

Exercise 6.5 Schwarzschild spacetime in isotropic coordinates

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1 {t, r, \theta, \varphi}::Coordinate.
2 {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4 \partial{#}::PartialDerivative.
5
6 g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
7
8 Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}}).
11
12 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
13                        - \partial_{d}{\Gamma^{a}_{b c}}
14                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17 Rab := R_{a b} -> R^{c}_{c a b}.
18
19 gab := { g_{t t}          = -((2*r-m)/(2*r+m))**2,
20         g_{r r}          = (1+m/(2*r))**4,
21         g_{\theta\theta}   = r**2 (1+m/(2*r))**4,
22         g_{\varphi\varphi} = r**2 \sin(\theta)**2 (1+m/(2*r))**4}. # cdb(ex-0605.101,gab)
23
24 complete (gab, $g^{a b}$) # cdb(ex-0605.102,gab)
25
26 substitute (Rabcd, Gamma)
27 substitute (Rab, Rabcd)
28
29 evaluate (Gamma, gab, rhsonly=True) # cdb(ex-0605.103,Gamma)
30 evaluate (Rabcd, gab, rhsonly=True) # cdb(ex-0605.104,Rabcd)
31 evaluate (Rab, gab, rhsonly=True) # cdb(ex-0605.105,Rab)

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$$\left[g_{tt} = -((2r - m)(2r + m)^{-1})^2, g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, g_{\theta\theta} = r^2 \left(1 + \frac{1}{2}mr^{-1}\right)^4, g_{\varphi\varphi} = r^2 (\sin \theta)^2 \left(1 + \frac{1}{2}mr^{-1}\right)^4 \right] \quad (\text{ex-0605.101})$$

$$\left[g_{tt} = -((2r - m)(2r + m)^{-1})^2, g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, g_{\theta\theta} = r^2 \left(1 + \frac{1}{2}mr^{-1}\right)^4, g_{\varphi\varphi} = r^2 (\sin \theta)^2 \left(1 + \frac{1}{2}mr^{-1}\right)^4, g^{tt} = (-m^2 - 4mr - 4r^2)(m^2 - 4mr + 4r^2)^{-1}, g^{rr} = 16r^4(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-1}, g^{\theta\theta} = 16r^2(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-1}, g^{\varphi\varphi} = 16r^2(\sin \theta)^2(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-1} \right]$$

$$\Gamma_{bc}^a \rightarrow \square_{cb}^a \left\{ \begin{array}{l} \square_{\varphi r}^{\varphi} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{\varphi \theta}^{\varphi} = (\tan \theta)^{-1} \\ \square_{\theta r}^{\theta} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{rr}^r = -2m(r(m + 2r))^{-1} \\ \square_{tr}^t = 4m(-m^2 + 4r^2)^{-1} \\ \square_{r\varphi}^{\varphi} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan \theta)^{-1} \\ \square_{r\theta}^{\theta} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{rt}^t = 4m(-m^2 + 4r^2)^{-1} \\ \square_{\varphi\varphi}^r = r(m - 2r)(\sin \theta)^2(m + 2r)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2} \sin(2\theta) \\ \square_{\theta\theta}^r = r(m - 2r)(m + 2r)^{-1} \\ \square_{tt}^r = -64mr^4(m - 2r)((m + 2r)^3(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4))^{-1} \end{array} \right. \quad (\text{ex-0605.103})$$

R_{bcd}^a

$$\rightarrow \square_{db}^a \begin{cases} \square_{tt}^r = 128 m r^3 (-m^2 + 4 m r - 4 r^2) (m^8 + 16 m^7 r + 112 m^6 r^2 + 448 m^5 r^3 + 1120 m^4 r^4 + 1792 m^3 r^5 + 1792 m^2 r^6 + 1024 m r^7 + 256 r^8)^{-1} \\ \square_{\theta\theta}^r = -4 m r (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{\varphi\varphi}^\theta = 8 m r (\sin \theta)^2 (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{\varphi\varphi}^r = -4 m r (\sin \theta)^2 (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{tr}^t = -8 m (r (m^2 + 4 m r + 4 r^2))^{-1} \\ \square_{\theta r}^\theta = 4 m (r (m^2 + 4 m r + 4 r^2))^{-1} \\ \square_{\varphi\theta}^\varphi = -8 m r (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{\varphi r}^\varphi = 4 m (r (m^2 + 4 m r + 4 r^2))^{-1} \\ \square_{rt}^r = 128 m r^3 (m^2 - 4 m r + 4 r^2) (m^8 + 16 m^7 r + 112 m^6 r^2 + 448 m^5 r^3 + 1120 m^4 r^4 + 1792 m^3 r^5 + 1792 m^2 r^6 + 1024 m r^7 + 256 r^8)^{-1} \\ \square_{r\theta}^r = 4 m r (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{\theta\varphi}^\theta = -8 m r (\sin \theta)^2 (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{r\varphi}^r = 4 m r (\sin \theta)^2 (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{rr}^t = 8 m (r (m^2 + 4 m r + 4 r^2))^{-1} \\ \square_{rr}^\theta = -4 m (r (m^2 + 4 m r + 4 r^2))^{-1} \\ \square_{\theta\theta}^\varphi = 8 m r (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{rr}^\varphi = -4 m (r (m^2 + 4 m r + 4 r^2))^{-1} \\ \square_{\varphi\varphi}^t = -4 m r (\sin \theta)^2 (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{\theta\theta}^t = -4 m r (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{tt}^\varphi = 64 m r^3 (m - 2 r)^2 (m^4 + 8 m^3 r + 24 m^2 r^2 + 32 m r^3 + 16 r^4)^{-2} \\ \square_{tt}^\theta = 64 m r^3 (m - 2 r)^2 (m^4 + 8 m^3 r + 24 m^2 r^2 + 32 m r^3 + 16 r^4)^{-2} \\ \square_{t\varphi}^t = 4 m r (\sin \theta)^2 (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{t\theta}^t = 4 m r (m^2 + 4 m r + 4 r^2)^{-1} \\ \square_{\varphi t}^\varphi = -64 m r^3 (m - 2 r)^2 (m^4 + 8 m^3 r + 24 m^2 r^2 + 32 m r^3 + 16 r^4)^{-2} \\ \square_{\theta t}^\theta = -64 m r^3 (m - 2 r)^2 (m^4 + 8 m^3 r + 24 m^2 r^2 + 32 m r^3 + 16 r^4)^{-2} \end{cases} \quad (\text{ex-0605.104})$$

 $R_{ab} \rightarrow 0$
 (ex-0605.105)