Exercise 6.7 Killing vectors of the Schwarzschild spacetime

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{t, r, \theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).

;::Symbol.

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} (   \partial_{g}{g_{b f}}
                                           + \partial_{f}{g_{b g}}
                                           - \partial_{b}{g_{f g}} ).

deriv := \xi_{a ; b} -> \partial_{b}{\xi_{a}} - \Gamma^{c}_{a b} \xi_{c}.
lower := \xi_{a} -> g_{a b} \xi^{b}.

expr  := \xi_{a ; b} + \xi_{b ; a}.                  # cdb(ex-0607.100,expr)

substitute   (expr, deriv)                           # cdb(ex-0607.101,expr)
substitute   (expr, lower)                           # cdb(ex-0607.102,expr)
substitute   (expr, Gamma)                           # cdb(ex-0607.103,expr)
distribute   (expr)                                  # cdb(ex-0607.104,expr)
product_rule (expr)                                  # cdb(ex-0607.105,expr)
canonicalise (expr)                                  # cdb(ex-0607.106,expr)

# choose a vector

# Kvect := {\xi^{t} = 1}.
# Kvect := {\xi^{\varphi} = 1}.
Kvect := {\xi^{\theta} = \sin(\varphi), \xi^{\varphi} = \cos(\theta)/\sin(\theta) \cos(\varphi)}.
# Kvect := {\xi^{\theta} = \cos(\varphi), \xi^{\varphi} = - \cos(\theta)/\sin(\theta) \sin(\varphi)}.
                                                      # cdb(ex-0607.107,Kvect)

gab := { g_{t t}            = -(1-2*m/r),
         g_{r r}            = 1/(1-(2*m/r)),
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            g_{\theta\theta}   = r**2,
            g_{\varphi\varphi} = r**2 \sin(\theta)**2}.  # cdb(ex-0607.108,gab)

complete   (gab, $g^{a b}$)                              # cdb(ex-0607.109,gab)

evaluate   (expr, join (gab,Kvect))                      # cdb(ex-0607.110,expr)
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$$[\xi^a] = \left[\xi^\theta = \sin\varphi, \xi^\varphi = \cos\theta(\sin\theta)^{-1}\cos\varphi\right] \tag{ex-0607.107}$$

$$[g_{ab}] = \left[g_{tt} = -1 + 2mr^{-1}, g_{rr} = \left(1 - 2mr^{-1}\right)^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-0607.108}$$

$$[g_{ab}, g^{ab}] = \Big[g_{tt} = -1 + 2mr^{-1}, g_{rr} = \left(1 - 2mr^{-1}\right)^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2, g^{tt} = \left(2mr^{-1} - 1\right)^{-1}, g^{rr} = -2mr^{-1} + 1, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\Big] \tag{ex-0607.109}$$

$$\begin{aligned}
\xi_{a;b} + \xi_{b;a} &= \partial_b\xi_a - \Gamma^c{}_{ab}\xi_c + \partial_a\xi_b - \Gamma^c{}_{ba}\xi_c && \text{(ex-0607.101)}\\
&= \partial_b\left(g_{ac}\xi^c\right) - \Gamma^c{}_{ab}g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) - \Gamma^c{}_{ba}g_{cd}\xi^d && \text{(ex-0607.102)}\\
&= \partial_b\left(g_{ac}\xi^c\right) - \frac{1}{2}g^{ce}\left(\partial_b g_{ea} + \partial_a g_{eb} - \partial_e g_{ab}\right)g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) - \frac{1}{2}g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ba}\right)g_{cd}\xi^d && \text{(ex-0607.103)}\\
&= \partial_b\left(g_{ac}\xi^c\right) - g^{ce}\partial_b g_{ea}g_{cd}\xi^d - g^{ce}\partial_a g_{eb}g_{cd}\xi^d + \frac{1}{2}g^{ce}\partial_e g_{ab}g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) + \frac{1}{2}g^{ce}\partial_e g_{ba}g_{cd}\xi^d && \text{(ex-0607.104)}\\
&= \partial_b g_{ac}\xi^c + g_{ac}\partial_b\xi^c - g^{ce}\partial_b g_{ea}g_{cd}\xi^d - g^{ce}\partial_a g_{eb}g_{cd}\xi^d + \frac{1}{2}g^{ce}\partial_e g_{ab}g_{cd}\xi^d + \partial_a g_{bc}\xi^c + g_{bc}\partial_a\xi^c + \frac{1}{2}g^{ce}\partial_e g_{ba}g_{cd}\xi^d && \text{(ex-0607.105)}\\
&= \partial_b g_{ac}\xi^c + g_{ac}\partial_b\xi^c - g^{cd}\partial_b g_{ac}g_{de}\xi^e - g^{cd}\partial_a g_{bc}g_{de}\xi^e + g^{cd}\partial_c g_{ab}g_{de}\xi^e + \partial_a g_{bc}\xi^c + g_{bc}\partial_a\xi^c && \text{(ex-0607.106)}\\
&= 0 && \text{(ex-0607.110)}
\end{aligned}$$