

## Example 14 The Weyl tensor is conformally invariant

This example shows that the Weyl tensor is conformally invariant. That is, for a pair of metrics  $g$  and  $\bar{g}$  related by a conformal transformation,  $\bar{g}_{ab} = \phi g_{ab}$  then  $\bar{C}_{bcd}^a = C_{bcd}^a$  or equally  $\bar{C}_{abcd} = \phi C_{abcd}$ .

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1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices (position=independent).
2
3  \partial{#}::PartialDerivative.
4
5  g_{a b}::Metric.
6  g^{a b}::InverseMetric.
7  g_{a}^{b}::KroneckerDelta.
8
9  GammaU := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
10                                     + \partial_{c}{g_{b d}}
11                                     - \partial_{d}{g_{b c}}).
12
13  GammaD := \Gamma_{a b c} -> 1/2 ( \partial_{b}{g_{a c}}
14                                     + \partial_{c}{g_{b a}}
15                                     - \partial_{a}{g_{b c}}).
16
17  Rabcd := R_{a b c d} -> \partial_{c}{\Gamma_{a b d}}
18                        - \partial_{d}{\Gamma_{a b c}}
19                        + \Gamma_{e a d} \Gamma^{e}_{b c}
20                        - \Gamma_{e a c} \Gamma^{e}_{b d}.
21
22  Rab      := R_{a b} -> g^{c d} R_{a c b d}.
23
24  Rscalar := R -> g^{a b} R_{a b}.
25
26  # Weyl in 4-dimensions
27
28  Cabcd := R_{a b c d} - (1/2) (R_{a c} g_{b d} - R_{a d} g_{b c})
29                        - (1/2) (g_{a c} R_{b d} - g_{a d} R_{b c})
30                        + (R/6) (g_{a c} g_{b d} - g_{a d} g_{b c}).
31
32  {\partial_{a b}{\phi},\partial_{a}{\phi},\phi}::SortOrder.
33  {\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.

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34
35 substitute (Cabcd,Rscalar)
36 substitute (Cabcd,Rab)
37 substitute (Cabcd,Rabcd)
38 substitute (Cabcd,GammaU)
39 substitute (Cabcd,GammaD)
40
41 distribute (Cabcd)
42
43 sort_product (Cabcd)
44 rename_dummies (Cabcd)
45 canonicalise (Cabcd)
46
47 # this is the Weyl tensor on the base metric
48 baseC := @(Cabcd).
49
50 conformal := {g_{a b} -> \phi g_{a b}, g^{a b} -> (1/phi) g^{a b}}.
51
52 substitute (Cabcd, conformal)
53 product_rule (Cabcd)
54 distribute (Cabcd)
55 product_rule (Cabcd)
56 distribute (Cabcd)
57
58 map_sympy (Cabcd, "simplify")
59
60 sort_product (Cabcd)
61 rename_dummies (Cabcd)
62 canonicalise (Cabcd)
63
64 # this is the Weyl tensor on the conformal metric
65 confC := @(Cabcd).
66
67 # their difference, should be zero
68 diff := @(confC) - \phi @(baseC). # cdb (ex-14.diff.100,diff)
69
70 distribute (diff)
71 sort_product (diff)

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72 rename_dummies (diff)
73 canonicalise (diff) # cdb (ex-14.diff.101,diff)
74
75 # this trick is not essential but it does reduce the number of terms in diff
76 substitute (diff,$\partial_{\{a\}\{b\}\{g_{\{c\} d\}}\} \rightarrow g_{\{c\} d\} b\{a\}\})
77 substitute (diff,$\partial_{\{a\}\{g_{\{b\} c\}}\} \rightarrow 0\})
78 substitute (diff,$g_{\{c\} d\} b\{a\} \rightarrow \partial_{\{a\}\{b\}\{g_{\{c\} d\}}\}}) # cdb (ex-14.diff.102,diff)
79
80 # standard expressions in 4-d
81 substitute (diff,$g_{\{a\} b\} g^{\{a\} b\} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.201,diff)
82 substitute (diff,$g_{\{a\} b\} g^{\{c\} b\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.202,diff)
83 substitute (diff,$g_{\{b\} a\} g^{\{b\} c\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.203,diff)
84 substitute (diff,$g_{\{a\}}^{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.204,diff)
85 substitute (diff,$g^{\{a\}}_{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.205,diff)
86 eliminate_kronecker (diff) # cdb (ex-14.diff.206,diff)
87
88 # need a second round since the above block introduces new terms that match those just eliminated
89 substitute (diff,$g_{\{a\} b\} g^{\{a\} b\} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.301,diff)
90 substitute (diff,$g_{\{a\} b\} g^{\{c\} b\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.302,diff)
91 substitute (diff,$g_{\{b\} a\} g^{\{b\} c\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.303,diff)
92 substitute (diff,$g_{\{a\}}^{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.304,diff)
93 substitute (diff,$g^{\{a\}}_{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.305,diff)
94 eliminate_kronecker (diff) # cdb (ex-14.diff.306,diff)
95
96 sort_product (diff)
97 rename_dummies (diff)
98 canonicalise (diff) # cdb (ex-14.diff.400,diff)
99
100 checkpoint.append (baseC)
101 checkpoint.append (confC)

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$$\begin{aligned}
\Delta = & \frac{1}{2}\partial_{bc}\phi g_{ad} - \frac{1}{2}\partial_{ac}\phi g_{bd} - \frac{1}{2}\partial_{bd}\phi g_{ac} + \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_{df}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g^{ef} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_{cf}g^{ef} \\
& + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g^{ef} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_{df}g^{ef} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g^{ef} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_{ac}\phi g_{bd}g_{ef}g^{ef} + \frac{1}{2}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{ef}g^{gh} - \frac{1}{2}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_{ad}\phi g_{bc}g_{ef}g^{ef} - \frac{1}{2}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{ef}g^{gh} + \frac{1}{2}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} + \frac{1}{4}\partial_{bd}\phi g_{ac}g_{ef}g^{ef} \\
& - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{8}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} - \frac{1}{4}\partial_{bc}\phi g_{ad}g_{ef}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} \\
& - \frac{1}{24}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij} + \frac{1}{24}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij}
\end{aligned}$$

(ex-14.diff.102)

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_{df}g^{ef} + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_{cf}g^{ef} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g^{ef} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g^{ef} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_{cf}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_{df}g^{ef} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g^{ef} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g^{ef} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g^{ef} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} \\
& - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} \\
& + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj}
\end{aligned}
\tag{ex-14.diff.201}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_{ef}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_{ef}g_g^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_{ef}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_{ef}g_g^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{ef} g^{gi} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{ef} g^{gi}
\end{aligned}
\tag{ex-14.diff.202}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^i g_i^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^i g_i^h g^{ef}
\end{aligned}
\tag{ex-14.diff.203}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^i g_i^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^i g_i^h g^{ef}
\end{aligned}
\tag{ex-14.diff.204}$$



$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{be}g_d^e + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ae}g_c^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\
& + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{be}g_c^e - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_c^e - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ae}g_d^e \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_a^e g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_c^e \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bd}g_c^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bd}g_c^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cf}g_g^f g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{af}g_{bd}g_g^f g^{eg} \\
& + \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_{de}\phi g_a^e g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_d^e + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ag}g_{bc}g_d^g g^{ef} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_a^e g_{bc}g_d^f - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{df}g_g^f g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{af}g_{bc}g_g^f g^{eg} - \frac{1}{6}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_b^e - \frac{1}{4}\partial_{be}\phi g_{ac}g_d^e \\
& - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^g g_g^f - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bg}g_d^g g^{ef} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_b^e g_d^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{df}g_g^f g^{eg} + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bf}g_g^f g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_b^e + \frac{1}{4}\partial_{be}\phi g_{ad}g_c^e + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^g g_g^f + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bg}g_c^g g^{ef} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_b^e g_c^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cf}g_g^f g^{eg} \\
& - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bf}g_g^f g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_g^f g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_g^f g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_i^h g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^i g_i^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^i g_i^h g^{ef} \tag{ex-14.diff.205}
\end{aligned}$$

$$\begin{aligned}
\Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad} \\
& + \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} + \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} - \frac{5}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\
& - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^f + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g_{cg}g^{eg} + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ag}g_{bd}g^{eg} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^f \\
& - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_{dg}g^{eg} - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ag}g_{bc}g^{eg} - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac}g_f^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_{dg}g^{eg} \\
& + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_{bg}g^{eg} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_{cg}g^{eg} - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_{bg}g^{eg} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} \\
& - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_{gi}g^{eg} g^{fi} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_{gi}g^{eg} g^{fi} + \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^h g^{ef} - \frac{1}{8}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^h g^{ef} \tag{ex-14.diff.206}
\end{aligned}$$



$$\begin{aligned} \Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad} \\ & + \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ & + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g^e_c + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g^e_a g_{bd} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g^e_d - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g^e_a g_{bc} \\ & - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g^e_d + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g^e_b + \frac{1}{4}\partial_{cb}\phi g_{ad} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g^e_c - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g^e_b \\ & + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^f_g g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^f_g g^{eg} \end{aligned} \quad (\text{ex-14.diff.304})$$

$$\begin{aligned} \Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_d\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_d\phi\partial_a\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad} \\ & + \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ & + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bd}g^e_c + \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_a^e g_{bd} + \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^e - \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} \\ & - \frac{1}{6}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^e + \frac{1}{4}\partial_d\phi\partial_e\phi\phi^{-1}g_{ac}g_b^e + \frac{1}{4}\partial_{cb}\phi g_{ad} - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^e - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^e \\ & + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} + \frac{1}{4}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} \end{aligned} \quad (\text{ex-14.diff.305})$$

$$\begin{aligned}\Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ & - \frac{1}{12}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} \\ & - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg}\end{aligned}\tag{ex-14.diff.306}$$

$$\Delta = 0 \quad (\text{ex-14.diff.400})$$