

Exercise 6.7 Killing vectors of the Schwarzschild spacetime

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1 {t, r, \theta, \varphi}::Coordinate.
2 {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4 ::Symbol.
5
6 \partial{#}::PartialDerivative.
7
8 g_{a b}::Metric.
9 g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
10
11 Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} ( \partial_{g}\{g_{b f}\}
12                                     + \partial_{f}\{g_{b g}\}
13                                     - \partial_{b}\{g_{f g}\} ).
14
15 deriv := \xi_{a ; b} -> \partial_{b}\{\xi_{a}\} - \Gamma^{c}_{a b} \xi_{c}.
16 lower := \xi_{a} -> g_{a b} \xi^{b}.
17
18 expr := \xi_{a ; b} + \xi_{b ; a}. # cdb(ex-0607.100,expr)
19
20 substitute (expr, deriv) # cdb(ex-0607.101,expr)
21 substitute (expr, lower) # cdb(ex-0607.102,expr)
22 substitute (expr, Gamma) # cdb(ex-0607.103,expr)
23 distribute (expr) # cdb(ex-0607.104,expr)
24 product_rule (expr) # cdb(ex-0607.105,expr)
25 canonicalise (expr) # cdb(ex-0607.106,expr)
26
27 # choose a vector
28
29 # Kvect := {\xi^{t} = 1}.
30 # Kvect := {\xi^{\varphi} = 1}.
31 Kvect := {\xi^{\theta} = \sin(\varphi), \xi^{\varphi} = \cos(\theta)/\sin(\theta) \cos(\varphi)}.
32 # Kvect := {\xi^{\theta} = \cos(\varphi), \xi^{\varphi} = - \cos(\theta)/\sin(\theta) \sin(\varphi)}.
33 # cdb(ex-0607.107,Kvect)
34
35 gab := { g_{t t} = -(1-2*m/r),
36          g_{r r} = 1/(1-(2*m/r)),

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37      g_{\theta\theta} = r**2,
38      g_{\varphi\varphi} = r**2 \sin(\theta)**2}. # cdb(ex-0607.108,gab)
39
40 complete (gab, $g^{a b}$) # cdb(ex-0607.109,gab)
41
42 evaluate (expr, gab+Kvect) # cdb(ex-0607.110,expr)

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$$[\xi^a] = [\xi^\theta = \sin \varphi, \xi^\varphi = \cos \theta (\sin \theta)^{-1} \cos \varphi] \quad (\text{ex-0607.107})$$

$$[g_{ab}] = [g_{tt} = -1 + 2mr^{-1}, g_{rr} = (1 - 2mr^{-1})^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin \theta)^2] \quad (\text{ex-0607.108})$$

$$[g_{ab}, g^{ab}] = [g_{tt} = -1 + 2mr^{-1}, g_{rr} = (1 - 2mr^{-1})^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin \theta)^2, g^{tt} = (2mr^{-1} - 1)^{-1}, g^{rr} = -2mr^{-1} + 1, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2(\sin \theta)^2)^{-1}] \quad (\text{ex-0607.109})$$

$$\xi_{a;b} + \xi_{b;a} = \partial_b \xi_a - \Gamma_{ab}^c \xi_c + \partial_a \xi_b - \Gamma_{ba}^c \xi_c \quad (\text{ex-0607.101})$$

$$= \partial_b (g_{ac} \xi^c) - \Gamma_{ab}^c g_{cd} \xi^d + \partial_a (g_{bc} \xi^c) - \Gamma_{ba}^c g_{cd} \xi^d \quad (\text{ex-0607.102})$$

$$= \partial_b (g_{ac} \xi^c) - \frac{1}{2} g^{ce} (\partial_b g_{ea} + \partial_a g_{eb} - \partial_e g_{ab}) g_{cd} \xi^d + \partial_a (g_{bc} \xi^c) - \frac{1}{2} g^{ce} (\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ba}) g_{cd} \xi^d \quad (\text{ex-0607.103})$$

$$= \partial_b (g_{ac} \xi^c) - g^{ce} \partial_b g_{ea} g_{cd} \xi^d - g^{ce} \partial_a g_{eb} g_{cd} \xi^d + \frac{1}{2} g^{ce} \partial_e g_{ab} g_{cd} \xi^d + \partial_a (g_{bc} \xi^c) + \frac{1}{2} g^{ce} \partial_e g_{ba} g_{cd} \xi^d \quad (\text{ex-0607.104})$$

$$= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{ce} \partial_b g_{ea} g_{cd} \xi^d - g^{ce} \partial_a g_{eb} g_{cd} \xi^d + \frac{1}{2} g^{ce} \partial_e g_{ab} g_{cd} \xi^d + \partial_a g_{bc} \xi^c + g_{bc} \partial_a \xi^c + \frac{1}{2} g^{ce} \partial_e g_{ba} g_{cd} \xi^d \quad (\text{ex-0607.105})$$

$$= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{cd} \partial_b g_{ac} g_{de} \xi^e - g^{cd} \partial_a g_{bc} g_{de} \xi^e + g^{cd} \partial_c g_{ab} g_{de} \xi^e + \partial_a g_{bc} \xi^c + g_{bc} \partial_a \xi^c \quad (\text{ex-0607.106})$$

$$= 0 \quad (\text{ex-0607.110})$$