Exercise 3.3 Computing R_{abcd}

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).

dgab := \partial_{c}{g_{a b}} ->   \Gamma^{d}_{a c} g_{d b}
                                 + \Gamma^{d}_{b c} g_{a d}.      # cdb(dgab.000,dgab)

RabcdU := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                           - \partial_{d}{\Gamma^{a}_{b c}}
                           + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                           - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.   # cdb(Rabcd.000,RabcdU)

GammaD := {g_{a e} \Gamma^{e}_{b c} -> \Gamma_{a b c},
           g_{e a} \Gamma^{e}_{b c} -> \Gamma_{a b c}}.           # cdb(Gamma.010,GammaD)

RabcdD := R_{a b c d} -> g_{a e} R^{e}_{b c d}.                  # cdb(Rabcd.010,RabcdD)

gabDGamma := g_{a e} \partial_{c}{\Gamma^{e}_{b d}} ->
                 \partial_{c}{g_{a e} \Gamma^{e}_{b d}}
               - \Gamma^{e}_{b d} \partial_{c}{g_{a e}}.         # cdb(gabDGamma.000,gabDGamma)

# this pair of rules needed to sort \Gamma_{a b c} to the very left
# this helps canonicalise spot the terms that cancel
bah := \Gamma_{a b c} -> A_{a b c}.
foo := A_{a b c} -> \Gamma_{a b c}.

expr := R_{a b c d}.                                             # cdb(ex-0303.101,expr)

substitute     (expr, RabcdD)                                    # cdb(ex-0303.102,expr)
substitute     (expr, RabcdU)                                    # cdb(ex-0303.103,expr)
distribute     (expr)                                            # cdb(ex-0303.104,expr)
substitute     (expr, gabDGamma)                                 # cdb(ex-0303.105,expr)
substitute     (expr, dgab)                                      # cdb(ex-0303.106,expr)
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substitute     (expr, GammaD)                    # cdb(ex-0303.107,expr)
distribute     (expr)                            # cdb(ex-0303.109,expr)
substitute     (expr, bah)                       # cdb(ex-0303.110,expr)
sort_product   (expr)                            # cdb(ex-0303.111,expr)
rename_dummies (expr)                            # cdb(ex-0303.112,expr)
substitute     (expr, foo)                       # cdb(ex-0303.113,expr)
canonicalise   (expr)                            # cdb(ex-0303.114,expr)
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$$R_{abcd} = g_{ae} R^{e}{}_{bcd} \tag{ex-0303.102}$$

$$= g_{ae} \left(\partial_{c} \Gamma^{e}{}_{bd} - \partial_{d} \Gamma^{e}{}_{bc} + \Gamma^{f}{}_{bd} \Gamma^{e}{}_{cf} - \Gamma^{f}{}_{bc} \Gamma^{e}{}_{df} \right) \tag{ex-0303.103}$$

$$= g_{ae} \partial_{c} \Gamma^{e}{}_{bd} - g_{ae} \partial_{d} \Gamma^{e}{}_{bc} + g_{ae} \Gamma^{f}{}_{bd} \Gamma^{e}{}_{cf} - g_{ae} \Gamma^{f}{}_{bc} \Gamma^{e}{}_{df} \tag{ex-0303.104}$$

$$= \partial_{c}(g_{ae} \Gamma^{e}{}_{bd}) - \Gamma^{e}{}_{bd} \partial_{c} g_{ae} - \partial_{d}(g_{ae} \Gamma^{e}{}_{bc}) + \Gamma^{e}{}_{bc} \partial_{d} g_{ae} + g_{ae} \Gamma^{f}{}_{bd} \Gamma^{e}{}_{cf} - g_{ae} \Gamma^{f}{}_{bc} \Gamma^{e}{}_{df} \tag{ex-0303.105}$$

$$= \partial_{c}(g_{ae} \Gamma^{e}{}_{bd}) - \Gamma^{e}{}_{bd} \left(\Gamma^{f}{}_{ac} g_{fe} + \Gamma^{f}{}_{ec} g_{af} \right) - \partial_{d}(g_{ae} \Gamma^{e}{}_{bc}) + \Gamma^{e}{}_{bc} \left(\Gamma^{f}{}_{ad} g_{fe} + \Gamma^{f}{}_{ed} g_{af} \right) + g_{ae} \Gamma^{f}{}_{bd} \Gamma^{e}{}_{cf} - g_{ae} \Gamma^{f}{}_{bc} \Gamma^{e}{}_{df} \tag{ex-0303.106}$$

$$= \partial_{c} \Gamma_{abd} - \Gamma^{e}{}_{bd} \left(\Gamma_{eac} + \Gamma_{aec} \right) - \partial_{d} \Gamma_{abc} + \Gamma^{e}{}_{bc} \left(\Gamma_{ead} + \Gamma_{aed} \right) + \Gamma_{acf} \Gamma^{f}{}_{bd} - \Gamma_{adf} \Gamma^{f}{}_{bc} \tag{ex-0303.107}$$

$$= \partial_{c} \Gamma_{abd} - \Gamma^{e}{}_{bd} \Gamma_{eac} - \Gamma^{e}{}_{bd} \Gamma_{aec} - \partial_{d} \Gamma_{abc} + \Gamma^{e}{}_{bc} \Gamma_{ead} + \Gamma^{e}{}_{bc} \Gamma_{aed} + \Gamma_{acf} \Gamma^{f}{}_{bd} - \Gamma_{adf} \Gamma^{f}{}_{bc} \tag{ex-0303.109}$$

$$= \partial_{c} A_{abd} - \Gamma^{e}{}_{bd} A_{eac} - \Gamma^{e}{}_{bd} A_{aec} - \partial_{d} A_{abc} + \Gamma^{e}{}_{bc} A_{ead} + \Gamma^{e}{}_{bc} A_{aed} + A_{acf} \Gamma^{f}{}_{bd} - A_{adf} \Gamma^{f}{}_{bc} \tag{ex-0303.110}$$

$$= \partial_{c} A_{abd} - A_{eac} \Gamma^{e}{}_{bd} - A_{aec} \Gamma^{e}{}_{bd} - \partial_{d} A_{abc} + A_{ead} \Gamma^{e}{}_{bc} + A_{aed} \Gamma^{e}{}_{bc} + A_{acf} \Gamma^{f}{}_{bd} - A_{adf} \Gamma^{f}{}_{bc} \tag{ex-0303.111}$$

$$= \partial_{c} A_{abd} - A_{eac} \Gamma^{e}{}_{bd} - A_{aec} \Gamma^{e}{}_{bd} - \partial_{d} A_{abc} + A_{ead} \Gamma^{e}{}_{bc} + A_{aed} \Gamma^{e}{}_{bc} + A_{ace} \Gamma^{e}{}_{bd} - A_{ade} \Gamma^{e}{}_{bc} \tag{ex-0303.112}$$

$$= \partial_{c} \Gamma_{abd} - \Gamma_{eac} \Gamma^{e}{}_{bd} - \Gamma_{aec} \Gamma^{e}{}_{bd} - \partial_{d} \Gamma_{abc} + \Gamma_{ead} \Gamma^{e}{}_{bc} + \Gamma_{aed} \Gamma^{e}{}_{bc} + \Gamma_{ace} \Gamma^{e}{}_{bd} - \Gamma_{ade} \Gamma^{e}{}_{bc} \tag{ex-0303.113}$$

$$= \partial_{c} \Gamma_{abd} - \Gamma_{eac} \Gamma^{e}{}_{bd} - \partial_{d} \Gamma_{abc} + \Gamma_{ead} \Gamma^{e}{}_{bc} \tag{ex-0303.114}$$