# Exercise 1.1 Verify symmetry of $\Gamma^a{}_{bc}$

```
{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

g_{a b}::Metric.

\partial{#}::PartialDerivative.

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
                                            + \partial_{c}{g_{b d}}
                                            - \partial_{d}{g_{b c}} ).

diff := \Gamma^{a}_{b c} - \Gamma^{a}_{c b}.   # cdb (ex-0101.101,diff)

substitute    (diff, Gamma)                    # cdb (ex-0101.102,diff)
distribute    (diff)                           # cdb (ex-0101.103,diff)
canonicalise  (diff)                           # cdb (ex-0101.104,diff)
```

$$\begin{aligned}
\Gamma^a{}_{bc} - \Gamma^a{}_{cb} &= \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) - \frac{1}{2} g^{ad} \left(\partial_c g_{db} + \partial_b g_{cd} - \partial_d g_{cb}\right) \\
&= \frac{1}{2} g^{ad} \partial_b g_{dc} + \frac{1}{2} g^{ad} \partial_c g_{bd} - \frac{1}{2} g^{ad} \partial_d g_{bc} - \frac{1}{2} g^{ad} \partial_c g_{db} - \frac{1}{2} g^{ad} \partial_b g_{cd} + \frac{1}{2} g^{ad} \partial_d g_{cb} \\
&= 0
\end{aligned}$$

# Exercise 1.2 Christoffel symbol and dg

```
{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

g_{a b}::Metric.
g_{a}^{b}::KroneckerDelta.

\partial{#}::PartialDerivative.

GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
                                            + \partial_{c}{g_{b d}}
                                            - \partial_{d}{g_{b c}} ).

GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.

expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}.   # cdb (ex-0102.101,expr)

substitute          (expr, GammaD)                                  # cdb (ex-0102.102,expr)
substitute          (expr, GammaU)                                  # cdb (ex-0102.103,expr)
distribute          (expr)                                          # cdb (ex-0102.104,expr)
eliminate_metric    (expr)                                          # cdb (ex-0102.105,expr)
eliminate_kronecker (expr)                                          # cdb (ex-0102.106,expr)
canonicalise        (expr)                                          # cdb (ex-0102.107,expr)
```

$$
\begin{aligned}
\Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} &= g_{ad}\Gamma^{d}{}_{bc} + g_{bd}\Gamma^{d}{}_{ac} - \partial_c g_{ab} \\
&= \frac{1}{2} g_{ad} g^{de} \left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right) + \frac{1}{2} g_{bd} g^{de} \left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - \partial_c g_{ab} \\
&= \frac{1}{2} g_{ad} g^{de} \partial_b g_{ec} + \frac{1}{2} g_{ad} g^{de} \partial_c g_{be} - \frac{1}{2} g_{ad} g^{de} \partial_e g_{bc} + \frac{1}{2} g_{bd} g^{de} \partial_a g_{ec} + \frac{1}{2} g_{bd} g^{de} \partial_c g_{ae} - \frac{1}{2} g_{bd} g^{de} \partial_e g_{ac} - \partial_c g_{ab} \\
&= \frac{1}{2} g_{a}{}^{e} \partial_b g_{ec} + \frac{1}{2} g_{a}{}^{e} \partial_c g_{be} - \frac{1}{2} g_{a}{}^{e} \partial_e g_{bc} + \frac{1}{2} g_{b}{}^{e} \partial_a g_{ec} + \frac{1}{2} g_{b}{}^{e} \partial_c g_{ae} - \frac{1}{2} g_{b}{}^{e} \partial_e g_{ac} - \partial_c g_{ab} \\
&= \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_c g_{ab} \\
&= 0
\end{aligned}
$$

10 Aug 2022

## Exercise 1.3 Christoffel symbol and dg with a single rule

```
{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

g_{a b}::Metric.
g_{a}^{b}::KroneckerDelta.

\partial{#}::PartialDerivative.

GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
                                             + \partial_{c}{g_{b d}}
                                             - \partial_{d}{g_{b c}} ).

GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.              # cdb (ex-0103.101,GammaD)

substitute          (GammaD, GammaU)                               # cdb (ex-0103.102,GammaD)  # requires Indices(position=independent)
distribute          (GammaD)                                       # cdb (ex-0103.103,GammaD)
eliminate_metric    (GammaD)                                       # cdb (ex-0103.104,GammaD)
eliminate_kronecker (GammaD)                                       # cdb (ex-0103.105,GammaD)

expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}.   # cdb (ex-0103.201,expr)

substitute          (expr, GammaD)                                 # cdb (ex-0103.202,expr)
canonicalise        (expr)                                         # cdb (ex-0103.203,expr)
```

$$\Gamma_{abc} \to g_{ad}\Gamma^{d}{}_{bc} \tag{ex-0103.101}$$

$$\Gamma_{abc} \to \frac{1}{2} g_{ad} g^{de} \left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right) \tag{ex-0103.102}$$

$$\Gamma_{abc} \to \frac{1}{2} g_{ad} g^{de} \partial_b g_{ec} + \frac{1}{2} g_{ad} g^{de} \partial_c g_{be} - \frac{1}{2} g_{ad} g^{de} \partial_e g_{bc} \tag{ex-0103.103}$$

$$\Gamma_{abc} \to \frac{1}{2} g_{a}{}^{e} \partial_b g_{ec} + \frac{1}{2} g_{a}{}^{e} \partial_c g_{be} - \frac{1}{2} g_{a}{}^{e} \partial_e g_{bc} \tag{ex-0103.104}$$

$$\Gamma_{abc} \to \frac{1}{2} \partial_b g_{ac} + \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_a g_{bc} \tag{ex-0103.105}$$

$$\Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} = \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_c g_{ab} \tag{ex-0103.202}$$

$$= 0 \tag{ex-0103.203}$$

## Exercise 1.3 Repeat but without `position=independent`

```
{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

g_{a b}::Metric.
g_{a}^{b}::KroneckerDelta.

\partial{#}::PartialDerivative.

GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
                                             + \partial_{c}{g_{b d}}
                                             - \partial_{d}{g_{b c}} ).

GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.               # cdb (ex-0103.301,GammaD)

substitute          (GammaD, GammaU)                                # cdb (ex-0103.302,GammaD)
distribute          (GammaD)                                        # cdb (ex-0103.303,GammaD)
eliminate_metric    (GammaD)                                        # cdb (ex-0103.304,GammaD)
eliminate_kronecker (GammaD)                                        # cdb (ex-0103.305,GammaD)

expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}.   # cdb (ex-0103.401,expr)

substitute          (expr, GammaD)                                  # cdb (ex-0103.402,expr)
canonicalise        (expr)                                          # cdb (ex-0103.403,expr)
```

$$\Gamma_{abc} \to g_{ad}\Gamma^{d}{}_{bc} \tag{ex-0103.301}$$

$$\frac{1}{2}g_{a}{}^{d}\left(\partial_{b}g_{dc} + \partial_{c}g_{bd} - \partial_{d}g_{bc}\right) \to \frac{1}{2}g_{ad}g^{de}\left(\partial_{b}g_{ec} + \partial_{c}g_{be} - \partial_{e}g_{bc}\right) \tag{ex-0103.302}$$

$$\frac{1}{2}g_{a}{}^{d}\partial_{b}g_{dc} + \frac{1}{2}g_{a}{}^{d}\partial_{c}g_{bd} - \frac{1}{2}g_{a}{}^{d}\partial_{d}g_{bc} \to \frac{1}{2}g_{ad}g^{de}\partial_{b}g_{ec} + \frac{1}{2}g_{ad}g^{de}\partial_{c}g_{be} - \frac{1}{2}g_{ad}g^{de}\partial_{e}g_{bc} \tag{ex-0103.303}$$

$$\frac{1}{2}\partial_{b}g_{ac} + \frac{1}{2}\partial_{c}g_{ba} - \frac{1}{2}\partial_{a}g_{bc} \to \frac{1}{2}g_{a}{}^{e}\partial_{b}g_{ec} + \frac{1}{2}g_{a}{}^{e}\partial_{c}g_{be} - \frac{1}{2}g_{a}{}^{e}\partial_{e}g_{bc} \tag{ex-0103.304}$$

$$\frac{1}{2}\partial_{b}g_{ac} + \frac{1}{2}\partial_{c}g_{ba} - \frac{1}{2}\partial_{a}g_{bc} \to \frac{1}{2}\partial_{b}g_{ac} + \frac{1}{2}\partial_{c}g_{ba} - \frac{1}{2}\partial_{a}g_{bc} \tag{ex-0103.305}$$

$$\Gamma_{abc} + \Gamma_{bac} - \partial_{c}g_{ab} = \Gamma_{abc} + \Gamma_{bac} - \partial_{c}g_{ab} \tag{ex-0103.402}$$

$$= \Gamma_{abc} + \Gamma_{bac} - \partial_{c}g_{ab} \tag{ex-0103.403}$$

## Exercise 1.4 Experiments with sorting

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z#}::Indices(position=independent).

\partial{#}::PartialDerivative.

expr := C^{f}
        w^{e}
        B^{d}
        v^{c}
        A^{b}
        u^{a}.                      # cdb (ex-0104.100,expr)

sort_product (expr)                 # cdb (ex-0104.101,expr)

expr := \Omega_{f}
        \gamma_{e}
        \Pi_{d}
        \beta_{c}
        \Gamma_{b}
        \alpha_{a}.                 # cdb (ex-0104.200,expr)

sort_product (expr)                 # cdb (ex-0104.201,expr)

expr := C^{f}
        w^{e}
        B^{d}
        v^{c}
        A^{b}
        u^{a}
        \Omega_{f}
        \gamma_{e}
        \Pi_{d}
        \beta_{c}
        \Gamma_{b}
        \alpha_{a}.                 # cdb (ex-0104.300,expr)

sort_product (expr)                 # cdb (ex-0104.301,expr)
```

```
expr := \partial_{f}{C^{f}}
        w^{l}
        \partial_{d}{B^{d}}
        v^{k}
        \partial_{b}{A^{b}}
        u^{j}
        \Omega_{i}
        \partial^{e}{ \gamma_{e}}
        \Pi_{h}
        \partial^{c}{\beta_{c}}
        \Gamma_{g}
        \partial^{a}{\alpha_{a}}.     # cdb (ex-0104.400,expr)

sort_product (expr)                   # cdb (ex-0104.401,expr)

expr := \partial{C}
        w
        \partial{B}
        v
        \partial{A}
        u
        \Omega
        \partial{ \gamma}
        \Pi
        \partial{\beta}
        \Gamma
        \partial{\alpha}.             # cdb (ex-0104.500,expr)

sort_product (expr)                   # cdb (ex-0104.501,expr)

expr := A_{b}
        A_{a}
        A_{c d e}
        A_{f g}.                      # cdb (ex-0104.600,expr)

sort_product (expr)                   # cdb (ex-0104.601,expr)
```

```
expr :=  A_{a} A^{a}
       + A^{a} A_{a}.                     # cdb (ex-0104.700,expr)

sort_product (expr)                       # cdb (ex-0104.701,expr)
```

$$\texttt{ex-0104.100} := C^f w^e B^d v^c A^b u^a$$

$$\texttt{ex-0104.101} := A^b B^d C^f u^a v^c w^e$$

$$\texttt{ex-0104.200} := \Omega_f \gamma_e \Pi_d \beta_c \Gamma_b \alpha_a$$

$$\texttt{ex-0104.201} := \Gamma_b \Omega_f \Pi_d \alpha_a \beta_c \gamma_e$$

$$\texttt{ex-0104.300} := C^f w^e B^d v^c A^b u^a \Omega_f \gamma_e \Pi_d \beta_c \Gamma_b \alpha_a$$

$$\texttt{ex-0104.301} := A^b B^d C^f \Gamma_b \Omega_f \Pi_d \alpha_a \beta_c \gamma_e u^a v^c w^e$$

$$\texttt{ex-0104.400} := \partial_f C^f w^l \partial_d B^d v^k \partial_b A^b u^j \Omega_i \partial^e \gamma_e \Pi_h \partial^c \beta_c \Gamma_g \partial^a \alpha_a$$

$$\texttt{ex-0104.401} := \Gamma_g \Omega_i \Pi_h \partial_b A^b \partial_d B^d \partial_f C^f \partial^a \alpha_a \partial^c \beta_c \partial^e \gamma_e u^j v^k w^l$$

$$\texttt{ex-0104.500} := \partial C w \partial B v \partial A u \Omega \partial \gamma \Pi \partial \beta \Gamma \partial \alpha$$

$$\texttt{ex-0104.501} := \Gamma \Omega \Pi \partial A \partial B \partial C \partial \alpha \partial \beta \partial \gamma u v w$$

$$\texttt{ex-0104.600} := A_b A_a A_{cde} A_{fg}$$

$$\texttt{ex-0104.601} := A_a A_b A_{fg} A_{cde}$$

$$\texttt{ex-0104.700} := A_a A^a + A^a A_a$$

$$\texttt{ex-0104.701} := 2 A_a A^a$$

# Exercise 1.5 A sort hack

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z#}::Indices(position=independent).

foo := A_{a} A^{a} + A^{a} A_{a}.              # cdb (ex-0105.100,foo)

sort_product (foo)                             # cdb (ex-0105.101,foo)

substitute   (foo, $A^{a} -> Z^{a}$)           # cdb (ex-0105.102,foo)
sort_product (foo)                             # cdb (ex-0105.103,foo)
substitute   (foo, $Z^{a} -> A^{a}$)           # cdb (ex-0105.104,foo)
```

$$\texttt{ex-0105.100} := A_a A^a + A^a A_a$$
$$\texttt{ex-0105.101} := 2A_a A^a$$
$$\texttt{ex-0105.102} := 2A_a Z^a$$
$$\texttt{ex-0105.103} := 2A_a Z^a$$
$$\texttt{ex-0105.104} := 2A_a A^a$$

## Exercise 1.6 Multiple `SortOrder` lists

```
{D,C,B,A}::SortOrder.  # first SortOrder list

foo := A B C D .       # cdb(ex-0106.101,foo)

sort_product (foo)     # cdb(ex-0106.102,foo)

{V,U}::SortOrder.      # second SortOrder list, all entries distinct from first list

foo := U V A B C D .   # cdb(ex-0106.201,foo)

sort_product (foo)     # cdb(ex-0106.202,foo)

{A,B,C,D}::SortOrder.  # all entries in this list appear in the
                       # first SortOrder so they will be effectively ignored

foo := U V D C B A .   # cdb(ex-0106.301,foo)

sort_product (foo)     # cdb(ex-0106.302,foo)
```

$$\texttt{ex-0106.101} := ABCD$$
$$\texttt{ex-0106.102} := DCBA$$

$$\texttt{ex-0106.201} := UVABCD$$
$$\texttt{ex-0106.202} := DCBAVU$$

$$\texttt{ex-0106.301} := UVDCBA$$
$$\texttt{ex-0106.302} := DCBAVU$$

# Exercise 1.7 Subtleties of `foo = bah` and `foo := @(bah)`

```
{a,b,c,d,e,f,h#}::Indices.

foo := B_{b} A_{a}.
bah := A_{a} C_{c}.

# cdbBeg(print.0107)
print("foo = "+str(foo))
print("bah = "+str(bah)+"\n")

print("type foo = "+str(type(foo)))
print("type bah = "+str(type(bah))+"\n")

print("id foo = "+str(id(foo)))
print("id bah = "+str(id(bah))+"\n")

bah = foo

print("foo = "+str(foo))
print("bah = "+str(bah)+"\n")

sort_product (foo)

print("bah = "+str(bah)+"\n")

print("id foo = "+str(id(foo)))
print("id bah = "+str(id(bah))+"\n")

bah := @(foo).

print("id foo = "+str(id(foo)))
print("id bah = "+str(id(bah))+"\n")
# cdbEnd(print.0107)
```

```
1  foo = B_{b} A_{a}
2  bah = A_{a} C_{c}
3
4  type foo = <class ’cadabra2.Ex’>
5  type bah = <class ’cadabra2.Ex’>
6
7  id foo = 4375746672
8  id bah = 4358937840
9
10 foo = B_{b} A_{a}
11 bah = B_{b} A_{a}
12
13 bah = A_{a} B_{b}
14
15 id foo = 4375746672
16 id bah = 4375746672
17
18 id foo = 4375746672
19 id bah = 4383473264
```

Note that the line numbers referenced in the following are those of the output above not those of the Cadabra source.

- Lines 7 and 8 show that the objects `foo` and `bah` point to distinct areas of memeory (i.e., they point to different objects).
- Lines 10 and 11 show the result of the statement `bah = foo`.
- Line 13 shows that `bah` has changed after the statement `sort_product (foo)`.
- Lines 15 and 16 verifies that `foo` and `bah` point to the same object (so changes in `foo` will be seen by `bah`, as just noted).
- Lines 18 and 19 shows that after `bah := @(foo)` the symbols `bah` and `foo` no longer point to the same object.

# Exercise 1.8 Syntax errors – original code

```
{a,b,c,d,e,f#}::Indices.
C{#}::Symmetric.

foo := A_{a} B_{b} + C_{ab}.                        # C_{ab} should be C_{a b}
bah := B_{b} A_{a} + C_{ba}.                        # C_{ba} should be C_{b a}
meh := @(foo) - @(bah)                              # missing dot or semi-colon terminator

if meh == 0:
   print ("meh is zero, and all is good")
      success = True.                               # indentation error, drop the dot
else:
   print ("meh is not zero, oops")
      success = False.                              # indentation error, drop the dot

canonicalise (meh).                                 # terminate with ; or nothing
sort_product (meh);

{\alpha\beta\gamma}::Indices.                       # separate list elements with commas

foo := Ex ("A_{ab} - A_{a b}");                     # use = for assignment, A_{ab} should be A_{a b}
bah := Ex ("A_{\alpha\beta} - A_{\alpha \beta}");   # use = for assignment, need raw string in Ex
```

# Exercise 1.8 Syntax errors – corrected code

```
{a,b,c,d,e,f#}::Indices.
C{#}::Symmetric.

foo := A_{a} B_{b} + C_{a b}.                          # cdb (ex-0108.101,foo)
bah := B_{b} A_{a} + C_{b a}.                          # cdb (ex-0108.102,bah)
meh := @(foo) - @(bah).                                # cdb (ex-0108.103,meh)

if meh == 0:
   print ("meh is zero, and all is good")
   success = True
else:
   print ("meh is not zero, oops")
   success = False

canonicalise (meh)                                     # cdb (ex-0108.104,meh)
sort_product (meh);                                    # cdb (ex-0108.105,meh)

{\alpha,\beta,\gamma}::Indices.

foo = Ex ("A_{a b} - A_{a b}");                        # cdb (ex-0108.106,foo)
bah = Ex (r"A_{\alpha\beta} - A_{\alpha \beta}");      # cdb (ex-0108.107,bah)
```

$$\texttt{ex-0108.101} := A_a B_b + C_{ab}$$
$$\texttt{ex-0108.102} := B_b A_a + C_{ba}$$
$$\texttt{ex-0108.103} := A_a B_b + C_{ab} - B_b A_a - C_{ba}$$
$$\texttt{ex-0108.104} := A_a B_b - B_b A_a$$
$$\texttt{ex-0108.105} := 0$$
$$\texttt{ex-0108.106} := 0$$
$$\texttt{ex-0108.107} := 0$$

# Exercise 1.9 No index clashes

```
{a,b,c,d,e,f,u,v,w}::Indices.

foo := A_{a c} C^{c}.                                          # cdb (ex-0109.101,foo)
bah := B_{b c} C^{c}.                                          # cdb (ex-0109.102,bah)

foobah := @(foo) @(bah).                                       # cdb (ex-0109.103,foobah)
```

$$A_{ac}C^{c} \tag{ex-0109.101}$$

$$B_{bc}C^{c} \tag{ex-0109.102}$$

$$A_{ac}C^{c}B_{bd}C^{d} \tag{ex-0109.103}$$

# Exercise 1.10 Relabel free indices

```
{a,b,c,d,e,f,u,v,w}::Indices.

\delta{#}::KroneckerDelta.

expr := A_{a b c}.                                              # cdb (ex-0110.101,expr)

expr := \delta^{a}_{u} \delta^{b}_{v} \delta^{c}_{w} @(expr).  # cdb (ex-0110.102,expr)

eliminate_kronecker (expr)                                      # cdb (ex-0110.103,expr)
```

$$A_{abc} \tag{ex-0110.101}$$

$$\delta^{a}{}_{u}\delta^{b}{}_{v}\delta^{c}{}_{w}A_{abc} \tag{ex-0110.102}$$

$$A_{uvw} \tag{ex-0110.103}$$

## Exercise 1.11 Cycling free indices – preferred solution

```
{a,b,c,d,e,f,u,v,w}::Indices.

expr := A_{a b c}.                                          # cdb (ex-0111.101,expr)

rule := T_{a b c} -> @(expr).
expr := T_{b c a}.                                          # cdb (ex-0111.102,expr)

substitute (expr, rule)                                     # cdb (ex-0111.103,expr)
```

$$A_{abc} \tag{ex-0111.101}$$

$$T_{bca} \tag{ex-0111.102}$$

$$A_{bca} \tag{ex-0111.103}$$

# Exercise 1.11 Cycling free indices – alternative solution

This alternative solution uses two rounds of Kroncker deltas. It does the job but is not as simple as the previous solution.

```
{a,b,c,d,e,f,u,v,w}::Indices.

\delta{#}::KroneckerDelta.

expr := A_{a b c}.                                            # cdb (ex-0111.201,expr)

expr := \delta^{a}_{u} \delta^{b}_{v} \delta^{c}_{w} @(expr).  # cdb (ex-0111.202,expr)

eliminate_kronecker (expr)                                    # cdb (ex-0111.203,expr)

expr := \delta^{u}_{b} \delta^{v}_{c} \delta^{w}_{a} @(expr).  # cdb (ex-0111.204,expr)

eliminate_kronecker (expr)                                    # cdb (ex-0111.205,expr)
```

$$A_{abc} \tag{ex-0111.201}$$

$$\delta^{a}{}_{u}\delta^{b}{}_{v}\delta^{c}{}_{w}A_{abc} \tag{ex-0111.202}$$

$$A_{uvw} \tag{ex-0111.203}$$

$$\delta^{u}{}_{b}\delta^{v}{}_{c}\delta^{w}{}_{a}A_{uvw} \tag{ex-0111.204}$$

$$A_{bca} \tag{ex-0111.205}$$

## Exercise 2.1 Using Cadabra's own product rule

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# templates for covariant derivatives

deriv1 := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}}
                               + \Gamma^{b}_{c a} A?^{c}.

deriv2 := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}}
                               - \Gamma^{c}_{b a} A?_{c}.

# create an object

uv := \nabla_{a}{v_{b} u^{b}}
    - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.101,uv)

# apply the rules, then simplify

product_rule   (uv)                   # cdb (ex-0201.102,uv)
substitute     (uv,deriv1)            # cdb (ex-0201.103,uv)
substitute     (uv,deriv2)            # cdb (ex-0201.104,uv)
distribute     (uv)                   # cdb (ex-0201.105,uv)
sort_product   (uv)                   # cdb (ex-0201.106,uv)
rename_dummies (uv)                   # cdb (ex-0201.107,uv)
```

$$
\begin{aligned}
\nabla_a \left(v_b u^b\right) - \partial_a \left(v_b u^b\right) &= \nabla_a v_b u^b + v_b \nabla_a u^b - \partial_a v_b u^b - v_b \partial_a u^b && \texttt{(ex-0201.102)} \\
&= \nabla_a v_b u^b + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c\right) - \partial_a v_b u^b - v_b \partial_a u^b && \texttt{(ex-0201.103)} \\
&= \left(\partial_a v_b - \Gamma^c{}_{ba} v_c\right) u^b + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c\right) - \partial_a v_b u^b - v_b \partial_a u^b && \texttt{(ex-0201.104)} \\
&= -\Gamma^c{}_{ba} v_c u^b + v_b \Gamma^b{}_{ca} u^c && \texttt{(ex-0201.105)} \\
&= -\Gamma^c{}_{ba} u^b v_c + \Gamma^b{}_{ca} u^c v_b && \texttt{(ex-0201.106)} \\
&= 0 && \texttt{(ex-0201.107)}
\end{aligned}
$$

## Exercise 2.1 Using hand crafted product rules

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# templates for covariant derivatives

deriv1 := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}}
                              + \Gamma^{b}_{c a} A?^{c}.

deriv2 := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}}
                              - \Gamma^{c}_{b a} A?_{c}.

# tempaltes for product rules

deriv3 := \nabla_{a}{A?_{b} B?^{c}} -> B?^{c} \nabla_{a}{A?_{b}}
                                     + A?_{b} \nabla_{a}{B?^{c}}.

deriv4 := \partial_{a}{A?_{b} B?^{c}} -> B?^{c} \partial_{a}{A?_{b}}
                                       + A?_{b} \partial_{a}{B?^{c}}.

# create an object

uv := \nabla_{a}{v_{b} u^{b}}
    - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.201,uv)

# apply the rules, then simplify

substitute     (uv,deriv3)            # cdb (ex-0201.202,uv)
substitute     (uv,deriv4)            # cdb (ex-0201.203,uv)
substitute     (uv,deriv1)            # cdb (ex-0201.204,uv)
substitute     (uv,deriv2)            # cdb (ex-0201.205,uv)
distribute     (uv)                   # cdb (ex-0201.206,uv)
sort_product   (uv)                   # cdb (ex-0201.207,uv)
rename_dummies (uv)                   # cdb (ex-0201.208,uv)
```

$$
\begin{aligned}
\nabla_a \left(v_b u^b\right) - \partial_a \left(v_b u^b\right) &= u^b \nabla_a v_b + v_b \nabla_a u^b - \partial_a \left(v_b u^b\right) && \texttt{(ex-0201.202)} \\
&= u^b \nabla_a v_b + v_b \nabla_a u^b - u^b \partial_a v_b - v_b \partial_a u^b && \texttt{(ex-0201.203)} \\
&= u^b \nabla_a v_b + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c\right) - u^b \partial_a v_b - v_b \partial_a u^b && \texttt{(ex-0201.204)} \\
&= u^b \left(\partial_a v_b - \Gamma^c{}_{ba} v_c\right) + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c\right) - u^b \partial_a v_b - v_b \partial_a u^b && \texttt{(ex-0201.205)} \\
&= -u^b \Gamma^c{}_{ba} v_c + v_b \Gamma^b{}_{ca} u^c && \texttt{(ex-0201.206)} \\
&= -\Gamma^c{}_{ba} u^b v_c + \Gamma^b{}_{ca} u^c v_b && \texttt{(ex-0201.207)} \\
&= 0 && \texttt{(ex-0201.208)}
\end{aligned}
$$

# Exercise 2.2 Covariant derivative of $v_{ab}$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# template for covariant derivative of a vector

derivU := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}} + \Gamma^{b}_{c a} A?^{c}.
derivD := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.

vab := v_{a b} -> A_{a} B_{b}.
iab := A_{a} B_{b} -> v_{a b}.

pab := \partial_{a}{A_{b}} B_{c} -> \partial_{a}{A_{b} B_{c}} - A_{b} \partial_{a}{B_{c}}.

# create an object

Dvab := \nabla_{a}{v_{b c}}.     # cdb (ex-0202.101,Dvab)

# apply the rule, then simplify

substitute     (Dvab,vab)        # cdb (ex-0202.102,Dvab)
product_rule   (Dvab)            # cdb (ex-0202.103,Dvab)
substitute     (Dvab,derivD)     # cdb (ex-0202.104,Dvab)
substitute     (Dvab,derivU)     # cdb (ex-0202.105,Dvab)
distribute     (Dvab)            # cdb (ex-0202.106,Dvab)
substitute     (Dvab,pab)        # cdb (ex-0202.107,Dvab)
canonicalise   (Dvab)            # cdb (ex-0202.108,Dvab)
substitute     (Dvab,iab)        # cdb (ex-0202.109,Dvab)
sort_product   (Dvab)            # cdb (ex-0202.110,Dvab)
```

$$
\begin{aligned}
\nabla_a v_{bc} &= \nabla_a \left(A_b B_c\right) && \texttt{(ex-0202.102)} \\
&= \nabla_a A_b B_c + A_b \nabla_a B_c && \texttt{(ex-0202.103)} \\
&= \left(\partial_a A_b - \Gamma^d{}_{ba} A_d\right) B_c + A_b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d\right) && \texttt{(ex-0202.104)} \\
&= \left(\partial_a A_b - \Gamma^d{}_{ba} A_d\right) B_c + A_b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d\right) && \texttt{(ex-0202.105)} \\
&= \partial_a A_b B_c - \Gamma^d{}_{ba} A_d B_c + A_b \partial_a B_c - A_b \Gamma^d{}_{ca} B_d && \texttt{(ex-0202.106)} \\
&= \partial_a \left(A_b B_c\right) - \Gamma^d{}_{ba} A_d B_c - A_b \Gamma^d{}_{ca} B_d && \texttt{(ex-0202.107)} \\
&= \partial_a \left(A_b B_c\right) - \Gamma^d{}_{ba} A_d B_c - A_b \Gamma^d{}_{ca} B_d && \texttt{(ex-0202.108)} \\
&= \partial_a v_{bc} - \Gamma^d{}_{ba} v_{dc} - v_{bd} \Gamma^d{}_{ca} && \texttt{(ex-0202.109)} \\
&= \partial_a v_{bc} - \Gamma^d{}_{ba} v_{dc} - \Gamma^d{}_{ca} v_{bd} && \texttt{(ex-0202.110)}
\end{aligned}
$$

# Exercise 2.3 Covariant derivative of $v^a{}_b$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# template for covariant derivative of a vector

derivU := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}} + \Gamma^{b}_{c a} A?^{c}.
derivD := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.

vab := v^{a}_{b} -> A^{a} B_{b}.
iab := A^{a} B_{b} -> v^{a}_{b}.

pab := \partial_{a}{A^{b}} B_{c} -> \partial_{a}{A^{b} B_{c}} - A^{b} \partial_{a}{B_{c}}.

# create an object

Dvab := \nabla_{a}{v^{b}_{c}}.   # cdb (ex-0203.101,Dvab)

# apply the rule, then simplify

substitute     (Dvab,vab)        # cdb (ex-0203.102,Dvab)
product_rule   (Dvab)            # cdb (ex-0203.103,Dvab)
substitute     (Dvab,derivD)     # cdb (ex-0203.104,Dvab)
substitute     (Dvab,derivU)     # cdb (ex-0203.105,Dvab)
distribute     (Dvab)            # cdb (ex-0203.106,Dvab)
substitute     (Dvab,pab)        # cdb (ex-0203.107,Dvab)
canonicalise   (Dvab)            # cdb (ex-0203.108,Dvab)
substitute     (Dvab,iab)        # cdb (ex-0203.109,Dvab)
sort_product   (Dvab)            # cdb (ex-0203.110,Dvab)
```

$$
\begin{aligned}
\nabla_a v^b{}_c &= \nabla_a \left(A^b B_c\right) && \texttt{(ex-0203.102)}\\
&= \nabla_a A^b B_c + A^b \nabla_a B_c && \texttt{(ex-0203.103)}\\
&= \nabla_a A^b B_c + A^b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d\right) && \texttt{(ex-0203.104)}\\
&= \left(\partial_a A^b + \Gamma^b{}_{da} A^d\right) B_c + A^b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d\right) && \texttt{(ex-0203.105)}\\
&= \partial_a A^b B_c + \Gamma^b{}_{da} A^d B_c + A^b \partial_a B_c - A^b \Gamma^d{}_{ca} B_d && \texttt{(ex-0203.106)}\\
&= \partial_a \left(A^b B_c\right) + \Gamma^b{}_{da} A^d B_c - A^b \Gamma^d{}_{ca} B_d && \texttt{(ex-0203.107)}\\
&= \partial_a \left(A^b B_c\right) + \Gamma^b{}_{da} A^d B_c - A^b \Gamma^d{}_{ca} B_d && \texttt{(ex-0203.108)}\\
&= \partial_a v^b{}_c + \Gamma^b{}_{da} v^d{}_c - v^b{}_d \Gamma^d{}_{ca} && \texttt{(ex-0203.109)}\\
&= \partial_a v^b{}_c + \Gamma^b{}_{da} v^d{}_c - \Gamma^d{}_{ca} v^b{}_d && \texttt{(ex-0203.110)}
\end{aligned}
$$

# Exercise 2.4 Combining rules – a problem

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# rules for covariant derivatives of v

deriv1 := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}}
                             + \Gamma^{b}_{d a} v^{d}.

deriv2 := \nabla_{a}{\nabla_{b}{v^{c}}} -> \partial_{a}{\nabla_{b}{v^{c}}}
                                          + \Gamma^{c}_{d a} \nabla_{b}{v^{d}}
                                          - \Gamma^{d}_{b a} \nabla_{d}{v^{c}}.

# attempt to combine both rules for second covariant derivative of v

substitute (deriv2,deriv1)       # cdb (ex-0204.101,deriv2)
```

Note that the call to `substitute` has made changes to both sides of the rule for `deriv2`. This is not ideal and a better method is developed in the following exercise.

$$\nabla_a \left(\partial_b v^c + \Gamma^c{}_{db} v^d\right) \rightarrow \partial_a \left(\partial_b v^c + \Gamma^c{}_{db} v^d\right) + \Gamma^c{}_{da} \left(\partial_b v^d + \Gamma^d{}_{eb} v^e\right) - \Gamma^d{}_{ba} \left(\partial_d v^c + \Gamma^c{}_{ed} v^e\right) \tag{ex-0204.101}$$

# Exercise 2.5 Combining rules – a solution

```
1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \nabla{#}::Derivative.
4  \partial{#}::PartialDerivative.
5
6  # rules for covariant derivatives of v
7
8  deriv1 := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}}
9                               + \Gamma^{b}_{d a} v^{d}.
10
11 deriv2 := \nabla_{a}{\nabla_{b}{v^{c}}} -> \partial_{a}{\nabla_{b}{v^{c}}}
12                                           + \Gamma^{c}_{d a} \nabla_{b}{v^{d}}
13                                           - \Gamma^{d}_{b a} \nabla_{d}{v^{c}}.
14
15 # second covariant derivative of v
16
17 expr := v^{c}_{b a} -> \nabla_{a}{\nabla_{b}{v^{c}}}.  # cdb (ex-0205.101,expr)
18 save := @(expr).
19
20 # apply the rules, then simplify
21
22 substitute     (expr,deriv2)       # cdb (ex-0205.102,expr)
23 substitute     (expr,deriv1)       # cdb (ex-0205.103,expr)
24 distribute     (expr)              # cdb (ex-0205.104,expr)
25 product_rule   (expr)              # cdb (ex-0205.105,expr)
26 canonicalise   (expr)              # cdb (ex-0205.107,expr)
27 substitute     (expr,save)         # cdb (ex-0205.108,expr)
```

The trick here is to introduce in line 17 a dummy left hand side, `v^{c}{}_{b a}`, that is invisible with respect to the substitution rules of lines 8 and 11. Thus lines 22 and 23 will only target the right hand side of `expr`.

Notice how a copy of the initial expression is made in 18. This is used later in line 27 to replace the dummy object `v^{c}{}_{b a}` with `\nabla_{a}{\nabla_{b}{v^{c}}}` but this time acting on the left hand side of the rule. The result is a rule for second covariant deriavtives.

$$v^{c}{}_{ba} \to \nabla_a \left(\nabla_b v^c\right) \tag{ex-0205.101}$$

$$v^{c}{}_{ba} \to \partial_a \left(\nabla_b v^c\right) + \Gamma^{c}{}_{da} \nabla_b v^d - \Gamma^{d}{}_{ba} \nabla_d v^c \tag{ex-0205.102}$$

$$v^{c}{}_{ba} \to \partial_a \left(\partial_b v^c + \Gamma^{c}{}_{db} v^d\right) + \Gamma^{c}{}_{da} \left(\partial_b v^d + \Gamma^{d}{}_{eb} v^e\right) - \Gamma^{d}{}_{ba} \left(\partial_d v^c + \Gamma^{c}{}_{ed} v^e\right) \tag{ex-0205.103}$$

$$v^{c}{}_{ba} \to \partial_{ab} v^c + \partial_a \left(\Gamma^{c}{}_{db} v^d\right) + \Gamma^{c}{}_{da} \partial_b v^d + \Gamma^{c}{}_{da} \Gamma^{d}{}_{eb} v^e - \Gamma^{d}{}_{ba} \partial_d v^c - \Gamma^{d}{}_{ba} \Gamma^{c}{}_{ed} v^e \tag{ex-0205.104}$$

$$v^{c}{}_{ba} \to \partial_{ab} v^c + \partial_a \Gamma^{c}{}_{db} v^d + \Gamma^{c}{}_{db} \partial_a v^d + \Gamma^{c}{}_{da} \partial_b v^d + \Gamma^{c}{}_{da} \Gamma^{d}{}_{eb} v^e - \Gamma^{d}{}_{ba} \partial_d v^c - \Gamma^{d}{}_{ba} \Gamma^{c}{}_{ed} v^e \tag{ex-0205.105}$$

$$v^{c}{}_{ba} \to \partial_{ab} v^c + \partial_a \Gamma^{c}{}_{db} v^d + \Gamma^{c}{}_{db} \partial_a v^d + \Gamma^{c}{}_{da} \partial_b v^d + \Gamma^{c}{}_{da} \Gamma^{d}{}_{eb} v^e - \Gamma^{d}{}_{ba} \partial_d v^c - \Gamma^{c}{}_{de} \Gamma^{e}{}_{ba} v^d \tag{ex-0205.107}$$

$$\nabla_a \left(\nabla_b v^c\right) \to \partial_{ab} v^c + \partial_a \Gamma^{c}{}_{db} v^d + \Gamma^{c}{}_{db} \partial_a v^d + \Gamma^{c}{}_{da} \partial_b v^d + \Gamma^{c}{}_{da} \Gamma^{d}{}_{eb} v^e - \Gamma^{d}{}_{ba} \partial_d v^c - \Gamma^{c}{}_{de} \Gamma^{e}{}_{ba} v^d \tag{ex-0205.108}$$

# Exercise 2.6 Cummutation of ∇ on a scalar

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# covariant derivative of \phi

dphi := \nabla_{a}{\phi} -> \partial_{a}{\phi}.

# rules to hide and reveal \partial\phi

hide   := \partial_{a}{\phi} -> w_{a}.
reveal := w_{a} -> \partial_{a}{\phi}.

# template for covariant derivative of a dual-vector

deriv := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.

# create an object

expr :=   \nabla_{a}{\nabla_{b}{\phi}}
        - \nabla_{b}{\nabla_{a}{\phi}}.   # cdb (ex-0206.101,expr)

# apply the rules, then simplify

substitute     (expr,dphi)                # cdb (ex-0206.102,expr)
substitute     (expr,hide)                # cdb (ex-0206.103,expr)
substitute     (expr,deriv)               # cdb (ex-0206.104,expr)
substitute     (expr,reveal)              # cdb (ex-0206.105,expr)
canonicalise   (expr)                     # cdb (ex-0206.106,expr)
```

$$\nabla_a (\nabla_b \phi) - \nabla_b (\nabla_a \phi) = \nabla_a (\partial_b \phi) - \nabla_b (\partial_a \phi) \tag{ex-0206.102}$$

$$= \nabla_a w_b - \nabla_b w_a \tag{ex-0206.103}$$

$$= \partial_a w_b - \Gamma^c{}_{ba} w_c - \partial_b w_a + \Gamma^c{}_{ab} w_c \tag{ex-0206.104}$$

$$= \partial_{ab} \phi - \Gamma^c{}_{ba} \partial_c \phi - \partial_{ba} \phi + \Gamma^c{}_{ab} \partial_c \phi \tag{ex-0206.105}$$

$$= -\Gamma^c{}_{ba} \partial_c \phi + \Gamma^c{}_{ab} \partial_c \phi \tag{ex-0206.106}$$

# Exercise 2.7 Selective kill

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative.

hide   := \partial_{d}{\Gamma^{a}_{b c}} -> Z_{d a b c}.
reveal := Z_{d a b c} -> \partial_{d}{\Gamma^{a}_{b c}}.

kill := \Gamma^{a}_{b c} -> 0.

Gamma := \Gamma^{a}_{b c}
       + x^{d} \partial_{d}{\Gamma^{a}_{b c}}.       # cdb (ex-0207.101,Gamma)

substitute (Gamma,hide)                              # cdb (ex-0207.102,Gamma)
substitute (Gamma,kill)                              # cdb (ex-0207.103,Gamma)
substitute (Gamma,reveal)                            # cdb (ex-0207.104,Gamma)
```

$$\Gamma^{a}{}_{bc}(x) = \Gamma^{a}{}_{bc} + x^{d}\partial_{d}\Gamma^{a}{}_{bc} \tag{ex-0207.101}$$

$$= \Gamma^{a}{}_{bc} + x^{d} Z_{dabc} \tag{ex-0207.102}$$

$$= x^{d} Z_{dabc} \tag{ex-0207.103}$$

$$= x^{d}\partial_{d}\Gamma^{a}{}_{bc} \tag{ex-0207.104}$$

## Exercise 2.7 Naive kill

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::Derivative.

kill := \Gamma^{a}_{b c} -> 0.

Gamma := \Gamma^{a}_{b c}
       + x^{d} \partial_{d}{\Gamma^{a}_{b c}}.       # cdb (ex-0207.201,Gamma)

substitute (Gamma,kill)                              # cdb (ex-0207.202,Gamma)
```

$$\Gamma^{a}{}_{bc}(x) = \Gamma^{a}{}_{bc} + x^{d}\partial_{d}\Gamma^{a}{}_{bc} \qquad \text{(ex-0207.201)}$$

$$= 0 \qquad \text{(ex-0207.202)}$$

## Exercise 2.7 No problem killing partial derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative.

kill := \partial_{c}{A_{a b}} -> 0.

Aab := A_{a b} + x^{c} \partial_{c}{A_{a b}}
               + x^{c} x^{d} \partial_{d}{\partial_{c}{A_{a b}}}.    # cdb (ex-0207.301,Aab)

substitute (Aab,kill)                                                # cdb (ex-0207.302,Aab)
```

$$A_{ab}(x) = A_{ab} + x^{c}\partial_{c}A_{ab} + x^{c}x^{d}\partial_{dc}A_{ab} \qquad \text{(ex-0207.301)}$$

$$= A_{ab} + x^{c}x^{d}\partial_{dc}A_{ab} \qquad \text{(ex-0207.302)}$$

# Exercise 2.8 Position keyword in `::Indices`

```
{a,b,c}::Indices(position=free).

foo := A_{a b} + A^{a b}.                          # cdb (ex-0208.101,foo)

substitute (foo, $A_{a b} -> B_{a b}$)             # cdb (ex-0208.102,foo)

{p,q,r}::Indices(position=fixed).

foo := A_{p q} B^{p q} + A^{p q} B_{p q}.          # cdb (ex-0208.201,foo)

canonicalise (foo)                                 # cdb (ex-0208.202,foo)

{u,v,w}::Indices(position=independent).

foo := A_{u v} B^{u v} + A^{u v} B_{u v}.          # cdb (ex-0208.301,foo)

canonicalise (foo)                                 # cdb (ex-0208.302,foo)
```

$$A_{ab} + A^{ab} = B_{ab} + B^{ab} \qquad (\texttt{ex-0208.102})$$

$$A_{pq}B^{pq} + A^{pq}B_{pq} = 2A^{pq}B_{pq} \qquad (\texttt{ex-0208.202})$$

$$A_{uv}B^{uv} + A^{uv}B_{uv} = A_{uv}B^{uv} + A^{uv}B_{uv} \qquad (\texttt{ex-0208.302})$$

# Exercise 3.1 Some symmetries of Riemann

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

;::Symbol;

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.       # cdb(Rabcd.000,Rabcd)

dRabcd := R^{a}_{b c d ; e} -> \partial_{e}{R^{a}_{b c d}}
                             + \Gamma^{a}_{f e} R^{f}_{b c d}
                             - \Gamma^{f}_{b e} R^{a}_{f c d}
                             - \Gamma^{f}_{c e} R^{a}_{b f d}
                             - \Gamma^{f}_{d e} R^{a}_{b c f}.       # cdb(dRabcd.000,dRabcd)
```

## Exercise 3.1 Antisymmetry on last pair of indices

```
expr := R^{a}_{b c d} + R^{a}_{b d c}.                 # cdb(ex-0301.101,expr)

substitute (expr, Rabcd)                                # cdb(ex-0301.102,expr)
```

$$R^{a}{}_{bcd} + R^{a}{}_{bdc} = 0 \tag{ex-0301.102}$$

## Exercise 3.1 First Bianchi identity

```
expr := R^{a}_{b c d} + R^{a}_{d b c} + R^{a}_{c d b}.              # cdb(ex-0301.201,expr)

substitute   (expr, Rabcd)                                           # cdb(ex-0301.202,expr)
canonicalise (expr)                                                  # cdb(ex-0301.203,expr)
```

$$
\begin{aligned}
R^{a}{}_{bcd} + R^{a}{}_{dbc} + R^{a}{}_{cdb} &= \partial_{c}\Gamma^{a}{}_{bd} - \partial_{d}\Gamma^{a}{}_{bc} + \Gamma^{e}{}_{bd}\Gamma^{a}{}_{ce} - \Gamma^{e}{}_{bc}\Gamma^{a}{}_{de} + \partial_{b}\Gamma^{a}{}_{dc} - \partial_{c}\Gamma^{a}{}_{db} + \Gamma^{e}{}_{dc}\Gamma^{a}{}_{be} - \Gamma^{e}{}_{db}\Gamma^{a}{}_{ce} + \partial_{d}\Gamma^{a}{}_{cb} - \partial_{b}\Gamma^{a}{}_{cd} + \Gamma^{e}{}_{cb}\Gamma^{a}{}_{de} \\
&\quad - \Gamma^{e}{}_{cd}\Gamma^{a}{}_{be} \qquad (\texttt{ex-0301.202}) \\
&= 0 \qquad (\texttt{ex-0301.203})
\end{aligned}
$$

## Exercise 3.1 Second Bianchi identity

```
expr := R^{a}_{b c d ; e} + R^{a}_{b e c ; d} + R^{a}_{b d e ; c}.   # cdb(ex-0301.301,expr)

substitute     (expr, dRabcd)                                        # cdb(ex-0301.302,expr)
substitute     (expr,  Rabcd)                                        # cdb(ex-0301.303,expr)
distribute     (expr)                                                # cdb(ex-0301.304,expr)
product_rule   (expr)                                                # cdb(ex-0301.305,expr)
sort_product   (expr)                                                # cdb(ex-0301.306,expr)
rename_dummies (expr)                                                # cdb(ex-0301.307,expr)
canonicalise   (expr)                                                # cdb(ex-0301.308,expr)
```

$$
\begin{aligned}
R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} &= \partial_e R^a{}_{bcd} + \Gamma^a{}_{fe} R^f{}_{bcd} - \Gamma^f{}_{be} R^a{}_{fcd} - \Gamma^f{}_{ce} R^a{}_{bfd} - \Gamma^f{}_{de} R^a{}_{bcf} + \partial_d R^a{}_{bec} + \Gamma^a{}_{fd} R^f{}_{bec} - \Gamma^f{}_{bd} R^a{}_{fec} - \Gamma^f{}_{ed} R^a{}_{bfc} - \Gamma^f{}_{cd} R^a{}_{bef} \\
&\quad + \partial_c R^a{}_{bde} + \Gamma^a{}_{fc} R^f{}_{bde} - \Gamma^f{}_{bc} R^a{}_{fde} - \Gamma^f{}_{dc} R^a{}_{bfe} - \Gamma^f{}_{ec} R^a{}_{bdf} \qquad \texttt{(ex-0301.302)} \\
&= \partial_e \left(\partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^f{}_{bd}\Gamma^a{}_{cf} - \Gamma^f{}_{bc}\Gamma^a{}_{df}\right) + \Gamma^a{}_{fe}\left(\partial_c \Gamma^f{}_{bd} - \partial_d \Gamma^f{}_{bc} + \Gamma^g{}_{bd}\Gamma^f{}_{cg} - \Gamma^g{}_{bc}\Gamma^f{}_{dg}\right) \\
&\quad - \Gamma^f{}_{be}\left(\partial_c \Gamma^a{}_{fd} - \partial_d \Gamma^a{}_{fc} + \Gamma^g{}_{fd}\Gamma^a{}_{cg} - \Gamma^g{}_{fc}\Gamma^a{}_{dg}\right) - \Gamma^f{}_{ce}\left(\partial_f \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bf} + \Gamma^g{}_{bd}\Gamma^a{}_{fg} - \Gamma^g{}_{bf}\Gamma^a{}_{dg}\right) \\
&\quad - \Gamma^f{}_{de}\left(\partial_c \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{bc} + \Gamma^g{}_{bf}\Gamma^a{}_{cg} - \Gamma^g{}_{bc}\Gamma^a{}_{fg}\right) + \partial_d\left(\partial_e \Gamma^a{}_{bc} - \partial_c \Gamma^a{}_{be} + \Gamma^f{}_{bc}\Gamma^a{}_{ef} - \Gamma^f{}_{be}\Gamma^a{}_{cf}\right) \\
&\quad + \Gamma^a{}_{fd}\left(\partial_e \Gamma^f{}_{bc} - \partial_c \Gamma^f{}_{be} + \Gamma^g{}_{bc}\Gamma^f{}_{eg} - \Gamma^g{}_{be}\Gamma^f{}_{cg}\right) - \Gamma^f{}_{bd}\left(\partial_e \Gamma^a{}_{fc} - \partial_c \Gamma^a{}_{fe} + \Gamma^g{}_{fc}\Gamma^a{}_{eg} - \Gamma^g{}_{fe}\Gamma^a{}_{cg}\right) \\
&\quad - \Gamma^f{}_{ed}\left(\partial_f \Gamma^a{}_{bc} - \partial_c \Gamma^a{}_{bf} + \Gamma^g{}_{bc}\Gamma^a{}_{fg} - \Gamma^g{}_{bf}\Gamma^a{}_{cg}\right) - \Gamma^f{}_{cd}\left(\partial_e \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{be} + \Gamma^g{}_{bf}\Gamma^a{}_{eg} - \Gamma^g{}_{be}\Gamma^a{}_{fg}\right) \\
&\quad + \partial_c\left(\partial_d \Gamma^a{}_{be} - \partial_e \Gamma^a{}_{bd} + \Gamma^f{}_{be}\Gamma^a{}_{df} - \Gamma^f{}_{bd}\Gamma^a{}_{ef}\right) + \Gamma^a{}_{fc}\left(\partial_d \Gamma^f{}_{be} - \partial_e \Gamma^f{}_{bd} + \Gamma^g{}_{be}\Gamma^f{}_{dg} - \Gamma^g{}_{bd}\Gamma^f{}_{eg}\right) \\
&\quad - \Gamma^f{}_{bc}\left(\partial_d \Gamma^a{}_{fe} - \partial_e \Gamma^a{}_{fd} + \Gamma^g{}_{fe}\Gamma^a{}_{dg} - \Gamma^g{}_{fd}\Gamma^a{}_{eg}\right) - \Gamma^f{}_{dc}\left(\partial_f \Gamma^a{}_{be} - \partial_e \Gamma^a{}_{bf} + \Gamma^g{}_{be}\Gamma^a{}_{fg} - \Gamma^g{}_{bf}\Gamma^a{}_{eg}\right) \\
&\quad - \Gamma^f{}_{ec}\left(\partial_d \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{bd} + \Gamma^g{}_{bf}\Gamma^a{}_{dg} - \Gamma^g{}_{bd}\Gamma^a{}_{fg}\right) \qquad \texttt{(ex-0301.303)} \\
&= \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \partial_e\left(\Gamma^f{}_{bd}\Gamma^a{}_{cf}\right) - \partial_e\left(\Gamma^f{}_{bc}\Gamma^a{}_{df}\right) + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^g{}_{bd}\Gamma^f{}_{cg} - \Gamma^a{}_{fe}\Gamma^g{}_{bc}\Gamma^f{}_{dg} \\
&\quad - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^f{}_{be}\Gamma^g{}_{fd}\Gamma^a{}_{cg} + \Gamma^f{}_{be}\Gamma^g{}_{fc}\Gamma^a{}_{dg} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^f{}_{ce}\Gamma^g{}_{bd}\Gamma^a{}_{fg} + \Gamma^f{}_{ce}\Gamma^g{}_{bf}\Gamma^a{}_{dg} \\
&\quad - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^f{}_{de}\Gamma^g{}_{bf}\Gamma^a{}_{cg} + \Gamma^f{}_{de}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \partial_d\left(\Gamma^f{}_{bc}\Gamma^a{}_{ef}\right) - \partial_d\left(\Gamma^f{}_{be}\Gamma^a{}_{cf}\right) \\
&\quad + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^g{}_{bc}\Gamma^f{}_{eg} - \Gamma^a{}_{fd}\Gamma^g{}_{be}\Gamma^f{}_{cg} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} - \Gamma^f{}_{bd}\Gamma^g{}_{fc}\Gamma^a{}_{eg} + \Gamma^f{}_{bd}\Gamma^g{}_{fe}\Gamma^a{}_{cg} \\
&\quad - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^f{}_{ed}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \Gamma^f{}_{ed}\Gamma^g{}_{bf}\Gamma^a{}_{cg} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} - \Gamma^f{}_{cd}\Gamma^g{}_{bf}\Gamma^a{}_{eg} + \Gamma^f{}_{cd}\Gamma^g{}_{be}\Gamma^a{}_{fg} \\
&\quad + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \partial_c\left(\Gamma^f{}_{be}\Gamma^a{}_{df}\right) - \partial_c\left(\Gamma^f{}_{bd}\Gamma^a{}_{ef}\right) + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^g{}_{be}\Gamma^f{}_{dg} - \Gamma^a{}_{fc}\Gamma^g{}_{bd}\Gamma^f{}_{eg} \\
&\quad - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^f{}_{bc}\Gamma^g{}_{fe}\Gamma^a{}_{dg} + \Gamma^f{}_{bc}\Gamma^g{}_{fd}\Gamma^a{}_{eg} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^f{}_{dc}\Gamma^g{}_{be}\Gamma^a{}_{fg} + \Gamma^f{}_{dc}\Gamma^g{}_{bf}\Gamma^a{}_{eg} \\
&\quad - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^f{}_{ec}\Gamma^g{}_{bf}\Gamma^a{}_{dg} + \Gamma^f{}_{ec}\Gamma^g{}_{bd}\Gamma^a{}_{fg} \qquad \texttt{(ex-0301.304)}
\end{aligned}
$$

$$
\begin{aligned}
R^{a}{}_{bcd;e} + R^{a}{}_{bec;d} + R^{a}{}_{bde;c} = {} & \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \partial_{e}\Gamma^{f}{}_{bd}\Gamma^{a}{}_{cf} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \partial_{e}\Gamma^{f}{}_{bc}\Gamma^{a}{}_{df} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{cg} \\
& - \Gamma^{a}{}_{fe}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{dg} - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{f}{}_{be}\Gamma^{g}{}_{fd}\Gamma^{a}{}_{cg} + \Gamma^{f}{}_{be}\Gamma^{g}{}_{fc}\Gamma^{a}{}_{dg} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd}\Gamma^{a}{}_{fg} \\
& + \Gamma^{f}{}_{ce}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{dg} - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{f}{}_{de}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{cg} + \Gamma^{f}{}_{de}\Gamma^{g}{}_{bc}\Gamma^{a}{}_{fg} + \partial_{de}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \partial_{d}\Gamma^{f}{}_{bc}\Gamma^{a}{}_{ef} \\
& + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} - \partial_{d}\Gamma^{f}{}_{be}\Gamma^{a}{}_{cf} - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{eg} - \Gamma^{a}{}_{fd}\Gamma^{g}{}_{be}\Gamma^{f}{}_{cg} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} \\
& + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} - \Gamma^{f}{}_{bd}\Gamma^{g}{}_{fc}\Gamma^{a}{}_{eg} + \Gamma^{f}{}_{bd}\Gamma^{g}{}_{fe}\Gamma^{a}{}_{cg} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc}\Gamma^{a}{}_{fg} + \Gamma^{f}{}_{ed}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{cg} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} \\
& + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} - \Gamma^{f}{}_{cd}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{eg} + \Gamma^{f}{}_{cd}\Gamma^{g}{}_{be}\Gamma^{a}{}_{fg} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} + \partial_{c}\Gamma^{f}{}_{be}\Gamma^{a}{}_{df} + \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{df} - \partial_{c}\Gamma^{f}{}_{bd}\Gamma^{a}{}_{ef} \\
& - \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{a}{}_{fc}\Gamma^{g}{}_{be}\Gamma^{f}{}_{dg} - \Gamma^{a}{}_{fc}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{eg} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{f}{}_{bc}\Gamma^{g}{}_{fe}\Gamma^{a}{}_{dg} \\
& + \Gamma^{f}{}_{bc}\Gamma^{g}{}_{fd}\Gamma^{a}{}_{eg} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{dc}\Gamma^{g}{}_{be}\Gamma^{a}{}_{fg} + \Gamma^{f}{}_{dc}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{eg} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} - \Gamma^{f}{}_{ec}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{dg} \\
& + \Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd}\Gamma^{a}{}_{fg}
\end{aligned}
\tag{ex-0301.305}
$$

$$
\begin{aligned}
= {} & \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \Gamma^{a}{}_{cf}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \Gamma^{a}{}_{df}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{bd} \\
& - \Gamma^{a}{}_{fe}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{bc} - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{a}{}_{cg}\Gamma^{f}{}_{be}\Gamma^{g}{}_{fd} + \Gamma^{a}{}_{dg}\Gamma^{f}{}_{be}\Gamma^{g}{}_{fc} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd} \\
& + \Gamma^{a}{}_{dg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{a}{}_{cg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bc} + \partial_{de}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \Gamma^{a}{}_{ef}\partial_{d}\Gamma^{f}{}_{bc} \\
& + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} - \Gamma^{a}{}_{cf}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bc} - \Gamma^{a}{}_{fd}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{be} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} \\
& + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} - \Gamma^{a}{}_{eg}\Gamma^{f}{}_{bd}\Gamma^{g}{}_{fc} + \Gamma^{a}{}_{cg}\Gamma^{f}{}_{bd}\Gamma^{g}{}_{fe} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc} + \Gamma^{a}{}_{cg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} \\
& + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} - \Gamma^{a}{}_{eg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{be} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} + \Gamma^{a}{}_{df}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{df} - \Gamma^{a}{}_{ef}\partial_{c}\Gamma^{f}{}_{bd} \\
& - \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{a}{}_{fc}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{be} - \Gamma^{a}{}_{fc}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bd} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{a}{}_{dg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{fe} \\
& + \Gamma^{a}{}_{eg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{fd} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{be} + \Gamma^{a}{}_{eg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} - \Gamma^{a}{}_{dg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bf} \\
& + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd}
\end{aligned}
\tag{ex-0301.306}
$$

$$
\begin{aligned}
R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} &= \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \Gamma^a{}_{cf}\partial_e\Gamma^f{}_{bd} + \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{cf} - \Gamma^a{}_{df}\partial_e\Gamma^f{}_{bc} - \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{df} + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^f{}_{cg}\Gamma^g{}_{bd} \\
&\quad - \Gamma^a{}_{fe}\Gamma^f{}_{dg}\Gamma^g{}_{bc} - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^a{}_{cf}\Gamma^g{}_{be}\Gamma^f{}_{gd} + \Gamma^a{}_{df}\Gamma^g{}_{be}\Gamma^f{}_{gc} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ce}\Gamma^g{}_{bd} \\
&\quad + \Gamma^a{}_{df}\Gamma^g{}_{ce}\Gamma^f{}_{bg} - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^a{}_{cf}\Gamma^g{}_{de}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{de}\Gamma^g{}_{bc} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \Gamma^a{}_{ef}\partial_d\Gamma^f{}_{bc} + \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{ef} \\
&\quad - \Gamma^a{}_{cf}\partial_d\Gamma^f{}_{be} - \Gamma^f{}_{be}\partial_d\Gamma^a{}_{cf} + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^f{}_{eg}\Gamma^g{}_{bc} - \Gamma^a{}_{fd}\Gamma^f{}_{cg}\Gamma^g{}_{be} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} \\
&\quad - \Gamma^a{}_{ef}\Gamma^g{}_{bd}\Gamma^f{}_{gc} + \Gamma^a{}_{cf}\Gamma^g{}_{bd}\Gamma^f{}_{ge} - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ed}\Gamma^g{}_{bc} + \Gamma^a{}_{cf}\Gamma^g{}_{ed}\Gamma^f{}_{bg} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} \\
&\quad - \Gamma^a{}_{ef}\Gamma^g{}_{cd}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{cd}\Gamma^g{}_{be} + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \Gamma^a{}_{df}\partial_c\Gamma^f{}_{be} + \Gamma^f{}_{be}\partial_c\Gamma^a{}_{df} - \Gamma^a{}_{ef}\partial_c\Gamma^f{}_{bd} - \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{ef} + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} \\
&\quad - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^f{}_{dg}\Gamma^g{}_{be} - \Gamma^a{}_{fc}\Gamma^f{}_{eg}\Gamma^g{}_{bd} - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^a{}_{df}\Gamma^g{}_{bc}\Gamma^f{}_{ge} + \Gamma^a{}_{ef}\Gamma^g{}_{bc}\Gamma^f{}_{gd} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} \\
&\quad + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{dc}\Gamma^g{}_{be} + \Gamma^a{}_{ef}\Gamma^g{}_{dc}\Gamma^f{}_{bg} - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^a{}_{df}\Gamma^g{}_{ec}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{ec}\Gamma^g{}_{bd} \qquad (\texttt{ex-0301.307}) \\
&= 0 \qquad (\texttt{ex-0301.308})
\end{aligned}
$$

## Exercise 3.2 Riemann tensor from commutation of $\nabla$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});

# rules for the first two covariant derivs of V^a

deriv1 := \nabla_{a}{V^{b}} -> \partial_{a}{V^{b}}
                             + \Gamma^{b}_{d a} V^{d}.          # cdb (ex-0302.101,deriv1)

deriv2 := \nabla_{a}{\nabla_{b}{V^{c}}} -> \partial_{a}{\nabla_{b}{V^{c}}}
                                         + \Gamma^{c}_{d a} \nabla_{b}{V^{d}}
                                         - \Gamma^{d}_{b a} \nabla_{d}{V^{c}}.
                                                                # cdb (ex-0302.102,deriv2)

Vabc :=   \nabla_{c}{\nabla_{b}{V^{a}}}
        - \nabla_{b}{\nabla_{c}{V^{a}}}.                        # cdb (ex-0302.103, Vabc)

substitute (Vabc,deriv2)                                        # cdb (ex-0302.104, Vabc)
substitute (Vabc,deriv1)                                        # cdb (ex-0302.105, Vabc)

distribute     (Vabc)                                           # cdb (ex-0302.106, Vabc)
product_rule   (Vabc)                                           # cdb (ex-0302.107, Vabc)

sort_product   (Vabc)                                           # cdb (ex-0302.108, Vabc)
rename_dummies (Vabc)                                           # cdb (ex-0302.109, Vabc)
canonicalise   (Vabc)                                           # cdb (ex-0302.110, Vabc)
factor_out     (Vabc,$V^{a?}$)                                  # cdb (ex-0302.111, Vabc)
```

$$\nabla_c\left(\nabla_b V^a\right) - \nabla_b\left(\nabla_c V^a\right) = \partial_c\left(\nabla_b V^a\right) + \Gamma^a{}_{dc}\nabla_b V^d - \Gamma^d{}_{bc}\nabla_d V^a - \partial_b\left(\nabla_c V^a\right) - \Gamma^a{}_{db}\nabla_c V^d + \Gamma^d{}_{cb}\nabla_d V^a \tag{ex-0302.104}$$

$$\begin{aligned} &= \partial_c\left(\partial_b V^a + \Gamma^a{}_{db}V^d\right) + \Gamma^a{}_{dc}\left(\partial_b V^d + \Gamma^d{}_{eb}V^e\right) - \Gamma^d{}_{bc}\left(\partial_d V^a + \Gamma^a{}_{ed}V^e\right) - \partial_b\left(\partial_c V^a + \Gamma^a{}_{dc}V^d\right) - \Gamma^a{}_{db}\left(\partial_c V^d + \Gamma^d{}_{ec}V^e\right) \\ &\quad + \Gamma^d{}_{cb}\left(\partial_d V^a + \Gamma^a{}_{ed}V^e\right) \end{aligned} \tag{ex-0302.105}$$

$$\begin{aligned} &= \partial_{cb}V^a + \partial_c\left(\Gamma^a{}_{db}V^d\right) + \Gamma^a{}_{dc}\partial_b V^d + \Gamma^a{}_{dc}\Gamma^d{}_{eb}V^e - \Gamma^d{}_{bc}\partial_d V^a - \Gamma^d{}_{bc}\Gamma^a{}_{ed}V^e - \partial_{bc}V^a - \partial_b\left(\Gamma^a{}_{dc}V^d\right) - \Gamma^a{}_{db}\partial_c V^d \\ &\quad - \Gamma^a{}_{db}\Gamma^d{}_{ec}V^e + \Gamma^d{}_{cb}\partial_d V^a + \Gamma^d{}_{cb}\Gamma^a{}_{ed}V^e \end{aligned} \tag{ex-0302.106}$$

$$\begin{aligned} &= \partial_{cb}V^a + \partial_c\Gamma^a{}_{db}V^d + \Gamma^a{}_{dc}\Gamma^d{}_{eb}V^e - \Gamma^d{}_{bc}\partial_d V^a - \Gamma^d{}_{bc}\Gamma^a{}_{ed}V^e - \partial_{bc}V^a - \partial_b\Gamma^a{}_{dc}V^d - \Gamma^a{}_{db}\Gamma^d{}_{ec}V^e + \Gamma^d{}_{cb}\partial_d V^a \\ &\quad + \Gamma^d{}_{cb}\Gamma^a{}_{ed}V^e \end{aligned} \tag{ex-0302.107}$$

$$\begin{aligned} &= \partial_{cb}V^a + V^d\partial_c\Gamma^a{}_{db} + V^e\Gamma^a{}_{dc}\Gamma^d{}_{eb} - \Gamma^d{}_{bc}\partial_d V^a - V^e\Gamma^a{}_{ed}\Gamma^d{}_{bc} - \partial_{bc}V^a - V^d\partial_b\Gamma^a{}_{dc} - V^e\Gamma^a{}_{db}\Gamma^d{}_{ec} + \Gamma^d{}_{cb}\partial_d V^a \\ &\quad + V^e\Gamma^a{}_{ed}\Gamma^d{}_{cb} \end{aligned} \tag{ex-0302.108}$$

$$\begin{aligned} &= \partial_{cb}V^a + V^d\partial_c\Gamma^a{}_{db} + V^d\Gamma^a{}_{ec}\Gamma^e{}_{db} - \Gamma^d{}_{bc}\partial_d V^a - V^d\Gamma^a{}_{de}\Gamma^e{}_{bc} - \partial_{bc}V^a - V^d\partial_b\Gamma^a{}_{dc} - V^d\Gamma^a{}_{eb}\Gamma^e{}_{dc} + \Gamma^d{}_{cb}\partial_d V^a \\ &\quad + V^d\Gamma^a{}_{de}\Gamma^e{}_{cb} \end{aligned} \tag{ex-0302.109}$$

$$= V^d\partial_c\Gamma^a{}_{bd} + V^d\Gamma^a{}_{ce}\Gamma^e{}_{bd} - V^d\partial_b\Gamma^a{}_{cd} - V^d\Gamma^a{}_{be}\Gamma^e{}_{cd} \tag{ex-0302.110}$$

$$= V^d\left(\partial_c\Gamma^a{}_{bd} + \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \partial_b\Gamma^a{}_{cd} - \Gamma^a{}_{be}\Gamma^e{}_{cd}\right) \tag{ex-0302.111}$$

$$= -R^a{}_{dbc}V^d$$

This result agrees with Misner, Thorne and Wheeler. pg. 266.

## Exercise 3.3 Computing $R_{abcd}$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).

dgab := \partial_{c}{g_{a b}} ->   \Gamma^{d}_{a c} g_{d b}
                                 + \Gamma^{d}_{b c} g_{a d}.      # cdb(dgab.000,dgab)

RabcdU := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                           - \partial_{d}{\Gamma^{a}_{b c}}
                           + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                           - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.   # cdb(Rabcd.000,RabcdU)

GammaD := {g_{a e} \Gamma^{e}_{b c} -> \Gamma_{a b c},
           g_{e a} \Gamma^{e}_{b c} -> \Gamma_{a b c}}.           # cdb(Gamma.010,GammaD)

RabcdD := R_{a b c d} -> g_{a e} R^{e}_{b c d}.                   # cdb(Rabcd.010,RabcdD)

gabDGamma := g_{a e} \partial_{c}{\Gamma^{e}_{b d}} ->
                   \partial_{c}{g_{a e} \Gamma^{e}_{b d}}
                 - \Gamma^{e}_{b d} \partial_{c}{g_{a e}}.        # cdb(gabDGamma.000,gabDGamma)

# this pair of rules needed to sort \Gamma_{a b c} to the very left
# this helps canonicalise spot the terms that cancel
bah := \Gamma_{a b c} -> A_{a b c}.
foo := A_{a b c} -> \Gamma_{a b c}.

expr := R_{a b c d}.                                              # cdb(ex-0303.101,expr)

substitute     (expr, RabcdD)                                     # cdb(ex-0303.102,expr)
substitute     (expr, RabcdU)                                     # cdb(ex-0303.103,expr)
distribute     (expr)                                             # cdb(ex-0303.104,expr)
substitute     (expr, gabDGamma)                                  # cdb(ex-0303.105,expr)
substitute     (expr, dgab)                                       # cdb(ex-0303.106,expr)
```

```
substitute     (expr, GammaD)                          # cdb(ex-0303.107,expr)
distribute     (expr)                                  # cdb(ex-0303.109,expr)
substitute     (expr, bah)                             # cdb(ex-0303.110,expr)
sort_product   (expr)                                  # cdb(ex-0303.111,expr)
rename_dummies (expr)                                  # cdb(ex-0303.112,expr)
substitute     (expr, foo)                             # cdb(ex-0303.113,expr)
canonicalise   (expr)                                  # cdb(ex-0303.114,expr)
```

$$
\begin{aligned}
R_{abcd} &= g_{ae} R^{e}{}_{bcd} \qquad \text{(ex-0303.102)}\\
&= g_{ae} \left(\partial_{c}\Gamma^{e}{}_{bd} - \partial_{d}\Gamma^{e}{}_{bc} + \Gamma^{f}{}_{bd}\Gamma^{e}{}_{cf} - \Gamma^{f}{}_{bc}\Gamma^{e}{}_{df}\right) \qquad \text{(ex-0303.103)}\\
&= g_{ae}\partial_{c}\Gamma^{e}{}_{bd} - g_{ae}\partial_{d}\Gamma^{e}{}_{bc} + g_{ae}\Gamma^{f}{}_{bd}\Gamma^{e}{}_{cf} - g_{ae}\Gamma^{f}{}_{bc}\Gamma^{e}{}_{df} \qquad \text{(ex-0303.104)}\\
&= \partial_{c}\left(g_{ae}\Gamma^{e}{}_{bd}\right) - \Gamma^{e}{}_{bd}\partial_{c}g_{ae} - \partial_{d}\left(g_{ae}\Gamma^{e}{}_{bc}\right) + \Gamma^{e}{}_{bc}\partial_{d}g_{ae} + g_{ae}\Gamma^{f}{}_{bd}\Gamma^{e}{}_{cf} - g_{ae}\Gamma^{f}{}_{bc}\Gamma^{e}{}_{df} \qquad \text{(ex-0303.105)}\\
&= \partial_{c}\left(g_{ae}\Gamma^{e}{}_{bd}\right) - \Gamma^{e}{}_{bd}\left(\Gamma^{f}{}_{ac}g_{fe} + \Gamma^{f}{}_{ec}g_{af}\right) - \partial_{d}\left(g_{ae}\Gamma^{e}{}_{bc}\right) + \Gamma^{e}{}_{bc}\left(\Gamma^{f}{}_{ad}g_{fe} + \Gamma^{f}{}_{ed}g_{af}\right) + g_{ae}\Gamma^{f}{}_{bd}\Gamma^{e}{}_{cf} - g_{ae}\Gamma^{f}{}_{bc}\Gamma^{e}{}_{df} \qquad \text{(ex-0303.106)}\\
&= \partial_{c}\Gamma_{abd} - \Gamma^{e}{}_{bd}\left(\Gamma_{eac} + \Gamma_{aec}\right) - \partial_{d}\Gamma_{abc} + \Gamma^{e}{}_{bc}\left(\Gamma_{ead} + \Gamma_{aed}\right) + \Gamma_{acf}\Gamma^{f}{}_{bd} - \Gamma_{adf}\Gamma^{f}{}_{bc} \qquad \text{(ex-0303.107)}\\
&= \partial_{c}\Gamma_{abd} - \Gamma^{e}{}_{bd}\Gamma_{eac} - \Gamma^{e}{}_{bd}\Gamma_{aec} - \partial_{d}\Gamma_{abc} + \Gamma^{e}{}_{bc}\Gamma_{ead} + \Gamma^{e}{}_{bc}\Gamma_{aed} + \Gamma_{acf}\Gamma^{f}{}_{bd} - \Gamma_{adf}\Gamma^{f}{}_{bc} \qquad \text{(ex-0303.109)}\\
&= \partial_{c}A_{abd} - \Gamma^{e}{}_{bd}A_{eac} - \Gamma^{e}{}_{bd}A_{aec} - \partial_{d}A_{abc} + \Gamma^{e}{}_{bc}A_{ead} + \Gamma^{e}{}_{bc}A_{aed} + A_{acf}\Gamma^{f}{}_{bd} - A_{adf}\Gamma^{f}{}_{bc} \qquad \text{(ex-0303.110)}\\
&= \partial_{c}A_{abd} - A_{eac}\Gamma^{e}{}_{bd} - A_{aec}\Gamma^{e}{}_{bd} - \partial_{d}A_{abc} + A_{ead}\Gamma^{e}{}_{bc} + A_{aed}\Gamma^{e}{}_{bc} + A_{acf}\Gamma^{f}{}_{bd} - A_{adf}\Gamma^{f}{}_{bc} \qquad \text{(ex-0303.111)}\\
&= \partial_{c}A_{abd} - A_{eac}\Gamma^{e}{}_{bd} - A_{aec}\Gamma^{e}{}_{bd} - \partial_{d}A_{abc} + A_{ead}\Gamma^{e}{}_{bc} + A_{aed}\Gamma^{e}{}_{bc} + A_{ace}\Gamma^{e}{}_{bd} - A_{ade}\Gamma^{e}{}_{bc} \qquad \text{(ex-0303.112)}\\
&= \partial_{c}\Gamma_{abd} - \Gamma_{eac}\Gamma^{e}{}_{bd} - \Gamma_{aec}\Gamma^{e}{}_{bd} - \partial_{d}\Gamma_{abc} + \Gamma_{ead}\Gamma^{e}{}_{bc} + \Gamma_{aed}\Gamma^{e}{}_{bc} + \Gamma_{ace}\Gamma^{e}{}_{bd} - \Gamma_{ade}\Gamma^{e}{}_{bc} \qquad \text{(ex-0303.113)}\\
&= \partial_{c}\Gamma_{abd} - \Gamma_{eac}\Gamma^{e}{}_{bd} - \partial_{d}\Gamma_{abc} + \Gamma_{ead}\Gamma^{e}{}_{bc} \qquad \text{(ex-0303.114)}
\end{aligned}
$$

## Exercise 3.4 More symmetries of Riemann

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Symmetric.
g^{a b}::Symmetric.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).

GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                             + \partial_{c}{g_{b d}}
                                             - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)

GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
                                   + \partial_{c}{g_{b a}}
                                   - \partial_{a}{g_{b c}}).           # cdb(Gamma.010,GammaD)

Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
                        - \partial_{d}{\Gamma_{a b c}}
                        + \Gamma_{e a d} \Gamma^{e}_{b c}
                        - \Gamma_{e a c} \Gamma^{e}_{b d}.             # cdb(Rabcd.000,Rabcd)
```

# Exercise 3.4 Antisymmetry on first pair of indices

```
expr := R_{a b c d} + R_{b a c d}.    # cdb(ex-0304.101,expr)

substitute     (expr, Rabcd)          # cdb(ex-0304.102,expr)
substitute     (expr, GammaU)         # cdb(ex-0304.103,expr)
substitute     (expr, GammaD)         # cdb(ex-0304.104,expr)
distribute     (expr)                 # cdb(ex-0304.105,expr)
product_rule   (expr)                 # cdb(ex-0304.106,expr)
sort_product   (expr)                 # cdb(ex-0304.107,expr)
rename_dummies (expr)                 # cdb(ex-0304.108,expr)
canonicalise   (expr)                 # cdb(ex-0304.109,expr)
```

$$
\begin{aligned}
R_{abcd}+R_{bacd} &= \partial_c\Gamma_{abd}-\partial_d\Gamma_{abc}+\Gamma_{ead}\Gamma^e{}_{bc}-\Gamma_{eac}\Gamma^e{}_{bd}+\partial_c\Gamma_{bad}-\partial_d\Gamma_{bac}+\Gamma_{ebd}\Gamma^e{}_{ac}-\Gamma_{ebc}\Gamma^e{}_{ad} && \texttt{(ex-0304.102)}\\
&= \partial_c\Gamma_{abd}-\partial_d\Gamma_{abc}+\frac{1}{2}\Gamma_{ead}g^{ef}\left(\partial_b g_{fc}+\partial_c g_{bf}-\partial_f g_{bc}\right)-\frac{1}{2}\Gamma_{eac}g^{ef}\left(\partial_b g_{fd}+\partial_d g_{bf}-\partial_f g_{bd}\right)+\partial_c\Gamma_{bad}-\partial_d\Gamma_{bac}\\
&\quad+\frac{1}{2}\Gamma_{ebd}g^{ef}\left(\partial_a g_{fc}+\partial_c g_{af}-\partial_f g_{ac}\right)-\frac{1}{2}\Gamma_{ebc}g^{ef}\left(\partial_a g_{fd}+\partial_d g_{af}-\partial_f g_{ad}\right) && \texttt{(ex-0304.103)}\\
&= \partial_c\left(\frac{1}{2}\partial_b g_{ad}+\frac{1}{2}\partial_d g_{ba}-\frac{1}{2}\partial_a g_{bd}\right)-\partial_d\left(\frac{1}{2}\partial_b g_{ac}+\frac{1}{2}\partial_c g_{ba}-\frac{1}{2}\partial_a g_{bc}\right)+\frac{1}{2}\left(\frac{1}{2}\partial_a g_{ed}+\frac{1}{2}\partial_d g_{ae}-\frac{1}{2}\partial_e g_{ad}\right)g^{ef}\left(\partial_b g_{fc}+\partial_c g_{bf}-\partial_f g_{bc}\right)\\
&\quad-\frac{1}{2}\left(\frac{1}{2}\partial_a g_{ec}+\frac{1}{2}\partial_c g_{ae}-\frac{1}{2}\partial_e g_{ac}\right)g^{ef}\left(\partial_b g_{fd}+\partial_d g_{bf}-\partial_f g_{bd}\right)+\partial_c\left(\frac{1}{2}\partial_a g_{bd}+\frac{1}{2}\partial_d g_{ab}-\frac{1}{2}\partial_b g_{ad}\right)\\
&\quad-\partial_d\left(\frac{1}{2}\partial_a g_{bc}+\frac{1}{2}\partial_c g_{ab}-\frac{1}{2}\partial_b g_{ac}\right)+\frac{1}{2}\left(\frac{1}{2}\partial_b g_{ed}+\frac{1}{2}\partial_d g_{be}-\frac{1}{2}\partial_e g_{bd}\right)g^{ef}\left(\partial_a g_{fc}+\partial_c g_{af}-\partial_f g_{ac}\right)\\
&\quad-\frac{1}{2}\left(\frac{1}{2}\partial_b g_{ec}+\frac{1}{2}\partial_c g_{be}-\frac{1}{2}\partial_e g_{bc}\right)g^{ef}\left(\partial_a g_{fd}+\partial_d g_{af}-\partial_f g_{ad}\right) && \texttt{(ex-0304.104)}\\
&= \frac{1}{2}\partial_{cd}g_{ba}-\frac{1}{2}\partial_{dc}g_{ba}+\frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc}+\frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf}-\frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc}+\frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc}+\frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf}-\frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc}\\
&\quad-\frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc}-\frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf}+\frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc}-\frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd}-\frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf}+\frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd}-\frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd}\\
&\quad-\frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf}+\frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd}+\frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd}+\frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf}-\frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd}+\frac{1}{2}\partial_{cd}g_{ab}-\frac{1}{2}\partial_{dc}g_{ab}+\frac{1}{4}\partial_b g_{ed}g^{ef}\partial_a g_{fc}\\
&\quad+\frac{1}{4}\partial_b g_{ed}g^{ef}\partial_c g_{af}-\frac{1}{4}\partial_b g_{ed}g^{ef}\partial_f g_{ac}+\frac{1}{4}\partial_d g_{be}g^{ef}\partial_a g_{fc}+\frac{1}{4}\partial_d g_{be}g^{ef}\partial_c g_{af}-\frac{1}{4}\partial_d g_{be}g^{ef}\partial_f g_{ac}-\frac{1}{4}\partial_e g_{bd}g^{ef}\partial_a g_{fc}-\frac{1}{4}\partial_e g_{bd}g^{ef}\partial_c g_{af}\\
&\quad+\frac{1}{4}\partial_e g_{bd}g^{ef}\partial_f g_{ac}-\frac{1}{4}\partial_b g_{ec}g^{ef}\partial_a g_{fd}-\frac{1}{4}\partial_b g_{ec}g^{ef}\partial_d g_{af}+\frac{1}{4}\partial_b g_{ec}g^{ef}\partial_f g_{ad}-\frac{1}{4}\partial_c g_{be}g^{ef}\partial_a g_{fd}-\frac{1}{4}\partial_c g_{be}g^{ef}\partial_d g_{af}+\frac{1}{4}\partial_c g_{be}g^{ef}\partial_f g_{ad}\\
&\quad+\frac{1}{4}\partial_e g_{bc}g^{ef}\partial_a g_{fd}+\frac{1}{4}\partial_e g_{bc}g^{ef}\partial_d g_{af}-\frac{1}{4}\partial_e g_{bc}g^{ef}\partial_f g_{ad} && \texttt{(ex-0304.105)}
\end{aligned}
$$

$$
\begin{aligned}
R_{abcd} + R_{bacd} = {} & \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_b g_{fc} + \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_f g_{bc} \\
& - \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_d g_{bf} + \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_b g_{fd} \\
& - \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_f g_{bd} + \frac{1}{2}\partial_{cd} g_{ab} - \frac{1}{2}\partial_{dc} g_{ab} + \frac{1}{4}\partial_b g_{ed} g^{ef} \partial_a g_{fc} \\
& + \frac{1}{4}\partial_b g_{ed} g^{ef} \partial_c g_{af} - \frac{1}{4}\partial_b g_{ed} g^{ef} \partial_f g_{ac} + \frac{1}{4}\partial_d g_{be} g^{ef} \partial_a g_{fc} + \frac{1}{4}\partial_d g_{be} g^{ef} \partial_c g_{af} - \frac{1}{4}\partial_d g_{be} g^{ef} \partial_f g_{ac} - \frac{1}{4}\partial_e g_{bd} g^{ef} \partial_a g_{fc} - \frac{1}{4}\partial_e g_{bd} g^{ef} \partial_c g_{af} \\
& + \frac{1}{4}\partial_e g_{bd} g^{ef} \partial_f g_{ac} - \frac{1}{4}\partial_b g_{ec} g^{ef} \partial_a g_{fd} - \frac{1}{4}\partial_b g_{ec} g^{ef} \partial_d g_{af} + \frac{1}{4}\partial_b g_{ec} g^{ef} \partial_f g_{ad} - \frac{1}{4}\partial_c g_{be} g^{ef} \partial_a g_{fd} - \frac{1}{4}\partial_c g_{be} g^{ef} \partial_d g_{af} + \frac{1}{4}\partial_c g_{be} g^{ef} \partial_f g_{ad} \\
& + \frac{1}{4}\partial_e g_{bc} g^{ef} \partial_a g_{fd} + \frac{1}{4}\partial_e g_{bc} g^{ef} \partial_d g_{af} - \frac{1}{4}\partial_e g_{bc} g^{ef} \partial_f g_{ad} \qquad (\texttt{ex-0304.106}) \\
= {} & \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{4}\partial_a g_{ed} \partial_b g_{fc} g^{ef} + \frac{1}{4}\partial_a g_{ed} \partial_c g_{bf} g^{ef} - \frac{1}{4}\partial_a g_{ed} \partial_f g_{bc} g^{ef} + \frac{1}{4}\partial_b g_{fc} \partial_d g_{ae} g^{ef} + \frac{1}{4}\partial_c g_{bf} \partial_d g_{ae} g^{ef} - \frac{1}{4}\partial_d g_{ae} \partial_f g_{bc} g^{ef} \\
& - \frac{1}{4}\partial_b g_{fc} \partial_e g_{ad} g^{ef} - \frac{1}{4}\partial_c g_{bf} \partial_e g_{ad} g^{ef} + \frac{1}{4}\partial_e g_{ad} \partial_f g_{bc} g^{ef} - \frac{1}{4}\partial_a g_{ec} \partial_b g_{fd} g^{ef} - \frac{1}{4}\partial_a g_{ec} \partial_d g_{bf} g^{ef} + \frac{1}{4}\partial_a g_{ec} \partial_f g_{bd} g^{ef} - \frac{1}{4}\partial_b g_{fd} \partial_c g_{ae} g^{ef} \\
& - \frac{1}{4}\partial_c g_{ae} \partial_d g_{bf} g^{ef} + \frac{1}{4}\partial_c g_{ae} \partial_f g_{bd} g^{ef} + \frac{1}{4}\partial_b g_{fd} \partial_e g_{ac} g^{ef} + \frac{1}{4}\partial_d g_{bf} \partial_e g_{ac} g^{ef} - \frac{1}{4}\partial_e g_{ac} \partial_f g_{bd} g^{ef} + \frac{1}{2}\partial_{cd} g_{ab} - \frac{1}{2}\partial_{dc} g_{ab} + \frac{1}{4}\partial_a g_{fc} \partial_b g_{ed} g^{ef} \\
& + \frac{1}{4}\partial_b g_{ed} \partial_c g_{af} g^{ef} - \frac{1}{4}\partial_b g_{ed} \partial_f g_{ac} g^{ef} + \frac{1}{4}\partial_a g_{fc} \partial_d g_{be} g^{ef} + \frac{1}{4}\partial_c g_{af} \partial_d g_{be} g^{ef} - \frac{1}{4}\partial_d g_{be} \partial_f g_{ac} g^{ef} - \frac{1}{4}\partial_a g_{fc} \partial_e g_{bd} g^{ef} - \frac{1}{4}\partial_c g_{af} \partial_e g_{bd} g^{ef} \\
& + \frac{1}{4}\partial_e g_{bd} \partial_f g_{ac} g^{ef} - \frac{1}{4}\partial_a g_{fd} \partial_b g_{ec} g^{ef} - \frac{1}{4}\partial_b g_{ec} \partial_d g_{af} g^{ef} + \frac{1}{4}\partial_b g_{ec} \partial_f g_{ad} g^{ef} - \frac{1}{4}\partial_a g_{fd} \partial_c g_{be} g^{ef} - \frac{1}{4}\partial_c g_{be} \partial_d g_{af} g^{ef} + \frac{1}{4}\partial_c g_{be} \partial_f g_{ad} g^{ef} \\
& + \frac{1}{4}\partial_a g_{fd} \partial_e g_{bc} g^{ef} + \frac{1}{4}\partial_d g_{af} \partial_e g_{bc} g^{ef} - \frac{1}{4}\partial_e g_{bc} \partial_f g_{ad} g^{ef} \qquad (\texttt{ex-0304.107})
\end{aligned}
$$

$$
\begin{aligned}
R_{abcd} + R_{bacd} &= \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{fe} + \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{fe} + \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{fe} - \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{fe} \\
&\quad - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{fe} - \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{fe} \\
&\quad - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{fe} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{fe} \\
&\quad + \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{fe} + \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{fe} + \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{fe} - \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{fe} - \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{ef} - \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{ef} \\
&\quad + \frac{1}{4}\partial_e g_{bd}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{fe} - \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{fe} - \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{fe} - \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{fe} \\
&\quad + \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{ef} + \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{ef} - \frac{1}{4}\partial_e g_{bc}\partial_f g_{ad}g^{ef} && \texttt{(ex-0304.108)} \\
&= 0 && \texttt{(ex-0304.109)}
\end{aligned}
$$

## Exercise 3.4 Symmetric on swapping first and second pair of indices

```
expr := R_{a b c d} - R_{c d a b}.   # cdb(ex-0304.201,expr)

substitute     (expr, Rabcd)         # cdb(ex-0304.202,expr)
substitute     (expr, GammaU)        # cdb(ex-0304.203,expr)
substitute     (expr, GammaD)        # cdb(ex-0304.204,expr)
distribute     (expr)                # cdb(ex-0304.205,expr)
product_rule   (expr)                # cdb(ex-0304.206,expr)
sort_product   (expr)                # cdb(ex-0304.207,expr)
rename_dummies (expr)                # cdb(ex-0304.208,expr)
canonicalise   (expr)                # cdb(ex-0304.209,expr)
```

$$
\begin{aligned}
R_{abcd} - R_{cdab} &= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead}\Gamma^e{}_{bc} - \Gamma_{eac}\Gamma^e{}_{bd} - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} - \Gamma_{ecb}\Gamma^e{}_{da} + \Gamma_{eca}\Gamma^e{}_{db} && \texttt{(ex-0304.202)} \\
&= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2}\Gamma_{ead} g^{ef} \left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) - \frac{1}{2}\Gamma_{eac} g^{ef} \left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} \\
&\quad - \frac{1}{2}\Gamma_{ecb} g^{ef} \left(\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}\right) + \frac{1}{2}\Gamma_{eca} g^{ef} \left(\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}\right) && \texttt{(ex-0304.203)} \\
&= \partial_c \left(\frac{1}{2}\partial_b g_{ad} + \frac{1}{2}\partial_d g_{ba} - \frac{1}{2}\partial_a g_{bd}\right) - \partial_d \left(\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc}\right) + \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ed} + \frac{1}{2}\partial_d g_{ae} - \frac{1}{2}\partial_e g_{ad}\right) g^{ef} \left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) \\
&\quad - \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ec} + \frac{1}{2}\partial_c g_{ae} - \frac{1}{2}\partial_e g_{ac}\right) g^{ef} \left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) - \partial_a \left(\frac{1}{2}\partial_d g_{cb} + \frac{1}{2}\partial_b g_{dc} - \frac{1}{2}\partial_c g_{db}\right) \\
&\quad + \partial_b \left(\frac{1}{2}\partial_d g_{ca} + \frac{1}{2}\partial_a g_{dc} - \frac{1}{2}\partial_c g_{da}\right) - \frac{1}{2}\left(\frac{1}{2}\partial_c g_{eb} + \frac{1}{2}\partial_b g_{ce} - \frac{1}{2}\partial_e g_{cb}\right) g^{ef} \left(\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{2}\partial_c g_{ea} + \frac{1}{2}\partial_a g_{ce} - \frac{1}{2}\partial_e g_{ca}\right) g^{ef} \left(\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}\right) && \texttt{(ex-0304.204)} \\
&= \frac{1}{2}\partial_{cb} g_{ad} + \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{ca} g_{bd} - \frac{1}{2}\partial_{db} g_{ac} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{2}\partial_{da} g_{bc} + \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_b g_{fc} \\
&\quad + \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_f g_{bc} - \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_d g_{bf} \\
&\quad + \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_b g_{fd} - \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_f g_{bd} \\
&\quad - \frac{1}{2}\partial_{ad} g_{cb} - \frac{1}{2}\partial_{ab} g_{dc} + \frac{1}{2}\partial_{ac} g_{db} + \frac{1}{2}\partial_{bd} g_{ca} + \frac{1}{2}\partial_{ba} g_{dc} - \frac{1}{2}\partial_{bc} g_{da} - \frac{1}{4}\partial_c g_{eb} g^{ef} \partial_d g_{fa} - \frac{1}{4}\partial_c g_{eb} g^{ef} \partial_a g_{df} + \frac{1}{4}\partial_c g_{eb} g^{ef} \partial_f g_{da} \\
&\quad - \frac{1}{4}\partial_b g_{ce} g^{ef} \partial_d g_{fa} - \frac{1}{4}\partial_b g_{ce} g^{ef} \partial_a g_{df} + \frac{1}{4}\partial_b g_{ce} g^{ef} \partial_f g_{da} + \frac{1}{4}\partial_e g_{cb} g^{ef} \partial_d g_{fa} + \frac{1}{4}\partial_e g_{cb} g^{ef} \partial_a g_{df} - \frac{1}{4}\partial_e g_{cb} g^{ef} \partial_f g_{da} + \frac{1}{4}\partial_c g_{ea} g^{ef} \partial_d g_{fb} \\
&\quad + \frac{1}{4}\partial_c g_{ea} g^{ef} \partial_b g_{df} - \frac{1}{4}\partial_c g_{ea} g^{ef} \partial_f g_{db} + \frac{1}{4}\partial_a g_{ce} g^{ef} \partial_d g_{fb} + \frac{1}{4}\partial_a g_{ce} g^{ef} \partial_b g_{df} - \frac{1}{4}\partial_a g_{ce} g^{ef} \partial_f g_{db} - \frac{1}{4}\partial_e g_{ca} g^{ef} \partial_d g_{fb} - \frac{1}{4}\partial_e g_{ca} g^{ef} \partial_b g_{df} \\
&\quad + \frac{1}{4}\partial_e g_{ca} g^{ef} \partial_f g_{db} && \texttt{(ex-0304.205)}
\end{aligned}
$$

$$
\begin{aligned}
R_{abcd} - R_{cdab} = {} & \frac{1}{2}\partial_{cb} g_{ad} + \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{ca} g_{bd} - \frac{1}{2}\partial_{db} g_{ac} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{2}\partial_{da} g_{bc} + \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed} g^{ef} \partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_b g_{fc} \\
& + \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae} g^{ef} \partial_f g_{bc} - \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad} g^{ef} \partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_d g_{bf} \\
& + \frac{1}{4}\partial_a g_{ec} g^{ef} \partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_b g_{fd} - \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae} g^{ef} \partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac} g^{ef} \partial_f g_{bd} \\
& - \frac{1}{2}\partial_{ad} g_{cb} - \frac{1}{2}\partial_{ab} g_{dc} + \frac{1}{2}\partial_{ac} g_{db} + \frac{1}{2}\partial_{bd} g_{ca} + \frac{1}{2}\partial_{ba} g_{dc} - \frac{1}{2}\partial_{bc} g_{da} - \frac{1}{4}\partial_c g_{eb} g^{ef} \partial_d g_{fa} - \frac{1}{4}\partial_c g_{eb} g^{ef} \partial_a g_{df} + \frac{1}{4}\partial_c g_{eb} g^{ef} \partial_f g_{da} \\
& - \frac{1}{4}\partial_b g_{ce} g^{ef} \partial_d g_{fa} - \frac{1}{4}\partial_b g_{ce} g^{ef} \partial_a g_{df} + \frac{1}{4}\partial_b g_{ce} g^{ef} \partial_f g_{da} + \frac{1}{4}\partial_e g_{cb} g^{ef} \partial_d g_{fa} + \frac{1}{4}\partial_e g_{cb} g^{ef} \partial_a g_{df} - \frac{1}{4}\partial_e g_{cb} g^{ef} \partial_f g_{da} + \frac{1}{4}\partial_c g_{ea} g^{ef} \partial_d g_{fb} \\
& + \frac{1}{4}\partial_c g_{ea} g^{ef} \partial_b g_{df} - \frac{1}{4}\partial_c g_{ea} g^{ef} \partial_f g_{db} + \frac{1}{4}\partial_a g_{ce} g^{ef} \partial_d g_{fb} + \frac{1}{4}\partial_a g_{ce} g^{ef} \partial_b g_{df} - \frac{1}{4}\partial_a g_{ce} g^{ef} \partial_f g_{db} - \frac{1}{4}\partial_e g_{ca} g^{ef} \partial_d g_{fb} - \frac{1}{4}\partial_e g_{ca} g^{ef} \partial_b g_{df} \\
& + \frac{1}{4}\partial_e g_{ca} g^{ef} \partial_f g_{db} \qquad \texttt{(ex-0304.206)}
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{1}{2}\partial_{cb} g_{ad} + \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{ca} g_{bd} - \frac{1}{2}\partial_{db} g_{ac} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{2}\partial_{da} g_{bc} + \frac{1}{4}\partial_a g_{ed} \partial_b g_{fc} g^{ef} + \frac{1}{4}\partial_a g_{ed} \partial_c g_{bf} g^{ef} - \frac{1}{4}\partial_a g_{ed} \partial_f g_{bc} g^{ef} + \frac{1}{4}\partial_b g_{fc} \partial_d g_{ae} g^{ef} \\
& + \frac{1}{4}\partial_c g_{bf} \partial_d g_{ae} g^{ef} - \frac{1}{4}\partial_d g_{ae} \partial_f g_{bc} g^{ef} - \frac{1}{4}\partial_b g_{fc} \partial_e g_{ad} g^{ef} - \frac{1}{4}\partial_c g_{bf} \partial_e g_{ad} g^{ef} + \frac{1}{4}\partial_e g_{ad} \partial_f g_{bc} g^{ef} - \frac{1}{4}\partial_a g_{ec} \partial_b g_{fd} g^{ef} - \frac{1}{4}\partial_a g_{ec} \partial_d g_{bf} g^{ef} \\
& + \frac{1}{4}\partial_a g_{ec} \partial_f g_{bd} g^{ef} - \frac{1}{4}\partial_b g_{fd} \partial_c g_{ae} g^{ef} - \frac{1}{4}\partial_c g_{ae} \partial_d g_{bf} g^{ef} + \frac{1}{4}\partial_c g_{ae} \partial_f g_{bd} g^{ef} + \frac{1}{4}\partial_b g_{fd} \partial_e g_{ac} g^{ef} + \frac{1}{4}\partial_d g_{bf} \partial_e g_{ac} g^{ef} - \frac{1}{4}\partial_e g_{ac} \partial_f g_{bd} g^{ef} \\
& - \frac{1}{2}\partial_{ad} g_{cb} - \frac{1}{2}\partial_{ab} g_{dc} + \frac{1}{2}\partial_{ac} g_{db} + \frac{1}{2}\partial_{bd} g_{ca} + \frac{1}{2}\partial_{ba} g_{dc} - \frac{1}{2}\partial_{bc} g_{da} - \frac{1}{4}\partial_c g_{eb} \partial_d g_{fa} g^{ef} - \frac{1}{4}\partial_a g_{df} \partial_c g_{eb} g^{ef} + \frac{1}{4}\partial_c g_{eb} \partial_f g_{da} g^{ef} \\
& - \frac{1}{4}\partial_b g_{ce} \partial_d g_{fa} g^{ef} - \frac{1}{4}\partial_a g_{df} \partial_b g_{ce} g^{ef} + \frac{1}{4}\partial_b g_{ce} \partial_f g_{da} g^{ef} + \frac{1}{4}\partial_d g_{fa} \partial_e g_{cb} g^{ef} + \frac{1}{4}\partial_a g_{df} \partial_e g_{cb} g^{ef} - \frac{1}{4}\partial_e g_{cb} \partial_f g_{da} g^{ef} + \frac{1}{4}\partial_c g_{ea} \partial_d g_{fb} g^{ef} \\
& + \frac{1}{4}\partial_b g_{df} \partial_c g_{ea} g^{ef} - \frac{1}{4}\partial_c g_{ea} \partial_f g_{db} g^{ef} + \frac{1}{4}\partial_a g_{ce} \partial_d g_{fb} g^{ef} + \frac{1}{4}\partial_a g_{ce} \partial_b g_{df} g^{ef} - \frac{1}{4}\partial_a g_{ce} \partial_f g_{db} g^{ef} - \frac{1}{4}\partial_d g_{fb} \partial_e g_{ca} g^{ef} - \frac{1}{4}\partial_b g_{df} \partial_e g_{ca} g^{ef} \\
& + \frac{1}{4}\partial_e g_{ca} \partial_f g_{db} g^{ef} \qquad \texttt{(ex-0304.207)}
\end{aligned}
$$

$$
\begin{aligned}
R_{abcd} - R_{cdab} &= \frac{1}{2}\partial_{cb} g_{ad} + \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{ca} g_{bd} - \frac{1}{2}\partial_{db} g_{ac} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{2}\partial_{da} g_{bc} + \frac{1}{4}\partial_a g_{ed} \partial_b g_{fc} g^{ef} + \frac{1}{4}\partial_a g_{ed} \partial_c g_{bf} g^{ef} - \frac{1}{4}\partial_a g_{fd} \partial_e g_{bc} g^{fe} + \frac{1}{4}\partial_b g_{ec} \partial_d g_{af} g^{fe} \\
&\quad + \frac{1}{4}\partial_c g_{be} \partial_d g_{af} g^{fe} - \frac{1}{4}\partial_d g_{af} \partial_e g_{bc} g^{fe} - \frac{1}{4}\partial_b g_{fc} \partial_e g_{ad} g^{ef} - \frac{1}{4}\partial_c g_{bf} \partial_e g_{ad} g^{ef} + \frac{1}{4}\partial_e g_{ad} \partial_f g_{bc} g^{ef} - \frac{1}{4}\partial_a g_{ec} \partial_b g_{fd} g^{ef} - \frac{1}{4}\partial_a g_{ec} \partial_d g_{bf} g^{ef} \\
&\quad + \frac{1}{4}\partial_a g_{fc} \partial_e g_{bd} g^{fe} - \frac{1}{4}\partial_b g_{ed} \partial_c g_{af} g^{fe} - \frac{1}{4}\partial_c g_{ae} \partial_d g_{bf} g^{ef} + \frac{1}{4}\partial_c g_{af} \partial_e g_{bd} g^{fe} + \frac{1}{4}\partial_b g_{fd} \partial_e g_{ac} g^{ef} + \frac{1}{4}\partial_d g_{bf} \partial_e g_{ac} g^{ef} - \frac{1}{4}\partial_e g_{ac} \partial_f g_{bd} g^{ef} \\
&\quad - \frac{1}{2}\partial_{ad} g_{cb} - \frac{1}{2}\partial_{ab} g_{dc} + \frac{1}{2}\partial_{ac} g_{db} + \frac{1}{2}\partial_{bd} g_{ca} + \frac{1}{2}\partial_{ba} g_{dc} - \frac{1}{2}\partial_{bc} g_{da} - \frac{1}{4}\partial_c g_{eb} \partial_d g_{fa} g^{ef} - \frac{1}{4}\partial_a g_{de} \partial_c g_{fb} g^{fe} + \frac{1}{4}\partial_c g_{fb} \partial_e g_{da} g^{fe} \\
&\quad - \frac{1}{4}\partial_b g_{ce} \partial_d g_{fa} g^{ef} - \frac{1}{4}\partial_a g_{de} \partial_b g_{cf} g^{fe} + \frac{1}{4}\partial_b g_{cf} \partial_e g_{da} g^{fe} + \frac{1}{4}\partial_d g_{fa} \partial_e g_{cb} g^{ef} + \frac{1}{4}\partial_a g_{df} \partial_e g_{cb} g^{ef} - \frac{1}{4}\partial_e g_{cb} \partial_f g_{da} g^{ef} + \frac{1}{4}\partial_c g_{ea} \partial_d g_{fb} g^{ef} \\
&\quad + \frac{1}{4}\partial_b g_{de} \partial_c g_{fa} g^{fe} - \frac{1}{4}\partial_c g_{fa} \partial_e g_{db} g^{fe} + \frac{1}{4}\partial_a g_{ce} \partial_d g_{fb} g^{ef} + \frac{1}{4}\partial_a g_{ce} \partial_b g_{df} g^{ef} - \frac{1}{4}\partial_a g_{cf} \partial_e g_{db} g^{fe} - \frac{1}{4}\partial_d g_{fb} \partial_e g_{ca} g^{ef} - \frac{1}{4}\partial_b g_{df} \partial_e g_{ca} g^{ef} \\
&\quad + \frac{1}{4}\partial_e g_{ca} \partial_f g_{db} g^{ef} \tag{ex-0304.208} \\
&= 0 \tag{ex-0304.209}
\end{aligned}
$$

# Exercise 3.5 Commutation of covariant derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

expr :=   \nabla_{d}{\nabla_{c}{A_{a} B_{b}}}
        - \nabla_{c}{\nabla_{d}{A_{a} B_{b}}}.  # cdb(ex-0305.100,expr)

product_rule (expr)                             # cdb(ex-0305.101,expr)
distribute   (expr)                             # cdb(ex-0305.102,expr)
product_rule (expr)                             # cdb(ex-0305.103,expr)
factor_out   (expr,$A_{a?},B_{b?}$)             # cdb(ex-0305.104,expr)
```

$$\begin{aligned}
\nabla_{d}\left(\nabla_{c}\left(A_{a}B_{b}\right)\right)-\nabla_{c}\left(\nabla_{d}\left(A_{a}B_{b}\right)\right) &= \nabla_{d}\left(\nabla_{c}A_{a}B_{b}+A_{a}\nabla_{c}B_{b}\right)-\nabla_{c}\left(\nabla_{d}A_{a}B_{b}+A_{a}\nabla_{d}B_{b}\right) && \texttt{(ex-0305.101)}\\
&= \nabla_{d}\left(\nabla_{c}A_{a}B_{b}\right)+\nabla_{d}\left(A_{a}\nabla_{c}B_{b}\right)-\nabla_{c}\left(\nabla_{d}A_{a}B_{b}\right)-\nabla_{c}\left(A_{a}\nabla_{d}B_{b}\right) && \texttt{(ex-0305.102)}\\
&= \nabla_{d}\left(\nabla_{c}A_{a}\right)B_{b}+A_{a}\nabla_{d}\left(\nabla_{c}B_{b}\right)-\nabla_{c}\left(\nabla_{d}A_{a}\right)B_{b}-A_{a}\nabla_{c}\left(\nabla_{d}B_{b}\right) && \texttt{(ex-0305.103)}\\
&= B_{b}\left(\nabla_{d}\left(\nabla_{c}A_{a}\right)-\nabla_{c}\left(\nabla_{d}A_{a}\right)\right)+A_{a}\left(\nabla_{d}\left(\nabla_{c}B_{b}\right)-\nabla_{c}\left(\nabla_{d}B_{b}\right)\right) && \texttt{(ex-0305.104)}
\end{aligned}$$

# Exercise 3.6 Commutation of $\nabla$ on the Riemann tensor – simple computation

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

DD{#}::Derivative.
\nabla{#}::Derivative.

RabcdF := R_{a b c d} -> A_{a} B_{b} C_{c} D_{d}.      # cdb(RabcdF.000,RabcdF)
RabcdB := A_{a} B_{b} C_{c} D_{d} -> R_{a b c d}.      # cdb(RabcdB.000,RabcdB)

derivDD := DD_{b c}{V?_{a}}} -> R^{d}_{a b c} V?_{d}.  # cdb(derivDD.000,derivDD)

nablaDD := \nabla_{f}{\nabla_{e}{R_{a b c d}}}
         - \nabla_{e}{\nabla_{f}{R_{a b c d}}} -> DD_{e f}{R_{a b c d}}.

# product rule for DD acting on A_{a} B_{b} C_{c} D_{d}
pruleDD := DD_{e f}{A_{a} B_{b} C_{c} D_{d}} -> DD_{e f}{A_{a}} B_{b} C_{c} D_{d}
                                              + A_{a} DD_{e f}{B_{b}} C_{c} D_{d}
                                              + A_{a} B_{b} DD_{e f}{C_{c}} D_{d}
                                              + A_{a} B_{b} C_{c} DD_{e f}{D_{d}}.
                                                       # cdb(pruleDD.000,pruleDD)

expr :=   \nabla_{f}{\nabla_{e}{R_{a b c d}}}
        - \nabla_{e}{\nabla_{f}{R_{a b c d}}}.         # cdb (ex-0306.100, expr)

substitute   (expr,nablaDD)                            # cdb (ex-0306.101, expr)
substitute   (expr,RabcdF)                             # cdb (ex-0306.102, expr)
substitute   (expr,pruleDD)                            # cdb (ex-0306.103, expr)
substitute   (expr,derivDD)                            # cdb (ex-0306.104, expr)
sort_product (expr)                                    # cdb (ex-0306.105, expr)
substitute   (expr,RabcdB)                             # cdb (ex-0306.106, expr)
```

$$\begin{aligned}
\nabla_f \left(\nabla_e R_{abcd}\right) - \nabla_e \left(\nabla_f R_{abcd}\right) &= DD_{ef} R_{abcd} && \texttt{(ex-0306.101)}\\
&= DD_{ef} \left(A_a B_b C_c D_d\right) && \texttt{(ex-0306.102)}\\
&= DD_{ef} A_a B_b C_c D_d + A_a DD_{ef} B_b C_c D_d + A_a B_b DD_{ef} C_c D_d + A_a B_b C_c DD_{ef} D_d && \texttt{(ex-0306.103)}\\
&= R^g{}_{aef} A_g B_b C_c D_d + A_a R^g{}_{bef} B_g C_c D_d + A_a B_b R^g{}_{cef} C_g D_d + A_a B_b C_c R^g{}_{def} D_g && \texttt{(ex-0306.104)}\\
&= A_g B_b C_c D_d R^g{}_{aef} + A_a B_g C_c D_d R^g{}_{bef} + A_a B_b C_g D_d R^g{}_{cef} + A_a B_b C_c D_g R^g{}_{def} && \texttt{(ex-0306.105)}\\
&= R_{gbcd} R^g{}_{aef} + R_{agcd} R^g{}_{bef} + R_{abgd} R^g{}_{cef} + R_{abcg} R^g{}_{def} && \texttt{(ex-0306.106)}
\end{aligned}$$

# Exercise 3.7 Commutation of $\nabla$ on the Riemann tensor – direct computation

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

;::Symbol;

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

RabcdD :=   \partial_{c}{\Gamma_{a b d}}
          - \partial_{d}{\Gamma_{a b c}}
          + \Gamma_{e a d} \Gamma^{e}_{b c}
          - \Gamma_{e a c} \Gamma^{e}_{b d} -> R_{a b c d}.            # cdb(Rabcd.010,RabcdD)

RabcdU :=   \partial_{c}{\Gamma^{a}_{b d}}
          - \partial_{d}{\Gamma^{a}_{b c}}
          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
          - \Gamma^{e}_{b c} \Gamma^{a}_{d e} -> R^{a}_{b c d}.        # cdb(Rabcd.000,RabcdU)

d1Rabcd := R_{a b c d ; e} ->   \partial_{e}{R_{a b c d}}
                              - \Gamma^{f}_{a e} R_{f b c d}
                              - \Gamma^{f}_{b e} R_{a f c d}
                              - \Gamma^{f}_{c e} R_{a b f d}
                              - \Gamma^{f}_{d e} R_{a b c f}.          # cdb(d1Rabcd.000,d1Rabcd)

d2Rabcd := R_{a b c d ; e ; f} ->   \partial_{f}{R_{a b c d ; e}}
                                  - \Gamma^{g}_{a f} R_{g b c d ; e}
                                  - \Gamma^{g}_{b f} R_{a g c d ; e}
                                  - \Gamma^{g}_{c f} R_{a b g d ; e}
                                  - \Gamma^{g}_{d f} R_{a b c g ; e}
                                  - \Gamma^{g}_{e f} R_{a b c d ; g}.  # cdb(d2Rabcd.000,d2Rabcd)

substitute (d2Rabcd,d1Rabcd)                                           # cdb (d2Rabcd.001, d2Rabcd)

expr := R_{a b c d ; e ; f} - R_{a b c d ; f ; e}.                     # cdb (ex-0307.100, expr)

substitute    (expr,d2Rabcd)                                           # cdb (ex-0307.101, expr)
```

```

distribute      (expr)                                                # cdb (ex-0307.102, expr)
product_rule    (expr)                                                # cdb (ex-0307.103, expr)

sort_product    (expr)                                                # cdb (ex-0307.104, expr)
rename_dummies  (expr)                                                # cdb (ex-0307.105, expr)
canonicalise    (expr)                                                # cdb (ex-0307.106, expr)
factor_out      (expr,$R_{a? b? c? d?}$)                              # cdb (ex-0307.107, expr)

substitute      (expr,RabcdU)                                         # cdb (ex-0307.108, expr)
substitute      (expr,$R^{a}_{b c d} -> -R^{a}_{b d c}$)              # cdb (ex-0307.109, expr)
```

$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_f \left(\partial_e R_{abcd} - \Gamma^g{}_{ae} R_{gbcd} - \Gamma^g{}_{be} R_{agcd} - \Gamma^g{}_{ce} R_{abgd} - \Gamma^g{}_{de} R_{abcg}\right) - \Gamma^g{}_{af} \left(\partial_e R_{gbcd} - \Gamma^h{}_{ge} R_{hbcd} - \Gamma^h{}_{be} R_{ghcd} - \Gamma^h{}_{ce} R_{gbhd} - \Gamma^h{}_{de} R_{gbch}\right) \\
& - \Gamma^g{}_{bf} \left(\partial_e R_{agcd} - \Gamma^h{}_{ae} R_{hgcd} - \Gamma^h{}_{ge} R_{ahcd} - \Gamma^h{}_{ce} R_{aghd} - \Gamma^h{}_{de} R_{agch}\right) \\
& - \Gamma^g{}_{cf} \left(\partial_e R_{abgd} - \Gamma^h{}_{ae} R_{hbgd} - \Gamma^h{}_{be} R_{ahgd} - \Gamma^h{}_{ge} R_{abhd} - \Gamma^h{}_{de} R_{abgh}\right) \\
& - \Gamma^g{}_{df} \left(\partial_e R_{abcg} - \Gamma^h{}_{ae} R_{hbcg} - \Gamma^h{}_{be} R_{ahcg} - \Gamma^h{}_{ce} R_{abhg} - \Gamma^h{}_{ge} R_{abch}\right) \\
& - \Gamma^g{}_{ef} \left(\partial_g R_{abcd} - \Gamma^h{}_{ag} R_{hbcd} - \Gamma^h{}_{bg} R_{ahcd} - \Gamma^h{}_{cg} R_{abhd} - \Gamma^h{}_{dg} R_{abch}\right) - \partial_e \left(\partial_f R_{abcd} - \Gamma^g{}_{af} R_{gbcd} - \Gamma^g{}_{bf} R_{agcd} - \Gamma^g{}_{cf} R_{abgd} - \Gamma^g{}_{df} R_{abcg}\right) \\
& + \Gamma^g{}_{ae} \left(\partial_f R_{gbcd} - \Gamma^h{}_{gf} R_{hbcd} - \Gamma^h{}_{bf} R_{ghcd} - \Gamma^h{}_{cf} R_{gbhd} - \Gamma^h{}_{df} R_{gbch}\right) \\
& + \Gamma^g{}_{be} \left(\partial_f R_{agcd} - \Gamma^h{}_{af} R_{hgcd} - \Gamma^h{}_{gf} R_{ahcd} - \Gamma^h{}_{cf} R_{aghd} - \Gamma^h{}_{df} R_{agch}\right) \\
& + \Gamma^g{}_{ce} \left(\partial_f R_{abgd} - \Gamma^h{}_{af} R_{hbgd} - \Gamma^h{}_{bf} R_{ahgd} - \Gamma^h{}_{gf} R_{abhd} - \Gamma^h{}_{df} R_{abgh}\right) \\
& + \Gamma^g{}_{de} \left(\partial_f R_{abcg} - \Gamma^h{}_{af} R_{hbcg} - \Gamma^h{}_{bf} R_{ahcg} - \Gamma^h{}_{cf} R_{abhg} - \Gamma^h{}_{gf} R_{abch}\right) \\
& + \Gamma^g{}_{fe} \left(\partial_g R_{abcd} - \Gamma^h{}_{ag} R_{hbcd} - \Gamma^h{}_{bg} R_{ahcd} - \Gamma^h{}_{cg} R_{abhd} - \Gamma^h{}_{dg} R_{abch}\right)
\end{aligned}
\tag{ex-0307.101}
$$

$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_{fe} R_{abcd} - \partial_f \left(\Gamma^g{}_{ae} R_{gbcd}\right) - \partial_f \left(\Gamma^g{}_{be} R_{agcd}\right) - \partial_f \left(\Gamma^g{}_{ce} R_{abgd}\right) - \partial_f \left(\Gamma^g{}_{de} R_{abcg}\right) - \Gamma^g{}_{af} \partial_e R_{gbcd} + \Gamma^g{}_{af} \Gamma^h{}_{ge} R_{hbcd} + \Gamma^g{}_{af} \Gamma^h{}_{be} R_{ghcd} \\
& + \Gamma^g{}_{af} \Gamma^h{}_{ce} R_{gbhd} + \Gamma^g{}_{af} \Gamma^h{}_{de} R_{gbch} - \Gamma^g{}_{bf} \partial_e R_{agcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ae} R_{hgcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ge} R_{ahcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ce} R_{aghd} + \Gamma^g{}_{bf} \Gamma^h{}_{de} R_{agch} \\
& - \Gamma^g{}_{cf} \partial_e R_{abgd} + \Gamma^g{}_{cf} \Gamma^h{}_{ae} R_{hbgd} + \Gamma^g{}_{cf} \Gamma^h{}_{be} R_{ahgd} + \Gamma^g{}_{cf} \Gamma^h{}_{ge} R_{abhd} + \Gamma^g{}_{cf} \Gamma^h{}_{de} R_{abgh} - \Gamma^g{}_{df} \partial_e R_{abcg} + \Gamma^g{}_{df} \Gamma^h{}_{ae} R_{hbcg} \\
& + \Gamma^g{}_{df} \Gamma^h{}_{be} R_{ahcg} + \Gamma^g{}_{df} \Gamma^h{}_{ce} R_{abhg} + \Gamma^g{}_{df} \Gamma^h{}_{ge} R_{abch} - \Gamma^g{}_{ef} \partial_g R_{abcd} + \Gamma^g{}_{ef} \Gamma^h{}_{ag} R_{hbcd} + \Gamma^g{}_{ef} \Gamma^h{}_{bg} R_{ahcd} + \Gamma^g{}_{ef} \Gamma^h{}_{cg} R_{abhd} \\
& + \Gamma^g{}_{ef} \Gamma^h{}_{dg} R_{abch} - \partial_{ef} R_{abcd} + \partial_e \left(\Gamma^g{}_{af} R_{gbcd}\right) + \partial_e \left(\Gamma^g{}_{bf} R_{agcd}\right) + \partial_e \left(\Gamma^g{}_{cf} R_{abgd}\right) + \partial_e \left(\Gamma^g{}_{df} R_{abcg}\right) + \Gamma^g{}_{ae} \partial_f R_{gbcd} - \Gamma^g{}_{ae} \Gamma^h{}_{gf} R_{hbcd} \\
& - \Gamma^g{}_{ae} \Gamma^h{}_{bf} R_{ghcd} - \Gamma^g{}_{ae} \Gamma^h{}_{cf} R_{gbhd} - \Gamma^g{}_{ae} \Gamma^h{}_{df} R_{gbch} + \Gamma^g{}_{be} \partial_f R_{agcd} - \Gamma^g{}_{be} \Gamma^h{}_{af} R_{hgcd} - \Gamma^g{}_{be} \Gamma^h{}_{gf} R_{ahcd} - \Gamma^g{}_{be} \Gamma^h{}_{cf} R_{aghd} \\
& - \Gamma^g{}_{be} \Gamma^h{}_{df} R_{agch} + \Gamma^g{}_{ce} \partial_f R_{abgd} - \Gamma^g{}_{ce} \Gamma^h{}_{af} R_{hbgd} - \Gamma^g{}_{ce} \Gamma^h{}_{bf} R_{ahgd} - \Gamma^g{}_{ce} \Gamma^h{}_{gf} R_{abhd} - \Gamma^g{}_{ce} \Gamma^h{}_{df} R_{abgh} + \Gamma^g{}_{de} \partial_f R_{abcg} \\
& - \Gamma^g{}_{de} \Gamma^h{}_{af} R_{hbcg} - \Gamma^g{}_{de} \Gamma^h{}_{bf} R_{ahcg} - \Gamma^g{}_{de} \Gamma^h{}_{cf} R_{abhg} - \Gamma^g{}_{de} \Gamma^h{}_{gf} R_{abch} + \Gamma^g{}_{fe} \partial_g R_{abcd} - \Gamma^g{}_{fe} \Gamma^h{}_{ag} R_{hbcd} - \Gamma^g{}_{fe} \Gamma^h{}_{bg} R_{ahcd} \\
& - \Gamma^g{}_{fe} \Gamma^h{}_{cg} R_{abhd} - \Gamma^g{}_{fe} \Gamma^h{}_{dg} R_{abch}
\end{aligned}
\tag{ex-0307.102}
$$

$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_{fe}R_{abcd} - \partial_f\Gamma^{g}{}_{ae}R_{gbcd} - \partial_f\Gamma^{g}{}_{be}R_{agcd} - \partial_f\Gamma^{g}{}_{ce}R_{abgd} - \partial_f\Gamma^{g}{}_{de}R_{abcg} + \Gamma^{g}{}_{af}\Gamma^{h}{}_{ge}R_{hbcd} + \Gamma^{g}{}_{af}\Gamma^{h}{}_{be}R_{ghcd} + \Gamma^{g}{}_{af}\Gamma^{h}{}_{ce}R_{gbhd} \\
& + \Gamma^{g}{}_{af}\Gamma^{h}{}_{de}R_{gbch} + \Gamma^{g}{}_{bf}\Gamma^{h}{}_{ae}R_{hgcd} + \Gamma^{g}{}_{bf}\Gamma^{h}{}_{ge}R_{ahcd} + \Gamma^{g}{}_{bf}\Gamma^{h}{}_{ce}R_{aghd} + \Gamma^{g}{}_{bf}\Gamma^{h}{}_{de}R_{agch} + \Gamma^{g}{}_{cf}\Gamma^{h}{}_{ae}R_{hbgd} + \Gamma^{g}{}_{cf}\Gamma^{h}{}_{be}R_{ahgd} \\
& + \Gamma^{g}{}_{cf}\Gamma^{h}{}_{ge}R_{abhd} + \Gamma^{g}{}_{cf}\Gamma^{h}{}_{de}R_{abgh} + \Gamma^{g}{}_{df}\Gamma^{h}{}_{ae}R_{hbcg} + \Gamma^{g}{}_{df}\Gamma^{h}{}_{be}R_{ahcg} + \Gamma^{g}{}_{df}\Gamma^{h}{}_{ce}R_{abhg} + \Gamma^{g}{}_{df}\Gamma^{h}{}_{ge}R_{abch} - \Gamma^{g}{}_{ef}\partial_g R_{abcd} \\
& + \Gamma^{g}{}_{ef}\Gamma^{h}{}_{ag}R_{hbcd} + \Gamma^{g}{}_{ef}\Gamma^{h}{}_{bg}R_{ahcd} + \Gamma^{g}{}_{ef}\Gamma^{h}{}_{cg}R_{abhd} + \Gamma^{g}{}_{ef}\Gamma^{h}{}_{dg}R_{abch} - \partial_{ef}R_{abcd} + \partial_e\Gamma^{g}{}_{af}R_{gbcd} + \partial_e\Gamma^{g}{}_{bf}R_{agcd} + \partial_e\Gamma^{g}{}_{cf}R_{abgd} \\
& + \partial_e\Gamma^{g}{}_{df}R_{abcg} - \Gamma^{g}{}_{ae}\Gamma^{h}{}_{gf}R_{hbcd} - \Gamma^{g}{}_{ae}\Gamma^{h}{}_{bf}R_{ghcd} - \Gamma^{g}{}_{ae}\Gamma^{h}{}_{cf}R_{gbhd} - \Gamma^{g}{}_{ae}\Gamma^{h}{}_{df}R_{gbch} - \Gamma^{g}{}_{be}\Gamma^{h}{}_{af}R_{hgcd} - \Gamma^{g}{}_{be}\Gamma^{h}{}_{gf}R_{ahcd} \\
& - \Gamma^{g}{}_{be}\Gamma^{h}{}_{cf}R_{aghd} - \Gamma^{g}{}_{be}\Gamma^{h}{}_{df}R_{agch} - \Gamma^{g}{}_{ce}\Gamma^{h}{}_{af}R_{hbgd} - \Gamma^{g}{}_{ce}\Gamma^{h}{}_{bf}R_{ahgd} - \Gamma^{g}{}_{ce}\Gamma^{h}{}_{gf}R_{abhd} - \Gamma^{g}{}_{ce}\Gamma^{h}{}_{df}R_{abgh} - \Gamma^{g}{}_{de}\Gamma^{h}{}_{af}R_{hbcg} \\
& - \Gamma^{g}{}_{de}\Gamma^{h}{}_{bf}R_{ahcg} - \Gamma^{g}{}_{de}\Gamma^{h}{}_{cf}R_{abhg} - \Gamma^{g}{}_{de}\Gamma^{h}{}_{gf}R_{abch} + \Gamma^{g}{}_{fe}\partial_g R_{abcd} - \Gamma^{g}{}_{fe}\Gamma^{h}{}_{ag}R_{hbcd} - \Gamma^{g}{}_{fe}\Gamma^{h}{}_{bg}R_{ahcd} - \Gamma^{g}{}_{fe}\Gamma^{h}{}_{cg}R_{abhd} \\
& - \Gamma^{g}{}_{fe}\Gamma^{h}{}_{dg}R_{abch}
\end{aligned}
\tag{ex-0307.103}
$$

$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_{fe}R_{abcd} - R_{gbcd}\partial_f\Gamma^{g}{}_{ae} - R_{agcd}\partial_f\Gamma^{g}{}_{be} - R_{abgd}\partial_f\Gamma^{g}{}_{ce} - R_{abcg}\partial_f\Gamma^{g}{}_{de} + R_{hbcd}\Gamma^{g}{}_{af}\Gamma^{h}{}_{ge} + R_{ghcd}\Gamma^{g}{}_{af}\Gamma^{h}{}_{be} + R_{gbhd}\Gamma^{g}{}_{af}\Gamma^{h}{}_{ce} \\
& + R_{gbch}\Gamma^{g}{}_{af}\Gamma^{h}{}_{de} + R_{hgcd}\Gamma^{g}{}_{bf}\Gamma^{h}{}_{ae} + R_{ahcd}\Gamma^{g}{}_{bf}\Gamma^{h}{}_{ge} + R_{aghd}\Gamma^{g}{}_{bf}\Gamma^{h}{}_{ce} + R_{agch}\Gamma^{g}{}_{bf}\Gamma^{h}{}_{de} + R_{hbgd}\Gamma^{g}{}_{cf}\Gamma^{h}{}_{ae} + R_{ahgd}\Gamma^{g}{}_{cf}\Gamma^{h}{}_{be} \\
& + R_{abhd}\Gamma^{g}{}_{cf}\Gamma^{h}{}_{ge} + R_{abgh}\Gamma^{g}{}_{cf}\Gamma^{h}{}_{de} + R_{hbcg}\Gamma^{g}{}_{df}\Gamma^{h}{}_{ae} + R_{ahcg}\Gamma^{g}{}_{df}\Gamma^{h}{}_{be} + R_{abhg}\Gamma^{g}{}_{df}\Gamma^{h}{}_{ce} + R_{abch}\Gamma^{g}{}_{df}\Gamma^{h}{}_{ge} - \Gamma^{g}{}_{ef}\partial_g R_{abcd} \\
& + R_{hbcd}\Gamma^{g}{}_{ef}\Gamma^{h}{}_{ag} + R_{ahcd}\Gamma^{g}{}_{ef}\Gamma^{h}{}_{bg} + R_{abhd}\Gamma^{g}{}_{ef}\Gamma^{h}{}_{cg} + R_{abch}\Gamma^{g}{}_{ef}\Gamma^{h}{}_{dg} - \partial_{ef}R_{abcd} + R_{gbcd}\partial_e\Gamma^{g}{}_{af} + R_{agcd}\partial_e\Gamma^{g}{}_{bf} + R_{abgd}\partial_e\Gamma^{g}{}_{cf} \\
& + R_{abcg}\partial_e\Gamma^{g}{}_{df} - R_{hbcd}\Gamma^{g}{}_{ae}\Gamma^{h}{}_{gf} - R_{ghcd}\Gamma^{g}{}_{ae}\Gamma^{h}{}_{bf} - R_{gbhd}\Gamma^{g}{}_{ae}\Gamma^{h}{}_{cf} - R_{gbch}\Gamma^{g}{}_{ae}\Gamma^{h}{}_{df} - R_{hgcd}\Gamma^{g}{}_{be}\Gamma^{h}{}_{af} - R_{ahcd}\Gamma^{g}{}_{be}\Gamma^{h}{}_{gf} \\
& - R_{aghd}\Gamma^{g}{}_{be}\Gamma^{h}{}_{cf} - R_{agch}\Gamma^{g}{}_{be}\Gamma^{h}{}_{df} - R_{hbgd}\Gamma^{g}{}_{ce}\Gamma^{h}{}_{af} - R_{ahgd}\Gamma^{g}{}_{ce}\Gamma^{h}{}_{bf} - R_{abhd}\Gamma^{g}{}_{ce}\Gamma^{h}{}_{gf} - R_{abgh}\Gamma^{g}{}_{ce}\Gamma^{h}{}_{df} - R_{hbcg}\Gamma^{g}{}_{de}\Gamma^{h}{}_{af} \\
& - R_{ahcg}\Gamma^{g}{}_{de}\Gamma^{h}{}_{bf} - R_{abhg}\Gamma^{g}{}_{de}\Gamma^{h}{}_{cf} - R_{abch}\Gamma^{g}{}_{de}\Gamma^{h}{}_{gf} + \Gamma^{g}{}_{fe}\partial_g R_{abcd} - R_{hbcd}\Gamma^{g}{}_{fe}\Gamma^{h}{}_{ag} - R_{ahcd}\Gamma^{g}{}_{fe}\Gamma^{h}{}_{bg} - R_{abhd}\Gamma^{g}{}_{fe}\Gamma^{h}{}_{cg} \\
& - R_{abch}\Gamma^{g}{}_{fe}\Gamma^{h}{}_{dg}
\end{aligned}
\tag{ex-0307.104}
$$

$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_{fe}R_{abcd} - R_{gbcd}\partial_f\Gamma^{g}{}_{ae} - R_{agcd}\partial_f\Gamma^{g}{}_{be} - R_{abgd}\partial_f\Gamma^{g}{}_{ce} - R_{abcg}\partial_f\Gamma^{g}{}_{de} + R_{gbcd}\Gamma^{h}{}_{af}\Gamma^{g}{}_{he} + R_{ghcd}\Gamma^{g}{}_{af}\Gamma^{h}{}_{be} + R_{gbhd}\Gamma^{g}{}_{af}\Gamma^{h}{}_{ce} \\
& + R_{gbch}\Gamma^{g}{}_{af}\Gamma^{h}{}_{de} + R_{ghcd}\Gamma^{h}{}_{bf}\Gamma^{g}{}_{ae} + R_{agcd}\Gamma^{h}{}_{bf}\Gamma^{g}{}_{he} + R_{aghd}\Gamma^{g}{}_{bf}\Gamma^{h}{}_{ce} + R_{agch}\Gamma^{g}{}_{bf}\Gamma^{h}{}_{de} + R_{gbhd}\Gamma^{h}{}_{cf}\Gamma^{g}{}_{ae} + R_{aghd}\Gamma^{h}{}_{cf}\Gamma^{g}{}_{be} \\
& + R_{abgd}\Gamma^{h}{}_{cf}\Gamma^{g}{}_{he} + R_{abgh}\Gamma^{g}{}_{cf}\Gamma^{h}{}_{de} + R_{gbch}\Gamma^{h}{}_{df}\Gamma^{g}{}_{ae} + R_{agch}\Gamma^{h}{}_{df}\Gamma^{g}{}_{be} + R_{abgh}\Gamma^{h}{}_{df}\Gamma^{g}{}_{ce} + R_{abcg}\Gamma^{h}{}_{df}\Gamma^{g}{}_{he} - \Gamma^{g}{}_{ef}\partial_g R_{abcd} \\
& + R_{gbcd}\Gamma^{h}{}_{ef}\Gamma^{g}{}_{ah} + R_{agcd}\Gamma^{h}{}_{ef}\Gamma^{g}{}_{bh} + R_{abgd}\Gamma^{h}{}_{ef}\Gamma^{g}{}_{ch} + R_{abcg}\Gamma^{h}{}_{ef}\Gamma^{g}{}_{dh} - \partial_{ef}R_{abcd} + R_{gbcd}\partial_e\Gamma^{g}{}_{af} + R_{agcd}\partial_e\Gamma^{g}{}_{bf} + R_{abgd}\partial_e\Gamma^{g}{}_{cf} \\
& + R_{abcg}\partial_e\Gamma^{g}{}_{df} - R_{gbcd}\Gamma^{h}{}_{ae}\Gamma^{g}{}_{hf} - R_{ghcd}\Gamma^{g}{}_{ae}\Gamma^{h}{}_{bf} - R_{gbhd}\Gamma^{g}{}_{ae}\Gamma^{h}{}_{cf} - R_{gbch}\Gamma^{g}{}_{ae}\Gamma^{h}{}_{df} - R_{ghcd}\Gamma^{h}{}_{be}\Gamma^{g}{}_{af} - R_{agcd}\Gamma^{h}{}_{be}\Gamma^{g}{}_{hf} \\
& - R_{aghd}\Gamma^{g}{}_{be}\Gamma^{h}{}_{cf} - R_{agch}\Gamma^{g}{}_{be}\Gamma^{h}{}_{df} - R_{gbhd}\Gamma^{h}{}_{ce}\Gamma^{g}{}_{af} - R_{aghd}\Gamma^{h}{}_{ce}\Gamma^{g}{}_{bf} - R_{abgd}\Gamma^{h}{}_{ce}\Gamma^{g}{}_{hf} - R_{abgh}\Gamma^{g}{}_{ce}\Gamma^{h}{}_{df} - R_{gbch}\Gamma^{h}{}_{de}\Gamma^{g}{}_{af} \\
& - R_{agch}\Gamma^{h}{}_{de}\Gamma^{g}{}_{bf} - R_{abgh}\Gamma^{h}{}_{de}\Gamma^{g}{}_{cf} - R_{abcg}\Gamma^{h}{}_{de}\Gamma^{g}{}_{hf} + \Gamma^{g}{}_{fe}\partial_g R_{abcd} - R_{gbcd}\Gamma^{h}{}_{fe}\Gamma^{g}{}_{ah} - R_{agcd}\Gamma^{h}{}_{fe}\Gamma^{g}{}_{bh} - R_{abgd}\Gamma^{h}{}_{fe}\Gamma^{g}{}_{ch} \\
& - R_{abcg}\Gamma^{h}{}_{fe}\Gamma^{g}{}_{dh}
\end{aligned}
\tag{ex-0307.105}
$$

$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & -R_{gbcd}\partial_f\Gamma^g{}_{ae} - R_{agcd}\partial_f\Gamma^g{}_{be} - R_{abgd}\partial_f\Gamma^g{}_{ce} - R_{abcg}\partial_f\Gamma^g{}_{de} + R_{gbcd}\Gamma^h{}_{af}\Gamma^g{}_{eh} + R_{agcd}\Gamma^h{}_{bf}\Gamma^g{}_{eh} + R_{abgd}\Gamma^h{}_{cf}\Gamma^g{}_{eh} + R_{abcg}\Gamma^h{}_{df}\Gamma^g{}_{eh} \\
& + R_{gbcd}\partial_e\Gamma^g{}_{af} + R_{agcd}\partial_e\Gamma^g{}_{bf} + R_{abgd}\partial_e\Gamma^g{}_{cf} + R_{abcg}\partial_e\Gamma^g{}_{df} - R_{gbcd}\Gamma^h{}_{ae}\Gamma^g{}_{fh} - R_{agcd}\Gamma^h{}_{be}\Gamma^g{}_{fh} - R_{abgd}\Gamma^h{}_{ce}\Gamma^g{}_{fh} \\
& - R_{abcg}\Gamma^h{}_{de}\Gamma^g{}_{fh} \qquad (\texttt{ex-0307.106})
\end{aligned}
$$

$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & R_{gbcd}\left(-\partial_f\Gamma^g{}_{ae} + \Gamma^h{}_{af}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{af} - \Gamma^h{}_{ae}\Gamma^g{}_{fh}\right) + R_{agcd}\left(-\partial_f\Gamma^g{}_{be} + \Gamma^h{}_{bf}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{bf} - \Gamma^h{}_{be}\Gamma^g{}_{fh}\right) \\
& + R_{abgd}\left(-\partial_f\Gamma^g{}_{ce} + \Gamma^h{}_{cf}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{cf} - \Gamma^h{}_{ce}\Gamma^g{}_{fh}\right) + R_{abcg}\left(-\partial_f\Gamma^g{}_{de} + \Gamma^h{}_{df}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{df} - \Gamma^h{}_{de}\Gamma^g{}_{fh}\right) \qquad (\texttt{ex-0307.107})
\end{aligned}
$$

$$
R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd}R^g{}_{afe} - R_{agcd}R^g{}_{bfe} - R_{abgd}R^g{}_{cfe} - R_{abcg}R^g{}_{dfe} \qquad (\texttt{ex-0307.108})
$$

$$
R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd}R^g{}_{aef} + R_{agcd}R^g{}_{bef} + R_{abgd}R^g{}_{cef} + R_{abcg}R^g{}_{def} \qquad (\texttt{ex-0307.109})
$$

## Exercise 3.8 Symmetry of $R_{ab}$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
                                                         # cdb (dgab.000,dgab)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                                             + \partial_{c}{g_{b e}}
                                             - \partial_{e}{g_{b c}}).
                                                         # cdb (Gamma.000,Gamma)

Rabcd := R^{a}_{b c d} ->
         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
       - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                         # cdb (Rabcd.000,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                         # cdb (Rab.000,Rab)

expr := 4 (R_{a b} - R_{b a}).                           # cdb (ex-0308.100,expr)

substitute   (expr, Rab)                                 # cdb (ex-0308.101,expr)
substitute   (expr, Rabcd)                               # cdb (ex-0308.102,expr)
substitute   (expr, Gamma)                               # cdb (ex-0308.103,expr)

distribute   (expr)                                      # cdb (ex-0308.104,expr)
product_rule (expr)                                      # cdb (ex-0308.105,expr)
canonicalise (expr)                                      # cdb (ex-0308.106,expr)

substitute   (expr, dgab)                                # cdb (ex-0308.107,expr)
canonicalise (expr)                                      # cdb (ex-0308.108,expr)
```

$$
\begin{align}
4R_{ab} - 4R_{ba} &= 4R^{c}{}_{acb} - 4R^{c}{}_{bca} \tag{ex-0308.101}\\
&= 4\partial_c \Gamma^{c}{}_{ab} + 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ab} - 4\partial_b \Gamma^{c}{}_{ac} - 4\Gamma^{c}{}_{eb}\Gamma^{e}{}_{ac} - 4\partial_c \Gamma^{c}{}_{ba} - 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ba} + 4\partial_a \Gamma^{c}{}_{bc} + 4\Gamma^{c}{}_{ea}\Gamma^{e}{}_{bc} \tag{ex-0308.102}\\
&= 2\partial_c \left(g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\right) + g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right) - 2\partial_b \left(g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right)\right) \nonumber\\
&\quad - g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right) g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) - 2\partial_c \left(g^{ce}\left(\partial_b g_{ea} + \partial_a g_{be} - \partial_e g_{ba}\right)\right) \nonumber\\
&\quad - g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_b g_{fa} + \partial_a g_{bf} - \partial_f g_{ba}\right) + 2\partial_a \left(g^{ce}\left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right)\right) \nonumber\\
&\quad + g^{cd}\left(\partial_e g_{da} + \partial_a g_{ed} - \partial_d g_{ea}\right) g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) \tag{ex-0308.103}\\
&= 2\partial_c\left(g^{ce}\partial_a g_{eb}\right) + 2\partial_c\left(g^{ce}\partial_b g_{ae}\right) - 2\partial_c\left(g^{ce}\partial_e g_{ab}\right) + g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ab} + g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{fb} \nonumber\\
&\quad + g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ab} - 2\partial_b\left(g^{ce}\partial_a g_{ec}\right) - 2\partial_b\left(g^{ce}\partial_c g_{ae}\right) \nonumber\\
&\quad + 2\partial_b\left(g^{ce}\partial_e g_{ac}\right) - g^{cd}\partial_e g_{db} g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db} g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db} g^{ef}\partial_f g_{ac} - g^{cd}\partial_b g_{ed} g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed} g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed} g^{ef}\partial_f g_{ac} \nonumber\\
&\quad + g^{cd}\partial_d g_{eb} g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb} g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb} g^{ef}\partial_f g_{ac} - 2\partial_c\left(g^{ce}\partial_b g_{ea}\right) - 2\partial_c\left(g^{ce}\partial_a g_{be}\right) + 2\partial_c\left(g^{ce}\partial_e g_{ba}\right) - g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{fa} \nonumber\\
&\quad - g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{bf} + g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ba} - g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{fa} - g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{bf} + g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ba} + g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{fa} + g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{bf} \nonumber\\
&\quad - g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ba} + 2\partial_a\left(g^{ce}\partial_b g_{ec}\right) + 2\partial_a\left(g^{ce}\partial_c g_{be}\right) - 2\partial_a\left(g^{ce}\partial_e g_{bc}\right) + g^{cd}\partial_e g_{da} g^{ef}\partial_b g_{fc} + g^{cd}\partial_e g_{da} g^{ef}\partial_c g_{bf} - g^{cd}\partial_e g_{da} g^{ef}\partial_f g_{bc} \nonumber\\
&\quad + g^{cd}\partial_a g_{ed} g^{ef}\partial_b g_{fc} + g^{cd}\partial_a g_{ed} g^{ef}\partial_c g_{bf} - g^{cd}\partial_a g_{ed} g^{ef}\partial_f g_{bc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_b g_{fc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_c g_{bf} + g^{cd}\partial_d g_{ea} g^{ef}\partial_f g_{bc} \tag{ex-0308.104}\\
&= 2\partial_c g^{ce}\partial_a g_{eb} + 2g^{ce}\partial_{ca} g_{eb} + 2\partial_c g^{ce}\partial_b g_{ae} + 2g^{ce}\partial_{cb} g_{ae} - 2\partial_c g^{ce}\partial_e g_{ab} - 2g^{ce}\partial_{ce} g_{ab} + g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ab} \nonumber\\
&\quad + g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{fb} + g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ab} - 2\partial_b g^{ce}\partial_a g_{ec} \nonumber\\
&\quad - 2g^{ce}\partial_{ba} g_{ec} - 2\partial_b g^{ce}\partial_c g_{ae} - 2g^{ce}\partial_{bc} g_{ae} + 2\partial_b g^{ce}\partial_e g_{ac} + 2g^{ce}\partial_{be} g_{ac} - g^{cd}\partial_e g_{db} g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db} g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db} g^{ef}\partial_f g_{ac} \nonumber\\
&\quad - g^{cd}\partial_b g_{ed} g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed} g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed} g^{ef}\partial_f g_{ac} + g^{cd}\partial_d g_{eb} g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb} g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb} g^{ef}\partial_f g_{ac} - 2\partial_c g^{ce}\partial_b g_{ea} \nonumber\\
&\quad - 2g^{ce}\partial_{cb} g_{ea} - 2\partial_c g^{ce}\partial_a g_{be} - 2g^{ce}\partial_{ca} g_{be} + 2\partial_c g^{ce}\partial_e g_{ba} + 2g^{ce}\partial_{ce} g_{ba} - g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{fa} - g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{bf} + g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ba} \nonumber\\
&\quad - g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{fa} - g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{bf} + g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ba} + g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{fa} + g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{bf} - g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ba} + 2\partial_a g^{ce}\partial_b g_{ec} \nonumber\\
&\quad + 2g^{ce}\partial_{ab} g_{ec} + 2\partial_a g^{ce}\partial_c g_{be} + 2g^{ce}\partial_{ac} g_{be} - 2\partial_a g^{ce}\partial_e g_{bc} - 2g^{ce}\partial_{ae} g_{bc} + g^{cd}\partial_e g_{da} g^{ef}\partial_b g_{fc} + g^{cd}\partial_e g_{da} g^{ef}\partial_c g_{bf} - g^{cd}\partial_e g_{da} g^{ef}\partial_f g_{bc} \nonumber\\
&\quad + g^{cd}\partial_a g_{ed} g^{ef}\partial_b g_{fc} + g^{cd}\partial_a g_{ed} g^{ef}\partial_c g_{bf} - g^{cd}\partial_a g_{ed} g^{ef}\partial_f g_{bc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_b g_{fc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_c g_{bf} + g^{cd}\partial_d g_{ea} g^{ef}\partial_f g_{bc} \tag{ex-0308.105}\\
&= -2\partial_b g^{ce}\partial_a g_{ce} + 2\partial_a g^{ce}\partial_b g_{ce} \tag{ex-0308.106}\\
&= 2g^{cd}g^{ef}\partial_b g_{df}\partial_a g_{ce} - 2g^{cd}g^{ef}\partial_a g_{df}\partial_b g_{ce} \tag{ex-0308.107}\\
&= 0 \tag{ex-0308.108}
\end{align}
$$

## Exercise 3.8 Symmetry of $R_{ab}$ alternative solution

This differs from the previous code by the inclusion of a call to `canonicalise` immediately after the first two substitutions and a declaration that ${\Gamma^a}_{bc}$ is symmetric in $bc$. This pair of changes produces a more compact set of results than given above. Incidently, this also shows that $\partial_a {\Gamma^c}_{bc} = \partial_b {\Gamma^c}_{ac}$.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
                                                       # cdb (dgab.000,dgab)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                                             + \partial_{c}{g_{b e}}
                                             - \partial_{e}{g_{b c}}).
                                                       # cdb (Gamma.000,Gamma)

Rabcd := R^{a}_{b c d} ->
          \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
        - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                       # cdb (Rabcd.000,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                       # cdb (Rab.000,Rab)

expr := 4 (R_{a b} - R_{b a}).                         # cdb (ex-0308.200,expr)

substitute   (expr, Rab)                               # cdb (ex-0308.201,expr)
substitute   (expr, Rabcd)                             # cdb (ex-0308.202,expr)
canonicalise (expr)                                    # cdb (ex-0308.203,expr)
substitute   (expr, Gamma)                             # cdb (ex-0308.204,expr)

distribute   (expr)                                    # cdb (ex-0308.205,expr)
```

```
product_rule (expr)                                    # cdb (ex-0308.206,expr)
canonicalise (expr)                                    # cdb (ex-0308.207,expr)

substitute   (expr, dgab)                              # cdb (ex-0308.208,expr)
canonicalise (expr)                                    # cdb (ex-0308.209,expr)
```

$$
\begin{aligned}
4R_{ab} - 4R_{ba} &= 4R^{c}{}_{acb} - 4R^{c}{}_{bca} && \texttt{(ex-0308.201)}\\
&= 4\partial_{c}\Gamma^{c}{}_{ab} + 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ab} - 4\partial_{b}\Gamma^{c}{}_{ac} - 4\Gamma^{c}{}_{eb}\Gamma^{e}{}_{ac} - 4\partial_{c}\Gamma^{c}{}_{ba} - 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ba} + 4\partial_{a}\Gamma^{c}{}_{bc} + 4\Gamma^{c}{}_{ea}\Gamma^{e}{}_{bc} && \texttt{(ex-0308.202)}\\
&= -4\partial_{b}\Gamma^{c}{}_{ac} + 4\partial_{a}\Gamma^{c}{}_{bc} && \texttt{(ex-0308.203)}\\
&= -2\partial_{b}\left(g^{ce}\left(\partial_{a}g_{ec} + \partial_{c}g_{ae} - \partial_{e}g_{ac}\right)\right) + 2\partial_{a}\left(g^{ce}\left(\partial_{b}g_{ec} + \partial_{c}g_{be} - \partial_{e}g_{bc}\right)\right) && \texttt{(ex-0308.204)}\\
&= -2\partial_{b}\left(g^{ce}\partial_{a}g_{ec}\right) - 2\partial_{b}\left(g^{ce}\partial_{c}g_{ae}\right) + 2\partial_{b}\left(g^{ce}\partial_{e}g_{ac}\right) + 2\partial_{a}\left(g^{ce}\partial_{b}g_{ec}\right) + 2\partial_{a}\left(g^{ce}\partial_{c}g_{be}\right) - 2\partial_{a}\left(g^{ce}\partial_{e}g_{bc}\right) && \texttt{(ex-0308.205)}\\
&= -2\partial_{b}g^{ce}\partial_{a}g_{ec} - 2g^{ce}\partial_{ba}g_{ec} - 2\partial_{b}g^{ce}\partial_{c}g_{ae} - 2g^{ce}\partial_{bc}g_{ae} + 2\partial_{b}g^{ce}\partial_{e}g_{ac} + 2g^{ce}\partial_{be}g_{ac} + 2\partial_{a}g^{ce}\partial_{b}g_{ec} + 2g^{ce}\partial_{ab}g_{ec} + 2\partial_{a}g^{ce}\partial_{c}g_{be}\\
&\quad + 2g^{ce}\partial_{ac}g_{be} - 2\partial_{a}g^{ce}\partial_{e}g_{bc} - 2g^{ce}\partial_{ae}g_{bc} && \texttt{(ex-0308.206)}\\
&= -2\partial_{b}g^{ce}\partial_{a}g_{ce} + 2\partial_{a}g^{ce}\partial_{b}g_{ce} && \texttt{(ex-0308.207)}\\
&= 2g^{cd}g^{ef}\partial_{b}g_{df}\partial_{a}g_{ce} - 2g^{cd}g^{ef}\partial_{a}g_{df}\partial_{b}g_{ce} && \texttt{(ex-0308.208)}\\
&= 0 && \texttt{(ex-0308.209)}
\end{aligned}
$$

## Exercise 3.9 Ricci in terms of the metric and its derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.      # cdb (ex-0309.dgab,dgab)

Gamma := \Gamma^{a}_{b c} ->
         (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                         + \partial_{c}{g_{b e}}
                         - \partial_{e}{g_{b c}}).                              # cdb (ex-0309.Gamma,Gamma)

Rabcd := R^{a}_{b c d} ->
         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
       - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.   # cdb (ex-0309.Rabcd,Rabcd)

FourRab := 4 R^{c}_{a c b}.                        # cdb (ex-0309.101,FourRab)

substitute     (FourRab, Rabcd)                    # cdb (ex-0309.102,FourRab)
substitute     (FourRab, Gamma)                    # cdb (ex-0309.103,FourRab)

product_rule   (FourRab)                           # cdb (ex-0309.104,FourRab)
distribute     (FourRab)                           # cdb (ex-0309.105,FourRab)

substitute     (FourRab, dgab)                     # cdb (ex-0309.106,FourRab)

sort_product   (FourRab)                           # cdb (ex-0309.107,FourRab)
rename_dummies (FourRab)                           # cdb (ex-0309.108,FourRab)
canonicalise   (FourRab)                           # cdb (ex-0309.109,FourRab)

# sort so that g to appearees before dg

substitute     (FourRab, $g^{a b} -> A^{a b}$)
sort_product   (FourRab)
```

```
rename_dummies (FourRab)
substitute     (FourRab, $A^{a b} -> g^{a b}$)    # cdb (ex-0309.110,FourRab)
```

$$
\begin{aligned}
4R_{ab} &= 4{R^c}_{acb} \qquad \texttt{(ex-0309.101)}\\
&= 4\partial_c {\Gamma^c}_{ab} + 4{\Gamma^c}_{ec}{\Gamma^e}_{ab} - 4\partial_b {\Gamma^c}_{ac} - 4{\Gamma^c}_{eb}{\Gamma^e}_{ac} \qquad \texttt{(ex-0309.102)}\\
&= 2\partial_c \left(g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\right) + g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right) - 2\partial_b \left(g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right)\right)\\
&\quad - g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right) g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) \qquad \texttt{(ex-0309.103)}\\
&= 2\partial_c g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right) + 2g^{ce}\partial_c\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right) + g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right)\\
&\quad - 2\partial_b g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - 2g^{ce}\partial_b\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right) g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) \qquad \texttt{(ex-0309.104)}\\
&= 2\partial_c g^{ce}\partial_a g_{eb} + 2\partial_c g^{ce}\partial_b g_{ae} - 2\partial_c g^{ce}\partial_e g_{ab} + 2g^{ce}\partial_{ca} g_{eb} + 2g^{ce}\partial_{cb} g_{ae} - 2g^{ce}\partial_{ce} g_{ab} + g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ab}\\
&\quad + g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{fb} + g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ab} - 2\partial_b g^{ce}\partial_a g_{ec}\\
&\quad - 2\partial_b g^{ce}\partial_c g_{ae} + 2\partial_b g^{ce}\partial_e g_{ac} - 2g^{ce}\partial_{ba} g_{ec} - 2g^{ce}\partial_{bc} g_{ae} + 2g^{ce}\partial_{be} g_{ac} - g^{cd}\partial_e g_{db} g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db} g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db} g^{ef}\partial_f g_{ac}\\
&\quad - g^{cd}\partial_b g_{ed} g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed} g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed} g^{ef}\partial_f g_{ac} + g^{cd}\partial_d g_{eb} g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb} g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb} g^{ef}\partial_f g_{ac} \qquad \texttt{(ex-0309.105)}\\
&= -2g^{cd}g^{ef}\partial_c g_{df}\partial_a g_{eb} - 2g^{cd}g^{ef}\partial_c g_{df}\partial_b g_{ae} + 2g^{cd}g^{ef}\partial_c g_{df}\partial_e g_{ab} + 2g^{ce}\partial_{ca} g_{eb} + 2g^{ce}\partial_{cb} g_{ae} - 2g^{ce}\partial_{ce} g_{ab} + g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{af}\\
&\quad - g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ab} + g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{fb} + g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ab}\\
&\quad + 2g^{cd}g^{ef}\partial_b g_{df}\partial_a g_{ec} + 2g^{cd}g^{ef}\partial_b g_{df}\partial_c g_{ae} - 2g^{cd}g^{ef}\partial_b g_{df}\partial_e g_{ac} - 2g^{ce}\partial_{ba} g_{ec} - 2g^{ce}\partial_{bc} g_{ae} + 2g^{ce}\partial_{be} g_{ac} - g^{cd}\partial_e g_{db} g^{ef}\partial_a g_{fc}\\
&\quad - g^{cd}\partial_e g_{db} g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db} g^{ef}\partial_f g_{ac} - g^{cd}\partial_b g_{ed} g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed} g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed} g^{ef}\partial_f g_{ac} + g^{cd}\partial_d g_{eb} g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb} g^{ef}\partial_c g_{af}\\
&\quad - g^{cd}\partial_d g_{eb} g^{ef}\partial_f g_{ac} \qquad \texttt{(ex-0309.106)}\\
&= -2\partial_a g_{eb}\partial_c g_{df} g^{cd}g^{ef} - 2\partial_b g_{ae}\partial_c g_{df} g^{cd}g^{ef} + 2\partial_c g_{df}\partial_e g_{ab} g^{cd}g^{ef} + 2\partial_{ca} g_{eb} g^{ce} + 2\partial_{cb} g_{ae} g^{ce} - 2\partial_{ce} g_{ab} g^{ce} + \partial_a g_{fb}\partial_e g_{dc} g^{cd}g^{ef} + \partial_b g_{af}\partial_e g_{dc} g^{cd}g^{ef}\\
&\quad - \partial_e g_{dc}\partial_f g_{ab} g^{cd}g^{ef} + \partial_a g_{fb}\partial_c g_{ed} g^{cd}g^{ef} + \partial_b g_{af}\partial_c g_{ed} g^{cd}g^{ef} - \partial_c g_{ed}\partial_f g_{ab} g^{cd}g^{ef} - \partial_a g_{fb}\partial_d g_{ec} g^{cd}g^{ef} - \partial_b g_{af}\partial_d g_{ec} g^{cd}g^{ef} + \partial_d g_{ec}\partial_f g_{ab} g^{cd}g^{ef}\\
&\quad + 2\partial_a g_{ec}\partial_b g_{df} g^{cd}g^{ef} + 2\partial_b g_{df}\partial_c g_{ae} g^{cd}g^{ef} - 2\partial_b g_{df}\partial_e g_{ac} g^{cd}g^{ef} - 2\partial_{ba} g_{ec} g^{ce} - 2\partial_{bc} g_{ae} g^{ce} + 2\partial_{be} g_{ac} g^{ce} - \partial_a g_{fc}\partial_e g_{db} g^{cd}g^{ef}\\
&\quad - \partial_c g_{af}\partial_e g_{db} g^{cd}g^{ef} + \partial_e g_{db}\partial_f g_{ac} g^{cd}g^{ef} - \partial_a g_{fc}\partial_b g_{ed} g^{cd}g^{ef} - \partial_b g_{ed}\partial_c g_{af} g^{cd}g^{ef} + \partial_b g_{ed}\partial_f g_{ac} g^{cd}g^{ef} + \partial_a g_{fc}\partial_d g_{eb} g^{cd}g^{ef} + \partial_c g_{af}\partial_d g_{eb} g^{cd}g^{ef}\\
&\quad - \partial_d g_{eb}\partial_f g_{ac} g^{cd}g^{ef} \qquad \texttt{(ex-0309.107)}\\
&= -2\partial_a g_{db}\partial_c g_{ef} g^{ce}g^{df} - 2\partial_b g_{ad}\partial_c g_{ef} g^{ce}g^{df} + 2\partial_c g_{ef}\partial_d g_{ab} g^{ce}g^{df} + 2\partial_{ca} g_{db} g^{cd} + 2\partial_{cb} g_{ad} g^{cd} - 2\partial_{cd} g_{ab} g^{cd} + \partial_a g_{db}\partial_c g_{ef} g^{fe}g^{cd} + \partial_b g_{ad}\partial_c g_{ef} g^{fe}g^{cd}\\
&\quad - \partial_c g_{ef}\partial_d g_{ab} g^{fe}g^{cd} + \partial_a g_{db}\partial_c g_{ef} g^{cf}g^{ed} + \partial_b g_{ad}\partial_c g_{ef} g^{cf}g^{ed} - \partial_c g_{ef}\partial_d g_{ab} g^{cf}g^{ed} - \partial_a g_{db}\partial_c g_{ef} g^{fc}g^{ed} - \partial_b g_{ad}\partial_c g_{ef} g^{fc}g^{ed} + \partial_c g_{ef}\partial_d g_{ab} g^{fc}g^{ed}\\
&\quad + 2\partial_a g_{cd}\partial_b g_{ef} g^{de}g^{cf} + 2\partial_b g_{de}\partial_c g_{af} g^{cd}g^{fe} - 2\partial_b g_{de}\partial_c g_{af} g^{fd}g^{ce} - 2\partial_{ba} g_{cd} g^{dc} - 2\partial_{bc} g_{ad} g^{cd} + 2\partial_{bc} g_{ad} g^{dc} - \partial_a g_{de}\partial_c g_{fb} g^{ef}g^{cd}\\
&\quad - \partial_c g_{ae}\partial_d g_{fb} g^{cf}g^{de} + \partial_c g_{eb}\partial_d g_{af} g^{fe}g^{cd} - \partial_a g_{cd}\partial_b g_{ef} g^{df}g^{ec} - \partial_b g_{de}\partial_c g_{af} g^{ce}g^{df} + \partial_b g_{de}\partial_c g_{af} g^{fe}g^{dc} + \partial_a g_{de}\partial_c g_{fb} g^{ec}g^{fd} + \partial_c g_{ae}\partial_d g_{fb} g^{cd}g^{fe}\\
&\quad - \partial_c g_{eb}\partial_d g_{af} g^{fc}g^{ed} \qquad \texttt{(ex-0309.108)}\\
&= -2\partial_a g_{bc}\partial_d g_{ef} g^{ce}g^{df} - 2\partial_b g_{ac}\partial_d g_{ef} g^{ce}g^{df} + 2\partial_c g_{ab}\partial_d g_{ef} g^{ce}g^{df} + 2\partial_{ac} g_{bd} g^{cd} + 2\partial_{bc} g_{ad} g^{cd} - 2\partial_{cd} g_{ab} g^{cd} + \partial_a g_{bc}\partial_d g_{ef} g^{cd}g^{ef} + \partial_b g_{ac}\partial_d g_{ef} g^{cd}g^{ef}\\
&\quad - \partial_c g_{ab}\partial_d g_{ef} g^{cd}g^{ef} + \partial_a g_{cd}\partial_b g_{ef} g^{ce}g^{df} - 2\partial_{ab} g_{cd} g^{cd} - 2\partial_c g_{ad}\partial_e g_{bf} g^{cf}g^{de} + 2\partial_c g_{ad}\partial_e g_{bf} g^{ce}g^{df} \qquad \texttt{(ex-0309.109)}\\
&= -2g^{cd}g^{ef}\partial_a g_{bc}\partial_e g_{df} - 2g^{cd}g^{ef}\partial_b g_{ac}\partial_e g_{df} + 2g^{cd}g^{ef}\partial_c g_{ab}\partial_e g_{df} + 2g^{cd}\partial_{ac} g_{bd} + 2g^{cd}\partial_{bc} g_{ad} - 2g^{cd}\partial_{cd} g_{ab} + g^{cd}g^{ef}\partial_a g_{bc}\partial_d g_{ef} + g^{cd}g^{ef}\partial_b g_{ac}\partial_d g_{ef}\\
&\quad - g^{cd}g^{ef}\partial_c g_{ab}\partial_d g_{ef} + g^{cd}g^{ef}\partial_a g_{ce}\partial_b g_{df} - 2g^{cd}\partial_{ab} g_{cd} - 2g^{cd}g^{ef}\partial_c g_{ae}\partial_f g_{bd} + 2g^{cd}g^{ef}\partial_c g_{ae}\partial_d g_{bf} \qquad \texttt{(ex-0309.110)}
\end{aligned}
$$

# Exercise 3.10 Example of `repeat=True` in a substitution

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent).

foo := A B + A B A B + A B A B A B + A B A B A B A B .   # cdb(ex-0310.foo.001,foo)
bah := @(foo).                                           # cdb(ex-0310.bah.001,bah)

substitute (foo,$A B -> A$)                              # cdb(ex-0310.foo.002,foo)
substitute (bah,$A B -> A$,repeat=True)                  # cdb(ex-0310.bah.002,bah)
```

Without `repeat=True` only the first match in a product will be susbstituted.

$$\texttt{ex-0310.foo.001} := AB + ABAB + ABABAB + ABABABAB$$
$$\texttt{ex-0310.foo.002} := A + AAB + AABAB + AABABAB$$

But with `repeat=True` then all matches in a product will be susbstituted.

$$\texttt{ex-0310.bah.001} := AB + ABAB + ABABAB + ABABABAB$$
$$\texttt{ex-0310.bah.002} := A + AA + AAA + AAAA$$

# Exercise 4.1 Differentiate a polynomial – a limited method

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

def deriv (poly):

    \delta^{a}::Weight(label=\epsilon).

    bah := @(poly).

    substitute     (bah,$x^{a} -> x^{a} + \delta^{a}$)
    distribute     (bah)

    foo := @(bah) - @(poly).

    keep_weight    (foo, $\epsilon = 1$)
    sort_product   (foo)
    rename_dummies (foo)
    factor_out     (foo, $\delta^{a?}$)
    substitute     (foo, $\delta^{a} -> 1$)

    return foo

# --------------------------------------------------------------

poly := c^{a}
      + c^{a}{}_{b} x^b
      + c^{a}{}_{b c} x^b x^c.    # cdb (ex-0401.100,poly)

dpoly = deriv (poly)              # cdb (ex-0401.101,dpoly)
```

$$p = c^{a} + c^{a}{}_{b}x^{b} + c^{a}{}_{bc}x^{b}x^{c} \qquad \text{(ex-0401.100)}$$

$$dp = c^{a}{}_{b} + c^{a}{}_{cb}x^{c} + c^{a}{}_{bc}x^{c} \qquad \text{(ex-0401.101)}$$

# Exercise 4.1 Differentiate a polynomial – a better method

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

def deriv (poly):

    \partial{#}::PartialDerivative.
    \delta^{a}_{b}::KroneckerDelta.

    x^{a}::Depends(\partial{#}).

    bah := \partial_{b}{@(poly)}.

    distribute     (bah)
    unwrap         (bah)  # drop all terms that don't explicitly depend on a derivative operator
    product_rule   (bah)
    distribute     (bah)
    substitute     (bah,$\partial_{b}{x^{a}}->\delta^{a}_{b}$)
    eliminate_kronecker (bah)

    sort_product   (bah)
    rename_dummies (bah)

    return bah

poly := c^{a}
      + c^{a}{}_{b} x^b
      + c^{a}{}_{b c} x^b x^c.    # cdb (ex-0401.200,poly)

dpoly = deriv (poly)              # cdb (ex-0401.201,dpoly)
```

$$p = c^{a} + c^{a}{}_{b}x^{b} + c^{a}{}_{bc}x^{b}x^{c} \qquad \text{(ex-0401.200)}$$

$$dp = c^{a}{}_{b} + c^{a}{}_{bc}x^{c} + c^{a}{}_{cb}x^{c} \qquad \text{(ex-0401.201)}$$

# Exercise 4.2 Inconsistent free indices

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

def deriv (poly):

    \delta^{a}::Weight(label=\epsilon).

    bah := @(poly).

    substitute     (bah,$x^{a} -> x^{a} + \delta^{a}$)
    distribute     (bah)

    foo := @(bah) - @(poly).

    keep_weight    (foo, $\epsilon = 1$)
    substitute     (foo, $\delta^{a} -> 1$)

    return foo

# ---------------------------------------------------------------

poly := c^{a}
      + c^{a}{}_{b} x^b
      + c^{a}{}_{b c} x^b x^c.    # cdb (ex-0402.100,poly)

dpoly = deriv (poly)              # cdb (ex-0402.101,dpoly)
```

$$p = c^{a} + c^{a}{}_{b}x^{b} + c^{a}{}_{bc}x^{b}x^{c} \tag{ex-0402.100}$$

$$dp = c^{a}{}_{b} + c^{a}{}_{bc}x^{b} + c^{a}{}_{bc}x^{c} \tag{ex-0402.101}$$

# Exercise 4.3 Polynomial products

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

def get_term (poly,n):

    x^{a}::Weight(label=xnum).     # assign weights to x^{a}

    foo := @(poly).                # make a copy of poly
    bah  = Ex("xnum = " + str(n))  # choose a target
    keep_weight (foo,bah)          # extract the target

    return foo

def poly_product (p,q,n):

    pq = Ex("0")

    for i in range (0,n+1):
       for j in range (0,i+1):
          termA = get_term (p,j)
          termB = get_term (q,i-j)
          termAB := @(termA) @(termB).
          pq = pq + termAB

    sort_product   (pq)
    rename_dummies (pq)
    factor_out     (pq,$x^{a?}$)

    return pq

# ---------------------------------------------------------------

# two polynomials

polyA := c^{a}
       + c^{a}_{b} x^b
```

```
        + c^{a}_{b c} x^b x^c
        + c^{a}_{b c d} x^b x^c x^d
        + c^{a}_{b c d e} x^b x^c x^d x^e.    # cdb(ex-0403.100,polyA)

polyB := d^{f}
        + d^{f}_{b} x^b
        + d^{f}_{b c} x^b x^c
        + d^{f}_{b c d} x^b x^c x^d
        + d^{f}_{b c d e} x^b x^c x^d x^e.    # cdb(ex-0403.101,polyB)

# multiply polynomials and truncate

polyAB = poly_product (polyA,polyB,3)        # cdb(ex-0403.102,polyAB)
```

$$p = c^{a} + c^{a}{}_{b}x^{b} + c^{a}{}_{bc}x^{b}x^{c} + c^{a}{}_{bcd}x^{b}x^{c}x^{d} + c^{a}{}_{bcde}x^{b}x^{c}x^{d}x^{e} \quad (\texttt{ex-0403.100})$$

$$q = d^{f} + d^{f}{}_{b}x^{b} + d^{f}{}_{bc}x^{b}x^{c} + d^{f}{}_{bcd}x^{b}x^{c}x^{d} + d^{f}{}_{bcde}x^{b}x^{c}x^{d}x^{e} \quad (\texttt{ex-0403.101})$$

$$pq = c^{a}d^{f} + x^{b}\left(c^{a}d^{f}{}_{b} + c^{a}{}_{b}d^{f}\right) + x^{b}x^{c}\left(c^{a}d^{f}{}_{bc} + c^{a}{}_{b}d^{f}{}_{c} + c^{a}{}_{bc}d^{f}\right) + x^{b}x^{c}x^{d}\left(c^{a}d^{f}{}_{bcd} + c^{a}{}_{b}d^{f}{}_{cd} + c^{a}{}_{bc}d^{f}{}_{d} + c^{a}{}_{bcd}d^{f}\right) \quad (\texttt{ex-0403.102})$$

# Exercise 4.4 Reformatting simple expressions

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

def reformat (obj,scale):

    {x^{a},A_{a b},B_{a b},C_{a b},g^{a b}}::SortOrder.  # choose a sort order

    foo  = Ex(str(scale))          # create a scale factor
    bah := @(foo) @(obj).          # apply the scale factor, clears all fractions

    distribute     (bah)           # only required if (bah) contains brackets
    sort_product   (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    factor_out     (bah,$x^{a?}$)

    ans := @(bah) / @(foo).        # undo previous scaling

    return ans

# ----------------------------------------------------------------

# a messy unformatted expression

expr := + (1/3) A_{a b} x^{a} x^{b}
        + (1/9) B_{e c} x^{c} x^{e}
        - (1/5) C_{p c} B_{d q} g^{c d} x^{p} x^{q}.  # cdb (ex-0404.100,expr)

# reformat terms and tidy fractions

expr = reformat (expr,45)                             # cdb(ex-0404.101,expr)
```

$$g = \frac{1}{3}A_{ab}x^a x^b + \frac{1}{9}B_{ec}x^c x^e - \frac{1}{5}C_{pc}B_{dq}g^{cd}x^p x^q \qquad (\texttt{ex-0404.100})$$

$$= \frac{1}{45}x^a x^b \left(15A_{ab} + 5B_{ab} - 9B_{ca}C_{bd}g^{dc}\right) \qquad (\texttt{ex-0404.101})$$

## Exercise 4.5 Reformatting complex expressions

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

def get_term (obj,n):

    x^{a}::Weight(label=xnum).     # assign weights to x^{a}

    foo := @(obj).                 # make a copy of obj
    bah  = Ex("xnum = " + str(n))  # choose a target
    keep_weight (foo,bah)          # extract the target

    return foo

def reformat (obj,scale):

    {x^{a},A_{a},B_{a},A_{a b},B_{a b},C_{a b},C_{a b c},g^{a b}}::SortOrder.  # choose a sort order

    foo  = Ex(str(scale))          # create a scale factor
    bah := @(foo) @(obj).          # apply the scale factor, clears all fractions

    distribute     (bah)           # only required if (bah) contains brackets
    sort_product   (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    factor_out     (bah,$x^{a?}$)

    ans := @(bah) / @(foo).        # undo previous scaling

    return ans

# ---------------------------------------------------------------

# a messy unformatted expression

```

```
expr :=    (1/7) A_{e} x^{e}
        -  (1/3) B_{f} x^{f}
        +  (1/3) A_{a b} x^{a} x^{b}
        +  (1/9) B_{e c} x^{c} x^{e}
        -  (1/5) C_{p c} B_{d q} g^{c d} x^{p} x^{q}
        +  (3/7) A_{a b c} x^{a} x^{b} x^{c}
        -  (1/5) B_{a b} C_{c d e} g^{c d} x^{a} x^{b} x^{e}
        + (7/11) B_{a b} B_{c d} C_{e f g} g^{b c} g^{d f} x^{a} x^{e} x^{g}.  # cdb (ex-0405.100,expr)

# split the expression into seprate terms

term1 = get_term (expr,1)        # cdb(term1.101,term1)
term2 = get_term (expr,2)        # cdb(term2.101,term2)
term3 = get_term (expr,3)        # cdb(term3.101,term3)

# reformat terms and tidy fractions

term1 = reformat (term1, 21)     # cdb(term1.102,term1)
term2 = reformat (term2, 45)     # cdb(term2.102,term2)
term3 = reformat (term3,385)     # cdb(term3.102,term3)

# rebuild the expression

expr := @(term1) + @(term2) + @(term3).  # cdb (ex-0405.101,expr)
```

$$g = \frac{1}{7}A_e x^e - \frac{1}{3}B_f x^f + \frac{1}{3}A_{ab}x^a x^b + \frac{1}{9}B_{ec}x^c x^e - \frac{1}{5}C_{pc}B_{dq}g^{cd}x^p x^q + \frac{3}{7}A_{abc}x^a x^b x^c - \frac{1}{5}B_{ab}C_{cde}g^{cd}x^a x^b x^e + \frac{7}{11}B_{ab}B_{cd}C_{efg}g^{bc}g^{df}x^a x^e x^g \quad \texttt{(ex-0405.100)}$$

$$= \frac{1}{21}x^a\left(3A_a - 7B_a\right) + \frac{1}{45}x^a x^b\left(15A_{ab} + 5B_{ab} - 9B_{ca}C_{bd}g^{dc}\right) + \frac{1}{385}x^a x^b x^c\left(165A_{abc} - 77B_{ab}C_{dec}g^{de} + 245B_{ad}B_{ef}C_{bgc}g^{de}g^{fg}\right) \quad \texttt{(ex-0405.101)}$$

# Exercise 4.6 Bespoke sort

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

def bespoke_sort (expr):

    substitute     (expr,$ x^{a}            -> AAA01^{a}      $)
    substitute     (expr,$ g_{a b}          -> AAA02_{a b}    $)
    substitute     (expr,$ \Gamma_{a b c}   -> AAA03_{a b c}  $)

    sort_product   (expr)

    substitute     (expr,$ AAA01^{a}        -> x^{a}          $)
    substitute     (expr,$ AAA02_{a b}      -> g_{a b}        $)
    substitute     (expr,$ AAA03_{a b c}    -> \Gamma_{a b c} $)

    return expr

# ---------------------------------------------------------------

expr := g_{a b} x^{a} x^{b} + \Gamma_{a b c} x^{a} x^{b} x^{c}.  # cdb(ex-0406.100,expr)

expr = bespoke_sort (expr)                                      # cdb(ex-0406.101,expr)
```

$$
\begin{aligned}
p &= g_{ab}x^a x^b + \Gamma_{abc}x^a x^b x^c && \text{(ex-0406.100)}\\
&= x^a x^b g_{ab} + x^a x^b x^c \Gamma_{abc} && \text{(ex-0406.101)}
\end{aligned}
$$

## Exercise 4.7 Return in functions

```
{a,b,c,d,e,f,g,h,i,j,k,l#}::Indices(position=independent).

# ---------------------------------------------------------------
# this function uses in-place changes for obj

def tidy (obj):

    sort_product   (obj)
    rename_dummies (obj)
    canonicalise   (obj)

foo := C^{f} B^{a} A_{f a}.                 # cdb (ex-0407.101,foo)
tidy (foo)                                  # cdb (ex-0407.102,foo)

# ---------------------------------------------------------------
# this function creates new objects,
# it will not give the correct result

def tidy (obj):

    bah := @(obj).

    sort_product   (bah)
    rename_dummies (bah)
    canonicalise   (bah)

    obj := @(bah).

foo := C^{f} B^{a} A_{f a}.                 # cdb (ex-0407.201,foo)
tidy (foo)                                  # cdb (ex-0407.202,foo)

# ---------------------------------------------------------------
# this function uses a return statement
# it will give the correct result

def tidy (obj):
```

```

    bah := @(obj).

    sort_product   (bah)
    rename_dummies (bah)
    canonicalise   (bah)

    obj := @(bah).

    return obj

foo := C^{f} B^{a} A_{f a}.                          # cdb (ex-0407.301,foo)
foo = tidy (foo)                                     # cdb (ex-0407.302,foo)
```

$$C^f B^a A_{fa} = A_{ab} B^b C^a \tag{ex-0407.102}$$

$$C^f B^a A_{fa} = C^f B^a A_{fa} \tag{ex-0407.202}$$

$$C^f B^a A_{fa} = A_{ab} B^b C^a \tag{ex-0407.302}$$

# Exercise 5.1 Swap terms

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

expr := A_{a} (P^{b}+Q^{b}) + C_{a} V^{b}.                              # cdb (ex-0501.100,expr)

substitute (expr, $A_{a} B?? + C_{a} D?? -> A_{a} D?? + C_{a} B??$)   # cdb (ex-0501.101,expr)
```

$$\texttt{ex-0501.100} := A_{a} \left(P^{b} + Q^{b}\right) + C_{a} V^{b}$$

$$\texttt{ex-0501.101} := A_{a} V^{b} + C_{a} \left(P^{b} + Q^{b}\right)$$

## Exercise 5.2 Leading factors forbidden in patterns

This exercise will raise a Cadabra run-time error – the scale factor on the left hand side of the rule (3 in this case) is not allowed.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

expr := 2 V_{a b} - 3 V_{b a}.                  # cdb (ex-0502.100,expr)

substitute (expr, $3 V_{b a} -> - 3 V_{a b}$)   # cdb (ex-0502.101,expr)
```

```
Traceback (most recent call last):
  File "/usr/local/bin/cadabra2", line 248, in <module>
    exec(cmp)
  File "ex-0502.py", line 18, in <module>
    substitute (expr, Ex(r'''3 V_{b a} -> - 3 V_{a b}''', False))
RuntimeError: substitute: Index error in replacement rule.
              substitute: No numerical pre-factors allowed on lhs of replacement rule.
```

# Exercise 5.3 Deleting a term using patterns

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

expr := A_{a b} B^{a b} + A_{a b} A_{c d} B^{a b} B^{c d} - C_{a b} B^{a b}.  # cdb (ex-0503.100,expr)

zoom       (expr, $A_{a b} A_{c d} Q??$)                                    # cdb (ex-0503.101,expr)
substitute (expr, $A_{a b} -> 0$)                                           # cdb (ex-0503.102,expr)
unzoom     (expr)                                                           # cdb (ex-0503.103,expr)
```

$$\texttt{ex-0503.100} := A_{ab}B^{ab} + A_{ab}A_{cd}B^{ab}B^{cd} - C_{ab}B^{ab}$$

$$\texttt{ex-0503.101} := \ldots + A_{ab}A_{cd}B^{ab}B^{cd} + \ldots$$

$$\texttt{ex-0503.102} := \ldots\ldots$$

$$\texttt{ex-0503.103} := A_{ab}B^{ab} - C_{ab}B^{ab}$$

# Exercise 5.4 Deleting a term using tags

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

def add_tags (obj,tag):
   n = 0
   ans = Ex('0')
   for i in obj.top().terms():
      foo = obj[i]
      bah = Ex(tag+'_{'+str(n)+'}')
      ans := @(ans) + @(bah) @(foo).
      n = n + 1
   return ans

def clear_tags (obj,tag):
   ans := @(obj).
   foo  = Ex(tag+'_{a?} -> 1')
   substitute (ans,foo)
   return ans

expr := A_{a b} B^{a b} + A_{a b} A_{c d} B^{a b} B^{c d} - C_{a b} B^{a b}.  # cdb (ex-0504.100,expr)

expr  = add_tags (expr,'\\mu')                                               # cdb (ex-0504.101,expr)

substitute (expr, $\mu_{1} -> 0$)                                            # cdb (ex-0504.102,expr)

expr = clear_tags (expr,'\\mu')                                              # cdb (ex-0504.103,expr)
```

$$\texttt{ex-0504.100} := A_{ab}B^{ab} + A_{ab}A_{cd}B^{ab}B^{cd} - C_{ab}B^{ab}$$

$$\texttt{ex-0504.101} := \mu_0 A_{ab}B^{ab} + \mu_1 A_{ab}A_{cd}B^{ab}B^{cd} - \mu_2 C_{ab}B^{ab}$$

$$\texttt{ex-0504.102} := \mu_0 A_{ab}B^{ab} - \mu_2 C_{ab}B^{ab}$$

$$\texttt{ex-0504.103} := A_{ab}B^{ab} - C_{ab}B^{ab}$$

## Exercise 5.5 Commuting covariant derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

;::Symbol.

def add_tags (obj,tag):
   n = 0
   ans = Ex('0')
   for i in obj.top().terms():
      foo = obj[i]
      bah = Ex(tag+'_{'+str(n)+'}')
      ans := @(ans) + @(bah) @(foo).
      n = n + 1
   return ans

def clear_tags (obj,tag):
   ans := @(obj).
   foo  = Ex(tag+'_{a?} -> 1')
   substitute (ans,foo)
   return ans

rule := V^{a}_{; b ; c} -> V^{a}_{; c ; b} - R^{a}_{d b c} V^{d}.

expr := V^{a}_{; b ; c} - V^{a}_{; c ; b}.   # cdb (ex-0505.100,expr)

expr  = add_tags (expr,'\\mu')               # cdb (ex-0505.101,expr)

zoom       (expr, $\mu_{0} Q??$)             # cdb (ex-0505.102,expr)
substitute (expr, rule)                      # cdb (ex-0505.103,expr)
unzoom     (expr)                            # cdb (ex-0505.104,expr)

expr = clear_tags (expr,'\\mu')              # cdb (ex-0505.105,expr)
```

$$
\begin{aligned}
V^a{}_{;b;c} - V^a{}_{;c;b} &= \mu_0 V^a{}_{;b;c} - \mu_1 V^a{}_{;c;b} && \texttt{(ex-0505.101)} \\
&= \mu_0 V^a{}_{;b;c} - \mu_1 V^a{}_{;c;b} && \texttt{(ex-0505.101)} \\
&= \mu_0 V^a{}_{;b;c} + \dots && \texttt{(ex-0505.102)} \\
&= \mu_0 \left( V^a{}_{;c;b} - R^a{}_{dbc} V^d \right) + \dots && \texttt{(ex-0505.103)} \\
&= \mu_0 \left( V^a{}_{;c;b} - R^a{}_{dbc} V^d \right) - \mu_1 V^a{}_{;c;b} && \texttt{(ex-0505.104)} \\
&= -R^a{}_{dbc} V^d && \texttt{(ex-0505.105)}
\end{aligned}
$$

## Exercise 6.1 Evaluate – without `rhsonly = True`

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

V  := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.   # cdb(ex-0601.100,V)
dV := dV_{a b} -> \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.  # cdb(ex-0601.101,dV)

evaluate (dV, V)                                              # cdb(ex-0601.102,dV)
```

Notice how `evaluate` has been applied to both the left and right hand sides of the rule.

$$V_a = [V_\theta = \varphi, V_\varphi = \sin\theta] \tag{ex-0601.100}$$

$$dV_{ab} \rightarrow \partial_b V_a - \partial_a V_b \tag{ex-0601.101}$$

$$\square_{ab} \begin{cases} \square_{\theta\theta} = dV_{\theta\theta} \\ \square_{\varphi\theta} = dV_{\varphi\theta} \\ \square_{\theta\varphi} = dV_{\theta\varphi} \\ \square_{\varphi\varphi} = dV_{\varphi\varphi} \end{cases} \rightarrow \square_{ab} \begin{cases} \square_{\varphi\theta} = \cos\theta - 1 \\ \square_{\theta\varphi} = 1 - \cos\theta \end{cases} \tag{ex-0601.102}$$

# Exercise 6.1 Evaluate – with `rhsonly = True`

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

V  := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.    # cdb(ex-0601.200,V)
dV := dV_{a b} -> \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.  # cdb(ex-0601.201,dV)

evaluate (dV, V, rhsonly=True)                                # cdb(ex-0601.202,dV)
```

This is an improvement, only the right hnd side has been expanded into components.

$$V_a = [V_\theta = \varphi, V_\varphi = \sin\theta] \tag{ex-0601.200}$$

$$dV_{ab} \rightarrow \partial_b V_a - \partial_a V_b \tag{ex-0601.201}$$

$$dV_{ab} \rightarrow \Box_{ab} \begin{cases} \Box_{\varphi\theta} = \cos\theta - 1 \\ \Box_{\theta\varphi} = 1 - \cos\theta \end{cases} \tag{ex-0601.202}$$

# Exercise 6.2 Evaluate on an expression (not a rule)

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

V  := { V_{\theta} = f(\theta,\varphi), V_{\varphi} = g(\theta,\varphi) }.  # cdb(ex-0602.100,V)
dV := \partial_{b}{V_{a}} + \partial_{a}{V_{b}}.                              # cdb(ex-0602.101,dV)

evaluate (dV, V)                                                              # cdb(ex-0602.102,dV)
```

$$V_a = [V_\theta = f(\theta,\varphi), V_\varphi = g(\theta,\varphi)] \tag{ex-0602.100}$$

$$\partial_b V_a + \partial_a V_b \tag{ex-0602.101}$$

$$\Box_{ab} \begin{cases} \Box_{\varphi\varphi} = 2\partial_\varphi g(\theta,\varphi) \\ \Box_{\varphi\theta} = \partial_\varphi f(\theta,\varphi) + \partial_\theta g(\theta,\varphi) \\ \Box_{\theta\varphi} = \partial_\varphi f(\theta,\varphi) + \partial_\theta g(\theta,\varphi) \\ \Box_{\theta\theta} = 2\partial_\theta f(\theta,\varphi) \end{cases} \tag{ex-0602.102}$$

# Exercise 6.3 Evaluate with undefined components

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

bah := {V_{\theta} = \varphi, V_{\varphi} = \sin(\theta)}.  # cdb(ex-0603.100,bah)
foo := U_{a} V_{b}.                                          # cdb(ex-0603.101,foo)

evaluate (foo, bah)                                          # cdb(ex-0603.102,foo)
```

$$[V_{\theta} = \varphi, V_{\varphi} = \sin\theta] \tag{ex-0603.100}$$

$$U_{a} V_{b} \tag{ex-0603.101}$$

$$\Box_{ab} \begin{cases} \Box_{\theta\theta} = \varphi U_{\theta} \\ \Box_{\theta\varphi} = U_{\theta} \sin\theta \\ \Box_{\varphi\theta} = \varphi U_{\varphi} \\ \Box_{\varphi\varphi} = U_{\varphi} \sin\theta \end{cases} \tag{ex-0603.102}$$

# Exercise 6.4 Scalar curavture of a 2-sphere

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

Rab := R_{a b} -> R^{c}_{a c b}.

R := R -> R_{a b} g^{a b}.

gab := { g_{\theta\theta}   = r**2,
         g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.      # cdb(ex-0604.101,gab)

complete   (gab, $g^{a b}$)                                # cdb(ex-0604.102,gab)

substitute (Rabcd, Gamma)
substitute (Rab, Rabcd)
substitute (R, Rab)

evaluate   (Gamma, gab, rhsonly=True)                      # cdb(ex-0604.103,Gamma)
evaluate   (Rabcd, gab, rhsonly=True)                      # cdb(ex-0604.104,Rabcd)
evaluate   (Rab,   gab, rhsonly=True)                      # cdb(ex-0604.105,Rab)
evaluate   (R,     gab, rhsonly=True)                      # cdb(ex-0604.106,R)
```

$$\left[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-0604.101}$$

$$\left[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\right] \tag{ex-0604.102}$$

$$\Gamma^a{}_{bc} \to \square_{cb}{}^a \begin{cases} \square_{\varphi\theta}{}^\varphi = (\tan\theta)^{-1} \\ \square_{\theta\varphi}{}^\varphi = (\tan\theta)^{-1} \\ \square_{\varphi\varphi}{}^\theta = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-0604.103}$$

$$R^a{}_{bcd} \to \square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^\theta{}_\theta = (\sin\theta)^2 \\ \square_{\varphi\theta}{}^\varphi{}_\theta = -1 \\ \square_{\theta\varphi}{}^\theta{}_\varphi = -(\sin\theta)^2 \\ \square_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \tag{ex-0604.104}$$

$$R_{ab} \to \square_{ba} \begin{cases} \square_{\varphi\varphi} = (\sin\theta)^2 \\ \square_{\theta\theta} = 1 \end{cases} \tag{ex-0604.105}$$

$$R \to 2r^{-2} \tag{ex-0604.106}$$

## Exercise 6.5 Schwarzschild spacetime in isotropic coordinates

```
{t, r, \theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

Rab := R_{a b} -> R^{c}_{a c b}.

gab := { g_{t t}            = -((2*r-m)/(2*r+m))**2,
         g_{r r}            = (1+m/(2*r))**4,
         g_{\theta\theta}   = r**2 (1+m/(2*r))**4,
         g_{\varphi\varphi} = r**2 \sin(\theta)**2 (1+m/(2*r))**4}.  # cdb(ex-0605.101,gab)

complete   (gab, $g^{a b}$)                                      # cdb(ex-0605.102,gab)

substitute (Rabcd, Gamma)
substitute (Rab, Rabcd)

evaluate   (Gamma, gab, rhsonly=True)                            # cdb(ex-0605.103,Gamma)
evaluate   (Rabcd, gab, rhsonly=True)                            # cdb(ex-0605.104,Rabcd)
evaluate   (Rab,   gab, rhsonly=True)                            # cdb(ex-0605.105,Rab)
```

$$\left[g_{tt} = -\left((2r-m)\,(2r+m)^{-1}\right)^2, g_{rr} = \left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\theta\theta} = r^2\left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1+\frac{1}{2}mr^{-1}\right)^4\right] \tag{ex-0605.101}$$

$$\left[g_{tt} = -\left((2r-m)\,(2r+m)^{-1}\right)^2, g_{rr} = \left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\theta\theta} = r^2\left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1+\frac{1}{2}mr^{-1}\right)^4, g^{tt} = \left(-m^2-4mr\right.\right.$$
$$\left.-4r^2\right)\left(m^2-4mr+4r^2\right)^{-1}, g^{rr} = 16r^4\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-1}, g^{\theta\theta} = 16r^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-1},$$
$$\left.g^{\varphi\varphi} = 16r^2\left(m^4(\sin\theta)^2+8m^3r(\sin\theta)^2+24m^2r^2(\sin\theta)^2+32mr^3(\sin\theta)^2+16r^4(\sin\theta)^2\right)^{-1}\right] \tag{ex-0605.102}$$

$$\Gamma^a{}_{bc} \to \square_{cb}{}^a \begin{cases} \square_{\varphi r}{}^{\varphi} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{\varphi\theta}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\theta r}{}^{\theta} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{rr}{}^{r} = -2m(r\,(m+2r))^{-1} \\ \square_{tr}{}^{t} = 4m\left(-m^2+4r^2\right)^{-1} \\ \square_{r\varphi}{}^{\varphi} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{\theta\varphi}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{r\theta}{}^{\theta} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{rt}{}^{t} = 4m\left(-m^2+4r^2\right)^{-1} \\ \square_{\varphi\varphi}{}^{r} = r\,(m-2r)\,(\sin\theta)^2(m+2r)^{-1} \\ \square_{\varphi\varphi}{}^{\theta} = -\frac{1}{2}\sin(2\theta) \\ \square_{\theta\theta}{}^{r} = r\,(m-2r)\,(m+2r)^{-1} \\ \square_{tt}{}^{r} = -64mr^4\,(m-2r)\left((m+2r)^3\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)\right)^{-1} \end{cases} \tag{ex-0605.103}$$

$R^a{}_{bcd}$

$$\to \Box_{db}{}^a{}_c \begin{cases} \Box_{tt}{}^r{}_r = 128mr^3\left(-m^2+4mr-4r^2\right)\left(m^8+16m^7r+112m^6r^2+448m^5r^3+1120m^4r^4+1792m^3r^5+1792m^2r^6+1024mr^7+256r^8\right)^{-1} \\ \Box_{\theta\theta}{}^r{}_r = -4mr\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{\varphi\varphi}{}^\theta{}_\theta = 8mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{\varphi\varphi}{}^r{}_r = -4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{tr}{}^t{}_r = -8m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\ \Box_{\theta r}{}^\theta{}_r = 4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\ \Box_{\varphi\theta}{}^\varphi{}_\theta = -8mr\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{\varphi r}{}^\varphi{}_r = 4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\ \Box_{rt}{}^r{}_t = 128mr^3\left(m^2-4mr+4r^2\right)\left(m^8+16m^7r+112m^6r^2+448m^5r^3+1120m^4r^4+1792m^3r^5+1792m^2r^6+1024mr^7+256r^8\right)^{-1} \\ \Box_{r\theta}{}^r{}_\theta = 4mr\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{\theta\varphi}{}^\theta{}_\varphi = -8mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{r\varphi}{}^r{}_\varphi = 4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{rr}{}^t{}_t = 8m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\ \Box_{rr}{}^\theta{}_\theta = -4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\ \Box_{\theta\theta}{}^\varphi{}_\varphi = 8mr\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{rr}{}^\varphi{}_\varphi = -4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\ \Box_{\varphi\varphi}{}^t{}_t = -4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{\theta\theta}{}^t{}_t = -4mr\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{tt}{}^\varphi{}_\varphi = 64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2} \\ \Box_{tt}{}^\theta{}_\theta = 64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2} \\ \Box_{t\varphi}{}^t{}_\varphi = 4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{t\theta}{}^t{}_\theta = 4mr\left(m^2+4mr+4r^2\right)^{-1} \\ \Box_{\varphi t}{}^\varphi{}_t = -64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2} \\ \Box_{\theta t}{}^\theta{}_t = -64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2} \end{cases} \quad \texttt{(ex-0605.104)}$$

$$R_{ab} \to 0 \quad \texttt{(ex-0605.105)}$$

# Exercise 6.6 The Kasner cosmology

```
{t, x, y, z}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={t, x, y, z}, position=independent).

\partial{#}::PartialDerivative.

p1::LaTeXForm("p_1").
p2::LaTeXForm("p_2").
p3::LaTeXForm("p_3").

g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

Rab := R_{a b} -> R^{c}_{a c b}.

gab := { g_{t t} = -1,
         g_{x x} = t**(2*p1),
         g_{y y} = t**(2*p2),
         g_{z z} = t**(2*p3)}.                            # cdb(ex-0606.101,gab)

complete   (gab, $g^{a b}$)                               # cdb(ex-0606.102,gab)

substitute (Rabcd, Gamma)
substitute (Rab, Rabcd)

evaluate   (Gamma, gab, rhsonly=True)                     # cdb(ex-0606.103,Gamma)
evaluate   (Rabcd, gab, rhsonly=True)                     # cdb(ex-0606.104,Rabcd)
evaluate   (Rab,   gab, rhsonly=True)                     # cdb(ex-0606.105,Rab)
```

$$\left[g_{tt} = -1, g_{xx} = t^{2p_1}, g_{yy} = t^{2p_2}, g_{zz} = t^{2p_3}\right] \qquad \texttt{(ex-0606.101)}$$

$$\left[g_{tt} = -1, g_{xx} = t^{2p_1}, g_{yy} = t^{2p_2}, g_{zz} = t^{2p_3}, g^{tt} = -1, g^{xx} = t^{-2p_1}, g^{yy} = t^{-2p_2}, g^{zz} = t^{-2p_3}\right] \qquad \texttt{(ex-0606.102)}$$

$$\Gamma^{a}{}_{bc} \rightarrow \square_{cb}{}^{a} \begin{cases} \square_{zt}{}^{z} = p_3 t^{-1} \\ \square_{yt}{}^{y} = p_2 t^{-1} \\ \square_{xt}{}^{x} = p_1 t^{-1} \\ \square_{tz}{}^{z} = p_3 t^{-1} \\ \square_{ty}{}^{y} = p_2 t^{-1} \\ \square_{tx}{}^{x} = p_1 t^{-1} \\ \square_{zz}{}^{t} = p_3 t^{(2p_3-1)} \\ \square_{yy}{}^{t} = p_2 t^{(2p_2-1)} \\ \square_{xx}{}^{t} = p_1 t^{(2p_1-1)} \end{cases} \qquad \texttt{(ex-0606.103)}$$

$$R^{a}{}_{bcd} \to \Box_{db}{}^{a}{}_{c} \begin{cases} \Box_{xx}{}^{t}{}_{t} = p_1 t^{(2p_1-2)}\left(p_1 - 1\right) \\ \Box_{yy}{}^{t}{}_{t} = p_2 t^{(2p_2-2)}\left(p_2 - 1\right) \\ \Box_{zz}{}^{t}{}_{t} = p_3 t^{(2p_3-2)}\left(p_3 - 1\right) \\ \Box_{xt}{}^{x}{}_{t} = p_1\left(p_1 - 1\right) t^{-2} \\ \Box_{yt}{}^{y}{}_{t} = p_2\left(p_2 - 1\right) t^{-2} \\ \Box_{zt}{}^{z}{}_{t} = p_3\left(p_3 - 1\right) t^{-2} \\ \Box_{tx}{}^{t}{}_{x} = p_1 t^{(2p_1-2)}\left(1 - p_1\right) \\ \Box_{ty}{}^{t}{}_{y} = p_2 t^{(2p_2-2)}\left(1 - p_2\right) \\ \Box_{tz}{}^{t}{}_{z} = p_3 t^{(2p_3-2)}\left(1 - p_3\right) \\ \Box_{tt}{}^{x}{}_{x} = p_1\left(1 - p_1\right) t^{-2} \\ \Box_{tt}{}^{y}{}_{y} = p_2\left(1 - p_2\right) t^{-2} \\ \Box_{tt}{}^{z}{}_{z} = p_3\left(1 - p_3\right) t^{-2} \\ \Box_{zz}{}^{y}{}_{y} = p_2 p_3 t^{(2p_3-2)} \\ \Box_{zz}{}^{x}{}_{x} = p_1 p_3 t^{(2p_3-2)} \\ \Box_{yy}{}^{z}{}_{z} = p_2 p_3 t^{(2p_2-2)} \\ \Box_{yy}{}^{x}{}_{x} = p_1 p_2 t^{(2p_2-2)} \\ \Box_{xx}{}^{z}{}_{z} = p_1 p_3 t^{(2p_1-2)} \\ \Box_{xx}{}^{y}{}_{y} = p_1 p_2 t^{(2p_1-2)} \\ \Box_{yz}{}^{y}{}_{z} = -p_2 p_3 t^{(2p_3-2)} \\ \Box_{xz}{}^{x}{}_{z} = -p_1 p_3 t^{(2p_3-2)} \\ \Box_{zy}{}^{z}{}_{y} = -p_2 p_3 t^{(2p_2-2)} \\ \Box_{xy}{}^{x}{}_{y} = -p_1 p_2 t^{(2p_2-2)} \\ \Box_{zx}{}^{z}{}_{x} = -p_1 p_3 t^{(2p_1-2)} \\ \Box_{yx}{}^{y}{}_{x} = -p_1 p_2 t^{(2p_1-2)} \end{cases} \qquad (\texttt{ex-0606.104})$$

$$R_{ab} \to \Box_{ba} \begin{cases} \Box_{xx} = p_1 t^{(2p_1-2)}\left(p_1 + p_2 + p_3 - 1\right) \\ \Box_{yy} = p_2 t^{(2p_2-2)}\left(p_1 + p_2 + p_3 - 1\right) \\ \Box_{zz} = p_3 t^{(2p_3-2)}\left(p_1 + p_2 + p_3 - 1\right) \\ \Box_{tt} = \left(-{p_1}^2 + p_1 - {p_2}^2 + p_2 - {p_3}^2 + p_3\right) t^{-2} \end{cases} \qquad (\texttt{ex-0606.105})$$

# Exercise 6.7 Killing vectors of the Schwarzschild spacetime

```
{t, r, \theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).

;::Symbol.

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} (   \partial_{g}{g_{b f}}
                                           + \partial_{f}{g_{b g}}
                                           - \partial_{b}{g_{f g}} ).

deriv := \xi_{a ; b} -> \partial_{b}{\xi_{a}} - \Gamma^{c}_{a b} \xi_{c}.
lower := \xi_{a} -> g_{a b} \xi^{b}.

expr  := \xi_{a ; b} + \xi_{b ; a}.                  # cdb(ex-0607.100,expr)

substitute   (expr, deriv)                           # cdb(ex-0607.101,expr)
substitute   (expr, lower)                           # cdb(ex-0607.102,expr)
substitute   (expr, Gamma)                           # cdb(ex-0607.103,expr)
distribute   (expr)                                  # cdb(ex-0607.104,expr)
product_rule (expr)                                  # cdb(ex-0607.105,expr)
canonicalise (expr)                                  # cdb(ex-0607.106,expr)

# choose a vector

# Kvect := {\xi^{t} = 1}.
# Kvect := {\xi^{\varphi} = 1}.
Kvect := {\xi^{\theta} = \sin(\varphi), \xi^{\varphi} = \cos(\theta)/\sin(\theta) \cos(\varphi)}.
# Kvect := {\xi^{\theta} = \cos(\varphi), \xi^{\varphi} = - \cos(\theta)/\sin(\theta) \sin(\varphi)}.
                                                      # cdb(ex-0607.107,Kvect)

gab := { g_{t t}            = -(1-2*m/r),
         g_{r r}            = 1/(1-(2*m/r)),
```

```
            g_{\theta\theta}   = r**2,
            g_{\varphi\varphi} = r**2 \sin(\theta)**2}.  # cdb(ex-0607.108,gab)

complete   (gab, $g^{a b}$)                          # cdb(ex-0607.109,gab)

evaluate   (expr, join (gab,Kvect))                  # cdb(ex-0607.110,expr)
```

$$[\xi^a] = \left[\xi^\theta = \sin\varphi, \xi^\varphi = \cos\theta(\sin\theta)^{-1}\cos\varphi\right] \tag{ex-0607.107}$$

$$[g_{ab}] = \left[g_{tt} = -1 + 2mr^{-1}, g_{rr} = \left(1 - 2mr^{-1}\right)^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-0607.108}$$

$$\begin{aligned}[g_{ab}, g^{ab}] = \Big[&g_{tt} = -1 + 2mr^{-1}, g_{rr} = \left(1 - 2mr^{-1}\right)^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2, g^{tt} = \left(2mr^{-1} - 1\right)^{-1}, g^{rr} = -2mr^{-1} + 1, g^{\theta\theta} = r^{-2},\\ &g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\Big]\end{aligned} \tag{ex-0607.109}$$

$$\begin{aligned}
\xi_{a;b} + \xi_{b;a} &= \partial_b\xi_a - \Gamma^c{}_{ab}\xi_c + \partial_a\xi_b - \Gamma^c{}_{ba}\xi_c && \text{(ex-0607.101)}\\
&= \partial_b\left(g_{ac}\xi^c\right) - \Gamma^c{}_{ab}g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) - \Gamma^c{}_{ba}g_{cd}\xi^d && \text{(ex-0607.102)}\\
&= \partial_b\left(g_{ac}\xi^c\right) - \frac{1}{2}g^{ce}\left(\partial_b g_{ea} + \partial_a g_{eb} - \partial_e g_{ab}\right)g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) - \frac{1}{2}g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ba}\right)g_{cd}\xi^d && \text{(ex-0607.103)}\\
&= \partial_b\left(g_{ac}\xi^c\right) - g^{ce}\partial_b g_{ea}g_{cd}\xi^d - g^{ce}\partial_a g_{eb}g_{cd}\xi^d + \frac{1}{2}g^{ce}\partial_e g_{ab}g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) + \frac{1}{2}g^{ce}\partial_e g_{ba}g_{cd}\xi^d && \text{(ex-0607.104)}\\
&= \partial_b g_{ac}\xi^c + g_{ac}\partial_b\xi^c - g^{ce}\partial_b g_{ea}g_{cd}\xi^d - g^{ce}\partial_a g_{eb}g_{cd}\xi^d + \frac{1}{2}g^{ce}\partial_e g_{ab}g_{cd}\xi^d + \partial_a g_{bc}\xi^c + g_{bc}\partial_a\xi^c + \frac{1}{2}g^{ce}\partial_e g_{ba}g_{cd}\xi^d && \text{(ex-0607.105)}\\
&= \partial_b g_{ac}\xi^c + g_{ac}\partial_b\xi^c - g^{cd}\partial_b g_{ac}g_{de}\xi^e - g^{cd}\partial_a g_{bc}g_{de}\xi^e + g^{cd}\partial_c g_{ab}g_{de}\xi^e + \partial_a g_{bc}\xi^c + g_{bc}\partial_a\xi^c && \text{(ex-0607.106)}\\
&= 0 && \text{(ex-0607.110)}
\end{aligned}$$

# Exercise 6.08a A problem with `evaluate`

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

V_{a}::Depends(\theta,\varphi,\partial{#}).

dVrule := { \partial_{\theta}{V_{\varphi}} = \sin(\theta),
            \partial_{\varphi}{V_{\theta}} = \cos(\theta)}.  # cdb(ex-0608.101,dVrule)
dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.             # cdb(ex-0608.102,dV)

evaluate (dV, dVrule)                                         # cdb(ex-0608.103,dV)
```

```
Traceback (most recent call last):
  File "/usr/local/bin/cadabra2", line 248, in <module>
    exec(cmp)
  File "ex-0608.py", line 27, in <module>
    evaluate (dV, dVrule)
RuntimeError: Dependencies on derivatives are not yet handled in the SymPy bridge
```

# Exercise 6.08b A work around

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

V_{a}::Depends(\theta,\varphi,\partial{#}).

hide := \partial_{a}{V_{b}} -> dV_{a b}.

dVrule := { dV_{\theta\varphi} = \sin(\theta),
            dV_{\varphi\theta} = \cos(\theta)}.             # cdb(ex-0608.201,dVrule)
dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.            # cdb(ex-0608.202,dV)

substitute (dV, hide)                                       # cdb(ex-0608.212,dV)
evaluate (dV, dVrule)                                       # cdb(ex-0608.203,dV)
```

The workaround here is to to hide the derivatives before calling `evaluate`.

$$dV_{ba} - dV_{ab} \tag{ex-0608.212}$$

$$dV_{ab} = \partial_{b} V_{a} - \partial_{a} V_{b} \tag{ex-0608.202}$$

$$= \Box_{ab} \begin{cases} \Box_{\varphi\theta} = \sin\theta - \cos\theta \\ \Box_{\theta\varphi} = -\sin\theta + \cos\theta \end{cases} \tag{ex-0608.203}$$

## Exercise 7.1 C-code for a $R_{ab}$ for a generic metric

```
{x,y,z}::Coordinate.
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(values={x,y,z},position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

import cdblib

FourRab = cdblib.get ('FourRab','ex-0309.json')

Rab := 1/4 @(FourRab).

substitute (Rab, $ \partial_{a b}{g_{c d}} -> dg_{c d a b} $)
substitute (Rab, $ \partial_{a}{g_{b c}}   -> dg_{b c a}   $)

# ----------------------------------------------------------------
# build rules to export Cadabra expressions to Python
# use known symmetries for g_{a b}, dg_{ab,c,d} etc.
# note: replacements must not contain underscores (reserved for subscripts),
#       so g_{x x}-> g_xx is not allowed

gabRule := {g_{x x} -> gxx, g_{x y} -> gxy, g_{x z} -> gxz,
            g_{y x} -> gxy, g_{y y} -> gyy, g_{y z} -> gyz,
            g_{z x} -> gxz, g_{z y} -> gyz, g_{z z} -> gzz}.

iabRule := {g^{x x} -> ixx, g^{x y} -> ixy, g^{x z} -> ixz,
            g^{y x} -> ixy, g^{y y} -> iyy, g^{y z} -> iyz,
            g^{z x} -> ixz, g^{z y} -> iyz, g^{z z} -> izz}.

d1gabRule := {dg_{x x x} -> dgxxx, dg_{x y x} -> dgxyx, dg_{x z x} -> dgxzx,
              dg_{y x x} -> dgxyx, dg_{y y x} -> dgyyx, dg_{y z x} -> dgyzx,
              dg_{z x x} -> dgxzx, dg_{z y x} -> dgyzx, dg_{z z x} -> dgzzx,

              dg_{x x y} -> dgxxy, dg_{x y y} -> dgxyy, dg_{x z y} -> dgxzy,
```

```
              dg_{y x y} -> dgxyy, dg_{y y y} -> dgyyy, dg_{y z y} -> dgyzy,
              dg_{z x y} -> dgxzy, dg_{z y y} -> dgyzy, dg_{z z y} -> dgzzy,

              dg_{x x z} -> dgxxz, dg_{x y z} -> dgxyz, dg_{x z z} -> dgxzz,
              dg_{y x z} -> dgxyz, dg_{y y z} -> dgyyz, dg_{y z z} -> dgyzz,
              dg_{z x z} -> dgxzz, dg_{z y z} -> dgyzz, dg_{z z z} -> dgzzz}.

d2gabRule := {dg_{x x x x} -> dgxxxx, dg_{x y x x} -> dgxyxx, dg_{x z x x} -> dgxzxx,
              dg_{y x x x} -> dgxyxx, dg_{y y x x} -> dgyyxx, dg_{y z x x} -> dgyzxx,
              dg_{z x x x} -> dgxzxx, dg_{z y x x} -> dgyzxx, dg_{z z x x} -> dgzzxx,
              dg_{x x y x} -> dgxxxy, dg_{x y y x} -> dgxyxy, dg_{x z y x} -> dgxzxy,
              dg_{y x y x} -> dgxyxy, dg_{y y y x} -> dgyyxy, dg_{y z y x} -> dgyzxy,
              dg_{z x y x} -> dgxzxy, dg_{z y y x} -> dgyzxy, dg_{z z y x} -> dgzzxy,
              dg_{x x z x} -> dgxxxz, dg_{x y z x} -> dgxyxz, dg_{x z z x} -> dgxzxz,
              dg_{y x z x} -> dgxyxz, dg_{y y z x} -> dgyyxz, dg_{y z z x} -> dgyzxz,
              dg_{z x z x} -> dgxzxz, dg_{z y z x} -> dgyzxz, dg_{z z z x} -> dgzzxz,

              dg_{x x x y} -> dgxxxy, dg_{x y x y} -> dgxyxy, dg_{x z x y} -> dgxzxy,
              dg_{y x x y} -> dgxyxy, dg_{y y x y} -> dgyyxy, dg_{y z x y} -> dgyzxy,
              dg_{z x x y} -> dgxzxy, dg_{z y x y} -> dgyzxy, dg_{z z x y} -> dgzzxy,
              dg_{x x y y} -> dgxxyy, dg_{x y y y} -> dgxyyy, dg_{x z y y} -> dgxzyy,
              dg_{y x y y} -> dgxyyy, dg_{y y y y} -> dgyyyy, dg_{y z y y} -> dgyzyy,
              dg_{z x y y} -> dgxzyy, dg_{z y y y} -> dgyzyy, dg_{z z y y} -> dgzzyy,
              dg_{x x z y} -> dgxxyz, dg_{x y z y} -> dgxyyz, dg_{x z z y} -> dgxzyz,
              dg_{y x z y} -> dgxyyz, dg_{y y z y} -> dgyyyz, dg_{y z z y} -> dgyzyz,
              dg_{z x z y} -> dgxzyz, dg_{z y z y} -> dgyzyz, dg_{z z z y} -> dgzzyz,

              dg_{x x x z} -> dgxxxz, dg_{x y x z} -> dgxyxz, dg_{x z x z} -> dgxzxz,
              dg_{y x x z} -> dgxyxz, dg_{y y x z} -> dgyyxz, dg_{y z x z} -> dgyzxz,
              dg_{z x x z} -> dgxzxz, dg_{z y x z} -> dgyzxz, dg_{z z x z} -> dgzzxz,
              dg_{x x y z} -> dgxxyz, dg_{x y y z} -> dgxyyz, dg_{x z y z} -> dgxzyz,
              dg_{y x y z} -> dgxyyz, dg_{y y y z} -> dgyyyz, dg_{y z y z} -> dgyzyz,
              dg_{z x y z} -> dgxzyz, dg_{z y y z} -> dgyzyz, dg_{z z y z} -> dgzzyz,
              dg_{x x z z} -> dgxxzz, dg_{x y z z} -> dgxyzz, dg_{x z z z} -> dgxzzz,
              dg_{y x z z} -> dgxyzz, dg_{y y z z} -> dgyyzz, dg_{y z z z} -> dgyzzz,
              dg_{z x z z} -> dgxzzz, dg_{z y z z} -> dgyzzz, dg_{z z z z} -> dgzzzz}.

def write_code (obj,name,filename,rank):
```

```
    import os

    from sympy.printing.c import C99CodePrinter as printer
    from sympy.codegen.ast import Assignment

    idx=[]  # indices in the form [{x, x}, {x, y} ...]
    lst=[]  # corresponding terms [termxx, termxy, ...]

    for i in range( len(obj[rank]) ):                  # rank = number of free indices
        idx.append( str(obj[rank][i][0]._sympy_()) )  # indices for this term
        lst.append( str(obj[rank][i][1]._sympy_()) )  # the matching term

    mat = sympy.Matrix([lst])                          # row vector of terms
    sub_exprs, simplified_rhs = sympy.cse(mat)         # optimise code

    with open(os.getcwd() + '/' + filename, 'w') as out:

        for lhs, rhs in sub_exprs:
            out.write(printer().doprint(Assignment(lhs, rhs))+'\n')

        for index, rhs in enumerate (simplified_rhs[0]):
            lhs = sympy.Symbol(name+' '+(idx[index]).replace(', ',']['))
            out.write(printer().doprint(Assignment(lhs, rhs))+'\n')

def JoinLists (obj):
    ans := {}.
    for i in range (len(obj)):
        ans = join (ans,obj[i])
    return ans

evaluate   (Rab,  JoinLists ([gabRule,d1gabRule,d2gabRule,iabRule]), simplify=False)

write_code (Rab, 'Rab', 'ex-0701-rab.c',2)
```

The code for $R_{ab}$ can be found in the file `ex-0701-rab.c`. It is long and it would require more work to turn it into something useful in a numerical code. For example, functions would be needed to compute the first and second partial derivatives of the metric. But that is not a Cadabra issue.