

Exercise 3.9 Ricci in terms of the metric and its derivatives

```

1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \partial{#}::PartialDerivative;
4
5 g_{a b}::Metric;
6 g^{a b}::InverseMetric;
7
8 dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.      # cdb (ex-0309.dgab,dgab)
9
10 Gamma := \Gamma^{a}_{b c} ->
11         (1/2) g^{a e} ( \partial_{b}{g_{e c}}
12                       + \partial_{c}{g_{b e}}
13                       - \partial_{e}{g_{b c}}).      # cdb (ex-0309.Gamma,Gamma)
14
15 Rabcd := R^{a}_{b c d} ->
16         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
17         - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.      # cdb (ex-0309.Rabcd,Rabcd)
18
19 FourRab := 4 R^{c}_{a c b}.      # cdb (ex-0309.101,FourRab)
20
21 substitute      (FourRab, Rabcd)      # cdb (ex-0309.102,FourRab)
22 substitute      (FourRab, Gamma)      # cdb (ex-0309.103,FourRab)
23
24 product_rule    (FourRab)      # cdb (ex-0309.104,FourRab)
25 distribute      (FourRab)      # cdb (ex-0309.105,FourRab)
26
27 substitute      (FourRab, dgab)      # cdb (ex-0309.106,FourRab)
28
29 sort_product    (FourRab)      # cdb (ex-0309.107,FourRab)
30 rename_dummies  (FourRab)      # cdb (ex-0309.108,FourRab)
31 canonicalise    (FourRab)      # cdb (ex-0309.109,FourRab)
32
33 # sort so that g to appeares before dg
34
35 substitute      (FourRab, $g^{a b} -> A^{a b}$)
36 sort_product    (FourRab)

```

```
37 rename_dummies (FourRab)
38 substitute      (FourRab, $A^{a b} -> g^{a b}$)    # cdb (ex-0309.110,FourRab)
```

$$\begin{aligned}
4R_{ab} &= 4R_{acb} & (\text{ex-0309.101}) \\
&= 4\partial_c \Gamma_{ab}^c + 4\Gamma_{ec}^c \Gamma_{ab}^e - 4\partial_b \Gamma_{ac}^c - 4\Gamma_{eb}^c \Gamma_{ac}^e & (\text{ex-0309.102}) \\
&= 2\partial_c (g^{ce} (\partial_{ag} eb + \partial_{ag} ae - \partial_{ag} ab)) + g^{cd} (\partial_{ag} dc + \partial_{ag} ed - \partial_{ag} ec) g^{ef} (\partial_{ag} fb + \partial_{ag} af - \partial_{ag} ab) - 2\partial_b (g^{ce} (\partial_{ag} ec + \partial_{ag} ae - \partial_{ag} ac)) \\
&\quad - g^{cd} (\partial_{ag} db + \partial_{ag} ed - \partial_{ag} eb) g^{ef} (\partial_{ag} fc + \partial_{ag} af - \partial_{ag} ac) & (\text{ex-0309.103}) \\
&= 2\partial_g^{ce} (\partial_{ag} eb + \partial_{ag} ae - \partial_{ag} ab) + 2g^{ce} \partial_c (\partial_{ag} eb + \partial_{ag} ae - \partial_{ag} ab) + g^{cd} (\partial_{ag} dc + \partial_{ag} ed - \partial_{ag} ec) g^{ef} (\partial_{ag} fb + \partial_{ag} af - \partial_{ag} ab) \\
&\quad - 2\partial_{ag}^{ce} (\partial_{ag} ec + \partial_{ag} ae - \partial_{ag} ac) - 2g^{ce} \partial_b (\partial_{ag} ec + \partial_{ag} ae - \partial_{ag} ac) - g^{cd} (\partial_{ag} db + \partial_{ag} ed - \partial_{ag} eb) g^{ef} (\partial_{ag} fc + \partial_{ag} af - \partial_{ag} ac) & (\text{ex-0309.104}) \\
&= 2\partial_g^{ce} \partial_{ag} eb + 2\partial_g^{ce} \partial_{ag} ae - 2\partial_g^{ce} \partial_{ag} ab + 2g^{ce} \partial_{ca} g^{eb} + 2g^{ce} \partial_{ct} g^{ae} - 2g^{ce} \partial_{ce} g^{ab} + g^{cd} \partial_{ag} dc g^{ef} \partial_{ag} fb + g^{cd} \partial_{ag} dc g^{ef} \partial_{ag} af - g^{cd} \partial_{ag} dc g^{ef} \partial_{ag} ab \\
&\quad + g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} fb + g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} af - g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} ab - g^{cd} \partial_{ag} ec g^{ef} \partial_{ag} fb - g^{cd} \partial_{ag} ec g^{ef} \partial_{ag} af + g^{cd} \partial_{ag} ec g^{ef} \partial_{ag} ab - 2\partial_{ag}^{ce} \partial_{ag} ec - 2\partial_{ag}^{ce} \partial_{ag} ae \\
&\quad + 2\partial_{ag}^{ce} \partial_{ag} ac - 2g^{ce} \partial_{ba} g^{ec} - 2g^{ce} \partial_{bg} ae + 2g^{ce} \partial_{bg} ac - g^{cd} \partial_{ag} db g^{ef} \partial_{ag} fc - g^{cd} \partial_{ag} db g^{ef} \partial_{ag} af + g^{cd} \partial_{ag} db g^{ef} \partial_{ag} ac - g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} fc \\
&\quad - g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} af + g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} ac + g^{cd} \partial_{ag} eb g^{ef} \partial_{ag} fc + g^{cd} \partial_{ag} eb g^{ef} \partial_{ag} af - g^{cd} \partial_{ag} eb g^{ef} \partial_{ag} ac & (\text{ex-0309.105}) \\
&= -2g^{cd} g^{ef} \partial_{ag} df \partial_{ag} eb - 2g^{cd} g^{ef} \partial_{ag} df \partial_{ag} ae + 2g^{cd} g^{ef} \partial_{ag} df \partial_{ag} ab + 2g^{ce} \partial_{ca} g^{eb} + 2g^{ce} \partial_{ct} g^{ae} - 2g^{ce} \partial_{ce} g^{ab} + g^{cd} \partial_{ag} dc g^{ef} \partial_{ag} fb + g^{cd} \partial_{ag} dc g^{ef} \partial_{ag} af \\
&\quad - g^{cd} \partial_{ag} dc g^{ef} \partial_{ag} ab + g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} fb + g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} af - g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} ab - g^{cd} \partial_{ag} ec g^{ef} \partial_{ag} fb - g^{cd} \partial_{ag} ec g^{ef} \partial_{ag} af + g^{cd} \partial_{ag} ec g^{ef} \partial_{ag} ab \\
&\quad + 2g^{cd} g^{ef} \partial_{ag} df \partial_{ag} ec + 2g^{cd} g^{ef} \partial_{ag} df \partial_{ag} ae - 2g^{cd} g^{ef} \partial_{ag} df \partial_{ag} ac - 2g^{ce} \partial_{ba} g^{ec} - 2g^{ce} \partial_{bg} ae + 2g^{ce} \partial_{bg} ac - g^{cd} \partial_{ag} db g^{ef} \partial_{ag} fc - g^{cd} \partial_{ag} db g^{ef} \partial_{ag} af \\
&\quad + g^{cd} \partial_{ag} db g^{ef} \partial_{ag} ac - g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} fc - g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} af + g^{cd} \partial_{ag} ed g^{ef} \partial_{ag} ac + g^{cd} \partial_{ag} eb g^{ef} \partial_{ag} fc + g^{cd} \partial_{ag} eb g^{ef} \partial_{ag} af - g^{cd} \partial_{ag} eb g^{ef} \partial_{ag} ac & (\text{ex-0309.106}) \\
&= -2\partial_{ag} eb \partial_{ag} df g^{cd} g^{ef} - 2\partial_{ag} ae \partial_{ag} df g^{cd} g^{ef} + 2\partial_{ag} df \partial_{ag} ab g^{cd} g^{ef} + 2\partial_{ca} g^{eb} g^{ce} + 2\partial_{ct} g^{ae} g^{ce} - 2\partial_{ce} g^{ab} g^{ce} + \partial_{ag} fb \partial_{ag} dc g^{cd} g^{ef} + \partial_{ag} af \partial_{ag} dc g^{cd} g^{ef} \\
&\quad - \partial_{ag} dc \partial_{ag} fb g^{cd} g^{ef} + \partial_{ag} fb \partial_{ag} ed g^{cd} g^{ef} + \partial_{ag} af \partial_{ag} ed g^{cd} g^{ef} - \partial_{ag} ed \partial_{ag} fb g^{cd} g^{ef} - \partial_{ag} fb \partial_{ag} ec g^{cd} g^{ef} - \partial_{ag} af \partial_{ag} ec g^{cd} g^{ef} + \partial_{ag} ec \partial_{ag} fb g^{cd} g^{ef} \\
&\quad + 2\partial_{ag} ec \partial_{ag} df g^{cd} g^{ef} + 2\partial_{ag} df \partial_{ag} ae g^{cd} g^{ef} - 2\partial_{ag} df \partial_{ag} ac g^{cd} g^{ef} - 2\partial_{ba} g^{ec} g^{ce} - 2\partial_{bg} ae g^{ce} + 2\partial_{bg} ac g^{ce} - \partial_{ag} fc \partial_{ag} db g^{cd} g^{ef} - \partial_{ag} af \partial_{ag} db g^{cd} g^{ef} \\
&\quad + \partial_{ag} db \partial_{ag} fc g^{cd} g^{ef} - \partial_{ag} fc \partial_{ag} ed g^{cd} g^{ef} - \partial_{ag} ed \partial_{ag} af g^{cd} g^{ef} + \partial_{ag} ed \partial_{ag} fc g^{cd} g^{ef} + \partial_{ag} fc \partial_{ag} eb g^{cd} g^{ef} + \partial_{ag} af \partial_{ag} eb g^{cd} g^{ef} - \partial_{ag} eb \partial_{ag} fc g^{cd} g^{ef} & (\text{ex-0309.107}) \\
&= -2\partial_{ag} db \partial_{ag} ef g^{ce} g^{df} - 2\partial_{ag} ad \partial_{ag} ef g^{ce} g^{df} + 2\partial_{ag} ef \partial_{ag} ab g^{ce} g^{df} + 2\partial_{ca} g^{ab} g^{cd} + 2\partial_{ct} g^{ad} g^{cd} - 2\partial_{ce} g^{ab} g^{cd} + \partial_{ag} ab \partial_{ag} ef g^{fe} g^{cd} + \partial_{ag} ad \partial_{ag} ef g^{fe} g^{cd} \\
&\quad - \partial_{ag} ef \partial_{ag} ab g^{fe} g^{cd} + \partial_{ag} ab \partial_{ag} ef g^{cf} g^{ed} + \partial_{ag} ad \partial_{ag} ef g^{cf} g^{ed} - \partial_{ag} ef \partial_{ag} ab g^{cf} g^{ed} - \partial_{ag} db \partial_{ag} ef g^{fc} g^{ed} - \partial_{ag} ad \partial_{ag} ef g^{fc} g^{ed} + \partial_{ag} ef \partial_{ag} ab g^{fc} g^{ed} \\
&\quad + 2\partial_{ag} cd \partial_{ag} ef g^{de} g^{cf} + 2\partial_{ag} de \partial_{ag} af g^{cd} g^{fe} - 2\partial_{ag} de \partial_{ag} af g^{fd} g^{ce} - 2\partial_{ba} g^{cd} g^{dc} - 2\partial_{bg} ad g^{cd} + 2\partial_{bg} ad g^{dc} - \partial_{ag} de \partial_{ag} fb g^{ef} g^{cd} - \partial_{ag} ae \partial_{ag} fb g^{cf} g^{de} \\
&\quad + \partial_{ag} eb \partial_{ag} af g^{fe} g^{cd} - \partial_{ag} cd \partial_{ag} ef g^{df} g^{ec} - \partial_{ag} de \partial_{ag} af g^{ce} g^{df} + \partial_{ag} de \partial_{ag} af g^{fe} g^{dc} + \partial_{ag} de \partial_{ag} fb g^{ec} g^{fd} + \partial_{ag} ae \partial_{ag} fb g^{cd} g^{fe} - \partial_{ag} eb \partial_{ag} af g^{fc} g^{ed} & (\text{ex-0309.108}) \\
&= -2\partial_{ag} bc \partial_{ag} ef g^{ce} g^{df} - 2\partial_{ag} ac \partial_{ag} ef g^{ce} g^{df} + 2\partial_{ag} ab \partial_{ag} ef g^{ce} g^{df} + 2\partial_{ac} g^{bd} g^{cd} + 2\partial_{bg} ad g^{cd} - 2\partial_{ce} g^{ab} g^{cd} + \partial_{ag} bc \partial_{ag} ef g^{cd} g^{ef} + \partial_{ag} ac \partial_{ag} ef g^{cd} g^{ef} \\
&\quad - \partial_{ag} ab \partial_{ag} ef g^{cd} g^{ef} + \partial_{ag} cd \partial_{ag} ef g^{ce} g^{df} - 2\partial_{at} g^{cd} g^{cd} - 2\partial_{ag} ad \partial_{ag} bf g^{cf} g^{de} + 2\partial_{ag} ad \partial_{ag} bf g^{ce} g^{df} & (\text{ex-0309.109}) \\
&= -2g^{cd} g^{ef} \partial_{ag} bc \partial_{ag} df - 2g^{cd} g^{ef} \partial_{ag} ac \partial_{ag} df + 2g^{cd} g^{ef} \partial_{ag} ab \partial_{ag} df + 2g^{cd} \partial_{ac} g^{bd} + 2g^{cd} \partial_{bg} ad - 2g^{cd} \partial_{ce} g^{ab} + g^{cd} g^{ef} \partial_{ag} bc \partial_{ag} ef + g^{cd} g^{ef} \partial_{ag} ac \partial_{ag} ef \\
&\quad - g^{cd} g^{ef} \partial_{ag} ab \partial_{ag} ef + g^{cd} g^{ef} \partial_{ag} ce \partial_{ag} df - 2g^{cd} \partial_{at} g^{cd} - 2g^{cd} g^{ef} \partial_{ag} ae \partial_{ag} fb + 2g^{cd} g^{ef} \partial_{ag} ac \partial_{ag} bf & (\text{ex-0309.110})
\end{aligned}$$