

Example 9 The Gauss equation

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4
5 K_{a b}::Symmetric.
6 g^{a}_{b}::KroneckerDelta.
7
8 # define the projection operator
9
10 hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}.
11
12 # 3-covariant derivative obtained by projection on 4-covariant derivative
13
14 vpq:=v_{p q} -> h^{a}_{p} h^{b}_{q} \nabla_{b}{v_{a}}.
15
16 # compute 3-curvature by commutation of covariant derivatives
17
18 vpqr:= h^{a}_{p} h^{b}_{q} h^{c}_{r} ( \nabla_{c}{v_{a b}} - \nabla_{b}{v_{a c}} ).
19
20 substitute (vpq,hab)
21 substitute (vpqr,vpq)
22
23 distribute (vpqr)
24 product_rule (vpqr)
25 distribute (vpqr)
26 eliminate_kronecker (vpqr)
27
28 # standard substitutions
29
30 substitute (vpqr,$h^{a}_{b} n^{b} -> 0$)
31 substitute (vpqr,$h^{a}_{b} n_{a} -> 0$)
32 substitute (vpqr,$\nabla_{a}{g^{b}_{c}} -> 0$)
33 substitute (vpqr,$n^{a} \nabla_{b}{v_{a}} -> -v_{a} \nabla_{b}{n^{a}}$)
34 substitute (vpqr,$v_{a} \nabla_{b}{n^{a}} -> v_{p} h^{p}_{a} \nabla_{b}{n^{a}}$)
35 substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n_{q}} -> K_{a b}$)
36 substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n^{b}} -> K_{a}^{q}$) # cdb(ex-09.095,vpqr)
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37
38 # tidy up
39
40 {v_{a},h^{a}_{b},K_{a}^{b},K_{a b},R^{a}_{b c d},\nabla_{a}\{v_{b}\}}::SortOrder.
41
42 sort_product      (vpqr)                # cdb(ex-09.096,vpqr)
43 rename_dummies    (vpqr)                # cdb(ex-09.097,vpqr)
44 canonicalise      (vpqr)                # cdb(ex-09.098,vpqr)
45 factor_out        (vpqr,$h^{a?}_{b?}$)   # cdb(ex-09.099,vpqr)
46 factor_out        (vpqr,$v_{a?}$)       # cdb(ex-09.101,vpqr)
47
48 checkpoint.append (vpqr)

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$$(D_r D_q - D_q D_r) v_p = h^e_p h^d_q h^c_r \nabla_c (\nabla_d v_e) - h^e_p K_{rq} n^d \nabla_d v_e + K_q^b K_{rp} v_b - h^d_p h^b_q h^e_r \nabla_b (\nabla_e v_d) + h^d_p K_{qr} n^e \nabla_e v_d - K_{qp} K_r^c v_c \quad (\text{ex-09.095})$$

$$= h^c_r h^d_q h^e_p \nabla_c (\nabla_d v_e) - h^e_p K_{rq} \nabla_d v_e n^d + v_b K_q^b K_{rp} - h^b_q h^d_p h^e_r \nabla_b (\nabla_e v_d) + h^d_p K_{qr} \nabla_e v_d n^e - v_c K_r^c K_{qp} \quad (\text{ex-09.096})$$

$$= h^a_r h^b_q h^c_p \nabla_a (\nabla_b v_c) - h^b_p K_{rq} \nabla_a v_b n^a + v_a K_q^a K_{rp} - h^a_q h^c_p h^b_r \nabla_a (\nabla_b v_c) + h^b_p K_{qr} \nabla_a v_b n^a - v_a K_r^a K_{qp} \quad (\text{ex-09.097})$$

$$= h^a_p h^b_q h^c_r \nabla_c (\nabla_b v_a) + v_a K_q^a K_{pr} - h^a_p h^b_q h^c_r \nabla_b (\nabla_c v_a) - v_a K_r^a K_{pq} \quad (\text{ex-09.098})$$

$$= v_a K_q^a K_{pr} - v_a K_r^a K_{pq} + h^a_p h^b_q h^c_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) \quad (\text{ex-09.099})$$

$$= h^a_p h^b_q h^c_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.101})$$

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1 R{#}::LaTeXForm("\{\strut\}^g R").
2
3 gRabcd := \nabla_{\{c\}}{\nabla_{\{b\}}{v_{\{a\}}}}
4         -\nabla_{\{b\}}{\nabla_{\{c\}}{v_{\{a\}}}} -> R^{\{d\}}_{\{a\} \{b\} \{c\}} v_{\{d\}}.
5
6 substitute      (vpqr,gRabcd)                # cdb(ex-09.102,vpqr)
7 distribute      (vpqr)                        # cdb(ex-09.103,vpqr)
8 substitute      (vpqr,$v_{\{a\}} -> h^{\{b\}}_{\{a\}} v_{\{b\}}$) # cdb(ex-09.104,vpqr)
9 substitute      (vpqr,$h^{\{b\}}_{\{a\}} K_{\{c\}}^{\{a\}} -> K_{\{c\}}^{\{b\}}$) # cdb(ex-09.105,vpqr)
10 sort_product   (vpqr)                        # cdb(ex-09.106,vpqr)
11 rename_dummies (vpqr)                        # cdb(ex-09.107,vpqr)
12 canonicalise   (vpqr)                        # cdb(ex-09.108,vpqr)
13 factor_out      (vpqr,$v_{\{a\}}$)           # cdb(ex-09.109,vpqr)
14 substitute      (vpqr,$v_{\{a\}}->1$)        # cdb(ex-09.110,vpqr)
15 sort_product   (vpqr)                        # cdb(ex-09.111,vpqr)
16
17 checkpoint.append (vpqr)

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$$(D_r D_q - D_q D_r) v_p = h^a_p h^b_q h^c_r (\nabla_c (\nabla_b v_a) - \nabla_b (\nabla_c v_a)) + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.101})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} v_d + v_a (K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.102})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} v_d + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.103})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} h^e_d v_e + h^b_a v_b K_q^a K_{pr} - h^b_a v_b K_r^a K_{pq} \quad (\text{ex-09.104})$$

$$= h^a_p h^b_q h^c_r {}^g R^d_{abc} h^e_d v_e + K_q^b v_b K_{pr} - K_r^b v_b K_{pq} \quad (\text{ex-09.105})$$

$$= v_e h^a_p h^b_q h^c_r h^e_d {}^g R^d_{abc} + v_b K_q^b K_{pr} - v_b K_r^b K_{pq} \quad (\text{ex-09.106})$$

$$= v_e h^b_p h^c_q h^d_r h^e_a {}^g R^a_{bcd} + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.107})$$

$$= v_a h^b_p h^c_q h^d_r h^e_a {}^g R^e_{bcd} + v_a K_q^a K_{pr} - v_a K_r^a K_{pq} \quad (\text{ex-09.108})$$

$$= v_a (h^b_p h^c_q h^d_r h^e_a {}^g R^e_{bcd} + K_q^a K_{pr} - K_r^a K_{pq}) \quad (\text{ex-09.109})$$

$${}^h R^a_{pqr} = h^b_p h^c_q h^d_r h^e_a {}^g R^e_{bcd} + K_q^a K_{pr} - K_r^a K_{pq} \quad (\text{ex-09.110})$$

$${}^h R^a_{pqr} = h^a_e h^b_p h^c_q h^d_r {}^g R^e_{bcd} + K_q^a K_{pr} - K_r^a K_{pq} \quad (\text{ex-09.111})$$