

## Exercise 3.8 Symmetry of $R_{ab}$

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative;
4
5  g_{a b}::Metric;
6  g^{a b}::InverseMetric;
7
8  dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
9                                     # cdb (dgab.000,dgab)
10
11 Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} ( \partial_{b}{g_{e c}}
12                                     + \partial_{c}{g_{b e}}
13                                     - \partial_{e}{g_{b c}}).
14                                     # cdb (Gamma.000,Gamma)
15
16 Rabcd := R^{a}_{b c d} ->
17     \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
18     - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
19                                     # cdb (Rabcd.000,Rabcd)
20
21 Rab := R_{a b} -> R^{c}_{c a b}.
22                                     # cdb (Rab.000,Rab)
23
24 expr := 4 (R_{a b} - R_{b a}).
25                                     # cdb (ex-0308.100,expr)
26
27 substitute (expr, Rab)
28                                     # cdb (ex-0308.101,expr)
29 substitute (expr, Rabcd)
30                                     # cdb (ex-0308.102,expr)
31 substitute (expr, Gamma)
32                                     # cdb (ex-0308.103,expr)
33
34 distribute (expr)
35                                     # cdb (ex-0308.104,expr)
36 product_rule (expr)
37                                     # cdb (ex-0308.105,expr)
38 canonicalise (expr)
39                                     # cdb (ex-0308.106,expr)
40
41 substitute (expr, dgab)
42                                     # cdb (ex-0308.107,expr)
43 canonicalise (expr)
44                                     # cdb (ex-0308.108,expr)

```

$$4R_{ab} - 4R_{ba} = 4R^c_{acb} - 4R^c_{bca} \quad (\text{ex-0308.101})$$

$$= 4\partial_c \Gamma^c_{ab} + 4\Gamma^c_{ec} \Gamma^e_{ab} - 4\partial_b \Gamma^c_{ac} - 4\Gamma^c_{eb} \Gamma^e_{ac} - 4\partial_c \Gamma^c_{ba} - 4\Gamma^c_{ec} \Gamma^e_{ba} + 4\partial_a \Gamma^c_{bc} + 4\Gamma^c_{ea} \Gamma^e_{bc} \quad (\text{ex-0308.102})$$

$$\begin{aligned} &= 2\partial_c (g^{ce} (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab})) + g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}) - 2\partial_b (g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac})) \\ &\quad - g^{cd} (\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) - 2\partial_c (g^{ce} (\partial_b g_{ea} + \partial_a g_{be} - \partial_e g_{ba})) \\ &\quad - g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_b g_{fa} + \partial_a g_{bf} - \partial_f g_{ba}) + 2\partial_a (g^{ce} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc})) \\ &\quad + g^{cd} (\partial_e g_{da} + \partial_a g_{ed} - \partial_d g_{ea}) g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) \end{aligned} \quad (\text{ex-0308.103})$$

$$\begin{aligned} &= 2\partial_c (g^{ce} \partial_a g_{eb}) + 2\partial_c (g^{ce} \partial_b g_{ae}) - 2\partial_c (g^{ce} \partial_e g_{ab}) + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} \\ &\quad + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} - 2\partial_b (g^{ce} \partial_a g_{ec}) - 2\partial_b (g^{ce} \partial_c g_{ae}) \\ &\quad + 2\partial_b (g^{ce} \partial_e g_{ac}) - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} \\ &\quad + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} - 2\partial_c (g^{ce} \partial_b g_{ea}) - 2\partial_c (g^{ce} \partial_a g_{be}) + 2\partial_c (g^{ce} \partial_e g_{ba}) - g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{fa} \\ &\quad - g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{bf} + g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ba} - g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{fa} - g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{bf} + g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ba} + g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{fa} + g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{bf} \\ &\quad - g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ba} + 2\partial_a (g^{ce} \partial_b g_{ec}) + 2\partial_a (g^{ce} \partial_c g_{be}) - 2\partial_a (g^{ce} \partial_e g_{bc}) + g^{cd} \partial_e g_{da} g^{ef} \partial_b g_{fc} + g^{cd} \partial_e g_{da} g^{ef} \partial_c g_{bf} - g^{cd} \partial_e g_{da} g^{ef} \partial_f g_{bc} \\ &\quad + g^{cd} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + g^{cd} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - g^{cd} \partial_a g_{ed} g^{ef} \partial_f g_{bc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_b g_{fc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_c g_{bf} + g^{cd} \partial_d g_{ea} g^{ef} \partial_f g_{bc} \end{aligned} \quad (\text{ex-0308.104})$$

$$\begin{aligned} &= 2\partial_c g^{ce} \partial_a g_{eb} + 2g^{ce} \partial_{ca} g_{eb} + 2\partial_c g^{ce} \partial_b g_{ae} + 2g^{ce} \partial_{cb} g_{ae} - 2\partial_c g^{ce} \partial_e g_{ab} - 2g^{ce} \partial_{ce} g_{ab} + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} \\ &\quad + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} - 2\partial_b g^{ce} \partial_a g_{ec} \\ &\quad - 2g^{ce} \partial_{ba} g_{ec} - 2\partial_b g^{ce} \partial_c g_{ae} - 2g^{ce} \partial_{bc} g_{ae} + 2\partial_b g^{ce} \partial_e g_{ac} + 2g^{ce} \partial_{be} g_{ac} - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} \\ &\quad - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} - 2\partial_c g^{ce} \partial_b g_{ea} \\ &\quad - 2g^{ce} \partial_{cb} g_{ea} - 2\partial_c g^{ce} \partial_a g_{be} - 2g^{ce} \partial_{ca} g_{be} + 2\partial_c g^{ce} \partial_e g_{ba} + 2g^{ce} \partial_{ce} g_{ba} - g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{fa} - g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{bf} + g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ba} \\ &\quad - g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{fa} - g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{bf} + g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ba} + g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{fa} + g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{bf} - g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ba} + 2\partial_a g^{ce} \partial_b g_{ec} \\ &\quad + 2g^{ce} \partial_{ab} g_{ec} + 2\partial_a g^{ce} \partial_c g_{be} + 2g^{ce} \partial_{ac} g_{be} - 2\partial_a g^{ce} \partial_e g_{bc} - 2g^{ce} \partial_{ae} g_{bc} + g^{cd} \partial_e g_{da} g^{ef} \partial_b g_{fc} + g^{cd} \partial_e g_{da} g^{ef} \partial_c g_{bf} - g^{cd} \partial_e g_{da} g^{ef} \partial_f g_{bc} \\ &\quad + g^{cd} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + g^{cd} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - g^{cd} \partial_a g_{ed} g^{ef} \partial_f g_{bc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_b g_{fc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_c g_{bf} + g^{cd} \partial_d g_{ea} g^{ef} \partial_f g_{bc} \end{aligned} \quad (\text{ex-0308.105})$$

$$= -2\partial_b g^{ce} \partial_a g_{ce} + 2\partial_a g^{ce} \partial_b g_{ce} \quad (\text{ex-0308.106})$$

$$= 2g^{cd} g^{ef} \partial_b g_{df} \partial_a g_{ce} - 2g^{cd} g^{ef} \partial_a g_{df} \partial_b g_{ce} \quad (\text{ex-0308.107})$$

$$= 0 \quad (\text{ex-0308.108})$$

## Exercise 3.8 Symmetry of $R_{ab}$ alternative solution

This differs from the previous code by the inclusion of a call to `canonicalise` immediately after the first two substitutions and a declaration that  $\Gamma^a_{bc}$  is symmetric in  $bc$ . This pair of changes produces a more compact set of results than given above. Incidentally, this also shows that  $\partial_a \Gamma^c_{bc} = \partial_b \Gamma^c_{ac}$ .

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative;
4
5  \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
6
7  g_{a b}::Metric;
8  g^{a b}::InverseMetric;
9
10 dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
11                                     # cdb (dgab.000,dgab)
12
13 Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} ( \partial_{b}{g_{e c}}
14                                     + \partial_{c}{g_{b e}}
15                                     - \partial_{e}{g_{b c}}).
16                                     # cdb (Gamma.000,Gamma)
17
18 Rabcd := R^{a}_{b c d} ->
19     \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
20     - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
21                                     # cdb (Rabcd.000,Rabcd)
22
23 Rab := R_{a b} -> R^{c}_{a c b}.
24                                     # cdb (Rab.000,Rab)
25
26 expr := 4 (R_{a b} - R_{b a}).
27                                     # cdb (ex-0308.200,expr)
28
29 substitute (expr, Rab)
30                                     # cdb (ex-0308.201,expr)
31 substitute (expr, Rabcd)
32                                     # cdb (ex-0308.202,expr)
33 canonicalise (expr)
34                                     # cdb (ex-0308.203,expr)
35 substitute (expr, Gamma)
36                                     # cdb (ex-0308.204,expr)
37
38 distribute (expr)
39                                     # cdb (ex-0308.205,expr)

```

```

33 product_rule (expr)                # cdb (ex-0308.206,expr)
34 canonicalise (expr)                # cdb (ex-0308.207,expr)
35
36 substitute (expr, dgab)            # cdb (ex-0308.208,expr)
37 canonicalise (expr)                # cdb (ex-0308.209,expr)

```

$$\begin{aligned}
4R_{ab} - 4R_{ba} &= 4R^c_{acb} - 4R^c_{bca} && (\text{ex-0308.201}) \\
&= 4\partial_c \Gamma^c_{ab} + 4\Gamma^c_{ec} \Gamma^e_{ab} - 4\partial_b \Gamma^c_{ac} - 4\Gamma^c_{eb} \Gamma^e_{ac} - 4\partial_c \Gamma^c_{ba} - 4\Gamma^c_{ec} \Gamma^e_{ba} + 4\partial_a \Gamma^c_{bc} + 4\Gamma^c_{ea} \Gamma^e_{bc} && (\text{ex-0308.202}) \\
&= -4\partial_b \Gamma^c_{ac} + 4\partial_a \Gamma^c_{bc} && (\text{ex-0308.203}) \\
&= -2\partial_b (g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac})) + 2\partial_a (g^{ce} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc})) && (\text{ex-0308.204}) \\
&= -2\partial_b (g^{ce} \partial_a g_{ec}) - 2\partial_b (g^{ce} \partial_c g_{ae}) + 2\partial_b (g^{ce} \partial_e g_{ac}) + 2\partial_a (g^{ce} \partial_b g_{ec}) + 2\partial_a (g^{ce} \partial_c g_{be}) - 2\partial_a (g^{ce} \partial_e g_{bc}) && (\text{ex-0308.205}) \\
&= -2\partial_b g^{ce} \partial_a g_{ec} - 2g^{ce} \partial_{ba} g_{ec} - 2\partial_b g^{ce} \partial_c g_{ae} - 2g^{ce} \partial_{bc} g_{ae} + 2\partial_b g^{ce} \partial_e g_{ac} + 2g^{ce} \partial_{be} g_{ac} + 2\partial_a g^{ce} \partial_b g_{ec} + 2g^{ce} \partial_{ab} g_{ec} + 2\partial_a g^{ce} \partial_c g_{be} \\
&\quad + 2g^{ce} \partial_{ac} g_{be} - 2\partial_a g^{ce} \partial_e g_{bc} - 2g^{ce} \partial_{ae} g_{bc} && (\text{ex-0308.206}) \\
&= -2\partial_b g^{ce} \partial_a g_{ce} + 2\partial_a g^{ce} \partial_b g_{ce} && (\text{ex-0308.207}) \\
&= 2g^{cd} g^{ef} \partial_b g_{df} \partial_a g_{ce} - 2g^{cd} g^{ef} \partial_a g_{df} \partial_b g_{ce} && (\text{ex-0308.208}) \\
&= 0 && (\text{ex-0308.209})
\end{aligned}$$