Exercise 3.1 Some symmetries of Riemann

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

;::Symbol;

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.       # cdb(Rabcd.000,Rabcd)

dRabcd := R^{a}_{b c d ; e} -> \partial_{e}{R^{a}_{b c d}}
                             + \Gamma^{a}_{f e} R^{f}_{b c d}
                             - \Gamma^{f}_{b e} R^{a}_{f c d}
                             - \Gamma^{f}_{c e} R^{a}_{b f d}
                             - \Gamma^{f}_{d e} R^{a}_{b c f}.       # cdb(dRabcd.000,dRabcd)
```

Exercise 3.1 Antisymmetry on last pair of indices

```
expr := R^{a}_{b c d} + R^{a}_{b d c}.                    # cdb(ex-0301.101,expr)

substitute (expr, Rabcd)                                  # cdb(ex-0301.102,expr)
```

$$R^{a}{}_{bcd} + R^{a}{}_{bdc} = 0 \tag{ex-0301.102}$$

Exercise 3.1 First Bianchi identity

```
expr := R^{a}_{b c d} + R^{a}_{d b c} + R^{a}_{c d b}.                    # cdb(ex-0301.201,expr)

substitute   (expr, Rabcd)                                               # cdb(ex-0301.202,expr)
canonicalise (expr)                                                      # cdb(ex-0301.203,expr)
```

$$R^{a}{}_{bcd} + R^{a}{}_{dbc} + R^{a}{}_{cdb} = \partial_c\Gamma^{a}{}_{bd} - \partial_d\Gamma^{a}{}_{bc} + \Gamma^{e}{}_{bd}\Gamma^{a}{}_{ce} - \Gamma^{e}{}_{bc}\Gamma^{a}{}_{de} + \partial_b\Gamma^{a}{}_{dc} - \partial_c\Gamma^{a}{}_{db} + \Gamma^{e}{}_{dc}\Gamma^{a}{}_{be} - \Gamma^{e}{}_{db}\Gamma^{a}{}_{ce} + \partial_d\Gamma^{a}{}_{cb} - \partial_b\Gamma^{a}{}_{cd} + \Gamma^{e}{}_{cb}\Gamma^{a}{}_{de} - \Gamma^{e}{}_{cd}\Gamma^{a}{}_{be} \quad \text{(ex-0301.202)}$$

$$= 0 \quad \text{(ex-0301.203)}$$

Exercise 3.1 Second Bianchi identity

```
expr := R^{a}_{b c d ; e} + R^{a}_{b e c ; d} + R^{a}_{b d e ; c}.   # cdb(ex-0301.301,expr)

substitute     (expr, dRabcd)                                       # cdb(ex-0301.302,expr)
substitute     (expr,  Rabcd)                                       # cdb(ex-0301.303,expr)
distribute     (expr)                                               # cdb(ex-0301.304,expr)
product_rule   (expr)                                               # cdb(ex-0301.305,expr)
sort_product   (expr)                                               # cdb(ex-0301.306,expr)
rename_dummies (expr)                                               # cdb(ex-0301.307,expr)
canonicalise   (expr)                                               # cdb(ex-0301.308,expr)
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$$
\begin{aligned}
R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} &= \partial_e R^a{}_{bcd} + \Gamma^a{}_{fe} R^f{}_{bcd} - \Gamma^f{}_{be} R^a{}_{fcd} - \Gamma^f{}_{ce} R^a{}_{bfd} - \Gamma^f{}_{de} R^a{}_{bcf} + \partial_d R^a{}_{bec} + \Gamma^a{}_{fd} R^f{}_{bec} - \Gamma^f{}_{bd} R^a{}_{fec} - \Gamma^f{}_{ed} R^a{}_{bfc} - \Gamma^f{}_{cd} R^a{}_{bef} \\
&\quad + \partial_c R^a{}_{bde} + \Gamma^a{}_{fc} R^f{}_{bde} - \Gamma^f{}_{bc} R^a{}_{fde} - \Gamma^f{}_{dc} R^a{}_{bfe} - \Gamma^f{}_{ec} R^a{}_{bdf} \qquad \text{(ex-0301.302)} \\
&= \partial_e \left(\partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^f{}_{bd} \Gamma^a{}_{cf} - \Gamma^f{}_{bc} \Gamma^a{}_{df} \right) + \Gamma^a{}_{fe} \left(\partial_c \Gamma^f{}_{bd} - \partial_d \Gamma^f{}_{bc} + \Gamma^g{}_{bd} \Gamma^f{}_{cg} - \Gamma^g{}_{bc} \Gamma^f{}_{dg} \right) \\
&\quad - \Gamma^f{}_{be} \left(\partial_c \Gamma^a{}_{fd} - \partial_d \Gamma^a{}_{fc} + \Gamma^g{}_{fd} \Gamma^a{}_{cg} - \Gamma^g{}_{fc} \Gamma^a{}_{dg} \right) - \Gamma^f{}_{ce} \left(\partial_f \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bf} + \Gamma^g{}_{bd} \Gamma^a{}_{fg} - \Gamma^g{}_{bf} \Gamma^a{}_{dg} \right) \\
&\quad - \Gamma^f{}_{de} \left(\partial_c \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{bc} + \Gamma^g{}_{bf} \Gamma^a{}_{cg} - \Gamma^g{}_{bc} \Gamma^a{}_{fg} \right) + \partial_d \left(\partial_e \Gamma^a{}_{bc} - \partial_c \Gamma^a{}_{be} + \Gamma^f{}_{bc} \Gamma^a{}_{ef} - \Gamma^f{}_{be} \Gamma^a{}_{cf} \right) \\
&\quad + \Gamma^a{}_{fd} \left(\partial_e \Gamma^f{}_{bc} - \partial_c \Gamma^f{}_{be} + \Gamma^g{}_{bc} \Gamma^f{}_{eg} - \Gamma^g{}_{be} \Gamma^f{}_{cg} \right) - \Gamma^f{}_{bd} \left(\partial_e \Gamma^a{}_{fc} - \partial_c \Gamma^a{}_{fe} + \Gamma^g{}_{fc} \Gamma^a{}_{eg} - \Gamma^g{}_{fe} \Gamma^a{}_{cg} \right) \\
&\quad - \Gamma^f{}_{ed} \left(\partial_f \Gamma^a{}_{bc} - \partial_c \Gamma^a{}_{bf} + \Gamma^g{}_{bc} \Gamma^a{}_{fg} - \Gamma^g{}_{bf} \Gamma^a{}_{cg} \right) - \Gamma^f{}_{cd} \left(\partial_e \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{be} + \Gamma^g{}_{bf} \Gamma^a{}_{eg} - \Gamma^g{}_{be} \Gamma^a{}_{fg} \right) \\
&\quad + \partial_c \left(\partial_d \Gamma^a{}_{be} - \partial_e \Gamma^a{}_{bd} + \Gamma^f{}_{be} \Gamma^a{}_{df} - \Gamma^f{}_{bd} \Gamma^a{}_{ef} \right) + \Gamma^a{}_{fc} \left(\partial_d \Gamma^f{}_{be} - \partial_e \Gamma^f{}_{bd} + \Gamma^g{}_{be} \Gamma^f{}_{dg} - \Gamma^g{}_{bd} \Gamma^f{}_{eg} \right) \\
&\quad - \Gamma^f{}_{bc} \left(\partial_d \Gamma^a{}_{fe} - \partial_e \Gamma^a{}_{fd} + \Gamma^g{}_{fe} \Gamma^a{}_{dg} - \Gamma^g{}_{fd} \Gamma^a{}_{eg} \right) - \Gamma^f{}_{dc} \left(\partial_f \Gamma^a{}_{be} - \partial_e \Gamma^a{}_{bf} + \Gamma^g{}_{be} \Gamma^a{}_{fg} - \Gamma^g{}_{bf} \Gamma^a{}_{eg} \right) \\
&\quad - \Gamma^f{}_{ec} \left(\partial_d \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{bd} + \Gamma^g{}_{bf} \Gamma^a{}_{dg} - \Gamma^g{}_{bd} \Gamma^a{}_{fg} \right) \qquad \text{(ex-0301.303)} \\
&= \partial_{ec} \Gamma^a{}_{bd} - \partial_{ed} \Gamma^a{}_{bc} + \partial_e \left(\Gamma^f{}_{bd} \Gamma^a{}_{cf} \right) - \partial_e \left(\Gamma^f{}_{bc} \Gamma^a{}_{df} \right) + \Gamma^a{}_{fe} \partial_c \Gamma^f{}_{bd} - \Gamma^a{}_{fe} \partial_d \Gamma^f{}_{bc} + \Gamma^a{}_{fe} \Gamma^g{}_{bd} \Gamma^f{}_{cg} - \Gamma^a{}_{fe} \Gamma^g{}_{bc} \Gamma^f{}_{dg} \\
&\quad - \Gamma^f{}_{be} \partial_c \Gamma^a{}_{fd} + \Gamma^f{}_{be} \partial_d \Gamma^a{}_{fc} - \Gamma^f{}_{be} \Gamma^g{}_{fd} \Gamma^a{}_{cg} + \Gamma^f{}_{be} \Gamma^g{}_{fc} \Gamma^a{}_{dg} - \Gamma^f{}_{ce} \partial_f \Gamma^a{}_{bd} + \Gamma^f{}_{ce} \partial_d \Gamma^a{}_{bf} - \Gamma^f{}_{ce} \Gamma^g{}_{bd} \Gamma^a{}_{fg} + \Gamma^f{}_{ce} \Gamma^g{}_{bf} \Gamma^a{}_{dg} \\
&\quad - \Gamma^f{}_{de} \partial_c \Gamma^a{}_{bf} + \Gamma^f{}_{de} \partial_f \Gamma^a{}_{bc} - \Gamma^f{}_{de} \Gamma^g{}_{bf} \Gamma^a{}_{cg} + \Gamma^f{}_{de} \Gamma^g{}_{bc} \Gamma^a{}_{fg} + \partial_{de} \Gamma^a{}_{bc} - \partial_{dc} \Gamma^a{}_{be} + \partial_d \left(\Gamma^f{}_{bc} \Gamma^a{}_{ef} \right) - \partial_d \left(\Gamma^f{}_{be} \Gamma^a{}_{cf} \right) \\
&\quad + \Gamma^a{}_{fd} \partial_e \Gamma^f{}_{bc} - \Gamma^a{}_{fd} \partial_c \Gamma^f{}_{be} + \Gamma^a{}_{fd} \Gamma^g{}_{bc} \Gamma^f{}_{eg} - \Gamma^a{}_{fd} \Gamma^g{}_{be} \Gamma^f{}_{cg} - \Gamma^f{}_{bd} \partial_e \Gamma^a{}_{fc} + \Gamma^f{}_{bd} \partial_c \Gamma^a{}_{fe} - \Gamma^f{}_{bd} \Gamma^g{}_{fc} \Gamma^a{}_{eg} + \Gamma^f{}_{bd} \Gamma^g{}_{fe} \Gamma^a{}_{cg} \\
&\quad - \Gamma^f{}_{ed} \partial_f \Gamma^a{}_{bc} + \Gamma^f{}_{ed} \partial_c \Gamma^a{}_{bf} - \Gamma^f{}_{ed} \Gamma^g{}_{bc} \Gamma^a{}_{fg} + \Gamma^f{}_{ed} \Gamma^g{}_{bf} \Gamma^a{}_{cg} - \Gamma^f{}_{cd} \partial_e \Gamma^a{}_{bf} + \Gamma^f{}_{cd} \partial_f \Gamma^a{}_{be} - \Gamma^f{}_{cd} \Gamma^g{}_{bf} \Gamma^a{}_{eg} + \Gamma^f{}_{cd} \Gamma^g{}_{be} \Gamma^a{}_{fg} \\
&\quad + \partial_{cd} \Gamma^a{}_{be} - \partial_{ce} \Gamma^a{}_{bd} + \partial_c \left(\Gamma^f{}_{be} \Gamma^a{}_{df} \right) - \partial_c \left(\Gamma^f{}_{bd} \Gamma^a{}_{ef} \right) + \Gamma^a{}_{fc} \partial_d \Gamma^f{}_{be} - \Gamma^a{}_{fc} \partial_e \Gamma^f{}_{bd} + \Gamma^a{}_{fc} \Gamma^g{}_{be} \Gamma^f{}_{dg} - \Gamma^a{}_{fc} \Gamma^g{}_{bd} \Gamma^f{}_{eg} \\
&\quad - \Gamma^f{}_{bc} \partial_d \Gamma^a{}_{fe} + \Gamma^f{}_{bc} \partial_e \Gamma^a{}_{fd} - \Gamma^f{}_{bc} \Gamma^g{}_{fe} \Gamma^a{}_{dg} + \Gamma^f{}_{bc} \Gamma^g{}_{fd} \Gamma^a{}_{eg} - \Gamma^f{}_{dc} \partial_f \Gamma^a{}_{be} + \Gamma^f{}_{dc} \partial_e \Gamma^a{}_{bf} - \Gamma^f{}_{dc} \Gamma^g{}_{be} \Gamma^a{}_{fg} + \Gamma^f{}_{dc} \Gamma^g{}_{bf} \Gamma^a{}_{eg} \\
&\quad - \Gamma^f{}_{ec} \partial_d \Gamma^a{}_{bf} + \Gamma^f{}_{ec} \partial_f \Gamma^a{}_{bd} - \Gamma^f{}_{ec} \Gamma^g{}_{bf} \Gamma^a{}_{dg} + \Gamma^f{}_{ec} \Gamma^g{}_{bd} \Gamma^a{}_{fg} \qquad \text{(ex-0301.304)}
\end{aligned}
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\begin{aligned}
R^{a}{}_{bcd;e} + R^{a}{}_{bec;d} + R^{a}{}_{bde;c} &= \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \partial_{e}\Gamma^{f}{}_{bd}\Gamma^{a}{}_{cf} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \partial_{e}\Gamma^{f}{}_{bc}\Gamma^{a}{}_{df} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{cg} \\
&\quad - \Gamma^{a}{}_{fe}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{dg} - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{f}{}_{be}\Gamma^{g}{}_{fd}\Gamma^{a}{}_{cg} + \Gamma^{f}{}_{be}\Gamma^{g}{}_{fc}\Gamma^{a}{}_{dg} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd}\Gamma^{a}{}_{fg} \\
&\quad + \Gamma^{f}{}_{ce}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{dg} - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{f}{}_{de}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{cg} + \Gamma^{f}{}_{de}\Gamma^{g}{}_{bc}\Gamma^{a}{}_{fg} + \partial_{de}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \partial_{d}\Gamma^{f}{}_{bc}\Gamma^{a}{}_{ef} + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} \\
&\quad - \partial_{d}\Gamma^{f}{}_{be}\Gamma^{a}{}_{cf} - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{eg} - \Gamma^{a}{}_{fd}\Gamma^{g}{}_{be}\Gamma^{f}{}_{cg} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} \\
&\quad - \Gamma^{f}{}_{bd}\Gamma^{g}{}_{fc}\Gamma^{a}{}_{eg} + \Gamma^{f}{}_{bd}\Gamma^{g}{}_{fe}\Gamma^{a}{}_{cg} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc}\Gamma^{a}{}_{fg} + \Gamma^{f}{}_{ed}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{cg} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} \\
&\quad - \Gamma^{f}{}_{cd}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{eg} + \Gamma^{f}{}_{cd}\Gamma^{g}{}_{be}\Gamma^{a}{}_{fg} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} + \partial_{c}\Gamma^{f}{}_{be}\Gamma^{a}{}_{df} + \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{df} - \partial_{c}\Gamma^{f}{}_{bd}\Gamma^{a}{}_{ef} - \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} \\
&\quad - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{a}{}_{fc}\Gamma^{g}{}_{be}\Gamma^{f}{}_{dg} - \Gamma^{a}{}_{fc}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{eg} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{f}{}_{bc}\Gamma^{g}{}_{fe}\Gamma^{a}{}_{dg} + \Gamma^{f}{}_{bc}\Gamma^{g}{}_{fd}\Gamma^{a}{}_{eg} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} \\
&\quad + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{dc}\Gamma^{g}{}_{be}\Gamma^{a}{}_{fg} + \Gamma^{f}{}_{dc}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{eg} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} - \Gamma^{f}{}_{ec}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{dg} + \Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd}\Gamma^{a}{}_{fg} \quad \texttt{(ex-0301.305)} \\
&= \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \Gamma^{a}{}_{cf}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \Gamma^{a}{}_{df}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{bd} \\
&\quad - \Gamma^{a}{}_{fe}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{bc} - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{a}{}_{cg}\Gamma^{f}{}_{be}\Gamma^{g}{}_{fd} + \Gamma^{a}{}_{dg}\Gamma^{f}{}_{be}\Gamma^{g}{}_{fc} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd} \\
&\quad + \Gamma^{a}{}_{dg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{a}{}_{cg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bc} + \partial_{de}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \Gamma^{a}{}_{ef}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} \\
&\quad - \Gamma^{a}{}_{cf}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bc} - \Gamma^{a}{}_{fd}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{be} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} \\
&\quad - \Gamma^{a}{}_{eg}\Gamma^{f}{}_{bd}\Gamma^{g}{}_{fc} + \Gamma^{a}{}_{cg}\Gamma^{f}{}_{bd}\Gamma^{g}{}_{fe} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc} + \Gamma^{a}{}_{cg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} \\
&\quad - \Gamma^{a}{}_{eg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{be} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} + \Gamma^{a}{}_{df}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{df} - \Gamma^{a}{}_{ef}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} \\
&\quad - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{a}{}_{fc}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{be} - \Gamma^{a}{}_{fc}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bd} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{a}{}_{dg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{fe} + \Gamma^{a}{}_{eg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{fd} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} \\
&\quad + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{be} + \Gamma^{a}{}_{eg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} - \Gamma^{a}{}_{dg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd} \quad \texttt{(ex-0301.306)}
\end{aligned}
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\begin{aligned}
R^{a}{}_{bcd;e} + R^{a}{}_{bec;d} + R^{a}{}_{bde;c} &= \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \Gamma^{a}{}_{cf}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \Gamma^{a}{}_{df}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{bd} \\
&\quad - \Gamma^{a}{}_{fe}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{bc} - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{a}{}_{cf}\Gamma^{g}{}_{be}\Gamma^{f}{}_{gd} + \Gamma^{a}{}_{df}\Gamma^{g}{}_{be}\Gamma^{f}{}_{gc} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd} \\
&\quad + \Gamma^{a}{}_{df}\Gamma^{g}{}_{ce}\Gamma^{f}{}_{bg} - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{a}{}_{cf}\Gamma^{g}{}_{de}\Gamma^{f}{}_{bg} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bc} + \partial_{de}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \Gamma^{a}{}_{ef}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} \\
&\quad - \Gamma^{a}{}_{cf}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bc} - \Gamma^{a}{}_{fd}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{be} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} \\
&\quad - \Gamma^{a}{}_{ef}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{gc} + \Gamma^{a}{}_{cf}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{ge} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc} + \Gamma^{a}{}_{cf}\Gamma^{g}{}_{ed}\Gamma^{f}{}_{bg} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} \\
&\quad - \Gamma^{a}{}_{ef}\Gamma^{g}{}_{cd}\Gamma^{f}{}_{bg} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{be} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} + \Gamma^{a}{}_{df}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{df} - \Gamma^{a}{}_{ef}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} \\
&\quad - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{a}{}_{fc}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{be} - \Gamma^{a}{}_{fc}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bd} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{a}{}_{df}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{ge} + \Gamma^{a}{}_{ef}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{gd} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} \\
&\quad + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{be} + \Gamma^{a}{}_{ef}\Gamma^{g}{}_{dc}\Gamma^{f}{}_{bg} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} - \Gamma^{a}{}_{df}\Gamma^{g}{}_{ec}\Gamma^{f}{}_{bg} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd} \quad \text{(ex-0301.307)} \\
&= 0 \quad \text{(ex-0301.308)}
\end{aligned}
$$