Exercise 3.8 Symmetry of R_{ab}

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
                                                    # cdb (dgab.000,dgab)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                                             + \partial_{c}{g_{b e}}
                                             - \partial_{e}{g_{b c}}).
                                                    # cdb (Gamma.000,Gamma)

Rabcd := R^{a}_{b c d} ->
         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
       - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                    # cdb (Rabcd.000,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                    # cdb (Rab.000,Rab)

expr := 4 (R_{a b} - R_{b a}).                      # cdb (ex-0308.100,expr)

substitute   (expr, Rab)                            # cdb (ex-0308.101,expr)
substitute   (expr, Rabcd)                          # cdb (ex-0308.102,expr)
substitute   (expr, Gamma)                          # cdb (ex-0308.103,expr)

distribute   (expr)                                 # cdb (ex-0308.104,expr)
product_rule (expr)                                 # cdb (ex-0308.105,expr)
canonicalise (expr)                                 # cdb (ex-0308.106,expr)

substitute   (expr, dgab)                           # cdb (ex-0308.107,expr)
canonicalise (expr)                                 # cdb (ex-0308.108,expr)
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$$4\,R_{ab} - 4\,R_{ba} = 4\,R^{c}{}_{acb} - 4\,R^{c}{}_{bca} \tag{ex-0308.101}$$

$$= 4\,\partial_{c}\Gamma^{c}{}_{ab} + 4\,\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ab} - 4\,\partial_{b}\Gamma^{c}{}_{ac} - 4\,\Gamma^{c}{}_{eb}\Gamma^{e}{}_{ac} - 4\,\partial_{c}\Gamma^{c}{}_{ba} - 4\,\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ba} + 4\,\partial_{a}\Gamma^{c}{}_{bc} + 4\,\Gamma^{c}{}_{ea}\Gamma^{e}{}_{bc} \tag{ex-0308.102}$$

$$\begin{aligned}= {} & 2\,\partial_{c}(g^{ce}\,(\partial_{d}g_{eb} + \partial_{b}g_{ae} - \partial_{e}g_{ab})) + g^{cd}\,(\partial_{e}g_{dc} + \partial_{d}g_{ed} - \partial_{d}g_{ec})\,g^{ef}\,(\partial_{d}g_{fb} + \partial_{b}g_{af} - \partial_{f}g_{ab}) - 2\,\partial_{b}(g^{ce}\,(\partial_{d}g_{ec} + \partial_{d}g_{ae} - \partial_{e}g_{ac})) \\ & - g^{cd}\,(\partial_{e}g_{db} + \partial_{b}g_{ed} - \partial_{d}g_{eb})\,g^{ef}\,(\partial_{d}g_{fc} + \partial_{d}g_{af} - \partial_{f}g_{ac}) - 2\,\partial_{c}(g^{ce}\,(\partial_{b}g_{ea} + \partial_{d}g_{be} - \partial_{e}g_{ba})) \\ & - g^{cd}\,(\partial_{e}g_{dc} + \partial_{d}g_{ed} - \partial_{d}g_{ec})\,g^{ef}\,(\partial_{b}g_{fa} + \partial_{d}g_{bf} - \partial_{f}g_{ba}) + 2\,\partial_{a}(g^{ce}\,(\partial_{b}g_{ec} + \partial_{d}g_{be} - \partial_{e}g_{bc})) \\ & + g^{cd}\,(\partial_{e}g_{da} + \partial_{d}g_{ed} - \partial_{d}g_{ea})\,g^{ef}\,(\partial_{b}g_{fc} + \partial_{d}g_{bf} - \partial_{f}g_{bc})\end{aligned} \tag{ex-0308.103}$$

$$\begin{aligned}= {} & 2\,\partial_{c}(g^{ce}\partial_{d}g_{eb}) + 2\,\partial_{c}(g^{ce}\partial_{b}g_{ae}) - 2\,\partial_{c}(g^{ce}\partial_{e}g_{ab}) + g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{d}g_{fb} + g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{b}g_{af} - g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{f}g_{ab} + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{d}g_{fb} \\ & + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{b}g_{af} - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{f}g_{ab} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{d}g_{fb} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{b}g_{af} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{f}g_{ab} - 2\,\partial_{b}(g^{ce}\partial_{d}g_{ec}) - 2\,\partial_{b}(g^{ce}\partial_{d}g_{ae}) \\ & + 2\,\partial_{b}(g^{ce}\partial_{e}g_{ac}) - g^{cd}\partial_{e}g_{db}g^{ef}\partial_{d}g_{fc} - g^{cd}\partial_{e}g_{db}g^{ef}\partial_{d}g_{af} + g^{cd}\partial_{e}g_{db}g^{ef}\partial_{f}g_{ac} - g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{d}g_{fc} - g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{d}g_{af} + g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{f}g_{ac} \\ & + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{d}g_{fc} + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{e}g_{af} - g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{f}g_{ac} - 2\,\partial_{c}(g^{ce}\partial_{b}g_{ea}) - 2\,\partial_{c}(g^{ce}\partial_{d}g_{be}) + 2\,\partial_{c}(g^{ce}\partial_{e}g_{ba}) - g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{b}g_{fa} \\ & - g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{d}g_{bf} + g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{f}g_{ba} - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{b}g_{fa} - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{d}g_{bf} + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{f}g_{ba} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{b}g_{fa} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{d}g_{bf} \\ & - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{f}g_{ba} + 2\,\partial_{a}(g^{ce}\partial_{b}g_{ec}) + 2\,\partial_{a}(g^{ce}\partial_{d}g_{be}) - 2\,\partial_{a}(g^{ce}\partial_{d}g_{bc}) + g^{cd}\partial_{e}g_{da}g^{ef}\partial_{b}g_{fc} + g^{cd}\partial_{e}g_{da}g^{ef}\partial_{d}g_{bf} - g^{cd}\partial_{e}g_{da}g^{ef}\partial_{f}g_{bc} \\ & + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{b}g_{fc} + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{d}g_{bf} - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{f}g_{bc} - g^{cd}\partial_{d}g_{ea}g^{ef}\partial_{b}g_{fc} - g^{cd}\partial_{d}g_{ea}g^{ef}\partial_{d}g_{bf} + g^{cd}\partial_{d}g_{ea}g^{ef}\partial_{f}g_{bc}\end{aligned} \tag{ex-0308.104}$$

$$\begin{aligned}= {} & 2\,\partial_{d}g^{ce}\partial_{d}g_{eb} + 2\,g^{ce}\partial_{c}\partial_{d}g_{eb} + 2\,\partial_{d}g^{ce}\partial_{b}g_{ae} + 2\,g^{ce}\partial_{c}\partial_{b}g_{ae} - 2\,\partial_{d}g^{ce}\partial_{e}g_{ab} - 2\,g^{ce}\partial_{c}\partial_{e}g_{ab} + g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{d}g_{fb} + g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{b}g_{af} - g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{f}g_{ab} \\ & + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{d}g_{fb} + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{b}g_{af} - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{f}g_{ab} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{d}g_{fb} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{b}g_{af} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{f}g_{ab} - 2\,\partial_{b}g^{ce}\partial_{d}g_{ec} \\ & - 2\,g^{ce}\partial_{b}\partial_{d}g_{ec} - 2\,\partial_{b}g^{ce}\partial_{e}g_{ae} - 2\,g^{ce}\partial_{b}\partial_{e}g_{ae} + 2\,\partial_{b}g^{ce}\partial_{e}g_{ac} + 2\,g^{ce}\partial_{b}\partial_{e}g_{ac} - g^{cd}\partial_{e}g_{db}g^{ef}\partial_{d}g_{fc} - g^{cd}\partial_{e}g_{db}g^{ef}\partial_{d}g_{af} + g^{cd}\partial_{e}g_{db}g^{ef}\partial_{f}g_{ac} \\ & - g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{d}g_{fc} - g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{e}g_{af} + g^{cd}\partial_{b}g_{ed}g^{ef}\partial_{f}g_{ac} + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{d}g_{fc} + g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{e}g_{af} - g^{cd}\partial_{d}g_{eb}g^{ef}\partial_{f}g_{ac} - 2\,\partial_{d}g^{ce}\partial_{b}g_{ea} \\ & - 2\,g^{ce}\partial_{c}\partial_{b}g_{ea} - 2\,\partial_{d}g^{ce}\partial_{d}g_{be} - 2\,g^{ce}\partial_{c}\partial_{d}g_{be} + 2\,\partial_{d}g^{ce}\partial_{e}g_{ba} + 2\,g^{ce}\partial_{c}\partial_{e}g_{ba} - g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{b}g_{fa} - g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{d}g_{bf} + g^{cd}\partial_{e}g_{dc}g^{ef}\partial_{f}g_{ba} \\ & - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{b}g_{fa} - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{d}g_{bf} + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{f}g_{ba} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{b}g_{fa} + g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{d}g_{bf} - g^{cd}\partial_{d}g_{ec}g^{ef}\partial_{f}g_{ba} + 2\,\partial_{d}g^{ce}\partial_{b}g_{ec} \\ & + 2\,g^{ce}\partial_{a}\partial_{b}g_{ec} + 2\,\partial_{d}g^{ce}\partial_{e}g_{be} + 2\,g^{ce}\partial_{a}g_{be} - 2\,\partial_{d}g^{ce}\partial_{e}g_{bc} - 2\,g^{ce}\partial_{a}g_{bc} + g^{cd}\partial_{e}g_{da}g^{ef}\partial_{b}g_{fc} + g^{cd}\partial_{e}g_{da}g^{ef}\partial_{d}g_{bf} - g^{cd}\partial_{e}g_{da}g^{ef}\partial_{f}g_{bc} \\ & + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{b}g_{fc} + g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{d}g_{bf} - g^{cd}\partial_{d}g_{ed}g^{ef}\partial_{f}g_{bc} - g^{cd}\partial_{d}g_{ea}g^{ef}\partial_{b}g_{fc} - g^{cd}\partial_{d}g_{ea}g^{ef}\partial_{d}g_{bf} + g^{cd}\partial_{d}g_{ea}g^{ef}\partial_{f}g_{bc}\end{aligned} \tag{ex-0308.105}$$

$$= -2\,\partial_{b}g^{ce}\partial_{d}g_{ce} + 2\,\partial_{d}g^{ce}\partial_{b}g_{ce} \tag{ex-0308.106}$$

$$= 2\,g^{cd}g^{ef}\partial_{b}g_{df}\partial_{d}g_{ce} - 2\,g^{cd}g^{ef}\partial_{d}g_{df}\partial_{b}g_{ce} \tag{ex-0308.107}$$

$$= 0 \tag{ex-0308.108}$$

Exercise 3.8 Symmetry of R_{ab} alternative solution

This differs from the previous code by the inclusion of a call to canonicalise immediately after the first two substitutions and a declaration that $\Gamma^{a}{}_{bc}$ is symmetric in bc. This pair of changes produces a more compact set of results than given above. Incidently, this also shows that $\partial_a \Gamma^{c}{}_{bc} = \partial_b \Gamma^{c}{}_{ac}$.

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
                                                    # cdb (dgab.000,dgab)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                                             + \partial_{c}{g_{b e}}
                                             - \partial_{e}{g_{b c}}).
                                                    # cdb (Gamma.000,Gamma)

Rabcd := R^{a}_{b c d} ->
         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
       - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                    # cdb (Rabcd.000,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                    # cdb (Rab.000,Rab)

expr := 4 (R_{a b} - R_{b a}).                      # cdb (ex-0308.200,expr)

substitute   (expr, Rab)                            # cdb (ex-0308.201,expr)
substitute   (expr, Rabcd)                          # cdb (ex-0308.202,expr)
canonicalise (expr)                                 # cdb (ex-0308.203,expr)
substitute   (expr, Gamma)                          # cdb (ex-0308.204,expr)

distribute   (expr)                                 # cdb (ex-0308.205,expr)
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33  product_rule (expr)                                  # cdb (ex-0308.206,expr)
34  canonicalise (expr)                                  # cdb (ex-0308.207,expr)
35
36  substitute   (expr, dgab)                            # cdb (ex-0308.208,expr)
37  canonicalise (expr)                                  # cdb (ex-0308.209,expr)
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$$
\begin{aligned}
4\,R_{ab} - 4\,R_{ba} &= 4\,R^{c}_{\ acb} - 4\,R^{c}_{\ bca} && \text{(ex-0308.201)}\\
&= 4\,\partial_{c}\Gamma^{c}_{\ ab} + 4\,\Gamma^{c}_{\ ec}\Gamma^{e}_{\ ab} - 4\,\partial_{b}\Gamma^{c}_{\ ac} - 4\,\Gamma^{c}_{\ eb}\Gamma^{e}_{\ ac} - 4\,\partial_{c}\Gamma^{c}_{\ ba} - 4\,\Gamma^{c}_{\ ec}\Gamma^{e}_{\ ba} + 4\,\partial_{a}\Gamma^{c}_{\ bc} + 4\,\Gamma^{c}_{\ ea}\Gamma^{e}_{\ bc} && \text{(ex-0308.202)}\\
&= -4\,\partial_{b}\Gamma^{c}_{\ ac} + 4\,\partial_{a}\Gamma^{c}_{\ bc} && \text{(ex-0308.203)}\\
&= -2\,\partial_{b}(g^{ce}\,(\partial_{d}g_{ec} + \partial_{c}g_{ae} - \partial_{e}g_{ac})) + 2\,\partial_{a}(g^{ce}\,(\partial_{b}g_{ec} + \partial_{c}g_{be} - \partial_{e}g_{bc})) && \text{(ex-0308.204)}\\
&= -2\,\partial_{b}(g^{ce}\partial_{d}g_{ec}) - 2\,\partial_{b}(g^{ce}\partial_{c}g_{ae}) + 2\,\partial_{b}(g^{ce}\partial_{e}g_{ac}) + 2\,\partial_{a}(g^{ce}\partial_{b}g_{ec}) + 2\,\partial_{a}(g^{ce}\partial_{c}g_{be}) - 2\,\partial_{a}(g^{ce}\partial_{e}g_{bc}) && \text{(ex-0308.205)}\\
&= -2\,\partial_{b}g^{ce}\partial_{d}g_{ec} - 2\,g^{ce}\partial_{bd}g_{ec} - 2\,\partial_{b}g^{ce}\partial_{c}g_{ae} - 2\,g^{ce}\partial_{bc}g_{ae} + 2\,\partial_{b}g^{ce}\partial_{e}g_{ac} + 2\,g^{ce}\partial_{be}g_{ac} + 2\,\partial_{d}g^{ce}\partial_{b}g_{ec} + 2\,g^{ce}\partial_{ab}g_{ec} + 2\,\partial_{d}g^{ce}\partial_{c}g_{be} + 2\,g^{ce}\partial_{ac}g_{be}\\
&\quad - 2\,\partial_{d}g^{ce}\partial_{e}g_{bc} - 2\,g^{ce}\partial_{ae}g_{bc} && \text{(ex-0308.206)}\\
&= -2\,\partial_{b}g^{ce}\partial_{d}g_{ce} + 2\,\partial_{d}g^{ce}\partial_{b}g_{ce} && \text{(ex-0308.207)}\\
&= 2\,g^{cd}g^{ef}\partial_{b}g_{df}\partial_{d}g_{ce} - 2\,g^{cd}g^{ef}\partial_{d}g_{df}\partial_{b}g_{ce} && \text{(ex-0308.208)}\\
&= 0 && \text{(ex-0308.209)}
\end{aligned}
$$