Example 1 The metric connection

```
1    # Define some properties
2
3    {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
4
5    g_{a b}::Metric.
6    g_{a}^{b}::KroneckerDelta.
7
8    \nabla{#}::Derivative.
9    \partial{#}::PartialDerivative.
10
11   # Define rules for covariant derivative and the Christoffel symbol
12
13   nabla := \nabla_{c}{g_{a b}} -> \partial_{c}{g_{a b}} - g_{a d}\Gamma^{d}_{b c}
14                                            - g_{d b}\Gamma^{d}_{a c}.    # cdb (nabla.100,nabla)
15
16   Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
17                                              + \partial_{c}{g_{b d}}
18                                              - \partial_{d}{g_{b c}} ).   # cdb (Gamma.100,Gamma)
19
20   # Start with a simple expression
21
22   cderiv := \nabla_{c}{g_{a b}}.                                    # cdb (ex-01.100,cderiv)
23
24   # Do the computations
25
26   substitute         (cderiv, nabla)                               # cdb (ex-01.101,cderiv)
27   substitute         (cderiv, Gamma)                               # cdb (ex-01.102,cderiv)
28   distribute         (cderiv)                                      # cdb (ex-01.103,cderiv)
29   eliminate_metric   (cderiv)                                      # cdb (ex-01.104,cderiv)
30   eliminate_kronecker (cderiv)                                     # cdb (ex-01.105,cderiv)
31   canonicalise       (cderiv)                                      # cdb (ex-01.106,cderiv)
32
33   checkpoint.append (cderiv)
```

$$\nabla_c g_{ab} \rightarrow \partial_c g_{ab} - g_{ad}\Gamma^d{}_{bc} - g_{db}\Gamma^d{}_{ac} \tag{nabla.100}$$

$$\Gamma^a{}_{bc} \rightarrow \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) \tag{Gamma.100}$$

$$\nabla_c g_{ab} = \partial_c g_{ab} - g_{ad}\Gamma^d{}_{bc} - g_{db}\Gamma^d{}_{ac} \tag{ex-01.101}$$

$$= \partial_c g_{ab} - \frac{1}{2}g_{ad}g^{de}\left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right) - \frac{1}{2}g_{db}g^{de}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) \tag{ex-01.102}$$

$$= \partial_c g_{ab} - \frac{1}{2}g_{ad}g^{de}\partial_b g_{ec} - \frac{1}{2}g_{ad}g^{de}\partial_c g_{be} + \frac{1}{2}g_{ad}g^{de}\partial_e g_{bc} - \frac{1}{2}g_{db}g^{de}\partial_a g_{ec} - \frac{1}{2}g_{db}g^{de}\partial_c g_{ae} + \frac{1}{2}g_{db}g^{de}\partial_e g_{ac} \tag{ex-01.103}$$

$$= \partial_c g_{ab} - \frac{1}{2}g_a{}^e\partial_b g_{ec} - \frac{1}{2}g_a{}^e\partial_c g_{be} + \frac{1}{2}g_a{}^e\partial_e g_{bc} - \frac{1}{2}g_b{}^e\partial_a g_{ec} - \frac{1}{2}g_b{}^e\partial_c g_{ae} + \frac{1}{2}g_b{}^e\partial_e g_{ac} \tag{ex-01.104}$$

$$= \frac{1}{2}\partial_c g_{ab} - \frac{1}{2}\partial_c g_{ba} \tag{ex-01.105}$$

$$= 0 \tag{ex-01.106}$$

Example 2 Covariant derivatives

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3   \nabla{#}::Derivative.
4   \partial{#}::PartialDerivative.
5
6   # rule for covariant derivative of v^{a}
7
8   deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.
9
10  # create an expression
11
12  foo := \nabla_{a}{v^{b}}.                    # cdb (ex-02.101,foo)
13
14  # apply the rule, then simplify
15
16  substitute   (foo,deriv)                     # cdb (ex-02.102,foo)
17  canonicalise (foo)                           # cdb (ex-02.103,foo)
18
19  checkpoint.append (foo)
```

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ca} v^c \qquad\qquad \text{(ex-02.102)}$$
$$= \partial_a v^b + \Gamma^{bc}{}_a v_c \qquad\qquad \text{(ex-02.103)}$$

Example 2 Covariant derivatives using "position=independent"

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4   \partial{#}::PartialDerivative.
5
6   # rule for covariant derivative of v^{a}
7
8   deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.
9
10  # create an expression
11
12  foo := \nabla_{a}{v^{b}}.                    # cdb (ex-02.201,foo)
13
14  # apply the rule, then simplify
15
16  substitute   (foo,deriv)                     # cdb (ex-02.202,foo)
17  canonicalise (foo)                           # cdb (ex-02.203,foo)
18
19  checkpoint.append (foo)
```

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ca} v^c \qquad \text{(ex-02.202)}$$
$$= \partial_a v^b + \Gamma^b{}_{ca} v^c \qquad \text{(ex-02.203)}$$

Example 2 Covariant derivatives using generic rule for deriv

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4   \partial{#}::PartialDerivative.
5
6   # template for covariant derivative of a vector
7
8   deriv := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}} + \Gamma^{b}_{c a} A?^{c}.
9
10  # create an expression
11
12  foo := \nabla_{a}{u^{b}} + \nabla_{a}{v^{b}}. # cdb (ex-02.301,foo)
13
14  # apply the rule, then simplify
15
16  substitute   (foo,deriv)                    # cdb (ex-02.302,foo)
17  canonicalise (foo)                          # cdb (ex-02.303,foo)
18
19  checkpoint.append (foo)
```

$$\nabla_a u^b + \nabla_a v^b = \partial_a u^b + \Gamma^b{}_{ca} u^c + \partial_a v^b + \Gamma^b{}_{ca} v^c \qquad \text{(ex-02.302)}$$

$$= \partial_a u^b + \Gamma^b{}_{ca} u^c + \partial_a v^b + \Gamma^b{}_{ca} v^c \qquad \text{(ex-02.303)}$$

Example 3a The Riemann curvature tensor

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \partial{#}::PartialDerivative.
4
5   \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});
6
7   ;::Symbol;   # Suggsted by Kasper as a way to make use of ; legal
8                # see https://cadabra.science/qa/473/is-this-legal-syntax
9                # this code works with and without this trick
10
11  # rules for the first two covariant derivs of V^a
12
13  deriv1 := V^{a}_{; b}        -> \partial_{b}{V^{a}}
14                                  + \Gamma^{a}_{c b} V^{c}.        # cdb (ex-03.101,deriv1)
15
16  deriv2 := V^{a}_{; b ; c}  -> \partial_{c}{V^{a}_{; b}}
17                                  + \Gamma^{a}_{d c} V^{d}_{; b}
18                                  - \Gamma^{d}_{b c} V^{a}_{; d}.  # cdb (ex-03.102,deriv2)
19
20  substitute (deriv2,deriv1)                    # cdb (ex-03.103, deriv2)
21
22  Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}.    # cdb (ex-03.104, Vabc)
23
24  substitute (Vabc,deriv2)                      # cdb (ex-03.105, Vabc)
25
26  distribute      (Vabc)                        # cdb (ex-03.106, Vabc)
27  product_rule    (Vabc)                        # cdb (ex-03.107, Vabc)
28
29  sort_product    (Vabc)                        # cdb (ex-03.108, Vabc)
30  rename_dummies  (Vabc)                        # cdb (ex-03.109, Vabc)
31  canonicalise    (Vabc)                        # cdb (ex-03.110, Vabc)
32
33  sort_sum        (Vabc)                        # cdb (ex-03.111, Vabc)
34  factor_out      (Vabc,$V^{a?}$)               # cdb (ex-03.112, Vabc)
35
36  checkpoint.append (Vabc)
```

```
37
38    # create rule for Riemann, export later (for use by lib/dgeom)
39
40    substitute (Vabc,$V^{a} -> -1$)              # cdb (ex-03.113, Vabc)
41                                                 # note use of -1 to get correct
42                                                 # signs when coupled with the rule
43                                                 # for Rabcd (next statement)
44
45    Rabcd := R^{a}_{d b c} -> @(Vabc).           # cdb (ex-03.114, Rabcd) #
46
47    foo    := R^{a}_{b c d}.                     # cdb (ex-03.115, foo)
48    substitute (foo, Rabcd)                      # cdb (ex-03.116, foo)
49
50    # update rule to use nice indices
51
52    Rabcd := R^{a}_{b c d} -> @(foo).
53
54    checkpoint.append (Rabcd)
```

$$V^a{}_{;b} \to \partial_b V^a + \Gamma^a{}_{cb} V^c \tag{ex-03.101}$$

$$V^a{}_{;b;c} \to \partial_c V^a{}_{;b} + \Gamma^a{}_{dc} V^d{}_{;b} - \Gamma^d{}_{bc} V^a{}_{;d} \tag{ex-03.102}$$

$$V^a{}_{;b;c} \to \partial_c \left(\partial_b V^a + \Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \left(\partial_b V^d + \Gamma^d{}_{eb} V^e \right) - \Gamma^d{}_{bc} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) \tag{ex-03.103}$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c \left(\partial_b V^a + \Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \left(\partial_b V^d + \Gamma^d{}_{eb} V^e \right) - \Gamma^d{}_{bc} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) - \partial_b \left(\partial_c V^a + \Gamma^a{}_{dc} V^d \right) - \Gamma^a{}_{db} \left(\partial_c V^d + \Gamma^d{}_{ec} V^e \right)$$
$$+ \Gamma^d{}_{cb} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) \tag{ex-03.105}$$

$$= \partial_{cb} V^a + \partial_c \left(\Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \left(\Gamma^a{}_{dc} V^d \right) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e$$
$$+ \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \tag{ex-03.106}$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \tag{ex-03.107}$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^e \Gamma^a{}_{dc} \Gamma^d{}_{eb} - \Gamma^d{}_{bc} \partial_d V^a - V^e \Gamma^a{}_{ed} \Gamma^d{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^e \Gamma^a{}_{db} \Gamma^d{}_{ec} + \Gamma^d{}_{cb} \partial_d V^a + V^e \Gamma^a{}_{ed} \Gamma^d{}_{cb} \tag{ex-03.108}$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^d \Gamma^a{}_{ec} \Gamma^e{}_{db} - \Gamma^d{}_{bc} \partial_d V^a - V^d \Gamma^a{}_{de} \Gamma^e{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^d \Gamma^a{}_{eb} \Gamma^e{}_{dc} + \Gamma^d{}_{cb} \partial_d V^a + V^d \Gamma^a{}_{de} \Gamma^e{}_{cb} \tag{ex-03.109}$$

$$= V^d \partial_c \Gamma^a{}_{bd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} \tag{ex-03.110}$$

$$= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} \tag{ex-03.111}$$

$$= V^d \left(\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} - \Gamma^a{}_{be} \Gamma^e{}_{cd} + \Gamma^a{}_{ce} \Gamma^e{}_{bd} \right) \tag{ex-03.112}$$

$$= -R^a{}_{dbc} V^d \quad \text{(MTW)}$$

$$R^a{}_{bcd} = -\partial_d \Gamma^a{}_{cb} + \partial_c \Gamma^a{}_{db} + \Gamma^a{}_{ce} \Gamma^e{}_{db} - \Gamma^a{}_{de} \Gamma^e{}_{cb}$$

Example 3b The Riemann curvature tensor

This differs from the above by not using the `::TableauSymmetry` property. It gives the same results as above but it does require a little bit more housekeeping.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative.

;::Symbol;   # Suggsted by Kasper as a way to make use of ; legal
             # see https://cadabra.science/qa/473/is-this-legal-syntax
             # this code works with and without this trick

# rules for the first two covariant derivs of V^a

deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
                          + \Gamma^{a}_{c b} V^{c}.         # cdb (ex-03.301,deriv1)

deriv2 := V^{a}_{; b ; c}  -> \partial_{c}{V^{a}_{; b}}
                          + \Gamma^{a}_{d c} V^{d}_{; b}
                          - \Gamma^{d}_{b c} V^{a}_{; d}.   # cdb (ex-03.302,deriv2)

substitute (deriv2,deriv1)                          # cdb (ex-03.303, deriv2)

Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}.          # cdb (ex-03.304, Vabc)

substitute (Vabc,deriv2)                            # cdb (ex-03.305, Vabc)

distribute    (Vabc)                                # cdb (ex-03.306, Vabc)
product_rule  (Vabc)                                # cdb (ex-03.307, Vabc)

# ----------------------------------------------------
# trick to obtain a symmetric connection

G_{a b}::Symmetric.

substitute    (Vabc,$\Gamma^{a}_{b c} ->  G^{a} G_{b c}$)
sort_product  (Vabc)                                # cdb (ex-03.308, Vabc)
```

```
34    rename_dummies (Vabc)                        # cdb (ex-03.309, Vabc)
35    canonicalise   (Vabc)                        # cdb (ex-03.310, Vabc)
36    substitute     (Vabc,$G^{a} G_{b c} -> \Gamma^{a}_{b c}$,repeat=True)
37    # ---------------------------------------------------
38
39    sort_product   (Vabc)
40    rename_dummies (Vabc)
41    canonicalise   (Vabc)
42
43    sort_sum       (Vabc)                         # cdb (ex-03.311, Vabc)
44    factor_out     (Vabc,$V^{a?}$)                # cdb (ex-03.312, Vabc)
45
46    checkpoint.append (Vabc)
```

$$V^a{}_{;b} \to \partial_b V^a + \Gamma^a{}_{cb} V^c \tag{ex-03.301}$$

$$V^a{}_{;b;c} \to \partial_c V^a{}_{;b} + \Gamma^a{}_{dc} V^d{}_{;b} - \Gamma^d{}_{bc} V^a{}_{;d} \tag{ex-03.302}$$

$$V^a{}_{;b;c} \to \partial_c \left(\partial_b V^a + \Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \left(\partial_b V^d + \Gamma^d{}_{eb} V^e \right) - \Gamma^d{}_{bc} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) \tag{ex-03.303}$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c \left(\partial_b V^a + \Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \left(\partial_b V^d + \Gamma^d{}_{eb} V^e \right) - \Gamma^d{}_{bc} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) - \partial_b \left(\partial_c V^a + \Gamma^a{}_{dc} V^d \right) - \Gamma^a{}_{db} \left(\partial_c V^d + \Gamma^d{}_{ec} V^e \right)$$

$$+ \Gamma^d{}_{cb} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) \tag{ex-03.305}$$

$$= \partial_{cb} V^a + \partial_c \left(\Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \left(\Gamma^a{}_{dc} V^d \right) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e$$

$$+ \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \tag{ex-03.306}$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \tag{ex-03.307}$$

$$= \partial_{cb} V^a + V^d \partial_c \left(G^a G_{db} \right) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b \left(G^a G_{dc} \right) - G^a G^d G_{db} G_{ec} V^e$$

$$+ G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \tag{ex-03.308}$$

$$= \partial_{cb} V^a + V^d \partial_c \left(G^a G_{db} \right) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b \left(G^a G_{dc} \right) - G^a G^d G_{db} G_{ec} V^e$$

$$+ G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \tag{ex-03.309}$$

$$= V^d \partial_c \left(G^a G_{bd} \right) + G^a G^d G_{be} G_{cd} V^e - V^d \partial_b \left(G^a G_{cd} \right) - G^a G^d G_{bd} G_{ce} V^e \tag{ex-03.310}$$

$$= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} + V^d \Gamma^a{}_{bd} \Gamma^e{}_{ce} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} \tag{ex-03.311}$$

$$= V^d \left(\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} + \Gamma^a{}_{bd} \Gamma^e{}_{ce} - \Gamma^a{}_{be} \Gamma^e{}_{cd} \right) \tag{ex-03.312}$$

$$= -R^a{}_{dbc} V^d \quad \text{(MTW)}$$

Example 4 Python functions

```
1   {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3   def truncate (poly,n):
4
5       # define the weight and give it a label
6       x^{a}::Weight(label=\epsilon).
7
8       # start with an empty espression
9       ans = Ex("0")
10
11      # loop over selected terms in the source
12      for i in range (0,n+1):
13
14          foo := @(poly).
15          bah  = Ex("\epsilon = " + str(i))
16
17          # extract a single term
18          keep_weight (foo, bah)
19
20          # update the running sum
21          ans = ans + foo
22
23      # all done, return final answer
24      return ans
25
26   Quartic := c^{a}
27           + c^{a}_{b} x^b
28           + c^{a}_{b c} x^b x^c
29           + c^{a}_{b c d} x^b x^c x^d
30           + c^{a}_{b c d e} x^b x^c x^d x^e.   # cdb (ex-04.100,Quartic)
31
32   Cubic = truncate (Quartic,3)                 # cdb (ex-04.101,Cubic)
33
34   checkpoint.append (Cubic)
```

$$p(x) = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c + c^a{}_{bcd} x^b x^c x^d + c^a{}_{bcde} x^b x^c x^d x^e \tag{ex-04.100}$$

$$q(x) = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c + c^a{}_{bcd} x^b x^c x^d \tag{ex-04.101}$$

Example 5a Keeping focused

```
1   {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3   expr := A_{a} v^{a} + B_{a} v^{a} + C_{a} v^{a}.  # cdb (ex-05.100,expr)
4
5   zoom (expr,$B_{a} Q??$)                          # cdb (ex-05.101,expr)
6   substitute (expr, $v^{a} -> w^{a}$)              # cdb (ex-05.102,expr)
7   unzoom (expr)                                    # cdb (ex-05.103,expr)
8
9   checkpoint.append (expr)
```

$$A_a v^a + B_a v^a + C_a v^a = \ldots + B_a v^a + \ldots \tag{ex-05.101}$$
$$= \ldots + B_a w^a + \ldots \tag{ex-05.102}$$
$$= A_a v^a + B_a w^a + C_a v^a \tag{ex-05.103}$$

Example 5b Tags

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3    def add_tags (obj,tag):
4        n = 0
5        ans = Ex('0')
6        for i in obj.top().terms():
7            foo = obj[i]
8            bah = Ex(tag+'_{'+str(n)+'}')
9            ans := @(ans) + @(bah) @(foo).
10           n = n + 1
11       return ans
12
13   def clear_tags (obj,tag):
14       ans := @(obj).
15       foo  = Ex(tag+'_{a?} -> 1')
16       substitute (ans,foo)
17       return ans
18
19   expr := 2 V_{p q} - 3 V_{q p}.                   # cdb (ex-05.200,expr)
20
21   expr = add_tags (expr,'\\mu')                    # cdb (ex-05.201,expr)
22
23   zoom       (expr, $\mu_{1} Q??$)                 # cdb (ex-05.202,expr)
24   substitute (expr, $V_{a b} -> - V_{b a}$)        # cdb (ex-05.203,expr)
25   unzoom     (expr)                                # cdb (ex-05.204,expr)
26
27   expr = clear_tags (expr,'\\mu')                  # cdb (ex-05.205,expr)
28
29   checkpoint.append (expr)
```

2

$$2V_{pq} - 3V_{qp} = 2\mu_0 V_{pq} - 3\mu_1 V_{qp} \tag{ex-05.201}$$
$$= \ldots - 3\mu_1 V_{qp} \tag{ex-05.202}$$
$$= \ldots + 3\mu_1 V_{pq} \tag{ex-05.203}$$
$$= 2\mu_0 V_{pq} + 3\mu_1 V_{pq} \tag{ex-05.204}$$
$$= 5V_{pq} \tag{ex-05.205}$$

Example 6-01 Evaluating components

```
1    {\theta, \varphi}::Coordinate.
2    {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4    \partial{#}::PartialDerivative.
5
6    V  := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.    # cdb(ex-06.100,V)
7    dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.              # cdb(ex-06.101,dV)
8
9    evaluate (dV, V)    # cdb(ex-06.102,dV)
```

$$V_a = [V_\theta = \varphi, \ V_\varphi = \sin\theta]$$

$$\text{(ex-06.100)}$$

$$\partial_b V_a - \partial_a V_b = \square_{ab} \begin{cases} \square_{\varphi\theta} = \cos\theta - 1 \\ \square_{\theta\varphi} = -\cos\theta + 1 \end{cases}$$

$$\text{(ex-06.102)}$$

Example 6-02 Riemann tensor of a 2-sphere

```
1   {\theta, \varphi}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4   \partial{#}::PartialDerivative.
5
6   Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
7                                             + \partial_{c}{g_{b d}}
8                                             - \partial_{d}{g_{b c}}).
9
10  Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
11                            - \partial_{d}{\Gamma^{a}_{b c}}
12                            + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
13                            - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
14
15  gab := { g_{\theta\theta}   = r**2,
16           g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.        # cdb(ex-06.201,gab)
17
18  iab := { g^{\theta\theta}   = 1/r**2,
19           g^{\varphi\varphi} = 1/(r**2 \sin(\theta)**2) }.  # cdb(ex-06.202,iab)
20
21  substitute (Rabcd, Gamma)                                  # cdb(ex-06.203,Gamma)
22
23  evaluate    (Gamma, gab+iab, rhsonly=True)                 # cdb(ex-06.204,Gamma)
24  evaluate    (Rabcd, gab+iab, rhsonly=True)                 # cdb(ex-06.205,Rabcd)
25
26  # convert from a rule to a simple expression
27  Riem := R^{a}_{b c d}.
28  substitute (Riem, Rabcd)                                   # cdb(ex-06.206,Riem)
29
30  from cdb.core.component import *
31
32  RiemCompt = get_component (Riem, $\theta, \varphi, \theta, \varphi$)   # cdb(ex-06.207,RiemCompt)
```

$$\left[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin\theta)^2 \right] \tag{ex-06.201}$$

$$\left[g^{\theta\theta} = r^{-2}, \ g^{\varphi\varphi} = \left(r^2(\sin\theta)^2 \right)^{-1} \right] \tag{ex-06.202}$$

$$\Gamma^a{}_{bc} \rightarrow \Box_{cb}{}^a \begin{cases} \Box_{\varphi\theta}{}^\varphi = (\tan\theta)^{-1} \\ \Box_{\theta\varphi}{}^\varphi = (\tan\theta)^{-1} \\ \Box_{\varphi\varphi}{}^\theta = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-06.204}$$

$$R^a{}_{bcd} \rightarrow \Box_{db}{}^a{}_c \begin{cases} \Box_{\varphi\varphi}{}^\theta{}_\theta = (\sin\theta)^2 \\ \Box_{\varphi\theta}{}^\varphi{}_\theta = -1 \\ \Box_{\theta\varphi}{}^\theta{}_\varphi = -(\sin\theta)^2 \\ \Box_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \tag{ex-06.205}$$

$$\Box_{db}{}^a{}_c \begin{cases} \Box_{\varphi\varphi}{}^\theta{}_\theta = (\sin\theta)^2 \\ \Box_{\varphi\theta}{}^\varphi{}_\theta = -1 \\ \Box_{\theta\varphi}{}^\theta{}_\varphi = -(\sin\theta)^2 \\ \Box_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \tag{ex-06.206}$$

$$R^\theta{}_{\varphi\varphi\theta} = -(\sin\theta)^2 \tag{ex-06.207}$$

Example 6-03 Using `complete` to compute the inverse metric

This version uses `complete` to compute the inverse metric.

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

gab := { g_{\theta\theta}   = r**2,
         g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.      # cdb(ex-06.301,gab)

complete   (gab, $g^{a b}$)                            # cdb(ex-06.302,gab)

substitute (Rabcd, Gamma)

evaluate   (Gamma, gab, rhsonly=True)                 # cdb(ex-06.303,Gamma)
evaluate   (Rabcd, gab, rhsonly=True)                 # cdb(ex-06.304,Rabcd)
```

$$\left[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-06.301}$$

$$\left[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin\theta)^2, \ g^{\theta\theta} = r^{-2}, \ g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\right] \tag{ex-06.302}$$

$$\Gamma^a{}_{bc} \to \square_{cb}{}^a \begin{cases} \square_{\varphi\theta}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\theta\varphi}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\varphi\varphi}{}^{\theta} = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-06.303}$$

$$R^a{}_{bcd} \to \square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^{\theta}{}_{\theta} = (\sin\theta)^2 \\ \square_{\varphi\theta}{}^{\varphi}{}_{\theta} = -1 \\ \square_{\theta\varphi}{}^{\theta}{}_{\varphi} = -(\sin\theta)^2 \\ \square_{\theta\theta}{}^{\varphi}{}_{\varphi} = 1 \end{cases} \tag{ex-06.304}$$

Example 6-04 Components by scalar projection

This example shows how one component of the Riemann tensor can be computed using a scalar projection.

```
1    {\theta, \varphi}::Coordinate.
2    {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4    theta{#}::LaTeXForm{"\theta"}.
5    varphi{#}::LaTeXForm{"\varphi"}.
6
7    # usual definitions for the connection and Riemann tensor
8
9    Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
10                                               + \partial_{c}{g_{b d}}
11                                               - \partial_{d}{g_{b c}}).
12
13   Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
14                             - \partial_{d}{\Gamma^{a}_{b c}}
15                             + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
16                             - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
17
18   gab := { g_{\theta \theta}   = r**2,
19            g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.   # cdb(ex-06.400,gab)
20
21   iab := { g^{\theta\theta}   = 1/r**2,
22            g^{\varphi\varphi} = 1/(r**2 \sin(\theta)**2) }.
23
24   substitute (Rabcd, Gamma)
25   evaluate   (Rabcd, gab+iab, rhsonly=True)
26
27   # above code just to compute Rabcd
28   # following code is all that is needed for the scalar projection method
29
30   # define the basis for vectors and dual vectors
31
32   basis := {theta^{\theta} = 1, varphi^{\varphi} = 1}.
33   dual  := {theta_{\theta} = 1, varphi_{\varphi} = 1}.
34
```

```
35      # obtain components by contracting with basis
36
37      compt := R^{a}_{b c d} theta_{a} varphi^{b} theta^{c} varphi^{d}.    # cdb(ex-06.401,compt)
38      substitute (compt,Rabcd)
39
40      evaluate (compt,basis+dual)                                          # cdb(ex-06.402,compt)
41
42      compt_sympy = compt._sympy_()
43
44      # cdbBeg(print.ex-06.04)
45      print ('type compt  = ' + str(type(compt)))          # shows that compt is a Cadabra object
46      print ('type gphiphi = ' + str(type(compt_sympy)))   # shows that gphiphi is a Python object
47      print ('     compt  = ' + str(compt))                # will contain LaTeX markup
48      print ('     gphiphi = ' + str(compt_sympy))         # will be pure Python/SymPy
49      # cdbEnd(print.ex-06.04)
50
51      checkpoint.append (compt)
```

$$R^{\theta}{}_{\varphi\theta\varphi} = R^{a}{}_{bcd}\theta_{a}\varphi^{b}\theta^{c}\varphi^{d} \qquad\qquad (\text{ex-06.401})$$

$$= (\sin\theta)^{2} \qquad\qquad (\text{ex-06.402})$$

```
1   type compt  = <class 'cadabra2.Ex'>
2   type gphiphi = <class 'sympy.core.power.Pow'>
3        compt  = (\sin(\theta))**2
4        gphiphi = sin(theta)**2
```

7

Example 6-05 Components by selection

This example shows how one component of the metric tensor can be computed by indexing the result of a call to `evaluate`.

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

theta{#}::LaTeXForm{"\theta"}.
varphi{#}::LaTeXForm{"\varphi"}.

gab := { g_{\theta \theta}   = r**2,
         g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.   # cdb(ex-06.500,gab)

metric := g_{a b}.

evaluate (metric,gab)

indcs = metric[2][1][0]                              # cdb(ex-06.501,indcs)
compt = metric[2][1][1]                              # cdb(ex-06.502,compt)

# cdbBeg(print.ex-06.05)
print ('metric = ' + str(metric.input_form()))+'\n')  # reveals Cadabra's internal structure for storing metric

print ('metric[0] = ' + str(metric[0]))
print ('metric[1] = ' + str(metric[1]))
print ('metric[2] = ' + str(metric[2])+'\n')

print ('metric[2][1] = '+ str(metric[2][1]))
print ('metric[2][1][0] = '+ str(metric[2][1][0]))
print ('metric[2][1][1] = '+ str(metric[2][1][1]))
# cdbEnd(print.ex-06.05)

checkpoint.append (indcs)
checkpoint.append (compt)
```

$$g_{\varphi\varphi} = g_{[\varphi, \, \varphi]} \tag{ex-06.501}$$
$$= r^2(\sin\theta)^2 \tag{ex-06.502}$$

```
1   metric = \components_{a b}({{\theta, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2})
2
3   metric[0] = a
4   metric[1] = b
5   metric[2] = {{\theta, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2}
6
7   metric[2][1] = {\varphi, \varphi} = (r)**2 (\sin(\theta))**2
8   metric[2][1][0] = {\varphi, \varphi}
9   metric[2][1][1] = (r)**2 (\sin(\theta))**2
```

Example 7 Export to C-code

```python
def write_code (obj,name,filename,rank):

    import os

    from sympy.printing.ccode import C99CodePrinter as printer
    from sympy.printing.codeprinter import Assignment

    idx=[]   # indices in the form [{x, x}, {x, y} ...]
    lst=[]   # corresponding terms [termxx, termxy, ...]

    for i in range( len(obj[rank]) ):                  # rank = number of free indices
        idx.append( str(obj[rank][i][0]._sympy_()) )   # indices for this term
        lst.append( str(obj[rank][i][1]._sympy_()) )   # the matching term

    mat = sympy.Matrix([lst])                          # row vector of terms
    sub_exprs, simplified_rhs = sympy.cse(mat)         # optimise code

    with open(os.getcwd() + '/' + filename, 'w') as out:

        for lhs, rhs in sub_exprs:
            out.write(printer().doprint(Assignment(lhs, rhs))+'\n')

        for index, rhs in enumerate (simplified_rhs[0]):
            lhs = sympy.Symbol(name+' '+(idx[index]).replace(', ',''][''))
            out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
```

```
1   {\theta, \varphi}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4   \partial{#}::PartialDerivative.
5
6   g_{a b}::Metric.
7   g^{a b}::InverseMetric.
8
9   Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} (   \partial_{g}{g_{b f}}
10                                            + \partial_{f}{g_{b g}}
11                                            - \partial_{b}{g_{f g}} ).
12
13  Rabcd := R^{d}_{e f g} ->   \partial_{f}{\Gamma^{d}_{e g}}
14                            - \partial_{g}{\Gamma^{d}_{e f}}
15                            + \Gamma^{d}_{b f} \Gamma^{b}_{e g}
16                            - \Gamma^{d}_{b g} \Gamma^{b}_{e f}.
17
18  Rab := R_{a b} -> R^{c}_{a c b}.
19
20  gab := { g_{\theta \theta}   = r**2,
21           g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.    # cdb(ex-07.101,gab)
22
23  complete (gab, $g^{a b}$)                                 # cdb(ex-07.102,gab)
24
25  substitute (Rabcd, Gamma)
26  substitute (Rab, Rabcd)
27
28  evaluate    (Gamma, gab, rhsonly=True)                    # cdb(ex-07.103,Gamma)
29  evaluate    (Rabcd, gab, rhsonly=True)                    # cdb(ex-07.104,Rabcd)
30  evaluate    (Rab,   gab, rhsonly=True)                    # cdb(ex-07.105,Rab)
31
32  write_code (Gamma[1],'myGamma','example-07-gamma.c',3)
33  write_code (Rabcd[1],'myRabcd','example-07-rabcd.c',4)
34  write_code (Rab[1],  'myRab',  'example-07-rab.c',2)
```

$$\left[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-07.101}$$

$$\left[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin\theta)^2, \ g^{\theta\theta} = r^{-2}, \ g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\right] \tag{ex-07.102}$$

$$\Gamma^a{}_{fg} \rightarrow \Box_{fg}{}^a \begin{cases} \Box_{\varphi\theta}{}^\varphi = (\tan\theta)^{-1} \\ \Box_{\theta\varphi}{}^\varphi = (\tan\theta)^{-1} \\ \Box_{\varphi\varphi}{}^\theta = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-07.103}$$

$$R^d{}_{efg} \rightarrow \Box_{eg}{}^d{}_f \begin{cases} \Box_{\varphi\varphi}{}^\theta{}_\theta = (\sin\theta)^2 \\ \Box_{\theta\varphi}{}^\varphi{}_\theta = -1 \\ \Box_{\varphi\theta}{}^\theta{}_\varphi = -(\sin\theta)^2 \\ \Box_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \tag{ex-07.104}$$

$$R_{ab} \rightarrow \Box_{ab} \begin{cases} \Box_{\varphi\varphi} = (\sin\theta)^2 \\ \Box_{\theta\theta} = 1 \end{cases} \tag{ex-07.105}$$

Example 8 Importing and exporting Cadabra expressions

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   def create (file_name):
4       import json, io, os, errno
5
6       try:
7           os.remove(file_name)                # delete the file if it exsists
8           with open(file_name, 'w'): pass     # create an empty file
9       except OSError as e:
10          if e.errno == errno.ENOENT:         # errno.ENOENT = no such file or directory
11              with open(file_name, 'w'): pass # create an empty file
12          else:
13              raise                           # report an exception
14
15      # Create and save an empty dict
16      data_out = {}
17      with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
18          out_file.write(json.dumps(data_out,
19                              indent=2,
20                              sort_keys=True,
21                              separators=(',', ': '),
22                              ensure_ascii=False)+'\n')
23
24  def put (key_name,object,file_name):
25      import json, io, os
26
27      # Read the current dict
28      with io.open(os.getcwd() + '/' + file_name) as inp_file:
29          data_out = json.load(inp_file)
30
31      # Add a new entry to the dict
32      data_out[key_name] = object.input_form()
33
34      # Save the updated dict
35      with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
36          out_file.write(json.dumps(data_out,
```

```
37                                     indent=2,
38                                     sort_keys=True,
39                                     separators=(',', ': '),
40                                     ensure_ascii=False)+'\n')
41
42  def get (key_name,file_name):
43      import json, io, os
44
45      # Read the current dict
46      with io.open(os.getcwd() + '/' + file_name) as inp_file:
47          data_inp = json.load(inp_file)
48
49      # Return one entry from the dict
50      return Ex (data_inp[key_name])
51
52  lib_name = 'example-08.json'
53
54  create (lib_name)
55
56  \nabla{#}::Derivative.
57
58  gab := g_{a b} - 1/3 x^{c} x^{d} R_{a c b d}
59              - 1/6 x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}.                # cdb (ex-08-02.101,gab)
60
61  iab := g^{a b} + 1/3 x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
62              + 1/6 x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}.   # cdb (ex-08-02.102,iab)
63
64  put ('g_ab',gab,lib_name)
65  put ('g^ab',iab,lib_name)
66
67  foo = get ('g_ab',lib_name)                        # cdb (ex-08-02.103,foo)
68  bah = get ('g^ab',lib_name)                        # cdb (ex-08-02.104,bah)
69
70  tmp := @(gab) - @(foo).                            # cdb (ex-08-02.105,tmp)
71  tmp := @(iab) - @(bah).                            # cdb (ex-08-02.106,tmp)
```

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} \qquad \text{(ex-08-02.101)}$$

$$g^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \qquad \text{(ex-08-02.102)}$$

$$\bar{g}_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} \qquad \text{(ex-08-02.103)}$$

$$\bar{g}^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \qquad \text{(ex-08-02.104)}$$

$$g_{ab}(x) - \bar{g}_{ab}(x) = 0 \qquad \text{(ex-08-02.105)}$$

$$g^{ab}(x) - \bar{g}^{ab}(x) = 0 \qquad \text{(ex-08-02.106)}$$

Example 9 The Gauss equation

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4
5   K_{a b}::Symmetric.
6   g^{a}_{b}::KroneckerDelta.
7
8   # define the projection operator
9
10  hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}.
11
12  # 3-covariant derivative obtained by projection on 4-covariant derivative
13
14  vpq:=v_{p q} -> h^{a}_{p} h^{b}_{q} \nabla_{b}{v_{a}}.
15
16  # compute 3-curvature by commutation of covariant derivatives
17
18  vpqr:= h^{a}_{p} h^{b}_{q} h^{c}_{r} ( \nabla_{c}{v_{a b}} - \nabla_{b}{v_{a c}} ).
19
20  substitute (vpq,hab)
21  substitute (vpqr,vpq)
22
23  distribute    (vpqr)
24  product_rule (vpqr)
25  distribute    (vpqr)
26  eliminate_kronecker (vpqr)
27
28  # standard substitutions
29
30  substitute (vpqr,$h^{a}_{b} n^{b} -> 0$)
31  substitute (vpqr,$h^{a}_{b} n_{a} -> 0$)
32  substitute (vpqr,$\nabla_{a}{g^{b}_{c}} -> 0$)
33  substitute (vpqr,$n^{a} \nabla_{b}{v_{a}} -> -v_{a} \nabla_{b}{n^{a}}$)
34  substitute (vpqr,$v_{a} \nabla_{b}{n^{a}} -> v_{p} h^{p}_{a}\nabla_{b}{n^{a}}$)
35  substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n_{q}} -> K_{a b}$)
36  substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n^{b}} -> K_{a}^{q}$)   # cdb(ex-09.095,vpqr)
```

```
37
38    # tidy up
39
40    {v_{a},h^{a}_{b},K_{a}^{b},K_{a b},R^{a}_{b c d},\nabla_{a}{v_{b}}}::SortOrder.
41
42    sort_product    (vpqr)                                    # cdb(ex-09.096,vpqr)
43    rename_dummies  (vpqr)                                    # cdb(ex-09.097,vpqr)
44    canonicalise    (vpqr)                                    # cdb(ex-09.098,vpqr)
45    factor_out      (vpqr,$h^{a?}_{b?}$)                      # cdb(ex-09.099,vpqr)
46    factor_out      (vpqr,$v_{a?}$)                          # cdb(ex-09.101,vpqr)
47
48    checkpoint.append (vpqr)
```

$$(D_r D_q - D_q D_r)v_p = h^e{}_p h^d{}_q h^c{}_r \nabla_c (\nabla_d v_e) - h^e{}_p K_{rq} n^d \nabla_d v_e + K_q{}^b K_{rp} v_b - h^d{}_p h^b{}_q h^e{}_r \nabla_b (\nabla_e v_d) + h^d{}_p K_{qr} n^e \nabla_e v_d - K_{qp} K_r{}^c v_c \qquad \text{(ex-09.095)}$$

$$= h^c{}_r h^d{}_q h^e{}_p \nabla_c (\nabla_d v_e) - h^e{}_p K_{rq} \nabla_d v_e n^d + v_b K_q{}^b K_{rp} - h^b{}_q h^d{}_p h^e{}_r \nabla_b (\nabla_e v_d) + h^d{}_p K_{qr} \nabla_e v_d n^e - v_c K_r{}^c K_{qp} \qquad \text{(ex-09.096)}$$

$$= h^a{}_r h^b{}_q h^c{}_p \nabla_a (\nabla_b v_c) - h^b{}_p K_{rq} \nabla_a v_b n^a + v_a K_q{}^a K_{rp} - h^a{}_q h^c{}_p h^b{}_r \nabla_a (\nabla_b v_c) + h^b{}_p K_{qr} \nabla_a v_b n^a - v_a K_r{}^a K_{qp} \qquad \text{(ex-09.097)}$$

$$= h^a{}_p h^b{}_q h^c{}_r \nabla_c (\nabla_b v_a) + v_a K_q{}^a K_{pr} - h^a{}_p h^b{}_q h^c{}_r \nabla_b (\nabla_c v_a) - v_a K_r{}^a K_{pq} \qquad \text{(ex-09.098)}$$

$$= v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} + h^a{}_p h^b{}_q h^c{}_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) \qquad \text{(ex-09.099)}$$

$$= h^a{}_p h^b{}_q h^c{}_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) + v_a (K_q{}^a K_{pr} - K_r{}^a K_{pq}) \qquad \text{(ex-09.101)}$$

```
1    R{#}::LaTeXForm("{{\strut}^g R}").
2
3    gRabcd := \nabla_{c}{\nabla_{b}{v_{a}}}
4             -\nabla_{b}{\nabla_{c}{v_{a}}} -> R^{d}_{a b c} v_{d}.
5
6    substitute      (vpqr,gRabcd)                          # cdb(ex-09.102,vpqr)
7    distribute      (vpqr)                                 # cdb(ex-09.103,vpqr)
8    substitute      (vpqr,$v_{a} -> h^{b}_{a} v_{b}$)      # cdb(ex-09.104,vpqr)
9    substitute      (vpqr,$h^{b}_{a} K_{c}^{a} -> K_{c}^{b}$)  # cdb(ex-09.105,vpqr)
10   sort_product    (vpqr)                                 # cdb(ex-09.106,vpqr)
11   rename_dummies  (vpqr)                                 # cdb(ex-09.107,vpqr)
12   canonicalise    (vpqr)                                 # cdb(ex-09.108,vpqr)
13   factor_out      (vpqr,$v_{a?}$)                        # cdb(ex-09.109,vpqr)
14   substitute      (vpqr,$v_{a}->1$)                      # cdb(ex-09.110,vpqr)
15   sort_product    (vpqr)                                 # cdb(ex-09.111,vpqr)
16
17   checkpoint.append (vpqr)
```

$$(D_r D_q - D_q D_r)v_p = h^a{}_p h^b{}_q h^c{}_r \left(\nabla_c \left(\nabla_b v_a \right) - \nabla_b \left(\nabla_c v_a \right) \right) + v_a \left(K_q{}^a K_{pr} - K_r{}^a K_{pq} \right) \tag{ex-09.101}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} v_d + v_a \left(K_q{}^a K_{pr} - K_r{}^a K_{pq} \right) \tag{ex-09.102}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} v_d + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} \tag{ex-09.103}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} h^e{}_d v_e + h^b{}_a v_b K_q{}^a K_{pr} - h^b{}_a v_b K_r{}^a K_{pq} \tag{ex-09.104}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} h^e{}_d v_e + K_q{}^b v_b K_{pr} - K_r{}^b v_b K_{pq} \tag{ex-09.105}$$

$$= v_e h^a{}_p h^b{}_q h^c{}_r h^e{}_d{}^g R^d{}_{abc} + v_b K_q{}^b K_{pr} - v_b K_r{}^b K_{pq} \tag{ex-09.106}$$

$$= v_e h^b{}_p h^c{}_q h^d{}_r h^e{}_a{}^g R^a{}_{bcd} + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} \tag{ex-09.107}$$

$$= v_a h^b{}_p h^c{}_q h^d{}_r h^a{}_e{}^g R^e{}_{bcd} + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} \tag{ex-09.108}$$

$$= v_a \left(h^b{}_p h^c{}_q h^d{}_r h^a{}_e{}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq} \right) \tag{ex-09.109}$$

$$^h R^a{}_{pqr} = h^b{}_p h^c{}_q h^d{}_r h^a{}_e{}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq} \tag{ex-09.110}$$

$$^h R^a{}_{pqr} = h^a{}_e h^b{}_p h^c{}_q h^d{}_r{}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq} \tag{ex-09.111}$$

Example 10 The determinant of the metric

Our game here is to compute (the leading terms) in $\det g$ of the metric in RNC form

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f \left(8g^{gh}R_{acdg}R_{befh} - 9\nabla_{cd}R_{aebf}\right) + \cdots$$

For the sake of simplicity let's assume that we are working in 3-dimensions. The following analysis is easily generalsied to other dimensions (and the final answers for $\det g$ and friends are unchanged).

Define ϵ^{abc}_{ijk} by

$$\epsilon^{abc}_{ijk} = \delta^a_i \delta^b_j \delta^c_k - \delta^b_i \delta^a_j \delta^c_k + \delta^c_i \delta^a_j \delta^b_k - \delta^c_i \delta^b_j \delta^a_k + \delta^b_i \delta^c_j \delta^a_k - \delta^a_i \delta^c_j \delta^b_k \tag{1}$$

It is easy to see that ϵ^{abc}_{ijk} is anti-symmetric in both its upper and lower indices. A trivial computation shows that for any 3×3 square matrix M_{ab},

$$\epsilon^{abc}_{123} M_{1a} M_{2b} M_{3c} = \left(\delta^a_1 \delta^b_2 \delta^c_3 - \delta^b_1 \delta^a_2 \delta^c_3 + \delta^c_1 \delta^a_2 \delta^b_3 - \delta^c_1 \delta^b_2 \delta^a_3 + \delta^b_1 \delta^c_2 \delta^a_3 - \delta^a_1 \delta^c_2 \delta^b_3\right) M_{1a} M_{2b} M_{3c} = \det M \tag{2}$$

This can be easily generalised to

$$\epsilon^{abc}_{ijk} M_{pa} M_{qb} M_{rc} = \begin{cases} \pm \det M & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The $\pm$ sign in the above depends on the particular permutations of (ijk) and (pqr). If both permutations are even or both odd then the sign is $+1$ otherwise the sign is -1. The same arguments can also be applied to a matrix inverse N^{-1} leading to

$$\epsilon^{ijk}_{uvw} N^{pu} N^{qv} M^{rw} = \begin{cases} \pm \det N^{-1} & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Note that the $\pm$ in this case will match exactly that for the case of $\det M$. Thus, multiplying both expressions and summing over all choices for (ijk) and (pqr) leads to

$$\sum_{\substack{(ijk) \\ (pqr)}} \left(\det N^{-1}\right) \det M = \epsilon^{ijk}_{uvw} N^{pu} N^{qv} M^{rw} \epsilon^{abc}_{ijk} M_{pa} M_{qb} M_{rc} \tag{5}$$

where the sum on the left hand side includes just those (ijk) and (prq) that are permutations of (123). There are 3! choices for (ijk) and 3! choices for (pqr) and thus the left hand side is easily reduced to $(3!)^2 \det M / \det N$ where $\det N = 1/\det(N^{-1})$. For the right hand side notice that

$$\epsilon^{ijk}_{uvw} \epsilon^{abc}_{ijk} = 3!\, \epsilon^{abc}_{uvw} \tag{6}$$

which leads to

$$\det M = \frac{1}{3!} \det N \epsilon^{abc}_{uvw} M_{pa} M_{qb} M_{rc} N^{pu} N^{qv} N^{rw} \tag{7}$$

For our RNC metric we will set $N^{ab} = g^{ab}$ and $M_{ij} = g_{ij}(x)$. Since g^{ab} is of the form $\mathrm{diag}(-1,1,1,1)$ we have $\det g = -1$ and thus

$$\det g(x) = -\frac{1}{3!} \epsilon^{abc}_{ijk} g_{pa}(x)\, g_{qb}(x)\, g_{rc}(x)\, g^{ip} g^{jq} g^{kr} \tag{8}$$

The ϵ^{abc}_{ijk} can be constructed in Cadabra by applying the `asym` algorithm to the upper indices of $\delta^a_i \delta^b_j \delta^c_k$. Note that `asym` will include the 1/3! coeffcient as part of its output.

The following code computes $-\det g$ rather than $\det g$.

Note that Calzetta etal. use an opposite sign for R_{abcd} so when comparing the following results against Calzetta do take note of this flipped sign in R_{abcd}.

The determinant of the metric

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Integer(1..3).
4
5   \nabla{#}::Derivative.
6
7   d{#}::KroneckerDelta.
8
9   g^{a b}::Symmetric.
10  g_{a b}::Symmetric.
11
12  R_{a b c d}::RiemannTensor.
13
14  x^{a}::Weight(label=num,value=1).
15
16  def truncate (obj,n):
17
18      ans = Ex("0")   # create a Cadabra object with value zero
19
20      for i in range (0,n+1):
21          foo := @(obj).
22          bah  = Ex("num = " + str(i))
23          distribute  (foo)
24          keep_weight (foo, bah)
25          ans = ans + foo
26
27      return ans
28
29  gab := g_{a b}
30          - (1/3)   x^{c} x^{d} R_{a c b d}
31          - (1/6)   x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
32          + (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
33                                           -9 \nabla_{c d}{R_{a e b f}} ).        # cdb (ex-10.gab.000,gab)
34
35  iab := g^{a b}
36          + (1/3)   x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
```

```
             + (1/6)  x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}
             + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
                             ( 4 g^{i j} R_{c g d i} R_{e h f j}
                             +3 \nabla_{c d}{R_{e g f h}} ).              # cdb(ex-10.iab.000,iab)

    distribute (gab)
    distribute (iab)


    gxab := gx_{a b} -> @(gab).


    eps := d^{a}_{i} d^{b}_{j} d^{c}_{k}.    # cdb (ex-10.eps.001,eps) # includes a factor of 1/3!
    asym (eps,$^{a},^{b},^{c}$)              # cdb (ex-10.eps.002,eps)


    # compute negative detg rather than det g, note 1/3! included in eps
    Ndetg := @(eps) gx_{p a} gx_{q b} gx_{r c} g^{i p} g^{j q} g^{k r}.   # cdb (ex-10.Ndetg.001,Ndetg)


    substitute       (Ndetg,gxab)                                        # cdb (ex-10.Ndetg.002,Ndetg)
    distribute       (Ndetg)                                             # cdb (ex-10.Ndetg.003,Ndetg)
    Ndetg = truncate (Ndetg,4)                                           # cdb (ex-10.Ndetg.004,Ndetg)
    substitute       (Ndetg,$g^{a b} g_{b c} -> d^{a}_{c}$,repeat=True)  # cdb (ex-10.Ndetg.005,Ndetg)
    eliminate_kronecker (Ndetg)                                          # cdb (ex-10.Ndetg.006,Ndetg)
    sort_product     (Ndetg)                                             # cdb (ex-10.Ndetg.007,Ndetg)
    rename_dummies   (Ndetg)                                             # cdb (ex-10.Ndetg.008,Ndetg)
    canonicalise     (Ndetg)                                             # cdb (ex-10.Ndetg.009,Ndetg)


    # introduce the Ricci tensor


    substitute (Ndetg,$R_{a b c d} g^{a c}              -> R_{b d}$,repeat=True)              # cdb (ex-10.Ndetg.101,Ndetg)
    substitute (Ndetg,$\nabla_{a}{R_{b c d e}} g^{b d}   -> \nabla_{a}{R_{c e}}$,repeat=True)   # cdb (ex-10.Ndetg.102,Ndetg)
    substitute (Ndetg,$\nabla_{a b}{R_{c d e f}} g^{c e} -> \nabla_{a b}{R_{d f}}$,repeat=True) # cdb (ex-10.Ndetg.103,Ndetg)


    # the following was based on sqrt-Ndetg.tex


    sqrtNdetg := 1/2 + (1/2) @(Ndetg)
               - (1/8) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
               - (1/4) (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
                                                    # cdb (ex-10.sqrtNdetg.001,sqrtNdetg)
```

```
75    sort_product   (sqrtNdetg)                                              # cdb (ex-10.sqrtNdetg.002,sqrtNdetg)
76    rename_dummies (sqrtNdetg)                                              # cdb (ex-10.sqrtNdetg.003,sqrtNdetg)
77    canonicalise   (sqrtNdetg)                                              # cdb (ex-10.sqrtNdetg.004,sqrtNdetg)

79    logNdetg := -1 + @(Ndetg)
80              - (1/2) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
81              - (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
82                                                                           # cdb (ex-10.logNdetg.001,logNdetg)

84    sort_product   (logNdetg)                                              # cdb (ex-10.logNdetg.002,logNdetg)
85    rename_dummies (logNdetg)                                              # cdb (ex-10.logNdetg.003,logNdetg)
86    canonicalise   (logNdetg)                                              # cdb (ex-10.logNdetg.004,logNdetg)

88    # ========================================================================================
89    # the remaining code is just for pretty printing

91    def product_sort (obj):
92        substitute (obj,$ x^{a}                      -> A000^{a}              $)
93        substitute (obj,$ g^{a b}                    -> A001^{a b}            $)
94        substitute (obj,$ \nabla_{c}{R_{a b}}        -> A004_{a b c}          $)
95        substitute (obj,$ \nabla_{c d}{R_{a b}}      -> A005_{a b c d}        $)
96        substitute (obj,$ \nabla_{c d e}{R_{a b}}    -> A006_{a b c d e}      $)
97        substitute (obj,$ \nabla_{c d e f}{R_{a b}}  -> A007_{a b c d e f}    $)
98        substitute (obj,$ \nabla_{e}{R_{a b c d}}    -> A008_{a b c d e}      $)
99        substitute (obj,$ \nabla_{e f}{R_{a b c d}}  -> A009_{a b c d e f}    $)
100       substitute (obj,$ \nabla_{e f g}{R_{a b c d}}   -> A010_{a b c d e f g}   $)
101       substitute (obj,$ \nabla_{e f g h}{R_{a b c d}} -> A011_{a b c d e f g h} $)
102       substitute (obj,$ R_{a b}                    -> A002_{a b}            $)
103       substitute (obj,$ R_{a b c d}                -> A003_{a b c d}        $)
104       sort_product   (obj)
105       rename_dummies (obj)
106       substitute (obj,$ A000^{a}              -> x^{a}                      $)
107       substitute (obj,$ A001^{a b}            -> g^{a b}                    $)
108       substitute (obj,$ A002_{a b}            -> R_{a b}                    $)
109       substitute (obj,$ A003_{a b c d}        -> R_{a b c d}                $)
110       substitute (obj,$ A004_{a b c}          -> \nabla_{c}{R_{a b}}        $)
111       substitute (obj,$ A005_{a b c d}        -> \nabla_{c d}{R_{a b}}      $)
112       substitute (obj,$ A006_{a b c d e}      -> \nabla_{c d e}{R_{a b}}    $)
```

```
        substitute (obj,$ A007_{a b c d e f}       -> \nabla_{c d e f}{R_{a b}}       $)
        substitute (obj,$ A008_{a b c d e}         -> \nabla_{e}{R_{a b c d}}         $)
        substitute (obj,$ A009_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}}       $)
        substitute (obj,$ A010_{a b c d e f g}     -> \nabla_{e f g}{R_{a b c d}}     $)
        substitute (obj,$ A011_{a b c d e f g h}   -> \nabla_{e f g h}{R_{a b c d}}   $)

    def get_term (obj,n):

        x^{a}::Weight(label=xnum).

        foo := @(obj).
        bah  = Ex("xnum = " + str(n))
        keep_weight (foo,bah)

        return foo

    def reformat (obj,scale):
        foo  = Ex(str(scale))
        bah := @(foo) @(obj).
        distribute      (bah)
        product_sort    (bah)
        rename_dummies (bah)
        canonicalise    (bah)
        sort_sum        (bah)
        factor_out      (bah,$x^{a?}$)
        ans := @(bah) / @(foo).
        return ans

    def rescale (obj,scale):
        foo  = Ex(str(scale))
        bah := @(foo) @(obj).
        distribute  (bah)
        factor_out  (bah,$x^{a?}$)
        return bah

    # ---------------------------------------------------------------
    # reformat Ndetg
```

```
151    Rterm0 = get_term (Ndetg,0)       # cdb (ex-10.Rterm0.701,Rterm0)
152    Rterm1 = get_term (Ndetg,1)       # cdb (ex-10.Rterm1.701,Rterm1)
153    Rterm2 = get_term (Ndetg,2)       # cdb (ex-10.Rterm2.701,Rterm2)
154    Rterm3 = get_term (Ndetg,3)       # cdb (ex-10.Rterm3.701,Rterm3)
155    Rterm4 = get_term (Ndetg,4)       # cdb (ex-10.Rterm4.701,Rterm4)
156
157    Rterm0 = reformat (Rterm0,  1)    # cdb (ex-10.Rterm0.702,Rterm0)
158    Rterm1 = reformat (Rterm1,  1)    # cdb (ex-10.Rterm1.702,Rterm1)
159    Rterm2 = reformat (Rterm2,  3)    # cdb (ex-10.Rterm2.702,Rterm2)
160    Rterm3 = reformat (Rterm3,  6)    # cdb (ex-10.Rterm3.702,Rterm3)
161    Rterm4 = reformat (Rterm4,180)    # cdb (ex-10.Rterm4.702,Rterm4)
162
163    Ndetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.Ndetg.701,Ndetg)
164
165    # --------------------------------------------------------------
166    # reformat sqrtNdetg
167
168    Rterm0 = get_term (sqrtNdetg,0)   # cdb (ex-10.Rterm0.801,Rterm0)
169    Rterm1 = get_term (sqrtNdetg,1)   # cdb (ex-10.Rterm1.801,Rterm1)
170    Rterm2 = get_term (sqrtNdetg,2)   # cdb (ex-10.Rterm2.801,Rterm2)
171    Rterm3 = get_term (sqrtNdetg,3)   # cdb (ex-10.Rterm3.801,Rterm3)
172    Rterm4 = get_term (sqrtNdetg,4)   # cdb (ex-10.Rterm4.801,Rterm4)
173
174    Rterm0 = reformat (Rterm0,  1)    # cdb (ex-10.Rterm0.802,Rterm0)
175    Rterm1 = reformat (Rterm1,  1)    # cdb (ex-10.Rterm1.802,Rterm1)
176    Rterm2 = reformat (Rterm2,  6)    # cdb (ex-10.Rterm2.802,Rterm2)
177    Rterm3 = reformat (Rterm3, 12)    # cdb (ex-10.Rterm3.802,Rterm3)
178    Rterm4 = reformat (Rterm4,360)    # cdb (ex-10.Rterm4.802,Rterm4)
179
180    sqrtNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.sqrtNdetg.801,sqrtNdetg)
181
182    # --------------------------------------------------------------
183    # reformat logNdetg
184
185    Rterm0 = get_term (logNdetg,0)    # cdb (ex-10.Rterm0.801,Rterm0)
186    Rterm1 = get_term (logNdetg,1)    # cdb (ex-10.Rterm1.801,Rterm1)
187    Rterm2 = get_term (logNdetg,2)    # cdb (ex-10.Rterm2.801,Rterm2)
188    Rterm3 = get_term (logNdetg,3)    # cdb (ex-10.Rterm3.801,Rterm3)
```

```
189    Rterm4 = get_term (logNdetg,4)      # cdb (ex-10.Rterm4.801,Rterm4)

190

191    Rterm0 = reformat (Rterm0,  1)      # cdb (ex-10.Rterm0.802,Rterm0)
192    Rterm1 = reformat (Rterm1,  1)      # cdb (ex-10.Rterm1.802,Rterm1)
193    Rterm2 = reformat (Rterm2,  3)      # cdb (ex-10.Rterm2.802,Rterm2)
194    Rterm3 = reformat (Rterm3,  6)      # cdb (ex-10.Rterm3.802,Rterm3)
195    Rterm4 = reformat (Rterm4,180)      # cdb (ex-10.Rterm4.802,Rterm4)

196

197    logNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.logNdetg.901,logNdetg)

198

199    checkpoint.append (Ndetg)
200    checkpoint.append (sqrtNdetg)
201    checkpoint.append (logNdetg)
```

The metric determinant in Riemann normal coordinates

$$-\det g(x) = 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) + \cdots$$

The volume element in RNC

If $-\det g(x)$ is non-negative then we also have

$$\sqrt{-\det g(x)} = 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) + \cdots$$

The log of -detg in RNC

$$\log\left(-\det g(x)\right) = -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) + \cdots$$

Apart from the signs, this matches exactly the expression given by Calzetta etal. (eq. A14)

Example 11 The RNC connection.

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent).
2
3   D{#}::PartialDerivative.
4   \nabla{#}::Derivative.
5
6   g_{a b}::Metric.
7   g^{a b}::InverseMetric.
8   g^{a b}::Weight(label=gnum,value=1).
9
10  \delta{#}::KroneckerDelta.
11
12  R_{a b c d}::RiemannTensor.
13  R_{a b c d}::Depends(\nabla{#}).
14
15  x^{a}::Depends(D{#}).
16  x^{a}::Weight(label=xnum,value=1).
17
18  Dx := D_{a}{x^{b}} -> \delta^{b}_{a}.    # cdb (ex-11.000,Dx)
19
20  gab := g_{a b} -> g_{a b}
21                 -  (1/3)   x^{c} x^{d} R_{a c b d}
22                 -  (1/6)   x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
23                 +  (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
24                                                  -9 \nabla_{c d}{R_{a e b f}} ).      # cdb (ex-11.001,gab)
25
26  iab := g^{a b} -> g^{a b}
27                 + (1/3)   x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
28                 + (1/6)   x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}
29                 + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
30                                         ( 4 g^{i j} R_{c g d i} R_{e h f j}
31                                         +3 \nabla_{c d}{R_{e g f h}} ).              # cdb(ex-11.002,iab)
32
33  distribute (gab)
34  distribute (iab)
35
36  ChrSym := \Gamma^{a}_{b c} -> 1/2 g^{a d} (  D_{b}{g_{d c}}
```

```
                                     + D_{c}{g_{b d}}
                                     - D_{d}{g_{b c}} ).    # cdb (ex-11.003,ChrSym)


Gamma := \Gamma^{a}_{b c}.                 # cdb (ex-11.100,Gamma)


substitute       (Gamma,ChrSym)            # cdb (ex-11.101,Gamma)
substitute       (Gamma,gab)               # cdb (ex-11.102,Gamma)
substitute       (Gamma,iab)               # cdb (ex-11.103,Gamma)
distribute       (Gamma)                    # cdb (ex-11.104,Gamma)
unwrap           (Gamma)                    # cdb (ex-11.105,Gamma)
product_rule     (Gamma)                    # cdb (ex-11.106,Gamma)
distribute       (Gamma)                    # cdb (ex-11.107,Gamma)
substitute       (Gamma,Dx)                 # cdb (ex-11.108,Gamma)
eliminate_kronecker (Gamma)                 # cdb (ex-11.109,Gamma)


def truncate (obj,n):

    ans = Ex("0")   # create a Cadabra object with value zero

    for i in range (0,n+1):
        foo := @(obj).
        bah  = Ex("xnum = " + str(i))
        distribute  (foo)
        keep_weight (foo, bah)
        ans = ans + foo

    return ans


checkpoint.append (Gamma)


# sort_product    (Gamma)  # 52.3 sec, 49 Mbyte
# rename_dummies (Gamma)  # 58.6 sec, 51 Mbyte
# canonicalise    (Gamma)  # killed after 20 mins and over 500 Mbyte


Gamma = truncate (Gamma,3)   # cdb (ex-11.110,Gamma)  # allow up to 3rd order in x^a


sort_product     (Gamma)
rename_dummies  (Gamma)
```

```
canonicalise    (Gamma)

checkpoint.append (Gamma)

# =============================================================================
# the remaining code is just for pretty printing

def product_sort (obj):
    substitute (obj,$ g^{a b}                   -> A001^{a b}           $)
    substitute (obj,$ x^{a}                     -> A002^{a}             $)
    substitute (obj,$ z^{a}                     -> A003^{a}             $)
    substitute (obj,$ R_{a b c d}               -> A004_{a b c d}       $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}   -> A005_{a b c d e}     $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}} -> A006_{a b c d e f}   $)
    sort_sum       (obj)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a b}                -> g^{a b}              $)
    substitute (obj,$ A002^{a}                  -> x^{a}                $)
    substitute (obj,$ A003^{a}                  -> z^{a}                $)
    substitute (obj,$ A004_{a b c d}            -> R_{a b c d}          $)
    substitute (obj,$ A005_{a b c d e}          -> \nabla_{e}{R_{a b c d}}   $)
    substitute (obj,$ A006_{a b c d e f}        -> \nabla_{e f}{R_{a b c d}} $)

def get_xterm (obj,n):

    foo := @(obj).
    bah  = Ex("xnum = " + str(n))
    distribute (foo)
    keep_weight (foo, bah)

    return foo

def get_gterm (obj,n):

    foo := @(obj).
    bah  = Ex("gnum = " + str(n))
    distribute (foo)
```

```
113        keep_weight (foo, bah)

114

115        return foo

116

117    def reformat (obj,scale):

118

119        foo  = Ex(str(scale))
120        bah := @(foo) @(obj).

121

122        distribute     (bah)
123        product_sort   (bah)
124        rename_dummies (bah)
125        canonicalise   (bah)
126        factor_out     (bah,$x^{a?},g^{b? c?}$)
127        ans := @(bah) / @(foo).

128

129        return ans

130

131    gam1  = get_xterm (Gamma, 1)                         # cdb (ex-11.200,gam1)
132    gam2  = get_xterm (Gamma, 2)                         # cdb (ex-11.201,gam2)
133    gam3  = get_xterm (Gamma, 3)                         # cdb (ex-11.202,gam3)

134

135    gam31 = get_gterm (gam3,  1)                         # cdb (ex-11.210,gam31)
136    gam32 = get_gterm (gam3,  2)                         # cdb (ex-11.211,gam31)

137

138    gam1  = reformat (gam1,   3)                         # cdb (ex-11.220,gam1)
139    gam2  = reformat (gam2,  12)                         # cdb (ex-11.221,gam2)

140

141    gam31 = reformat (gam31, 40)                         # cdb (ex-11.222,gam31)
142    gam32 = reformat (gam32, 45)                         # cdb (ex-11.223,gam32)

143

144    Gamma  := @(gam1) + @(gam2) + @(gam31) + @(gam32).   # cdb (ex-11.230,Gamma)
145    Scaled := 360 @(Gamma).                             # cdb (ex-11.231,Scaled)

146

147    checkpoint.append (Gamma)
```

$$\Gamma^a{}_{bc}(x) = \frac{1}{3}g^{ad}x^e\left(R_{bdce} + R_{becd}\right) + \frac{1}{12}g^{ad}x^e x^f\left(-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}\right)$$

$$+ \frac{1}{40}g^{ad}x^e x^f x^g\left(-\nabla_{ce} R_{bfdg} - \nabla_{ec} R_{bfdg} + \nabla_{de} R_{bfcg} + \nabla_{ed} R_{bfcg} + 2\nabla_{ef} R_{bdcg} + 2\nabla_{ef} R_{bgcd} - \nabla_{be} R_{cfdg} - \nabla_{eb} R_{cfdg}\right)$$

$$+ \frac{1}{45}g^{ad}g^{ef}x^g x^h x^i\left(4R_{becg}R_{dhfi} + 4R_{bgce}R_{dhfi} - 2R_{bdeg}R_{chfi} - R_{bedg}R_{chfi} + R_{bgde}R_{chfi} - 2R_{bgeh}R_{cdfi} - R_{bgeh}R_{cfdi}\right.$$

$$\left. + R_{bgeh}R_{cidf}\right) \qquad \text{(ex-11.230)}$$

$$360\Gamma^a{}_{bc}(x) = 120g^{ad}x^e\left(R_{bdce} + R_{becd}\right) + 30g^{ad}x^e x^f\left(-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}\right)$$

$$+ 9g^{ad}x^e x^f x^g\left(-\nabla_{ce} R_{bfdg} - \nabla_{ec} R_{bfdg} + \nabla_{de} R_{bfcg} + \nabla_{ed} R_{bfcg} + 2\nabla_{ef} R_{bdcg} + 2\nabla_{ef} R_{bgcd} - \nabla_{be} R_{cfdg} - \nabla_{eb} R_{cfdg}\right)$$

$$+ 8g^{ad}g^{ef}x^g x^h x^i\left(4R_{becg}R_{dhfi} + 4R_{bgce}R_{dhfi} - 2R_{bdeg}R_{chfi} - R_{bedg}R_{chfi} + R_{bgde}R_{chfi} - 2R_{bgeh}R_{cdfi} - R_{bgeh}R_{cfdi}\right.$$

$$\left. + R_{bgeh}R_{cidf}\right) \qquad \text{(ex-11.231)}$$

Save $\Gamma^a{}_{bc}$ for later use in Example 12.

```
1   jsonfile = 'example-11.json'
2   cdblib.create (jsonfile)
3   cdblib.put ('Gamma',Gamma,jsonfile)
```

Example 12 Checking the 2nd and 3rd order terms of Calzetta etal.

The following calculations show that my results for the RNC connection agree with those of Calzetta etal. to third order terms.

Note that I take ∇_{ab} to be $\nabla_a (\nabla_b)$.

Note also that $(LCB) R_{abcd} = -(Calzetta) R_{abcd}$. Consequently, I replace R_{abcd} with $-R_{abcd}$ in the Calzetta expressions (done as a Cadabra substitution rule).

This is relatively straightforward. We just apply a few carefully chosen applications of the first and second Bianchi identities.

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,u,v#}::Indices("latin",position=independent).
2   {\mu,\nu,\rho,\sigma,\tau,\lambda,\xi#}::Indices("greek",position=independent).
3
4   \nabla{#}::Derivative.
5
6   g_{a b}::Metric.
7   g^{a b}::InverseMetric.
8   g^{a b}::Weight(label=gnum,value=1).
9
10  \delta{#}::KroneckerDelta.
11
12  R_{a b c d}::RiemannTensor.
13  R_{a b c d}::Depends(\nabla{#}).
14
15  x^{a}::Weight(label=xnum,value=1).
16
17  def add_tags (obj,tag):
18
19      n = 0
20      ans = Ex('0')
21
22      for i in obj.top().terms():
23          foo = obj[i]
24          bah = Ex(tag+'_{'+str(n)+'}')
25          ans := @(ans) + @(bah) @(foo).
26          n = n + 1
27
28      return ans
29
30  def clear_tags (obj,tag):
31
32      ans := @(obj).
33      foo  = Ex(tag+'_{a?} -> 1')
34      substitute (ans,foo)
35
36      return ans
37
38  def get_xterm (obj,n):
```

2

```
39
40      foo := @(obj).
41      bah  = Ex("xnum = " + str(n))
42      distribute  (foo)
43      keep_weight (foo, bah)
44
45      return foo
46
47   def get_gterm (obj,n):
48
49      foo := @(obj).
50      bah  = Ex("gnum = " + str(n))
51      distribute   (foo)
52      keep_weight (foo, bah)
53
54      return foo
55
56   def product_sort (obj):
57      substitute (obj,$ g^{a b}                    -> A001^{a b}             $)
58      substitute (obj,$ x^{a}                      -> A002^{a}              $)
59      substitute (obj,$ z^{a}                      -> A003^{a}              $)
60      substitute (obj,$ R_{a b c d}                -> A004_{a b c d}        $)
61      substitute (obj,$ \nabla_{e}{R_{a b c d}}    -> A005_{a b c d e}      $)
62      substitute (obj,$ \nabla_{e f}{R_{a b c d}}  -> A006_{a b c d e f}    $)
63      sort_sum        (obj)
64      sort_product    (obj)
65      rename_dummies (obj)
66      substitute (obj,$ A001^{a b}                 -> g^{a b}               $)
67      substitute (obj,$ A002^{a}                   -> x^{a}                 $)
68      substitute (obj,$ A003^{a}                   -> z^{a}                 $)
69      substitute (obj,$ A004_{a b c d}             -> R_{a b c d}           $)
70      substitute (obj,$ A005_{a b c d e}           -> \nabla_{e}{R_{a b c d}}   $)
71      substitute (obj,$ A006_{a b c d e f}         -> \nabla_{e f}{R_{a b c d}} $)
72
73   def reformat (obj,scaleA,scaleB):
74
75      foo  = Ex(str(scaleA))
76      moo  = Ex(str(scaleB))
```

```
77      bah := @(foo) @(obj) / @(moo).

78

79      distribute      (bah)
80      product_sort    (bah)
81      rename_dummies  (bah)
82      canonicalise    (bah)
83      factor_out      (bah,$g^{c? d?}$)
84      factor_out      (bah,$x^{a?},z^{b?}$)
85      ans := @(moo) @(bah) / @(foo).

86

87      return ans

88

89  # ===========================================================================
90  # LCB

91

92  import cdblib
93  Gamma  = cdblib.get ('Gamma','example-11.json')              # cdb(ex-12.100,Gamma)

94

95  # note that the next two lines require careful inspection of the free indices on Gamma
96  # expecting Gamma = \Gamma^{a}_{bc}
97  Gamma := z^{b} z^{c} @(Gamma).

98

99  # lower index ^{a} to _{v}

100

101 Gamma := g_{v a} @(Gamma).

102

103 distribute (Gamma)
104 substitute (Gamma, $g_{a d} g^{d b} -> \delta_{a}^{b}$)
105 eliminate_kronecker (Gamma)                                  # cdb(ex-12.101,Gamma)

106

107 # change free index _{v} to _{a}

108

109 foo := tmp_{v} -> @(Gamma).                                  # cdb(ex-12.191,foo)
110 bah := tmp_{a}.                                              # cdb(ex-12.192,bah)
111 substitute (bah, foo)                                        # cdb(ex-12.193,bah)

112

113 Gamma := @(bah).                                             # cdb(ex-12.102,Gamma)

114
```

4

```
115    product_sort (Gamma)                                        # cdb(ex-12.103,Gamma)

116

117    checkpoint.append (Gamma)

118

119    gam1  = get_xterm (Gamma,1)                                 # cdb(ex-12.200,gam1)
120    gam2  = get_xterm (Gamma,2)                                 # cdb(ex-12.201,gam2)
121    gam3  = get_xterm (Gamma,3)                                 # cdb(ex-12.202,gam3)

122

123    gam30 = get_gterm (gam3,0)                                  # cdb(ex-12.203,gam30)
124    gam31 = get_gterm (gam3,1)                                  # cdb(ex-12.204,gam31)

125

126    gam1  = reformat (gam1, 3,1)                                # cdb(ex-12.300,gam1)
127    gam2  = reformat (gam2,12,1)                                # cdb(ex-12.301,gam2)

128

129    gam30 = reformat (gam30,40,1)                               # cdb(ex-12.302,gam30)
130    gam31 = reformat (gam31,45,2)                               # cdb(ex-12.303,gam31)

131

132    gam3  := @(gam30) + @(gam31).                               # cdb(ex-12.304,gam3)

133

134    Gamma := @(gam1) + @(gam2) + @(gam3).                       # cdb(ex-12.305,Gamma)

135

136    checkpoint.append (Gamma)
```

$$\texttt{ex-12.100} := \frac{1}{3}g^{ad}x^e\left(R_{bdce}+R_{becd}\right) + \frac{1}{12}g^{ad}x^e x^f\left(-\nabla_c R_{bedf}+\nabla_d R_{becf}+2\nabla_e R_{bdcf}+2\nabla_e R_{bfcd}-\nabla_b R_{cedf}\right)$$

$$+\frac{1}{40}g^{ad}x^e x^f x^g\left(-\nabla_{ce}R_{bfdg}-\nabla_{ec}R_{bfdg}+\nabla_{de}R_{bfcg}+\nabla_{ed}R_{bfcg}+2\nabla_{ef}R_{bdcg}+2\nabla_{ef}R_{bgcd}-\nabla_{be}R_{cfdg}-\nabla_{eb}R_{cfdg}\right)$$

$$+\frac{1}{45}g^{ad}g^{ef}x^g x^h x^i\left(4R_{becg}R_{dhfi}+4R_{bgce}R_{dhfi}-2R_{bdeg}R_{chfi}-R_{bedg}R_{chfi}+R_{bgde}R_{chfi}-2R_{bgeh}R_{cdfi}-R_{bgeh}R_{cfdi}+R_{bgeh}R_{cidf}\right)$$

$$\texttt{ex-12.191} := tmp_v$$

$$\rightarrow \frac{1}{3}z^b z^c x^e R_{bvce}+\frac{1}{3}z^b z^c x^e R_{becv}-\frac{1}{12}z^b z^c x^e x^f\nabla_c R_{bevf}+\frac{1}{12}z^b z^c x^e x^f\nabla_v R_{becf}+\frac{1}{6}z^b z^c x^e x^f\nabla_e R_{bvcf}+\frac{1}{6}z^b z^c x^e x^f\nabla_e R_{bfcv}$$

$$-\frac{1}{12}z^b z^c x^e x^f\nabla_b R_{cevf}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ce}R_{bfvg}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ec}R_{bfvg}+\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ve}R_{bfcg}+\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ev}R_{bfcg}$$

$$+\frac{1}{20}z^b z^c x^e x^f x^g\nabla_{ef}R_{bvcg}+\frac{1}{20}z^b z^c x^e x^f x^g\nabla_{ef}R_{bgcv}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{be}R_{cfvg}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{eb}R_{cfvg}$$

$$+\frac{4}{45}z^b z^c g^{ef}x^g x^h x^i R_{becg}R_{vhfi}+\frac{4}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgce}R_{vhfi}-\frac{2}{45}z^b z^c g^{ef}x^g x^h x^i R_{bveg}R_{chfi}-\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{bevg}R_{chfi}$$

$$+\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgve}R_{chfi}-\frac{2}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgeh}R_{cvfi}-\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgeh}R_{cfvi}+\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgeh}R_{civf}$$

$$\texttt{ex-12.192} := tmp_a$$

$$\texttt{ex-12.193} := \frac{1}{3}z^b z^c x^e R_{bace}+\frac{1}{3}z^b z^c x^e R_{beca}-\frac{1}{12}z^b z^c x^e x^f\nabla_c R_{beaf}+\frac{1}{12}z^b z^c x^e x^f\nabla_a R_{becf}+\frac{1}{6}z^b z^c x^e x^f\nabla_e R_{bacf}+\frac{1}{6}z^b z^c x^e x^f\nabla_e R_{bfca}$$

$$-\frac{1}{12}z^b z^c x^e x^f\nabla_b R_{ceaf}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ce}R_{bfag}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ec}R_{bfag}+\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ae}R_{bfcg}+\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{ea}R_{bfcg}$$

$$+\frac{1}{20}z^b z^c x^e x^f x^g\nabla_{ef}R_{bacg}+\frac{1}{20}z^b z^c x^e x^f x^g\nabla_{ef}R_{bgca}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{be}R_{cfag}-\frac{1}{40}z^b z^c x^e x^f x^g\nabla_{eb}R_{cfag}$$

$$+\frac{4}{45}z^b z^c g^{ef}x^g x^h x^i R_{becg}R_{ahfi}+\frac{4}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgce}R_{ahfi}-\frac{2}{45}z^b z^c g^{ef}x^g x^h x^i R_{baeg}R_{chfi}-\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{beag}R_{chfi}$$

$$+\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgae}R_{chfi}-\frac{2}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgeh}R_{cafi}-\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgeh}R_{cfai}+\frac{1}{45}z^b z^c g^{ef}x^g x^h x^i R_{bgeh}R_{ciaf}$$

$$\text{ex-12.101} := \frac{1}{3}z^b z^c x^e R_{bvce} + \frac{1}{3}z^b z^c x^e R_{becv} - \frac{1}{12}z^b z^c x^e x^f \nabla_c R_{bevf} + \frac{1}{12}z^b z^c x^e x^f \nabla_v R_{becf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bvcf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bfcv}$$

$$- \frac{1}{12}z^b z^c x^e x^f \nabla_b R_{cevf} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ce} R_{bfvg} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ec} R_{bfvg} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ve} R_{bfcg} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ev} R_{bfcg}$$

$$+ \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bvcg} + \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bgcv} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{be} R_{cfvg} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{eb} R_{cfvg}$$

$$+ \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{becg} R_{vhfi} + \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgce} R_{vhfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{bveg} R_{chfi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bevg} R_{chfi}$$

$$+ \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgve} R_{chfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cvfi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cfvi} + \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{civf}$$

$$\text{ex-12.102} := \frac{1}{3}z^b z^c x^e R_{bace} + \frac{1}{3}z^b z^c x^e R_{beca} - \frac{1}{12}z^b z^c x^e x^f \nabla_c R_{beaf} + \frac{1}{12}z^b z^c x^e x^f \nabla_a R_{becf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bacf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bfca}$$

$$- \frac{1}{12}z^b z^c x^e x^f \nabla_b R_{ceaf} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ce} R_{bfag} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ec} R_{bfag} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ae} R_{bfcg} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ea} R_{bfcg}$$

$$+ \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bacg} + \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bgca} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{be} R_{cfag} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{eb} R_{cfag}$$

$$+ \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{becg} R_{ahfi} + \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgce} R_{ahfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{baeg} R_{chfi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{beag} R_{chfi}$$

$$+ \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgae} R_{chfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cafi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cfai} + \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{ciaf}$$

$$\text{ex-12.103} := \frac{1}{3}x^b z^c z^d R_{cadb} + \frac{1}{3}x^b z^c z^d R_{cbda} + \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{daec} - \frac{1}{12}x^b x^c z^d z^e \nabla_e R_{dbac} + \frac{1}{12}x^b x^c z^d z^e \nabla_a R_{dbec} + \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{dcea}$$

$$- \frac{1}{12}x^b x^c z^d z^e \nabla_d R_{ebac} + \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{eafd} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fb} R_{ecad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{bf} R_{ecad} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{ecfd}$$

$$+ \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{ecfd} + \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{edfa} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{fcad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{fcad}$$

$$- \frac{2}{45}g^{bc} x^d x^e x^f z^g z^h R_{gabd} R_{hecf} - \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gbad} R_{hecf} + \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{aecf} R_{gbhd} + \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdab} R_{hecf}$$

$$+ \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{aecf} R_{gdhb} - \frac{2}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdbe} R_{hacf} - \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdbe} R_{hcaf} + \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdbe} R_{hfac}$$

$$\text{ex-12.200} := \frac{1}{3}x^b z^c z^d R_{cadb} + \frac{1}{3}x^b z^c z^d R_{cbda}$$

$$\text{ex-12.201} := \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{daec} - \frac{1}{12}x^b x^c z^d z^e \nabla_e R_{dbac} + \frac{1}{12}x^b x^c z^d z^e \nabla_a R_{dbec} + \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{dcea} - \frac{1}{12}x^b x^c z^d z^e \nabla_d R_{ebac}$$

$$\texttt{ex-12.202} := \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{eafd} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fb}R_{ecad} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{bf}R_{ecad} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{ecfd}$$

$$+ \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{ecfd} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{edfa} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{eb}R_{fcad} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{be}R_{fcad}$$

$$- \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gabd}R_{hecf} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gbad}R_{hecf} + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{aecf}R_{gbhd} + \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdab}R_{hecf}$$

$$+ \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{aecf}R_{gdhb} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdbe}R_{hacf} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdbe}R_{hcaf} + \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdbe}R_{hfac}$$

$$\texttt{ex-12.203} := \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{eafd} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fb}R_{ecad} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{bf}R_{ecad} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{ecfd}$$

$$+ \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{ecfd} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{edfa} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{eb}R_{fcad} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{be}R_{fcad}$$

$$\texttt{ex-12.204} := -\frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gabd}R_{hecf} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gbad}R_{hecf} + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{aecf}R_{gbhd} + \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdab}R_{hecf}$$

$$+ \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{aecf}R_{gdhb} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdbe}R_{hacf} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdbe}R_{hcaf} + \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gdbe}R_{hfac}$$

$$\texttt{ex-12.300} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\texttt{ex-12.301} := \frac{1}{12}x^bx^cz^dz^e\left(4\nabla_bR_{adce} + 2\nabla_dR_{abce} + \nabla_aR_{bdce}\right)$$

$$\texttt{ex-12.302} := \frac{1}{40}x^bx^cx^dz^ez^f\left(4\nabla_{bc}R_{aedf} + 2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf}\right)$$

$$\texttt{ex-12.303} := \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h\left(4R_{adbe}R_{cgfh} - 2R_{agbd}R_{cefh} - R_{adbg}R_{cefh} + R_{abdg}R_{cefh}\right)$$

$$\texttt{ex-12.304} := \frac{1}{40}x^bx^cx^dz^ez^f\left(4\nabla_{bc}R_{aedf} + 2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf}\right)$$

$$+ \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h\left(4R_{adbe}R_{cgfh} - 2R_{agbd}R_{cefh} - R_{adbg}R_{cefh} + R_{abdg}R_{cefh}\right)$$

$$\texttt{ex-12.305} := \frac{2}{3}x^bz^cz^dR_{acbd} + \frac{1}{12}x^bx^cz^dz^e\left(4\nabla_bR_{adce} + 2\nabla_dR_{abce} + \nabla_aR_{bdce}\right)$$

$$+ \frac{1}{40}x^bx^cx^dz^ez^f\left(4\nabla_{bc}R_{aedf} + 2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf}\right)$$

$$+ \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h\left(4R_{adbe}R_{cgfh} - 2R_{agbd}R_{cefh} - R_{adbg}R_{cefh} + R_{abdg}R_{cefh}\right)$$

```
1    # ===========================================================================
2    # Calzetta
3    # note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}
4
5    GammaBar := z^{\nu} z^{\rho} (
6                   (2/3) R^{\mu}_{\nu\rho\sigma} x^{\sigma}
7              + (1/12) (5 \nabla_{\lambda}{R^{\mu}_{\nu\rho\sigma}}
8                   + \nabla_{\rho}{R^{\mu}_{\sigma\nu\lambda}}) x^{\sigma} x^{\lambda}
9              + (1/6) (   (9/10) \nabla_{\tau\lambda}{R^{\mu}_{\rho\nu\sigma}}
10                  + (3/20) (   \nabla_{\tau\rho}{R^{\mu}_{\sigma\nu\lambda}}
11                       + \nabla_{\rho\tau}{R^{\mu}_{\sigma\nu\lambda}} )
12                  + (1/60) (   21 R^{\mu}_{\lambda\xi\rho} R^{\xi}_{\sigma\nu\tau}
13                       + 48 R^{\mu}_{\xi\rho\lambda} R^{\xi}_{\sigma\nu\tau}
14                       - 37 R^{\mu}_{\sigma\xi\lambda} R^{\xi}_{\nu\rho\tau} ) ) x^{\sigma} x^{\lambda} x^{\tau} ).
15                                                      # cdb(ex-12.400,GammaBar)
16
17   # convert from Greek to Latin indices
18
19   distribute (GammaBar)
20   rename_dummies (GammaBar,"greek","latin")                # cdb(ex-12.401,GammaBar)
21
22   # lower the \mu index
23
24   GammaBar := \delta_{a \mu} @(GammaBar).                  # cdb(ex-12.402,GammaBar)
25   distribute (GammaBar)                                    # cdb(ex-12.403,GammaBar)
26   eliminate_kronecker (GammaBar)                           # cdb(ex-12.404,GammaBar)
27
28   # sort products
29
30   product_sort (GammaBar)                                  # cdb(ex-12.405,GammaBar)
31
32   checkpoint.append (GammaBar)
33
34   # Replace R with - R (Calzetta uses the non-MTW convention for Riemann)
35
36   substitute (GammaBar, $R_{a b c d} -> - R_{a b c d}$)    # cdb(ex-12.406,GammaBar)
37   substitute (GammaBar, $R^{a}_{b c d} -> - R^{a}_{b c d}$)  # cdb(ex-12.407,GammaBar)
38
```

```
39    substitute (GammaBar, $R^{a}_{b c d} -> g^{a e} R_{e b c d}$)   # cdb(ex-12.408,GammaBar)
40
41    cal1 = get_xterm (GammaBar,1)                                   # cdb(ex-12.500,cal1)
42    cal2 = get_xterm (GammaBar,2)                                   # cdb(ex-12.501,cal2)
43    cal3 = get_xterm (GammaBar,3)                                   # cdb(ex-12.502,cal3)
44
45    cal1 = reformat (cal1,3,1)                                      # cdb(ex-12.600,cal1)
46    cal2 = reformat (cal2,12,1)                                     # cdb(ex-12.601,cal2)
47    # cal3 = reformat (cal3,360,1)                                     # cdb(ex-12.602,cal3)
48
49    cal30 = get_gterm (cal3,0)                                      # cdb(ex-12.602,cal30)
50    cal31 = get_gterm (cal3,1)                                      # cdb(ex-12.603,cal31)
51
52    cal1  = reformat (cal1, 3,1)                                    # cdb(ex-12.604,cal1)
53    cal2  = reformat (cal2,12,1)                                    # cdb(ex-12.605,cal2)
54
55    cal30 = reformat (cal30,40,1)                                   # cdb(ex-12.606,cal30)
56    cal31 = reformat (cal31,360,1)                                  # cdb(ex-12.607,cal31)
57
58    cal3 := @(cal30) + @(cal31).                                    # cdb(ex-12.608,cal3)
59
60    GammaBar := @(cal1) + @(cal2) + @(cal3).                        # cdb(ex-12.409,GammaBar)
61
62    checkpoint.append (GammaBar)
```

$$\texttt{ex-12.400} := z^\nu z^\rho \left(\frac{2}{3} R^\mu{}_{\nu\rho\sigma} x^\sigma + \frac{1}{12} \left(5\nabla_\lambda R^\mu{}_{\nu\rho\sigma} + \nabla_\rho R^\mu{}_{\sigma\nu\lambda} \right) x^\sigma x^\lambda \right.$$
$$\left. + \frac{1}{6} \left(\frac{9}{10} \nabla_{\tau\lambda} R^\mu{}_{\rho\nu\sigma} + \frac{3}{20} \nabla_{\tau\rho} R^\mu{}_{\sigma\nu\lambda} + \frac{3}{20} \nabla_{\rho\tau} R^\mu{}_{\sigma\nu\lambda} + \frac{7}{20} R^\mu{}_{\lambda\xi\rho} R^\xi{}_{\sigma\nu\tau} + \frac{4}{5} R^\mu{}_{\xi\rho\lambda} R^\xi{}_{\sigma\nu\tau} - \frac{37}{60} R^\mu{}_{\sigma\xi\lambda} R^\xi{}_{\nu\rho\tau} \right) x^\sigma x^\lambda x^\tau \right)$$

$$\texttt{ex-12.401} := \frac{2}{3} z^a z^b R^\mu{}_{abc} x^c + \frac{5}{12} z^a z^b \nabla_d R^\mu{}_{abc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu{}_{abc} x^a x^c + \frac{3}{20} z^b z^a \nabla_{de} R^\mu{}_{abc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu{}_{abc} x^a x^c x^d$$
$$+ \frac{1}{40} z^b z^d \nabla_{de} R^\mu{}_{abc} x^a x^c x^e + \frac{7}{120} z^e z^c R^\mu{}_{abc} R^b{}_{def} x^d x^a x^f + \frac{2}{15} z^e z^b R^\mu{}_{abc} R^a{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu{}_{abc} R^b{}_{def} x^a x^c x^f$$

$$\texttt{ex-12.402} := \delta_{a\mu} \left(\frac{2}{3} z^g z^b R^\mu{}_{gbc} x^c + \frac{5}{12} z^g z^b \nabla_d R^\mu{}_{gbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu{}_{gbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R^\mu{}_{gbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu{}_{gbc} x^g x^c x^d \right.$$
$$\left. + \frac{1}{40} z^b z^d \nabla_{de} R^\mu{}_{gbc} x^g x^c x^e + \frac{7}{120} z^e z^c R^\mu{}_{gbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R^\mu{}_{gbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu{}_{gbc} R^b{}_{def} x^g x^c x^f \right)$$

$$\texttt{ex-12.403} := \frac{2}{3} \delta_{a\mu} z^g z^b R^\mu{}_{gbc} x^c + \frac{5}{12} \delta_{a\mu} z^g z^b \nabla_d R^\mu{}_{gbc} x^c x^d + \frac{1}{12} \delta_{a\mu} z^b z^d \nabla_d R^\mu{}_{gbc} x^g x^c + \frac{3}{20} \delta_{a\mu} z^b z^g \nabla_{de} R^\mu{}_{gbc} x^c x^e x^d + \frac{1}{40} \delta_{a\mu} z^b z^e \nabla_{de} R^\mu{}_{gbc} x^g x^c x^d$$
$$+ \frac{1}{40} \delta_{a\mu} z^b z^d \nabla_{de} R^\mu{}_{gbc} x^g x^c x^e + \frac{7}{120} \delta_{a\mu} z^e z^c R^\mu{}_{gbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} \delta_{a\mu} z^e z^b R^\mu{}_{gbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} \delta_{a\mu} z^d z^e R^\mu{}_{gbc} R^b{}_{def} x^g x^c x^f$$

$$\texttt{ex-12.404} := \frac{2}{3} z^g z^b R_{agbc} x^c + \frac{5}{12} z^g z^b \nabla_d R_{agbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R_{agbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R_{agbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R_{agbc} x^g x^c x^d$$
$$+ \frac{1}{40} z^b z^d \nabla_{de} R_{agbc} x^g x^c x^e + \frac{7}{120} z^e z^c R_{agbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R_{agbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R_{agbc} R^b{}_{def} x^g x^c x^f$$

$$\texttt{ex-12.405} := \frac{2}{3} x^b z^c z^d R_{adcb} + \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} + \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb}$$
$$+ \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc}$$

$$\texttt{ex-12.406} := -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb}$$
$$- \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} + \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} - \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} - \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc}$$

$$\texttt{ex-12.407} := -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb}$$
$$- \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc}$$

$$\texttt{ex-12.408} := -\frac{2}{3}x^b z^c z^d R_{adcb} - \frac{1}{12}x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12}x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb}$$

$$- \frac{3}{20}x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360}x^b x^c x^d z^e z^f R_{adgb}g^{gh}R_{hefc} + \frac{2}{15}x^b x^c x^d z^e z^f R_{ageb}g^{gh}R_{hcfd} + \frac{7}{120}x^b x^c x^d z^e z^f R_{adge}g^{gh}R_{hbfc}$$

$$\texttt{ex-12.500} := -\frac{2}{3}x^b z^c z^d R_{adcb}$$

$$\texttt{ex-12.501} := -\frac{1}{12}x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12}x^b x^c z^d z^e \nabla_c R_{aedb}$$

$$\texttt{ex-12.502} := -\frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20}x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb}$$

$$- \frac{37}{360}x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15}x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcfd} + \frac{7}{120}x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}$$

$$\text{ex-12.600} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.601} := \frac{1}{12} x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5 \nabla_b R_{adce} \right)$$

$$\text{ex-12.602} := -\frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb}$$

$$\text{ex-12.603} := -\frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}$$

$$\text{ex-12.604} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.605} := \frac{1}{12} x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5 \nabla_b R_{adce} \right)$$

$$\text{ex-12.606} := \frac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf} \right)$$

$$\text{ex-12.607} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh} \right)$$

$$\text{ex-12.608} := \frac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf} \right) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh} \right)$$

$$\texttt{ex-12.409} := \frac{2}{3}x^b z^c z^d R_{acbd} + \frac{1}{12}x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5\nabla_b R_{adce}\right) + \frac{1}{40}x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}\right)$$

$$+ \frac{1}{360}g^{bc}x^d x^e x^f z^g z^h \left(37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh}\right)$$

The fun begins $\Gamma - \bar{\Gamma}$

It's now time to compute the difference $\Gamma - \bar{\Gamma}$. Here it is.

```
1    def reformat_diff (obj):
2
3        distribute (obj)
4
5        obj1  = get_xterm (obj,1)
6        obj2  = get_xterm (obj,2)
7        obj3  = get_xterm (obj,3)
8
9        obj30 = get_gterm (obj3,0)
10       obj31 = get_gterm (obj3,1)
11
12       obj1  = reformat (obj1, 3,1)
13       obj2  = reformat (obj2,12,1)
14
15       obj30 = reformat (obj30,40,1)
16       obj31 = reformat (obj31,360,1)
17
18       obj3 := @(obj30) + @(obj31).
19
20       ans := @(obj1) + @(obj2) + @(obj3).
21
22       return ans
23
24   # We could use reformat_diff here but instead we'll do it one step at a time so that
25   # we can see exactly what's going on. Later on we will use reformat_diff to do the job.
26
27   diff := @(Gamma) - @(GammaBar).                        # cdb(ex-12.diff.100,diff)
28   distribute (diff)
29
30   diff1  = get_xterm (diff,1)                            # cdb(ex-12.diff.200,diff1)
31   diff2  = get_xterm (diff,2)                            # cdb(ex-12.diff.201,diff2)
32   diff3  = get_xterm (diff,3)                            # cdb(ex-12.diff.202,diff3)
33
34   diff30 = get_gterm (diff3,0)                           # cdb(ex-12.diff.203,diff30)
```

```
35    diff31 = get_gterm (diff3,1)                              # cdb(ex-12.diff.204,diff31)

36

37    diff1  = reformat (diff1, 3,1)                            # cdb(ex-12.diff.300,diff1)
38    diff2  = reformat (diff2,12,1)                            # cdb(ex-12.diff.301,diff2)

39

40    diff30 = reformat (diff30,40,1)                           # cdb(ex-12.diff.302,diff30)
41    diff31 = reformat (diff31,360,1)                          # cdb(ex-12.diff.303,diff31)

42

43    diff3 := @(diff30) + @(diff31).                          # cdb(ex-12.diff.304,diff3)

44

45    diff := @(diff1) + @(diff2) + @(diff3).                  # cdb(ex-12.diff.305,diff)
```

$$
\texttt{ex-12.diff.100} := \frac{1}{12}x^b x^c z^d z^e \left(4\nabla_b R_{adce} + 2\nabla_d R_{abce} + \nabla_a R_{bdce}\right) + \frac{1}{40}x^b x^c x^d z^e z^f \left(4\nabla_{bc} R_{aedf} + 2\nabla_{be} R_{acdf} + 2\nabla_{eb} R_{acdf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right)
$$
$$
+ \frac{2}{45}g^{bc} x^d x^e x^f z^g z^h \left(4 R_{adbe} R_{cgfh} - 2 R_{agbd} R_{cefh} - R_{adbg} R_{cefh} + R_{abdg} R_{cefh}\right) - \frac{1}{12}x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5\nabla_b R_{adce}\right)
$$
$$
- \frac{1}{40}x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}\right) - \frac{1}{360}g^{bc} x^d x^e x^f z^g z^h \left(37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh}\right)
$$

$\texttt{ex-12.diff.200} := 0$

$$
\texttt{ex-12.diff.201} := -\frac{1}{12}x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12}x^b x^c z^d z^e \nabla_d R_{abce} + \frac{1}{12}x^b x^c z^d z^e \nabla_a R_{bdce}
$$

$$
\texttt{ex-12.diff.202} := -\frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}
$$
$$
+ \frac{3}{40}g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} + \frac{1}{72}g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh}
$$

$$
\texttt{ex-12.diff.203} := -\frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}
$$

$$
\texttt{ex-12.diff.204} := \frac{3}{40}g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} + \frac{1}{72}g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh}
$$

ex-12.diff.300 := 0

ex-12.diff.301 := $\dfrac{1}{12} x^b x^c z^d z^e \left(\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce} \right)$

ex-12.diff.302 := $\dfrac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf} \right)$

ex-12.diff.303 := $\dfrac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh} \right)$

ex-12.diff.304 := $\dfrac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf} \right)$

$\qquad + \dfrac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh} \right)$

ex-12.diff.305 := $\dfrac{1}{12} x^b x^c z^d z^e \left(\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce} \right) + \dfrac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf} \right)$

$\qquad + \dfrac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh} \right)$

Second order terms

```
1   diff2  = get_xterm (diff,2)
2   diff2 := 12 @(diff2).                                            # cdb (ex-12.701,diff2)
3   distribute (diff2)                                              # cdb (ex-12.702,diff2)
4
5   diff2 = add_tags (diff2,'\\mu')                                  # cdb (ex-12.711,diff2)
6
7   # swap indices on middle term, then apply 2nd Bianchi identity
8
9   zoom      (diff2, $\mu_{1} Q??$)                                # cdb (ex-12.712,diff2)
10  substitute (diff2, $\nabla_{b}{R_{a d c e}} -> - \nabla_{b}{R_{d a c e}}$)   # cdb (ex-12.713,diff2)
11  unzoom    (diff2)
12
13  substitute (diff2, $\mu_{1} -> \mu_{0}, \mu_{2} -> \mu_{0}$)     # cdb (ex-12.714,diff2)
14  substitute (diff2, $\mu_{0} -> 0$)                              # cdb (ex-12.715,diff2)
15
16  diff2 = clear_tags (diff2,'\\mu')                               # cdb (ex-12.716,diff2)
17
18  diff2 := @(diff2) / 12 .
19
20  diff := @(diff1) + @(diff2) + @(diff3).
21
22  diff  = reformat_diff (diff)                                    # cdb(ex-12.diff.306,diff)
```

$$\texttt{ex-12.701} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\texttt{ex-12.702} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\texttt{ex-12.711} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_2 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\texttt{ex-12.712} := \ldots - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \ldots$$

$$\texttt{ex-12.713} := \ldots + \mu_1 x^b x^c z^d z^e \nabla_b R_{dace} + \ldots$$

$$\texttt{ex-12.714} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} + \mu_0 x^b x^c z^d z^e \nabla_b R_{dace} + \mu_0 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\texttt{ex-12.715} := 0$$

$$\texttt{ex-12.716} := 0$$

$$\texttt{ex-12.diff.306} := \frac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf} \right)$$
$$+ \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh} \right)$$

Third order terms, commute $\nabla\nabla R$ terms

```
1   diff3   = get_xterm (diff,3)
2   diff3 := 360 @(diff3).                            # cdb (ex-12.801,diff3)
3   distribute (diff3)                                # cdb (ex-12.802,diff3)
4
5   # commutation rule for covariant derivs on Rabcd, see exrecise 3.6
6   # note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}
7   CommuteNablaRiemann := \nabla_{f e}(R_{a b c d}) -> \nabla_{e f}(R_{a b c d})
8                                                 + g^{u v} R_{u a e f} R_{v b c d}
9                                                 + g^{u v} R_{u b e f} R_{a v c d}
10                                                + g^{u v} R_{u c e f} R_{a b v d}
11                                                + g^{u v} R_{u d e f} R_{a b c v}.
12
13  diff3 = add_tags (diff3,'\\mu')                   # cdb (ex-12.901,diff3)
14
15  # commute derivs on Rabcd so that each double deriv is of the form \nabla_{b*}
16
17  substitute (diff3, $\mu_{3} -> \mu_{1}$)          # cdb (ex-12.902,diff3)
18
19  zoom       (diff3, $\mu_{1} Q??$)                 # cdb (ex-12.903,diff3)
20  substitute (diff3, CommuteNablaRiemann)          # cdb (ex-12.904,diff3)
21  unzoom     (diff3)
22
23  diff3 = clear_tags (diff3,'\\mu')
24  diff3 := @(diff3) / 360 .
25
26  distribute   (diff3)
27  canonicalise (diff3)                             # cdb (ex-12.905,diff3)
28
29  diff := @(diff1) + @(diff2) + @(diff3).
30
31  diff  = reformat_diff (diff)                      # cdb(ex-12.diff.307,diff)
```

$$\texttt{ex-12.801} := 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$
$$- 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\texttt{ex-12.802} := 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$
$$- 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\texttt{ex-12.901} := 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_3 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$
$$- 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\texttt{ex-12.902} := 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$
$$- 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\texttt{ex-12.903} := \ldots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \ldots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \ldots$$

$$\texttt{ex-12.904} := \ldots + 9\mu_1 x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + g^{uv} R_{uabe} R_{vcdf} + g^{uv} R_{ucbe} R_{avdf} + g^{uv} R_{udbe} R_{acvf} + g^{uv} R_{ufbe} R_{acdv} \right) + \ldots$$
$$+ 9\mu_1 x^b x^c x^d z^e z^f \left(\nabla_{ba} R_{cedf} + g^{uv} R_{ucba} R_{vedf} + g^{uv} R_{ueba} R_{cvdf} + g^{uv} R_{udba} R_{cevf} + g^{uv} R_{ufba} R_{cedv} \right) + \ldots$$

$$\texttt{ex-12.905} := \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abeu} R_{cfdv} - \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abcu} R_{defv} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$
$$- \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\texttt{ex-12.diff.307} := \frac{1}{40} x^b x^c x^d z^e z^f \left(2\nabla_{be} R_{acdf} - 2\nabla_{bc} R_{aedf} + 2\nabla_{ba} R_{cedf} \right) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 32 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh} \right)$$

Third order terms, use 2nd Bianchi identity on $\nabla\nabla R$ terms

```
1    diff3  = get_xterm (diff,3)
2    diff3 := 360 @(diff3).                                             # cdb (ex-12.910,diff3)
3    distribute (diff3)                                                 # cdb (ex-12.911,diff3)
4
5    diff3 = add_tags (diff3,'\\mu')                                    # cdb (ex-12.912,diff3)
6
7    # swap indices on middle second deriv term, then apply 2nd Bianchi identity
8
9    zoom       (diff3, $\mu_{1} Q??$)                                  # cdb (ex-12.913,diff3)
10   substitute (diff3, $\nabla_{b c}{R_{a e d f}} -> - \nabla_{b c}{R_{e a d f}}$)  # cdb (ex-12.914,diff3)
11   unzoom     (diff3)
12
13   substitute (diff3, $\mu_{1} -> \mu_{0}, \mu_{2} -> \mu_{0}$)       # cdb (ex-12.915,diff3)
14   substitute (diff3, $\mu_{0} -> 0$)                                 # cdb (ex-12.916,diff3)
15
16   diff3 = clear_tags (diff3,'\\mu')
17   diff3 := @(diff3) / 360 .
18
19   distribute   (diff3)
20   canonicalise (diff3)                                               # cdb (ex-12.917,diff3)
21
22   diff := @(diff1) + @(diff2) + @(diff3).
23
24   diff  = reformat_diff (diff)                                       # cdb(ex-12.diff.308,diff)
```

$\texttt{ex-12.910} := 18x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$
$\qquad - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$\texttt{ex-12.911} := 18x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$
$\qquad - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$\texttt{ex-12.912} := 18\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18\mu_2 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$
$\qquad - 32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$\texttt{ex-12.913} := \ldots - 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \ldots$

$\texttt{ex-12.914} := \ldots + 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + \ldots$

$\texttt{ex-12.915} := 18\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 18\mu_0 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + 18\mu_0 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$
$\qquad - 32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$\texttt{ex-12.916} := -32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$\texttt{ex-12.917} := -\dfrac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \dfrac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \dfrac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$$\texttt{ex-12.diff.308} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 32 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh} \right)$$

Third order terms, use 1st Bianchi identity on RR terms

```
1   diff3  = get_xterm (diff,3)
2   diff3 := 360 @(diff3).
3   distribute (diff3)
4
5   diff3 = add_tags (diff3,'\\mu')                                      # cdb (ex-12.921,diff3)
6
7   # swap indices on middle term, then apply 1st Bianchi identity
8
9   zoom       (diff3, $\mu_{1} Q??$)                                    # cdb (ex-12.922,diff3)
10  substitute (diff3, $R_{a d b g} R_{c e f h} -> - R_{a d g b} R_{c e f h}$)   # cdb (ex-12.923,diff3)
11  unzoom     (diff3)
12
13  substitute (diff3, $\mu_{1} -> \mu_{0}, \mu_{2} -> \mu_{0}$)          # cdb (ex-12.924,diff3)
14  substitute (diff3, $\mu_{0} -> 0$)                                   # cdb (ex-12.925,diff3)
15
16  diff3 = clear_tags (diff3,'\\mu')                                    # cdb (ex-12.926,diff3)
17
18  diff := @(diff1) + @(diff2) + @(diff3).
19
20  diff  = reformat_diff (diff)                                         # cdb(ex-12.diff.309,diff)
```

25

$\texttt{ex-12.921} := -32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_2 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$\texttt{ex-12.922} := \ldots + 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} + \ldots$

$\texttt{ex-12.923} := \ldots - 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} + \ldots$

$\texttt{ex-12.924} := -32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} - 32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} - 32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$

$\texttt{ex-12.925} := 0$

$\texttt{ex-12.926} := 0$

$$\texttt{ex-12.diff.309} := 0$$

Example 13a The Weyl tensor vanishes in 3d – direct proof

```
1    {x,y,z}::Coordinate.
2    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices (values={x,y,z},position=independent).
3
4    \partial{#}::PartialDerivative.
5
6    g_{a b}::Metric.
7    g^{a b}::InverseMetric.
8
9    {\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.
10
11   GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
12                                               + \partial_{c}{g_{b d}}
13                                               - \partial_{d}{g_{b c}}).  # cdb(Gamma.000,GammaU)
14
15   GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
16                                      + \partial_{c}{g_{b a}}
17                                      - \partial_{a}{g_{b c}}).          # cdb(Gamma.010,GammaD)
18
19   Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
20                           - \partial_{d}{\Gamma_{a b c}}
21                           + \Gamma_{e a d} \Gamma^{e}_{b c}
22                           - \Gamma_{e a c} \Gamma^{e}_{b d}.            # cdb (Rabcd.000,Rabcd)
23
24   Rab     := R_{a b} -> g^{c d} R_{a c b d}.                            # cdb (Rab.000,Rab)
25
26   Rscalar := R -> g^{a b} R_{a b}.                                      # cdb (R.000,Rscalar)
27
28   # Weyl in 3-dimensions
29
30   Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
31                        - (g_{a c} R_{b d} - g_{a d} R_{b c})
32                   + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).          # cdb (ex-13a.100,Cabcd)
33
34   # Use 8 Cabcd to clear the fractions
35
36   EightCabcd := 8 @(Cabcd).                                             # cdb (ex-13a.110,EightCabcd)
```

```
37
38    substitute      (Cabcd,Rscalar)
39    substitute      (Cabcd,Rab)
40    substitute      (Cabcd,Rabcd)
41    substitute      (Cabcd,GammaU)
42    substitute      (Cabcd,GammaD)
43
44    distribute      (Cabcd)
45
46    sort_product    (Cabcd)
47    rename_dummies  (Cabcd)
48    canonicalise    (Cabcd)                                          # cdb (ex-13a.101,Cabcd)
49
50    EightCabcd := 8 @(Cabcd).                                        # cdb (ex-13a.111,EightCabcd)
51
52    gab := {g_{x x} = gxx, g_{x y} = gxy, g_{x z} = gxz,
53            g_{y x} = gxy, g_{y y} = gyy, g_{y z} = gyz,
54            g_{z x} = gxz, g_{z y} = gyz, g_{z z} = gzz}.
55
56    complete  (gab, $g^{a b}$)
57    evaluate  (Cabcd,gab)                                           # cdb (ex-13a.102,Cabcd)
58    evaluate  (EightCabcd,gab)                                      # cdb (ex-13a.112,EightCabcd)
```

$$8C_{abcd} = 8R_{abcd} - 8R_{ac}g_{bd} + 8R_{ad}g_{bc} - 8g_{ac}R_{bd} + 8g_{ad}R_{bc} + 4R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \tag{ex-13a.110}$$

$$
\begin{aligned}
= {} & 4\partial_{bc}g_{ad} - 4\partial_{ac}g_{bd} - 4\partial_{bd}g_{ac} + 4\partial_{ad}g_{bc} + 2\partial_a g_{de}\partial_b g_{cf}g^{ef} + 2\partial_a g_{de}\partial_c g_{bf}g^{ef} - 2\partial_a g_{de}\partial_f g_{bc}g^{ef} + 2\partial_b g_{ce}\partial_d g_{af}g^{ef} + 2\partial_c g_{be}\partial_d g_{af}g^{ef} \\
& - 2\partial_d g_{ae}\partial_f g_{bc}g^{ef} - 2\partial_b g_{ce}\partial_f g_{ad}g^{ef} - 2\partial_c g_{be}\partial_f g_{ad}g^{ef} + 2\partial_e g_{ad}\partial_f g_{bc}g^{ef} - 2\partial_a g_{ce}\partial_b g_{df}g^{ef} - 2\partial_a g_{ce}\partial_d g_{bf}g^{ef} + 2\partial_a g_{ce}\partial_f g_{bd}g^{ef} - 2\partial_b g_{de}\partial_c g_{af}g^{ef} \\
& - 2\partial_c g_{ae}\partial_d g_{bf}g^{ef} + 2\partial_c g_{ae}\partial_f g_{bd}g^{ef} + 2\partial_b g_{de}\partial_f g_{ac}g^{ef} + 2\partial_d g_{be}\partial_f g_{ac}g^{ef} - 2\partial_e g_{ac}\partial_f g_{bd}g^{ef} - 4\partial_{ce}g_{af}g_{bd}g^{ef} + 4\partial_{ac}g_{ef}g_{bd}g^{ef} + 4\partial_{ef}g_{ac}g_{bd}g^{ef} \\
& - 4\partial_{ae}g_{cf}g_{bd}g^{ef} - 2\partial_a g_{ef}\partial_c g_{gh}g_{bd}g^{eg}g^{fh} - 4\partial_e g_{af}\partial_g g_{ch}g_{bd}g^{eg}g^{fh} + 4\partial_e g_{af}\partial_g g_{ch}g_{bd}g^{eh}g^{fg} + 4\partial_a g_{ce}\partial_f g_{gh}g_{bd}g^{eg}g^{fh} - 2\partial_a g_{ce}\partial_f g_{gh}g_{bd}g^{ef}g^{gh} \\
& + 4\partial_c g_{ae}\partial_f g_{gh}g_{bd}g^{eg}g^{fh} - 2\partial_c g_{ae}\partial_f g_{gh}g_{bd}g^{ef}g^{gh} - 4\partial_e g_{ac}\partial_f g_{gh}g_{bd}g^{eg}g^{fh} + 2\partial_e g_{ac}\partial_f g_{gh}g_{bd}g^{ef}g^{gh} + 4\partial_{de}g_{af}g_{bc}g^{ef} - 4\partial_{ad}g_{ef}g_{bc}g^{ef} \\
& - 4\partial_{ef}g_{ad}g_{bc}g^{ef} + 4\partial_{ae}g_{df}g_{bc}g^{ef} + 2\partial_a g_{ef}\partial_d g_{gh}g_{bc}g^{eg}g^{fh} + 4\partial_e g_{af}\partial_g g_{dh}g_{bc}g^{eg}g^{fh} - 4\partial_e g_{af}\partial_g g_{dh}g_{bc}g^{eh}g^{fg} - 4\partial_a g_{de}\partial_f g_{gh}g_{bc}g^{eg}g^{fh} \\
& + 2\partial_a g_{de}\partial_f g_{gh}g_{bc}g^{ef}g^{gh} - 4\partial_d g_{ae}\partial_f g_{gh}g_{bc}g^{eg}g^{fh} + 2\partial_d g_{ae}\partial_f g_{gh}g_{bc}g^{ef}g^{gh} + 4\partial_e g_{ad}\partial_f g_{gh}g_{bc}g^{eg}g^{fh} - 2\partial_e g_{ad}\partial_f g_{gh}g_{bc}g^{ef}g^{gh} - 4\partial_{de}g_{bf}g_{ac}g^{ef} \\
& + 4\partial_{bd}g_{ef}g_{ac}g^{ef} + 4\partial_{ef}g_{bd}g_{ac}g^{ef} - 4\partial_{be}g_{df}g_{ac}g^{ef} - 2\partial_b g_{ef}\partial_d g_{gh}g_{ac}g^{eg}g^{fh} - 4\partial_e g_{bf}\partial_g g_{dh}g_{ac}g^{eg}g^{fh} + 4\partial_e g_{bf}\partial_g g_{dh}g_{ac}g^{eh}g^{fg} + 4\partial_b g_{de}\partial_f g_{gh}g_{ac}g^{eg}g^{fh} \\
& - 2\partial_b g_{de}\partial_f g_{gh}g_{ac}g^{ef}g^{gh} + 4\partial_d g_{be}\partial_f g_{gh}g_{ac}g^{eg}g^{fh} - 2\partial_d g_{be}\partial_f g_{gh}g_{ac}g^{ef}g^{gh} - 4\partial_e g_{bd}\partial_f g_{gh}g_{ac}g^{eg}g^{fh} + 2\partial_e g_{bd}\partial_f g_{gh}g_{ac}g^{ef}g^{gh} + 4\partial_{ce}g_{bf}g_{ad}g^{ef} \\
& - 4\partial_{bc}g_{ef}g_{ad}g^{ef} - 4\partial_{ef}g_{bc}g_{ad}g^{ef} + 4\partial_{be}g_{cf}g_{ad}g^{ef} + 2\partial_b g_{ef}\partial_c g_{gh}g_{ad}g^{eg}g^{fh} + 4\partial_e g_{bf}\partial_g g_{ch}g_{ad}g^{eg}g^{fh} - 4\partial_e g_{bf}\partial_g g_{ch}g_{ad}g^{eh}g^{fg} - 4\partial_b g_{ce}\partial_f g_{gh}g_{ad}g^{eg}g^{fh} \\
& + 2\partial_b g_{ce}\partial_f g_{gh}g_{ad}g^{ef}g^{gh} - 4\partial_c g_{be}\partial_f g_{gh}g_{ad}g^{eg}g^{fh} + 2\partial_c g_{be}\partial_f g_{gh}g_{ad}g^{ef}g^{gh} + 4\partial_e g_{bc}\partial_f g_{gh}g_{ad}g^{eg}g^{fh} - 2\partial_e g_{bc}\partial_f g_{gh}g_{ad}g^{ef}g^{gh} + 4\partial_{ef}g_{gh}g_{ac}g_{bd}g^{eg}g^{fh} \\
& - 4\partial_{ef}g_{gh}g_{ad}g_{bc}g^{eg}g^{fh} - 4\partial_{ef}g_{gh}g_{ac}g_{bd}g^{ef}g^{gh} + 4\partial_{ef}g_{gh}g_{ad}g_{bc}g^{ef}g^{gh} - 2\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{ei}g^{fh}g^{gj} + 2\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{ei}g^{fh}g^{gj} \\
& + 3\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{eh}g^{fi}g^{gj} - 3\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{eh}g^{fi}g^{gj} - 4\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{ef}g^{gi}g^{hj} + 4\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{ef}g^{gi}g^{hj} \\
& + 4\partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{ef}g^{gh}g^{ij} - 4\partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{ef}g^{gh}g^{ij} - \partial_e g_{fg}\partial_h g_{ij}g_{ac}g_{bd}g^{eh}g^{fg}g^{ij} + \partial_e g_{fg}\partial_h g_{ij}g_{ad}g_{bc}g^{eh}g^{fg}g^{ij} \tag{ex-13a.111} \\
= {} & 0 \tag{ex-13a.112}
\end{aligned}
$$

Example 13b The Weyl tensor vanishes in 3d − orthonormal basis

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3   g_{a b}::Metric.
4   g^{a b}::InverseMetric.
5
6   R_{a b c d}::RiemannTensor.
7
8   ex{#}::LaTeXForm{"e_x"}.
9   ey{#}::LaTeXForm{"e_y"}.
10  ez{#}::LaTeXForm{"e_z"}.
11
12  {R_{a b c d}, g_{a b}, g^{a b}}::SortOrder.
13
14  Rab     := R_{a b} -> g^{c d} R_{a c b d}.
15
16  Rscalar := R -> g^{a b} R_{a b}.
17
18  gab := g^{a b} -> ex^{a} ex^{b} + ey^{a} ey^{b} + ez^{a} ez^{b}.
19
20  ortho := {ex^{a} ex^{b} g_{a b} -> 1, ey^{a} ey^{b} g_{a b} -> 1, ez^{a} ez^{b} g_{a b} -> 1,
21            ex^{a} ey^{b} g_{a b} -> 0, ex^{a} ez^{b} g_{a b} -> 0,
22            ey^{a} ex^{b} g_{a b} -> 0, ey^{a} ez^{b} g_{a b} -> 0,
23            ez^{a} ex^{b} g_{a b} -> 0, ez^{a} ey^{b} g_{a b} -> 0}.
24
25  # Weyl in 3-dimensions
26
27  Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
28                       - (g_{a c} R_{b d} - g_{a d} R_{b c})
29               + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).   # cdb (ex-13b.100,Cabcd)
30
31
32  substitute      (Cabcd, Rscalar)                              # cdb(ex-13b.101,Cabcd)
33  substitute      (Cabcd, Rab)                                  # cdb(ex-13b.102,Cabcd)
34  distribute      (Cabcd)                                       # cdb(ex-13b.103,Cabcd)
35
36  Cabcd := C_{a b c d} -> @(Cabcd).
```

```
37
38   Cxyxy := C_{a b c d} ex^{a} ey^{b} ex^{c} ey^{d}.                    # cdb(ex-13b.104,Cxyxy)
39
40   substitute      (Cxyxy,Cabcd)                                        # cdb(ex-13b.105,Cxyxy)
41   distribute      (Cxyxy)                                              # cdb(ex-13b.106,Cxyxy)
42
43   substitute      (Cxyxy, ortho, repeat=True)                         # cdb(ex-13b.107,Cxyxy)
44
45   substitute      (Cxyxy, gab)                                         # cdb(ex-13b.108,Cxyxy)
46   distribute      (Cxyxy)                                              # cdb(ex-13b.109,Cxyxy)
47
48   sort_product    (Cxyxy)                                              # cdb(ex-13b.110,Cxyxy)
49   rename_dummies  (Cxyxy)                                              # cdb(ex-13b.111,Cxyxy)
50   canonicalise    (Cxyxy)                                              # cdb(ex-13b.112,Cxyxy)
```

$$\texttt{ex-13b.101} := R_{abcd} - R_{ac}g_{bd} + R_{ad}g_{bc} - g_{ac}R_{bd} + g_{ad}R_{bc} + \frac{1}{2}g^{ef}R_{ef}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

$$\texttt{ex-13b.102} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

$$\texttt{ex-13b.103} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}$$

$$C_{abcd}e_x^a e_y^b e_x^c e_y^d = \left(R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc} \right) e_x^a e_y^b e_x^c e_y^d \quad \text{(ex-13b.105)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - g^{ef}R_{aecf}g_{bd}e_x^a e_y^b e_x^c e_y^d + g^{fe}R_{afde}g_{bc}e_x^a e_y^b e_x^c e_y^d - g_{ac}g^{fe}R_{bfde}e_x^a e_y^b e_x^c e_y^d + g_{ad}g^{ef}R_{becf}e_x^a e_y^b e_x^c e_y^d$$

$$+ \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd}e_x^a e_y^b e_x^c e_y^d - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}e_x^a e_y^b e_x^c e_y^d \quad \text{(ex-13b.106)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - g^{ef}R_{aecf}e_x^a e_x^c - g^{fe}R_{bfde}e_y^b e_y^d + \frac{1}{2}g^{ef}g^{gh}R_{egfh} \quad \text{(ex-13b.107)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - \left(e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f \right) R_{aecf}e_x^a e_x^c - \left(e_x^f e_x^e + e_y^f e_y^e + e_z^f e_z^e \right) R_{bfde}e_y^b e_y^d$$

$$+ \frac{1}{2}\left(e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f \right)\left(e_x^g e_x^h + e_y^g e_y^h + e_z^g e_z^h \right) R_{egfh} \quad \text{(ex-13b.108)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_y^d - e_x^e e_x^f R_{aecf}e_x^a e_x^c - e_y^e e_y^f R_{aecf}e_x^a e_x^c - e_z^e e_z^f R_{aecf}e_x^a e_x^c - e_x^f e_x^e R_{bfde}e_y^b e_y^d - e_y^f e_y^e R_{bfde}e_y^b e_y^d - e_z^f e_z^e R_{bfde}e_y^b e_y^d$$

$$+ \frac{1}{2}e_x^e e_x^f e_x^g e_x^h R_{egfh} + \frac{1}{2}e_x^e e_x^f e_y^g e_y^h R_{egfh} + \frac{1}{2}e_x^e e_x^f e_z^g e_z^h R_{egfh} + \frac{1}{2}e_y^e e_y^f e_x^g e_x^h R_{egfh} + \frac{1}{2}e_y^e e_y^f e_y^g e_y^h R_{egfh} + \frac{1}{2}e_y^e e_y^f e_z^g e_z^h R_{egfh} + \frac{1}{2}e_z^e e_z^f e_x^g e_x^h R_{egfh}$$

$$+ \frac{1}{2}e_z^e e_z^f e_y^g e_y^h R_{egfh} + \frac{1}{2}e_z^e e_z^f e_z^g e_z^h R_{egfh} \quad \text{(ex-13b.109)}$$

$$= R_{abcd}e_x^a e_x^c e_y^b e_y^d - R_{aecf}e_x^a e_x^c e_x^e e_x^f - R_{aecf}e_x^a e_x^c e_y^e e_y^f - R_{aecf}e_x^a e_x^c e_z^e e_z^f - R_{bfde}e_x^e e_x^f e_y^b e_y^d - R_{bfde}e_y^b e_y^d e_y^e e_y^f - R_{bfde}e_y^b e_y^d e_z^e e_z^f$$

$$+ \frac{1}{2}R_{egfh}e_x^e e_x^f e_x^g e_x^h + \frac{1}{2}R_{egfh}e_x^e e_x^f e_y^g e_y^h + \frac{1}{2}R_{egfh}e_x^e e_x^f e_z^g e_z^h + \frac{1}{2}R_{egfh}e_x^g e_x^h e_y^e e_y^f + \frac{1}{2}R_{egfh}e_y^e e_y^f e_y^g e_y^h + \frac{1}{2}R_{egfh}e_y^e e_y^f e_z^g e_z^h + \frac{1}{2}R_{egfh}e_x^g e_x^h e_z^e e_z^f$$

$$+ \frac{1}{2}R_{egfh}e_y^g e_y^h e_z^e e_z^f + \frac{1}{2}R_{egfh}e_z^e e_z^f e_z^g e_z^h \quad \text{(ex-13b.110)}$$

$$= \frac{1}{2}R_{abcd}e_x^a e_x^c e_y^b e_y^d - \frac{1}{2}R_{abcd}e_x^a e_x^c e_x^b e_x^d - \frac{1}{2}R_{abcd}e_x^a e_x^c e_z^b e_z^d - R_{abcd}e_x^d e_x^b e_y^a e_y^c - R_{abcd}e_y^a e_y^c e_y^d e_y^b - R_{abcd}e_y^a e_y^c e_z^e e_z^b + \frac{1}{2}R_{abcd}e_x^b e_x^d e_y^a e_y^c$$

$$+ \frac{1}{2}R_{abcd}e_y^a e_y^c e_y^b e_y^d + \frac{1}{2}R_{abcd}e_y^a e_y^c e_z^b e_z^d + \frac{1}{2}R_{abcd}e_x^b e_x^d e_z^a e_z^c + \frac{1}{2}R_{abcd}e_y^b e_y^d e_z^a e_z^c + \frac{1}{2}R_{abcd}e_z^a e_z^c e_z^b e_z^d \quad \text{(ex-13b.111)}$$

$$= 0 \quad \text{(ex-13b.112)}$$

7

Example 13c The Weyl tensor vanishes in 3d – orthonormal basis

```
1   Cxyxz := C_{a b c d} ex^{a} ey^{b} ex^{c} ez^{d}.              # cdb(ex-13c.101,Cxyxz)
2
3   substitute      (Cxyxz,Cabcd)                                  # cdb(ex-13c.102,Cxyxz)
4
5   distribute      (Cxyxz)                                        # cdb(ex-13c.103,Cxyxz)
6
7   substitute      (Cxyxz, ortho, repeat=True)                    # cdb(ex-13c.104,Cxyxz)
8
9   substitute      (Cxyxz, gab)                                   # cdb(ex-13c.105,Cxyxz)
10  distribute      (Cxyxz)                                        # cdb(ex-13c.106,Cxyxz)
11
12  sort_product    (Cxyxz)                                        # cdb(ex-13c.107,Cxyxz)
13  rename_dummies  (Cxyxz)                                        # cdb(ex-13c.108,Cxyxz)
14  canonicalise    (Cxyxz)                                        # cdb(ex-13c.109,Cxyxz)
```

$$C_{abcd}e^a_x e^b_y e^c_x e^d_z = \left(R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc} \right) e^a_x e^b_y e^c_x e^d_z \quad \text{(ex-13c.102)}$$

$$= R_{abcd}e^a_x e^b_y e^c_x e^d_z - g^{ef}R_{aecf}g_{bd}e^a_x e^b_y e^c_x e^d_z + g^{fe}R_{afde}g_{bc}e^a_x e^b_y e^c_x e^d_z - g_{ac}g^{fe}R_{bfde}e^a_x e^b_y e^c_x e^d_z + g_{ad}g^{ef}R_{becf}e^a_x e^b_y e^c_x e^d_z$$

$$+ \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd}e^a_x e^b_y e^c_x e^d_z - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}e^a_x e^b_y e^c_x e^d_z \quad \text{(ex-13c.103)}$$

$$= R_{abcd}e^a_x e^b_y e^c_x e^d_z - g^{fe}R_{bfde}e^b_y e^d_z \quad \text{(ex-13c.104)}$$

$$= R_{abcd}e^a_x e^b_y e^c_x e^d_z - \left(e^f_x e^e_x + e^f_y e^e_y + e^f_z e^e_z \right) R_{bfde}e^b_y e^d_z \quad \text{(ex-13c.105)}$$

$$= R_{abcd}e^a_x e^b_y e^c_x e^d_z - e^f_x e^e_x R_{bfde}e^b_y e^d_z - e^f_y e^e_y R_{bfde}e^b_y e^d_z - e^f_z e^e_z R_{bfde}e^b_y e^d_z \quad \text{(ex-13c.106)}$$

$$= R_{abcd}e^a_x e^c_x e^b_y e^d_z - R_{bfde}e^e_x e^f_x e^b_y e^d_z - R_{bfde}e^b_y e^e_y e^f_y e^d_z - R_{bfde}e^b_y e^d_z e^e_z e^f_z \quad \text{(ex-13c.107)}$$

$$= R_{abcd}e^a_x e^c_x e^b_y e^d_z - R_{abcd}e^d_x e^b_x e^a_y e^c_z - R_{abcd}e^a_y e^d_y e^b_y e^c_z - R_{abcd}e^a_y e^c_z e^d_z e^b_z \quad \text{(ex-13c.108)}$$

$$= 0 \quad \text{(ex-13c.109)}$$

Example 14 The Weyl tensor is conformally invariant

This example shows that the Weyl tensor is conformally invariant. That is, for a pair of metrics g and $\bar{g}$ related by a conformal transformation, $\bar{g}_{ab} = \phi g_{ab}$ then $\overline{C}^a_{bcd} = C^a_{bcd}$ or equally $\overline{C}_{abcd} = \phi C_{abcd}$.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices (position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.

GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                            + \partial_{c}{g_{b d}}
                                            - \partial_{d}{g_{b c}}).

GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
                                   + \partial_{c}{g_{b a}}
                                   - \partial_{a}{g_{b c}}).

Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
                        - \partial_{d}{\Gamma_{a b c}}
                        + \Gamma_{e a d} \Gamma^{e}_{b c}
                        - \Gamma_{e a c} \Gamma^{e}_{b d}.

Rab     := R_{a b} -> g^{c d} R_{a c b d}.

Rscalar := R -> g^{a b} R_{a b}.

# Weyl in 4-dimensions

Cabcd := R_{a b c d} - (1/2) (R_{a c} g_{b d} - R_{a d} g_{b c})
                     - (1/2) (g_{a c} R_{b d} - g_{a d} R_{b c})
                     + (R/6) (g_{a c} g_{b d} - g_{a d} g_{b c}).

{\partial_{a b}{\phi},\partial_{a}{\phi},\phi}::SortOrder.
{\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.
```

```
34
35    substitute  (Cabcd,Rscalar)
36    substitute  (Cabcd,Rab)
37    substitute  (Cabcd,Rabcd)
38    substitute  (Cabcd,GammaU)
39    substitute  (Cabcd,GammaD)
40
41    distribute      (Cabcd)
42
43    sort_product    (Cabcd)
44    rename_dummies  (Cabcd)
45    canonicalise    (Cabcd)
46
47    # this is the Weyl tensor on the base metric
48    baseC := @(Cabcd).
49
50    conformal := {g_{a b} -> \phi g_{a b}, g^{a b} -> (1/phi) g^{a b}}.
51
52    substitute      (Cabcd, conformal)
53    product_rule    (Cabcd)
54    distribute      (Cabcd)
55    product_rule    (Cabcd)
56    distribute      (Cabcd)
57
58    map_sympy       (Cabcd, "simplify")
59
60    sort_product    (Cabcd)
61    rename_dummies  (Cabcd)
62    canonicalise    (Cabcd)
63
64    # this is the Weyl tensor on the conformal metric
65    confC := @(Cabcd).
66
67    # their difference, should be zero
68    diff := @(confC) - \phi @(baseC).   # cdb (ex-14.diff.100,diff)
69
70    distribute      (diff)
71    sort_product    (diff)
```

```
72    rename_dummies (diff)
73    canonicalise   (diff)  # cdb (ex-14.diff.101,diff)

74
75    # this trick is not essential but it does reduce the number of terms in diff
76    substitute       (diff,$\partial_{a}{\partial_{b}{g_{c d}}} -> g_{c d b a}$)
77    substitute       (diff,$\partial_{a}{g_{b c}} -> 0$)
78    substitute       (diff,$g_{c d b a} -> \partial_{a}{\partial_{b}{g_{c d}}}$)  # cdb (ex-14.diff.102,diff)

79
80    # standard expressions in 4-d
81    substitute       (diff,$g_{a b} g^{a b} -> 4$,repeat=True)          # cdb (ex-14.diff.201,diff)
82    substitute       (diff,$g_{a b} g^{c b} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.202,diff)
83    substitute       (diff,$g_{b a} g^{b c} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.203,diff)
84    substitute       (diff,$g_{a}^{a} -> 4$,repeat=True)                # cdb (ex-14.diff.204,diff)
85    substitute       (diff,$g^{a}_{a} -> 4$,repeat=True)                # cdb (ex-14.diff.205,diff)
86    eliminate_kronecker (diff)                                         # cdb (ex-14.diff.206,diff)

87
88    # need a second round since the above block introduces new terms that match those just eliminated
89    substitute       (diff,$g_{a b} g^{a b} -> 4$,repeat=True)          # cdb (ex-14.diff.301,diff)
90    substitute       (diff,$g_{a b} g^{c b} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.302,diff)
91    substitute       (diff,$g_{b a} g^{b c} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.303,diff)
92    substitute       (diff,$g_{a}^{a} -> 4$,repeat=True)                # cdb (ex-14.diff.304,diff)
93    substitute       (diff,$g^{a}_{a} -> 4$,repeat=True)                # cdb (ex-14.diff.305,diff)
94    eliminate_kronecker (diff)                                         # cdb (ex-14.diff.306,diff)

95
96    sort_product   (diff)
97    rename_dummies (diff)
98    canonicalise   (diff)  # cdb (ex-14.diff.400,diff)

99
100   checkpoint.append (baseC)
101   checkpoint.append (confC)
```

$$\begin{aligned}
\Delta = {} & \frac{1}{2}\partial_{bc}\phi g_{ad} - \frac{1}{2}\partial_{ac}\phi g_{bd} - \frac{1}{2}\partial_{bd}\phi g_{ac} + \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_{df}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g^{ef} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_{cf}g^{ef} \\
& + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g^{ef} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_{df}g^{ef} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g^{ef} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_{ac}\phi g_{bd}g_{ef}g^{ef} + \frac{1}{2}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{ef}g^{gh} - \frac{1}{2}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_{ad}\phi g_{bc}g_{ef}g^{ef} - \frac{1}{2}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{ef}g^{gh} + \frac{1}{2}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} + \frac{1}{4}\partial_{bd}\phi g_{ac}g_{ef}g^{ef} \\
& - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{8}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} - \frac{1}{4}\partial_{bc}\phi g_{ad}g_{ef}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}h^{hj} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}h^{hj} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}h^{hj} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}h^{hj} + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} \\
& - \frac{1}{24}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij} + \frac{1}{24}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij} \tag{ex-14.diff.102}
\end{aligned}$$

$$
\begin{aligned}
\Delta = {} & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_{df}g^{ef} + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_{cf}g^{ef} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g^{ef} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g^{ef} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_{cf}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_{df}g^{ef} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g^{ef} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g^{ef} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g^{ef} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} \\
& - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} \\
& + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} \qquad\text{(ex-14.diff.201)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d{}^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc} \\
&+ \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c{}^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d{}^e \\
&- \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a{}^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c{}^e \\
&- \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_g{}^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bd}g_c{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g{}^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g{}^f g^{eg} \\
&+ \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a{}^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d{}^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_g{}^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d{}^g g^{ef} \\
&- \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bc}g_d{}^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g{}^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g{}^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b{}^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d{}^e \\
&- \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_g{}^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b{}^e g_d{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g{}^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g{}^f g^{eg} \\
&+ \frac{1}{4}\partial_{ce}\phi g_{ad}g_b{}^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c{}^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_g{}^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c{}^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b{}^e g_c{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g{}^f g^{eg} \\
&- \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g{}^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g{}^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i{}^h g^{eg}g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i{}^h g^{eg}g^{fi} \\
&+ \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i{}^h g^{ef}g^{gi} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i{}^h g^{ef}g^{gi} \tag{ex-14.diff.202}
\end{aligned}
$$

$$\Delta = -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d{}^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc}$$

$$+ \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c{}^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d{}^e$$

$$- \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a{}^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c{}^e$$

$$- \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f{}^g g_g{}^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bd}g_c{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g{}^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g{}^f g^{eg}$$

$$+ \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a{}^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d{}^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f{}^g g_g{}^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d{}^g g^{ef}$$

$$- \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bc}g_d{}^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g{}^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g{}^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b{}^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d{}^e$$

$$- \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f{}^g g_g{}^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b{}^e g_d{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g{}^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g{}^f g^{eg}$$

$$+ \frac{1}{4}\partial_{ce}\phi g_{ad}g_b{}^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c{}^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f{}^g g_g{}^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c{}^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b{}^e g_c{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g{}^f g^{eg}$$

$$- \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g{}^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g{}^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i{}^h g^{eg}g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i{}^h g^{eg}g^{fi}$$

$$+ \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h{}^i g_i{}^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h{}^i g_i{}^h g^{ef} \qquad\qquad (\texttt{ex-14.diff.203})$$

$$
\begin{aligned}
\Delta = {} & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d{}^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c{}^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d{}^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a{}^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c{}^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f{}^g g_g{}^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bd}g_c{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g{}^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g{}^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a{}^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d{}^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f{}^g g_g{}^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d{}^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bc}g_d{}^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g{}^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g{}^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b{}^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d{}^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f{}^g g_g{}^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b{}^e g_d{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g{}^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g{}^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b{}^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c{}^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f{}^g g_g{}^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c{}^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b{}^e g_c{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g{}^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g{}^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g{}^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i{}^h g^{eg}g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i{}^h g^{eg}g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h{}^i g_i{}^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h{}^i g_i{}^h g^{ef}
\end{aligned}
\tag{ex-14.diff.204}
$$

$$\Delta = -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d{}^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc}$$

$$+ \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c{}^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d{}^e$$

$$- \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a{}^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c{}^e$$

$$- \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f{}^g g_g{}^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bd}g_c{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g{}^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g{}^f g^{eg}$$

$$+ \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a{}^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d{}^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f{}^g g_g{}^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d{}^g g^{ef}$$

$$- \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a{}^e g_{bc}g_d{}^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g{}^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g{}^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b{}^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d{}^e$$

$$- \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f{}^g g_g{}^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d{}^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b{}^e g_d{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g{}^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g{}^f g^{eg}$$

$$+ \frac{1}{4}\partial_{ce}\phi g_{ad}g_b{}^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c{}^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f{}^g g_g{}^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c{}^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b{}^e g_c{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g{}^f g^{eg}$$

$$- \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g{}^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g{}^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g{}^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i{}^h g^{eg}g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i{}^h g^{eg}g^{fi}$$

$$+ \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h{}^i g_i{}^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h{}^i g_i{}^h g^{ef} \tag{ex-14.diff.205}$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}$$

$$+ \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} + \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} - \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$

$$- \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cg}g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ag}g_{bd}g^{eg} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f{}^f$$

$$- \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{dg}g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ag}g_{bc}g^{eg} - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{dg}g^{eg}$$

$$+ \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bg}g^{eg} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cg}g^{eg} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bg}g^{eg} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg}$$

$$- \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gi}g^{eg}g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gi}g^{eg}g^{fi} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h{}^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h{}^h g^{ef} \tag{ex-14.diff.206}$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} + \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} - \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cg}g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ag}g_{bd}g^{eg} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f{}^f$$
$$- \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{dg}g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ag}g_{bc}g^{eg} - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{dg}g^{eg}$$
$$+ \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bg}g^{eg} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cg}g^{eg} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bg}g^{eg} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg}$$
$$- \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gi}g^{eg}g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gi}g^{eg}g^{fi} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h{}^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h{}^h g^{ef} \tag{ex-14.diff.301}$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} + \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} - \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f{}^f$$
$$- \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc} - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e$$
$$+ \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg}$$
$$+ \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h{}^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h{}^h g^{ef} \tag{ex-14.diff.302}$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} + \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} - \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f{}^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f{}^f$$
$$- \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc} - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f{}^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e$$
$$+ \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f{}^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg}$$
$$+ \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h{}^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h{}^h g^{ef} \tag{ex-14.diff.303}$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$+ \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc}$$
$$- \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e + \frac{1}{4}\partial_{cb}\phi g_{ad} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e$$
$$+ \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} \tag{ex-14.diff.304}$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$+ \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c{}^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bd} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d{}^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a{}^e g_{bc}$$
$$- \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d{}^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b{}^e + \frac{1}{4}\partial_{cb}\phi g_{ad} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c{}^e - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b{}^e$$
$$+ \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g{}^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g{}^f g^{eg} \tag{ex-14.diff.305}$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{12}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg}$$
$$- \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} \tag{ex-14.diff.306}$$

$$\Delta = 0 \tag{ex-14.diff.400}$$

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Example 15 Verifying the BSSN equations

This is short example verifies two of the main equations in the Phys Rev D paper by Miguel Alcubierre, Bernd Brugmann etal. (Phys.Rev.D. (62) 044034 (2000)).

The code for the full set of BSSN equations can be found at `https://github.com/leo-brewin/adm-bssn-equations`

```
{a,b,c,d,e,f,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent,values={t,x,y,z}).
{t,x,y,z}::Coordinate.

\partial{#}::PartialDerivative.
D{#}::Derivative.
DBar{#}::Derivative.

N::Depends(t,x,y,z).

g_{a b}::Symmetric.
g^{a b}::Symmetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

g_{a b}::Depends(t,x,y,z).
g^{a b}::Depends(t,x,y,z).

gBar_{a b}::Symmetric.
gBar^{a b}::Symmetric.
gBar_{a}^{b}::KroneckerDelta.
gBar^{a}_{b}::KroneckerDelta.

gBar_{a b}::Depends(t,x,y,z).
gBar^{a b}::Depends(t,x,y,z).

trK::LaTeXForm("K").
detg::LaTeXForm("g").
ABar{#}::LaTeXForm("{\bar{A}}").
DBar{#}::LaTeXForm("{\bar{D}}").
```

15.1 Evolution equation for ϕ

```
1    phi      := \phi -> (1/12) \log(detg).
2    gdotK    := g^{i j} K_{i j} -> trK.
3    DdetgDt := \partial_{t}{detg} -> detg g^{i j} \partial_{t}{g_{i j}}.
4
5    DgijDt   := \partial_{t}{g_{i j}} -> -2 N K_{i j}.
6
7    dlog     := \partial_{a?}{\log(A?)} -> (1/A?)\partial_{a?}{A?}.
8    dexp     := \partial_{a?}{\exp(A?)} -> \exp(A?)\partial_{a?}{A?}.
9
10   dotphi  := \partial_{t}{\phi}.
11
12   substitute (dotphi, phi)            # cdb (ex-15-02.101,dotphi)
13   substitute (dotphi, dlog)           # cdb (ex-15-02.102,dotphi)
14   substitute (dotphi, DdetgDt)        # cdb (ex-15-02.103,dotphi)
15   substitute (dotphi, DgijDt)         # cdb (ex-15-02.104,dotphi)
16   substitute (dotphi, gdotK)          # cdb (ex-15-02.105,dotphi)
17   map_sympy  (dotphi, "simplify")     # cdb (ex-15-02.106,dotphi)
18
19   DphiDt := \partial_{t}{\phi} -> @(dotphi).
20
21   checkpoint.append (dotphi)
```

$$\frac{d\phi}{dt} = \frac{1}{12}\partial_t\left(\log\left(g\right)\right) \tag{ex-15-02.101}$$

$$= \frac{1}{12}g^{-1}\partial_t g \tag{ex-15-02.102}$$

$$= \frac{1}{12}g^{-1}gg^{ij}\partial_t g_{ij} \tag{ex-15-02.103}$$

$$= -\frac{1}{6}g^{-1}gg^{ij}NK_{ij} \tag{ex-15-02.104}$$

$$= -\frac{1}{6}g^{-1}gKN \tag{ex-15-02.105}$$

$$= -\frac{1}{6}KN \tag{ex-15-02.106}$$

15.2 Evolution equation for $\bar{g}_{ij}$

```
1   gBarij := gBar_{i j} -> \exp(-4\phi) g_{i j}.
2   Kij    := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.
3   A2ABar := \exp(-4\phi) A_{i j} -> ABar_{i j}.
4   ABar2A := ABar_{i j} -> \exp(-4\phi) A_{i j}.
5
6   dotgBarij := \partial_{t}{gBar_{i j}}.
7
8   substitute   (dotgBarij, gBarij)        # cdb (ex-15-03.101,dotgBarij)
9   product_rule (dotgBarij)                # cdb (ex-15-03.102,dotgBarij)
10  substitute   (dotgBarij, dexp)          # cdb (ex-15-03.103,dotgBarij)
11  substitute   (dotgBarij, DgijDt)        # cdb (ex-15-03.104,dotgBarij)
12  substitute   (dotgBarij, DphiDt)        # cdb (ex-15-03.105,dotgBarij)
13  substitute   (dotgBarij, Kij)           # cdb (ex-15-03.106,dotgBarij)
14  distribute   (dotgBarij)                # cdb (ex-15-03.107,dotgBarij)
15  map_sympy    (dotgBarij, "simplify")    # cdb (ex-15-03.108,dotgBarij)
16  substitute   (dotgBarij, A2ABar)        # cdb (ex-15-03.109,dotgBarij)
17
18  DgBarijDt := \partial_{t}{gBar_{i j}} -> @(dotgBarij).
19
20  checkpoint.append (dotgBarij)
```

$$\frac{d\bar{g}_{ij}}{dt} = \partial_t \left(\exp\left(-4\phi\right) g_{ij} \right) \tag{ex-15-03.101}$$

$$= \partial_t \left(\exp\left(-4\phi\right) \right) g_{ij} + \exp\left(-4\phi\right) \partial_t g_{ij} \tag{ex-15-03.102}$$

$$= -4\exp\left(-4\phi\right) \partial_t \phi g_{ij} + \exp\left(-4\phi\right) \partial_t g_{ij} \tag{ex-15-03.103}$$

$$= -4\exp\left(-4\phi\right) \partial_t \phi g_{ij} - 2\exp\left(-4\phi\right) N K_{ij} \tag{ex-15-03.104}$$

$$= \frac{2}{3} \exp\left(-4\phi\right) K N g_{ij} - 2\exp\left(-4\phi\right) N K_{ij} \tag{ex-15-03.105}$$

$$= \frac{2}{3} \exp\left(-4\phi\right) K N g_{ij} - 2\exp\left(-4\phi\right) N \left(A_{ij} + \frac{1}{3} g_{ij} K \right) \tag{ex-15-03.106}$$

$$= \frac{2}{3} \exp\left(-4\phi\right) K N g_{ij} - 2\exp\left(-4\phi\right) N A_{ij} - \frac{2}{3} \exp\left(-4\phi\right) N g_{ij} K \tag{ex-15-03.107}$$

$$= -2N \exp\left(-4\phi\right) A_{ij} \tag{ex-15-03.108}$$

$$= -2N \bar{A}_{ij} \tag{ex-15-03.109}$$