Exercise 3.4 More symmetries of Riemann

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Symmetric.
g^{a b}::Symmetric.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).

GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                            + \partial_{c}{g_{b d}}
                                            - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)

GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
                                   + \partial_{c}{g_{b a}}
                                   - \partial_{a}{g_{b c}}).          # cdb(Gamma.010,GammaD)

Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
                        - \partial_{d}{\Gamma_{a b c}}
                        + \Gamma_{e a d} \Gamma^{e}_{b c}
                        - \Gamma_{e a c} \Gamma^{e}_{b d}.            # cdb(Rabcd.000,Rabcd)
```

Exercise 3.4 Antisymmetry on first pair of indices

```
expr := R_{a b c d} + R_{b a c d}.   # cdb(ex-0304.101,expr)

substitute     (expr, Rabcd)         # cdb(ex-0304.102,expr)
substitute     (expr, GammaU)        # cdb(ex-0304.103,expr)
substitute     (expr, GammaD)        # cdb(ex-0304.104,expr)
distribute     (expr)                # cdb(ex-0304.105,expr)
product_rule   (expr)                # cdb(ex-0304.106,expr)
sort_product   (expr)                # cdb(ex-0304.107,expr)
rename_dummies (expr)                # cdb(ex-0304.108,expr)
canonicalise   (expr)                # cdb(ex-0304.109,expr)
```

$$R_{abcd} + R_{bacd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead}\Gamma^e_{bc} - \Gamma_{eac}\Gamma^e_{bd} + \partial_c \Gamma_{bad} - \partial_d \Gamma_{bac} + \Gamma_{ebd}\Gamma^e_{ac} - \Gamma_{ebc}\Gamma^e_{ad} \tag{ex-0304.102}$$

$$= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2}\Gamma_{ead} g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) - \frac{1}{2}\Gamma_{eac} g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) + \partial_c \Gamma_{bad} - \partial_d \Gamma_{bac}$$
$$+ \frac{1}{2}\Gamma_{ebd} g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) - \frac{1}{2}\Gamma_{ebc} g^{ef}\left(\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}\right) \tag{ex-0304.103}$$

$$= \partial_c\left(\frac{1}{2}\partial_b g_{ad} + \frac{1}{2}\partial_d g_{ba} - \frac{1}{2}\partial_a g_{bd}\right) - \partial_d\left(\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc}\right) + \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ed} + \frac{1}{2}\partial_d g_{ae} - \frac{1}{2}\partial_e g_{ad}\right) g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right)$$
$$- \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ec} + \frac{1}{2}\partial_c g_{ae} - \frac{1}{2}\partial_e g_{ac}\right) g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) + \partial_c\left(\frac{1}{2}\partial_a g_{bd} + \frac{1}{2}\partial_d g_{ab} - \frac{1}{2}\partial_b g_{ad}\right)$$
$$- \partial_d\left(\frac{1}{2}\partial_a g_{bc} + \frac{1}{2}\partial_c g_{ab} - \frac{1}{2}\partial_b g_{ac}\right) + \frac{1}{2}\left(\frac{1}{2}\partial_b g_{ed} + \frac{1}{2}\partial_d g_{be} - \frac{1}{2}\partial_e g_{bd}\right) g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right)$$
$$- \frac{1}{2}\left(\frac{1}{2}\partial_b g_{ec} + \frac{1}{2}\partial_c g_{be} - \frac{1}{2}\partial_e g_{bc}\right) g^{ef}\left(\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}\right) \tag{ex-0304.104}$$

$$= \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{4}\partial_a g_{ed} g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed} g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed} g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae} g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_d g_{ae} g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae} g^{ef}\partial_f g_{bc}$$
$$- \frac{1}{4}\partial_e g_{ad} g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad} g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad} g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec} g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec} g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_a g_{ec} g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_e g_{ae} g^{ef}\partial_b g_{fd}$$
$$- \frac{1}{4}\partial_e g_{ae} g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_e g_{ae} g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac} g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac} g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac} g^{ef}\partial_f g_{bd} + \frac{1}{2}\partial_{cd} g_{ab} - \frac{1}{2}\partial_{dc} g_{ab} + \frac{1}{4}\partial_b g_{ed} g^{ef}\partial_a g_{fc}$$
$$+ \frac{1}{4}\partial_b g_{ed} g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_b g_{ed} g^{ef}\partial_f g_{ac} + \frac{1}{4}\partial_d g_{be} g^{ef}\partial_a g_{fc} + \frac{1}{4}\partial_d g_{be} g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_d g_{be} g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_c g_{bd} g^{ef}\partial_a g_{fc} - \frac{1}{4}\partial_c g_{bd} g^{ef}\partial_c g_{af}$$
$$+ \frac{1}{4}\partial_c g_{bd} g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_b g_{ec} g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_b g_{ec} g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_b g_{ec} g^{ef}\partial_f g_{ad} - \frac{1}{4}\partial_c g_{be} g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_c g_{be} g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_c g_{be} g^{ef}\partial_f g_{ad}$$
$$+ \frac{1}{4}\partial_c g_{bc} g^{ef}\partial_a g_{fd} + \frac{1}{4}\partial_c g_{bc} g^{ef}\partial_d g_{af} - \frac{1}{4}\partial_c g_{bc} g^{ef}\partial_f g_{ad} \tag{ex-0304.105}$$

$$
\begin{aligned}
R_{abcd} + R_{bacd} = {} & \frac{1}{2}\,\partial_{cd}g_{ba} - \frac{1}{2}\,\partial_{dc}g_{ba} + \frac{1}{4}\,\partial_{d}g_{ed}g^{ef}\partial_{b}g_{fc} + \frac{1}{4}\,\partial_{c}g_{ed}g^{ef}\partial_{b}g_{bf} - \frac{1}{4}\,\partial_{d}g_{ed}g^{ef}\partial_{f}g_{bc} + \frac{1}{4}\,\partial_{d}g_{ae}g^{ef}\partial_{b}g_{fc} + \frac{1}{4}\,\partial_{c}g_{ae}g^{ef}\partial_{d}g_{bf} - \frac{1}{4}\,\partial_{d}g_{ae}g^{ef}\partial_{f}g_{bc} \\
& - \frac{1}{4}\,\partial_{c}g_{ad}g^{ef}\partial_{b}g_{fc} - \frac{1}{4}\,\partial_{c}g_{ad}g^{ef}\partial_{c}g_{bf} + \frac{1}{4}\,\partial_{c}g_{ad}g^{ef}\partial_{f}g_{bc} - \frac{1}{4}\,\partial_{d}g_{ec}g^{ef}\partial_{b}g_{fd} - \frac{1}{4}\,\partial_{d}g_{ec}g^{ef}\partial_{d}g_{bf} + \frac{1}{4}\,\partial_{d}g_{ec}g^{ef}\partial_{f}g_{bd} - \frac{1}{4}\,\partial_{c}g_{ae}g^{ef}\partial_{b}g_{fd} \\
& - \frac{1}{4}\,\partial_{c}g_{ae}g^{ef}\partial_{d}g_{bf} + \frac{1}{4}\,\partial_{c}g_{ae}g^{ef}\partial_{f}g_{bd} + \frac{1}{4}\,\partial_{c}g_{ac}g^{ef}\partial_{b}g_{fd} + \frac{1}{4}\,\partial_{c}g_{ac}g^{ef}\partial_{d}g_{bf} - \frac{1}{4}\,\partial_{c}g_{ac}g^{ef}\partial_{f}g_{bd} + \frac{1}{2}\,\partial_{cd}g_{ab} - \frac{1}{2}\,\partial_{dc}g_{ab} + \frac{1}{4}\,\partial_{b}g_{ed}g^{ef}\partial_{d}g_{fc} \\
& + \frac{1}{4}\,\partial_{b}g_{ed}g^{ef}\partial_{c}g_{af} - \frac{1}{4}\,\partial_{b}g_{ed}g^{ef}\partial_{f}g_{ac} + \frac{1}{4}\,\partial_{d}g_{be}g^{ef}\partial_{d}g_{fc} + \frac{1}{4}\,\partial_{d}g_{be}g^{ef}\partial_{c}g_{af} - \frac{1}{4}\,\partial_{d}g_{be}g^{ef}\partial_{f}g_{ac} - \frac{1}{4}\,\partial_{c}g_{bd}g^{ef}\partial_{d}g_{fc} - \frac{1}{4}\,\partial_{c}g_{bd}g^{ef}\partial_{d}g_{af} \\
& + \frac{1}{4}\,\partial_{c}g_{bd}g^{ef}\partial_{f}g_{ac} - \frac{1}{4}\,\partial_{b}g_{ec}g^{ef}\partial_{d}g_{fd} - \frac{1}{4}\,\partial_{b}g_{ec}g^{ef}\partial_{d}g_{af} + \frac{1}{4}\,\partial_{b}g_{ec}g^{ef}\partial_{f}g_{ad} - \frac{1}{4}\,\partial_{c}g_{be}g^{ef}\partial_{d}g_{fd} - \frac{1}{4}\,\partial_{c}g_{be}g^{ef}\partial_{d}g_{af} + \frac{1}{4}\,\partial_{c}g_{be}g^{ef}\partial_{f}g_{ad} \\
& + \frac{1}{4}\,\partial_{c}g_{bc}g^{ef}\partial_{d}g_{fd} + \frac{1}{4}\,\partial_{c}g_{bc}g^{ef}\partial_{d}g_{af} - \frac{1}{4}\,\partial_{c}g_{bc}g^{ef}\partial_{f}g_{ad} \qquad \texttt{(ex-0304.106)} \\
= {} & \frac{1}{2}\,\partial_{cd}g_{ba} - \frac{1}{2}\,\partial_{dc}g_{ba} + \frac{1}{4}\,\partial_{d}g_{ed}\partial_{b}g_{fc}g^{ef} + \frac{1}{4}\,\partial_{c}g_{ed}\partial_{c}g_{bf}g^{ef} - \frac{1}{4}\,\partial_{d}g_{ed}\partial_{f}g_{bc}g^{ef} + \frac{1}{4}\,\partial_{b}g_{fc}\partial_{d}g_{ae}g^{ef} + \frac{1}{4}\,\partial_{c}g_{bf}\partial_{d}g_{ae}g^{ef} - \frac{1}{4}\,\partial_{d}g_{ae}\partial_{f}g_{bc}g^{ef} \\
& - \frac{1}{4}\,\partial_{b}g_{fc}\partial_{c}g_{ad}g^{ef} - \frac{1}{4}\,\partial_{c}g_{bf}\partial_{c}g_{ad}g^{ef} + \frac{1}{4}\,\partial_{c}g_{ad}\partial_{f}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{d}g_{ec}\partial_{b}g_{fd}g^{ef} - \frac{1}{4}\,\partial_{d}g_{ec}\partial_{d}g_{bf}g^{ef} + \frac{1}{4}\,\partial_{d}g_{ec}\partial_{f}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{b}g_{fd}\partial_{c}g_{ae}g^{ef} \\
& - \frac{1}{4}\,\partial_{c}g_{ae}\partial_{d}g_{bf}g^{ef} + \frac{1}{4}\,\partial_{c}g_{ae}\partial_{f}g_{bd}g^{ef} + \frac{1}{4}\,\partial_{b}g_{fd}\partial_{c}g_{ac}g^{ef} + \frac{1}{4}\,\partial_{d}g_{bf}\partial_{c}g_{ac}g^{ef} - \frac{1}{4}\,\partial_{c}g_{ac}\partial_{f}g_{bd}g^{ef} + \frac{1}{2}\,\partial_{cd}g_{ab} - \frac{1}{2}\,\partial_{dc}g_{ab} + \frac{1}{4}\,\partial_{d}g_{fc}\partial_{b}g_{ed}g^{ef} \\
& + \frac{1}{4}\,\partial_{b}g_{ed}\partial_{c}g_{af}g^{ef} - \frac{1}{4}\,\partial_{b}g_{ed}\partial_{f}g_{ac}g^{ef} + \frac{1}{4}\,\partial_{d}g_{fc}\partial_{d}g_{be}g^{ef} + \frac{1}{4}\,\partial_{c}g_{af}\partial_{d}g_{be}g^{ef} - \frac{1}{4}\,\partial_{d}g_{be}g^{ef}\partial_{f}g_{ac} - \frac{1}{4}\,\partial_{c}g_{bd}\partial_{d}g_{fc}g^{ef} - \frac{1}{4}\,\partial_{c}g_{bd}\partial_{d}g_{af}g^{ef} \\
& + \frac{1}{4}\,\partial_{c}g_{bd}\partial_{f}g_{ac}g^{ef} - \frac{1}{4}\,\partial_{d}g_{fd}\partial_{b}g_{ec}g^{ef} - \frac{1}{4}\,\partial_{b}g_{ec}\partial_{d}g_{af}g^{ef} + \frac{1}{4}\,\partial_{b}g_{ec}\partial_{f}g_{ad}g^{ef} - \frac{1}{4}\,\partial_{d}g_{fd}\partial_{c}g_{be}g^{ef} - \frac{1}{4}\,\partial_{c}g_{be}\partial_{d}g_{af}g^{ef} + \frac{1}{4}\,\partial_{c}g_{be}\partial_{f}g_{ad}g^{ef} \\
& + \frac{1}{4}\,\partial_{d}g_{fd}\partial_{c}g_{bc}g^{ef} + \frac{1}{4}\,\partial_{d}g_{af}\partial_{c}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{c}g_{bc}\partial_{f}g_{ad}g^{ef} \qquad \texttt{(ex-0304.107)}
\end{aligned}
$$

$$
\begin{aligned}
R_{abcd} + R_{bacd} = {} & \frac{1}{2}\,\partial_{cd}g_{ba} - \frac{1}{2}\,\partial_{dc}g_{ba} + \frac{1}{4}\,\partial_{a}g_{ed}\partial_{b}g_{fc}g^{ef} + \frac{1}{4}\,\partial_{a}g_{ed}\partial_{g}g_{bf}g^{ef} - \frac{1}{4}\,\partial_{a}g_{fd}\partial_{e}g_{bc}g^{fe} + \frac{1}{4}\,\partial_{b}g_{ec}\partial_{a}g_{af}g^{fe} + \frac{1}{4}\,\partial_{g}g_{be}\partial_{a}g_{af}g^{fe} - \frac{1}{4}\,\partial_{a}g_{af}\partial_{e}g_{bc}g^{fe} \\
& - \frac{1}{4}\,\partial_{b}g_{fc}\partial_{e}g_{ad}g^{ef} - \frac{1}{4}\,\partial_{e}g_{bf}\partial_{e}g_{ad}g^{ef} + \frac{1}{4}\,\partial_{e}g_{ad}\partial_{f}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{a}g_{ec}\partial_{b}g_{fd}g^{ef} - \frac{1}{4}\,\partial_{a}g_{ec}\partial_{a}g_{bf}g^{ef} + \frac{1}{4}\,\partial_{a}g_{fc}\partial_{e}g_{bd}g^{fe} - \frac{1}{4}\,\partial_{b}g_{ed}\partial_{e}g_{af}g^{fe} \\
& - \frac{1}{4}\,\partial_{e}g_{ae}\partial_{a}g_{bf}g^{ef} + \frac{1}{4}\,\partial_{e}g_{af}\partial_{e}g_{bd}g^{fe} + \frac{1}{4}\,\partial_{b}g_{fd}\partial_{e}g_{ac}g^{ef} + \frac{1}{4}\,\partial_{a}g_{bf}\partial_{e}g_{ac}g^{ef} - \frac{1}{4}\,\partial_{e}g_{ac}\partial_{f}g_{bd}g^{ef} + \frac{1}{2}\,\partial_{cd}g_{ab} - \frac{1}{2}\,\partial_{dc}g_{ab} + \frac{1}{4}\,\partial_{a}g_{ec}\partial_{b}g_{fd}g^{fe} \\
& + \frac{1}{4}\,\partial_{b}g_{ed}\partial_{e}g_{af}g^{ef} - \frac{1}{4}\,\partial_{b}g_{fd}\partial_{e}g_{ac}g^{fe} + \frac{1}{4}\,\partial_{a}g_{ec}\partial_{a}g_{bf}g^{fe} + \frac{1}{4}\,\partial_{e}g_{ae}\partial_{a}g_{bf}g^{fe} - \frac{1}{4}\,\partial_{a}g_{bf}\partial_{e}g_{ac}g^{fe} - \frac{1}{4}\,\partial_{a}g_{fc}\partial_{e}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{e}g_{af}\partial_{e}g_{bd}g^{ef} \\
& + \frac{1}{4}\,\partial_{e}g_{bd}\partial_{f}g_{ac}g^{ef} - \frac{1}{4}\,\partial_{a}g_{ed}\partial_{b}g_{fc}g^{fe} - \frac{1}{4}\,\partial_{b}g_{ec}\partial_{a}g_{af}g^{ef} + \frac{1}{4}\,\partial_{b}g_{fc}\partial_{e}g_{ad}g^{fe} - \frac{1}{4}\,\partial_{a}g_{ed}\partial_{e}g_{bf}g^{fe} - \frac{1}{4}\,\partial_{e}g_{be}\partial_{a}g_{af}g^{ef} + \frac{1}{4}\,\partial_{e}g_{bf}\partial_{e}g_{ad}g^{fe} \\
& + \frac{1}{4}\,\partial_{a}g_{fd}\partial_{e}g_{bc}g^{ef} + \frac{1}{4}\,\partial_{a}g_{af}\partial_{e}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{e}g_{bc}\partial_{f}g_{ad}g^{ef} \qquad \texttt{(ex-0304.108)} \\
= {} & 0 \qquad \texttt{(ex-0304.109)}
\end{aligned}
$$

Exercise 3.4 Symmetric on swapping first and second pair of indices

```
expr := R_{a b c d} - R_{c d a b}.   # cdb(ex-0304.201,expr)

substitute     (expr, Rabcd)         # cdb(ex-0304.202,expr)
substitute     (expr, GammaU)        # cdb(ex-0304.203,expr)
substitute     (expr, GammaD)        # cdb(ex-0304.204,expr)
distribute     (expr)                # cdb(ex-0304.205,expr)
product_rule   (expr)                # cdb(ex-0304.206,expr)
sort_product   (expr)                # cdb(ex-0304.207,expr)
rename_dummies (expr)                # cdb(ex-0304.208,expr)
canonicalise   (expr)                # cdb(ex-0304.209,expr)
```

$$R_{abcd} - R_{cdab} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead}\Gamma^e{}_{bc} - \Gamma_{eac}\Gamma^e{}_{bd} - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} - \Gamma_{ecb}\Gamma^e{}_{da} + \Gamma_{eca}\Gamma^e{}_{db} \tag{ex-0304.202}$$

$$= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2}\Gamma_{ead} g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) - \frac{1}{2}\Gamma_{eac} g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda}$$
$$- \frac{1}{2}\Gamma_{ecb} g^{ef}\left(\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}\right) + \frac{1}{2}\Gamma_{eca} g^{ef}\left(\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}\right) \tag{ex-0304.203}$$

$$= \partial_c\left(\frac{1}{2}\partial_b g_{ad} + \frac{1}{2}\partial_d g_{ba} - \frac{1}{2}\partial_a g_{bd}\right) - \partial_d\left(\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc}\right) + \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ed} + \frac{1}{2}\partial_d g_{ae} - \frac{1}{2}\partial_e g_{ad}\right) g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right)$$
$$- \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ec} + \frac{1}{2}\partial_c g_{ae} - \frac{1}{2}\partial_e g_{ac}\right) g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) - \partial_a\left(\frac{1}{2}\partial_d g_{cb} + \frac{1}{2}\partial_b g_{dc} - \frac{1}{2}\partial_c g_{db}\right)$$
$$+ \partial_b\left(\frac{1}{2}\partial_d g_{ca} + \frac{1}{2}\partial_a g_{dc} - \frac{1}{2}\partial_c g_{da}\right) - \frac{1}{2}\left(\frac{1}{2}\partial_c g_{eb} + \frac{1}{2}\partial_b g_{ce} - \frac{1}{2}\partial_e g_{cb}\right) g^{ef}\left(\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2}\partial_c g_{ea} + \frac{1}{2}\partial_a g_{ce} - \frac{1}{2}\partial_e g_{ca}\right) g^{ef}\left(\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}\right) \tag{ex-0304.204}$$

$$= \frac{1}{2}\partial_{cb} g_{ad} + \frac{1}{2}\partial_{cd} g_{ba} - \frac{1}{2}\partial_{ca} g_{bd} - \frac{1}{2}\partial_{db} g_{ac} - \frac{1}{2}\partial_{dc} g_{ba} + \frac{1}{2}\partial_{da} g_{bc} + \frac{1}{4}\partial_a g_{ed} g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed} g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed} g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae} g^{ef}\partial_b g_{fc}$$
$$+ \frac{1}{4}\partial_d g_{ae} g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae} g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_e g_{ad} g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad} g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad} g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec} g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec} g^{ef}\partial_d g_{bf}$$
$$+ \frac{1}{4}\partial_d g_{ec} g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae} g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_c g_{ae} g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae} g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac} g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac} g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac} g^{ef}\partial_f g_{bd}$$
$$- \frac{1}{2}\partial_{ad} g_{cb} - \frac{1}{2}\partial_{ab} g_{dc} + \frac{1}{2}\partial_{ac} g_{db} + \frac{1}{2}\partial_{bd} g_{ca} + \frac{1}{2}\partial_{bc} g_{dc} - \frac{1}{2}\partial_{bc} g_{da} - \frac{1}{4}\partial_c g_{eb} g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_c g_{eb} g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_c g_{eb} g^{ef}\partial_f g_{da}$$
$$- \frac{1}{4}\partial_b g_{ce} g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_b g_{ce} g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_b g_{ce} g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_c g_{cb} g^{ef}\partial_d g_{fa} + \frac{1}{4}\partial_c g_{cb} g^{ef}\partial_a g_{df} - \frac{1}{4}\partial_c g_{cb} g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_c g_{ea} g^{ef}\partial_d g_{fb}$$
$$+ \frac{1}{4}\partial_c g_{ea} g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_c g_{ea} g^{ef}\partial_f g_{db} + \frac{1}{4}\partial_a g_{ce} g^{ef}\partial_d g_{fb} + \frac{1}{4}\partial_a g_{ce} g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_a g_{ce} g^{ef}\partial_f g_{db} - \frac{1}{4}\partial_c g_{ca} g^{ef}\partial_d g_{fb} - \frac{1}{4}\partial_c g_{ca} g^{ef}\partial_b g_{df}$$
$$+ \frac{1}{4}\partial_c g_{ca} g^{ef}\partial_f g_{db} \tag{ex-0304.205}$$

$$
\begin{aligned}
R_{abcd} - R_{cdab} &= \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{ca}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_{a}g_{ed}g^{ef}\partial_{b}g_{fc} + \frac{1}{4}\partial_{a}g_{ed}g^{ef}\partial_{c}g_{bf} - \frac{1}{4}\partial_{a}g_{ed}g^{ef}\partial_{f}g_{bc} + \frac{1}{4}\partial_{a}g_{ae}g^{ef}\partial_{b}g_{fc} \\
&+ \frac{1}{4}\partial_{a}g_{ae}g^{ef}\partial_{c}g_{bf} - \frac{1}{4}\partial_{a}g_{ae}g^{ef}\partial_{f}g_{bc} - \frac{1}{4}\partial_{e}g_{ad}g^{ef}\partial_{b}g_{fc} - \frac{1}{4}\partial_{e}g_{ad}g^{ef}\partial_{c}g_{bf} + \frac{1}{4}\partial_{e}g_{ad}g^{ef}\partial_{f}g_{bc} - \frac{1}{4}\partial_{a}g_{ec}g^{ef}\partial_{b}g_{fd} - \frac{1}{4}\partial_{a}g_{ec}g^{ef}\partial_{d}g_{bf} \\
&+ \frac{1}{4}\partial_{a}g_{ec}g^{ef}\partial_{f}g_{bd} - \frac{1}{4}\partial_{e}g_{ae}g^{ef}\partial_{b}g_{fd} - \frac{1}{4}\partial_{e}g_{ae}g^{ef}\partial_{d}g_{bf} + \frac{1}{4}\partial_{e}g_{ae}g^{ef}\partial_{f}g_{bd} + \frac{1}{4}\partial_{e}g_{ac}g^{ef}\partial_{b}g_{fd} + \frac{1}{4}\partial_{e}g_{ac}g^{ef}\partial_{d}g_{bf} - \frac{1}{4}\partial_{e}g_{ac}g^{ef}\partial_{f}g_{bd} \\
&- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{bc}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_{e}g_{eb}g^{ef}\partial_{d}g_{fa} - \frac{1}{4}\partial_{e}g_{eb}g^{ef}\partial_{a}g_{df} + \frac{1}{4}\partial_{e}g_{eb}g^{ef}\partial_{f}g_{da} \\
&- \frac{1}{4}\partial_{b}g_{ce}g^{ef}\partial_{a}g_{fa} - \frac{1}{4}\partial_{b}g_{ce}g^{ef}\partial_{a}g_{df} + \frac{1}{4}\partial_{b}g_{ce}g^{ef}\partial_{f}g_{da} + \frac{1}{4}\partial_{e}g_{cb}g^{ef}\partial_{a}g_{fa} + \frac{1}{4}\partial_{e}g_{cb}g^{ef}\partial_{a}g_{df} - \frac{1}{4}\partial_{e}g_{cb}g^{ef}\partial_{f}g_{da} + \frac{1}{4}\partial_{e}g_{ea}g^{ef}\partial_{a}g_{fb} \\
&+ \frac{1}{4}\partial_{e}g_{ea}g^{ef}\partial_{b}g_{df} - \frac{1}{4}\partial_{e}g_{ea}g^{ef}\partial_{f}g_{db} + \frac{1}{4}\partial_{a}g_{ce}g^{ef}\partial_{a}g_{fb} + \frac{1}{4}\partial_{a}g_{ce}g^{ef}\partial_{b}g_{df} - \frac{1}{4}\partial_{a}g_{ce}g^{ef}\partial_{f}g_{db} - \frac{1}{4}\partial_{e}g_{ca}g^{ef}\partial_{a}g_{fb} - \frac{1}{4}\partial_{e}g_{ca}g^{ef}\partial_{b}g_{df} \\
&+ \frac{1}{4}\partial_{e}g_{ca}g^{ef}\partial_{f}g_{db} \qquad \texttt{(ex-0304.206)} \\
&= \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{ca}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_{a}g_{ed}\partial_{b}g_{fc}g^{ef} + \frac{1}{4}\partial_{a}g_{ed}\partial_{c}g_{bf}g^{ef} - \frac{1}{4}\partial_{a}g_{ed}\partial_{f}g_{bc}g^{ef} + \frac{1}{4}\partial_{b}g_{fc}\partial_{a}g_{ae}g^{ef} \\
&+ \frac{1}{4}\partial_{c}g_{bf}\partial_{a}g_{ae}g^{ef} - \frac{1}{4}\partial_{a}g_{ae}\partial_{f}g_{bc}g^{ef} - \frac{1}{4}\partial_{b}g_{fc}\partial_{e}g_{ad}g^{ef} - \frac{1}{4}\partial_{c}g_{bf}\partial_{e}g_{ad}g^{ef} + \frac{1}{4}\partial_{e}g_{ad}\partial_{f}g_{bc}g^{ef} - \frac{1}{4}\partial_{a}g_{ec}\partial_{b}g_{fd}g^{ef} - \frac{1}{4}\partial_{a}g_{ec}\partial_{d}g_{bf}g^{ef} \\
&+ \frac{1}{4}\partial_{a}g_{ec}\partial_{f}g_{bd}g^{ef} - \frac{1}{4}\partial_{b}g_{fd}\partial_{e}g_{ae}g^{ef} - \frac{1}{4}\partial_{e}g_{ae}\partial_{d}g_{bf}g^{ef} + \frac{1}{4}\partial_{e}g_{ae}\partial_{f}g_{bd}g^{ef} + \frac{1}{4}\partial_{b}g_{fd}\partial_{e}g_{ac}g^{ef} + \frac{1}{4}\partial_{d}g_{bf}\partial_{e}g_{ac}g^{ef} - \frac{1}{4}\partial_{e}g_{ac}\partial_{f}g_{bd}g^{ef} \\
&- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{bc}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_{e}g_{eb}\partial_{d}g_{fa}g^{ef} - \frac{1}{4}\partial_{d}g_{df}\partial_{e}g_{eb}g^{ef} + \frac{1}{4}\partial_{e}g_{eb}\partial_{f}g_{da}g^{ef} \\
&- \frac{1}{4}\partial_{b}g_{ce}\partial_{d}g_{fa}g^{ef} - \frac{1}{4}\partial_{d}g_{df}\partial_{b}g_{ce}g^{ef} + \frac{1}{4}\partial_{b}g_{ce}\partial_{f}g_{da}g^{ef} + \frac{1}{4}\partial_{d}g_{fa}\partial_{e}g_{cb}g^{ef} + \frac{1}{4}\partial_{d}g_{df}\partial_{e}g_{cb}g^{ef} - \frac{1}{4}\partial_{e}g_{cb}\partial_{f}g_{da}g^{ef} + \frac{1}{4}\partial_{e}g_{ea}\partial_{d}g_{fb}g^{ef} \\
&+ \frac{1}{4}\partial_{b}g_{df}\partial_{e}g_{ea}g^{ef} - \frac{1}{4}\partial_{e}g_{ea}\partial_{f}g_{db}g^{ef} + \frac{1}{4}\partial_{d}g_{ce}\partial_{d}g_{fb}g^{ef} + \frac{1}{4}\partial_{d}g_{ce}\partial_{b}g_{df}g^{ef} - \frac{1}{4}\partial_{d}g_{ce}\partial_{f}g_{db}g^{ef} - \frac{1}{4}\partial_{e}g_{ca}\partial_{d}g_{fb}g^{ef} - \frac{1}{4}\partial_{e}g_{ca}\partial_{b}g_{df}g^{ef} \\
&+ \frac{1}{4}\partial_{e}g_{ca}\partial_{f}g_{db}g^{ef} \qquad \texttt{(ex-0304.207)}
\end{aligned}
$$

$$
\begin{aligned}
R_{abcd} - R_{cdab} &= \frac{1}{2}\,\partial_{cb}g_{ad} + \frac{1}{2}\,\partial_{ca}g_{bd} - \frac{1}{2}\,\partial_{cd}g_{ba} - \frac{1}{2}\,\partial_{db}g_{ac} - \frac{1}{2}\,\partial_{da}g_{bc} + \frac{1}{2}\,\partial_{ba}g_{dc} + \frac{1}{4}\,\partial_{a}g_{ed}\partial_{b}g_{fc}g^{ef} + \frac{1}{4}\,\partial_{a}g_{ed}\partial_{c}g_{bf}g^{ef} - \frac{1}{4}\,\partial_{a}g_{fd}\partial_{c}g_{bc}g^{fe} + \frac{1}{4}\,\partial_{b}g_{ec}\partial_{a}g_{af}g^{fe} \\
&\quad + \frac{1}{4}\,\partial_{c}g_{be}\partial_{a}g_{af}g^{fe} - \frac{1}{4}\,\partial_{a}g_{af}\partial_{c}g_{bc}g^{fe} - \frac{1}{4}\,\partial_{b}g_{fc}\partial_{c}g_{ad}g^{ef} - \frac{1}{4}\,\partial_{c}g_{bf}\partial_{c}g_{ad}g^{ef} + \frac{1}{4}\,\partial_{c}g_{ad}\partial_{f}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{a}g_{ec}\partial_{b}g_{fd}g^{ef} - \frac{1}{4}\,\partial_{a}g_{ec}\partial_{a}g_{bf}g^{ef} \\
&\quad + \frac{1}{4}\,\partial_{a}g_{fc}\partial_{c}g_{bd}g^{fe} - \frac{1}{4}\,\partial_{b}g_{ed}\partial_{c}g_{af}g^{fe} - \frac{1}{4}\,\partial_{c}g_{ae}\partial_{a}g_{bf}g^{ef} + \frac{1}{4}\,\partial_{c}g_{af}\partial_{c}g_{bd}g^{fe} + \frac{1}{4}\,\partial_{b}g_{fd}\partial_{c}g_{ac}g^{ef} + \frac{1}{4}\,\partial_{a}g_{bf}\partial_{c}g_{ac}g^{ef} - \frac{1}{4}\,\partial_{c}g_{ac}\partial_{f}g_{bd}g^{ef} \\
&\quad - \frac{1}{2}\,\partial_{ad}g_{cb} - \frac{1}{2}\,\partial_{ab}g_{dc} + \frac{1}{2}\,\partial_{ac}g_{db} + \frac{1}{2}\,\partial_{bd}g_{ca} + \frac{1}{2}\,\partial_{bc}g_{dc} - \frac{1}{2}\,\partial_{bc}g_{da} - \frac{1}{4}\,\partial_{c}g_{eb}\partial_{a}g_{fa}g^{ef} - \frac{1}{4}\,\partial_{a}g_{de}\partial_{c}g_{fb}g^{fe} + \frac{1}{4}\,\partial_{c}g_{fb}\partial_{c}g_{da}g^{fe} \\
&\quad - \frac{1}{4}\,\partial_{b}g_{ce}\partial_{a}g_{fa}g^{ef} - \frac{1}{4}\,\partial_{a}g_{de}\partial_{b}g_{cf}g^{fe} + \frac{1}{4}\,\partial_{b}g_{cf}\partial_{c}g_{da}g^{fe} + \frac{1}{4}\,\partial_{a}g_{fa}\partial_{c}g_{cb}g^{ef} + \frac{1}{4}\,\partial_{a}g_{df}\partial_{c}g_{cb}g^{ef} - \frac{1}{4}\,\partial_{c}g_{cb}\partial_{f}g_{da}g^{ef} + \frac{1}{4}\,\partial_{c}g_{ea}\partial_{a}g_{fb}g^{ef} \\
&\quad + \frac{1}{4}\,\partial_{b}g_{de}\partial_{c}g_{fa}g^{fe} - \frac{1}{4}\,\partial_{c}g_{fa}\partial_{c}g_{db}g^{fe} + \frac{1}{4}\,\partial_{a}g_{ce}\partial_{a}g_{fb}g^{ef} + \frac{1}{4}\,\partial_{a}g_{ce}\partial_{b}g_{df}g^{ef} - \frac{1}{4}\,\partial_{a}g_{cf}\partial_{c}g_{db}g^{fe} - \frac{1}{4}\,\partial_{a}g_{fb}\partial_{c}g_{ca}g^{ef} - \frac{1}{4}\,\partial_{b}g_{df}\partial_{c}g_{ca}g^{ef} \\
&\quad + \frac{1}{4}\,\partial_{c}g_{ca}\partial_{f}g_{db}g^{ef} && \texttt{(ex-0304.208)} \\
&= 0 && \texttt{(ex-0304.209)}
\end{aligned}
$$