Exercise 3.1 Some symmetries of Riemann

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

;::Symbol;

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.       # cdb(Rabcd.000,Rabcd)

dRabcd := R^{a}_{b c d ; e} -> \partial_{e}{R^{a}_{b c d}}
                             + \Gamma^{a}_{f e} R^{f}_{b c d}
                             - \Gamma^{f}_{b e} R^{a}_{f c d}
                             - \Gamma^{f}_{c e} R^{a}_{b f d}
                             - \Gamma^{f}_{d e} R^{a}_{b c f}.       # cdb(dRabcd.000,dRabcd)
```

Exercise 3.1 Antisymmetry on last pair of indices

```
expr := R^{a}_{b c d} + R^{a}_{b d c}.                    # cdb(ex-0301.101,expr)

substitute (expr, Rabcd)                                  # cdb(ex-0301.102,expr)
```

$$R^{a}_{\ bcd} + R^{a}_{\ bdc} = 0 \qquad \text{(ex-0301.102)}$$

Exercise 3.1 First Bianchi identity

```
expr := R^{a}_{b c d} + R^{a}_{d b c} + R^{a}_{c d b}.                    # cdb(ex-0301.201,expr)

substitute   (expr, Rabcd)                                               # cdb(ex-0301.202,expr)
canonicalise (expr)                                                      # cdb(ex-0301.203,expr)
```

$$R^{a}{}_{bcd} + R^{a}{}_{dbc} + R^{a}{}_{cdb} = \partial_{c}\Gamma^{a}{}_{bd} - \partial_{d}\Gamma^{a}{}_{bc} + \Gamma^{e}{}_{bd}\Gamma^{a}{}_{ce} - \Gamma^{e}{}_{bc}\Gamma^{a}{}_{de} + \partial_{b}\Gamma^{a}{}_{dc} - \partial_{c}\Gamma^{a}{}_{db} + \Gamma^{e}{}_{dc}\Gamma^{a}{}_{be} - \Gamma^{e}{}_{db}\Gamma^{a}{}_{ce} + \partial_{d}\Gamma^{a}{}_{cb} - \partial_{b}\Gamma^{a}{}_{cd} + \Gamma^{e}{}_{cb}\Gamma^{a}{}_{de} - \Gamma^{e}{}_{cd}\Gamma^{a}{}_{be} \quad \text{(ex-0301.202)}$$

$$= 0 \quad \text{(ex-0301.203)}$$

Exercise 3.1 Second Bianchi identity

```
expr := R^{a}_{b c d ; e} + R^{a}_{b e c ; d} + R^{a}_{b d e ; c}.   # cdb(ex-0301.301,expr)

substitute     (expr, dRabcd)                                       # cdb(ex-0301.302,expr)
substitute     (expr,  Rabcd)                                       # cdb(ex-0301.303,expr)
distribute     (expr)                                               # cdb(ex-0301.304,expr)
product_rule   (expr)                                               # cdb(ex-0301.305,expr)
sort_product   (expr)                                               # cdb(ex-0301.306,expr)
rename_dummies (expr)                                               # cdb(ex-0301.307,expr)
canonicalise   (expr)                                               # cdb(ex-0301.308,expr)
```

$$
\begin{aligned}
R^{a}_{bcd;e} + R^{a}_{bec;d} + R^{a}_{bde;c} &= \partial_{e}R^{a}_{bcd} + \Gamma^{a}_{fe}R^{f}_{bcd} - \Gamma^{f}_{be}R^{a}_{fcd} - \Gamma^{f}_{ce}R^{a}_{bfd} - \Gamma^{f}_{de}R^{a}_{bcf} + \partial_{d}R^{a}_{bec} + \Gamma^{a}_{fd}R^{f}_{bec} - \Gamma^{f}_{bd}R^{a}_{fec} - \Gamma^{f}_{ed}R^{a}_{bfc} - \Gamma^{f}_{cd}R^{a}_{bef} \\
&\quad + \partial_{c}R^{a}_{bde} + \Gamma^{a}_{fc}R^{f}_{bde} - \Gamma^{f}_{bc}R^{a}_{fde} - \Gamma^{f}_{dc}R^{a}_{bfe} - \Gamma^{f}_{ec}R^{a}_{bdf} \qquad \text{(ex-0301.302)} \\
&= \partial_{e}\left(\partial_{c}\Gamma^{a}_{bd} - \partial_{d}\Gamma^{a}_{bc} + \Gamma^{f}_{bd}\Gamma^{a}_{cf} - \Gamma^{f}_{bc}\Gamma^{a}_{df}\right) + \Gamma^{a}_{fe}\left(\partial_{c}\Gamma^{f}_{bd} - \partial_{d}\Gamma^{f}_{bc} + \Gamma^{g}_{bd}\Gamma^{f}_{cg} - \Gamma^{g}_{bc}\Gamma^{f}_{dg}\right) \\
&\quad - \Gamma^{f}_{be}\left(\partial_{c}\Gamma^{a}_{fd} - \partial_{d}\Gamma^{a}_{fc} + \Gamma^{g}_{fd}\Gamma^{a}_{cg} - \Gamma^{g}_{fc}\Gamma^{a}_{dg}\right) - \Gamma^{f}_{ce}\left(\partial_{f}\Gamma^{a}_{bd} - \partial_{d}\Gamma^{a}_{bf} + \Gamma^{g}_{bd}\Gamma^{a}_{fg} - \Gamma^{g}_{bf}\Gamma^{a}_{dg}\right) \\
&\quad - \Gamma^{f}_{de}\left(\partial_{c}\Gamma^{a}_{bf} - \partial_{f}\Gamma^{a}_{bc} + \Gamma^{g}_{bf}\Gamma^{a}_{cg} - \Gamma^{g}_{bc}\Gamma^{a}_{fg}\right) + \partial_{d}\left(\partial_{e}\Gamma^{a}_{bc} - \partial_{c}\Gamma^{a}_{be} + \Gamma^{f}_{bc}\Gamma^{a}_{ef} - \Gamma^{f}_{be}\Gamma^{a}_{cf}\right) \\
&\quad + \Gamma^{a}_{fd}\left(\partial_{e}\Gamma^{f}_{bc} - \partial_{c}\Gamma^{f}_{be} + \Gamma^{g}_{bc}\Gamma^{f}_{eg} - \Gamma^{g}_{be}\Gamma^{f}_{cg}\right) - \Gamma^{f}_{bd}\left(\partial_{e}\Gamma^{a}_{fc} - \partial_{c}\Gamma^{a}_{fe} + \Gamma^{g}_{fc}\Gamma^{a}_{eg} - \Gamma^{g}_{fe}\Gamma^{a}_{cg}\right) \\
&\quad - \Gamma^{f}_{ed}\left(\partial_{f}\Gamma^{a}_{bc} - \partial_{c}\Gamma^{a}_{bf} + \Gamma^{g}_{bc}\Gamma^{a}_{fg} - \Gamma^{g}_{bf}\Gamma^{a}_{cg}\right) - \Gamma^{f}_{cd}\left(\partial_{e}\Gamma^{a}_{bf} - \partial_{f}\Gamma^{a}_{be} + \Gamma^{g}_{bf}\Gamma^{a}_{eg} - \Gamma^{g}_{be}\Gamma^{a}_{fg}\right) \\
&\quad + \partial_{c}\left(\partial_{d}\Gamma^{a}_{be} - \partial_{e}\Gamma^{a}_{bd} + \Gamma^{f}_{be}\Gamma^{a}_{df} - \Gamma^{f}_{bd}\Gamma^{a}_{ef}\right) + \Gamma^{a}_{fc}\left(\partial_{d}\Gamma^{f}_{be} - \partial_{e}\Gamma^{f}_{bd} + \Gamma^{g}_{be}\Gamma^{f}_{dg} - \Gamma^{g}_{bd}\Gamma^{f}_{eg}\right) \\
&\quad - \Gamma^{f}_{bc}\left(\partial_{d}\Gamma^{a}_{fe} - \partial_{e}\Gamma^{a}_{fd} + \Gamma^{g}_{fe}\Gamma^{a}_{dg} - \Gamma^{g}_{fd}\Gamma^{a}_{eg}\right) - \Gamma^{f}_{dc}\left(\partial_{f}\Gamma^{a}_{be} - \partial_{e}\Gamma^{a}_{bf} + \Gamma^{g}_{be}\Gamma^{a}_{fg} - \Gamma^{g}_{bf}\Gamma^{a}_{eg}\right) \\
&\quad - \Gamma^{f}_{ec}\left(\partial_{d}\Gamma^{a}_{bf} - \partial_{f}\Gamma^{a}_{bd} + \Gamma^{g}_{bf}\Gamma^{a}_{dg} - \Gamma^{g}_{bd}\Gamma^{a}_{fg}\right) \qquad \text{(ex-0301.303)} \\
&= \partial_{ec}\Gamma^{a}_{bd} - \partial_{ed}\Gamma^{a}_{bc} + \partial_{e}\left(\Gamma^{f}_{bd}\Gamma^{a}_{cf}\right) - \partial_{e}\left(\Gamma^{f}_{bc}\Gamma^{a}_{df}\right) + \Gamma^{a}_{fe}\partial_{c}\Gamma^{f}_{bd} - \Gamma^{a}_{fe}\partial_{d}\Gamma^{f}_{bc} + \Gamma^{a}_{fe}\Gamma^{g}_{bd}\Gamma^{f}_{cg} - \Gamma^{a}_{fe}\Gamma^{g}_{bc}\Gamma^{f}_{dg} - \Gamma^{f}_{be}\partial_{c}\Gamma^{a}_{fd} \\
&\quad + \Gamma^{f}_{be}\partial_{d}\Gamma^{a}_{fc} - \Gamma^{f}_{be}\Gamma^{g}_{fd}\Gamma^{a}_{cg} + \Gamma^{f}_{be}\Gamma^{g}_{fc}\Gamma^{a}_{dg} - \Gamma^{f}_{ce}\partial_{f}\Gamma^{a}_{bd} + \Gamma^{f}_{ce}\partial_{d}\Gamma^{a}_{bf} - \Gamma^{f}_{ce}\Gamma^{g}_{bd}\Gamma^{a}_{fg} + \Gamma^{f}_{ce}\Gamma^{g}_{bf}\Gamma^{a}_{dg} - \Gamma^{f}_{de}\partial_{c}\Gamma^{a}_{bf} + \Gamma^{f}_{de}\partial_{f}\Gamma^{a}_{bc} \\
&\quad - \Gamma^{f}_{de}\Gamma^{g}_{bf}\Gamma^{a}_{cg} + \Gamma^{f}_{de}\Gamma^{g}_{bc}\Gamma^{a}_{fg} + \partial_{dc}\Gamma^{a}_{bc} - \partial_{dc}\Gamma^{a}_{be} + \partial_{d}\left(\Gamma^{f}_{bc}\Gamma^{a}_{ef}\right) - \partial_{d}\left(\Gamma^{f}_{be}\Gamma^{a}_{cf}\right) + \Gamma^{a}_{fd}\partial_{e}\Gamma^{f}_{bc} - \Gamma^{a}_{fd}\partial_{c}\Gamma^{f}_{be} + \Gamma^{a}_{fd}\Gamma^{g}_{bc}\Gamma^{f}_{eg} \\
&\quad - \Gamma^{a}_{fd}\Gamma^{g}_{be}\Gamma^{f}_{cg} - \Gamma^{f}_{bd}\partial_{e}\Gamma^{a}_{fc} + \Gamma^{f}_{bd}\partial_{c}\Gamma^{a}_{fe} - \Gamma^{f}_{bd}\Gamma^{g}_{fc}\Gamma^{a}_{eg} + \Gamma^{f}_{bd}\Gamma^{g}_{fe}\Gamma^{a}_{cg} - \Gamma^{f}_{ed}\partial_{f}\Gamma^{a}_{bc} + \Gamma^{f}_{ed}\partial_{c}\Gamma^{a}_{bf} - \Gamma^{f}_{ed}\Gamma^{g}_{bc}\Gamma^{a}_{fg} \\
&\quad + \Gamma^{f}_{ed}\Gamma^{g}_{bf}\Gamma^{a}_{cg} - \Gamma^{f}_{cd}\partial_{e}\Gamma^{a}_{bf} + \Gamma^{f}_{cd}\partial_{f}\Gamma^{a}_{be} - \Gamma^{f}_{cd}\Gamma^{g}_{bf}\Gamma^{a}_{eg} + \Gamma^{f}_{cd}\Gamma^{g}_{be}\Gamma^{a}_{fg} + \partial_{cd}\Gamma^{a}_{be} - \partial_{ce}\Gamma^{a}_{bd} + \partial_{c}\left(\Gamma^{f}_{be}\Gamma^{a}_{df}\right) - \partial_{c}\left(\Gamma^{f}_{bd}\Gamma^{a}_{ef}\right) \\
&\quad + \Gamma^{a}_{fc}\partial_{d}\Gamma^{f}_{be} - \Gamma^{a}_{fc}\partial_{e}\Gamma^{f}_{bd} + \Gamma^{a}_{fc}\Gamma^{g}_{be}\Gamma^{f}_{dg} - \Gamma^{a}_{fc}\Gamma^{g}_{bd}\Gamma^{f}_{eg} - \Gamma^{f}_{bc}\partial_{d}\Gamma^{a}_{fe} + \Gamma^{f}_{bc}\partial_{e}\Gamma^{a}_{fd} - \Gamma^{f}_{bc}\Gamma^{g}_{fe}\Gamma^{a}_{dg} + \Gamma^{f}_{bc}\Gamma^{g}_{fd}\Gamma^{a}_{eg} \\
&\quad - \Gamma^{f}_{dc}\partial_{f}\Gamma^{a}_{be} + \Gamma^{f}_{dc}\partial_{e}\Gamma^{a}_{bf} - \Gamma^{f}_{dc}\Gamma^{g}_{be}\Gamma^{a}_{fg} + \Gamma^{f}_{dc}\Gamma^{g}_{bf}\Gamma^{a}_{eg} - \Gamma^{f}_{ec}\partial_{d}\Gamma^{a}_{bf} + \Gamma^{f}_{ec}\partial_{f}\Gamma^{a}_{bd} - \Gamma^{f}_{ec}\Gamma^{g}_{bf}\Gamma^{a}_{dg} + \Gamma^{f}_{ec}\Gamma^{g}_{bd}\Gamma^{a}_{fg} \qquad \text{(ex-0301.304)}
\end{aligned}
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\begin{aligned}
R^{a}{}_{bcd;e} + R^{a}{}_{bec;d} + R^{a}{}_{bde;c} = {} & \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \partial_{e}\Gamma^{f}{}_{bd}\Gamma^{a}{}_{cf} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \partial_{e}\Gamma^{f}{}_{bc}\Gamma^{a}{}_{df} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{cg} - \Gamma^{a}{}_{fe}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{dg} \\
& - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{f}{}_{be}\Gamma^{g}{}_{fd}\Gamma^{a}{}_{cg} + \Gamma^{f}{}_{be}\Gamma^{g}{}_{fc}\Gamma^{a}{}_{dg} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd}\Gamma^{a}{}_{fg} + \Gamma^{f}{}_{ce}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{dg} - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} \\
& + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{f}{}_{de}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{cg} + \Gamma^{f}{}_{de}\Gamma^{g}{}_{bc}\Gamma^{a}{}_{fg} + \partial_{de}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \partial_{d}\Gamma^{f}{}_{bc}\Gamma^{a}{}_{ef} + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} - \partial_{d}\Gamma^{f}{}_{be}\Gamma^{a}{}_{cf} - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} \\
& + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{eg} - \Gamma^{a}{}_{fd}\Gamma^{g}{}_{be}\Gamma^{f}{}_{cg} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} - \Gamma^{f}{}_{bd}\Gamma^{g}{}_{fc}\Gamma^{a}{}_{eg} + \Gamma^{f}{}_{bd}\Gamma^{g}{}_{fe}\Gamma^{a}{}_{cg} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} \\
& + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc}\Gamma^{a}{}_{fg} + \Gamma^{f}{}_{ed}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{cg} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} - \Gamma^{f}{}_{cd}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{eg} + \Gamma^{f}{}_{cd}\Gamma^{g}{}_{be}\Gamma^{a}{}_{fg} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} \\
& + \partial_{e}\Gamma^{f}{}_{be}\Gamma^{a}{}_{df} + \Gamma^{f}{}_{be}\partial_{e}\Gamma^{a}{}_{df} - \partial_{e}\Gamma^{f}{}_{bd}\Gamma^{a}{}_{ef} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{a}{}_{fc}\Gamma^{g}{}_{be}\Gamma^{f}{}_{dg} - \Gamma^{a}{}_{fc}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{eg} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} \\
& + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{f}{}_{bc}\Gamma^{g}{}_{fe}\Gamma^{a}{}_{dg} + \Gamma^{f}{}_{bc}\Gamma^{g}{}_{fd}\Gamma^{a}{}_{eg} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} - \Gamma^{f}{}_{dc}\Gamma^{g}{}_{be}\Gamma^{a}{}_{fg} + \Gamma^{f}{}_{dc}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{eg} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} \\
& - \Gamma^{f}{}_{ec}\Gamma^{g}{}_{bf}\Gamma^{a}{}_{dg} + \Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd}\Gamma^{a}{}_{fg} \qquad \texttt{(ex-0301.305)} \\
= {} & \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \Gamma^{a}{}_{cf}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \Gamma^{a}{}_{df}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{bd} - \Gamma^{a}{}_{fe}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{bc} \\
& - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{a}{}_{cg}\Gamma^{f}{}_{be}\Gamma^{g}{}_{fd} + \Gamma^{a}{}_{dg}\Gamma^{f}{}_{be}\Gamma^{g}{}_{fc} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd} + \Gamma^{a}{}_{dg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} \\
& + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{a}{}_{cg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bc} + \partial_{de}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \Gamma^{a}{}_{ef}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} - \Gamma^{a}{}_{cf}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} \\
& + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bc} - \Gamma^{a}{}_{fd}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{be} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} - \Gamma^{a}{}_{eg}\Gamma^{f}{}_{bd}\Gamma^{g}{}_{fc} + \Gamma^{a}{}_{cg}\Gamma^{f}{}_{bd}\Gamma^{g}{}_{fe} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} \\
& + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc} + \Gamma^{a}{}_{cg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} - \Gamma^{a}{}_{eg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{be} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} \\
& + \Gamma^{a}{}_{df}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{df} - \Gamma^{a}{}_{ef}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{a}{}_{fc}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{be} - \Gamma^{a}{}_{fc}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bd} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} \\
& + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{a}{}_{dg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{fe} + \Gamma^{a}{}_{eg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{fd} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{be} + \Gamma^{a}{}_{eg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{bf} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} \\
& - \Gamma^{a}{}_{dg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bf} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd} \qquad \texttt{(ex-0301.306)}
\end{aligned}
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\begin{aligned}
R^{a}{}_{bcd;e} + R^{a}{}_{bec;d} + R^{a}{}_{bde;c} = {} & \partial_{ec}\Gamma^{a}{}_{bd} - \partial_{ed}\Gamma^{a}{}_{bc} + \Gamma^{a}{}_{cf}\partial_{e}\Gamma^{f}{}_{bd} + \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{cf} - \Gamma^{a}{}_{df}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{df} + \Gamma^{a}{}_{fe}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{a}{}_{fe}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{a}{}_{fe}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{bd} - \Gamma^{a}{}_{fe}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{bc} \\
& - \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{fd} + \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{fc} - \Gamma^{a}{}_{cf}\Gamma^{g}{}_{be}\Gamma^{f}{}_{gd} + \Gamma^{a}{}_{df}\Gamma^{g}{}_{be}\Gamma^{f}{}_{gc} - \Gamma^{f}{}_{ce}\partial_{f}\Gamma^{a}{}_{bd} + \Gamma^{f}{}_{ce}\partial_{d}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ce}\Gamma^{g}{}_{bd} + \Gamma^{a}{}_{df}\Gamma^{g}{}_{ce}\Gamma^{f}{}_{bg} \\
& - \Gamma^{f}{}_{de}\partial_{c}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{de}\partial_{f}\Gamma^{a}{}_{bc} - \Gamma^{a}{}_{cf}\Gamma^{g}{}_{de}\Gamma^{f}{}_{bg} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{de}\Gamma^{g}{}_{bc} + \partial_{dc}\Gamma^{a}{}_{bc} - \partial_{dc}\Gamma^{a}{}_{be} + \Gamma^{a}{}_{ef}\partial_{d}\Gamma^{f}{}_{bc} + \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} - \Gamma^{a}{}_{cf}\partial_{d}\Gamma^{f}{}_{be} \\
& - \Gamma^{f}{}_{be}\partial_{d}\Gamma^{a}{}_{cf} + \Gamma^{a}{}_{fd}\partial_{e}\Gamma^{f}{}_{bc} - \Gamma^{a}{}_{fd}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{a}{}_{fd}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bc} - \Gamma^{a}{}_{fd}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{be} - \Gamma^{f}{}_{bd}\partial_{e}\Gamma^{a}{}_{fc} + \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{fe} - \Gamma^{a}{}_{ef}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{gc} \\
& + \Gamma^{a}{}_{cf}\Gamma^{g}{}_{bd}\Gamma^{f}{}_{ge} - \Gamma^{f}{}_{ed}\partial_{f}\Gamma^{a}{}_{bc} + \Gamma^{f}{}_{ed}\partial_{c}\Gamma^{a}{}_{bf} - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ed}\Gamma^{g}{}_{bc} + \Gamma^{a}{}_{cf}\Gamma^{g}{}_{ed}\Gamma^{f}{}_{bg} - \Gamma^{f}{}_{cd}\partial_{e}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{cd}\partial_{f}\Gamma^{a}{}_{be} - \Gamma^{a}{}_{ef}\Gamma^{g}{}_{cd}\Gamma^{f}{}_{bg} \\
& + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{cd}\Gamma^{g}{}_{be} + \partial_{cd}\Gamma^{a}{}_{be} - \partial_{ce}\Gamma^{a}{}_{bd} + \Gamma^{a}{}_{df}\partial_{c}\Gamma^{f}{}_{be} + \Gamma^{f}{}_{be}\partial_{c}\Gamma^{a}{}_{df} - \Gamma^{a}{}_{ef}\partial_{c}\Gamma^{f}{}_{bd} - \Gamma^{f}{}_{bd}\partial_{c}\Gamma^{a}{}_{ef} + \Gamma^{a}{}_{fc}\partial_{d}\Gamma^{f}{}_{be} - \Gamma^{a}{}_{fc}\partial_{e}\Gamma^{f}{}_{bd} \\
& + \Gamma^{a}{}_{fc}\Gamma^{f}{}_{dg}\Gamma^{g}{}_{be} - \Gamma^{a}{}_{fc}\Gamma^{f}{}_{eg}\Gamma^{g}{}_{bd} - \Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{fe} + \Gamma^{f}{}_{bc}\partial_{e}\Gamma^{a}{}_{fd} - \Gamma^{a}{}_{df}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{ge} + \Gamma^{a}{}_{ef}\Gamma^{g}{}_{bc}\Gamma^{f}{}_{gd} - \Gamma^{f}{}_{dc}\partial_{f}\Gamma^{a}{}_{be} + \Gamma^{f}{}_{dc}\partial_{e}\Gamma^{a}{}_{bf} \\
& - \Gamma^{a}{}_{fg}\Gamma^{f}{}_{dc}\Gamma^{g}{}_{be} + \Gamma^{a}{}_{ef}\Gamma^{g}{}_{dc}\Gamma^{f}{}_{bg} - \Gamma^{f}{}_{ec}\partial_{d}\Gamma^{a}{}_{bf} + \Gamma^{f}{}_{ec}\partial_{f}\Gamma^{a}{}_{bd} - \Gamma^{a}{}_{df}\Gamma^{g}{}_{ec}\Gamma^{f}{}_{bg} + \Gamma^{a}{}_{fg}\Gamma^{f}{}_{ec}\Gamma^{g}{}_{bd} \qquad \texttt{(ex-0301.307)} \\
= {} & 0 \qquad \texttt{(ex-0301.308)}
\end{aligned}
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