Exercise 3.9 Ricci in terms of the metric and its derivatives

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.      # cdb (ex-0309.dgab,dgab)

Gamma := \Gamma^{a}_{b c} ->
         (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                         + \partial_{c}{g_{b e}}
                         - \partial_{e}{g_{b c}}).                             # cdb (ex-0309.Gamma,Gamma)

Rabcd := R^{a}_{b c d} ->
         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
       - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.   # cdb (ex-0309.Rabcd,Rabcd)

FourRab := 4 R^{c}_{a c b}.                        # cdb (ex-0309.101,FourRab)

substitute     (FourRab, Rabcd)                    # cdb (ex-0309.102,FourRab)
substitute     (FourRab, Gamma)                    # cdb (ex-0309.103,FourRab)

product_rule   (FourRab)                           # cdb (ex-0309.104,FourRab)
distribute     (FourRab)                           # cdb (ex-0309.105,FourRab)

substitute     (FourRab, dgab)                     # cdb (ex-0309.106,FourRab)

sort_product   (FourRab)                           # cdb (ex-0309.107,FourRab)
rename_dummies (FourRab)                           # cdb (ex-0309.108,FourRab)
canonicalise   (FourRab)                           # cdb (ex-0309.109,FourRab)

# sort so that g to appeares before dg

substitute     (FourRab, $g^{a b} -> A^{a b}$)
sort_product   (FourRab)
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37  rename_dummies (FourRab)
38  substitute     (FourRab, $A^{a b} -> g^{a b}$)   # cdb (ex-0309.110,FourRab)
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$$4R_{ab} = 4\,R^{c}_{\ acb} \tag{ex-0309.101}$$

$$= 4\,\partial_{c}\Gamma^{c}_{\ ab} + 4\,\Gamma^{c}_{\ ec}\Gamma^{e}_{\ ab} - 4\,\partial_{b}\Gamma^{c}_{\ ac} - 4\,\Gamma^{c}_{\ eb}\Gamma^{e}_{\ ac} \tag{ex-0309.102}$$

$$= 2\,\partial_{c}(g^{ce}\left(\partial_{d}g_{eb} + \partial_{b}g_{ae} - \partial_{e}g_{ab}\right)) + g^{cd}\left(\partial_{d}g_{dc} + \partial_{c}g_{ed} - \partial_{d}g_{ec}\right)g^{ef}\left(\partial_{d}g_{fb} + \partial_{b}g_{af} - \partial_{f}g_{ab}\right) - 2\,\partial_{b}(g^{ce}\left(\partial_{d}g_{ec} + \partial_{c}g_{ae} - \partial_{e}g_{ac}\right))$$
$$- g^{cd}\left(\partial_{c}g_{db} + \partial_{b}g_{ed} - \partial_{d}g_{eb}\right)g^{ef}\left(\partial_{d}g_{fc} + \partial_{c}g_{af} - \partial_{f}g_{ac}\right) \tag{ex-0309.103}$$

$$= 2\,\partial_{c}g^{ce}\left(\partial_{d}g_{eb} + \partial_{b}g_{ae} - \partial_{e}g_{ab}\right) + 2\,g^{ce}\partial_{c}(\partial_{d}g_{eb} + \partial_{b}g_{ae} - \partial_{e}g_{ab}) + g^{cd}\left(\partial_{c}g_{dc} + \partial_{c}g_{ed} - \partial_{d}g_{ec}\right)g^{ef}\left(\partial_{d}g_{fb} + \partial_{b}g_{af} - \partial_{f}g_{ab}\right)$$
$$- 2\,\partial_{b}g^{ce}\left(\partial_{d}g_{ec} + \partial_{c}g_{ae} - \partial_{e}g_{ac}\right) - 2\,g^{ce}\partial_{b}(\partial_{d}g_{ec} + \partial_{c}g_{ae} - \partial_{e}g_{ac}) - g^{cd}\left(\partial_{c}g_{db} + \partial_{b}g_{ed} - \partial_{d}g_{eb}\right)g^{ef}\left(\partial_{d}g_{fc} + \partial_{c}g_{af} - \partial_{f}g_{ac}\right) \tag{ex-0309.104}$$

$$= 2\,\partial_{c}g^{ce}\partial_{d}g_{eb} + 2\,\partial_{c}g^{ce}\partial_{b}g_{ae} - 2\,\partial_{c}g^{ce}\partial_{e}g_{ab} + 2\,g^{ce}\partial_{cd}g_{eb} + 2\,g^{ce}\partial_{cb}g_{ae} - 2\,g^{ce}\partial_{ce}g_{ab} + g^{cd}\partial_{c}g_{dc}g^{ef}\partial_{d}g_{fb} + g^{cd}\partial_{c}g_{dc}g^{ef}\partial_{b}g_{af} - g^{cd}\partial_{c}g_{dc}g^{ef}\partial_{f}g_{ab}$$
$$+ g^{cd}\partial_{c}g_{ed}g^{ef}\partial_{d}g_{fb} + g^{cd}\partial_{c}g_{ed}g^{ef}\partial_{b}g_{af} - g^{cd}\partial_{c}g_{ed}g^{ef}\partial_{f}g_{ab} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{d}g_{fb} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{b}g_{af} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{f}g_{ab} - 2\,\partial_{b}g^{ce}\partial_{d}g_{ec} - 2\,\partial_{b}g^{ce}\partial_{c}g_{ae}$$
$$+ 2\,\partial_{b}g^{ce}\partial_{e}g_{ac} - 2\,g^{ce}\partial_{bd}g_{ec} - 2\,g^{ce}\partial_{bc}g_{ae} + 2\,g^{ce}\partial_{be}g_{ac} - g^{cd}\partial_{c}g_{db}g^{ef}\partial_{d}g_{fc} - g^{cd}\partial_{c}g_{db}g^{ef}\partial_{c}g_{af} + g^{cd}\partial_{c}g_{db}g^{ef}\partial_{f}g_{ac} - g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{d}g_{fc}$$
$$- g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{c}g_{af} + g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{f}g_{ac} + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{d}g_{fc} + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{c}g_{af} - g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{f}g_{ac} \tag{ex-0309.105}$$

$$= -2\,g^{cd}g^{ef}\partial_{c}g_{df}\partial_{d}g_{eb} - 2\,g^{cd}g^{ef}\partial_{c}g_{df}\partial_{b}g_{ae} + 2\,g^{cd}g^{ef}\partial_{c}g_{df}\partial_{c}g_{ab} + 2\,g^{ce}\partial_{cd}g_{eb} + 2\,g^{ce}\partial_{cb}g_{ae} - 2\,g^{ce}\partial_{ce}g_{ab} + g^{cd}\partial_{c}g_{dc}g^{ef}\partial_{d}g_{fb} + g^{cd}\partial_{c}g_{dc}g^{ef}\partial_{b}g_{af}$$
$$- g^{cd}\partial_{c}g_{dc}g^{ef}\partial_{f}g_{ab} + g^{cd}\partial_{c}g_{ed}g^{ef}\partial_{d}g_{fb} + g^{cd}\partial_{c}g_{ed}g^{ef}\partial_{b}g_{af} - g^{cd}\partial_{c}g_{ed}g^{ef}\partial_{f}g_{ab} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{d}g_{fb} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{b}g_{af} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{f}g_{ab}$$
$$+ 2\,g^{cd}g^{ef}\partial_{b}g_{df}\partial_{d}g_{ec} + 2\,g^{cd}g^{ef}\partial_{b}g_{df}\partial_{c}g_{ae} - 2\,g^{cd}g^{ef}\partial_{b}g_{df}\partial_{e}g_{ac} - 2\,g^{ce}\partial_{bd}g_{ec} - 2\,g^{ce}\partial_{bc}g_{ae} + 2\,g^{ce}\partial_{be}g_{ac} - g^{cd}\partial_{c}g_{db}g^{ef}\partial_{d}g_{fc} - g^{cd}\partial_{c}g_{db}g^{ef}\partial_{c}g_{af}$$
$$+ g^{cd}\partial_{c}g_{db}g^{ef}\partial_{f}g_{ac} - g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{d}g_{fc} - g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{c}g_{af} + g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{f}g_{ac} + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{d}g_{fc} + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{c}g_{af} - g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{f}g_{ac} \tag{ex-0309.106}$$

$$= -2\,\partial_{d}g_{eb}\partial_{c}g_{df}g^{cd}g^{ef} - 2\,\partial_{b}g_{ae}\partial_{c}g_{df}g^{cd}g^{ef} + 2\,\partial_{c}g_{df}\partial_{e}g_{ab}g^{cd}g^{ef} + 2\,\partial_{cd}g_{eb}g^{ce} + 2\,\partial_{cb}g_{ae}g^{ce} - 2\,\partial_{ce}g_{ab}g^{ce} + \partial_{d}g_{fb}\partial_{e}g_{dc}g^{cd}g^{ef} + \partial_{b}g_{af}\partial_{e}g_{dc}g^{cd}g^{ef}$$
$$- \partial_{e}g_{dc}\partial_{f}g_{ab}g^{cd}g^{ef} + \partial_{d}g_{fb}\partial_{c}g_{ed}g^{cd}g^{ef} + \partial_{b}g_{af}\partial_{c}g_{ed}g^{cd}g^{ef} - \partial_{c}g_{ed}\partial_{f}g_{ab}g^{cd}g^{ef} - \partial_{d}g_{fb}\partial_{d}g_{ec}g^{cd}g^{ef} - \partial_{b}g_{af}\partial_{d}g_{ec}g^{cd}g^{ef} + \partial_{d}g_{ec}\partial_{f}g_{ab}g^{cd}g^{ef}$$
$$+ 2\,\partial_{d}g_{ec}\partial_{b}g_{df}g^{cd}g^{ef} + 2\,\partial_{b}g_{df}\partial_{c}g_{ae}g^{cd}g^{ef} - 2\,\partial_{b}g_{df}\partial_{c}g_{ac}g^{cd}g^{ef} - 2\,\partial_{bd}g_{ec}g^{ce} - 2\,\partial_{bc}g_{ae}g^{ce} + 2\,\partial_{be}g_{ac}g^{ce} - \partial_{d}g_{fc}\partial_{e}g_{db}g^{cd}g^{ef} - \partial_{c}g_{af}\partial_{e}g_{db}g^{cd}g^{ef}$$
$$+ \partial_{e}g_{db}\partial_{f}g_{ac}g^{cd}g^{ef} - \partial_{d}g_{fc}\partial_{b}g_{ed}g^{cd}g^{ef} - \partial_{b}g_{ed}\partial_{c}g_{af}g^{cd}g^{ef} + \partial_{b}g_{ed}\partial_{f}g_{ac}g^{cd}g^{ef} + \partial_{d}g_{fc}\partial_{d}g_{eb}g^{cd}g^{ef} + \partial_{c}g_{af}\partial_{d}g_{eb}g^{cd}g^{ef} - \partial_{d}g_{eb}\partial_{f}g_{ac}g^{cd}g^{ef} \tag{ex-0309.107}$$

$$= -2\,\partial_{d}g_{db}\partial_{c}g_{ef}g^{ce}g^{df} - 2\,\partial_{b}g_{ad}\partial_{c}g_{ef}g^{ce}g^{df} + 2\,\partial_{c}g_{ef}\partial_{d}g_{ab}g^{ce}g^{df} + 2\,\partial_{cd}g_{db}g^{cd} + 2\,\partial_{cb}g_{ad}g^{cd} - 2\,\partial_{cd}g_{ab}g^{cd} + \partial_{d}g_{db}\partial_{c}g_{ef}g^{fe}g^{cd} + \partial_{b}g_{ad}\partial_{c}g_{ef}g^{fe}g^{cd}$$
$$- \partial_{c}g_{ef}\partial_{d}g_{ab}g^{fe}g^{cd} + \partial_{d}g_{db}\partial_{c}g_{ef}g^{cf}g^{ed} + \partial_{b}g_{ad}\partial_{c}g_{ef}g^{cf}g^{ed} - \partial_{c}g_{ef}\partial_{d}g_{ab}g^{cf}g^{ed} - \partial_{d}g_{db}\partial_{c}g_{ef}g^{fc}g^{ed} - \partial_{b}g_{ad}\partial_{c}g_{ef}g^{fc}g^{ed} + \partial_{c}g_{ef}\partial_{d}g_{ab}g^{fc}g^{ed}$$
$$+ 2\,\partial_{d}g_{cd}\partial_{b}g_{ef}g^{de}g^{cf} + 2\,\partial_{b}g_{de}\partial_{c}g_{af}g^{cd}g^{fe} - 2\,\partial_{b}g_{de}\partial_{c}g_{af}g^{fd}g^{ce} - 2\,\partial_{bd}g_{cd}g^{dc} - 2\,\partial_{bc}g_{ad}g^{cd} + 2\,\partial_{bc}g_{ad}g^{dc} - \partial_{d}g_{de}\partial_{c}g_{fb}g^{ef}g^{cd} - \partial_{c}g_{ae}\partial_{d}g_{fb}g^{cf}g^{de}$$
$$+ \partial_{c}g_{eb}\partial_{d}g_{af}g^{fe}g^{cd} - \partial_{d}g_{cd}\partial_{b}g_{ef}g^{df}g^{ec} - \partial_{b}g_{de}\partial_{c}g_{af}g^{ce}g^{df} + \partial_{b}g_{de}\partial_{c}g_{af}g^{fe}g^{dc} + \partial_{d}g_{de}\partial_{c}g_{fb}g^{ec}g^{fd} + \partial_{c}g_{ae}\partial_{d}g_{fb}g^{cd}g^{fe} - \partial_{c}g_{eb}\partial_{d}g_{af}g^{fc}g^{ed} \tag{ex-0309.108}$$

$$= -2\,\partial_{d}g_{bc}\partial_{d}g_{ef}g^{ce}g^{df} - 2\,\partial_{b}g_{ac}\partial_{d}g_{ef}g^{ce}g^{df} + 2\,\partial_{c}g_{ab}\partial_{d}g_{ef}g^{ce}g^{df} + 2\,\partial_{ac}g_{bd}g^{cd} + 2\,\partial_{bc}g_{ad}g^{cd} - 2\,\partial_{cd}g_{ab}g^{cd} + \partial_{d}g_{bc}\partial_{d}g_{ef}g^{cd}g^{ef} + \partial_{b}g_{ac}\partial_{d}g_{ef}g^{cd}g^{ef}$$
$$- \partial_{c}g_{ab}\partial_{d}g_{ef}g^{cd}g^{ef} + \partial_{d}g_{cd}\partial_{b}g_{ef}g^{ce}g^{df} - 2\,\partial_{ab}g_{cd}g^{cd} - 2\,\partial_{c}g_{ad}\partial_{c}g_{bf}g^{cf}g^{de} + 2\,\partial_{c}g_{ad}\partial_{e}g_{bf}g^{ce}g^{df} \tag{ex-0309.109}$$

$$= -2\,g^{cd}g^{ef}\partial_{d}g_{bc}\partial_{c}g_{df} - 2\,g^{cd}g^{ef}\partial_{b}g_{ac}\partial_{c}g_{df} + 2\,g^{cd}g^{ef}\partial_{c}g_{ab}\partial_{e}g_{df} + 2\,g^{cd}\partial_{ac}g_{bd} + 2\,g^{cd}\partial_{bc}g_{ad} - 2\,g^{cd}\partial_{cd}g_{ab} + g^{cd}g^{ef}\partial_{d}g_{bc}\partial_{d}g_{ef} + g^{cd}g^{ef}\partial_{b}g_{ac}\partial_{d}g_{ef}$$
$$- g^{cd}g^{ef}\partial_{c}g_{ab}\partial_{d}g_{ef} + g^{cd}g^{ef}\partial_{d}g_{ce}\partial_{b}g_{df} - 2\,g^{cd}\partial_{ab}g_{cd} - 2\,g^{cd}g^{ef}\partial_{c}g_{ae}\partial_{f}g_{bd} + 2\,g^{cd}g^{ef}\partial_{c}g_{ae}\partial_{d}g_{bf} \tag{ex-0309.110}$$