

## Exercise 1.1 Verify symmetry of $\Gamma^a_{bc}$

```

1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 g_{a b}::Metric.
4
5 \partialial{#}::PartialDerivative.
6
7 Gamma := \Gamma^a_{b c} -> (1/2) g^a d ( \partialial_{b}{g_{d c}}
8                                     + \partialial_{c}{g_{b d}}
9                                     - \partialial_{d}{g_{b c}} ).
10
11 diff := \Gamma^a_{b c} - \Gamma^a_{c b}.    # cdb (ex-0101.101,diff)
12
13 substitute (diff, Gamma)                  # cdb (ex-0101.102,diff)
14 distribute (diff)                         # cdb (ex-0101.103,diff)
15 canonicalise (diff)                       # cdb (ex-0101.104,diff)

```

$$\begin{aligned}
 \Gamma^a_{bc} - \Gamma^a_{cb} &= \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) - \frac{1}{2}g^{ad}(\partial_c g_{db} + \partial_b g_{cd} - \partial_d g_{cb}) \\
 &= \frac{1}{2}g^{ad}\partial_b g_{dc} + \frac{1}{2}g^{ad}\partial_c g_{bd} - \frac{1}{2}g^{ad}\partial_d g_{bc} - \frac{1}{2}g^{ad}\partial_c g_{db} - \frac{1}{2}g^{ad}\partial_b g_{cd} + \frac{1}{2}g^{ad}\partial_d g_{cb} \\
 &= 0
 \end{aligned}$$

## Exercise 1.2 Christoffel symbol and dg

```

1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 g_{a b}::Metric.
4 g_{a}^{b}::KroneckerDelta.
5
6 \partial_{#}::PartialDerivative.
7
8 GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} ( \partial_{b}{g_{d c}}
9               + \partial_{c}{g_{b d}}
10              - \partial_{d}{g_{b c}} ).
11
12 GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.
13
14 expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}. # cdb (ex-0102.101,expr)
15
16 substitute      (expr, GammaD) # cdb (ex-0102.102,expr)
17 substitute      (expr, GammaU) # cdb (ex-0102.103,expr)
18 distribute      (expr) # cdb (ex-0102.104,expr)
19 eliminate_metric (expr) # cdb (ex-0102.105,expr)
20 eliminate_kronecker (expr) # cdb (ex-0102.106,expr)
21 canonicalise     (expr) # cdb (ex-0102.107,expr)

```

$$\begin{aligned}
 \Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} &= g_{ad} \Gamma^d_{bc} + g_{bd} \Gamma^d_{ac} - \partial_c g_{ab} \\
 &= \frac{1}{2} g_{ad} g^{de} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) + \frac{1}{2} g_{bd} g^{de} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}) - \partial_c g_{ab} \\
 &= \frac{1}{2} g_{ad} g^{de} \partial_b g_{ec} + \frac{1}{2} g_{ad} g^{de} \partial_c g_{be} - \frac{1}{2} g_{ad} g^{de} \partial_e g_{bc} + \frac{1}{2} g_{bd} g^{de} \partial_a g_{ec} + \frac{1}{2} g_{bd} g^{de} \partial_c g_{ae} - \frac{1}{2} g_{bd} g^{de} \partial_e g_{ac} - \partial_c g_{ab} \\
 &= \frac{1}{2} g_a^e \partial_b g_{ec} + \frac{1}{2} g_a^e \partial_c g_{be} - \frac{1}{2} g_a^e \partial_e g_{bc} + \frac{1}{2} g_b^e \partial_a g_{ec} + \frac{1}{2} g_b^e \partial_c g_{ae} - \frac{1}{2} g_b^e \partial_e g_{ac} - \partial_c g_{ab} \\
 &= \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_c g_{ab} \\
 &= 0
 \end{aligned}$$

### Exercise 1.3 Christoffel symbol and dg with a single rule

```

1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 g_{a b}::Metric.
4 g_{a}^{b}::KroneckerDelta.
5
6 \partial{#}::PartialDerivative.
7
8 GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}} ).
11
12 GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}. # cdb (ex-0103.101,GammaD)
13
14 substitute (GammaD, GammaU) # cdb (ex-0103.102,GammaD) # requires Indices(position=independent)
15 distribute (GammaD) # cdb (ex-0103.103,GammaD)
16 eliminate_metric (GammaD) # cdb (ex-0103.104,GammaD)
17 eliminate_kronecker (GammaD) # cdb (ex-0103.105,GammaD)
18
19 expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}. # cdb (ex-0103.201,expr)
20
21 substitute (expr, GammaD) # cdb (ex-0103.202,expr)
22 canonicalise (expr) # cdb (ex-0103.203,expr)

```

$$\Gamma_{abc} \rightarrow g_{ad}\Gamma^d_{bc} \quad (\text{ex-0103.101})$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}g_{ad}g^{de}(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) \quad (\text{ex-0103.102})$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}g_{ad}g^{de}\partial_b g_{ec} + \frac{1}{2}g_{ad}g^{de}\partial_c g_{be} - \frac{1}{2}g_{ad}g^{de}\partial_e g_{bc} \quad (\text{ex-0103.103})$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}g_a^e \partial_b g_{ec} + \frac{1}{2}g_a^e \partial_c g_{be} - \frac{1}{2}g_a^e \partial_e g_{bc} \quad (\text{ex-0103.104})$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc} \quad (\text{ex-0103.105})$$

$$\Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} = \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_c g_{ab} \quad (\text{ex-0103.202})$$

$$= 0 \quad (\text{ex-0103.203})$$

## Exercise 1.3 Repeat but without position=independent

```
1 {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3 g_{a b}::Metric.
4 g_{a}^{b}::KroneckerDelta.
5
6 \partial{#}::PartialDerivative.
7
8 GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}} ).
11
12 GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.          # cdb (ex-0103.301,GammaD)
13
14 substitute      (GammaD, GammaU)          # cdb (ex-0103.302,GammaD)
15 distribute      (GammaD)                  # cdb (ex-0103.303,GammaD)
16 eliminate_metric (GammaD)                  # cdb (ex-0103.304,GammaD)
17 eliminate_kronecker (GammaD)              # cdb (ex-0103.305,GammaD)
18
19 expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}.  # cdb (ex-0103.401,expr)
20
21 substitute      (expr, GammaD)             # cdb (ex-0103.402,expr)
22 canonicalise    (expr)                     # cdb (ex-0103.403,expr)
```

$$\Gamma_{abc} \rightarrow g_{ad}\Gamma_{bc}^d \quad (\text{ex-0103.301})$$

$$\frac{1}{2}g_a^d(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \rightarrow \frac{1}{2}g_{ad}g^{de}(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) \quad (\text{ex-0103.302})$$

$$\frac{1}{2}g_a^d\partial_b g_{dc} + \frac{1}{2}g_a^d\partial_c g_{bd} - \frac{1}{2}g_a^d\partial_d g_{bc} \rightarrow \frac{1}{2}g_{ad}g^{de}\partial_b g_{ec} + \frac{1}{2}g_{ad}g^{de}\partial_c g_{be} - \frac{1}{2}g_{ad}g^{de}\partial_e g_{bc} \quad (\text{ex-0103.303})$$

$$\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc} \rightarrow \frac{1}{2}g_a^e\partial_b g_{ec} + \frac{1}{2}g_a^e\partial_c g_{be} - \frac{1}{2}g_a^e\partial_e g_{bc} \quad (\text{ex-0103.304})$$

$$\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc} \rightarrow \frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc} \quad (\text{ex-0103.305})$$

$$\Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} \quad (\text{ex-0103.402})$$

$$= \Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} \quad (\text{ex-0103.403})$$

## Exercise 1.4 Experiments with sorting

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 expr := C^{f}
6         w^{e}
7         B^{d}
8         v^{c}
9         A^{b}
10        u^{a}.                                # cdb (ex-0104.100,expr)
11
12 sort_product (expr)                          # cdb (ex-0104.101,expr)
13
14 expr := \Omega_{f}
15         \gamma_{e}
16         \Pi_{d}
17         \beta_{c}
18         \Gamma_{b}
19         \alpha_{a}.                          # cdb (ex-0104.200,expr)
20
21 sort_product (expr)                          # cdb (ex-0104.201,expr)
22
23 expr := C^{f}
24         w^{e}
25         B^{d}
26         v^{c}
27         A^{b}
28         u^{a}
29         \Omega_{f}
30         \gamma_{e}
31         \Pi_{d}
32         \beta_{c}
33         \Gamma_{b}
34         \alpha_{a}.                          # cdb (ex-0104.300,expr)
35
36 sort_product (expr)                          # cdb (ex-0104.301,expr)
```

```

37
38 expr := \partial_{f}{C^{f}}
39         w^{l}
40         \partial_{d}{B^{d}}
41         v^{k}
42         \partial_{b}{A^{b}}
43         u^{j}
44         \Omega_{i}
45         \partial^{e}{\gamma_{e}}
46         \Pi_{h}
47         \partial^{c}{\beta_{c}}
48         \Gamma_{g}
49         \partial^{a}{\alpha_{a}}.      # cdb (ex-0104.400,expr)
50
51 sort_product (expr)          # cdb (ex-0104.401,expr)
52
53 expr := \partial{C}
54         w
55         \partial{B}
56         v
57         \partial{A}
58         u
59         \Omega
60         \partial{ \gamma}
61         \Pi
62         \partial{\beta}
63         \Gamma
64         \partial{\alpha}.      # cdb (ex-0104.500,expr)
65
66 sort_product (expr)          # cdb (ex-0104.501,expr)
67
68 expr := A_{b}
69         A_{a}
70         A_{c d e}
71         A_{f g}.      # cdb (ex-0104.600,expr)
72
73 sort_product (expr)          # cdb (ex-0104.601,expr)
74

```



```

75 expr := A_{a} A^{a}
76      + A^{a} A_{a}.
77
78 sort_product (expr)

```

```

# cdb (ex-0104.700,expr)

```

```

# cdb (ex-0104.701,expr)

```

$$\text{ex-0104.100} := C^f w^e B^d v^c A^b u^a$$

$$\text{ex-0104.101} := A^b B^d C^f u^a v^c w^e$$

$$\text{ex-0104.200} := \Omega_f \gamma_e \Pi_d \beta_c \Gamma_b \alpha_a$$

$$\text{ex-0104.201} := \Gamma_b \Omega_f \Pi_d \alpha_a \beta_c \gamma_e$$

$$\text{ex-0104.300} := C^f w^e B^d v^c A^b u^a \Omega_f \gamma_e \Pi_d \beta_c \Gamma_b \alpha_a$$

$$\text{ex-0104.301} := A^b B^d C^f \Gamma_b \Omega_f \Pi_d \alpha_a \beta_c \gamma_e u^a v^c w^e$$

$$\text{ex-0104.400} := \partial_f C^f w^l \partial_d B^d v^k \partial_b A^b u^j \Omega_i \partial^e \gamma_e \Pi_h \partial^c \beta_c \Gamma_g \partial^a \alpha_a$$

$$\text{ex-0104.401} := \Gamma_g \Omega_i \Pi_h \partial_b A^b \partial_d B^d \partial_f C^f \partial^a \alpha_a \partial^c \beta_c \partial^e \gamma_e u^j v^k w^l$$

$$\text{ex-0104.500} := \partial C w \partial B v \partial A u \Omega \partial \gamma \Pi \partial \beta \Gamma \partial \alpha$$

$$\text{ex-0104.501} := \Gamma \Omega \Pi \partial A \partial B \partial C \partial \alpha \partial \beta \partial \gamma u v w$$

$$\text{ex-0104.600} := A_b A_a A_{cde} A_{fg}$$

$$\text{ex-0104.601} := A_a A_b A_{fg} A_{cde}$$

$$\text{ex-0104.700} := A_a A^a + A^a A_a$$

$$\text{ex-0104.701} := A_a A^a + A^a A_a$$

## Exercise 1.5 A sort hack

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z#}::Indices(position=independent).
2
3 foo := A_{a} A^{a} + A^{a} A_{a}.           # cdb (ex-0105.100,foo)
4
5 sort_product (foo)                          # cdb (ex-0105.101,foo)
6
7 substitute (foo, $A^{a} -> Z^{a}$)         # cdb (ex-0105.102,foo)
8 sort_product (foo)                          # cdb (ex-0105.103,foo)
9 substitute (foo, $Z^{a} -> A^{a}$)         # cdb (ex-0105.104,foo)
```

$$\text{ex-0105.100} := A_a A^a + A^a A_a$$

$$\text{ex-0105.101} := A_a A^a + A^a A_a$$

$$\text{ex-0105.102} := A_a Z^a + Z^a A_a$$

$$\text{ex-0105.103} := 2A_a Z^a$$

$$\text{ex-0105.104} := 2A_a A^a$$

## Exercise 1.6 Multiple SortOrder lists

```
1 {D,C,B,A}::SortOrder. # first SortOrder list
2
3 foo := A B C D .      # cdb(ex-0106.101,foo)
4
5 sort_product (foo)    # cdb(ex-0106.102,foo)
6
7 {V,U}::SortOrder.     # second SortOrder list, all entries distinct from first list
8
9 foo := U V A B C D .  # cdb(ex-0106.201,foo)
10
11 sort_product (foo)    # cdb(ex-0106.202,foo)
12
13 {A,B,C,D}::SortOrder. # all entries in this list appear in the
14                       # first SortOrder so they will be effectively ignored
15
16 foo := U V D C B A .  # cdb(ex-0106.301,foo)
17
18 sort_product (foo)    # cdb(ex-0106.302,foo)
```

ex-0106.101 := *ABCD*

ex-0106.102 := *DCBA*

ex-0106.201 := *UVABCD*

ex-0106.202 := *DCBAUV*

ex-0106.301 := *UVDCBA*

ex-0106.302 := *DCBAUV*

## Exercise 1.7 Subtleties of `foo = bah` and `foo := @(bah)`

```
1 {a,b,c,d,e,f,h#}::Indices.
2
3 foo := B_{b} A_{a}.
4 bah := A_{a} C_{c}.
5
6 # cdbBeg(print.0107)
7 print("foo = "+str(foo))
8 print("bah = "+str(bah)+"\n")
9
10 print("type foo = "+str(type(foo)))
11 print("type bah = "+str(type(bah))+"\n")
12
13 print("id foo = "+str(id(foo)))
14 print("id bah = "+str(id(bah))+"\n")
15
16 bah = foo
17
18 print("foo = "+str(foo))
19 print("bah = "+str(bah)+"\n")
20
21 sort_product (foo)
22
23 print("bah = "+str(bah)+"\n")
24
25 print("id foo = "+str(id(foo)))
26 print("id bah = "+str(id(bah))+"\n")
27
28 bah := @(foo).
29
30 print("id foo = "+str(id(foo)))
31 print("id bah = "+str(id(bah))+"\n")
32 # cdbEnd(print.0107)
```

```

1  foo = B_{b} A_{a}
2  bah = A_{a} C_{c}
3
4  type foo = <class 'cadabra2.Ex'>
5  type bah = <class 'cadabra2.Ex'>
6
7  id foo = 4439454080
8  id bah = 4403326456
9
10 foo = B_{b} A_{a}
11 bah = B_{b} A_{a}
12
13 bah = A_{a} B_{b}
14
15 id foo = 4439454080
16 id bah = 4439454080
17
18 id foo = 4439454080
19 id bah = 4443249104

```

Note that the line numbers referenced in the following are those of the output above not those of the Cadabra source.

- Lines 7 and 8 show that the objects `foo` and `bah` point to distinct areas of memory (i.e., they point to different objects).
- Lines 10 and 11 show the result of the statement `bah = foo`.
- Line 13 shows that `bah` has changed after the statement `sort_product (foo)`.
- Lines 15 and 16 verifies that `foo` and `bah` point to the same object (so changes in `foo` will be seen by `bah`, as just noted).
- Lines 18 and 19 shows that after `bah := @(foo)` the symbols `bah` and `foo` no longer point to the same object.

## Exercise 1.8 Syntax errors – original code

```
1 {a,b,c,d,e,f#}::Indices.
2 C{#}::Symmetric.
3
4 foo := A_{a} B_{b} + C_{ab}.           # C_{ab} should be C_{a b}
5 bah := B_{b} A_{a} + C_{ba}.           # C_{ba} should be C_{b a}
6 meh := @(foo) - @(bah)                 # missing dot or semi-colon terminator
7
8 if meh == 0:
9     print ("meh is zero, and all is good")
10     success = True.                   # indentation error, drop the dot
11 else:
12     print ("meh is not zero, oops")
13     success = False.                 # indentation error, drop the dot
14
15 canonicalise (meh).                  # terminate with ; or nothing
16 sort_product (meh);
17
18 {\alpha\beta\gamma}::Indices.         # separate list elements with commas
19
20 foo := Ex ("A_{ab} - A_{a b}");       # use = for assignment, A_{ab} should be A_{a b}
21 bah := Ex ("A_{\alpha\beta} - A_{\alpha \beta}"); # use = for assignment, need raw string in Ex
```

## Exercise 1.8 Syntax errors – corrected code

```

1  {a,b,c,d,e,f#}::Indices.
2  C{#}::Symmetric.
3
4  foo := A_{a} B_{b} + C_{a b}.           # cdb (ex-0108.101,foo)
5  bah := B_{b} A_{a} + C_{b a}.           # cdb (ex-0108.102,bah)
6  meh := @(foo) - @(bah).                 # cdb (ex-0108.103,meh)
7
8  if meh == 0:
9      print ("meh is zero, and all is good")
10     success = True
11 else:
12     print ("meh is not zero, oops")
13     success = False
14
15 canonicalise (meh)                       # cdb (ex-0108.104,meh)
16 sort_product (meh);                     # cdb (ex-0108.105,meh)
17
18 {\alpha,\beta,\gamma}::Indices.
19
20 foo = Ex ("A_{a b} - A_{a b}");          # cdb (ex-0108.106,foo)
21 bah = Ex (r"A_{\alpha\beta} - A_{\alpha \beta}"); # cdb (ex-0108.107,bah)

```

```

ex-0108.101 :=  $A_a B_b + C_{ab}$ 
ex-0108.102 :=  $B_b A_a + C_{ba}$ 
ex-0108.103 :=  $A_a B_b + C_{ab} - B_b A_a - C_{ba}$ 
ex-0108.104 :=  $A_a B_b - B_b A_a$ 
ex-0108.105 := 0
ex-0108.106 := 0
ex-0108.107 := 0

```

## Exercise 1.9 No index clashes

```
1 {a,b,c,d,e,f,u,v,w}::Indices.
2
3 foo := A_{a c} C^c.           # cdb (ex-0109.101,foo)
4 bah := B_{b c} C^c.           # cdb (ex-0109.102,bah)
5
6 foobah := @(foo) @(bah).       # cdb (ex-0109.103,foobah)
```

$$A_{ac}C^c \quad (\text{ex-0109.101})$$

$$B_{bc}C^c \quad (\text{ex-0109.102})$$

$$A_{ac}C^c B_{bd}C^d \quad (\text{ex-0109.103})$$



## Exercise 1.10 Relabel free indices

```
1 {a,b,c,d,e,f,u,v,w}::Indices.  
2  
3 \delta{#}::KroneckerDelta.  
4  
5 expr := A_{a b c}. # cdb (ex-0110.101,expr)  
6  
7 expr := \delta^{a}_{u} \delta^{b}_{v} \delta^{c}_{w} @ (expr). # cdb (ex-0110.102,expr)  
8  
9 eliminate_kronecker (expr) # cdb (ex-0110.103,expr)
```

$$A_{abc} \quad (ex-0110.101)$$

$$\delta^a_u \delta^b_v \delta^c_w A_{abc} \quad (ex-0110.102)$$

$$A_{uvw} \quad (ex-0110.103)$$

## Exercise 1.11 Cycling free indices – preferred solution

```
1 {a,b,c,d,e,f,u,v,w}::Indices.
2
3 expr := A_{a b c}.                                # cdb (ex-0111.101,expr)
4
5 rule := T_{a b c} -> @(expr).
6 expr := T_{b c a}.                                # cdb (ex-0111.102,expr)
7
8 substitute (expr, rule)                            # cdb (ex-0111.103,expr)
```

$A_{abc}$	(ex-0111.101)
$T_{bca}$	(ex-0111.102)
$A_{bca}$	(ex-0111.103)

## Exercise 1.11 Cycling free indices – alternative solution

This alternative solution uses two rounds of Kronecker deltas. It does the job but is not as simple as the previous solution.

```

1  {a,b,c,d,e,f,u,v,w}::Indices.
2
3  \delta{#}::KroneckerDelta.
4
5  expr := A_{a b c}.                                # cdb (ex-0111.201,expr)
6
7  expr := \delta^{a}_{u} \delta^{b}_{v} \delta^{c}_{w} @(expr). # cdb (ex-0111.202,expr)
8
9  eliminate_kronecker (expr)                        # cdb (ex-0111.203,expr)
10
11 expr := \delta^{u}_{b} \delta^{v}_{c} \delta^{w}_{a} @(expr). # cdb (ex-0111.204,expr)
12
13 eliminate_kronecker (expr)                        # cdb (ex-0111.205,expr)

```

$A_{abc}$	(ex-0111.201)
$\delta^a_u \delta^b_v \delta^c_w A_{abc}$	(ex-0111.202)
$A_{uvw}$	(ex-0111.203)
$\delta^u_b \delta^v_c \delta^w_a A_{uvw}$	(ex-0111.204)
$A_{bca}$	(ex-0111.205)

## Exercise 2.1 Using Cadabra's own product rule

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # templates for covariant derivatives
7
8 deriv1 := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}}
9         + \Gamma^{b}_{c a} A^{c}.
10
11 deriv2 := \nabla_{a}{A_{b}} -> \partial_{a}{A_{b}}
12         - \Gamma^{c}_{b a} A_{c}.
13
14 # create an object
15
16 uv := \nabla_{a}{v_{b} u^{b}}
17      - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.101,uv)
18
19 # apply the rules, then simplify
20
21 product_rule (uv)      # cdb (ex-0201.102,uv)
22 substitute (uv,deriv1) # cdb (ex-0201.103,uv)
23 substitute (uv,deriv2) # cdb (ex-0201.104,uv)
24 distribute (uv)        # cdb (ex-0201.105,uv)
25 sort_product (uv)      # cdb (ex-0201.106,uv)
26 rename_dummies (uv)    # cdb (ex-0201.107,uv)
```

$$\nabla_a (v_b u^b) - \partial_a (v_b u^b) = \nabla_a v_b u^b + v_b \nabla_a u^b - \partial_a v_b u^b - v_b \partial_a u^b \quad (\text{ex-0201.102})$$

$$= \nabla_a v_b u^b + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - \partial_a v_b u^b - v_b \partial_a u^b \quad (\text{ex-0201.103})$$

$$= (\partial_a v_b - \Gamma^c_{ba} v_c) u^b + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - \partial_a v_b u^b - v_b \partial_a u^b \quad (\text{ex-0201.104})$$

$$= -\Gamma^c_{ba} v_c u^b + v_b \Gamma^b_{ca} u^c \quad (\text{ex-0201.105})$$

$$= -\Gamma^c_{ba} u^b v_c + \Gamma^b_{ca} u^c v_b \quad (\text{ex-0201.106})$$

$$= 0 \quad (\text{ex-0201.107})$$

## Exercise 2.1 Using hand crafted product rules

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # templates for covariant derivatives
7
8 deriv1 := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}}
9         + \Gamma^{b}_{c a} A^{c}.
10
11 deriv2 := \nabla_{a}{A_{b}} -> \partial_{a}{A_{b}}
12         - \Gamma^{c}_{b a} A_{c}.
13
14 # templates for product rules
15
16 deriv3 := \nabla_{a}{A_{b} B^{c}} -> B^{c} \nabla_{a}{A_{b}}
17         + A_{b} \nabla_{a}{B^{c}}.
18
19 deriv4 := \partial_{a}{A_{b} B^{c}} -> B^{c} \partial_{a}{A_{b}}
20         + A_{b} \partial_{a}{B^{c}}.
21
22 # create an object
23
24 uv := \nabla_{a}{v_{b} u^{b}}
25     - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.201,uv)
26
27 # apply the rules, then simplify
28
29 substitute (uv,deriv3)          # cdb (ex-0201.202,uv)
30 substitute (uv,deriv4)          # cdb (ex-0201.203,uv)
31 substitute (uv,deriv1)          # cdb (ex-0201.204,uv)
32 substitute (uv,deriv2)          # cdb (ex-0201.205,uv)
33 distribute (uv)                 # cdb (ex-0201.206,uv)
34 sort_product (uv)               # cdb (ex-0201.207,uv)
35 rename_dummies (uv)             # cdb (ex-0201.208,uv)

```

$$\nabla_a (v_b u^b) - \partial_a (v_b u^b) = u^b \nabla_a v_b + v_b \nabla_a u^b - \partial_a (v_b u^b) \quad (\text{ex-0201.202})$$

$$= u^b \nabla_a v_b + v_b \nabla_a u^b - u^b \partial_a v_b - v_b \partial_a u^b \quad (\text{ex-0201.203})$$

$$= u^b \nabla_a v_b + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - u^b \partial_a v_b - v_b \partial_a u^b \quad (\text{ex-0201.204})$$

$$= u^b (\partial_a v_b - \Gamma^c_{ba} v_c) + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - u^b \partial_a v_b - v_b \partial_a u^b \quad (\text{ex-0201.205})$$

$$= -u^b \Gamma^c_{ba} v_c + v_b \Gamma^b_{ca} u^c \quad (\text{ex-0201.206})$$

$$= -\Gamma^c_{ba} u^b v_c + \Gamma^b_{ca} u^c v_b \quad (\text{ex-0201.207})$$

$$= 0 \quad (\text{ex-0201.208})$$

## Exercise 2.2 Covariant derivative of $v_{ab}$

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # template for covariant derivative of a vector
7
8 derivU := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}} + \Gamma^{b}_{c a} A^{c}.
9 derivD := \nabla_{a}{A_{b}} -> \partial_{a}{A_{b}} - \Gamma^{c}_{c b} A_{c}.
10
11 vab := v_{a b} -> A_{a} B_{b}.
12 iab := A_{a} B_{b} -> v_{a b}.
13
14 pab := \partial_{a}{A_{b}} B_{c} -> \partial_{a}{A_{b} B_{c}} - A_{b} \partial_{a}{B_{c}}.
15
16 # create an object
17
18 Dvab := \nabla_{a}{v_{b c}}. # cdb (ex-0202.101,Dvab)
19
20 # apply the rule, then simplify
21
22 substitute (Dvab,vab) # cdb (ex-0202.102,Dvab)
23 product_rule (Dvab) # cdb (ex-0202.103,Dvab)
24 substitute (Dvab,derivD) # cdb (ex-0202.104,Dvab)
25 substitute (Dvab,derivU) # cdb (ex-0202.105,Dvab)
26 distribute (Dvab) # cdb (ex-0202.106,Dvab)
27 substitute (Dvab,pab) # cdb (ex-0202.107,Dvab)
28 canonicalise (Dvab) # cdb (ex-0202.108,Dvab)
29 substitute (Dvab,iab) # cdb (ex-0202.109,Dvab)
30 sort_product (Dvab) # cdb (ex-0202.110,Dvab)
```



$$\begin{aligned}\nabla_a v_{bc} &= \nabla_a (A_b B_c) && (\text{ex-0202.102}) \\ &= \nabla_a A_b B_c + A_b \nabla_a B_c && (\text{ex-0202.103}) \\ &= (\partial_a A_b - \Gamma^d_{ba} A_d) B_c + A_b (\partial_a B_c - \Gamma^d_{ca} B_d) && (\text{ex-0202.104}) \\ &= (\partial_a A_b - \Gamma^d_{ba} A_d) B_c + A_b (\partial_a B_c - \Gamma^d_{ca} B_d) && (\text{ex-0202.105}) \\ &= \partial_a A_b B_c - \Gamma^d_{ba} A_d B_c + A_b \partial_a B_c - A_b \Gamma^d_{ca} B_d && (\text{ex-0202.106}) \\ &= \partial_a (A_b B_c) - \Gamma^d_{ba} A_d B_c - A_b \Gamma^d_{ca} B_d && (\text{ex-0202.107}) \\ &= \partial_a (A_b B_c) - \Gamma^d_{ba} A_d B_c - A_b \Gamma^d_{ca} B_d && (\text{ex-0202.108}) \\ &= \partial_a v_{bc} - \Gamma^d_{ba} v_{dc} - v_{bd} \Gamma^d_{ca} && (\text{ex-0202.109}) \\ &= \partial_a v_{bc} - \Gamma^d_{ba} v_{dc} - \Gamma^d_{ca} v_{bd} && (\text{ex-0202.110})\end{aligned}$$

## Exercise 2.3 Covariant derivative of $v^a_b$

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # template for covariant derivative of a vector
7
8 derivU := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}} + \Gamma^{b}_{c a} A^{c}.
9 derivD := \nabla_{a}{A_{b}} -> \partial_{a}{A_{b}} - \Gamma^{c}_{c b} A_{c}.
10
11 vab := v^{a}_{b} -> A^{a} B_{b}.
12 iab := A^{a} B_{b} -> v^{a}_{b}.
13
14 pab := \partial_{a}{A^{b}} B_{c} -> \partial_{a}{A^{b} B_{c}} - A^{b} \partial_{a}{B_{c}}.
15
16 # create an object
17
18 Dvab := \nabla_{a}{v^{b}_{c}}. # cdb (ex-0203.101,Dvab)
19
20 # apply the rule, then simplify
21
22 substitute (Dvab,vab) # cdb (ex-0203.102,Dvab)
23 product_rule (Dvab) # cdb (ex-0203.103,Dvab)
24 substitute (Dvab,derivD) # cdb (ex-0203.104,Dvab)
25 substitute (Dvab,derivU) # cdb (ex-0203.105,Dvab)
26 distribute (Dvab) # cdb (ex-0203.106,Dvab)
27 substitute (Dvab,pab) # cdb (ex-0203.107,Dvab)
28 canonicalise (Dvab) # cdb (ex-0203.108,Dvab)
29 substitute (Dvab,iab) # cdb (ex-0203.109,Dvab)
30 sort_product (Dvab) # cdb (ex-0203.110,Dvab)
```

$$\nabla_a v^b{}_c = \nabla_a (A^b B_c) \quad (\text{ex-0203.102})$$

$$= \nabla_a A^b B_c + A^b \nabla_a B_c \quad (\text{ex-0203.103})$$

$$= \nabla_a A^b B_c + A^b (\partial_a B_c - \Gamma^d{}_{ca} B_d) \quad (\text{ex-0203.104})$$

$$= (\partial_a A^b + \Gamma^b{}_{da} A^d) B_c + A^b (\partial_a B_c - \Gamma^d{}_{ca} B_d) \quad (\text{ex-0203.105})$$

$$= \partial_a A^b B_c + \Gamma^b{}_{da} A^d B_c + A^b \partial_a B_c - A^b \Gamma^d{}_{ca} B_d \quad (\text{ex-0203.106})$$

$$= \partial_a (A^b B_c) + \Gamma^b{}_{da} A^d B_c - A^b \Gamma^d{}_{ca} B_d \quad (\text{ex-0203.107})$$

$$= \partial_a (A^b B_c) + \Gamma^b{}_{da} A^d B_c - A^b \Gamma^d{}_{ca} B_d \quad (\text{ex-0203.108})$$

$$= \partial_a v^b{}_c + \Gamma^b{}_{da} v^d{}_c - v^b{}_d \Gamma^d{}_{ca} \quad (\text{ex-0203.109})$$

$$= \partial_a v^b{}_c + \Gamma^b{}_{da} v^d{}_c - \Gamma^d{}_{ca} v^b{}_d \quad (\text{ex-0203.110})$$

## Exercise 2.4 Combining rules – a problem

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # rules for covariant derivatives of v
7
8 deriv1 := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}}
9          + \Gamma^{b}_{d a} v^{d}.
10
11 deriv2 := \nabla_{a}{\nabla_{b}{v^{c}}} -> \partial_{a}{\nabla_{b}{v^{c}}}
12          + \Gamma^{c}_{d a} \nabla_{b}{v^{d}}
13          - \Gamma^{d}_{d b a} \nabla_{d}{v^{c}}.
14
15 # attempt to combine both rules for second covariant derivative of v
16
17 substitute (deriv2,deriv1)      # cdb (ex-0204.101,deriv2)
```

Note that the call to `substitute` has made changes to both sides of the rule for `deriv2`. This is not ideal and a better method is developed in the following exercise.

$$\nabla_a (\partial_b v^c + \Gamma^c_{db} v^d) \rightarrow \partial_a (\partial_b v^c + \Gamma^c_{db} v^d) + \Gamma^c_{da} (\partial_b v^d + \Gamma^d_{eb} v^e) - \Gamma^d_{ba} (\partial_d v^c + \Gamma^c_{ed} v^e) \quad (\text{ex-0204.101})$$

## Exercise 2.5 Combining rules – a solution

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # rules for covariant derivatives of v
7
8 deriv1 := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}}
9         + \Gamma^{b}_{d a} v^{d}.
10
11 deriv2 := \nabla_{a}{\nabla_{b}{v^{c}}} -> \partial_{a}{\nabla_{b}{v^{c}}}
12         + \Gamma^{c}_{d a} \nabla_{b}{v^{d}}
13         - \Gamma^{d}_{d b a} \nabla_{d}{v^{c}}.
14
15 # second covariant derivative of v
16
17 expr := v^{c}_{b a} -> \nabla_{a}{\nabla_{b}{v^{c}}}. # cdb (ex-0205.101,expr)
18 save := @(expr).
19
20 # apply the rules, then simplify
21
22 substitute (expr,deriv2) # cdb (ex-0205.102,expr)
23 substitute (expr,deriv1) # cdb (ex-0205.103,expr)
24 distribute (expr) # cdb (ex-0205.104,expr)
25 product_rule (expr) # cdb (ex-0205.105,expr)
26 canonicalise (expr) # cdb (ex-0205.107,expr)
27 substitute (expr,save) # cdb (ex-0205.108,expr)
```

The trick here is to introduce in line 17 a dummy left hand side,  $v^{c}_{b a}$ , that is invisible with respect to the substitution rules of lines 8 and 11. Thus lines 22 and 23 will only target the right hand side of `expr`.

Notice how a copy of the initial expression is made in 18. This is used later in line 27 to replace the dummy object  $v^{c}_{b a}$  with  $\nabla_{a}{\nabla_{b}{v^{c}}}$  but this time acting on the left hand side of the rule. The result is a rule for second covariant derivatives.

$$v^c{}_{ba} \rightarrow \nabla_a (\nabla_b v^c) \quad (\text{ex-0205.101})$$

$$v^c{}_{ba} \rightarrow \partial_a (\nabla_b v^c) + \Gamma^c{}_{da} \nabla_b v^d - \Gamma^d{}_{ba} \nabla_d v^c \quad (\text{ex-0205.102})$$

$$v^c{}_{ba} \rightarrow \partial_a (\partial_b v^c + \Gamma^c{}_{db} v^d) + \Gamma^c{}_{da} (\partial_b v^d + \Gamma^d{}_{eb} v^e) - \Gamma^d{}_{ba} (\partial_d v^c + \Gamma^c{}_{ed} v^e) \quad (\text{ex-0205.103})$$

$$v^c{}_{ba} \rightarrow \partial_{ab} v^c + \partial_a (\Gamma^c{}_{db} v^d) + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^d{}_{ba} \Gamma^c{}_{ed} v^e \quad (\text{ex-0205.104})$$

$$v^c{}_{ba} \rightarrow \partial_{ab} v^c + \partial_a \Gamma^c{}_{db} v^d + \Gamma^c{}_{db} \partial_a v^d + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^d{}_{ba} \Gamma^c{}_{ed} v^e \quad (\text{ex-0205.105})$$

$$v^c{}_{ba} \rightarrow \partial_{ab} v^c + \partial_a \Gamma^c{}_{db} v^d + \Gamma^c{}_{db} \partial_a v^d + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^c{}_{de} \Gamma^e{}_{ba} v^d \quad (\text{ex-0205.107})$$

$$\nabla_a (\nabla_b v^c) \rightarrow \partial_{ab} v^c + \partial_a \Gamma^c{}_{db} v^d + \Gamma^c{}_{db} \partial_a v^d + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^c{}_{de} \Gamma^e{}_{ba} v^d \quad (\text{ex-0205.108})$$

## Exercise 2.6 Commutation of $\nabla$ on a scalar

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # covariant derivative of \phi
7
8 dphi := \nabla_{a}{\phi} -> \partial_{a}{\phi}.
9
10 # rules to hide and reveal \partial\phi
11
12 hide := \partial_{a}{\phi} -> w_{a}.
13 reveal := w_{a} -> \partial_{a}{\phi}.
14
15 # template for covariant derivative of a dual-vector
16
17 deriv := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.
18
19 # create an object
20
21 expr := \nabla_{a}{\nabla_{b}{\phi}}
22 - \nabla_{b}{\nabla_{a}{\phi}}. # cdb (ex-0206.101,expr)
23
24 # apply the rules, then simplify
25
26 substitute (expr,dphi) # cdb (ex-0206.102,expr)
27 substitute (expr,hide) # cdb (ex-0206.103,expr)
28 substitute (expr,deriv) # cdb (ex-0206.104,expr)
29 substitute (expr,reveal) # cdb (ex-0206.105,expr)
30 canonicalise (expr) # cdb (ex-0206.106,expr)
```

$$\nabla_a (\nabla_b \phi) - \nabla_b (\nabla_a \phi) = \nabla_a (\partial_b \phi) - \nabla_b (\partial_a \phi) \quad (\text{ex-0206.102})$$

$$= \nabla_a w_b - \nabla_b w_a \quad (\text{ex-0206.103})$$

$$= \partial_a w_b - \Gamma_{ba}^c w_c - \partial_b w_a + \Gamma_{ab}^c w_c \quad (\text{ex-0206.104})$$

$$= \partial_{ab} \phi - \Gamma_{ba}^c \partial_c \phi - \partial_{ba} \phi + \Gamma_{ab}^c \partial_c \phi \quad (\text{ex-0206.105})$$

$$= -\Gamma_{ba}^c \partial_c \phi + \Gamma_{ab}^c \partial_c \phi \quad (\text{ex-0206.106})$$



## Exercise 2.7 Selective kill

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 hide    := \partial_{d}{\Gamma^{a}_{b c}} -> Z_{d a b c}.
6 reveal := Z_{d a b c} -> \partial_{d}{\Gamma^{a}_{b c}}.
7
8 kill := \Gamma^{a}_{b c} -> 0.
9
10 Gamma := \Gamma^{a}_{b c}
11         + x^{d} \partial_{d}{\Gamma^{a}_{b c}}.      # cdb (ex-0207.101,Gamma)
12
13 substitute (Gamma,hide)      # cdb (ex-0207.102,Gamma)
14 substitute (Gamma,kill)     # cdb (ex-0207.103,Gamma)
15 substitute (Gamma,reveal)   # cdb (ex-0207.104,Gamma)

```

$$\begin{aligned}
 \Gamma^a_{bc}(x) &= \Gamma^a_{bc} + x^d \partial_d \Gamma^a_{bc} && (\text{ex-0207.101}) \\
 &= \Gamma^a_{bc} + x^d Z_{dabc} && (\text{ex-0207.102}) \\
 &= x^d Z_{dabc} && (\text{ex-0207.103}) \\
 &= x^d \partial_d \Gamma^a_{bc} && (\text{ex-0207.104})
 \end{aligned}$$

## Exercise 2.7 Naive kill

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::Derivative.
4
5 kill := \Gamma^{a}_{b c} -> 0.
6
7 Gamma := \Gamma^{a}_{b c}
8         + x^{d} \partial_{d}\{\Gamma^{a}_{b c}\}.      # cdb (ex-0207.201,Gamma)
9
10 substitute (Gamma,kill)                        # cdb (ex-0207.202,Gamma)

```

$$\Gamma^a_{bc}(x) = \Gamma^a_{bc} + x^d \partial_d \Gamma^a_{bc} \quad (\text{ex-0207.201})$$

$$= 0 \quad (\text{ex-0207.202})$$

## Exercise 2.7 No problem killing partial derivatives

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 kill := \partial_{c}\{A_{a b}\} -> 0.
6
7 Aab := A_{a b} + x^{c} \partial_{c}\{A_{a b}\}
8         + x^{c} x^{d} \partial_{d}\{\partial_{c}\{A_{a b}\}\}.      # cdb (ex-0207.301,Aab)
9
10 substitute (Aab,kill)                            # cdb (ex-0207.302,Aab)

```

$$A_{ab}(x) = A_{ab} + x^c \partial_c A_{ab} + x^c x^d \partial_{dc} A_{ab} \quad (\text{ex-0207.301})$$

$$= A_{ab} + x^c x^d \partial_{dc} A_{ab} \quad (\text{ex-0207.302})$$

## Exercise 2.8 Position keyword in ::Indices

```

1 {a,b,c}::Indices(position=free).
2
3 foo := A_{a b} + A^{a b}.           # cdb (ex-0208.101,foo)
4
5 substitute (foo, $A_{a b} -> B_{a b}$)   # cdb (ex-0208.102,foo)
6
7 {p,q,r}::Indices(position=fixed).
8
9 foo := A_{p q} B^{p q} + A^{p q} B_{p q}.   # cdb (ex-0208.201,foo)
10
11 canonicalise (foo)                       # cdb (ex-0208.202,foo)
12
13 {u,v,w}::Indices(position=independent).
14
15 foo := A_{u v} B^{u v} + A^{u v} B_{u v}.   # cdb (ex-0208.301,foo)
16
17 canonicalise (foo)                       # cdb (ex-0208.302,foo)

```

$$A_{ab} + A^{ab} = B_{ab} + B^{ab} \quad (\text{ex-0208.102})$$

$$A_{pq}B^{pq} + A^{pq}B_{pq} = 2A^{pq}B_{pq} \quad (\text{ex-0208.202})$$

$$A_{uv}B^{uv} + A^{uv}B_{uv} = A_{uv}B^{uv} + A^{uv}B_{uv} \quad (\text{ex-0208.302})$$

## Exercise 3.1 Some symmetries of Riemann

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 ::Symbol;
4
5 \partial{#}::PartialDerivative.
6
7 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
8
9 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
10                      - \partial_{d}{\Gamma^{a}_{b c}}
11                      + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
12                      - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.      # cdb(Rabcd.000,Rabcd)
13
14 dRabcd := R^{a}_{b c d ; e} -> \partial_{e}{R^{a}_{b c d}}
15                      + \Gamma^{a}_{f e} R^{f}_{b c d}
16                      - \Gamma^{f}_{b e} R^{a}_{f c d}
17                      - \Gamma^{f}_{c e} R^{a}_{b f d}
18                      - \Gamma^{f}_{d e} R^{a}_{b c f}.      # cdb(dRabcd.000,dRabcd)
```

## Exercise 3.1 Antisymmetry on last pair of indices

```
1  expr := R^{a}_{b c d} + R^{a}_{b d c}.                                # cdb(ex-0301.101,expr)
2
3  substitute (expr, Rabcd)                                           # cdb(ex-0301.102,expr)
```

$$R^a_{bcd} + R^a_{bdc} = 0 \qquad (\text{ex-0301.102})$$

## Exercise 3.1 First Bianchi identity

```

1  expr := R^{a}_{b c d} + R^{a}_{d b c} + R^{a}_{c d b}.           # cdb(ex-0301.201,expr)
2
3  substitute (expr, Rabcd)                                     # cdb(ex-0301.202,expr)
4  canonicalise (expr)                                          # cdb(ex-0301.203,expr)

```

$$\begin{aligned}
 R^a_{bcd} + R^a_{dbc} + R^a_{cdb} &= \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ce} - \Gamma^e_{bc} \Gamma^a_{de} + \partial_b \Gamma^a_{dc} - \partial_c \Gamma^a_{db} + \Gamma^e_{dc} \Gamma^a_{be} - \Gamma^e_{db} \Gamma^a_{ce} + \partial_d \Gamma^a_{cb} - \partial_b \Gamma^a_{cd} + \Gamma^e_{cb} \Gamma^a_{de} \\
 &\quad - \Gamma^e_{cd} \Gamma^a_{be} \qquad \qquad \qquad \text{(ex-0301.202)} \\
 &= 0 \qquad \qquad \qquad \text{(ex-0301.203)}
 \end{aligned}$$

## Exercise 3.1 Second Bianchi identity

```
1  expr := R^{a}_{b c d ; e} + R^{a}_{b e c ; d} + R^{a}_{b d e ; c}.    # cdb(ex-0301.301,expr)
2
3  substitute      (expr, dRabcd)      # cdb(ex-0301.302,expr)
4  substitute      (expr, Rabcd)       # cdb(ex-0301.303,expr)
5  distribute      (expr)              # cdb(ex-0301.304,expr)
6  product_rule    (expr)              # cdb(ex-0301.305,expr)
7  sort_product    (expr)              # cdb(ex-0301.306,expr)
8  rename_dummies  (expr)              # cdb(ex-0301.307,expr)
9  canonicalise    (expr)              # cdb(ex-0301.308,expr)
```

$$\begin{aligned}
R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} &= \partial_e R^a_{bcd} + \Gamma^a_{fe} R^f_{bcd} - \Gamma^f_{be} R^a_{fcd} - \Gamma^f_{ce} R^a_{bfd} - \Gamma^f_{de} R^a_{bcf} + \partial_d R^a_{bec} + \Gamma^a_{fd} R^f_{bec} - \Gamma^f_{bd} R^a_{fec} - \Gamma^f_{ed} R^a_{bfc} - \Gamma^f_{cd} R^a_{bef} \\
&\quad + \partial_c R^a_{bde} + \Gamma^a_{fc} R^f_{bde} - \Gamma^f_{bc} R^a_{fde} - \Gamma^f_{dc} R^a_{bfe} - \Gamma^f_{ec} R^a_{bdf} \tag{ex-0301.302}
\end{aligned}$$

$$\begin{aligned}
&= \partial_e \left( \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^f_{bd} \Gamma^a_{cf} - \Gamma^f_{bc} \Gamma^a_{df} \right) + \Gamma^a_{fe} \left( \partial_c \Gamma^f_{bd} - \partial_d \Gamma^f_{bc} + \Gamma^g_{bd} \Gamma^f_{cg} - \Gamma^g_{bc} \Gamma^f_{dg} \right) \\
&\quad - \Gamma^f_{be} \left( \partial_c \Gamma^a_{fd} - \partial_d \Gamma^a_{fc} + \Gamma^g_{fd} \Gamma^a_{cg} - \Gamma^g_{fc} \Gamma^a_{dg} \right) - \Gamma^f_{ce} \left( \partial_f \Gamma^a_{bd} - \partial_d \Gamma^a_{bf} + \Gamma^g_{bd} \Gamma^a_{fg} - \Gamma^g_{bf} \Gamma^a_{dg} \right) \\
&\quad - \Gamma^f_{de} \left( \partial_c \Gamma^a_{bf} - \partial_f \Gamma^a_{bc} + \Gamma^g_{bf} \Gamma^a_{cg} - \Gamma^g_{bc} \Gamma^a_{fg} \right) + \partial_d \left( \partial_e \Gamma^a_{bc} - \partial_c \Gamma^a_{be} + \Gamma^f_{bc} \Gamma^a_{ef} - \Gamma^f_{be} \Gamma^a_{cf} \right) \\
&\quad + \Gamma^a_{fd} \left( \partial_e \Gamma^f_{bc} - \partial_c \Gamma^f_{be} + \Gamma^g_{bc} \Gamma^f_{eg} - \Gamma^g_{be} \Gamma^f_{cg} \right) - \Gamma^f_{bd} \left( \partial_e \Gamma^a_{fc} - \partial_c \Gamma^a_{fe} + \Gamma^g_{fc} \Gamma^a_{eg} - \Gamma^g_{fe} \Gamma^a_{cg} \right) \\
&\quad - \Gamma^f_{ed} \left( \partial_f \Gamma^a_{bc} - \partial_c \Gamma^a_{bf} + \Gamma^g_{bc} \Gamma^a_{fg} - \Gamma^g_{bf} \Gamma^a_{cg} \right) - \Gamma^f_{cd} \left( \partial_e \Gamma^a_{bf} - \partial_f \Gamma^a_{be} + \Gamma^g_{bf} \Gamma^a_{eg} - \Gamma^g_{be} \Gamma^a_{fg} \right) \\
&\quad + \partial_c \left( \partial_d \Gamma^a_{be} - \partial_e \Gamma^a_{bd} + \Gamma^f_{be} \Gamma^a_{df} - \Gamma^f_{bd} \Gamma^a_{ef} \right) + \Gamma^a_{fc} \left( \partial_d \Gamma^f_{be} - \partial_e \Gamma^f_{bd} + \Gamma^g_{be} \Gamma^f_{dg} - \Gamma^g_{bd} \Gamma^f_{eg} \right) \\
&\quad - \Gamma^f_{bc} \left( \partial_d \Gamma^a_{fe} - \partial_e \Gamma^a_{fd} + \Gamma^g_{fe} \Gamma^a_{dg} - \Gamma^g_{fd} \Gamma^a_{eg} \right) - \Gamma^f_{dc} \left( \partial_f \Gamma^a_{be} - \partial_e \Gamma^a_{bf} + \Gamma^g_{be} \Gamma^a_{fg} - \Gamma^g_{bf} \Gamma^a_{eg} \right) \\
&\quad - \Gamma^f_{ec} \left( \partial_d \Gamma^a_{bf} - \partial_f \Gamma^a_{bd} + \Gamma^g_{bf} \Gamma^a_{dg} - \Gamma^g_{bd} \Gamma^a_{fg} \right) \tag{ex-0301.303}
\end{aligned}$$

$$\begin{aligned}
&= \partial_{ec} \Gamma^a_{bd} - \partial_{ed} \Gamma^a_{bc} + \partial_e \left( \Gamma^f_{bd} \Gamma^a_{cf} \right) - \partial_e \left( \Gamma^f_{bc} \Gamma^a_{df} \right) + \Gamma^a_{fe} \partial_c \Gamma^f_{bd} - \Gamma^a_{fe} \partial_d \Gamma^f_{bc} + \Gamma^a_{fe} \Gamma^g_{bd} \Gamma^f_{cg} - \Gamma^a_{fe} \Gamma^g_{bc} \Gamma^f_{dg} \\
&\quad - \Gamma^f_{be} \partial_c \Gamma^a_{fd} + \Gamma^f_{be} \partial_d \Gamma^a_{fc} - \Gamma^f_{be} \Gamma^g_{fd} \Gamma^a_{cg} + \Gamma^f_{be} \Gamma^g_{fc} \Gamma^a_{dg} - \Gamma^f_{ce} \partial_f \Gamma^a_{bd} + \Gamma^f_{ce} \partial_d \Gamma^a_{bf} - \Gamma^f_{ce} \Gamma^g_{bd} \Gamma^a_{fg} + \Gamma^f_{ce} \Gamma^g_{bf} \Gamma^a_{dg} \\
&\quad - \Gamma^f_{de} \partial_c \Gamma^a_{bf} + \Gamma^f_{de} \partial_f \Gamma^a_{bc} - \Gamma^f_{de} \Gamma^g_{bf} \Gamma^a_{cg} + \Gamma^f_{de} \Gamma^g_{bc} \Gamma^a_{fg} + \partial_{de} \Gamma^a_{bc} - \partial_{dc} \Gamma^a_{be} + \partial_d \left( \Gamma^f_{bc} \Gamma^a_{ef} \right) - \partial_d \left( \Gamma^f_{be} \Gamma^a_{cf} \right) \\
&\quad + \Gamma^a_{fd} \partial_e \Gamma^f_{bc} - \Gamma^a_{fd} \partial_c \Gamma^f_{be} + \Gamma^a_{fd} \Gamma^g_{bc} \Gamma^f_{eg} - \Gamma^a_{fd} \Gamma^g_{be} \Gamma^f_{cg} - \Gamma^f_{bd} \partial_e \Gamma^a_{fc} + \Gamma^f_{bd} \partial_c \Gamma^a_{fe} - \Gamma^f_{bd} \Gamma^g_{fc} \Gamma^a_{eg} + \Gamma^f_{bd} \Gamma^g_{fe} \Gamma^a_{cg} \\
&\quad - \Gamma^f_{ed} \partial_f \Gamma^a_{bc} + \Gamma^f_{ed} \partial_c \Gamma^a_{bf} - \Gamma^f_{ed} \Gamma^g_{bc} \Gamma^a_{fg} + \Gamma^f_{ed} \Gamma^g_{bf} \Gamma^a_{cg} - \Gamma^f_{cd} \partial_e \Gamma^a_{bf} + \Gamma^f_{cd} \partial_f \Gamma^a_{be} - \Gamma^f_{cd} \Gamma^g_{bf} \Gamma^a_{eg} + \Gamma^f_{cd} \Gamma^g_{be} \Gamma^a_{fg} \\
&\quad + \partial_{cd} \Gamma^a_{be} - \partial_{ce} \Gamma^a_{bd} + \partial_c \left( \Gamma^f_{be} \Gamma^a_{df} \right) - \partial_c \left( \Gamma^f_{bd} \Gamma^a_{ef} \right) + \Gamma^a_{fc} \partial_d \Gamma^f_{be} - \Gamma^a_{fc} \partial_e \Gamma^f_{bd} + \Gamma^a_{fc} \Gamma^g_{be} \Gamma^f_{dg} - \Gamma^a_{fc} \Gamma^g_{bd} \Gamma^f_{eg} \\
&\quad - \Gamma^f_{bc} \partial_d \Gamma^a_{fe} + \Gamma^f_{bc} \partial_e \Gamma^a_{fd} - \Gamma^f_{bc} \Gamma^g_{fe} \Gamma^a_{dg} + \Gamma^f_{bc} \Gamma^g_{fd} \Gamma^a_{eg} - \Gamma^f_{dc} \partial_f \Gamma^a_{be} + \Gamma^f_{dc} \partial_e \Gamma^a_{bf} - \Gamma^f_{dc} \Gamma^g_{be} \Gamma^a_{fg} + \Gamma^f_{dc} \Gamma^g_{bf} \Gamma^a_{eg} \\
&\quad - \Gamma^f_{ec} \partial_d \Gamma^a_{bf} + \Gamma^f_{ec} \partial_f \Gamma^a_{bd} - \Gamma^f_{ec} \Gamma^g_{bf} \Gamma^a_{dg} + \Gamma^f_{ec} \Gamma^g_{bd} \Gamma^a_{fg} \tag{ex-0301.304}
\end{aligned}$$



$$\begin{aligned}
R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} = & \partial_{ec}\Gamma^a_{bd} - \partial_{ed}\Gamma^a_{bc} + \partial_e\Gamma^f_{bd}\Gamma^a_{cf} + \Gamma^f_{bd}\partial_e\Gamma^a_{cf} - \partial_e\Gamma^f_{bc}\Gamma^a_{df} - \Gamma^f_{bc}\partial_e\Gamma^a_{df} + \Gamma^a_{fe}\partial_c\Gamma^f_{bd} - \Gamma^a_{fe}\partial_d\Gamma^f_{bc} + \Gamma^a_{fe}\Gamma^g_{bd}\Gamma^f_{cg} \\
& - \Gamma^a_{fe}\Gamma^g_{bc}\Gamma^f_{dg} - \Gamma^f_{be}\partial_c\Gamma^a_{fd} + \Gamma^f_{be}\partial_d\Gamma^a_{fc} - \Gamma^f_{be}\Gamma^g_{fd}\Gamma^a_{cg} + \Gamma^f_{be}\Gamma^g_{fc}\Gamma^a_{dg} - \Gamma^f_{ce}\partial_f\Gamma^a_{bd} + \Gamma^f_{ce}\partial_d\Gamma^a_{bf} - \Gamma^f_{ce}\Gamma^g_{bd}\Gamma^a_{fg} \\
& + \Gamma^f_{ce}\Gamma^g_{bf}\Gamma^a_{dg} - \Gamma^f_{de}\partial_c\Gamma^a_{bf} + \Gamma^f_{de}\partial_f\Gamma^a_{bc} - \Gamma^f_{de}\Gamma^g_{bf}\Gamma^a_{cg} + \Gamma^f_{de}\Gamma^g_{bc}\Gamma^a_{fg} + \partial_{de}\Gamma^a_{bc} - \partial_{dc}\Gamma^a_{be} + \partial_d\Gamma^f_{bc}\Gamma^a_{ef} \\
& + \Gamma^f_{bc}\partial_d\Gamma^a_{ef} - \partial_d\Gamma^f_{be}\Gamma^a_{cf} - \Gamma^f_{be}\partial_d\Gamma^a_{cf} + \Gamma^a_{fd}\partial_e\Gamma^f_{bc} - \Gamma^a_{fd}\partial_c\Gamma^f_{be} + \Gamma^a_{fd}\Gamma^g_{bc}\Gamma^f_{eg} - \Gamma^a_{fd}\Gamma^g_{be}\Gamma^f_{cg} - \Gamma^f_{bd}\partial_e\Gamma^a_{fc} \\
& + \Gamma^f_{bd}\partial_c\Gamma^a_{fe} - \Gamma^f_{bd}\Gamma^g_{fc}\Gamma^a_{eg} + \Gamma^f_{bd}\Gamma^g_{fe}\Gamma^a_{cg} - \Gamma^f_{ed}\partial_f\Gamma^a_{bc} + \Gamma^f_{ed}\partial_c\Gamma^a_{bf} - \Gamma^f_{ed}\Gamma^g_{bc}\Gamma^a_{fg} + \Gamma^f_{ed}\Gamma^g_{bf}\Gamma^a_{cg} - \Gamma^f_{cd}\partial_e\Gamma^a_{bf} \\
& + \Gamma^f_{cd}\partial_f\Gamma^a_{be} - \Gamma^f_{cd}\Gamma^g_{bf}\Gamma^a_{eg} + \Gamma^f_{cd}\Gamma^g_{be}\Gamma^a_{fg} + \partial_{cd}\Gamma^a_{be} - \partial_{ce}\Gamma^a_{bd} + \partial_c\Gamma^f_{be}\Gamma^a_{df} + \Gamma^f_{be}\partial_c\Gamma^a_{df} - \partial_c\Gamma^f_{bd}\Gamma^a_{ef} \\
& - \Gamma^f_{bd}\partial_c\Gamma^a_{ef} + \Gamma^a_{fc}\partial_d\Gamma^f_{be} - \Gamma^a_{fc}\partial_e\Gamma^f_{bd} + \Gamma^a_{fc}\Gamma^g_{be}\Gamma^f_{dg} - \Gamma^a_{fc}\Gamma^g_{bd}\Gamma^f_{eg} - \Gamma^f_{bc}\partial_d\Gamma^a_{fe} + \Gamma^f_{bc}\partial_e\Gamma^a_{fd} - \Gamma^f_{bc}\Gamma^g_{fe}\Gamma^a_{dg} \\
& + \Gamma^f_{bc}\Gamma^g_{fd}\Gamma^a_{eg} - \Gamma^f_{dc}\partial_f\Gamma^a_{be} + \Gamma^f_{dc}\partial_e\Gamma^a_{bf} - \Gamma^f_{dc}\Gamma^g_{be}\Gamma^a_{fg} + \Gamma^f_{dc}\Gamma^g_{bf}\Gamma^a_{eg} - \Gamma^f_{ec}\partial_d\Gamma^a_{bf} + \Gamma^f_{ec}\partial_f\Gamma^a_{bd} - \Gamma^f_{ec}\Gamma^g_{bf}\Gamma^a_{dg} \\
& + \Gamma^f_{ec}\Gamma^g_{bd}\Gamma^a_{fg} \quad (\text{ex-0301.305}) \\
= & \partial_{ec}\Gamma^a_{bd} - \partial_{ed}\Gamma^a_{bc} + \Gamma^a_{cf}\partial_e\Gamma^f_{bd} + \Gamma^f_{bd}\partial_e\Gamma^a_{cf} - \Gamma^a_{df}\partial_e\Gamma^f_{bc} - \Gamma^f_{bc}\partial_e\Gamma^a_{df} + \Gamma^a_{fe}\partial_c\Gamma^f_{bd} - \Gamma^a_{fe}\partial_d\Gamma^f_{bc} + \Gamma^a_{fe}\Gamma^f_{cg}\Gamma^g_{bd} \\
& - \Gamma^a_{fe}\Gamma^f_{dg}\Gamma^g_{bc} - \Gamma^f_{be}\partial_c\Gamma^a_{fd} + \Gamma^f_{be}\partial_d\Gamma^a_{fc} - \Gamma^a_{cg}\Gamma^f_{be}\Gamma^g_{fd} + \Gamma^a_{dg}\Gamma^f_{be}\Gamma^g_{fc} - \Gamma^f_{ce}\partial_f\Gamma^a_{bd} + \Gamma^f_{ce}\partial_d\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ce}\Gamma^g_{bd} \\
& + \Gamma^a_{dg}\Gamma^f_{ce}\Gamma^g_{bf} - \Gamma^f_{de}\partial_c\Gamma^a_{bf} + \Gamma^f_{de}\partial_f\Gamma^a_{bc} - \Gamma^a_{cg}\Gamma^f_{de}\Gamma^g_{bf} + \Gamma^a_{fg}\Gamma^f_{de}\Gamma^g_{bc} + \partial_{de}\Gamma^a_{bc} - \partial_{dc}\Gamma^a_{be} + \Gamma^a_{ef}\partial_d\Gamma^f_{bc} \\
& + \Gamma^f_{bc}\partial_d\Gamma^a_{ef} - \Gamma^a_{cf}\partial_d\Gamma^f_{be} - \Gamma^f_{be}\partial_d\Gamma^a_{cf} + \Gamma^a_{fd}\partial_e\Gamma^f_{bc} - \Gamma^a_{fd}\partial_c\Gamma^f_{be} + \Gamma^a_{fd}\Gamma^f_{eg}\Gamma^g_{bc} - \Gamma^a_{fd}\Gamma^f_{cg}\Gamma^g_{be} - \Gamma^f_{bd}\partial_e\Gamma^a_{fc} \\
& + \Gamma^f_{bd}\partial_c\Gamma^a_{fe} - \Gamma^a_{eg}\Gamma^f_{bd}\Gamma^g_{fc} + \Gamma^a_{cg}\Gamma^f_{bd}\Gamma^g_{fe} - \Gamma^f_{ed}\partial_f\Gamma^a_{bc} + \Gamma^f_{ed}\partial_c\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ed}\Gamma^g_{bc} + \Gamma^a_{cg}\Gamma^f_{ed}\Gamma^g_{bf} - \Gamma^f_{cd}\partial_e\Gamma^a_{bf} \\
& + \Gamma^f_{cd}\partial_f\Gamma^a_{be} - \Gamma^a_{eg}\Gamma^f_{cd}\Gamma^g_{bf} + \Gamma^a_{fg}\Gamma^f_{cd}\Gamma^g_{be} + \partial_{cd}\Gamma^a_{be} - \partial_{ce}\Gamma^a_{bd} + \Gamma^a_{df}\partial_c\Gamma^f_{be} + \Gamma^f_{be}\partial_c\Gamma^a_{df} - \Gamma^a_{ef}\partial_c\Gamma^f_{bd} \\
& - \Gamma^f_{bd}\partial_c\Gamma^a_{ef} + \Gamma^a_{fc}\partial_d\Gamma^f_{be} - \Gamma^a_{fc}\partial_e\Gamma^f_{bd} + \Gamma^a_{fc}\Gamma^f_{dg}\Gamma^g_{be} - \Gamma^a_{fc}\Gamma^f_{eg}\Gamma^g_{bd} - \Gamma^f_{bc}\partial_d\Gamma^a_{fe} + \Gamma^f_{bc}\partial_e\Gamma^a_{fd} - \Gamma^a_{dg}\Gamma^f_{bc}\Gamma^g_{fe} \\
& + \Gamma^a_{eg}\Gamma^f_{bc}\Gamma^g_{fd} - \Gamma^f_{dc}\partial_f\Gamma^a_{be} + \Gamma^f_{dc}\partial_e\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{dc}\Gamma^g_{be} + \Gamma^a_{eg}\Gamma^f_{dc}\Gamma^g_{bf} - \Gamma^f_{ec}\partial_d\Gamma^a_{bf} + \Gamma^f_{ec}\partial_f\Gamma^a_{bd} - \Gamma^a_{dg}\Gamma^f_{ec}\Gamma^g_{bf} \\
& + \Gamma^a_{fg}\Gamma^f_{ec}\Gamma^g_{bd} \quad (\text{ex-0301.306})
\end{aligned}$$

$$\begin{aligned}
R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} = & \partial_{ec}\Gamma^a_{bd} - \partial_{ed}\Gamma^a_{bc} + \Gamma^a_{cf}\partial_e\Gamma^f_{bd} + \Gamma^f_{bd}\partial_e\Gamma^a_{cf} - \Gamma^a_{df}\partial_e\Gamma^f_{bc} - \Gamma^f_{bc}\partial_e\Gamma^a_{df} + \Gamma^a_{fe}\partial_c\Gamma^f_{bd} - \Gamma^a_{fe}\partial_d\Gamma^f_{bc} + \Gamma^a_{fe}\Gamma^f_{cg}\Gamma^g_{bd} \\
& - \Gamma^a_{fe}\Gamma^f_{dg}\Gamma^g_{bc} - \Gamma^f_{be}\partial_c\Gamma^a_{fd} + \Gamma^f_{be}\partial_d\Gamma^a_{fc} - \Gamma^a_{cf}\Gamma^g_{be}\Gamma^f_{gd} + \Gamma^a_{df}\Gamma^g_{be}\Gamma^f_{gc} - \Gamma^f_{ce}\partial_f\Gamma^a_{bd} + \Gamma^f_{ce}\partial_d\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ce}\Gamma^g_{bd} \\
& + \Gamma^a_{df}\Gamma^g_{ce}\Gamma^f_{bg} - \Gamma^f_{de}\partial_c\Gamma^a_{bf} + \Gamma^f_{de}\partial_f\Gamma^a_{bc} - \Gamma^a_{cf}\Gamma^g_{de}\Gamma^f_{bg} + \Gamma^a_{fg}\Gamma^f_{de}\Gamma^g_{bc} + \partial_{de}\Gamma^a_{bc} - \partial_{dc}\Gamma^a_{be} + \Gamma^a_{ef}\partial_d\Gamma^f_{bc} + \Gamma^f_{bc}\partial_d\Gamma^a_{ef} \\
& - \Gamma^a_{cf}\partial_d\Gamma^f_{be} - \Gamma^f_{be}\partial_d\Gamma^a_{cf} + \Gamma^a_{fd}\partial_e\Gamma^f_{bc} - \Gamma^a_{fd}\partial_c\Gamma^f_{be} + \Gamma^a_{fd}\Gamma^f_{eg}\Gamma^g_{bc} - \Gamma^a_{fd}\Gamma^f_{cg}\Gamma^g_{be} - \Gamma^f_{bd}\partial_e\Gamma^a_{fc} + \Gamma^f_{bd}\partial_c\Gamma^a_{fe} \\
& - \Gamma^a_{ef}\Gamma^g_{bd}\Gamma^f_{gc} + \Gamma^a_{cf}\Gamma^g_{bd}\Gamma^f_{ge} - \Gamma^f_{ed}\partial_f\Gamma^a_{bc} + \Gamma^f_{ed}\partial_c\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ed}\Gamma^g_{bc} + \Gamma^a_{cf}\Gamma^g_{ed}\Gamma^f_{bg} - \Gamma^f_{cd}\partial_e\Gamma^a_{bf} + \Gamma^f_{cd}\partial_f\Gamma^a_{be} \\
& - \Gamma^a_{ef}\Gamma^g_{cd}\Gamma^f_{bg} + \Gamma^a_{fg}\Gamma^f_{cd}\Gamma^g_{be} + \partial_{cd}\Gamma^a_{be} - \partial_{ce}\Gamma^a_{bd} + \Gamma^a_{df}\partial_c\Gamma^f_{be} + \Gamma^f_{be}\partial_c\Gamma^a_{df} - \Gamma^a_{ef}\partial_c\Gamma^f_{bd} - \Gamma^f_{bd}\partial_c\Gamma^a_{ef} + \Gamma^a_{fc}\partial_d\Gamma^f_{be} \\
& - \Gamma^a_{fc}\partial_e\Gamma^f_{bd} + \Gamma^a_{fc}\Gamma^f_{dg}\Gamma^g_{be} - \Gamma^a_{fc}\Gamma^f_{eg}\Gamma^g_{bd} - \Gamma^f_{bc}\partial_d\Gamma^a_{fe} + \Gamma^f_{bc}\partial_e\Gamma^a_{fd} - \Gamma^a_{df}\Gamma^g_{bc}\Gamma^f_{ge} + \Gamma^a_{ef}\Gamma^g_{bc}\Gamma^f_{gd} - \Gamma^f_{dc}\partial_f\Gamma^a_{be} \\
& + \Gamma^f_{dc}\partial_e\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{dc}\Gamma^g_{be} + \Gamma^a_{ef}\Gamma^g_{dc}\Gamma^f_{bg} - \Gamma^f_{ec}\partial_d\Gamma^a_{bf} + \Gamma^f_{ec}\partial_f\Gamma^a_{bd} - \Gamma^a_{df}\Gamma^g_{ec}\Gamma^f_{bg} + \Gamma^a_{fg}\Gamma^f_{ec}\Gamma^g_{bd} \quad (\text{ex-0301.307}) \\
= 0 & \quad (\text{ex-0301.308})
\end{aligned}$$

## Exercise 3.2 Riemann tensor from commutation of $\nabla$

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});
7
8 # rules for the first two covariant derivs of V^a
9
10 deriv1 := \nabla_{a}{V^{b}} -> \partial_{a}{V^{b}}
11          + \Gamma^{b}_{d a} V^{d}.          # cdb (ex-0302.101,deriv1)
12
13 deriv2 := \nabla_{a}{\nabla_{b}{V^{c}}} -> \partial_{a}{\nabla_{b}{V^{c}}}
14          + \Gamma^{c}_{d a} \nabla_{b}{V^{d}}
15          - \Gamma^{d}_{b a} \nabla_{d}{V^{c}}.
16          # cdb (ex-0302.102,deriv2)
17
18 Vabc := \nabla_{c}{\nabla_{b}{V^{a}}}
19         - \nabla_{b}{\nabla_{c}{V^{a}}}.          # cdb (ex-0302.103, Vabc)
20
21 substitute (Vabc,deriv2)          # cdb (ex-0302.104, Vabc)
22 substitute (Vabc,deriv1)          # cdb (ex-0302.105, Vabc)
23
24 distribute      (Vabc)            # cdb (ex-0302.106, Vabc)
25 product_rule    (Vabc)            # cdb (ex-0302.107, Vabc)
26
27 sort_product    (Vabc)            # cdb (ex-0302.108, Vabc)
28 rename_dummies  (Vabc)            # cdb (ex-0302.109, Vabc)
29 canonicalise    (Vabc)            # cdb (ex-0302.110, Vabc)
30 factor_out      (Vabc,$V^{a?}$)   # cdb (ex-0302.111, Vabc)

```

$$\nabla_c (\nabla_b V^a) - \nabla_b (\nabla_c V^a) = \partial_c (\nabla_b V^a) + \Gamma^a_{dc} \nabla_b V^d - \Gamma^d_{bc} \nabla_d V^a - \partial_b (\nabla_c V^a) - \Gamma^a_{db} \nabla_c V^d + \Gamma^d_{cb} \nabla_d V^a \quad (\text{ex-0302.104})$$

$$\begin{aligned} &= \partial_c (\partial_b V^a + \Gamma^a_{db} V^d) + \Gamma^a_{dc} (\partial_b V^d + \Gamma^d_{eb} V^e) - \Gamma^d_{bc} (\partial_d V^a + \Gamma^a_{ed} V^e) - \partial_b (\partial_c V^a + \Gamma^a_{dc} V^d) - \Gamma^a_{db} (\partial_c V^d + \Gamma^d_{ec} V^e) \\ &\quad + \Gamma^d_{cb} (\partial_d V^a + \Gamma^a_{ed} V^e) \end{aligned} \quad (\text{ex-0302.105})$$

$$\begin{aligned} &= \partial_{cb} V^a + \partial_c (\Gamma^a_{db} V^d) + \Gamma^a_{dc} \partial_b V^d + \Gamma^a_{dc} \Gamma^d_{eb} V^e - \Gamma^d_{bc} \partial_d V^a - \Gamma^d_{bc} \Gamma^a_{ed} V^e - \partial_{bc} V^a - \partial_b (\Gamma^a_{dc} V^d) - \Gamma^a_{db} \partial_c V^d \\ &\quad - \Gamma^a_{db} \Gamma^d_{ec} V^e + \Gamma^d_{cb} \partial_d V^a + \Gamma^d_{cb} \Gamma^a_{ed} V^e \end{aligned} \quad (\text{ex-0302.106})$$

$$\begin{aligned} &= \partial_{cb} V^a + \partial_c \Gamma^a_{db} V^d + \Gamma^a_{dc} \Gamma^d_{eb} V^e - \Gamma^d_{bc} \partial_d V^a - \Gamma^d_{bc} \Gamma^a_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a_{dc} V^d - \Gamma^a_{db} \Gamma^d_{ec} V^e + \Gamma^d_{cb} \partial_d V^a \\ &\quad + \Gamma^d_{cb} \Gamma^a_{ed} V^e \end{aligned} \quad (\text{ex-0302.107})$$

$$\begin{aligned} &= \partial_{cb} V^a + V^d \partial_c \Gamma^a_{db} + V^e \Gamma^a_{dc} \Gamma^d_{eb} - \Gamma^d_{bc} \partial_d V^a - V^e \Gamma^a_{ed} \Gamma^d_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a_{dc} - V^e \Gamma^a_{db} \Gamma^d_{ec} + \Gamma^d_{cb} \partial_d V^a \\ &\quad + V^e \Gamma^a_{ed} \Gamma^d_{cb} \end{aligned} \quad (\text{ex-0302.108})$$

$$\begin{aligned} &= \partial_{cb} V^a + V^d \partial_c \Gamma^a_{db} + V^d \Gamma^a_{ec} \Gamma^e_{db} - \Gamma^d_{bc} \partial_d V^a - V^d \Gamma^a_{de} \Gamma^e_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a_{dc} - V^d \Gamma^a_{eb} \Gamma^e_{dc} + \Gamma^d_{cb} \partial_d V^a \\ &\quad + V^d \Gamma^a_{de} \Gamma^e_{cb} \end{aligned} \quad (\text{ex-0302.109})$$

$$= V^d \partial_c \Gamma^a_{bd} + V^d \Gamma^a_{ce} \Gamma^e_{bd} - V^d \partial_b \Gamma^a_{cd} - V^d \Gamma^a_{be} \Gamma^e_{cd} \quad (\text{ex-0302.110})$$

$$= V^d (\partial_c \Gamma^a_{bd} + \Gamma^a_{ce} \Gamma^e_{bd} - \partial_b \Gamma^a_{cd} - \Gamma^a_{be} \Gamma^e_{cd}) \quad (\text{ex-0302.111})$$

$$= -R^a_{dbc} V^d$$

This result agrees with Misner, Thorne and Wheeler. pg. 266.

## Exercise 3.3 Computing $R_{abcd}$

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative.
4
5  \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
6  \Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).
7
8  dgab := \partial_{c}{g_{a b}} ->  \Gamma^{d}_{a c} g_{d b}
9                                     + \Gamma^{d}_{b c} g_{a d}.      # cdb(dgab.000,dgab)
10
11 RabcdU := R^{a}_{b c d} ->  \partial_{c}{\Gamma^{a}_{b d}}
12                             - \partial_{d}{\Gamma^{a}_{b c}}
13                             + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
14                             - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.      # cdb(Rabcd.000,RabcdU)
15
16 GammaD := {g_{a e} \Gamma^{e}_{b c} -> \Gamma_{a b c},
17            g_{e a} \Gamma^{e}_{b c} -> \Gamma_{a b c}}.      # cdb(Gamma.010,GammaD)
18
19 RabcdD := R_{a b c d} -> g_{a e} R^{e}_{b c d}.      # cdb(Rabcd.010,RabcdD)
20
21 gabDGamma := g_{a e} \partial_{c}{\Gamma^{e}_{b d}} ->
22              \partial_{c}{g_{a e} \Gamma^{e}_{b d}}
23              - \Gamma^{e}_{b d} \partial_{c}{g_{a e}}.      # cdb(gabDGamma.000,gabDGamma)
24
25 # this pair of rules needed to sort \Gamma_{a b c} to the very left
26 # this helps canonicalise spot the terms that cancel
27 bah := \Gamma_{a b c} -> A_{a b c}.
28 foo := A_{a b c} -> \Gamma_{a b c}.
29
30 expr := R_{a b c d}.      # cdb(ex-0303.101,expr)
31
32 substitute      (expr, RabcdD)      # cdb(ex-0303.102,expr)
33 substitute      (expr, RabcdU)      # cdb(ex-0303.103,expr)
34 distribute      (expr)              # cdb(ex-0303.104,expr)
35 substitute      (expr, gabDGamma)    # cdb(ex-0303.105,expr)
36 substitute      (expr, dgab)         # cdb(ex-0303.106,expr)

```

```

37 substitute      (expr, GammaD)          # cdb(ex-0303.107,expr)
38 distribute      (expr)                  # cdb(ex-0303.109,expr)
39 substitute      (expr, bah)             # cdb(ex-0303.110,expr)
40 sort_product    (expr)                  # cdb(ex-0303.111,expr)
41 rename_dummies  (expr)                  # cdb(ex-0303.112,expr)
42 substitute      (expr, foo)             # cdb(ex-0303.113,expr)
43 canonicalise    (expr)                  # cdb(ex-0303.114,expr)

```

$$R_{abcd} = g_{ae} R^e_{bcd} \quad (\text{ex-0303.102})$$

$$= g_{ae} (\partial_c \Gamma^e_{bd} - \partial_d \Gamma^e_{bc} + \Gamma^f_{bd} \Gamma^e_{cf} - \Gamma^f_{bc} \Gamma^e_{df}) \quad (\text{ex-0303.103})$$

$$= g_{ae} \partial_c \Gamma^e_{bd} - g_{ae} \partial_d \Gamma^e_{bc} + g_{ae} \Gamma^f_{bd} \Gamma^e_{cf} - g_{ae} \Gamma^f_{bc} \Gamma^e_{df} \quad (\text{ex-0303.104})$$

$$= \partial_c (g_{ae} \Gamma^e_{bd}) - \Gamma^e_{bd} \partial_c g_{ae} - \partial_d (g_{ae} \Gamma^e_{bc}) + \Gamma^e_{bc} \partial_d g_{ae} + g_{ae} \Gamma^f_{bd} \Gamma^e_{cf} - g_{ae} \Gamma^f_{bc} \Gamma^e_{df} \quad (\text{ex-0303.105})$$

$$= \partial_c (g_{ae} \Gamma^e_{bd}) - \Gamma^e_{bd} (\Gamma^f_{ac} g_{fe} + \Gamma^f_{ec} g_{af}) - \partial_d (g_{ae} \Gamma^e_{bc}) + \Gamma^e_{bc} (\Gamma^f_{ad} g_{fe} + \Gamma^f_{ed} g_{af}) + g_{ae} \Gamma^f_{bd} \Gamma^e_{cf} - g_{ae} \Gamma^f_{bc} \Gamma^e_{df} \quad (\text{ex-0303.106})$$

$$= \partial_c \Gamma_{abd} - \Gamma^e_{bd} (\Gamma_{eac} + \Gamma_{aec}) - \partial_d \Gamma_{abc} + \Gamma^e_{bc} (\Gamma_{ead} + \Gamma_{aed}) + \Gamma_{acf} \Gamma^f_{bd} - \Gamma_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.107})$$

$$= \partial_c \Gamma_{abd} - \Gamma^e_{bd} \Gamma_{eac} - \Gamma^e_{bd} \Gamma_{aec} - \partial_d \Gamma_{abc} + \Gamma^e_{bc} \Gamma_{ead} + \Gamma^e_{bc} \Gamma_{aed} + \Gamma_{acf} \Gamma^f_{bd} - \Gamma_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.109})$$

$$= \partial_c A_{abd} - \Gamma^e_{bd} A_{eac} - \Gamma^e_{bd} A_{aec} - \partial_d A_{abc} + \Gamma^e_{bc} A_{ead} + \Gamma^e_{bc} A_{aed} + A_{acf} \Gamma^f_{bd} - A_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.110})$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e_{bd} - A_{aec} \Gamma^e_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e_{bc} + A_{aed} \Gamma^e_{bc} + A_{acf} \Gamma^f_{bd} - A_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.111})$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e_{bd} - A_{aec} \Gamma^e_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e_{bc} + A_{aed} \Gamma^e_{bc} + A_{ace} \Gamma^e_{bd} - A_{ade} \Gamma^e_{bc} \quad (\text{ex-0303.112})$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e_{bd} - \Gamma_{aec} \Gamma^e_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e_{bc} + \Gamma_{aed} \Gamma^e_{bc} + \Gamma_{ace} \Gamma^e_{bd} - \Gamma_{ade} \Gamma^e_{bc} \quad (\text{ex-0303.113})$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e_{bc} \quad (\text{ex-0303.114})$$

## Exercise 3.4 More symmetries of Riemann

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative.
4
5 g_{a b}::Symmetric.
6 g^{a b}::Symmetric.
7
8 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
9 \Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).
10
11 GammaU := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
12                                     + \partial_{c}{g_{b d}}
13                                     - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)
14
15 GammaD := \Gamma_{a b c} -> 1/2 ( \partial_{b}{g_{a c}}
16                                     + \partial_{c}{g_{b a}}
17                                     - \partial_{a}{g_{b c}}). # cdb(Gamma.010,GammaD)
18
19 Rabcd := R_{a b c d} -> \partial_{c}{\Gamma_{a b d}}
20                       - \partial_{d}{\Gamma_{a b c}}
21                       + \Gamma_{e a d} \Gamma^{e}_{b c}
22                       - \Gamma_{e a c} \Gamma^{e}_{b d}. # cdb(Rabcd.000,Rabcd)
```

## Exercise 3.4 Antisymmetry on first pair of indices

```
1  expr := R_{a b c d} + R_{b a c d}.    # cdb(ex-0304.101,expr)
2
3  substitute      (expr, Rabcd)         # cdb(ex-0304.102,expr)
4  substitute      (expr, GammaU)        # cdb(ex-0304.103,expr)
5  substitute      (expr, GammaD)        # cdb(ex-0304.104,expr)
6  distribute      (expr)                # cdb(ex-0304.105,expr)
7  product_rule    (expr)                # cdb(ex-0304.106,expr)
8  sort_product    (expr)                # cdb(ex-0304.107,expr)
9  rename_dummies  (expr)                # cdb(ex-0304.108,expr)
10 canonicalise    (expr)                # cdb(ex-0304.109,expr)
```



$$R_{abcd} + R_{bacd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e_{bc} - \Gamma_{eac} \Gamma^e_{bd} + \partial_c \Gamma_{bad} - \partial_d \Gamma_{bac} + \Gamma_{ebd} \Gamma^e_{ac} - \Gamma_{ebc} \Gamma^e_{ad} \quad (\text{ex-0304.102})$$

$$\begin{aligned} &= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2} \Gamma_{ead} g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) - \frac{1}{2} \Gamma_{eac} g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) + \partial_c \Gamma_{bad} - \partial_d \Gamma_{bac} \\ &\quad + \frac{1}{2} \Gamma_{ebd} g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) - \frac{1}{2} \Gamma_{ebc} g^{ef} (\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}) \end{aligned} \quad (\text{ex-0304.103})$$

$$\begin{aligned} &= \partial_c \left( \frac{1}{2} \partial_b g_{ad} + \frac{1}{2} \partial_d g_{ba} - \frac{1}{2} \partial_a g_{bd} \right) - \partial_d \left( \frac{1}{2} \partial_b g_{ac} + \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_a g_{bc} \right) + \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ed} + \frac{1}{2} \partial_d g_{ae} - \frac{1}{2} \partial_e g_{ad} \right) g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) \\ &\quad - \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ec} + \frac{1}{2} \partial_c g_{ae} - \frac{1}{2} \partial_e g_{ac} \right) g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) + \partial_c \left( \frac{1}{2} \partial_a g_{bd} + \frac{1}{2} \partial_d g_{ab} - \frac{1}{2} \partial_b g_{ad} \right) \\ &\quad - \partial_d \left( \frac{1}{2} \partial_a g_{bc} + \frac{1}{2} \partial_c g_{ab} - \frac{1}{2} \partial_b g_{ac} \right) + \frac{1}{2} \left( \frac{1}{2} \partial_b g_{ed} + \frac{1}{2} \partial_d g_{be} - \frac{1}{2} \partial_e g_{bd} \right) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) \\ &\quad - \frac{1}{2} \left( \frac{1}{2} \partial_b g_{ec} + \frac{1}{2} \partial_c g_{be} - \frac{1}{2} \partial_e g_{bc} \right) g^{ef} (\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}) \end{aligned} \quad (\text{ex-0304.104})$$

$$\begin{aligned} &= \frac{1}{2} \partial_{cd} g_{ba} - \frac{1}{2} \partial_{dc} g_{ba} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_f g_{bc} + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_b g_{fc} + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_f g_{bc} \\ &\quad - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_b g_{fc} - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_c g_{bf} + \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_f g_{bc} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_b g_{fd} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_d g_{bf} + \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_f g_{bd} - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_b g_{fd} \\ &\quad - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_d g_{bf} + \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_f g_{bd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_b g_{fd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_d g_{bf} - \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_f g_{bd} + \frac{1}{2} \partial_{cd} g_{ab} - \frac{1}{2} \partial_{dc} g_{ab} + \frac{1}{4} \partial_b g_{ed} g^{ef} \partial_a g_{fc} \\ &\quad + \frac{1}{4} \partial_b g_{ed} g^{ef} \partial_c g_{af} - \frac{1}{4} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + \frac{1}{4} \partial_d g_{be} g^{ef} \partial_a g_{fc} + \frac{1}{4} \partial_d g_{be} g^{ef} \partial_c g_{af} - \frac{1}{4} \partial_d g_{be} g^{ef} \partial_f g_{ac} - \frac{1}{4} \partial_e g_{bd} g^{ef} \partial_a g_{fc} - \frac{1}{4} \partial_e g_{bd} g^{ef} \partial_c g_{af} \\ &\quad + \frac{1}{4} \partial_e g_{bd} g^{ef} \partial_f g_{ac} - \frac{1}{4} \partial_b g_{ec} g^{ef} \partial_a g_{fd} - \frac{1}{4} \partial_b g_{ec} g^{ef} \partial_d g_{af} + \frac{1}{4} \partial_b g_{ec} g^{ef} \partial_f g_{ad} - \frac{1}{4} \partial_c g_{be} g^{ef} \partial_a g_{fd} - \frac{1}{4} \partial_c g_{be} g^{ef} \partial_d g_{af} + \frac{1}{4} \partial_c g_{be} g^{ef} \partial_f g_{ad} \\ &\quad + \frac{1}{4} \partial_e g_{bc} g^{ef} \partial_a g_{fd} + \frac{1}{4} \partial_e g_{bc} g^{ef} \partial_d g_{af} - \frac{1}{4} \partial_e g_{bc} g^{ef} \partial_f g_{ad} \end{aligned} \quad (\text{ex-0304.105})$$

$$\begin{aligned}
R_{abcd} + R_{bacd} = & \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc} \\
& - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd} \\
& - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_a g_{fc} \\
& + \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_f g_{ac} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_a g_{fc} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_d g_{be}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_a g_{fc} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_c g_{af} \\
& + \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_f g_{ad} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_c g_{be}g^{ef}\partial_f g_{ad} \\
& + \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_a g_{fd} + \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_d g_{af} - \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_f g_{ad} \tag{ex-0304.106} \\
= & \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_f g_{bc}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_d g_{ae}g^{ef} + \frac{1}{4}\partial_c g_{bf}\partial_d g_{ae}g^{ef} - \frac{1}{4}\partial_d g_{ae}\partial_f g_{bc}g^{ef} \\
& - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_a g_{ec}\partial_f g_{bd}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_c g_{ae}g^{ef} \\
& - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{ae}\partial_f g_{bd}g^{ef} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_a g_{fc}\partial_b g_{ed}g^{ef} \\
& + \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{ef} - \frac{1}{4}\partial_b g_{ed}\partial_f g_{ac}g^{ef} + \frac{1}{4}\partial_a g_{fc}\partial_d g_{be}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_d g_{be}g^{ef} - \frac{1}{4}\partial_d g_{be}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{ef} - \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{ef} \\
& + \frac{1}{4}\partial_e g_{bd}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_b g_{ec}g^{ef} - \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_b g_{ec}\partial_f g_{ad}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_c g_{be}g^{ef} - \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_c g_{be}\partial_f g_{ad}g^{ef} \\
& + \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{ef} + \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{ef} - \frac{1}{4}\partial_e g_{bc}\partial_f g_{ad}g^{ef} \tag{ex-0304.107}
\end{aligned}$$

$$\begin{aligned}
R_{abcd} + R_{bacd} = & \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{fe} + \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{fe} + \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{fe} - \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{fe} \\
& - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{fe} - \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{fe} \\
& - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{fe} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{fe} \\
& + \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{fe} + \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{fe} + \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{fe} - \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{fe} - \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{ef} - \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{ef} \\
& + \frac{1}{4}\partial_e g_{bd}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{fe} - \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{fe} - \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{fe} - \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{fe} \\
& + \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{ef} + \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{ef} - \frac{1}{4}\partial_e g_{bc}\partial_f g_{ad}g^{ef} \tag{ex-0304.108}
\end{aligned}$$

$$= 0 \tag{ex-0304.109}$$

## Exercise 3.4 Symmetric on swapping first and second pair of indices

```
1  expr := R_{a b c d} - R_{c d a b}.    # cdb(ex-0304.201,expr)
2
3  substitute      (expr, Rabcd)         # cdb(ex-0304.202,expr)
4  substitute      (expr, GammaU)        # cdb(ex-0304.203,expr)
5  substitute      (expr, GammaD)        # cdb(ex-0304.204,expr)
6  distribute      (expr)                # cdb(ex-0304.205,expr)
7  product_rule    (expr)                # cdb(ex-0304.206,expr)
8  sort_product    (expr)                # cdb(ex-0304.207,expr)
9  rename_dummies  (expr)                # cdb(ex-0304.208,expr)
10 canonicalise    (expr)                # cdb(ex-0304.209,expr)
```

$$R_{abcd} - R_{cdab} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma_{bc}^e - \Gamma_{eac} \Gamma_{bd}^e - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} - \Gamma_{ecb} \Gamma_{da}^e + \Gamma_{eca} \Gamma_{db}^e \quad (\text{ex-0304.202})$$

$$\begin{aligned} &= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2} \Gamma_{ead} g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) - \frac{1}{2} \Gamma_{eac} g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} \\ &\quad - \frac{1}{2} \Gamma_{ecb} g^{ef} (\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}) + \frac{1}{2} \Gamma_{eca} g^{ef} (\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}) \end{aligned} \quad (\text{ex-0304.203})$$

$$\begin{aligned} &= \partial_c \left( \frac{1}{2} \partial_b g_{ad} + \frac{1}{2} \partial_d g_{ba} - \frac{1}{2} \partial_a g_{bd} \right) - \partial_d \left( \frac{1}{2} \partial_b g_{ac} + \frac{1}{2} \partial_c g_{ba} - \frac{1}{2} \partial_a g_{bc} \right) + \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ed} + \frac{1}{2} \partial_d g_{ae} - \frac{1}{2} \partial_e g_{ad} \right) g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) \\ &\quad - \frac{1}{2} \left( \frac{1}{2} \partial_a g_{ec} + \frac{1}{2} \partial_c g_{ae} - \frac{1}{2} \partial_e g_{ac} \right) g^{ef} (\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}) - \partial_a \left( \frac{1}{2} \partial_d g_{cb} + \frac{1}{2} \partial_b g_{dc} - \frac{1}{2} \partial_c g_{db} \right) \\ &\quad + \partial_b \left( \frac{1}{2} \partial_d g_{ca} + \frac{1}{2} \partial_a g_{dc} - \frac{1}{2} \partial_c g_{da} \right) - \frac{1}{2} \left( \frac{1}{2} \partial_c g_{eb} + \frac{1}{2} \partial_b g_{ce} - \frac{1}{2} \partial_e g_{cb} \right) g^{ef} (\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}) \\ &\quad + \frac{1}{2} \left( \frac{1}{2} \partial_c g_{ea} + \frac{1}{2} \partial_a g_{ce} - \frac{1}{2} \partial_e g_{ca} \right) g^{ef} (\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}) \end{aligned} \quad (\text{ex-0304.204})$$

$$\begin{aligned} &= \frac{1}{2} \partial_{cb} g_{ad} + \frac{1}{2} \partial_{cd} g_{ba} - \frac{1}{2} \partial_{ca} g_{bd} - \frac{1}{2} \partial_{db} g_{ac} - \frac{1}{2} \partial_{dc} g_{ba} + \frac{1}{2} \partial_{da} g_{bc} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_a g_{ed} g^{ef} \partial_f g_{bc} + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_b g_{fc} \\ &\quad + \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_c g_{bf} - \frac{1}{4} \partial_d g_{ae} g^{ef} \partial_f g_{bc} - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_b g_{fc} - \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_c g_{bf} + \frac{1}{4} \partial_e g_{ad} g^{ef} \partial_f g_{bc} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_b g_{fd} - \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_d g_{bf} \\ &\quad + \frac{1}{4} \partial_a g_{ec} g^{ef} \partial_f g_{bd} - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_b g_{fd} - \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_d g_{bf} + \frac{1}{4} \partial_c g_{ae} g^{ef} \partial_f g_{bd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_b g_{fd} + \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_d g_{bf} - \frac{1}{4} \partial_e g_{ac} g^{ef} \partial_f g_{bd} \\ &\quad - \frac{1}{2} \partial_{ad} g_{cb} - \frac{1}{2} \partial_{ab} g_{dc} + \frac{1}{2} \partial_{ac} g_{db} + \frac{1}{2} \partial_{bd} g_{ca} + \frac{1}{2} \partial_{ba} g_{dc} - \frac{1}{2} \partial_{bc} g_{da} - \frac{1}{4} \partial_c g_{eb} g^{ef} \partial_d g_{fa} - \frac{1}{4} \partial_c g_{eb} g^{ef} \partial_a g_{df} + \frac{1}{4} \partial_c g_{eb} g^{ef} \partial_f g_{da} \\ &\quad - \frac{1}{4} \partial_b g_{ce} g^{ef} \partial_d g_{fa} - \frac{1}{4} \partial_b g_{ce} g^{ef} \partial_a g_{df} + \frac{1}{4} \partial_b g_{ce} g^{ef} \partial_f g_{da} + \frac{1}{4} \partial_e g_{cb} g^{ef} \partial_d g_{fa} + \frac{1}{4} \partial_e g_{cb} g^{ef} \partial_a g_{df} - \frac{1}{4} \partial_e g_{cb} g^{ef} \partial_f g_{da} + \frac{1}{4} \partial_c g_{ea} g^{ef} \partial_d g_{fb} \\ &\quad + \frac{1}{4} \partial_c g_{ea} g^{ef} \partial_b g_{df} - \frac{1}{4} \partial_c g_{ea} g^{ef} \partial_f g_{db} + \frac{1}{4} \partial_a g_{ce} g^{ef} \partial_d g_{fb} + \frac{1}{4} \partial_a g_{ce} g^{ef} \partial_b g_{df} - \frac{1}{4} \partial_a g_{ce} g^{ef} \partial_f g_{db} - \frac{1}{4} \partial_e g_{ca} g^{ef} \partial_d g_{fb} - \frac{1}{4} \partial_e g_{ca} g^{ef} \partial_b g_{df} \\ &\quad + \frac{1}{4} \partial_e g_{ca} g^{ef} \partial_f g_{db} \end{aligned} \quad (\text{ex-0304.205})$$

$$\begin{aligned}
R_{abcd} - R_{cdab} &= \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc} \\
&+ \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf} \\
&+ \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd} \\
&- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_f g_{da} \\
&- \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_d g_{fa} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_a g_{df} - \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_d g_{fb} \\
&+ \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_f g_{db} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_d g_{fb} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_f g_{db} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_d g_{fb} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_b g_{df} \\
&+ \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_f g_{db} \tag{ex-0304.206} \\
&= \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_f g_{bc}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_d g_{ae}g^{ef} \\
&+ \frac{1}{4}\partial_c g_{bf}\partial_d g_{ae}g^{ef} - \frac{1}{4}\partial_d g_{ae}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} \\
&+ \frac{1}{4}\partial_a g_{ec}\partial_f g_{bd}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_c g_{ae}g^{ef} - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{ae}\partial_f g_{bd}g^{ef} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} \\
&- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{df}\partial_c g_{eb}g^{ef} + \frac{1}{4}\partial_c g_{eb}\partial_f g_{da}g^{ef} \\
&- \frac{1}{4}\partial_b g_{ce}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{df}\partial_b g_{ce}g^{ef} + \frac{1}{4}\partial_b g_{ce}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_d g_{fa}\partial_e g_{cb}g^{ef} + \frac{1}{4}\partial_a g_{df}\partial_e g_{cb}g^{ef} - \frac{1}{4}\partial_e g_{cb}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_c g_{ea}\partial_d g_{fb}g^{ef} \\
&+ \frac{1}{4}\partial_b g_{df}\partial_c g_{ea}g^{ef} - \frac{1}{4}\partial_c g_{ea}\partial_f g_{db}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_d g_{fb}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_b g_{df}g^{ef} - \frac{1}{4}\partial_a g_{ce}\partial_f g_{db}g^{ef} - \frac{1}{4}\partial_d g_{fb}\partial_e g_{ca}g^{ef} - \frac{1}{4}\partial_b g_{df}\partial_e g_{ca}g^{ef} \\
&+ \frac{1}{4}\partial_e g_{ca}\partial_f g_{db}g^{ef} \tag{ex-0304.207}
\end{aligned}$$

$$\begin{aligned}
R_{abcd} - R_{cdab} = & \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{fe} + \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{fe} \\
& + \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{fe} - \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{fe} - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} \\
& + \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{fe} - \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{fe} - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{fe} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} \\
& - \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{de}\partial_c g_{fb}g^{fe} + \frac{1}{4}\partial_c g_{fb}\partial_e g_{da}g^{fe} \\
& - \frac{1}{4}\partial_b g_{ce}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{de}\partial_b g_{cf}g^{fe} + \frac{1}{4}\partial_b g_{cf}\partial_e g_{da}g^{fe} + \frac{1}{4}\partial_d g_{fa}\partial_e g_{cb}g^{ef} + \frac{1}{4}\partial_a g_{df}\partial_e g_{cb}g^{ef} - \frac{1}{4}\partial_e g_{cb}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_c g_{ea}\partial_d g_{fb}g^{ef} \\
& + \frac{1}{4}\partial_b g_{de}\partial_c g_{fa}g^{fe} - \frac{1}{4}\partial_c g_{fa}\partial_e g_{db}g^{fe} + \frac{1}{4}\partial_a g_{ce}\partial_d g_{fb}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_b g_{df}g^{ef} - \frac{1}{4}\partial_a g_{cf}\partial_e g_{db}g^{fe} - \frac{1}{4}\partial_d g_{fb}\partial_e g_{ca}g^{ef} - \frac{1}{4}\partial_b g_{df}\partial_e g_{ca}g^{ef} \\
& + \frac{1}{4}\partial_e g_{ca}\partial_f g_{db}g^{ef} \tag{ex-0304.208} \\
= 0 \tag{ex-0304.209}
\end{aligned}$$

## Exercise 3.5 Commutation of covariant derivatives

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4
5 expr := \nabla_{d}{\nabla_{c}{A_{a} B_{b}}}
6         - \nabla_{c}{\nabla_{d}{A_{a} B_{b}}}. # cdb(ex-0305.100,expr)
7
8 product_rule (expr) # cdb(ex-0305.101,expr)
9 distribute (expr) # cdb(ex-0305.102,expr)
10 product_rule (expr) # cdb(ex-0305.103,expr)
11 factor_out (expr,$A_{a?},B_{b?}$) # cdb(ex-0305.104,expr)

```

$$\nabla_d (\nabla_c (A_a B_b)) - \nabla_c (\nabla_d (A_a B_b)) = \nabla_d (\nabla_c A_a B_b + A_a \nabla_c B_b) - \nabla_c (\nabla_d A_a B_b + A_a \nabla_d B_b) \quad (\text{ex-0305.101})$$

$$= \nabla_d (\nabla_c A_a B_b) + \nabla_d (A_a \nabla_c B_b) - \nabla_c (\nabla_d A_a B_b) - \nabla_c (A_a \nabla_d B_b) \quad (\text{ex-0305.102})$$

$$= \nabla_d (\nabla_c A_a) B_b + A_a \nabla_d (\nabla_c B_b) - \nabla_c (\nabla_d A_a) B_b - A_a \nabla_c (\nabla_d B_b) \quad (\text{ex-0305.103})$$

$$= B_b (\nabla_d (\nabla_c A_a) - \nabla_c (\nabla_d A_a)) + A_a (\nabla_d (\nabla_c B_b) - \nabla_c (\nabla_d B_b)) \quad (\text{ex-0305.104})$$



## Exercise 3.6 Commutation of $\nabla$ on the Riemann tensor – simple computation

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3  DD{#}::Derivative.
4  \nabla{#}::Derivative.
5
6  RabcdF := R_{a b c d} -> A_{a} B_{b} C_{c} D_{d}.      # cdb(RabcdF.000,RabcdF)
7  RabcdB := A_{a} B_{b} C_{c} D_{d} -> R_{a b c d}.      # cdb(RabcdB.000,RabcdB)
8
9  derivDD := DD_{b c}{V?_{a}} -> R^{d}_{a b c} V?_{d}.    # cdb(derivDD.000,derivDD)
10
11  nablaDD := \nabla_{f}{\nabla_{e}{R_{a b c d}}}
12            - \nabla_{e}{\nabla_{f}{R_{a b c d}}} -> DD_{e f}{R_{a b c d}}.
13
14  # product rule for DD acting on A_{a} B_{b} C_{c} D_{d}
15  pruleDD := DD_{e f}{A_{a} B_{b} C_{c} D_{d}} -> DD_{e f}{A_{a}} B_{b} C_{c} D_{d}
16            + A_{a} DD_{e f}{B_{b}} C_{c} D_{d}
17            + A_{a} B_{b} DD_{e f}{C_{c}} D_{d}
18            + A_{a} B_{b} C_{c} DD_{e f}{D_{d}}.
19            # cdb(pruleDD.000,pruleDD)
20
21  expr := \nabla_{f}{\nabla_{e}{R_{a b c d}}}
22            - \nabla_{e}{\nabla_{f}{R_{a b c d}}}.      # cdb (ex-0306.100, expr)
23
24  substitute (expr,nablaDD)      # cdb (ex-0306.101, expr)
25  substitute (expr,RabcdF)      # cdb (ex-0306.102, expr)
26  substitute (expr,pruleDD)     # cdb (ex-0306.103, expr)
27  substitute (expr,derivDD)     # cdb (ex-0306.104, expr)
28  sort_product (expr)           # cdb (ex-0306.105, expr)
29  substitute (expr,RabcdB)      # cdb (ex-0306.106, expr)

```

$$\nabla_f (\nabla_e R_{abcd}) - \nabla_e (\nabla_f R_{abcd}) = DD_{ef} R_{abcd} \quad (\text{ex-0306.101})$$

$$= DD_{ef} (A_a B_b C_c D_d) \quad (\text{ex-0306.102})$$

$$= DD_{ef} A_a B_b C_c D_d + A_a DD_{ef} B_b C_c D_d + A_a B_b DD_{ef} C_c D_d + A_a B_b C_c DD_{ef} D_d \quad (\text{ex-0306.103})$$

$$= R^g_{aef} A_g B_b C_c D_d + A_a R^g_{bef} B_g C_c D_d + A_a B_b R^g_{cef} C_g D_d + A_a B_b C_c R^g_{def} D_g \quad (\text{ex-0306.104})$$

$$= A_g B_b C_c D_d R^g_{aef} + A_a B_g C_c D_d R^g_{bef} + A_a B_b C_g D_d R^g_{cef} + A_a B_b C_c D_g R^g_{def} \quad (\text{ex-0306.105})$$

$$= R_{ghcd} R^g_{aef} + R_{agcd} R^g_{bef} + R_{abgd} R^g_{cef} + R_{abcg} R^g_{def} \quad (\text{ex-0306.106})$$

## Exercise 3.7 Commutation of $\nabla$ on the Riemann tensor – direct computation

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 ::Symbol;
4
5 \partial{#}::PartialDerivative.
6
7 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
8
9 RabcdD := \partial_{c}{\Gamma_{a b d}}
10          - \partial_{d}{\Gamma_{a b c}}
11          + \Gamma_{e a d} \Gamma^{e}_{b c}
12          - \Gamma_{e a c} \Gamma^{e}_{b d} -> R_{a b c d}.          # cdb(Rabcd.010,RabcdD)
13
14 RabcdU := \partial_{c}{\Gamma^{a}_{b d}}
15          - \partial_{d}{\Gamma^{a}_{b c}}
16          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
17          - \Gamma^{e}_{b c} \Gamma^{a}_{d e} -> R^{a}_{b c d}.          # cdb(Rabcd.000,RabcdU)
18
19 d1Rabcd := R_{a b c d ; e} -> \partial_{e}{R_{a b c d}}
20          - \Gamma^{f}_{a e} R_{f b c d}
21          - \Gamma^{f}_{b e} R_{a f c d}
22          - \Gamma^{f}_{c e} R_{a b f d}
23          - \Gamma^{f}_{d e} R_{a b c f}.          # cdb(d1Rabcd.000,d1Rabcd)
24
25 d2Rabcd := R_{a b c d ; e ; f} -> \partial_{f}{R_{a b c d ; e}}
26          - \Gamma^{g}_{a f} R_{g b c d ; e}
27          - \Gamma^{g}_{b f} R_{a g c d ; e}
28          - \Gamma^{g}_{c f} R_{a b g d ; e}
29          - \Gamma^{g}_{d f} R_{a b c g ; e}
30          - \Gamma^{g}_{e f} R_{a b c d ; g}.          # cdb(d2Rabcd.000,d2Rabcd)
31
32 substitute (d2Rabcd,d1Rabcd)          # cdb (d2Rabcd.001, d2Rabcd)
33
34 expr := R_{a b c d ; e ; f} - R_{a b c d ; f ; e}.          # cdb (ex-0307.100, expr)
35
36 substitute (expr,d2Rabcd)          # cdb (ex-0307.101, expr)

```

```

37
38 distribute      (expr)                      # cdb (ex-0307.102, expr)
39 product_rule    (expr)                      # cdb (ex-0307.103, expr)
40
41 sort_product    (expr)                      # cdb (ex-0307.104, expr)
42 rename_dummies  (expr)                      # cdb (ex-0307.105, expr)
43 canonicalise    (expr)                      # cdb (ex-0307.106, expr)
44 factor_out      (expr,$R_{a? b? c? d?}$)    # cdb (ex-0307.107, expr)
45
46 substitute      (expr,RabcdU)               # cdb (ex-0307.108, expr)
47 substitute      (expr,$R^{a}_{b c d} -> -R^{a}_{b d c}$) # cdb (ex-0307.109, expr)

```

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_f (\partial_e R_{abcd} - \Gamma_{ae}^g R_{gbcd} - \Gamma_{be}^g R_{agcd} - \Gamma_{ce}^g R_{abgd} - \Gamma_{de}^g R_{abcg}) - \Gamma_{af}^g (\partial_e R_{gbcd} - \Gamma_{ge}^h R_{hbcd} - \Gamma_{be}^h R_{ghcd} - \Gamma_{ce}^h R_{gbhd} - \Gamma_{de}^h R_{gbch}) \\
& - \Gamma_{bf}^g (\partial_e R_{agcd} - \Gamma_{ae}^h R_{hgcd} - \Gamma_{ge}^h R_{ahcd} - \Gamma_{ce}^h R_{aghd} - \Gamma_{de}^h R_{agch}) \\
& - \Gamma_{cf}^g (\partial_e R_{abgd} - \Gamma_{ae}^h R_{hbgd} - \Gamma_{be}^h R_{ahgd} - \Gamma_{ge}^h R_{abhd} - \Gamma_{de}^h R_{abgh}) \\
& - \Gamma_{df}^g (\partial_e R_{abcg} - \Gamma_{ae}^h R_{hbcg} - \Gamma_{be}^h R_{ahcg} - \Gamma_{ce}^h R_{abhg} - \Gamma_{ge}^h R_{abch}) \\
& - \Gamma_{ef}^g (\partial_g R_{abcd} - \Gamma_{ag}^h R_{hbcd} - \Gamma_{bg}^h R_{ahcd} - \Gamma_{cg}^h R_{abhd} - \Gamma_{dg}^h R_{abch}) - \partial_e (\partial_f R_{abcd} - \Gamma_{af}^g R_{gbcd} - \Gamma_{bf}^g R_{agcd} - \Gamma_{cf}^g R_{abgd} - \Gamma_{df}^g R_{abcg}) \\
& + \Gamma_{ae}^g (\partial_f R_{gbcd} - \Gamma_{gf}^h R_{hbcd} - \Gamma_{bf}^h R_{ghcd} - \Gamma_{cf}^h R_{gbhd} - \Gamma_{df}^h R_{gbch}) \\
& + \Gamma_{be}^g (\partial_f R_{agcd} - \Gamma_{af}^h R_{hgcd} - \Gamma_{gf}^h R_{ahcd} - \Gamma_{cf}^h R_{aghd} - \Gamma_{df}^h R_{agch}) \\
& + \Gamma_{ce}^g (\partial_f R_{abgd} - \Gamma_{af}^h R_{hbgd} - \Gamma_{bf}^h R_{ahgd} - \Gamma_{gf}^h R_{abhd} - \Gamma_{df}^h R_{abgh}) \\
& + \Gamma_{de}^g (\partial_f R_{abcg} - \Gamma_{af}^h R_{hbcg} - \Gamma_{bf}^h R_{ahcg} - \Gamma_{cf}^h R_{abhg} - \Gamma_{gf}^h R_{abch}) \\
& + \Gamma_{fe}^g (\partial_g R_{abcd} - \Gamma_{ag}^h R_{hbcd} - \Gamma_{bg}^h R_{ahcd} - \Gamma_{cg}^h R_{abhd} - \Gamma_{dg}^h R_{abch}) \quad (\text{ex-0307.101})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - \partial_f (\Gamma_{ae}^g R_{gbcd}) - \partial_f (\Gamma_{be}^g R_{agcd}) - \partial_f (\Gamma_{ce}^g R_{abgd}) - \partial_f (\Gamma_{de}^g R_{abcg}) - \Gamma_{af}^g \partial_e R_{gbcd} + \Gamma_{af}^g \Gamma_{ge}^h R_{hbcd} + \Gamma_{af}^g \Gamma_{be}^h R_{ghcd} \\
& + \Gamma_{af}^g \Gamma_{ce}^h R_{gbhd} + \Gamma_{af}^g \Gamma_{de}^h R_{gbch} - \Gamma_{bf}^g \partial_e R_{agcd} + \Gamma_{bf}^g \Gamma_{ae}^h R_{hgcd} + \Gamma_{bf}^g \Gamma_{ge}^h R_{ahcd} + \Gamma_{bf}^g \Gamma_{ce}^h R_{aghd} + \Gamma_{bf}^g \Gamma_{de}^h R_{agch} \\
& - \Gamma_{cf}^g \partial_e R_{abgd} + \Gamma_{cf}^g \Gamma_{ae}^h R_{hbgd} + \Gamma_{cf}^g \Gamma_{be}^h R_{ahgd} + \Gamma_{cf}^g \Gamma_{ge}^h R_{abhd} + \Gamma_{cf}^g \Gamma_{de}^h R_{abgh} - \Gamma_{df}^g \partial_e R_{abcg} + \Gamma_{df}^g \Gamma_{ae}^h R_{hbcg} \\
& + \Gamma_{df}^g \Gamma_{be}^h R_{ahcg} + \Gamma_{df}^g \Gamma_{ce}^h R_{abhg} + \Gamma_{df}^g \Gamma_{ge}^h R_{abch} - \Gamma_{ef}^g \partial_g R_{abcd} + \Gamma_{ef}^g \Gamma_{ag}^h R_{hbcd} + \Gamma_{ef}^g \Gamma_{bg}^h R_{ahcd} + \Gamma_{ef}^g \Gamma_{cg}^h R_{abhd} \\
& + \Gamma_{ef}^g \Gamma_{dg}^h R_{abch} - \partial_{ef} R_{abcd} + \partial_e (\Gamma_{af}^g R_{gbcd}) + \partial_e (\Gamma_{bf}^g R_{agcd}) + \partial_e (\Gamma_{cf}^g R_{abgd}) + \partial_e (\Gamma_{df}^g R_{abcg}) + \Gamma_{ae}^g \partial_f R_{gbcd} - \Gamma_{ae}^g \Gamma_{gf}^h R_{hbcd} \\
& - \Gamma_{ae}^g \Gamma_{bf}^h R_{ghcd} - \Gamma_{ae}^g \Gamma_{cf}^h R_{gbhd} - \Gamma_{ae}^g \Gamma_{df}^h R_{gbch} + \Gamma_{be}^g \partial_f R_{agcd} - \Gamma_{be}^g \Gamma_{af}^h R_{hgcd} - \Gamma_{be}^g \Gamma_{gf}^h R_{ahcd} - \Gamma_{be}^g \Gamma_{cf}^h R_{aghd} \\
& - \Gamma_{be}^g \Gamma_{df}^h R_{agch} + \Gamma_{ce}^g \partial_f R_{abgd} - \Gamma_{ce}^g \Gamma_{af}^h R_{hbgd} - \Gamma_{ce}^g \Gamma_{bf}^h R_{ahgd} - \Gamma_{ce}^g \Gamma_{gf}^h R_{abhd} - \Gamma_{ce}^g \Gamma_{df}^h R_{abgh} + \Gamma_{de}^g \partial_f R_{abcg} \\
& - \Gamma_{de}^g \Gamma_{af}^h R_{hbcg} - \Gamma_{de}^g \Gamma_{bf}^h R_{ahcg} - \Gamma_{de}^g \Gamma_{cf}^h R_{abhg} - \Gamma_{de}^g \Gamma_{gf}^h R_{abch} + \Gamma_{fe}^g \partial_g R_{abcd} - \Gamma_{fe}^g \Gamma_{ag}^h R_{hbcd} - \Gamma_{fe}^g \Gamma_{bg}^h R_{ahcd} \\
& - \Gamma_{fe}^g \Gamma_{cg}^h R_{abhd} - \Gamma_{fe}^g \Gamma_{dg}^h R_{abch} \quad (\text{ex-0307.102})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - \partial_f \Gamma^g_{ae} R_{gbcd} - \partial_f \Gamma^g_{be} R_{agcd} - \partial_f \Gamma^g_{ce} R_{abgd} - \partial_f \Gamma^g_{de} R_{abcg} + \Gamma^g_{af} \Gamma^h_{ge} R_{hbcd} + \Gamma^g_{af} \Gamma^h_{be} R_{ghcd} + \Gamma^g_{af} \Gamma^h_{ce} R_{gbhd} \\
& + \Gamma^g_{af} \Gamma^h_{de} R_{gbch} + \Gamma^g_{bf} \Gamma^h_{ae} R_{hgcd} + \Gamma^g_{bf} \Gamma^h_{ge} R_{ahcd} + \Gamma^g_{bf} \Gamma^h_{ce} R_{aghd} + \Gamma^g_{bf} \Gamma^h_{de} R_{agch} + \Gamma^g_{cf} \Gamma^h_{ae} R_{hbcd} + \Gamma^g_{cf} \Gamma^h_{be} R_{ahgd} \\
& + \Gamma^g_{cf} \Gamma^h_{ge} R_{abhd} + \Gamma^g_{cf} \Gamma^h_{de} R_{abgh} + \Gamma^g_{df} \Gamma^h_{ae} R_{hbcg} + \Gamma^g_{df} \Gamma^h_{be} R_{ahcg} + \Gamma^g_{df} \Gamma^h_{ce} R_{abhg} + \Gamma^g_{df} \Gamma^h_{ge} R_{abch} - \Gamma^g_{ef} \partial_g R_{abcd} \\
& + \Gamma^g_{ef} \Gamma^h_{ag} R_{hbcd} + \Gamma^g_{ef} \Gamma^h_{bg} R_{ahcd} + \Gamma^g_{ef} \Gamma^h_{cg} R_{abhd} + \Gamma^g_{ef} \Gamma^h_{dg} R_{abch} - \partial_{ef} R_{abcd} + \partial_e \Gamma^g_{af} R_{gbcd} + \partial_e \Gamma^g_{bf} R_{agcd} + \partial_e \Gamma^g_{cf} R_{abgd} \\
& + \partial_e \Gamma^g_{df} R_{abcg} - \Gamma^g_{ae} \Gamma^h_{gf} R_{hbcd} - \Gamma^g_{ae} \Gamma^h_{bf} R_{ghcd} - \Gamma^g_{ae} \Gamma^h_{cf} R_{gbhd} - \Gamma^g_{ae} \Gamma^h_{df} R_{gbch} - \Gamma^g_{be} \Gamma^h_{af} R_{hgcd} - \Gamma^g_{be} \Gamma^h_{gf} R_{ahcd} \\
& - \Gamma^g_{be} \Gamma^h_{cf} R_{aghd} - \Gamma^g_{be} \Gamma^h_{df} R_{agch} - \Gamma^g_{ce} \Gamma^h_{af} R_{hbcd} - \Gamma^g_{ce} \Gamma^h_{bf} R_{ahgd} - \Gamma^g_{ce} \Gamma^h_{gf} R_{abhd} - \Gamma^g_{ce} \Gamma^h_{df} R_{abgh} - \Gamma^g_{de} \Gamma^h_{af} R_{hbcg} \\
& - \Gamma^g_{de} \Gamma^h_{bf} R_{ahcg} - \Gamma^g_{de} \Gamma^h_{cf} R_{abhg} - \Gamma^g_{de} \Gamma^h_{gf} R_{abch} + \Gamma^g_{fe} \partial_g R_{abcd} - \Gamma^g_{fe} \Gamma^h_{ag} R_{hbcd} - \Gamma^g_{fe} \Gamma^h_{bg} R_{ahcd} - \Gamma^g_{fe} \Gamma^h_{cg} R_{abhd} \\
& - \Gamma^g_{fe} \Gamma^h_{dg} R_{abch}
\end{aligned} \tag{ex-0307.103}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma^g_{ae} - R_{agcd} \partial_f \Gamma^g_{be} - R_{abgd} \partial_f \Gamma^g_{ce} - R_{abcg} \partial_f \Gamma^g_{de} + R_{hbcd} \Gamma^g_{af} \Gamma^h_{ge} + R_{ghcd} \Gamma^g_{af} \Gamma^h_{be} + R_{gbhd} \Gamma^g_{af} \Gamma^h_{ce} \\
& + R_{gbch} \Gamma^g_{af} \Gamma^h_{de} + R_{hgcd} \Gamma^g_{bf} \Gamma^h_{ae} + R_{ahcd} \Gamma^g_{bf} \Gamma^h_{ge} + R_{aghd} \Gamma^g_{bf} \Gamma^h_{ce} + R_{agch} \Gamma^g_{bf} \Gamma^h_{de} + R_{hbcd} \Gamma^g_{cf} \Gamma^h_{ae} + R_{ahgd} \Gamma^g_{cf} \Gamma^h_{be} \\
& + R_{abhd} \Gamma^g_{cf} \Gamma^h_{ge} + R_{abgh} \Gamma^g_{cf} \Gamma^h_{de} + R_{hbcg} \Gamma^g_{df} \Gamma^h_{ae} + R_{ahcg} \Gamma^g_{df} \Gamma^h_{be} + R_{abhg} \Gamma^g_{df} \Gamma^h_{ce} + R_{abch} \Gamma^g_{df} \Gamma^h_{ge} - \Gamma^g_{ef} \partial_g R_{abcd} \\
& + R_{hbcd} \Gamma^g_{ef} \Gamma^h_{ag} + R_{ahcd} \Gamma^g_{ef} \Gamma^h_{bg} + R_{abhd} \Gamma^g_{ef} \Gamma^h_{cg} + R_{abch} \Gamma^g_{ef} \Gamma^h_{dg} - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma^g_{af} + R_{agcd} \partial_e \Gamma^g_{bf} + R_{abgd} \partial_e \Gamma^g_{cf} \\
& + R_{abcg} \partial_e \Gamma^g_{df} - R_{hbcd} \Gamma^g_{ae} \Gamma^h_{gf} - R_{ghcd} \Gamma^g_{ae} \Gamma^h_{bf} - R_{gbhd} \Gamma^g_{ae} \Gamma^h_{cf} - R_{gbch} \Gamma^g_{ae} \Gamma^h_{df} - R_{hgcd} \Gamma^g_{be} \Gamma^h_{af} - R_{ahcd} \Gamma^g_{be} \Gamma^h_{gf} \\
& - R_{aghd} \Gamma^g_{be} \Gamma^h_{cf} - R_{agch} \Gamma^g_{be} \Gamma^h_{df} - R_{hbcd} \Gamma^g_{ce} \Gamma^h_{af} - R_{ahgd} \Gamma^g_{ce} \Gamma^h_{bf} - R_{abhd} \Gamma^g_{ce} \Gamma^h_{gf} - R_{abgh} \Gamma^g_{ce} \Gamma^h_{df} - R_{hbcg} \Gamma^g_{de} \Gamma^h_{af} \\
& - R_{ahcg} \Gamma^g_{de} \Gamma^h_{bf} - R_{abhg} \Gamma^g_{de} \Gamma^h_{cf} - R_{abch} \Gamma^g_{de} \Gamma^h_{gf} + \Gamma^g_{fe} \partial_g R_{abcd} - R_{hbcd} \Gamma^g_{fe} \Gamma^h_{ag} - R_{ahcd} \Gamma^g_{fe} \Gamma^h_{bg} - R_{abhd} \Gamma^g_{fe} \Gamma^h_{cg} \\
& - R_{abch} \Gamma^g_{fe} \Gamma^h_{dg}
\end{aligned} \tag{ex-0307.104}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma^g_{ae} - R_{agcd} \partial_f \Gamma^g_{be} - R_{abgd} \partial_f \Gamma^g_{ce} - R_{abcg} \partial_f \Gamma^g_{de} + R_{gbcd} \Gamma^h_{af} \Gamma^g_{he} + R_{ghcd} \Gamma^h_{af} \Gamma^g_{be} + R_{gbhd} \Gamma^h_{af} \Gamma^g_{ce} \\
& + R_{gbch} \Gamma^h_{af} \Gamma^g_{de} + R_{ghcd} \Gamma^h_{bf} \Gamma^g_{ae} + R_{agcd} \Gamma^h_{bf} \Gamma^g_{he} + R_{aghd} \Gamma^h_{bf} \Gamma^g_{ce} + R_{agch} \Gamma^h_{bf} \Gamma^g_{de} + R_{gbhd} \Gamma^h_{cf} \Gamma^g_{ae} + R_{aghd} \Gamma^h_{cf} \Gamma^g_{be} \\
& + R_{abgd} \Gamma^h_{cf} \Gamma^g_{he} + R_{abgh} \Gamma^h_{cf} \Gamma^g_{de} + R_{gbch} \Gamma^h_{df} \Gamma^g_{ae} + R_{agch} \Gamma^h_{df} \Gamma^g_{be} + R_{abgh} \Gamma^h_{df} \Gamma^g_{ce} + R_{abcg} \Gamma^h_{df} \Gamma^g_{he} - \Gamma^g_{ef} \partial_g R_{abcd} \\
& + R_{gbcd} \Gamma^h_{ef} \Gamma^g_{ah} + R_{agcd} \Gamma^h_{ef} \Gamma^g_{bh} + R_{abgd} \Gamma^h_{ef} \Gamma^g_{ch} + R_{abcg} \Gamma^h_{ef} \Gamma^g_{dh} - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma^g_{af} + R_{agcd} \partial_e \Gamma^g_{bf} + R_{abgd} \partial_e \Gamma^g_{cf} \\
& + R_{abcg} \partial_e \Gamma^g_{df} - R_{gbcd} \Gamma^h_{ae} \Gamma^g_{hf} - R_{ghcd} \Gamma^h_{ae} \Gamma^g_{bf} - R_{gbhd} \Gamma^h_{ae} \Gamma^g_{cf} - R_{gbch} \Gamma^h_{ae} \Gamma^g_{df} - R_{ghcd} \Gamma^h_{be} \Gamma^g_{af} - R_{agcd} \Gamma^h_{be} \Gamma^g_{hf} \\
& - R_{aghd} \Gamma^h_{be} \Gamma^g_{cf} - R_{agch} \Gamma^h_{be} \Gamma^g_{df} - R_{gbhd} \Gamma^h_{ce} \Gamma^g_{af} - R_{aghd} \Gamma^h_{ce} \Gamma^g_{bf} - R_{abgd} \Gamma^h_{ce} \Gamma^g_{hf} - R_{abgh} \Gamma^h_{ce} \Gamma^g_{df} - R_{gbch} \Gamma^h_{de} \Gamma^g_{af} \\
& - R_{agch} \Gamma^h_{de} \Gamma^g_{bf} - R_{abgh} \Gamma^h_{de} \Gamma^g_{cf} - R_{abcg} \Gamma^h_{de} \Gamma^g_{hf} + \Gamma^g_{fe} \partial_g R_{abcd} - R_{gbcd} \Gamma^h_{fe} \Gamma^g_{ah} - R_{agcd} \Gamma^h_{fe} \Gamma^g_{bh} - R_{abgd} \Gamma^h_{fe} \Gamma^g_{ch} \\
& - R_{abcg} \Gamma^h_{fe} \Gamma^g_{dh}
\end{aligned} \tag{ex-0307.105}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & -R_{gbcd}\partial_f\Gamma^g_{ae} - R_{agcd}\partial_f\Gamma^g_{be} - R_{abgd}\partial_f\Gamma^g_{ce} - R_{abcg}\partial_f\Gamma^g_{de} + R_{gbcd}\Gamma^h_{af}\Gamma^g_{eh} + R_{agcd}\Gamma^h_{bf}\Gamma^g_{eh} + R_{abgd}\Gamma^h_{cf}\Gamma^g_{eh} + R_{abcg}\Gamma^h_{df}\Gamma^g_{eh} \\
& + R_{gbcd}\partial_e\Gamma^g_{af} + R_{agcd}\partial_e\Gamma^g_{bf} + R_{abgd}\partial_e\Gamma^g_{cf} + R_{abcg}\partial_e\Gamma^g_{df} - R_{gbcd}\Gamma^h_{ae}\Gamma^g_{fh} - R_{agcd}\Gamma^h_{be}\Gamma^g_{fh} - R_{abgd}\Gamma^h_{ce}\Gamma^g_{fh} \\
& - R_{abcg}\Gamma^h_{de}\Gamma^g_{fh}
\end{aligned} \tag{ex-0307.106}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & R_{gbcd}\left(-\partial_f\Gamma^g_{ae} + \Gamma^h_{af}\Gamma^g_{eh} + \partial_e\Gamma^g_{af} - \Gamma^h_{ae}\Gamma^g_{fh}\right) + R_{agcd}\left(-\partial_f\Gamma^g_{be} + \Gamma^h_{bf}\Gamma^g_{eh} + \partial_e\Gamma^g_{bf} - \Gamma^h_{be}\Gamma^g_{fh}\right) \\
& + R_{abgd}\left(-\partial_f\Gamma^g_{ce} + \Gamma^h_{cf}\Gamma^g_{eh} + \partial_e\Gamma^g_{cf} - \Gamma^h_{ce}\Gamma^g_{fh}\right) + R_{abcg}\left(-\partial_f\Gamma^g_{de} + \Gamma^h_{df}\Gamma^g_{eh} + \partial_e\Gamma^g_{df} - \Gamma^h_{de}\Gamma^g_{fh}\right)
\end{aligned} \tag{ex-0307.107}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd}R^g_{afe} - R_{agcd}R^g_{bfe} - R_{abgd}R^g_{cfe} - R_{abcg}R^g_{dfe} \tag{ex-0307.108}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd}R^g_{aef} + R_{agcd}R^g_{bef} + R_{abgd}R^g_{cef} + R_{abcg}R^g_{def} \tag{ex-0307.109}$$

### Exercise 3.8 Symmetry of $R_{ab}$

[illegible]



$$4R_{ab} - 4R_{ba} = 4R^c_{acb} - 4R^c_{bca} \quad (\text{ex-0308.101})$$

$$= 4\partial_c \Gamma^c_{ab} + 4\Gamma^c_{ec} \Gamma^e_{ab} - 4\partial_b \Gamma^c_{ac} - 4\Gamma^c_{eb} \Gamma^e_{ac} - 4\partial_c \Gamma^c_{ba} - 4\Gamma^c_{ec} \Gamma^e_{ba} + 4\partial_a \Gamma^c_{bc} + 4\Gamma^c_{ea} \Gamma^e_{bc} \quad (\text{ex-0308.102})$$

$$\begin{aligned} &= 2\partial_c (g^{ce} (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab})) + g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}) - 2\partial_b (g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac})) \\ &\quad - g^{cd} (\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) - 2\partial_c (g^{ce} (\partial_b g_{ea} + \partial_a g_{be} - \partial_e g_{ba})) \\ &\quad - g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_b g_{fa} + \partial_a g_{bf} - \partial_f g_{ba}) + 2\partial_a (g^{ce} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc})) \\ &\quad + g^{cd} (\partial_e g_{da} + \partial_a g_{ed} - \partial_d g_{ea}) g^{ef} (\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}) \end{aligned} \quad (\text{ex-0308.103})$$

$$\begin{aligned} &= 2\partial_c (g^{ce} \partial_a g_{eb}) + 2\partial_c (g^{ce} \partial_b g_{ae}) - 2\partial_c (g^{ce} \partial_e g_{ab}) + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} \\ &\quad + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} - 2\partial_b (g^{ce} \partial_a g_{ec}) - 2\partial_b (g^{ce} \partial_c g_{ae}) \\ &\quad + 2\partial_b (g^{ce} \partial_e g_{ac}) - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} \\ &\quad + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} - 2\partial_c (g^{ce} \partial_b g_{ea}) - 2\partial_c (g^{ce} \partial_a g_{be}) + 2\partial_c (g^{ce} \partial_e g_{ba}) - g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{fa} \\ &\quad - g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{bf} + g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ba} - g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{fa} - g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{bf} + g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ba} + g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{fa} + g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{bf} \\ &\quad - g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ba} + 2\partial_a (g^{ce} \partial_b g_{ec}) + 2\partial_a (g^{ce} \partial_c g_{be}) - 2\partial_a (g^{ce} \partial_e g_{bc}) + g^{cd} \partial_e g_{da} g^{ef} \partial_b g_{fc} + g^{cd} \partial_e g_{da} g^{ef} \partial_c g_{bf} - g^{cd} \partial_e g_{da} g^{ef} \partial_f g_{bc} \\ &\quad + g^{cd} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + g^{cd} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - g^{cd} \partial_a g_{ed} g^{ef} \partial_f g_{bc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_b g_{fc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_c g_{bf} + g^{cd} \partial_d g_{ea} g^{ef} \partial_f g_{bc} \end{aligned} \quad (\text{ex-0308.104})$$

$$\begin{aligned} &= 2\partial_c g^{ce} \partial_a g_{eb} + 2g^{ce} \partial_{ca} g_{eb} + 2\partial_c g^{ce} \partial_b g_{ae} + 2g^{ce} \partial_{cb} g_{ae} - 2\partial_c g^{ce} \partial_e g_{ab} - 2g^{ce} \partial_{ce} g_{ab} + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} \\ &\quad + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} - 2\partial_b g^{ce} \partial_a g_{ec} \\ &\quad - 2g^{ce} \partial_{ba} g_{ec} - 2\partial_b g^{ce} \partial_c g_{ae} - 2g^{ce} \partial_{bc} g_{ae} + 2\partial_b g^{ce} \partial_e g_{ac} + 2g^{ce} \partial_{be} g_{ac} - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} \\ &\quad - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} - 2\partial_c g^{ce} \partial_b g_{ea} \\ &\quad - 2g^{ce} \partial_{cb} g_{ea} - 2\partial_c g^{ce} \partial_a g_{be} - 2g^{ce} \partial_{ca} g_{be} + 2\partial_c g^{ce} \partial_e g_{ba} + 2g^{ce} \partial_{ce} g_{ba} - g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{fa} - g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{bf} + g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ba} \\ &\quad - g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{fa} - g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{bf} + g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ba} + g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{fa} + g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{bf} - g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ba} + 2\partial_a g^{ce} \partial_b g_{ec} \\ &\quad + 2g^{ce} \partial_{ab} g_{ec} + 2\partial_a g^{ce} \partial_c g_{be} + 2g^{ce} \partial_{ac} g_{be} - 2\partial_a g^{ce} \partial_e g_{bc} - 2g^{ce} \partial_{ae} g_{bc} + g^{cd} \partial_e g_{da} g^{ef} \partial_b g_{fc} + g^{cd} \partial_e g_{da} g^{ef} \partial_c g_{bf} - g^{cd} \partial_e g_{da} g^{ef} \partial_f g_{bc} \\ &\quad + g^{cd} \partial_a g_{ed} g^{ef} \partial_b g_{fc} + g^{cd} \partial_a g_{ed} g^{ef} \partial_c g_{bf} - g^{cd} \partial_a g_{ed} g^{ef} \partial_f g_{bc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_b g_{fc} - g^{cd} \partial_d g_{ea} g^{ef} \partial_c g_{bf} + g^{cd} \partial_d g_{ea} g^{ef} \partial_f g_{bc} \end{aligned} \quad (\text{ex-0308.105})$$

$$= -2\partial_b g^{ce} \partial_a g_{ce} + 2\partial_a g^{ce} \partial_b g_{ce} \quad (\text{ex-0308.106})$$

$$= 2g^{cd} g^{ef} \partial_b g_{df} \partial_a g_{ce} - 2g^{cd} g^{ef} \partial_a g_{df} \partial_b g_{ce} \quad (\text{ex-0308.107})$$

$$= 0 \quad (\text{ex-0308.108})$$

## Exercise 3.8 Symmetry of $R_{ab}$ alternative solution

This differs from the previous code by the inclusion of a call to `canonicalise` immediately after the first two substitutions and a declaration that  $\Gamma^a_{bc}$  is symmetric in  $bc$ . This pair of changes produces a more compact set of results than given above. Incidentally, this also shows that  $\partial_a \Gamma^c_{bc} = \partial_b \Gamma^c_{ac}$ .

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative;
4
5  \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
6
7  g_{a b}::Metric;
8  g^{a b}::InverseMetric;
9
10 dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
11                                     # cdb (dgab.000,dgab)
12
13 Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} ( \partial_{b}{g_{e c}}
14                                     + \partial_{c}{g_{b e}}
15                                     - \partial_{e}{g_{b c}}).
16                                     # cdb (Gamma.000,Gamma)
17
18 Rabcd := R^{a}_{b c d} ->
19     \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
20     - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
21                                     # cdb (Rabcd.000,Rabcd)
22
23 Rab := R_{a b} -> R^{c}_{a c b}.
24                                     # cdb (Rab.000,Rab)
25
26 expr := 4 (R_{a b} - R_{b a}).
27                                     # cdb (ex-0308.200,expr)
28
29 substitute (expr, Rab)
30                                     # cdb (ex-0308.201,expr)
31 substitute (expr, Rabcd)
32                                     # cdb (ex-0308.202,expr)
33 canonicalise (expr)
34                                     # cdb (ex-0308.203,expr)
35 substitute (expr, Gamma)
36                                     # cdb (ex-0308.204,expr)
37
38 distribute (expr)
39                                     # cdb (ex-0308.205,expr)

```

```

33 product_rule (expr)                # cdb (ex-0308.206,expr)
34 canonicalise (expr)                # cdb (ex-0308.207,expr)
35
36 substitute   (expr, dgab)          # cdb (ex-0308.208,expr)
37 canonicalise (expr)                # cdb (ex-0308.209,expr)

```

$$\begin{aligned}
4R_{ab} - 4R_{ba} &= 4R^c_{acb} - 4R^c_{bca} && (\text{ex-0308.201}) \\
&= 4\partial_c \Gamma^c_{ab} + 4\Gamma^c_{ec} \Gamma^e_{ab} - 4\partial_b \Gamma^c_{ac} - 4\Gamma^c_{eb} \Gamma^e_{ac} - 4\partial_c \Gamma^c_{ba} - 4\Gamma^c_{ec} \Gamma^e_{ba} + 4\partial_a \Gamma^c_{bc} + 4\Gamma^c_{ea} \Gamma^e_{bc} && (\text{ex-0308.202}) \\
&= -4\partial_b \Gamma^c_{ac} + 4\partial_a \Gamma^c_{bc} && (\text{ex-0308.203}) \\
&= -2\partial_b (g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac})) + 2\partial_a (g^{ce} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc})) && (\text{ex-0308.204}) \\
&= -2\partial_b (g^{ce} \partial_a g_{ec}) - 2\partial_b (g^{ce} \partial_c g_{ae}) + 2\partial_b (g^{ce} \partial_e g_{ac}) + 2\partial_a (g^{ce} \partial_b g_{ec}) + 2\partial_a (g^{ce} \partial_c g_{be}) - 2\partial_a (g^{ce} \partial_e g_{bc}) && (\text{ex-0308.205}) \\
&= -2\partial_b g^{ce} \partial_a g_{ec} - 2g^{ce} \partial_{ba} g_{ec} - 2\partial_b g^{ce} \partial_c g_{ae} - 2g^{ce} \partial_{bc} g_{ae} + 2\partial_b g^{ce} \partial_e g_{ac} + 2g^{ce} \partial_{be} g_{ac} + 2\partial_a g^{ce} \partial_b g_{ec} + 2g^{ce} \partial_{ab} g_{ec} + 2\partial_a g^{ce} \partial_c g_{be} \\
&\quad + 2g^{ce} \partial_{ac} g_{be} - 2\partial_a g^{ce} \partial_e g_{bc} - 2g^{ce} \partial_{ae} g_{bc} && (\text{ex-0308.206}) \\
&= -2\partial_b g^{ce} \partial_a g_{ce} + 2\partial_a g^{ce} \partial_b g_{ce} && (\text{ex-0308.207}) \\
&= 2g^{cd} g^{ef} \partial_b g_{df} \partial_a g_{ce} - 2g^{cd} g^{ef} \partial_a g_{df} \partial_b g_{ce} && (\text{ex-0308.208}) \\
&= 0 && (\text{ex-0308.209})
\end{aligned}$$

## Exercise 3.9 Ricci in terms of the metric and its derivatives

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative;
4
5  g_{a b}::Metric;
6  g^{a b}::InverseMetric;
7
8  dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.      # cdb (ex-0309.dgab,dgab)
9
10 Gamma := \Gamma^{a}_{b c} ->
11         (1/2) g^{a e} ( \partial_{b}{g_{e c}}
12                        + \partial_{c}{g_{b e}}
13                        - \partial_{e}{g_{b c}}).      # cdb (ex-0309.Gamma,Gamma)
14
15 Rabcd := R^{a}_{b c d} ->
16         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
17         - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.      # cdb (ex-0309.Rabcd,Rabcd)
18
19 FourRab := 4 R^{c}_{a c b}.      # cdb (ex-0309.101,FourRab)
20
21 substitute      (FourRab, Rabcd)      # cdb (ex-0309.102,FourRab)
22 substitute      (FourRab, Gamma)      # cdb (ex-0309.103,FourRab)
23
24 product_rule    (FourRab)      # cdb (ex-0309.104,FourRab)
25 distribute      (FourRab)      # cdb (ex-0309.105,FourRab)
26
27 substitute      (FourRab, dgab)      # cdb (ex-0309.106,FourRab)
28
29 sort_product    (FourRab)      # cdb (ex-0309.107,FourRab)
30 rename_dummies  (FourRab)      # cdb (ex-0309.108,FourRab)
31 canonicalise    (FourRab)      # cdb (ex-0309.109,FourRab)
32
33 # sort so that g to appeares before dg
34
35 substitute      (FourRab, $g^{a b} -> A^{a b}$)
36 sort_product    (FourRab)

```

```
37 rename_dummies (FourRab)
38 substitute      (FourRab, $A^{a b} -> g^{a b}$)    # cdb (ex-0309.110,FourRab)
```

$$\begin{aligned}
4R_{ab} &= 4R_{acb} & (\text{ex-0309.101}) \\
&= 4\partial_c \Gamma^c_{ab} + 4\Gamma^c_{ec} \Gamma^e_{ab} - 4\partial_b \Gamma^c_{ac} - 4\Gamma^c_{eb} \Gamma^e_{ac} & (\text{ex-0309.102}) \\
&= 2\partial_c (g^{ce} (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab})) + g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}) - 2\partial_b (g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac})) \\
&\quad - g^{cd} (\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) & (\text{ex-0309.103}) \\
&= 2\partial_c g^{ce} (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}) + 2g^{ce} \partial_c (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}) + g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}) \\
&\quad - 2\partial_b g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}) - 2g^{ce} \partial_b (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}) - g^{cd} (\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) & (\text{ex-0309.104}) \\
&= 2\partial_c g^{ce} \partial_a g_{eb} + 2\partial_c g^{ce} \partial_b g_{ae} - 2\partial_c g^{ce} \partial_e g_{ab} + 2g^{ce} \partial_{ca} g_{eb} + 2g^{ce} \partial_{cb} g_{ae} - 2g^{ce} \partial_{ce} g_{ab} + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} \\
&\quad + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} - 2\partial_b g^{ce} \partial_a g_{ec} \\
&\quad - 2\partial_b g^{ce} \partial_c g_{ae} + 2\partial_b g^{ce} \partial_e g_{ac} - 2g^{ce} \partial_{ba} g_{ec} - 2g^{ce} \partial_{bc} g_{ae} + 2g^{ce} \partial_{be} g_{ac} - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} \\
&\quad - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} & (\text{ex-0309.105}) \\
&= -2g^{cd} g^{ef} \partial_c g_{df} \partial_a g_{eb} - 2g^{cd} g^{ef} \partial_c g_{df} \partial_b g_{ae} + 2g^{cd} g^{ef} \partial_c g_{df} \partial_e g_{ab} + 2g^{ce} \partial_{ca} g_{eb} + 2g^{ce} \partial_{cb} g_{ae} - 2g^{ce} \partial_{ce} g_{ab} + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} \\
&\quad - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} \\
&\quad + 2g^{cd} g^{ef} \partial_b g_{df} \partial_a g_{ec} + 2g^{cd} g^{ef} \partial_b g_{df} \partial_c g_{ae} - 2g^{cd} g^{ef} \partial_b g_{df} \partial_e g_{ac} - 2g^{ce} \partial_{ba} g_{ec} - 2g^{ce} \partial_{bc} g_{ae} + 2g^{ce} \partial_{be} g_{ac} - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} \\
&\quad - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} \\
&\quad - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} & (\text{ex-0309.106}) \\
&= -2\partial_a g_{eb} \partial_c g_{df} g^{cd} g^{ef} - 2\partial_b g_{ae} \partial_c g_{df} g^{cd} g^{ef} + 2\partial_c g_{df} \partial_e g_{ab} g^{cd} g^{ef} + 2\partial_{ca} g_{eb} g^{ce} + 2\partial_{cb} g_{ae} g^{ce} - 2\partial_{ce} g_{ab} g^{ce} + \partial_a g_{fb} \partial_e g_{dc} g^{cd} g^{ef} + \partial_b g_{af} \partial_e g_{dc} g^{cd} g^{ef} \\
&\quad - \partial_e g_{dc} \partial_f g_{ab} g^{cd} g^{ef} + \partial_a g_{fb} \partial_c g_{ed} g^{cd} g^{ef} + \partial_b g_{af} \partial_c g_{ed} g^{cd} g^{ef} - \partial_c g_{ed} \partial_f g_{ab} g^{cd} g^{ef} - \partial_a g_{fb} \partial_d g_{ec} g^{cd} g^{ef} - \partial_b g_{af} \partial_d g_{ec} g^{cd} g^{ef} + \partial_d g_{ec} \partial_f g_{ab} g^{cd} g^{ef} \\
&\quad + 2\partial_a g_{ec} \partial_b g_{df} g^{cd} g^{ef} + 2\partial_b g_{df} \partial_c g_{ae} g^{cd} g^{ef} - 2\partial_b g_{df} \partial_e g_{ac} g^{cd} g^{ef} - 2\partial_{ba} g_{ec} g^{ce} - 2\partial_{bc} g_{ae} g^{ce} + 2\partial_{be} g_{ac} g^{ce} - \partial_a g_{fc} \partial_e g_{db} g^{cd} g^{ef} \\
&\quad - \partial_c g_{af} \partial_e g_{db} g^{cd} g^{ef} + \partial_e g_{db} \partial_f g_{ac} g^{cd} g^{ef} - \partial_a g_{fc} \partial_b g_{ed} g^{cd} g^{ef} - \partial_b g_{ed} \partial_c g_{af} g^{cd} g^{ef} + \partial_b g_{ed} \partial_f g_{ac} g^{cd} g^{ef} + \partial_a g_{fc} \partial_d g_{eb} g^{cd} g^{ef} + \partial_c g_{af} \partial_d g_{eb} g^{cd} g^{ef} \\
&\quad - \partial_d g_{eb} \partial_f g_{ac} g^{cd} g^{ef} & (\text{ex-0309.107}) \\
&= -2\partial_a g_{db} \partial_c g_{ef} g^{ce} g^{df} - 2\partial_b g_{ad} \partial_c g_{ef} g^{ce} g^{df} + 2\partial_c g_{ef} \partial_d g_{ab} g^{ce} g^{df} + 2\partial_{ca} g_{db} g^{cd} + 2\partial_{cb} g_{ad} g^{cd} - 2\partial_{cd} g_{ab} g^{cd} + \partial_a g_{db} \partial_c g_{ef} g^{fe} g^{cd} + \partial_b g_{ad} \partial_c g_{ef} g^{fe} g^{cd} \\
&\quad - \partial_c g_{ef} \partial_d g_{ab} g^{fe} g^{cd} + \partial_a g_{db} \partial_c g_{ef} g^{cf} g^{ed} + \partial_b g_{ad} \partial_c g_{ef} g^{cf} g^{ed} - \partial_c g_{ef} \partial_d g_{ab} g^{cf} g^{ed} - \partial_a g_{db} \partial_c g_{ef} g^{fc} g^{ed} - \partial_b g_{ad} \partial_c g_{ef} g^{fc} g^{ed} + \partial_c g_{ef} \partial_d g_{ab} g^{fc} g^{ed} \\
&\quad + 2\partial_a g_{cd} \partial_b g_{ef} g^{de} g^{cf} + 2\partial_b g_{de} \partial_c g_{af} g^{cd} g^{fe} - 2\partial_b g_{de} \partial_c g_{af} g^{fd} g^{ce} - 2\partial_{ba} g_{cd} g^{dc} - 2\partial_{bc} g_{ad} g^{cd} + 2\partial_{bc} g_{ad} g^{dc} - \partial_a g_{de} \partial_c g_{fb} g^{ef} g^{cd} \\
&\quad - \partial_c g_{ae} \partial_d g_{fb} g^{cf} g^{de} + \partial_c g_{eb} \partial_d g_{af} g^{fe} g^{cd} - \partial_a g_{cd} \partial_b g_{ef} g^{df} g^{ec} - \partial_b g_{de} \partial_c g_{af} g^{ce} g^{df} + \partial_b g_{de} \partial_c g_{af} g^{fe} g^{dc} + \partial_a g_{de} \partial_c g_{fb} g^{ec} g^{fd} + \partial_c g_{ae} \partial_d g_{fb} g^{cd} g^{fe} \\
&\quad - \partial_c g_{eb} \partial_d g_{af} g^{fc} g^{ed} & (\text{ex-0309.108}) \\
&= -2\partial_a g_{bc} \partial_d g_{ef} g^{ce} g^{df} - 2\partial_b g_{ac} \partial_d g_{ef} g^{ce} g^{df} + 2\partial_c g_{ab} \partial_d g_{ef} g^{ce} g^{df} + 2\partial_{ac} g_{bd} g^{cd} + 2\partial_{bc} g_{ad} g^{cd} - 2\partial_{cd} g_{ab} g^{cd} + \partial_a g_{bc} \partial_d g_{ef} g^{cd} g^{ef} + \partial_b g_{ac} \partial_d g_{ef} g^{cd} g^{ef} \\
&\quad - \partial_c g_{ab} \partial_d g_{ef} g^{cd} g^{ef} + \partial_a g_{cd} \partial_b g_{ef} g^{ce} g^{df} - 2\partial_{ab} g_{cd} g^{cd} - 2\partial_c g_{ad} \partial_e g_{bf} g^{cf} g^{de} + 2\partial_c g_{ad} \partial_e g_{bf} g^{ce} g^{df} & (\text{ex-0309.109}) \\
&= -2g^{cd} g^{ef} \partial_a g_{bc} \partial_e g_{df} - 2g^{cd} g^{ef} \partial_b g_{ac} \partial_e g_{df} + 2g^{cd} g^{ef} \partial_c g_{ab} \partial_e g_{df} + 2g^{cd} \partial_{ac} g_{bd} + 2g^{cd} \partial_{bc} g_{ad} - 2g^{cd} \partial_{cd} g_{ab} + g^{cd} g^{ef} \partial_a g_{bc} \partial_d g_{ef} + g^{cd} g^{ef} \partial_b g_{ac} \partial_d g_{ef} \\
&\quad - g^{cd} g^{ef} \partial_c g_{ab} \partial_d g_{ef} + g^{cd} g^{ef} \partial_a g_{ce} \partial_b g_{df} - 2g^{cd} \partial_{ab} g_{cd} - 2g^{cd} g^{ef} \partial_c g_{ae} \partial_f g_{bd} + 2g^{cd} g^{ef} \partial_c g_{ae} \partial_d g_{bf} & (\text{ex-0309.110})
\end{aligned}$$

## Exercise 3.10 Example of repeat=True in a substitution

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent).
2
3 foo := A B + A B A B + A B A B A B + A B A B A B A B .      # cdb(ex-0310.foo.001,foo)
4 bah := @(foo).                                                  # cdb(ex-0310.bah.001,bah)
5
6 substitute (foo,$A B -> A$)                                     # cdb(ex-0310.foo.002,foo)
7 substitute (bah,$A B -> A$,repeat=True)                         # cdb(ex-0310.bah.002,bah)
```

Without `repeat=True` only the first match in a product will be substituted.

$$\begin{aligned}\text{ex-0310.foo.001} &:= AB + ABAB + ABABAB + ABABABAB \\ \text{ex-0310.foo.002} &:= A + AAB + AABAB + AABABAB\end{aligned}$$

But with `repeat=True` then all matches in a product will be substituted.

$$\begin{aligned}\text{ex-0310.bah.001} &:= AB + ABAB + ABABAB + ABABABAB \\ \text{ex-0310.bah.002} &:= A + AA + AAA + AAAA\end{aligned}$$

## Exercise 4.1 Differentiate a polynomial – a limited method

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 def deriv (poly):
4
5     \delta^{a}::Weight(label=\epsilon).
6
7     bah := @(poly).
8
9     substitute      (bah,$x^{a} -> x^{a} + \delta^{a}$)
10    distribute      (bah)
11
12    foo := @(bah) - @(poly).
13
14    keep_weight      (foo, $\epsilon = 1$)
15    sort_product     (foo)
16    rename_dummies   (foo)
17    factor_out       (foo, $\delta^{a?}$)
18    substitute       (foo, $\delta^{a} -> 1$)
19
20    return foo
21
22 # -----
23
24 poly := c^{a}
25       + c^{a}_{b} x^b
26       + c^{a}_{b c} x^b x^c.    # cdb (ex-0401.100,poly)
27
28 dpoly = deriv (poly)           # cdb (ex-0401.101,dpoly)

```

$$p = c^a + c^a_b x^b + c^a_{bc} x^b x^c \quad (\text{ex-0401.100})$$

$$dp = c^a_b + c^a_{cb} x^c + c^a_{bc} x^c \quad (\text{ex-0401.101})$$



## Exercise 4.1 Differentiate a polynomial – a better method

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 def deriv (poly):
4
5     \partial{#}::PartialDerivative.
6     \delta^{a}_{b}::KroneckerDelta.
7
8     x^{a}::Depends(\partial{#}).
9
10    bah := \partial_{b}{@(poly)}.
11
12    distribute      (bah)
13    unwrap          (bah)  # drop all terms that don't explicitly depend on a derivative operator
14    product_rule    (bah)
15    distribute      (bah)
16    substitute      (bah,$\partial_{b}{x^{a}}->\delta^{a}_{b}$)
17    eliminate_kronecker (bah)
18
19    sort_product    (bah)
20    rename_dummies  (bah)
21
22    return bah
23
24 poly := c^{a}
25       + c^{a}{}_{b} x^{b}
26       + c^{a}{}_{b c} x^{b} x^{c}.    # cdb (ex-0401.200,poly)
27
28 dpoly = deriv (poly)                # cdb (ex-0401.201,dpoly)

```

$$p = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c \quad (\text{ex-0401.200})$$

$$dp = c^a{}_b + c^a{}_{bc} x^c + c^a{}_{cb} x^c \quad (\text{ex-0401.201})$$

## Exercise 4.2 Inconsistent free indices

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 def deriv (poly):
4
5     \delta^{a}::Weight(label=\epsilon).
6
7     bah := @(poly).
8
9     substitute      (bah,$x^{a} -> x^{a} + \delta^{a}$)
10    distribute      (bah)
11
12    foo := @(bah) - @(poly).
13
14    keep_weight      (foo, $\epsilon = 1$)
15    substitute      (foo, $\delta^{a} -> 1$)
16
17    return foo
18
19 # -----
20
21 poly := c^{a}
22        + c^{a}_{b} x^b
23        + c^{a}_{b c} x^b x^c.      # cdb (ex-0402.100,poly)
24
25 dpoly = deriv (poly)              # cdb (ex-0402.101,dpoly)

```

$$p = c^a + c^a_b x^b + c^a_{bc} x^b x^c \quad (\text{ex-0402.100})$$

$$dp = c^a_b + c^a_{bc} x^b + c^a_{bc} x^c \quad (\text{ex-0402.101})$$

## Exercise 4.3 Polynomial products

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3
4 def get_term (poly,n):
5
6     x^{a}::Weight(label=xnum).      # assign weights to x^{a}
7
8     foo := @(poly).                  # make a copy of poly
9     bah = Ex("xnum = " + str(n))    # choose a target
10    keep_weight (foo,bah)             # extract the target
11
12    return foo
13
14 def poly_product (p,q,n):
15
16    pq = Ex("0")
17
18    for i in range (0,n+1):
19        for j in range (0,i+1):
20            termA = get_term (p,j)
21            termB = get_term (q,i-j)
22            termAB := @(termA) @(termB).
23            pq = pq + termAB
24
25    sort_product (pq)
26    rename_dummies (pq)
27    factor_out (pq,$x^{a?}$)
28
29    return pq
30
31 # -----
32
33 # two polynomials
34
35 polyA := c^{a}
36         + c^{a}_{b} x^b
```

```

37      + c^{a}_{b c} x^b x^c
38      + c^{a}_{b c d} x^b x^c x^d
39      + c^{a}_{b c d e} x^b x^c x^d x^e.      # cdb(ex-0403.100,polyA)
40
41 polyB := d^{f}
42      + d^{f}_{b} x^b
43      + d^{f}_{b c} x^b x^c
44      + d^{f}_{b c d} x^b x^c x^d
45      + d^{f}_{b c d e} x^b x^c x^d x^e.      # cdb(ex-0403.101,polyB)
46
47 # multiply polynomials and truncate
48
49 polyAB = poly_product (polyA,polyB,3)      # cdb(ex-0403.102,polyAB)

```

$$p = c^a + c^a_b x^b + c^a_{bc} x^b x^c + c^a_{bcd} x^b x^c x^d + c^a_{bcde} x^b x^c x^d x^e \quad (\text{ex-0403.100})$$

$$q = d^f + d^f_b x^b + d^f_{bc} x^b x^c + d^f_{bcd} x^b x^c x^d + d^f_{bcde} x^b x^c x^d x^e \quad (\text{ex-0403.101})$$

$$pq = c^a d^f + x^b (c^a d^f_b + c^a_b d^f) + x^b x^c (c^a d^f_{bc} + c^a_b d^f_c + c^a_{bc} d^f) + x^b x^c x^d (c^a d^f_{bcd} + c^a_b d^f_{cd} + c^a_{bc} d^f_d + c^a_{bcd} d^f) \quad (\text{ex-0403.102})$$

## Exercise 4.4 Reformatting simple expressions

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3
4 \nabla{#}::Derivative.
5
6 def reformat (obj,scale):
7
8     {x^{a},A_{a b},B_{a b},C_{a b},g^{a b}}::SortOrder. # choose a sort order
9
10    foo = Ex(str(scale)) # create a scale factor
11    bah := @(foo) @(obj). # apply the scale factor, clears all fractions
12
13    distribute (bah) # only required if (bah) contains brackets
14    sort_product (bah)
15    rename_dummies (bah)
16    canonicalise (bah)
17    factor_out (bah,$x^{a?}$)
18
19    ans := @(bah) / @(foo). # undo previous scaling
20
21    return ans
22
23 # -----
24
25 # a messy unformatted expression
26
27 expr := + (1/3) A_{a b} x^{a} x^{b}
28         + (1/9) B_{e c} x^{c} x^{e}
29         - (1/5) C_{p c} B_{d q} g^{c d} x^{p} x^{q}. # cdb (ex-0404.100,expr)
30
31 # reformat terms and tidy fractions
32
33 expr = reformat (expr,45) # cdb(ex-0404.101,expr)
```

$$g = \frac{1}{3}A_{ab}x^ax^b + \frac{1}{9}B_{ec}x^cx^e - \frac{1}{5}C_{pc}B_{dq}g^{cd}x^px^q \quad (\text{ex-0404.100})$$

$$= \frac{1}{45}x^ax^b(15A_{ab} + 5B_{ab} - 9B_{ca}C_{bd}g^{dc}) \quad (\text{ex-0404.101})$$

## Exercise 4.5 Reformatting complex expressions

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3
4 \nabla{#}::Derivative.
5
6 def get_term (obj,n):
7
8     x^{a}::Weight(label=xnum).      # assign weights to x^{a}
9
10    foo := @(obj).                  # make a copy of obj
11    bah = Ex("xnum = " + str(n))   # choose a target
12    keep_weight (foo,bah)           # extract the target
13
14    return foo
15
16 def reformat (obj,scale):
17
18    {x^{a},A_{a},B_{a},A_{a b},B_{a b},C_{a b},C_{a b c},g^{a b}}::SortOrder. # choose a sort order
19
20    foo = Ex(str(scale))            # create a scale factor
21    bah := @(foo) @(obj).            # apply the scale factor, clears all fractions
22
23    distribute      (bah)            # only required if (bah) contains brackets
24    sort_product    (bah)
25    rename_dummies  (bah)
26    canonicalise    (bah)
27    factor_out      (bah,$x^{a?}$)
28
29    ans := @(bah) / @(foo).          # undo previous scaling
30
31    return ans
32
33 # -----
34
35 # a messy unformatted expression
36
```

```

37  expr :=      (1/7) A_{e} x^{e}
38            -   (1/3) B_{f} x^{f}
39            +   (1/3) A_{a b} x^{a} x^{b}
40            +   (1/9) B_{e c} x^{c} x^{e}
41            -   (1/5) C_{p c} B_{d q} g^{c d} x^{p} x^{q}
42            +   (3/7) A_{a b c} x^{a} x^{b} x^{c}
43            -   (1/5) B_{a b} C_{c d e} g^{c d} x^{a} x^{b} x^{e}
44            +   (7/11) B_{a b} B_{c d} C_{e f g} g^{b c} g^{d f} x^{a} x^{e} x^{g}. # cdb (ex-0405.100,expr)
45
46  # split the expression into seprate terms
47
48  term1 = get_term (expr,1)      # cdb(term1.101,term1)
49  term2 = get_term (expr,2)      # cdb(term2.101,term2)
50  term3 = get_term (expr,3)      # cdb(term3.101,term3)
51
52  # reformat terms and tidy fractions
53
54  term1 = reformat (term1, 21)    # cdb(term1.102,term1)
55  term2 = reformat (term2, 45)    # cdb(term2.102,term2)
56  term3 = reformat (term3,385)    # cdb(term3.102,term3)
57
58  # rebuild the expression
59
60  expr := @(term1) + @(term2) + @(term3). # cdb (ex-0405.101,expr)

```



$$\begin{aligned}
g &= \frac{1}{7}A_e x^e - \frac{1}{3}B_f x^f + \frac{1}{3}A_{ab}x^a x^b + \frac{1}{9}B_{ec}x^c x^e - \frac{1}{5}C_{pc}B_{dq}g^{cd}x^p x^q + \frac{3}{7}A_{abc}x^a x^b x^c - \frac{1}{5}B_{ab}C_{cde}g^{cd}x^a x^b x^e + \frac{7}{11}B_{ab}B_{cd}C_{efg}g^{bc}g^{df}x^a x^e x^g \quad (\text{ex-0405.100}) \\
&= \frac{1}{21}x^a (3A_a - 7B_a) + \frac{1}{45}x^a x^b (15A_{ab} + 5B_{ab} - 9B_{ca}C_{bd}g^{dc}) + \frac{1}{385}x^a x^b x^c (165A_{abc} - 77B_{ab}C_{dec}g^{de} + 245B_{ad}B_{ef}C_{bge}g^{de}g^{fg}) \quad (\text{ex-0405.101})
\end{aligned}$$

## Exercise 4.6 Bespoke sort

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3
4 def bespoke_sort (expr):
5
6     substitute      (expr,$ x^{a}          -> AAA01^{a}      $)
7     substitute      (expr,$ g_{a b}        -> AAA02_{a b}    $)
8     substitute      (expr,$ \Gamma_{a b c}  -> AAA03_{a b c}    $)
9
10    sort_product    (expr)
11
12    substitute      (expr,$ AAA01^{a}      -> x^{a}          $)
13    substitute      (expr,$ AAA02_{a b}    -> g_{a b}        $)
14    substitute      (expr,$ AAA03_{a b c}  -> \Gamma_{a b c} $)
15
16    return expr
17
18 # -----
19
20 expr := g_{a b} x^{a} x^{b} + \Gamma_{a b c} x^{a} x^{b} x^{c}. # cdb(ex-0406.100,expr)
21
22 expr = bespoke_sort (expr)                                     # cdb(ex-0406.101,expr)

```

$$p = g_{ab}x^ax^b + \Gamma_{abc}x^ax^bx^c \quad (\text{ex-0406.100})$$

$$= x^ax^bg_{ab} + x^ax^bx^c\Gamma_{abc} \quad (\text{ex-0406.101})$$

## Exercise 4.7 Return in functions

```
1 {a,b,c,d,e,f,g,h,i,j,k,l#}::Indices(position=independent).
2
3 # -----
4 # this function uses in-place changes for obj
5
6 def tidy (obj):
7
8     sort_product    (obj)
9     rename_dummies  (obj)
10    canonicalise     (obj)
11
12    foo := C^{f} B^{a} A_{f a}.           # cdb (ex-0407.101,foo)
13    tidy (foo)                             # cdb (ex-0407.102,foo)
14
15 # -----
16 # this function creates new objects,
17 # it will not give the correct result
18
19 def tidy (obj):
20
21    bah := @(obj).
22
23    sort_product    (bah)
24    rename_dummies  (bah)
25    canonicalise     (bah)
26
27    obj := @(bah).
28
29    foo := C^{f} B^{a} A_{f a}.           # cdb (ex-0407.201,foo)
30    tidy (foo)                             # cdb (ex-0407.202,foo)
31
32 # -----
33 # this function uses a return statement
34 # it will give the correct result
35
36 def tidy (obj):
```

```

37
38   bah := @(obj).
39
40   sort_product   (bah)
41   rename_dummies (bah)
42   canonicalise   (bah)
43
44   obj := @(bah).
45
46   return obj
47
48   foo := C^{f} B^a A_{fa} A_{f a}.           # cdb (ex-0407.301,foo)
49   foo = tidy (foo)                          # cdb (ex-0407.302,foo)

```

$$C^f B^a A_{fa} = A_{ab} B^b C^a \quad (\text{ex-0407.102})$$

$$C^f B^a A_{fa} = C^f B^a A_{fa} \quad (\text{ex-0407.202})$$

$$C^f B^a A_{fa} = A_{ab} B^b C^a \quad (\text{ex-0407.302})$$

## Exercise 5.1 Swap terms

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 expr := A_{a} (P^{b}+Q^{b}) + C_{a} V^{b}. # cdb (ex-0501.100,expr)
4
5 substitute (expr, $A_{a} B?? + C_{a} D?? -> A_{a} D?? + C_{a} B??$) # cdb (ex-0501.101,expr)
```

$$\text{ex-0501.100} := A_a (P^b + Q^b) + C_a V^b$$

$$\text{ex-0501.101} := A_a V^b + C_a (P^b + Q^b)$$

## Exercise 5.2 Leading factors forbidden in patterns

This exercise will raise a Cadabra run-time error – the scale factor on the left hand side of the rule (3 in this case) is not allowed.

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 expr := 2 V_{a b} - 3 V_{b a}. # cdb (ex-0502.100,expr)
4
5 substitute (expr, $3 V_{b a} -> - 3 V_{a b}$) # cdb (ex-0502.101,expr)
```

Traceback (most recent call last):

File "/usr/local/bin/cadabra2", line 248, in <module>

exec(cmp)

File "ex-0502.py", line 18, in <module>

substitute (expr, Ex(r'''3 V\_{b a} -> - 3 V\_{a b}''', False))

RuntimeError: substitute: Index error in replacement rule.

substitute: No numerical pre-factors allowed on lhs of replacement rule.

## Exercise 5.3 Deleting a term using patterns

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 expr := A_{a b} B^{a b} + A_{a b} A_{c d} B^{a b} B^{c d} - C_{a b} B^{a b}. # cdb (ex-0503.100,expr)
4
5 zoom      (expr, $A_{a b} A_{c d} Q???) # cdb (ex-0503.101,expr)
6 substitute (expr, $A_{a b} -> 0$) # cdb (ex-0503.102,expr)
7 unzoom    (expr) # cdb (ex-0503.103,expr)
```

$$\text{ex-0503.100} := A_{ab}B^{ab} + A_{ab}A_{cd}B^{ab}B^{cd} - C_{ab}B^{ab}$$

$$\text{ex-0503.101} := \dots + A_{ab}A_{cd}B^{ab}B^{cd} + \dots$$

$$\text{ex-0503.102} := \dots$$

$$\text{ex-0503.103} := A_{ab}B^{ab} - C_{ab}B^{ab}$$

## Exercise 5.4 Deleting a term using tags

```

1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  def add_tags (obj,tag):
4      n = 0
5      ans = Ex('0')
6      for i in obj.top().terms():
7          foo = obj[i]
8          bah = Ex(tag+'_'+str(n)+'')
9          ans := @(ans) + @(bah) @(foo).
10         n = n + 1
11     return ans
12
13 def clear_tags (obj,tag):
14     ans := @(obj).
15     foo = Ex(tag+'_{a?} -> 1')
16     substitute (ans,foo)
17     return ans
18
19 expr := A_{a b} B^{a b} + A_{a b} A_{c d} B^{a b} B^{c d} - C_{a b} B^{a b}. # cdb (ex-0504.100,expr)
20
21 expr = add_tags (expr,'\mu') # cdb (ex-0504.101,expr)
22
23 substitute (expr, $\mu_{1} -> 0$) # cdb (ex-0504.102,expr)
24
25 expr = clear_tags (expr,'\mu') # cdb (ex-0504.103,expr)

```

$$\text{ex-0504.100} := A_{ab}B^{ab} + A_{ab}A_{cd}B^{ab}B^{cd} - C_{ab}B^{ab}$$

$$\text{ex-0504.101} := \mu_0 A_{ab}B^{ab} + \mu_1 A_{ab}A_{cd}B^{ab}B^{cd} - \mu_2 C_{ab}B^{ab}$$

$$\text{ex-0504.102} := \mu_0 A_{ab}B^{ab} - \mu_2 C_{ab}B^{ab}$$

$$\text{ex-0504.103} := A_{ab}B^{ab} - C_{ab}B^{ab}$$



## Exercise 5.5 Commuting covariant derivatives

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 ::Symbol.
4
5 def add_tags (obj,tag):
6     n = 0
7     ans = Ex('0')
8     for i in obj.top().terms():
9         foo = obj[i]
10        bah = Ex(tag+'_'+str(n)+'')
11        ans := @ (ans) + @ (bah) @ (foo).
12        n = n + 1
13    return ans
14
15 def clear_tags (obj,tag):
16     ans := @ (obj).
17     foo = Ex(tag+'_{a?} -> 1')
18     substitute (ans,foo)
19     return ans
20
21 rule := V^{a}_{; b ; c} -> V^{a}_{; c ; b} - R^{a}_{d b c} V^{d}.
22
23 expr := V^{a}_{; b ; c} - V^{a}_{; c ; b}. # cdb (ex-0505.100,expr)
24
25 expr = add_tags (expr,'\mu') # cdb (ex-0505.101,expr)
26
27 zoom (expr, $\mu_{0} Q??$) # cdb (ex-0505.102,expr)
28 substitute (expr, rule) # cdb (ex-0505.103,expr)
29 unzoom (expr) # cdb (ex-0505.104,expr)
30
31 expr = clear_tags (expr,'\mu') # cdb (ex-0505.105,expr)
```

$$\begin{aligned}
V^a{}_{;b;c} - V^a{}_{;c;b} &= \mu_0 V^a{}_{;b;c} - \mu_1 V^a{}_{;c;b} && (\text{ex-0505.101}) \\
&= \mu_0 V^a{}_{;b;c} - \mu_1 V^a{}_{;c;b} && (\text{ex-0505.101}) \\
&= \mu_0 V^a{}_{;b;c} + \dots && (\text{ex-0505.102}) \\
&= \mu_0 (V^a{}_{;c;b} - R^a{}_{dbc} V^d) + \dots && (\text{ex-0505.103}) \\
&= \mu_0 (V^a{}_{;c;b} - R^a{}_{dbc} V^d) - \mu_1 V^a{}_{;c;b} && (\text{ex-0505.104}) \\
&= -R^a{}_{dbc} V^d && (\text{ex-0505.105})
\end{aligned}$$

## Exercise 6.1 Evaluate – without rhsonly = True

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  V := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }. # cdb(ex-0601.100,V)
7  dV := dV_{a b} -> \partial_{b}{V_{a}} - \partial_{a}{V_{b}}. # cdb(ex-0601.101,dV)
8
9  evaluate (dV, V) # cdb(ex-0601.102,dV)

```

Notice how `evaluate` has been applied to both the left and right hand sides of the rule.

$$V_a = [V_\theta = \varphi, V_\varphi = \sin \theta] \quad (\text{ex-0601.100})$$

$$dV_{ab} \rightarrow \partial_b V_a - \partial_a V_b \quad (\text{ex-0601.101})$$

$$\square_{ab} \begin{cases} \square_{\theta\theta} = dV_{\theta\theta} \\ \square_{\varphi\theta} = dV_{\varphi\theta} \\ \square_{\theta\varphi} = dV_{\theta\varphi} \\ \square_{\varphi\varphi} = dV_{\varphi\varphi} \end{cases} \rightarrow \square_{ab} \begin{cases} \square_{\varphi\theta} = \cos \theta - 1 \\ \square_{\theta\varphi} = -\cos \theta + 1 \end{cases} \quad (\text{ex-0601.102})$$

## Exercise 6.1 Evaluate – with rhsonly = True

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  V := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }. # cdb(ex-0601.200,V)
7  dV := dV_{a b} -> \partial_{b}{V_{a}} - \partial_{a}{V_{b}}. # cdb(ex-0601.201,dV)
8
9  evaluate (dV, V, rhsonly=True) # cdb(ex-0601.202,dV)

```

This is an improvement, only the right hnd side has been expanded into components.

$$V_a = [V_\theta = \varphi, V_\varphi = \sin \theta] \quad (\text{ex-0601.200})$$

$$dV_{ab} \rightarrow \partial_b V_a - \partial_a V_b \quad (\text{ex-0601.201})$$

$$dV_{ab} \rightarrow \square_{ab} \begin{cases} \square_{\varphi\theta} = \cos \theta - 1 \\ \square_{\theta\varphi} = -\cos \theta + 1 \end{cases} \quad (\text{ex-0601.202})$$

## Exercise 6.2 Evaluate on an expression (not a rule)

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  V := { V_{\theta} = f(\theta,\varphi), V_{\varphi} = g(\theta,\varphi) }. # cdb(ex-0602.100,V)
7  dV := \partial_{b}{V_{a}} + \partial_{a}{V_{b}}. # cdb(ex-0602.101,dV)
8
9  evaluate (dV, V) # cdb(ex-0602.102,dV)

```

$$V_a = [V_\theta = f(\theta, \varphi), V_\varphi = g(\theta, \varphi)] \quad (\text{ex-0602.100})$$

$$\partial_b V_a + \partial_a V_b \quad (\text{ex-0602.101})$$

$$\square_{ab} \begin{cases} \square_{\varphi\varphi} = 2\partial_\varphi g(\theta, \varphi) \\ \square_{\varphi\theta} = \partial_\varphi f(\theta, \varphi) + \partial_\theta g(\theta, \varphi) \\ \square_{\theta\varphi} = \partial_\varphi f(\theta, \varphi) + \partial_\theta g(\theta, \varphi) \\ \square_{\theta\theta} = 2\partial_\theta f(\theta, \varphi) \end{cases} \quad (\text{ex-0602.102})$$

## Exercise 6.3 Evaluate with undefined components

```
1  {\theta, \varphi}::Coordinate.  
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).  
3  
4  bah := {V_{\theta} = \varphi, V_{\varphi} = \sin(\theta)}. # cdb(ex-0603.100,bah)  
5  foo := U_{a} V_{b}. # cdb(ex-0603.101,foo)  
6  
7  evaluate (foo, bah) # cdb(ex-0603.102,foo)
```

$$[V_{\theta} = \varphi, V_{\varphi} = \sin \theta] \quad (\text{ex-0603.100})$$

$$U_a V_b \quad (\text{ex-0603.101})$$

$$\square_{ab} \begin{cases} \square_{\theta\theta} = \varphi U_{\theta} \\ \square_{\theta\varphi} = U_{\theta} \sin \theta \\ \square_{\varphi\theta} = \varphi U_{\varphi} \\ \square_{\varphi\varphi} = U_{\varphi} \sin \theta \end{cases} \quad (\text{ex-0603.102})$$

## Exercise 6.4 Scalar curavture of a 2-sphere

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)
7
8  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}}).
11
12  Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
13                        - \partial_{d}{\Gamma^{a}_{b c}}
14                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17  Rab := R_{a b} -> R^{c}_{c}_{a b}.
18
19  R := R -> R_{a b} g^{a b}.
20
21  gab := { g_{\theta\theta} = r**2,
22          g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.  # cdb(ex-0604.101,gab)
23
24  complete (gab, $g^{a b}$)  # cdb(ex-0604.102,gab)
25
26  substitute (Rabcd, Gamma)
27  substitute (Rab, Rabcd)
28  substitute (R, Rab)
29
30  evaluate (Gamma, gab, rhsonly=True)  # cdb(ex-0604.103,Gamma)
31  evaluate (Rabcd, gab, rhsonly=True)  # cdb(ex-0604.104,Rabcd)
32  evaluate (Rab, gab, rhsonly=True)  # cdb(ex-0604.105,Rab)
33  evaluate (R, gab, rhsonly=True)  # cdb(ex-0604.106,R)

```

$$[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin \theta)^2] \quad (\text{ex-0604.101})$$

$$\left[ g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin \theta)^2, \ g^{\theta\theta} = r^{-2}, \ g^{\varphi\varphi} = (r^2(\sin \theta)^2)^{-1} \right] \quad (\text{ex-0604.102})$$

$$\Gamma^a_{bc} \rightarrow \square_{cb}{}^a \begin{cases} \square_{\varphi\theta}{}^\varphi = (\tan \theta)^{-1} \\ \square_{\theta\varphi}{}^\varphi = (\tan \theta)^{-1} \\ \square_{\varphi\varphi}{}^\theta = -\frac{1}{2} \sin(2\theta) \end{cases} \quad (\text{ex-0604.103})$$

$$R^a_{bcd} \rightarrow \square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^\theta{}_\theta = (\sin \theta)^2 \\ \square_{\varphi\theta}{}^\varphi{}_\theta = -1 \\ \square_{\theta\varphi}{}^\theta{}_\varphi = -(\sin \theta)^2 \\ \square_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \quad (\text{ex-0604.104})$$

$$R_{ab} \rightarrow \square_{ba} \begin{cases} \square_{\varphi\varphi} = (\sin \theta)^2 \\ \square_{\theta\theta} = 1 \end{cases} \quad (\text{ex-0604.105})$$

$$R \rightarrow 2r^{-2} \quad (\text{ex-0604.106})$$



## Exercise 6.5 Schwarzschild spacetime in isotropic coordinates

```

1 {t, r, \theta, \varphi}::Coordinate.
2 {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4 \partial{#}::PartialDerivative.
5
6 g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
7
8 Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}}).
11
12 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
13                        - \partial_{d}{\Gamma^{a}_{b c}}
14                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17 Rab := R_{a b} -> R^{c}_{c a b}.
18
19 gab := { g_{t t}          = -((2*r-m)/(2*r+m))**2,
20         g_{r r}          = (1+m/(2*r))**4,
21         g_{\theta\theta}   = r**2 (1+m/(2*r))**4,
22         g_{\varphi\varphi} = r**2 \sin(\theta)**2 (1+m/(2*r))**4}. # cdb(ex-0605.101,gab)
23
24 complete (gab, $g^{a b}$) # cdb(ex-0605.102,gab)
25
26 substitute (Rabcd, Gamma)
27 substitute (Rab, Rabcd)
28
29 evaluate (Gamma, gab, rhsonly=True) # cdb(ex-0605.103,Gamma)
30 evaluate (Rabcd, gab, rhsonly=True) # cdb(ex-0605.104,Rabcd)
31 evaluate (Rab, gab, rhsonly=True) # cdb(ex-0605.105,Rab)

```

$$\left[ g_{tt} = -((2r - m)(2r + m)^{-1})^2, \quad g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g_{\theta\theta} = r^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1 + \frac{1}{2}mr^{-1}\right)^4 \right] \quad (\text{ex-0605.101})$$

$$\left[ g_{tt} = -((2r - m)(2r + m)^{-1})^2, \quad g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g_{\theta\theta} = r^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g^{tt} = - (m + 2r)^2(-m + 2r)^{-2}, \quad g^{rr} = \left(\frac{1}{2}mr^{-1} + 1\right)^{-4}, \quad g^{\theta\theta} = \left(r^2\left(\frac{1}{2}mr^{-1} + 1\right)^4\right)^{-1}, \quad g^{\varphi\varphi} = \left(r^2\left(\frac{1}{2}mr^{-1} + 1\right)^4(\sin\theta)^2\right)^{-1} \right] \quad (\text{ex-0605.102})$$

$$\Gamma^a_{bc} \rightarrow \square_{cb}^a \left\{ \begin{array}{l} \square_{\varphi r}^{\varphi} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{\varphi\theta}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\theta r}^{\theta} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{rr}^r = -2m(r(m + 2r))^{-1} \\ \square_{tr}^t = 4m(-m^2 + 4r^2)^{-1} \\ \square_{r\varphi}^{\varphi} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan\theta)^{-1} \\ \square_{r\theta}^{\theta} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{rt}^t = 4m(-m^2 + 4r^2)^{-1} \\ \square_{\varphi\varphi}^r = r(m - 2r)(\sin\theta)^2(m + 2r)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2}\sin(2\theta) \\ \square_{\theta\theta}^r = r(m - 2r)(m + 2r)^{-1} \\ \square_{tt}^r = -64mr^4(m - 2r)(m + 2r)^{-7} \end{array} \right. \quad (\text{ex-0605.103})$$

$$R^a{}_{bcd} \rightarrow \square_{db}{}^a{}_c \left\{ \begin{array}{l} \square_{tt}{}^r{}_r = -128m^3r^3(m+2r)^{-8} + 512m^2r^4(m+2r)^{-8} - 512mr^5(m+2r)^{-8} \\ \square_{\theta\theta}{}^r{}_r = -4mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\varphi\varphi}{}^\theta{}_\theta = 8mr(\sin\theta)^2(m+2r)^{-2} \\ \square_{\varphi\varphi}{}^r{}_r = -4mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{tr}{}^t{}_r = -8m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\theta r}{}^\theta{}_r = 4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\varphi\theta}{}^\varphi{}_\theta = (m-2r)^2(m+2r)^{-2} - 1 \\ \square_{\varphi r}{}^\varphi{}_r = 4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{rt}{}^r{}_t = 128m^3r^3(m+2r)^{-8} - 512m^2r^4(m+2r)^{-8} + 512mr^5(m+2r)^{-8} \\ \square_{r\theta}{}^r{}_r = 4mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\theta\varphi}{}^\theta{}_\varphi = (m-2r)^2(\sin\theta)^2(m+2r)^{-2} - (\sin\theta)^2 \\ \square_{r\varphi}{}^r{}_r = 4mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{rr}{}^t{}_t = 8m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{rr}{}^\theta{}_\theta = -4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\theta\theta}{}^\varphi{}_\varphi = 8mr(m+2r)^{-2} \\ \square_{rr}{}^\varphi{}_\varphi = -4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\varphi\varphi}{}^t{}_t = -4mr(\sin\theta)^2(m+2r)^{-2} \\ \square_{\theta\theta}{}^t{}_t = -4mr(m+2r)^{-2} \\ \square_{tt}{}^\varphi{}_\varphi = 64mr^3(m-2r)^2(m+2r)^{-8} \\ \square_{tt}{}^\theta{}_\theta = 64mr^3(m-2r)^2(m+2r)^{-8} \\ \square_{t\varphi}{}^t{}_\varphi = 4mr(\sin\theta)^2(m+2r)^{-2} \\ \square_{t\theta}{}^t{}_\theta = 4mr(m+2r)^{-2} \\ \square_{\varphi t}{}^\varphi{}_t = -64mr^3(m-2r)^2(m+2r)^{-8} \\ \square_{\theta t}{}^\theta{}_t = -64mr^3(m-2r)^2(m+2r)^{-8} \end{array} \right. \quad (\text{ex-0605.104})$$

$$R_{ab} \rightarrow 0 \quad (\text{ex-0605.105})$$

## Exercise 6.6 The Kasner cosmology

```
1 {t, x, y, z}::Coordinate.
2 {a,b,c,d,e,f,g,h#}::Indices(values={t, x, y, z}, position=independent).
3
4 \partial{#}::PartialDerivative.
5
6 p1::LaTeXForm("p_1").
7 p2::LaTeXForm("p_2").
8 p3::LaTeXForm("p_3").
9
10 g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
11
12 Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
13                                     + \partial_{c}{g_{b d}}
14                                     - \partial_{d}{g_{b c}}).
15
16 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
17                             - \partial_{d}{\Gamma^{a}_{b c}}
18                             + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
19                             - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
20
21 Rab := R_{a b} -> R^{c}_{c}_{a b}.
22
23 gab := { g_{t t} = -1,
24          g_{x x} = t**(2*p1),
25          g_{y y} = t**(2*p2),
26          g_{z z} = t**(2*p3)}. # cdb(ex-0606.101,gab)
27
28 complete (gab, $g^{a b}$) # cdb(ex-0606.102,gab)
29
30 substitute (Rabcd, Gamma)
31 substitute (Rab, Rabcd)
32
33 evaluate (Gamma, gab, rhsonly=True) # cdb(ex-0606.103,Gamma)
34 evaluate (Rabcd, gab, rhsonly=True) # cdb(ex-0606.104,Rabcd)
35 evaluate (Rab, gab, rhsonly=True) # cdb(ex-0606.105,Rab)
```

$$[g_{tt} = -1, g_{xx} = t^{2p_1}, g_{yy} = t^{2p_2}, g_{zz} = t^{2p_3}] \quad (\text{ex-0606.101})$$

$$[g_{tt} = -1, g_{xx} = t^{2p_1}, g_{yy} = t^{2p_2}, g_{zz} = t^{2p_3}, g^{tt} = -1, g^{xx} = t^{-2p_1}, g^{yy} = t^{-2p_2}, g^{zz} = t^{-2p_3}] \quad (\text{ex-0606.102})$$

$$\Gamma^a_{bc} \rightarrow \square_{cb}^a \left\{ \begin{array}{l} \square_{zt}^z = p_3 t^{-1} \\ \square_{yt}^y = p_2 t^{-1} \\ \square_{xt}^x = p_1 t^{-1} \\ \square_{tz}^z = p_3 t^{-1} \\ \square_{ty}^y = p_2 t^{-1} \\ \square_{tx}^x = p_1 t^{-1} \\ \square_{zz}^t = p_3 t^{(2p_3-1)} \\ \square_{yy}^t = p_2 t^{(2p_2-1)} \\ \square_{xx}^t = p_1 t^{(2p_1-1)} \end{array} \right. \quad (\text{ex-0606.103})$$

$$R^a{}_{bcd} \rightarrow \square_{db}{}^a{}_c \left\{ \begin{array}{l} \square_{xx}{}^t{}_t = p_1 t^{(2p_1-2)} (p_1 - 1) \\ \square_{yy}{}^t{}_t = p_2 t^{(2p_2-2)} (p_2 - 1) \\ \square_{zz}{}^t{}_t = p_3 t^{(2p_3-2)} (p_3 - 1) \\ \square_{xt}{}^x{}_t = p_1 (p_1 - 1) t^{-2} \\ \square_{yt}{}^y{}_t = p_2 (p_2 - 1) t^{-2} \\ \square_{zt}{}^z{}_t = p_3 (p_3 - 1) t^{-2} \\ \square_{tx}{}^t{}_x = -p_1 t^{(2p_1-2)} (p_1 - 1) \\ \square_{ty}{}^t{}_y = -p_2 t^{(2p_2-2)} (p_2 - 1) \\ \square_{tz}{}^t{}_z = -p_3 t^{(2p_3-2)} (p_3 - 1) \\ \square_{tt}{}^x{}_x = p_1 (-p_1 + 1) t^{-2} \\ \square_{tt}{}^y{}_y = p_2 (-p_2 + 1) t^{-2} \\ \square_{tt}{}^z{}_z = p_3 (-p_3 + 1) t^{-2} \\ \square_{zz}{}^y{}_y = p_2 p_3 t^{(2p_3-2)} \\ \square_{zz}{}^x{}_x = p_1 p_3 t^{(2p_3-2)} \\ \square_{yy}{}^z{}_z = p_2 p_3 t^{(2p_2-2)} \\ \square_{yy}{}^x{}_x = p_1 p_2 t^{(2p_2-2)} \\ \square_{xx}{}^z{}_z = p_1 p_3 t^{(2p_1-2)} \\ \square_{xx}{}^y{}_y = p_1 p_2 t^{(2p_1-2)} \\ \square_{yz}{}^y{}_z = -p_2 p_3 t^{(2p_3-2)} \\ \square_{xz}{}^x{}_z = -p_1 p_3 t^{(2p_3-2)} \\ \square_{zy}{}^z{}_y = -p_2 p_3 t^{(2p_2-2)} \\ \square_{xy}{}^x{}_y = -p_1 p_2 t^{(2p_2-2)} \\ \square_{zx}{}^z{}_x = -p_1 p_3 t^{(2p_1-2)} \\ \square_{yx}{}^y{}_x = -p_1 p_2 t^{(2p_1-2)} \end{array} \right. \quad (\text{ex-0606.104})$$

$$R_{ab} \rightarrow \square_{ba} \left\{ \begin{array}{l} \square_{xx} = p_1 t^{(2p_1-2)} (p_1 + p_2 + p_3 - 1) \\ \square_{yy} = p_2 t^{(2p_2-2)} (p_1 + p_2 + p_3 - 1) \\ \square_{zz} = p_3 t^{(2p_3-2)} (p_1 + p_2 + p_3 - 1) \\ \square_{tt} = (-p_1^2 + p_1 - p_2^2 + p_2 - p_3^2 + p_3) t^{-2} \end{array} \right. \quad (\text{ex-0606.105})$$

## Exercise 6.7 Killing vectors of the Schwarzschild spacetime

```

1 {t, r, \theta, \varphi}::Coordinate.
2 {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4 ::Symbol.
5
6 \partial{#}::PartialDerivative.
7
8 g_{a b}::Metric.
9 g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
10
11 Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} ( \partial_{g}\{g_{b f}\}
12                                     + \partial_{f}\{g_{b g}\}
13                                     - \partial_{b}\{g_{f g}\} ).
14
15 deriv := \xi_{a ; b} -> \partial_{b}\{\xi_{a}\} - \Gamma^{c}_{a b} \xi_{c}.
16 lower := \xi_{a} -> g_{a b} \xi^{b}.
17
18 expr := \xi_{a ; b} + \xi_{b ; a}. # cdb(ex-0607.100,expr)
19
20 substitute (expr, deriv) # cdb(ex-0607.101,expr)
21 substitute (expr, lower) # cdb(ex-0607.102,expr)
22 substitute (expr, Gamma) # cdb(ex-0607.103,expr)
23 distribute (expr) # cdb(ex-0607.104,expr)
24 product_rule (expr) # cdb(ex-0607.105,expr)
25 canonicalise (expr) # cdb(ex-0607.106,expr)
26
27 # choose a vector
28
29 # Kvect := {\xi^{t} = 1}.
30 # Kvect := {\xi^{\varphi} = 1}.
31 Kvect := {\xi^{\theta} = \sin(\varphi), \xi^{\varphi} = \cos(\theta)/\sin(\theta) \cos(\varphi)}.
32 # Kvect := {\xi^{\theta} = \cos(\varphi), \xi^{\varphi} = - \cos(\theta)/\sin(\theta) \sin(\varphi)}.
33 # cdb(ex-0607.107,Kvect)
34
35 gab := { g_{t t} = -(1-2*m/r),
36          g_{r r} = 1/(1-(2*m/r)),

```

```

37      g_{\theta\theta} = r**2,
38      g_{\varphi\varphi} = r**2 \sin(\theta)**2}. # cdb(ex-0607.108,gab)
39
40 complete (gab, $g^{a b}$) # cdb(ex-0607.109,gab)
41
42 evaluate (expr, gab+Kvect) # cdb(ex-0607.110,expr)

```

$$[\xi^a] = [\xi^\theta = \sin(\varphi), \xi^\varphi = \cos\theta(\sin\theta)^{-1}\cos(\varphi)] \quad (\text{ex-0607.107})$$

$$[g_{ab}] = [g_{tt} = -1 + 2mr^{-1}, g_{rr} = (1 - 2mr^{-1})^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2] \quad (\text{ex-0607.108})$$

$$[g_{ab}, g^{ab}] = [g_{tt} = -1 + 2mr^{-1}, g_{rr} = (1 - 2mr^{-1})^{-1}, g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2(\sin\theta)^2, g^{tt} = (2mr^{-1} - 1)^{-1}, g^{rr} = -2mr^{-1} + 1, g^{\theta\theta} = r^{-2}, g^{\varphi\varphi} = (r^2(\sin\theta)^2)^{-1}] \quad (\text{ex-0607.109})$$

$$\xi_{a;b} + \xi_{b;a} = \partial_b \xi_a - \Gamma_{ab}^c \xi_c + \partial_a \xi_b - \Gamma_{ba}^c \xi_c \quad (\text{ex-0607.101})$$

$$= \partial_b (g_{ac} \xi^c) - \Gamma_{ab}^c g_{cd} \xi^d + \partial_a (g_{bc} \xi^c) - \Gamma_{ba}^c g_{cd} \xi^d \quad (\text{ex-0607.102})$$

$$= \partial_b (g_{ac} \xi^c) - \frac{1}{2} g^{ce} (\partial_b g_{ea} + \partial_a g_{eb} - \partial_e g_{ab}) g_{cd} \xi^d + \partial_a (g_{bc} \xi^c) - \frac{1}{2} g^{ce} (\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ba}) g_{cd} \xi^d \quad (\text{ex-0607.103})$$

$$= \partial_b (g_{ac} \xi^c) - g^{ce} \partial_b g_{ea} g_{cd} \xi^d - g^{ce} \partial_a g_{eb} g_{cd} \xi^d + \frac{1}{2} g^{ce} \partial_e g_{ab} g_{cd} \xi^d + \partial_a (g_{bc} \xi^c) + \frac{1}{2} g^{ce} \partial_e g_{ba} g_{cd} \xi^d \quad (\text{ex-0607.104})$$

$$= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{ce} \partial_b g_{ea} g_{cd} \xi^d - g^{ce} \partial_a g_{eb} g_{cd} \xi^d + \frac{1}{2} g^{ce} \partial_e g_{ab} g_{cd} \xi^d + \partial_a g_{bc} \xi^c + g_{bc} \partial_a \xi^c + \frac{1}{2} g^{ce} \partial_e g_{ba} g_{cd} \xi^d \quad (\text{ex-0607.105})$$

$$= \partial_b g_{ac} \xi^c + g_{ac} \partial_b \xi^c - g^{cd} \partial_b g_{ac} g_{de} \xi^e - g^{cd} \partial_a g_{bc} g_{de} \xi^e + g^{cd} \partial_c g_{ab} g_{de} \xi^e + \partial_a g_{bc} \xi^c + g_{bc} \partial_a \xi^c \quad (\text{ex-0607.106})$$

$$= 0 \quad (\text{ex-0607.110})$$



## Exercise 6.08a A problem with evaluate

```
1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  V_{a}::Depends(\theta,\varphi,\partial{#}).
7
8  dVrule := { \partial_{\theta}V_{\varphi} = \sin(\theta),
9              \partial_{\varphi}V_{\theta} = \cos(\theta)}. # cdb(ex-0608.101,dVrule)
10 dV := \partial_bV_a - \partial_aV_b. # cdb(ex-0608.102,dV)
11
12 evaluate (dV, dVrule) # cdb(ex-0608.103,dV)
```

Traceback (most recent call last):

File "/usr/local/bin/cadabra2", line 248, in <module>

exec(cmp)

File "ex-0608.py", line 27, in <module>

evaluate (dV, dVrule)

RuntimeError: Dependencies on derivatives are not yet handled in the SymPy bridge

## Exercise 6.08b A work around

```

1  {\theta, \varphi}::Coordinate.
2  {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4  \partial{#}::PartialDerivative.
5
6  V_{a}::Depends(\theta,\varphi,\partial{#}).
7
8  hide := \partial_{a}{V_{b}} -> dV_{a b}.
9
10 dVrule := { dV_{\theta\varphi} = \sin(\theta),
11             dV_{\varphi\theta} = \cos(\theta)}.      # cdb(ex-0608.201,dVrule)
12 dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.        # cdb(ex-0608.202,dV)
13
14 substitute (dV, hide)                            # cdb(ex-0608.212,dV)
15 evaluate (dV, dVrule)                            # cdb(ex-0608.203,dV)

```

The workaround here is to to hide the derivatives before calling `evaluate`.

$$dV_{ba} - dV_{ab} \quad (\text{ex-0608.212})$$

$$dV_{ab} = \partial_b V_a - \partial_a V_b \quad (\text{ex-0608.202})$$

$$= \square_{ab} \begin{cases} \square_{\varphi\theta} = \sin \theta - \cos \theta \\ \square_{\theta\varphi} = -\sin \theta + \cos \theta \end{cases} \quad (\text{ex-0608.203})$$

## Exercise 7.1 C-code for a $R_{ab}$ for a generic metric

```
1 {x,y,z}::Coordinate.
2 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(values={x,y,z},position=independent).
3
4 \partial{#}::PartialDerivative.
5
6 g_{a b}::Metric.
7 g^{a b}::InverseMetric.
8
9 import cdblib
10
11 FourRab = cdblib.get ('FourRab','ex-0309.json')
12
13 Rab := 1/4 @(FourRab).
14
15 substitute (Rab, $ \partial_{a b}{g_{c d}} -> dg_{c d a b} $)
16 substitute (Rab, $ \partial_a{g_{b c}} -> dg_{b c a} $)
17
18 # -----
19 # build rules to export Cadabra expressions to Python
20 # use known symmetries for g_{a b}, dg_{ab,c,d} etc.
21 # note: replacements must not contain underscores (reserved for subscripts),
22 #      so g_{x x}-> g_xx is not allowed
23
24 gabRule := {g_{x x} -> gxx, g_{x y} -> gxy, g_{x z} -> gxz,
25             g_{y x} -> gxy, g_{y y} -> gyy, g_{y z} -> gyz,
26             g_{z x} -> gxz, g_{z y} -> gyz, g_{z z} -> gzz}.
27
28 iabRule := {g^{x x} -> ixx, g^{x y} -> ixy, g^{x z} -> ixz,
29             g^{y x} -> ixy, g^{y y} -> iyy, g^{y z} -> iyz,
30             g^{z x} -> ixz, g^{z y} -> iyz, g^{z z} -> izz}.
31
32 d1gabRule := {dg_{x x x} -> dgxxx, dg_{x y x} -> dgxyx, dg_{x z x} -> dgxzx,
33              dg_{y x x} -> dgxyx, dg_{y y x} -> dgyyx, dg_{y z x} -> dgyzx,
34              dg_{z x x} -> dgxzx, dg_{z y x} -> dgyzx, dg_{z z x} -> dgzzx,
35
36              dg_{x x y} -> dgxxy, dg_{x y y} -> dgxyy, dg_{x z y} -> dgxzy,
```

```

37         dg_{y x y} -> dgxyy, dg_{y y y} -> dgyyy, dg_{y z y} -> dgyzy,
38         dg_{z x y} -> dgxzy, dg_{z y y} -> dgyzy, dg_{z z y} -> dgzzy,
39
40         dg_{x x z} -> dgxxz, dg_{x y z} -> dgxyz, dg_{x z z} -> dgxzz,
41         dg_{y x z} -> dgxyz, dg_{y y z} -> dgyyz, dg_{y z z} -> dgyzz,
42         dg_{z x z} -> dgxzz, dg_{z y z} -> dgyzz, dg_{z z z} -> dgzzz}.
43
44     d2gabRule := {dg_{x x x x} -> dgxxxx, dg_{x y x x} -> dgxyxx, dg_{x z x x} -> dgxxxx,
45         dg_{y x x x} -> dgxyxx, dg_{y y x x} -> dgyyxx, dg_{y z x x} -> dgyzxx,
46         dg_{z x x x} -> dgxzzx, dg_{z y x x} -> dgyzxx, dg_{z z x x} -> dgzzxx,
47         dg_{x x y x} -> dgxxxy, dg_{x y y x} -> dgxyxy, dg_{x z y x} -> dgxzxy,
48         dg_{y x y x} -> dgxyxy, dg_{y y y x} -> dgyyxy, dg_{y z y x} -> dgyzxy,
49         dg_{z x y x} -> dgxzxy, dg_{z y y x} -> dgyzxy, dg_{z z y x} -> dgzzxy,
50         dg_{x x z x} -> dgxxxz, dg_{x y z x} -> dgxyxz, dg_{x z z x} -> dgxzzx,
51         dg_{y x z x} -> dgxyxz, dg_{y y z x} -> dgyyxz, dg_{y z z x} -> dgyzzx,
52         dg_{z x z x} -> dgxzzx, dg_{z y z x} -> dgyzzx, dg_{z z z x} -> dgzzxz,
53
54         dg_{x x x y} -> dgxxxy, dg_{x y x y} -> dgxyxy, dg_{x z x y} -> dgxzxy,
55         dg_{y x x y} -> dgxyxy, dg_{y y x y} -> dgyyxy, dg_{y z x y} -> dgyzxy,
56         dg_{z x x y} -> dgxzxy, dg_{z y x y} -> dgyzxy, dg_{z z x y} -> dgzzxy,
57         dg_{x x y y} -> dgxxyy, dg_{x y y y} -> dgxyyy, dg_{x z y y} -> dgxzyy,
58         dg_{y x y y} -> dgxyyy, dg_{y y y y} -> dgyyyy, dg_{y z y y} -> dgyzyy,
59         dg_{z x y y} -> dgxzyy, dg_{z y y y} -> dgyzyy, dg_{z z y y} -> dgzzyy,
60         dg_{x x z y} -> dgxxyz, dg_{x y z y} -> dgxyyz, dg_{x z z y} -> dgxzyz,
61         dg_{y x z y} -> dgxyyz, dg_{y y z y} -> dgyyyz, dg_{y z z y} -> dgyzyz,
62         dg_{z x z y} -> dgxzyz, dg_{z y z y} -> dgyzyz, dg_{z z z y} -> dgzzyz,
63
64         dg_{x x x z} -> dgxxxz, dg_{x y x z} -> dgxyxz, dg_{x z x z} -> dgxzzz,
65         dg_{y x x z} -> dgxyxz, dg_{y y x z} -> dgyyxz, dg_{y z x z} -> dgyzzz,
66         dg_{z x x z} -> dgxzzz, dg_{z y x z} -> dgyzzz, dg_{z z x z} -> dgzzzz,
67         dg_{x x y z} -> dgxxyz, dg_{x y y z} -> dgxyyz, dg_{x z y z} -> dgxzyz,
68         dg_{y x y z} -> dgxyyz, dg_{y y y z} -> dgyyyz, dg_{y z y z} -> dgyzyz,
69         dg_{z x y z} -> dgxzyz, dg_{z y y z} -> dgyzyz, dg_{z z y z} -> dgzzyz,
70         dg_{x x z z} -> dgxxxz, dg_{x y z z} -> dgxyzz, dg_{x z z z} -> dgxzzz,
71         dg_{y x z z} -> dgxyzz, dg_{y y z z} -> dgyyzz, dg_{y z z z} -> dgyzzz,
72         dg_{z x z z} -> dgxzzz, dg_{z y z z} -> dgyzzz, dg_{z z z z} -> dgzzzz}.
73
74     def write_code (obj,name,filename,rank):

```

```

75
76 import os
77
78 from sympy.printing.ccode import C99CodePrinter as printer
79 from sympy.printing.codeprinter import Assignment
80
81 idx=[] # indices in the form [{x, x}, {x, y} ...]
82 lst=[] # corresponding terms [termxx, termxy, ...]
83
84 for i in range( len(obj[rank]) ): # rank = number of free indices
85     idx.append( str(obj[rank][i][0]._sympy_()) ) # indices for this term
86     lst.append( str(obj[rank][i][1]._sympy_()) ) # the matching term
87
88 mat = sympy.Matrix([lst]) # row vector of terms
89 sub_exprs, simplified_rhs = sympy.cse(mat) # optimise code
90
91 with open(os.getcwd() + '/' + filename, 'w') as out:
92
93     for lhs, rhs in sub_exprs:
94         out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
95
96     for index, rhs in enumerate (simplified_rhs[0]):
97         lhs = sympy.Symbol(name+' '+(idx[index]).replace(' ', ',')['])
98         out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
99
100 evaluate (Rab, gabRule+d1gabRule+d2gabRule+iabRule, simplify=False)
101
102 write_code (Rab, 'Rab', 'ex-0701-rab.c',2)

```

The code for  $R_{ab}$  can be found in the file `ex-0701-rab.c`. It is long and it would require more work to turn it into something useful in a numerical code. For example, functions would be needed to compute the first and second partial derivatives of the metric. But that is not a Cadabra issue.