Exercise 3.7 Commutation of ∇ on the Riemann tensor – direct computation

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

;::Symbol;

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

RabcdD :=   \partial_{c}{\Gamma_{a b d}}
          - \partial_{d}{\Gamma_{a b c}}
          + \Gamma_{e a d} \Gamma^{e}_{b c}
          - \Gamma_{e a c} \Gamma^{e}_{b d} -> R_{a b c d}.            # cdb(Rabcd.010,RabcdD)

RabcdU :=   \partial_{c}{\Gamma^{a}_{b d}}
          - \partial_{d}{\Gamma^{a}_{b c}}
          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
          - \Gamma^{e}_{b c} \Gamma^{a}_{d e} -> R^{a}_{b c d}.       # cdb(Rabcd.000,RabcdU)

d1Rabcd := R_{a b c d ; e} ->   \partial_{e}{R_{a b c d}}
                             - \Gamma^{f}_{a e} R_{f b c d}
                             - \Gamma^{f}_{b e} R_{a f c d}
                             - \Gamma^{f}_{c e} R_{a b f d}
                             - \Gamma^{f}_{d e} R_{a b c f}.         # cdb(d1Rabcd.000,d1Rabcd)

d2Rabcd := R_{a b c d ; e ; f} ->   \partial_{f}{R_{a b c d ; e}}
                                 - \Gamma^{g}_{a f} R_{g b c d ; e}
                                 - \Gamma^{g}_{b f} R_{a g c d ; e}
                                 - \Gamma^{g}_{c f} R_{a b g d ; e}
                                 - \Gamma^{g}_{d f} R_{a b c g ; e}
                                 - \Gamma^{g}_{e f} R_{a b c d ; g}. # cdb(d2Rabcd.000,d2Rabcd)

substitute (d2Rabcd,d1Rabcd)                                         # cdb (d2Rabcd.001, d2Rabcd)

expr := R_{a b c d ; e ; f} - R_{a b c d ; f ; e}.                   # cdb (ex-0307.100, expr)

substitute     (expr,d2Rabcd)                                        # cdb (ex-0307.101, expr)
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distribute      (expr)                                          # cdb (ex-0307.102, expr)
product_rule    (expr)                                          # cdb (ex-0307.103, expr)

sort_product    (expr)                                          # cdb (ex-0307.104, expr)
rename_dummies (expr)                                           # cdb (ex-0307.105, expr)
canonicalise    (expr)                                          # cdb (ex-0307.106, expr)
factor_out      (expr,$R_{a? b? c? d?}$)                        # cdb (ex-0307.107, expr)

substitute      (expr,RabcdU)                                   # cdb (ex-0307.108, expr)
substitute      (expr,$R^{a}_{b c d} -> -R^{a}_{b d c}$)        # cdb (ex-0307.109, expr)
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\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_f \left(\partial_e R_{abcd} - \Gamma^g{}_{ae} R_{gbcd} - \Gamma^g{}_{be} R_{agcd} - \Gamma^g{}_{ce} R_{abgd} - \Gamma^g{}_{de} R_{abcg} \right) - \Gamma^g{}_{af} \left(\partial_e R_{gbcd} - \Gamma^h{}_{ge} R_{hbcd} - \Gamma^h{}_{be} R_{ghcd} - \Gamma^h{}_{ce} R_{gbhd} - \Gamma^h{}_{de} R_{gbch} \right) \\
& - \Gamma^g{}_{bf} \left(\partial_e R_{agcd} - \Gamma^h{}_{ae} R_{hgcd} - \Gamma^h{}_{ge} R_{ahcd} - \Gamma^h{}_{ce} R_{aghd} - \Gamma^h{}_{de} R_{agch} \right) \\
& - \Gamma^g{}_{cf} \left(\partial_e R_{abgd} - \Gamma^h{}_{ae} R_{hbgd} - \Gamma^h{}_{be} R_{ahgd} - \Gamma^h{}_{ge} R_{abhd} - \Gamma^h{}_{de} R_{abgh} \right) \\
& - \Gamma^g{}_{df} \left(\partial_e R_{abcg} - \Gamma^h{}_{ae} R_{hbcg} - \Gamma^h{}_{be} R_{ahcg} - \Gamma^h{}_{ce} R_{abhg} - \Gamma^h{}_{ge} R_{abch} \right) \\
& - \Gamma^g{}_{ef} \left(\partial_g R_{abcd} - \Gamma^h{}_{ag} R_{hbcd} - \Gamma^h{}_{bg} R_{ahcd} - \Gamma^h{}_{cg} R_{abhd} - \Gamma^h{}_{dg} R_{abch} \right) - \partial_e \left(\partial_f R_{abcd} - \Gamma^g{}_{af} R_{gbcd} - \Gamma^g{}_{bf} R_{agcd} - \Gamma^g{}_{cf} R_{abgd} - \Gamma^g{}_{df} R_{abcg} \right) \\
& + \Gamma^g{}_{ae} \left(\partial_f R_{gbcd} - \Gamma^h{}_{gf} R_{hbcd} - \Gamma^h{}_{bf} R_{ghcd} - \Gamma^h{}_{cf} R_{gbhd} - \Gamma^h{}_{df} R_{gbch} \right) \\
& + \Gamma^g{}_{be} \left(\partial_f R_{agcd} - \Gamma^h{}_{af} R_{hgcd} - \Gamma^h{}_{gf} R_{ahcd} - \Gamma^h{}_{cf} R_{aghd} - \Gamma^h{}_{df} R_{agch} \right) \\
& + \Gamma^g{}_{ce} \left(\partial_f R_{abgd} - \Gamma^h{}_{af} R_{hbgd} - \Gamma^h{}_{bf} R_{ahgd} - \Gamma^h{}_{gf} R_{abhd} - \Gamma^h{}_{df} R_{abgh} \right) \\
& + \Gamma^g{}_{de} \left(\partial_f R_{abcg} - \Gamma^h{}_{af} R_{hbcg} - \Gamma^h{}_{bf} R_{ahcg} - \Gamma^h{}_{cf} R_{abhg} - \Gamma^h{}_{gf} R_{abch} \right) \\
& + \Gamma^g{}_{fe} \left(\partial_g R_{abcd} - \Gamma^h{}_{ag} R_{hbcd} - \Gamma^h{}_{bg} R_{ahcd} - \Gamma^h{}_{cg} R_{abhd} - \Gamma^h{}_{dg} R_{abch} \right)
\end{aligned}
\tag{ex-0307.101}
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$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_{fe} R_{abcd} - \partial_f \left(\Gamma^g{}_{ae} R_{gbcd} \right) - \partial_f \left(\Gamma^g{}_{be} R_{agcd} \right) - \partial_f \left(\Gamma^g{}_{ce} R_{abgd} \right) - \partial_f \left(\Gamma^g{}_{de} R_{abcg} \right) - \Gamma^g{}_{af} \partial_e R_{gbcd} + \Gamma^g{}_{af} \Gamma^h{}_{ge} R_{hbcd} + \Gamma^g{}_{af} \Gamma^h{}_{be} R_{ghcd} \\
& + \Gamma^g{}_{af} \Gamma^h{}_{ce} R_{gbhd} + \Gamma^g{}_{af} \Gamma^h{}_{de} R_{gbch} - \Gamma^g{}_{bf} \partial_e R_{agcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ae} R_{hgcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ge} R_{ahcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ce} R_{aghd} + \Gamma^g{}_{bf} \Gamma^h{}_{de} R_{agch} \\
& - \Gamma^g{}_{cf} \partial_e R_{abgd} + \Gamma^g{}_{cf} \Gamma^h{}_{ae} R_{hbgd} + \Gamma^g{}_{cf} \Gamma^h{}_{be} R_{ahgd} + \Gamma^g{}_{cf} \Gamma^h{}_{ge} R_{abhd} + \Gamma^g{}_{cf} \Gamma^h{}_{de} R_{abgh} - \Gamma^g{}_{df} \partial_e R_{abcg} + \Gamma^g{}_{df} \Gamma^h{}_{ae} R_{hbcg} + \Gamma^g{}_{df} \Gamma^h{}_{be} R_{ahcg} \\
& + \Gamma^g{}_{df} \Gamma^h{}_{ce} R_{abhg} + \Gamma^g{}_{df} \Gamma^h{}_{ge} R_{abch} - \Gamma^g{}_{ef} \partial_g R_{abcd} + \Gamma^g{}_{ef} \Gamma^h{}_{ag} R_{hbcd} + \Gamma^g{}_{ef} \Gamma^h{}_{bg} R_{ahcd} + \Gamma^g{}_{ef} \Gamma^h{}_{cg} R_{abhd} + \Gamma^g{}_{ef} \Gamma^h{}_{dg} R_{abch} - \partial_{ef} R_{abcd} \\
& + \partial_e \left(\Gamma^g{}_{af} R_{gbcd} \right) + \partial_e \left(\Gamma^g{}_{bf} R_{agcd} \right) + \partial_e \left(\Gamma^g{}_{cf} R_{abgd} \right) + \partial_e \left(\Gamma^g{}_{df} R_{abcg} \right) + \Gamma^g{}_{ae} \partial_f R_{gbcd} - \Gamma^g{}_{ae} \Gamma^h{}_{gf} R_{hbcd} - \Gamma^g{}_{ae} \Gamma^h{}_{bf} R_{ghcd} \\
& - \Gamma^g{}_{ae} \Gamma^h{}_{cf} R_{gbhd} - \Gamma^g{}_{ae} \Gamma^h{}_{df} R_{gbch} + \Gamma^g{}_{be} \partial_f R_{agcd} - \Gamma^g{}_{be} \Gamma^h{}_{af} R_{hgcd} - \Gamma^g{}_{be} \Gamma^h{}_{gf} R_{ahcd} - \Gamma^g{}_{be} \Gamma^h{}_{cf} R_{aghd} - \Gamma^g{}_{be} \Gamma^h{}_{df} R_{agch} \\
& + \Gamma^g{}_{ce} \partial_f R_{abgd} - \Gamma^g{}_{ce} \Gamma^h{}_{af} R_{hbgd} - \Gamma^g{}_{ce} \Gamma^h{}_{bf} R_{ahgd} - \Gamma^g{}_{ce} \Gamma^h{}_{gf} R_{abhd} - \Gamma^g{}_{ce} \Gamma^h{}_{df} R_{abgh} + \Gamma^g{}_{de} \partial_f R_{abcg} - \Gamma^g{}_{de} \Gamma^h{}_{af} R_{hbcg} - \Gamma^g{}_{de} \Gamma^h{}_{bf} R_{ahcg} \\
& - \Gamma^g{}_{de} \Gamma^h{}_{cf} R_{abhg} - \Gamma^g{}_{de} \Gamma^h{}_{gf} R_{abch} + \Gamma^g{}_{fe} \partial_g R_{abcd} - \Gamma^g{}_{fe} \Gamma^h{}_{ag} R_{hbcd} - \Gamma^g{}_{fe} \Gamma^h{}_{bg} R_{ahcd} - \Gamma^g{}_{fe} \Gamma^h{}_{cg} R_{abhd} - \Gamma^g{}_{fe} \Gamma^h{}_{dg} R_{abch}
\end{aligned}
\tag{ex-0307.102}
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$$
\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = {} & \partial_{fe} R_{abcd} - \partial_f \Gamma^g{}_{ae} R_{gbcd} - \partial_f \Gamma^g{}_{be} R_{agcd} - \partial_f \Gamma^g{}_{ce} R_{abgd} - \partial_f \Gamma^g{}_{de} R_{abcg} + \Gamma^g{}_{af} \Gamma^h{}_{ge} R_{hbcd} + \Gamma^g{}_{af} \Gamma^h{}_{be} R_{ghcd} + \Gamma^g{}_{af} \Gamma^h{}_{ce} R_{gbhd} \\
& + \Gamma^g{}_{af} \Gamma^h{}_{de} R_{gbch} + \Gamma^g{}_{bf} \Gamma^h{}_{ae} R_{hgcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ge} R_{ahcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ce} R_{aghd} + \Gamma^g{}_{bf} \Gamma^h{}_{de} R_{agch} + \Gamma^g{}_{cf} \Gamma^h{}_{ae} R_{hbgd} + \Gamma^g{}_{cf} \Gamma^h{}_{be} R_{ahgd} \\
& + \Gamma^g{}_{cf} \Gamma^h{}_{ge} R_{abhd} + \Gamma^g{}_{cf} \Gamma^h{}_{de} R_{abgh} + \Gamma^g{}_{df} \Gamma^h{}_{ae} R_{hbcg} + \Gamma^g{}_{df} \Gamma^h{}_{be} R_{ahcg} + \Gamma^g{}_{df} \Gamma^h{}_{ce} R_{abhg} + \Gamma^g{}_{df} \Gamma^h{}_{ge} R_{abch} - \Gamma^g{}_{ef} \partial_g R_{abcd} \\
& + \Gamma^g{}_{ef} \Gamma^h{}_{ag} R_{hbcd} + \Gamma^g{}_{ef} \Gamma^h{}_{bg} R_{ahcd} + \Gamma^g{}_{ef} \Gamma^h{}_{cg} R_{abhd} + \Gamma^g{}_{ef} \Gamma^h{}_{dg} R_{abch} - \partial_{ef} R_{abcd} + \partial_e \Gamma^g{}_{af} R_{gbcd} + \partial_e \Gamma^g{}_{bf} R_{agcd} + \partial_e \Gamma^g{}_{cf} R_{abgd} \\
& + \partial_e \Gamma^g{}_{df} R_{abcg} - \Gamma^g{}_{ae} \Gamma^h{}_{gf} R_{hbcd} - \Gamma^g{}_{ae} \Gamma^h{}_{bf} R_{ghcd} - \Gamma^g{}_{ae} \Gamma^h{}_{cf} R_{gbhd} - \Gamma^g{}_{ae} \Gamma^h{}_{df} R_{gbch} - \Gamma^g{}_{be} \Gamma^h{}_{af} R_{hgcd} - \Gamma^g{}_{be} \Gamma^h{}_{gf} R_{ahcd} \\
& - \Gamma^g{}_{be} \Gamma^h{}_{cf} R_{aghd} - \Gamma^g{}_{be} \Gamma^h{}_{df} R_{agch} - \Gamma^g{}_{ce} \Gamma^h{}_{af} R_{hbgd} - \Gamma^g{}_{ce} \Gamma^h{}_{bf} R_{ahgd} - \Gamma^g{}_{ce} \Gamma^h{}_{gf} R_{abhd} - \Gamma^g{}_{ce} \Gamma^h{}_{df} R_{abgh} - \Gamma^g{}_{de} \Gamma^h{}_{af} R_{hbcg} \\
& - \Gamma^g{}_{de} \Gamma^h{}_{bf} R_{ahcg} - \Gamma^g{}_{de} \Gamma^h{}_{cf} R_{abhg} - \Gamma^g{}_{de} \Gamma^h{}_{gf} R_{abch} + \Gamma^g{}_{fe} \partial_g R_{abcd} - \Gamma^g{}_{fe} \Gamma^h{}_{ag} R_{hbcd} - \Gamma^g{}_{fe} \Gamma^h{}_{bg} R_{ahcd} - \Gamma^g{}_{fe} \Gamma^h{}_{cg} R_{abhd} \\
& - \Gamma^g{}_{fe} \Gamma^h{}_{dg} R_{abch}
\end{aligned}
\tag{ex-0307.103}
$$

$$R_{abcd;e;f} - R_{abcd;f;e} = \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma^g{}_{ae} - R_{agcd} \partial_f \Gamma^g{}_{be} - R_{abgd} \partial_f \Gamma^g{}_{ce} - R_{abcg} \partial_f \Gamma^g{}_{de} + R_{hbcd} \Gamma^g{}_{af} \Gamma^h{}_{ge} + R_{ghcd} \Gamma^g{}_{af} \Gamma^h{}_{be} + R_{gbhd} \Gamma^g{}_{af} \Gamma^h{}_{ce}$$
$$+ R_{gbch} \Gamma^g{}_{af} \Gamma^h{}_{de} + R_{hgcd} \Gamma^g{}_{bf} \Gamma^h{}_{ae} + R_{ahcd} \Gamma^g{}_{bf} \Gamma^h{}_{ge} + R_{aghd} \Gamma^g{}_{bf} \Gamma^h{}_{ce} + R_{agch} \Gamma^g{}_{bf} \Gamma^h{}_{de} + R_{hbgd} \Gamma^g{}_{cf} \Gamma^h{}_{ae} + R_{ahgd} \Gamma^g{}_{cf} \Gamma^h{}_{be}$$
$$+ R_{abhd} \Gamma^g{}_{cf} \Gamma^h{}_{ge} + R_{abgh} \Gamma^g{}_{cf} \Gamma^h{}_{de} + R_{hbcg} \Gamma^g{}_{df} \Gamma^h{}_{ae} + R_{ahcg} \Gamma^g{}_{df} \Gamma^h{}_{be} + R_{abhg} \Gamma^g{}_{df} \Gamma^h{}_{ce} + R_{abch} \Gamma^g{}_{df} \Gamma^h{}_{ge} - \Gamma^g{}_{ef} \partial_g R_{abcd}$$
$$+ R_{hbcd} \Gamma^g{}_{ef} \Gamma^h{}_{ag} + R_{ahcd} \Gamma^g{}_{ef} \Gamma^h{}_{bg} + R_{abhd} \Gamma^g{}_{ef} \Gamma^h{}_{cg} + R_{abch} \Gamma^g{}_{ef} \Gamma^h{}_{dg} - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma^g{}_{af} + R_{agcd} \partial_e \Gamma^g{}_{bf} + R_{abgd} \partial_e \Gamma^g{}_{cf}$$
$$+ R_{abcg} \partial_e \Gamma^g{}_{df} - R_{hbcd} \Gamma^g{}_{ae} \Gamma^h{}_{gf} - R_{ghcd} \Gamma^g{}_{ae} \Gamma^h{}_{bf} - R_{gbhd} \Gamma^g{}_{ae} \Gamma^h{}_{cf} - R_{gbch} \Gamma^g{}_{ae} \Gamma^h{}_{df} - R_{hgcd} \Gamma^g{}_{be} \Gamma^h{}_{af} - R_{ahcd} \Gamma^g{}_{be} \Gamma^h{}_{gf}$$
$$- R_{aghd} \Gamma^g{}_{be} \Gamma^h{}_{cf} - R_{agch} \Gamma^g{}_{be} \Gamma^h{}_{df} - R_{hbgd} \Gamma^g{}_{ce} \Gamma^h{}_{af} - R_{ahgd} \Gamma^g{}_{ce} \Gamma^h{}_{bf} - R_{abhd} \Gamma^g{}_{ce} \Gamma^h{}_{gf} - R_{abgh} \Gamma^g{}_{ce} \Gamma^h{}_{df} - R_{hbcg} \Gamma^g{}_{de} \Gamma^h{}_{af}$$
$$- R_{ahcg} \Gamma^g{}_{de} \Gamma^h{}_{bf} - R_{abhg} \Gamma^g{}_{de} \Gamma^h{}_{cf} - R_{abch} \Gamma^g{}_{de} \Gamma^h{}_{gf} + \Gamma^g{}_{fe} \partial_g R_{abcd} - R_{hbcd} \Gamma^g{}_{fe} \Gamma^h{}_{ag} - R_{ahcd} \Gamma^g{}_{fe} \Gamma^h{}_{bg} - R_{abhd} \Gamma^g{}_{fe} \Gamma^h{}_{cg}$$
$$- R_{abch} \Gamma^g{}_{fe} \Gamma^h{}_{dg} \tag{ex-0307.104}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma^g{}_{ae} - R_{agcd} \partial_f \Gamma^g{}_{be} - R_{abgd} \partial_f \Gamma^g{}_{ce} - R_{abcg} \partial_f \Gamma^g{}_{de} + R_{gbcd} \Gamma^h{}_{af} \Gamma^g{}_{he} + R_{ghcd} \Gamma^g{}_{af} \Gamma^h{}_{be} + R_{gbhd} \Gamma^g{}_{af} \Gamma^h{}_{ce}$$
$$+ R_{gbch} \Gamma^g{}_{af} \Gamma^h{}_{de} + R_{ghcd} \Gamma^h{}_{bf} \Gamma^g{}_{ae} + R_{agcd} \Gamma^h{}_{bf} \Gamma^g{}_{he} + R_{aghd} \Gamma^g{}_{bf} \Gamma^h{}_{ce} + R_{agch} \Gamma^g{}_{bf} \Gamma^h{}_{de} + R_{gbhd} \Gamma^h{}_{cf} \Gamma^g{}_{ae} + R_{aghd} \Gamma^h{}_{cf} \Gamma^g{}_{be}$$
$$+ R_{abgd} \Gamma^h{}_{cf} \Gamma^g{}_{he} + R_{abgh} \Gamma^g{}_{cf} \Gamma^h{}_{de} + R_{gbch} \Gamma^h{}_{df} \Gamma^g{}_{ae} + R_{agch} \Gamma^h{}_{df} \Gamma^g{}_{be} + R_{abgh} \Gamma^h{}_{df} \Gamma^g{}_{ce} + R_{abcg} \Gamma^h{}_{df} \Gamma^g{}_{he} - \Gamma^g{}_{ef} \partial_g R_{abcd}$$
$$+ R_{gbcd} \Gamma^h{}_{ef} \Gamma^g{}_{ah} + R_{agcd} \Gamma^h{}_{ef} \Gamma^g{}_{bh} + R_{abgd} \Gamma^h{}_{ef} \Gamma^g{}_{ch} + R_{abcg} \Gamma^h{}_{ef} \Gamma^g{}_{dh} - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma^g{}_{af} + R_{agcd} \partial_e \Gamma^g{}_{bf} + R_{abgd} \partial_e \Gamma^g{}_{cf}$$
$$+ R_{abcg} \partial_e \Gamma^g{}_{df} - R_{gbcd} \Gamma^h{}_{ae} \Gamma^g{}_{hf} - R_{ghcd} \Gamma^g{}_{ae} \Gamma^h{}_{bf} - R_{gbhd} \Gamma^g{}_{ae} \Gamma^h{}_{cf} - R_{gbch} \Gamma^g{}_{ae} \Gamma^h{}_{df} - R_{ghcd} \Gamma^h{}_{be} \Gamma^g{}_{af} - R_{agcd} \Gamma^h{}_{be} \Gamma^g{}_{hf}$$
$$- R_{aghd} \Gamma^g{}_{be} \Gamma^h{}_{cf} - R_{agch} \Gamma^g{}_{be} \Gamma^h{}_{df} - R_{gbhd} \Gamma^h{}_{ce} \Gamma^g{}_{af} - R_{aghd} \Gamma^h{}_{ce} \Gamma^g{}_{bf} - R_{abgd} \Gamma^h{}_{ce} \Gamma^g{}_{hf} - R_{abgh} \Gamma^g{}_{ce} \Gamma^h{}_{df} - R_{gbch} \Gamma^h{}_{de} \Gamma^g{}_{af}$$
$$- R_{agch} \Gamma^h{}_{de} \Gamma^g{}_{bf} - R_{abgh} \Gamma^h{}_{de} \Gamma^g{}_{cf} - R_{abcg} \Gamma^h{}_{de} \Gamma^g{}_{hf} + \Gamma^g{}_{fe} \partial_g R_{abcd} - R_{gbcd} \Gamma^h{}_{fe} \Gamma^g{}_{ah} - R_{agcd} \Gamma^h{}_{fe} \Gamma^g{}_{bh} - R_{abgd} \Gamma^h{}_{fe} \Gamma^g{}_{ch}$$
$$- R_{abcg} \Gamma^h{}_{fe} \Gamma^g{}_{dh} \tag{ex-0307.105}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd} \partial_f \Gamma^g{}_{ae} - R_{agcd} \partial_f \Gamma^g{}_{be} - R_{abgd} \partial_f \Gamma^g{}_{ce} - R_{abcg} \partial_f \Gamma^g{}_{de} + R_{gbcd} \Gamma^h{}_{af} \Gamma^g{}_{eh} + R_{agcd} \Gamma^h{}_{bf} \Gamma^g{}_{eh} + R_{abgd} \Gamma^h{}_{cf} \Gamma^g{}_{eh} + R_{abcg} \Gamma^h{}_{df} \Gamma^g{}_{eh}$$
$$+ R_{gbcd} \partial_e \Gamma^g{}_{af} + R_{agcd} \partial_e \Gamma^g{}_{bf} + R_{abgd} \partial_e \Gamma^g{}_{cf} + R_{abcg} \partial_e \Gamma^g{}_{df} - R_{gbcd} \Gamma^h{}_{ae} \Gamma^g{}_{fh} - R_{agcd} \Gamma^h{}_{be} \Gamma^g{}_{fh} - R_{abgd} \Gamma^h{}_{ce} \Gamma^g{}_{fh}$$
$$- R_{abcg} \Gamma^h{}_{de} \Gamma^g{}_{fh} \tag{ex-0307.106}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd} \left(-\partial_f \Gamma^g{}_{ae} + \Gamma^h{}_{af} \Gamma^g{}_{eh} + \partial_e \Gamma^g{}_{af} - \Gamma^h{}_{ae} \Gamma^g{}_{fh} \right) + R_{agcd} \left(-\partial_f \Gamma^g{}_{be} + \Gamma^h{}_{bf} \Gamma^g{}_{eh} + \partial_e \Gamma^g{}_{bf} - \Gamma^h{}_{be} \Gamma^g{}_{fh} \right)$$
$$+ R_{abgd} \left(-\partial_f \Gamma^g{}_{ce} + \Gamma^h{}_{cf} \Gamma^g{}_{eh} + \partial_e \Gamma^g{}_{cf} - \Gamma^h{}_{ce} \Gamma^g{}_{fh} \right) + R_{abcg} \left(-\partial_f \Gamma^g{}_{de} + \Gamma^h{}_{df} \Gamma^g{}_{eh} + \partial_e \Gamma^g{}_{df} - \Gamma^h{}_{de} \Gamma^g{}_{fh} \right) \tag{ex-0307.107}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd} R^g{}_{afe} - R_{agcd} R^g{}_{bfe} - R_{abgd} R^g{}_{cfe} - R_{abcg} R^g{}_{dfe} \tag{ex-0307.108}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd} R^g{}_{aef} + R_{agcd} R^g{}_{bef} + R_{abgd} R^g{}_{cef} + R_{abcg} R^g{}_{def} \tag{ex-0307.109}$$