

Exercise 3.3 Computing R_{abcd}

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1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative.
4
5  \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
6  \Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).
7
8  dgab := \partial_{c}{g_{a b}} ->  \Gamma^{d}_{a c} g_{d b}
9                                     + \Gamma^{d}_{b c} g_{a d}.      # cdb(dgab.000,dgab)
10
11 RabcdU := R^{a}_{b c d} ->  \partial_{c}{\Gamma^{a}_{b d}}
12                             - \partial_{d}{\Gamma^{a}_{b c}}
13                             + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
14                             - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.      # cdb(Rabcd.000,RabcdU)
15
16 GammaD := {g_{a e} \Gamma^{e}_{b c} -> \Gamma_{a b c},
17            g_{e a} \Gamma^{e}_{b c} -> \Gamma_{a b c}}.      # cdb(Gamma.010,GammaD)
18
19 RabcdD := R_{a b c d} -> g_{a e} R^{e}_{b c d}.      # cdb(Rabcd.010,RabcdD)
20
21 gabDGamma := g_{a e} \partial_{c}{\Gamma^{e}_{b d}} ->
22              \partial_{c}{g_{a e} \Gamma^{e}_{b d}}
23              - \Gamma^{e}_{b d} \partial_{c}{g_{a e}}.      # cdb(gabDGamma.000,gabDGamma)
24
25 # this pair of rules needed to sort \Gamma_{a b c} to the very left
26 # this helps canonicalise spot the terms that cancel
27 bah := \Gamma_{a b c} -> A_{a b c}.
28 foo := A_{a b c} -> \Gamma_{a b c}.
29
30 expr := R_{a b c d}.      # cdb(ex-0303.101,expr)
31
32 substitute      (expr, RabcdD)      # cdb(ex-0303.102,expr)
33 substitute      (expr, RabcdU)      # cdb(ex-0303.103,expr)
34 distribute      (expr)              # cdb(ex-0303.104,expr)
35 substitute      (expr, gabDGamma)    # cdb(ex-0303.105,expr)
36 substitute      (expr, dgab)         # cdb(ex-0303.106,expr)

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37 substitute      (expr, GammaD)          # cdb(ex-0303.107,expr)
38 distribute      (expr)                  # cdb(ex-0303.109,expr)
39 substitute      (expr, bah)             # cdb(ex-0303.110,expr)
40 sort_product    (expr)                  # cdb(ex-0303.111,expr)
41 rename_dummies  (expr)                  # cdb(ex-0303.112,expr)
42 substitute      (expr, foo)             # cdb(ex-0303.113,expr)
43 canonicalise    (expr)                  # cdb(ex-0303.114,expr)

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$$R_{abcd} = g_{ae} R^e_{bcd} \quad (\text{ex-0303.102})$$

$$= g_{ae} (\partial_c \Gamma^e_{bd} - \partial_d \Gamma^e_{bc} + \Gamma^f_{bd} \Gamma^e_{cf} - \Gamma^f_{bc} \Gamma^e_{df}) \quad (\text{ex-0303.103})$$

$$= g_{ae} \partial_c \Gamma^e_{bd} - g_{ae} \partial_d \Gamma^e_{bc} + g_{ae} \Gamma^f_{bd} \Gamma^e_{cf} - g_{ae} \Gamma^f_{bc} \Gamma^e_{df} \quad (\text{ex-0303.104})$$

$$= \partial_c (g_{ae} \Gamma^e_{bd}) - \Gamma^e_{bd} \partial_c g_{ae} - \partial_d (g_{ae} \Gamma^e_{bc}) + \Gamma^e_{bc} \partial_d g_{ae} + g_{ae} \Gamma^f_{bd} \Gamma^e_{cf} - g_{ae} \Gamma^f_{bc} \Gamma^e_{df} \quad (\text{ex-0303.105})$$

$$= \partial_c (g_{ae} \Gamma^e_{bd}) - \Gamma^e_{bd} (\Gamma^f_{ac} g_{fe} + \Gamma^f_{ec} g_{af}) - \partial_d (g_{ae} \Gamma^e_{bc}) + \Gamma^e_{bc} (\Gamma^f_{ad} g_{fe} + \Gamma^f_{ed} g_{af}) + g_{ae} \Gamma^f_{bd} \Gamma^e_{cf} - g_{ae} \Gamma^f_{bc} \Gamma^e_{df} \quad (\text{ex-0303.106})$$

$$= \partial_c \Gamma_{abd} - \Gamma^e_{bd} (\Gamma_{eac} + \Gamma_{aec}) - \partial_d \Gamma_{abc} + \Gamma^e_{bc} (\Gamma_{ead} + \Gamma_{aed}) + \Gamma_{acf} \Gamma^f_{bd} - \Gamma_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.107})$$

$$= \partial_c \Gamma_{abd} - \Gamma^e_{bd} \Gamma_{eac} - \Gamma^e_{bd} \Gamma_{aec} - \partial_d \Gamma_{abc} + \Gamma^e_{bc} \Gamma_{ead} + \Gamma^e_{bc} \Gamma_{aed} + \Gamma_{acf} \Gamma^f_{bd} - \Gamma_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.109})$$

$$= \partial_c A_{abd} - \Gamma^e_{bd} A_{eac} - \Gamma^e_{bd} A_{aec} - \partial_d A_{abc} + \Gamma^e_{bc} A_{ead} + \Gamma^e_{bc} A_{aed} + A_{acf} \Gamma^f_{bd} - A_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.110})$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e_{bd} - A_{aec} \Gamma^e_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e_{bc} + A_{aed} \Gamma^e_{bc} + A_{acf} \Gamma^f_{bd} - A_{adf} \Gamma^f_{bc} \quad (\text{ex-0303.111})$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e_{bd} - A_{aec} \Gamma^e_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e_{bc} + A_{aed} \Gamma^e_{bc} + A_{ace} \Gamma^e_{bd} - A_{ade} \Gamma^e_{bc} \quad (\text{ex-0303.112})$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e_{bd} - \Gamma_{aec} \Gamma^e_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e_{bc} + \Gamma_{aed} \Gamma^e_{bc} + \Gamma_{ace} \Gamma^e_{bd} - \Gamma_{ade} \Gamma^e_{bc} \quad (\text{ex-0303.113})$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e_{bc} \quad (\text{ex-0303.114})$$