

## Exercise 3.4 More symmetries of Riemann

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 g_{a b}::Symmetric.
6 g^{a b}::Symmetric.
7
8 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
9 \Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).
10
11 GammaU := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
12                                     + \partial_{c}{g_{b d}}
13                                     - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)
14
15 GammaD := \Gamma_{a b c} -> 1/2 ( \partial_{b}{g_{a c}}
16                                     + \partial_{c}{g_{b a}}
17                                     - \partial_{a}{g_{b c}}). # cdb(Gamma.010,GammaD)
18
19 Rabcd := R_{a b c d} -> \partial_{c}{\Gamma_{a b d}}
20                       - \partial_{d}{\Gamma_{a b c}}
21                       + \Gamma_{e a d} \Gamma^{e}_{b c}
22                       - \Gamma_{e a c} \Gamma^{e}_{b d}. # cdb(Rabcd.000,Rabcd)
```

## Exercise 3.4 Antisymmetry on first pair of indices

```
1  expr := R_{a b c d} + R_{b a c d}.    # cdb(ex-0304.101,expr)
2
3  substitute      (expr, Rabcd)         # cdb(ex-0304.102,expr)
4  substitute      (expr, GammaU)        # cdb(ex-0304.103,expr)
5  substitute      (expr, GammaD)        # cdb(ex-0304.104,expr)
6  distribute      (expr)                # cdb(ex-0304.105,expr)
7  product_rule    (expr)                # cdb(ex-0304.106,expr)
8  sort_product    (expr)                # cdb(ex-0304.107,expr)
9  rename_dummies  (expr)                # cdb(ex-0304.108,expr)
10 canonicalise    (expr)                # cdb(ex-0304.109,expr)
```

$$R_{abcd} + R_{bacd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e_{bc} - \Gamma_{eac} \Gamma^e_{bd} + \partial_c \Gamma_{bad} - \partial_d \Gamma_{bac} + \Gamma_{ebd} \Gamma^e_{ac} - \Gamma_{ebc} \Gamma^e_{ad} \quad (\text{ex-0304.102})$$

$$\begin{aligned} &= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2} \Gamma_{ead} g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) - \frac{1}{2} \Gamma_{eac} g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) + \partial_c \Gamma_{bad} - \partial_d \Gamma_{bac} \\ &\quad + \frac{1}{2} \Gamma_{ebd} g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) - \frac{1}{2} \Gamma_{ebc} g^{ef} (\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}) \end{aligned} \quad (\text{ex-0304.103})$$

$$\begin{aligned} &= \partial_c \left( \frac{1}{2} \partial_b g_{ad} + \frac{1}{2} \partial_d g_{ba} - \frac{1}{2} \partial_a g_{bd} \right) - \partial_d \left( \frac{1}{2} \partial_b g_{ac} + \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_a g_{bc} \right) + \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ed} + \frac{1}{2} \partial_d g_{ae} - \frac{1}{2} \partial_e g_{ad} \right) g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) \\ &\quad - \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ec} + \frac{1}{2} \partial_c g_{ae} - \frac{1}{2} \partial_e g_{ac} \right) g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) + \partial_c \left( \frac{1}{2} \partial_a g_{bd} + \frac{1}{2} \partial_d g_{ab} - \frac{1}{2} \partial_b g_{ad} \right) \\ &\quad - \partial_d \left( \frac{1}{2} \partial_a g_{bc} + \frac{1}{2} \partial_c g_{ab} - \frac{1}{2} \partial_b g_{ac} \right) + \frac{1}{2} \left( \frac{1}{2} \partial_b g_{ed} + \frac{1}{2} \partial_d g_{be} - \frac{1}{2} \partial_e g_{bd} \right) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) \\ &\quad - \frac{1}{2} \left( \frac{1}{2} \partial_b g_{ec} + \frac{1}{2} \partial_c g_{be} - \frac{1}{2} \partial_e g_{bc} \right) g^{ef} (\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}) \end{aligned} \quad (\text{ex-0304.104})$$

$$\begin{aligned} &= \frac{1}{2} \partial_{cd} g_{ba} - \frac{1}{2} \partial_{dc} g_{ba} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_f g_{bc} + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_b g_{fc} + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_f g_{bc} \\ &\quad - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_b g_{fc} - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_c g_{bf} + \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_f g_{bc} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_b g_{fd} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_d g_{bf} + \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_f g_{bd} - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_b g_{fd} \\ &\quad - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_d g_{bf} + \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_f g_{bd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_b g_{fd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_d g_{bf} - \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_f g_{bd} + \frac{1}{2} \partial_{cd} g_{ab} - \frac{1}{2} \partial_{dc} g_{ab} + \frac{1}{4} \partial_b g_{ed} g^{ef} \partial_a g_{fc} \\ &\quad + \frac{1}{4} \partial_b g_{ed} g^{ef} \partial_c g_{af} - \frac{1}{4} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + \frac{1}{4} \partial_d g_{be} g^{ef} \partial_a g_{fc} + \frac{1}{4} \partial_d g_{be} g^{ef} \partial_c g_{af} - \frac{1}{4} \partial_d g_{be} g^{ef} \partial_f g_{ac} - \frac{1}{4} \partial_e g_{bd} g^{ef} \partial_a g_{fc} - \frac{1}{4} \partial_e g_{bd} g^{ef} \partial_c g_{af} \\ &\quad + \frac{1}{4} \partial_e g_{bd} g^{ef} \partial_f g_{ac} - \frac{1}{4} \partial_b g_{ec} g^{ef} \partial_a g_{fd} - \frac{1}{4} \partial_b g_{ec} g^{ef} \partial_d g_{af} + \frac{1}{4} \partial_b g_{ec} g^{ef} \partial_f g_{ad} - \frac{1}{4} \partial_c g_{be} g^{ef} \partial_a g_{fd} - \frac{1}{4} \partial_c g_{be} g^{ef} \partial_d g_{af} + \frac{1}{4} \partial_c g_{be} g^{ef} \partial_f g_{ad} \\ &\quad + \frac{1}{4} \partial_e g_{bc} g^{ef} \partial_a g_{fd} + \frac{1}{4} \partial_e g_{bc} g^{ef} \partial_d g_{af} - \frac{1}{4} \partial_e g_{bc} g^{ef} \partial_f g_{ad} \end{aligned} \quad (\text{ex-0304.105})$$

$$\begin{aligned}
R_{abcd} + R_{bacd} = & \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc} \\
& - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd} \\
& - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_a g_{fc} \\
& + \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_f g_{ac} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_a g_{fc} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_d g_{be}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_a g_{fc} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_c g_{af} \\
& + \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_f g_{ad} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_c g_{be}g^{ef}\partial_f g_{ad} \\
& + \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_a g_{fd} + \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_d g_{af} - \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_f g_{ad} \tag{ex-0304.106} \\
= & \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_f g_{bc}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_d g_{ae}g^{ef} + \frac{1}{4}\partial_c g_{bf}\partial_d g_{ae}g^{ef} - \frac{1}{4}\partial_d g_{ae}\partial_f g_{bc}g^{ef} \\
& - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_a g_{ec}\partial_f g_{bd}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_c g_{ae}g^{ef} \\
& - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{ae}\partial_f g_{bd}g^{ef} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_a g_{fc}\partial_b g_{ed}g^{ef} \\
& + \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{ef} - \frac{1}{4}\partial_b g_{ed}\partial_f g_{ac}g^{ef} + \frac{1}{4}\partial_a g_{fc}\partial_d g_{be}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_d g_{be}g^{ef} - \frac{1}{4}\partial_d g_{be}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{ef} - \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{ef} \\
& + \frac{1}{4}\partial_e g_{bd}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_b g_{ec}g^{ef} - \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_b g_{ec}\partial_f g_{ad}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_c g_{be}g^{ef} - \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_c g_{be}\partial_f g_{ad}g^{ef} \\
& + \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{ef} + \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{ef} - \frac{1}{4}\partial_e g_{bc}\partial_f g_{ad}g^{ef} \tag{ex-0304.107}
\end{aligned}$$

$$\begin{aligned}
R_{abcd} + R_{bacd} = & \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{fe} + \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{fe} + \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{fe} - \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{fe} \\
& - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{fe} - \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{fe} \\
& - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{fe} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{fe} \\
& + \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{fe} + \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{fe} + \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{fe} - \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{fe} - \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{ef} - \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{ef} \\
& + \frac{1}{4}\partial_e g_{bd}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{fe} - \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{fe} - \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{fe} - \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{fe} \\
& + \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{ef} + \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{ef} - \frac{1}{4}\partial_e g_{bc}\partial_f g_{ad}g^{ef}
\end{aligned} \tag{ex-0304.108}$$

$$= 0 \tag{ex-0304.109}$$

## Exercise 3.4 Symmetric on swapping first and second pair of indices

```
1  expr := R_{a b c d} - R_{c d a b}.    # cdb(ex-0304.201,expr)
2
3  substitute      (expr, Rabcd)         # cdb(ex-0304.202,expr)
4  substitute      (expr, GammaU)        # cdb(ex-0304.203,expr)
5  substitute      (expr, GammaD)        # cdb(ex-0304.204,expr)
6  distribute      (expr)                # cdb(ex-0304.205,expr)
7  product_rule    (expr)                # cdb(ex-0304.206,expr)
8  sort_product    (expr)                # cdb(ex-0304.207,expr)
9  rename_dummies  (expr)                # cdb(ex-0304.208,expr)
10 canonicalise    (expr)                # cdb(ex-0304.209,expr)
```

$$R_{abcd} - R_{cdab} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma_{bc}^e - \Gamma_{eac} \Gamma_{bd}^e - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} - \Gamma_{ecb} \Gamma_{da}^e + \Gamma_{eca} \Gamma_{db}^e \quad (\text{ex-0304.202})$$

$$\begin{aligned} &= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2} \Gamma_{ead} g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) - \frac{1}{2} \Gamma_{eac} g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} \\ &\quad - \frac{1}{2} \Gamma_{ecb} g^{ef} (\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}) + \frac{1}{2} \Gamma_{eca} g^{ef} (\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}) \end{aligned} \quad (\text{ex-0304.203})$$

$$\begin{aligned} &= \partial_c \left( \frac{1}{2} \partial_b g_{ad} + \frac{1}{2} \partial_d g_{ba} - \frac{1}{2} \partial_a g_{bd} \right) - \partial_d \left( \frac{1}{2} \partial_b g_{ac} + \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_a g_{bc} \right) + \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ed} + \frac{1}{2} \partial_d g_{ae} - \frac{1}{2} \partial_e g_{ad} \right) g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) \\ &\quad - \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ec} + \frac{1}{2} \partial_c g_{ae} - \frac{1}{2} \partial_e g_{ac} \right) g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) - \partial_a \left( \frac{1}{2} \partial_d g_{cb} + \frac{1}{2} \partial_b g_{dc} - \frac{1}{2} \partial_c g_{db} \right) \\ &\quad + \partial_b \left( \frac{1}{2} \partial_d g_{ca} + \frac{1}{2} \partial_a g_{dc} - \frac{1}{2} \partial_c g_{da} \right) - \frac{1}{2} \left( \frac{1}{2} \partial_c g_{eb} + \frac{1}{2} \partial_b g_{ce} - \frac{1}{2} \partial_e g_{cb} \right) g^{ef} (\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}) \\ &\quad + \frac{1}{2} \left( \frac{1}{2} \partial_c g_{ea} + \frac{1}{2} \partial_a g_{ce} - \frac{1}{2} \partial_e g_{ca} \right) g^{ef} (\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}) \end{aligned} \quad (\text{ex-0304.204})$$

$$\begin{aligned} &= \frac{1}{2} \partial_{cb} g_{ad} + \frac{1}{2} \partial_{cd} g_{ba} - \frac{1}{2} \partial_{ca} g_{bd} - \frac{1}{2} \partial_{db} g_{ac} - \frac{1}{2} \partial_{dc} g_{ba} + \frac{1}{2} \partial_{da} g_{bc} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_f g_{bc} + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_b g_{fc} \\ &\quad + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_f g_{bc} - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_b g_{fc} - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_c g_{bf} + \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_f g_{bc} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_b g_{fd} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_d g_{bf} \\ &\quad + \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_f g_{bd} - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_b g_{fd} - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_d g_{bf} + \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_f g_{bd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_b g_{fd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_d g_{bf} - \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_f g_{bd} \\ &\quad - \frac{1}{2} \partial_{ad} g_{cb} - \frac{1}{2} \partial_{ab} g_{dc} + \frac{1}{2} \partial_{ac} g_{db} + \frac{1}{2} \partial_{bd} g_{ca} + \frac{1}{2} \partial_{ba} g_{dc} - \frac{1}{2} \partial_{bc} g_{da} - \frac{1}{4} \partial_c g_{eb} g^{ef} \partial_d g_{fa} - \frac{1}{4} \partial_c g_{eb} g^{ef} \partial_a g_{df} + \frac{1}{4} \partial_c g_{eb} g^{ef} \partial_f g_{da} \\ &\quad - \frac{1}{4} \partial_b g_{ce} g^{ef} \partial_d g_{fa} - \frac{1}{4} \partial_b g_{ce} g^{ef} \partial_a g_{df} + \frac{1}{4} \partial_b g_{ce} g^{ef} \partial_f g_{da} + \frac{1}{4} \partial_e g_{cb} g^{ef} \partial_d g_{fa} + \frac{1}{4} \partial_e g_{cb} g^{ef} \partial_a g_{df} - \frac{1}{4} \partial_e g_{cb} g^{ef} \partial_f g_{da} + \frac{1}{4} \partial_c g_{ea} g^{ef} \partial_d g_{fb} \\ &\quad + \frac{1}{4} \partial_c g_{ea} g^{ef} \partial_b g_{df} - \frac{1}{4} \partial_c g_{ea} g^{ef} \partial_f g_{db} + \frac{1}{4} \partial_a g_{ce} g^{ef} \partial_d g_{fb} + \frac{1}{4} \partial_a g_{ce} g^{ef} \partial_b g_{df} - \frac{1}{4} \partial_a g_{ce} g^{ef} \partial_f g_{db} - \frac{1}{4} \partial_e g_{ca} g^{ef} \partial_d g_{fb} - \frac{1}{4} \partial_e g_{ca} g^{ef} \partial_b g_{df} \\ &\quad + \frac{1}{4} \partial_e g_{ca} g^{ef} \partial_f g_{db} \end{aligned} \quad (\text{ex-0304.205})$$

$$\begin{aligned}
R_{abcd} - R_{cdab} = & \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc} \\
& + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf} \\
& + \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd} \\
& - \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_f g_{da} \\
& - \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_d g_{fa} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_a g_{df} - \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_d g_{fb} \\
& + \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_f g_{db} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_d g_{fb} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_f g_{db} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_d g_{fb} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_b g_{df} \\
& + \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_f g_{db} \tag{ex-0304.206} \\
= & \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_f g_{bc}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_d g_{ae}g^{ef} \\
& + \frac{1}{4}\partial_c g_{bf}\partial_d g_{ae}g^{ef} - \frac{1}{4}\partial_d g_{ae}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} \\
& + \frac{1}{4}\partial_a g_{ec}\partial_f g_{bd}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_c g_{ae}g^{ef} - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{ae}\partial_f g_{bd}g^{ef} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} \\
& - \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{df}\partial_c g_{eb}g^{ef} + \frac{1}{4}\partial_c g_{eb}\partial_f g_{da}g^{ef} \\
& - \frac{1}{4}\partial_b g_{ce}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{df}\partial_b g_{ce}g^{ef} + \frac{1}{4}\partial_b g_{ce}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_d g_{fa}\partial_e g_{cb}g^{ef} + \frac{1}{4}\partial_a g_{df}\partial_e g_{cb}g^{ef} - \frac{1}{4}\partial_e g_{cb}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_c g_{ea}\partial_d g_{fb}g^{ef} \\
& + \frac{1}{4}\partial_b g_{df}\partial_c g_{ea}g^{ef} - \frac{1}{4}\partial_c g_{ea}\partial_f g_{db}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_d g_{fb}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_b g_{df}g^{ef} - \frac{1}{4}\partial_a g_{ce}\partial_f g_{db}g^{ef} - \frac{1}{4}\partial_d g_{fb}\partial_e g_{ca}g^{ef} - \frac{1}{4}\partial_b g_{df}\partial_e g_{ca}g^{ef} \\
& + \frac{1}{4}\partial_e g_{ca}\partial_f g_{db}g^{ef} \tag{ex-0304.207}
\end{aligned}$$



$$\begin{aligned}
R_{abcd} - R_{cdab} = & \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{fe} + \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{fe} \\
& + \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{fe} - \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{fe} - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} \\
& + \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{fe} - \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{fe} - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{fe} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} \\
& - \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{de}\partial_c g_{fb}g^{fe} + \frac{1}{4}\partial_c g_{fb}\partial_e g_{da}g^{fe} \\
& - \frac{1}{4}\partial_b g_{ce}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{de}\partial_b g_{cf}g^{fe} + \frac{1}{4}\partial_b g_{cf}\partial_e g_{da}g^{fe} + \frac{1}{4}\partial_d g_{fa}\partial_e g_{cb}g^{ef} + \frac{1}{4}\partial_a g_{df}\partial_e g_{cb}g^{ef} - \frac{1}{4}\partial_e g_{cb}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_c g_{ea}\partial_d g_{fb}g^{ef} \\
& + \frac{1}{4}\partial_b g_{de}\partial_c g_{fa}g^{fe} - \frac{1}{4}\partial_c g_{fa}\partial_e g_{db}g^{fe} + \frac{1}{4}\partial_a g_{ce}\partial_d g_{fb}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_b g_{df}g^{ef} - \frac{1}{4}\partial_a g_{cf}\partial_e g_{db}g^{fe} - \frac{1}{4}\partial_d g_{fb}\partial_e g_{ca}g^{ef} - \frac{1}{4}\partial_b g_{df}\partial_e g_{ca}g^{ef} \\
& + \frac{1}{4}\partial_e g_{ca}\partial_f g_{db}g^{ef} \tag{ex-0304.208} \\
= 0 \tag{ex-0304.209}
\end{aligned}$$