Example 9 The Gauss equation

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.

K_{a b}::Symmetric.
g^{a}_{b}::KroneckerDelta.

# define the projection operator

hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}.

# 3-covariant derivative obtained by projection on 4-covariant derivative

vpq:=v_{p q} -> h^{a}_{p} h^{b}_{q} \nabla_{b}{v_{a}}.

# compute 3-curvature by commutation of covariant derivatives

vpqr:= h^{a}_{p} h^{b}_{q} h^{c}_{r} ( \nabla_{c}{v_{a b}} - \nabla_{b}{v_{a c}} ).

substitute (vpq,hab)
substitute (vpqr,vpq)

distribute   (vpqr)
product_rule (vpqr)
distribute   (vpqr)
eliminate_kronecker (vpqr)

# standard substitutions

substitute (vpqr,$h^{a}_{b} n^{b} -> 0$)
substitute (vpqr,$h^{a}_{b} n_{a} -> 0$)
substitute (vpqr,$\nabla_{a}{g^{b}_{c}} -> 0$)
substitute (vpqr,$n^{a} \nabla_{b}{v_{a}} -> -v_{a} \nabla_{b}{n^{a}}$)
substitute (vpqr,$v_{a} \nabla_{b}{n^{a}} -> v_{p} h^{p}_{a}\nabla_{b}{n^{a}}$)
substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n_{q}} -> K_{a b}$)
substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n^{b}} -> K_{a}^{q}$)  # cdb(ex-09.095,vpqr)
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```
# tidy up

{v_{a},h^{a}_{b},K_{a}^{b},K_{a b},R^{a}_{b c d},\nabla_{a}{v_{b}}}::SortOrder.

sort_product    (vpqr)                                  # cdb(ex-09.096,vpqr)
rename_dummies (vpqr)                                   # cdb(ex-09.097,vpqr)
canonicalise    (vpqr)                                  # cdb(ex-09.098,vpqr)
factor_out      (vpqr,$h^{a?}_{b?}$)                    # cdb(ex-09.099,vpqr)
factor_out      (vpqr,$v_{a?}$)                         # cdb(ex-09.101,vpqr)

json.append (vpqr)
```

$$(D_rD_q - D_qD_r)v_p = h^e{}_p h^d{}_q h^c{}_r \nabla_c(\nabla_d v_e) - h^e{}_p K_{rq} n^d \nabla_d v_e + K_q{}^b K_{rp} v_b - h^d{}_p h^b{}_q h^e{}_r \nabla_b(\nabla_e v_d) + h^d{}_p K_{qr} n^e \nabla_e v_d - K_{qp} K_r{}^c v_c \quad \text{(ex-09.095)}$$

$$= h^c{}_r h^d{}_q h^e{}_p \nabla_c(\nabla_d v_e) - h^e{}_p K_{rq} \nabla_d v_e n^d + v_b K_q{}^b K_{rp} - h^b{}_q h^d{}_p h^e{}_r \nabla_b(\nabla_e v_d) + h^d{}_p K_{qr} \nabla_e v_d n^e - v_c K_r{}^c K_{qp} \quad \text{(ex-09.096)}$$

$$= h^a{}_r h^b{}_q h^c{}_p \nabla_a(\nabla_b v_c) - h^b{}_p K_{rq} \nabla_a v_b n^a + v_a K_q{}^a K_{rp} - h^a{}_q h^c{}_p h^b{}_r \nabla_a(\nabla_b v_c) + h^b{}_p K_{qr} \nabla_a v_b n^a - v_a K_r{}^a K_{qp} \quad \text{(ex-09.097)}$$

$$= h^a{}_p h^b{}_q h^c{}_r \nabla_c(\nabla_b v_a) + v_a K_q{}^a K_{pr} - h^a{}_p h^b{}_q h^c{}_r \nabla_b(\nabla_c v_a) - v_a K_r{}^a K_{pq} \quad \text{(ex-09.098)}$$

$$= v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} + h^a{}_p h^b{}_q h^c{}_r (\nabla_c(\nabla_b v_a) - \nabla_b(\nabla_c v_a)) \quad \text{(ex-09.099)}$$

$$= h^a{}_p h^b{}_q h^c{}_r (\nabla_c(\nabla_b v_a) - \nabla_b(\nabla_c v_a)) + v_a (K_q{}^a K_{pr} - K_r{}^a K_{pq}) \quad \text{(ex-09.101)}$$

```
R{#}::LaTeXForm("{{\strut}^g R}").

gRabcd := \nabla_{c}{\nabla_{b}{v_{a}}}
         -\nabla_{b}{\nabla_{c}{v_{a}}} -> R^{d}_{a b c} v_{d}.

substitute      (vpqr,gRabcd)                                  # cdb(ex-09.102,vpqr)
distribute      (vpqr)                                         # cdb(ex-09.103,vpqr)
substitute      (vpqr,$v_{a} -> h^{b}_{a} v_{b}$)              # cdb(ex-09.104,vpqr)
substitute      (vpqr,$h^{b}_{a} K_{c}^{a} -> K_{c}^{b}$)      # cdb(ex-09.105,vpqr)
sort_product    (vpqr)                                         # cdb(ex-09.106,vpqr)
rename_dummies  (vpqr)                                         # cdb(ex-09.107,vpqr)
canonicalise    (vpqr)                                         # cdb(ex-09.108,vpqr)
factor_out      (vpqr,$v_{a?}$)                                # cdb(ex-09.109,vpqr)
substitute      (vpqr,$v_{a}->1$)                              # cdb(ex-09.110,vpqr)
sort_product    (vpqr)                                         # cdb(ex-09.111,vpqr)

json.append (vpqr)
```

$$(D_r D_q - D_q D_r) v_p = h^a{}_p h^b{}_q h^c{}_r \left(\nabla_c \left(\nabla_b v_a\right) - \nabla_b \left(\nabla_c v_a\right)\right) + v_a \left(K_q{}^a K_{pr} - K_r{}^a K_{pq}\right) \tag{ex-09.101}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} v_d + v_a \left(K_q{}^a K_{pr} - K_r{}^a K_{pq}\right) \tag{ex-09.102}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} v_d + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} \tag{ex-09.103}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} h^e{}_d v_e + h^b{}_a v_b K_q{}^a K_{pr} - h^b{}_a v_b K_r{}^a K_{pq} \tag{ex-09.104}$$

$$= h^a{}_p h^b{}_q h^c{}_r{}^g R^d{}_{abc} h^e{}_d v_e + K_q{}^b v_b K_{pr} - K_r{}^b v_b K_{pq} \tag{ex-09.105}$$

$$= v_e h^a{}_p h^b{}_q h^c{}_r h^e{}_d{}^g R^d{}_{abc} + v_b K_q{}^b K_{pr} - v_b K_r{}^b K_{pq} \tag{ex-09.106}$$

$$= v_e h^b{}_p h^c{}_q h^d{}_r h^e{}_a{}^g R^a{}_{bcd} + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} \tag{ex-09.107}$$

$$= v_a h^b{}_p h^c{}_q h^d{}_r h^a{}_e{}^g R^e{}_{bcd} + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} \tag{ex-09.108}$$

$$= v_a \left(h^b{}_p h^c{}_q h^d{}_r h^a{}_e{}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq}\right) \tag{ex-09.109}$$

$${}^h R^a{}_{pqr} = h^b{}_p h^c{}_q h^d{}_r h^a{}_e{}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq} \tag{ex-09.110}$$

$${}^h R^a{}_{pqr} = h^a{}_e h^b{}_p h^c{}_q h^d{}_r{}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq} \tag{ex-09.111}$$