factor_out (DABarDt,$\exp(-4\phi)$)           # cdb(eq12.13,DABarDt)
```

```
foo := ABar^{a}_{b} -> gBar^{a c} ABar_{c b}.

substitute (DABarDt, foo)

DABarDt = product_sort (DABarDt)                # cdb(eq12.14,DABarDt)

substitute (DABarDt,defGammaBarU)               # cdb(eq12.15,DABarDt)
distribute (DABarDt)

DABarDt = product_sort (DABarDt)                # cdb(eq12.16,DABarDt)

canonicalise (DABarDt)                          # cdb(eq12.17,DABarDt)

foo := gBar^{b c} \partial_{a}{gBar_{b c}} -> 0.   # follows from det(g) = 1
bah := gBar^{e b} gBar^{f c} \partial_{a}{gBar_{b c}} -> - \partial_{a}{gBar^{e f}}.

substitute (DABarDt,foo)                        # cdb(eq12.18,DABarDt)
substitute (DABarDt,bah)                        # cdb(eq12.19,DABarDt)

DABarDt = product_sort (DABarDt)

canonicalise (DABarDt)                          # cdb(eq12.20,DABarDt)
factor_out   (DABarDt, $\exp(-4\phi)$)          # cdb(eq12.21,DABarDt)

                                                # cdb(eq12.99,DABarDt)

cdblib.put ('DABarDt',DABarDt,jsonfile)
```

$$\partial_t \bar{A}_{ij} = N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - D_{ij}N - \frac{1}{3}g_{ij}\left(NR_{ab} - D_{ab}N\right)g^{ab}\right) \qquad \texttt{(eq12.01)}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \Gamma^c{}_{ij}\partial_c N - \frac{1}{3}g_{ij}\left(NR_{ab} - \partial_{ab}N + \Gamma^c{}_{ab}\partial_c N\right)g^{ab}\right) \qquad \texttt{(eq12.02)}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\Bigg(NR_{ij} - \partial_{ij}N + \left(\bar{\Gamma}^c{}_{ij} + 2\bar{g}^c{}_j\partial_i\phi + 2\bar{g}^c{}_i\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N$$
$$- \frac{1}{3}g_{ij}\left(NR_{ab} - \partial_{ab}N + \left(\bar{\Gamma}^c{}_{ab} + 2\bar{g}^c{}_b\partial_a\phi + 2\bar{g}^c{}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c N\right)g^{ab}\Bigg) \qquad \texttt{(eq12.03)}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\Bigg(NR_{ij} - \partial_{ij}N + \left(\bar{\Gamma}^c{}_{ij} + 2\bar{g}^c{}_j\partial_i\phi + 2\bar{g}^c{}_i\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N$$
$$- \frac{1}{3}\exp\left(4\phi\right)\bar{g}_{ij}\left(NR_{ab} - \partial_{ab}N + \left(\bar{\Gamma}^c{}_{ab} + 2\bar{g}^c{}_b\partial_a\phi + 2\bar{g}^c{}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c N\right)g^{ab}\Bigg) \qquad \texttt{(eq12.04)}$$

$$= N\left(\mathrm{tr}K\bar{A}_{ij} - 2\bar{A}_{ia}\bar{A}^a{}_j\right) + \exp\left(-4\phi\right)\Bigg(NR_{ij} - \partial_{ij}N + \left(\bar{\Gamma}^c{}_{ij} + 2\bar{g}^c{}_j\partial_i\phi + 2\bar{g}^c{}_i\partial_j\phi - 2\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\right)\partial_c N$$
$$- \frac{1}{3}\exp\left(4\phi\right)\bar{g}_{ij}\left(NR_{ab} - \partial_{ab}N + \left(\bar{\Gamma}^c{}_{ab} + 2\bar{g}^c{}_b\partial_a\phi + 2\bar{g}^c{}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c N\right)\exp\left(-4\phi\right)\bar{g}^{ab}\Bigg) \qquad \texttt{(eq12.05)}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + \exp\left(-4\phi\right)NR_{ij} - \exp\left(-4\phi\right)\partial_{ij}N + \exp\left(-4\phi\right)\bar{\Gamma}^c{}_{ij}\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^c{}_j\partial_i\phi\partial_c N + 2\exp\left(-4\phi\right)\bar{g}^c{}_i\partial_j\phi\partial_c N$$
$$- 2\exp\left(-4\phi\right)\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N - \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}NR_{ab}\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_{ab}N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{\Gamma}^c{}_{ab}\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} - \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^c{}_b\partial_a\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^c{}_a\partial_b\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} \qquad \texttt{(eq12.06)}$$

$$= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + \exp\left(-4\phi\right)NR_{ij} - \exp\left(-4\phi\right)\partial_{ij}N + \exp\left(-4\phi\right)\bar{\Gamma}^c{}_{ij}\partial_c N + 2\exp\left(-4\phi\right)\partial_i\phi\partial_j N + 2\exp\left(-4\phi\right)\partial_j\phi\partial_i N$$
$$- 2\exp\left(-4\phi\right)\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N - \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}NR_{ab}\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_{ab}N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{\Gamma}^c{}_{ab}\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} - \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_a\phi\partial_b N\exp\left(-4\phi\right)\bar{g}^{ab}$$
$$- \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_b\phi\partial_a N\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} \qquad \texttt{(eq12.07)}$$

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + \exp\left(-4\phi\right)NR_{ij} - \exp\left(-4\phi\right)\partial_{ij}N + \exp\left(-4\phi\right)\bar{\Gamma}^c{}_{ij}\partial_c N + 2\exp\left(-4\phi\right)\partial_i\phi\partial_j N + 2\exp\left(-4\phi\right)\partial_j\phi\partial_i N \\
&\quad - 2\exp\left(-4\phi\right)\bar{g}^{ce}\bar{g}_{ij}\partial_e\phi\partial_c N - \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}NR_{ab}\exp\left(-4\phi\right)\bar{g}^{ab} + \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_{ab}N\exp\left(-4\phi\right)\bar{g}^{ab} \\
&\quad - \frac{1}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{\Gamma}^c{}_{ab}\partial_c N\exp\left(-4\phi\right)\bar{g}^{ab} - \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_a\phi\partial_b N\exp\left(-4\phi\right)\bar{g}^{ab} \\
&\quad - \frac{2}{3}\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\partial_b\phi\partial_a N\exp\left(-4\phi\right)\bar{g}^{ab} + 2\exp\left(-4\phi\right)\exp\left(4\phi\right)\bar{g}_{ij}\bar{g}^{ce}\partial_e\phi\partial_c N\exp\left(-4\phi\right) && \texttt{(eq12.08)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp\left(-4\phi\right)\partial_a N + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N \\
&\quad - 2\bar{g}_{ij}\bar{g}^{ab}\partial_b\phi\exp\left(-4\phi\right)\partial_a N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right) + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_{ab}N \\
&\quad - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_c N - \frac{2}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_b N \\
&\quad + \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_b\phi\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_a N && \texttt{(eq12.09)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp\left(-4\phi\right)\partial_a N + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N \\
&\quad - 2\bar{g}_{ij}\bar{g}^{ba}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right) + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_{ab}N \\
&\quad - \frac{1}{3}\bar{g}_{ij}\bar{g}^{bc}\bar{\Gamma}^a{}_{bc}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_a N - \frac{2}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_b N \\
&\quad + \frac{4}{3}\bar{g}_{ij}\bar{g}^{ba}\partial_a\phi\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_b N && \texttt{(eq12.10)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp\left(-4\phi\right)\partial_a N + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N \\
&\quad - 2\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right) + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_{ab}N \\
&\quad - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_c N + \frac{2}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\exp\left(-4\phi\right)\exp\left(4\phi\right)\partial_b N && \texttt{(eq12.11)} \\
&= \mathrm{tr}KN\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j + N\exp\left(-4\phi\right)R_{ij} - \exp\left(-4\phi\right)\partial_{ij}N + \bar{\Gamma}^a{}_{ij}\exp\left(-4\phi\right)\partial_a N + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N \\
&\quad - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\exp\left(-4\phi\right)\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\exp\left(-4\phi\right)\partial_c N && \texttt{(eq12.12)} \\
&= \mathrm{tr}KN\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}^a{}_j \\
&\quad + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \bar{\Gamma}^a{}_{ij}\partial_a N + 2\partial_i\phi\partial_j N + 2\partial_j\phi\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_{ab}N - \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\bar{\Gamma}^c{}_{ab}\partial_c N\right) && \texttt{(eq12.13)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= N \mathrm{tr} K \bar{A}_{ij} - 2N \bar{A}_{aj} \bar{A}_{ib} \bar{g}^{ba} \\
&\quad + \exp\left(-4\phi\right) \left(N R_{ij} - \partial_{ij} N + \bar{\Gamma}^{a}{}_{ij} \partial_a N + 2\partial_i \phi \partial_j N + 2\partial_j \phi \partial_i N - \frac{4}{3} \bar{g}_{ij} \bar{g}^{ab} \partial_a \phi \partial_b N - \frac{1}{3} N \bar{g}_{ij} \bar{g}^{ab} R_{ab} + \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \partial_{ab} N - \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \bar{\Gamma}^{c}{}_{ab} \partial_c N \right) && \texttt{(eq12.14)} \\
&= N \mathrm{tr} K \bar{A}_{ij} - 2N \bar{A}_{aj} \bar{A}_{ib} \bar{g}^{ba} + \exp\left(-4\phi\right) \Bigg(N R_{ij} - \partial_{ij} N + \frac{1}{2} \bar{g}^{ae} \left(\partial_i \bar{g}_{ej} + \partial_j \bar{g}_{ie} - \partial_e \bar{g}_{ij} \right) \partial_a N + 2\partial_i \phi \partial_j N + 2\partial_j \phi \partial_i N - \frac{4}{3} \bar{g}_{ij} \bar{g}^{ab} \partial_a \phi \partial_b N \\
&\qquad - \frac{1}{3} N \bar{g}_{ij} \bar{g}^{ab} R_{ab} + \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \partial_{ab} N - \frac{1}{6} \bar{g}_{ij} \bar{g}^{ab} \bar{g}^{ce} \left(\partial_a \bar{g}_{eb} + \partial_b \bar{g}_{ae} - \partial_e \bar{g}_{ab} \right) \partial_c N \Bigg) && \texttt{(eq12.15)} \\
&= N \mathrm{tr} K \bar{A}_{ij} - 2N \bar{A}_{aj} \bar{A}_{ib} \bar{g}^{ba} + N R_{ij} \exp\left(-4\phi\right) - \exp\left(-4\phi\right) \partial_{ij} N + \frac{1}{2} \bar{g}^{ba} \exp\left(-4\phi\right) \partial_b N \partial_i \bar{g}_{aj} + \frac{1}{2} \bar{g}^{ba} \exp\left(-4\phi\right) \partial_b N \partial_j \bar{g}_{ia} \\
&\quad - \frac{1}{2} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_a N \partial_b \bar{g}_{ij} + 2\partial_i \phi \exp\left(-4\phi\right) \partial_j N + 2\partial_j \phi \exp\left(-4\phi\right) \partial_i N - \frac{4}{3} \bar{g}_{ij} \bar{g}^{ab} \partial_a \phi \exp\left(-4\phi\right) \partial_b N - \frac{1}{3} N \bar{g}_{ij} \bar{g}^{ab} R_{ab} \exp\left(-4\phi\right) \\
&\quad + \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_{ab} N - \frac{1}{6} \bar{g}_{ij} \bar{g}^{cb} \bar{g}^{da} \exp\left(-4\phi\right) \partial_d N \partial_c \bar{g}_{ab} - \frac{1}{6} \bar{g}_{ij} \bar{g}^{ac} \bar{g}^{db} \exp\left(-4\phi\right) \partial_d N \partial_c \bar{g}_{ab} + \frac{1}{6} \bar{g}_{ij} \bar{g}^{ab} \bar{g}^{cd} \exp\left(-4\phi\right) \partial_c N \partial_d \bar{g}_{ab} && \texttt{(eq12.16)} \\
&= N \mathrm{tr} K \bar{A}_{ij} - 2N \bar{A}_{ia} \bar{A}_{jb} \bar{g}^{ab} + N R_{ij} \exp\left(-4\phi\right) - \exp\left(-4\phi\right) \partial_{ij} N + \frac{1}{2} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_a N \partial_i \bar{g}_{jb} + \frac{1}{2} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_a N \partial_j \bar{g}_{ib} \\
&\quad - \frac{1}{2} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_a N \partial_b \bar{g}_{ij} + 2\partial_i \phi \exp\left(-4\phi\right) \partial_j N + 2\partial_j \phi \exp\left(-4\phi\right) \partial_i N - \frac{4}{3} \bar{g}_{ij} \bar{g}^{ab} \partial_a \phi \exp\left(-4\phi\right) \partial_b N - \frac{1}{3} N \bar{g}_{ij} \bar{g}^{ab} R_{ab} \exp\left(-4\phi\right) \\
&\quad + \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_{ab} N - \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \bar{g}^{cd} \exp\left(-4\phi\right) \partial_a N \partial_c \bar{g}_{bd} + \frac{1}{6} \bar{g}_{ij} \bar{g}^{ab} \bar{g}^{cd} \exp\left(-4\phi\right) \partial_a N \partial_b \bar{g}_{cd} && \texttt{(eq12.17)} \\
&= N \mathrm{tr} K \bar{A}_{ij} - 2N \bar{A}_{ia} \bar{A}_{jb} \bar{g}^{ab} + N R_{ij} \exp\left(-4\phi\right) - \exp\left(-4\phi\right) \partial_{ij} N + \frac{1}{2} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_a N \partial_i \bar{g}_{jb} + \frac{1}{2} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_a N \partial_j \bar{g}_{ib} \\
&\quad - \frac{1}{2} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_a N \partial_b \bar{g}_{ij} + 2\partial_i \phi \exp\left(-4\phi\right) \partial_j N + 2\partial_j \phi \exp\left(-4\phi\right) \partial_i N - \frac{4}{3} \bar{g}_{ij} \bar{g}^{ab} \partial_a \phi \exp\left(-4\phi\right) \partial_b N - \frac{1}{3} N \bar{g}_{ij} \bar{g}^{ab} R_{ab} \exp\left(-4\phi\right) \\
&\quad + \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \exp\left(-4\phi\right) \partial_{ab} N - \frac{1}{3} \bar{g}_{ij} \bar{g}^{ab} \bar{g}^{cd} \exp\left(-4\phi\right) \partial_a N \partial_c \bar{g}_{bd} && \texttt{(eq12.18)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \bar{A}_{ij} &= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_j\bar{g}_{ib} \\
&\quad - \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right) \\
&\quad + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N + \frac{1}{3}\bar{g}_{ij}\partial_c\bar{g}^{ac}\exp\left(-4\phi\right)\partial_a N \qquad \texttt{(eq12.19)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + NR_{ij}\exp\left(-4\phi\right) - \exp\left(-4\phi\right)\partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_j\bar{g}_{ib} \\
&\quad - \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\exp\left(-4\phi\right)\partial_j N + 2\partial_j\phi\exp\left(-4\phi\right)\partial_i N - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\exp\left(-4\phi\right)\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab}\exp\left(-4\phi\right) \\
&\quad + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{ab}N + \frac{1}{3}\bar{g}_{ij}\exp\left(-4\phi\right)\partial_a N\partial_b\bar{g}^{ab} \qquad \texttt{(eq12.20)} \\
&= N\mathrm{tr}K\bar{A}_{ij} - 2N\bar{A}_{ia}\bar{A}_{jb}\bar{g}^{ab} + \exp\left(-4\phi\right)\left(NR_{ij} - \partial_{ij}N + \frac{1}{2}\bar{g}^{ab}\partial_a N\partial_i\bar{g}_{jb} + \frac{1}{2}\bar{g}^{ab}\partial_a N\partial_j\bar{g}_{ib} - \frac{1}{2}\bar{g}^{ab}\partial_a N\partial_b\bar{g}_{ij} + 2\partial_i\phi\partial_j N + 2\partial_j\phi\partial_i N\right. \\
&\quad \left. - \frac{4}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_a\phi\partial_b N - \frac{1}{3}N\bar{g}_{ij}\bar{g}^{ab}R_{ab} + \frac{1}{3}\bar{g}_{ij}\bar{g}^{ab}\partial_{ab}N + \frac{1}{3}\bar{g}_{ij}\partial_a N\partial_b\bar{g}^{ab}\right) \qquad \texttt{(eq12.21)}
\end{aligned}
$$

PhysRevD.62.044034 equation (14)

The advice given by Miguel Alcubierre, Bernd Brugmann etal (Phys Rev D (67) 084023, 2nd-3rd paragraph on pg. 084023-4)

> ... if one wants to achieve numerical stability. In the computer code we do not use the numerically evolved $\bar{\Gamma}^i$ in all places, but we follow this rule:
>
> Partial derivatives $\partial_j \bar{\Gamma}^i$ are computed as finite differences of the independent variables $\bar{\Gamma}^i$ that are evolved using ...

The Einstein Toolkit code uses the same rule – the only place where the *evolved* $\bar{\Gamma}^i$ are used is in computing the $\partial_j \bar{\Gamma}^i$ terms in the equation for $\bar{R}_{ij}$, that is equation (18) of the Phys Rev D (62) 044034 paper.

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-14.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------

Rphi := -2 DBar_{a b}{\phi} - 2 gBar_{a b} gBar^{c d} DBar_{c d}{\phi}
        +4 DBar_{a}{\phi} DBar_{b}{\phi} - 4 gBar_{a b} gBar^{c d} DBar_{c}{\phi} DBar_{d}{\phi}.

                                                        # cdb(eq15.prd,Rphi)

RBar := - (1/2) gBar^{l m} \partial_{l m}{gBar_{a b}}
        + (1/2) gBar_{k a} \partial_{b}{GammaBar^{k}}
        + (1/2) gBar_{k b} \partial_{a}{GammaBar^{k}}
        + (1/2) GammaBar^{k} GammaBar_{a b k}
        + (1/2) GammaBar^{k} GammaBar_{b a k}
        + gBar^{l m} gBar^{k e} (  GammaBar_{e l a} GammaBar_{b k m}
                                 + GammaBar_{e l b} GammaBar_{a k m}
                                 + GammaBar_{k a m} GammaBar_{e l b}).

                                                        # cdb(eq18.prd,RBar)

defRab := R_{a b} -> @(Rphi) + @(RBar).

Rab := RBar_{a b} + Rphi_{a b}.                         # cdb(eq14.01,Rab)
Rab := R_{a b}.                                         # cdb(eq14.00,Rab)

substitute (Rab, defRab)                                # cdb(eq14.02,Rab)
substitute (Rab, defDBar1)                              # cdb(eq14.03,Rab)
substitute (Rab, defDBar2)                              # cdb(eq14.04,Rab)
substitute (Rab, defGamma2GammaBar)                     # cdb(eq14.05,Rab)
distribute (Rab)                                        # cdb(eq14.06,Rab)
eliminate_kronecker (Rab)                               # cdb(eq14.07,Rab)

Rab = product_sort (Rab)                                # cdb(eq14.08,Rab)
```

```
rename_dummies (Rab)                                    # cdb(eq14.09,Rab)
canonicalise   (Rab)                                    # cdb(eq14.10,Rab)

foo := GammaBar^{a} GammaBar_{b c a} -> gBar^{d e} GammaBar^{a}_{d e} GammaBar_{b c a}.

substitute (Rab, foo)                                   # cdb(eq14.11,Rab)
substitute (Rab, defGBarSq)                             # cdb(eq14.12,Rab)
substitute (Rab, defGammaBarD)                          # cdb(eq14.13,Rab)
substitute (Rab, defGammaBarU)                          # cdb(eq14.14,Rab)
distribute (Rab)                                        # cdb(eq14.15,Rab)

foo := \partial_{a}{gBar_{b c}} gBar^{b c} -> 0.   # follows from det(g) = 1

substitute   (Rab,foo)                                  # cdb(eq14.16,Rab)
canonicalise (Rab)                                      # cdb(eq14.17,Rab)

foo := gBar^{b e} gBar^{c f} \partial_{a}{gBar_{b c}}  -> - \partial_{a}{gBar^{e f}}.
bah := gBar^{e b} gBar^{f c} \partial_{a}{gBar_{b c}}  -> - \partial_{a}{gBar^{e f}}.
moo := gBar^{e b} gBar^{c f} \partial_{a}{gBar_{b c}}  -> - \partial_{a}{gBar^{e f}}.

substitute (Rab,foo)                                    # cdb(eq14.18,Rab)
substitute (Rab,bah)                                    # cdb(eq14.19,Rab)
substitute (Rab,moo)                                    # cdb(eq14.20,Rab)

Rab = product_sort (Rab)                                # cdb(eq14.21,Rab)
                                                        # cdb(eq14.99,Rab)

defRab := R_{a b} -> @(Rab).   # used later in bssn-ricci-scalar.tex

cdblib.put ('Rab',Rab,jsonfile)
cdblib.put ('defRab',defRab,jsonfile)
```

$$
\begin{aligned}
R_{ab} &= \bar{R}_{ab} + R^{\phi}{}_{ab} && \texttt{(eq14.01)}\\
&= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\bar{D}_a\phi\bar{D}_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{D}_c\phi\bar{D}_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak}\\
&\quad + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) && \texttt{(eq14.02)}\\
&= -2\bar{D}_{ab}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\bar{D}_{cd}\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak}\\
&\quad + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) && \texttt{(eq14.03)}\\
&= -2\partial_{ab}\phi + 2\Gamma^c{}_{ab}\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\left(\partial_{cd}\phi - \Gamma^e{}_{cd}\partial_e\phi\right) + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak}\\
&\quad + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) && \texttt{(eq14.04)}\\
&= -2\partial_{ab}\phi + 2\left(\bar{\Gamma}^c{}_{ab} + 2\bar{g}^c{}_b\partial_a\phi + 2\bar{g}^c{}_a\partial_b\phi - 2\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\right)\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\left(\partial_{cd}\phi - \left(\bar{\Gamma}^e{}_{cd} + 2\bar{g}^e{}_d\partial_c\phi + 2\bar{g}^e{}_c\partial_d\phi - 2\bar{g}^{ef}\bar{g}_{cd}\partial_f\phi\right)\partial_e\phi\right) + 4\partial_a\phi\partial_b\phi\\
&\quad - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\left(\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{\Gamma}_{kam}\bar{\Gamma}_{elb}\right) && \texttt{(eq14.05)}\\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 4\bar{g}^c{}_b\partial_a\phi\partial_c\phi + 4\bar{g}^c{}_a\partial_b\phi\partial_c\phi - 4\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{cd}\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{cd}\bar{g}^e{}_d\partial_c\phi\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{cd}\bar{g}^e{}_c\partial_d\phi\partial_e\phi\\
&\quad - 4\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\bar{g}_{cd}\partial_f\phi\partial_e\phi + 4\partial_a\phi\partial_b\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm}\\
&\quad + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{kam}\bar{\Gamma}_{elb} && \texttt{(eq14.06)}\\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 8\partial_a\phi\partial_b\phi + 4\partial_b\phi\partial_a\phi - 4\bar{g}^{ce}\bar{g}_{ab}\partial_e\phi\partial_c\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{cd}\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{ce}\partial_c\phi\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{ed}\partial_d\phi\partial_e\phi - 4\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\bar{g}_{cd}\partial_f\phi\partial_e\phi\\
&\quad - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{lm}\partial_{lm}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ka}\partial_b\bar{\Gamma}^k + \frac{1}{2}\bar{g}_{kb}\partial_a\bar{\Gamma}^k + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{abk} + \frac{1}{2}\bar{\Gamma}^k\bar{\Gamma}_{bak} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{ela}\bar{\Gamma}_{bkm} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{elb}\bar{\Gamma}_{akm} + \bar{g}^{lm}\bar{g}^{ke}\bar{\Gamma}_{kam}\bar{\Gamma}_{elb} && \texttt{(eq14.07)}\\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{cd}\partial_e\phi + 4\bar{g}_{ab}\bar{g}^{cd}\partial_d\phi\partial_c\phi - 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ef}\partial_e\phi\partial_f\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}\\
&\quad + \frac{1}{2}\bar{g}_{ca}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{cb}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bcf}\bar{\Gamma}_{dea} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{acf}\bar{\Gamma}_{deb} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{deb}\bar{\Gamma}_{caf} && \texttt{(eq14.08)}\\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{de}\bar{\Gamma}^c{}_{de}\partial_c\phi + 4\bar{g}_{ab}\bar{g}^{dc}\partial_c\phi\partial_d\phi - 4\bar{g}_{ab}\bar{g}_{ef}\bar{g}^{ef}\bar{g}^{cd}\partial_c\phi\partial_d\phi - 4\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}\\
&\quad + \frac{1}{2}\bar{g}_{ca}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{cb}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc} + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} + \bar{g}^{ce}\bar{g}^{fd}\bar{\Gamma}_{bcd}\bar{\Gamma}_{efa} + \bar{g}^{ce}\bar{g}^{fd}\bar{\Gamma}_{acd}\bar{\Gamma}_{efb} + \bar{g}^{ec}\bar{g}^{df}\bar{\Gamma}_{cdb}\bar{\Gamma}_{eaf} && \texttt{(eq14.09)}\\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{cd}\partial_e\phi - 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{abc}\\
&\quad + \frac{1}{2}\bar{\Gamma}^c\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} && \texttt{(eq14.10)}
\end{aligned}
$$

$$
\begin{aligned}
R_{ab} &= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{cd}\partial_e\phi - 4\bar{g}_{ab}\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^c{}_{de}\bar{\Gamma}_{abc} \\
&\quad + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^c{}_{de}\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} && \texttt{(eq14.11)} \\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{cd}\partial_e\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^c{}_{de}\bar{\Gamma}_{abc} \\
&\quad + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^c{}_{de}\bar{\Gamma}_{bac} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{bce}\bar{\Gamma}_{daf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{ace}\bar{\Gamma}_{dbf} + \bar{g}^{cd}\bar{g}^{ef}\bar{\Gamma}_{cae}\bar{\Gamma}_{dbf} && \texttt{(eq14.12)} \\
&= -2\partial_{ab}\phi + 2\bar{\Gamma}^c{}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{\Gamma}^e{}_{cd}\partial_e\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^c{}_{de}\left(\frac{1}{2}\partial_b\bar{g}_{ac} + \frac{1}{2}\partial_c\bar{g}_{ba} - \frac{1}{2}\partial_a\bar{g}_{bc}\right) + \frac{1}{2}\bar{g}^{de}\bar{\Gamma}^c{}_{de}\left(\frac{1}{2}\partial_a\bar{g}_{bc} + \frac{1}{2}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_b\bar{g}_{ac}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{be} + \frac{1}{2}\partial_e\bar{g}_{cb} - \frac{1}{2}\partial_b\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_a\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{ad} - \frac{1}{2}\partial_d\bar{g}_{af}\right) + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{ae} + \frac{1}{2}\partial_e\bar{g}_{ca} - \frac{1}{2}\partial_a\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_a\bar{g}_{ce} + \frac{1}{2}\partial_e\bar{g}_{ac} - \frac{1}{2}\partial_c\bar{g}_{ae}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) && \texttt{(eq14.13)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\left(\partial_a\bar{g}_{eb} + \partial_b\bar{g}_{ae} - \partial_e\bar{g}_{ab}\right)\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\left(\partial_c\bar{g}_{fd} + \partial_d\bar{g}_{cf} - \partial_f\bar{g}_{cd}\right)\partial_e\phi \\
&\quad - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{4}\bar{g}^{de}\bar{g}^{cf}\left(\partial_d\bar{g}_{fe} + \partial_e\bar{g}_{df} - \partial_f\bar{g}_{de}\right)\left(\frac{1}{2}\partial_b\bar{g}_{ac} + \frac{1}{2}\partial_c\bar{g}_{ba} - \frac{1}{2}\partial_a\bar{g}_{bc}\right) \\
&\quad + \frac{1}{4}\bar{g}^{de}\bar{g}^{cf}\left(\partial_d\bar{g}_{fe} + \partial_e\bar{g}_{df} - \partial_f\bar{g}_{de}\right)\left(\frac{1}{2}\partial_a\bar{g}_{bc} + \frac{1}{2}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_b\bar{g}_{ac}\right) + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{be} + \frac{1}{2}\partial_e\bar{g}_{cb} - \frac{1}{2}\partial_b\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_a\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{ad} - \frac{1}{2}\partial_d\bar{g}_{af}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_c\bar{g}_{ae} + \frac{1}{2}\partial_e\bar{g}_{ca} - \frac{1}{2}\partial_a\bar{g}_{ce}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) \\
&\quad + \bar{g}^{cd}\bar{g}^{ef}\left(\frac{1}{2}\partial_a\bar{g}_{ce} + \frac{1}{2}\partial_e\bar{g}_{ac} - \frac{1}{2}\partial_c\bar{g}_{ae}\right)\left(\frac{1}{2}\partial_b\bar{g}_{df} + \frac{1}{2}\partial_f\bar{g}_{bd} - \frac{1}{2}\partial_d\bar{g}_{bf}\right) && \texttt{(eq14.14)} \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{eb}\partial_c\phi + \bar{g}^{ce}\partial_b\bar{g}_{ae}\partial_c\phi - \bar{g}^{ce}\partial_e\bar{g}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{fd}\partial_e\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_d\bar{g}_{cf}\partial_e\phi - \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_f\bar{g}_{cd}\partial_e\phi \\
&\quad - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ba} - \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_f\bar{g}_{de}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ab} \\
&\quad + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ab} - \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_f\bar{g}_{de}\partial_c\bar{g}_{ab} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_a\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_a\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_d\bar{g}_{af} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_a\bar{g}_{df} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_f\bar{g}_{ad} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_b\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_d\bar{g}_{bf} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_b\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_d\bar{g}_{bf} && \texttt{(eq14.15)}
\end{aligned}
$$

$$
\begin{aligned}
R_{ab} &= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{eb}\partial_c\phi + \bar{g}^{ce}\partial_b\bar{g}_{ae}\partial_c\phi - \bar{g}^{ce}\partial_e\bar{g}_{ab}\partial_c\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{fd}\partial_e\phi + \bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_d\bar{g}_{cf}\partial_e\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi \\
&\quad - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ba} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_d\bar{g}_{fe}\partial_c\bar{g}_{ab} + \frac{1}{8}\bar{g}^{de}\bar{g}^{cf}\partial_e\bar{g}_{df}\partial_c\bar{g}_{ab} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_a\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{be}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_a\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_f\bar{g}_{ad} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{cb}\partial_d\bar{g}_{af} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_a\bar{g}_{df} \\
&\quad - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_f\bar{g}_{ad} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_d\bar{g}_{af} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_b\bar{g}_{df} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ca}\partial_d\bar{g}_{bf} + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_b\bar{g}_{df} \\
&\quad + \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_e\bar{g}_{ac}\partial_d\bar{g}_{bf} \qquad (\texttt{eq14.16}) \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{de}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ab}\partial_e\bar{g}_{df} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_a\bar{g}_{ce}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} - \frac{1}{4}\bar{g}^{cd}\bar{g}^{ef}\partial_a\bar{g}_{ce}\partial_b\bar{g}_{df} \\
&\quad + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_b\bar{g}_{ce}\partial_d\bar{g}_{af} \qquad (\texttt{eq14.17}) \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{de}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ab}\partial_e\bar{g}_{df} - \frac{1}{2}\partial_a\bar{g}^{df}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} + \frac{1}{4}\partial_a\bar{g}^{df}\partial_b\bar{g}_{df} - \frac{1}{2}\partial_b\bar{g}^{df}\partial_d\bar{g}_{af} \qquad (\texttt{eq14.18}) \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi + 2\bar{g}_{ab}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{de}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} \\
&\quad + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c - \frac{1}{2}\partial_e\bar{g}^{ce}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_a\bar{g}^{df}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} + \frac{1}{4}\partial_a\bar{g}^{df}\partial_b\bar{g}_{df} - \frac{1}{2}\partial_b\bar{g}^{df}\partial_d\bar{g}_{af} \qquad (\texttt{eq14.19}) \\
&= -2\partial_{ab}\phi + \bar{g}^{ce}\partial_a\bar{g}_{bc}\partial_e\phi + \bar{g}^{ce}\partial_b\bar{g}_{ac}\partial_e\phi - \bar{g}^{ce}\partial_c\bar{g}_{ab}\partial_e\phi + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}_{ab}\partial_c\bar{g}^{cf}\partial_f\phi - 12\bar{g}_{ab}\bar{g}^{ef}\partial_e\phi\partial_f\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c \\
&\quad + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c - \frac{1}{2}\partial_e\bar{g}^{ce}\partial_c\bar{g}_{ab} - \frac{1}{2}\partial_a\bar{g}^{df}\partial_d\bar{g}_{bf} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\partial_c\bar{g}_{ae}\partial_d\bar{g}_{bf} + \frac{1}{4}\partial_a\bar{g}^{df}\partial_b\bar{g}_{df} - \frac{1}{2}\partial_b\bar{g}^{df}\partial_d\bar{g}_{af} \qquad (\texttt{eq14.20}) \\
&= -2\partial_{ab}\phi + \bar{g}^{cd}\partial_d\phi\partial_a\bar{g}_{bc} + \bar{g}^{cd}\partial_d\phi\partial_b\bar{g}_{ac} - \bar{g}^{cd}\partial_d\phi\partial_c\bar{g}_{ab} + 12\partial_a\phi\partial_b\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}_{ab}\partial_d\phi\partial_c\bar{g}^{cd} - 12\bar{g}_{ab}\bar{g}^{cd}\partial_c\phi\partial_d\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_b\bar{\Gamma}^c \\
&\quad + \frac{1}{2}\bar{g}_{bc}\partial_a\bar{\Gamma}^c - \frac{1}{2}\partial_c\bar{g}_{ab}\partial_d\bar{g}^{cd} - \frac{1}{2}\partial_d\bar{g}_{bc}\partial_a\bar{g}^{dc} - \frac{1}{2}\bar{g}^{ed}\bar{g}^{cf}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} + \frac{1}{2}\bar{g}^{ef}\bar{g}^{cd}\partial_e\bar{g}_{ac}\partial_f\bar{g}_{bd} + \frac{1}{4}\partial_b\bar{g}_{cd}\partial_a\bar{g}^{cd} - \frac{1}{2}\partial_d\bar{g}_{ac}\partial_b\bar{g}^{dc} \qquad (\texttt{eq14.21})
\end{aligned}
$$

There is a single term in this final expression that appears to be neither symmetric in ab nor part of a symmetric pair, namely

$$\partial_b \bar{g}_{cd} \partial_a \bar{g}^{cd}$$

It is, however, easy to show that this term is symmetric in ab. Start by noting that, for any $\bar{g}_{ab}$,

$$\partial_a \bar{g}^{cd} = -\bar{g}^{ce} \bar{g}^{df} \partial_a \bar{g}_{ef}$$

Now contract both sides with $\partial_b \bar{g}_{cd}$ to obtain

$$\partial_a \bar{g}^{cd} \partial_b \bar{g}_{cd} = -\bar{g}^{ce} \bar{g}^{df} \partial_a \bar{g}_{ef} \partial_b \bar{g}_{cd}$$

The right hand side is clearly symmetric in ab and thus the left hand must also be symmtric in ab.

PhysRevD.62.044034 equation (20)

```
from shared import *
import cdblib

jsonfile = 'bssn-eqtns-20.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------

DGiBarDt := \partial_{t}{GammaBar^{i}}.                 # cdb(eq20.00,DGiBarDt)
DGiBarDt := - 2 ABar^{i j} \partial_{j}{N}
            + 2 N (  GammaBar^{i}_{j k} ABar^{k j}
                   - (2/3) gBar^{i j} \partial_{j}{trK}
                   + 6 ABar^{i j} \partial_{j}{\phi}).  # cdb(eq20.01,DGiBarDt)

substitute   (DGiBarDt,defGammaBarU)                    # cdb(eq20.02,DGiBarDt)

distribute   (DGiBarDt)
DGiBarDt = product_sort (DGiBarDt)                      # cdb(eq20.03,DGiBarDt)

canonicalise (DGiBarDt)                                 # cdb(eq20.04,DGiBarDt)
                                                        # cdb(eq20.99,DGiBarDt)

cdblib.put ('DGiBarDt',DGiBarDt,jsonfile)
```

$$\begin{aligned}
\partial_t \bar{\Gamma}^i &= -2\bar{A}^{ij}\partial_j N + 2N\left(\bar{\Gamma}^i{}_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{g}^{ij}\partial_j \mathrm{tr}K + 6\bar{A}^{ij}\partial_j\phi\right) && \text{(eq20.01)}\\
&= -2\bar{A}^{ij}\partial_j N + 2N\left(\frac{1}{2}\bar{g}^{ie}\left(\partial_j\bar{g}_{ek} + \partial_k\bar{g}_{je} - \partial_e\bar{g}_{jk}\right)\bar{A}^{kj} - \frac{2}{3}\bar{g}^{ij}\partial_j \mathrm{tr}K + 6\bar{A}^{ij}\partial_j\phi\right) && \text{(eq20.02)}\\
&= -2\bar{A}^{ia}\partial_a N + N\bar{A}^{ab}\bar{g}^{ic}\partial_b\bar{g}_{ca} + N\bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{bc} - N\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ba} - \frac{4}{3}N\bar{g}^{ia}\partial_a \mathrm{tr}K + 12N\bar{A}^{ia}\partial_a\phi && \text{(eq20.03)}
\end{aligned}$$

Lapse function

This is about as easy it gets – choose a static lapse (which is fine for the Kasner spacetime).

```
from shared import *
import cdblib

jsonfile = 'bssn-lapse.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------

# DNDt := -2 N trK.  # 1+log
# DNDt := -N*N trK.  # Harmonic
DNDt := 0.           # Static lapse

cdblib.put ('DNDt',DNDt,jsonfile)
```

Ricci scalar

Here we compute the Ricci scalar R in terms of the BSSN data.

Note that this expression for R will only be used when evaluating the constraints. It will *not* be used in the evolution equations so the advice that the evolved $\bar{\Gamma}^i$ should be expressed in terms of $\bar{g}_{ij}$ does not apply here.

```
from shared import *
import cdblib

jsonfile = 'bssn-ricci-scalar.json'
cdblib.create (jsonfile)

defRab = cdblib.get ('defRab','bssn-eqtns-14.json')

# --------------------------------------------------------------------------

defG2GBarU := g^{a b} -> \exp(-4\phi) gBar^{a b}.

Rscalar := R.                                       # cdb(Rscalar.00,Rscalar)
Rscalar := g^{a b} R_{a b}.                         # cdb(Rscalar.01,Rscalar)

substitute (Rscalar, defRab)                        # cdb(Rscalar.02,Rscalar)
substitute (Rscalar, defG2GBarU)                    # cdb(Rscalar.03,Rscalar)
distribute (Rscalar)                                # cdb(Rscalar.04,Rscalar)

Rscalar = product_sort (Rscalar)                    # cdb(Rscalar.05,Rscalar)

rename_dummies (Rscalar)                            # cdb(Rscalar.06,Rscalar)
canonicalise   (Rscalar)                            # cdb(Rscalar.07,Rscalar)

foo := gBar^{b c} \partial_{a}{gBar_{b c}} -> 0.   # follows from det(g) = 1

substitute (Rscalar, foo)                           # cdb(Rscalar.08,Rscalar)

foo := gBar_{a b} gBar^{a b} -> 3.
bah := gBar_{a b} gBar^{a c} -> gBar_{b}^{c}.
moo := gBar^{c d} gBar^{e f} \partial_{a}{gBar_{c e}} -> - \partial_{a}{gBar^{d f}}.
```

```

substitute (Rscalar, foo)                               # cdb(Rscalar.09,Rscalar)
substitute (Rscalar, bah)                               # cdb(Rscalar.10,Rscalar)
substitute (Rscalar, moo)                               # cdb(Rscalar.11,Rscalar)
eliminate_kronecker (Rscalar)                           # cdb(Rscalar.12,Rscalar)
rename_dummies (Rscalar)                                # cdb(Rscalar.13,Rscalar)
canonicalise   (Rscalar)                                # cdb(Rscalar.14,Rscalar)

foo := gBar^{a b} gBar^{c d} \partial_{c}{gBar_{b d}} -> - \partial_{c}{gBar^{a c}}.
bah := \partial_{b}{gBar^{a b}} -> - GammaBar^{a}.  # prd62.eqn17

substitute (Rscalar, foo)                               # cdb(Rscalar.15,Rscalar)
substitute (Rscalar, bah)                               # cdb(Rscalar.16,Rscalar)

Rscalar = product_sort (Rscalar)                        # cdb(Rscalar.17,Rscalar)

rename_dummies (Rscalar)                                # cdb(Rscalar.18,Rscalar)
canonicalise   (Rscalar)                                # cdb(Rscalar.19,Rscalar)

foo := gBar^{a b} gBar^{c d} \partial_{a b}{gBar_{c d}} ->
     - gBar^{a b} \partial_{a}{gBar_{c d}} \partial_{b}{gBar^{c d}}. # follows from det(g) = 1

substitute (Rscalar, foo)                               # cdb(Rscalar.20,Rscalar)
factor_out (Rscalar, $\exp(-4\phi)$)                    # cdb(Rscalar.21,Rscalar)

cdblib.put ('Rscalar',Rscalar,jsonfile)
```

$$
\begin{aligned}
R &= g^{ab} R_{ab} && \texttt{(Rscalar.01)}\\
&= g^{ab} \Big(-2\partial_{ab}\phi + \bar{g}^{cd}\partial_{d}\phi\partial_{a}\bar{g}_{bc} + \bar{g}^{cd}\partial_{d}\phi\partial_{b}\bar{g}_{ac} - \bar{g}^{cd}\partial_{d}\phi\partial_{c}\bar{g}_{ab} + 12\partial_{a}\phi\partial_{b}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}_{ab}\partial_{d}\phi\partial_{c}\bar{g}^{cd} - 12\bar{g}_{ab}\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ac}\partial_{b}\bar{\Gamma}^{c}\\
&\qquad + \frac{1}{2}\bar{g}_{bc}\partial_{a}\bar{\Gamma}^{c} - \frac{1}{2}\partial_{c}\bar{g}_{ab}\partial_{d}\bar{g}^{cd} - \frac{1}{2}\partial_{d}\bar{g}_{bc}\partial_{a}\bar{g}^{dc} - \frac{1}{2}\bar{g}^{ed}\bar{g}^{cf}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} + \frac{1}{2}\bar{g}^{ef}\bar{g}^{cd}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} + \frac{1}{4}\partial_{b}\bar{g}_{cd}\partial_{a}\bar{g}^{cd} - \frac{1}{2}\partial_{d}\bar{g}_{ac}\partial_{b}\bar{g}^{dc}\Big) && \texttt{(Rscalar.02)}\\
&= \exp(-4\phi)\,\bar{g}^{ab} \Big(-2\partial_{ab}\phi + \bar{g}^{cd}\partial_{d}\phi\partial_{a}\bar{g}_{bc} + \bar{g}^{cd}\partial_{d}\phi\partial_{b}\bar{g}_{ac} - \bar{g}^{cd}\partial_{d}\phi\partial_{c}\bar{g}_{ab} + 12\partial_{a}\phi\partial_{b}\phi - 2\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\bar{g}_{ab}\partial_{d}\phi\partial_{c}\bar{g}^{cd} - 12\bar{g}_{ab}\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi - \frac{1}{2}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}\\
&\qquad + \frac{1}{2}\bar{g}_{ac}\partial_{b}\bar{\Gamma}^{c} + \frac{1}{2}\bar{g}_{bc}\partial_{a}\bar{\Gamma}^{c} - \frac{1}{2}\partial_{c}\bar{g}_{ab}\partial_{d}\bar{g}^{cd} - \frac{1}{2}\partial_{d}\bar{g}_{bc}\partial_{a}\bar{g}^{dc} - \frac{1}{2}\bar{g}^{ed}\bar{g}^{cf}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} + \frac{1}{2}\bar{g}^{ef}\bar{g}^{cd}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} + \frac{1}{4}\partial_{b}\bar{g}_{cd}\partial_{a}\bar{g}^{cd} - \frac{1}{2}\partial_{d}\bar{g}_{ac}\partial_{b}\bar{g}^{dc}\Big) && \texttt{(Rscalar.03)}\\
&= -2\exp(-4\phi)\,\bar{g}^{ab}\partial_{ab}\phi + \exp(-4\phi)\,\bar{g}^{ab}\bar{g}^{cd}\partial_{d}\phi\partial_{a}\bar{g}_{bc} + \exp(-4\phi)\,\bar{g}^{ab}\bar{g}^{cd}\partial_{d}\phi\partial_{b}\bar{g}_{ac} - \exp(-4\phi)\,\bar{g}^{ab}\bar{g}^{cd}\partial_{d}\phi\partial_{c}\bar{g}_{ab} + 12\exp(-4\phi)\,\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\\
&\quad - 2\exp(-4\phi)\,\bar{g}^{ab}\bar{g}_{ab}\bar{g}^{cd}\partial_{cd}\phi - 2\exp(-4\phi)\,\bar{g}^{ab}\bar{g}_{ab}\partial_{d}\phi\partial_{c}\bar{g}^{cd} - 12\exp(-4\phi)\,\bar{g}^{ab}\bar{g}_{ab}\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi - \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\bar{g}^{cd}\partial_{cd}\bar{g}_{ab}\\
&\quad + \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\bar{g}_{ac}\partial_{b}\bar{\Gamma}^{c} + \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\bar{g}_{bc}\partial_{a}\bar{\Gamma}^{c} - \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\partial_{c}\bar{g}_{ab}\partial_{d}\bar{g}^{cd} - \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\partial_{d}\bar{g}_{bc}\partial_{a}\bar{g}^{dc}\\
&\quad - \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\bar{g}^{ed}\bar{g}^{cf}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} + \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\bar{g}^{ef}\bar{g}^{cd}\partial_{e}\bar{g}_{ac}\partial_{f}\bar{g}_{bd} + \frac{1}{4}\exp(-4\phi)\,\bar{g}^{ab}\partial_{b}\bar{g}_{cd}\partial_{a}\bar{g}^{cd} - \frac{1}{2}\exp(-4\phi)\,\bar{g}^{ab}\partial_{d}\bar{g}_{ac}\partial_{b}\bar{g}^{dc} && \texttt{(Rscalar.04)}\\
&= -2\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + \bar{g}^{ca}\bar{g}^{bd}\partial_{d}\phi\exp(-4\phi)\,\partial_{c}\bar{g}_{ab} + \bar{g}^{ac}\bar{g}^{bd}\partial_{d}\phi\exp(-4\phi)\,\partial_{c}\bar{g}_{ab} - \bar{g}^{ab}\bar{g}^{cd}\partial_{d}\phi\exp(-4\phi)\,\partial_{c}\bar{g}_{ab} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp(-4\phi)\\
&\quad - 2\bar{g}_{ab}\bar{g}^{ab}\bar{g}^{cd}\partial_{cd}\phi\exp(-4\phi) - 2\bar{g}_{ab}\bar{g}^{ab}\partial_{d}\phi\exp(-4\phi)\,\partial_{c}\bar{g}^{cd} - 12\bar{g}_{ab}\bar{g}^{ab}\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi\exp(-4\phi) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\,\partial_{cd}\bar{g}_{ab} + \frac{1}{2}\bar{g}_{ab}\bar{g}^{ac}\exp(-4\phi)\,\partial_{c}\bar{\Gamma}^{b}\\
&\quad + \frac{1}{2}\bar{g}_{ab}\bar{g}^{ca}\exp(-4\phi)\,\partial_{c}\bar{\Gamma}^{b} - \frac{1}{2}\bar{g}^{ab}\exp(-4\phi)\,\partial_{c}\bar{g}_{ab}\partial_{d}\bar{g}^{cd} - \frac{1}{2}\bar{g}^{ca}\exp(-4\phi)\,\partial_{d}\bar{g}_{ab}\partial_{c}\bar{g}^{db} - \frac{1}{2}\bar{g}^{ac}\bar{g}^{be}\bar{g}^{fd}\exp(-4\phi)\,\partial_{f}\bar{g}_{ab}\partial_{e}\bar{g}_{cd}\\
&\quad + \frac{1}{2}\bar{g}^{ac}\bar{g}^{bd}\bar{g}^{ef}\exp(-4\phi)\,\partial_{e}\bar{g}_{ab}\partial_{f}\bar{g}_{cd} + \frac{1}{4}\bar{g}^{cd}\exp(-4\phi)\,\partial_{d}\bar{g}_{ab}\partial_{c}\bar{g}^{ab} - \frac{1}{2}\bar{g}^{ac}\exp(-4\phi)\,\partial_{d}\bar{g}_{ab}\partial_{c}\bar{g}^{db} && \texttt{(Rscalar.05)}\\
&= -2\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + \bar{g}^{bc}\bar{g}^{da}\partial_{a}\phi\exp(-4\phi)\,\partial_{b}\bar{g}_{cd} + \bar{g}^{cb}\bar{g}^{da}\partial_{a}\phi\exp(-4\phi)\,\partial_{b}\bar{g}_{cd} - \bar{g}^{cd}\bar{g}^{ba}\partial_{a}\phi\exp(-4\phi)\,\partial_{b}\bar{g}_{cd} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp(-4\phi)\\
&\quad - 2\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) - 2\bar{g}_{cd}\bar{g}^{cd}\partial_{a}\phi\exp(-4\phi)\,\partial_{b}\bar{g}^{ba} - 12\bar{g}_{cd}\bar{g}^{cd}\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp(-4\phi) - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ab}\exp(-4\phi)\,\partial_{ab}\bar{g}_{cd} + \frac{1}{2}\bar{g}_{ca}\bar{g}^{cb}\exp(-4\phi)\,\partial_{b}\bar{\Gamma}^{a}\\
&\quad + \frac{1}{2}\bar{g}_{ca}\bar{g}^{bc}\exp(-4\phi)\,\partial_{b}\bar{\Gamma}^{a} - \frac{1}{2}\bar{g}^{cd}\exp(-4\phi)\,\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{ab} - \frac{1}{2}\bar{g}^{bc}\exp(-4\phi)\,\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{ad} - \frac{1}{2}\bar{g}^{cd}\bar{g}^{eb}\bar{g}^{af}\exp(-4\phi)\,\partial_{a}\bar{g}_{ce}\partial_{b}\bar{g}_{df}\\
&\quad + \frac{1}{2}\bar{g}^{cd}\bar{g}^{ef}\bar{g}^{ab}\exp(-4\phi)\,\partial_{a}\bar{g}_{ce}\partial_{b}\bar{g}_{df} + \frac{1}{4}\bar{g}^{ba}\exp(-4\phi)\,\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{cd} - \frac{1}{2}\bar{g}^{cb}\exp(-4\phi)\,\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{ad} && \texttt{(Rscalar.06)}
\end{aligned}
$$

$$
\begin{aligned}
R &= -2\bar{g}^{ab}\partial_{ab}\phi\exp\left(-4\phi\right) + 2\bar{g}^{ab}\bar{g}^{cd}\partial_{a}\phi\exp\left(-4\phi\right)\partial_{c}\bar{g}_{bd} - \bar{g}^{ab}\bar{g}^{cd}\partial_{a}\phi\exp\left(-4\phi\right)\partial_{b}\bar{g}_{cd} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp\left(-4\phi\right) - 2\bar{g}_{ab}\bar{g}^{ab}\bar{g}^{cd}\partial_{cd}\phi\exp\left(-4\phi\right)\\
&\quad - 2\bar{g}_{ab}\bar{g}^{ab}\partial_{c}\phi\exp\left(-4\phi\right)\partial_{d}\bar{g}^{cd} - 12\bar{g}_{ab}\bar{g}^{ab}\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi\exp\left(-4\phi\right) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_{ab}\bar{g}_{cd} + \bar{g}_{ab}\bar{g}^{ac}\exp\left(-4\phi\right)\partial_{c}\bar{\Gamma}^{b} - \frac{1}{2}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{c}\bar{g}_{ab}\partial_{d}\bar{g}^{cd}\\
&\quad - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{c}\bar{g}_{ad}\partial_{b}\bar{g}^{cd} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{d}\bar{g}_{bf} + \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{b}\bar{g}_{df} + \frac{1}{4}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{cd} \qquad \texttt{(Rscalar.07)}\\
&= -2\bar{g}^{ab}\partial_{ab}\phi\exp\left(-4\phi\right) + 2\bar{g}^{ab}\bar{g}^{cd}\partial_{a}\phi\exp\left(-4\phi\right)\partial_{c}\bar{g}_{bd} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp\left(-4\phi\right) - 2\bar{g}_{ab}\bar{g}^{ab}\bar{g}^{cd}\partial_{cd}\phi\exp\left(-4\phi\right) - 2\bar{g}_{ab}\bar{g}^{ab}\partial_{c}\phi\exp\left(-4\phi\right)\partial_{d}\bar{g}^{cd}\\
&\quad - 12\bar{g}_{ab}\bar{g}^{ab}\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi\exp\left(-4\phi\right) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_{ab}\bar{g}_{cd} + \bar{g}_{ab}\bar{g}^{ac}\exp\left(-4\phi\right)\partial_{c}\bar{\Gamma}^{b} - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{c}\bar{g}_{ad}\partial_{b}\bar{g}^{cd}\\
&\quad - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{d}\bar{g}_{bf} + \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{b}\bar{g}_{df} + \frac{1}{4}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{cd} \qquad \texttt{(Rscalar.08)}\\
&= -2\bar{g}^{ab}\partial_{ab}\phi\exp\left(-4\phi\right) + 2\bar{g}^{ab}\bar{g}^{cd}\partial_{a}\phi\exp\left(-4\phi\right)\partial_{c}\bar{g}_{bd} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp\left(-4\phi\right) - 6\bar{g}^{cd}\partial_{cd}\phi\exp\left(-4\phi\right) - 6\partial_{c}\phi\exp\left(-4\phi\right)\partial_{d}\bar{g}^{cd}\\
&\quad - 36\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi\exp\left(-4\phi\right) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_{ab}\bar{g}_{cd} + \bar{g}_{ab}\bar{g}^{ac}\exp\left(-4\phi\right)\partial_{c}\bar{\Gamma}^{b} - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{c}\bar{g}_{ad}\partial_{b}\bar{g}^{cd} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{d}\bar{g}_{bf}\\
&\quad + \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{b}\bar{g}_{df} + \frac{1}{4}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{cd} \qquad \texttt{(Rscalar.09)}\\
&= -2\bar{g}^{ab}\partial_{ab}\phi\exp\left(-4\phi\right) + 2\bar{g}^{ab}\bar{g}^{cd}\partial_{a}\phi\exp\left(-4\phi\right)\partial_{c}\bar{g}_{bd} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp\left(-4\phi\right) - 6\bar{g}^{cd}\partial_{cd}\phi\exp\left(-4\phi\right) - 6\partial_{c}\phi\exp\left(-4\phi\right)\partial_{d}\bar{g}^{cd}\\
&\quad - 36\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi\exp\left(-4\phi\right) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_{ab}\bar{g}_{cd} + \bar{g}_{b}{}^{c}\exp\left(-4\phi\right)\partial_{c}\bar{\Gamma}^{b} - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{c}\bar{g}_{ad}\partial_{b}\bar{g}^{cd} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{d}\bar{g}_{bf}\\
&\quad + \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\bar{g}^{ef}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{ce}\partial_{b}\bar{g}_{df} + \frac{1}{4}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{cd} \qquad \texttt{(Rscalar.10)}\\
&= -2\bar{g}^{ab}\partial_{ab}\phi\exp\left(-4\phi\right) + 2\bar{g}^{ab}\bar{g}^{cd}\partial_{a}\phi\exp\left(-4\phi\right)\partial_{c}\bar{g}_{bd} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp\left(-4\phi\right) - 6\bar{g}^{cd}\partial_{cd}\phi\exp\left(-4\phi\right) - 6\partial_{c}\phi\exp\left(-4\phi\right)\partial_{d}\bar{g}^{cd}\\
&\quad - 36\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi\exp\left(-4\phi\right) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_{ab}\bar{g}_{cd} + \bar{g}_{b}{}^{c}\exp\left(-4\phi\right)\partial_{c}\bar{\Gamma}^{b} - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{c}\bar{g}_{ad}\partial_{b}\bar{g}^{cd} + \frac{1}{2}\bar{g}^{ab}\partial_{a}\bar{g}^{df}\exp\left(-4\phi\right)\partial_{d}\bar{g}_{bf}\\
&\quad - \frac{1}{2}\bar{g}^{ab}\partial_{a}\bar{g}^{df}\exp\left(-4\phi\right)\partial_{b}\bar{g}_{df} + \frac{1}{4}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{cd} \qquad \texttt{(Rscalar.11)}\\
&= -2\bar{g}^{ab}\partial_{ab}\phi\exp\left(-4\phi\right) + 2\bar{g}^{ab}\bar{g}^{cd}\partial_{a}\phi\exp\left(-4\phi\right)\partial_{c}\bar{g}_{bd} + 12\bar{g}^{ab}\partial_{a}\phi\partial_{b}\phi\exp\left(-4\phi\right) - 6\bar{g}^{cd}\partial_{cd}\phi\exp\left(-4\phi\right) - 6\partial_{c}\phi\exp\left(-4\phi\right)\partial_{d}\bar{g}^{cd}\\
&\quad - 36\bar{g}^{cd}\partial_{c}\phi\partial_{d}\phi\exp\left(-4\phi\right) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp\left(-4\phi\right)\partial_{ab}\bar{g}_{cd} + \exp\left(-4\phi\right)\partial_{b}\bar{\Gamma}^{b} - \bar{g}^{ab}\exp\left(-4\phi\right)\partial_{c}\bar{g}_{ad}\partial_{b}\bar{g}^{cd} + \frac{1}{2}\bar{g}^{ab}\partial_{a}\bar{g}^{df}\exp\left(-4\phi\right)\partial_{d}\bar{g}_{bf}\\
&\quad - \frac{1}{2}\bar{g}^{ab}\partial_{a}\bar{g}^{df}\exp\left(-4\phi\right)\partial_{b}\bar{g}_{df} + \frac{1}{4}\bar{g}^{ab}\exp\left(-4\phi\right)\partial_{a}\bar{g}_{cd}\partial_{b}\bar{g}^{cd} \qquad \texttt{(Rscalar.12)}
\end{aligned}
$$

$$
\begin{aligned}
R &= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + 2\bar{g}^{ac}\bar{g}^{bd}\partial_a\phi\exp(-4\phi)\,\partial_b\bar{g}_{cd} - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) - 6\partial_a\phi\exp(-4\phi)\,\partial_b\bar{g}^{ab} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\,\partial_{ab}\bar{g}_{cd} \\
&\quad + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a - \bar{g}^{cb}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{ad} + \frac{1}{2}\bar{g}^{ac}\partial_a\bar{g}^{bd}\exp(-4\phi)\,\partial_b\bar{g}_{cd} - \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}^{cd}\exp(-4\phi)\,\partial_b\bar{g}_{cd} + \frac{1}{4}\bar{g}^{ab}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} && \texttt{(Rscalar.13)} \\
&= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + 2\bar{g}^{ab}\bar{g}^{cd}\partial_a\phi\exp(-4\phi)\,\partial_c\bar{g}_{bd} - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) - 6\partial_a\phi\exp(-4\phi)\,\partial_b\bar{g}^{ab} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\,\partial_{ab}\bar{g}_{cd} \\
&\quad + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a - \bar{g}^{ab}\exp(-4\phi)\,\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} + \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}^{cd}\exp(-4\phi)\,\partial_c\bar{g}_{bd} - \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}^{cd}\exp(-4\phi)\,\partial_b\bar{g}_{cd} + \frac{1}{4}\bar{g}^{ab}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} && \texttt{(Rscalar.14)} \\
&= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) - 2\partial_c\bar{g}^{ac}\partial_a\phi\exp(-4\phi) - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) - 6\partial_a\phi\exp(-4\phi)\,\partial_b\bar{g}^{ab} - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\,\partial_{ab}\bar{g}_{cd} + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a \\
&\quad - \bar{g}^{ab}\exp(-4\phi)\,\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} + \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}^{cd}\exp(-4\phi)\,\partial_c\bar{g}_{bd} - \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}^{cd}\exp(-4\phi)\,\partial_b\bar{g}_{cd} + \frac{1}{4}\bar{g}^{ab}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} && \texttt{(Rscalar.15)} \\
&= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + 2\bar{\Gamma}^a\partial_a\phi\exp(-4\phi) - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) + 6\partial_a\phi\exp(-4\phi)\,\bar{\Gamma}^a - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\,\partial_{ab}\bar{g}_{cd} + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a \\
&\quad - \bar{g}^{ab}\exp(-4\phi)\,\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} + \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}^{cd}\exp(-4\phi)\,\partial_c\bar{g}_{bd} - \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}^{cd}\exp(-4\phi)\,\partial_b\bar{g}_{cd} + \frac{1}{4}\bar{g}^{ab}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} && \texttt{(Rscalar.16)} \\
&= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + 8\bar{\Gamma}^a\partial_a\phi\exp(-4\phi) - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) - \frac{1}{2}\bar{g}^{cd}\bar{g}^{ab}\exp(-4\phi)\,\partial_{cd}\bar{g}_{ab} + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a - \bar{g}^{ac}\exp(-4\phi)\,\partial_d\bar{g}_{ab}\partial_c\bar{g}^{db} \\
&\quad + \frac{1}{2}\bar{g}^{ca}\exp(-4\phi)\,\partial_d\bar{g}_{ab}\partial_c\bar{g}^{db} - \frac{1}{2}\bar{g}^{cd}\exp(-4\phi)\,\partial_d\bar{g}_{ab}\partial_c\bar{g}^{ab} + \frac{1}{4}\bar{g}^{cd}\exp(-4\phi)\,\partial_c\bar{g}_{ab}\partial_d\bar{g}^{ab} && \texttt{(Rscalar.17)} \\
&= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + 8\bar{\Gamma}^a\partial_a\phi\exp(-4\phi) - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\,\partial_{ab}\bar{g}_{cd} + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a - \bar{g}^{cb}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{ad} \\
&\quad + \frac{1}{2}\bar{g}^{bc}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{ad} - \frac{1}{2}\bar{g}^{ba}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} + \frac{1}{4}\bar{g}^{ab}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} && \texttt{(Rscalar.18)} \\
&= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + 8\bar{\Gamma}^a\partial_a\phi\exp(-4\phi) - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) - \frac{1}{2}\bar{g}^{ab}\bar{g}^{cd}\exp(-4\phi)\,\partial_{ab}\bar{g}_{cd} + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a - \frac{1}{2}\bar{g}^{ab}\exp(-4\phi)\,\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} \\
&\quad - \frac{1}{4}\bar{g}^{ab}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} && \texttt{(Rscalar.19)} \\
&= -8\bar{g}^{ab}\partial_{ab}\phi\exp(-4\phi) + 8\bar{\Gamma}^a\partial_a\phi\exp(-4\phi) - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi\exp(-4\phi) + \frac{1}{2}\bar{g}^{ab}\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd}\exp(-4\phi) + \exp(-4\phi)\,\partial_a\bar{\Gamma}^a \\
&\quad - \frac{1}{2}\bar{g}^{ab}\exp(-4\phi)\,\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd} - \frac{1}{4}\bar{g}^{ab}\exp(-4\phi)\,\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} && \texttt{(Rscalar.20)} \\
&= \exp(-4\phi)\left(-8\bar{g}^{ab}\partial_{ab}\phi + 8\bar{\Gamma}^a\partial_a\phi - 24\bar{g}^{ab}\partial_a\phi\partial_b\phi + \frac{1}{4}\bar{g}^{ab}\partial_a\bar{g}_{cd}\partial_b\bar{g}^{cd} + \partial_a\bar{\Gamma}^a - \frac{1}{2}\bar{g}^{ab}\partial_c\bar{g}_{ad}\partial_b\bar{g}^{cd}\right) && \texttt{(Rscalar.21)}
\end{aligned}
$$

PhysRevD.67.084023 equation (19)

```
from shared import *
import cdblib

jsonfile = 'bssn-constraints.json'
cdblib.create (jsonfile)

# ----------------------------------------------------------------------------
# Hamiltonian constraint

Ham := R + K_{a b} g^{a b} K_{c d} g^{c d} - K_{a b} K_{c d} g^{a c} g^{b d}.
                                                          # cdb(Ham.101,Ham)

Ham := R + (2/3) (trK)**2 - ABar_{a b} ABar^{a b}.        # cdb(Ham.102,Ham)
```

$$\mathcal{H} = R + K_{ab} g^{ab} K_{cd} g^{cd} - K_{ab} K_{cd} g^{ac} g^{bd} \tag{Ham.101}$$

$$= R + \frac{2}{3} \mathrm{tr} K^2 - \bar{A}_{ab} \bar{A}^{ab} \tag{Ham.102}$$

PhysRevD.67.084023 equation (20)

```
# ----------------------------------------------------------------------------
# Momentum constraint

confMom := 6 ABar^{i a} \partial_{a}{\phi}
           + \partial_{a}{ABar^{i a}}
           + ABar^{a b} GammaBar^{i}_{a b}
           - (2/3) gBar^{i a} \partial_{a}{trK}.

defGammaBar := GammaBar^{a}_{b c} ->
               (1/2) gBar^{a e} (   \partial_{b}{gBar_{e c}}
                                  + \partial_{c}{gBar_{b e}}
                                  - \partial_{e}{gBar_{b c}}).

substitute (confMom, defGammaBar)                      # cdb(confMom.101,confMom)
distribute (confMom)                                   # cdb(confMom.102,confMom)

confMom = product_sort (confMom)                       # cdb(confMom.103,confMom)

rename_dummies (confMom)                               # cdb(confMom.104,confMom)
canonicalise   (confMom)                               # cdb(confMom.105,confMom)

foo := \partial_{a}{ABar^{i a}} -> \partial_{a}{gBar^{i c} gBar^{a d} ABar_{c d}}.

substitute   (confMom, foo)                            # cdb(confMom.106,confMom)
product_rule (confMom)                                 # cdb(confMom.107,confMom)

confMom = product_sort (confMom)                       # cdb(confMom.108,confMom)

rename_dummies (confMom)                               # cdb(confMom.109,confMom)
canonicalise   (confMom)                               # cdb(confMom.110,confMom)

cdblib.put ('Ham',Ham,jsonfile)
cdblib.put ('confMom',confMom,jsonfile)
```

$$
\begin{aligned}
\exp(4\phi)\mathcal{D}^j &= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \frac{1}{2}\bar{A}^{ab}\bar{g}^{ie}\left(\partial_a\bar{g}_{eb} + \partial_b\bar{g}_{ae} - \partial_e\bar{g}_{ab}\right) - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.101}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \frac{1}{2}\bar{A}^{ab}\bar{g}^{ie}\partial_a\bar{g}_{eb} + \frac{1}{2}\bar{A}^{ab}\bar{g}^{ie}\partial_b\bar{g}_{ae} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ie}\partial_e\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.102}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{cb} + \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_b\bar{g}_{ac} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.103}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{cb} + \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_b\bar{g}_{ac} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.104}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{A}^{ia} + \bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{bc} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.105}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \partial_a\left(\bar{g}^{ic}\bar{g}^{ad}\bar{A}_{cd}\right) + \bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{bc} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.106}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \partial_a\bar{g}^{ic}\bar{g}^{ad}\bar{A}_{cd} + \bar{g}^{ic}\partial_a\bar{g}^{ad}\bar{A}_{cd} + \bar{g}^{ic}\bar{g}^{ad}\partial_a\bar{A}_{cd} + \bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{bc} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.107}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \bar{A}_{ab}\bar{g}^{cb}\partial_c\bar{g}^{ia} + \bar{A}_{ab}\bar{g}^{ia}\partial_c\bar{g}^{cb} + \bar{g}^{cb}\bar{g}^{ia}\partial_c\bar{A}_{ab} + \bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{bc} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.108}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \bar{A}_{ab}\bar{g}^{cb}\partial_c\bar{g}^{ia} + \bar{A}_{ab}\bar{g}^{ia}\partial_c\bar{g}^{cb} + \bar{g}^{cb}\bar{g}^{ia}\partial_c\bar{A}_{ab} + \bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{bc} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.109}) \\
&= 6\bar{A}^{ia}\partial_a\phi + \bar{A}_{ab}\bar{g}^{ac}\partial_c\bar{g}^{ib} + \bar{A}_{ab}\bar{g}^{ia}\partial_c\bar{g}^{bc} + \bar{g}^{ia}\bar{g}^{bc}\partial_b\bar{A}_{ac} + \bar{A}^{ab}\bar{g}^{ic}\partial_a\bar{g}_{bc} - \frac{1}{2}\bar{A}^{ab}\bar{g}^{ic}\partial_c\bar{g}_{ab} - \frac{2}{3}\bar{g}^{ia}\partial_a\mathrm{tr}K && (\texttt{confMom.110})
\end{aligned}
$$