

Example 3a The Riemann curvature tensor

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});
6
7 ::Symbol; # Suggested by Kasper as a way to make use of ; legal
8           # see https://cadabra.science/qa/473/is-this-legal-syntax
9           # this code works with and without this trick
10
11 # rules for the first two covariant derivs of V^a
12
13 deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
14         + \Gamma^{a}_{c b} V^{c}.      # cdb (ex-03.101,deriv1)
15
16 deriv2 := V^{a}_{; b ; c} -> \partial_{c}{V^{a}_{; b}}
17         + \Gamma^{a}_{d c} V^{d}_{; b}
18         - \Gamma^{d}_{b c} V^{a}_{; d}. # cdb (ex-03.102,deriv2)
19
20 substitute (deriv2,deriv1)      # cdb (ex-03.103, deriv2)
21
22 Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}. # cdb (ex-03.104, Vabc)
23
24 substitute (Vabc,deriv2)      # cdb (ex-03.105, Vabc)
25
26 distribute      (Vabc)      # cdb (ex-03.106, Vabc)
27 product_rule    (Vabc)      # cdb (ex-03.107, Vabc)
28
29 sort_product    (Vabc)      # cdb (ex-03.108, Vabc)
30 rename_dummies  (Vabc)      # cdb (ex-03.109, Vabc)
31 canonicalise    (Vabc)      # cdb (ex-03.110, Vabc)
32
33 sort_sum        (Vabc)      # cdb (ex-03.111, Vabc)
34 factor_out      (Vabc,$V^{a?}$) # cdb (ex-03.112, Vabc)
35
36 checkpoint.append (Vabc)
```

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37
38 # create rule for Riemann, export later (for use by lib/dgeom)
39
40 substitute (Vabc,$V^{a} -> -1$)           # cdb (ex-03.113, Vabc)
41                                           # note use of -1 to get correct
42                                           # signs when coupled with the rule
43                                           # for Rabcd (next statement)
44
45 Rabcd := R^{a}_{[d b c]} -> @(Vabc).       # cdb (ex-03.114, Rabcd) #
46
47 foo    := R^{a}_{[b c d]}.                 # cdb (ex-03.115, foo)
48 substitute (foo, Rabcd)                   # cdb (ex-03.116, foo)
49
50 # update rule to use nice indices
51
52 Rabcd := R^{a}_{[b c d]} -> @(foo).
53
54 checkpoint.append (Rabcd)

```

$$V^a{}_{;b} \rightarrow \partial_b V^a + \Gamma^a{}_{cb} V^c \quad (\text{ex-03.101})$$

$$V^a{}_{;b;c} \rightarrow \partial_c V^a{}_{;b} + \Gamma^a{}_{dc} V^d{}_{;b} - \Gamma^d{}_{bc} V^a{}_{;d} \quad (\text{ex-03.102})$$

$$V^a{}_{;b;c} \rightarrow \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.103})$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) - \partial_b (\partial_c V^a + \Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} (\partial_c V^d + \Gamma^d{}_{ec} V^e) + \Gamma^d{}_{cb} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.105})$$

$$= \partial_{cb} V^a + \partial_c (\Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b (\Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.106})$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.107})$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^e \Gamma^a{}_{dc} \Gamma^d{}_{eb} - \Gamma^d{}_{bc} \partial_d V^a - V^e \Gamma^a{}_{ed} \Gamma^d{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^e \Gamma^a{}_{db} \Gamma^d{}_{ec} + \Gamma^d{}_{cb} \partial_d V^a + V^e \Gamma^a{}_{ed} \Gamma^d{}_{cb} \quad (\text{ex-03.108})$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^d \Gamma^a{}_{ec} \Gamma^e{}_{db} - \Gamma^d{}_{bc} \partial_d V^a - V^d \Gamma^a{}_{de} \Gamma^e{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^d \Gamma^a{}_{eb} \Gamma^e{}_{dc} + \Gamma^d{}_{cb} \partial_d V^a + V^d \Gamma^a{}_{de} \Gamma^e{}_{cb} \quad (\text{ex-03.109})$$

$$= V^d \partial_c \Gamma^a{}_{bd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} \quad (\text{ex-03.110})$$

$$= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} \quad (\text{ex-03.111})$$

$$= V^d (\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} - \Gamma^a{}_{be} \Gamma^e{}_{cd} + \Gamma^a{}_{ce} \Gamma^e{}_{bd}) \quad (\text{ex-03.112})$$

$$= -R^a{}_{dbc} V^d \quad (\text{MTW})$$

$$R^a{}_{bcd} = -\partial_d \Gamma^a{}_{cb} + \partial_c \Gamma^a{}_{db} + \Gamma^a{}_{ce} \Gamma^e{}_{db} - \Gamma^a{}_{de} \Gamma^e{}_{cb}$$

Example 3b The Riemann curvature tensor

This differs from the above by not using the `::TableauSymmetry` property. It gives the same results as above but it does require a little bit more housekeeping.

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 ::Symbol; # Suggsted by Kasper as a way to make use of ; legal
6           # see https://cadabra.science/qa/473/is-this-legal-syntax
7           # this code works with and without this trick
8
9 # rules for the first two covariant derivs of V^a
10
11 deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
12         + \Gamma^{a}_{c b} V^{c}.      # cdb (ex-03.301,deriv1)
13
14 deriv2 := V^{a}_{; b ; c} -> \partial_{c}{V^{a}_{; b}}
15         + \Gamma^{a}_{d c} V^{d}_{; b}
16         - \Gamma^{d}_{b c} V^{a}_{; d}. # cdb (ex-03.302,deriv2)
17
18 substitute (deriv2,deriv1)      # cdb (ex-03.303, deriv2)
19
20 Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}. # cdb (ex-03.304, Vabc)
21
22 substitute (Vabc,deriv2)      # cdb (ex-03.305, Vabc)
23
24 distribute      (Vabc)      # cdb (ex-03.306, Vabc)
25 product_rule    (Vabc)      # cdb (ex-03.307, Vabc)
26
27 # -----
28 # trick to obtain a symmetric connection
29
30 G_{a b}::Symmetric.
31
32 substitute      (Vabc,$\Gamma^{a}_{b c} -> G^{a} G_{b c}$)
33 sort_product    (Vabc)      # cdb (ex-03.308, Vabc)
```

```

34 rename_dummies (Vabc) # cdb (ex-03.309, Vabc)
35 canonicalise (Vabc) # cdb (ex-03.310, Vabc)
36 substitute (Vabc,$G^{a} G_{b c} -> \Gamma^{a}_{b c}$,repeat=True)
37 # -----
38
39 sort_product (Vabc)
40 rename_dummies (Vabc)
41 canonicalise (Vabc)
42
43 sort_sum (Vabc) # cdb (ex-03.311, Vabc)
44 factor_out (Vabc,$V^{a?}$) # cdb (ex-03.312, Vabc)
45
46 checkpoint.append (Vabc)

```

$$V^a{}_{;b} \rightarrow \partial_b V^a + \Gamma^a{}_{cb} V^c \quad (\text{ex-03.301})$$

$$V^a{}_{;b;c} \rightarrow \partial_c V^a{}_{;b} + \Gamma^a{}_{dc} V^d{}_{;b} - \Gamma^d{}_{bc} V^a{}_{;d} \quad (\text{ex-03.302})$$

$$V^a{}_{;b;c} \rightarrow \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.303})$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c (\partial_b V^a + \Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} (\partial_b V^d + \Gamma^d{}_{eb} V^e) - \Gamma^d{}_{bc} (\partial_d V^a + \Gamma^a{}_{ed} V^e) - \partial_b (\partial_c V^a + \Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} (\partial_c V^d + \Gamma^d{}_{ec} V^e) + \Gamma^d{}_{cb} (\partial_d V^a + \Gamma^a{}_{ed} V^e) \quad (\text{ex-03.305})$$

$$= \partial_{cb} V^a + \partial_c (\Gamma^a{}_{db} V^d) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b (\Gamma^a{}_{dc} V^d) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.306})$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \quad (\text{ex-03.307})$$

$$= \partial_{cb} V^a + V^d \partial_c (G^a G_{db}) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b (G^a G_{dc}) - G^a G^d G_{db} G_{ec} V^e + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \quad (\text{ex-03.308})$$

$$= \partial_{cb} V^a + V^d \partial_c (G^a G_{db}) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b (G^a G_{dc}) - G^a G^d G_{db} G_{ec} V^e + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \quad (\text{ex-03.309})$$

$$= V^d \partial_c (G^a G_{bd}) + G^a G^d G_{be} G_{cd} V^e - V^d \partial_b (G^a G_{cd}) - G^a G^d G_{bd} G_{ce} V^e \quad (\text{ex-03.310})$$

$$= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} + V^d \Gamma^a{}_{bd} \Gamma^e{}_{ce} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} \quad (\text{ex-03.311})$$

$$= V^d (\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} + \Gamma^a{}_{bd} \Gamma^e{}_{ce} - \Gamma^a{}_{be} \Gamma^e{}_{cd}) \quad (\text{ex-03.312})$$

$$= -R^a{}_{dbc} V^d \quad (\text{MTW})$$