Exercise 6.7 Killing vectors of the Schwarzschild spacetime

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{t, r, \theta, \varphi}::Coordinate.
     {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
     ;::Symbol.
     \partial{#}::PartialDerivative.
     g_{a b}::Metric.
     g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
10
     Gamma := Gamma^{a}_{f g} \rightarrow 1/2 g^{a b} ( partial_{g}_{g_b f})
11
                                                      + \partial_{f}{g_{b g}}
12
                                                      - \partial_{b}{g_{f g}} ).
13
14
     deriv := \xi_{a ; b} -> \partial_{b}{\xi_{a}} - \Gamma^{c}_{a b} \xi_{c}.
15
     lower := xi_{a} \rightarrow g_{a b} xi_{b}.
16
17
     expr := xi_{a ; b} + xi_{b ; a}.
                                                                  # cdb(ex-0607.100,expr)
19
     substitute (expr, deriv)
                                                                  # cdb(ex-0607.101,expr)
     substitute (expr, lower)
                                                                  # cdb(ex-0607.102,expr)
21
                                                                  # cdb(ex-0607.103,expr)
     substitute (expr, Gamma)
22
     distribute (expr)
                                                                  # cdb(ex-0607.104,expr)
     product_rule (expr)
                                                                  # cdb(ex-0607.105,expr)
                                                                  # cdb(ex-0607.106,expr)
     canonicalise (expr)
26
     # choose a vector
27
28
     # Kvect := {\langle xi^{t} \rangle = 1 \rangle}.
29
     # Kvect := {\langle xi^{\langle varphi \rangle} = 1 \rangle}.
     Kvect := \{ xi^{\theta} = \sin(\alpha), xi^{\phi} = \cos(\theta) / \sin(\theta) \}.
31
     # Kvect := {\langle xi^{\hat{t}} = \langle cos(\langle rheta) = \langle varphi \rangle, \langle rheta = \langle varphi \rangle = - \langle cos(\langle rheta) \rangle = - \langle varphi \rangle \}.
32
                                                                   # cdb(ex-0607.107, Kvect)
33
34
     gab := \{ g_{t} t \}
                                     = -(1-2*m/r),
35
                g_{r r}
                                     = 1/(1-(2*m/r)),
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```
g_{\theta\theta} = r**2,
g_{\varphi\varphi} = r**2 \sin(\theta)**2}. # cdb(ex-0607.108,gab)

complete (gab, $g^{a b}$) # cdb(ex-0607.109,gab)

evaluate (expr, join (gab,Kvect)) # cdb(ex-0607.110,expr)
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$$\begin{split} [\xi^a] &= \left[\xi^\theta = \sin \varphi, \; \xi^\varphi = \cos \theta (\sin \theta)^{-1} \cos \varphi \right] & (\text{ex-0607.107}) \\ [g_{ab}] &= \left[g_{tt} = -1 + 2mr^{-1}, \; g_{rr} = \left(1 - 2mr^{-1} \right)^{-1}, \; g_{\theta\theta} = r^2, \; g_{\varphi\varphi} = r^2 (\sin \theta)^2 \right] & (\text{ex-0607.108}) \\ [g_{ab}, g^{ab}] &= \left[g_{tt} = -1 + 2mr^{-1}, \; g_{rr} = \left(1 - 2mr^{-1} \right)^{-1}, \; g_{\theta\theta} = r^2, \; g_{\varphi\varphi} = r^2 (\sin \theta)^2, \; g^{tt} = \left(2mr^{-1} - 1 \right)^{-1}, \; g^{rr} = -2mr^{-1} + 1, \; g^{\theta\theta} = r^{-2}, \\ g^{\varphi\varphi} &= \left(r^2 (\sin \theta)^2 \right)^{-1} \right] & (\text{ex-0607.109}) \\ \xi_{a;b} + \xi_{b;a} &= \partial_b \xi_a - \Gamma^c_{ab} \xi_c + \partial_a \xi_b - \Gamma^c_{ba} \xi_c \\ &= \partial_b \left(g_{ac} \xi^c \right) - \Gamma^c_{ab} g_{cd} \xi^d + \partial_a \left(g_{bc} \xi^c \right) - \Gamma^c_{ba} g_{cd} \xi^d \\ &= \partial_b \left(g_{ac} \xi^c \right) - \frac{1}{2} g^{ce} \left(\partial_b g_{ea} + \partial_a g_{eb} - \partial_e g_{ab} \right) g_{cd} \xi^d + \partial_a \left(g_{bc} \xi^c \right) - \frac{1}{2} g^{ce} \left(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ba} \right) g_{cd} \xi^d \\ &= \partial_b \left(g_{ac} \xi^c \right) - g^{ce} \partial_b g_{ea} g_{cd} \xi^d - g^{ce} \partial_a g_{eb} g_{cd} \xi^d + \partial_a \left(g_{bc} \xi^c \right) + \frac{1}{2} g^{ce} \partial_e g_{ba} g_{cd} \xi^d \\ &= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{ce} \partial_b g_{ea} g_{cd} \xi^d - g^{ce} \partial_a g_{eb} g_{cd} \xi^d + \frac{1}{2} g^{ce} \partial_e g_{ab} g_{cd} \xi^d + \partial_a g_{bc} \xi^c + g_{bc} \partial_a \xi^c + \frac{1}{2} g^{ce} \partial_e g_{ba} g_{cd} \xi^d \\ &= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{cd} \partial_b g_{ac} g_{de} \xi^c - g^{cd} \partial_a g_{bc} g_{de} \xi^c + g^{cd} \partial_c g_{ab} g_{cd} \xi^d + \partial_a g_{bc} \xi^c + g_{bc} \partial_a \xi^c + \frac{1}{2} g^{ce} \partial_e g_{ba} g_{cd} \xi^d \\ &= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{cd} \partial_b g_{ac} g_{de} \xi^c - g^{cd} \partial_a g_{bc} g_{de} \xi^c + g^{cd} \partial_c g_{ab} g_{cd} \xi^d + \partial_a g_{bc} \xi^c + g_{bc} \partial_a \xi^c \\ &= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{cd} \partial_b g_{ac} g_{de} \xi^c - g^{cd} \partial_a g_{bc} g_{de} \xi^c + g^{cd} \partial_c g_{ab} g_{cd} \xi^c + g_{bc} \partial_a \xi^c \\ &= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{cd} \partial_b g_{ac} g_{de} \xi^c - g^{cd} \partial_a g_{bc} g_{de} \xi^c + g^{cd} \partial_c g_{ab} g_{cd} \xi^c + g_{bc} \partial_a \xi^c \\ &= (\text{ex-0607.106}) \end{split}$$

(ex-0607.110)