Example 9 The Gauss equation

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\{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u\#\}::Indices(position=independent).
    \nabla{#}::Derivative.
    K_{a b}::Symmetric.
    g^{a}_{b}::KroneckerDelta.
    # define the projection operator
    hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}.
10
11
    # 3-covariant derivative obtained by projection on 4-covariant derivative
12
13
    vpq:=v_{p q} \rightarrow h^{a}_{p} h^{b}_{q} \nabla_{b}{v_{a}}.
14
15
    # compute 3-curvature by commutation of covariant derivatives
16
17
     vpqr:= h^{a}_{p} h^{b}_{q} h^{c}_{r} ( \lambda_{c}^{c}_{v} - \lambda_{c}^{c}) . 
19
    substitute (vpq,hab)
    substitute (vpqr,vpq)
21
    distribute (vpqr)
    product_rule (vpqr)
    distribute (vpqr)
    eliminate_kronecker (vpqr)
26
27
    # standard substitutions
28
29
    substitute (vpqr,h^{a}_{b} -> 0)
    substitute (vpqr,h^{a}_{b} = 0)
31
    substitute (vpqr,\alpha_{a}{g^{b}_{c}} -> 0)
32
    33
    substitute (vpqr,h^{p}_{a} h^{q}_{b} \quad (p_{q}) -> K_{a b})
    substitute (vpqr,h^{p}_{a} h^{q}_{b} \ nabla_{p}{n^{b}} -> K_{a}^{q}$) # cdb(ex-09.095, vpqr)
```

```
37
                                 # tidy up
39
                                 \{v_{a}, h^{a}_{b}, K_{a}^{b}, K
40
41
                                                                                                                                                                                                                                                                                                                                                                                                                                     # cdb(ex-09.096, vpqr)
                                 sort_product
                                                                                                                                       (vpqr)
42
                                                                                                                                                                                                                                                                                                                                                                                                                                     # cdb(ex-09.097, vpqr)
                                 rename_dummies (vpqr)
                                                                                                                                                                                                                                                                                                                                                                                                                                     # cdb(ex-09.098, vpgr)
                                  canonicalise
                                                                                                                                       (vpqr)
                                                                                                                                      (vpqr,$h^{a?}_{b?}$)
                                                                                                                                                                                                                                                                                                                                                                                                                                    # cdb(ex-09.099, vpqr)
                                 factor_out
                                                                                                                                       (vpqr,$v_{a?}$)
                                                                                                                                                                                                                                                                                                                                                                                                                                     # cdb(ex-09.101, vpqr)
                                 factor_out
46
47
                                 checkpoint.append (vpqr)
```

$$(D_{r}D_{q} - D_{q}D_{r})v_{p} = h^{e}_{\ p}h^{d}_{\ q}h^{c}_{\ r}\nabla_{c}\left(\nabla_{d}v_{e}\right) - h^{e}_{\ p}K_{rq}n^{d}\nabla_{d}v_{e} + K_{q}{}^{b}K_{rp}v_{b} - h^{d}_{\ p}h^{b}_{\ q}h^{e}_{\ r}\nabla_{b}\left(\nabla_{e}v_{d}\right) + h^{d}_{\ p}K_{qr}n^{e}\nabla_{e}v_{d} - K_{qp}K_{r}{}^{c}v_{c}$$
 (ex-09.095)
$$= h^{c}_{\ r}h^{d}_{\ q}h^{e}_{\ p}\nabla_{c}\left(\nabla_{d}v_{e}\right) - h^{e}_{\ p}K_{rq}\nabla_{d}v_{e}n^{d} + v_{b}K_{q}{}^{b}K_{rp} - h^{b}_{\ q}h^{d}_{\ p}h^{e}_{\ r}\nabla_{b}\left(\nabla_{e}v_{d}\right) + h^{d}_{\ p}K_{qr}\nabla_{e}v_{d}n^{e} - v_{c}K_{r}{}^{c}K_{qp}$$
 (ex-09.096)
$$= h^{a}_{\ r}h^{b}_{\ q}h^{c}_{\ p}\nabla_{a}\left(\nabla_{b}v_{c}\right) - h^{b}_{\ p}K_{rq}\nabla_{a}v_{b}n^{a} + v_{a}K_{q}{}^{a}K_{rp} - h^{a}_{\ q}h^{c}_{\ p}h^{b}_{\ r}\nabla_{a}\left(\nabla_{b}v_{c}\right) + h^{b}_{\ p}K_{qr}\nabla_{a}v_{b}n^{a} - v_{a}K_{r}{}^{a}K_{qp}$$
 (ex-09.097)
$$= h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}\nabla_{c}\left(\nabla_{b}v_{a}\right) + v_{a}K_{q}{}^{a}K_{pr} - h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}\nabla_{b}\left(\nabla_{c}v_{a}\right) - v_{a}K_{r}{}^{a}K_{pq}$$
 (ex-09.098)
$$= v_{a}K_{q}{}^{a}K_{pr} - v_{a}K_{r}{}^{a}K_{pq} + h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}\left(\nabla_{c}\left(\nabla_{b}v_{a}\right) - \nabla_{b}\left(\nabla_{c}v_{a}\right)\right)$$
 (ex-09.101)

```
R{#}::LaTeXForm("{{\strut}^g R}").
     gRabcd := \\nabla_{c}\\nabla_{b}{v_{a}}
              -\nabla_{b}{\nabla_{c}_{v_{a}}} - R^{d}_{a b c} v_{d}.
     substitute
                     (vpqr,gRabcd)
                                                                  # cdb(ex-09.102, vpqr)
                                                                  # cdb(ex-09.103, vpqr)
     distribute
                     (vpqr)
                     (vpqr, v_{a} -> h^{b}_{a} v_{b})
                                                                  # cdb(ex-09.104, vpqr)
     substitute
                     (vpqr, h^{b}_{a} K_{c}^{a} -> K_{c}^{b})
     substitute
                                                                  # cdb(ex-09.105, vpqr)
     sort_product
                                                                  # cdb(ex-09.106, vpqr)
                     (vpqr)
10
     rename_dummies (vpqr)
                                                                  # cdb(ex-09.107, vpqr)
11
                                                                  # cdb(ex-09.108, vpqr)
     canonicalise
                     (vpqr)
                     (vpqr,$v_{a?}$)
                                                                  # cdb(ex-09.109, vpqr)
     factor_out
13
                     (vpqr, v_{a}->1)
                                                                  # cdb(ex-09.110, vpqr)
     substitute
14
                                                                  # cdb(ex-09.111, vpgr)
     sort_product
                     (vpqr)
15
16
     checkpoint.append (vpqr)
17
```

$$(D_{r}D_{q} - D_{q}D_{r})v_{p} = h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r} (\nabla_{c} (\nabla_{b}v_{a}) - \nabla_{b} (\nabla_{c}v_{a})) + v_{a} (K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq})$$

$$= h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}^{\ g}R^{d}_{\ abc}v_{d} + v_{a} (K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq})$$

$$= h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}^{\ g}R^{d}_{\ abc}v_{d} + v_{a}K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}^{\ g}R^{d}_{\ abc}h^{c}_{\ d}v_{e} + h^{b}_{\ a}v_{b}K_{q}{}^{a}K_{pr} - h^{b}_{\ a}v_{b}K_{r}{}^{a}K_{pq}$$

$$= h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}^{\ g}R^{d}_{\ abc}h^{c}_{\ d}v_{e} + K_{q}{}^{b}v_{b}K_{pr} - K_{r}{}^{b}v_{b}K_{pq}$$

$$= h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}^{\ g}R^{d}_{\ abc}h^{c}_{\ d}v_{e} + K_{q}{}^{b}v_{b}K_{pr} - K_{r}{}^{b}v_{b}K_{pq}$$

$$= v_{e}h^{a}_{\ p}h^{b}_{\ q}h^{c}_{\ r}h^{e}_{\ d}^{\ g}R^{d}_{\ abc} + v_{b}K_{q}{}^{b}K_{pr} - v_{b}K_{r}{}^{b}K_{pq}$$

$$= v_{e}h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{e}_{\ a}^{\ g}R^{d}_{\ bcd} + v_{a}K_{q}{}^{a}K_{pr} - v_{a}K_{r}{}^{a}K_{pq}$$

$$= v_{a}h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + v_{a}K_{q}{}^{a}K_{pr} - v_{a}K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{q}{}^{a}K_{pr} - K_{r}{}^{a}K_{pq}$$

$$= v_{a}(h^{b}_{\ p}h^{c}_{\ q}h^{d}_{\ r}h^{a}_{\ e}^{\ g}R^{e}_{\ bcd} + K_{$$

(ex-09.111)

 ${}^{h}R^{a}_{par} = h^{a}_{e}h^{b}_{p}h^{c}_{a}h^{d}_{r}{}^{g}R^{e}_{pcd} + K_{a}{}^{a}K_{pr} - K_{r}{}^{a}K_{pa}$