

Example 1 The metric connection

```
1  # Define some properties
2
3  {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
4
5  g_{a b}::Metric.
6  g_{a}^{b}::KroneckerDelta.
7
8  \nabla{#}::Derivative.
9  \partial{#}::PartialDerivative.
10
11 # Define rules for covariant derivative and the Christoffel symbol
12
13 nabla := \nabla_{c}{g_{a b}} -> \partial_{c}{g_{a b}} - g_{a d}\Gamma^{d}_{b c}
14                                     - g_{d b}\Gamma^{d}_{a c}.    # cdb (nabla.100,nabla)
15
16 Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a d} ( \partial_{b}{g_{d c}}
17                                     + \partial_{c}{g_{b d}}
18                                     - \partial_{d}{g_{b c}} ).    # cdb (Gamma.100,Gamma)
19
20 # Start with a simple expression
21
22 cderiv := \nabla_{c}{g_{a b}}.    # cdb (ex-01.100,cderiv)
23
24 # Do the computations
25
26 substitute      (cderiv, nabla)    # cdb (ex-01.101,cderiv)
27 substitute      (cderiv, Gamma)    # cdb (ex-01.102,cderiv)
28 distribute      (cderiv)            # cdb (ex-01.103,cderiv)
29 eliminate_metric (cderiv)           # cdb (ex-01.104,cderiv)
30 eliminate_kronecker (cderiv)        # cdb (ex-01.105,cderiv)
31 canonicalise     (cderiv)           # cdb (ex-01.106,cderiv)
32
33 checkpoint.append (cderiv)
```

$$\nabla g_{ab} \rightarrow \partial g_{ab} - g_{ad}\Gamma_{bc}^d - g_{db}\Gamma_{ac}^d \quad (\text{nabla.100})$$

$$\Gamma_{bc}^a \rightarrow \frac{1}{2}g^{ad}(\partial_{\mathfrak{b}}g_{dc} + \partial_{\mathfrak{c}}g_{bd} - \partial_{\mathfrak{d}}g_{bc}) \quad (\text{Gamma.100})$$

$$\nabla g_{ab} = \partial g_{ab} - g_{ad}\Gamma_{bc}^d - g_{db}\Gamma_{ac}^d \quad (\text{ex-01.101})$$

$$= \partial g_{ab} - \frac{1}{2}g_{ad}g^{de}(\partial_{\mathfrak{b}}g_{ec} + \partial_{\mathfrak{c}}g_{be} - \partial_{\mathfrak{d}}g_{bc}) - \frac{1}{2}g_{db}g^{de}(\partial_{\mathfrak{a}}g_{ec} + \partial_{\mathfrak{c}}g_{ae} - \partial_{\mathfrak{d}}g_{ac}) \quad (\text{ex-01.102})$$

$$= \partial g_{ab} - \frac{1}{2}g_{ad}g^{de}\partial_{\mathfrak{b}}g_{ec} - \frac{1}{2}g_{ad}g^{de}\partial_{\mathfrak{c}}g_{be} + \frac{1}{2}g_{ad}g^{de}\partial_{\mathfrak{d}}g_{bc} - \frac{1}{2}g_{db}g^{de}\partial_{\mathfrak{a}}g_{ec} - \frac{1}{2}g_{db}g^{de}\partial_{\mathfrak{c}}g_{ae} + \frac{1}{2}g_{db}g^{de}\partial_{\mathfrak{d}}g_{ac} \quad (\text{ex-01.103})$$

$$= \partial g_{ab} - \frac{1}{2}g_a^e\partial_{\mathfrak{b}}g_{ec} - \frac{1}{2}g_a^e\partial_{\mathfrak{c}}g_{be} + \frac{1}{2}g_a^e\partial_{\mathfrak{d}}g_{bc} - \frac{1}{2}g_b^e\partial_{\mathfrak{a}}g_{ec} - \frac{1}{2}g_b^e\partial_{\mathfrak{c}}g_{ae} + \frac{1}{2}g_b^e\partial_{\mathfrak{d}}g_{ac} \quad (\text{ex-01.104})$$

$$= \frac{1}{2}\partial g_{ab} - \frac{1}{2}\partial g_{ba} \quad (\text{ex-01.105})$$

$$= 0 \quad (\text{ex-01.106})$$

Example 2 Covariant derivatives

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # rule for covariant derivative of v^{a}
7
8 deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.
9
10 # create an expression
11
12 foo := \nabla_{a}{v^{b}}. # cdb (ex-02.101,foo)
13
14 # apply the rule, then simplify
15
16 substitute (foo,deriv) # cdb (ex-02.102,foo)
17 canonicalise (foo) # cdb (ex-02.103,foo)
18
19 checkpoint.append (foo)
```

$$\nabla_a v^b = \partial_a v^b + \Gamma_{ca}^b v^c \quad (\text{ex-02.102})$$

$$= \partial_a v^b + \Gamma_a^{bc} v_c \quad (\text{ex-02.103})$$

Example 2 Covariant derivatives using “position=independent”

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # rule for covariant derivative of v^{a}
7
8 deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.
9
10 # create an expression
11
12 foo := \nabla_{a}{v^{b}}. # cdb (ex-02.201,foo)
13
14 # apply the rule, then simplify
15
16 substitute (foo,deriv) # cdb (ex-02.202,foo)
17 canonicalise (foo) # cdb (ex-02.203,foo)
18
19 checkpoint.append (foo)
```

$$\nabla_a v^b = \partial_a v^b + \Gamma_{ca}^b v^c \quad (\text{ex-02.202})$$

$$= \partial_a v^b + \Gamma_{ca}^b v^c \quad (\text{ex-02.203})$$

Example 2 Covariant derivatives using generic rule for deriv

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # template for covariant derivative of a vector
7
8 deriv := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}} + \Gamma^{b}_{c a} A^{c}.
9
10 # create an expression
11
12 foo := \nabla_{a}{u^{b}} + \nabla_{a}{v^{b}}. # cdb (ex-02.301,foo)
13
14 # apply the rule, then simplify
15
16 substitute (foo,deriv) # cdb (ex-02.302,foo)
17 canonicalise (foo) # cdb (ex-02.303,foo)
18
19 checkpoint.append (foo)

```

$$\nabla_a u^b + \nabla_a v^b = \partial_a u^b + \Gamma_{ca}^b u^c + \partial_a v^b + \Gamma_{ca}^b v^c \quad (\text{ex-02.302})$$

$$= \partial_a u^b + \Gamma_{ca}^b u^c + \partial_a v^b + \Gamma_{ca}^b v^c \quad (\text{ex-02.303})$$

Example 3a The Riemann curvature tensor

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});
6
7 ::Symbol; # Suggested by Kasper as a way to make use of ; legal
8           # see https://cadabra.science/qa/473/is-this-legal-syntax
9           # this code works with and without this trick
10
11 # rules for the first two covariant derivs of V^a
12
13 deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
14         + \Gamma^{a}_{c b} V^{c}.      # cdb (ex-03.101,deriv1)
15
16 deriv2 := V^{a}_{; b ; c} -> \partial_{c}{V^{a}_{; b}}
17         + \Gamma^{a}_{d c} V^{d}_{; b}
18         - \Gamma^{d}_{b c} V^{a}_{; d}. # cdb (ex-03.102,deriv2)
19
20 substitute (deriv2,deriv1)      # cdb (ex-03.103, deriv2)
21
22 Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}. # cdb (ex-03.104, Vabc)
23
24 substitute (Vabc,deriv2)      # cdb (ex-03.105, Vabc)
25
26 distribute      (Vabc)      # cdb (ex-03.106, Vabc)
27 product_rule    (Vabc)      # cdb (ex-03.107, Vabc)
28
29 sort_product    (Vabc)      # cdb (ex-03.108, Vabc)
30 rename_dummies  (Vabc)      # cdb (ex-03.109, Vabc)
31 canonicalise    (Vabc)      # cdb (ex-03.110, Vabc)
32
33 sort_sum        (Vabc)      # cdb (ex-03.111, Vabc)
34 factor_out      (Vabc,$V^{a?}$) # cdb (ex-03.112, Vabc)
35
36 checkpoint.append (Vabc)
```

```

37
38 # create rule for Riemann, export later (for use by lib/dgeom)
39
40 substitute (Vabc,$V^{a} -> -1$)           # cdb (ex-03.113, Vabc)
41                                           # note use of -1 to get correct
42                                           # signs when coupled with the rule
43                                           # for Rabcd (next statement)
44
45 Rabcd := R^{a}_{[d b c]} -> @(Vabc).       # cdb (ex-03.114, Rabcd) #
46
47 foo    := R^{a}_{[b c d]}.                 # cdb (ex-03.115, foo)
48 substitute (foo, Rabcd)                   # cdb (ex-03.116, foo)
49
50 # update rule to use nice indices
51
52 Rabcd := R^{a}_{[b c d]} -> @(foo).
53
54 checkpoint.append (Rabcd)

```

$$V^a_{;b} \rightarrow \partial_b V^a + \Gamma^a_{cb} V^c \quad (\text{ex-03.101})$$

$$V^a_{;b;c} \rightarrow \partial_c V^a_{;b} + \Gamma^a_{dc} V^d_{;b} - \Gamma^d_{bc} V^a_{;d} \quad (\text{ex-03.102})$$

$$V^a_{;b;c} \rightarrow \partial_c (\partial_b V^a + \Gamma^a_{db} V^d) + \Gamma^a_{dc} (\partial_b V^d + \Gamma^d_{eb} V^e) - \Gamma^d_{bc} (\partial_d V^a + \Gamma^a_{ed} V^e) \quad (\text{ex-03.103})$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c(\partial_b V^a + \Gamma^a_{db} V^d) + \Gamma^a_{dc}(\partial_b V^d + \Gamma^d_{eb} V^e) - \Gamma^d_{bc}(\partial_d V^a + \Gamma^a_{ed} V^e) - \partial_b(\partial_c V^a + \Gamma^a_{dc} V^d) - \Gamma^a_{db}(\partial_c V^d + \Gamma^d_{ec} V^e) + \Gamma^d_{cb}(\partial_d V^a + \Gamma^a_{ed} V^e) \quad (\text{ex-03.105})$$

$$= \partial_{cb} V^a + \partial_c(\Gamma^a_{db} V^d) + \Gamma^a_{dc} \partial_b V^d + \Gamma^a_{dc} \Gamma^d_{eb} V^e - \Gamma^d_{bc} \partial_d V^a - \Gamma^d_{bc} \Gamma^a_{ed} V^e - \partial_{bc} V^a - \partial_b(\Gamma^a_{dc} V^d) - \Gamma^a_{db} \partial_c V^d - \Gamma^a_{db} \Gamma^d_{ec} V^e + \Gamma^d_{cb} \partial_d V^a + \Gamma^d_{cb} \Gamma^a_{ed} V^e \quad (\text{ex-03.106})$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a_{db} V^d + \Gamma^a_{dc} \Gamma^d_{eb} V^e - \Gamma^d_{bc} \partial_d V^a - \Gamma^d_{bc} \Gamma^a_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a_{dc} V^d - \Gamma^a_{db} \Gamma^d_{ec} V^e + \Gamma^d_{cb} \partial_d V^a + \Gamma^d_{cb} \Gamma^a_{ed} V^e \quad (\text{ex-03.107})$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a_{db} + V^e \Gamma^a_{dc} \Gamma^d_{eb} - \Gamma^d_{bc} \partial_d V^a - V^e \Gamma^a_{ed} \Gamma^d_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a_{dc} - V^e \Gamma^a_{db} \Gamma^d_{ec} + \Gamma^d_{cb} \partial_d V^a + V^e \Gamma^a_{ed} \Gamma^d_{cb} \quad (\text{ex-03.108})$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a_{db} + V^d \Gamma^a_{ec} \Gamma^e_{db} - \Gamma^d_{bc} \partial_d V^a - V^d \Gamma^a_{de} \Gamma^e_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a_{dc} - V^d \Gamma^a_{eb} \Gamma^e_{dc} + \Gamma^d_{cb} \partial_d V^a + V^d \Gamma^a_{de} \Gamma^e_{cb} \quad (\text{ex-03.109})$$

$$= V^d \partial_c \Gamma^a_{bd} + V^d \Gamma^a_{ce} \Gamma^e_{bd} - V^d \partial_b \Gamma^a_{cd} - V^d \Gamma^a_{be} \Gamma^e_{cd} \quad (\text{ex-03.110})$$

$$= V^d \partial_c \Gamma^a_{bd} - V^d \partial_b \Gamma^a_{cd} - V^d \Gamma^a_{be} \Gamma^e_{cd} + V^d \Gamma^a_{ce} \Gamma^e_{bd} \quad (\text{ex-03.111})$$

$$= V^d (\partial_c \Gamma^a_{bd} - \partial_b \Gamma^a_{cd} - \Gamma^a_{be} \Gamma^e_{cd} + \Gamma^a_{ce} \Gamma^e_{bd}) \quad (\text{ex-03.112})$$

$$= -R^a{}_{dbc} V^d \quad (\text{MTW})$$

$$R^a{}_{bcd} = -\partial_d \Gamma^a_{cb} + \partial_c \Gamma^a_{db} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb}$$

Example 3b The Riemann curvature tensor

This differs from the above by not using the `::TableauSymmetry` property. It gives the same results as above but it does require a little bit more housekeeping.

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 ::Symbol; # Suggsted by Kasper as a way to make use of ; legal
6           # see https://cadabra.science/qa/473/is-this-legal-syntax
7           # this code works with and without this trick
8
9 # rules for the first two covariant derivs of V^a
10
11 deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
12         + \Gamma^{a}_{c b} V^{c}.      # cdb (ex-03.301,deriv1)
13
14 deriv2 := V^{a}_{; b ; c} -> \partial_{c}{V^{a}_{; b}}
15         + \Gamma^{a}_{d c} V^{d}_{; b}
16         - \Gamma^{d}_{b c} V^{a}_{; d}. # cdb (ex-03.302,deriv2)
17
18 substitute (deriv2,deriv1)      # cdb (ex-03.303, deriv2)
19
20 Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}. # cdb (ex-03.304, Vabc)
21
22 substitute (Vabc,deriv2)      # cdb (ex-03.305, Vabc)
23
24 distribute      (Vabc)      # cdb (ex-03.306, Vabc)
25 product_rule    (Vabc)      # cdb (ex-03.307, Vabc)
26
27 # -----
28 # trick to obtain a symmetric connection
29
30 G_{a b}::Symmetric.
31
32 substitute      (Vabc,$\Gamma^{a}_{b c} -> G^{a} G_{b c}$)
33 sort_product    (Vabc)      # cdb (ex-03.308, Vabc)
```

```

34 rename_dummies (Vabc) # cdb (ex-03.309, Vabc)
35 canonicalise (Vabc) # cdb (ex-03.310, Vabc)
36 substitute (Vabc,$G^{a} G_{b c} -> \Gamma^{a}_{b c}$,repeat=True)
37 # -----
38
39 sort_product (Vabc)
40 rename_dummies (Vabc)
41 canonicalise (Vabc)
42
43 sort_sum (Vabc) # cdb (ex-03.311, Vabc)
44 factor_out (Vabc,$V^{a?}$) # cdb (ex-03.312, Vabc)
45
46 checkpoint.append (Vabc)

```

$$V^a_{;b} \rightarrow \partial_b V^a + \Gamma^a_{cb} V^c \quad (\text{ex-03.301})$$

$$V^a_{;b;c} \rightarrow \partial_c V^a_{;b} + \Gamma^a_{dc} V^d_{;b} - \Gamma^d_{bc} V^a_{;d} \quad (\text{ex-03.302})$$

$$V^a_{;b;c} \rightarrow \partial_c (\partial_b V^a + \Gamma^a_{db} V^d) + \Gamma^a_{dc} (\partial_b V^d + \Gamma^d_{eb} V^e) - \Gamma^d_{bc} (\partial_d V^a + \Gamma^a_{ed} V^e) \quad (\text{ex-03.303})$$

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \partial_c(\partial_b V^a + \Gamma^a_{db} V^d) + \Gamma^a_{dc}(\partial_b V^d + \Gamma^d_{eb} V^e) - \Gamma^d_{bc}(\partial_d V^a + \Gamma^a_{ed} V^e) - \partial_b(\partial_c V^a + \Gamma^a_{dc} V^d) - \Gamma^a_{db}(\partial_c V^d + \Gamma^d_{ec} V^e) + \Gamma^d_{cb}(\partial_d V^a + \Gamma^a_{ed} V^e) \quad (\text{ex-03.305})$$

$$= \partial_{cb} V^a + \partial_c(\Gamma^a_{db} V^d) + \Gamma^a_{dc} \partial_b V^d + \Gamma^a_{dc} \Gamma^d_{eb} V^e - \Gamma^d_{bc} \partial_d V^a - \Gamma^d_{bc} \Gamma^a_{ed} V^e - \partial_{bc} V^a - \partial_b(\Gamma^a_{dc} V^d) - \Gamma^a_{db} \partial_c V^d - \Gamma^a_{db} \Gamma^d_{ec} V^e + \Gamma^d_{cb} \partial_d V^a + \Gamma^d_{cb} \Gamma^a_{ed} V^e \quad (\text{ex-03.306})$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a_{db} V^d + \Gamma^a_{dc} \Gamma^d_{eb} V^e - \Gamma^d_{bc} \partial_d V^a - \Gamma^d_{bc} \Gamma^a_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a_{dc} V^d - \Gamma^a_{db} \Gamma^d_{ec} V^e + \Gamma^d_{cb} \partial_d V^a + \Gamma^d_{cb} \Gamma^a_{ed} V^e \quad (\text{ex-03.307})$$

$$= \partial_{cb} V^a + V^d \partial_c(G^a G_{db}) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b(G^a G_{dc}) - G^a G^d G_{db} G_{ec} V^e + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \quad (\text{ex-03.308})$$

$$= \partial_{cb} V^a + V^d \partial_c(G^a G_{db}) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b(G^a G_{dc}) - G^a G^d G_{db} G_{ec} V^e + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e \quad (\text{ex-03.309})$$

$$= V^d \partial_c(G^a G_{bd}) + G^a G^d G_{be} G_{cd} V^e - V^d \partial_b(G^a G_{cd}) - G^a G^d G_{bd} G_{ce} V^e \quad (\text{ex-03.310})$$

$$= V^d \partial_c \Gamma^a_{bd} - V^d \partial_b \Gamma^a_{cd} + V^d \Gamma^a_{bd} \Gamma^e_{ce} - V^d \Gamma^a_{be} \Gamma^e_{cd} \quad (\text{ex-03.311})$$

$$= V^d (\partial_c \Gamma^a_{bd} - \partial_b \Gamma^a_{cd} + \Gamma^a_{bd} \Gamma^e_{ce} - \Gamma^a_{be} \Gamma^e_{cd}) \quad (\text{ex-03.312})$$

$$= -R^a{}_{dbc} V^d \quad (\text{MTW})$$

Example 4 Python functions

```
1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 def truncate (poly,n):
4
5     # define the weight and give it a label
6     x^{a}::Weight(label=\epsilon).
7
8     # start with an empty expression
9     ans = Ex("0")
10
11    # loop over selected terms in the source
12    for i in range (0,n+1):
13
14        foo := @ (poly).
15        bah = Ex("\epsilon = " + str(i))
16
17        # extract a single term
18        keep_weight (foo, bah)
19
20        # update the running sum
21        ans = ans + foo
22
23    # all done, return final answer
24    return ans
25
26    Quartic := c^{a}
27              + c^{a}_{b} x^b
28              + c^{a}_{b c} x^b x^c
29              + c^{a}_{b c d} x^b x^c x^d
30              + c^{a}_{b c d e} x^b x^c x^d x^e.    # cdb (ex-04.100,Quartic)
31
32    Cubic = truncate (Quartic,3)                    # cdb (ex-04.101,Cubic)
33
34    checkpoint.append (Cubic)
```

$$p(x) = c^a + c^a_b x^b + c^a_{bc} x^b x^c + c^a_{bcd} x^b x^c x^d + c^a_{bcde} x^b x^c x^d x^e \quad (\text{ex-04.100})$$

$$q(x) = c^a + c^a_b x^b + c^a_{bc} x^b x^c + c^a_{bcd} x^b x^c x^d \quad (\text{ex-04.101})$$

Example 5a Keeping focused

```
1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 expr := A_{a} v^{a} + B_{a} v^{a} + C_{a} v^{a}. # cdb (ex-05.100,expr)
4
5 zoom (expr,$B_{a} Q???) # cdb (ex-05.101,expr)
6 substitute (expr, $v^{a} -> w^{a}$) # cdb (ex-05.102,expr)
7 unzoom (expr) # cdb (ex-05.103,expr)
8
9 checkpoint.append (expr)
```

$$A_a v^a + B_a v^a + C_a v^a = \dots + B_a v^a + \dots \quad (\text{ex-05.101})$$

$$= \dots + B_a w^a + \dots \quad (\text{ex-05.102})$$

$$= A_a v^a + B_a w^a + C_a v^a \quad (\text{ex-05.103})$$

Example 5b Tags

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 def add_tags (obj,tag):
4     n = 0
5     ans = Ex('0')
6     for i in obj.top().terms():
7         foo = obj[i]
8         bah = Ex(tag+'_'+str(n)+'')
9         ans := @(ans) + @(bah) @(foo).
10        n = n + 1
11    return ans
12
13 def clear_tags (obj,tag):
14     ans := @(obj).
15     foo = Ex(tag+'_{a?} -> 1')
16     substitute (ans,foo)
17     return ans
18
19 expr := 2 V_{p q} - 3 V_{q p}. # cdb (ex-05.200,expr)
20
21 expr = add_tags (expr,'\mu') # cdb (ex-05.201,expr)
22
23 zoom (expr, $\mu_{1} Q??$) # cdb (ex-05.202,expr)
24 substitute (expr, $V_{a b} -> - V_{b a}$) # cdb (ex-05.203,expr)
25 unzoom (expr) # cdb (ex-05.204,expr)
26
27 expr = clear_tags (expr,'\mu') # cdb (ex-05.205,expr)
28
29 checkpoint.append (expr)
```

$$2 V_{pq} - 3 V_{qp} = 2 \mu_0 V_{pq} - 3 \mu_1 V_{qp} \quad (\text{ex-05.201})$$

$$= \dots - 3 \mu_1 V_{qp} \quad (\text{ex-05.202})$$

$$= \dots + 3 \mu_1 V_{pq} \quad (\text{ex-05.203})$$

$$= 2 \mu_0 V_{pq} + 3 \mu_1 V_{pq} \quad (\text{ex-05.204})$$

$$= 5 V_{pq} \quad (\text{ex-05.205})$$

Example 6-01 Evaluating components

```
1  {\theta, \varphi}::Coordinate.  
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).  
3  
4  \partial{#}::PartialDerivative.  
5  
6  V := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.      # cdb(ex-06.100,V)  
7  dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.              # cdb(ex-06.101,dV)  
8  
9  evaluate (dV, V)    # cdb(ex-06.102,dV)
```

$$V_a = [V_\theta = \varphi, V_\varphi = \sin \theta] \quad (\text{ex-06.100})$$

$$\partial_b V_a - \partial_a V_b = \square_{ab} \begin{cases} \square_{\varphi\theta} = \cos \theta - 1 \\ \square_{\theta\varphi} = 1 - \cos \theta \end{cases} \quad (\text{ex-06.102})$$

Example 6-02 Riemann tensor of a 2-sphere

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
7                                     + \partial_{c}{g_{b d}}
8                                     - \partial_{d}{g_{b c}}).
9
10 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
11                        - \partial_{d}{\Gamma^{a}_{b c}}
12                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
13                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
14
15 gab := { g_{\theta\theta} = r**2,
16          g_{\varphi\varphi} = r**2 \sin(\theta)**2 }. # cdb(ex-06.201,gab)
17
18 iab := { g^{\theta\theta} = 1/r**2,
19          g^{\varphi\varphi} = 1/(r**2 \sin(\theta)**2) }. # cdb(ex-06.202,iab)
20
21 substitute (Rabcd, Gamma) # cdb(ex-06.203,Gamma)
22
23 evaluate (Gamma, join (gab,iab), rhsonly=True) # cdb(ex-06.204,Gamma)
24 evaluate (Rabcd, join (gab,iab), rhsonly=True) # cdb(ex-06.205,Rabcd)
25
26 # convert from a rule to a simple expression
27 Riem := R^{a}_{b c d}.
28 substitute (Riem, Rabcd) # cdb(ex-06.206,Riem)
29
30 from cdb.core.component import *
31
32 RiemCompt = get_component (Riem, $\theta, \varphi, \theta, \varphi$) # cdb(ex-06.207,RiemCompt)

```

$$[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2 (\sin \theta)^2] \quad (\text{ex-06.201})$$

$$[g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2 (\sin \theta)^2)^{-1}] \quad (\text{ex-06.202})$$

$$\Gamma_{bc}^a \rightarrow \square_{cb}^a \begin{cases} \square_{\varphi\theta}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2} \sin(2\theta) \end{cases} \quad (\text{ex-06.204})$$

$$R_{bcd}^a \rightarrow \square_{db}^a{}_c \begin{cases} \square_{\varphi\varphi}^{\theta}{}_{\theta} = (\sin \theta)^2 \\ \square_{\varphi\theta}^{\varphi}{}_{\theta} = -1 \\ \square_{\theta\varphi}^{\theta}{}_{\varphi} = -(\sin \theta)^2 \\ \square_{\theta\theta}^{\varphi}{}_{\varphi} = 1 \end{cases} \quad (\text{ex-06.205})$$

$$\square_{db}^a{}_c \begin{cases} \square_{\varphi\varphi}^{\theta}{}_{\theta} = (\sin \theta)^2 \\ \square_{\varphi\theta}^{\varphi}{}_{\theta} = -1 \\ \square_{\theta\varphi}^{\theta}{}_{\varphi} = -(\sin \theta)^2 \\ \square_{\theta\theta}^{\varphi}{}_{\varphi} = 1 \end{cases} \quad (\text{ex-06.206})$$

$$R^{\theta}{}_{\varphi\varphi\theta} = -(\sin \theta)^2 \quad (\text{ex-06.207})$$

Example 6-03 Using complete to compute the inverse metric

This version uses `complete` to compute the inverse metric.

```
1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)
7
8  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}}).
11
12  Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
13                        - \partial_{d}{\Gamma^{a}_{b c}}
14                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17  gab := { g_{\theta\theta} = r**2,
18          g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.      # cdb(ex-06.301,gab)
19
20  complete (gab, $g^{a b}$)                                # cdb(ex-06.302,gab)
21
22  substitute (Rabcd, Gamma)
23
24  evaluate (Gamma, gab, rhsonly=True)                       # cdb(ex-06.303,Gamma)
25  evaluate (Rabcd, gab, rhsonly=True)                       # cdb(ex-06.304,Rabcd)
```

$$[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2 (\sin \theta)^2] \quad (\text{ex-06.301})$$

$$\left[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2 (\sin \theta)^2, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2 (\sin \theta)^2)^{-1} \right] \quad (\text{ex-06.302})$$

$$\Gamma_{bc}^a \rightarrow \square_{cb}^a \begin{cases} \square_{\varphi\theta}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2} \sin(2\theta) \end{cases} \quad (\text{ex-06.303})$$

$$R_{bcd}^a \rightarrow \square_{db}^a \begin{cases} \square_{\varphi\varphi}^{\theta} = (\sin \theta)^2 \\ \square_{\varphi\theta}^{\varphi} = -1 \\ \square_{\theta\varphi}^{\theta} = -(\sin \theta)^2 \\ \square_{\theta\theta}^{\varphi} = 1 \end{cases} \quad (\text{ex-06.304})$$

Example 6-04 Components by scalar projection

This example shows how one component of the Riemann tensor can be computed using a scalar projection.

```
1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  theta{#}::LaTeXForm("\theta").
5  varphi{#}::LaTeXForm("\varphi").
6
7  # usual definitions for the connection and Riemann tensor
8
9  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
10                                     + \partial_{c}{g_{b d}}
11                                     - \partial_{d}{g_{b c}}).
12
13  Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
14                                     - \partial_{d}{\Gamma^{a}_{b c}}
15                                     + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
16                                     - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
17
18  gab := { g_{\theta \theta} = r**2,
19           g_{\varphi \varphi} = r**2 \sin(\theta)**2 }. # cdb(ex-06.400,gab)
20
21  iab := { g^{\theta \theta} = 1/r**2,
22           g^{\varphi \varphi} = 1/(r**2 \sin(\theta)**2) }.
23
24  substitute (Rabcd, Gamma)
25  evaluate (Rabcd, join (gab,iab), rhsonly=True)
26
27  # above code just to compute Rabcd
28  # following code is all that is needed for the scalar projection method
29
30  # define the basis for vectors and dual vectors
31
32  basis := {theta^{\theta} = 1, varphi^{\varphi} = 1}.
33  dual := {theta_{\theta} = 1, varphi_{\varphi} = 1}.
34
```

```

35 # obtain components by contracting with basis
36
37 compt := R^{a}_{[b c d]} theta_{a} varphi^{b} theta^{c} varphi^{d}. # cdb(ex-06.401,compt)
38 substitute (compt,Rabcd)
39
40 evaluate (compt, join (basis,dual)) # cdb(ex-06.402,compt)
41
42 compt_sympy = compt._sympy_()
43
44 # cdbBeg(print.ex-06.04)
45 print ('type compt = ' + str(type(compt))) # shows that compt is a Cadabra object
46 print ('type ghiphi = ' + str(type(compt_sympy))) # shows that ghiphi is a Python object
47 print ('      compt = ' + str(compt)) # will contain LaTeX markup
48 print ('      ghiphi = ' + str(compt_sympy)) # will be pure Python/SymPy
49 # cdbEnd(print.ex-06.04)
50
51 checkpoint.append (compt)

```

$$R^{\theta}_{\varphi\theta\varphi} = R^a_{bcd}\theta_a\varphi^b\theta^c\varphi^d \quad (\text{ex-06.401})$$

$$= (\sin\theta)^2 \quad (\text{ex-06.402})$$

```

1 type compt = <class 'cadabra2.Ex'>
2 type ghiphi = <class 'sympy.core.power.Pow'>
3     compt = (\sin(\theta))**2
4     ghiphi = sin(theta)**2

```

Example 6-05 Components by selection

This example shows how one component of the metric tensor can be computed by indexing the result of a call to `evaluate`.

```
1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  theta{#}::LaTeXForm("\theta").
5  varphi{#}::LaTeXForm("\varphi").
6
7  gab := { g_{\theta \theta}    = r**2,
8           g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.  # cdb(ex-06.500,gab)
9
10 metric := g_{a b}.
11
12 evaluate (metric,gab)
13
14 indcs = metric[2][1][0]          # cdb(ex-06.501,indcs)
15 compt = metric[2][1][1]         # cdb(ex-06.502,compt)
16
17 # cdbBeg(print.ex-06.05)
18 print ('metric = ' + str(metric.input_form())+'\n')  # reveals Cadabra's internal structure for storing metric
19
20 print ('metric[0] = ' + str(metric[0]))
21 print ('metric[1] = ' + str(metric[1]))
22 print ('metric[2] = ' + str(metric[2])+'\n')
23
24 print ('metric[2][1] = '+ str(metric[2][1]))
25 print ('metric[2][1][0] = '+ str(metric[2][1][0]))
26 print ('metric[2][1][1] = '+ str(metric[2][1][1]))
27 # cdbEnd(print.ex-06.05)
28
29 checkpoint.append (indcs)
30 checkpoint.append (compt)
```


$$g_{\varphi\varphi} = g_{[\varphi,\varphi]} \quad (\text{ex-06.501})$$

$$= r^2 (\sin \theta)^2 \quad (\text{ex-06.502})$$

```

1  metric = \components_{a b}({{\theta}, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2)
2
3  metric[0] = a
4  metric[1] = b
5  metric[2] = {{{\theta}, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2}
6
7  metric[2][1] = {\varphi, \varphi} = (r)**2 (\sin(\theta))**2
8  metric[2][1][0] = {\varphi, \varphi}
9  metric[2][1][1] = (r)**2 (\sin(\theta))**2

```

Example 7 Export to C-code

```
1 def write_code (obj,name,filename,rank):
2
3     import os
4
5     from sympy.printing.c import C99CodePrinter as printer
6     from sympy.codegen.ast import Assignment
7
8     idx=[] # indices in the form [{x, x}, {x, y} ...]
9     lst=[] # corresponding terms [termxx, termxy, ...]
10
11     for i in range( len(obj[rank]) ): # rank = number of free indices
12         idx.append( str(obj[rank][i][0]._sympy_()) ) # indices for this term
13         lst.append( str(obj[rank][i][1]._sympy_()) ) # the matching term
14
15     mat = sympy.Matrix([lst]) # row vector of terms
16     sub_exprs, simplified_rhs = sympy.cse(mat) # optimise code
17
18     with open(os.getcwd() + '/' + filename, 'w') as out:
19
20         for lhs, rhs in sub_exprs:
21             out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
22
23         for index, rhs in enumerate (simplified_rhs[0]):
24             lhs = sympy.Symbol(name+' '+idx[index]).replace(', ', '[')
25             out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
```

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  g_{a b}::Metric.
7  g^{a b}::InverseMetric.
8
9  Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} ( \partial_{g}{g_{b f}}
10                                     + \partial_{f}{g_{b g}}
11                                     - \partial_{b}{g_{f g}} ).
12
13  Rabcd := R^{d}_{e f g} -> \partial_{f}{\Gamma^{d}_{e g}}
14                               - \partial_{g}{\Gamma^{d}_{e f}}
15                               + \Gamma^{d}_{b f} \Gamma^{b}_{e g}
16                               - \Gamma^{d}_{b g} \Gamma^{b}_{e f}.
17
18  Rab := R_{a b} -> R^{c}_{c} {a c b}.
19
20  gab := { g_{\theta \theta} = r**2,
21           g_{\varphi \varphi} = r**2 \sin(\theta)**2 }. # cdb(ex-07.101,gab)
22
23  complete (gab, $g^{a b}$) # cdb(ex-07.102,gab)
24
25  substitute (Rabcd, Gamma)
26  substitute (Rab, Rabcd)
27
28  evaluate (Gamma, gab, rhsonly=True) # cdb(ex-07.103,Gamma)
29  evaluate (Rabcd, gab, rhsonly=True) # cdb(ex-07.104,Rabcd)
30  evaluate (Rab, gab, rhsonly=True) # cdb(ex-07.105,Rab)
31
32  write_code (Gamma[1], 'myGamma', 'example-07-gamma.c', 3)
33  write_code (Rabcd[1], 'myRabcd', 'example-07-rabcd.c', 4)
34  write_code (Rab[1], 'myRab', 'example-07-rab.c', 2)

```

$$[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2 (\sin \theta)^2] \quad (\text{ex-07.101})$$

$$\left[g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2 (\sin \theta)^2, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2 (\sin \theta)^2)^{-1} \right] \quad (\text{ex-07.102})$$

$$\Gamma_{fg}^a \rightarrow \square_{fg}^a \begin{cases} \square_{\varphi\theta}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2} \sin(2\theta) \end{cases} \quad (\text{ex-07.103})$$

$$R_{efg}^d \rightarrow \square_{eg}^d \begin{cases} \square_{\varphi\varphi}^{\theta} = (\sin \theta)^2 \\ \square_{\theta\varphi}^{\varphi} = -1 \\ \square_{\varphi\theta}^{\theta} = -(\sin \theta)^2 \\ \square_{\theta\theta}^{\varphi} = 1 \end{cases} \quad (\text{ex-07.104})$$

$$R_{ab} \rightarrow \square_{ab} \begin{cases} \square_{\varphi\varphi} = (\sin \theta)^2 \\ \square_{\theta\theta} = 1 \end{cases} \quad (\text{ex-07.105})$$

Example 8 Importing and exporting Cadabra expressions

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 def create (file_name):
4     import json, io, os, errno
5
6     try:
7         os.remove(file_name)           # delete the file if it exists
8         with open(file_name, 'w'): pass # create an empty file
9     except OSError as e:
10        if e.errno == errno.ENOENT:     # errno.ENOENT = no such file or directory
11            with open(file_name, 'w'): pass # create an empty file
12        else:
13            raise                       # report an exception
14
15    # Create and save an empty dict
16    data_out = {}
17    with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
18        out_file.write(json.dumps(data_out,
19                                indent=2,
20                                sort_keys=True,
21                                separators=(',', ' '),
22                                ensure_ascii=False)+'\n')
23
24 def put (key_name, object, file_name):
25     import json, io, os
26
27     # Read the current dict
28     with io.open(os.getcwd() + '/' + file_name) as inp_file:
29         data_out = json.load(inp_file)
30
31     # Add a new entry to the dict
32     data_out[key_name] = object.input_form()
33
34     # Save the updated dict
35     with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
36         out_file.write(json.dumps(data_out,
```

```

37         indent=2,
38         sort_keys=True,
39         separators=(',', ' '),
40         ensure_ascii=False)+'\n')
41
42 def get (key_name,file_name):
43     import json, io, os
44
45     # Read the current dict
46     with io.open(os.getcwd() + '/' + file_name) as inp_file:
47         data_inp = json.load(inp_file)
48
49     # Return one entry from the dict
50     return Ex (data_inp[key_name])
51
52 lib_name = 'example-08.json'
53
54 create (lib_name)
55
56 \nabla{#}::Derivative.
57
58 gab := g_{a b} - 1/3 x^{c} x^{d} R_{a c b d}
59         - 1/6 x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}. # cdb (ex-08-02.101,gab)
60
61 iab := g^{a b} + 1/3 x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
62         + 1/6 x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}. # cdb (ex-08-02.102,iab)
63
64 put ('g_ab',gab,lib_name)
65 put ('g^ab',iab,lib_name)
66
67 foo = get ('g_ab',lib_name) # cdb (ex-08-02.103,foo)
68 bah = get ('g^ab',lib_name) # cdb (ex-08-02.104,bah)
69
70 tmp := @(gab) - @(foo). # cdb (ex-08-02.105,tmp)
71 tmp := @(iab) - @(bah). # cdb (ex-08-02.106,tmp)

```

$$g_{ab}(x) = g_{ab} - \frac{1}{3} x^c x^d R_{acbd} - \frac{1}{6} x^c x^d x^e \nabla_c R_{adbe} \quad (\text{ex-08-02.101})$$

$$g^{ab}(x) = g^{ab} + \frac{1}{3} x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6} x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \quad (\text{ex-08-02.102})$$

$$\bar{g}_{ab}(x) = g_{ab} - \frac{1}{3} x^c x^d R_{acbd} - \frac{1}{6} x^c x^d x^e \nabla_c R_{adbe} \quad (\text{ex-08-02.103})$$

$$\bar{g}^{ab}(x) = g^{ab} + \frac{1}{3} x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6} x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \quad (\text{ex-08-02.104})$$

$$g_{ab}(x) - \bar{g}_{ab}(x) = 0 \quad (\text{ex-08-02.105})$$

$$g^{ab}(x) - \bar{g}^{ab}(x) = 0 \quad (\text{ex-08-02.106})$$

Example 9 The Gauss equation

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4
5 K_{a b}::Symmetric.
6 g^{a}_{b}::KroneckerDelta.
7
8 # define the projection operator
9
10 hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}.
11
12 # 3-covariant derivative obtained by projection on 4-covariant derivative
13
14 vpq:=v_{p q} -> h^{a}_{p} h^{b}_{q} \nabla_{b}{v_{a}}.
15
16 # compute 3-curvature by commutation of covariant derivatives
17
18 vpqr:= h^{a}_{p} h^{b}_{q} h^{c}_{r} ( \nabla_{c}{v_{a b}} - \nabla_{b}{v_{a c}} ).
19
20 substitute (vpq,hab)
21 substitute (vpqr,vpq)
22
23 distribute (vpqr)
24 product_rule (vpqr)
25 distribute (vpqr)
26 eliminate_kronecker (vpqr)
27
28 # standard substitutions
29
30 substitute (vpqr,$h^{a}_{b} n^{b} -> 0$)
31 substitute (vpqr,$h^{a}_{b} n_{a} -> 0$)
32 substitute (vpqr,$\nabla_{a}{g^{b}_{c}} -> 0$)
33 substitute (vpqr,$n^{a} \nabla_{b}{v_{a}} -> -v_{a} \nabla_{b}{n^{a}}$)
34 substitute (vpqr,$v_{a} \nabla_{b}{n^{a}} -> v_{p} h^{p}_{a} \nabla_{b}{n^{a}}$)
35 substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n_{q}} -> K_{a b}$)
36 substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n^{b}} -> K_{a}^{q}$) # cdb(ex-09.095,vpqr)
```



```

37
38 # tidy up
39
40 {v_{a},h^{a}_{b},K_{a}^{b},K_{a b},R^{a}_{b c d},\nabla_{a}\{v_{b}\}}::SortOrder.
41
42 sort_product      (vpqr)                # cdb(ex-09.096,vpqr)
43 rename_dummies    (vpqr)                # cdb(ex-09.097,vpqr)
44 canonicalise      (vpqr)                # cdb(ex-09.098,vpqr)
45 factor_out        (vpqr,$h^{a?}_{b?}$)   # cdb(ex-09.099,vpqr)
46 factor_out        (vpqr,$v_{a?}$)       # cdb(ex-09.101,vpqr)
47
48 checkpoint.append (vpqr)

```

$$(D_r D_q - D_q D_r) v_p = h_p^e h_q^d h_r^c \nabla_c (\nabla_d v_e) - h_p^e K_{rq} n^d \nabla_d v_e + K_q^b K_{rp} v_b - h_p^d h_q^b h_r^e \nabla_b (\nabla_d v_e) + h_p^d K_{qr} n^e \nabla_d v_e - K_{qp} K_r^c v_c \quad (\text{ex-09.095})$$

$$= h_r^c h_q^d h_p^e \nabla_c (\nabla_d v_e) - h_p^e K_{rq} \nabla_d v_e n^d + v_b K_q^b K_{rp} - h_q^b h_p^d h_r^e \nabla_b (\nabla_d v_e) + h_p^d K_{qr} \nabla_d v_e n^e - v_c K_r^c K_{qp} \quad (\text{ex-09.096})$$

$$= h_r^a h_q^b h_p^c \nabla_a (\nabla_b v_c) - h_p^b K_{rq} \nabla_a v_b n^a + v_a K_q^a K_{rp} - h_q^a h_p^c h_r^b \nabla_a (\nabla_b v_c) + h_p^b K_{qr} \nabla_a v_b n^a - v_a K_r^a K_{qp} \quad (\text{ex-09.097})$$

$$= h_p^a h_q^b h_r^c \nabla_c (\nabla_b v_a) + v_a K_q^a K_{pr} - h_p^a h_q^b h_r^c \nabla_b (\nabla_a v_c) - v_a K_r^a K_{pq} \quad (\text{ex-09.098})$$

$$= v_a K_q^a K_{pr} - v_a K_r^a K_{pq} + h_p^a h_q^b h_r^c (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) \quad (\text{ex-09.099})$$

$$= h_p^a h_q^b h_r^c (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.101})$$

```

1 R{#}::LaTeXForm("\{\strut\}^g R").
2
3 gRabcd := \nabla_{\{c\}}{\nabla_{\{b\}}{v_{\{a\}}}}
4          -\nabla_{\{b\}}{\nabla_{\{c\}}{v_{\{a\}}}} -> R^{\{d\}}_{\{a\} \{b\} \{c\}} v_{\{d\}}.
5
6 substitute      (vpqr,gRabcd)                # cdb(ex-09.102,vpqr)
7 distribute      (vpqr)                        # cdb(ex-09.103,vpqr)
8 substitute      (vpqr,$v_{\{a\}} -> h^{\{b\}}_{\{a\}} v_{\{b\}}$) # cdb(ex-09.104,vpqr)
9 substitute      (vpqr,$h^{\{b\}}_{\{a\}} K_{\{c\}}^{\{a\}} -> K_{\{c\}}^{\{b\}}$) # cdb(ex-09.105,vpqr)
10 sort_product   (vpqr)                        # cdb(ex-09.106,vpqr)
11 rename_dummies (vpqr)                        # cdb(ex-09.107,vpqr)
12 canonicalise   (vpqr)                        # cdb(ex-09.108,vpqr)
13 factor_out     (vpqr,$v_{\{a\}}$)            # cdb(ex-09.109,vpqr)
14 substitute      (vpqr,$v_{\{a\}}->1$)        # cdb(ex-09.110,vpqr)
15 sort_product   (vpqr)                        # cdb(ex-09.111,vpqr)
16
17 checkpoint.append (vpqr)

```

$$(D_r D_q - D_q D_r) v_p = h_p^a h_q^b h_r^c (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.101})$$

$$= h_p^a h_q^b h_r^c R_{abc}^d v_d + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.102})$$

$$= h_p^a h_q^b h_r^c R_{abc}^d v_d + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.103})$$

$$= h_p^a h_q^b h_r^c R_{abc}^d h_d^e v_e + h_a^b v_b K_q^a K_{pr} - h_a^b v_b K_r^a K_{pq} \quad (\text{ex-09.104})$$

$$= h_p^a h_q^b h_r^c R_{abc}^d h_d^e v_e + K_q^b v_b K_{pr} - K_r^b v_b K_{pq} \quad (\text{ex-09.105})$$

$$= v_e h_p^a h_q^b h_r^c h_d^e R_{abc}^d + v_b K_q^b K_{pr} - v_b K_r^b K_{pq} \quad (\text{ex-09.106})$$

$$= v_e h_p^b h_q^c h_r^d h_a^e R_{bcd}^a + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.107})$$

$$= v_a h_p^b h_q^c h_r^d h_e^a R_{bcd}^e + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.108})$$

$$= v_a (h_p^b h_q^c h_r^d h_e^a R_{bcd}^e + K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.109})$$

$${}^h R_{pqr}^a = h_p^b h_q^c h_r^d h_e^a R_{bcd}^e + K_q^a K_{pr} - K_r^a K_{pq} \quad (\text{ex-09.110})$$

$${}^h R_{pqr}^a = h_e^a h_p^b h_q^c h_r^d R_{bcd}^e + K_q^a K_{pr} - K_r^a K_{pq} \quad (\text{ex-09.111})$$

Example 10 The determinant of the metric

Our game here is to compute (the leading terms) in $\det g$ of the metric in RNC form

$$g_{ab}(x) = g_{ab} - \frac{1}{3} x^c x^d R_{acbd} - \frac{1}{6} x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180} x^c x^d x^e x^f (8 g^{gh} R_{acd g} R_{b e f h} - 9 \nabla_{cd} R_{a e b f}) + \dots$$

For the sake of simplicity let's assume that we are working in 3-dimensions. The following analysis is easily generalised to other dimensions (and the final answers for $\det g$ and friends are unchanged).

Define ϵ_{ijk}^{abc} by

$$\epsilon_{ijk}^{abc} = \delta_i^a \delta_j^b \delta_k^c - \delta_i^b \delta_j^a \delta_k^c + \delta_i^c \delta_j^a \delta_k^b - \delta_i^c \delta_j^b \delta_k^a + \delta_i^b \delta_j^c \delta_k^a - \delta_i^a \delta_j^c \delta_k^b \quad (1)$$

It is easy to see that ϵ_{ijk}^{abc} is anti-symmetric in both its upper and lower indices. A trivial computation shows that for any 3×3 square matrix M_{ab} ,

$$\epsilon_{123}^{abc} M_{1a} M_{2b} M_{3c} = (\delta_1^a \delta_2^b \delta_3^c - \delta_1^b \delta_2^a \delta_3^c + \delta_1^c \delta_2^a \delta_3^b - \delta_1^c \delta_2^b \delta_3^a + \delta_1^b \delta_2^c \delta_3^a - \delta_1^a \delta_2^c \delta_3^b) M_{1a} M_{2b} M_{3c} = \det M \quad (2)$$

This can be easily generalised to

$$\epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} = \begin{cases} \pm \det M & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The \pm sign in the above depends on the particular permutations of (ijk) and (pqr) . If both permutations are even or both odd then the sign is $+1$ otherwise the sign is -1 . The same arguments can also be applied to a matrix inverse N^{-1} leading to

$$\epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} = \begin{cases} \pm \det N^{-1} & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Note that the \pm in this case will match exactly that for the case of $\det M$. Thus, multiplying both expressions and summing over all choices for (ijk) and (pqr) leads to

$$\sum_{\substack{(ijk) \\ (pqr)}} (\det N^{-1}) \det M = \epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} \epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} \quad (5)$$

where the sum on the left hand side includes just those (ijk) and (prq) that are permutations of (123) . There are $3!$ choices for (ijk) and $3!$ choices for (prq) and thus the left hand side is easily reduced to $(3!)^2 \det M / \det N$ where $\det N = 1 / \det(N^{-1})$. For the right hand side notice that

$$\epsilon_{uvw}^{ijk} \epsilon_{ijk}^{abc} = 3! \epsilon_{uvw}^{abc} \quad (6)$$

which leads to

$$\det M = \frac{1}{3!} \det N \epsilon_{uvw}^{abc} M_{pa} M_{qb} M_{rc} N^{pu} N^{qv} N^{rw} \quad (7)$$

For our RNC metric we will set $N^{ab} = g^{ab}$ and $M_{ij} = g_{ij}(x)$. Since g^{ab} is of the form $\text{diag}(-1, 1, 1, 1)$ we have $\det g = -1$ and thus

$$\det g(x) = -\frac{1}{3!} \epsilon_{ijk}^{abc} g_{pa}(x) g_{qb}(x) g_{rc}(x) g^{ip} g^{jq} g^{kr} \quad (8)$$

The ϵ_{ijk}^{abc} can be constructed in Cadabra by applying the `asym` algorithm to the upper indices of $\delta_i^a \delta_j^b \delta_k^c$. Note that `asym` will include the $1/3!$ coefficient as part of its output.

The following code computes $-\det g$ rather than $\det g$.

Note that Calzetta et al. use an opposite sign for R_{abcd} so when comparing the following results against Calzetta do take note of this flipped sign in R_{abcd} .

The determinant of the metric

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Integer(1..3).
4
5 \nabla{#}::Derivative.
6
7 d{#}::KroneckerDelta.
8
9 g^{a b}::Symmetric.
10 g_{a b}::Symmetric.
11
12 R_{a b c d}::RiemannTensor.
13
14 x^{a}::Weight(label=num,value=1).
15
16 def truncate (obj,n):
17
18     ans = Ex("0") # create a Cadabra object with value zero
19
20     for i in range (0,n+1):
21         foo := @(obj).
22         bah = Ex("num = " + str(i))
23         distribute (foo)
24         keep_weight (foo, bah)
25         ans = ans + foo
26
27     return ans
28
29 gab := g_{a b}
30         - (1/3) x^{c} x^{d} R_{a c b d}
31         - (1/6) x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
32         + (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
33                                         -9 \nabla_{c d}{R_{a e b f}} ). # cdb (ex-10.gab.000,gab)
34
35 iab := g^{a b}
36         + (1/3) x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
```

```

37      + (1/6) x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}\{R_{d f e} g\}
38      + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
39          ( 4 g^{i j} R_{c g d i} R_{e h f j}
40          + 3 \nabla_{c d}\{R_{e g f h}\} ).
41
42      distribute (gab)
43      distribute (iab)
44
45      gxab := gx_{a b} -> @(gab).
46
47      eps := d^{a}_{i} d^{b}_{j} d^{c}_{k}.      # cdb (ex-10.eps.001,eps) # includes a factor of 1/3!
48      asym (eps,$^{a},^{b},^{c}$)             # cdb (ex-10.eps.002,eps)
49
50      # compute negative detg rather than det g, note 1/3! included in eps
51      Ndetg := @(eps) gx_{p a} gx_{q b} gx_{r c} g^{i p} g^{j q} g^{k r}.      # cdb (ex-10.Ndetg.001,Ndetg)
52
53      substitute      (Ndetg,gxab)                                     # cdb (ex-10.Ndetg.002,Ndetg)
54      distribute      (Ndetg)                                         # cdb (ex-10.Ndetg.003,Ndetg)
55      Ndetg = truncate (Ndetg,4)                                       # cdb (ex-10.Ndetg.004,Ndetg)
56      substitute      (Ndetg,$g^{a b} g_{b c} -> d^{a}_{c}$,repeat=True) # cdb (ex-10.Ndetg.005,Ndetg)
57      eliminate_kronecker (Ndetg)                                     # cdb (ex-10.Ndetg.006,Ndetg)
58      sort_product     (Ndetg)                                         # cdb (ex-10.Ndetg.007,Ndetg)
59      rename_dummies   (Ndetg)                                         # cdb (ex-10.Ndetg.008,Ndetg)
60      canonicalise     (Ndetg)                                         # cdb (ex-10.Ndetg.009,Ndetg)
61
62      # introduce the Ricci tensor
63
64      substitute (Ndetg,$R_{a b c d} g^{a c} -> R_{b d}$,repeat=True)      # cdb (ex-10.Ndetg.101,Ndetg)
65      substitute (Ndetg,$\nabla_{a}\{R_{b c d e}\} g^{b d} -> \nabla_{a}\{R_{c e}\}$,repeat=True) # cdb (ex-10.Ndetg.102,Ndetg)
66      substitute (Ndetg,$\nabla_{a b}\{R_{c d e f}\} g^{c e} -> \nabla_{a b}\{R_{d f}\}$,repeat=True) # cdb (ex-10.Ndetg.103,Ndetg)
67
68      # the following was based on sqrt-Ndetg.tex
69
70      sqrtNdetg := 1/2 + (1/2) @(Ndetg)
71          - (1/8) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
72          - (1/4) (1/18) R_{a b} \nabla_{c}\{R_{d e}\} x^{a} x^{b} x^{c} x^{d} x^{e}.
73
74          # cdb (ex-10.sqrtNdetg.001,sqrtNdetg)

```

```

75 sort_product (sqrtNdetg) # cdb (ex-10.sqrtNdetg.002,sqrtNdetg)
76 rename_dummies (sqrtNdetg) # cdb (ex-10.sqrtNdetg.003,sqrtNdetg)
77 canonicalise (sqrtNdetg) # cdb (ex-10.sqrtNdetg.004,sqrtNdetg)
78
79 logNdetg := -1 + @(Ndetg)
80 - (1/2) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
81 - (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
82 # cdb (ex-10.logNdetg.001,logNdetg)
83
84 sort_product (logNdetg) # cdb (ex-10.logNdetg.002,logNdetg)
85 rename_dummies (logNdetg) # cdb (ex-10.logNdetg.003,logNdetg)
86 canonicalise (logNdetg) # cdb (ex-10.logNdetg.004,logNdetg)
87
88 # =====
89 # the remaining code is just for pretty printing
90
91 def product_sort (obj):
92     substitute (obj,$ x^{a} -> A000^{a} $)
93     substitute (obj,$ g^{a b} -> A001^{a b} $)
94     substitute (obj,$ \nabla_{c}{R_{a b}} -> A004_{a b c} $)
95     substitute (obj,$ \nabla_{c d}{R_{a b}} -> A005_{a b c d} $)
96     substitute (obj,$ \nabla_{c d e}{R_{a b}} -> A006_{a b c d e} $)
97     substitute (obj,$ \nabla_{c d e f}{R_{a b}} -> A007_{a b c d e f} $)
98     substitute (obj,$ \nabla_{e}{R_{a b c d}} -> A008_{a b c d e} $)
99     substitute (obj,$ \nabla_{e f}{R_{a b c d}} -> A009_{a b c d e f} $)
100    substitute (obj,$ \nabla_{e f g}{R_{a b c d}} -> A010_{a b c d e f g} $)
101    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}} -> A011_{a b c d e f g h} $)
102    substitute (obj,$ R_{a b} -> A002_{a b} $)
103    substitute (obj,$ R_{a b c d} -> A003_{a b c d} $)
104    sort_product (obj)
105    rename_dummies (obj)
106    substitute (obj,$ A000^{a} -> x^{a} $)
107    substitute (obj,$ A001^{a b} -> g^{a b} $)
108    substitute (obj,$ A002_{a b} -> R_{a b} $)
109    substitute (obj,$ A003_{a b c d} -> R_{a b c d} $)
110    substitute (obj,$ A004_{a b c} -> \nabla_{c}{R_{a b}} $)
111    substitute (obj,$ A005_{a b c d} -> \nabla_{c d}{R_{a b}} $)
112    substitute (obj,$ A006_{a b c d e} -> \nabla_{c d e}{R_{a b}} $)

```

```

113 substitute (obj,$ A007_{a b c d e f}      -> \nabla_{c d e f}{R_{a b}}      $)
114 substitute (obj,$ A008_{a b c d e}        -> \nabla_{e}{R_{a b c d}}      $)
115 substitute (obj,$ A009_{a b c d e f}      -> \nabla_{e f}{R_{a b c d}}      $)
116 substitute (obj,$ A010_{a b c d e f g}    -> \nabla_{e f g}{R_{a b c d}}    $)
117 substitute (obj,$ A011_{a b c d e f g h}  -> \nabla_{e f g h}{R_{a b c d}}  $)
118
119 def get_term (obj,n):
120
121     x^{a}::Weight(label=xnum).
122
123     foo := @(obj).
124     bah = Ex("xnum = " + str(n))
125     keep_weight (foo,bah)
126
127     return foo
128
129 def reformat (obj,scale):
130     foo = Ex(str(scale))
131     bah := @(foo) @(obj).
132     distribute      (bah)
133     product_sort    (bah)
134     rename_dummies  (bah)
135     canonicalise    (bah)
136     sort_sum        (bah)
137     factor_out      (bah,$x^{a?}$)
138     ans := @(bah) / @(foo).
139     return ans
140
141 def rescale (obj,scale):
142     foo = Ex(str(scale))
143     bah := @(foo) @(obj).
144     distribute      (bah)
145     factor_out      (bah,$x^{a?}$)
146     return bah
147
148 # -----
149 # reformat Ndetg
150

```



```

151 Rterm0 = get_term (Ndetg,0)      # cdb (ex-10.Rterm0.701,Rterm0)
152 Rterm1 = get_term (Ndetg,1)      # cdb (ex-10.Rterm1.701,Rterm1)
153 Rterm2 = get_term (Ndetg,2)      # cdb (ex-10.Rterm2.701,Rterm2)
154 Rterm3 = get_term (Ndetg,3)      # cdb (ex-10.Rterm3.701,Rterm3)
155 Rterm4 = get_term (Ndetg,4)      # cdb (ex-10.Rterm4.701,Rterm4)
156
157 Rterm0 = reformat (Rterm0, 1)     # cdb (ex-10.Rterm0.702,Rterm0)
158 Rterm1 = reformat (Rterm1, 1)     # cdb (ex-10.Rterm1.702,Rterm1)
159 Rterm2 = reformat (Rterm2, 3)     # cdb (ex-10.Rterm2.702,Rterm2)
160 Rterm3 = reformat (Rterm3, 6)     # cdb (ex-10.Rterm3.702,Rterm3)
161 Rterm4 = reformat (Rterm4,180)    # cdb (ex-10.Rterm4.702,Rterm4)
162
163 Ndetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.Ndetg.701,Ndetg)
164
165 # -----
166 # reformat sqrtNdetg
167
168 Rterm0 = get_term (sqrtNdetg,0)    # cdb (ex-10.Rterm0.801,Rterm0)
169 Rterm1 = get_term (sqrtNdetg,1)    # cdb (ex-10.Rterm1.801,Rterm1)
170 Rterm2 = get_term (sqrtNdetg,2)    # cdb (ex-10.Rterm2.801,Rterm2)
171 Rterm3 = get_term (sqrtNdetg,3)    # cdb (ex-10.Rterm3.801,Rterm3)
172 Rterm4 = get_term (sqrtNdetg,4)    # cdb (ex-10.Rterm4.801,Rterm4)
173
174 Rterm0 = reformat (Rterm0, 1)      # cdb (ex-10.Rterm0.802,Rterm0)
175 Rterm1 = reformat (Rterm1, 1)      # cdb (ex-10.Rterm1.802,Rterm1)
176 Rterm2 = reformat (Rterm2, 6)      # cdb (ex-10.Rterm2.802,Rterm2)
177 Rterm3 = reformat (Rterm3, 12)     # cdb (ex-10.Rterm3.802,Rterm3)
178 Rterm4 = reformat (Rterm4,360)     # cdb (ex-10.Rterm4.802,Rterm4)
179
180 sqrtNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.sqrtNdetg.801,sqrtNdetg)
181
182 # -----
183 # reformat logNdetg
184
185 Rterm0 = get_term (logNdetg,0)     # cdb (ex-10.Rterm0.801,Rterm0)
186 Rterm1 = get_term (logNdetg,1)     # cdb (ex-10.Rterm1.801,Rterm1)
187 Rterm2 = get_term (logNdetg,2)     # cdb (ex-10.Rterm2.801,Rterm2)
188 Rterm3 = get_term (logNdetg,3)     # cdb (ex-10.Rterm3.801,Rterm3)

```

```

189 Rterm4 = get_term (logNdetg,4)      # cdb (ex-10.Rterm4.801,Rterm4)
190
191 Rterm0 = reformat (Rterm0, 1)      # cdb (ex-10.Rterm0.802,Rterm0)
192 Rterm1 = reformat (Rterm1, 1)      # cdb (ex-10.Rterm1.802,Rterm1)
193 Rterm2 = reformat (Rterm2, 3)      # cdb (ex-10.Rterm2.802,Rterm2)
194 Rterm3 = reformat (Rterm3, 6)      # cdb (ex-10.Rterm3.802,Rterm3)
195 Rterm4 = reformat (Rterm4,180)     # cdb (ex-10.Rterm4.802,Rterm4)
196
197 logNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.logNdetg.901,logNdetg)
198
199 checkpoint.append (Ndetg)
200 checkpoint.append (sqrtNdetg)
201 checkpoint.append (logNdetg)

```

The metric determinant in Riemann normal coordinates

$$-\det g(x) = 1 - \frac{1}{3} x^a x^b R_{ab} - \frac{1}{6} x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180} x^a x^b x^c x^d (-9 \nabla_{ab} R_{cd} + 10 R_{ab} R_{cd} - 2 g^{ef} g^{gh} R_{aebg} R_{cf dh}) + \dots$$

The volume element in RNC

If $-\det g(x)$ is non-negative then we also have

$$\sqrt{-\det g(x)} = 1 - \frac{1}{6} x^a x^b R_{ab} - \frac{1}{12} x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360} x^a x^b x^c x^d (-9 \nabla_{ab} R_{cd} + 5 R_{ab} R_{cd} - 2 g^{ef} g^{gh} R_{aebg} R_{cf dh}) + \dots$$

The log of -detg in RNC

$$\log(-\det g(x)) = -\frac{1}{3} x^a x^b R_{ab} - \frac{1}{6} x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180} x^a x^b x^c x^d (-9 \nabla_{ab} R_{cd} - 2 g^{ef} g^{gh} R_{aebg} R_{cf dh}) + \dots$$

Apart from the signs, this matches exactly the expression given by Calzetta et al. (eq. A14)

Example 11 The RNC connection.

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent).
2
3  D{#}::PartialDerivative.
4  \nabla{#}::Derivative.
5
6  g_{a b}::Metric.
7  g^{a b}::InverseMetric.
8  g^{a b}::Weight(label=gnum,value=1).
9
10 \delta{#}::KroneckerDelta.
11
12 R_{a b c d}::RiemannTensor.
13 R_{a b c d}::Depends(\nabla{#}).
14
15 x^{a}::Depends(D{#}).
16 x^{a}::Weight(label=xnum,value=1).
17
18 Dx := D_{a}{x^{b}} -> \delta^{b}_{a}.    # cdb (ex-11.000,Dx)
19
20 gab := g_{a b} -> g_{a b}
21      - (1/3)  x^{c} x^{d} R_{a c b d}
22      - (1/6)  x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
23      + (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
24      - 9 \nabla_{c d}{R_{a e b f}} ).    # cdb (ex-11.001,gab)
25
26 iab := g^{a b} -> g^{a b}
27      + (1/3)  x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
28      + (1/6)  x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}
29      + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
30      ( 4 g^{i j} R_{c g d i} R_{e h f j}
31      + 3 \nabla_{c d}{R_{e g f h}} ).    # cdb(ex-11.002,iab)
32
33 distribute (gab)
34 distribute (iab)
35
36 ChrSym := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( D_{b}{g_{d c}}

```

```

37         + D_{c}{g_{b d}}
38         - D_{d}{g_{b c}} ). # cdb (ex-11.003,ChrSym)
39
40 Gamma := \Gamma^{a}_{b c}. # cdb (ex-11.100,Gamma)
41
42 substitute (Gamma,ChrSym) # cdb (ex-11.101,Gamma)
43 substitute (Gamma,gab) # cdb (ex-11.102,Gamma)
44 substitute (Gamma,iab) # cdb (ex-11.103,Gamma)
45 distribute (Gamma) # cdb (ex-11.104,Gamma)
46 unwrap (Gamma) # cdb (ex-11.105,Gamma)
47 product_rule (Gamma) # cdb (ex-11.106,Gamma)
48 distribute (Gamma) # cdb (ex-11.107,Gamma)
49 substitute (Gamma,Dx) # cdb (ex-11.108,Gamma)
50 eliminate_kronecker (Gamma) # cdb (ex-11.109,Gamma)
51
52 def truncate (obj,n):
53
54     ans = Ex("0") # create a Cadabra object with value zero
55
56     for i in range (0,n+1):
57         foo := @obj.
58         bah = Ex("xnum = " + str(i))
59         distribute (foo)
60         keep_weight (foo, bah)
61         ans = ans + foo
62
63     return ans
64
65 checkpoint.append (Gamma)
66
67 # sort_product (Gamma) # 52.3 sec, 49 Mbyte
68 # rename_dummies (Gamma) # 58.6 sec, 51 Mbyte
69 # canonicalise (Gamma) # killed after 20 mins and over 500 Mbyte
70
71 Gamma = truncate (Gamma,3) # cdb (ex-11.110,Gamma) # allow up to 3rd order in x^a
72
73 sort_product (Gamma)
74 rename_dummies (Gamma)

```

```

75 canonicalise (Gamma)
76
77 checkpoint.append (Gamma)
78
79 # =====
80 # the remaining code is just for pretty printing
81
82 def product_sort (obj):
83     substitute (obj,$ g^{a b}          -> A001^{a b}          $)
84     substitute (obj,$ x^{a}            -> A002^{a}          $)
85     substitute (obj,$ z^{a}            -> A003^{a}          $)
86     substitute (obj,$ R_{a b c d}       -> A004_{a b c d}     $)
87     substitute (obj,$ \nabla_{e}\{R_{a b c d}\} -> A005_{a b c d e}     $)
88     substitute (obj,$ \nabla_{e f}\{R_{a b c d}\} -> A006_{a b c d e f}   $)
89     sort_sum      (obj)
90     sort_product  (obj)
91     rename_dummies (obj)
92     substitute (obj,$ A001^{a b}          -> g^{a b}          $)
93     substitute (obj,$ A002^{a}            -> x^{a}            $)
94     substitute (obj,$ A003^{a}            -> z^{a}            $)
95     substitute (obj,$ A004_{a b c d}       -> R_{a b c d}       $)
96     substitute (obj,$ A005_{a b c d e}     -> \nabla_{e}\{R_{a b c d}\} $)
97     substitute (obj,$ A006_{a b c d e f}   -> \nabla_{e f}\{R_{a b c d}\} $)
98
99 def get_xterm (obj,n):
100
101     foo := @(obj).
102     bah = Ex("xnum = " + str(n))
103     distribute (foo)
104     keep_weight (foo, bah)
105
106     return foo
107
108 def get_gterm (obj,n):
109
110     foo := @(obj).
111     bah = Ex("gnum = " + str(n))
112     distribute (foo)

```

```

113     keep_weight (foo, bah)
114
115     return foo
116
117 def reformat (obj,scale):
118
119     foo = Ex(str(scale))
120     bah := @(foo) @(obj).
121
122     distribute      (bah)
123     product_sort    (bah)
124     rename_dummies  (bah)
125     canonicalise     (bah)
126     factor_out      (bah,$x^{a?},g^{b? c?}$)
127     ans := @(bah) / @(foo).
128
129     return ans
130
131 gam1 = get_xterm (Gamma, 1)           # cdb (ex-11.200,gam1)
132 gam2 = get_xterm (Gamma, 2)           # cdb (ex-11.201,gam2)
133 gam3 = get_xterm (Gamma, 3)           # cdb (ex-11.202,gam3)
134
135 gam31 = get_gterm (gam3, 1)           # cdb (ex-11.210,gam31)
136 gam32 = get_gterm (gam3, 2)           # cdb (ex-11.211,gam31)
137
138 gam1 = reformat (gam1, 3)             # cdb (ex-11.220,gam1)
139 gam2 = reformat (gam2, 12)            # cdb (ex-11.221,gam2)
140
141 gam31 = reformat (gam31, 40)          # cdb (ex-11.222,gam31)
142 gam32 = reformat (gam32, 45)          # cdb (ex-11.223,gam32)
143
144 Gamma := @(gam1) + @(gam2) + @(gam31) + @(gam32). # cdb (ex-11.230,Gamma)
145 Scaled := 360 @(Gamma).               # cdb (ex-11.231,Scaled)
146
147 checkpoint.append (Gamma)

```

$$\begin{aligned}
\Gamma_{bc}^a(x) = & \frac{1}{3} g^{ad} x^e (R_{bdce} + R_{becd}) + \frac{1}{12} g^{ad} x^e x^f (-\nabla_e R_{bedf} + \nabla_d R_{becf} + 2 \nabla_e R_{bdcf} + 2 \nabla_e R_{bfcd} - \nabla_b R_{cedf}) \\
& + \frac{1}{40} g^{ad} x^e x^f x^g (-\nabla_{ce} R_{bfdg} - \nabla_{ec} R_{bfdg} + \nabla_{de} R_{bfcg} + \nabla_{ed} R_{bfcg} + 2 \nabla_{ef} R_{bdcg} + 2 \nabla_{ef} R_{bgcd} - \nabla_{be} R_{cfdg} - \nabla_{eb} R_{cfdg}) \\
& + \frac{1}{45} g^{ad} g^{ef} x^g x^h x^i (4 R_{becg} R_{dhfi} + 4 R_{bgce} R_{dhfi} - 2 R_{bdeg} R_{chfi} - R_{bedg} R_{chfi} + R_{bgde} R_{chfi} - 2 R_{bgeh} R_{cdfi} - R_{bgeh} R_{cfdi} \\
& \qquad \qquad \qquad + R_{bgeh} R_{cidf}) \quad (\text{ex-11.230})
\end{aligned}$$

$$\begin{aligned}
360\Gamma_{bc}^a(x) = & 120 g^{ad} x^e (R_{bdce} + R_{becd}) + 30 g^{ad} x^e x^f (-\nabla_e R_{bedf} + \nabla_d R_{becf} + 2 \nabla_e R_{bdcf} + 2 \nabla_e R_{bfcd} - \nabla_b R_{cedf}) \\
& + 9 g^{ad} x^e x^f x^g (-\nabla_{ce} R_{bfdg} - \nabla_{ec} R_{bfdg} + \nabla_{de} R_{bfcg} + \nabla_{ed} R_{bfcg} + 2 \nabla_{ef} R_{bdcg} + 2 \nabla_{ef} R_{bgcd} - \nabla_{be} R_{cfdg} - \nabla_{eb} R_{cfdg}) \\
& + 8 g^{ad} g^{ef} x^g x^h x^i (4 R_{becg} R_{dhfi} + 4 R_{bgce} R_{dhfi} - 2 R_{bdeg} R_{chfi} - R_{bedg} R_{chfi} + R_{bgde} R_{chfi} - 2 R_{bgeh} R_{cdfi} - R_{bgeh} R_{cfdi} \\
& \qquad \qquad \qquad + R_{bgeh} R_{cidf}) \quad (\text{ex-11.231})
\end{aligned}$$

Save Γ_{bc}^a for later use in Example 12.

```
1 jsonfile = 'example-11.json'
2 cdblib.create (jsonfile)
3 cdblib.put ('Gamma',Gamma,jsonfile)
```

Example 12 Checking the 2nd and 3rd order terms of Calzetta etal.

The following calculations show that my results for the RNC connection agree with those of Calzetta etal. to third order terms.

Note that I take ∇_{ab} to be $\nabla_a(\nabla_b)$.

Note also that $(LCB) R_{abcd} = -(Calzetta) R_{abcd}$. Consequently, I replace R_{abcd} with $-R_{abcd}$ in the Calzetta expressions (done as a Cadabra substitution rule).

This is relatively straightforward. We just apply a few carefully chosen applications of the first and second Bianchi identities.

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,u,v#}::Indices("latin",position=independent).
2 {\mu,\nu,\rho,\sigma,\tau,\lambda,\xi#}::Indices("greek",position=independent).
3
4 \nabla{#}::Derivative.
5
6 g_{a b}::Metric.
7 g^{a b}::InverseMetric.
8 g^{a b}::Weight(label=gnum,value=1).
9
10 \delta{#}::KroneckerDelta.
11
12 R_{a b c d}::RiemannTensor.
13 R_{a b c d}::Depends(\nabla{#}).
14
15 x^{a}::Weight(label=xnum,value=1).
16
17 def add_tags (obj,tag):
18
19     n = 0
20     ans = Ex('0')
21
22     for i in obj.top().terms():
23         foo = obj[i]
24         bah = Ex(tag+'_{'+str(n)+'}')
25         ans := @(ans) + @(bah) @(foo).
26         n = n + 1
27
28     return ans
29
30 def clear_tags (obj,tag):
31
32     ans := @(obj).
33     foo = Ex(tag+'_{a?} -> 1')
34     substitute (ans,foo)
35
36     return ans
37
38 def get_xterm (obj,n):

```

```

39
40     foo := @(obj).
41     bah  = Ex("xnum = " + str(n))
42     distribute (foo)
43     keep_weight (foo, bah)
44
45     return foo
46
47 def get_gterm (obj,n):
48
49     foo := @(obj).
50     bah  = Ex("gnum = " + str(n))
51     distribute (foo)
52     keep_weight (foo, bah)
53
54     return foo
55
56 def product_sort (obj):
57     substitute (obj,$ g^{a b}                -> A001^{a b}                $)
58     substitute (obj,$ x^{a}                  -> A002^{a}                  $)
59     substitute (obj,$ z^{a}                  -> A003^{a}                  $)
60     substitute (obj,$ R_{a b c d}            -> A004_{a b c d}            $)
61     substitute (obj,$ \nabla_{e}\{R_{a b c d}\} -> A005_{a b c d e}        $)
62     substitute (obj,$ \nabla_{e f}\{R_{a b c d}\} -> A006_{a b c d e f}      $)
63     sort_sum      (obj)
64     sort_product  (obj)
65     rename_dummies (obj)
66     substitute (obj,$ A001^{a b}              -> g^{a b}              $)
67     substitute (obj,$ A002^{a}                -> x^{a}                $)
68     substitute (obj,$ A003^{a}                -> z^{a}                $)
69     substitute (obj,$ A004_{a b c d}          -> R_{a b c d}          $)
70     substitute (obj,$ A005_{a b c d e}        -> \nabla_{e}\{R_{a b c d}\} $)
71     substitute (obj,$ A006_{a b c d e f}      -> \nabla_{e f}\{R_{a b c d}\} $)
72
73 def reformat (obj,scaleA,scaleB):
74
75     foo  = Ex(str(scaleA))
76     moo  = Ex(str(scaleB))

```

```

77     bah := @(foo) @(obj) / @(moo).
78
79     distribute      (bah)
80     product_sort    (bah)
81     rename_dummies  (bah)
82     canonicalise     (bah)
83     factor_out       (bah,$g^{c? d?}$)
84     factor_out       (bah,$x^{a?},z^{b?}$)
85     ans := @(moo) @(bah) / @(foo).
86
87     return ans
88
89     # =====
90     # LCB
91
92     import cdblib
93     Gamma = cdblib.get ('Gamma','example-11.json')           # cdb(ex-12.100,Gamma)
94
95     # -----
96     # note: in versions prior to 24 may 2024, the first \Gamma^a shown in example 12
97     #       of the tutorial was actually the *downstairs* version
98     #       this block added to create reformed version of \Gamma^a for the tutorial
99
100    # note that the next line requires careful inspection of the free indices on Gamma
101    # expecting Gamma = \Gamma^a_{bc}
102
103    GammaUp := z^{b} z^{c} @(Gamma).                          # cdb(ex-12.110,GammaUp)
104
105    product_sort (GammaUp)                                     # cdb(ex-12.113,GammaUp)
106
107    checkpoint.append (GammaUp)
108
109    gam1 = get_xterm (GammaUp,1)                               # cdb(ex-12.210,gam1)
110    gam2 = get_xterm (GammaUp,2)                               # cdb(ex-12.211,gam2)
111    gam3 = get_xterm (GammaUp,3)                               # cdb(ex-12.212,gam3)
112
113    gam30 = get_gterm (gam3,1)                                 # cdb(ex-12.213,gam30)
114    gam31 = get_gterm (gam3,2)                                 # cdb(ex-12.214,gam31)

```

```

115
116 gam1 = reformat (gam1, 3,1) # cdb(ex-12.310,gam1)
117 gam2 = reformat (gam2,12,1) # cdb(ex-12.311,gam2)
118
119 gam30 = reformat (gam30,40,1) # cdb(ex-12.312,gam30)
120 gam31 = reformat (gam31,45,2) # cdb(ex-12.313,gam31)
121
122 gam3 := @(gam30) + @(gam31). # cdb(ex-12.314,gam3)
123
124 GammaUp := @(gam1) + @(gam2) + @(gam3). # cdb(ex-12.315,GammaUp)
125
126 checkpoint.append (GammaUp)
127 # -----
128
129 # lower index ^{a} to _{v}
130
131 Gamma := g_{v a} @(GammaUp).
132
133 distribute (Gamma)
134 substitute (Gamma, $g_{a d} g^{d b} -> \delta_{a}^{b}$)
135 eliminate_kronecker (Gamma) # cdb(ex-12.101,Gamma)
136
137 # change free index _{v} to _{a}
138
139 foo := tmp_{v} -> @(Gamma). # cdb(ex-12.191,foo)
140 bah := tmp_{a}. # cdb(ex-12.192,bah)
141 substitute (bah, foo) # cdb(ex-12.193,bah)
142
143 Gamma := @(bah). # cdb(ex-12.102,Gamma)
144
145 product_sort (Gamma) # cdb(ex-12.103,Gamma)
146
147 checkpoint.append (Gamma)
148
149 gam1 = get_xterm (Gamma,1) # cdb(ex-12.200,gam1)
150 gam2 = get_xterm (Gamma,2) # cdb(ex-12.201,gam2)
151 gam3 = get_xterm (Gamma,3) # cdb(ex-12.202,gam3)
152

```

```

153 gam30 = get_gterm (gam3,0)           # cdb(ex-12.203,gam30)
154 gam31 = get_gterm (gam3,1)           # cdb(ex-12.204,gam31)
155
156 gam1  = reformat (gam1, 3,1)           # cdb(ex-12.300,gam1)
157 gam2  = reformat (gam2,12,1)           # cdb(ex-12.301,gam2)
158
159 gam30 = reformat (gam30,40,1)           # cdb(ex-12.302,gam30)
160 gam31 = reformat (gam31,45,2)           # cdb(ex-12.303,gam31)
161
162 gam3  := @(gam30) + @(gam31).           # cdb(ex-12.304,gam3)
163
164 Gamma := @(gam1) + @(gam2) + @(gam3).   # cdb(ex-12.305,Gamma)
165
166 checkpoint.append (Gamma)

```

$$\begin{aligned}
\text{ex-12.100} := & \frac{1}{3} g^{ad} x^e (R_{bdce} + R_{becd}) + \frac{1}{12} g^{ad} x^e x^f (-\nabla_e R_{bedf} + \nabla_d R_{becf} + 2 \nabla_e R_{bdcf} + 2 \nabla_e R_{bfcd} - \nabla_b R_{cedf}) \\
& + \frac{1}{40} g^{ad} x^e x^f x^g (-\nabla_{ce} R_{bfdg} - \nabla_{ec} R_{bfdg} + \nabla_{de} R_{bfce} + \nabla_{ed} R_{bfce} + 2 \nabla_{ef} R_{bdce} + 2 \nabla_{ef} R_{bgcd} - \nabla_{be} R_{cfdg} - \nabla_{eb} R_{cfdg}) \\
& + \frac{1}{45} g^{ad} g^{ef} x^g x^h x^i (4 R_{becg} R_{dhfi} + 4 R_{bgce} R_{dhfi} - 2 R_{bdeg} R_{chfi} - R_{bedg} R_{chfi} + R_{bgde} R_{chfi} - 2 R_{bgeh} R_{cdfi} - R_{bgeh} R_{cfdi} + R_{bgeh} R_{cidf})
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.191} := & tmp_v \\
\rightarrow & \frac{2}{3} x^c x^d z^e R_{vdce} + \frac{1}{3} x^c x^d z^e z^f \nabla_e R_{vedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{vcdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_v R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{vfe} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{vdeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{vdeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{vc} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{cv} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{vhce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{vecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vceh} R_{dfgi}
\end{aligned}$$

$$\text{ex-12.192} := tmp_a$$

$$\begin{aligned}
\text{ex-12.193} := & \frac{2}{3} x^c x^d z^e R_{adce} + \frac{1}{3} x^c x^d z^e z^f \nabla_e R_{aedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{acdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_a R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{afe} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{adeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{adeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ac} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ca} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{ahce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{aecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aceh} R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.101} := & \frac{2}{3} x^c x^d z^e R_{vdce} + \frac{1}{3} x^c x^d z^e z^f \nabla_e R_{vedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{vcdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_v R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{vfe} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{vdeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{vdeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{vc} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{cv} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{vhce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{vecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vceh} R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.102} := & \frac{2}{3} x^c x^d z^e R_{adce} + \frac{1}{3} x^c x^d z^e z^f \nabla_e R_{aedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{acdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_a R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{afe} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{adeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{adeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ac} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ca} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{ahce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{aecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aceh} R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.103} := & \frac{2}{3} x^b z^c z^d R_{acbd} + \frac{1}{6} x^b x^c z^d z^e \nabla_d R_{abce} + \frac{1}{3} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} \\
& + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{10} x^b x^c x^d z^e z^f \nabla_{be} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{8}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}
\end{aligned}$$

$$\text{ex-12.200} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.201} := \frac{1}{6} x^b x^c z^d z^e \nabla_d R_{abce} + \frac{1}{3} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\begin{aligned}
\text{ex-12.202} := & \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{10} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{8}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}
\end{aligned}$$

$$\text{ex-12.203} := \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{10} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$

$$\text{ex-12.204} := \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{8}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.300} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.301} := \frac{1}{12} x^b x^c z^d z^e (2 \nabla_d R_{abce} + 4 \nabla_b R_{adce} + \nabla_a R_{bdce})$$

$$\text{ex-12.302} := \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf})$$

$$\text{ex-12.303} := \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2 R_{agbd} R_{cefh})$$

$$\begin{aligned}
\text{ex-12.304} := & \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2 R_{agbd} R_{cefh})
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.305} := & \frac{2}{3} x^b z^c z^d R_{acbd} + \frac{1}{12} x^b x^c z^d z^e (2 \nabla_d R_{abce} + 4 \nabla_b R_{adce} + \nabla_a R_{bdce}) \\
& + \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adb g} R_{cefh} - 2 R_{agbd} R_{cefh})
\end{aligned}$$

$$\begin{aligned} \text{ex-12.110} := & z^b z^c \left(\frac{1}{3} g^{ad} x^e (R_{bdce} + R_{becd}) + \frac{1}{12} g^{ad} x^e x^f (-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}) \right. \\ & + \frac{1}{40} g^{ad} x^e x^f x^g (-\nabla_{ce} R_{bf dg} - \nabla_{ec} R_{bf dg} + \nabla_{de} R_{bf cg} + \nabla_{ed} R_{bf cg} + 2\nabla_{ef} R_{bd cg} + 2\nabla_{ef} R_{bg cd} - \nabla_{be} R_{cf dg} - \nabla_{eb} R_{cf dg}) \\ & \left. + \frac{1}{45} g^{ad} g^{ef} x^g x^h x^i (4 R_{becg} R_{dh fi} + 4 R_{bg ce} R_{dh fi} - 2 R_{bdeg} R_{ch fi} - R_{bedg} R_{ch fi} + R_{bg de} R_{ch fi} - 2 R_{bge h} R_{cdf i} - R_{bge h} R_{cf di} + R_{bge h} R_{cid f}) \right) \end{aligned}$$

$$\begin{aligned} \text{ex-12.113} := & z^h z^i \left(\frac{1}{3} g^{ab} x^c (R_{hbic} + R_{hcib}) + \frac{1}{12} g^{ab} x^c x^d (2\nabla_c R_{hbid} + \nabla_b R_{hcid} - \nabla_i R_{hcbd} + 2\nabla_c R_{hdib} - \nabla_h R_{icbd}) \right. \\ & + \frac{1}{40} g^{ab} x^c x^d x^e (2\nabla_{cd} R_{hb ie} + \nabla_{bc} R_{hd ie} + \nabla_{cb} R_{hd ie} - \nabla_{ic} R_{hd be} - \nabla_{ci} R_{hd be} + 2\nabla_{cd} R_{he ib} - \nabla_{he} R_{id be} - \nabla_{ch} R_{id be}) \\ & \left. + \frac{1}{45} g^{ab} g^{cd} x^e x^f x^g (-2 R_{hbce} R_{if dg} + 4 R_{hc ie} R_{bf dg} - R_{hc be} R_{if dg} + 4 R_{he ic} R_{bf dg} + R_{he bc} R_{if dg} - 2 R_{he cf} R_{ib dg} - R_{he cf} R_{id bg} + R_{he cf} R_{ig bd}) \right) \end{aligned}$$

$$\text{ex-12.210} := \frac{1}{3} z^h z^i g^{ab} x^c R_{hbic} + \frac{1}{3} z^h z^i g^{ab} x^c R_{hcib}$$

$$\text{ex-12.211} := \frac{1}{6} z^h z^i g^{ab} x^c x^d \nabla_c R_{hbid} + \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_b R_{hcid} - \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_i R_{hcbd} + \frac{1}{6} z^h z^i g^{ab} x^c x^d \nabla_c R_{hdib} - \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_h R_{icbd}$$

$$\begin{aligned} \text{ex-12.212} := & \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{hb ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{bc} R_{hd ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{cb} R_{hd ie} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ic} R_{hd be} \\ & - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ci} R_{hd be} + \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{he ib} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{he} R_{id be} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ch} R_{id be} \\ & - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hbce} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc ie} R_{bf dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc be} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he ic} R_{bf dg} \\ & + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he bc} R_{if dg} - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ib dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{id bg} + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ig bd} \end{aligned}$$

$$\begin{aligned} \text{ex-12.213} := & \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{hb ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{bc} R_{hd ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{cb} R_{hd ie} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ic} R_{hd be} \\ & - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ci} R_{hd be} + \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{he ib} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{he} R_{id be} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ch} R_{id be} \end{aligned}$$

$$\begin{aligned} \text{ex-12.214} := & -\frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hbce} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc ie} R_{bf dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc be} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he ic} R_{bf dg} \\ & + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he bc} R_{if dg} - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ib dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{id bg} + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ig bd} \end{aligned}$$

$$\begin{aligned}
\text{ex-12.310} &:= \frac{2}{3} g^{ab} x^c z^d z^e R_{bdce} \\
\text{ex-12.311} &:= \frac{1}{12} g^{ab} x^c x^d z^e z^f (4 \nabla_c R_{bedf} + 2 \nabla_e R_{bcd f} + \nabla_b R_{cedf}) \\
\text{ex-12.312} &:= \frac{1}{40} g^{ab} x^c x^d x^e z^f z^g (4 \nabla_{cd} R_{bfeg} + 2 \nabla_{cf} R_{bdeg} + 2 \nabla_{fc} R_{bdeg} + \nabla_{bc} R_{dfeg} + \nabla_{cb} R_{dfeg}) \\
\text{ex-12.313} &:= \frac{2}{45} g^{ab} g^{cd} x^e x^f x^g z^h z^i (-2 R_{bhce} R_{dfgi} - R_{bec h} R_{dfgi} + 4 R_{bec f} R_{dhgi} + R_{bce h} R_{dfgi}) \\
\text{ex-12.314} &:= \frac{1}{40} g^{ab} x^c x^d x^e z^f z^g (4 \nabla_{cd} R_{bfeg} + 2 \nabla_{cf} R_{bdeg} + 2 \nabla_{fc} R_{bdeg} + \nabla_{bc} R_{dfeg} + \nabla_{cb} R_{dfeg}) \\
&\quad + \frac{2}{45} g^{ab} g^{cd} x^e x^f x^g z^h z^i (-2 R_{bhce} R_{dfgi} - R_{bec h} R_{dfgi} + 4 R_{bec f} R_{dhgi} + R_{bce h} R_{dfgi}) \\
\text{ex-12.315} &:= \frac{2}{3} g^{ab} x^c z^d z^e R_{bdce} + \frac{1}{12} g^{ab} x^c x^d z^e z^f (4 \nabla_c R_{bedf} + 2 \nabla_e R_{bcd f} + \nabla_b R_{cedf}) \\
&\quad + \frac{1}{40} g^{ab} x^c x^d x^e z^f z^g (4 \nabla_{cd} R_{bfeg} + 2 \nabla_{cf} R_{bdeg} + 2 \nabla_{fc} R_{bdeg} + \nabla_{bc} R_{dfeg} + \nabla_{cb} R_{dfeg}) \\
&\quad + \frac{2}{45} g^{ab} g^{cd} x^e x^f x^g z^h z^i (-2 R_{bhce} R_{dfgi} - R_{bec h} R_{dfgi} + 4 R_{bec f} R_{dhgi} + R_{bce h} R_{dfgi})
\end{aligned}$$

```

1  # =====
2  # Calzetta
3  # note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}
4
5  GammaBar := z^{\nu} z^{\rho} (
6      (2/3) R^{\mu}_{\nu\rho\sigma} x^{\sigma}
7      + (1/12) (5 \nabla_{\lambda}\{R^{\mu}_{\nu\rho\sigma}\}
8          + \nabla_{\rho}\{R^{\mu}_{\sigma\nu\lambda}\}) x^{\sigma} x^{\lambda}
9      + (1/6) ( (9/10) \nabla_{\tau\lambda}\{R^{\mu}_{\rho\nu\sigma}\}
10          + (3/20) ( \nabla_{\tau\rho}\{R^{\mu}_{\sigma\nu\lambda}\}
11              + \nabla_{\rho\tau}\{R^{\mu}_{\sigma\nu\lambda}\} )
12          + (1/60) ( 21 R^{\mu}_{\lambda\xi\rho} R^{\xi}_{\sigma\nu\tau}
13              + 48 R^{\mu}_{\xi\rho\lambda} R^{\xi}_{\sigma\nu\tau}
14              - 37 R^{\mu}_{\sigma\xi\lambda} R^{\xi}_{\nu\rho\tau} ) ) x^{\sigma} x^{\lambda} x^{\tau} ).
15      # cdb(ex-12.400,GammaBar)
16
17  # convert from Greek to Latin indices
18
19  distribute (GammaBar)
20  rename_dummies (GammaBar,"greek","latin") # cdb(ex-12.401,GammaBar)
21
22  # lower the \mu index
23
24  GammaBar := \delta_{a \mu} @(GammaBar). # cdb(ex-12.402,GammaBar)
25  distribute (GammaBar) # cdb(ex-12.403,GammaBar)
26  eliminate_kronecker (GammaBar) # cdb(ex-12.404,GammaBar)
27
28  # sort products
29
30  product_sort (GammaBar) # cdb(ex-12.405,GammaBar)
31
32  checkpoint.append (GammaBar)
33
34  # Replace R with - R (Calzetta uses the non-MTW convention for Riemann)
35
36  substitute (GammaBar, $R_{a b c d} -> - R_{a b c d}$) # cdb(ex-12.406,GammaBar)
37  substitute (GammaBar, $R^{\{a\}_{b c d} -> - R^{\{a\}_{b c d}$) # cdb(ex-12.407,GammaBar)
38

```

```

39 substitute (GammaBar, $R^{a}_{b c d} -> g^{a e} R_{e b c d}$) # cdb(ex-12.408,GammaBar)
40
41 cal1 = get_xterm (GammaBar,1) # cdb(ex-12.500,cal1)
42 cal2 = get_xterm (GammaBar,2) # cdb(ex-12.501,cal2)
43 cal3 = get_xterm (GammaBar,3) # cdb(ex-12.502,cal3)
44
45 cal1 = reformat (cal1,3,1) # cdb(ex-12.600,cal1)
46 cal2 = reformat (cal2,12,1) # cdb(ex-12.601,cal2)
47 # cal3 = reformat (cal3,360,1) # cdb(ex-12.602,cal3)
48
49 cal30 = get_gterm (cal3,0) # cdb(ex-12.602,cal30)
50 cal31 = get_gterm (cal3,1) # cdb(ex-12.603,cal31)
51
52 cal1 = reformat (cal1, 3,1) # cdb(ex-12.604,cal1)
53 cal2 = reformat (cal2,12,1) # cdb(ex-12.605,cal2)
54
55 cal30 = reformat (cal30,40,1) # cdb(ex-12.606,cal30)
56 cal31 = reformat (cal31,360,1) # cdb(ex-12.607,cal31)
57
58 cal3 := @(cal30) + @(cal31). # cdb(ex-12.608,cal3)
59
60 GammaBar := @(cal1) + @(cal2) + @(cal3). # cdb(ex-12.409,GammaBar)
61
62 checkpoint.append (GammaBar)

```

$$\begin{aligned}
\text{ex-12.400} &:= z^\nu z^\rho \left(\frac{2}{3} R^\mu_{\nu\rho\sigma} x^\sigma + \frac{1}{12} (5 \nabla_\lambda R^\mu_{\nu\rho\sigma} + \nabla_\rho R^\mu_{\sigma\nu\lambda}) x^\sigma x^\lambda \right. \\
&\quad \left. + \frac{1}{6} \left(\frac{9}{10} \nabla_\tau R^\mu_{\rho\nu\sigma} + \frac{3}{20} \nabla_\tau R^\mu_{\sigma\nu\lambda} + \frac{3}{20} \nabla_\rho R^\mu_{\sigma\nu\lambda} + \frac{7}{20} R^\mu_{\lambda\xi\rho} R^\xi_{\sigma\nu\tau} + \frac{4}{5} R^\mu_{\xi\rho\lambda} R^\xi_{\sigma\nu\tau} - \frac{37}{60} R^\mu_{\sigma\xi\lambda} R^\xi_{\nu\rho\tau} \right) x^\sigma x^\lambda x^\tau \right) \\
\text{ex-12.401} &:= \frac{2}{3} z^a z^b R^\mu_{abc} x^c + \frac{5}{12} z^a z^b \nabla_d R^\mu_{abc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu_{abc} x^a x^c + \frac{3}{20} z^b z^a \nabla_{de} R^\mu_{abc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu_{abc} x^a x^c x^d \\
&\quad + \frac{1}{40} z^b z^d \nabla_{de} R^\mu_{abc} x^a x^c x^e + \frac{7}{120} z^e z^c R^\mu_{abc} R^b_{def} x^d x^a x^f + \frac{2}{15} z^e z^b R^\mu_{abc} R^a_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu_{abc} R^b_{def} x^a x^c x^f \\
\text{ex-12.402} &:= \delta_{a\mu} \left(\frac{2}{3} z^g z^b R^\mu_{gbc} x^c + \frac{5}{12} z^g z^b \nabla_d R^\mu_{gbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu_{gbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R^\mu_{gbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu_{gbc} x^g x^c x^d \right. \\
&\quad \left. + \frac{1}{40} z^b z^d \nabla_{de} R^\mu_{gbc} x^g x^c x^e + \frac{7}{120} z^e z^c R^\mu_{gbc} R^b_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R^\mu_{gbc} R^g_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu_{gbc} R^b_{def} x^g x^c x^f \right) \\
\text{ex-12.403} &:= \frac{2}{3} \delta_{a\mu} z^g z^b R^\mu_{gbc} x^c + \frac{5}{12} \delta_{a\mu} z^g z^b \nabla_d R^\mu_{gbc} x^c x^d + \frac{1}{12} \delta_{a\mu} z^b z^d \nabla_d R^\mu_{gbc} x^g x^c + \frac{3}{20} \delta_{a\mu} z^b z^g \nabla_{de} R^\mu_{gbc} x^c x^e x^d + \frac{1}{40} \delta_{a\mu} z^b z^e \nabla_{de} R^\mu_{gbc} x^g x^c x^d \\
&\quad + \frac{1}{40} \delta_{a\mu} z^b z^d \nabla_{de} R^\mu_{gbc} x^g x^c x^e + \frac{7}{120} \delta_{a\mu} z^e z^c R^\mu_{gbc} R^b_{def} x^d x^g x^f + \frac{2}{15} \delta_{a\mu} z^e z^b R^\mu_{gbc} R^g_{def} x^d x^c x^f - \frac{37}{360} \delta_{a\mu} z^d z^e R^\mu_{gbc} R^b_{def} x^g x^c x^f \\
\text{ex-12.404} &:= \frac{2}{3} z^g z^b R_{agbc} x^c + \frac{5}{12} z^g z^b \nabla_d R_{agbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R_{agbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R_{agbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R_{agbc} x^g x^c x^d \\
&\quad + \frac{1}{40} z^b z^d \nabla_{de} R_{agbc} x^g x^c x^e + \frac{7}{120} z^e z^c R_{agbc} R^b_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R_{agbc} R^g_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R_{agbc} R^b_{def} x^g x^c x^f \\
\text{ex-12.405} &:= \frac{2}{3} x^b z^c z^d R_{adcb} + \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} + \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
&\quad + \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g_{bfc} \\
\text{ex-12.406} &:= -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} - \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
&\quad - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} + \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g_{efc} - \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g_{cfd} - \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g_{bfc} \\
\text{ex-12.407} &:= -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} - \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
&\quad - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g_{bfc}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.408} := & -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_f R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
& - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcf d} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}
\end{aligned}$$

$$\text{ex-12.500} := -\frac{2}{3} x^b z^c z^d R_{adcb}$$

$$\text{ex-12.501} := -\frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb}$$

$$\begin{aligned} \text{ex-12.502} := & -\frac{1}{40} x^b x^c x^d z^e z^f \nabla_f R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} \\ & - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcf d} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc} \end{aligned}$$

$$\text{ex-12.600} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.601} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce})$$

$$\text{ex-12.602} := -\frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb}$$

$$\text{ex-12.603} := -\frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcf d} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}$$

$$\text{ex-12.604} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.605} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce})$$

$$\text{ex-12.606} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf})$$

$$\text{ex-12.607} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adb g} R_{cefh} + 48 R_{abdg} R_{cefh})$$

$$\text{ex-12.608} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf}) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adb g} R_{cefh} + 48 R_{abdg} R_{cefh})$$

$$\begin{aligned}
\mathbf{ex-12.409} := & \frac{2}{3} x^b z^c z^d R_{acbd} + \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce}) + \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf}) \\
& + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh})
\end{aligned}$$

The fun begins $\Gamma - \bar{\Gamma}$

It's now time to compute the difference $\Gamma - \bar{\Gamma}$. Here it is.

```
1  def reformat_diff (obj):
2
3      distribute (obj)
4
5      obj1  = get_xterm (obj,1)
6      obj2  = get_xterm (obj,2)
7      obj3  = get_xterm (obj,3)
8
9      obj30 = get_gterm (obj3,0)
10     obj31 = get_gterm (obj3,1)
11
12     obj1  = reformat (obj1, 3,1)
13     obj2  = reformat (obj2,12,1)
14
15     obj30 = reformat (obj30,40,1)
16     obj31 = reformat (obj31,360,1)
17
18     obj3  := @(obj30) + @(obj31).
19
20     ans  := @(obj1) + @(obj2) + @(obj3).
21
22     return ans
23
24     # We could use reformat_diff here but instead we'll do it one step at a time so that
25     # we can see exactly what's going on. Later on we will use reformat_diff to do the job.
26
27     diff := @(Gamma) - @(GammaBar).                # cdb(ex-12.diff.100,diff)
28     distribute (diff)
29
30     diff1  = get_xterm (diff,1)                    # cdb(ex-12.diff.200,diff1)
31     diff2  = get_xterm (diff,2)                    # cdb(ex-12.diff.201,diff2)
32     diff3  = get_xterm (diff,3)                    # cdb(ex-12.diff.202,diff3)
33
34     diff30 = get_gterm (diff3,0)                    # cdb(ex-12.diff.203,diff30)
```

```

35 diff31 = get_gterm (diff3,1) # cdb(ex-12.diff.204,diff31)
36
37 diff1 = reformat (diff1, 3,1) # cdb(ex-12.diff.300,diff1)
38 diff2 = reformat (diff2,12,1) # cdb(ex-12.diff.301,diff2)
39
40 diff30 = reformat (diff30,40,1) # cdb(ex-12.diff.302,diff30)
41 diff31 = reformat (diff31,360,1) # cdb(ex-12.diff.303,diff31)
42
43 diff3 := @(diff30) + @(diff31). # cdb(ex-12.diff.304,diff3)
44
45 diff := @(diff1) + @(diff2) + @(diff3). # cdb(ex-12.diff.305,diff)

```

$$\begin{aligned}
\text{ex-12.diff.100} := & \frac{1}{12} x^b x^c z^d z^e (2 \nabla_d R_{abce} + 4 \nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2 R_{agbd} R_{cefh}) - \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce}) \\
& - \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf}) - \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh})
\end{aligned}$$

$$\text{ex-12.diff.200} := 0$$

$$\text{ex-12.diff.201} := \frac{1}{12} x^b x^c z^d z^e \nabla_d R_{abce} - \frac{1}{12} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\begin{aligned}
\text{ex-12.diff.202} := & \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}
\end{aligned}$$

$$\text{ex-12.diff.203} := \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$

$$\text{ex-12.diff.204} := -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.diff.300} := 0$$

$$\text{ex-12.diff.301} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce})$$

$$\text{ex-12.diff.302} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf})$$

$$\text{ex-12.diff.303} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh})$$

$$\begin{aligned} \text{ex-12.diff.304} := & \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

$$\begin{aligned} \text{ex-12.diff.305} := & \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

Second order terms

```

1  diff2 = get_xterm (diff,2)
2  diff2 := 12 @(diff2).                                     # cdb (ex-12.701,diff2)
3  distribute (diff2)                                       # cdb (ex-12.702,diff2)
4
5  diff2 = add_tags (diff2,'\\mu')                          # cdb (ex-12.711,diff2)
6
7  # swap indices on middle term, then apply 2nd Bianchi identity
8
9  zoom (diff2, $\\mu_{1} Q??)$                             # cdb (ex-12.712,diff2)
10 substitute (diff2, $\\nabla_{b}\\{R_{a d c e}\\} -> - \\nabla_{b}\\{R_{d a c e}\\}$) # cdb (ex-12.713,diff2)
11 unzoom (diff2)
12
13 substitute (diff2, $\\mu_{1} -> \\mu_{0}, \\mu_{2} -> \\mu_{0}$) # cdb (ex-12.714,diff2)
14 substitute (diff2, $\\mu_{0} -> 0$)                       # cdb (ex-12.715,diff2)
15
16 diff2 = clear_tags (diff2,'\\mu')                        # cdb (ex-12.716,diff2)
17
18 diff2 := @(diff2) / 12 .
19
20 diff := @(diff1) + @(diff2) + @(diff3).
21
22 diff = reformat_diff (diff)                             # cdb(ex-12.diff.306,diff)

```

$$\text{ex-12.701} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.702} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.711} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_2 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.712} := \dots - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.713} := \dots + \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.714} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} + \mu_0 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_0 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.715} := 0$$

$$\text{ex-12.716} := 0$$

$$\begin{aligned} \text{ex-12.diff.306} := & \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adb g} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

Third order terms, commute $\nabla\nabla R$ terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3).                                # cdb (ex-12.801,diff3)
3  distribute (diff3)                                    # cdb (ex-12.802,diff3)
4
5  # commutation rule for covariant derivs on Rabcd, see exrecise 3.6
6  # note: \nabla_{a b} defined as \nabla_a \nabla_b
7  CommuteNablaRiemann := \nabla_{f e}(R_{a b c d}) -> \nabla_{e f}(R_{a b c d})
8                                     + g^{u v} R_{u a e f} R_{v b c d}
9                                     + g^{u v} R_{u b e f} R_{a v c d}
10                                    + g^{u v} R_{u c e f} R_{a b v d}
11                                    + g^{u v} R_{u d e f} R_{a b c v}.
12
13 diff3 = add_tags (diff3, '\\mu')                      # cdb (ex-12.901,diff3)
14
15 # commute derivs on Rabcd so that each double deriv is of the form \nabla_{b*}
16
17 substitute (diff3, $\\mu_{3} -> \\mu_{1}$)            # cdb (ex-12.902,diff3)
18
19 zoom (diff3, $\\mu_{1} Q??$)                          # cdb (ex-12.903,diff3)
20 substitute (diff3, CommuteNablaRiemann)               # cdb (ex-12.904,diff3)
21 unzoom (diff3)
22
23 diff3 = clear_tags (diff3, '\\mu')
24 diff3 := @(diff3) / 360 .
25
26 distribute (diff3)
27 canonicalise (diff3)                                  # cdb (ex-12.905,diff3)
28
29 diff := @(diff1) + @(diff2) + @(diff3).
30
31 diff = reformat_diff (diff)                            # cdb(ex-12.diff.307,diff)

```

$$\begin{aligned} \text{ex-12.801} := & 9 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.802} := & 9 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.901} := & 9 \mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9 \mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18 \mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9 \mu_3 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9 \mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27 \mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5 \mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.902} := & 9 \mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9 \mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18 \mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9 \mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9 \mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27 \mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5 \mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.903} := \dots + 9 \mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \dots + 9 \mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \dots$$

$$\begin{aligned} \text{ex-12.904} := & \dots + 9 \mu_1 x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + g^{uv} R_{uabe} R_{vcdf} + g^{uv} R_{ucbe} R_{avdf} + g^{uv} R_{udbe} R_{acvf} + g^{uv} R_{ufbe} R_{acdv}) + \dots \\ & + 9 \mu_1 x^b x^c x^d z^e z^f (\nabla_{ba} R_{cedf} + g^{uv} R_{ucba} R_{vedf} + g^{uv} R_{ueba} R_{cvdf} + g^{uv} R_{udba} R_{cevf} + g^{uv} R_{ufba} R_{cedv}) + \dots \end{aligned}$$

$$\begin{aligned} \text{ex-12.905} := & \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abeu} R_{cfdv} - \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abcu} R_{defv} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} \\ & + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} \\ & - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.diff.307} := \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} - 2 \nabla_{bc} R_{aedf} + 2 \nabla_{ba} R_{cedf}) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 32 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh})$$

Third order terms, use 2nd Bianchi identity on $\nabla\nabla R$ terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3). # cdb (ex-12.910,diff3)
3  distribute (diff3) # cdb (ex-12.911,diff3)
4
5  diff3 = add_tags (diff3,'\mu') # cdb (ex-12.912,diff3)
6
7  # swap indices on middle second deriv term, then apply 2nd Bianchi identity
8
9  zoom (diff3, $\mu_{1}$ Q??$) # cdb (ex-12.913,diff3)
10 substitute (diff3, $\nabla_{b c}\{R_{a e d f}\} \rightarrow - \nabla_{a c}\{R_{b e d f}\}$) # cdb (ex-12.914,diff3)
11 unzoom (diff3)
12
13 substitute (diff3, $\mu_{1} \rightarrow \mu_{0}$, $\mu_{2} \rightarrow \mu_{0}$) # cdb (ex-12.915,diff3)
14 substitute (diff3, $\mu_{0} \rightarrow 0$) # cdb (ex-12.916,diff3)
15
16 diff3 = clear_tags (diff3,'\mu')
17 diff3 := @(diff3) / 360 .
18
19 distribute (diff3)
20 canonicalise (diff3) # cdb (ex-12.917,diff3)
21
22 diff := @(diff1) + @(diff2) + @(diff3).
23
24 diff = reformat_diff (diff) # cdb(ex-12.diff.308,diff)

```

$$\begin{aligned}
\text{ex-12.910} &:= 18 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.911} &:= 18 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.912} &:= 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18 \mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18 \mu_2 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 \mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.913} &:= \dots - 18 \mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \dots \\
\text{ex-12.914} &:= \dots + 18 \mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + \dots \\
\text{ex-12.915} &:= 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 \mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.916} &:= -32 \mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.917} &:= -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.diff.308} &:= \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 32 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh})
\end{aligned}$$

Third order terms, use 1st Bianchi identity on RR terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3).
3  distribute (diff3)
4
5  diff3 = add_tags (diff3,'\mu') # cdb (ex-12.921,diff3)
6
7  # swap indices on middle term, then apply 1st Bianchi identity
8
9  zoom (diff3, $\mu_{1}$ Q??$) # cdb (ex-12.922,diff3)
10 substitute (diff3, $R_{a d b g} R_{c e f h} \rightarrow - R_{a d g b} R_{c e f h}$) # cdb (ex-12.923,diff3)
11 unzoom (diff3)
12
13 substitute (diff3, $\mu_{1} \rightarrow \mu_{0}$, $\mu_{2} \rightarrow \mu_{0}$) # cdb (ex-12.924,diff3)
14 substitute (diff3, $\mu_{0} \rightarrow 0$) # cdb (ex-12.925,diff3)
15
16 diff3 = clear_tags (diff3,'\mu') # cdb (ex-12.926,diff3)
17
18 diff := @(diff1) + @(diff2) + @(diff3).
19
20 diff = reformat_diff (diff) # cdb(ex-12.diff.309,diff)

```

$$\text{ex-12.921} := -32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_2 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.922} := \dots + 32 \mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} + \dots$$

$$\text{ex-12.923} := \dots - 32 \mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} + \dots$$

$$\text{ex-12.924} := -32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} - 32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} - 32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.925} := 0$$

$$\text{ex-12.926} := 0$$

$$\text{ex-12.diff.309} := 0$$

Example 13a The Weyl tensor vanishes in 3d – direct proof

```

1 {x,y,z}::Coordinate.
2 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices(values={x,y,z},position=independent).
3
4 \partial{#}::PartialDerivative.
5
6 g_{a b}::Metric.
7 g^{a b}::InverseMetric.
8
9 {\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.
10
11 GammaU := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
12                                     + \partial_{c}{g_{b d}}
13                                     - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)
14
15 GammaD := \Gamma_{a b c} -> 1/2 ( \partial_{b}{g_{a c}}
16                               + \partial_{c}{g_{b a}}
17                               - \partial_{a}{g_{b c}}). # cdb(Gamma.010,GammaD)
18
19 Rabcd := R_{a b c d} -> \partial_{c}{\Gamma_{a b d}}
20                       - \partial_{d}{\Gamma_{a b c}}
21                       + \Gamma_{e a d} \Gamma^{e}_{b c}
22                       - \Gamma_{e a c} \Gamma^{e}_{b d}. # cdb (Rabcd.000,Rabcd)
23
24 Rab := R_{a b} -> g^{c d} R_{a c b d}. # cdb (Rab.000,Rab)
25
26 Rscalar := R -> g^{a b} R_{a b}. # cdb (R.000,Rscalar)
27
28 # Weyl in 3-dimensions
29
30 Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
31         - (g_{a c} R_{b d} - g_{a d} R_{b c})
32         + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}). # cdb (ex-13a.100,Cabcd)
33
34 # Use 8 Cabcd to clear the fractions
35
36 EightCabcd := 8 @(Cabcd). # cdb (ex-13a.110,EightCabcd)

```

```

37
38 substitute      (Cabcd,Rscalar)
39 substitute      (Cabcd,Rab)
40 substitute      (Cabcd,Rabcd)
41 substitute      (Cabcd,GammaU)
42 substitute      (Cabcd,GammaD)
43
44 distribute      (Cabcd)
45
46 sort_product    (Cabcd)
47 rename_dummies  (Cabcd)
48 canonicalise    (Cabcd)                                # cdb (ex-13a.101,Cabcd)
49
50 EightCabcd := 8 @(Cabcd).                                # cdb (ex-13a.111,EightCabcd)
51
52 gab := {g_{x x} = gxx, g_{x y} = gxy, g_{x z} = gxz,
53         g_{y x} = gxy, g_{y y} = gyy, g_{y z} = gyz,
54         g_{z x} = gxz, g_{z y} = gyz, g_{z z} = gzz}.
55
56 complete (gab, $g^{a b}$)
57 evaluate (Cabcd,gab)                                    # cdb (ex-13a.102,Cabcd)
58 evaluate (EightCabcd,gab)                               # cdb (ex-13a.112,EightCabcd)

```


$$8C_{abcd} = 8R_{abcd} - 8R_{acgbd} + 8R_{adgbc} - 8g_{ac}R_{bd} + 8g_{ad}R_{bc} + 4R(g_{ac}g_{bd} - g_{ad}g_{bc}) \quad (\text{ex-13a.110})$$

$$\begin{aligned}
&= 4\partial_{bc}g_{ad} - 4\partial_{ac}g_{bd} - 4\partial_{bd}g_{ac} + 4\partial_{ad}g_{bc} + 2\partial_{gde}\partial_{gcf}g^{ef} + 2\partial_{gde}\partial_{gbf}g^{ef} - 2\partial_{gde}\partial_{fgbc}g^{ef} + 2\partial_{gce}\partial_{gaf}g^{ef} + 2\partial_{gbe}\partial_{gaf}g^{ef} - 2\partial_{gae}\partial_{fgbc}g^{ef} \\
&\quad - 2\partial_{gce}\partial_{fgad}g^{ef} - 2\partial_{gbe}\partial_{fgad}g^{ef} + 2\partial_{gae}\partial_{fgbc}g^{ef} - 2\partial_{gce}\partial_{gdf}g^{ef} - 2\partial_{gce}\partial_{gbf}g^{ef} + 2\partial_{gce}\partial_{fgbd}g^{ef} - 2\partial_{gde}\partial_{gaf}g^{ef} - 2\partial_{gae}\partial_{gbf}g^{ef} \\
&\quad + 2\partial_{gae}\partial_{fgbd}g^{ef} + 2\partial_{gde}\partial_{fgac}g^{ef} + 2\partial_{gbe}\partial_{fgac}g^{ef} - 2\partial_{gae}\partial_{fgbd}g^{ef} - 4\partial_{cegaf}g_{bd}g^{ef} + 4\partial_{cegaf}g_{bd}g^{ef} + 4\partial_{efgac}g_{bd}g^{ef} - 4\partial_{acegf}g_{bd}g^{ef} \\
&\quad - 2\partial_{gef}\partial_{ggh}g_{bd}g^{eg}g^{fh} - 4\partial_{gaf}\partial_{gch}g_{bd}g^{eg}g^{fh} + 4\partial_{gaf}\partial_{gch}g_{bd}g^{eh}g^{fg} + 4\partial_{gce}\partial_{fggh}g_{bd}g^{eg}g^{fh} - 2\partial_{gce}\partial_{fggh}g_{bd}g^{ef}g^{gh} + 4\partial_{gae}\partial_{fggh}g_{bd}g^{eg}g^{fh} \\
&\quad - 2\partial_{gae}\partial_{fggh}g_{bd}g^{ef}g^{gh} - 4\partial_{gae}\partial_{fggh}g_{bd}g^{eg}g^{fh} + 2\partial_{gae}\partial_{fggh}g_{bd}g^{ef}g^{gh} + 4\partial_{degaf}g_{bc}g^{ef} - 4\partial_{degaf}g_{bc}g^{ef} - 4\partial_{efgad}g_{bc}g^{ef} + 4\partial_{aegdf}g_{bc}g^{ef} \\
&\quad + 2\partial_{gef}\partial_{ggh}g_{bc}g^{eg}g^{fh} + 4\partial_{gaf}\partial_{ggh}g_{bc}g^{eg}g^{fh} - 4\partial_{gaf}\partial_{ggh}g_{bc}g^{eh}g^{fg} - 4\partial_{gde}\partial_{fggh}g_{bc}g^{eg}g^{fh} + 2\partial_{gde}\partial_{fggh}g_{bc}g^{ef}g^{gh} - 4\partial_{gae}\partial_{fggh}g_{bc}g^{eg}g^{fh} \\
&\quad + 2\partial_{gae}\partial_{fggh}g_{bc}g^{ef}g^{gh} + 4\partial_{gae}\partial_{fggh}g_{bc}g^{eg}g^{fh} - 2\partial_{gae}\partial_{fggh}g_{bc}g^{ef}g^{gh} - 4\partial_{degbf}g_{ac}g^{ef} + 4\partial_{degbf}g_{ac}g^{ef} + 4\partial_{efgbd}g_{ac}g^{ef} - 4\partial_{begdf}g_{ac}g^{ef} \\
&\quad - 2\partial_{gef}\partial_{ggh}g_{ac}g^{eg}g^{fh} - 4\partial_{gbf}\partial_{ggh}g_{ac}g^{eg}g^{fh} + 4\partial_{gbf}\partial_{ggh}g_{ac}g^{eh}g^{fg} + 4\partial_{gde}\partial_{fggh}g_{ac}g^{eg}g^{fh} - 2\partial_{gde}\partial_{fggh}g_{ac}g^{ef}g^{gh} \\
&\quad + 4\partial_{gbe}\partial_{fggh}g_{ac}g^{eg}g^{fh} - 2\partial_{gbe}\partial_{fggh}g_{ac}g^{ef}g^{gh} - 4\partial_{gbd}\partial_{fggh}g_{ac}g^{eg}g^{fh} + 2\partial_{gbd}\partial_{fggh}g_{ac}g^{ef}g^{gh} + 4\partial_{cegbf}g_{ad}g^{ef} - 4\partial_{begbf}g_{ad}g^{ef} \\
&\quad - 4\partial_{efgbc}g_{ad}g^{ef} + 4\partial_{begcf}g_{ad}g^{ef} + 2\partial_{gef}\partial_{ggh}g_{ad}g^{eg}g^{fh} + 4\partial_{gbf}\partial_{gch}g_{ad}g^{eg}g^{fh} - 4\partial_{gbf}\partial_{gch}g_{ad}g^{eh}g^{fg} - 4\partial_{gce}\partial_{fggh}g_{ad}g^{eg}g^{fh} \\
&\quad + 2\partial_{gce}\partial_{fggh}g_{ad}g^{ef}g^{gh} - 4\partial_{gbe}\partial_{fggh}g_{ad}g^{eg}g^{fh} + 2\partial_{gbe}\partial_{fggh}g_{ad}g^{ef}g^{gh} + 4\partial_{gbc}\partial_{fggh}g_{ad}g^{eg}g^{fh} - 2\partial_{gbc}\partial_{fggh}g_{ad}g^{ef}g^{gh} + 4\partial_{efggh}g_{ac}g_{bd}g^{eg}g^{fh} \\
&\quad - 4\partial_{efggh}g_{ad}g_{bc}g^{eg}g^{fh} - 4\partial_{efggh}g_{ac}g_{bd}g^{ef}g^{gh} + 4\partial_{efggh}g_{ad}g_{bc}g^{ef}g^{gh} - 2\partial_{gfg}\partial_{hgij}g_{ac}g_{bd}g^{ei}g^{fh}g^{gj} + 2\partial_{gfg}\partial_{hgij}g_{ad}g_{bc}g^{ei}g^{fh}g^{gj} \\
&\quad + 3\partial_{gfg}\partial_{hgij}g_{ac}g_{bd}g^{eh}g^{fi}g^{gj} - 3\partial_{gfg}\partial_{hgij}g_{ad}g_{bc}g^{eh}g^{fi}g^{gj} - 4\partial_{gfg}\partial_{hgij}g_{ac}g_{bd}g^{ef}g^{gi}g^{hj} + 4\partial_{gfg}\partial_{hgij}g_{ad}g_{bc}g^{ef}g^{gi}g^{hj} \\
&\quad + 4\partial_{gfg}\partial_{hgij}g_{ac}g_{bd}g^{ef}g^{gh}g^{ij} - 4\partial_{gfg}\partial_{hgij}g_{ad}g_{bc}g^{ef}g^{gh}g^{ij} - \partial_{gfg}\partial_{hgij}g_{ac}g_{bd}g^{eh}g^{fg}g^{ij} + \partial_{gfg}\partial_{hgij}g_{ad}g_{bc}g^{eh}g^{fg}g^{ij} \quad (\text{ex-13a.111})
\end{aligned}$$

$$= 0 \quad (\text{ex-13a.112})$$

Example 13b The Weyl tensor vanishes in 3d – orthonormal basis

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3  g_{a b}::Metric.
4  g^{a b}::InverseMetric.
5
6  R_{a b c d}::RiemannTensor.
7
8  ex{#}::LaTeXForm("e_x").
9  ey{#}::LaTeXForm("e_y").
10 ez{#}::LaTeXForm("e_z").
11
12 {R_{a b c d}, g_{a b}, g^{a b}}::SortOrder.
13
14 Rab      := R_{a b} -> g^{c d} R_{a c b d}.
15
16 Rscalar := R -> g^{a b} R_{a b}.
17
18 gab := g^{a b} -> ex^{a} ex^{b} + ey^{a} ey^{b} + ez^{a} ez^{b}.
19
20 ortho := {ex^{a} ex^{b} g_{a b} -> 1, ey^{a} ey^{b} g_{a b} -> 1, ez^{a} ez^{b} g_{a b} -> 1,
21           ex^{a} ey^{b} g_{a b} -> 0, ex^{a} ez^{b} g_{a b} -> 0,
22           ey^{a} ex^{b} g_{a b} -> 0, ey^{a} ez^{b} g_{a b} -> 0,
23           ez^{a} ex^{b} g_{a b} -> 0, ez^{a} ey^{b} g_{a b} -> 0}.
24
25 # Weyl in 3-dimensions
26
27 Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
28         - (g_{a c} R_{b d} - g_{a d} R_{b c})
29         + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).    # cdb (ex-13b.100,Cabcd)
30
31
32 substitute (Cabcd, Rscalar)                # cdb(ex-13b.101,Cabcd)
33 substitute (Cabcd, Rab)                    # cdb(ex-13b.102,Cabcd)
34 distribute (Cabcd)                        # cdb(ex-13b.103,Cabcd)
35
36 Cabcd := C_{a b c d} -> @(Cabcd).

```

```

37 Cxyxy := C_{a b c d} ex^{a} ey^{b} ex^{c} ey^{d}.
38                                     # cdb(ex-13b.104,Cxyxy)
39
40 substitute      (Cxyxy,Cabcd)      # cdb(ex-13b.105,Cxyxy)
41 distribute      (Cxyxy)             # cdb(ex-13b.106,Cxyxy)
42
43 substitute      (Cxyxy, ortho, repeat=True) # cdb(ex-13b.107,Cxyxy)
44
45 substitute      (Cxyxy, gab)        # cdb(ex-13b.108,Cxyxy)
46 distribute      (Cxyxy)             # cdb(ex-13b.109,Cxyxy)
47
48 sort_product    (Cxyxy)             # cdb(ex-13b.110,Cxyxy)
49 rename_dummies  (Cxyxy)             # cdb(ex-13b.111,Cxyxy)
50 canonicalise    (Cxyxy)             # cdb(ex-13b.112,Cxyxy)

```

$$\text{ex-13b.101} := R_{abcd} - R_{ac}g_{bd} + R_{ad}g_{bc} - g_{ac}R_{bd} + g_{ad}R_{bc} + \frac{1}{2}g^{ef}R_{ef}(g_{ac}g_{bd} - g_{ad}g_{bc})$$

$$\text{ex-13b.102} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}(g_{ac}g_{bd} - g_{ad}g_{bc})$$

$$\text{ex-13b.103} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}$$

$$C_{abcd}e_x^a e_y^b e_x^c e_y^d = \left(R_{abcd} - g^{ef} R_{aecf} g_{bd} + g^{fe} R_{afde} g_{bc} - g_{ac} g^{fe} R_{bfde} + g_{ad} g^{ef} R_{becf} + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} \right) e_x^a e_y^b e_x^c e_y^d \quad (\text{ex-13b.105})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_y^d - g^{ef} R_{aecf} g_{bd} e_x^a e_y^b e_x^c e_y^d + g^{fe} R_{afde} g_{bc} e_x^a e_y^b e_x^c e_y^d - g_{ac} g^{fe} R_{bfde} e_x^a e_y^b e_x^c e_y^d + g_{ad} g^{ef} R_{becf} e_x^a e_y^b e_x^c e_y^d \\ + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} e_x^a e_y^b e_x^c e_y^d - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} e_x^a e_y^b e_x^c e_y^d \quad (\text{ex-13b.106})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_y^d - g^{ef} R_{aecf} e_x^a e_x^c - g^{fe} R_{bfde} e_y^b e_y^d + \frac{1}{2} g^{ef} g^{gh} R_{egfh} \quad (\text{ex-13b.107})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_y^d - (e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f) R_{aecf} e_x^a e_x^c - (e_x^f e_x^e + e_y^f e_y^e + e_z^f e_z^e) R_{bfde} e_y^b e_y^d \\ + \frac{1}{2} (e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f) (e_x^g e_x^h + e_y^g e_y^h + e_z^g e_z^h) R_{egfh} \quad (\text{ex-13b.108})$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_y^d - e_x^e e_x^f R_{aecf} e_x^a e_x^c - e_y^e e_y^f R_{aecf} e_x^a e_x^c - e_z^e e_z^f R_{aecf} e_x^a e_x^c - e_x^f e_x^e R_{bfde} e_y^b e_y^d - e_y^f e_y^e R_{bfde} e_y^b e_y^d - e_z^f e_z^e R_{bfde} e_y^b e_y^d \\ + \frac{1}{2} e_x^e e_x^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_x^e e_x^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_x^e e_x^f e_z^g e_z^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_z^g e_z^h R_{egfh} \\ + \frac{1}{2} e_z^e e_z^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_z^e e_z^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_z^e e_z^f e_z^g e_z^h R_{egfh} \quad (\text{ex-13b.109})$$

$$= R_{abcd} e_x^a e_x^c e_y^b e_y^d - R_{aecf} e_x^a e_x^c e_x^e e_x^f - R_{aecf} e_x^a e_x^c e_y^e e_y^f - R_{aecf} e_x^a e_x^c e_z^e e_z^f - R_{bfde} e_x^e e_x^f e_y^b e_y^d - R_{bfde} e_y^b e_y^d e_y^e e_y^f - R_{bfde} e_y^b e_y^d e_z^e e_z^f \\ + \frac{1}{2} R_{egfh} e_x^e e_x^f e_x^g e_x^h + \frac{1}{2} R_{egfh} e_x^e e_x^f e_y^g e_y^h + \frac{1}{2} R_{egfh} e_x^e e_x^f e_z^g e_z^h + \frac{1}{2} R_{egfh} e_x^g e_x^h e_y^e e_y^f + \frac{1}{2} R_{egfh} e_y^e e_y^f e_y^g e_y^h + \frac{1}{2} R_{egfh} e_y^e e_y^f e_z^g e_z^h \\ + \frac{1}{2} R_{egfh} e_x^g e_x^h e_z^e e_z^f + \frac{1}{2} R_{egfh} e_y^g e_y^h e_z^e e_z^f + \frac{1}{2} R_{egfh} e_z^g e_z^h e_z^e e_z^f \quad (\text{ex-13b.110})$$

$$= \frac{1}{2} R_{abcd} e_x^a e_x^c e_y^b e_y^d - \frac{1}{2} R_{abcd} e_x^a e_x^c e_x^b e_x^d - \frac{1}{2} R_{abcd} e_x^a e_x^c e_z^b e_z^d - R_{abcd} e_x^d e_x^b e_y^a e_y^c - R_{abcd} e_y^a e_y^c e_y^d e_y^b - R_{abcd} e_y^a e_y^c e_z^d e_z^b + \frac{1}{2} R_{abcd} e_x^b e_x^d e_y^a e_y^c \\ + \frac{1}{2} R_{abcd} e_y^a e_y^c e_y^b e_y^d + \frac{1}{2} R_{abcd} e_y^a e_y^c e_z^b e_z^d + \frac{1}{2} R_{abcd} e_x^b e_x^d e_z^a e_z^c + \frac{1}{2} R_{abcd} e_y^b e_y^d e_z^a e_z^c + \frac{1}{2} R_{abcd} e_z^a e_z^c e_z^b e_z^d \quad (\text{ex-13b.111})$$

$$= 0 \quad (\text{ex-13b.112})$$

Example 13c The Weyl tensor vanishes in 3d – orthonormal basis

```
1 Cxyz := C_{a b c d} ex^{a} ey^{b} ex^{c} ez^{d}. # cdb(ex-13c.101,Cxyz)
2
3 substitute (Cxyz,Cabcd) # cdb(ex-13c.102,Cxyz)
4
5 distribute (Cxyz) # cdb(ex-13c.103,Cxyz)
6
7 substitute (Cxyz, ortho, repeat=True) # cdb(ex-13c.104,Cxyz)
8
9 substitute (Cxyz, gab) # cdb(ex-13c.105,Cxyz)
10 distribute (Cxyz) # cdb(ex-13c.106,Cxyz)
11
12 sort_product (Cxyz) # cdb(ex-13c.107,Cxyz)
13 rename_dummies (Cxyz) # cdb(ex-13c.108,Cxyz)
14 canonicalise (Cxyz) # cdb(ex-13c.109,Cxyz)
```

$$C_{abcd}e_x^ae_y^be_x^ce_z^d = \left(R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc} \right) e_x^ae_y^be_x^ce_z^d \quad (\text{ex-13c.102})$$

$$= R_{abcd}e_x^ae_y^be_x^ce_z^d - g^{ef}R_{aecf}g_{bd}e_x^ae_y^be_x^ce_z^d + g^{fe}R_{afde}g_{bc}e_x^ae_y^be_x^ce_z^d - g_{ac}g^{fe}R_{bfde}e_x^ae_y^be_x^ce_z^d + g_{ad}g^{ef}R_{becf}e_x^ae_y^be_x^ce_z^d + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd}e_x^ae_y^be_x^ce_z^d - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}e_x^ae_y^be_x^ce_z^d \quad (\text{ex-13c.103})$$

$$= R_{abcd}e_x^ae_y^be_x^ce_z^d - g^{fe}R_{bfde}e_y^be_z^d \quad (\text{ex-13c.104})$$

$$= R_{abcd}e_x^ae_y^be_x^ce_z^d - (e_x^fe_x^e + e_y^fe_y^e + e_z^fe_z^e) R_{bfde}e_y^be_z^d \quad (\text{ex-13c.105})$$

$$= R_{abcd}e_x^ae_y^be_x^ce_z^d - e_x^fe_x^e R_{bfde}e_y^be_z^d - e_y^fe_y^e R_{bfde}e_y^be_z^d - e_z^fe_z^e R_{bfde}e_y^be_z^d \quad (\text{ex-13c.106})$$

$$= R_{abcd}e_x^ae_x^ce_y^be_z^d - R_{bfde}e_x^ce_x^fe_y^be_z^d - R_{bfde}e_y^be_y^ce_y^fe_z^d - R_{bfde}e_y^be_z^de_z^fe_z^e \quad (\text{ex-13c.107})$$

$$= R_{abcd}e_x^ae_x^ce_y^be_z^d - R_{abcd}e_x^de_x^be_y^ae_z^c - R_{abcd}e_y^ae_y^de_y^be_z^c - R_{abcd}e_y^ae_z^ce_z^de_z^b \quad (\text{ex-13c.108})$$

$$= 0 \quad (\text{ex-13c.109})$$

Example 14 The Weyl tensor is conformally invariant

This example shows that the Weyl tensor is conformally invariant. That is, for a pair of metrics g and \bar{g} related by a conformal transformation, $\bar{g}_{ab} = \phi g_{ab}$ then $\bar{C}_{bcd}^a = C_{bcd}^a$ or equally $\bar{C}_{abcd} = \phi C_{abcd}$.

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative.
4
5  g_{a b}::Metric.
6  g^{a b}::InverseMetric.
7  g_{a}^{b}::KroneckerDelta.
8
9  GammaU := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
10                                     + \partial_{c}{g_{b d}}
11                                     - \partial_{d}{g_{b c}}).
12
13  GammaD := \Gamma_{a b c} -> 1/2 ( \partial_{b}{g_{a c}}
14                                     + \partial_{c}{g_{b a}}
15                                     - \partial_{a}{g_{b c}}).
16
17  Rabcd := R_{a b c d} -> \partial_{c}{\Gamma_{a b d}}
18                        - \partial_{d}{\Gamma_{a b c}}
19                        + \Gamma_{e a d} \Gamma^{e}_{b c}
20                        - \Gamma_{e a c} \Gamma^{e}_{b d}.
21
22  Rab      := R_{a b} -> g^{c d} R_{a c b d}.
23
24  Rscalar := R -> g^{a b} R_{a b}.
25
26  # Weyl in 4-dimensions
27
28  Cabcd := R_{a b c d} - (1/2) (R_{a c} g_{b d} - R_{a d} g_{b c})
29                - (1/2) (g_{a c} R_{b d} - g_{a d} R_{b c})
30                + (R/6) (g_{a c} g_{b d} - g_{a d} g_{b c}).
31
32  {\partial_{a b}{\phi},\partial_{a}{\phi},\phi}::SortOrder.
33  {\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.

```



```

34
35 substitute (Cabcd,Rscalar)
36 substitute (Cabcd,Rab)
37 substitute (Cabcd,Rabcd)
38 substitute (Cabcd,GammaU)
39 substitute (Cabcd,GammaD)
40
41 distribute (Cabcd)
42
43 sort_product (Cabcd)
44 rename_dummies (Cabcd)
45 canonicalise (Cabcd)
46
47 # this is the Weyl tensor on the base metric
48 baseC := @(Cabcd).
49
50 conformal := {g_{a b} -> \phi g_{a b}, g^{a b} -> (1/phi) g^{a b}}.
51
52 substitute (Cabcd, conformal)
53 product_rule (Cabcd)
54 distribute (Cabcd)
55 product_rule (Cabcd)
56 distribute (Cabcd)
57
58 map_sympy (Cabcd, "simplify")
59
60 sort_product (Cabcd)
61 rename_dummies (Cabcd)
62 canonicalise (Cabcd)
63
64 # this is the Weyl tensor on the conformal metric
65 confC := @(Cabcd).
66
67 # their difference, should be zero
68 diff := @(confC) - \phi @(baseC). # cdb (ex-14.diff.100,diff)
69
70 distribute (diff)
71 sort_product (diff)

```

```

72 rename_dummies (diff)
73 canonicalise (diff) # cdb (ex-14.diff.101,diff)
74
75 # this trick is not essential but it does reduce the number of terms in diff
76 substitute (diff,$\partial_{\{a\}\{b\}\{g_{\{c\} d\}}\} \rightarrow g_{\{c\} d\} b\{a\}\})
77 substitute (diff,$\partial_{\{a\}\{g_{\{b\} c\}}\} \rightarrow 0\})
78 substitute (diff,$g_{\{c\} d\} b\{a\} \rightarrow \partial_{\{a\}\{b\}\{g_{\{c\} d\}}\}}) # cdb (ex-14.diff.102,diff)
79
80 # standard expressions in 4-d
81 substitute (diff,$g_{\{a\} b\} g^{\{a\} b\} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.201,diff)
82 substitute (diff,$g_{\{a\} b\} g^{\{c\} b\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.202,diff)
83 substitute (diff,$g_{\{b\} a\} g^{\{b\} c\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.203,diff)
84 substitute (diff,$g_{\{a\}}^{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.204,diff)
85 substitute (diff,$g^{\{a\}}_{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.205,diff)
86 eliminate_kronecker (diff) # cdb (ex-14.diff.206,diff)
87
88 # need a second round since the above block introduces new terms that match those just eliminated
89 substitute (diff,$g_{\{a\} b\} g^{\{a\} b\} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.301,diff)
90 substitute (diff,$g_{\{a\} b\} g^{\{c\} b\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.302,diff)
91 substitute (diff,$g_{\{b\} a\} g^{\{b\} c\} \rightarrow g_{\{a\}}^{\{c\}}$,repeat=True) # cdb (ex-14.diff.303,diff)
92 substitute (diff,$g_{\{a\}}^{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.304,diff)
93 substitute (diff,$g^{\{a\}}_{\{a\}} \rightarrow 4$,repeat=True) # cdb (ex-14.diff.305,diff)
94 eliminate_kronecker (diff) # cdb (ex-14.diff.306,diff)
95
96 sort_product (diff)
97 rename_dummies (diff)
98 canonicalise (diff) # cdb (ex-14.diff.400,diff)
99
100 checkpoint.append (baseC)
101 checkpoint.append (confC)

```

$$\begin{aligned}
\Delta = & \frac{1}{2} \partial_b \phi g_{ad} - \frac{1}{2} \partial_a \phi g_{bd} - \frac{1}{2} \partial_{bd} \phi g_{ac} + \frac{1}{2} \partial_{ad} \phi g_{bc} + \frac{1}{4} \partial_a \phi \partial_c \phi \phi^{-1} g_{be} g_{df} g^{ef} - \frac{1}{4} \partial_a \phi \partial_c \phi \phi^{-1} g_{bc} g_{df} g^{ef} + \frac{1}{4} \partial_b \phi \partial_a \phi \phi^{-1} g_{ae} g_{cf} g^{ef} \\
& - \frac{1}{4} \partial_a \phi \partial_c \phi \phi^{-1} g_{af} g_{bc} g^{ef} - \frac{1}{4} \partial_b \phi \partial_c \phi \phi^{-1} g_{ad} g_{cf} g^{ef} - \frac{1}{4} \partial_c \phi \partial_e \phi \phi^{-1} g_{ad} g_{bf} g^{ef} + \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4} \partial_a \phi \partial_d \phi \phi^{-1} g_{be} g_{cf} g^{ef} \\
& + \frac{1}{4} \partial_a \phi \partial_e \phi \phi^{-1} g_{bd} g_{cf} g^{ef} - \frac{1}{4} \partial_b \phi \partial_c \phi \phi^{-1} g_{ae} g_{df} g^{ef} + \frac{1}{4} \partial_c \phi \partial_e \phi \phi^{-1} g_{af} g_{bd} g^{ef} + \frac{1}{4} \partial_b \phi \partial_e \phi \phi^{-1} g_{ac} g_{df} g^{ef} + \frac{1}{4} \partial_a \phi \partial_e \phi \phi^{-1} g_{ac} g_{bf} g^{ef} \\
& - \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ce} \phi g_{af} g_{bd} g^{ef} + \frac{1}{4} \partial_{ac} \phi g_{bd} g_{ef} g^{ef} + \frac{1}{2} \partial_{ef} \phi g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ae} \phi g_{bd} g_{cf} g^{ef} - \frac{1}{8} \partial_a \phi \partial_c \phi \phi^{-1} g_{bd} g_{ef} g_{gh} g^{eg} g^{fh} \\
& - \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ag} g_{bd} g_{ch} g^{ef} g^{gh} + \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ag} g_{bd} g_{ch} g^{eg} g^{fh} + \frac{1}{4} \partial_a \phi \partial_e \phi \phi^{-1} g_{bd} g_{cf} g_{gh} g^{eg} g^{fh} - \frac{1}{8} \partial_a \phi \partial_e \phi \phi^{-1} g_{bd} g_{cf} g_{gh} g^{ef} g^{gh} \\
& + \frac{1}{4} \partial_c \phi \partial_e \phi \phi^{-1} g_{af} g_{bd} g_{gh} g^{eg} g^{fh} - \frac{1}{8} \partial_c \phi \partial_e \phi \phi^{-1} g_{af} g_{bd} g_{gh} g^{ef} g^{gh} - \frac{1}{2} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bd} g_{gh} g^{eg} g^{fh} + \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bd} g_{gh} g^{ef} g^{gh} \\
& + \frac{1}{4} \partial_d \phi g_{af} g_{bc} g^{ef} - \frac{1}{4} \partial_{ad} \phi g_{bc} g_{ef} g^{ef} - \frac{1}{2} \partial_{ef} \phi g_{ad} g_{bc} g^{ef} + \frac{1}{4} \partial_{ae} \phi g_{bc} g_{df} g^{ef} + \frac{1}{8} \partial_a \phi \partial_d \phi \phi^{-1} g_{bc} g_{ef} g_{gh} g^{eg} g^{fh} + \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ag} g_{bc} g_{dh} g^{ef} g^{gh} \\
& - \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ag} g_{bc} g_{dh} g^{eg} g^{fh} - \frac{1}{4} \partial_a \phi \partial_e \phi \phi^{-1} g_{bc} g_{df} g_{gh} g^{eg} g^{fh} + \frac{1}{8} \partial_a \phi \partial_e \phi \phi^{-1} g_{bc} g_{df} g_{gh} g^{ef} g^{gh} - \frac{1}{4} \partial_a \phi \partial_e \phi \phi^{-1} g_{af} g_{bc} g_{gh} g^{eg} g^{fh} \\
& + \frac{1}{8} \partial_a \phi \partial_e \phi \phi^{-1} g_{af} g_{bc} g_{gh} g^{ef} g^{gh} + \frac{1}{2} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bc} g_{gh} g^{eg} g^{fh} - \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bc} g_{gh} g^{ef} g^{gh} - \frac{1}{4} \partial_{de} \phi g_{ac} g_{bf} g^{ef} + \frac{1}{4} \partial_{bd} \phi g_{ac} g_{ef} g^{ef} \\
& - \frac{1}{4} \partial_{be} \phi g_{ac} g_{df} g^{ef} - \frac{1}{8} \partial_b \phi \partial_a \phi \phi^{-1} g_{ac} g_{ef} g_{gh} g^{eg} g^{fh} - \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bg} g_{dh} g^{ef} g^{gh} + \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bg} g_{dh} g^{eg} g^{fh} + \frac{1}{4} \partial_b \phi \partial_e \phi \phi^{-1} g_{ac} g_{df} g_{gh} g^{eg} g^{fh} \\
& - \frac{1}{8} \partial_b \phi \partial_e \phi \phi^{-1} g_{ac} g_{df} g_{gh} g^{ef} g^{gh} + \frac{1}{4} \partial_a \phi \partial_e \phi \phi^{-1} g_{ac} g_{bf} g_{gh} g^{eg} g^{fh} - \frac{1}{8} \partial_a \phi \partial_e \phi \phi^{-1} g_{ac} g_{bf} g_{gh} g^{ef} g^{gh} + \frac{1}{4} \partial_{ce} \phi g_{ad} g_{bf} g^{ef} - \frac{1}{4} \partial_b \phi g_{ad} g_{ef} g^{ef} \\
& + \frac{1}{4} \partial_{be} \phi g_{ad} g_{cf} g^{ef} + \frac{1}{8} \partial_b \phi \partial_c \phi \phi^{-1} g_{ad} g_{ef} g_{gh} g^{eg} g^{fh} + \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bg} g_{ch} g^{ef} g^{gh} - \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bg} g_{ch} g^{eg} g^{fh} - \frac{1}{4} \partial_b \phi \partial_e \phi \phi^{-1} g_{ad} g_{cf} g_{gh} g^{eg} g^{fh} \\
& + \frac{1}{8} \partial_b \phi \partial_e \phi \phi^{-1} g_{ad} g_{cf} g_{gh} g^{ef} g^{gh} - \frac{1}{4} \partial_c \phi \partial_e \phi \phi^{-1} g_{ad} g_{bf} g_{gh} g^{eg} g^{fh} + \frac{1}{8} \partial_c \phi \partial_e \phi \phi^{-1} g_{ad} g_{bf} g_{gh} g^{ef} g^{gh} + \frac{1}{6} \partial_{ef} \phi g_{ac} g_{bd} g_{gh} g^{eg} g^{fh} - \frac{1}{6} \partial_{ef} \phi g_{ad} g_{bc} g_{gh} g^{eg} g^{fh} \\
& - \frac{1}{6} \partial_{ef} \phi g_{ac} g_{bd} g_{gh} g^{ef} g^{gh} + \frac{1}{6} \partial_{ef} \phi g_{ad} g_{bc} g_{gh} g^{ef} g^{gh} - \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bd} g_{gh} g_{ij} g^{eg} g^i g^{hj} + \frac{1}{4} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bc} g_{gh} g_{ij} g^{eg} g^i g^{hj} \\
& + \frac{1}{8} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bd} g_{gh} g_{ij} g^{ef} g^{gi} g^{hj} - \frac{1}{8} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bc} g_{gh} g_{ij} g^{ef} g^{gi} g^{hj} + \frac{1}{6} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bd} g_{gh} g_{ij} g^{eg} g^{fh} g^{ij} - \frac{1}{6} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bc} g_{gh} g_{ij} g^{eg} g^{fh} g^{ij} \\
& - \frac{1}{24} \partial_c \phi \partial_f \phi \phi^{-1} g_{ac} g_{bd} g_{gh} g_{ij} g^{ef} g^{gh} g^{ij} + \frac{1}{24} \partial_c \phi \partial_f \phi \phi^{-1} g_{ad} g_{bc} g_{gh} g_{ij} g^{ef} g^{gh} g^{ij}
\end{aligned}$$

(ex-14.diff.102)

$$\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{ba}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{be}g_{df}g^{ef} + \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_{b\phi}\partial_{a\phi}\phi^{-1}g_{ae}g_{cf}g^{ef} + \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{af}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_{b\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ad}g_{cf}g^{ef} + \frac{1}{4}\partial_{c\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ad}g_{bf}g^{ef} - \frac{1}{12}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{be}g_{cf}g^{ef} - \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{bd}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_{b\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ae}g_{df}g^{ef} - \frac{1}{4}\partial_{c\phi}\partial_{\epsilon}\phi\phi^{-1}g_{af}g_{bd}g^{ef} - \frac{1}{4}\partial_{b\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ac}g_{df}g^{ef} - \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ac}g_{bf}g^{ef} + \frac{1}{12}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} \\
& - \frac{1}{6}\partial_{\epsilon f\phi}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ac}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} \\
& + \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} + \frac{1}{6}\partial_{\epsilon f\phi}g_{ad}g_{bc}g^{ef} \\
& + \frac{1}{4}\partial_{ac}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} - \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_{b\phi}\partial_{a\phi}\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_{b\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_{b\phi}\partial_{a\phi}\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_{b\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_{c\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_{\epsilon f\phi}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{\epsilon f\phi}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{4}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{8}\partial_{a\phi}\partial_{\epsilon}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_{\epsilon}\phi\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj}
\end{aligned}
\tag{ex-14.diff.201}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2} \partial_{bc} \phi g_{ad} + \frac{1}{2} \partial_{ac} \phi g_{bd} + \frac{1}{2} \partial_{bd} \phi g_{ac} - \frac{1}{2} \partial_{ad} \phi g_{bc} + \frac{1}{4} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{be} g_d^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_d^e + \frac{1}{4} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ae} g_c^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bc} \\
& + \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_c^e + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_b^e - \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4} \partial_{a\phi} \partial_{a\phi} \phi^{-1} g_{be} g_c^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_c^e - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ae} g_d^e \\
& - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bd} - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_d^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_b^e + \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ce\phi} g_a^e g_{bd} - \frac{1}{6} \partial_{ef\phi} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ae\phi} g_{bd} g_c^e \\
& - \frac{1}{8} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bd} g_{ef} g_g^f g^{eg} - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bd} g_c^g g^{ef} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bd} g_c^f + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_{cf} g_g^f g^{eg} + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bd} g_g^f g^{eg} \\
& + \frac{1}{6} \partial_{a\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_g^f g^{eg} + \frac{1}{4} \partial_{de\phi} g_a^e g_{bc} + \frac{1}{6} \partial_{ef\phi} g_{ad} g_{bc} g^{ef} + \frac{1}{4} \partial_{ae\phi} g_{bc} g_d^e + \frac{1}{8} \partial_{a\phi} \partial_{a\phi} \phi^{-1} g_{bc} g_{ef} g_g^f g^{eg} + \frac{1}{4} \partial_{a\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bc} g_d^g g^{ef} \\
& - \frac{1}{4} \partial_{a\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bc} g_d^f - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_{df} g_g^f g^{eg} - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bc} g_g^f g^{eg} - \frac{1}{6} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{de\phi} g_{ac} g_b^e - \frac{1}{4} \partial_{be\phi} g_{ac} g_d^e \\
& - \frac{1}{8} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ac} g_{ef} g_g^f g^{eg} - \frac{1}{4} \partial_{a\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bg} g_d^g g^{ef} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_b^e g_d^f + \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{df} g_g^f g^{eg} + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{bf} g_g^f g^{eg} \\
& + \frac{1}{4} \partial_{ce\phi} g_{ad} g_b^e + \frac{1}{4} \partial_{be\phi} g_{ad} g_c^e + \frac{1}{8} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_{ef} g_g^f g^{eg} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bg} g_c^g g^{ef} - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_b^e g_c^f - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{cf} g_g^f g^{eg} \\
& - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{bf} g_g^f g^{eg} + \frac{1}{6} \partial_{ef\phi} g_{ac} g_{bd} g_g^f g^{eg} - \frac{1}{6} \partial_{ef\phi} g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{a\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_{gh} g_i^h g^{eg} g^{fi} + \frac{1}{4} \partial_{a\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_{gh} g_i^h g^{eg} g^{fi} \\
& + \frac{1}{8} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_{gh} g_i^h g^{ef} g^{gi} - \frac{1}{8} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_{gh} g_i^h g^{ef} g^{gi}
\end{aligned}
\tag{ex-14.diff.202}$$

$$\begin{aligned}
\Delta = & -\frac{1}{2} \partial_{bc} \phi g_{ad} + \frac{1}{2} \partial_{ac} \phi g_{bd} + \frac{1}{2} \partial_{ba} \phi g_{ac} - \frac{1}{2} \partial_{ad} \phi g_{bc} + \frac{1}{4} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bc} g_d^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_d^e + \frac{1}{4} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ae} g_c^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bc} \\
& + \frac{1}{4} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_c^e + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_b^e - \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} g_c^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_c^e - \frac{1}{4} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ae} g_d^e \\
& - \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bd} - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_d^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_b^e + \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ce\phi} g_a^e g_{bd} - \frac{1}{6} \partial_{ef\phi} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ac\phi} g_{bd} g_c^e \\
& - \frac{1}{8} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bd} g_f^g g_g^f - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bd} g_c^g g^{ef} + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bd} g_c^f + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_{cf} g_g^f g^{eg} + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bd} g_g^f g^{eg} \\
& + \frac{1}{6} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_g^f g^{eg} + \frac{1}{4} \partial_{de\phi} g_a^e g_{bc} + \frac{1}{6} \partial_{ef\phi} g_{ad} g_{bc} g^{ef} + \frac{1}{4} \partial_{ae\phi} g_{bc} g_d^e + \frac{1}{8} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} g_f^g g_g^f + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bc} g_d^g g^{ef} \\
& - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bc} g_d^f - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_{df} g_g^f g^{eg} - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bc} g_g^f g^{eg} - \frac{1}{6} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{de\phi} g_{ac} g_b^e - \frac{1}{4} \partial_{be\phi} g_{ac} g_d^e \\
& - \frac{1}{8} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ac} g_f^g g_g^f - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bg} g_d^g g^{ef} + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_b^e g_d^f + \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{df} g_g^f g^{eg} + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{bf} g_g^f g^{eg} + \frac{1}{4} \partial_{ce\phi} g_{ad} g_b^e \\
& + \frac{1}{4} \partial_{be\phi} g_{ad} g_c^e + \frac{1}{8} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_f^g g_g^f + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bg} g_c^g g^{ef} - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_b^e g_c^f - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{cf} g_g^f g^{eg} - \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{bf} g_g^f g^{eg} \\
& + \frac{1}{6} \partial_{ef\phi} g_{ac} g_{bd} g_g^f g^{eg} - \frac{1}{6} \partial_{ef\phi} g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_{gh} g_i^h g^{eg} g^{fi} + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_{gh} g_i^h g^{eg} g^{fi} + \frac{1}{8} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_h^i g_i^h g^{ef} \\
& - \frac{1}{8} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_h^i g_i^h g^{ef}
\end{aligned}$$

(ex-14.diff.203)

$$\begin{aligned}
\Delta = & -\frac{1}{2} \partial_{bc} \phi g_{ad} + \frac{1}{2} \partial_{ac} \phi g_{bd} + \frac{1}{2} \partial_{ba} \phi g_{ac} - \frac{1}{2} \partial_{ad} \phi g_{bc} + \frac{1}{4} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bc} g_d^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_d^e + \frac{1}{4} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ae} g_c^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bc} \\
& + \frac{1}{4} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_c^e + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_b^e - \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} g_c^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_c^e - \frac{1}{4} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ae} g_d^e \\
& - \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bd} - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_d^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_b^e + \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ce\phi} g_a^e g_{bd} - \frac{1}{6} \partial_{ef\phi} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ac\phi} g_{bd} g_c^e \\
& - \frac{1}{8} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bd} g_f^g g_g^f - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bd} g_c^g g^{ef} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bd} g_c^f + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_{cf} g_g^f g^{eg} + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bd} g_g^f g^{eg} \\
& + \frac{1}{6} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_g^f g^{eg} + \frac{1}{4} \partial_{de\phi} g_a^e g_{bc} + \frac{1}{6} \partial_{ef\phi} g_{ad} g_{bc} g^{ef} + \frac{1}{4} \partial_{ae\phi} g_{bc} g_d^e + \frac{1}{8} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} g_f^g g_g^f + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bc} g_d^g g^{ef} \\
& - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bc} g_d^f - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_{df} g_g^f g^{eg} - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bc} g_g^f g^{eg} - \frac{1}{6} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{de\phi} g_{ac} g_b^e - \frac{1}{4} \partial_{be\phi} g_{ac} g_d^e \\
& - \frac{1}{8} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ac} g_f^g g_g^f - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bg} g_d^g g^{ef} + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_b^e g_d^f + \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{df} g_g^f g^{eg} + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{bf} g_g^f g^{eg} + \frac{1}{4} \partial_{ce\phi} g_{ad} g_b^e \\
& + \frac{1}{4} \partial_{be\phi} g_{ad} g_c^e + \frac{1}{8} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_f^g g_g^f + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bg} g_c^g g^{ef} - \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_b^e g_c^f - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{cf} g_g^f g^{eg} - \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{bf} g_g^f g^{eg} \\
& + \frac{1}{6} \partial_{ef\phi} g_{ac} g_{bd} g_g^f g^{eg} - \frac{1}{6} \partial_{ef\phi} g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_{gh} g_i^h g^{eg} g^{fi} + \frac{1}{4} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_{gh} g_i^h g^{eg} g^{fi} + \frac{1}{8} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_{gh} g_i^h g^{ef} \\
& - \frac{1}{8} \partial_{c\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_{gh} g_i^h g^{ef}
\end{aligned}$$

(ex-14.diff.204)

$$\begin{aligned}
\Delta = & -\frac{1}{2} \partial_{bc} \phi g_{ad} + \frac{1}{2} \partial_{ac} \phi g_{bd} + \frac{1}{2} \partial_{ba} \phi g_{ac} - \frac{1}{2} \partial_{ad} \phi g_{bc} + \frac{1}{4} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bc} g_d^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_d^e + \frac{1}{4} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ae} g_c^e + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bc} \\
& + \frac{1}{4} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_c^e + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_b^e - \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} g_c^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_c^e - \frac{1}{4} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ae} g_d^e \\
& - \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_a^e g_{bd} - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_d^e - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_b^e + \frac{1}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ce} \phi g_a^e g_{bd} - \frac{1}{6} \partial_{ef} \phi g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ac} \phi g_{bd} g_c^e \\
& - \frac{1}{8} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bd} g_f^g g_g^f - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bd} g_c^g g^{ef} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bd} g_c^f + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_{cf} g_g^f g^{eg} + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bd} g_g^f g^{eg} \\
& + \frac{1}{6} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_g^f g^{eg} + \frac{1}{4} \partial_{de} \phi g_a^e g_{bc} + \frac{1}{6} \partial_{ef} \phi g_{ad} g_{bc} g^{ef} + \frac{1}{4} \partial_{ac} \phi g_{bc} g_d^e + \frac{1}{8} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} g_f^g g_g^f + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ag} g_{bc} g_d^g g^{ef} \\
& - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_a^e g_{bc} g_d^f - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_{df} g_g^f g^{eg} - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{af} g_{bc} g_g^f g^{eg} - \frac{1}{6} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{de} \phi g_{ac} g_b^e - \frac{1}{4} \partial_{bc} \phi g_{ac} g_d^e \\
& - \frac{1}{8} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ac} g_f^g g_g^f - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bg} g_d^g g^{ef} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_b^e g_d^f + \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{df} g_g^f g^{eg} + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{bf} g_g^f g^{eg} + \frac{1}{4} \partial_{ce} \phi g_{ad} g_b^e \\
& + \frac{1}{4} \partial_{bc} \phi g_{ad} g_c^e + \frac{1}{8} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_f^g g_g^f + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bg} g_c^g g^{ef} - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_b^e g_c^f - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{cf} g_g^f g^{eg} - \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{bf} g_g^f g^{eg} \\
& + \frac{1}{6} \partial_{ef} \phi g_{ac} g_{bd} g_g^f g^{eg} - \frac{1}{6} \partial_{ef} \phi g_{ad} g_{bc} g_g^f g^{eg} - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_{gh} g_i^h g^{eg} g^{fi} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_{gh} g_i^h g^{eg} g^{fi} + \frac{1}{8} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_h^i g_i^h g^{ef} \\
& - \frac{1}{8} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_h^i g_i^h g^{ef} \tag{ex-14.diff.205}
\end{aligned}$$

$$\begin{aligned}
\Delta = & -\frac{1}{4} \partial_{bc} \phi g_{ad} + \frac{1}{4} \partial_{ac} \phi g_{bd} + \frac{1}{4} \partial_{ba} \phi g_{ac} - \frac{1}{4} \partial_{ad} \phi g_{bc} + \frac{1}{4} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bd} - \frac{1}{4} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} + \frac{1}{4} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ac} + \frac{1}{4} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} - \frac{1}{4} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} \\
& + \frac{1}{4} \partial_{c\phi} \partial_{b\phi} \phi^{-1} g_{ad} + \frac{5}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4} \partial_{c\phi} \partial_{a\phi} \phi^{-1} g_{bd} - \frac{1}{4} \partial_{a\phi} \partial_{b\phi} \phi^{-1} g_{ac} - \frac{5}{12} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4} \partial_{ca} \phi g_{bd} - \frac{1}{6} \partial_{ef} \phi g_{ac} g_{bd} g^{ef} \\
& - \frac{1}{8} \partial_{a\phi} \partial_{c\phi} \phi^{-1} g_{bd} g_f^f + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bd} g_{cg} g^{eg} + \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ag} g_{bd} g^{eg} + \frac{1}{6} \partial_{e\phi} \partial_{g\phi} \phi^{-1} g_{ac} g_{bd} g^{eg} + \frac{1}{4} \partial_{da} \phi g_{bc} + \frac{1}{6} \partial_{ef} \phi g_{ad} g_{bc} g^{ef} + \frac{1}{8} \partial_{a\phi} \partial_{d\phi} \phi^{-1} g_{bc} g_f^f \\
& - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{bc} g_{dg} g^{eg} - \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ag} g_{bc} g^{eg} - \frac{1}{6} \partial_{e\phi} \partial_{g\phi} \phi^{-1} g_{ad} g_{bc} g^{eg} - \frac{1}{4} \partial_{db} \phi g_{ac} - \frac{1}{8} \partial_{b\phi} \partial_{a\phi} \phi^{-1} g_{ac} g_f^f + \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{dg} g^{eg} \\
& + \frac{1}{4} \partial_{a\phi} \partial_{e\phi} \phi^{-1} g_{ac} g_{bg} g^{eg} + \frac{1}{4} \partial_{cb} \phi g_{ad} + \frac{1}{8} \partial_{b\phi} \partial_{c\phi} \phi^{-1} g_{ad} g_f^f - \frac{1}{4} \partial_{b\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{cg} g^{eg} - \frac{1}{4} \partial_{c\phi} \partial_{e\phi} \phi^{-1} g_{ad} g_{bg} g^{eg} + \frac{1}{6} \partial_{eg} \phi g_{ac} g_{bd} g^{eg} - \frac{1}{6} \partial_{eg} \phi g_{ad} g_{bc} g^{eg} \\
& - \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_{gi} g^{eg} g^{fi} + \frac{1}{4} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_{gi} g^{eg} g^{fi} + \frac{1}{8} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ac} g_{bd} g_h^h g^{ef} - \frac{1}{8} \partial_{e\phi} \partial_{f\phi} \phi^{-1} g_{ad} g_{bc} g_h^h g^{ef} \tag{ex-14.diff.206}
\end{aligned}$$

$$\begin{aligned} \Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{ba}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a\phi}\partial_{c\phi}\phi^{-1}g_{bd} - \frac{1}{4}\partial_{a\phi}\partial_{a\phi}\phi^{-1}g_{bc} + \frac{1}{4}\partial_{b\phi}\partial_{a\phi}\phi^{-1}g_{ac} + \frac{1}{4}\partial_{a\phi}\partial_{a\phi}\phi^{-1}g_{bc} - \frac{1}{4}\partial_{b\phi}\partial_{c\phi}\phi^{-1}g_{ad} \\ & + \frac{1}{4}\partial_{c\phi}\partial_{b\phi}\phi^{-1}g_{ad} + \frac{5}{12}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{c\phi}\partial_{a\phi}\phi^{-1}g_{bd} - \frac{1}{4}\partial_{a\phi}\partial_{b\phi}\phi^{-1}g_{ac} - \frac{5}{12}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{e\phi}\partial_{ac}g_{bd}g^{ef} \\ & - \frac{1}{8}\partial_{a\phi}\partial_{c\phi}\phi^{-1}g_{bd}g_f^f + \frac{1}{4}\partial_{a\phi}\partial_{e\phi}\phi^{-1}g_{bd}g_{cg}g^{eg} + \frac{1}{4}\partial_{c\phi}\partial_{e\phi}\phi^{-1}g_{ag}g_{bd}g^{eg} + \frac{1}{6}\partial_{e\phi}\partial_{g\phi}\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{e\phi}\partial_{ac}g_{bd}g^{ef} + \frac{1}{8}\partial_{a\phi}\partial_{a\phi}\phi^{-1}g_{bc}g_f^f \\ & - \frac{1}{4}\partial_{a\phi}\partial_{e\phi}\phi^{-1}g_{bc}g_{dg}g^{eg} - \frac{1}{4}\partial_{a\phi}\partial_{e\phi}\phi^{-1}g_{ag}g_{bc}g^{eg} - \frac{1}{6}\partial_{e\phi}\partial_{g\phi}\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{ab}\phi g_{ac} - \frac{1}{8}\partial_{b\phi}\partial_{a\phi}\phi^{-1}g_{ac}g_f^f + \frac{1}{4}\partial_{b\phi}\partial_{e\phi}\phi^{-1}g_{ac}g_{dg}g^{eg} \\ & + \frac{1}{4}\partial_{a\phi}\partial_{e\phi}\phi^{-1}g_{ac}g_{bg}g^{eg} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_{b\phi}\partial_{c\phi}\phi^{-1}g_{ad}g_f^f - \frac{1}{4}\partial_{b\phi}\partial_{e\phi}\phi^{-1}g_{ad}g_{cg}g^{eg} - \frac{1}{4}\partial_{c\phi}\partial_{e\phi}\phi^{-1}g_{ad}g_{bg}g^{eg} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} \\ & - \frac{1}{4}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g_{gi}g^{eg}g^{fi} + \frac{1}{4}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g_{gi}g^{eg}g^{fi} + \frac{1}{8}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g_h^h g^{ef} - \frac{1}{8}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g_h^h g^{ef} \quad (\text{ex-14.diff.301}) \end{aligned}$$

$$\begin{aligned} \Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_b\phi\partial_a\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad} \\ & + \frac{1}{4}\partial_c\phi\partial_b\phi\phi^{-1}g_{ad} + \frac{5}{12}\partial_c\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_c\phi\partial_d\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_d\phi\partial_b\phi\phi^{-1}g_{ac} - \frac{5}{12}\partial_c\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{cd}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ & - \frac{1}{8}\partial_a\phi\partial_c\phi\phi^{-1}g_{bd}g_f^f + \frac{1}{4}\partial_a\phi\partial_d\phi\phi^{-1}g_{bd}g_c^c + \frac{1}{4}\partial_c\phi\partial_d\phi\phi^{-1}g_a^c g_{bd} + \frac{1}{6}\partial_c\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_a\phi\partial_d\phi\phi^{-1}g_{bc}g_f^f \\ & - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{bc}g_d^d - \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_a^e g_{bc} - \frac{1}{6}\partial_c\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{ab}\phi g_{ac} - \frac{1}{8}\partial_b\phi\partial_a\phi\phi^{-1}g_{ac}g_f^f + \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ac}g_d^d + \frac{1}{4}\partial_a\phi\partial_e\phi\phi^{-1}g_{ac}g_b^b \\ & + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_b\phi\partial_c\phi\phi^{-1}g_{ad}g_f^f - \frac{1}{4}\partial_b\phi\partial_e\phi\phi^{-1}g_{ad}g_c^c - \frac{1}{4}\partial_c\phi\partial_e\phi\phi^{-1}g_{ad}g_b^b + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_c\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} \\ & + \frac{1}{4}\partial_c\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} + \frac{1}{8}\partial_c\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g_h^h g^{ef} - \frac{1}{8}\partial_c\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g_h^h g^{ef} \end{aligned} \quad (\text{ex-14.diff.302})$$

$$\begin{aligned} \Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a\phi}\partial_{c\phi}\phi^{-1}g_{bd} - \frac{1}{4}\partial_{a\phi}\partial_{a\phi}\phi^{-1}g_{bc} + \frac{1}{4}\partial_{b\phi}\partial_{a\phi}\phi^{-1}g_{ac} + \frac{1}{4}\partial_{a\phi}\partial_{a\phi}\phi^{-1}g_{bc} - \frac{1}{4}\partial_{b\phi}\partial_{c\phi}\phi^{-1}g_{ad} \\ & + \frac{1}{4}\partial_{c\phi}\partial_{b\phi}\phi^{-1}g_{ad} + \frac{5}{12}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{c\phi}\partial_{a\phi}\phi^{-1}g_{bd} - \frac{1}{4}\partial_{a\phi}\partial_{b\phi}\phi^{-1}g_{ac} - \frac{5}{12}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{cd}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ & - \frac{1}{8}\partial_{a\phi}\partial_{c\phi}\phi^{-1}g_{bd}g_f^f + \frac{1}{4}\partial_{a\phi}\partial_{c\phi}\phi^{-1}g_{bd}g_c^e + \frac{1}{4}\partial_{c\phi}\partial_{e\phi}\phi^{-1}g_a^e g_{bd} + \frac{1}{6}\partial_{c\phi}\partial_{g\phi}\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{dd}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\partial_{a\phi}\partial_{a\phi}\phi^{-1}g_{bc}g_f^f \\ & - \frac{1}{4}\partial_{a\phi}\partial_{e\phi}\phi^{-1}g_{bc}g_d^e - \frac{1}{4}\partial_{a\phi}\partial_{e\phi}\phi^{-1}g_a^e g_{bc} - \frac{1}{6}\partial_{c\phi}\partial_{g\phi}\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{ab}\phi g_{ac} - \frac{1}{8}\partial_{b\phi}\partial_{a\phi}\phi^{-1}g_{ac}g_f^f + \frac{1}{4}\partial_{b\phi}\partial_{e\phi}\phi^{-1}g_{ac}g_d^e + \frac{1}{4}\partial_{a\phi}\partial_{e\phi}\phi^{-1}g_{ac}g_b^e \\ & + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_{b\phi}\partial_{c\phi}\phi^{-1}g_{ad}g_f^f - \frac{1}{4}\partial_{b\phi}\partial_{e\phi}\phi^{-1}g_{ad}g_c^e - \frac{1}{4}\partial_{c\phi}\partial_{e\phi}\phi^{-1}g_{ad}g_b^e + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g_g^f g^{eg} \\ & + \frac{1}{4}\partial_{c\phi}\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g_g^f g^{eg} + \frac{1}{8}\partial_{c\phi}\partial_{f\phi}\phi^{-1}g_{ac}g_{bd}g_h^h g^{ef} - \frac{1}{8}\partial_{e\phi}\partial_{f\phi}\phi^{-1}g_{ad}g_{bc}g_h^h g^{ef} \end{aligned} \quad (\text{ex-14.diff.303})$$

$$\begin{aligned} \Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{ba}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_{a\phi}\partial_c\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_{a\phi}\partial_d\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_{b\phi}\partial_a\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_{a\phi}\partial_b\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_{b\phi}\partial_c\phi\phi^{-1}g_{ad} \\ & + \frac{1}{4}\partial_{a\phi}\partial_{b\phi}\phi^{-1}g_{ad} - \frac{1}{12}\partial_{e\phi}\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{e\phi}\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_{a\phi}\partial_{b\phi}\phi^{-1}g_{ac} + \frac{1}{12}\partial_{e\phi}\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{e,f}\phi g_{ac}g_{bd}g^{ef} \\ & + \frac{1}{4}\partial_{a\phi}\partial_e\phi\phi^{-1}g_{bd}g^e_c + \frac{1}{4}\partial_{e\phi}\partial_c\phi\phi^{-1}g^e_a g_{bd} + \frac{1}{6}\partial_{e\phi}\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{e,f}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a\phi}\partial_e\phi\phi^{-1}g_{bc}g^e_d - \frac{1}{4}\partial_{a\phi}\partial_c\phi\phi^{-1}g^e_a g_{bc} \\ & - \frac{1}{6}\partial_{e\phi}\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{ab}\phi g_{ac} + \frac{1}{4}\partial_{b\phi}\partial_e\phi\phi^{-1}g_{ac}g^e_d + \frac{1}{4}\partial_{a\phi}\partial_e\phi\phi^{-1}g_{ac}g^e_b + \frac{1}{4}\partial_{cd}\phi g_{ad} - \frac{1}{4}\partial_{b\phi}\partial_e\phi\phi^{-1}g_{ad}g^e_c - \frac{1}{4}\partial_{c\phi}\partial_e\phi\phi^{-1}g_{ad}g^e_b \\ & + \frac{1}{6}\partial_{e,g}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{e,g}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{e\phi}\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^f_g g^{eg} + \frac{1}{4}\partial_{e\phi}\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^f_g g^{eg} \end{aligned} \quad (\text{ex-14.diff.304})$$

$$\begin{aligned} \Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{ba}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_{a\phi}\partial_c\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_{a\phi}\partial_d\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_{b\phi}\partial_a\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_{a\phi}\partial_b\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_{b\phi}\partial_c\phi\phi^{-1}g_{ad} \\ & + \frac{1}{4}\partial_{a\phi}\partial_{b\phi}\phi^{-1}g_{ad} - \frac{1}{12}\partial_{e\phi}\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{c\phi}\partial_a\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_{a\phi}\partial_{b\phi}\phi^{-1}g_{ac} + \frac{1}{12}\partial_{d\phi}\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{cd}\phi g_{bd} - \frac{1}{6}\partial_{e,f}\phi g_{ac}g_{bd}g^{ef} \\ & + \frac{1}{4}\partial_{a\phi}\partial_e\phi\phi^{-1}g_{bd}g^e_c + \frac{1}{4}\partial_{c\phi}\partial_e\phi\phi^{-1}g^e_a g_{bd} + \frac{1}{6}\partial_{e\phi}\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{e,f}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a\phi}\partial_e\phi\phi^{-1}g_{bc}g^e_d - \frac{1}{4}\partial_{a\phi}\partial_c\phi\phi^{-1}g^e_a g_{bc} \\ & - \frac{1}{6}\partial_{e\phi}\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{ab}\phi g_{ac} + \frac{1}{4}\partial_{b\phi}\partial_e\phi\phi^{-1}g_{ac}g^e_d + \frac{1}{4}\partial_{a\phi}\partial_e\phi\phi^{-1}g_{ac}g^e_b + \frac{1}{4}\partial_{cd}\phi g_{ad} - \frac{1}{4}\partial_{b\phi}\partial_e\phi\phi^{-1}g_{ad}g^e_c - \frac{1}{4}\partial_{c\phi}\partial_e\phi\phi^{-1}g_{ad}g^e_b \\ & + \frac{1}{6}\partial_{e,g}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{e,g}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{e\phi}\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^f_g g^{eg} + \frac{1}{4}\partial_{e\phi}\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^f_g g^{eg} \end{aligned} \quad (\text{ex-14.diff.305})$$

$$\begin{aligned}\Delta = & -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_e\phi\partial_f\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{cd}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ & - \frac{1}{12}\partial_e\phi\partial_g\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_e\phi\partial_g\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} \\ & - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg}\end{aligned}\tag{ex-14.diff.306}$$

$$\Delta = 0 \qquad \qquad \qquad (\text{ex-14.diff.400})$$

Example 15 Verifying the BSSN equations

This is short example verifies two of the main equations in the Phys Rev D paper by Miguel Alcubierre, Bernd Bruggmann et al. (Phys.Rev.D. (62) 044034 (2000)).

The code for the full set of BSSN equations can be found at <https://github.com/leo-brewin/adm-bssn-equations>

```
1 {a,b,c,d,e,f,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent,values={t,x,y,z}).
2 {t,x,y,z}::Coordinate.
3
4 \partial{#}::PartialDerivative.
5 D{#}::Derivative.
6 DBar{#}::Derivative.
7
8 N::Depends(t,x,y,z).
9
10 g_{a b}::Symmetric.
11 g^{a b}::Symmetric.
12 g_{a}^{b}::KroneckerDelta.
13 g^{a}_{b}::KroneckerDelta.
14
15 g_{a b}::Depends(t,x,y,z).
16 g^{a b}::Depends(t,x,y,z).
17
18 gBar_{a b}::Symmetric.
19 gBar^{a b}::Symmetric.
20 gBar_{a}^{b}::KroneckerDelta.
21 gBar^{a}_{b}::KroneckerDelta.
22
23 gBar_{a b}::Depends(t,x,y,z).
24 gBar^{a b}::Depends(t,x,y,z).
25
26 trK::LaTeXForm("K").
27 detg::LaTeXForm("g").
28 ABar{#}::LaTeXForm("{\bar{A}}").
29 DBar{#}::LaTeXForm("{\bar{D}}").
```

15.1 Evolution equation for ϕ

```
1  phi      := \phi -> (1/12) \log(detg).
2  gdotK    := g^{i j} K_{i j} -> trK.
3  DdetgDt  := \partial_t\{detg\} -> detg g^{i j} \partial_t\{g_{i j}\}.
4
5  DgijDt   := \partial_t\{g_{i j}\} -> -2 N K_{i j}.
6
7  dlog      := \partial_{a?}\{\log(A?)\} -> (1/A?)\partial_{a?}\{A?\}.
8  dexp      := \partial_{a?}\{\exp(A?)\} -> \exp(A?)\partial_{a?}\{A?\}.
9
10 dotphi    := \partial_t\{\phi\}.
11
12 substitute (dotphi, phi)           # cdb (ex-15-02.101,dotphi)
13 substitute (dotphi, dlog)          # cdb (ex-15-02.102,dotphi)
14 substitute (dotphi, DdetgDt)       # cdb (ex-15-02.103,dotphi)
15 substitute (dotphi, DgijDt)        # cdb (ex-15-02.104,dotphi)
16 substitute (dotphi, gdotK)         # cdb (ex-15-02.105,dotphi)
17 map_sympy (dotphi, "simplify")     # cdb (ex-15-02.106,dotphi)
18
19 DphiDt := \partial_t\{\phi\} -> @(dotphi).
20
21 checkpoint.append (dotphi)
```

$$\frac{d\phi}{dt} = \frac{1}{12} \partial_t (\log(g)) \quad (\text{ex-15-02.101})$$

$$= \frac{1}{12} g^{-1} \partial_t g \quad (\text{ex-15-02.102})$$

$$= \frac{1}{12} g^{-1} g g^{ij} \partial_t g_{ij} \quad (\text{ex-15-02.103})$$

$$= -\frac{1}{6} g^{-1} g g^{ij} N K_{ij} \quad (\text{ex-15-02.104})$$

$$= -\frac{1}{6} g^{-1} g K N \quad (\text{ex-15-02.105})$$

$$= -\frac{1}{6} K N \quad (\text{ex-15-02.106})$$

15.2 Evolution equation for \bar{g}_{ij}

```
1  gBarij := gBar_{i j} -> \exp(-4\phi) g_{i j}.
2  Kij     := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.
3  A2ABar := \exp(-4\phi) A_{i j} -> ABar_{i j}.
4  ABar2A  := ABar_{i j} -> \exp(-4\phi) A_{i j}.
5
6  dotgBarij := \partial_t{gBar_{i j}}.
7
8  substitute (dotgBarij, gBarij)          # cdb (ex-15-03.101,dotgBarij)
9  product_rule (dotgBarij)                # cdb (ex-15-03.102,dotgBarij)
10 substitute (dotgBarij, dexp)             # cdb (ex-15-03.103,dotgBarij)
11 substitute (dotgBarij, DgijDt)           # cdb (ex-15-03.104,dotgBarij)
12 substitute (dotgBarij, DphiDt)           # cdb (ex-15-03.105,dotgBarij)
13 substitute (dotgBarij, Kij)              # cdb (ex-15-03.106,dotgBarij)
14 distribute (dotgBarij)                  # cdb (ex-15-03.107,dotgBarij)
15 map_sympy (dotgBarij, "simplify")        # cdb (ex-15-03.108,dotgBarij)
16 substitute (dotgBarij, A2ABar)           # cdb (ex-15-03.109,dotgBarij)
17
18 DgBarijDt := \partial_t{gBar_{i j}} -> @(dotgBarij).
19
20 checkpoint.append (dotgBarij)
```

$$\frac{d\bar{g}_{ij}}{dt} = \partial_t(\exp(-4\phi) g_{ij}) \quad (\text{ex-15-03.101})$$

$$= \partial_t(\exp(-4\phi)) g_{ij} + \exp(-4\phi) \partial_t g_{ij} \quad (\text{ex-15-03.102})$$

$$= -4 \exp(-4\phi) \partial_t \phi g_{ij} + \exp(-4\phi) \partial_t g_{ij} \quad (\text{ex-15-03.103})$$

$$= -4 \exp(-4\phi) \partial_t \phi g_{ij} - 2 \exp(-4\phi) N K_{ij} \quad (\text{ex-15-03.104})$$

$$= \frac{2}{3} \exp(-4\phi) K N g_{ij} - 2 \exp(-4\phi) N K_{ij} \quad (\text{ex-15-03.105})$$

$$= \frac{2}{3} \exp(-4\phi) K N g_{ij} - 2 \exp(-4\phi) N \left(A_{ij} + \frac{1}{3} g_{ij} K \right) \quad (\text{ex-15-03.106})$$

$$= \frac{2}{3} \exp(-4\phi) K N g_{ij} - 2 \exp(-4\phi) N A_{ij} - \frac{2}{3} \exp(-4\phi) N g_{ij} K \quad (\text{ex-15-03.107})$$

$$= -2 N \exp(-4\phi) A_{ij} \quad (\text{ex-15-03.108})$$

$$= -2 N \bar{A}_{ij} \quad (\text{ex-15-03.109})$$