

## Exercise 3.7 Commutation of $\nabla$ on the Riemann tensor – direct computation

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 ::Symbol;
4
5 \partial{#}::PartialDerivative.
6
7 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
8
9 RabcdD := \partial_{c}{\Gamma_{a b d}}
10          - \partial_{d}{\Gamma_{a b c}}
11          + \Gamma_{e a d} \Gamma^{e}_{b c}
12          - \Gamma_{e a c} \Gamma^{e}_{b d} -> R_{a b c d}.          # cdb(Rabcd.010,RabcdD)
13
14 RabcdU := \partial_{c}{\Gamma^{a}_{b d}}
15          - \partial_{d}{\Gamma^{a}_{b c}}
16          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
17          - \Gamma^{e}_{b c} \Gamma^{a}_{d e} -> R^{a}_{b c d}.          # cdb(Rabcd.000,RabcdU)
18
19 d1Rabcd := R_{a b c d ; e} -> \partial_{e}{R_{a b c d}}
20          - \Gamma^{f}_{a e} R_{f b c d}
21          - \Gamma^{f}_{b e} R_{a f c d}
22          - \Gamma^{f}_{c e} R_{a b f d}
23          - \Gamma^{f}_{d e} R_{a b c f}.          # cdb(d1Rabcd.000,d1Rabcd)
24
25 d2Rabcd := R_{a b c d ; e ; f} -> \partial_{f}{R_{a b c d ; e}}
26          - \Gamma^{g}_{a f} R_{g b c d ; e}
27          - \Gamma^{g}_{b f} R_{a g c d ; e}
28          - \Gamma^{g}_{c f} R_{a b g d ; e}
29          - \Gamma^{g}_{d f} R_{a b c g ; e}
30          - \Gamma^{g}_{e f} R_{a b c d ; g}.          # cdb(d2Rabcd.000,d2Rabcd)
31
32 substitute (d2Rabcd,d1Rabcd)          # cdb (d2Rabcd.001, d2Rabcd)
33
34 expr := R_{a b c d ; e ; f} - R_{a b c d ; f ; e}.          # cdb (ex-0307.100, expr)
35
36 substitute (expr,d2Rabcd)          # cdb (ex-0307.101, expr)

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37
38 distribute      (expr)                      # cdb (ex-0307.102, expr)
39 product_rule    (expr)                      # cdb (ex-0307.103, expr)
40
41 sort_product    (expr)                      # cdb (ex-0307.104, expr)
42 rename_dummies  (expr)                      # cdb (ex-0307.105, expr)
43 canonicalise    (expr)                      # cdb (ex-0307.106, expr)
44 factor_out      (expr,$R_{a? b? c? d?}$)    # cdb (ex-0307.107, expr)
45
46 substitute      (expr,RabcdU)               # cdb (ex-0307.108, expr)
47 substitute      (expr,$R^{a}_{b c d} -> -R^{a}_{b d c}$) # cdb (ex-0307.109, expr)

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$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_f (\partial_e R_{abcd} - \Gamma_{ae}^g R_{gbcd} - \Gamma_{be}^g R_{agcd} - \Gamma_{ce}^g R_{abgd} - \Gamma_{de}^g R_{abcg}) - \Gamma_{af}^g (\partial_e R_{gbcd} - \Gamma_{ge}^h R_{hbcd} - \Gamma_{be}^h R_{ghcd} - \Gamma_{ce}^h R_{gbhd} - \Gamma_{de}^h R_{gbch}) \\
& - \Gamma_{bf}^g (\partial_e R_{agcd} - \Gamma_{ae}^h R_{hgcd} - \Gamma_{ge}^h R_{ahcd} - \Gamma_{ce}^h R_{aghd} - \Gamma_{de}^h R_{agch}) \\
& - \Gamma_{cf}^g (\partial_e R_{abgd} - \Gamma_{ae}^h R_{hbgd} - \Gamma_{be}^h R_{ahgd} - \Gamma_{ge}^h R_{abhd} - \Gamma_{de}^h R_{abgh}) \\
& - \Gamma_{df}^g (\partial_e R_{abcg} - \Gamma_{ae}^h R_{hbcg} - \Gamma_{be}^h R_{ahcg} - \Gamma_{ce}^h R_{abhg} - \Gamma_{ge}^h R_{abch}) \\
& - \Gamma_{ef}^g (\partial_g R_{abcd} - \Gamma_{ag}^h R_{hbcd} - \Gamma_{bg}^h R_{ahcd} - \Gamma_{cg}^h R_{abhd} - \Gamma_{dg}^h R_{abch}) - \partial_e (\partial_f R_{abcd} - \Gamma_{af}^g R_{gbcd} - \Gamma_{bf}^g R_{agcd} - \Gamma_{cf}^g R_{abgd} - \Gamma_{df}^g R_{abcg}) \\
& + \Gamma_{ae}^g (\partial_f R_{gbcd} - \Gamma_{gf}^h R_{hbcd} - \Gamma_{bf}^h R_{ghcd} - \Gamma_{cf}^h R_{gbhd} - \Gamma_{df}^h R_{gbch}) \\
& + \Gamma_{be}^g (\partial_f R_{agcd} - \Gamma_{af}^h R_{hgcd} - \Gamma_{gf}^h R_{ahcd} - \Gamma_{cf}^h R_{aghd} - \Gamma_{df}^h R_{agch}) \\
& + \Gamma_{ce}^g (\partial_f R_{abgd} - \Gamma_{af}^h R_{hbgd} - \Gamma_{bf}^h R_{ahgd} - \Gamma_{gf}^h R_{abhd} - \Gamma_{df}^h R_{abgh}) \\
& + \Gamma_{de}^g (\partial_f R_{abcg} - \Gamma_{af}^h R_{hbcg} - \Gamma_{bf}^h R_{ahcg} - \Gamma_{cf}^h R_{abhg} - \Gamma_{gf}^h R_{abch}) \\
& + \Gamma_{fe}^g (\partial_g R_{abcd} - \Gamma_{ag}^h R_{hbcd} - \Gamma_{bg}^h R_{ahcd} - \Gamma_{cg}^h R_{abhd} - \Gamma_{dg}^h R_{abch}) \quad (\text{ex-0307.101})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - \partial_f (\Gamma_{ae}^g R_{gbcd}) - \partial_f (\Gamma_{be}^g R_{agcd}) - \partial_f (\Gamma_{ce}^g R_{abgd}) - \partial_f (\Gamma_{de}^g R_{abcg}) - \Gamma_{af}^g \partial_e R_{gbcd} + \Gamma_{af}^g \Gamma_{ge}^h R_{hbcd} + \Gamma_{af}^g \Gamma_{be}^h R_{ghcd} \\
& + \Gamma_{af}^g \Gamma_{ce}^h R_{gbhd} + \Gamma_{af}^g \Gamma_{de}^h R_{gbch} - \Gamma_{bf}^g \partial_e R_{agcd} + \Gamma_{bf}^g \Gamma_{ae}^h R_{hgcd} + \Gamma_{bf}^g \Gamma_{ge}^h R_{ahcd} + \Gamma_{bf}^g \Gamma_{ce}^h R_{aghd} + \Gamma_{bf}^g \Gamma_{de}^h R_{agch} \\
& - \Gamma_{cf}^g \partial_e R_{abgd} + \Gamma_{cf}^g \Gamma_{ae}^h R_{hbgd} + \Gamma_{cf}^g \Gamma_{be}^h R_{ahgd} + \Gamma_{cf}^g \Gamma_{ge}^h R_{abhd} + \Gamma_{cf}^g \Gamma_{de}^h R_{abgh} - \Gamma_{df}^g \partial_e R_{abcg} + \Gamma_{df}^g \Gamma_{ae}^h R_{hbcg} \\
& + \Gamma_{df}^g \Gamma_{be}^h R_{ahcg} + \Gamma_{df}^g \Gamma_{ce}^h R_{abhg} + \Gamma_{df}^g \Gamma_{ge}^h R_{abch} - \Gamma_{ef}^g \partial_g R_{abcd} + \Gamma_{ef}^g \Gamma_{ag}^h R_{hbcd} + \Gamma_{ef}^g \Gamma_{bg}^h R_{ahcd} + \Gamma_{ef}^g \Gamma_{cg}^h R_{abhd} \\
& + \Gamma_{ef}^g \Gamma_{dg}^h R_{abch} - \partial_{ef} R_{abcd} + \partial_e (\Gamma_{af}^g R_{gbcd}) + \partial_e (\Gamma_{bf}^g R_{agcd}) + \partial_e (\Gamma_{cf}^g R_{abgd}) + \partial_e (\Gamma_{df}^g R_{abcg}) + \Gamma_{ae}^g \partial_f R_{gbcd} - \Gamma_{ae}^g \Gamma_{gf}^h R_{hbcd} \\
& - \Gamma_{ae}^g \Gamma_{bf}^h R_{ghcd} - \Gamma_{ae}^g \Gamma_{cf}^h R_{gbhd} - \Gamma_{ae}^g \Gamma_{df}^h R_{gbch} + \Gamma_{be}^g \partial_f R_{agcd} - \Gamma_{be}^g \Gamma_{af}^h R_{hgcd} - \Gamma_{be}^g \Gamma_{gf}^h R_{ahcd} - \Gamma_{be}^g \Gamma_{cf}^h R_{aghd} \\
& - \Gamma_{be}^g \Gamma_{df}^h R_{agch} + \Gamma_{ce}^g \partial_f R_{abgd} - \Gamma_{ce}^g \Gamma_{af}^h R_{hbgd} - \Gamma_{ce}^g \Gamma_{bf}^h R_{ahgd} - \Gamma_{ce}^g \Gamma_{gf}^h R_{abhd} - \Gamma_{ce}^g \Gamma_{df}^h R_{abgh} + \Gamma_{de}^g \partial_f R_{abcg} \\
& - \Gamma_{de}^g \Gamma_{af}^h R_{hbcg} - \Gamma_{de}^g \Gamma_{bf}^h R_{ahcg} - \Gamma_{de}^g \Gamma_{cf}^h R_{abhg} - \Gamma_{de}^g \Gamma_{gf}^h R_{abch} + \Gamma_{fe}^g \partial_g R_{abcd} - \Gamma_{fe}^g \Gamma_{ag}^h R_{hbcd} - \Gamma_{fe}^g \Gamma_{bg}^h R_{ahcd} \\
& - \Gamma_{fe}^g \Gamma_{cg}^h R_{abhd} - \Gamma_{fe}^g \Gamma_{dg}^h R_{abch} \quad (\text{ex-0307.102})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - \partial_f \Gamma^g_{ae} R_{gbcd} - \partial_f \Gamma^g_{be} R_{agcd} - \partial_f \Gamma^g_{ce} R_{abgd} - \partial_f \Gamma^g_{de} R_{abcg} + \Gamma^g_{af} \Gamma^h_{ge} R_{hbcd} + \Gamma^g_{af} \Gamma^h_{be} R_{ghcd} + \Gamma^g_{af} \Gamma^h_{ce} R_{gbhd} \\
& + \Gamma^g_{af} \Gamma^h_{de} R_{gbch} + \Gamma^g_{bf} \Gamma^h_{ae} R_{hgcd} + \Gamma^g_{bf} \Gamma^h_{ge} R_{ahcd} + \Gamma^g_{bf} \Gamma^h_{ce} R_{aghd} + \Gamma^g_{bf} \Gamma^h_{de} R_{agch} + \Gamma^g_{cf} \Gamma^h_{ae} R_{hbcd} + \Gamma^g_{cf} \Gamma^h_{be} R_{ahgd} \\
& + \Gamma^g_{cf} \Gamma^h_{ge} R_{abhd} + \Gamma^g_{cf} \Gamma^h_{de} R_{abgh} + \Gamma^g_{df} \Gamma^h_{ae} R_{hbcd} + \Gamma^g_{df} \Gamma^h_{be} R_{ahcg} + \Gamma^g_{df} \Gamma^h_{ce} R_{abhg} + \Gamma^g_{df} \Gamma^h_{ge} R_{abch} - \Gamma^g_{ef} \partial_g R_{abcd} \\
& + \Gamma^g_{ef} \Gamma^h_{ag} R_{hbcd} + \Gamma^g_{ef} \Gamma^h_{bg} R_{ahcd} + \Gamma^g_{ef} \Gamma^h_{cg} R_{abhd} + \Gamma^g_{ef} \Gamma^h_{dg} R_{abch} - \partial_{ef} R_{abcd} + \partial_e \Gamma^g_{af} R_{gbcd} + \partial_e \Gamma^g_{bf} R_{agcd} + \partial_e \Gamma^g_{cf} R_{abgd} \\
& + \partial_e \Gamma^g_{df} R_{abcg} - \Gamma^g_{ae} \Gamma^h_{gf} R_{hbcd} - \Gamma^g_{ae} \Gamma^h_{bf} R_{ghcd} - \Gamma^g_{ae} \Gamma^h_{cf} R_{gbhd} - \Gamma^g_{ae} \Gamma^h_{df} R_{gbch} - \Gamma^g_{be} \Gamma^h_{af} R_{hgcd} - \Gamma^g_{be} \Gamma^h_{gf} R_{ahcd} \\
& - \Gamma^g_{be} \Gamma^h_{cf} R_{aghd} - \Gamma^g_{be} \Gamma^h_{df} R_{agch} - \Gamma^g_{ce} \Gamma^h_{af} R_{hbcd} - \Gamma^g_{ce} \Gamma^h_{bf} R_{ahgd} - \Gamma^g_{ce} \Gamma^h_{gf} R_{abhd} - \Gamma^g_{ce} \Gamma^h_{df} R_{abgh} - \Gamma^g_{de} \Gamma^h_{af} R_{hbcd} \\
& - \Gamma^g_{de} \Gamma^h_{bf} R_{ahcg} - \Gamma^g_{de} \Gamma^h_{cf} R_{abhg} - \Gamma^g_{de} \Gamma^h_{gf} R_{abch} + \Gamma^g_{fe} \partial_g R_{abcd} - \Gamma^g_{fe} \Gamma^h_{ag} R_{hbcd} - \Gamma^g_{fe} \Gamma^h_{bg} R_{ahcd} - \Gamma^g_{fe} \Gamma^h_{cg} R_{abhd} \\
& - \Gamma^g_{fe} \Gamma^h_{dg} R_{abch}
\end{aligned} \tag{ex-0307.103}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma^g_{ae} - R_{agcd} \partial_f \Gamma^g_{be} - R_{abgd} \partial_f \Gamma^g_{ce} - R_{abcg} \partial_f \Gamma^g_{de} + R_{hbcd} \Gamma^g_{af} \Gamma^h_{ge} + R_{ghcd} \Gamma^g_{af} \Gamma^h_{be} + R_{gbhd} \Gamma^g_{af} \Gamma^h_{ce} \\
& + R_{gbch} \Gamma^g_{af} \Gamma^h_{de} + R_{hgcd} \Gamma^g_{bf} \Gamma^h_{ae} + R_{ahcd} \Gamma^g_{bf} \Gamma^h_{ge} + R_{aghd} \Gamma^g_{bf} \Gamma^h_{ce} + R_{agch} \Gamma^g_{bf} \Gamma^h_{de} + R_{hbcd} \Gamma^g_{cf} \Gamma^h_{ae} + R_{ahgd} \Gamma^g_{cf} \Gamma^h_{be} \\
& + R_{abhd} \Gamma^g_{cf} \Gamma^h_{ge} + R_{abgh} \Gamma^g_{cf} \Gamma^h_{de} + R_{hbcd} \Gamma^g_{df} \Gamma^h_{ae} + R_{ahcg} \Gamma^g_{df} \Gamma^h_{be} + R_{abhg} \Gamma^g_{df} \Gamma^h_{ce} + R_{abch} \Gamma^g_{df} \Gamma^h_{ge} - \Gamma^g_{ef} \partial_g R_{abcd} \\
& + R_{hbcd} \Gamma^g_{ef} \Gamma^h_{ag} + R_{ahcd} \Gamma^g_{ef} \Gamma^h_{bg} + R_{abhd} \Gamma^g_{ef} \Gamma^h_{cg} + R_{abch} \Gamma^g_{ef} \Gamma^h_{dg} - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma^g_{af} + R_{agcd} \partial_e \Gamma^g_{bf} + R_{abgd} \partial_e \Gamma^g_{cf} \\
& + R_{abcg} \partial_e \Gamma^g_{df} - R_{hbcd} \Gamma^g_{ae} \Gamma^h_{gf} - R_{ghcd} \Gamma^g_{ae} \Gamma^h_{bf} - R_{gbhd} \Gamma^g_{ae} \Gamma^h_{cf} - R_{gbch} \Gamma^g_{ae} \Gamma^h_{df} - R_{hgcd} \Gamma^g_{be} \Gamma^h_{af} - R_{ahcd} \Gamma^g_{be} \Gamma^h_{gf} \\
& - R_{aghd} \Gamma^g_{be} \Gamma^h_{cf} - R_{agch} \Gamma^g_{be} \Gamma^h_{df} - R_{hbcd} \Gamma^g_{ce} \Gamma^h_{af} - R_{ahgd} \Gamma^g_{ce} \Gamma^h_{bf} - R_{abhd} \Gamma^g_{ce} \Gamma^h_{gf} - R_{abgh} \Gamma^g_{ce} \Gamma^h_{df} - R_{hbcd} \Gamma^g_{de} \Gamma^h_{af} \\
& - R_{ahcg} \Gamma^g_{de} \Gamma^h_{bf} - R_{abhg} \Gamma^g_{de} \Gamma^h_{cf} - R_{abch} \Gamma^g_{de} \Gamma^h_{gf} + \Gamma^g_{fe} \partial_g R_{abcd} - R_{hbcd} \Gamma^g_{fe} \Gamma^h_{ag} - R_{ahcd} \Gamma^g_{fe} \Gamma^h_{bg} - R_{abhd} \Gamma^g_{fe} \Gamma^h_{cg} \\
& - R_{abch} \Gamma^g_{fe} \Gamma^h_{dg}
\end{aligned} \tag{ex-0307.104}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma^g_{ae} - R_{agcd} \partial_f \Gamma^g_{be} - R_{abgd} \partial_f \Gamma^g_{ce} - R_{abcg} \partial_f \Gamma^g_{de} + R_{gbcd} \Gamma^h_{af} \Gamma^g_{he} + R_{ghcd} \Gamma^h_{af} \Gamma^g_{be} + R_{gbhd} \Gamma^h_{af} \Gamma^g_{ce} \\
& + R_{gbch} \Gamma^h_{af} \Gamma^g_{de} + R_{ghcd} \Gamma^h_{bf} \Gamma^g_{ae} + R_{agcd} \Gamma^h_{bf} \Gamma^g_{he} + R_{aghd} \Gamma^h_{bf} \Gamma^g_{ce} + R_{agch} \Gamma^h_{bf} \Gamma^g_{de} + R_{gbhd} \Gamma^h_{cf} \Gamma^g_{ae} + R_{aghd} \Gamma^h_{cf} \Gamma^g_{be} \\
& + R_{abgd} \Gamma^h_{cf} \Gamma^g_{he} + R_{abgh} \Gamma^h_{cf} \Gamma^g_{de} + R_{gbch} \Gamma^h_{df} \Gamma^g_{ae} + R_{agch} \Gamma^h_{df} \Gamma^g_{be} + R_{abgh} \Gamma^h_{df} \Gamma^g_{ce} + R_{abcg} \Gamma^h_{df} \Gamma^g_{he} - \Gamma^g_{ef} \partial_g R_{abcd} \\
& + R_{gbcd} \Gamma^h_{ef} \Gamma^g_{ah} + R_{agcd} \Gamma^h_{ef} \Gamma^g_{bh} + R_{abgd} \Gamma^h_{ef} \Gamma^g_{ch} + R_{abcg} \Gamma^h_{ef} \Gamma^g_{dh} - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma^g_{af} + R_{agcd} \partial_e \Gamma^g_{bf} + R_{abgd} \partial_e \Gamma^g_{cf} \\
& + R_{abcg} \partial_e \Gamma^g_{df} - R_{gbcd} \Gamma^h_{ae} \Gamma^g_{hf} - R_{ghcd} \Gamma^h_{ae} \Gamma^g_{bf} - R_{gbhd} \Gamma^h_{ae} \Gamma^g_{cf} - R_{gbch} \Gamma^h_{ae} \Gamma^g_{df} - R_{ghcd} \Gamma^h_{be} \Gamma^g_{af} - R_{agcd} \Gamma^h_{be} \Gamma^g_{hf} \\
& - R_{aghd} \Gamma^h_{be} \Gamma^g_{cf} - R_{agch} \Gamma^h_{be} \Gamma^g_{df} - R_{gbhd} \Gamma^h_{ce} \Gamma^g_{af} - R_{aghd} \Gamma^h_{ce} \Gamma^g_{bf} - R_{abgd} \Gamma^h_{ce} \Gamma^g_{hf} - R_{abgh} \Gamma^h_{ce} \Gamma^g_{df} - R_{gbch} \Gamma^h_{de} \Gamma^g_{af} \\
& - R_{agch} \Gamma^h_{de} \Gamma^g_{bf} - R_{abgh} \Gamma^h_{de} \Gamma^g_{cf} - R_{abcg} \Gamma^h_{de} \Gamma^g_{hf} + \Gamma^g_{fe} \partial_g R_{abcd} - R_{gbcd} \Gamma^h_{fe} \Gamma^g_{ah} - R_{agcd} \Gamma^h_{fe} \Gamma^g_{bh} - R_{abgd} \Gamma^h_{fe} \Gamma^g_{ch} \\
& - R_{abcg} \Gamma^h_{fe} \Gamma^g_{dh}
\end{aligned} \tag{ex-0307.105}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & -R_{gbcd}\partial_f\Gamma^g_{ae} - R_{agcd}\partial_f\Gamma^g_{be} - R_{abgd}\partial_f\Gamma^g_{ce} - R_{abcg}\partial_f\Gamma^g_{de} + R_{gbcd}\Gamma^h_{af}\Gamma^g_{eh} + R_{agcd}\Gamma^h_{bf}\Gamma^g_{eh} + R_{abgd}\Gamma^h_{cf}\Gamma^g_{eh} + R_{abcg}\Gamma^h_{df}\Gamma^g_{eh} \\
& + R_{gbcd}\partial_e\Gamma^g_{af} + R_{agcd}\partial_e\Gamma^g_{bf} + R_{abgd}\partial_e\Gamma^g_{cf} + R_{abcg}\partial_e\Gamma^g_{df} - R_{gbcd}\Gamma^h_{ae}\Gamma^g_{fh} - R_{agcd}\Gamma^h_{be}\Gamma^g_{fh} - R_{abgd}\Gamma^h_{ce}\Gamma^g_{fh} \\
& - R_{abcg}\Gamma^h_{de}\Gamma^g_{fh}
\end{aligned} \tag{ex-0307.106}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & R_{gbcd}(-\partial_f\Gamma^g_{ae} + \Gamma^h_{af}\Gamma^g_{eh} + \partial_e\Gamma^g_{af} - \Gamma^h_{ae}\Gamma^g_{fh}) + R_{agcd}(-\partial_f\Gamma^g_{be} + \Gamma^h_{bf}\Gamma^g_{eh} + \partial_e\Gamma^g_{bf} - \Gamma^h_{be}\Gamma^g_{fh}) \\
& + R_{abgd}(-\partial_f\Gamma^g_{ce} + \Gamma^h_{cf}\Gamma^g_{eh} + \partial_e\Gamma^g_{cf} - \Gamma^h_{ce}\Gamma^g_{fh}) + R_{abcg}(-\partial_f\Gamma^g_{de} + \Gamma^h_{df}\Gamma^g_{eh} + \partial_e\Gamma^g_{df} - \Gamma^h_{de}\Gamma^g_{fh})
\end{aligned} \tag{ex-0307.107}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd}R^g_{afe} - R_{agcd}R^g_{bfe} - R_{abgd}R^g_{cfe} - R_{abcg}R^g_{dfe} \tag{ex-0307.108}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd}R^g_{aef} + R_{agcd}R^g_{bef} + R_{abgd}R^g_{cef} + R_{abcg}R^g_{def} \tag{ex-0307.109}$$