Example 13a The Weyl tensor vanishes in 3d – direct proof

```
{x,y,z}::Coordinate.
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices(values={x,y,z},position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

{\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.

GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                             + \partial_{c}{g_{b d}}
                                             - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)

GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
                                   + \partial_{c}{g_{b a}}
                                   - \partial_{a}{g_{b c}}).           # cdb(Gamma.010,GammaD)

Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
                        - \partial_{d}{\Gamma_{a b c}}
                        + \Gamma_{e a d} \Gamma^{e}_{b c}
                        - \Gamma_{e a c} \Gamma^{e}_{b d}.             # cdb (Rabcd.000,Rabcd)

Rab     := R_{a b} -> g^{c d} R_{a c b d}.                             # cdb (Rab.000,Rab)

Rscalar := R -> g^{a b} R_{a b}.                                       # cdb (R.000,Rscalar)

# Weyl in 3-dimensions

Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
                     - (g_{a c} R_{b d} - g_{a d} R_{b c})
               + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).            # cdb (ex-13a.100,Cabcd)

# Use 8 Cabcd to clear the fractions

EightCabcd := 8 @(Cabcd).                                              # cdb (ex-13a.110,EightCabcd)
```

```
substitute     (Cabcd,Rscalar)
substitute     (Cabcd,Rab)
substitute     (Cabcd,Rabcd)
substitute     (Cabcd,GammaU)
substitute     (Cabcd,GammaD)

distribute     (Cabcd)

sort_product   (Cabcd)
rename_dummies (Cabcd)
canonicalise   (Cabcd)                                              # cdb (ex-13a.101,Cabcd)

EightCabcd := 8 @(Cabcd).                                          # cdb (ex-13a.111,EightCabcd)

gab := {g_{x x} = gxx, g_{x y} = gxy, g_{x z} = gxz,
        g_{y x} = gxy, g_{y y} = gyy, g_{y z} = gyz,
        g_{z x} = gxz, g_{z y} = gyz, g_{z z} = gzz}.

complete  (gab, $g^{a b}$)
evaluate  (Cabcd,gab)                                               # cdb (ex-13a.102,Cabcd)
evaluate  (EightCabcd,gab)                                          # cdb (ex-13a.112,EightCabcd)
```

$$
\begin{aligned}
8C_{abcd} &= 8\,R_{abcd} - 8\,R_{ac}g_{bd} + 8\,R_{ad}g_{bc} - 8\,g_{ac}R_{bd} + 8\,g_{ad}R_{bc} + 4\,R\,(g_{ac}g_{bd} - g_{ad}g_{bc}) && \texttt{(ex-13a.110)}\\
&= 4\,\partial_{bc}g_{ad} - 4\,\partial_{ac}g_{bd} - 4\,\partial_{bd}g_{ac} + 4\,\partial_{ad}g_{bc} + 2\,\partial_{a}g_{de}\partial_{b}g_{cf}g^{ef} + 2\,\partial_{a}g_{de}\partial_{c}g_{bf}g^{ef} - 2\,\partial_{a}g_{de}\partial_{f}g_{bc}g^{ef} + 2\,\partial_{b}g_{ce}\partial_{d}g_{af}g^{ef} + 2\,\partial_{c}g_{be}\partial_{d}g_{af}g^{ef} - 2\,\partial_{d}g_{ae}\partial_{f}g_{bc}g^{ef}\\
&\quad - 2\,\partial_{b}g_{ce}\partial_{f}g_{ad}g^{ef} - 2\,\partial_{c}g_{be}\partial_{f}g_{ad}g^{ef} + 2\,\partial_{e}g_{ad}\partial_{f}g_{bc}g^{ef} - 2\,\partial_{a}g_{ce}\partial_{b}g_{df}g^{ef} - 2\,\partial_{a}g_{ce}\partial_{d}g_{bf}g^{ef} + 2\,\partial_{a}g_{ce}\partial_{f}g_{bd}g^{ef} - 2\,\partial_{b}g_{de}\partial_{c}g_{af}g^{ef} - 2\,\partial_{c}g_{ae}\partial_{d}g_{bf}g^{ef}\\
&\quad + 2\,\partial_{c}g_{ae}\partial_{f}g_{bd}g^{ef} + 2\,\partial_{b}g_{de}\partial_{f}g_{ac}g^{ef} + 2\,\partial_{d}g_{be}\partial_{f}g_{ac}g^{ef} - 2\,\partial_{e}g_{ac}\partial_{f}g_{bd}g^{ef} - 4\,\partial_{ce}g_{af}g_{bd}g^{ef} + 4\,\partial_{ac}g_{ef}g_{bd}g^{ef} + 4\,\partial_{ef}g_{ac}g_{bd}g^{ef} - 4\,\partial_{ae}g_{cf}g_{bd}g^{ef}\\
&\quad - 2\,\partial_{a}g_{ef}\partial_{c}g_{gh}g_{bd}g^{eg}g^{fh} - 4\,\partial_{e}g_{af}\partial_{g}g_{ch}g_{bd}g^{eg}g^{fh} + 4\,\partial_{e}g_{af}\partial_{g}g_{ch}g_{bd}g^{eh}g^{fg} + 4\,\partial_{a}g_{ce}\partial_{f}g_{gh}g_{bd}g^{eg}g^{fh} - 2\,\partial_{a}g_{ce}\partial_{f}g_{gh}g_{bd}g^{ef}g^{gh} + 4\,\partial_{c}g_{ae}\partial_{f}g_{gh}g_{bd}g^{eg}g^{fh}\\
&\quad - 2\,\partial_{c}g_{ae}\partial_{f}g_{gh}g_{bd}g^{ef}g^{gh} - 4\,\partial_{e}g_{ac}\partial_{f}g_{gh}g_{bd}g^{eg}g^{fh} + 2\,\partial_{e}g_{ac}\partial_{f}g_{gh}g_{bd}g^{ef}g^{gh} + 4\,\partial_{de}g_{af}g_{bc}g^{ef} - 4\,\partial_{ad}g_{ef}g_{bc}g^{ef} - 4\,\partial_{ef}g_{ad}g_{bc}g^{ef} + 4\,\partial_{ae}g_{df}g_{bc}g^{ef}\\
&\quad + 2\,\partial_{a}g_{ef}\partial_{d}g_{gh}g_{bc}g^{eg}g^{fh} + 4\,\partial_{e}g_{af}\partial_{g}g_{dh}g_{bc}g^{eg}g^{fh} - 4\,\partial_{e}g_{af}\partial_{g}g_{dh}g_{bc}g^{eh}g^{fg} - 4\,\partial_{a}g_{de}\partial_{f}g_{gh}g_{bc}g^{eg}g^{fh} + 2\,\partial_{a}g_{de}\partial_{f}g_{gh}g_{bc}g^{ef}g^{gh} - 4\,\partial_{d}g_{ae}\partial_{f}g_{gh}g_{bc}g^{eg}g^{fh}\\
&\quad + 2\,\partial_{d}g_{ae}\partial_{f}g_{gh}g_{bc}g^{ef}g^{gh} + 4\,\partial_{e}g_{ad}\partial_{f}g_{gh}g_{bc}g^{eg}g^{fh} - 2\,\partial_{e}g_{ad}\partial_{f}g_{gh}g_{bc}g^{ef}g^{gh} - 4\,\partial_{de}g_{bf}g_{ac}g^{ef} + 4\,\partial_{bd}g_{ef}g_{ac}g^{ef} + 4\,\partial_{ef}g_{bd}g_{ac}g^{ef} - 4\,\partial_{be}g_{df}g_{ac}g^{ef}\\
&\quad - 2\,\partial_{b}g_{ef}\partial_{d}g_{gh}g_{ac}g^{eg}g^{fh} - 4\,\partial_{e}g_{bf}\partial_{g}g_{dh}g_{ac}g^{eg}g^{fh} + 4\,\partial_{e}g_{bf}\partial_{g}g_{dh}g_{ac}g^{eh}g^{fg} + 4\,\partial_{b}g_{de}\partial_{f}g_{gh}g_{ac}g^{eg}g^{fh} - 2\,\partial_{b}g_{de}\partial_{f}g_{gh}g_{ac}g^{ef}g^{gh}\\
&\quad + 4\,\partial_{d}g_{be}\partial_{f}g_{gh}g_{ac}g^{eg}g^{fh} - 2\,\partial_{d}g_{be}\partial_{f}g_{gh}g_{ac}g^{ef}g^{gh} - 4\,\partial_{e}g_{bd}\partial_{f}g_{gh}g_{ac}g^{eg}g^{fh} + 2\,\partial_{e}g_{bd}\partial_{f}g_{gh}g_{ac}g^{ef}g^{gh} + 4\,\partial_{ce}g_{bf}g_{ad}g^{ef} - 4\,\partial_{bc}g_{ef}g_{ad}g^{ef}\\
&\quad - 4\,\partial_{ef}g_{bc}g_{ad}g^{ef} + 4\,\partial_{be}g_{cf}g_{ad}g^{ef} + 2\,\partial_{b}g_{ef}\partial_{c}g_{gh}g_{ad}g^{eg}g^{fh} + 4\,\partial_{e}g_{bf}\partial_{g}g_{ch}g_{ad}g^{eg}g^{fh} - 4\,\partial_{e}g_{bf}\partial_{g}g_{ch}g_{ad}g^{eh}g^{fg} - 4\,\partial_{b}g_{ce}\partial_{f}g_{gh}g_{ad}g^{eg}g^{fh}\\
&\quad + 2\,\partial_{b}g_{ce}\partial_{f}g_{gh}g_{ad}g^{ef}g^{gh} - 4\,\partial_{c}g_{be}\partial_{f}g_{gh}g_{ad}g^{eg}g^{fh} + 2\,\partial_{c}g_{be}\partial_{f}g_{gh}g_{ad}g^{ef}g^{gh} + 4\,\partial_{e}g_{bc}\partial_{f}g_{gh}g_{ad}g^{eg}g^{fh} - 2\,\partial_{e}g_{bc}\partial_{f}g_{gh}g_{ad}g^{ef}g^{gh} + 4\,\partial_{ef}g_{gh}g_{ac}g_{bd}g^{eg}g^{fh}\\
&\quad - 4\,\partial_{ef}g_{gh}g_{ad}g_{bc}g^{eg}g^{fh} - 4\,\partial_{ef}g_{gh}g_{ac}g_{bd}g^{ef}g^{gh} + 4\,\partial_{ef}g_{gh}g_{ad}g_{bc}g^{ef}g^{gh} - 2\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{ei}g^{fh}g^{gj} + 2\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{ei}g^{fh}g^{gj}\\
&\quad + 3\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{eh}g^{fi}g^{gj} - 3\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{eh}g^{fi}g^{gj} - 4\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{ef}g^{gi}g^{hj} + 4\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{ef}g^{gi}g^{hj}\\
&\quad + 4\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{ef}g^{gh}g^{ij} - 4\,\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{ef}g^{gh}g^{ij} - \partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{eh}g^{fg}g^{ij} + \partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{eh}g^{fg}g^{ij} && \texttt{(ex-13a.111)}\\
&= 0 && \texttt{(ex-13a.112)}
\end{aligned}
$$

Example 13b The Weyl tensor vanishes in 3d – orthonormal basis

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.

ex{#}::LaTeXForm("e_x").
ey{#}::LaTeXForm("e_y").
ez{#}::LaTeXForm("e_z").

{R_{a b c d}, g_{a b}, g^{a b}}::SortOrder.

Rab     := R_{a b} -> g^{c d} R_{a c b d}.

Rscalar := R -> g^{a b} R_{a b}.

gab := g^{a b} -> ex^{a} ex^{b} + ey^{a} ey^{b} + ez^{a} ez^{b}.

ortho := {ex^{a} ex^{b} g_{a b} -> 1, ey^{a} ey^{b} g_{a b} -> 1, ez^{a} ez^{b} g_{a b} -> 1,
          ex^{a} ey^{b} g_{a b} -> 0, ex^{a} ez^{b} g_{a b} -> 0,
          ey^{a} ex^{b} g_{a b} -> 0, ey^{a} ez^{b} g_{a b} -> 0,
          ez^{a} ex^{b} g_{a b} -> 0, ez^{a} ey^{b} g_{a b} -> 0}.

# Weyl in 3-dimensions

Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
                     - (g_{a c} R_{b d} - g_{a d} R_{b c})
               + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).   # cdb (ex-13b.100,Cabcd)


substitute     (Cabcd, Rscalar)                               # cdb(ex-13b.101,Cabcd)
substitute     (Cabcd, Rab)                                   # cdb(ex-13b.102,Cabcd)
distribute     (Cabcd)                                        # cdb(ex-13b.103,Cabcd)

Cabcd := C_{a b c d} -> @(Cabcd).
```

```
Cxyxy := C_{a b c d} ex^{a} ey^{b} ex^{c} ey^{d}.            # cdb(ex-13b.104,Cxyxy)

substitute     (Cxyxy,Cabcd)                                 # cdb(ex-13b.105,Cxyxy)
distribute     (Cxyxy)                                       # cdb(ex-13b.106,Cxyxy)

substitute     (Cxyxy, ortho, repeat=True)                   # cdb(ex-13b.107,Cxyxy)

substitute     (Cxyxy, gab)                                  # cdb(ex-13b.108,Cxyxy)
distribute     (Cxyxy)                                       # cdb(ex-13b.109,Cxyxy)

sort_product   (Cxyxy)                                       # cdb(ex-13b.110,Cxyxy)
rename_dummies (Cxyxy)                                       # cdb(ex-13b.111,Cxyxy)
canonicalise   (Cxyxy)                                       # cdb(ex-13b.112,Cxyxy)
```

$$\texttt{ex-13b.101} := R_{abcd} - R_{ac}g_{bd} + R_{ad}g_{bc} - g_{ac}R_{bd} + g_{ad}R_{bc} + \frac{1}{2}g^{ef}R_{ef}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

$$\texttt{ex-13b.102} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

$$\texttt{ex-13b.103} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}$$

$$
\begin{aligned}
C_{abcd}e^a_x e^b_y e^c_x e^d_y &= \left(R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}\,g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}\,g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}\right) e^a_x e^b_y e^c_x e^d_y && \texttt{(ex-13b.105)}\\
&= R_{abcd}e^a_x e^b_y e^c_x e^d_y - g^{ef}R_{aecf}g_{bd}e^a_x e^b_y e^c_x e^d_y + g^{fe}R_{afde}g_{bc}e^a_x e^b_y e^c_x e^d_y - g_{ac}g^{fe}R_{bfde}e^a_x e^b_y e^c_x e^d_y + g_{ad}g^{ef}R_{becf}e^a_x e^b_y e^c_x e^d_y\\
&\quad + \frac{1}{2}\,g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd}e^a_x e^b_y e^c_x e^d_y - \frac{1}{2}\,g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}e^a_x e^b_y e^c_x e^d_y && \texttt{(ex-13b.106)}\\
&= R_{abcd}e^a_x e^b_y e^c_x e^d_y - g^{ef}R_{aecf}e^a_x e^c_x - g^{fe}R_{bfde}e^b_y e^d_y + \frac{1}{2}\,g^{ef}g^{gh}R_{egfh} && \texttt{(ex-13b.107)}\\
&= R_{abcd}e^a_x e^b_y e^c_x e^d_y - \left(e^e_x e^f_x + e^e_y e^f_y + e^e_z e^f_z\right) R_{aecf}e^a_x e^c_x - \left(e^f_x e^e_x + e^f_y e^e_y + e^f_z e^e_z\right) R_{bfde}e^b_y e^d_y\\
&\quad + \frac{1}{2}\,\left(e^e_x e^f_x + e^e_y e^f_y + e^e_z e^f_z\right)\left(e^g_x e^h_x + e^g_y e^h_y + e^g_z e^h_z\right) R_{egfh} && \texttt{(ex-13b.108)}\\
&= R_{abcd}e^a_x e^b_y e^c_x e^d_y - e^e_x e^f_x R_{aecf}e^a_x e^c_x - e^e_y e^f_y R_{aecf}e^a_x e^c_x - e^e_z e^f_z R_{aecf}e^a_x e^c_x - e^f_x e^e_x R_{bfde}e^b_y e^d_y - e^f_y e^e_y R_{bfde}e^b_y e^d_y - e^f_z e^e_z R_{bfde}e^b_y e^d_y\\
&\quad + \frac{1}{2}\,e^e_x e^f_x e^g_x e^h_x R_{egfh} + \frac{1}{2}\,e^e_x e^f_x e^g_y e^h_y R_{egfh} + \frac{1}{2}\,e^e_x e^f_x e^g_z e^h_z R_{egfh} + \frac{1}{2}\,e^e_y e^f_y e^g_x e^h_x R_{egfh} + \frac{1}{2}\,e^e_y e^f_y e^g_y e^h_y R_{egfh} + \frac{1}{2}\,e^e_y e^f_y e^g_z e^h_z R_{egfh}\\
&\quad + \frac{1}{2}\,e^e_z e^f_z e^g_x e^h_x R_{egfh} + \frac{1}{2}\,e^e_z e^f_z e^g_y e^h_y R_{egfh} + \frac{1}{2}\,e^e_z e^f_z e^g_z e^h_z R_{egfh} && \texttt{(ex-13b.109)}\\
&= R_{abcd}e^a_x e^c_x e^b_y e^d_y - R_{aecf}e^a_x e^c_x e^e_x e^f_x - R_{aecf}e^a_x e^c_x e^e_y e^f_y - R_{aecf}e^a_x e^c_x e^e_z e^f_z - R_{bfde}e^e_x e^f_x e^b_y e^d_y - R_{bfde}e^b_y e^d_y e^e_y e^f_y - R_{bfde}e^b_y e^d_y e^e_z e^f_z\\
&\quad + \frac{1}{2}\,R_{egfh}e^e_x e^f_x e^g_x e^h_x + \frac{1}{2}\,R_{egfh}e^e_x e^f_x e^g_y e^h_y + \frac{1}{2}\,R_{egfh}e^e_x e^f_x e^g_z e^h_z + \frac{1}{2}\,R_{egfh}e^g_x e^h_x e^e_y e^f_y + \frac{1}{2}\,R_{egfh}e^e_y e^f_y e^g_y e^h_y + \frac{1}{2}\,R_{egfh}e^e_y e^f_y e^g_z e^h_z\\
&\quad + \frac{1}{2}\,R_{egfh}e^g_x e^h_x e^e_z e^f_z + \frac{1}{2}\,R_{egfh}e^g_y e^h_y e^e_z e^f_z + \frac{1}{2}\,R_{egfh}e^e_z e^f_z e^g_z e^h_z && \texttt{(ex-13b.110)}\\
&= \frac{1}{2}\,R_{abcd}e^a_x e^c_x e^b_y e^d_y - \frac{1}{2}\,R_{abcd}e^a_x e^c_x e^b_x e^d_x - \frac{1}{2}\,R_{abcd}e^a_x e^c_x e^b_z e^d_z - R_{abcd}e^d_x e^b_x e^a_y e^c_y - R_{abcd}e^a_y e^c_y e^d_y e^b_y - R_{abcd}e^a_y e^c_y e^d_z e^b_z + \frac{1}{2}\,R_{abcd}e^b_x e^d_x e^a_y e^c_y\\
&\quad + \frac{1}{2}\,R_{abcd}e^a_y e^c_y e^b_y e^d_y + \frac{1}{2}\,R_{abcd}e^a_y e^c_y e^b_z e^d_z + \frac{1}{2}\,R_{abcd}e^b_x e^d_x e^a_z e^c_z + \frac{1}{2}\,R_{abcd}e^b_y e^d_y e^a_z e^c_z + \frac{1}{2}\,R_{abcd}e^a_z e^c_z e^b_z e^d_z && \texttt{(ex-13b.111)}\\
&= 0 && \texttt{(ex-13b.112)}
\end{aligned}
$$

Example 13c The Weyl tensor vanishes in 3d – orthonormal basis

```
Cxyxz := C_{a b c d} ex^{a} ey^{b} ex^{c} ez^{d}.              # cdb(ex-13c.101,Cxyxz)

substitute     (Cxyxz,Cabcd)                                    # cdb(ex-13c.102,Cxyxz)

distribute     (Cxyxz)                                          # cdb(ex-13c.103,Cxyxz)

substitute     (Cxyxz, ortho, repeat=True)                      # cdb(ex-13c.104,Cxyxz)

substitute     (Cxyxz, gab)                                     # cdb(ex-13c.105,Cxyxz)
distribute     (Cxyxz)                                          # cdb(ex-13c.106,Cxyxz)

sort_product   (Cxyxz)                                          # cdb(ex-13c.107,Cxyxz)
rename_dummies (Cxyxz)                                          # cdb(ex-13c.108,Cxyxz)
canonicalise   (Cxyxz)                                          # cdb(ex-13c.109,Cxyxz)
```

$$
\begin{aligned}
C_{abcd}e_x^a e_y^b e_x^c e_z^d &= \left(R_{abcd} - g^{ef} R_{aecf} g_{bd} + g^{fe} R_{afde} g_{bc} - g_{ac} g^{fe} R_{bfde} + g_{ad} g^{ef} R_{becf} + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} \right) e_x^a e_y^b e_x^c e_z^d && \texttt{(ex-13c.102)} \\
&= R_{abcd}e_x^a e_y^b e_x^c e_z^d - g^{ef} R_{aecf} g_{bd} e_x^a e_y^b e_x^c e_z^d + g^{fe} R_{afde} g_{bc} e_x^a e_y^b e_x^c e_z^d - g_{ac} g^{fe} R_{bfde} e_x^a e_y^b e_x^c e_z^d + g_{ad} g^{ef} R_{becf} e_x^a e_y^b e_x^c e_z^d \\
&\quad + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} e_x^a e_y^b e_x^c e_z^d - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} e_x^a e_y^b e_x^c e_z^d && \texttt{(ex-13c.103)} \\
&= R_{abcd}e_x^a e_y^b e_x^c e_z^d - g^{fe} R_{bfde} e_y^b e_z^d && \texttt{(ex-13c.104)} \\
&= R_{abcd}e_x^a e_y^b e_x^c e_z^d - \left(e_x^f e_x^e + e_y^f e_y^e + e_z^f e_z^e \right) R_{bfde} e_y^b e_z^d && \texttt{(ex-13c.105)} \\
&= R_{abcd}e_x^a e_y^b e_x^c e_z^d - e_x^f e_x^e R_{bfde} e_y^b e_z^d - e_y^f e_y^e R_{bfde} e_y^b e_z^d - e_z^f e_z^e R_{bfde} e_y^b e_z^d && \texttt{(ex-13c.106)} \\
&= R_{abcd}e_x^a e_x^c e_y^b e_z^d - R_{bfde} e_x^e e_x^f e_y^b e_z^d - R_{bfde} e_y^b e_y^e e_y^f e_z^d - R_{bfde} e_y^b e_z^d e_z^e e_z^f && \texttt{(ex-13c.107)} \\
&= R_{abcd}e_x^a e_x^c e_y^b e_z^d - R_{abcd} e_x^d e_x^b e_y^a e_z^c - R_{abcd} e_y^a e_y^d e_y^b e_z^c - R_{abcd} e_y^a e_z^c e_z^d e_z^b && \texttt{(ex-13c.108)} \\
&= 0 && \texttt{(ex-13c.109)}
\end{aligned}
$$