

Example 1 The metric connection

```
1  # Define some properties
2
3  {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
4
5  g_{a b}::Metric.
6  g_{a}^{b}::KroneckerDelta.
7
8  \nabla{#}::Derivative.
9  \partial{#}::PartialDerivative.
10
11 # Define rules for covariant derivative and the Christoffel symbol
12
13 nabla := \nabla_{c}{g_{a b}} -> \partial_{c}{g_{a b}} - g_{a d}\Gamma^{d}_{b c}
14                                     - g_{d b}\Gamma^{d}_{a c}.    # cdb (nabla.100,nabla)
15
16 Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a d} ( \partial_{b}{g_{d c}}
17                                     + \partial_{c}{g_{b d}}
18                                     - \partial_{d}{g_{b c}} ).    # cdb (Gamma.100,Gamma)
19
20 # Start with a simple expression
21
22 cderiv := \nabla_{c}{g_{a b}}.    # cdb (ex-01.100,cderiv)
23
24 # Do the computations
25
26 substitute      (cderiv, nabla)    # cdb (ex-01.101,cderiv)
27 substitute      (cderiv, Gamma)    # cdb (ex-01.102,cderiv)
28 distribute      (cderiv)           # cdb (ex-01.103,cderiv)
29 eliminate_metric (cderiv)          # cdb (ex-01.104,cderiv)
30 eliminate_kronecker (cderiv)       # cdb (ex-01.105,cderiv)
31 canonicalise     (cderiv)          # cdb (ex-01.106,cderiv)
32
33 checkpoint.append (cderiv)
```

$$\nabla_c g_{ab} \rightarrow \partial_c g_{ab} - g_{ad} \Gamma_{bc}^d - g_{db} \Gamma_{ac}^d \quad (\text{nabla a.100})$$

$$\Gamma_{bc}^a \rightarrow \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \quad (\text{Gamma.100})$$

$$\nabla_c g_{ab} = \partial_c g_{ab} - g_{ad} \Gamma_{bc}^d - g_{db} \Gamma_{ac}^d \quad (\text{ex-01.101})$$

$$= \partial_c g_{ab} - \frac{1}{2} g_{ad} g^{de} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) - \frac{1}{2} g_{db} g^{de} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}) \quad (\text{ex-01.102})$$

$$= \partial_c g_{ab} - \frac{1}{2} g_{ad} g^{de} \partial_b g_{ec} - \frac{1}{2} g_{ad} g^{de} \partial_c g_{be} + \frac{1}{2} g_{ad} g^{de} \partial_e g_{bc} - \frac{1}{2} g_{db} g^{de} \partial_a g_{ec} - \frac{1}{2} g_{db} g^{de} \partial_c g_{ae} + \frac{1}{2} g_{db} g^{de} \partial_e g_{ac} \quad (\text{ex-01.103})$$

$$= \partial_c g_{ab} - \frac{1}{2} g_a^e \partial_b g_{ec} - \frac{1}{2} g_a^e \partial_c g_{be} + \frac{1}{2} g_a^e \partial_e g_{bc} - \frac{1}{2} g_b^e \partial_a g_{ec} - \frac{1}{2} g_b^e \partial_c g_{ae} + \frac{1}{2} g_b^e \partial_e g_{ac} \quad (\text{ex-01.104})$$

$$= \frac{1}{2} \partial_c g_{ab} - \frac{1}{2} \partial_c g_{ba} \quad (\text{ex-01.105})$$

$$= 0 \quad (\text{ex-01.106})$$

Example 2 Covariant derivatives

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # rule for covariant derivative of v^{a}
7
8 deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.
9
10 # create an expression
11
12 foo := \nabla_{a}{v^{b}}. # cdb (ex-02.101,foo)
13
14 # apply the rule, then simplify
15
16 substitute (foo,deriv) # cdb (ex-02.102,foo)
17 canonicalise (foo) # cdb (ex-02.103,foo)
18
19 checkpoint.append (foo)
```

$$\nabla_a v^b = \partial_a v^b + \Gamma^b_{ca} v^c \quad (\text{ex-02.102})$$

$$= \partial_a v^b + \Gamma^{bc}_a v_c \quad (\text{ex-02.103})$$

Example 2 Covariant derivatives using “position=independent”

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # rule for covariant derivative of v^{a}
7
8 deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.
9
10 # create an expression
11
12 foo := \nabla_{a}{v^{b}}. # cdb (ex-02.201,foo)
13
14 # apply the rule, then simplify
15
16 substitute (foo,deriv) # cdb (ex-02.202,foo)
17 canonicalise (foo) # cdb (ex-02.203,foo)
18
19 checkpoint.append (foo)
```

$$\nabla_a v^b = \partial_a v^b + \Gamma^b_{ca} v^c \quad (\text{ex-02.202})$$

$$= \partial_a v^b + \Gamma^b_{ca} v^c \quad (\text{ex-02.203})$$

Example 2 Covariant derivatives using generic rule for deriv

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # template for covariant derivative of a vector
7
8 deriv := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}} + \Gamma^{b}_{c a} A^{c}.
9
10 # create an expression
11
12 foo := \nabla_{a}{u^{b}} + \nabla_{a}{v^{b}}. # cdb (ex-02.301,foo)
13
14 # apply the rule, then simplify
15
16 substitute (foo,deriv) # cdb (ex-02.302,foo)
17 canonicalise (foo) # cdb (ex-02.303,foo)
18
19 checkpoint.append (foo)

```

$$\nabla_a u^b + \nabla_a v^b = \partial_a u^b + \Gamma^b_{ca} u^c + \partial_a v^b + \Gamma^b_{ca} v^c \quad (\text{ex-02.302})$$

$$= \partial_a u^b + \Gamma^b_{ca} u^c + \partial_a v^b + \Gamma^b_{ca} v^c \quad (\text{ex-02.303})$$

Example 3a The Riemann curvature tensor

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});
6
7 ::Symbol; # Suggested by Kasper as a way to make use of ; legal
8           # see https://cadabra.science/qa/473/is-this-legal-syntax
9           # this code works with and without this trick
10
11 # rules for the first two covariant derivs of V^a
12
13 deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
14         + \Gamma^{a}_{c b} V^{c}.      # cdb (ex-03.101,deriv1)
15
16 deriv2 := V^{a}_{; b ; c} -> \partial_{c}{V^{a}_{; b}}
17         + \Gamma^{a}_{d c} V^{d}_{; b}
18         - \Gamma^{d}_{b c} V^{a}_{; d}. # cdb (ex-03.102,deriv2)
19
20 substitute (deriv2,deriv1)      # cdb (ex-03.103, deriv2)
21
22 Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}. # cdb (ex-03.104, Vabc)
23
24 substitute (Vabc,deriv2)      # cdb (ex-03.105, Vabc)
25
26 distribute      (Vabc)      # cdb (ex-03.106, Vabc)
27 product_rule    (Vabc)      # cdb (ex-03.107, Vabc)
28
29 sort_product    (Vabc)      # cdb (ex-03.108, Vabc)
30 rename_dummies  (Vabc)      # cdb (ex-03.109, Vabc)
31 canonicalise    (Vabc)      # cdb (ex-03.110, Vabc)
32
33 sort_sum        (Vabc)      # cdb (ex-03.111, Vabc)
34 factor_out      (Vabc,$V^{a?}$) # cdb (ex-03.112, Vabc)
35
36 checkpoint.append (Vabc)
```

```

37
38 # create rule for Riemann, export later (for use by lib/dgeom)
39
40 substitute (Vabc,$V^{a} -> -1$)           # cdb (ex-03.113, Vabc)
41                                           # note use of -1 to get correct
42                                           # signs when coupled with the rule
43                                           # for Rabcd (next statement)
44
45 Rabcd := R^{a}_{[d b c]} -> @(Vabc).       # cdb (ex-03.114, Rabcd) #
46
47 foo    := R^{a}_{[b c d]}.                 # cdb (ex-03.115, foo)
48 substitute (foo, Rabcd)                   # cdb (ex-03.116, foo)
49
50 # update rule to use nice indices
51
52 Rabcd := R^{a}_{[b c d]} -> @(foo).
53
54 checkpoint.append (Rabcd)

```

$$V^a{}_{;b} \rightarrow \partial_b V^a + \Gamma^a{}_{cb} V^c \quad (\text{ex-03.101})$$

$$V^a{}_{;b;c} \rightarrow \partial_c V^a{}_{;b} + \Gamma^a{}_{dc} V^d{}_{;b} - \Gamma^d{}_{bc} V^a{}_{;d} \quad (\text{ex-03.102})$$

$$V^a{}_{;b;c} \rightarrow \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.103})$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) - \partial_b (\partial_c V^a + \Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} (\partial_c V^d + \Gamma^d{}_{ec} V^e) + \Gamma^d{}_{cb} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.105})$$

$$= \partial_{cb} V^a + \partial_c (\Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b (\Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.106})$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.107})$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^e \Gamma^a{}_{dc} \Gamma^d{}_{eb} - \Gamma^d{}_{bc} \partial_d V^a - V^e \Gamma^a{}_{ed} \Gamma^d{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^e \Gamma^a{}_{db} \Gamma^d{}_{ec} + \Gamma^d{}_{cb} \partial_d V^a + V^e \Gamma^a{}_{ed} \Gamma^d{}_{cb} \quad (\text{ex-03.108})$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^d \Gamma^a{}_{ec} \Gamma^e{}_{db} - \Gamma^d{}_{bc} \partial_d V^a - V^d \Gamma^a{}_{de} \Gamma^e{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^d \Gamma^a{}_{eb} \Gamma^e{}_{dc} + \Gamma^d{}_{cb} \partial_d V^a + V^d \Gamma^a{}_{de} \Gamma^e{}_{cb} \quad (\text{ex-03.109})$$

$$= V^d \partial_c \Gamma^a{}_{bd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} \quad (\text{ex-03.110})$$

$$= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} \quad (\text{ex-03.111})$$

$$= V^d (\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} - \Gamma^a{}_{be} \Gamma^e{}_{cd} + \Gamma^a{}_{ce} \Gamma^e{}_{bd}) \quad (\text{ex-03.112})$$

$$= -R^a{}_{dbc} V^d \quad (\text{MTW})$$

$$R^a{}_{bcd} = -\partial_d \Gamma^a{}_{cb} + \partial_c \Gamma^a{}_{db} + \Gamma^a{}_{ce} \Gamma^e{}_{db} - \Gamma^a{}_{de} \Gamma^e{}_{cb}$$

Example 3b The Riemann curvature tensor

This differs from the above by not using the `::TableauSymmetry` property. It gives the same results as above but it does require a little bit more housekeeping.

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 ::Symbol; # Suggsted by Kasper as a way to make use of ; legal
6           # see https://cadabra.science/qa/473/is-this-legal-syntax
7           # this code works with and without this trick
8
9 # rules for the first two covariant derivs of V^a
10
11 deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
12         + \Gamma^{a}_{c b} V^{c}.      # cdb (ex-03.301,deriv1)
13
14 deriv2 := V^{a}_{; b ; c} -> \partial_{c}{V^{a}_{; b}}
15         + \Gamma^{a}_{d c} V^{d}_{; b}
16         - \Gamma^{d}_{b c} V^{a}_{; d}. # cdb (ex-03.302,deriv2)
17
18 substitute (deriv2,deriv1)      # cdb (ex-03.303, deriv2)
19
20 Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}. # cdb (ex-03.304, Vabc)
21
22 substitute (Vabc,deriv2)      # cdb (ex-03.305, Vabc)
23
24 distribute      (Vabc)      # cdb (ex-03.306, Vabc)
25 product_rule    (Vabc)      # cdb (ex-03.307, Vabc)
26
27 # -----
28 # trick to obtain a symmetric connection
29
30 G_{a b}::Symmetric.
31
32 substitute      (Vabc,$\Gamma^{a}_{b c} -> G^{a} G_{b c}$)
33 sort_product    (Vabc)      # cdb (ex-03.308, Vabc)
```

```

34 rename_dummies (Vabc) # cdb (ex-03.309, Vabc)
35 canonicalise (Vabc) # cdb (ex-03.310, Vabc)
36 substitute (Vabc,$G^{a} G_{b c} -> \Gamma^{a}_{b c}$,repeat=True)
37 # -----
38
39 sort_product (Vabc)
40 rename_dummies (Vabc)
41 canonicalise (Vabc)
42
43 sort_sum (Vabc) # cdb (ex-03.311, Vabc)
44 factor_out (Vabc,$V^{a?}$) # cdb (ex-03.312, Vabc)
45
46 checkpoint.append (Vabc)

```

$$V^a{}_{;b} \rightarrow \partial_b V^a + \Gamma^a{}_{cb} V^c \quad (\text{ex-03.301})$$

$$V^a{}_{;b;c} \rightarrow \partial_c V^a{}_{;b} + \Gamma^a{}_{dc} V^d{}_{;b} - \Gamma^d{}_{bc} V^a{}_{;d} \quad (\text{ex-03.302})$$

$$V^a{}_{;b;c} \rightarrow \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.303})$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) - \partial_b (\partial_c V^a + \Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} (\partial_c V^d + \Gamma^d{}_{ec} V^e) + \Gamma^d{}_{cb} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.305})$$

$$= \partial_{cb} V^a + \partial_c (\Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b (\Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.306})$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.307})$$

$$= \partial_{cb} V^a + V^d \partial_c (G^a G_{db}) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b (G^a G_{dc}) - G^a G^d G_{db} G_{ec} V^e + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \quad (\text{ex-03.308})$$

$$= \partial_{cb} V^a + V^d \partial_c (G^a G_{db}) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b (G^a G_{dc}) - G^a G^d G_{db} G_{ec} V^e + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \quad (\text{ex-03.309})$$

$$= V^d \partial_c (G^a G_{bd}) + G^a G^d G_{be} G_{cd} V^e - V^d \partial_b (G^a G_{cd}) - G^a G^d G_{bd} G_{ce} V^e \quad (\text{ex-03.310})$$

$$= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} + V^d \Gamma^a{}_{bd} \Gamma^e{}_{ce} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} \quad (\text{ex-03.311})$$

$$= V^d (\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} + \Gamma^a{}_{bd} \Gamma^e{}_{ce} - \Gamma^a{}_{be} \Gamma^e{}_{cd}) \quad (\text{ex-03.312})$$

$$= -R^a{}_{dbc} V^d \quad (\text{MTW})$$

Example 4 Python functions

```
1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 def truncate (poly,n):
4
5     # define the weight and give it a label
6     x^{a}::Weight(label=\epsilon).
7
8     # start with an empty expression
9     ans = Ex("0")
10
11    # loop over selected terms in the source
12    for i in range (0,n+1):
13
14        foo := @ (poly).
15        bah = Ex("\epsilon = " + str(i))
16
17        # extract a single term
18        keep_weight (foo, bah)
19
20        # update the running sum
21        ans = ans + foo
22
23    # all done, return final answer
24    return ans
25
26    Quartic := c^{a}
27              + c^{a}_{b} x^b
28              + c^{a}_{b c} x^b x^c
29              + c^{a}_{b c d} x^b x^c x^d
30              + c^{a}_{b c d e} x^b x^c x^d x^e.    # cdb (ex-04.100,Quartic)
31
32    Cubic = truncate (Quartic,3)                    # cdb (ex-04.101,Cubic)
33
34    checkpoint.append (Cubic)
```

$$p(x) = c^a + c^a_b x^b + c^a_{bc} x^b x^c + c^a_{bcd} x^b x^c x^d + c^a_{bcde} x^b x^c x^d x^e \quad (\text{ex-04.100})$$

$$q(x) = c^a + c^a_b x^b + c^a_{bc} x^b x^c + c^a_{bcd} x^b x^c x^d \quad (\text{ex-04.101})$$

Example 5a Keeping focused

```
1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.  
2  
3 expr := A_{a} v^{a} + B_{a} v^{a} + C_{a} v^{a}. # cdb (ex-05.100,expr)  
4  
5 zoom (expr,$B_{a} Q???) # cdb (ex-05.101,expr)  
6 substitute (expr, $v^{a} -> w^{a}$) # cdb (ex-05.102,expr)  
7 unzoom (expr) # cdb (ex-05.103,expr)  
8  
9 checkpoint.append (expr)
```

$$A_a v^a + B_a v^a + C_a v^a = \dots + B_a v^a + \dots \quad (\text{ex-05.101})$$

$$= \dots + B_a w^a + \dots \quad (\text{ex-05.102})$$

$$= A_a v^a + B_a w^a + C_a v^a \quad (\text{ex-05.103})$$

Example 5b Tags

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 def add_tags (obj,tag):
4     n = 0
5     ans = Ex('0')
6     for i in obj.top().terms():
7         foo = obj[i]
8         bah = Ex(tag+'_'+str(n)+'')
9         ans := @(ans) + @(bah) @(foo).
10        n = n + 1
11    return ans
12
13 def clear_tags (obj,tag):
14     ans := @(obj).
15     foo = Ex(tag+'_{a?} -> 1')
16     substitute (ans,foo)
17     return ans
18
19 expr := 2 V_{p q} - 3 V_{q p}. # cdb (ex-05.200,expr)
20
21 expr = add_tags (expr,'\mu') # cdb (ex-05.201,expr)
22
23 zoom (expr, $\mu_{1} Q??$) # cdb (ex-05.202,expr)
24 substitute (expr, $V_{a b} -> - V_{b a}$) # cdb (ex-05.203,expr)
25 unzoom (expr) # cdb (ex-05.204,expr)
26
27 expr = clear_tags (expr,'\mu') # cdb (ex-05.205,expr)
28
29 checkpoint.append (expr)
```

$$2V_{pq} - 3V_{qp} = 2\mu_0 V_{pq} - 3\mu_1 V_{qp} \quad (\text{ex-05.201})$$

$$= \dots - 3\mu_1 V_{qp} \quad (\text{ex-05.202})$$

$$= \dots + 3\mu_1 V_{pq} \quad (\text{ex-05.203})$$

$$= 2\mu_0 V_{pq} + 3\mu_1 V_{pq} \quad (\text{ex-05.204})$$

$$= 5V_{pq} \quad (\text{ex-05.205})$$

Example 6-01 Evaluating components

```
1  {\theta, \varphi}::Coordinate.  
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).  
3  
4  \partial{#}::PartialDerivative.  
5  
6  V := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.      # cdb(ex-06.100,V)  
7  dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.             # cdb(ex-06.101,dV)  
8  
9  evaluate (dV, V)    # cdb(ex-06.102,dV)
```

$$V_a = [V_\theta = \varphi, V_\varphi = \sin \theta] \quad (\text{ex-06.100})$$

$$\partial_b V_a - \partial_a V_b = \square_{ab} \begin{cases} \square_{\varphi\theta} = \cos \theta - 1 \\ \square_{\theta\varphi} = 1 - \cos \theta \end{cases} \quad (\text{ex-06.102})$$

Example 6-02 Riemann tensor of a 2-sphere

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
7                                         + \partial_{c}{g_{b d}}
8                                         - \partial_{d}{g_{b c}}).
9
10 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
11                        - \partial_{d}{\Gamma^{a}_{b c}}
12                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
13                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
14
15 gab := { g_{\theta\theta} = r**2,
16          g_{\varphi\varphi} = r**2 \sin(\theta)**2 }. # cdb(ex-06.201,gab)
17
18 iab := { g^{\theta\theta} = 1/r**2,
19          g^{\varphi\varphi} = 1/(r**2 \sin(\theta)**2) }. # cdb(ex-06.202,iab)
20
21 substitute (Rabcd, Gamma) # cdb(ex-06.203,Gamma)
22
23 evaluate (Gamma, join (gab,iab), rhsonly=True) # cdb(ex-06.204,Gamma)
24 evaluate (Rabcd, join (gab,iab), rhsonly=True) # cdb(ex-06.205,Rabcd)
25
26 # convert from a rule to a simple expression
27 Riem := R^{a}_{b c d}.
28 substitute (Riem, Rabcd) # cdb(ex-06.206,Riem)
29
30 from cdb.core.component import *
31
32 RiemCompt = get_component (Riem, $\theta, \varphi, \theta, \varphi$) # cdb(ex-06.207,RiemCompt)

```

$$[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin \theta)^2] \quad (\text{ex-06.201})$$

$$[g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2(\sin \theta)^2)^{-1}] \quad (\text{ex-06.202})$$

$$\Gamma^a_{bc} \rightarrow \square_{cb}{}^a \begin{cases} \square_{\varphi\theta}{}^\varphi = (\tan \theta)^{-1} \\ \square_{\theta\varphi}{}^\varphi = (\tan \theta)^{-1} \\ \square_{\varphi\varphi}{}^\theta = -\frac{1}{2} \sin(2\theta) \end{cases} \quad (\text{ex-06.204})$$

$$R^a_{bcd} \rightarrow \square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^\theta{}_\theta = (\sin \theta)^2 \\ \square_{\varphi\theta}{}^\varphi{}_\theta = -1 \\ \square_{\theta\varphi}{}^\theta{}_\varphi = -(\sin \theta)^2 \\ \square_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \quad (\text{ex-06.205})$$

$$\square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^\theta{}_\theta = (\sin \theta)^2 \\ \square_{\varphi\theta}{}^\varphi{}_\theta = -1 \\ \square_{\theta\varphi}{}^\theta{}_\varphi = -(\sin \theta)^2 \\ \square_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \quad (\text{ex-06.206})$$

$$R^\theta_{\varphi\varphi\theta} = -(\sin \theta)^2 \quad (\text{ex-06.207})$$

Example 6-03 Using complete to compute the inverse metric

This version uses `complete` to compute the inverse metric.

```
1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
7
8  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}}).
11
12  Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
13                        - \partial_{d}{\Gamma^{a}_{b c}}
14                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17  gab := { g_{\theta\theta} = r**2,
18          g_{\varphi\varphi} = r**2 \sin(\theta)**2 }. # cdb(ex-06.301,gab)
19
20  complete (gab, $g^{a b}$) # cdb(ex-06.302,gab)
21
22  substitute (Rabcd, Gamma)
23
24  evaluate (Gamma, gab, rhsonly=True) # cdb(ex-06.303,Gamma)
25  evaluate (Rabcd, gab, rhsonly=True) # cdb(ex-06.304,Rabcd)
```

$$[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin \theta)^2] \quad (\text{ex-06.301})$$

$$\left[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin \theta)^2, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2(\sin \theta)^2)^{-1} \right] \quad (\text{ex-06.302})$$

$$\Gamma^a_{bc} \rightarrow \square_{cb}^a \begin{cases} \square_{\varphi\theta}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2} \sin(2\theta) \end{cases} \quad (\text{ex-06.303})$$

$$R^a_{bcd} \rightarrow \square_{db}^a{}_c \begin{cases} \square_{\varphi\varphi}^{\theta}{}_{\theta} = (\sin \theta)^2 \\ \square_{\varphi\theta}^{\varphi}{}_{\theta} = -1 \\ \square_{\theta\varphi}^{\theta}{}_{\varphi} = -(\sin \theta)^2 \\ \square_{\theta\theta}^{\varphi}{}_{\varphi} = 1 \end{cases} \quad (\text{ex-06.304})$$

Example 6-04 Components by scalar projection

This example shows how one component of the Riemann tensor can be computed using a scalar projection.

```
1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  theta{#}::LaTeXForm("\theta").
5  varphi{#}::LaTeXForm("\varphi").
6
7  # usual definitions for the connection and Riemann tensor
8
9  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
10                                     + \partial_{c}{g_{b d}}
11                                     - \partial_{d}{g_{b c}}).
12
13  Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
14                                     - \partial_{d}{\Gamma^{a}_{b c}}
15                                     + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
16                                     - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
17
18  gab := { g_{\theta \theta} = r**2,
19           g_{\varphi \varphi} = r**2 \sin(\theta)**2 }. # cdb(ex-06.400,gab)
20
21  iab := { g^{\theta \theta} = 1/r**2,
22           g^{\varphi \varphi} = 1/(r**2 \sin(\theta)**2) }.
23
24  substitute (Rabcd, Gamma)
25  evaluate (Rabcd, join (gab,iab), rhsonly=True)
26
27  # above code just to compute Rabcd
28  # following code is all that is needed for the scalar projection method
29
30  # define the basis for vectors and dual vectors
31
32  basis := {theta^{\theta} = 1, varphi^{\varphi} = 1}.
33  dual := {theta_{\theta} = 1, varphi_{\varphi} = 1}.
34
```

```

35 # obtain components by contracting with basis
36
37 compt := R^{a}_{[b c d]} theta_{a} varphi^{b} theta^{c} varphi^{d}. # cdb(ex-06.401,compt)
38 substitute (compt,Rabcd)
39
40 evaluate (compt, join (basis,dual)) # cdb(ex-06.402,compt)
41
42 compt_sympy = compt._sympy_()
43
44 # cdbBeg(print.ex-06.04)
45 print ('type compt = ' + str(type(compt))) # shows that compt is a Cadabra object
46 print ('type ghiphi = ' + str(type(compt_sympy))) # shows that ghiphi is a Python object
47 print ('      compt = ' + str(compt)) # will contain LaTeX markup
48 print ('      ghiphi = ' + str(compt_sympy)) # will be pure Python/SymPy
49 # cdbEnd(print.ex-06.04)
50
51 checkpoint.append (compt)

```

$$R^{\theta}_{\varphi\theta\varphi} = R^a_{bcd}\theta_a\varphi^b\theta^c\varphi^d \quad (\text{ex-06.401})$$

$$= (\sin\theta)^2 \quad (\text{ex-06.402})$$

```

1 type compt = <class 'cadabra2.Ex'>
2 type ghiphi = <class 'sympy.core.power.Pow'>
3     compt = (\sin(\theta))**2
4     ghiphi = sin(theta)**2

```

Example 6-05 Components by selection

This example shows how one component of the metric tensor can be computed by indexing the result of a call to `evaluate`.

```
1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  theta{#}::LaTeXForm("\theta").
5  varphi{#}::LaTeXForm("\varphi").
6
7  gab := { g_{\theta \theta}    = r**2,
8           g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.  # cdb(ex-06.500,gab)
9
10 metric := g_{a b}.
11
12 evaluate (metric,gab)
13
14 indcs = metric[2][1][0]          # cdb(ex-06.501,indcs)
15 compt = metric[2][1][1]         # cdb(ex-06.502,compt)
16
17 # cdbBeg(print.ex-06.05)
18 print ('metric = ' + str(metric.input_form())+'\n')  # reveals Cadabra's internal structure for storing metric
19
20 print ('metric[0] = ' + str(metric[0]))
21 print ('metric[1] = ' + str(metric[1]))
22 print ('metric[2] = ' + str(metric[2])+'\n')
23
24 print ('metric[2][1] = '+ str(metric[2][1]))
25 print ('metric[2][1][0] = '+ str(metric[2][1][0]))
26 print ('metric[2][1][1] = '+ str(metric[2][1][1]))
27 # cdbEnd(print.ex-06.05)
28
29 checkpoint.append (indcs)
30 checkpoint.append (compt)
```


$$g_{\varphi\varphi} = g_{[\varphi,\varphi]} \quad (\text{ex-06.501})$$

$$= r^2 (\sin \theta)^2 \quad (\text{ex-06.502})$$

```

1  metric = \components_{a b}({{\theta}, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2)
2
3  metric[0] = a
4  metric[1] = b
5  metric[2] = {{{\theta}, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2}
6
7  metric[2][1] = {\varphi, \varphi} = (r)**2 (\sin(\theta))**2
8  metric[2][1][0] = {\varphi, \varphi}
9  metric[2][1][1] = (r)**2 (\sin(\theta))**2

```

Example 7 Export to C-code

```
1 def write_code (obj,name,filename,rank):
2
3     import os
4
5     from sympy.printing.c import C99CodePrinter as printer
6     from sympy.codegen.ast import Assignment
7
8     idx=[]  # indices in the form [{x, x}, {x, y} ...]
9     lst=[]  # corresponding terms [termxx, termxy, ...]
10
11     for i in range( len(obj[rank]) ):          # rank = number of free indices
12         idx.append( str(obj[rank][i][0]._sympy_()) ) # indices for this term
13         lst.append( str(obj[rank][i][1]._sympy_()) ) # the matching term
14
15     mat = sympy.Matrix([lst])                  # row vector of terms
16     sub_exprs, simplified_rhs = sympy.cse(mat)   # optimise code
17
18     with open(os.getcwd() + '/' + filename, 'w') as out:
19
20         for lhs, rhs in sub_exprs:
21             out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
22
23         for index, rhs in enumerate (simplified_rhs[0]):
24             lhs = sympy.Symbol(name+' '+idx[index]).replace(', ',')['])
25             out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
```

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  g_{a b}::Metric.
7  g^{a b}::InverseMetric.
8
9  Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} ( \partial_{g}{g_{b f}}
10                                     + \partial_{f}{g_{b g}}
11                                     - \partial_{b}{g_{f g}} ).
12
13  Rabcd := R^{d}_{e f g} -> \partial_{f}{\Gamma^{d}_{e g}}
14                               - \partial_{g}{\Gamma^{d}_{e f}}
15                               + \Gamma^{d}_{b f} \Gamma^{b}_{e g}
16                               - \Gamma^{d}_{b g} \Gamma^{b}_{e f}.
17
18  Rab := R_{a b} -> R^{c}_{c} {a c b}.
19
20  gab := { g_{\theta \theta} = r**2,
21           g_{\varphi \varphi} = r**2 \sin(\theta)**2 }. # cdb(ex-07.101,gab)
22
23  complete (gab, $g^{a b}$) # cdb(ex-07.102,gab)
24
25  substitute (Rabcd, Gamma)
26  substitute (Rab, Rabcd)
27
28  evaluate (Gamma, gab, rhsonly=True) # cdb(ex-07.103,Gamma)
29  evaluate (Rabcd, gab, rhsonly=True) # cdb(ex-07.104,Rabcd)
30  evaluate (Rab, gab, rhsonly=True) # cdb(ex-07.105,Rab)
31
32  write_code (Gamma[1], 'myGamma', 'example-07-gamma.c', 3)
33  write_code (Rabcd[1], 'myRabcd', 'example-07-rabcd.c', 4)
34  write_code (Rab[1], 'myRab', 'example-07-rab.c', 2)

```

$$[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin \theta)^2] \quad (\text{ex-07.101})$$

$$\left[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin \theta)^2, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2(\sin \theta)^2)^{-1} \right] \quad (\text{ex-07.102})$$

$$\Gamma^a_{fg} \rightarrow \square_{fg}^a \begin{cases} \square_{\varphi\theta}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2} \sin(2\theta) \end{cases} \quad (\text{ex-07.103})$$

$$R^d_{efg} \rightarrow \square_{eg}^d \begin{cases} \square_{\varphi\varphi}^{\theta} = (\sin \theta)^2 \\ \square_{\theta\varphi}^{\varphi} = -1 \\ \square_{\varphi\theta}^{\theta} = -(\sin \theta)^2 \\ \square_{\theta\theta}^{\varphi} = 1 \end{cases} \quad (\text{ex-07.104})$$

$$R_{ab} \rightarrow \square_{ab} \begin{cases} \square_{\varphi\varphi} = (\sin \theta)^2 \\ \square_{\theta\theta} = 1 \end{cases} \quad (\text{ex-07.105})$$

Example 8 Importing and exporting Cadabra expressions

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 def create (file_name):
4     import json, io, os, errno
5
6     try:
7         os.remove(file_name)           # delete the file if it exists
8         with open(file_name, 'w'): pass # create an empty file
9     except OSError as e:
10        if e.errno == errno.ENOENT:      # errno.ENOENT = no such file or directory
11            with open(file_name, 'w'): pass # create an empty file
12        else:
13            raise                       # report an exception
14
15    # Create and save an empty dict
16    data_out = {}
17    with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
18        out_file.write(json.dumps(data_out,
19                                indent=2,
20                                sort_keys=True,
21                                separators=(',', ' '),
22                                ensure_ascii=False)+'\n')
23
24 def put (key_name, object, file_name):
25     import json, io, os
26
27     # Read the current dict
28     with io.open(os.getcwd() + '/' + file_name) as inp_file:
29         data_out = json.load(inp_file)
30
31     # Add a new entry to the dict
32     data_out[key_name] = object.input_form()
33
34     # Save the updated dict
35     with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
36         out_file.write(json.dumps(data_out,
```

```

37         indent=2,
38         sort_keys=True,
39         separators=(',', ' '),
40         ensure_ascii=False)+'\n')
41
42 def get (key_name,file_name):
43     import json, io, os
44
45     # Read the current dict
46     with io.open(os.getcwd() + '/' + file_name) as inp_file:
47         data_inp = json.load(inp_file)
48
49     # Return one entry from the dict
50     return Ex (data_inp[key_name])
51
52 lib_name = 'example-08.json'
53
54 create (lib_name)
55
56 \nabla{#}::Derivative.
57
58 gab := g_{a b} - 1/3 x^{c} x^{d} R_{a c b d}
59         - 1/6 x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}. # cdb (ex-08-02.101,gab)
60
61 iab := g^{a b} + 1/3 x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
62         + 1/6 x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}. # cdb (ex-08-02.102,iab)
63
64 put ('g_ab',gab,lib_name)
65 put ('g^ab',iab,lib_name)
66
67 foo = get ('g_ab',lib_name) # cdb (ex-08-02.103,foo)
68 bah = get ('g^ab',lib_name) # cdb (ex-08-02.104,bah)
69
70 tmp := @(gab) - @(foo). # cdb (ex-08-02.105,tmp)
71 tmp := @(iab) - @(bah). # cdb (ex-08-02.106,tmp)

```

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} \quad (\text{ex-08-02.101})$$

$$g^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \quad (\text{ex-08-02.102})$$

$$\bar{g}_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} \quad (\text{ex-08-02.103})$$

$$\bar{g}^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \quad (\text{ex-08-02.104})$$

$$g_{ab}(x) - \bar{g}_{ab}(x) = 0 \quad (\text{ex-08-02.105})$$

$$g^{ab}(x) - \bar{g}^{ab}(x) = 0 \quad (\text{ex-08-02.106})$$

Example 9 The Gauss equation

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4
5 K_{a b}::Symmetric.
6 g^{a}_{b}::KroneckerDelta.
7
8 # define the projection operator
9
10 hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}.
11
12 # 3-covariant derivative obtained by projection on 4-covariant derivative
13
14 vpq:=v_{p q} -> h^{a}_{p} h^{b}_{q} \nabla_{b}{v_{a}}.
15
16 # compute 3-curvature by commutation of covariant derivatives
17
18 vpqr:= h^{a}_{p} h^{b}_{q} h^{c}_{r} ( \nabla_{c}{v_{a b}} - \nabla_{b}{v_{a c}} ).
19
20 substitute (vpq,hab)
21 substitute (vpqr,vpq)
22
23 distribute (vpqr)
24 product_rule (vpqr)
25 distribute (vpqr)
26 eliminate_kronecker (vpqr)
27
28 # standard substitutions
29
30 substitute (vpqr,$h^{a}_{b} n^{b} -> 0$)
31 substitute (vpqr,$h^{a}_{b} n_{a} -> 0$)
32 substitute (vpqr,$\nabla_{a}{g^{b}_{c}} -> 0$)
33 substitute (vpqr,$n^{a} \nabla_{b}{v_{a}} -> -v_{a} \nabla_{b}{n^{a}}$)
34 substitute (vpqr,$v_{a} \nabla_{b}{n^{a}} -> v_{p} h^{p}_{a} \nabla_{b}{n^{a}}$)
35 substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n_{q}} -> K_{a b}$)
36 substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n^{b}} -> K_{a}^{q}$) # cdb(ex-09.095,vpqr)
```



```

37
38 # tidy up
39
40 {v_{a},h^{a}_{b},K_{a}^{b},K_{a b},R^{a}_{b c d},\nabla_{a}\{v_{b}\}}::SortOrder.
41
42 sort_product      (vpqr)                # cdb(ex-09.096,vpqr)
43 rename_dummies    (vpqr)                # cdb(ex-09.097,vpqr)
44 canonicalise      (vpqr)                # cdb(ex-09.098,vpqr)
45 factor_out        (vpqr,$h^{a?}_{b?}$)   # cdb(ex-09.099,vpqr)
46 factor_out        (vpqr,$v_{a?}$)       # cdb(ex-09.101,vpqr)
47
48 checkpoint.append (vpqr)

```

$$(D_r D_q - D_q D_r) v_p = h^e_p h^d_q h^c_r \nabla_c (\nabla_d v_e) - h^e_p K_{rq} n^d \nabla_d v_e + K_q^b K_{rp} v_b - h^d_p h^b_q h^e_r \nabla_b (\nabla_e v_d) + h^d_p K_{qr} n^e \nabla_e v_d - K_{qp} K_r^c v_c \quad (\text{ex-09.095})$$

$$= h^c_r h^d_q h^e_p \nabla_c (\nabla_d v_e) - h^e_p K_{rq} \nabla_d v_e n^d + v_b K_q^b K_{rp} - h^b_q h^d_p h^e_r \nabla_b (\nabla_e v_d) + h^d_p K_{qr} \nabla_e v_d n^e - v_c K_r^c K_{qp} \quad (\text{ex-09.096})$$

$$= h^a_r h^b_q h^c_p \nabla_a (\nabla_b v_c) - h^b_p K_{rq} \nabla_a v_b n^a + v_a K_q^a K_{rp} - h^a_q h^c_p h^b_r \nabla_a (\nabla_b v_c) + h^b_p K_{qr} \nabla_a v_b n^a - v_a K_r^a K_{qp} \quad (\text{ex-09.097})$$

$$= h^a_p h^b_q h^c_r \nabla_c (\nabla_b v_a) + v_a K_q^a K_{pr} - h^a_p h^b_q h^c_r \nabla_b (\nabla_c v_a) - v_a K_r^a K_{pq} \quad (\text{ex-09.098})$$

$$= v_a K_q^a K_{pr} - v_a K_r^a K_{pq} + h^a_p h^b_q h^c_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) \quad (\text{ex-09.099})$$

$$= h^a_p h^b_q h^c_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.101})$$

```

1 R{#}::LaTeXForm("\{\strut\}^g R").
2
3 gRabcd := \nabla_{\{c\}}{\nabla_{\{b\}}{v_{\{a\}}}}
4          -\nabla_{\{b\}}{\nabla_{\{c\}}{v_{\{a\}}}} -> R^{\{d\}}_{\{a\} \{b\} \{c\}} v_{\{d\}}.
5
6 substitute      (vpqr,gRabcd)                # cdb(ex-09.102,vpqr)
7 distribute      (vpqr)                        # cdb(ex-09.103,vpqr)
8 substitute      (vpqr,$v_{\{a\}} -> h^{\{b\}}_{\{a\}} v_{\{b\}}$) # cdb(ex-09.104,vpqr)
9 substitute      (vpqr,$h^{\{b\}}_{\{a\}} K_{\{c\}}^{\{a\}} -> K_{\{c\}}^{\{b\}}$) # cdb(ex-09.105,vpqr)
10 sort_product   (vpqr)                        # cdb(ex-09.106,vpqr)
11 rename_dummies (vpqr)                        # cdb(ex-09.107,vpqr)
12 canonicalise   (vpqr)                        # cdb(ex-09.108,vpqr)
13 factor_out     (vpqr,$v_{\{a\}}$)            # cdb(ex-09.109,vpqr)
14 substitute      (vpqr,$v_{\{a\}}->1$)        # cdb(ex-09.110,vpqr)
15 sort_product   (vpqr)                        # cdb(ex-09.111,vpqr)
16
17 checkpoint.append (vpqr)

```

$$(D_r D_q - D_q D_r) v_p = h^a_p h^b_q h^c_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.101})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} v_d + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.102})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} v_d + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.103})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} h^e_d v_e + h^b_a v_b K_q^a K_{pr} - h^b_a v_b K_r^a K_{pq} \quad (\text{ex-09.104})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} h^e_d v_e + K_q^b v_b K_{pr} - K_r^b v_b K_{pq} \quad (\text{ex-09.105})$$

$$= v_e h^a_p h^b_q h^c_r h^e_d {}^g R^d_{abc} + v_b K_q^b K_{pr} - v_b K_r^b K_{pq} \quad (\text{ex-09.106})$$

$$= v_e h^b_p h^c_q h^d_r h^a_e {}^g R^a_{bcd} + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.107})$$

$$= v_a h^b_p h^c_q h^d_r h^a_e {}^g R^e_{bcd} + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.108})$$

$$= v_a (h^b_p h^c_q h^d_r h^a_e {}^g R^e_{bcd} + K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.109})$$

$${}^h R^a_{pqr} = h^b_p h^c_q h^d_r h^a_e {}^g R^e_{bcd} + K_q^a K_{pr} - K_r^a K_{pq} \quad (\text{ex-09.110})$$

$${}^h R^a_{pqr} = h^a_e h^b_p h^c_q h^d_r {}^g R^e_{bcd} + K_q^a K_{pr} - K_r^a K_{pq} \quad (\text{ex-09.111})$$

Example 10 The determinant of the metric

Our game here is to compute (the leading terms) in $\det g$ of the metric in RNC form

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f (8g^{gh} R_{acd g} R_{bef h} - 9\nabla_{cd} R_{aebf}) + \dots$$

For the sake of simplicity let's assume that we are working in 3-dimensions. The following analysis is easily generalised to other dimensions (and the final answers for $\det g$ and friends are unchanged).

Define ϵ_{ijk}^{abc} by

$$\epsilon_{ijk}^{abc} = \delta_i^a \delta_j^b \delta_k^c - \delta_i^b \delta_j^a \delta_k^c + \delta_i^c \delta_j^a \delta_k^b - \delta_i^c \delta_j^b \delta_k^a + \delta_i^b \delta_j^c \delta_k^a - \delta_i^a \delta_j^c \delta_k^b \quad (1)$$

It is easy to see that ϵ_{ijk}^{abc} is anti-symmetric in both its upper and lower indices. A trivial computation shows that for any 3×3 square matrix M_{ab} ,

$$\epsilon_{123}^{abc} M_{1a} M_{2b} M_{3c} = (\delta_1^a \delta_2^b \delta_3^c - \delta_1^b \delta_2^a \delta_3^c + \delta_1^c \delta_2^a \delta_3^b - \delta_1^c \delta_2^b \delta_3^a + \delta_1^b \delta_2^c \delta_3^a - \delta_1^a \delta_2^c \delta_3^b) M_{1a} M_{2b} M_{3c} = \det M \quad (2)$$

This can be easily generalised to

$$\epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} = \begin{cases} \pm \det M & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The \pm sign in the above depends on the particular permutations of (ijk) and (pqr) . If both permutations are even or both odd then the sign is $+1$ otherwise the sign is -1 . The same arguments can also be applied to a matrix inverse N^{-1} leading to

$$\epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} = \begin{cases} \pm \det N^{-1} & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Note that the \pm in this case will match exactly that for the case of $\det M$. Thus, multiplying both expressions and summing over all choices for (ijk) and (pqr) leads to

$$\sum_{\substack{(ijk) \\ (pqr)}} (\det N^{-1}) \det M = \epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} \epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} \quad (5)$$

where the sum on the left hand side includes just those (ijk) and (prq) that are permutations of (123) . There are $3!$ choices for (ijk) and $3!$ choices for (prq) and thus the left hand side is easily reduced to $(3!)^2 \det M / \det N$ where $\det N = 1/\det(N^{-1})$. For the right hand side notice that

$$\epsilon_{uvw}^{ijk} \epsilon_{ijk}^{abc} = 3! \epsilon_{uvw}^{abc} \quad (6)$$

which leads to

$$\det M = \frac{1}{3!} \det N \epsilon_{uvw}^{abc} M_{pa} M_{qb} M_{rc} N^{pu} N^{qv} N^{rw} \quad (7)$$

For our RNC metric we will set $N^{ab} = g^{ab}$ and $M_{ij} = g_{ij}(x)$. Since g^{ab} is of the form $\text{diag}(-1, 1, 1, 1)$ we have $\det g = -1$ and thus

$$\det g(x) = -\frac{1}{3!} \epsilon_{ijk}^{abc} g_{pa}(x) g_{qb}(x) g_{rc}(x) g^{ip} g^{jq} g^{kr} \quad (8)$$

The ϵ_{ijk}^{abc} can be constructed in Cadabra by applying the `asym` algorithm to the upper indices of $\delta_i^a \delta_j^b \delta_k^c$. Note that `asym` will include the $1/3!$ coefficient as part of its output.

The following code computes $-\det g$ rather than $\det g$.

Note that Calzetta et al. use an opposite sign for R_{abcd} so when comparing the following results against Calzetta do take note of this flipped sign in R_{abcd} .

The determinant of the metric

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Integer(1..3).
4
5 \nabla{#}::Derivative.
6
7 d{#}::KroneckerDelta.
8
9 g^{a b}::Symmetric.
10 g_{a b}::Symmetric.
11
12 R_{a b c d}::RiemannTensor.
13
14 x^{a}::Weight(label=num,value=1).
15
16 def truncate (obj,n):
17
18     ans = Ex("0") # create a Cadabra object with value zero
19
20     for i in range (0,n+1):
21         foo := @(obj).
22         bah = Ex("num = " + str(i))
23         distribute (foo)
24         keep_weight (foo, bah)
25         ans = ans + foo
26
27     return ans
28
29 gab := g_{a b}
30         - (1/3) x^{c} x^{d} R_{a c b d}
31         - (1/6) x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
32         + (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
33                                         -9 \nabla_{c d}{R_{a e b f}} ). # cdb (ex-10.gab.000,gab)
34
35 iab := g^{a b}
36         + (1/3) x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
```

```

37      + (1/6) x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}
38      + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
39          ( 4 g^{i j} R_{c g d i} R_{e h f j}
40          + 3 \nabla_{c d}{R_{e g f h}} ). # cdb(ex-10.iab.000,iab)
41
42 distribute (gab)
43 distribute (iab)
44
45 gxab := gx_{a b} -> @(gab).
46
47 eps := d^{a}_{i} d^{b}_{j} d^{c}_{k}. # cdb (ex-10.eps.001,eps) # includes a factor of 1/3!
48 asym (eps,$^{a},^{b},^{c}$) # cdb (ex-10.eps.002,eps)
49
50 # compute negative detg rather than det g, note 1/3! included in eps
51 Ndetg := @(eps) gx_{p a} gx_{q b} gx_{r c} g^{i p} g^{j q} g^{k r}. # cdb (ex-10.Ndetg.001,Ndetg)
52
53 substitute (Ndetg,gxab) # cdb (ex-10.Ndetg.002,Ndetg)
54 distribute (Ndetg) # cdb (ex-10.Ndetg.003,Ndetg)
55 Ndetg = truncate (Ndetg,4) # cdb (ex-10.Ndetg.004,Ndetg)
56 substitute (Ndetg,$g^{a b} g_{b c} -> d^{a}_{c}$,repeat=True) # cdb (ex-10.Ndetg.005,Ndetg)
57 eliminate_kronecker (Ndetg) # cdb (ex-10.Ndetg.006,Ndetg)
58 sort_product (Ndetg) # cdb (ex-10.Ndetg.007,Ndetg)
59 rename_dummies (Ndetg) # cdb (ex-10.Ndetg.008,Ndetg)
60 canonicalise (Ndetg) # cdb (ex-10.Ndetg.009,Ndetg)
61
62 # introduce the Ricci tensor
63
64 substitute (Ndetg,$R_{a b c d} g^{a c} -> R_{b d}$,repeat=True) # cdb (ex-10.Ndetg.101,Ndetg)
65 substitute (Ndetg,$\nabla_{a}{R_{b c d e}} g^{b d} -> \nabla_{a}{R_{c e}}$,repeat=True) # cdb (ex-10.Ndetg.102,Ndetg)
66 substitute (Ndetg,$\nabla_{a b}{R_{c d e f}} g^{c e} -> \nabla_{a b}{R_{d f}}$,repeat=True) # cdb (ex-10.Ndetg.103,Ndetg)
67
68 # the following was based on sqrt-Ndetg.tex
69
70 sqrtNdetg := 1/2 + (1/2) @(Ndetg)
71      - (1/8) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
72      - (1/4) (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
73      # cdb (ex-10.sqrtNdetg.001,sqrtNdetg)
74

```

```

75 sort_product (sqrtNdetg) # cdb (ex-10.sqrtNdetg.002,sqrtNdetg)
76 rename_dummies (sqrtNdetg) # cdb (ex-10.sqrtNdetg.003,sqrtNdetg)
77 canonicalise (sqrtNdetg) # cdb (ex-10.sqrtNdetg.004,sqrtNdetg)
78
79 logNdetg := -1 + @(Ndetg)
80 - (1/2) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
81 - (1/18) R_{a b} \nabla_{c}\{R_{d e}\} x^{a} x^{b} x^{c} x^{d} x^{e}.
82 # cdb (ex-10.logNdetg.001,logNdetg)
83
84 sort_product (logNdetg) # cdb (ex-10.logNdetg.002,logNdetg)
85 rename_dummies (logNdetg) # cdb (ex-10.logNdetg.003,logNdetg)
86 canonicalise (logNdetg) # cdb (ex-10.logNdetg.004,logNdetg)
87
88 # =====
89 # the remaining code is just for pretty printing
90
91 def product_sort (obj):
92     substitute (obj,$ x^{a} -> A000^{a} $)
93     substitute (obj,$ g^{a b} -> A001^{a b} $)
94     substitute (obj,$ \nabla_{c}\{R_{a b}\} -> A004_{a b c} $)
95     substitute (obj,$ \nabla_{c d}\{R_{a b}\} -> A005_{a b c d} $)
96     substitute (obj,$ \nabla_{c d e}\{R_{a b}\} -> A006_{a b c d e} $)
97     substitute (obj,$ \nabla_{c d e f}\{R_{a b}\} -> A007_{a b c d e f} $)
98     substitute (obj,$ \nabla_{e}\{R_{a b c d}\} -> A008_{a b c d e} $)
99     substitute (obj,$ \nabla_{e f}\{R_{a b c d}\} -> A009_{a b c d e f} $)
100    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\} -> A010_{a b c d e f g} $)
101    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\} -> A011_{a b c d e f g h} $)
102    substitute (obj,$ R_{a b} -> A002_{a b} $)
103    substitute (obj,$ R_{a b c d} -> A003_{a b c d} $)
104    sort_product (obj)
105    rename_dummies (obj)
106    substitute (obj,$ A000^{a} -> x^{a} $)
107    substitute (obj,$ A001^{a b} -> g^{a b} $)
108    substitute (obj,$ A002_{a b} -> R_{a b} $)
109    substitute (obj,$ A003_{a b c d} -> R_{a b c d} $)
110    substitute (obj,$ A004_{a b c} -> \nabla_{c}\{R_{a b}\} $)
111    substitute (obj,$ A005_{a b c d} -> \nabla_{c d}\{R_{a b}\} $)
112    substitute (obj,$ A006_{a b c d e} -> \nabla_{c d e}\{R_{a b}\} $)

```

```

113 substitute (obj,$ A007_{a b c d e f}      -> \nabla_{c d e f}{R_{a b}}      $)
114 substitute (obj,$ A008_{a b c d e}        -> \nabla_{e}{R_{a b c d}}      $)
115 substitute (obj,$ A009_{a b c d e f}      -> \nabla_{e f}{R_{a b c d}}      $)
116 substitute (obj,$ A010_{a b c d e f g}    -> \nabla_{e f g}{R_{a b c d}}    $)
117 substitute (obj,$ A011_{a b c d e f g h}  -> \nabla_{e f g h}{R_{a b c d}}  $)
118
119 def get_term (obj,n):
120
121     x^{a}::Weight(label=xnum).
122
123     foo := @(obj).
124     bah = Ex("xnum = " + str(n))
125     keep_weight (foo,bah)
126
127     return foo
128
129 def reformat (obj,scale):
130     foo = Ex(str(scale))
131     bah := @(foo) @(obj).
132     distribute      (bah)
133     product_sort    (bah)
134     rename_dummies  (bah)
135     canonicalise    (bah)
136     sort_sum        (bah)
137     factor_out      (bah,$x^{a?}$)
138     ans := @(bah) / @(foo).
139     return ans
140
141 def rescale (obj,scale):
142     foo = Ex(str(scale))
143     bah := @(foo) @(obj).
144     distribute      (bah)
145     factor_out      (bah,$x^{a?}$)
146     return bah
147
148 # -----
149 # reformat Ndetg
150

```



```

151 Rterm0 = get_term (Ndetg,0)      # cdb (ex-10.Rterm0.701,Rterm0)
152 Rterm1 = get_term (Ndetg,1)      # cdb (ex-10.Rterm1.701,Rterm1)
153 Rterm2 = get_term (Ndetg,2)      # cdb (ex-10.Rterm2.701,Rterm2)
154 Rterm3 = get_term (Ndetg,3)      # cdb (ex-10.Rterm3.701,Rterm3)
155 Rterm4 = get_term (Ndetg,4)      # cdb (ex-10.Rterm4.701,Rterm4)
156
157 Rterm0 = reformat (Rterm0, 1)     # cdb (ex-10.Rterm0.702,Rterm0)
158 Rterm1 = reformat (Rterm1, 1)     # cdb (ex-10.Rterm1.702,Rterm1)
159 Rterm2 = reformat (Rterm2, 3)     # cdb (ex-10.Rterm2.702,Rterm2)
160 Rterm3 = reformat (Rterm3, 6)     # cdb (ex-10.Rterm3.702,Rterm3)
161 Rterm4 = reformat (Rterm4,180)    # cdb (ex-10.Rterm4.702,Rterm4)
162
163 Ndetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.Ndetg.701,Ndetg)
164
165 # -----
166 # reformat sqrtNdetg
167
168 Rterm0 = get_term (sqrtNdetg,0)   # cdb (ex-10.Rterm0.801,Rterm0)
169 Rterm1 = get_term (sqrtNdetg,1)   # cdb (ex-10.Rterm1.801,Rterm1)
170 Rterm2 = get_term (sqrtNdetg,2)   # cdb (ex-10.Rterm2.801,Rterm2)
171 Rterm3 = get_term (sqrtNdetg,3)   # cdb (ex-10.Rterm3.801,Rterm3)
172 Rterm4 = get_term (sqrtNdetg,4)   # cdb (ex-10.Rterm4.801,Rterm4)
173
174 Rterm0 = reformat (Rterm0, 1)     # cdb (ex-10.Rterm0.802,Rterm0)
175 Rterm1 = reformat (Rterm1, 1)     # cdb (ex-10.Rterm1.802,Rterm1)
176 Rterm2 = reformat (Rterm2, 6)     # cdb (ex-10.Rterm2.802,Rterm2)
177 Rterm3 = reformat (Rterm3, 12)    # cdb (ex-10.Rterm3.802,Rterm3)
178 Rterm4 = reformat (Rterm4,360)    # cdb (ex-10.Rterm4.802,Rterm4)
179
180 sqrtNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.sqrtNdetg.801,sqrtNdetg)
181
182 # -----
183 # reformat logNdetg
184
185 Rterm0 = get_term (logNdetg,0)    # cdb (ex-10.Rterm0.801,Rterm0)
186 Rterm1 = get_term (logNdetg,1)    # cdb (ex-10.Rterm1.801,Rterm1)
187 Rterm2 = get_term (logNdetg,2)    # cdb (ex-10.Rterm2.801,Rterm2)
188 Rterm3 = get_term (logNdetg,3)    # cdb (ex-10.Rterm3.801,Rterm3)

```

```

189 Rterm4 = get_term (logNdetg,4)      # cdb (ex-10.Rterm4.801,Rterm4)
190
191 Rterm0 = reformat (Rterm0, 1)      # cdb (ex-10.Rterm0.802,Rterm0)
192 Rterm1 = reformat (Rterm1, 1)      # cdb (ex-10.Rterm1.802,Rterm1)
193 Rterm2 = reformat (Rterm2, 3)      # cdb (ex-10.Rterm2.802,Rterm2)
194 Rterm3 = reformat (Rterm3, 6)      # cdb (ex-10.Rterm3.802,Rterm3)
195 Rterm4 = reformat (Rterm4,180)     # cdb (ex-10.Rterm4.802,Rterm4)
196
197 logNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.logNdetg.901,logNdetg)
198
199 checkpoint.append (Ndetg)
200 checkpoint.append (sqrtNdetg)
201 checkpoint.append (logNdetg)

```

The metric determinant in Riemann normal coordinates

$$-\det g(x) = 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) + \dots$$

The volume element in RNC

If $-\det g(x)$ is non-negative then we also have

$$\sqrt{-\det g(x)} = 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) + \dots$$

The log of -detg in RNC

$$\log(-\det g(x)) = -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) + \dots$$

Apart from the signs, this matches exactly the expression given by Calzetta et al. (eq. A14)

Example 11 The RNC connection.

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent).
2
3  D{#}::PartialDerivative.
4  \nabla{#}::Derivative.
5
6  g_{a b}::Metric.
7  g^{a b}::InverseMetric.
8  g^{a b}::Weight(label=gnum,value=1).
9
10 \delta{#}::KroneckerDelta.
11
12 R_{a b c d}::RiemannTensor.
13 R_{a b c d}::Depends(\nabla{#}).
14
15 x^{a}::Depends(D{#}).
16 x^{a}::Weight(label=xnum,value=1).
17
18 Dx := D_{a}{x^{b}} -> \delta^{b}_{a}.    # cdb (ex-11.000,Dx)
19
20 gab := g_{a b} -> g_{a b}
21      - (1/3)  x^{c} x^{d} R_{a c b d}
22      - (1/6)  x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
23      + (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
24      - 9 \nabla_{c d}{R_{a e b f}} ).    # cdb (ex-11.001,gab)
25
26 iab := g^{a b} -> g^{a b}
27      + (1/3)  x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
28      + (1/6)  x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}
29      + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
30      ( 4 g^{i j} R_{c g d i} R_{e h f j}
31      + 3 \nabla_{c d}{R_{e g f h}} ).    # cdb(ex-11.002,iab)
32
33 distribute (gab)
34 distribute (iab)
35
36 ChrSym := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( D_{b}{g_{d c}}

```

```

37         + D_{c}{g_{b d}}
38         - D_{d}{g_{b c}} ). # cdb (ex-11.003,ChrSym)
39
40 Gamma := \Gamma^{a}_{b c}. # cdb (ex-11.100,Gamma)
41
42 substitute (Gamma,ChrSym) # cdb (ex-11.101,Gamma)
43 substitute (Gamma,gab) # cdb (ex-11.102,Gamma)
44 substitute (Gamma,iab) # cdb (ex-11.103,Gamma)
45 distribute (Gamma) # cdb (ex-11.104,Gamma)
46 unwrap (Gamma) # cdb (ex-11.105,Gamma)
47 product_rule (Gamma) # cdb (ex-11.106,Gamma)
48 distribute (Gamma) # cdb (ex-11.107,Gamma)
49 substitute (Gamma,Dx) # cdb (ex-11.108,Gamma)
50 eliminate_kronecker (Gamma) # cdb (ex-11.109,Gamma)
51
52 def truncate (obj,n):
53
54     ans = Ex("0") # create a Cadabra object with value zero
55
56     for i in range (0,n+1):
57         foo := @(obj).
58         bah = Ex("xnum = " + str(i))
59         distribute (foo)
60         keep_weight (foo, bah)
61         ans = ans + foo
62
63     return ans
64
65 checkpoint.append (Gamma)
66
67 # sort_product (Gamma) # 52.3 sec, 49 Mbyte
68 # rename_dummies (Gamma) # 58.6 sec, 51 Mbyte
69 # canonicalise (Gamma) # killed after 20 mins and over 500 Mbyte
70
71 Gamma = truncate (Gamma,3) # cdb (ex-11.110,Gamma) # allow up to 3rd order in x^a
72
73 sort_product (Gamma)
74 rename_dummies (Gamma)

```

```

75 canonicalise (Gamma)
76
77 checkpoint.append (Gamma)
78
79 # =====
80 # the remaining code is just for pretty printing
81
82 def product_sort (obj):
83     substitute (obj,$ g^{a b}          -> A001^{a b}          $)
84     substitute (obj,$ x^{a}            -> A002^{a}          $)
85     substitute (obj,$ z^{a}            -> A003^{a}          $)
86     substitute (obj,$ R_{a b c d}       -> A004_{a b c d}     $)
87     substitute (obj,$ \nabla_{e}\{R_{a b c d}\} -> A005_{a b c d e}     $)
88     substitute (obj,$ \nabla_{e f}\{R_{a b c d}\} -> A006_{a b c d e f}   $)
89     sort_sum      (obj)
90     sort_product  (obj)
91     rename_dummies (obj)
92     substitute (obj,$ A001^{a b}          -> g^{a b}          $)
93     substitute (obj,$ A002^{a}            -> x^{a}            $)
94     substitute (obj,$ A003^{a}            -> z^{a}            $)
95     substitute (obj,$ A004_{a b c d}       -> R_{a b c d}       $)
96     substitute (obj,$ A005_{a b c d e}     -> \nabla_{e}\{R_{a b c d}\} $)
97     substitute (obj,$ A006_{a b c d e f}   -> \nabla_{e f}\{R_{a b c d}\} $)
98
99 def get_xterm (obj,n):
100
101     foo := @(obj).
102     bah = Ex("xnum = " + str(n))
103     distribute (foo)
104     keep_weight (foo, bah)
105
106     return foo
107
108 def get_gterm (obj,n):
109
110     foo := @(obj).
111     bah = Ex("gnum = " + str(n))
112     distribute (foo)

```

```

113     keep_weight (foo, bah)
114
115     return foo
116
117 def reformat (obj,scale):
118
119     foo = Ex(str(scale))
120     bah := @(foo) @(obj).
121
122     distribute      (bah)
123     product_sort    (bah)
124     rename_dummies  (bah)
125     canonicalise     (bah)
126     factor_out      (bah,$x^{a?},g^{b? c?}$)
127     ans := @(bah) / @(foo).
128
129     return ans
130
131 gam1 = get_xterm (Gamma, 1)           # cdb (ex-11.200,gam1)
132 gam2 = get_xterm (Gamma, 2)           # cdb (ex-11.201,gam2)
133 gam3 = get_xterm (Gamma, 3)           # cdb (ex-11.202,gam3)
134
135 gam31 = get_gterm (gam3, 1)           # cdb (ex-11.210,gam31)
136 gam32 = get_gterm (gam3, 2)           # cdb (ex-11.211,gam31)
137
138 gam1 = reformat (gam1, 3)              # cdb (ex-11.220,gam1)
139 gam2 = reformat (gam2, 12)             # cdb (ex-11.221,gam2)
140
141 gam31 = reformat (gam31, 40)           # cdb (ex-11.222,gam31)
142 gam32 = reformat (gam32, 45)           # cdb (ex-11.223,gam32)
143
144 Gamma := @(gam1) + @(gam2) + @(gam31) + @(gam32). # cdb (ex-11.230,Gamma)
145 Scaled := 360 @(Gamma).                # cdb (ex-11.231,Scaled)
146
147 checkpoint.append (Gamma)

```

$$\begin{aligned}
\Gamma^a{}_{bc}(x) = & \frac{1}{3}g^{ad}x^e(R_{bdce} + R_{becd}) + \frac{1}{12}g^{ad}x^ex^f(-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}) \\
& + \frac{1}{40}g^{ad}x^ex^fx^g(-\nabla_{ce}R_{bfdg} - \nabla_{ec}R_{bfdg} + \nabla_{de}R_{bfcg} + \nabla_{ed}R_{bfcg} + 2\nabla_{ef}R_{bdcg} + 2\nabla_{ef}R_{bgcd} - \nabla_{be}R_{cfdg} - \nabla_{eb}R_{cfdg}) \\
& + \frac{1}{45}g^{ad}g^{ef}x^gx^hx^i(4R_{becg}R_{dhfi} + 4R_{bgce}R_{dhfi} - 2R_{bdeg}R_{chfi} - R_{bedg}R_{chfi} + R_{bgde}R_{chfi} - 2R_{bgeh}R_{cdfi} - R_{bgeh}R_{cfdi} \\
& \hspace{15em} + R_{bgeh}R_{cidf}) \quad (\text{ex-11.230})
\end{aligned}$$

$$\begin{aligned}
360\Gamma^a{}_{bc}(x) = & 120g^{ad}x^e(R_{bdce} + R_{becd}) + 30g^{ad}x^ex^f(-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}) \\
& + 9g^{ad}x^ex^fx^g(-\nabla_{ce}R_{bfdg} - \nabla_{ec}R_{bfdg} + \nabla_{de}R_{bfcg} + \nabla_{ed}R_{bfcg} + 2\nabla_{ef}R_{bdcg} + 2\nabla_{ef}R_{bgcd} - \nabla_{be}R_{cfdg} - \nabla_{eb}R_{cfdg}) \\
& + 8g^{ad}g^{ef}x^gx^hx^i(4R_{becg}R_{dhfi} + 4R_{bgce}R_{dhfi} - 2R_{bdeg}R_{chfi} - R_{bedg}R_{chfi} + R_{bgde}R_{chfi} - 2R_{bgeh}R_{cdfi} - R_{bgeh}R_{cfdi} \\
& \hspace{15em} + R_{bgeh}R_{cidf}) \quad (\text{ex-11.231})
\end{aligned}$$

Save Γ_{bc}^a for later use in Example 12.

```
1  jsonfile = 'example-11.json'
2  cdblib.create (jsonfile)
3  cdblib.put ('Gamma',Gamma,jsonfile)
```

Example 12 Checking the 2nd and 3rd order terms of Calzetta etal.

The following calculations show that my results for the RNC connection agree with those of Calzetta etal. to third order terms.

Note that I take ∇_{ab} to be $\nabla_a(\nabla_b)$.

Note also that $(LCB) R_{abcd} = -(Calzetta) R_{abcd}$. Consequently, I replace R_{abcd} with $-R_{abcd}$ in the Calzetta expressions (done as a Cadabra substitution rule).

This is relatively straightforward. We just apply a few carefully chosen applications of the first and second Bianchi identities.

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,u,v#}::Indices("latin",position=independent).
2 {\mu,\nu,\rho,\sigma,\tau,\lambda,\xi#}::Indices("greek",position=independent).
3
4 \nabla{#}::Derivative.
5
6 g_{a b}::Metric.
7 g^{a b}::InverseMetric.
8 g^{a b}::Weight(label=gnum,value=1).
9
10 \delta{#}::KroneckerDelta.
11
12 R_{a b c d}::RiemannTensor.
13 R_{a b c d}::Depends(\nabla{#}).
14
15 x^{a}::Weight(label=xnum,value=1).
16
17 def add_tags (obj,tag):
18
19     n = 0
20     ans = Ex('0')
21
22     for i in obj.top().terms():
23         foo = obj[i]
24         bah = Ex(tag+'_{'+str(n)+'}')
25         ans := @(ans) + @(bah) @(foo).
26         n = n + 1
27
28     return ans
29
30 def clear_tags (obj,tag):
31
32     ans := @(obj).
33     foo = Ex(tag+'_{a?} -> 1')
34     substitute (ans,foo)
35
36     return ans
37
38 def get_xterm (obj,n):

```

```

39
40     foo := @(obj).
41     bah = Ex("xnum = " + str(n))
42     distribute (foo)
43     keep_weight (foo, bah)
44
45     return foo
46
47 def get_gterm (obj,n):
48
49     foo := @(obj).
50     bah = Ex("gnum = " + str(n))
51     distribute (foo)
52     keep_weight (foo, bah)
53
54     return foo
55
56 def product_sort (obj):
57     substitute (obj,$ g^{a b}                                -> A001^{a b}                $)
58     substitute (obj,$ x^{a}                                   -> A002^{a}                $)
59     substitute (obj,$ z^{a}                                   -> A003^{a}                $)
60     substitute (obj,$ R_{a b c d}                             -> A004_{a b c d}          $)
61     substitute (obj,$ \nabla_{e}\{R_{a b c d}\}                -> A005_{a b c d e}        $)
62     substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}              -> A006_{a b c d e f}      $)
63     sort_sum (obj)
64     sort_product (obj)
65     rename_dummies (obj)
66     substitute (obj,$ A001^{a b}                               -> g^{a b}                $)
67     substitute (obj,$ A002^{a}                                 -> x^{a}                  $)
68     substitute (obj,$ A003^{a}                                 -> z^{a}                  $)
69     substitute (obj,$ A004_{a b c d}                           -> R_{a b c d}            $)
70     substitute (obj,$ A005_{a b c d e}                         -> \nabla_{e}\{R_{a b c d}\} $)
71     substitute (obj,$ A006_{a b c d e f}                       -> \nabla_{e f}\{R_{a b c d}\} $)
72
73 def reformat (obj,scaleA,scaleB):
74
75     foo = Ex(str(scaleA))
76     moo = Ex(str(scaleB))

```

```

77     bah := @(foo) @(obj) / @(moo).
78
79     distribute      (bah)
80     product_sort    (bah)
81     rename_dummies  (bah)
82     canonicalise     (bah)
83     factor_out       (bah,$g^{c? d?}$)
84     factor_out       (bah,$x^{a?},z^{b?}$)
85     ans := @(moo) @(bah) / @(foo).
86
87     return ans
88
89     # =====
90     # LCB
91
92     import cdblib
93     Gamma = cdblib.get ('Gamma','example-11.json')           # cdb(ex-12.100,Gamma)
94
95     # -----
96     # note: in versions prior to 24 may 2024, the first \Gamma^a shown in example 12
97     #       of the tutorial was actually the *downstairs* version
98     #       this block added to create reformed version of \Gamma^a for the tutorial
99
100    # note that the next line requires careful inspection of the free indices on Gamma
101    # expecting Gamma = \Gamma^a_{bc}
102
103    GammaUp := z^{b} z^{c} @(Gamma).                          # cdb(ex-12.110,GammaUp)
104
105    product_sort (GammaUp)                                     # cdb(ex-12.113,GammaUp)
106
107    checkpoint.append (GammaUp)
108
109    gam1 = get_xterm (GammaUp,1)                               # cdb(ex-12.210,gam1)
110    gam2 = get_xterm (GammaUp,2)                               # cdb(ex-12.211,gam2)
111    gam3 = get_xterm (GammaUp,3)                               # cdb(ex-12.212,gam3)
112
113    gam30 = get_gterm (gam3,1)                                 # cdb(ex-12.213,gam30)
114    gam31 = get_gterm (gam3,2)                                 # cdb(ex-12.214,gam31)

```

```

115
116 gam1 = reformat (gam1, 3,1) # cdb(ex-12.310,gam1)
117 gam2 = reformat (gam2,12,1) # cdb(ex-12.311,gam2)
118
119 gam30 = reformat (gam30,40,1) # cdb(ex-12.312,gam30)
120 gam31 = reformat (gam31,45,2) # cdb(ex-12.313,gam31)
121
122 gam3 := @(gam30) + @(gam31). # cdb(ex-12.314,gam3)
123
124 GammaUp := @(gam1) + @(gam2) + @(gam3). # cdb(ex-12.315,GammaUp)
125
126 checkpoint.append (GammaUp)
127 # -----
128
129 # lower index ^{a} to _{v}
130
131 Gamma := g_{v a} @(GammaUp).
132
133 distribute (Gamma)
134 substitute (Gamma, $g_{a d} g^{d b} -> \delta_{a}^{b}$)
135 eliminate_kronecker (Gamma) # cdb(ex-12.101,Gamma)
136
137 # change free index _{v} to _{a}
138
139 foo := tmp_{v} -> @(Gamma). # cdb(ex-12.191,foo)
140 bah := tmp_{a}. # cdb(ex-12.192,bah)
141 substitute (bah, foo) # cdb(ex-12.193,bah)
142
143 Gamma := @(bah). # cdb(ex-12.102,Gamma)
144
145 product_sort (Gamma) # cdb(ex-12.103,Gamma)
146
147 checkpoint.append (Gamma)
148
149 gam1 = get_xterm (Gamma,1) # cdb(ex-12.200,gam1)
150 gam2 = get_xterm (Gamma,2) # cdb(ex-12.201,gam2)
151 gam3 = get_xterm (Gamma,3) # cdb(ex-12.202,gam3)
152

```

```

153 gam30 = get_gterm (gam3,0)           # cdb(ex-12.203,gam30)
154 gam31 = get_gterm (gam3,1)           # cdb(ex-12.204,gam31)
155
156 gam1  = reformat (gam1, 3,1)           # cdb(ex-12.300,gam1)
157 gam2  = reformat (gam2,12,1)           # cdb(ex-12.301,gam2)
158
159 gam30 = reformat (gam30,40,1)           # cdb(ex-12.302,gam30)
160 gam31 = reformat (gam31,45,2)           # cdb(ex-12.303,gam31)
161
162 gam3  := @(gam30) + @(gam31).           # cdb(ex-12.304,gam3)
163
164 Gamma := @(gam1) + @(gam2) + @(gam3).   # cdb(ex-12.305,Gamma)
165
166 checkpoint.append (Gamma)

```

$$\begin{aligned}
\text{ex-12.100} &:= \frac{1}{3}g^{ad}x^e(R_{bdce} + R_{becd}) + \frac{1}{12}g^{ad}x^ex^f(-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}) \\
&+ \frac{1}{40}g^{ad}x^ex^fx^g(-\nabla_{ce}R_{bfdg} - \nabla_{ec}R_{bfdg} + \nabla_{de}R_{bfcg} + \nabla_{ed}R_{bfcg} + 2\nabla_{ef}R_{bdcg} + 2\nabla_{ef}R_{bgcd} - \nabla_{be}R_{cfdg} - \nabla_{eb}R_{cfdg}) \\
&+ \frac{1}{45}g^{ad}g^{ef}x^gx^hx^i(4R_{becg}R_{dhfi} + 4R_{bgce}R_{dhfi} - 2R_{bdeg}R_{chfi} - R_{bedg}R_{chfi} + R_{bgde}R_{chfi} - 2R_{bgeh}R_{cdfi} - R_{bgeh}R_{cfdi} + R_{bgeh}R_{cidf})
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.191} &:= tmp_v \\
&\rightarrow \frac{2}{3}x^cx^dz^eR_{vdce} + \frac{1}{3}x^cx^dz^ez^f\nabla_c R_{vedf} + \frac{1}{6}x^cx^dz^ez^f\nabla_e R_{vcd f} + \frac{1}{12}x^cx^dz^ez^f\nabla_v R_{cedf} + \frac{1}{10}x^cx^dx^ez^fz^g\nabla_{cd}R_{vfeg} \\
&+ \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{cf}R_{vdeg} + \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{fc}R_{vdeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{vc}R_{dfeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{cv}R_{dfeg} \\
&- \frac{4}{45}g^{cd}x^ex^fx^gz^hz^iR_{vhce}R_{dfgi} - \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{vech}R_{dfgi} + \frac{8}{45}g^{cd}x^ex^fx^gz^hz^iR_{vecf}R_{dhgi} + \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{vech}R_{dfgi}
\end{aligned}$$

$$\text{ex-12.192} := tmp_a$$

$$\begin{aligned}
\text{ex-12.193} &:= \frac{2}{3}x^cx^dz^eR_{adce} + \frac{1}{3}x^cx^dz^ez^f\nabla_c R_{aedf} + \frac{1}{6}x^cx^dz^ez^f\nabla_e R_{acd f} + \frac{1}{12}x^cx^dz^ez^f\nabla_a R_{cedf} + \frac{1}{10}x^cx^dx^ez^fz^g\nabla_{cd}R_{afeg} \\
&+ \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{cf}R_{adeg} + \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{fc}R_{adeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{ac}R_{dfeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{ca}R_{dfeg} \\
&- \frac{4}{45}g^{cd}x^ex^fx^gz^hz^iR_{ahce}R_{dfgi} - \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{aech}R_{dfgi} + \frac{8}{45}g^{cd}x^ex^fx^gz^hz^iR_{aecf}R_{dhgi} + \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{aech}R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.101} &:= \frac{2}{3}x^cx^dz^eR_{vdce} + \frac{1}{3}x^cx^dz^ez^f\nabla_c R_{vedf} + \frac{1}{6}x^cx^dz^ez^f\nabla_e R_{vcd f} + \frac{1}{12}x^cx^dz^ez^f\nabla_v R_{cedf} + \frac{1}{10}x^cx^dx^ez^fz^g\nabla_{cd}R_{vfeg} \\
&+ \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{cf}R_{vdeg} + \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{fc}R_{vdeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{vc}R_{dfeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{cv}R_{dfeg} \\
&- \frac{4}{45}g^{cd}x^ex^fx^gz^hz^iR_{vhce}R_{dfgi} - \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{vech}R_{dfgi} + \frac{8}{45}g^{cd}x^ex^fx^gz^hz^iR_{vecf}R_{dhgi} + \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{vech}R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.102} &:= \frac{2}{3}x^cx^dz^eR_{adce} + \frac{1}{3}x^cx^dz^ez^f\nabla_c R_{aedf} + \frac{1}{6}x^cx^dz^ez^f\nabla_e R_{acd f} + \frac{1}{12}x^cx^dz^ez^f\nabla_a R_{cedf} + \frac{1}{10}x^cx^dx^ez^fz^g\nabla_{cd}R_{afeg} \\
&+ \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{cf}R_{adeg} + \frac{1}{20}x^cx^dx^ez^fz^g\nabla_{fc}R_{adeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{ac}R_{dfeg} + \frac{1}{40}x^cx^dx^ez^fz^g\nabla_{ca}R_{dfeg} \\
&- \frac{4}{45}g^{cd}x^ex^fx^gz^hz^iR_{ahce}R_{dfgi} - \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{aech}R_{dfgi} + \frac{8}{45}g^{cd}x^ex^fx^gz^hz^iR_{aecf}R_{dhgi} + \frac{2}{45}g^{cd}x^ex^fx^gz^hz^iR_{aech}R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.103} := & \frac{2}{3}x^bz^cz^dR_{acbd} + \frac{1}{6}x^bx^cz^dz^e\nabla_dR_{abce} + \frac{1}{3}x^bx^cz^dz^e\nabla_bR_{adce} + \frac{1}{12}x^bx^cz^dz^e\nabla_aR_{bdce} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} \\
& + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} + \frac{1}{10}x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf} \\
& + \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + \frac{8}{45}g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}
\end{aligned}$$

$$\text{ex-12.200} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\text{ex-12.201} := \frac{1}{6}x^bx^cz^dz^e\nabla_dR_{abce} + \frac{1}{3}x^bx^cz^dz^e\nabla_bR_{adce} + \frac{1}{12}x^bx^cz^dz^e\nabla_aR_{bdce}$$

$$\begin{aligned}
\text{ex-12.202} := & \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} + \frac{1}{10}x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf} \\
& + \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + \frac{8}{45}g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}
\end{aligned}$$

$$\text{ex-12.203} := \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} + \frac{1}{10}x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf}$$

$$\text{ex-12.204} := \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + \frac{8}{45}g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}$$

$$\text{ex-12.300} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\text{ex-12.301} := \frac{1}{12}x^bx^cz^dz^e(2\nabla_dR_{abce} + 4\nabla_bR_{adce} + \nabla_aR_{bdce})$$

$$\text{ex-12.302} := \frac{1}{40}x^bx^cx^dz^ez^f(2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + 4\nabla_{bc}R_{aedf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf})$$

$$\text{ex-12.303} := \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h(R_{abdg}R_{cefh} + 4R_{adbe}R_{cgfh} - R_{adbg}R_{cefh} - 2R_{agbd}R_{cefh})$$

$$\begin{aligned}
\text{ex-12.304} := & \frac{1}{40}x^bx^cx^dz^ez^f(2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + 4\nabla_{bc}R_{aedf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf}) \\
& + \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h(R_{abdg}R_{cefh} + 4R_{adbe}R_{cgfh} - R_{adbg}R_{cefh} - 2R_{agbd}R_{cefh})
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.305} := & \frac{2}{3}x^bz^cz^dR_{acbd} + \frac{1}{12}x^bx^cz^dz^e(2\nabla_dR_{abce} + 4\nabla_bR_{adce} + \nabla_aR_{bdce}) \\
& + \frac{1}{40}x^bx^cx^dz^ez^f(2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + 4\nabla_{bc}R_{aedf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf}) \\
& + \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h(R_{abdg}R_{cefh} + 4R_{adbe}R_{cgfh} - R_{adbg}R_{cefh} - 2R_{agbd}R_{cefh})
\end{aligned}$$

$$\begin{aligned} \text{ex-12.110} := & z^b z^c \left(\frac{1}{3} g^{ad} x^e (R_{bdce} + R_{becd}) + \frac{1}{12} g^{ad} x^e x^f (-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}) + \frac{1}{40} g^{ad} x^e x^f x^g (-\nabla_{ce} R_{bf dg} \right. \\ & - \nabla_{ec} R_{bf dg} + \nabla_{de} R_{bf cg} + \nabla_{ed} R_{bf cg} + 2\nabla_{ef} R_{bd cg} + 2\nabla_{ef} R_{bg cd} - \nabla_{be} R_{cf dg} - \nabla_{eb} R_{cf dg}) + \frac{1}{45} g^{ad} g^{ef} x^g x^h x^i (4R_{becg} R_{dh fi} + 4R_{bg ce} R_{dh fi} \\ & \left. - 2R_{bdeg} R_{ch fi} - R_{bedg} R_{ch fi} + R_{bg de} R_{ch fi} - 2R_{bge h} R_{cdf i} - R_{bge h} R_{cdf i} + R_{bge h} R_{cid f}) \right) \end{aligned}$$

$$\begin{aligned} \text{ex-12.113} := & z^h z^i \left(\frac{1}{3} g^{ab} x^c (R_{hbic} + R_{hcib}) + \frac{1}{12} g^{ab} x^c x^d (2\nabla_c R_{hbid} + \nabla_b R_{hcid} - \nabla_i R_{hcbd} + 2\nabla_c R_{hdib} - \nabla_h R_{icbd}) + \frac{1}{40} g^{ab} x^c x^d x^e (2\nabla_{cd} R_{hb ie} \right. \\ & + \nabla_{bc} R_{hd ie} + \nabla_{cb} R_{hd ie} - \nabla_{ic} R_{hd be} - \nabla_{ci} R_{hd be} + 2\nabla_{cd} R_{he ib} - \nabla_{hc} R_{id be} - \nabla_{ch} R_{id be}) + \frac{1}{45} g^{ab} g^{cd} x^e x^f x^g (-2R_{hb ce} R_{if dg} + 4R_{hc ie} R_{bf dg} \\ & \left. - R_{hc be} R_{if dg} + 4R_{he ic} R_{bf dg} + R_{he bc} R_{if dg} - 2R_{he cf} R_{ib dg} - R_{he cf} R_{ib dg} + R_{he cf} R_{ig bd}) \right) \end{aligned}$$

$$\text{ex-12.210} := \frac{1}{3} z^h z^i g^{ab} x^c R_{hbic} + \frac{1}{3} z^h z^i g^{ab} x^c R_{hcib}$$

$$\text{ex-12.211} := \frac{1}{6} z^h z^i g^{ab} x^c x^d \nabla_c R_{hbid} + \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_b R_{hcid} - \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_i R_{hcbd} + \frac{1}{6} z^h z^i g^{ab} x^c x^d \nabla_c R_{hdib} - \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_h R_{icbd}$$

$$\begin{aligned} \text{ex-12.212} := & \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{hb ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{bc} R_{hd ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{cb} R_{hd ie} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ic} R_{hd be} \\ & - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ci} R_{hd be} + \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{he ib} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{hc} R_{id be} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ch} R_{id be} \\ & - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hb ce} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc ie} R_{bf dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc be} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he ic} R_{bf dg} \\ & + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he bc} R_{if dg} - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ib dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{id bg} + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ig bd} \end{aligned}$$

$$\begin{aligned} \text{ex-12.213} := & \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{hb ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{bc} R_{hd ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{cb} R_{hd ie} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ic} R_{hd be} \\ & - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ci} R_{hd be} + \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{he ib} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{hc} R_{id be} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ch} R_{id be} \end{aligned}$$

$$\begin{aligned} \text{ex-12.214} := & -\frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hb ce} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc ie} R_{bf dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc be} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he ic} R_{bf dg} \\ & + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he bc} R_{if dg} - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ib dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{id bg} + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ig bd} \end{aligned}$$

$$\text{ex-12.310} := \frac{2}{3}g^{ab}x^cz^dz^eR_{bdce}$$

$$\text{ex-12.311} := \frac{1}{12}g^{ab}x^cx^dz^ez^f(4\nabla_cR_{bedf} + 2\nabla_eR_{bcd f} + \nabla_bR_{cedf})$$

$$\text{ex-12.312} := \frac{1}{40}g^{ab}x^cx^dx^ez^fz^g(4\nabla_{cd}R_{bfeg} + 2\nabla_{cf}R_{bdeg} + 2\nabla_{fc}R_{bdeg} + \nabla_{bc}R_{dfeg} + \nabla_{cb}R_{dfeg})$$

$$\text{ex-12.313} := \frac{2}{45}g^{ab}g^{cd}x^ex^fx^gz^hz^i(-2R_{bhce}R_{dfgi} - R_{bec h}R_{dfgi} + 4R_{becf}R_{dhgi} + R_{bce h}R_{dfgi})$$

$$\begin{aligned} \text{ex-12.314} := & \frac{1}{40}g^{ab}x^cx^dx^ez^fz^g(4\nabla_{cd}R_{bfeg} + 2\nabla_{cf}R_{bdeg} + 2\nabla_{fc}R_{bdeg} + \nabla_{bc}R_{dfeg} + \nabla_{cb}R_{dfeg}) \\ & + \frac{2}{45}g^{ab}g^{cd}x^ex^fx^gz^hz^i(-2R_{bhce}R_{dfgi} - R_{bec h}R_{dfgi} + 4R_{becf}R_{dhgi} + R_{bce h}R_{dfgi}) \end{aligned}$$

$$\begin{aligned} \text{ex-12.315} := & \frac{2}{3}g^{ab}x^cz^dz^eR_{bdce} + \frac{1}{12}g^{ab}x^cx^dz^ez^f(4\nabla_cR_{bedf} + 2\nabla_eR_{bcd f} + \nabla_bR_{cedf}) \\ & + \frac{1}{40}g^{ab}x^cx^dx^ez^fz^g(4\nabla_{cd}R_{bfeg} + 2\nabla_{cf}R_{bdeg} + 2\nabla_{fc}R_{bdeg} + \nabla_{bc}R_{dfeg} + \nabla_{cb}R_{dfeg}) \\ & + \frac{2}{45}g^{ab}g^{cd}x^ex^fx^gz^hz^i(-2R_{bhce}R_{dfgi} - R_{bec h}R_{dfgi} + 4R_{becf}R_{dhgi} + R_{bce h}R_{dfgi}) \end{aligned}$$

```

1  # =====
2  # Calzetta
3  # note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}
4
5  GammaBar := z^{\nu} z^{\rho} (
6      (2/3) R^{\mu}_{\nu\rho\sigma} x^{\sigma}
7      + (1/12) (5 \nabla_{\lambda}\{R^{\mu}_{\nu\rho\sigma}\}
8          + \nabla_{\rho}\{R^{\mu}_{\sigma\nu\lambda}\}) x^{\sigma} x^{\lambda}
9      + (1/6) ( (9/10) \nabla_{\tau\lambda}\{R^{\mu}_{\rho\nu\sigma}\}
10          + (3/20) ( \nabla_{\tau\rho}\{R^{\mu}_{\sigma\nu\lambda}\}
11              + \nabla_{\rho\tau}\{R^{\mu}_{\sigma\nu\lambda}\} )
12          + (1/60) ( 21 R^{\mu}_{\lambda\xi\rho} R^{\xi}_{\sigma\nu\tau}
13              + 48 R^{\mu}_{\xi\rho\lambda} R^{\xi}_{\sigma\nu\tau}
14              - 37 R^{\mu}_{\sigma\xi\lambda} R^{\xi}_{\nu\rho\tau} ) ) x^{\sigma} x^{\lambda} x^{\tau} ).
15      # cdb(ex-12.400,GammaBar)
16
17  # convert from Greek to Latin indices
18
19  distribute (GammaBar)
20  rename_dummies (GammaBar,"greek","latin") # cdb(ex-12.401,GammaBar)
21
22  # lower the \mu index
23
24  GammaBar := \delta_{a \mu} @(GammaBar). # cdb(ex-12.402,GammaBar)
25  distribute (GammaBar) # cdb(ex-12.403,GammaBar)
26  eliminate_kronecker (GammaBar) # cdb(ex-12.404,GammaBar)
27
28  # sort products
29
30  product_sort (GammaBar) # cdb(ex-12.405,GammaBar)
31
32  checkpoint.append (GammaBar)
33
34  # Replace R with - R (Calzetta uses the non-MTW convention for Riemann)
35
36  substitute (GammaBar, $R_{a b c d} -> - R_{a b c d}$) # cdb(ex-12.406,GammaBar)
37  substitute (GammaBar, $R^{\{a\}_{b c d} -> - R^{\{a\}_{b c d}$) # cdb(ex-12.407,GammaBar)
38

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39 substitute (GammaBar, $R^{a}_{b c d} -> g^{a e} R_{e b c d}$) # cdb(ex-12.408,GammaBar)
40
41 cal1 = get_xterm (GammaBar,1) # cdb(ex-12.500,cal1)
42 cal2 = get_xterm (GammaBar,2) # cdb(ex-12.501,cal2)
43 cal3 = get_xterm (GammaBar,3) # cdb(ex-12.502,cal3)
44
45 cal1 = reformat (cal1,3,1) # cdb(ex-12.600,cal1)
46 cal2 = reformat (cal2,12,1) # cdb(ex-12.601,cal2)
47 # cal3 = reformat (cal3,360,1) # cdb(ex-12.602,cal3)
48
49 cal30 = get_gterm (cal3,0) # cdb(ex-12.602,cal30)
50 cal31 = get_gterm (cal3,1) # cdb(ex-12.603,cal31)
51
52 cal1 = reformat (cal1, 3,1) # cdb(ex-12.604,cal1)
53 cal2 = reformat (cal2,12,1) # cdb(ex-12.605,cal2)
54
55 cal30 = reformat (cal30,40,1) # cdb(ex-12.606,cal30)
56 cal31 = reformat (cal31,360,1) # cdb(ex-12.607,cal31)
57
58 cal3 := @(cal30) + @(cal31). # cdb(ex-12.608,cal3)
59
60 GammaBar := @(cal1) + @(cal2) + @(cal3). # cdb(ex-12.409,GammaBar)
61
62 checkpoint.append (GammaBar)

```

$$\text{ex-12.400} := z^\nu z^\rho \left(\frac{2}{3} R^\mu{}_{\nu\rho\sigma} x^\sigma + \frac{1}{12} (5 \nabla_\lambda R^\mu{}_{\nu\rho\sigma} + \nabla_\rho R^\mu{}_{\sigma\nu\lambda}) x^\sigma x^\lambda + \frac{1}{6} \left(\frac{9}{10} \nabla_{\tau\lambda} R^\mu{}_{\rho\nu\sigma} + \frac{3}{20} \nabla_{\tau\rho} R^\mu{}_{\sigma\nu\lambda} + \frac{3}{20} \nabla_{\rho\tau} R^\mu{}_{\sigma\nu\lambda} + \frac{7}{20} R^\mu{}_{\lambda\xi\rho} R^\xi{}_{\sigma\nu\tau} \right. \right. \\ \left. \left. + \frac{4}{5} R^\mu{}_{\xi\rho\lambda} R^\xi{}_{\sigma\nu\tau} - \frac{37}{60} R^\mu{}_{\sigma\xi\lambda} R^\xi{}_{\nu\rho\tau} \right) x^\sigma x^\lambda x^\tau \right)$$

$$\text{ex-12.401} := \frac{2}{3} z^a z^b R^\mu{}_{abc} x^c + \frac{5}{12} z^a z^b \nabla_d R^\mu{}_{abc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu{}_{abc} x^a x^c + \frac{3}{20} z^b z^a \nabla_{de} R^\mu{}_{abc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu{}_{abc} x^a x^c x^d \\ + \frac{1}{40} z^b z^d \nabla_{de} R^\mu{}_{abc} x^a x^c x^e + \frac{7}{120} z^e z^c R^\mu{}_{abc} R^b{}_{def} x^d x^a x^f + \frac{2}{15} z^e z^b R^\mu{}_{abc} R^a{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu{}_{abc} R^b{}_{def} x^a x^c x^f$$

$$\text{ex-12.402} := \delta_{a\mu} \left(\frac{2}{3} z^g z^b R^\mu{}_{gbc} x^c + \frac{5}{12} z^g z^b \nabla_d R^\mu{}_{gbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu{}_{gbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R^\mu{}_{gbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu{}_{gbc} x^g x^c x^d \right. \\ \left. + \frac{1}{40} z^b z^d \nabla_{de} R^\mu{}_{gbc} x^g x^c x^e + \frac{7}{120} z^e z^c R^\mu{}_{gbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R^\mu{}_{gbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu{}_{gbc} R^b{}_{def} x^g x^c x^f \right)$$

$$\text{ex-12.403} := \frac{2}{3} \delta_{a\mu} z^g z^b R^\mu{}_{gbc} x^c + \frac{5}{12} \delta_{a\mu} z^g z^b \nabla_d R^\mu{}_{gbc} x^c x^d + \frac{1}{12} \delta_{a\mu} z^b z^d \nabla_d R^\mu{}_{gbc} x^g x^c + \frac{3}{20} \delta_{a\mu} z^b z^g \nabla_{de} R^\mu{}_{gbc} x^c x^e x^d + \frac{1}{40} \delta_{a\mu} z^b z^e \nabla_{de} R^\mu{}_{gbc} x^g x^c x^d \\ + \frac{1}{40} \delta_{a\mu} z^b z^d \nabla_{de} R^\mu{}_{gbc} x^g x^c x^e + \frac{7}{120} \delta_{a\mu} z^e z^c R^\mu{}_{gbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} \delta_{a\mu} z^e z^b R^\mu{}_{gbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} \delta_{a\mu} z^d z^e R^\mu{}_{gbc} R^b{}_{def} x^g x^c x^f$$

$$\text{ex-12.404} := \frac{2}{3} z^g z^b R_{agbc} x^c + \frac{5}{12} z^g z^b \nabla_d R_{agbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R_{agbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R_{agbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R_{agbc} x^g x^c x^d \\ + \frac{1}{40} z^b z^d \nabla_{de} R_{agbc} x^g x^c x^e + \frac{7}{120} z^e z^c R_{agbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R_{agbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R_{agbc} R^b{}_{def} x^g x^c x^f$$

$$\text{ex-12.405} := \frac{2}{3} x^b z^c z^d R_{adcb} + \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} + \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\ + \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc}$$

$$\text{ex-12.406} := -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\ - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} + \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} - \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} - \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc}$$

$$\text{ex-12.407} := -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\ - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc}$$

$$\begin{aligned}
\text{ex-12.408} := & -\frac{2}{3}x^bz^cz^dR_{adcb} - \frac{1}{12}x^bx^cz^dz^e\nabla_eR_{acdb} - \frac{5}{12}x^bx^cz^dz^e\nabla_cR_{aedb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fc}R_{adeb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{cf}R_{adeb} \\
& - \frac{3}{20}x^bx^cx^dz^ez^f\nabla_{cd}R_{afeb} - \frac{37}{360}x^bx^cx^dz^ez^fR_{adgb}g^{gh}R_{hefc} + \frac{2}{15}x^bx^cx^dz^ez^fR_{ageb}g^{gh}R_{hcf d} + \frac{7}{120}x^bx^cx^dz^ez^fR_{adge}g^{gh}R_{hbfc}
\end{aligned}$$

$$\text{ex-12.500} := -\frac{2}{3}x^bz^cz^dR_{adcb}$$

$$\text{ex-12.501} := -\frac{1}{12}x^bx^cz^dz^e\nabla_eR_{acdb} - \frac{5}{12}x^bx^cz^dz^e\nabla_cR_{aedb}$$

$$\begin{aligned}\text{ex-12.502} := & -\frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fc}R_{adeb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{cf}R_{adeb} - \frac{3}{20}x^bx^cx^dz^ez^f\nabla_{cd}R_{afeb} \\ & - \frac{37}{360}x^bx^cx^dz^ez^fR_{adgb}g^{gh}R_{hefc} + \frac{2}{15}x^bx^cx^dz^ez^fR_{ageb}g^{gh}R_{hcf d} + \frac{7}{120}x^bx^cx^dz^ez^fR_{adge}g^{gh}R_{hbfc}\end{aligned}$$

$$\text{ex-12.600} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\text{ex-12.601} := \frac{1}{12}x^bx^cx^dz^e(\nabla_dR_{abce} + 5\nabla_bR_{adce})$$

$$\text{ex-12.602} := -\frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fc}R_{adeb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{cf}R_{adeb} - \frac{3}{20}x^bx^cx^dz^ez^f\nabla_{cd}R_{afeb}$$

$$\text{ex-12.603} := -\frac{37}{360}x^bx^cx^dz^ez^fR_{adgb}g^{gh}R_{hefc} + \frac{2}{15}x^bx^cx^dz^ez^fR_{ageb}g^{gh}R_{hcf d} + \frac{7}{120}x^bx^cx^dz^ez^fR_{adge}g^{gh}R_{hbfc}$$

$$\text{ex-12.604} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\text{ex-12.605} := \frac{1}{12}x^bx^cx^dz^e(\nabla_dR_{abce} + 5\nabla_bR_{adce})$$

$$\text{ex-12.606} := \frac{1}{40}x^bx^cx^dz^ez^f(\nabla_{be}R_{acdf} + \nabla_{eb}R_{acdf} + 6\nabla_{bc}R_{aedf})$$

$$\text{ex-12.607} := \frac{1}{360}g^{bc}x^dx^ex^fz^gz^h(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh})$$

$$\text{ex-12.608} := \frac{1}{40}x^bx^cx^dz^ez^f(\nabla_{be}R_{acdf} + \nabla_{eb}R_{acdf} + 6\nabla_{bc}R_{aedf}) + \frac{1}{360}g^{bc}x^dx^ex^fz^gz^h(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh})$$

$$\begin{aligned}
\text{ex-12.409} := & \frac{2}{3}x^bz^cz^dR_{acbd} + \frac{1}{12}x^bx^cz^dz^e(\nabla_dR_{abce} + 5\nabla_bR_{adce}) + \frac{1}{40}x^bx^cx^dz^ez^f(\nabla_{be}R_{acdf} + \nabla_{eb}R_{acdf} + 6\nabla_{bc}R_{aedf}) \\
& + \frac{1}{360}g^{bc}x^dx^ex^fz^gz^h(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh})
\end{aligned}$$

The fun begins $\Gamma - \bar{\Gamma}$

It's now time to compute the difference $\Gamma - \bar{\Gamma}$. Here it is.

```
1  def reformat_diff (obj):
2
3      distribute (obj)
4
5      obj1  = get_xterm (obj,1)
6      obj2  = get_xterm (obj,2)
7      obj3  = get_xterm (obj,3)
8
9      obj30 = get_gterm (obj3,0)
10     obj31 = get_gterm (obj3,1)
11
12     obj1  = reformat (obj1, 3,1)
13     obj2  = reformat (obj2,12,1)
14
15     obj30 = reformat (obj30,40,1)
16     obj31 = reformat (obj31,360,1)
17
18     obj3  := @(obj30) + @(obj31).
19
20     ans  := @(obj1) + @(obj2) + @(obj3).
21
22     return ans
23
24     # We could use reformat_diff here but instead we'll do it one step at a time so that
25     # we can see exactly what's going on. Later on we will use reformat_diff to do the job.
26
27     diff := @(Gamma) - @(GammaBar).                # cdb(ex-12.diff.100,diff)
28     distribute (diff)
29
30     diff1  = get_xterm (diff,1)                    # cdb(ex-12.diff.200,diff1)
31     diff2  = get_xterm (diff,2)                    # cdb(ex-12.diff.201,diff2)
32     diff3  = get_xterm (diff,3)                    # cdb(ex-12.diff.202,diff3)
33
34     diff30 = get_gterm (diff3,0)                    # cdb(ex-12.diff.203,diff30)
```

```

35 diff31 = get_gterm (diff3,1) # cdb(ex-12.diff.204,diff31)
36
37 diff1 = reformat (diff1, 3,1) # cdb(ex-12.diff.300,diff1)
38 diff2 = reformat (diff2,12,1) # cdb(ex-12.diff.301,diff2)
39
40 diff30 = reformat (diff30,40,1) # cdb(ex-12.diff.302,diff30)
41 diff31 = reformat (diff31,360,1) # cdb(ex-12.diff.303,diff31)
42
43 diff3 := @(diff30) + @(diff31). # cdb(ex-12.diff.304,diff3)
44
45 diff := @(diff1) + @(diff2) + @(diff3). # cdb(ex-12.diff.305,diff)

```

$$\begin{aligned}
\text{ex-12.diff.100} := & \frac{1}{12} x^b x^c z^d z^e (2\nabla_d R_{abce} + 4\nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (2\nabla_{be} R_{acdf} + 2\nabla_{eb} R_{acdf} + 4\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2R_{agbd} R_{cefh}) - \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5\nabla_b R_{adce}) \\
& - \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}) - \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37R_{adbe} R_{cgfh} - 21R_{adbg} R_{cefh} + 48R_{abdg} R_{cefh})
\end{aligned}$$

$$\text{ex-12.diff.200} := 0$$

$$\text{ex-12.diff.201} := \frac{1}{12} x^b x^c z^d z^e \nabla_d R_{abce} - \frac{1}{12} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\begin{aligned}
\text{ex-12.diff.202} := & \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}
\end{aligned}$$

$$\text{ex-12.diff.203} := \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$

$$\text{ex-12.diff.204} := -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.diff.300} := 0$$

$$\text{ex-12.diff.301} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce})$$

$$\text{ex-12.diff.302} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf})$$

$$\text{ex-12.diff.303} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh})$$

$$\begin{aligned} \text{ex-12.diff.304} := & \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

$$\begin{aligned} \text{ex-12.diff.305} := & \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

Second order terms

```
1 diff2 = get_xterm (diff,2)
2 diff2 := 12 @(diff2). # cdb (ex-12.701,diff2)
3 distribute (diff2) # cdb (ex-12.702,diff2)
4
5 diff2 = add_tags (diff2,'\\mu') # cdb (ex-12.711,diff2)
6
7 # swap indices on middle term, then apply 2nd Bianchi identity
8
9 zoom (diff2, $\\mu_{1} Q??$) # cdb (ex-12.712,diff2)
10 substitute (diff2, $\\nabla_{b}\\{R_{a d c e}\\} -> - \\nabla_{b}\\{R_{d a c e}\\}$) # cdb (ex-12.713,diff2)
11 unzoom (diff2)
12
13 substitute (diff2, $\\mu_{1} -> \\mu_{0}, \\mu_{2} -> \\mu_{0}$) # cdb (ex-12.714,diff2)
14 substitute (diff2, $\\mu_{0} -> 0$) # cdb (ex-12.715,diff2)
15
16 diff2 = clear_tags (diff2,'\\mu') # cdb (ex-12.716,diff2)
17
18 diff2 := @(diff2) / 12 .
19
20 diff := @(diff1) + @(diff2) + @(diff3).
21
22 diff = reformat_diff (diff) # cdb(ex-12.diff.306,diff)
```

$$\text{ex-12.701} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.702} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.711} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_2 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.712} := \dots - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.713} := \dots + \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.714} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} + \mu_0 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_0 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.715} := 0$$

$$\text{ex-12.716} := 0$$

$$\begin{aligned} \text{ex-12.diff.306} := & \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32R_{abdg} R_{cefh} + 27R_{adbe} R_{cgfh} + 5R_{adbg} R_{cefh} - 32R_{agbd} R_{cefh}) \end{aligned}$$

Third order terms, commute $\nabla\nabla R$ terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3).                                # cdb (ex-12.801,diff3)
3  distribute (diff3)                                    # cdb (ex-12.802,diff3)
4
5  # commutation rule for covariant derivs on Rabcd, see exrecise 3.6
6  # note: \nabla_{a b} defined as \nabla_a \nabla_b
7  CommuteNablaRiemann := \nabla_{f e}(R_{a b c d}) -> \nabla_{e f}(R_{a b c d})
8                                     + g^{u v} R_{u a e f} R_{v b c d}
9                                     + g^{u v} R_{u b e f} R_{a v c d}
10                                    + g^{u v} R_{u c e f} R_{a b v d}
11                                    + g^{u v} R_{u d e f} R_{a b c v}.
12
13 diff3 = add_tags (diff3, '\\mu')                      # cdb (ex-12.901,diff3)
14
15 # commute derivs on Rabcd so that each double deriv is of the form \nabla_{b*}
16
17 substitute (diff3, $\mu_{\{3\} } -> \mu_{\{1\}}$)      # cdb (ex-12.902,diff3)
18
19 zoom          (diff3, $\mu_{\{1\} } Q??$)              # cdb (ex-12.903,diff3)
20 substitute (diff3, CommuteNablaRiemann)                # cdb (ex-12.904,diff3)
21 unzoom        (diff3)
22
23 diff3 = clear_tags (diff3, '\\mu')
24 diff3 := @(diff3) / 360 .
25
26 distribute    (diff3)
27 canonicalise  (diff3)                                    # cdb (ex-12.905,diff3)
28
29 diff := @(diff1) + @(diff2) + @(diff3).
30
31 diff = reformat_diff (diff)                             # cdb(ex-12.diff.307,diff)

```

$$\begin{aligned} \text{ex-12.801} := & 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.802} := & 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.901} := & 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_3 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.902} := & 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.903} := \dots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \dots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \dots$$

$$\begin{aligned} \text{ex-12.904} := & \dots + 9\mu_1 x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + g^{uv} R_{uabe} R_{vcdf} + g^{uv} R_{ucbe} R_{avdf} + g^{uv} R_{udbe} R_{acvf} + g^{uv} R_{ufbe} R_{acdv}) + \dots \\ & + 9\mu_1 x^b x^c x^d z^e z^f (\nabla_{ba} R_{cedf} + g^{uv} R_{ucba} R_{vedf} + g^{uv} R_{ueba} R_{cvdf} + g^{uv} R_{udba} R_{cevf} + g^{uv} R_{ufba} R_{cedv}) + \dots \end{aligned}$$

$$\begin{aligned} \text{ex-12.905} := & \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abeu} R_{cfdv} - \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abcu} R_{defv} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.diff.307} := \frac{1}{40} x^b x^c x^d z^e z^f (2\nabla_{be} R_{acdf} - 2\nabla_{bc} R_{aedf} + 2\nabla_{ba} R_{cedf}) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32R_{abdg} R_{cefh} + 32R_{adbg} R_{cefh} - 32R_{agbd} R_{cefh})$$

Third order terms, use 2nd Bianchi identity on $\nabla\nabla R$ terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3). # cdb (ex-12.910,diff3)
3  distribute (diff3) # cdb (ex-12.911,diff3)
4
5  diff3 = add_tags (diff3,'\mu') # cdb (ex-12.912,diff3)
6
7  # swap indices on middle second deriv term, then apply 2nd Bianchi identity
8
9  zoom (diff3, $\mu_{1}$ Q??$) # cdb (ex-12.913,diff3)
10 substitute (diff3, $\nabla_{b c}\{R_{a e d f}\} \rightarrow - \nabla_{a c}\{R_{b e d f}\}$) # cdb (ex-12.914,diff3)
11 unzoom (diff3)
12
13 substitute (diff3, $\mu_{1} \rightarrow \mu_{0}$, $\mu_{2} \rightarrow \mu_{0}$) # cdb (ex-12.915,diff3)
14 substitute (diff3, $\mu_{0} \rightarrow 0$) # cdb (ex-12.916,diff3)
15
16 diff3 = clear_tags (diff3,'\mu')
17 diff3 := @(diff3) / 360 .
18
19 distribute (diff3)
20 canonicalise (diff3) # cdb (ex-12.917,diff3)
21
22 diff := @(diff1) + @(diff2) + @(diff3).
23
24 diff = reformat_diff (diff) # cdb(ex-12.diff.308,diff)

```

$$\begin{aligned}
\text{ex-12.910} &:= 18x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.911} &:= 18x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.912} &:= 18\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18\mu_2 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.913} &:= \dots - 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \dots \\
\text{ex-12.914} &:= \dots + 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + \dots \\
\text{ex-12.915} &:= 18\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 18\mu_0 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + 18\mu_0 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.916} &:= -32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.917} &:= -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.diff.308} &:= \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32R_{abdg} R_{cefh} + 32R_{adbg} R_{cefh} - 32R_{agbd} R_{cefh})
\end{aligned}$$

Third order terms, use 1st Bianchi identity on RR terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3).
3  distribute (diff3)
4
5  diff3 = add_tags (diff3,'\\mu') # cdb (ex-12.921,diff3)
6
7  # swap indices on middle term, then apply 1st Bianchi identity
8
9  zoom (diff3, $\\mu_{1} Q??$) # cdb (ex-12.922,diff3)
10 substitute (diff3, $R_{a d b g} R_{c e f h} -> - R_{a d g b} R_{c e f h}$) # cdb (ex-12.923,diff3)
11 unzoom (diff3)
12
13 substitute (diff3, $\\mu_{1} -> \\mu_{0}, \\mu_{2} -> \\mu_{0}$) # cdb (ex-12.924,diff3)
14 substitute (diff3, $\\mu_{0} -> 0$) # cdb (ex-12.925,diff3)
15
16 diff3 = clear_tags (diff3,'\\mu') # cdb (ex-12.926,diff3)
17
18 diff := @(diff1) + @(diff2) + @(diff3).
19
20 diff = reformat_diff (diff) # cdb(ex-12.diff.309,diff)

```

$$\text{ex-12.921} := -32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_2 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.922} := \dots + 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} + \dots$$

$$\text{ex-12.923} := \dots - 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} + \dots$$

$$\text{ex-12.924} := -32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} - 32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} - 32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.925} := 0$$

$$\text{ex-12.926} := 0$$

$$\text{ex-12.diff.309} := 0$$

Example 13a The Weyl tensor vanishes in 3d – direct proof

```

1 {x,y,z}::Coordinate.
2 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices(values={x,y,z},position=independent).
3
4 \partial{#}::PartialDerivative.
5
6 g_{a b}::Metric.
7 g^{a b}::InverseMetric.
8
9 {\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.
10
11 GammaU := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
12                                     + \partial_{c}{g_{b d}}
13                                     - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)
14
15 GammaD := \Gamma_{a b c} -> 1/2 ( \partial_{b}{g_{a c}}
16                               + \partial_{c}{g_{b a}}
17                               - \partial_{a}{g_{b c}}). # cdb(Gamma.010,GammaD)
18
19 Rabcd := R_{a b c d} -> \partial_{c}{\Gamma_{a b d}}
20                       - \partial_{d}{\Gamma_{a b c}}
21                       + \Gamma_{e a d} \Gamma^{e}_{b c}
22                       - \Gamma_{e a c} \Gamma^{e}_{b d}. # cdb (Rabcd.000,Rabcd)
23
24 Rab := R_{a b} -> g^{c d} R_{a c b d}. # cdb (Rab.000,Rab)
25
26 Rscalar := R -> g^{a b} R_{a b}. # cdb (R.000,Rscalar)
27
28 # Weyl in 3-dimensions
29
30 Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
31         - (g_{a c} R_{b d} - g_{a d} R_{b c})
32         + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}). # cdb (ex-13a.100,Cabcd)
33
34 # Use 8 Cabcd to clear the fractions
35
36 EightCabcd := 8 @(Cabcd). # cdb (ex-13a.110,EightCabcd)

```

```

37
38 substitute      (Cabcd,Rscalar)
39 substitute      (Cabcd,Rab)
40 substitute      (Cabcd,Rabcd)
41 substitute      (Cabcd,GammaU)
42 substitute      (Cabcd,GammaD)
43
44 distribute      (Cabcd)
45
46 sort_product    (Cabcd)
47 rename_dummies  (Cabcd)
48 canonicalise    (Cabcd)                                # cdb (ex-13a.101,Cabcd)
49
50 EightCabcd := 8 @(Cabcd).                                # cdb (ex-13a.111,EightCabcd)
51
52 gab := {g_{x x} = gxx, g_{x y} = gxy, g_{x z} = gxz,
53         g_{y x} = gxy, g_{y y} = gyy, g_{y z} = gyz,
54         g_{z x} = gxz, g_{z y} = gyz, g_{z z} = gzz}.
55
56 complete (gab, $g^{a b}$)
57 evaluate (Cabcd,gab)                                # cdb (ex-13a.102,Cabcd)
58 evaluate (EightCabcd,gab)                            # cdb (ex-13a.112,EightCabcd)

```


$$8C_{abcd} = 8R_{abcd} - 8R_{ac}g_{bd} + 8R_{ad}g_{bc} - 8g_{ac}R_{bd} + 8g_{ad}R_{bc} + 4R(g_{ac}g_{bd} - g_{ad}g_{bc}) \quad (\text{ex-13a.110})$$

$$\begin{aligned}
&= 4\partial_{bc}g_{ad} - 4\partial_{ac}g_{bd} - 4\partial_{bd}g_{ac} + 4\partial_{ad}g_{bc} + 2\partial_a g_{de}\partial_b g_{cf}g^{ef} + 2\partial_a g_{de}\partial_c g_{bf}g^{ef} - 2\partial_a g_{de}\partial_f g_{bc}g^{ef} + 2\partial_b g_{ce}\partial_d g_{af}g^{ef} + 2\partial_c g_{be}\partial_d g_{af}g^{ef} \\
&\quad - 2\partial_d g_{ae}\partial_f g_{bc}g^{ef} - 2\partial_b g_{ce}\partial_f g_{ad}g^{ef} - 2\partial_c g_{be}\partial_f g_{ad}g^{ef} + 2\partial_e g_{ad}\partial_f g_{bc}g^{ef} - 2\partial_a g_{ce}\partial_b g_{df}g^{ef} - 2\partial_a g_{ce}\partial_d g_{bf}g^{ef} + 2\partial_a g_{ce}\partial_f g_{bd}g^{ef} - 2\partial_b g_{de}\partial_c g_{af}g^{ef} \\
&\quad - 2\partial_c g_{ae}\partial_d g_{bf}g^{ef} + 2\partial_c g_{ae}\partial_f g_{bd}g^{ef} + 2\partial_b g_{de}\partial_f g_{ac}g^{ef} + 2\partial_d g_{be}\partial_f g_{ac}g^{ef} - 2\partial_e g_{ac}\partial_f g_{bd}g^{ef} - 4\partial_{ce}g_{af}g_{bd}g^{ef} + 4\partial_{ac}g_{ef}g_{bd}g^{ef} + 4\partial_{ef}g_{ac}g_{bd}g^{ef} \\
&\quad - 4\partial_{ae}g_{cf}g_{bd}g^{ef} - 2\partial_a g_{ef}\partial_c g_{gh}g_{bd}g^{eg}g^{fh} - 4\partial_e g_{af}\partial_g g_{ch}g_{bd}g^{eg}g^{fh} + 4\partial_e g_{af}\partial_g g_{ch}g_{bd}g^{eh}g^{fg} + 4\partial_a g_{ce}\partial_f g_{gh}g_{bd}g^{eg}g^{fh} - 2\partial_a g_{ce}\partial_f g_{gh}g_{bd}g^{ef}g^{gh} \\
&\quad + 4\partial_c g_{ae}\partial_f g_{gh}g_{bd}g^{eg}g^{fh} - 2\partial_c g_{ae}\partial_f g_{gh}g_{bd}g^{ef}g^{gh} - 4\partial_e g_{ac}\partial_f g_{gh}g_{bd}g^{eg}g^{fh} + 2\partial_e g_{ac}\partial_f g_{gh}g_{bd}g^{ef}g^{gh} + 4\partial_{de}g_{af}g_{bc}g^{ef} - 4\partial_{ad}g_{ef}g_{bc}g^{ef} \\
&\quad - 4\partial_{ef}g_{ad}g_{bc}g^{ef} + 4\partial_{ae}g_{df}g_{bc}g^{ef} + 2\partial_a g_{ef}\partial_d g_{gh}g_{bc}g^{eg}g^{fh} + 4\partial_e g_{af}\partial_g g_{dh}g_{bc}g^{eg}g^{fh} - 4\partial_e g_{af}\partial_g g_{dh}g_{bc}g^{eh}g^{fg} - 4\partial_a g_{de}\partial_f g_{gh}g_{bc}g^{eg}g^{fh} \\
&\quad + 2\partial_a g_{de}\partial_f g_{gh}g_{bc}g^{ef}g^{gh} - 4\partial_d g_{ae}\partial_f g_{gh}g_{bc}g^{eg}g^{fh} + 2\partial_d g_{ae}\partial_f g_{gh}g_{bc}g^{ef}g^{gh} + 4\partial_e g_{ad}\partial_f g_{gh}g_{bc}g^{eg}g^{fh} - 2\partial_e g_{ad}\partial_f g_{gh}g_{bc}g^{ef}g^{gh} - 4\partial_{de}g_{bf}g_{ac}g^{ef} \\
&\quad + 4\partial_{bd}g_{ef}g_{ac}g^{ef} + 4\partial_{ef}g_{bd}g_{ac}g^{ef} - 4\partial_{be}g_{df}g_{ac}g^{ef} - 2\partial_b g_{ef}\partial_d g_{gh}g_{ac}g^{eg}g^{fh} - 4\partial_e g_{bf}\partial_g g_{dh}g_{ac}g^{eg}g^{fh} + 4\partial_e g_{bf}\partial_g g_{dh}g_{ac}g^{eh}g^{fg} + 4\partial_b g_{de}\partial_f g_{gh}g_{ac}g^{eg}g^{fh} \\
&\quad - 2\partial_b g_{de}\partial_f g_{gh}g_{ac}g^{ef}g^{gh} + 4\partial_d g_{be}\partial_f g_{gh}g_{ac}g^{eg}g^{fh} - 2\partial_d g_{be}\partial_f g_{gh}g_{ac}g^{ef}g^{gh} - 4\partial_e g_{bd}\partial_f g_{gh}g_{ac}g^{eg}g^{fh} + 2\partial_e g_{bd}\partial_f g_{gh}g_{ac}g^{ef}g^{gh} + 4\partial_{ce}g_{bf}g_{ad}g^{ef} \\
&\quad - 4\partial_{bc}g_{ef}g_{ad}g^{ef} - 4\partial_{ef}g_{bc}g_{ad}g^{ef} + 4\partial_{be}g_{cf}g_{ad}g^{ef} + 2\partial_b g_{ef}\partial_c g_{gh}g_{ad}g^{eg}g^{fh} + 4\partial_e g_{bf}\partial_g g_{ch}g_{ad}g^{eg}g^{fh} - 4\partial_e g_{bf}\partial_g g_{ch}g_{ad}g^{eh}g^{fg} - 4\partial_b g_{ce}\partial_f g_{gh}g_{ad}g^{eg}g^{fh} \\
&\quad + 2\partial_b g_{ce}\partial_f g_{gh}g_{ad}g^{ef}g^{gh} - 4\partial_c g_{be}\partial_f g_{gh}g_{ad}g^{eg}g^{fh} + 2\partial_c g_{be}\partial_f g_{gh}g_{ad}g^{ef}g^{gh} + 4\partial_e g_{bc}\partial_f g_{gh}g_{ad}g^{eg}g^{fh} - 2\partial_e g_{bc}\partial_f g_{gh}g_{ad}g^{ef}g^{gh} + 4\partial_{ef}g_{gh}g_{ac}g_{bd}g^{eg}g^{fh} \\
&\quad - 4\partial_{ef}g_{gh}g_{ad}g_{bc}g^{eg}g^{fh} - 4\partial_{ef}g_{gh}g_{ac}g_{bd}g^{ef}g^{gh} + 4\partial_{ef}g_{gh}g_{ad}g_{bc}g^{ef}g^{gh} - 2\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{ei}g^{fh}g^{gj} + 2\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{ei}g^{fh}g^{gj} \\
&\quad + 3\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{eh}g^{fi}g^{gj} - 3\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{eh}g^{fi}g^{gj} - 4\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{ef}g^{gi}g^{hj} + 4\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{ef}g^{gi}g^{hj} \\
&\quad + 4\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{ef}g^{gh}g^{ij} - 4\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{ef}g^{gh}g^{ij} - \partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{eh}g^{fg}g^{ij} + \partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{eh}g^{fg}g^{ij} \quad (\text{ex-13a.111})
\end{aligned}$$

$$= 0 \quad (\text{ex-13a.112})$$

Example 13b The Weyl tensor vanishes in 3d – orthonormal basis

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3  g_{a b}::Metric.
4  g^{a b}::InverseMetric.
5
6  R_{a b c d}::RiemannTensor.
7
8  ex{#}::LaTeXForm("e_x").
9  ey{#}::LaTeXForm("e_y").
10 ez{#}::LaTeXForm("e_z").
11
12 {R_{a b c d}, g_{a b}, g^{a b}}::SortOrder.
13
14 Rab      := R_{a b} -> g^{c d} R_{a c b d}.
15
16 Rscalar := R -> g^{a b} R_{a b}.
17
18 gab := g^{a b} -> ex^{a} ex^{b} + ey^{a} ey^{b} + ez^{a} ez^{b}.
19
20 ortho := {ex^{a} ex^{b} g_{a b} -> 1, ey^{a} ey^{b} g_{a b} -> 1, ez^{a} ez^{b} g_{a b} -> 1,
21           ex^{a} ey^{b} g_{a b} -> 0, ex^{a} ez^{b} g_{a b} -> 0,
22           ey^{a} ex^{b} g_{a b} -> 0, ey^{a} ez^{b} g_{a b} -> 0,
23           ez^{a} ex^{b} g_{a b} -> 0, ez^{a} ey^{b} g_{a b} -> 0}.
24
25 # Weyl in 3-dimensions
26
27 Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
28         - (g_{a c} R_{b d} - g_{a d} R_{b c})
29         + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).    # cdb (ex-13b.100,Cabcd)
30
31
32 substitute (Cabcd, Rscalar)                # cdb(ex-13b.101,Cabcd)
33 substitute (Cabcd, Rab)                    # cdb(ex-13b.102,Cabcd)
34 distribute (Cabcd)                        # cdb(ex-13b.103,Cabcd)
35
36 Cabcd := C_{a b c d} -> @(Cabcd).

```

```

37 Cxyxy := C_{a b c d} ex^{a} ey^{b} ex^{c} ey^{d}.
38                                     # cdb(ex-13b.104,Cxyxy)
39
40 substitute      (Cxyxy,Cabcd)      # cdb(ex-13b.105,Cxyxy)
41 distribute      (Cxyxy)             # cdb(ex-13b.106,Cxyxy)
42
43 substitute      (Cxyxy, ortho, repeat=True) # cdb(ex-13b.107,Cxyxy)
44
45 substitute      (Cxyxy, gab)        # cdb(ex-13b.108,Cxyxy)
46 distribute      (Cxyxy)             # cdb(ex-13b.109,Cxyxy)
47
48 sort_product    (Cxyxy)             # cdb(ex-13b.110,Cxyxy)
49 rename_dummies  (Cxyxy)             # cdb(ex-13b.111,Cxyxy)
50 canonicalise    (Cxyxy)             # cdb(ex-13b.112,Cxyxy)

```

$$\text{ex-13b.101} := R_{abcd} - R_{ac}g_{bd} + R_{ad}g_{bc} - g_{ac}R_{bd} + g_{ad}R_{bc} + \frac{1}{2}g^{ef}R_{ef}(g_{ac}g_{bd} - g_{ad}g_{bc})$$

$$\text{ex-13b.102} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}(g_{ac}g_{bd} - g_{ad}g_{bc})$$

$$\text{ex-13b.103} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}$$

$$C_{abcd}e_x^a e_y^b e_x^c e_y^d = \left(R_{abcd} - g^{ef} R_{aecf} g_{bd} + g^{fe} R_{afde} g_{bc} - g_{ac} g^{fe} R_{bfde} + g_{ad} g^{ef} R_{becf} + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} \right) e_x^a e_y^b e_x^c e_y^d \quad (\text{ex-13b.105})$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - g^{ef} R_{aecf} g_{bd} e_x^a e_y^b e_x^c e_y^d + g^{fe} R_{afde} g_{bc} e_x^a e_y^b e_x^c e_y^d - g_{ac} g^{fe} R_{bfde} e_x^a e_y^b e_x^c e_y^d + g_{ad} g^{ef} R_{becf} e_x^a e_y^b e_x^c e_y^d + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} e_x^a e_y^b e_x^c e_y^d - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} e_x^a e_y^b e_x^c e_y^d \quad (\text{ex-13b.106})$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - g^{ef} R_{aecf} e_x^a e_x^c - g^{fe} R_{bfde} e_y^b e_y^d + \frac{1}{2} g^{ef} g^{gh} R_{egfh} \quad (\text{ex-13b.107})$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - (e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f) R_{aecf} e_x^a e_x^c - (e_x^f e_x^e + e_y^f e_y^e + e_z^f e_z^e) R_{bfde} e_y^b e_y^d + \frac{1}{2} (e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f) (e_x^g e_x^h + e_y^g e_y^h + e_z^g e_z^h) R_{egfh} \quad (\text{ex-13b.108})$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - e_x^e e_x^f R_{aecf} e_x^a e_x^c - e_y^e e_y^f R_{aecf} e_x^a e_x^c - e_z^e e_z^f R_{aecf} e_x^a e_x^c - e_x^f e_x^e R_{bfde} e_y^b e_y^d - e_y^f e_y^e R_{bfde} e_y^b e_y^d - e_z^f e_z^e R_{bfde} e_y^b e_y^d + \frac{1}{2} e_x^e e_x^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_x^e e_x^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_x^e e_x^f e_z^g e_z^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_z^g e_z^h R_{egfh} + \frac{1}{2} e_z^e e_z^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_z^e e_z^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_z^e e_z^f e_z^g e_z^h R_{egfh} \quad (\text{ex-13b.109})$$

$$= R_{abcd}e_x^a e_x^c e_y^b e_y^d - R_{aecf} e_x^a e_x^c e_y^e e_y^f - R_{aecf} e_x^a e_x^c e_y^f e_y^e - R_{aecf} e_x^a e_x^c e_z^e e_z^f - R_{bfde} e_x^e e_x^f e_y^b e_y^d - R_{bfde} e_y^b e_y^d e_y^e e_y^f - R_{bfde} e_y^b e_y^d e_z^e e_z^f + \frac{1}{2} R_{egfh} e_x^e e_x^f e_x^g e_x^h + \frac{1}{2} R_{egfh} e_x^e e_x^f e_y^g e_y^h + \frac{1}{2} R_{egfh} e_x^e e_x^f e_z^g e_z^h + \frac{1}{2} R_{egfh} e_x^g e_x^h e_y^e e_y^f + \frac{1}{2} R_{egfh} e_y^e e_y^f e_y^g e_y^h + \frac{1}{2} R_{egfh} e_y^e e_y^f e_z^g e_z^h + \frac{1}{2} R_{egfh} e_x^g e_x^h e_z^e e_z^f + \frac{1}{2} R_{egfh} e_y^g e_y^h e_z^e e_z^f + \frac{1}{2} R_{egfh} e_z^g e_z^h e_z^e e_z^f \quad (\text{ex-13b.110})$$

$$= \frac{1}{2} R_{abcd} e_x^a e_x^c e_y^b e_y^d - \frac{1}{2} R_{abcd} e_x^a e_x^c e_x^b e_x^d - \frac{1}{2} R_{abcd} e_x^a e_x^c e_z^b e_z^d - R_{abcd} e_x^d e_x^b e_y^a e_y^c - R_{abcd} e_y^a e_y^c e_y^d e_y^b - R_{abcd} e_y^a e_y^c e_z^d e_z^b + \frac{1}{2} R_{abcd} e_x^b e_x^d e_y^a e_y^c + \frac{1}{2} R_{abcd} e_y^a e_y^c e_y^b e_y^d + \frac{1}{2} R_{abcd} e_y^a e_y^c e_z^b e_z^d + \frac{1}{2} R_{abcd} e_x^b e_x^d e_z^a e_z^c + \frac{1}{2} R_{abcd} e_y^b e_y^d e_z^a e_z^c + \frac{1}{2} R_{abcd} e_z^a e_z^c e_z^b e_z^d \quad (\text{ex-13b.111})$$

$$= 0 \quad (\text{ex-13b.112})$$

Example 13c The Weyl tensor vanishes in 3d – orthonormal basis

```
1 Cxyz := C_{a b c d} ex^{a} ey^{b} ex^{c} ez^{d}. # cdb(ex-13c.101,Cxyz)
2
3 substitute (Cxyz,Cabcd) # cdb(ex-13c.102,Cxyz)
4
5 distribute (Cxyz) # cdb(ex-13c.103,Cxyz)
6
7 substitute (Cxyz, ortho, repeat=True) # cdb(ex-13c.104,Cxyz)
8
9 substitute (Cxyz, gab) # cdb(ex-13c.105,Cxyz)
10 distribute (Cxyz) # cdb(ex-13c.106,Cxyz)
11
12 sort_product (Cxyz) # cdb(ex-13c.107,Cxyz)
13 rename_dummies (Cxyz) # cdb(ex-13c.108,Cxyz)
14 canonicalise (Cxyz) # cdb(ex-13c.109,Cxyz)
```

$$C_{abcd}e_x^a e_y^b e_x^c e_z^d = \left(R_{abcd} - g^{ef} R_{aecf} g_{bd} + g^{fe} R_{afde} g_{bc} - g_{ac} g^{fe} R_{bfde} + g_{ad} g^{ef} R_{becf} + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} \right) e_x^a e_y^b e_x^c e_z^d \quad (\text{ex-13c.102})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_z^d - g^{ef} R_{aecf} g_{bd} e_x^a e_y^b e_x^c e_z^d + g^{fe} R_{afde} g_{bc} e_x^a e_y^b e_x^c e_z^d - g_{ac} g^{fe} R_{bfde} e_x^a e_y^b e_x^c e_z^d + g_{ad} g^{ef} R_{becf} e_x^a e_y^b e_x^c e_z^d \\ + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} e_x^a e_y^b e_x^c e_z^d - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} e_x^a e_y^b e_x^c e_z^d \quad (\text{ex-13c.103})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_z^d - g^{fe} R_{bfde} e_y^b e_z^d \quad (\text{ex-13c.104})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_z^d - (e_x^f e_x^e + e_y^f e_y^e + e_z^f e_z^e) R_{bfde} e_y^b e_z^d \quad (\text{ex-13c.105})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_z^d - e_x^f e_x^e R_{bfde} e_y^b e_z^d - e_y^f e_y^e R_{bfde} e_y^b e_z^d - e_z^f e_z^e R_{bfde} e_y^b e_z^d \quad (\text{ex-13c.106})$$

$$= R_{abcd} e_x^a e_x^c e_y^b e_z^d - R_{bfde} e_x^e e_x^f e_y^b e_z^d - R_{bfde} e_y^e e_y^f e_y^b e_z^d - R_{bfde} e_y^b e_z^d e_z^e e_z^f \quad (\text{ex-13c.107})$$

$$= R_{abcd} e_x^a e_x^c e_y^b e_z^d - R_{abcd} e_x^d e_x^b e_y^a e_z^c - R_{abcd} e_y^a e_y^d e_y^b e_z^c - R_{abcd} e_y^a e_z^c e_z^d e_z^b \quad (\text{ex-13c.108})$$

$$= 0 \quad (\text{ex-13c.109})$$

Example 14 The Weyl tensor is conformally invariant

This example shows that the Weyl tensor is conformally invariant. That is, for a pair of metrics g and \bar{g} related by a conformal transformation, $\bar{g}_{ab} = \phi g_{ab}$ then $\bar{C}_{bcd}^a = C_{bcd}^a$ or equally $\bar{C}_{abcd} = \phi C_{abcd}$.

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative.
4
5  g_{a b}::Metric.
6  g^{a b}::InverseMetric.
7  g_{a}^{b}::KroneckerDelta.
8
9  GammaU := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
10                                     + \partial_{c}{g_{b d}}
11                                     - \partial_{d}{g_{b c}}).
12
13  GammaD := \Gamma_{a b c} -> 1/2 ( \partial_{b}{g_{a c}}
14                                     + \partial_{c}{g_{b a}}
15                                     - \partial_{a}{g_{b c}}).
16
17  Rabcd := R_{a b c d} -> \partial_{c}{\Gamma_{a b d}}
18                        - \partial_{d}{\Gamma_{a b c}}
19                        + \Gamma_{e a d} \Gamma^{e}_{b c}
20                        - \Gamma_{e a c} \Gamma^{e}_{b d}.
21
22  Rab      := R_{a b} -> g^{c d} R_{a c b d}.
23
24  Rscalar := R -> g^{a b} R_{a b}.
25
26  # Weyl in 4-dimensions
27
28  Cabcd := R_{a b c d} - (1/2) (R_{a c} g_{b d} - R_{a d} g_{b c})
29                        - (1/2) (g_{a c} R_{b d} - g_{a d} R_{b c})
30                        + (R/6) (g_{a c} g_{b d} - g_{a d} g_{b c}).
31
32  {\partial_{a b}{\phi},\partial_{a}{\phi},\phi}::SortOrder.
33  {\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.

```



```

34
35 substitute (Cabcd,Rscalar)
36 substitute (Cabcd,Rab)
37 substitute (Cabcd,Rabcd)
38 substitute (Cabcd,GammaU)
39 substitute (Cabcd,GammaD)
40
41 distribute (Cabcd)
42
43 sort_product (Cabcd)
44 rename_dummies (Cabcd)
45 canonicalise (Cabcd)
46
47 # this is the Weyl tensor on the base metric
48 baseC := @(Cabcd).
49
50 conformal := {g_{a b} -> \phi g_{a b}, g^{a b} -> (1/phi) g^{a b}}.
51
52 substitute (Cabcd, conformal)
53 product_rule (Cabcd)
54 distribute (Cabcd)
55 product_rule (Cabcd)
56 distribute (Cabcd)
57
58 map_sympy (Cabcd, "simplify")
59
60 sort_product (Cabcd)
61 rename_dummies (Cabcd)
62 canonicalise (Cabcd)
63
64 # this is the Weyl tensor on the conformal metric
65 confC := @(Cabcd).
66
67 # their difference, should be zero
68 diff := @(confC) - \phi @(baseC). # cdb (ex-14.diff.100,diff)
69
70 distribute (diff)
71 sort_product (diff)

```

```

72 rename_dummies (diff)
73 canonicalise (diff) # cdb (ex-14.diff.101,diff)
74
75 # this trick is not essential but it does reduce the number of terms in diff
76 substitute (diff,$\partial_{\{a\}\{b\}\{g_{\{c\} d\}}\} \rightarrow g_{\{c\} d\} b\{a\}\})
77 substitute (diff,$\partial_{\{a\}\{g_{\{b\} c\}}\} \rightarrow 0\})
78 substitute (diff,$g_{\{c\} d\} b\{a\} \rightarrow \partial_{\{a\}\{b\}\{g_{\{c\} d\}}\}}) # cdb (ex-14.diff.102,diff)
79
80 # standard expressions in 4-d
81 substitute (diff,$g_{\{a\} b\} g^{\{a\} b\} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.201,diff)
82 substitute (diff,$g_{\{a\} b\} g^{\{c\} b\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.202,diff)
83 substitute (diff,$g_{\{b\} a\} g^{\{b\} c\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.203,diff)
84 substitute (diff,$g_{\{a\}}^{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.204,diff)
85 substitute (diff,$g^{\{a\}}_{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.205,diff)
86 eliminate_kronecker (diff) # cdb (ex-14.diff.206,diff)
87
88 # need a second round since the above block introduces new terms that match those just eliminated
89 substitute (diff,$g_{\{a\} b\} g^{\{a\} b\} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.301,diff)
90 substitute (diff,$g_{\{a\} b\} g^{\{c\} b\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.302,diff)
91 substitute (diff,$g_{\{b\} a\} g^{\{b\} c\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.303,diff)
92 substitute (diff,$g_{\{a\}}^{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.304,diff)
93 substitute (diff,$g^{\{a\}}_{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.305,diff)
94 eliminate_kronecker (diff) # cdb (ex-14.diff.306,diff)
95
96 sort_product (diff)
97 rename_dummies (diff)
98 canonicalise (diff) # cdb (ex-14.diff.400,diff)
99
100 checkpoint.append (baseC)
101 checkpoint.append (confC)

```

$$\begin{aligned}
\Delta = & \frac{1}{2}\partial_{bc}\phi g_{ad} - \frac{1}{2}\partial_{ac}\phi g_{bd} - \frac{1}{2}\partial_{bd}\phi g_{ac} + \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_{df}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g^{ef} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_{cf}g^{ef} \\
& + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g^{ef} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_{df}g^{ef} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g^{ef} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_{ac}\phi g_{bd}g_{ef}g^{ef} + \frac{1}{2}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{ef}g^{gh} - \frac{1}{2}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_{ad}\phi g_{bc}g_{ef}g^{ef} - \frac{1}{2}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{ef}g^{gh} + \frac{1}{2}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} + \frac{1}{4}\partial_{bd}\phi g_{ac}g_{ef}g^{ef} \\
& - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{8}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} - \frac{1}{4}\partial_{bc}\phi g_{ad}g_{ef}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} \\
& - \frac{1}{24}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij} + \frac{1}{24}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij}
\end{aligned}$$

(ex-14.diff.102)

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_{df}g^{ef} + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_{cf}g^{ef} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g^{ef} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g^{ef} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_{cf}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_{df}g^{ef} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g^{ef} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g^{ef} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g^{ef} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} \\
& - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} \\
& + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj}
\end{aligned}
\tag{ex-14.diff.201}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_g^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_g^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{ef} g^{gi} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{ef} g^{gi}
\end{aligned}
\tag{ex-14.diff.202}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^i g_i^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^i g_i^h g^{ef}
\end{aligned}
\tag{ex-14.diff.203}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^i g_i^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^i g_i^h g^{ef}
\end{aligned}
\tag{ex-14.diff.204}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^i g_i^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^i g_i^h g^{ef} \tag{ex-14.diff.205}
\end{aligned}$$

$$\begin{aligned}
\Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad} \\
& + \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} + \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} - \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cg}g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ag}g_{bd}g^{eg} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^f \\
& - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{dg}g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ag}g_{bc}g^{eg} - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{dg}g^{eg} \\
& + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bg}g^{eg} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cg}g^{eg} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bg}g^{eg} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gi}g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gi}g^{eg} g^{fi} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^h g^{ef} \tag{ex-14.diff.206}
\end{aligned}$$

Example 15 Verifying the BSSN equations

This is short example verifies two of the main equations in the Phys Rev D paper by Miguel Alcubierre, Bernd Bruggmann et al. (Phys.Rev.D. (62) 044034 (2000)).

The code for the full set of BSSN equations can be found at <https://github.com/leo-brewin/adm-bssn-equations>

```
1 {a,b,c,d,e,f,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent,values={t,x,y,z}).
2 {t,x,y,z}::Coordinate.
3
4 \partial{#}::PartialDerivative.
5 D{#}::Derivative.
6 DBar{#}::Derivative.
7
8 N::Depends(t,x,y,z).
9
10 g_{a b}::Symmetric.
11 g^{a b}::Symmetric.
12 g_{a}^{b}::KroneckerDelta.
13 g^{a}_{b}::KroneckerDelta.
14
15 g_{a b}::Depends(t,x,y,z).
16 g^{a b}::Depends(t,x,y,z).
17
18 gBar_{a b}::Symmetric.
19 gBar^{a b}::Symmetric.
20 gBar_{a}^{b}::KroneckerDelta.
21 gBar^{a}_{b}::KroneckerDelta.
22
23 gBar_{a b}::Depends(t,x,y,z).
24 gBar^{a b}::Depends(t,x,y,z).
25
26 trK::LaTeXForm("K").
27 detg::LaTeXForm("g").
28 ABar{#}::LaTeXForm("{\bar{A}}").
29 DBar{#}::LaTeXForm("{\bar{D}}").
```

15.1 Evolution equation for ϕ

```
1  phi      := \phi -> (1/12) \log(detg).
2  gdotK    := g^{i j} K_{i j} -> trK.
3  DdetgDt  := \partial_t\{detg\} -> detg g^{i j} \partial_t\{g_{i j}\}.
4
5  DgijDt   := \partial_t\{g_{i j}\} -> -2 N K_{i j}.
6
7  dlog      := \partial_{a?}\{\log(A?)\} -> (1/A?)\partial_{a?}\{A?\}.
8  dexp      := \partial_{a?}\{\exp(A?)\} -> \exp(A?)\partial_{a?}\{A?\}.
9
10 dotphi    := \partial_t\{\phi\}.
11
12 substitute (dotphi, phi)           # cdb (ex-15-02.101,dotphi)
13 substitute (dotphi, dlog)          # cdb (ex-15-02.102,dotphi)
14 substitute (dotphi, DdetgDt)       # cdb (ex-15-02.103,dotphi)
15 substitute (dotphi, DgijDt)        # cdb (ex-15-02.104,dotphi)
16 substitute (dotphi, gdotK)         # cdb (ex-15-02.105,dotphi)
17 map_sympy (dotphi, "simplify")     # cdb (ex-15-02.106,dotphi)
18
19 DphiDt := \partial_t\{\phi\} -> @(dotphi).
20
21 checkpoint.append (dotphi)
```

$$\frac{d\phi}{dt} = \frac{1}{12} \partial_t (\log(g)) \quad (\text{ex-15-02.101})$$

$$= \frac{1}{12} g^{-1} \partial_t g \quad (\text{ex-15-02.102})$$

$$= \frac{1}{12} g^{-1} g g^{ij} \partial_t g_{ij} \quad (\text{ex-15-02.103})$$

$$= -\frac{1}{6} g^{-1} g g^{ij} N K_{ij} \quad (\text{ex-15-02.104})$$

$$= -\frac{1}{6} g^{-1} g K N \quad (\text{ex-15-02.105})$$

$$= -\frac{1}{6} K N \quad (\text{ex-15-02.106})$$

15.2 Evolution equation for \bar{g}_{ij}

```
1  gBarij := gBar_{i j} -> \exp(-4\phi) g_{i j}.
2  Kij     := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.
3  A2ABar := \exp(-4\phi) A_{i j} -> ABar_{i j}.
4  ABar2A := ABar_{i j} -> \exp(-4\phi) A_{i j}.
5
6  dotgBarij := \partial_t\{gBar_{i j}\}.
7
8  substitute (dotgBarij, gBarij)          # cdb (ex-15-03.101,dotgBarij)
9  product_rule (dotgBarij)                # cdb (ex-15-03.102,dotgBarij)
10 substitute (dotgBarij, dexp)             # cdb (ex-15-03.103,dotgBarij)
11 substitute (dotgBarij, DgijDt)           # cdb (ex-15-03.104,dotgBarij)
12 substitute (dotgBarij, DphiDt)           # cdb (ex-15-03.105,dotgBarij)
13 substitute (dotgBarij, Kij)              # cdb (ex-15-03.106,dotgBarij)
14 distribute (dotgBarij)                  # cdb (ex-15-03.107,dotgBarij)
15 map_sympy (dotgBarij, "simplify")        # cdb (ex-15-03.108,dotgBarij)
16 substitute (dotgBarij, A2ABar)           # cdb (ex-15-03.109,dotgBarij)
17
18 DgBarijDt := \partial_t\{gBar_{i j}\} -> @(dotgBarij).
19
20 checkpoint.append (dotgBarij)
```

$$\frac{d\bar{g}_{ij}}{dt} = \partial_t (\exp(-4\phi) g_{ij}) \quad (\text{ex-15-03.101})$$

$$= \partial_t (\exp(-4\phi)) g_{ij} + \exp(-4\phi) \partial_t g_{ij} \quad (\text{ex-15-03.102})$$

$$= -4 \exp(-4\phi) \partial_t \phi g_{ij} + \exp(-4\phi) \partial_t g_{ij} \quad (\text{ex-15-03.103})$$

$$= -4 \exp(-4\phi) \partial_t \phi g_{ij} - 2 \exp(-4\phi) N K_{ij} \quad (\text{ex-15-03.104})$$

$$= \frac{2}{3} \exp(-4\phi) K N g_{ij} - 2 \exp(-4\phi) N K_{ij} \quad (\text{ex-15-03.105})$$

$$= \frac{2}{3} \exp(-4\phi) K N g_{ij} - 2 \exp(-4\phi) N \left(A_{ij} + \frac{1}{3} g_{ij} K \right) \quad (\text{ex-15-03.106})$$

$$= \frac{2}{3} \exp(-4\phi) K N g_{ij} - 2 \exp(-4\phi) N A_{ij} - \frac{2}{3} \exp(-4\phi) N g_{ij} K \quad (\text{ex-15-03.107})$$

$$= -2N \exp(-4\phi) A_{ij} \quad (\text{ex-15-03.108})$$

$$= -2N \bar{A}_{ij} \quad (\text{ex-15-03.109})$$