Exercise 1.1 Verify symmetry of $\Gamma^a{}_{bc}$

```
1   {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3   g_{a b}::Metric.
4
5   \partial{#}::PartialDerivative.
6
7   Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
8                                              + \partial_{c}{g_{b d}}
9                                              - \partial_{d}{g_{b c}} ).
10
11  diff := \Gamma^{a}_{b c} - \Gamma^{a}_{c b}.    # cdb (ex-0101.101,diff)
12
13  substitute    (diff, Gamma)                     # cdb (ex-0101.102,diff)
14  distribute    (diff)                            # cdb (ex-0101.103,diff)
15  canonicalise  (diff)                            # cdb (ex-0101.104,diff)
```

$$\Gamma^a{}_{bc} - \Gamma^a{}_{cb} = \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) - \frac{1}{2}g^{ad}\left(\partial_c g_{db} + \partial_b g_{cd} - \partial_d g_{cb}\right)$$

$$= \frac{1}{2}g^{ad}\partial_b g_{dc} + \frac{1}{2}g^{ad}\partial_c g_{bd} - \frac{1}{2}g^{ad}\partial_d g_{bc} - \frac{1}{2}g^{ad}\partial_c g_{db} - \frac{1}{2}g^{ad}\partial_b g_{cd} + \frac{1}{2}g^{ad}\partial_d g_{cb}$$

$$= 0$$

Exercise 1.2 Christoffel symbol and dg

```
1   {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3   g_{a b}::Metric.
4   g_{a}^{b}::KroneckerDelta.
5
6   \partial{#}::PartialDerivative.
7
8   GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
9                                             + \partial_{c}{g_{b d}}
10                                            - \partial_{d}{g_{b c}} ).
11
12  GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.
13
14  expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}.   # cdb (ex-0102.101,expr)
15
16  substitute          (expr, GammaD)                                 # cdb (ex-0102.102,expr)
17  substitute          (expr, GammaU)                                 # cdb (ex-0102.103,expr)
18  distribute          (expr)                                         # cdb (ex-0102.104,expr)
19  eliminate_metric    (expr)                                         # cdb (ex-0102.105,expr)
20  eliminate_kronecker (expr)                                         # cdb (ex-0102.106,expr)
21  canonicalise        (expr)                                         # cdb (ex-0102.107,expr)
```

$$\Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} = g_{ad}\Gamma^d{}_{bc} + g_{bd}\Gamma^d{}_{ac} - \partial_c g_{ab}$$

$$= \frac{1}{2}g_{ad}g^{de}\left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right) + \frac{1}{2}g_{bd}g^{de}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - \partial_c g_{ab}$$

$$= \frac{1}{2}g_{ad}g^{de}\partial_b g_{ec} + \frac{1}{2}g_{ad}g^{de}\partial_c g_{be} - \frac{1}{2}g_{ad}g^{de}\partial_e g_{bc} + \frac{1}{2}g_{bd}g^{de}\partial_a g_{ec} + \frac{1}{2}g_{bd}g^{de}\partial_c g_{ae} - \frac{1}{2}g_{bd}g^{de}\partial_e g_{ac} - \partial_c g_{ab}$$

$$= \frac{1}{2}g_a{}^e\partial_b g_{ec} + \frac{1}{2}g_a{}^e\partial_c g_{be} - \frac{1}{2}g_a{}^e\partial_e g_{bc} + \frac{1}{2}g_b{}^e\partial_a g_{ec} + \frac{1}{2}g_b{}^e\partial_c g_{ae} - \frac{1}{2}g_b{}^e\partial_e g_{ac} - \partial_c g_{ab}$$

$$= \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_c g_{ab}$$

$$= 0$$

Exercise 1.3 Christoffel symbol and dg with a single rule

```
1    {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3    g_{a b}::Metric.
4    g_{a}^{b}::KroneckerDelta.
5
6    \partial{#}::PartialDerivative.
7
8    GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
9                                             + \partial_{c}{g_{b d}}
10                                            - \partial_{d}{g_{b c}} ).
11
12   GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.          # cdb (ex-0103.101,GammaD)
13
14   substitute         (GammaD, GammaU)                          # cdb (ex-0103.102,GammaD)  # requires Indices(position=independent)
15   distribute         (GammaD)                                  # cdb (ex-0103.103,GammaD)
16   eliminate_metric   (GammaD)                                  # cdb (ex-0103.104,GammaD)
17   eliminate_kronecker (GammaD)                                 # cdb (ex-0103.105,GammaD)
18
19   expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}.   # cdb (ex-0103.201,expr)
20
21   substitute         (expr, GammaD)                            # cdb (ex-0103.202,expr)
22   canonicalise       (expr)                                    # cdb (ex-0103.203,expr)
```

$$\Gamma_{abc} \rightarrow g_{ad}\Gamma^{d}{}_{bc} \tag{ex-0103.101}$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}g_{ad}g^{de}\left(\partial_{b}g_{ec} + \partial_{c}g_{be} - \partial_{e}g_{bc}\right) \tag{ex-0103.102}$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}g_{ad}g^{de}\partial_{b}g_{ec} + \frac{1}{2}g_{ad}g^{de}\partial_{c}g_{be} - \frac{1}{2}g_{ad}g^{de}\partial_{e}g_{bc} \tag{ex-0103.103}$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}g_{a}{}^{e}\partial_{b}g_{ec} + \frac{1}{2}g_{a}{}^{e}\partial_{c}g_{be} - \frac{1}{2}g_{a}{}^{e}\partial_{e}g_{bc} \tag{ex-0103.104}$$

$$\Gamma_{abc} \rightarrow \frac{1}{2}\partial_{b}g_{ac} + \frac{1}{2}\partial_{c}g_{ba} - \frac{1}{2}\partial_{a}g_{bc} \tag{ex-0103.105}$$

$$\Gamma_{abc} + \Gamma_{bac} - \partial_{c}g_{ab} = \frac{1}{2}\partial_{c}g_{ba} - \frac{1}{2}\partial_{c}g_{ab} \tag{ex-0103.202}$$

$$= 0 \tag{ex-0103.203}$$

Exercise 1.3 Repeat but without `position=independent`

```
1   {a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.
2
3   g_{a b}::Metric.
4   g_{a}^{b}::KroneckerDelta.
5
6   \partial{#}::PartialDerivative.
7
8   GammaU := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
9                                              + \partial_{c}{g_{b d}}
10                                             - \partial_{d}{g_{b c}} ).
11
12  GammaD := \Gamma_{a b c} -> g_{a d} \Gamma^{d}_{b c}.              # cdb (ex-0103.301,GammaD)
13
14  substitute         (GammaD, GammaU)                              # cdb (ex-0103.302,GammaD)
15  distribute         (GammaD)                                      # cdb (ex-0103.303,GammaD)
16  eliminate_metric   (GammaD)                                      # cdb (ex-0103.304,GammaD)
17  eliminate_kronecker (GammaD)                                     # cdb (ex-0103.305,GammaD)
18
19  expr := \Gamma_{a b c} + \Gamma_{b a c} - \partial_{c}{g_{a b}}. # cdb (ex-0103.401,expr)
20
21  substitute         (expr, GammaD)                                # cdb (ex-0103.402,expr)
22  canonicalise       (expr)                                        # cdb (ex-0103.403,expr)
```

$$\Gamma_{abc} \to g_{ad}\Gamma^d{}_{bc} \tag{ex-0103.301}$$

$$\frac{1}{2}g_a{}^d\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) \to \frac{1}{2}g_{ad}g^{de}\left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right) \tag{ex-0103.302}$$

$$\frac{1}{2}g_a{}^d\partial_b g_{dc} + \frac{1}{2}g_a{}^d\partial_c g_{bd} - \frac{1}{2}g_a{}^d\partial_d g_{bc} \to \frac{1}{2}g_{ad}g^{de}\partial_b g_{ec} + \frac{1}{2}g_{ad}g^{de}\partial_c g_{be} - \frac{1}{2}g_{ad}g^{de}\partial_e g_{bc} \tag{ex-0103.303}$$

$$\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc} \to \frac{1}{2}g_a{}^e\partial_b g_{ec} + \frac{1}{2}g_a{}^e\partial_c g_{be} - \frac{1}{2}g_a{}^e\partial_e g_{bc} \tag{ex-0103.304}$$

$$\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc} \to \frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc} \tag{ex-0103.305}$$

$$\Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} \tag{ex-0103.402}$$

$$= \Gamma_{abc} + \Gamma_{bac} - \partial_c g_{ab} \tag{ex-0103.403}$$

4

Exercise 1.4 Experiments with sorting

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z#}::Indices(position=independent).
2
3    \partial{#}::PartialDerivative.
4
5    expr := C^{f}
6            w^{e}
7            B^{d}
8            v^{c}
9            A^{b}
10           u^{a}.                          # cdb (ex-0104.100,expr)
11
12   sort_product (expr)                     # cdb (ex-0104.101,expr)
13
14   expr := \Omega_{f}
15           \gamma_{e}
16           \Pi_{d}
17           \beta_{c}
18           \Gamma_{b}
19           \alpha_{a}.                     # cdb (ex-0104.200,expr)
20
21   sort_product (expr)                     # cdb (ex-0104.201,expr)
22
23   expr := C^{f}
24           w^{e}
25           B^{d}
26           v^{c}
27           A^{b}
28           u^{a}
29           \Omega_{f}
30           \gamma_{e}
31           \Pi_{d}
32           \beta_{c}
33           \Gamma_{b}
34           \alpha_{a}.                     # cdb (ex-0104.300,expr)
35
36   sort_product (expr)                     # cdb (ex-0104.301,expr)
```

```
37
38   expr := \partial_{f}{C^{f}}
39          w^{l}
40          \partial_{d}{B^{d}}
41          v^{k}
42          \partial_{b}{A^{b}}
43          u^{j}
44          \Omega_{i}
45          \partial^{e}{ \gamma_{e}}
46          \Pi_{h}
47          \partial^{c}{\beta_{c}}
48          \Gamma_{g}
49          \partial^{a}{\alpha_{a}}.      # cdb (ex-0104.400,expr)
50
51   sort_product (expr)                   # cdb (ex-0104.401,expr)
52
53   expr := \partial{C}
54          w
55          \partial{B}
56          v
57          \partial{A}
58          u
59          \Omega
60          \partial{ \gamma}
61          \Pi
62          \partial{\beta}
63          \Gamma
64          \partial{\alpha}.             # cdb (ex-0104.500,expr)
65
66   sort_product (expr)                   # cdb (ex-0104.501,expr)
67
68   expr := A_{b}
69          A_{a}
70          A_{c d e}
71          A_{f g}.                       # cdb (ex-0104.600,expr)
72
73   sort_product (expr)                   # cdb (ex-0104.601,expr)
74
```

```
75  expr :=  A_{a} A^{a}
76         + A^{a} A_{a}.                    # cdb (ex-0104.700,expr)
77
78  sort_product (expr)                      # cdb (ex-0104.701,expr)
```

$$\texttt{ex-0104.100} := C^f w^e B^d v^c A^b u^a$$

$$\texttt{ex-0104.101} := A^b B^d C^f u^a v^c w^e$$

$$\texttt{ex-0104.200} := \Omega_f \gamma_e \Pi_d \beta_c \Gamma_b \alpha_a$$

$$\texttt{ex-0104.201} := \Gamma_b \Omega_f \Pi_d \alpha_a \beta_c \gamma_e$$

$$\texttt{ex-0104.300} := C^f w^e B^d v^c A^b u^a \Omega_f \gamma_e \Pi_d \beta_c \Gamma_b \alpha_a$$

$$\texttt{ex-0104.301} := A^b B^d C^f \Gamma_b \Omega_f \Pi_d \alpha_a \beta_c \gamma_e u^a v^c w^e$$

$$\texttt{ex-0104.400} := \partial_f C^f w^l \partial_d B^d v^k \partial_b A^b u^j \Omega_i \partial^e \gamma_e \Pi_h \partial^c \beta_c \Gamma_g \partial^a \alpha_a$$

$$\texttt{ex-0104.401} := \Gamma_g \Omega_i \Pi_h \partial_b A^b \partial_d B^d \partial_f C^f \partial^a \alpha_a \partial^c \beta_c \partial^e \gamma_e u^j v^k w^l$$

$$\texttt{ex-0104.500} := \partial C w \partial B v \partial A u \Omega \partial \gamma \Pi \partial \beta \Gamma \partial \alpha$$

$$\texttt{ex-0104.501} := \Gamma \Omega \Pi \partial A \partial B \partial C \partial \alpha \partial \beta \partial \gamma u v w$$

$$\texttt{ex-0104.600} := A_b A_a A_{cde} A_{fg}$$

$$\texttt{ex-0104.601} := A_a A_b A_{fg} A_{cde}$$

$$\texttt{ex-0104.700} := A_a A^a + A^a A_a$$

$$\texttt{ex-0104.701} := A_a A^a + A^a A_a$$

Exercise 1.5 A sort hack

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z#}::Indices(position=independent).
2
3   foo := A_{a} A^{a} + A^{a} A_{a}.              # cdb (ex-0105.100,foo)
4
5   sort_product (foo)                            # cdb (ex-0105.101,foo)
6
7   substitute   (foo, $A^{a} -> Z^{a}$)          # cdb (ex-0105.102,foo)
8   sort_product (foo)                            # cdb (ex-0105.103,foo)
9   substitute   (foo, $Z^{a} -> A^{a}$)          # cdb (ex-0105.104,foo)
```

$$\text{ex-0105.100} := A_a A^a + A^a A_a$$
$$\text{ex-0105.101} := A_a A^a + A^a A_a$$
$$\text{ex-0105.102} := A_a Z^a + Z^a A_a$$
$$\text{ex-0105.103} := 2 A_a Z^a$$
$$\text{ex-0105.104} := 2 A_a A^a$$

Exercise 1.6 Multiple `SortOrder` lists

```
1    {D,C,B,A}::SortOrder.   # first SortOrder list
2
3    foo := A B C D .        # cdb(ex-0106.101,foo)
4
5    sort_product (foo)      # cdb(ex-0106.102,foo)
6
7    {V,U}::SortOrder.       # second SortOrder list, all entries distinct from first list
8
9    foo := U V A B C D .    # cdb(ex-0106.201,foo)
10
11   sort_product (foo)      # cdb(ex-0106.202,foo)
12
13   {A,B,C,D}::SortOrder.   # all entries in this list appear in the
14                           # first SortOrder so they will be effectively ignored
15
16   foo := U V D C B A .    # cdb(ex-0106.301,foo)
17
18   sort_product (foo)      # cdb(ex-0106.302,foo)
```

$$\texttt{ex-0106.101} := ABCD$$
$$\texttt{ex-0106.102} := DCBA$$

$$\texttt{ex-0106.201} := UVABCD$$
$$\texttt{ex-0106.202} := DCBAVU$$

$$\texttt{ex-0106.301} := UVDCBA$$
$$\texttt{ex-0106.302} := DCBAVU$$

Exercise 1.7 Subtleties of `foo = bah` and `foo := @(bah)`

```
1    {a,b,c,d,e,f,h#}::Indices.
2
3    foo := B_{b} A_{a}.
4    bah := A_{a} C_{c}.
5
6    # cdbBeg(print.0107)
7    print("foo = "+str(foo))
8    print("bah = "+str(bah)+"\n")
9
10   print("type foo = "+str(type(foo)))
11   print("type bah = "+str(type(bah))+"\n")
12
13   print("id foo = "+str(id(foo)))
14   print("id bah = "+str(id(bah))+"\n")
15
16   bah = foo
17
18   print("foo = "+str(foo))
19   print("bah = "+str(bah)+"\n")
20
21   sort_product (foo)
22
23   print("bah = "+str(bah)+"\n")
24
25   print("id foo = "+str(id(foo)))
26   print("id bah = "+str(id(bah))+"\n")
27
28   bah := @(foo).
29
30   print("id foo = "+str(id(foo)))
31   print("id bah = "+str(id(bah))+"\n")
32   # cdbEnd(print.0107)
```

```
1    foo = B_{b} A_{a}
2    bah = A_{a} C_{c}
3
4    type foo = <class 'cadabra2.Ex'>
5    type bah = <class 'cadabra2.Ex'>
6
7    id foo = 4589928496
8    id bah = 4589928552
9
10   foo = B_{b} A_{a}
11   bah = B_{b} A_{a}
12
13   bah = A_{a} B_{b}
14
15   id foo = 4589928496
16   id bah = 4589928496
17
18   id foo = 4589928496
19   id bah = 4589929336
```

Note that the line numbers referenced in the following are those of the output above not those of the Cadabra source.

- Lines 7 and 8 show that the objects `foo` and `bah` point to distinct areas of memeory (i.e., they point to different objects).

- Lines 10 and 11 show the result of the statement `bah = foo`.

- Line 13 shows that `bah` has changed after the statement `sort_product (foo)`.

- Lines 15 and 16 verifies that `foo` and `bah` point to the same object (so changes in `foo` will be seen by `bah`, as just noted).

- Lines 18 and 19 shows that after `bah := @(foo)` the symbols `bah` and `foo` no longer point to the same object.

Exercise 1.8 Syntax errors – original code

```
1    {a,b,c,d,e,f#}::Indices.
2    C{#}::Symmetric.
3
4    foo := A_{a} B_{b} + C_{ab}.                      # C_{ab} should be C_{a b}
5    bah := B_{b} A_{a} + C_{ba}.                      # C_{ba} should be C_{b a}
6    meh := @(foo) - @(bah)                            # missing dot or semi-colon terminator
7
8    if meh == 0:
9        print ("meh is zero, and all is good")
10           success = True.                           # indentation error, drop the dot
11   else:
12       print ("meh is not zero, oops")
13           success = False.                          # indentation error, drop the dot
14
15   canonicalise (meh).                               # terminate with ; or nothing
16   sort_product (meh);
17
18   {\alpha\beta\gamma}::Indices.                     # separate list elements with commas
19
20   foo := Ex ("A_{ab} - A_{a b}");                   # use = for assignment, A_{ab} should be A_{a b}
21   bah := Ex ("A_{\alpha\beta} - A_{\alpha \beta}"); # use = for assignment, need raw string in Ex
```

Exercise 1.8 Syntax errors – corrected code

```
1   {a,b,c,d,e,f#}::Indices.
2   C{#}::Symmetric.
3
4   foo := A_{a} B_{b} + C_{a b}.                     # cdb (ex-0108.101,foo)
5   bah := B_{b} A_{a} + C_{b a}.                     # cdb (ex-0108.102,bah)
6   meh := @(foo) - @(bah).                           # cdb (ex-0108.103,meh)
7
8   if meh == 0:
9       print ("meh is zero, and all is good")
10      success = True
11  else:
12      print ("meh is not zero, oops")
13      success = False
14
15  canonicalise (meh)                                # cdb (ex-0108.104,meh)
16  sort_product (meh);                               # cdb (ex-0108.105,meh)
17
18  {\alpha,\beta,\gamma}::Indices.
19
20  foo = Ex ("A_{a b} - A_{a b}");                   # cdb (ex-0108.106,foo)
21  bah = Ex (r"A_{\alpha\beta} - A_{\alpha \beta}"); # cdb (ex-0108.107,bah)
```

$$\texttt{ex-0108.101} := A_a B_b + C_{ab}$$

$$\texttt{ex-0108.102} := B_b A_a + C_{ba}$$

$$\texttt{ex-0108.103} := A_a B_b + C_{ab} - B_b A_a - C_{ba}$$

$$\texttt{ex-0108.104} := A_a B_b - B_b A_a$$

$$\texttt{ex-0108.105} := 0$$

$$\texttt{ex-0108.106} := 0$$

$$\texttt{ex-0108.107} := 0$$

Exercise 1.9 No index clashes

```
1  {a,b,c,d,e,f,u,v,w}::Indices.
2
3  foo := A_{a c} C^{c}.                        # cdb (ex-0109.101,foo)
4  bah := B_{b c} C^{c}.                        # cdb (ex-0109.102,bah)
5
6  foobah := @(foo) @(bah).                     # cdb (ex-0109.103,foobah)
```

$$A_{ac}C^c \tag{ex-0109.101}$$

$$B_{bc}C^c \tag{ex-0109.102}$$

$$A_{ac}C^c B_{bd}C^d \tag{ex-0109.103}$$

Exercise 1.10 Relabel free indices

```
1  {a,b,c,d,e,f,u,v,w}::Indices.
2
3  \delta{#}::KroneckerDelta.
4
5  expr := A_{a b c}.                                    # cdb (ex-0110.101,expr)
6
7  expr := \delta^{a}_{u} \delta^{b}_{v} \delta^{c}_{w} @(expr).  # cdb (ex-0110.102,expr)
8
9  eliminate_kronecker (expr)                            # cdb (ex-0110.103,expr)
```

$$A_{abc} \hspace{6cm} \text{(ex-0110.101)}$$
$$\delta^a{}_u \delta^b{}_v \delta^c{}_w A_{abc} \hspace{4cm} \text{(ex-0110.102)}$$
$$A_{uvw} \hspace{6cm} \text{(ex-0110.103)}$$

Exercise 1.11 Cycling free indices – preferred solution

```
1    {a,b,c,d,e,f,u,v,w}::Indices.
2
3    expr := A_{a b c}.                          # cdb (ex-0111.101,expr)
4
5    rule := T_{a b c} -> @(expr).
6    expr := T_{b c a}.                          # cdb (ex-0111.102,expr)
7
8    substitute (expr, rule)                     # cdb (ex-0111.103,expr)
```

$$A_{abc} \tag{ex-0111.101}$$
$$T_{bca} \tag{ex-0111.102}$$
$$A_{bca} \tag{ex-0111.103}$$

Exercise 1.11 Cycling free indices – alternative solution

This alternative solution uses two rounds of Kroncker deltas. It does the job but is not as simple as the previous solution.

```
{a,b,c,d,e,f,u,v,w}::Indices.

\delta{#}::KroneckerDelta.

expr := A_{a b c}.                                         # cdb (ex-0111.201,expr)

expr := \delta^{a}_{u} \delta^{b}_{v} \delta^{c}_{w} @(expr).  # cdb (ex-0111.202,expr)

eliminate_kronecker (expr)                                 # cdb (ex-0111.203,expr)

expr := \delta^{u}_{b} \delta^{v}_{c} \delta^{w}_{a} @(expr).  # cdb (ex-0111.204,expr)

eliminate_kronecker (expr)                                 # cdb (ex-0111.205,expr)
```

$$A_{abc} \tag{ex-0111.201}$$

$$\delta^{a}{}_{u}\delta^{b}{}_{v}\delta^{c}{}_{w}A_{abc} \tag{ex-0111.202}$$

$$A_{uvw} \tag{ex-0111.203}$$

$$\delta^{u}{}_{b}\delta^{v}{}_{c}\delta^{w}{}_{a}A_{uvw} \tag{ex-0111.204}$$

$$A_{bca} \tag{ex-0111.205}$$

Exercise 2.1 Using Cadabra's own product rule

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3    \nabla{#}::Derivative.
4    \partial{#}::PartialDerivative.
5
6    # templates for covariant derivatives
7
8    deriv1 := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}}
9                                   + \Gamma^{b}_{c a} A?^{c}.
10
11   deriv2 := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}}
12                                   - \Gamma^{c}_{b a} A?_{c}.
13
14   # create an object
15
16   uv := \nabla_{a}{v_{b} u^{b}}
17        - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.101,uv)
18
19   # apply the rules, then simplify
20
21   product_rule   (uv)                    # cdb (ex-0201.102,uv)
22   substitute     (uv,deriv1)             # cdb (ex-0201.103,uv)
23   substitute     (uv,deriv2)             # cdb (ex-0201.104,uv)
24   distribute     (uv)                    # cdb (ex-0201.105,uv)
25   sort_product   (uv)                    # cdb (ex-0201.106,uv)
26   rename_dummies (uv)                    # cdb (ex-0201.107,uv)
```

$$\nabla_a \left(v_b u^b \right) - \partial_a \left(v_b u^b \right) = \nabla_a v_b u^b + v_b \nabla_a u^b - \partial_a v_b u^b - v_b \partial_a u^b \tag{ex-0201.102}$$

$$= \nabla_a v_b u^b + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c \right) - \partial_a v_b u^b - v_b \partial_a u^b \tag{ex-0201.103}$$

$$= \left(\partial_a v_b - \Gamma^c{}_{ba} v_c \right) u^b + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c \right) - \partial_a v_b u^b - v_b \partial_a u^b \tag{ex-0201.104}$$

$$= -\Gamma^c{}_{ba} v_c u^b + v_b \Gamma^b{}_{ca} u^c \tag{ex-0201.105}$$

$$= -\Gamma^c{}_{ba} u^b v_c + \Gamma^b{}_{ca} u^c v_b \tag{ex-0201.106}$$

$$= 0 \tag{ex-0201.107}$$

Exercise 2.1 Using hand crafted product rules

```
1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \nabla{#}::Derivative.
4  \partial{#}::PartialDerivative.
5
6  # templates for covariant derivatives
7
8  deriv1 := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}}
9                                 + \Gamma^{b}_{c a} A?^{c}.
10
11 deriv2 := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}}
12                                 - \Gamma^{c}_{b a} A?_{c}.
13
14 # tempaltes for product rules
15
16 deriv3 := \nabla_{a}{A?_{b} B?^{c}} -> B?^{c} \nabla_{a}{A?_{b}}
17                                        + A?_{b} \nabla_{a}{B?^{c}}.
18
19 deriv4 := \partial_{a}{A?_{b} B?^{c}} -> B?^{c} \partial_{a}{A?_{b}}
20                                          + A?_{b} \partial_{a}{B?^{c}}.
21
22 # create an object
23
24 uv := \nabla_{a}{v_{b} u^{b}}
25     - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.201,uv)
26
27 # apply the rules, then simplify
28
29 substitute      (uv,deriv3)             # cdb (ex-0201.202,uv)
30 substitute      (uv,deriv4)             # cdb (ex-0201.203,uv)
31 substitute      (uv,deriv1)             # cdb (ex-0201.204,uv)
32 substitute      (uv,deriv2)             # cdb (ex-0201.205,uv)
33 distribute      (uv)                    # cdb (ex-0201.206,uv)
34 sort_product    (uv)                    # cdb (ex-0201.207,uv)
35 rename_dummies  (uv)                    # cdb (ex-0201.208,uv)
```

$$\nabla_a \left(v_b u^b \right) - \partial_a \left(v_b u^b \right) = u^b \nabla_a v_b + v_b \nabla_a u^b - \partial_a \left(v_b u^b \right) \tag{ex-0201.202}$$

$$= u^b \nabla_a v_b + v_b \nabla_a u^b - u^b \partial_a v_b - v_b \partial_a u^b \tag{ex-0201.203}$$

$$= u^b \nabla_a v_b + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c \right) - u^b \partial_a v_b - v_b \partial_a u^b \tag{ex-0201.204}$$

$$= u^b \left(\partial_a v_b - \Gamma^c{}_{ba} v_c \right) + v_b \left(\partial_a u^b + \Gamma^b{}_{ca} u^c \right) - u^b \partial_a v_b - v_b \partial_a u^b \tag{ex-0201.205}$$

$$= -u^b \Gamma^c{}_{ba} v_c + v_b \Gamma^b{}_{ca} u^c \tag{ex-0201.206}$$

$$= -\Gamma^c{}_{ba} u^b v_c + \Gamma^b{}_{ca} u^c v_b \tag{ex-0201.207}$$

$$= 0 \tag{ex-0201.208}$$

Exercise 2.2 Covariant derivative of v_{ab}

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3    \nabla{#}::Derivative.
4    \partial{#}::PartialDerivative.
5
6    # template for covariant derivative of a vector
7
8    derivU := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}} + \Gamma^{b}_{c a} A?^{c}.
9    derivD := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.
10
11   vab := v_{a b} -> A_{a} B_{b}.
12   iab := A_{a} B_{b} -> v_{a b}.
13
14   pab := \partial_{a}{A_{b}} B_{c} -> \partial_{a}{A_{b} B_{c}} - A_{b} \partial_{a}{B_{c}}.
15
16   # create an object
17
18   Dvab := \nabla_{a}{v_{b c}}.        # cdb (ex-0202.101,Dvab)
19
20   # apply the rule, then simplify
21
22   substitute      (Dvab,vab)          # cdb (ex-0202.102,Dvab)
23   product_rule    (Dvab)              # cdb (ex-0202.103,Dvab)
24   substitute      (Dvab,derivD)       # cdb (ex-0202.104,Dvab)
25   substitute      (Dvab,derivU)       # cdb (ex-0202.105,Dvab)
26   distribute      (Dvab)              # cdb (ex-0202.106,Dvab)
27   substitute      (Dvab,pab)          # cdb (ex-0202.107,Dvab)
28   canonicalise    (Dvab)              # cdb (ex-0202.108,Dvab)
29   substitute      (Dvab,iab)          # cdb (ex-0202.109,Dvab)
30   sort_product    (Dvab)              # cdb (ex-0202.110,Dvab)
```

$$\nabla_a v_{bc} = \nabla_a \left(A_b B_c \right) \tag{ex-0202.102}$$

$$= \nabla_a A_b B_c + A_b \nabla_a B_c \tag{ex-0202.103}$$

$$= \left(\partial_a A_b - \Gamma^d{}_{ba} A_d \right) B_c + A_b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d \right) \tag{ex-0202.104}$$

$$= \left(\partial_a A_b - \Gamma^d{}_{ba} A_d \right) B_c + A_b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d \right) \tag{ex-0202.105}$$

$$= \partial_a A_b B_c - \Gamma^d{}_{ba} A_d B_c + A_b \partial_a B_c - A_b \Gamma^d{}_{ca} B_d \tag{ex-0202.106}$$

$$= \partial_a \left(A_b B_c \right) - \Gamma^d{}_{ba} A_d B_c - A_b \Gamma^d{}_{ca} B_d \tag{ex-0202.107}$$

$$= \partial_a \left(A_b B_c \right) - \Gamma^d{}_{ba} A_d B_c - A_b \Gamma^d{}_{ca} B_d \tag{ex-0202.108}$$

$$= \partial_a v_{bc} - \Gamma^d{}_{ba} v_{dc} - v_{bd} \Gamma^d{}_{ca} \tag{ex-0202.109}$$

$$= \partial_a v_{bc} - \Gamma^d{}_{ba} v_{dc} - \Gamma^d{}_{ca} v_{bd} \tag{ex-0202.110}$$

Exercise 2.3 Covariant derivative of $v^a{}_b$

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4   \partial{#}::PartialDerivative.
5
6   # template for covariant derivative of a vector
7
8   derivU := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}} + \Gamma^{b}_{c a} A?^{c}.
9   derivD := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.
10
11  vab := v^{a}_{b} -> A^{a} B_{b}.
12  iab := A^{a} B_{b} -> v^{a}_{b}.
13
14  pab := \partial_{a}{A^{b}} B_{c} -> \partial_{a}{A^{b} B_{c}} - A^{b} \partial_{a}{B_{c}}.
15
16  # create an object
17
18  Dvab := \nabla_{a}{v^{b}_{c}}.      # cdb (ex-0203.101,Dvab)
19
20  # apply the rule, then simplify
21
22  substitute      (Dvab,vab)          # cdb (ex-0203.102,Dvab)
23  product_rule    (Dvab)              # cdb (ex-0203.103,Dvab)
24  substitute      (Dvab,derivD)       # cdb (ex-0203.104,Dvab)
25  substitute      (Dvab,derivU)       # cdb (ex-0203.105,Dvab)
26  distribute      (Dvab)              # cdb (ex-0203.106,Dvab)
27  substitute      (Dvab,pab)          # cdb (ex-0203.107,Dvab)
28  canonicalise    (Dvab)              # cdb (ex-0203.108,Dvab)
29  substitute      (Dvab,iab)          # cdb (ex-0203.109,Dvab)
30  sort_product    (Dvab)              # cdb (ex-0203.110,Dvab)
```

$$\nabla_a v^b{}_c = \nabla_a \left(A^b B_c \right) \tag{ex-0203.102}$$

$$= \nabla_a A^b B_c + A^b \nabla_a B_c \tag{ex-0203.103}$$

$$= \nabla_a A^b B_c + A^b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d \right) \tag{ex-0203.104}$$

$$= \left(\partial_a A^b + \Gamma^b{}_{da} A^d \right) B_c + A^b \left(\partial_a B_c - \Gamma^d{}_{ca} B_d \right) \tag{ex-0203.105}$$

$$= \partial_a A^b B_c + \Gamma^b{}_{da} A^d B_c + A^b \partial_a B_c - A^b \Gamma^d{}_{ca} B_d \tag{ex-0203.106}$$

$$= \partial_a \left(A^b B_c \right) + \Gamma^b{}_{da} A^d B_c - A^b \Gamma^d{}_{ca} B_d \tag{ex-0203.107}$$

$$= \partial_a \left(A^b B_c \right) + \Gamma^b{}_{da} A^d B_c - A^b \Gamma^d{}_{ca} B_d \tag{ex-0203.108}$$

$$= \partial_a v^b{}_c + \Gamma^b{}_{da} v^d{}_c - v^b{}_d \Gamma^d{}_{ca} \tag{ex-0203.109}$$

$$= \partial_a v^b{}_c + \Gamma^b{}_{da} v^d{}_c - \Gamma^d{}_{ca} v^b{}_d \tag{ex-0203.110}$$

Exercise 2.4 Combining rules – a problem

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4   \partial{#}::PartialDerivative.
5
6   # rules for covariant derivatives of v
7
8   deriv1 := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}}
9                              + \Gamma^{b}_{d a} v^{d}.
10
11  deriv2 := \nabla_{a}{\nabla_{b}{v^{c}}} -> \partial_{a}{\nabla_{b}{v^{c}}}
12                                          + \Gamma^{c}_{d a} \nabla_{b}{v^{d}}
13                                          - \Gamma^{d}_{b a} \nabla_{d}{v^{c}}.
14
15  # attempt to combine both rules for second covariant derivative of v
16
17  substitute (deriv2,deriv1)      # cdb (ex-0204.101,deriv2)
```

Note that the call to `substitute` has made changes to both sides of the rule for `deriv2`. This is not ideal and a better method is developed in the following exercise.

$$\nabla_a \left(\partial_b v^c + \Gamma^c{}_{db} v^d \right) \to \partial_a \left(\partial_b v^c + \Gamma^c{}_{db} v^d \right) + \Gamma^c{}_{da} \left(\partial_b v^d + \Gamma^d{}_{eb} v^e \right) - \Gamma^d{}_{ba} \left(\partial_d v^c + \Gamma^c{}_{ed} v^e \right) \qquad \text{(ex-0204.101)}$$

Exercise 2.5 Combining rules – a solution

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4   \partial{#}::PartialDerivative.
5
6   # rules for covariant derivatives of v
7
8   deriv1 := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}}
9                                 + \Gamma^{b}_{d a} v^{d}.
10
11  deriv2 := \nabla_{a}{\nabla_{b}{v^{c}}} -> \partial_{a}{\nabla_{b}{v^{c}}}
12                                          + \Gamma^{c}_{d a} \nabla_{b}{v^{d}}
13                                          - \Gamma^{d}_{b a} \nabla_{d}{v^{c}}.
14
15  # second covariant derivative of v
16
17  expr := v^{c}_{b a} -> \nabla_{a}{\nabla_{b}{v^{c}}}.   # cdb (ex-0205.101,expr)
18  save := @(expr).
19
20  # apply the rules, then simplify
21
22  substitute      (expr,deriv2)         # cdb (ex-0205.102,expr)
23  substitute      (expr,deriv1)         # cdb (ex-0205.103,expr)
24  distribute      (expr)                # cdb (ex-0205.104,expr)
25  product_rule    (expr)                # cdb (ex-0205.105,expr)
26  canonicalise    (expr)                # cdb (ex-0205.107,expr)
27  substitute      (expr,save)           # cdb (ex-0205.108,expr)
```

The trick here is to introduce in line 17 a dummy left hand side, `v^{c}{}_{b a}`, that is invisible with respect to the substitution rules of lines 8 and 11. Thus lines 22 and 23 will only target the right hand side of `expr`.

Notice how a copy of the initial expression is made in 18. This is used later in line 27 to replace the dummy object `v^{c}{}_{b a}` with `\nabla_{a}{\nabla_{b}{v^{c}}}` but this time acting on the left hand side of the rule. The result is a rule for second covariant deriavtives.

$$v^c{}_{ba} \rightarrow \nabla_a \left(\nabla_b v^c \right) \tag{ex-0205.101}$$

$$v^c{}_{ba} \rightarrow \partial_a \left(\nabla_b v^c \right) + \Gamma^c{}_{da} \nabla_b v^d - \Gamma^d{}_{ba} \nabla_d v^c \tag{ex-0205.102}$$

$$v^c{}_{ba} \rightarrow \partial_a \left(\partial_b v^c + \Gamma^c{}_{db} v^d \right) + \Gamma^c{}_{da} \left(\partial_b v^d + \Gamma^d{}_{eb} v^e \right) - \Gamma^d{}_{ba} \left(\partial_d v^c + \Gamma^c{}_{ed} v^e \right) \tag{ex-0205.103}$$

$$v^c{}_{ba} \rightarrow \partial_{ab} v^c + \partial_a \left(\Gamma^c{}_{db} v^d \right) + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^d{}_{ba} \Gamma^c{}_{ed} v^e \tag{ex-0205.104}$$

$$v^c{}_{ba} \rightarrow \partial_{ab} v^c + \partial_a \Gamma^c{}_{db} v^d + \Gamma^c{}_{db} \partial_a v^d + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^d{}_{ba} \Gamma^c{}_{ed} v^e \tag{ex-0205.105}$$

$$v^c{}_{ba} \rightarrow \partial_{ab} v^c + \partial_a \Gamma^c{}_{db} v^d + \Gamma^c{}_{db} \partial_a v^d + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^c{}_{de} \Gamma^e{}_{ba} v^d \tag{ex-0205.107}$$

$$\nabla_a \left(\nabla_b v^c \right) \rightarrow \partial_{ab} v^c + \partial_a \Gamma^c{}_{db} v^d + \Gamma^c{}_{db} \partial_a v^d + \Gamma^c{}_{da} \partial_b v^d + \Gamma^c{}_{da} \Gamma^d{}_{eb} v^e - \Gamma^d{}_{ba} \partial_d v^c - \Gamma^c{}_{de} \Gamma^e{}_{ba} v^d \tag{ex-0205.108}$$

Exercise 2.6 Cummutation of ∇ on a scalar

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4   \partial{#}::PartialDerivative.
5
6   # covariant derivative of \phi
7
8   dphi := \nabla_{a}{\phi} -> \partial_{a}{\phi}.
9
10  # rules to hide and reveal \partial\phi
11
12  hide    := \partial_{a}{\phi} -> w_{a}.
13  reveal := w_{a} -> \partial_{a}{\phi}.
14
15  # template for covariant derivative of a dual-vector
16
17  deriv := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.
18
19  # create an object
20
21  expr :=   \nabla_{a}{\nabla_{b}{\phi}}
22          - \nabla_{b}{\nabla_{a}{\phi}}.   # cdb (ex-0206.101,expr)
23
24  # apply the rules, then simplify
25
26  substitute      (expr,dphi)                 # cdb (ex-0206.102,expr)
27  substitute      (expr,hide)                 # cdb (ex-0206.103,expr)
28  substitute      (expr,deriv)                # cdb (ex-0206.104,expr)
29  substitute      (expr,reveal)               # cdb (ex-0206.105,expr)
30  canonicalise    (expr)                      # cdb (ex-0206.106,expr)
```

$$\nabla_a\left(\nabla_b\phi\right) - \nabla_b\left(\nabla_a\phi\right) = \nabla_a\left(\partial_b\phi\right) - \nabla_b\left(\partial_a\phi\right) \tag{ex-0206.102}$$

$$= \nabla_a w_b - \nabla_b w_a \tag{ex-0206.103}$$

$$= \partial_a w_b - \Gamma^c{}_{ba} w_c - \partial_b w_a + \Gamma^c{}_{ab} w_c \tag{ex-0206.104}$$

$$= \partial_{ab}\phi - \Gamma^c{}_{ba}\partial_c\phi - \partial_{ba}\phi + \Gamma^c{}_{ab}\partial_c\phi \tag{ex-0206.105}$$

$$= -\Gamma^c{}_{ba}\partial_c\phi + \Gamma^c{}_{ab}\partial_c\phi \tag{ex-0206.106}$$

Exercise 2.7 Selective kill

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \partial{#}::PartialDerivative.
4
5   hide   := \partial_{d}{\Gamma^{a}_{b c}} -> Z_{d a b c}.
6   reveal := Z_{d a b c} -> \partial_{d}{\Gamma^{a}_{b c}}.
7
8   kill := \Gamma^{a}_{b c} -> 0.
9
10  Gamma := \Gamma^{a}_{b c}
11         + x^{d} \partial_{d}{\Gamma^{a}_{b c}}.        # cdb (ex-0207.101,Gamma)
12
13  substitute (Gamma,hide)                              # cdb (ex-0207.102,Gamma)
14  substitute (Gamma,kill)                              # cdb (ex-0207.103,Gamma)
15  substitute (Gamma,reveal)                            # cdb (ex-0207.104,Gamma)
```

$$\Gamma^{a}{}_{bc}(x) = \Gamma^{a}{}_{bc} + x^{d}\partial_{d}\Gamma^{a}{}_{bc} \qquad \text{(ex-0207.101)}$$

$$= \Gamma^{a}{}_{bc} + x^{d}Z_{dabc} \qquad \text{(ex-0207.102)}$$

$$= x^{d}Z_{dabc} \qquad \text{(ex-0207.103)}$$

$$= x^{d}\partial_{d}\Gamma^{a}{}_{bc} \qquad \text{(ex-0207.104)}$$

Exercise 2.7 Naive kill

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \partial{#}::Derivative.
4
5   kill := \Gamma^{a}_{b c} -> 0.
6
7   Gamma := \Gamma^{a}_{b c}
8           + x^{d} \partial_{d}{\Gamma^{a}_{b c}}.        # cdb (ex-0207.201,Gamma)
9
10  substitute (Gamma,kill)                              # cdb (ex-0207.202,Gamma)
```

$$\Gamma^a{}_{bc}(x) = \Gamma^a{}_{bc} + x^d \partial_d \Gamma^a{}_{bc} \qquad \text{(ex-0207.201)}$$
$$= 0 \qquad \text{(ex-0207.202)}$$

Exercise 2.7 No problem killing partial derivatives

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   \partial{#}::PartialDerivative.
4
5   kill := \partial_{c}{A_{a b}} -> 0.
6
7   Aab := A_{a b} + x^{c} \partial_{c}{A_{a b}}
8               + x^{c} x^{d} \partial_{d}{\partial_{c}{A_{a b}}}.    # cdb (ex-0207.301,Aab)
9
10  substitute (Aab,kill)                              # cdb (ex-0207.302,Aab)
```

$$A_{ab}(x) = A_{ab} + x^c \partial_c A_{ab} + x^c x^d \partial_{dc} A_{ab} \qquad \text{(ex-0207.301)}$$
$$= A_{ab} + x^c x^d \partial_{dc} A_{ab} \qquad \text{(ex-0207.302)}$$

Exercise 2.8 Position keyword in `::Indices`

```
1    {a,b,c}::Indices(position=free).
2
3    foo := A_{a b} + A^{a b}.                    # cdb (ex-0208.101,foo)
4
5    substitute (foo, $A_{a b} -> B_{a b}$)       # cdb (ex-0208.102,foo)
6
7    {p,q,r}::Indices(position=fixed).
8
9    foo := A_{p q} B^{p q} + A^{p q} B_{p q}.     # cdb (ex-0208.201,foo)
10
11   canonicalise (foo)                           # cdb (ex-0208.202,foo)
12
13   {u,v,w}::Indices(position=independent).
14
15   foo := A_{u v} B^{u v} + A^{u v} B_{u v}.     # cdb (ex-0208.301,foo)
16
17   canonicalise (foo)                           # cdb (ex-0208.302,foo)
```

$$A_{ab} + A^{ab} = B_{ab} + B^{ab} \tag{ex-0208.102}$$

$$A_{pq}B^{pq} + A^{pq}B_{pq} = 2A^{pq}B_{pq} \tag{ex-0208.202}$$

$$A_{uv}B^{uv} + A^{uv}B_{uv} = A_{uv}B^{uv} + A^{uv}B_{uv} \tag{ex-0208.302}$$

Exercise 3.1 Some symmetries of Riemann

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

2

3   ;::Symbol;

4

5   \partial{#}::PartialDerivative.

6

7   \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

8

9   Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
10                            - \partial_{d}{\Gamma^{a}_{b c}}
11                            + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
12                            - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.      # cdb(Rabcd.000,Rabcd)

13

14  dRabcd := R^{a}_{b c d ; e} -> \partial_{e}{R^{a}_{b c d}}
15                              + \Gamma^{a}_{f e} R^{f}_{b c d}
16                              - \Gamma^{f}_{b e} R^{a}_{f c d}
17                              - \Gamma^{f}_{c e} R^{a}_{b f d}
18                              - \Gamma^{f}_{d e} R^{a}_{b c f}.      # cdb(dRabcd.000,dRabcd)
```

Exercise 3.1 Antisymmetry on last pair of indices

```
1    expr := R^{a}_{b c d} + R^{a}_{b d c}.                    # cdb(ex-0301.101,expr)
2
3    substitute (expr, Rabcd)                                  # cdb(ex-0301.102,expr)
```

$$R^a{}_{bcd} + R^a{}_{bdc} = 0$$

Exercise 3.1 First Bianchi identity

```
1    expr := R^{a}_{b c d} + R^{a}_{d b c} + R^{a}_{c d b}.          # cdb(ex-0301.201,expr)
2
3    substitute   (expr, Rabcd)                                      # cdb(ex-0301.202,expr)
4    canonicalise (expr)                                             # cdb(ex-0301.203,expr)
```

$$R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de} + \partial_b\Gamma^a{}_{dc} - \partial_c\Gamma^a{}_{db} + \Gamma^e{}_{dc}\Gamma^a{}_{be} - \Gamma^e{}_{db}\Gamma^a{}_{ce} + \partial_d\Gamma^a{}_{cb} - \partial_b\Gamma^a{}_{cd} + \Gamma^e{}_{cb}\Gamma^a{}_{de}$$
$$- \Gamma^e{}_{cd}\Gamma^a{}_{be} \qquad\qquad \text{(ex-0301.202)}$$
$$= 0 \qquad\qquad \text{(ex-0301.203)}$$

Exercise 3.1 Second Bianchi identity

```
1  expr := R^{a}_{b c d ; e} + R^{a}_{b e c ; d} + R^{a}_{b d e ; c}.    # cdb(ex-0301.301,expr)
2
3  substitute     (expr, dRabcd)                                         # cdb(ex-0301.302,expr)
4  substitute     (expr,  Rabcd)                                         # cdb(ex-0301.303,expr)
5  distribute     (expr)                                                 # cdb(ex-0301.304,expr)
6  product_rule   (expr)                                                 # cdb(ex-0301.305,expr)
7  sort_product   (expr)                                                 # cdb(ex-0301.306,expr)
8  rename_dummies (expr)                                                 # cdb(ex-0301.307,expr)
9  canonicalise   (expr)                                                 # cdb(ex-0301.308,expr)
```

4

$$R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = \partial_e R^a{}_{bcd} + \Gamma^a{}_{fe} R^f{}_{bcd} - \Gamma^f{}_{be} R^a{}_{fcd} - \Gamma^f{}_{ce} R^a{}_{bfd} - \Gamma^f{}_{de} R^a{}_{bcf} + \partial_d R^a{}_{bec} + \Gamma^a{}_{fd} R^f{}_{bec} - \Gamma^f{}_{bd} R^a{}_{fec} - \Gamma^f{}_{ed} R^a{}_{bfc} - \Gamma^f{}_{cd} R^a{}_{bef}$$

$$+ \partial_c R^a{}_{bde} + \Gamma^a{}_{fc} R^f{}_{bde} - \Gamma^f{}_{bc} R^a{}_{fde} - \Gamma^f{}_{dc} R^a{}_{bfe} - \Gamma^f{}_{ec} R^a{}_{bdf} \tag{ex-0301.302}$$

$$= \partial_e \left(\partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^f{}_{bd} \Gamma^a{}_{cf} - \Gamma^f{}_{bc} \Gamma^a{}_{df} \right) + \Gamma^a{}_{fe} \left(\partial_c \Gamma^f{}_{bd} - \partial_d \Gamma^f{}_{bc} + \Gamma^g{}_{bd} \Gamma^f{}_{cg} - \Gamma^g{}_{bc} \Gamma^f{}_{dg} \right)$$

$$- \Gamma^f{}_{be} \left(\partial_c \Gamma^a{}_{fd} - \partial_d \Gamma^a{}_{fc} + \Gamma^g{}_{fd} \Gamma^a{}_{cg} - \Gamma^g{}_{fc} \Gamma^a{}_{dg} \right) - \Gamma^f{}_{ce} \left(\partial_f \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bf} + \Gamma^g{}_{bd} \Gamma^a{}_{fg} - \Gamma^g{}_{bf} \Gamma^a{}_{dg} \right)$$

$$- \Gamma^f{}_{de} \left(\partial_c \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{bc} + \Gamma^g{}_{bf} \Gamma^a{}_{cg} - \Gamma^g{}_{bc} \Gamma^a{}_{fg} \right) + \partial_d \left(\partial_e \Gamma^a{}_{bc} - \partial_c \Gamma^a{}_{be} + \Gamma^f{}_{bc} \Gamma^a{}_{ef} - \Gamma^f{}_{be} \Gamma^a{}_{cf} \right)$$

$$+ \Gamma^a{}_{fd} \left(\partial_e \Gamma^f{}_{bc} - \partial_c \Gamma^f{}_{be} + \Gamma^g{}_{bc} \Gamma^f{}_{eg} - \Gamma^g{}_{be} \Gamma^f{}_{cg} \right) - \Gamma^f{}_{bd} \left(\partial_e \Gamma^a{}_{fc} - \partial_c \Gamma^a{}_{fe} + \Gamma^g{}_{fc} \Gamma^a{}_{eg} - \Gamma^g{}_{fe} \Gamma^a{}_{cg} \right)$$

$$- \Gamma^f{}_{ed} \left(\partial_f \Gamma^a{}_{bc} - \partial_c \Gamma^a{}_{bf} + \Gamma^g{}_{bc} \Gamma^a{}_{fg} - \Gamma^g{}_{bf} \Gamma^a{}_{cg} \right) - \Gamma^f{}_{cd} \left(\partial_e \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{be} + \Gamma^g{}_{bf} \Gamma^a{}_{eg} - \Gamma^g{}_{be} \Gamma^a{}_{fg} \right)$$

$$+ \partial_c \left(\partial_d \Gamma^a{}_{be} - \partial_e \Gamma^a{}_{bd} + \Gamma^f{}_{be} \Gamma^a{}_{df} - \Gamma^f{}_{bd} \Gamma^a{}_{ef} \right) + \Gamma^a{}_{fc} \left(\partial_d \Gamma^f{}_{be} - \partial_e \Gamma^f{}_{bd} + \Gamma^g{}_{be} \Gamma^f{}_{dg} - \Gamma^g{}_{bd} \Gamma^f{}_{eg} \right)$$

$$- \Gamma^f{}_{bc} \left(\partial_d \Gamma^a{}_{fe} - \partial_e \Gamma^a{}_{fd} + \Gamma^g{}_{fe} \Gamma^a{}_{dg} - \Gamma^g{}_{fd} \Gamma^a{}_{eg} \right) - \Gamma^f{}_{dc} \left(\partial_f \Gamma^a{}_{be} - \partial_e \Gamma^a{}_{bf} + \Gamma^g{}_{be} \Gamma^a{}_{fg} - \Gamma^g{}_{bf} \Gamma^a{}_{eg} \right)$$

$$- \Gamma^f{}_{ec} \left(\partial_d \Gamma^a{}_{bf} - \partial_f \Gamma^a{}_{bd} + \Gamma^g{}_{bf} \Gamma^a{}_{dg} - \Gamma^g{}_{bd} \Gamma^a{}_{fg} \right) \tag{ex-0301.303}$$

$$= \partial_{ec} \Gamma^a{}_{bd} - \partial_{ed} \Gamma^a{}_{bc} + \partial_e \left(\Gamma^f{}_{bd} \Gamma^a{}_{cf} \right) - \partial_e \left(\Gamma^f{}_{bc} \Gamma^a{}_{df} \right) + \Gamma^a{}_{fe} \partial_c \Gamma^f{}_{bd} - \Gamma^a{}_{fe} \partial_d \Gamma^f{}_{bc} + \Gamma^a{}_{fe} \Gamma^g{}_{bd} \Gamma^f{}_{cg} - \Gamma^a{}_{fe} \Gamma^g{}_{bc} \Gamma^f{}_{dg}$$

$$- \Gamma^f{}_{be} \partial_c \Gamma^a{}_{fd} + \Gamma^f{}_{be} \partial_d \Gamma^a{}_{fc} - \Gamma^f{}_{be} \Gamma^g{}_{fd} \Gamma^a{}_{cg} + \Gamma^f{}_{be} \Gamma^g{}_{fc} \Gamma^a{}_{dg} - \Gamma^f{}_{ce} \partial_f \Gamma^a{}_{bd} + \Gamma^f{}_{ce} \partial_d \Gamma^a{}_{bf} - \Gamma^f{}_{ce} \Gamma^g{}_{bd} \Gamma^a{}_{fg} + \Gamma^f{}_{ce} \Gamma^g{}_{bf} \Gamma^a{}_{dg}$$

$$- \Gamma^f{}_{de} \partial_c \Gamma^a{}_{bf} + \Gamma^f{}_{de} \partial_f \Gamma^a{}_{bc} - \Gamma^f{}_{de} \Gamma^g{}_{bf} \Gamma^a{}_{cg} + \Gamma^f{}_{de} \Gamma^g{}_{bc} \Gamma^a{}_{fg} + \partial_{de} \Gamma^a{}_{bc} - \partial_{dc} \Gamma^a{}_{be} + \partial_d \left(\Gamma^f{}_{bc} \Gamma^a{}_{ef} \right) - \partial_d \left(\Gamma^f{}_{be} \Gamma^a{}_{cf} \right)$$

$$+ \Gamma^a{}_{fd} \partial_e \Gamma^f{}_{bc} - \Gamma^a{}_{fd} \partial_c \Gamma^f{}_{be} + \Gamma^a{}_{fd} \Gamma^g{}_{bc} \Gamma^f{}_{eg} - \Gamma^a{}_{fd} \Gamma^g{}_{be} \Gamma^f{}_{cg} - \Gamma^f{}_{bd} \partial_e \Gamma^a{}_{fc} + \Gamma^f{}_{bd} \partial_c \Gamma^a{}_{fe} - \Gamma^f{}_{bd} \Gamma^g{}_{fc} \Gamma^a{}_{eg} + \Gamma^f{}_{bd} \Gamma^g{}_{fe} \Gamma^a{}_{cg}$$

$$- \Gamma^f{}_{ed} \partial_f \Gamma^a{}_{bc} + \Gamma^f{}_{ed} \partial_c \Gamma^a{}_{bf} - \Gamma^f{}_{ed} \Gamma^g{}_{bc} \Gamma^a{}_{fg} + \Gamma^f{}_{ed} \Gamma^g{}_{bf} \Gamma^a{}_{cg} - \Gamma^f{}_{cd} \partial_e \Gamma^a{}_{bf} + \Gamma^f{}_{cd} \partial_f \Gamma^a{}_{be} - \Gamma^f{}_{cd} \Gamma^g{}_{bf} \Gamma^a{}_{eg} + \Gamma^f{}_{cd} \Gamma^g{}_{be} \Gamma^a{}_{fg}$$

$$+ \partial_{cd} \Gamma^a{}_{be} - \partial_{ce} \Gamma^a{}_{bd} + \partial_c \left(\Gamma^f{}_{be} \Gamma^a{}_{df} \right) - \partial_c \left(\Gamma^f{}_{bd} \Gamma^a{}_{ef} \right) + \Gamma^a{}_{fc} \partial_d \Gamma^f{}_{be} - \Gamma^a{}_{fc} \partial_e \Gamma^f{}_{bd} + \Gamma^a{}_{fc} \Gamma^g{}_{be} \Gamma^f{}_{dg} - \Gamma^a{}_{fc} \Gamma^g{}_{bd} \Gamma^f{}_{eg}$$

$$- \Gamma^f{}_{bc} \partial_d \Gamma^a{}_{fe} + \Gamma^f{}_{bc} \partial_e \Gamma^a{}_{fd} - \Gamma^f{}_{bc} \Gamma^g{}_{fe} \Gamma^a{}_{dg} + \Gamma^f{}_{bc} \Gamma^g{}_{fd} \Gamma^a{}_{eg} - \Gamma^f{}_{dc} \partial_f \Gamma^a{}_{be} + \Gamma^f{}_{dc} \partial_e \Gamma^a{}_{bf} - \Gamma^f{}_{dc} \Gamma^g{}_{be} \Gamma^a{}_{fg} + \Gamma^f{}_{dc} \Gamma^g{}_{bf} \Gamma^a{}_{eg}$$

$$- \Gamma^f{}_{ec} \partial_d \Gamma^a{}_{bf} + \Gamma^f{}_{ec} \partial_f \Gamma^a{}_{bd} - \Gamma^f{}_{ec} \Gamma^g{}_{bf} \Gamma^a{}_{dg} + \Gamma^f{}_{ec} \Gamma^g{}_{bd} \Gamma^a{}_{fg} \tag{ex-0301.304}$$

$$
\begin{aligned}
R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = {} & \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \partial_e\Gamma^f{}_{bd}\Gamma^a{}_{cf} + \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{cf} - \partial_e\Gamma^f{}_{bc}\Gamma^a{}_{df} - \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{df} + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^g{}_{bd}\Gamma^f{}_{cg} \\
& - \Gamma^a{}_{fe}\Gamma^g{}_{bc}\Gamma^f{}_{dg} - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^f{}_{be}\Gamma^g{}_{fd}\Gamma^a{}_{cg} + \Gamma^f{}_{be}\Gamma^g{}_{fc}\Gamma^a{}_{dg} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^f{}_{ce}\Gamma^g{}_{bd}\Gamma^a{}_{fg} \\
& + \Gamma^f{}_{ce}\Gamma^g{}_{bf}\Gamma^a{}_{dg} - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^f{}_{de}\Gamma^g{}_{bf}\Gamma^a{}_{cg} + \Gamma^f{}_{de}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \partial_d\Gamma^f{}_{bc}\Gamma^a{}_{ef} \\
& + \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{ef} - \partial_d\Gamma^f{}_{be}\Gamma^a{}_{cf} - \Gamma^f{}_{be}\partial_d\Gamma^a{}_{cf} + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^g{}_{bc}\Gamma^f{}_{eg} - \Gamma^a{}_{fd}\Gamma^g{}_{be}\Gamma^f{}_{cg} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} \\
& + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} - \Gamma^f{}_{bd}\Gamma^g{}_{fc}\Gamma^a{}_{eg} + \Gamma^f{}_{bd}\Gamma^g{}_{fe}\Gamma^a{}_{cg} - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^f{}_{ed}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \Gamma^f{}_{ed}\Gamma^g{}_{bf}\Gamma^a{}_{cg} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} \\
& + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} - \Gamma^f{}_{cd}\Gamma^g{}_{bf}\Gamma^a{}_{eg} + \Gamma^f{}_{cd}\Gamma^g{}_{be}\Gamma^a{}_{fg} + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \partial_c\Gamma^f{}_{be}\Gamma^a{}_{df} + \Gamma^f{}_{be}\partial_c\Gamma^a{}_{df} - \partial_c\Gamma^f{}_{bd}\Gamma^a{}_{ef} \\
& - \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{ef} + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^g{}_{be}\Gamma^f{}_{dg} - \Gamma^a{}_{fc}\Gamma^g{}_{bd}\Gamma^f{}_{eg} - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^f{}_{bc}\Gamma^g{}_{fe}\Gamma^a{}_{dg} \\
& + \Gamma^f{}_{bc}\Gamma^g{}_{fd}\Gamma^a{}_{eg} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^f{}_{dc}\Gamma^g{}_{be}\Gamma^a{}_{fg} + \Gamma^f{}_{dc}\Gamma^g{}_{bf}\Gamma^a{}_{eg} - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^f{}_{ec}\Gamma^g{}_{bf}\Gamma^a{}_{dg} \\
& + \Gamma^f{}_{ec}\Gamma^g{}_{bd}\Gamma^a{}_{fg} && (\text{ex-0301.305}) \\
= {} & \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \Gamma^a{}_{cf}\partial_e\Gamma^f{}_{bd} + \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{cf} - \Gamma^a{}_{df}\partial_e\Gamma^f{}_{bc} - \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{df} + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^f{}_{cg}\Gamma^g{}_{bd} \\
& - \Gamma^a{}_{fe}\Gamma^f{}_{dg}\Gamma^g{}_{bc} - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^a{}_{cg}\Gamma^f{}_{be}\Gamma^g{}_{fd} + \Gamma^a{}_{dg}\Gamma^f{}_{be}\Gamma^g{}_{fc} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ce}\Gamma^g{}_{bd} \\
& + \Gamma^a{}_{dg}\Gamma^f{}_{ce}\Gamma^g{}_{bf} - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^a{}_{cg}\Gamma^f{}_{de}\Gamma^g{}_{bf} + \Gamma^a{}_{fg}\Gamma^f{}_{de}\Gamma^g{}_{bc} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \Gamma^a{}_{ef}\partial_d\Gamma^f{}_{bc} \\
& + \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{ef} - \Gamma^a{}_{cf}\partial_d\Gamma^f{}_{be} - \Gamma^f{}_{be}\partial_d\Gamma^a{}_{cf} + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^f{}_{eg}\Gamma^g{}_{bc} - \Gamma^a{}_{fd}\Gamma^f{}_{cg}\Gamma^g{}_{be} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} \\
& + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} - \Gamma^a{}_{eg}\Gamma^f{}_{bd}\Gamma^g{}_{fc} + \Gamma^a{}_{cg}\Gamma^f{}_{bd}\Gamma^g{}_{fe} - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ed}\Gamma^g{}_{bc} + \Gamma^a{}_{cg}\Gamma^f{}_{ed}\Gamma^g{}_{bf} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} \\
& + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} - \Gamma^a{}_{eg}\Gamma^f{}_{cd}\Gamma^g{}_{bf} + \Gamma^a{}_{fg}\Gamma^f{}_{cd}\Gamma^g{}_{be} + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \Gamma^a{}_{df}\partial_c\Gamma^f{}_{be} + \Gamma^f{}_{be}\partial_c\Gamma^a{}_{df} - \Gamma^a{}_{ef}\partial_c\Gamma^f{}_{bd} \\
& - \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{ef} + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^f{}_{dg}\Gamma^g{}_{be} - \Gamma^a{}_{fc}\Gamma^f{}_{eg}\Gamma^g{}_{bd} - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^a{}_{dg}\Gamma^f{}_{bc}\Gamma^g{}_{fe} \\
& + \Gamma^a{}_{eg}\Gamma^f{}_{bc}\Gamma^g{}_{fd} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{dc}\Gamma^g{}_{be} + \Gamma^a{}_{eg}\Gamma^f{}_{dc}\Gamma^g{}_{bf} - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^a{}_{dg}\Gamma^f{}_{ec}\Gamma^g{}_{bf} \\
& + \Gamma^a{}_{fg}\Gamma^f{}_{ec}\Gamma^g{}_{bd} && (\text{ex-0301.306})
\end{aligned}
$$

$$\begin{aligned}
R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = & \; \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \Gamma^a{}_{cf}\partial_e\Gamma^f{}_{bd} + \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{cf} - \Gamma^a{}_{df}\partial_e\Gamma^f{}_{bc} - \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{df} + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^f{}_{cg}\Gamma^g{}_{bd} \\
& - \Gamma^a{}_{fe}\Gamma^f{}_{dg}\Gamma^g{}_{bc} - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^a{}_{cf}\Gamma^g{}_{be}\Gamma^f{}_{gd} + \Gamma^a{}_{df}\Gamma^g{}_{be}\Gamma^f{}_{gc} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ce}\Gamma^g{}_{bd} \\
& + \Gamma^a{}_{df}\Gamma^g{}_{ce}\Gamma^f{}_{bg} - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^a{}_{cf}\Gamma^g{}_{de}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{de}\Gamma^g{}_{bc} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \Gamma^a{}_{ef}\partial_d\Gamma^f{}_{bc} + \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{ef} \\
& - \Gamma^a{}_{cf}\partial_d\Gamma^f{}_{be} - \Gamma^f{}_{be}\partial_d\Gamma^a{}_{cf} + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^f{}_{eg}\Gamma^g{}_{bc} - \Gamma^a{}_{fd}\Gamma^f{}_{cg}\Gamma^g{}_{be} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} \\
& - \Gamma^a{}_{ef}\Gamma^g{}_{bd}\Gamma^f{}_{gc} + \Gamma^a{}_{cf}\Gamma^g{}_{bd}\Gamma^f{}_{ge} - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ed}\Gamma^g{}_{bc} + \Gamma^a{}_{cf}\Gamma^g{}_{ed}\Gamma^f{}_{bg} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} \\
& - \Gamma^a{}_{ef}\Gamma^g{}_{cd}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{cd}\Gamma^g{}_{be} + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \Gamma^a{}_{df}\partial_c\Gamma^f{}_{be} + \Gamma^f{}_{be}\partial_c\Gamma^a{}_{df} - \Gamma^a{}_{ef}\partial_c\Gamma^f{}_{bd} - \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{ef} + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} \\
& - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^f{}_{dg}\Gamma^g{}_{be} - \Gamma^a{}_{fc}\Gamma^f{}_{eg}\Gamma^g{}_{bd} - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^a{}_{df}\Gamma^g{}_{bc}\Gamma^f{}_{ge} + \Gamma^a{}_{ef}\Gamma^g{}_{bc}\Gamma^f{}_{gd} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} \\
& + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{dc}\Gamma^g{}_{be} + \Gamma^a{}_{ef}\Gamma^g{}_{dc}\Gamma^f{}_{bg} - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^a{}_{df}\Gamma^g{}_{ec}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{ec}\Gamma^g{}_{bd} \qquad \text{(ex-0301.307)} \\
= & \; 0 \qquad \text{(ex-0301.308)}
\end{aligned}$$

Exercise 3.2 Riemann tensor from commutation of ∇

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

2

3    \nabla{#}::Derivative.
4    \partial{#}::PartialDerivative.

5

6    \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});

7

8    # rules for the first two covariant derivs of V^a

9

10   deriv1 := \nabla_{a}{V^{b}} -> \partial_{a}{V^{b}}
11                          + \Gamma^{b}_{d a} V^{d}.          # cdb (ex-0302.101,deriv1)

12

13   deriv2 := \nabla_{a}{\nabla_{b}{V^{c}}} -> \partial_{a}{\nabla_{b}{V^{c}}}
14                             + \Gamma^{c}_{d a} \nabla_{b}{V^{d}}
15                             - \Gamma^{d}_{b a} \nabla_{d}{V^{c}}.
16                                                              # cdb (ex-0302.102,deriv2)

17

18   Vabc :=   \nabla_{c}{\nabla_{b}{V^{a}}}
19             - \nabla_{b}{\nabla_{c}{V^{a}}}.                 # cdb (ex-0302.103, Vabc)

20

21   substitute (Vabc,deriv2)                                  # cdb (ex-0302.104, Vabc)
22   substitute (Vabc,deriv1)                                  # cdb (ex-0302.105, Vabc)

23

24   distribute    (Vabc)                                      # cdb (ex-0302.106, Vabc)
25   product_rule  (Vabc)                                      # cdb (ex-0302.107, Vabc)

26

27   sort_product  (Vabc)                                      # cdb (ex-0302.108, Vabc)
28   rename_dummies (Vabc)                                     # cdb (ex-0302.109, Vabc)
29   canonicalise  (Vabc)                                      # cdb (ex-0302.110, Vabc)
30   factor_out    (Vabc,$V^{a?}$)                             # cdb (ex-0302.111, Vabc)
```

$$\nabla_c \left(\nabla_b V^a \right) - \nabla_b \left(\nabla_c V^a \right) = \partial_c \left(\nabla_b V^a \right) + \Gamma^a{}_{dc} \nabla_b V^d - \Gamma^d{}_{bc} \nabla_d V^a - \partial_b \left(\nabla_c V^a \right) - \Gamma^a{}_{db} \nabla_c V^d + \Gamma^d{}_{cb} \nabla_d V^a \qquad \text{(ex-0302.104)}$$

$$= \partial_c \left(\partial_b V^a + \Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \left(\partial_b V^d + \Gamma^d{}_{eb} V^e \right) - \Gamma^d{}_{bc} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) - \partial_b \left(\partial_c V^a + \Gamma^a{}_{dc} V^d \right) - \Gamma^a{}_{db} \left(\partial_c V^d + \Gamma^d{}_{ec} V^e \right)$$

$$+ \Gamma^d{}_{cb} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e \right) \qquad \text{(ex-0302.105)}$$

$$= \partial_{cb} V^a + \partial_c \left(\Gamma^a{}_{db} V^d \right) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \left(\Gamma^a{}_{dc} V^d \right) - \Gamma^a{}_{db} \partial_c V^d$$

$$- \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \qquad \text{(ex-0302.106)}$$

$$= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a$$

$$+ \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e \qquad \text{(ex-0302.107)}$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^e \Gamma^a{}_{dc} \Gamma^d{}_{eb} - \Gamma^d{}_{bc} \partial_d V^a - V^e \Gamma^a{}_{ed} \Gamma^d{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^e \Gamma^a{}_{db} \Gamma^d{}_{ec} + \Gamma^d{}_{cb} \partial_d V^a$$

$$+ V^e \Gamma^a{}_{ed} \Gamma^d{}_{cb} \qquad \text{(ex-0302.108)}$$

$$= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^d \Gamma^a{}_{ec} \Gamma^e{}_{db} - \Gamma^d{}_{bc} \partial_d V^a - V^d \Gamma^a{}_{de} \Gamma^e{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^d \Gamma^a{}_{eb} \Gamma^e{}_{dc} + \Gamma^d{}_{cb} \partial_d V^a$$

$$+ V^d \Gamma^a{}_{de} \Gamma^e{}_{cb} \qquad \text{(ex-0302.109)}$$

$$= V^d \partial_c \Gamma^a{}_{bd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} \qquad \text{(ex-0302.110)}$$

$$= V^d \left(\partial_c \Gamma^a{}_{bd} + \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \partial_b \Gamma^a{}_{cd} - \Gamma^a{}_{be} \Gamma^e{}_{cd} \right) \qquad \text{(ex-0302.111)}$$

$$= -R^a{}_{dbc} V^d$$

This result agrees with Misner, Thorne and Wheeler. pg. 266.

Exercise 3.3 Computing R_{abcd}

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3   \partial{#}::PartialDerivative.
4
5   \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
6   \Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).
7
8   dgab := \partial_{c}{g_{a b}} ->    \Gamma^{d}_{a c} g_{d b}
9                                    + \Gamma^{d}_{b c} g_{a d}.        # cdb(dgab.000,dgab)
10
11  RabcdU := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
12                            - \partial_{d}{\Gamma^{a}_{b c}}
13                            + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
14                            - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.   # cdb(Rabcd.000,RabcdU)
15
16  GammaD := {g_{a e} \Gamma^{e}_{b c} -> \Gamma_{a b c},
17             g_{e a} \Gamma^{e}_{b c} -> \Gamma_{a b c}}.        # cdb(Gamma.010,GammaD)
18
19  RabcdD := R_{a b c d} -> g_{a e} R^{e}_{b c d}.               # cdb(Rabcd.010,RabcdD)
20
21  gabDGamma := g_{a e} \partial_{c}{\Gamma^{e}_{b d}} ->
22                   \partial_{c}{g_{a e} \Gamma^{e}_{b d}}
23                 - \Gamma^{e}_{b d} \partial_{c}{g_{a e}}.       # cdb(gabDGamma.000,gabDGamma)
24
25  # this pair of rules needed to sort \Gamma_{a b c} to the very left
26  # this helps canonicalise spot the terms that cancel
27  bah := \Gamma_{a b c} -> A_{a b c}.
28  foo := A_{a b c} -> \Gamma_{a b c}.
29
30  expr := R_{a b c d}.                                        # cdb(ex-0303.101,expr)
31
32  substitute      (expr, RabcdD)                              # cdb(ex-0303.102,expr)
33  substitute      (expr, RabcdU)                              # cdb(ex-0303.103,expr)
34  distribute      (expr)                                      # cdb(ex-0303.104,expr)
35  substitute      (expr, gabDGamma)                           # cdb(ex-0303.105,expr)
36  substitute      (expr, dgab)                                # cdb(ex-0303.106,expr)
```

```
37  substitute      (expr, GammaD)                          # cdb(ex-0303.107,expr)
38  distribute      (expr)                                  # cdb(ex-0303.109,expr)
39  substitute      (expr, bah)                             # cdb(ex-0303.110,expr)
40  sort_product    (expr)                                  # cdb(ex-0303.111,expr)
41  rename_dummies  (expr)                                  # cdb(ex-0303.112,expr)
42  substitute      (expr, foo)                             # cdb(ex-0303.113,expr)
43  canonicalise    (expr)                                  # cdb(ex-0303.114,expr)
```

$$R_{abcd} = g_{ae} R^e{}_{bcd} \tag{ex-0303.102}$$

$$= g_{ae} \left(\partial_c \Gamma^e{}_{bd} - \partial_d \Gamma^e{}_{bc} + \Gamma^f{}_{bd} \Gamma^e{}_{cf} - \Gamma^f{}_{bc} \Gamma^e{}_{df} \right) \tag{ex-0303.103}$$

$$= g_{ae} \partial_c \Gamma^e{}_{bd} - g_{ae} \partial_d \Gamma^e{}_{bc} + g_{ae} \Gamma^f{}_{bd} \Gamma^e{}_{cf} - g_{ae} \Gamma^f{}_{bc} \Gamma^e{}_{df} \tag{ex-0303.104}$$

$$= \partial_c \left(g_{ae} \Gamma^e{}_{bd} \right) - \Gamma^e{}_{bd} \partial_c g_{ae} - \partial_d \left(g_{ae} \Gamma^e{}_{bc} \right) + \Gamma^e{}_{bc} \partial_d g_{ae} + g_{ae} \Gamma^f{}_{bd} \Gamma^e{}_{cf} - g_{ae} \Gamma^f{}_{bc} \Gamma^e{}_{df} \tag{ex-0303.105}$$

$$= \partial_c \left(g_{ae} \Gamma^e{}_{bd} \right) - \Gamma^e{}_{bd} \left(\Gamma^f{}_{ac} g_{fe} + \Gamma^f{}_{ec} g_{af} \right) - \partial_d \left(g_{ae} \Gamma^e{}_{bc} \right) + \Gamma^e{}_{bc} \left(\Gamma^f{}_{ad} g_{fe} + \Gamma^f{}_{ed} g_{af} \right) + g_{ae} \Gamma^f{}_{bd} \Gamma^e{}_{cf} - g_{ae} \Gamma^f{}_{bc} \Gamma^e{}_{df} \tag{ex-0303.106}$$

$$= \partial_c \Gamma_{abd} - \Gamma^e{}_{bd} \left(\Gamma_{eac} + \Gamma_{aec} \right) - \partial_d \Gamma_{abc} + \Gamma^e{}_{bc} \left(\Gamma_{ead} + \Gamma_{aed} \right) + \Gamma_{acf} \Gamma^f{}_{bd} - \Gamma_{adf} \Gamma^f{}_{bc} \tag{ex-0303.107}$$

$$= \partial_c \Gamma_{abd} - \Gamma^e{}_{bd} \Gamma_{eac} - \Gamma^e{}_{bd} \Gamma_{aec} - \partial_d \Gamma_{abc} + \Gamma^e{}_{bc} \Gamma_{ead} + \Gamma^e{}_{bc} \Gamma_{aed} + \Gamma_{acf} \Gamma^f{}_{bd} - \Gamma_{adf} \Gamma^f{}_{bc} \tag{ex-0303.109}$$

$$= \partial_c A_{abd} - \Gamma^e{}_{bd} A_{eac} - \Gamma^e{}_{bd} A_{aec} - \partial_d A_{abc} + \Gamma^e{}_{bc} A_{ead} + \Gamma^e{}_{bc} A_{aed} + A_{acf} \Gamma^f{}_{bd} - A_{adf} \Gamma^f{}_{bc} \tag{ex-0303.110}$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e{}_{bd} - A_{aec} \Gamma^e{}_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e{}_{bc} + A_{aed} \Gamma^e{}_{bc} + A_{acf} \Gamma^f{}_{bd} - A_{adf} \Gamma^f{}_{bc} \tag{ex-0303.111}$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e{}_{bd} - A_{aec} \Gamma^e{}_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e{}_{bc} + A_{aed} \Gamma^e{}_{bc} + A_{ace} \Gamma^e{}_{bd} - A_{ade} \Gamma^e{}_{bc} \tag{ex-0303.112}$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e{}_{bd} - \Gamma_{aec} \Gamma^e{}_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e{}_{bc} + \Gamma_{aed} \Gamma^e{}_{bc} + \Gamma_{ace} \Gamma^e{}_{bd} - \Gamma_{ade} \Gamma^e{}_{bc} \tag{ex-0303.113}$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e{}_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e{}_{bc} \tag{ex-0303.114}$$

Exercise 3.4 More symmetries of Riemann

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3   \partial{#}::PartialDerivative.
4
5   g_{a b}::Symmetric.
6   g^{a b}::Symmetric.
7
8   \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
9   \Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).
10
11  GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
12                                              + \partial_{c}{g_{b d}}
13                                              - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)
14
15  GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
16                                     + \partial_{c}{g_{b a}}
17                                     - \partial_{a}{g_{b c}}).          # cdb(Gamma.010,GammaD)
18
19  Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
20                          - \partial_{d}{\Gamma_{a b c}}
21                          + \Gamma_{e a d} \Gamma^{e}_{b c}
22                          - \Gamma_{e a c} \Gamma^{e}_{b d}.            # cdb(Rabcd.000,Rabcd)
```

Exercise 3.4 Antisymmetry on first pair of indices

```
1    expr := R_{a b c d} + R_{b a c d}.    # cdb(ex-0304.101,expr)

2
3    substitute      (expr, Rabcd)         # cdb(ex-0304.102,expr)
4    substitute      (expr, GammaU)        # cdb(ex-0304.103,expr)
5    substitute      (expr, GammaD)        # cdb(ex-0304.104,expr)
6    distribute      (expr)                # cdb(ex-0304.105,expr)
7    product_rule    (expr)                # cdb(ex-0304.106,expr)
8    sort_product    (expr)                # cdb(ex-0304.107,expr)
9    rename_dummies  (expr)                # cdb(ex-0304.108,expr)
10   canonicalise    (expr)                # cdb(ex-0304.109,expr)
```

$$R_{abcd} + R_{bacd} = \partial_c\Gamma_{abd} - \partial_d\Gamma_{abc} + \Gamma_{ead}\Gamma^e{}_{bc} - \Gamma_{eac}\Gamma^e{}_{bd} + \partial_c\Gamma_{bad} - \partial_d\Gamma_{bac} + \Gamma_{ebd}\Gamma^e{}_{ac} - \Gamma_{ebc}\Gamma^e{}_{ad} \tag{ex-0304.102}$$

$$= \partial_c\Gamma_{abd} - \partial_d\Gamma_{abc} + \frac{1}{2}\Gamma_{ead}g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) - \frac{1}{2}\Gamma_{eac}g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) + \partial_c\Gamma_{bad} - \partial_d\Gamma_{bac}$$

$$+ \frac{1}{2}\Gamma_{ebd}g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) - \frac{1}{2}\Gamma_{ebc}g^{ef}\left(\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}\right) \tag{ex-0304.103}$$

$$= \partial_c\left(\frac{1}{2}\partial_b g_{ad} + \frac{1}{2}\partial_d g_{ba} - \frac{1}{2}\partial_a g_{bd}\right) - \partial_d\left(\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc}\right) + \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ed} + \frac{1}{2}\partial_d g_{ae} - \frac{1}{2}\partial_e g_{ad}\right)g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right)$$

$$- \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ec} + \frac{1}{2}\partial_c g_{ae} - \frac{1}{2}\partial_e g_{ac}\right)g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) + \partial_c\left(\frac{1}{2}\partial_a g_{bd} + \frac{1}{2}\partial_d g_{ab} - \frac{1}{2}\partial_b g_{ad}\right)$$

$$- \partial_d\left(\frac{1}{2}\partial_a g_{bc} + \frac{1}{2}\partial_c g_{ab} - \frac{1}{2}\partial_b g_{ac}\right) + \frac{1}{2}\left(\frac{1}{2}\partial_b g_{ed} + \frac{1}{2}\partial_d g_{be} - \frac{1}{2}\partial_e g_{bd}\right)g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right)$$

$$- \frac{1}{2}\left(\frac{1}{2}\partial_b g_{ec} + \frac{1}{2}\partial_c g_{be} - \frac{1}{2}\partial_e g_{bc}\right)g^{ef}\left(\partial_a g_{fd} + \partial_d g_{af} - \partial_f g_{ad}\right) \tag{ex-0304.104}$$

$$= \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc}$$

$$- \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd}$$

$$- \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_a g_{fc}$$

$$+ \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_f g_{ac} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_a g_{fc} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_d g_{be}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_a g_{fc} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_c g_{af}$$

$$+ \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_f g_{ad} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_c g_{be}g^{ef}\partial_f g_{ad}$$

$$+ \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_a g_{fd} + \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_d g_{af} - \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_f g_{ad} \tag{ex-0304.105}$$

$$R_{abcd} + R_{bacd} = \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc}$$

$$- \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd}$$

$$- \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_a g_{fc}$$

$$+ \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_b g_{ed}g^{ef}\partial_f g_{ac} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_a g_{fc} + \frac{1}{4}\partial_d g_{be}g^{ef}\partial_c g_{af} - \frac{1}{4}\partial_d g_{be}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_a g_{fc} - \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_c g_{af}$$

$$+ \frac{1}{4}\partial_e g_{bd}g^{ef}\partial_f g_{ac} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_b g_{ec}g^{ef}\partial_f g_{ad} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_a g_{fd} - \frac{1}{4}\partial_c g_{be}g^{ef}\partial_d g_{af} + \frac{1}{4}\partial_c g_{be}g^{ef}\partial_f g_{ad}$$

$$+ \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_a g_{fd} + \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_d g_{af} - \frac{1}{4}\partial_e g_{bc}g^{ef}\partial_f g_{ad} \tag{ex-0304.106}$$

$$= \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_f g_{bc}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_d g_{ae}g^{ef} + \frac{1}{4}\partial_c g_{bf}\partial_d g_{ae}g^{ef} - \frac{1}{4}\partial_d g_{ae}\partial_f g_{bc}g^{ef}$$

$$- \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_a g_{ec}\partial_f g_{bd}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_c g_{ae}g^{ef}$$

$$- \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{ae}\partial_f g_{bd}g^{ef} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_a g_{fc}\partial_b g_{ed}g^{ef}$$

$$+ \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{ef} - \frac{1}{4}\partial_b g_{ed}\partial_f g_{ac}g^{ef} + \frac{1}{4}\partial_a g_{fc}\partial_d g_{be}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_d g_{be}g^{ef} - \frac{1}{4}\partial_d g_{be}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{ef} - \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{ef}$$

$$+ \frac{1}{4}\partial_e g_{bd}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_b g_{ec}g^{ef} - \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_b g_{ec}\partial_f g_{ad}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_c g_{be}g^{ef} - \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_c g_{be}\partial_f g_{ad}g^{ef}$$

$$+ \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{ef} + \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{ef} - \frac{1}{4}\partial_e g_{bc}\partial_f g_{ad}g^{ef} \tag{ex-0304.107}$$

$$R_{abcd} + R_{bacd} = \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{fe} + \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{fe} + \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{fe} - \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{fe}$$

$$- \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{fe} - \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{fe}$$

$$- \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{fe} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef} + \frac{1}{2}\partial_{cd}g_{ab} - \frac{1}{2}\partial_{dc}g_{ab} + \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{fe}$$

$$+ \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{fe} + \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{fe} + \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{fe} - \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{fe} - \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{ef} - \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{ef}$$

$$+ \frac{1}{4}\partial_e g_{bd}\partial_f g_{ac}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{fe} - \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{fe} - \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{fe} - \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{ef} + \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{fe}$$

$$+ \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{ef} + \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{ef} - \frac{1}{4}\partial_e g_{bc}\partial_f g_{ad}g^{ef} \tag{ex-0304.108}$$

$$= 0 \tag{ex-0304.109}$$

5

Exercise 3.4 Symmetric on swapping first and second pair of indices

```
1   expr := R_{a b c d} - R_{c d a b}.    # cdb(ex-0304.201,expr)
2
3   substitute      (expr, Rabcd)         # cdb(ex-0304.202,expr)
4   substitute      (expr, GammaU)        # cdb(ex-0304.203,expr)
5   substitute      (expr, GammaD)        # cdb(ex-0304.204,expr)
6   distribute      (expr)                # cdb(ex-0304.205,expr)
7   product_rule    (expr)                # cdb(ex-0304.206,expr)
8   sort_product    (expr)                # cdb(ex-0304.207,expr)
9   rename_dummies  (expr)                # cdb(ex-0304.208,expr)
10  canonicalise    (expr)                # cdb(ex-0304.209,expr)
```

$$R_{abcd} - R_{cdab} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{ead}\Gamma^e{}_{bc} - \Gamma_{eac}\Gamma^e{}_{bd} - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda} - \Gamma_{ecb}\Gamma^e{}_{da} + \Gamma_{eca}\Gamma^e{}_{db} \qquad \text{(ex-0304.202)}$$

$$= \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \frac{1}{2}\Gamma_{ead}g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) - \frac{1}{2}\Gamma_{eac}g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) - \partial_a \Gamma_{cdb} + \partial_b \Gamma_{cda}$$

$$- \frac{1}{2}\Gamma_{ecb}g^{ef}\left(\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}\right) + \frac{1}{2}\Gamma_{eca}g^{ef}\left(\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}\right) \qquad \text{(ex-0304.203)}$$

$$= \partial_c \left(\frac{1}{2}\partial_b g_{ad} + \frac{1}{2}\partial_d g_{ba} - \frac{1}{2}\partial_a g_{bd}\right) - \partial_d \left(\frac{1}{2}\partial_b g_{ac} + \frac{1}{2}\partial_c g_{ba} - \frac{1}{2}\partial_a g_{bc}\right) + \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ed} + \frac{1}{2}\partial_d g_{ae} - \frac{1}{2}\partial_e g_{ad}\right)g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right)$$

$$- \frac{1}{2}\left(\frac{1}{2}\partial_a g_{ec} + \frac{1}{2}\partial_c g_{ae} - \frac{1}{2}\partial_e g_{ac}\right)g^{ef}\left(\partial_b g_{fd} + \partial_d g_{bf} - \partial_f g_{bd}\right) - \partial_a \left(\frac{1}{2}\partial_d g_{cb} + \frac{1}{2}\partial_b g_{dc} - \frac{1}{2}\partial_c g_{db}\right)$$

$$+ \partial_b \left(\frac{1}{2}\partial_d g_{ca} + \frac{1}{2}\partial_a g_{dc} - \frac{1}{2}\partial_c g_{da}\right) - \frac{1}{2}\left(\frac{1}{2}\partial_c g_{eb} + \frac{1}{2}\partial_b g_{ce} - \frac{1}{2}\partial_e g_{cb}\right)g^{ef}\left(\partial_d g_{fa} + \partial_a g_{df} - \partial_f g_{da}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{2}\partial_c g_{ea} + \frac{1}{2}\partial_a g_{ce} - \frac{1}{2}\partial_e g_{ca}\right)g^{ef}\left(\partial_d g_{fb} + \partial_b g_{df} - \partial_f g_{db}\right) \qquad \text{(ex-0304.204)}$$

$$= \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc}$$

$$+ \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf}$$

$$+ \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd}$$

$$- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_f g_{da}$$

$$- \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_d g_{fa} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_a g_{df} - \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_d g_{fb}$$

$$+ \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_f g_{db} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_d g_{fb} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_f g_{db} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_d g_{fb} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_b g_{df}$$

$$+ \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_f g_{db} \qquad \text{(ex-0304.205)}$$

$$R_{abcd} - R_{cdab} = \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_a g_{ed}g^{ef}\partial_f g_{bc} + \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_b g_{fc}$$

$$+ \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_c g_{bf} - \frac{1}{4}\partial_d g_{ae}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_b g_{fc} - \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_c g_{bf} + \frac{1}{4}\partial_e g_{ad}g^{ef}\partial_f g_{bc} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_d g_{bf}$$

$$+ \frac{1}{4}\partial_a g_{ec}g^{ef}\partial_f g_{bd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_b g_{fd} - \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_d g_{bf} + \frac{1}{4}\partial_c g_{ae}g^{ef}\partial_f g_{bd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_b g_{fd} + \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_d g_{bf} - \frac{1}{4}\partial_e g_{ac}g^{ef}\partial_f g_{bd}$$

$$- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_c g_{eb}g^{ef}\partial_f g_{da}$$

$$- \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_d g_{fa} - \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_a g_{df} + \frac{1}{4}\partial_b g_{ce}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_d g_{fa} + \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_a g_{df} - \frac{1}{4}\partial_e g_{cb}g^{ef}\partial_f g_{da} + \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_d g_{fb}$$

$$+ \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_c g_{ea}g^{ef}\partial_f g_{db} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_d g_{fb} + \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_b g_{df} - \frac{1}{4}\partial_a g_{ce}g^{ef}\partial_f g_{db} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_d g_{fb} - \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_b g_{df}$$

$$+ \frac{1}{4}\partial_e g_{ca}g^{ef}\partial_f g_{db} \tag{ex-0304.206}$$

$$= \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{ed}\partial_f g_{bc}g^{ef} + \frac{1}{4}\partial_b g_{fc}\partial_d g_{ae}g^{ef}$$

$$+ \frac{1}{4}\partial_c g_{bf}\partial_d g_{ae}g^{ef} - \frac{1}{4}\partial_d g_{ae}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef}$$

$$+ \frac{1}{4}\partial_a g_{ec}\partial_f g_{bd}g^{ef} - \frac{1}{4}\partial_b g_{fd}\partial_c g_{ae}g^{ef} - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{ae}\partial_f g_{bd}g^{ef} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef}$$

$$- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{df}\partial_c g_{eb}g^{ef} + \frac{1}{4}\partial_c g_{eb}\partial_f g_{da}g^{ef}$$

$$- \frac{1}{4}\partial_b g_{ce}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{df}\partial_b g_{ce}g^{ef} + \frac{1}{4}\partial_b g_{ce}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_d g_{fa}\partial_e g_{cb}g^{ef} + \frac{1}{4}\partial_a g_{df}\partial_e g_{cb}g^{ef} - \frac{1}{4}\partial_e g_{cb}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_c g_{ea}\partial_d g_{fb}g^{ef}$$

$$+ \frac{1}{4}\partial_b g_{df}\partial_c g_{ea}g^{ef} - \frac{1}{4}\partial_c g_{ea}\partial_f g_{db}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_d g_{fb}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_b g_{df}g^{ef} - \frac{1}{4}\partial_a g_{ce}\partial_f g_{db}g^{ef} - \frac{1}{4}\partial_d g_{fb}\partial_e g_{ca}g^{ef} - \frac{1}{4}\partial_b g_{df}\partial_e g_{ca}g^{ef}$$

$$+ \frac{1}{4}\partial_e g_{ca}\partial_f g_{db}g^{ef} \tag{ex-0304.207}$$

$$R_{abcd} - R_{cdab} = \frac{1}{2}\partial_{cb}g_{ad} + \frac{1}{2}\partial_{cd}g_{ba} - \frac{1}{2}\partial_{ca}g_{bd} - \frac{1}{2}\partial_{db}g_{ac} - \frac{1}{2}\partial_{dc}g_{ba} + \frac{1}{2}\partial_{da}g_{bc} + \frac{1}{4}\partial_a g_{ed}\partial_b g_{fc}g^{ef} + \frac{1}{4}\partial_a g_{ed}\partial_c g_{bf}g^{ef} - \frac{1}{4}\partial_a g_{fd}\partial_e g_{bc}g^{fe} + \frac{1}{4}\partial_b g_{ec}\partial_d g_{af}g^{fe}$$

$$+ \frac{1}{4}\partial_c g_{be}\partial_d g_{af}g^{fe} - \frac{1}{4}\partial_d g_{af}\partial_e g_{bc}g^{fe} - \frac{1}{4}\partial_b g_{fc}\partial_e g_{ad}g^{ef} - \frac{1}{4}\partial_c g_{bf}\partial_e g_{ad}g^{ef} + \frac{1}{4}\partial_e g_{ad}\partial_f g_{bc}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_b g_{fd}g^{ef} - \frac{1}{4}\partial_a g_{ec}\partial_d g_{bf}g^{ef}$$

$$+ \frac{1}{4}\partial_a g_{fc}\partial_e g_{bd}g^{fe} - \frac{1}{4}\partial_b g_{ed}\partial_c g_{af}g^{fe} - \frac{1}{4}\partial_c g_{ae}\partial_d g_{bf}g^{ef} + \frac{1}{4}\partial_c g_{af}\partial_e g_{bd}g^{fe} + \frac{1}{4}\partial_b g_{fd}\partial_e g_{ac}g^{ef} + \frac{1}{4}\partial_d g_{bf}\partial_e g_{ac}g^{ef} - \frac{1}{4}\partial_e g_{ac}\partial_f g_{bd}g^{ef}$$

$$- \frac{1}{2}\partial_{ad}g_{cb} - \frac{1}{2}\partial_{ab}g_{dc} + \frac{1}{2}\partial_{ac}g_{db} + \frac{1}{2}\partial_{bd}g_{ca} + \frac{1}{2}\partial_{ba}g_{dc} - \frac{1}{2}\partial_{bc}g_{da} - \frac{1}{4}\partial_c g_{eb}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{de}\partial_c g_{fb}g^{fe} + \frac{1}{4}\partial_c g_{fb}\partial_e g_{da}g^{fe}$$

$$- \frac{1}{4}\partial_b g_{ce}\partial_d g_{fa}g^{ef} - \frac{1}{4}\partial_a g_{de}\partial_b g_{cf}g^{fe} + \frac{1}{4}\partial_b g_{cf}\partial_e g_{da}g^{fe} + \frac{1}{4}\partial_d g_{fa}\partial_e g_{cb}g^{ef} + \frac{1}{4}\partial_a g_{df}\partial_e g_{cb}g^{ef} - \frac{1}{4}\partial_e g_{cb}\partial_f g_{da}g^{ef} + \frac{1}{4}\partial_c g_{ea}\partial_d g_{fb}g^{ef}$$

$$+ \frac{1}{4}\partial_b g_{de}\partial_c g_{fa}g^{fe} - \frac{1}{4}\partial_c g_{fa}\partial_e g_{db}g^{fe} + \frac{1}{4}\partial_a g_{ce}\partial_d g_{fb}g^{ef} + \frac{1}{4}\partial_a g_{ce}\partial_b g_{df}g^{ef} - \frac{1}{4}\partial_a g_{cf}\partial_e g_{db}g^{fe} - \frac{1}{4}\partial_d g_{fb}\partial_e g_{ca}g^{ef} - \frac{1}{4}\partial_b g_{df}\partial_e g_{ca}g^{ef}$$

$$+ \frac{1}{4}\partial_e g_{ca}\partial_f g_{db}g^{ef} \tag{ex-0304.208}$$

$$= 0 \tag{ex-0304.209}$$

Exercise 3.5 Commutation of covariant derivatives

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3   \nabla{#}::Derivative.
4
5   expr :=   \nabla_{d}{\nabla_{c}{A_{a} B_{b}}}
6          - \nabla_{c}{\nabla_{d}{A_{a} B_{b}}}.   # cdb(ex-0305.100,expr)
7
8   product_rule (expr)                            # cdb(ex-0305.101,expr)
9   distribute   (expr)                            # cdb(ex-0305.102,expr)
10  product_rule (expr)                            # cdb(ex-0305.103,expr)
11  factor_out   (expr,$A_{a?},B_{b?}$)            # cdb(ex-0305.104,expr)
```

$$\nabla_d\left(\nabla_c\left(A_a B_b\right)\right) - \nabla_c\left(\nabla_d\left(A_a B_b\right)\right) = \nabla_d\left(\nabla_c A_a B_b + A_a \nabla_c B_b\right) - \nabla_c\left(\nabla_d A_a B_b + A_a \nabla_d B_b\right) \tag{ex-0305.101}$$

$$= \nabla_d\left(\nabla_c A_a B_b\right) + \nabla_d\left(A_a \nabla_c B_b\right) - \nabla_c\left(\nabla_d A_a B_b\right) - \nabla_c\left(A_a \nabla_d B_b\right) \tag{ex-0305.102}$$

$$= \nabla_d\left(\nabla_c A_a\right) B_b + A_a \nabla_d\left(\nabla_c B_b\right) - \nabla_c\left(\nabla_d A_a\right) B_b - A_a \nabla_c\left(\nabla_d B_b\right) \tag{ex-0305.103}$$

$$= B_b\left(\nabla_d\left(\nabla_c A_a\right) - \nabla_c\left(\nabla_d A_a\right)\right) + A_a\left(\nabla_d\left(\nabla_c B_b\right) - \nabla_c\left(\nabla_d B_b\right)\right) \tag{ex-0305.104}$$

Exercise 3.6 Commutation of ∇ on the Riemann tensor – simple computation

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3   DD{#}::Derivative.
4   \nabla{#}::Derivative.
5
6   RabcdF := R_{a b c d} -> A_{a} B_{b} C_{c} D_{d}.        # cdb(RabcdF.000,RabcdF)
7   RabcdB := A_{a} B_{b} C_{c} D_{d} -> R_{a b c d}.        # cdb(RabcdB.000,RabcdB)
8
9   derivDD := DD_{b c}{V?_{a}}} -> R^{d}_{a b c} V?_{d}.  # cdb(derivDD.000,derivDD)
10
11  nablaDD := \nabla_{f}{\nabla_{e}{R_{a b c d}}}
12          - \nabla_{e}{\nabla_{f}{R_{a b c d}}} -> DD_{e f}{R_{a b c d}}.
13
14  # product rule for DD acting on A_{a} B_{b} C_{c} D_{d}
15  pruleDD := DD_{e f}{A_{a} B_{b} C_{c} D_{d}} -> DD_{e f}{A_{a}} B_{b} C_{c} D_{d}
16                                               + A_{a} DD_{e f}{B_{b}} C_{c} D_{d}
17                                               + A_{a} B_{b} DD_{e f}{C_{c}} D_{d}
18                                               + A_{a} B_{b} C_{c} DD_{e f}{D_{d}}.
19                                                            # cdb(pruleDD.000,pruleDD)
20
21  expr :=   \nabla_{f}{\nabla_{e}{R_{a b c d}}}
22          - \nabla_{e}{\nabla_{f}{R_{a b c d}}}.          # cdb (ex-0306.100, expr)
23
24  substitute   (expr,nablaDD)                             # cdb (ex-0306.101, expr)
25  substitute   (expr,RabcdF)                              # cdb (ex-0306.102, expr)
26  substitute   (expr,pruleDD)                             # cdb (ex-0306.103, expr)
27  substitute   (expr,derivDD)                             # cdb (ex-0306.104, expr)
28  sort_product (expr)                                     # cdb (ex-0306.105, expr)
29  substitute   (expr,RabcdB)                              # cdb (ex-0306.106, expr)
```

$$\nabla_f \left(\nabla_e R_{abcd} \right) - \nabla_e \left(\nabla_f R_{abcd} \right) = DD_{ef} R_{abcd} \tag{ex-0306.101}$$

$$= DD_{ef} \left(A_a B_b C_c D_d \right) \tag{ex-0306.102}$$

$$= DD_{ef} A_a B_b C_c D_d + A_a DD_{ef} B_b C_c D_d + A_a B_b DD_{ef} C_c D_d + A_a B_b C_c DD_{ef} D_d \tag{ex-0306.103}$$

$$= R^g{}_{aef} A_g B_b C_c D_d + A_a R^g{}_{bef} B_g C_c D_d + A_a B_b R^g{}_{cef} C_g D_d + A_a B_b C_c R^g{}_{def} D_g \tag{ex-0306.104}$$

$$= A_g B_b C_c D_d R^g{}_{aef} + A_a B_g C_c D_d R^g{}_{bef} + A_a B_b C_g D_d R^g{}_{cef} + A_a B_b C_c D_g R^g{}_{def} \tag{ex-0306.105}$$

$$= R_{gbcd} R^g{}_{aef} + R_{agcd} R^g{}_{bef} + R_{abgd} R^g{}_{cef} + R_{abcg} R^g{}_{def} \tag{ex-0306.106}$$

Exercise 3.7 Commutation of ∇ on the Riemann tensor – direct computation

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

;::Symbol;

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

RabcdD :=    \partial_{c}{\Gamma_{a b d}}
           - \partial_{d}{\Gamma_{a b c}}
           + \Gamma_{e a d} \Gamma^{e}_{b c}
           - \Gamma_{e a c} \Gamma^{e}_{b d} -> R_{a b c d}.            # cdb(Rabcd.010,RabcdD)

RabcdU :=    \partial_{c}{\Gamma^{a}_{b d}}
           - \partial_{d}{\Gamma^{a}_{b c}}
           + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
           - \Gamma^{e}_{b c} \Gamma^{a}_{d e} -> R^{a}_{b c d}.        # cdb(Rabcd.000,RabcdU)

d1Rabcd := R_{a b c d ; e} ->   \partial_{e}{R_{a b c d}}
                              - \Gamma^{f}_{a e} R_{f b c d}
                              - \Gamma^{f}_{b e} R_{a f c d}
                              - \Gamma^{f}_{c e} R_{a b f d}
                              - \Gamma^{f}_{d e} R_{a b c f}.           # cdb(d1Rabcd.000,d1Rabcd)

d2Rabcd := R_{a b c d ; e ; f} ->   \partial_{f}{R_{a b c d ; e}}
                                  - \Gamma^{g}_{a f} R_{g b c d ; e}
                                  - \Gamma^{g}_{b f} R_{a g c d ; e}
                                  - \Gamma^{g}_{c f} R_{a b g d ; e}
                                  - \Gamma^{g}_{d f} R_{a b c g ; e}
                                  - \Gamma^{g}_{e f} R_{a b c d ; g}.   # cdb(d2Rabcd.000,d2Rabcd)

substitute (d2Rabcd,d1Rabcd)                                           # cdb (d2Rabcd.001, d2Rabcd)

expr := R_{a b c d ; e ; f} - R_{a b c d ; f ; e}.                     # cdb (ex-0307.100, expr)

substitute    (expr,d2Rabcd)                                           # cdb (ex-0307.101, expr)
```

```
37
38   distribute       (expr)                                          # cdb (ex-0307.102, expr)
39   product_rule     (expr)                                          # cdb (ex-0307.103, expr)
40
41   sort_product     (expr)                                          # cdb (ex-0307.104, expr)
42   rename_dummies   (expr)                                          # cdb (ex-0307.105, expr)
43   canonicalise     (expr)                                          # cdb (ex-0307.106, expr)
44   factor_out       (expr,$R_{a? b? c? d?}$)                        # cdb (ex-0307.107, expr)
45
46   substitute       (expr,RabcdU)                                   # cdb (ex-0307.108, expr)
47   substitute       (expr,$R^{a}_{b c d} -> -R^{a}_{b d c}$)        # cdb (ex-0307.109, expr)
```

$$R_{abcd;e;f} - R_{abcd;f;e} = \partial_f \left(\partial_e R_{abcd} - \Gamma^g{}_{ae} R_{gbcd} - \Gamma^g{}_{be} R_{agcd} - \Gamma^g{}_{ce} R_{abgd} - \Gamma^g{}_{de} R_{abcg} \right) - \Gamma^g{}_{af} \left(\partial_e R_{gbcd} - \Gamma^h{}_{ge} R_{hbcd} - \Gamma^h{}_{be} R_{ghcd} - \Gamma^h{}_{ce} R_{gbhd} - \Gamma^h{}_{de} R_{gbch} \right)$$

$$- \Gamma^g{}_{bf} \left(\partial_e R_{agcd} - \Gamma^h{}_{ae} R_{hgcd} - \Gamma^h{}_{ge} R_{ahcd} - \Gamma^h{}_{ce} R_{aghd} - \Gamma^h{}_{de} R_{agch} \right)$$

$$- \Gamma^g{}_{cf} \left(\partial_e R_{abgd} - \Gamma^h{}_{ae} R_{hbgd} - \Gamma^h{}_{be} R_{ahgd} - \Gamma^h{}_{ge} R_{abhd} - \Gamma^h{}_{de} R_{abgh} \right)$$

$$- \Gamma^g{}_{df} \left(\partial_e R_{abcg} - \Gamma^h{}_{ae} R_{hbcg} - \Gamma^h{}_{be} R_{ahcg} - \Gamma^h{}_{ce} R_{abhg} - \Gamma^h{}_{ge} R_{abch} \right)$$

$$- \Gamma^g{}_{ef} \left(\partial_g R_{abcd} - \Gamma^h{}_{ag} R_{hbcd} - \Gamma^h{}_{bg} R_{ahcd} - \Gamma^h{}_{cg} R_{abhd} - \Gamma^h{}_{dg} R_{abch} \right) - \partial_e \left(\partial_f R_{abcd} - \Gamma^g{}_{af} R_{gbcd} - \Gamma^g{}_{bf} R_{agcd} - \Gamma^g{}_{cf} R_{abgd} - \Gamma^g{}_{df} R_{abcg} \right)$$

$$+ \Gamma^g{}_{ae} \left(\partial_f R_{gbcd} - \Gamma^h{}_{gf} R_{hbcd} - \Gamma^h{}_{bf} R_{ghcd} - \Gamma^h{}_{cf} R_{gbhd} - \Gamma^h{}_{df} R_{gbch} \right)$$

$$+ \Gamma^g{}_{be} \left(\partial_f R_{agcd} - \Gamma^h{}_{af} R_{hgcd} - \Gamma^h{}_{gf} R_{ahcd} - \Gamma^h{}_{cf} R_{aghd} - \Gamma^h{}_{df} R_{agch} \right)$$

$$+ \Gamma^g{}_{ce} \left(\partial_f R_{abgd} - \Gamma^h{}_{af} R_{hbgd} - \Gamma^h{}_{bf} R_{ahgd} - \Gamma^h{}_{gf} R_{abhd} - \Gamma^h{}_{df} R_{abgh} \right)$$

$$+ \Gamma^g{}_{de} \left(\partial_f R_{abcg} - \Gamma^h{}_{af} R_{hbcg} - \Gamma^h{}_{bf} R_{ahcg} - \Gamma^h{}_{cf} R_{abhg} - \Gamma^h{}_{gf} R_{abch} \right)$$

$$+ \Gamma^g{}_{fe} \left(\partial_g R_{abcd} - \Gamma^h{}_{ag} R_{hbcd} - \Gamma^h{}_{bg} R_{ahcd} - \Gamma^h{}_{cg} R_{abhd} - \Gamma^h{}_{dg} R_{abch} \right) \tag{ex-0307.101}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = \partial_{fe} R_{abcd} - \partial_f \left(\Gamma^g{}_{ae} R_{gbcd} \right) - \partial_f \left(\Gamma^g{}_{be} R_{agcd} \right) - \partial_f \left(\Gamma^g{}_{ce} R_{abgd} \right) - \partial_f \left(\Gamma^g{}_{de} R_{abcg} \right) - \Gamma^g{}_{af} \partial_e R_{gbcd} + \Gamma^g{}_{af} \Gamma^h{}_{ge} R_{hbcd} + \Gamma^g{}_{af} \Gamma^h{}_{be} R_{ghcd}$$

$$+ \Gamma^g{}_{af} \Gamma^h{}_{ce} R_{gbhd} + \Gamma^g{}_{af} \Gamma^h{}_{de} R_{gbch} - \Gamma^g{}_{bf} \partial_e R_{agcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ae} R_{hgcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ge} R_{ahcd} + \Gamma^g{}_{bf} \Gamma^h{}_{ce} R_{aghd} + \Gamma^g{}_{bf} \Gamma^h{}_{de} R_{agch}$$

$$- \Gamma^g{}_{cf} \partial_e R_{abgd} + \Gamma^g{}_{cf} \Gamma^h{}_{ae} R_{hbgd} + \Gamma^g{}_{cf} \Gamma^h{}_{be} R_{ahgd} + \Gamma^g{}_{cf} \Gamma^h{}_{ge} R_{abhd} + \Gamma^g{}_{cf} \Gamma^h{}_{de} R_{abgh} - \Gamma^g{}_{df} \partial_e R_{abcg} + \Gamma^g{}_{df} \Gamma^h{}_{ae} R_{hbcg}$$

$$+ \Gamma^g{}_{df} \Gamma^h{}_{be} R_{ahcg} + \Gamma^g{}_{df} \Gamma^h{}_{ce} R_{abhg} + \Gamma^g{}_{df} \Gamma^h{}_{ge} R_{abch} - \Gamma^g{}_{ef} \partial_g R_{abcd} + \Gamma^g{}_{ef} \Gamma^h{}_{ag} R_{hbcd} + \Gamma^g{}_{ef} \Gamma^h{}_{bg} R_{ahcd} + \Gamma^g{}_{ef} \Gamma^h{}_{cg} R_{abhd}$$

$$+ \Gamma^g{}_{ef} \Gamma^h{}_{dg} R_{abch} - \partial_{ef} R_{abcd} + \partial_e \left(\Gamma^g{}_{af} R_{gbcd} \right) + \partial_e \left(\Gamma^g{}_{bf} R_{agcd} \right) + \partial_e \left(\Gamma^g{}_{cf} R_{abgd} \right) + \partial_e \left(\Gamma^g{}_{df} R_{abcg} \right) + \Gamma^g{}_{ae} \partial_f R_{gbcd} - \Gamma^g{}_{ae} \Gamma^h{}_{gf} R_{hbcd}$$

$$- \Gamma^g{}_{ae} \Gamma^h{}_{bf} R_{ghcd} - \Gamma^g{}_{ae} \Gamma^h{}_{cf} R_{gbhd} - \Gamma^g{}_{ae} \Gamma^h{}_{df} R_{gbch} + \Gamma^g{}_{be} \partial_f R_{agcd} - \Gamma^g{}_{be} \Gamma^h{}_{af} R_{hgcd} - \Gamma^g{}_{be} \Gamma^h{}_{gf} R_{ahcd} - \Gamma^g{}_{be} \Gamma^h{}_{cf} R_{aghd}$$

$$- \Gamma^g{}_{be} \Gamma^h{}_{df} R_{agch} + \Gamma^g{}_{ce} \partial_f R_{abgd} - \Gamma^g{}_{ce} \Gamma^h{}_{af} R_{hbgd} - \Gamma^g{}_{ce} \Gamma^h{}_{bf} R_{ahgd} - \Gamma^g{}_{ce} \Gamma^h{}_{gf} R_{abhd} - \Gamma^g{}_{ce} \Gamma^h{}_{df} R_{abgh} + \Gamma^g{}_{de} \partial_f R_{abcg}$$

$$- \Gamma^g{}_{de} \Gamma^h{}_{af} R_{hbcg} - \Gamma^g{}_{de} \Gamma^h{}_{bf} R_{ahcg} - \Gamma^g{}_{de} \Gamma^h{}_{cf} R_{abhg} - \Gamma^g{}_{de} \Gamma^h{}_{gf} R_{abch} + \Gamma^g{}_{fe} \partial_g R_{abcd} - \Gamma^g{}_{fe} \Gamma^h{}_{ag} R_{hbcd} - \Gamma^g{}_{fe} \Gamma^h{}_{bg} R_{ahcd}$$

$$- \Gamma^g{}_{fe} \Gamma^h{}_{cg} R_{abhd} - \Gamma^g{}_{fe} \Gamma^h{}_{dg} R_{abch} \tag{ex-0307.102}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = \partial_{fe}R_{abcd} - \partial_f\Gamma^g{}_{ae}R_{gbcd} - \partial_f\Gamma^g{}_{be}R_{agcd} - \partial_f\Gamma^g{}_{ce}R_{abgd} - \partial_f\Gamma^g{}_{de}R_{abcg} + \Gamma^g{}_{af}\Gamma^h{}_{ge}R_{hbcd} + \Gamma^g{}_{af}\Gamma^h{}_{be}R_{ghcd} + \Gamma^g{}_{af}\Gamma^h{}_{ce}R_{gbhd}$$
$$+ \Gamma^g{}_{af}\Gamma^h{}_{de}R_{gbch} + \Gamma^g{}_{bf}\Gamma^h{}_{ae}R_{hgcd} + \Gamma^g{}_{bf}\Gamma^h{}_{ge}R_{ahcd} + \Gamma^g{}_{bf}\Gamma^h{}_{ce}R_{aghd} + \Gamma^g{}_{bf}\Gamma^h{}_{de}R_{agch} + \Gamma^g{}_{cf}\Gamma^h{}_{ae}R_{hbgd} + \Gamma^g{}_{cf}\Gamma^h{}_{be}R_{ahgd}$$
$$+ \Gamma^g{}_{cf}\Gamma^h{}_{ge}R_{abhd} + \Gamma^g{}_{cf}\Gamma^h{}_{de}R_{abgh} + \Gamma^g{}_{df}\Gamma^h{}_{ae}R_{hbcg} + \Gamma^g{}_{df}\Gamma^h{}_{be}R_{ahcg} + \Gamma^g{}_{df}\Gamma^h{}_{ce}R_{abhg} + \Gamma^g{}_{df}\Gamma^h{}_{ge}R_{abch} - \Gamma^g{}_{ef}\partial_g R_{abcd}$$
$$+ \Gamma^g{}_{ef}\Gamma^h{}_{ag}R_{hbcd} + \Gamma^g{}_{ef}\Gamma^h{}_{bg}R_{ahcd} + \Gamma^g{}_{ef}\Gamma^h{}_{cg}R_{abhd} + \Gamma^g{}_{ef}\Gamma^h{}_{dg}R_{abch} - \partial_{ef}R_{abcd} + \partial_e\Gamma^g{}_{af}R_{gbcd} + \partial_e\Gamma^g{}_{bf}R_{agcd} + \partial_e\Gamma^g{}_{cf}R_{abgd}$$
$$+ \partial_e\Gamma^g{}_{df}R_{abcg} - \Gamma^g{}_{ae}\Gamma^h{}_{gf}R_{hbcd} - \Gamma^g{}_{ae}\Gamma^h{}_{bf}R_{ghcd} - \Gamma^g{}_{ae}\Gamma^h{}_{cf}R_{gbhd} - \Gamma^g{}_{ae}\Gamma^h{}_{df}R_{gbch} - \Gamma^g{}_{be}\Gamma^h{}_{af}R_{hgcd} - \Gamma^g{}_{be}\Gamma^h{}_{gf}R_{ahcd}$$
$$- \Gamma^g{}_{be}\Gamma^h{}_{cf}R_{aghd} - \Gamma^g{}_{be}\Gamma^h{}_{df}R_{agch} - \Gamma^g{}_{ce}\Gamma^h{}_{af}R_{hbgd} - \Gamma^g{}_{ce}\Gamma^h{}_{bf}R_{ahgd} - \Gamma^g{}_{ce}\Gamma^h{}_{gf}R_{abhd} - \Gamma^g{}_{ce}\Gamma^h{}_{df}R_{abgh} - \Gamma^g{}_{de}\Gamma^h{}_{af}R_{hbcg}$$
$$- \Gamma^g{}_{de}\Gamma^h{}_{bf}R_{ahcg} - \Gamma^g{}_{de}\Gamma^h{}_{cf}R_{abhg} - \Gamma^g{}_{de}\Gamma^h{}_{gf}R_{abch} + \Gamma^g{}_{fe}\partial_g R_{abcd} - \Gamma^g{}_{fe}\Gamma^h{}_{ag}R_{hbcd} - \Gamma^g{}_{fe}\Gamma^h{}_{bg}R_{ahcd} - \Gamma^g{}_{fe}\Gamma^h{}_{cg}R_{abhd}$$
$$- \Gamma^g{}_{fe}\Gamma^h{}_{dg}R_{abch} \tag{ex-0307.103}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = \partial_{fe}R_{abcd} - R_{gbcd}\partial_f\Gamma^g{}_{ae} - R_{agcd}\partial_f\Gamma^g{}_{be} - R_{abgd}\partial_f\Gamma^g{}_{ce} - R_{abcg}\partial_f\Gamma^g{}_{de} + R_{hbcd}\Gamma^g{}_{af}\Gamma^h{}_{ge} + R_{ghcd}\Gamma^g{}_{af}\Gamma^h{}_{be} + R_{gbhd}\Gamma^g{}_{af}\Gamma^h{}_{ce}$$
$$+ R_{gbch}\Gamma^g{}_{af}\Gamma^h{}_{de} + R_{hgcd}\Gamma^g{}_{bf}\Gamma^h{}_{ae} + R_{ahcd}\Gamma^g{}_{bf}\Gamma^h{}_{ge} + R_{aghd}\Gamma^g{}_{bf}\Gamma^h{}_{ce} + R_{agch}\Gamma^g{}_{bf}\Gamma^h{}_{de} + R_{hbgd}\Gamma^g{}_{cf}\Gamma^h{}_{ae} + R_{ahgd}\Gamma^g{}_{cf}\Gamma^h{}_{be}$$
$$+ R_{abhd}\Gamma^g{}_{cf}\Gamma^h{}_{ge} + R_{abgh}\Gamma^g{}_{cf}\Gamma^h{}_{de} + R_{hbcg}\Gamma^g{}_{df}\Gamma^h{}_{ae} + R_{ahcg}\Gamma^g{}_{df}\Gamma^h{}_{be} + R_{abhg}\Gamma^g{}_{df}\Gamma^h{}_{ce} + R_{abch}\Gamma^g{}_{df}\Gamma^h{}_{ge} - \Gamma^g{}_{ef}\partial_g R_{abcd}$$
$$+ R_{hbcd}\Gamma^g{}_{ef}\Gamma^h{}_{ag} + R_{ahcd}\Gamma^g{}_{ef}\Gamma^h{}_{bg} + R_{abhd}\Gamma^g{}_{ef}\Gamma^h{}_{cg} + R_{abch}\Gamma^g{}_{ef}\Gamma^h{}_{dg} - \partial_{ef}R_{abcd} + R_{gbcd}\partial_e\Gamma^g{}_{af} + R_{agcd}\partial_e\Gamma^g{}_{bf} + R_{abgd}\partial_e\Gamma^g{}_{cf}$$
$$+ R_{abcg}\partial_e\Gamma^g{}_{df} - R_{hbcd}\Gamma^g{}_{ae}\Gamma^h{}_{gf} - R_{ghcd}\Gamma^g{}_{ae}\Gamma^h{}_{bf} - R_{gbhd}\Gamma^g{}_{ae}\Gamma^h{}_{cf} - R_{gbch}\Gamma^g{}_{ae}\Gamma^h{}_{df} - R_{hgcd}\Gamma^g{}_{be}\Gamma^h{}_{af} - R_{ahcd}\Gamma^g{}_{be}\Gamma^h{}_{gf}$$
$$- R_{aghd}\Gamma^g{}_{be}\Gamma^h{}_{cf} - R_{agch}\Gamma^g{}_{be}\Gamma^h{}_{df} - R_{hbgd}\Gamma^g{}_{ce}\Gamma^h{}_{af} - R_{ahgd}\Gamma^g{}_{ce}\Gamma^h{}_{bf} - R_{abhd}\Gamma^g{}_{ce}\Gamma^h{}_{gf} - R_{abgh}\Gamma^g{}_{ce}\Gamma^h{}_{df} - R_{hbcg}\Gamma^g{}_{de}\Gamma^h{}_{af}$$
$$- R_{ahcg}\Gamma^g{}_{de}\Gamma^h{}_{bf} - R_{abhg}\Gamma^g{}_{de}\Gamma^h{}_{cf} - R_{abch}\Gamma^g{}_{de}\Gamma^h{}_{gf} + \Gamma^g{}_{fe}\partial_g R_{abcd} - R_{hbcd}\Gamma^g{}_{fe}\Gamma^h{}_{ag} - R_{ahcd}\Gamma^g{}_{fe}\Gamma^h{}_{bg} - R_{abhd}\Gamma^g{}_{fe}\Gamma^h{}_{cg}$$
$$- R_{abch}\Gamma^g{}_{fe}\Gamma^h{}_{dg} \tag{ex-0307.104}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = \partial_{fe}R_{abcd} - R_{gbcd}\partial_f\Gamma^g{}_{ae} - R_{agcd}\partial_f\Gamma^g{}_{be} - R_{abgd}\partial_f\Gamma^g{}_{ce} - R_{abcg}\partial_f\Gamma^g{}_{de} + R_{gbcd}\Gamma^h{}_{af}\Gamma^g{}_{he} + R_{ghcd}\Gamma^g{}_{af}\Gamma^h{}_{be} + R_{gbhd}\Gamma^g{}_{af}\Gamma^h{}_{ce}$$
$$+ R_{gbch}\Gamma^g{}_{af}\Gamma^h{}_{de} + R_{ghcd}\Gamma^h{}_{bf}\Gamma^g{}_{ae} + R_{agcd}\Gamma^h{}_{bf}\Gamma^g{}_{he} + R_{aghd}\Gamma^g{}_{bf}\Gamma^h{}_{ce} + R_{agch}\Gamma^g{}_{bf}\Gamma^h{}_{de} + R_{gbhd}\Gamma^h{}_{cf}\Gamma^g{}_{ae} + R_{aghd}\Gamma^h{}_{cf}\Gamma^g{}_{be}$$
$$+ R_{abgd}\Gamma^h{}_{cf}\Gamma^g{}_{he} + R_{abgh}\Gamma^g{}_{cf}\Gamma^h{}_{de} + R_{gbch}\Gamma^h{}_{df}\Gamma^g{}_{ae} + R_{agch}\Gamma^h{}_{df}\Gamma^g{}_{be} + R_{abgh}\Gamma^h{}_{df}\Gamma^g{}_{ce} + R_{abcg}\Gamma^h{}_{df}\Gamma^g{}_{he} - \Gamma^g{}_{ef}\partial_g R_{abcd}$$
$$+ R_{gbcd}\Gamma^h{}_{ef}\Gamma^g{}_{ah} + R_{agcd}\Gamma^h{}_{ef}\Gamma^g{}_{bh} + R_{abgd}\Gamma^h{}_{ef}\Gamma^g{}_{ch} + R_{abcg}\Gamma^h{}_{ef}\Gamma^g{}_{dh} - \partial_{ef}R_{abcd} + R_{gbcd}\partial_e\Gamma^g{}_{af} + R_{agcd}\partial_e\Gamma^g{}_{bf} + R_{abgd}\partial_e\Gamma^g{}_{cf}$$
$$+ R_{abcg}\partial_e\Gamma^g{}_{df} - R_{gbcd}\Gamma^h{}_{ae}\Gamma^g{}_{hf} - R_{ghcd}\Gamma^g{}_{ae}\Gamma^h{}_{bf} - R_{gbhd}\Gamma^g{}_{ae}\Gamma^h{}_{cf} - R_{gbch}\Gamma^g{}_{ae}\Gamma^h{}_{df} - R_{ghcd}\Gamma^h{}_{be}\Gamma^g{}_{af} - R_{agcd}\Gamma^h{}_{be}\Gamma^g{}_{hf}$$
$$- R_{aghd}\Gamma^g{}_{be}\Gamma^h{}_{cf} - R_{agch}\Gamma^g{}_{be}\Gamma^h{}_{df} - R_{gbhd}\Gamma^h{}_{ce}\Gamma^g{}_{af} - R_{aghd}\Gamma^h{}_{ce}\Gamma^g{}_{bf} - R_{abgd}\Gamma^h{}_{ce}\Gamma^g{}_{hf} - R_{abgh}\Gamma^g{}_{ce}\Gamma^h{}_{df} - R_{gbch}\Gamma^h{}_{de}\Gamma^g{}_{af}$$
$$- R_{agch}\Gamma^h{}_{de}\Gamma^g{}_{bf} - R_{abgh}\Gamma^h{}_{de}\Gamma^g{}_{cf} - R_{abcg}\Gamma^h{}_{de}\Gamma^g{}_{hf} + \Gamma^g{}_{fe}\partial_g R_{abcd} - R_{gbcd}\Gamma^h{}_{fe}\Gamma^g{}_{ah} - R_{agcd}\Gamma^h{}_{fe}\Gamma^g{}_{bh} - R_{abgd}\Gamma^h{}_{fe}\Gamma^g{}_{ch}$$
$$- R_{abcg}\Gamma^h{}_{fe}\Gamma^g{}_{dh} \tag{ex-0307.105}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd}\partial_f\Gamma^g{}_{ae} - R_{agcd}\partial_f\Gamma^g{}_{be} - R_{abgd}\partial_f\Gamma^g{}_{ce} - R_{abcg}\partial_f\Gamma^g{}_{de} + R_{gbcd}\Gamma^h{}_{af}\Gamma^g{}_{eh} + R_{agcd}\Gamma^h{}_{bf}\Gamma^g{}_{eh} + R_{abgd}\Gamma^h{}_{cf}\Gamma^g{}_{eh} + R_{abcg}\Gamma^h{}_{df}\Gamma^g{}_{eh}$$
$$+ R_{gbcd}\partial_e\Gamma^g{}_{af} + R_{agcd}\partial_e\Gamma^g{}_{bf} + R_{abgd}\partial_e\Gamma^g{}_{cf} + R_{abcg}\partial_e\Gamma^g{}_{df} - R_{gbcd}\Gamma^h{}_{ae}\Gamma^g{}_{fh} - R_{agcd}\Gamma^h{}_{be}\Gamma^g{}_{fh} - R_{abgd}\Gamma^h{}_{ce}\Gamma^g{}_{fh}$$
$$- R_{abcg}\Gamma^h{}_{de}\Gamma^g{}_{fh} \tag{ex-0307.106}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd}\left(-\partial_f\Gamma^g{}_{ae} + \Gamma^h{}_{af}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{af} - \Gamma^h{}_{ae}\Gamma^g{}_{fh}\right) + R_{agcd}\left(-\partial_f\Gamma^g{}_{be} + \Gamma^h{}_{bf}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{bf} - \Gamma^h{}_{be}\Gamma^g{}_{fh}\right)$$
$$+ R_{abgd}\left(-\partial_f\Gamma^g{}_{ce} + \Gamma^h{}_{cf}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{cf} - \Gamma^h{}_{ce}\Gamma^g{}_{fh}\right) + R_{abcg}\left(-\partial_f\Gamma^g{}_{de} + \Gamma^h{}_{df}\Gamma^g{}_{eh} + \partial_e\Gamma^g{}_{df} - \Gamma^h{}_{de}\Gamma^g{}_{fh}\right) \tag{ex-0307.107}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd}R^g{}_{afe} - R_{agcd}R^g{}_{bfe} - R_{abgd}R^g{}_{cfe} - R_{abcg}R^g{}_{dfe} \tag{ex-0307.108}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd}R^g{}_{aef} + R_{agcd}R^g{}_{bef} + R_{abgd}R^g{}_{cef} + R_{abcg}R^g{}_{def} \tag{ex-0307.109}$$

5

Exercise 3.8 Symmetry of R_{ab}

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

2

3    \partial{#}::PartialDerivative;

4

5    g_{a b}::Metric;
6    g^{a b}::InverseMetric;

7

8    dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
9                                                      # cdb (dgab.000,dgab)

10

11   Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (    \partial_{b}{g_{e c}}
12                                               + \partial_{c}{g_{b e}}
13                                               - \partial_{e}{g_{b c}}).
14                                                   # cdb (Gamma.000,Gamma)

15

16   Rabcd := R^{a}_{b c d} ->
17            \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
18          - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
19                                                   # cdb (Rabcd.000,Rabcd)

20

21   Rab := R_{a b} -> R^{c}_{a c b}.                 # cdb (Rab.000,Rab)

22

23   expr := 4 (R_{a b} - R_{b a}).                   # cdb (ex-0308.100,expr)

24

25   substitute     (expr, Rab)                       # cdb (ex-0308.101,expr)
26   substitute     (expr, Rabcd)                     # cdb (ex-0308.102,expr)
27   substitute     (expr, Gamma)                     # cdb (ex-0308.103,expr)

28

29   distribute     (expr)                            # cdb (ex-0308.104,expr)
30   product_rule   (expr)                            # cdb (ex-0308.105,expr)
31   canonicalise   (expr)                            # cdb (ex-0308.106,expr)

32

33   substitute     (expr, dgab)                      # cdb (ex-0308.107,expr)
34   canonicalise   (expr)                            # cdb (ex-0308.108,expr)
```

$$4R_{ab} - 4R_{ba} = 4R^c{}_{acb} - 4R^c{}_{bca} \tag{ex-0308.101}$$

$$= 4\partial_c\Gamma^c{}_{ab} + 4\Gamma^c{}_{ec}\Gamma^e{}_{ab} - 4\partial_b\Gamma^c{}_{ac} - 4\Gamma^c{}_{eb}\Gamma^e{}_{ac} - 4\partial_c\Gamma^c{}_{ba} - 4\Gamma^c{}_{ec}\Gamma^e{}_{ba} + 4\partial_a\Gamma^c{}_{bc} + 4\Gamma^c{}_{ea}\Gamma^e{}_{bc} \tag{ex-0308.102}$$

$$= 2\partial_c\left(g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\right) + g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right)g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right) - 2\partial_b\left(g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right)\right)$$
$$- g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right)g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) - 2\partial_c\left(g^{ce}\left(\partial_b g_{ea} + \partial_a g_{be} - \partial_e g_{ba}\right)\right)$$
$$- g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right)g^{ef}\left(\partial_b g_{fa} + \partial_a g_{bf} - \partial_f g_{ba}\right) + 2\partial_a\left(g^{ce}\left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right)\right)$$
$$+ g^{cd}\left(\partial_e g_{da} + \partial_a g_{ed} - \partial_d g_{ea}\right)g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right) \tag{ex-0308.103}$$

$$= 2\partial_c\left(g^{ce}\partial_a g_{eb}\right) + 2\partial_c\left(g^{ce}\partial_b g_{ae}\right) - 2\partial_c\left(g^{ce}\partial_e g_{ab}\right) + g^{cd}\partial_e g_{dc}g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc}g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc}g^{ef}\partial_f g_{ab} + g^{cd}\partial_c g_{ed}g^{ef}\partial_a g_{fb}$$
$$+ g^{cd}\partial_c g_{ed}g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed}g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec}g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec}g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec}g^{ef}\partial_f g_{ab} - 2\partial_b\left(g^{ce}\partial_a g_{ec}\right) - 2\partial_b\left(g^{ce}\partial_c g_{ae}\right)$$
$$+ 2\partial_b\left(g^{ce}\partial_e g_{ac}\right) - g^{cd}\partial_e g_{db}g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db}g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db}g^{ef}\partial_f g_{ac} - g^{cd}\partial_b g_{ed}g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed}g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed}g^{ef}\partial_f g_{ac}$$
$$+ g^{cd}\partial_d g_{eb}g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb}g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb}g^{ef}\partial_f g_{ac} - 2\partial_c\left(g^{ce}\partial_b g_{ea}\right) - 2\partial_c\left(g^{ce}\partial_a g_{be}\right) + 2\partial_c\left(g^{ce}\partial_e g_{ba}\right) - g^{cd}\partial_e g_{dc}g^{ef}\partial_b g_{fa}$$
$$- g^{cd}\partial_e g_{dc}g^{ef}\partial_a g_{bf} + g^{cd}\partial_e g_{dc}g^{ef}\partial_f g_{ba} - g^{cd}\partial_c g_{ed}g^{ef}\partial_b g_{fa} - g^{cd}\partial_c g_{ed}g^{ef}\partial_a g_{bf} + g^{cd}\partial_c g_{ed}g^{ef}\partial_f g_{ba} + g^{cd}\partial_d g_{ec}g^{ef}\partial_b g_{fa} + g^{cd}\partial_d g_{ec}g^{ef}\partial_a g_{bf}$$
$$- g^{cd}\partial_d g_{ec}g^{ef}\partial_f g_{ba} + 2\partial_a\left(g^{ce}\partial_b g_{ec}\right) + 2\partial_a\left(g^{ce}\partial_c g_{be}\right) - 2\partial_a\left(g^{ce}\partial_e g_{bc}\right) + g^{cd}\partial_e g_{da}g^{ef}\partial_b g_{fc} + g^{cd}\partial_e g_{da}g^{ef}\partial_c g_{bf} - g^{cd}\partial_e g_{da}g^{ef}\partial_f g_{bc}$$
$$+ g^{cd}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + g^{cd}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - g^{cd}\partial_a g_{ed}g^{ef}\partial_f g_{bc} - g^{cd}\partial_d g_{ea}g^{ef}\partial_b g_{fc} - g^{cd}\partial_d g_{ea}g^{ef}\partial_c g_{bf} + g^{cd}\partial_d g_{ea}g^{ef}\partial_f g_{bc} \tag{ex-0308.104}$$

$$= 2\partial_c g^{ce}\partial_a g_{eb} + 2g^{ce}\partial_{ca}g_{eb} + 2\partial_c g^{ce}\partial_b g_{ae} + 2g^{ce}\partial_{cb}g_{ae} - 2\partial_c g^{ce}\partial_e g_{ab} - 2g^{ce}\partial_{ce}g_{ab} + g^{cd}\partial_e g_{dc}g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc}g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc}g^{ef}\partial_f g_{ab}$$
$$+ g^{cd}\partial_c g_{ed}g^{ef}\partial_a g_{fb} + g^{cd}\partial_c g_{ed}g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed}g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec}g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec}g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec}g^{ef}\partial_f g_{ab} - 2\partial_b g^{ce}\partial_a g_{ec}$$
$$- 2g^{ce}\partial_{ba}g_{ec} - 2\partial_b g^{ce}\partial_c g_{ae} - 2g^{ce}\partial_{bc}g_{ae} + 2\partial_b g^{ce}\partial_e g_{ac} + 2g^{ce}\partial_{be}g_{ac} - g^{cd}\partial_e g_{db}g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db}g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db}g^{ef}\partial_f g_{ac}$$
$$- g^{cd}\partial_b g_{ed}g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed}g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed}g^{ef}\partial_f g_{ac} + g^{cd}\partial_d g_{eb}g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb}g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb}g^{ef}\partial_f g_{ac} - 2\partial_c g^{ce}\partial_b g_{ea}$$
$$- 2g^{ce}\partial_{cb}g_{ea} - 2\partial_c g^{ce}\partial_a g_{be} - 2g^{ce}\partial_{ca}g_{be} + 2\partial_c g^{ce}\partial_e g_{ba} + 2g^{ce}\partial_{ce}g_{ba} - g^{cd}\partial_e g_{dc}g^{ef}\partial_b g_{fa} - g^{cd}\partial_e g_{dc}g^{ef}\partial_a g_{bf} + g^{cd}\partial_e g_{dc}g^{ef}\partial_f g_{ba}$$
$$- g^{cd}\partial_c g_{ed}g^{ef}\partial_b g_{fa} - g^{cd}\partial_c g_{ed}g^{ef}\partial_a g_{bf} + g^{cd}\partial_c g_{ed}g^{ef}\partial_f g_{ba} + g^{cd}\partial_d g_{ec}g^{ef}\partial_b g_{fa} + g^{cd}\partial_d g_{ec}g^{ef}\partial_a g_{bf} - g^{cd}\partial_d g_{ec}g^{ef}\partial_f g_{ba} + 2\partial_a g^{ce}\partial_b g_{ec}$$
$$+ 2g^{ce}\partial_{ab}g_{ec} + 2\partial_a g^{ce}\partial_c g_{be} + 2g^{ce}\partial_{ac}g_{be} - 2\partial_a g^{ce}\partial_e g_{bc} - 2g^{ce}\partial_{ae}g_{bc} + g^{cd}\partial_e g_{da}g^{ef}\partial_b g_{fc} + g^{cd}\partial_e g_{da}g^{ef}\partial_c g_{bf} - g^{cd}\partial_e g_{da}g^{ef}\partial_f g_{bc}$$
$$+ g^{cd}\partial_a g_{ed}g^{ef}\partial_b g_{fc} + g^{cd}\partial_a g_{ed}g^{ef}\partial_c g_{bf} - g^{cd}\partial_a g_{ed}g^{ef}\partial_f g_{bc} - g^{cd}\partial_d g_{ea}g^{ef}\partial_b g_{fc} - g^{cd}\partial_d g_{ea}g^{ef}\partial_c g_{bf} + g^{cd}\partial_d g_{ea}g^{ef}\partial_f g_{bc} \tag{ex-0308.105}$$

$$= -2\partial_b g^{ce}\partial_a g_{ce} + 2\partial_a g^{ce}\partial_b g_{ce} \tag{ex-0308.106}$$

$$= 2g^{cd}g^{ef}\partial_b g_{df}\partial_a g_{ce} - 2g^{cd}g^{ef}\partial_a g_{df}\partial_b g_{ce} \tag{ex-0308.107}$$

$$= 0 \tag{ex-0308.108}$$

Exercise 3.8 Symmetry of R_{ab} alternative solution

This differs from the previous code by the inclusion of a call to `canonicalise` immediately after the first two substitutions and a declaration that $\Gamma^a{}_{bc}$ is symmetric in bc. This pair of changes produces a more compact set of results than given above. Incidently, this also shows that $\partial_a \Gamma^c{}_{bc} = \partial_b \Gamma^c{}_{ac}$.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
                                            # cdb (dgab.000,dgab)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                                             + \partial_{c}{g_{b e}}
                                             - \partial_{e}{g_{b c}}).
                                                    # cdb (Gamma.000,Gamma)

Rabcd := R^{a}_{b c d} ->
            \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
        - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                    # cdb (Rabcd.000,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                    # cdb (Rab.000,Rab)

expr := 4 (R_{a b} - R_{b a}).                      # cdb (ex-0308.200,expr)

substitute   (expr, Rab)                            # cdb (ex-0308.201,expr)
substitute   (expr, Rabcd)                          # cdb (ex-0308.202,expr)
canonicalise (expr)                                 # cdb (ex-0308.203,expr)
substitute   (expr, Gamma)                          # cdb (ex-0308.204,expr)

distribute   (expr)                                 # cdb (ex-0308.205,expr)
```

```
33   product_rule (expr)                                  # cdb (ex-0308.206,expr)
34   canonicalise (expr)                                  # cdb (ex-0308.207,expr)
35
36   substitute   (expr, dgab)                            # cdb (ex-0308.208,expr)
37   canonicalise (expr)                                  # cdb (ex-0308.209,expr)
```

$$4R_{ab} - 4R_{ba} = 4R^{c}{}_{acb} - 4R^{c}{}_{bca} \tag{ex-0308.201}$$

$$= 4\partial_c \Gamma^c{}_{ab} + 4\Gamma^c{}_{ec}\Gamma^e{}_{ab} - 4\partial_b \Gamma^c{}_{ac} - 4\Gamma^c{}_{eb}\Gamma^e{}_{ac} - 4\partial_c \Gamma^c{}_{ba} - 4\Gamma^c{}_{ec}\Gamma^e{}_{ba} + 4\partial_a \Gamma^c{}_{bc} + 4\Gamma^c{}_{ea}\Gamma^e{}_{bc} \tag{ex-0308.202}$$

$$= -4\partial_b \Gamma^c{}_{ac} + 4\partial_a \Gamma^c{}_{bc} \tag{ex-0308.203}$$

$$= -2\partial_b \left(g^{ce} \left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac} \right) \right) + 2\partial_a \left(g^{ce} \left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc} \right) \right) \tag{ex-0308.204}$$

$$= -2\partial_b \left(g^{ce} \partial_a g_{ec} \right) - 2\partial_b \left(g^{ce} \partial_c g_{ae} \right) + 2\partial_b \left(g^{ce} \partial_e g_{ac} \right) + 2\partial_a \left(g^{ce} \partial_b g_{ec} \right) + 2\partial_a \left(g^{ce} \partial_c g_{be} \right) - 2\partial_a \left(g^{ce} \partial_e g_{bc} \right) \tag{ex-0308.205}$$

$$= -2\partial_b g^{ce} \partial_a g_{ec} - 2g^{ce} \partial_{ba} g_{ec} - 2\partial_b g^{ce} \partial_c g_{ae} - 2g^{ce} \partial_{bc} g_{ae} + 2\partial_b g^{ce} \partial_e g_{ac} + 2g^{ce} \partial_{be} g_{ac} + 2\partial_a g^{ce} \partial_b g_{ec} + 2g^{ce} \partial_{ab} g_{ec} + 2\partial_a g^{ce} \partial_c g_{be}$$
$$+ 2g^{ce} \partial_{ac} g_{be} - 2\partial_a g^{ce} \partial_e g_{bc} - 2g^{ce} \partial_{ae} g_{bc} \tag{ex-0308.206}$$

$$= -2\partial_b g^{ce} \partial_a g_{ce} + 2\partial_a g^{ce} \partial_b g_{ce} \tag{ex-0308.207}$$

$$= 2g^{cd} g^{ef} \partial_b g_{df} \partial_a g_{ce} - 2g^{cd} g^{ef} \partial_a g_{df} \partial_b g_{ce} \tag{ex-0308.208}$$

$$= 0 \tag{ex-0308.209}$$

Exercise 3.9 Ricci in terms of the metric and its derivatives

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3    \partial{#}::PartialDerivative;
4
5    g_{a b}::Metric;
6    g^{a b}::InverseMetric;
7
8    dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.       # cdb (ex-0309.dgab,dgab)
9
10   Gamma := \Gamma^{a}_{b c} ->
11           (1/2) g^{a e} (   \partial_{b}{g_{e c}}
12                           + \partial_{c}{g_{b e}}
13                           - \partial_{e}{g_{b c}}).                              # cdb (ex-0309.Gamma,Gamma)
14
15   Rabcd := R^{a}_{b c d} ->
16           \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
17         - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.    # cdb (ex-0309.Rabcd,Rabcd)
18
19   FourRab := 4 R^{c}_{a c b}.                        # cdb (ex-0309.101,FourRab)
20
21   substitute     (FourRab, Rabcd)                    # cdb (ex-0309.102,FourRab)
22   substitute     (FourRab, Gamma)                    # cdb (ex-0309.103,FourRab)
23
24   product_rule   (FourRab)                           # cdb (ex-0309.104,FourRab)
25   distribute     (FourRab)                           # cdb (ex-0309.105,FourRab)
26
27   substitute     (FourRab, dgab)                     # cdb (ex-0309.106,FourRab)
28
29   sort_product   (FourRab)                           # cdb (ex-0309.107,FourRab)
30   rename_dummies (FourRab)                           # cdb (ex-0309.108,FourRab)
31   canonicalise   (FourRab)                           # cdb (ex-0309.109,FourRab)
32
33   # sort so that g to appeares before dg
34
35   substitute     (FourRab, $g^{a b} -> A^{a b}$)
36   sort_product   (FourRab)
```

```
37   rename_dummies (FourRab)
38   substitute      (FourRab, $A^{a b} -> g^{a b}$)   # cdb (ex-0309.110,FourRab)
```

$$4R_{ab} = 4R^c{}_{acb} \tag{ex-0309.101}$$

$$= 4\partial_c \Gamma^c{}_{ab} + 4\Gamma^c{}_{ec}\Gamma^e{}_{ab} - 4\partial_b \Gamma^c{}_{ac} - 4\Gamma^c{}_{eb}\Gamma^e{}_{ac} \tag{ex-0309.102}$$

$$= 2\partial_c\left(g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\right) + g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right)g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right) - 2\partial_b\left(g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right)\right)$$
$$- g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right)g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) \tag{ex-0309.103}$$

$$= 2\partial_c g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right) + 2g^{ce}\partial_c\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right) + g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right)g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right)$$
$$- 2\partial_b g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - 2g^{ce}\partial_b\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) - g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right)g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) \tag{ex-0309.104}$$

$$= 2\partial_c g^{ce}\partial_a g_{eb} + 2\partial_c g^{ce}\partial_b g_{ae} - 2\partial_c g^{ce}\partial_e g_{ab} + 2g^{ce}\partial_{ca}g_{eb} + 2g^{ce}\partial_{cb}g_{ae} - 2g^{ce}\partial_{ce}g_{ab} + g^{cd}\partial_e g_{dc}g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc}g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc}g^{ef}\partial_f g_{ab}$$
$$+ g^{cd}\partial_c g_{ed}g^{ef}\partial_a g_{fb} + g^{cd}\partial_c g_{ed}g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed}g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec}g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec}g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec}g^{ef}\partial_f g_{ab} - 2\partial_b g^{ce}\partial_a g_{ec}$$
$$- 2\partial_b g^{ce}\partial_c g_{ae} + 2\partial_b g^{ce}\partial_e g_{ac} - 2g^{ce}\partial_{ba}g_{ec} - 2g^{ce}\partial_{bc}g_{ae} + 2g^{ce}\partial_{be}g_{ac} - g^{cd}\partial_e g_{db}g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db}g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db}g^{ef}\partial_f g_{ac}$$
$$- g^{cd}\partial_b g_{ed}g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed}g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed}g^{ef}\partial_f g_{ac} + g^{cd}\partial_d g_{eb}g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb}g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb}g^{ef}\partial_f g_{ac} \tag{ex-0309.105}$$

$$= -2g^{cd}g^{ef}\partial_c g_{df}\partial_a g_{eb} - 2g^{cd}g^{ef}\partial_c g_{df}\partial_b g_{ae} + 2g^{cd}g^{ef}\partial_c g_{df}\partial_e g_{ab} + 2g^{ce}\partial_{ca}g_{eb} + 2g^{ce}\partial_{cb}g_{ae} - 2g^{ce}\partial_{ce}g_{ab} + g^{cd}\partial_e g_{dc}g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc}g^{ef}\partial_b g_{af}$$
$$- g^{cd}\partial_e g_{dc}g^{ef}\partial_f g_{ab} + g^{cd}\partial_c g_{ed}g^{ef}\partial_a g_{fb} + g^{cd}\partial_c g_{ed}g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed}g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec}g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec}g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec}g^{ef}\partial_f g_{ab}$$
$$+ 2g^{cd}g^{ef}\partial_b g_{df}\partial_a g_{ec} + 2g^{cd}g^{ef}\partial_b g_{df}\partial_c g_{ae} - 2g^{cd}g^{ef}\partial_b g_{df}\partial_e g_{ac} - 2g^{ce}\partial_{ba}g_{ec} - 2g^{ce}\partial_{bc}g_{ae} + 2g^{ce}\partial_{be}g_{ac} - g^{cd}\partial_e g_{db}g^{ef}\partial_a g_{fc}$$
$$- g^{cd}\partial_e g_{db}g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db}g^{ef}\partial_f g_{ac} - g^{cd}\partial_b g_{ed}g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed}g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed}g^{ef}\partial_f g_{ac} + g^{cd}\partial_d g_{eb}g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb}g^{ef}\partial_c g_{af}$$
$$- g^{cd}\partial_d g_{eb}g^{ef}\partial_f g_{ac} \tag{ex-0309.106}$$

$$= -2\partial_a g_{eb}\partial_c g_{df}g^{cd}g^{ef} - 2\partial_b g_{ae}\partial_c g_{df}g^{cd}g^{ef} + 2\partial_c g_{df}\partial_e g_{ab}g^{cd}g^{ef} + 2\partial_{ca}g_{eb}g^{ce} + 2\partial_{cb}g_{ae}g^{ce} - 2\partial_{ce}g_{ab}g^{ce} + \partial_a g_{fb}\partial_e g_{dc}g^{cd}g^{ef} + \partial_b g_{af}\partial_e g_{dc}g^{cd}g^{ef}$$
$$- \partial_e g_{dc}\partial_f g_{ab}g^{cd}g^{ef} + \partial_a g_{fb}\partial_c g_{ed}g^{cd}g^{ef} + \partial_b g_{af}\partial_c g_{ed}g^{cd}g^{ef} - \partial_c g_{ed}\partial_f g_{ab}g^{cd}g^{ef} - \partial_a g_{fb}\partial_d g_{ec}g^{cd}g^{ef} - \partial_b g_{af}\partial_d g_{ec}g^{cd}g^{ef} + \partial_d g_{ec}\partial_f g_{ab}g^{cd}g^{ef}$$
$$+ 2\partial_a g_{ec}\partial_b g_{df}g^{cd}g^{ef} + 2\partial_b g_{df}\partial_c g_{ae}g^{cd}g^{ef} - 2\partial_b g_{df}\partial_e g_{ac}g^{cd}g^{ef} - 2\partial_{ba}g_{ec}g^{ce} - 2\partial_{bc}g_{ae}g^{ce} + 2\partial_{be}g_{ac}g^{ce} - \partial_a g_{fc}\partial_e g_{db}g^{cd}g^{ef}$$
$$- \partial_c g_{af}\partial_e g_{db}g^{cd}g^{ef} + \partial_e g_{db}\partial_f g_{ac}g^{cd}g^{ef} - \partial_a g_{fc}\partial_b g_{ed}g^{cd}g^{ef} - \partial_b g_{ed}\partial_c g_{af}g^{cd}g^{ef} + \partial_b g_{ed}\partial_f g_{ac}g^{cd}g^{ef} + \partial_a g_{fc}\partial_d g_{eb}g^{cd}g^{ef} + \partial_c g_{af}\partial_d g_{eb}g^{cd}g^{ef}$$
$$- \partial_d g_{eb}\partial_f g_{ac}g^{cd}g^{ef} \tag{ex-0309.107}$$

$$= -2\partial_a g_{db}\partial_c g_{ef}g^{ce}g^{df} - 2\partial_b g_{ad}\partial_c g_{ef}g^{ce}g^{df} + 2\partial_c g_{ef}\partial_d g_{ab}g^{ce}g^{df} + 2\partial_{ca}g_{db}g^{cd} + 2\partial_{cb}g_{ad}g^{cd} - 2\partial_{cd}g_{ab}g^{cd} + \partial_a g_{db}\partial_c g_{ef}g^{fe}g^{cd} + \partial_b g_{ad}\partial_c g_{ef}g^{fe}g^{cd}$$
$$- \partial_c g_{ef}\partial_d g_{ab}g^{fe}g^{cd} + \partial_a g_{db}\partial_c g_{ef}g^{cf}g^{ed} + \partial_b g_{ad}\partial_c g_{ef}g^{cf}g^{ed} - \partial_c g_{ef}\partial_d g_{ab}g^{cf}g^{ed} - \partial_a g_{db}\partial_c g_{ef}g^{fc}g^{ed} - \partial_b g_{ad}\partial_c g_{ef}g^{fc}g^{ed} + \partial_c g_{ef}\partial_d g_{ab}g^{fc}g^{ed}$$
$$+ 2\partial_a g_{cd}\partial_b g_{ef}g^{de}g^{cf} + 2\partial_b g_{de}\partial_c g_{af}g^{cd}g^{fe} - 2\partial_b g_{de}\partial_c g_{af}g^{fd}g^{ce} - 2\partial_{ba}g_{cd}g^{dc} - 2\partial_{bc}g_{ad}g^{cd} + 2\partial_{bc}g_{ad}g^{dc} - \partial_a g_{de}\partial_c g_{fb}g^{ef}g^{cd}$$
$$- \partial_c g_{ae}\partial_d g_{fb}g^{cf}g^{de} + \partial_c g_{eb}\partial_d g_{af}g^{fe}g^{cd} - \partial_a g_{cd}\partial_b g_{ef}g^{df}g^{ec} - \partial_b g_{de}\partial_c g_{af}g^{ce}g^{df} + \partial_b g_{de}\partial_c g_{af}g^{fe}g^{dc} + \partial_a g_{de}\partial_c g_{fb}g^{ec}g^{fd} + \partial_c g_{ae}\partial_d g_{fb}g^{cd}g^{fe}$$
$$- \partial_c g_{eb}\partial_d g_{af}g^{fc}g^{ed} \tag{ex-0309.108}$$

$$= -2\partial_a g_{bc}\partial_d g_{ef}g^{ce}g^{df} - 2\partial_b g_{ac}\partial_d g_{ef}g^{ce}g^{df} + 2\partial_c g_{ab}\partial_d g_{ef}g^{ce}g^{df} + 2\partial_{ac}g_{bd}g^{cd} + 2\partial_{bc}g_{ad}g^{cd} - 2\partial_{cd}g_{ab}g^{cd} + \partial_a g_{bc}\partial_d g_{ef}g^{cd}g^{ef} + \partial_b g_{ac}\partial_d g_{ef}g^{cd}g^{ef}$$
$$- \partial_c g_{ab}\partial_d g_{ef}g^{cd}g^{ef} + \partial_a g_{cd}\partial_b g_{ef}g^{ce}g^{df} - 2\partial_{ab}g_{cd}g^{cd} - 2\partial_c g_{ad}\partial_e g_{bf}g^{cf}g^{de} + 2\partial_c g_{ad}\partial_e g_{bf}g^{ce}g^{df} \tag{ex-0309.109}$$

$$= -2g^{cd}g^{ef}\partial_a g_{bc}\partial_e g_{df} - 2g^{cd}g^{ef}\partial_b g_{ac}\partial_e g_{df} + 2g^{cd}g^{ef}\partial_c g_{ab}\partial_e g_{df} + 2g^{cd}\partial_{ac}g_{bd} + 2g^{cd}\partial_{bc}g_{ad} - 2g^{cd}\partial_{cd}g_{ab} + g^{cd}g^{ef}\partial_a g_{bc}\partial_d g_{ef} + g^{cd}g^{ef}\partial_b g_{ac}\partial_d g_{ef}$$
$$- g^{cd}g^{ef}\partial_c g_{ab}\partial_d g_{ef} + g^{cd}g^{ef}\partial_a g_{ce}\partial_b g_{df} - 2g^{cd}\partial_{ab}g_{cd} - 2g^{cd}g^{ef}\partial_c g_{ae}\partial_f g_{bd} + 2g^{cd}g^{ef}\partial_c g_{ae}\partial_d g_{bf} \tag{ex-0309.110}$$

3

Exercise 3.10 Example of `repeat=True` in a substitution

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent).
2
3   foo := A B + A B A B + A B A B A B + A B A B A B A B .    # cdb(ex-0310.foo.001,foo)
4   bah := @(foo).                                            # cdb(ex-0310.bah.001,bah)
5
6   substitute (foo,$A B -> A$)                               # cdb(ex-0310.foo.002,foo)
7   substitute (bah,$A B -> A$,repeat=True)                   # cdb(ex-0310.bah.002,bah)
```

Without `repeat=True` only the first match in a product will be susbstituted.

$$\texttt{ex-0310.foo.001} := AB + ABAB + ABABAB + ABABABAB$$

$$\texttt{ex-0310.foo.002} := A + AAB + AABAB + AABABAB$$

But with `repeat=True` then all matches in a product will be susbstituted.

$$\texttt{ex-0310.bah.001} := AB + ABAB + ABABAB + ABABABAB$$

$$\texttt{ex-0310.bah.002} := A + AA + AAA + AAAA$$

Exercise 4.1 Differentiate a polynomial – a limited method

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3   def deriv (poly):
4
5       \delta^{a}::Weight(label=\epsilon).
6
7       bah := @(poly).
8
9       substitute      (bah,$x^{a} -> x^{a} + \delta^{a}$)
10      distribute      (bah)
11
12      foo := @(bah) - @(poly).
13
14      keep_weight     (foo, $\epsilon = 1$)
15      sort_product    (foo)
16      rename_dummies  (foo)
17      factor_out      (foo, $\delta^{a?}$)
18      substitute      (foo, $\delta^{a} -> 1$)
19
20      return foo
21
22  # ------------------------------------------------------------
23
24  poly := c^{a}
25        + c^{a}{}_{b} x^b
26        + c^{a}{}_{b c} x^b x^c.    # cdb (ex-0401.100,poly)
27
28  dpoly = deriv (poly)              # cdb (ex-0401.101,dpoly)
```

$$p = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c \tag{ex-0401.100}$$

$$dp = c^a{}_b + c^a{}_{cb} x^c + c^a{}_{bc} x^c \tag{ex-0401.101}$$

Exercise 4.1 Differentiate a polynomial – a better method

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3   def deriv (poly):
4
5       \partial{#}::PartialDerivative.
6       \delta^{a}_{b}::KroneckerDelta.
7
8       x^{a}::Depends(\partial{#}).
9
10      bah := \partial_{b}{@(poly)}.
11
12      distribute       (bah)
13      unwrap           (bah)   # drop all terms that don't explicitly depend on a derivative operator
14      product_rule     (bah)
15      distribute       (bah)
16      substitute       (bah,$\partial_{b}{x^{a}}->\delta^{a}_{b}$)
17      eliminate_kronecker (bah)
18
19      sort_product     (bah)
20      rename_dummies   (bah)
21
22      return bah
23
24  poly := c^{a}
25        + c^{a}{}_{b} x^b
26        + c^{a}{}_{b c} x^b x^c.    # cdb (ex-0401.200,poly)
27
28  dpoly = deriv (poly)              # cdb (ex-0401.201,dpoly)
```

$$p = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c \tag{ex-0401.200}$$

$$dp = c^a{}_b + c^a{}_{bc} x^c + c^a{}_{cb} x^c \tag{ex-0401.201}$$

Exercise 4.2 Inconsistent free indices

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3    def deriv (poly):
4
5        \delta^{a}::Weight(label=\epsilon).
6
7        bah := @(poly).
8
9        substitute      (bah,$x^{a} -> x^{a} + \delta^{a}$)
10       distribute      (bah)
11
12       foo := @(bah) - @(poly).
13
14       keep_weight     (foo, $\epsilon = 1$)
15       substitute      (foo, $\delta^{a} -> 1$)
16
17       return foo
18
19   # ------------------------------------------------------------
20
21   poly := c^{a}
22         + c^{a}{}_{b} x^b
23         + c^{a}{}_{b c} x^b x^c.      # cdb (ex-0402.100,poly)
24
25   dpoly = deriv (poly)                # cdb (ex-0402.101,dpoly)
```

$$p = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c \tag{ex-0402.100}$$

$$dp = c^a{}_b + c^a{}_{bc} x^b + c^a{}_{bc} x^c \tag{ex-0402.101}$$

Exercise 4.3 Polynomial products

```
1
2    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
3
4    def get_term (poly,n):
5
6        x^{a}::Weight(label=xnum).      # assign weights to x^{a}
7
8        foo := @(poly).                 # make a copy of poly
9        bah  = Ex("xnum = " + str(n))   # choose a target
10       keep_weight (foo,bah)           # extract the target
11
12       return foo
13
14   def poly_product (p,q,n):
15
16       pq = Ex("0")
17
18       for i in range (0,n+1):
19           for j in range (0,i+1):
20               termA = get_term (p,j)
21               termB = get_term (q,i-j)
22               termAB := @(termA) @(termB).
23               pq = pq + termAB
24
25       sort_product    (pq)
26       rename_dummies (pq)
27       factor_out      (pq,$x^{a?}$)
28
29       return pq
30
31   # ---------------------------------------------------------------
32
33   # two polynomials
34
35   polyA := c^{a}
36          + c^{a}_{b} x^b
```

```
37          + c^{a}_{b c} x^b x^c
38          + c^{a}_{b c d} x^b x^c x^d
39          + c^{a}_{b c d e} x^b x^c x^d x^e.     # cdb(ex-0403.100,polyA)
40
41   polyB := d^{f}
42          + d^{f}_{b} x^b
43          + d^{f}_{b c} x^b x^c
44          + d^{f}_{b c d} x^b x^c x^d
45          + d^{f}_{b c d e} x^b x^c x^d x^e.     # cdb(ex-0403.101,polyB)
46
47   # multiply polynomials and truncate
48
49   polyAB = poly_product (polyA,polyB,3)         # cdb(ex-0403.102,polyAB)
```

$$p = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c + c^a{}_{bcd} x^b x^c x^d + c^a{}_{bcde} x^b x^c x^d x^e \tag{ex-0403.100}$$

$$q = d^f + d^f{}_b x^b + d^f{}_{bc} x^b x^c + d^f{}_{bcd} x^b x^c x^d + d^f{}_{bcde} x^b x^c x^d x^e \tag{ex-0403.101}$$

$$pq = c^a d^f + x^b \left(c^a d^f{}_b + c^a{}_b d^f \right) + x^b x^c \left(c^a d^f{}_{bc} + c^a{}_b d^f{}_c + c^a{}_{bc} d^f \right) + x^b x^c x^d \left(c^a d^f{}_{bcd} + c^a{}_b d^f{}_{cd} + c^a{}_{bc} d^f{}_d + c^a{}_{bcd} d^f \right) \tag{ex-0403.102}$$

2

Exercise 4.4 Reformatting simple expressions

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

def reformat (obj,scale):

    {x^{a},A_{a b},B_{a b},C_{a b},g^{a b}}::SortOrder.   # choose a sort order

    foo  = Ex(str(scale))              # create a scale factor
    bah := @(foo) @(obj).              # apply the scale factor, clears all fractions

    distribute      (bah)              # only required if (bah) contains brackets
    sort_product    (bah)
    rename_dummies (bah)
    canonicalise    (bah)
    factor_out      (bah,$x^{a?}$)

    ans := @(bah) / @(foo).         # undo previous scaling

    return ans

# ------------------------------------------------------------

# a messy unformatted expression

expr := + (1/3) A_{a b} x^{a} x^{b}
        + (1/9) B_{e c} x^{c} x^{e}
        - (1/5) C_{p c} B_{d q} g^{c d} x^{p} x^{q}.   # cdb (ex-0404.100,expr)

# reformat terms and tidy fractions

expr = reformat (expr,45)                              # cdb(ex-0404.101,expr)
```

$$g = \frac{1}{3} A_{ab} x^a x^b + \frac{1}{9} B_{ec} x^c x^e - \frac{1}{5} C_{pc} B_{dq} g^{cd} x^p x^q \qquad \text{(ex-0404.100)}$$

$$= \frac{1}{45} x^a x^b \left(15 A_{ab} + 5 B_{ab} - 9 B_{ca} C_{bd} g^{dc} \right) \qquad \text{(ex-0404.101)}$$

Exercise 4.5 Reformatting complex expressions

```
1
2    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
3
4    \nabla{#}::Derivative.
5
6    def get_term (obj,n):
7
8        x^{a}::Weight(label=xnum).      # assign weights to x^{a}
9
10       foo := @(obj).                  # make a copy of obj
11       bah  = Ex("xnum = " + str(n))   # choose a target
12       keep_weight (foo,bah)           # extract the target
13
14       return foo
15
16   def reformat (obj,scale):
17
18       {x^{a},A_{a},B_{a},A_{a b},B_{a b},C_{a b},C_{a b c},g^{a b}}::SortOrder.   # choose a sort order
19
20       foo  = Ex(str(scale))           # create a scale factor
21       bah := @(foo) @(obj).           # apply the scale factor, clears all fractions
22
23       distribute      (bah)           # only required if (bah) contains brackets
24       sort_product    (bah)
25       rename_dummies  (bah)
26       canonicalise    (bah)
27       factor_out      (bah,$x^{a?}$)
28
29       ans := @(bah) / @(foo).         # undo previous scaling
30
31       return ans
32
33   # ------------------------------------------------------------
34
35   # a messy unformatted expression
36
```

```
37    expr :=    (1/7) A_{e} x^{e}
38           -  (1/3) B_{f} x^{f}
39           +  (1/3) A_{a b} x^{a} x^{b}
40           +  (1/9) B_{e c} x^{c} x^{e}
41           -  (1/5) C_{p c} B_{d q} g^{c d} x^{p} x^{q}
42           +  (3/7) A_{a b c} x^{a} x^{b} x^{c}
43           -  (1/5) B_{a b} C_{c d e} g^{c d} x^{a} x^{b} x^{e}
44           + (7/11) B_{a b} B_{c d} C_{e f g} g^{b c} g^{d f} x^{a} x^{e} x^{g}.  # cdb (ex-0405.100,expr)

46    # split the expression into seprate terms

48    term1 = get_term (expr,1)         # cdb(term1.101,term1)
49    term2 = get_term (expr,2)         # cdb(term2.101,term2)
50    term3 = get_term (expr,3)         # cdb(term3.101,term3)

52    # reformat terms and tidy fractions

54    term1 = reformat (term1, 21)      # cdb(term1.102,term1)
55    term2 = reformat (term2, 45)      # cdb(term2.102,term2)
56    term3 = reformat (term3,385)      # cdb(term3.102,term3)

58    # rebuild the expression

60    expr := @(term1) + @(term2) + @(term3).  # cdb (ex-0405.101,expr)
```

2

$$g = \frac{1}{7}A_e x^e - \frac{1}{3}B_f x^f + \frac{1}{3}A_{ab}x^a x^b + \frac{1}{9}B_{ec}x^c x^e - \frac{1}{5}C_{pc}B_{dq}g^{cd}x^p x^q + \frac{3}{7}A_{abc}x^a x^b x^c - \frac{1}{5}B_{ab}C_{cde}g^{cd}x^a x^b x^e + \frac{7}{11}B_{ab}B_{cd}C_{efg}g^{bc}g^{df}x^a x^e x^g \qquad \text{(ex-0405.100)}$$

$$= \frac{1}{21}x^a\left(3A_a - 7B_a\right) + \frac{1}{45}x^a x^b\left(15A_{ab} + 5B_{ab} - 9B_{ca}C_{bd}g^{dc}\right) + \frac{1}{385}x^a x^b x^c\left(165A_{abc} - 77B_{ab}C_{dec}g^{de} + 245B_{ad}B_{ef}C_{bgc}g^{de}g^{fg}\right) \qquad \text{(ex-0405.101)}$$

Exercise 4.6 Bespoke sort

```
1
2    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
3
4    def bespoke_sort (expr):
5
6        substitute      (expr,$ x^{a}            -> AAA01^{a}        $)
7        substitute      (expr,$ g_{a b}          -> AAA02_{a b}     $)
8        substitute      (expr,$ \Gamma_{a b c}   -> AAA03_{a b c}   $)
9
10       sort_product    (expr)
11
12       substitute      (expr,$ AAA01^{a}        -> x^{a}           $)
13       substitute      (expr,$ AAA02_{a b}      -> g_{a b}         $)
14       substitute      (expr,$ AAA03_{a b c}    -> \Gamma_{a b c} $)
15
16       return expr
17
18   # -------------------------------------------------------------
19
20   expr := g_{a b} x^{a} x^{b} + \Gamma_{a b c} x^{a} x^{b} x^{c}.  # cdb(ex-0406.100,expr)
21
22   expr = bespoke_sort (expr)                                      # cdb(ex-0406.101,expr)
```

$$p = g_{ab}x^a x^b + \Gamma_{abc}x^a x^b x^c \qquad \text{(ex-0406.100)}$$
$$= x^a x^b g_{ab} + x^a x^b x^c \Gamma_{abc} \qquad \text{(ex-0406.101)}$$

Exercise 4.7 Return in functions

```
1    {a,b,c,d,e,f,g,h,i,j,k,l#}::Indices(position=independent).
2
3    # ------------------------------------------------------------
4    # this function uses in-place changes for obj
5
6    def tidy (obj):
7
8        sort_product    (obj)
9        rename_dummies (obj)
10       canonicalise    (obj)
11
12   foo := C^{f} B^{a} A_{f a}.                    # cdb (ex-0407.101,foo)
13   tidy (foo)                                     # cdb (ex-0407.102,foo)
14
15   # ------------------------------------------------------------
16   # this function creates new objects,
17   # it will not give the correct result
18
19   def tidy (obj):
20
21       bah := @(obj).
22
23       sort_product    (bah)
24       rename_dummies (bah)
25       canonicalise    (bah)
26
27       obj := @(bah).
28
29   foo := C^{f} B^{a} A_{f a}.                    # cdb (ex-0407.201,foo)
30   tidy (foo)                                     # cdb (ex-0407.202,foo)
31
32   # ------------------------------------------------------------
33   # this function uses a return statement
34   # it will give the correct result
35
36   def tidy (obj):
```

```
37
38      bah := @(obj).
39
40      sort_product    (bah)
41      rename_dummies  (bah)
42      canonicalise    (bah)
43
44      obj := @(bah).
45
46      return obj
47
48  foo := C^{f} B^{a} A_{f a}.                # cdb (ex-0407.301,foo)
49  foo = tidy (foo)                           # cdb (ex-0407.302,foo)
```

$$C^f B^a A_{fa} = A_{ab} B^b C^a \qquad\qquad \text{(ex-0407.102)}$$

$$C^f B^a A_{fa} = C^f B^a A_{fa} \qquad\qquad \text{(ex-0407.202)}$$

$$C^f B^a A_{fa} = A_{ab} B^b C^a \qquad\qquad \text{(ex-0407.302)}$$

Exercise 5.1 Swap terms

```
1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3    expr := A_{a} (P^{b}+Q^{b}) + C_{a} V^{b}.                        # cdb (ex-0501.100,expr)
4
5    substitute (expr, $A_{a} B?? + C_{a} D?? -> A_{a} D?? + C_{a} B??$)    # cdb (ex-0501.101,expr)
```

$$\text{ex-0501.100} := A_a \left(P^b + Q^b \right) + C_a V^b$$

$$\text{ex-0501.101} := A_a V^b + C_a \left(P^b + Q^b \right)$$

Exercise 5.2 Leading factors forbidden in patterns

This exercise will raise a Cadabra run-time error – the scale factor on the left hand side of the rule (3 in this case) is not allowed.

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   expr := 2 V_{a b} - 3 V_{b a}.                    # cdb (ex-0502.100,expr)
4
5   substitute (expr, $3 V_{b a} -> - 3 V_{a b}$)   # cdb (ex-0502.101,expr)
```

```
Traceback (most recent call last):
  File "/usr/local/bin/cadabra2", line 248, in <module>
    exec(cmp)
  File "ex-0502.py", line 18, in <module>
    substitute (expr, Ex(r'''3 V_{b a} -> - 3 V_{a b}''', False))
RuntimeError: substitute: Index error in replacement rule.
            substitute: No numerical pre-factors allowed on lhs of replacement rule.
```

Exercise 5.3 Deleting a term using patterns

```
1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  expr := A_{a b} B^{a b} + A_{a b} A_{c d} B^{a b} B^{c d} - C_{a b} B^{a b}.  # cdb (ex-0503.100,expr)
4
5  zoom       (expr, $A_{a b} A_{c d} Q??$)                              # cdb (ex-0503.101,expr)
6  substitute (expr, $A_{a b} -> 0$)                                     # cdb (ex-0503.102,expr)
7  unzoom     (expr)                                                     # cdb (ex-0503.103,expr)
```

$$\texttt{ex-0503.100} := A_{ab}B^{ab} + A_{ab}A_{cd}B^{ab}B^{cd} - C_{ab}B^{ab}$$

$$\texttt{ex-0503.101} := \ldots + A_{ab}A_{cd}B^{ab}B^{cd} + \ldots$$

$$\texttt{ex-0503.102} := \ldots\ldots$$

$$\texttt{ex-0503.103} := A_{ab}B^{ab} - C_{ab}B^{ab}$$

Exercise 5.4 Deleting a term using tags

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   def add_tags (obj,tag):
4       n = 0
5       ans = Ex('0')
6       for i in obj.top().terms():
7           foo = obj[i]
8           bah = Ex(tag+'_{'+str(n)+'}')
9           ans := @(ans) + @(bah) @(foo).
10          n = n + 1
11      return ans
12
13  def clear_tags (obj,tag):
14      ans := @(obj).
15      foo  = Ex(tag+'_{a?} -> 1')
16      substitute (ans,foo)
17      return ans
18
19  expr := A_{a b} B^{a b} + A_{a b} A_{c d} B^{a b} B^{c d} - C_{a b} B^{a b}.   # cdb (ex-0504.100,expr)
20
21  expr  = add_tags (expr,'\\mu')                                                # cdb (ex-0504.101,expr)
22
23  substitute (expr, $\mu_{1} -> 0$)                                             # cdb (ex-0504.102,expr)
24
25  expr = clear_tags (expr,'\\mu')                                              # cdb (ex-0504.103,expr)
```

$$\text{ex-0504.100} := A_{ab}B^{ab} + A_{ab}A_{cd}B^{ab}B^{cd} - C_{ab}B^{ab}$$

$$\text{ex-0504.101} := \mu_0 A_{ab}B^{ab} + \mu_1 A_{ab}A_{cd}B^{ab}B^{cd} - \mu_2 C_{ab}B^{ab}$$

$$\text{ex-0504.102} := \mu_0 A_{ab}B^{ab} - \mu_2 C_{ab}B^{ab}$$

$$\text{ex-0504.103} := A_{ab}B^{ab} - C_{ab}B^{ab}$$

Exercise 5.5 Commuting covariant derivatives

```
1   {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3   ;::Symbol.
4
5   def add_tags (obj,tag):
6       n = 0
7       ans = Ex('0')
8       for i in obj.top().terms():
9           foo = obj[i]
10          bah = Ex(tag+'_{'+str(n)+'}')
11          ans := @(ans) + @(bah) @(foo).
12          n = n + 1
13      return ans
14
15  def clear_tags (obj,tag):
16      ans := @(obj).
17      foo  = Ex(tag+'_{a?} -> 1')
18      substitute (ans,foo)
19      return ans
20
21  rule := V^{a}_{; b ; c} -> V^{a}_{; c ; b} - R^{a}_{d b c} V^{d}.
22
23  expr := V^{a}_{; b ; c} - V^{a}_{; c ; b}.    # cdb (ex-0505.100,expr)
24
25  expr  = add_tags (expr,'\\mu')               # cdb (ex-0505.101,expr)
26
27  zoom        (expr, $\mu_{0} Q??$)            # cdb (ex-0505.102,expr)
28  substitute (expr, rule)                      # cdb (ex-0505.103,expr)
29  unzoom      (expr)                           # cdb (ex-0505.104,expr)
30
31  expr = clear_tags (expr,'\\mu')              # cdb (ex-0505.105,expr)
```

$$V^a{}_{;b;c} - V^a{}_{;c;b} = \mu_0 V^a{}_{;b;c} - \mu_1 V^a{}_{;c;b} \qquad \text{(ex-0505.101)}$$
$$= \mu_0 V^a{}_{;b;c} - \mu_1 V^a{}_{;c;b} \qquad \text{(ex-0505.101)}$$
$$= \mu_0 V^a{}_{;b;c} + \ldots \qquad \text{(ex-0505.102)}$$
$$= \mu_0 \left(V^a{}_{;c;b} - R^a{}_{dbc} V^d \right) + \ldots \qquad \text{(ex-0505.103)}$$
$$= \mu_0 \left(V^a{}_{;c;b} - R^a{}_{dbc} V^d \right) - \mu_1 V^a{}_{;c;b} \qquad \text{(ex-0505.104)}$$
$$= -R^a{}_{dbc} V^d \qquad \text{(ex-0505.105)}$$

Exercise 6.1 Evaluate – without `rhsonly = True`

```
1   {\theta, \varphi}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4   \partial{#}::PartialDerivative.
5
6   V  := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.    # cdb(ex-0601.100,V)
7   dV := dV_{a b} -> \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.   # cdb(ex-0601.101,dV)
8
9   evaluate (dV, V)                                              # cdb(ex-0601.102,dV)
```

Notice how `evaluate` has been applied to both the left and right hand sides of the rule.

$$V_a = [V_\theta = \varphi, \ V_\varphi = \sin\theta] \tag{ex-0601.100}$$

$$dV_{ab} \rightarrow \partial_b V_a - \partial_a V_b \tag{ex-0601.101}$$

$$\Box_{ab} \begin{cases} \Box_{\theta\theta} = dV_{\theta\theta} \\ \Box_{\varphi\theta} = dV_{\varphi\theta} \\ \Box_{\theta\varphi} = dV_{\theta\varphi} \\ \Box_{\varphi\varphi} = dV_{\varphi\varphi} \end{cases} \rightarrow \Box_{ab} \begin{cases} \Box_{\varphi\theta} = \cos\theta - 1 \\ \Box_{\theta\varphi} = -\cos\theta + 1 \end{cases} \tag{ex-0601.102}$$

Exercise 6.1 Evaluate – with rhsonly = True

```
1   {\theta, \varphi}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4   \partial{#}::PartialDerivative.
5
6   V  := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.    # cdb(ex-0601.200,V)
7   dV := dV_{a b} -> \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.   # cdb(ex-0601.201,dV)
8
9   evaluate (dV, V, rhsonly=True)                                # cdb(ex-0601.202,dV)
```

This is an improvement, only the right hnd side has been expanded into components.

$$V_a = [V_\theta = \varphi, \ V_\varphi = \sin\theta] \tag{ex-0601.200}$$

$$dV_{ab} \to \partial_b V_a - \partial_a V_b \tag{ex-0601.201}$$

$$dV_{ab} \to \Box_{ab} \begin{cases} \Box_{\varphi\theta} = \cos\theta - 1 \\ \Box_{\theta\varphi} = -\cos\theta + 1 \end{cases} \tag{ex-0601.202}$$

Exercise 6.2 Evaluate on an expression (not a rule)

```
1    {\theta, \varphi}::Coordinate.
2    {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4    \partial{#}::PartialDerivative.
5
6    V  := { V_{\theta} = f(\theta,\varphi), V_{\varphi} = g(\theta,\varphi) }.   # cdb(ex-0602.100,V)
7    dV := \partial_{b}{V_{a}} + \partial_{a}{V_{b}}.                            # cdb(ex-0602.101,dV)
8
9    evaluate (dV, V)                                                           # cdb(ex-0602.102,dV)
```

$$V_a = \left[V_\theta = f\left(\theta, \varphi\right),\ V_\varphi = g\left(\theta, \varphi\right) \right] \tag{ex-0602.100}$$

$$\partial_b V_a + \partial_a V_b \tag{ex-0602.101}$$

$$\square_{ab} \begin{cases} \square_{\varphi\varphi} = 2\partial_\varphi g\left(\theta, \varphi\right) \\ \square_{\varphi\theta} = \partial_\varphi f\left(\theta, \varphi\right) + \partial_\theta g\left(\theta, \varphi\right) \\ \square_{\theta\varphi} = \partial_\varphi f\left(\theta, \varphi\right) + \partial_\theta g\left(\theta, \varphi\right) \\ \square_{\theta\theta} = 2\partial_\theta f\left(\theta, \varphi\right) \end{cases} \tag{ex-0602.102}$$

Exercise 6.3 Evaluate with undefined components

```
1   {\theta, \varphi}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4   bah := {V_{\theta} = \varphi, V_{\varphi} = \sin(\theta)}.   # cdb(ex-0603.100,bah)
5   foo := U_{a} V_{b}.                                          # cdb(ex-0603.101,foo)
6
7   evaluate (foo, bah)                                          # cdb(ex-0603.102,foo)
```

$$[V_\theta = \varphi, \ V_\varphi = \sin\theta] \tag{ex-0603.100}$$

$$U_a V_b \tag{ex-0603.101}$$

$$\square_{ab} \begin{cases} \square_{\theta\theta} = \varphi U_\theta \\ \square_{\theta\varphi} = U_\theta \sin\theta \\ \square_{\varphi\theta} = \varphi U_\varphi \\ \square_{\varphi\varphi} = U_\varphi \sin\theta \end{cases} \tag{ex-0603.102}$$

Exercise 6.4 Scalar curavture of a 2-sphere

```
1    {\theta, \varphi}::Coordinate.
2    {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4    \partial{#}::PartialDerivative.
5
6    g^{a b}::InverseMetric.   # essential when using complete (gab, $g^{a b}$)
7
8    Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
9                                              + \partial_{c}{g_{b d}}
10                                              - \partial_{d}{g_{b c}}).
11
12   Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
13                              - \partial_{d}{\Gamma^{a}_{b c}}
14                              + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                              - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17   Rab := R_{a b} -> R^{c}_{a c b}.
18
19   R  := R -> R_{a b} g^{a b}.
20
21   gab := { g_{\theta\theta}   = r**2,
22            g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.     # cdb(ex-0604.101,gab)
23
24   complete   (gab, $g^{a b}$)                              # cdb(ex-0604.102,gab)
25
26   substitute (Rabcd, Gamma)
27   substitute (Rab, Rabcd)
28   substitute (R, Rab)
29
30   evaluate   (Gamma, gab, rhsonly=True)                    # cdb(ex-0604.103,Gamma)
31   evaluate   (Rabcd, gab, rhsonly=True)                    # cdb(ex-0604.104,Rabcd)
32   evaluate   (Rab,   gab, rhsonly=True)                    # cdb(ex-0604.105,Rab)
33   evaluate   (R,     gab, rhsonly=True)                    # cdb(ex-0604.106,R)
```

$$\left[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin\theta)^2 \right] \tag{ex-0604.101}$$

$$\left[g_{\theta\theta} = r^2, \ g_{\varphi\varphi} = r^2(\sin\theta)^2, \ g^{\theta\theta} = r^{-2}, \ g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1} \right] \tag{ex-0604.102}$$

$$\Gamma^a{}_{bc} \rightarrow \square_{cb}{}^a \begin{cases} \square_{\varphi\theta}{}^\varphi = (\tan\theta)^{-1} \\ \square_{\theta\varphi}{}^\varphi = (\tan\theta)^{-1} \\ \square_{\varphi\varphi}{}^\theta = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-0604.103}$$

$$R^a{}_{bcd} \rightarrow \square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^\theta{}_\theta = (\sin\theta)^2 \\ \square_{\varphi\theta}{}^\varphi{}_\theta = -1 \\ \square_{\theta\varphi}{}^\theta{}_\varphi = -(\sin\theta)^2 \\ \square_{\theta\theta}{}^\varphi{}_\varphi = 1 \end{cases} \tag{ex-0604.104}$$

$$R_{ab} \rightarrow \square_{ba} \begin{cases} \square_{\varphi\varphi} = (\sin\theta)^2 \\ \square_{\theta\theta} = 1 \end{cases} \tag{ex-0604.105}$$

$$R \rightarrow 2r^{-2} \tag{ex-0604.106}$$

Exercise 6.5 Schwarzschild spacetime in isotropic coordinates

```
1    {t, r, \theta, \varphi}::Coordinate.
2    {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4    \partial{#}::PartialDerivative.
5
6    g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)
7
8    Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
9                                              + \partial_{c}{g_{b d}}
10                                             - \partial_{d}{g_{b c}}).
11
12   Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
13                             - \partial_{d}{\Gamma^{a}_{b c}}
14                             + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                             - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17   Rab := R_{a b} -> R^{c}_{a c b}.
18
19   gab := { g_{t t}           = -((2*r-m)/(2*r+m))**2,
20            g_{r r}           = (1+m/(2*r))**4,
21            g_{\theta\theta}   = r**2 (1+m/(2*r))**4,
22            g_{\varphi\varphi} = r**2 \sin(\theta)**2 (1+m/(2*r))**4}.  # cdb(ex-0605.101,gab)
23
24   complete   (gab, $g^{a b}$)                                        # cdb(ex-0605.102,gab)
25
26   substitute (Rabcd, Gamma)
27   substitute (Rab, Rabcd)
28
29   evaluate   (Gamma, gab, rhsonly=True)                             # cdb(ex-0605.103,Gamma)
30   evaluate   (Rabcd, gab, rhsonly=True)                             # cdb(ex-0605.104,Rabcd)
31   evaluate   (Rab,   gab, rhsonly=True)                             # cdb(ex-0605.105,Rab)
```

$$\left[g_{tt} = -\left((2r-m)\,(2r+m)^{-1}\right)^2, \ g_{rr} = \left(1+\frac{1}{2}mr^{-1}\right)^4, \ g_{\theta\theta} = r^2\left(1+\frac{1}{2}mr^{-1}\right)^4, \ g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1+\frac{1}{2}mr^{-1}\right)^4 \right] \qquad \text{(ex-0605.101)}$$

$$\left[g_{tt} = -\left((2r-m)\,(2r+m)^{-1}\right)^2, \ g_{rr} = \left(1+\frac{1}{2}mr^{-1}\right)^4, \ g_{\theta\theta} = r^2\left(1+\frac{1}{2}mr^{-1}\right)^4, \ g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1+\frac{1}{2}mr^{-1}\right)^4, \ g^{tt} = \right.$$

$$\left. -(m+2r)^2(-m+2r)^{-2}, \ g^{rr} = \left(\frac{1}{2}mr^{-1}+1\right)^{-4}, \ g^{\theta\theta} = \left(r^2\left(\frac{1}{2}mr^{-1}+1\right)^4\right)^{-1}, \ g^{\varphi\varphi} = \left(r^2\left(\frac{1}{2}mr^{-1}+1\right)^4(\sin\theta)^2\right)^{-1} \right] \qquad \text{(ex-0605.102)}$$

$$\Gamma^a{}_{bc} \rightarrow \Box_{cb}{}^a \begin{cases}
\Box_{\varphi r}{}^\varphi = (-m+2r)\,(r\,(m+2r))^{-1} \\[4pt]
\Box_{\varphi\theta}{}^\varphi = (\tan\theta)^{-1} \\[4pt]
\Box_{\theta r}{}^\theta = (-m+2r)\,(r\,(m+2r))^{-1} \\[4pt]
\Box_{rr}{}^r = -2m(r\,(m+2r))^{-1} \\[4pt]
\Box_{tr}{}^t = 4m\left(-m^2+4r^2\right)^{-1} \\[4pt]
\Box_{r\varphi}{}^\varphi = (-m+2r)\,(r\,(m+2r))^{-1} \\[4pt]
\Box_{\theta\varphi}{}^\varphi = (\tan\theta)^{-1} \\[4pt]
\Box_{r\theta}{}^\theta = (-m+2r)\,(r\,(m+2r))^{-1} \\[4pt]
\Box_{rt}{}^t = 4m\left(-m^2+4r^2\right)^{-1} \\[4pt]
\Box_{\varphi\varphi}{}^r = r\,(m-2r)\,(\sin\theta)^2(m+2r)^{-1} \\[4pt]
\Box_{\varphi\varphi}{}^\theta = -\frac{1}{2}\sin(2\theta) \\[4pt]
\Box_{\theta\theta}{}^r = r\,(m-2r)\,(m+2r)^{-1} \\[4pt]
\Box_{tt}{}^r = -64mr^4\,(m-2r)\,(m+2r)^{-7}
\end{cases} \qquad \text{(ex-0605.103)}$$

$$R^a{}_{bcd} \to \Box_{db}{}^a{}_c \begin{cases} \Box_{tt}{}^r{}_r = -128m^3r^3(m+2r)^{-8} + 512m^2r^4(m+2r)^{-8} - 512mr^5(m+2r)^{-8} \\ \Box_{\theta\theta}{}^r{}_r = -4mr\left(m^2 + 4mr + 4r^2\right)^{-1} \\ \Box_{\varphi\varphi}{}^\theta{}_\theta = 8mr(\sin\theta)^2(m+2r)^{-2} \\ \Box_{\varphi\varphi}{}^r{}_r = -4mr(\sin\theta)^2\left(m^2 + 4mr + 4r^2\right)^{-1} \\ \Box_{tr}{}^t{}_r = -8m\left(r\left(m^2 + 4mr + 4r^2\right)\right)^{-1} \\ \Box_{\theta r}{}^\theta{}_r = 4m\left(r\left(m^2 + 4mr + 4r^2\right)\right)^{-1} \\ \Box_{\varphi\theta}{}^\varphi{}_\theta = (m-2r)^2(m+2r)^{-2} - 1 \\ \Box_{\varphi r}{}^\varphi{}_r = 4m\left(r\left(m^2 + 4mr + 4r^2\right)\right)^{-1} \\ \Box_{rt}{}^r{}_t = 128m^3r^3(m+2r)^{-8} - 512m^2r^4(m+2r)^{-8} + 512mr^5(m+2r)^{-8} \\ \Box_{r\theta}{}^r{}_\theta = 4mr\left(m^2 + 4mr + 4r^2\right)^{-1} \\ \Box_{\theta\varphi}{}^\theta{}_\varphi = (m-2r)^2(\sin\theta)^2(m+2r)^{-2} - (\sin\theta)^2 \\ \Box_{r\varphi}{}^r{}_\varphi = 4mr(\sin\theta)^2\left(m^2 + 4mr + 4r^2\right)^{-1} \\ \Box_{rr}{}^t{}_t = 8m\left(r\left(m^2 + 4mr + 4r^2\right)\right)^{-1} \\ \Box_{rr}{}^\theta{}_\theta = -4m\left(r\left(m^2 + 4mr + 4r^2\right)\right)^{-1} \\ \Box_{\theta\theta}{}^\varphi{}_\varphi = 8mr(m+2r)^{-2} \\ \Box_{rr}{}^\varphi{}_\varphi = -4m\left(r\left(m^2 + 4mr + 4r^2\right)\right)^{-1} \\ \Box_{\varphi\varphi}{}^t{}_t = -4mr(\sin\theta)^2(m+2r)^{-2} \\ \Box_{\theta\theta}{}^t{}_t = -4mr(m+2r)^{-2} \\ \Box_{tt}{}^\varphi{}_\varphi = 64mr^3(m-2r)^2(m+2r)^{-8} \\ \Box_{tt}{}^\theta{}_\theta = 64mr^3(m-2r)^2(m+2r)^{-8} \\ \Box_{t\varphi}{}^t{}_\varphi = 4mr(\sin\theta)^2(m+2r)^{-2} \\ \Box_{t\theta}{}^t{}_\theta = 4mr(m+2r)^{-2} \\ \Box_{\varphi t}{}^\varphi{}_t = -64mr^3(m-2r)^2(m+2r)^{-8} \\ \Box_{\theta t}{}^\theta{}_t = -64mr^3(m-2r)^2(m+2r)^{-8} \end{cases}$$

(ex-0605.104)

$$R_{ab} \to 0 \tag{ex-0605.105}$$

Exercise 6.6 The Kasner cosmology

```
1   {t, x, y, z}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={t, x, y, z}, position=independent).
3
4   \partial{#}::PartialDerivative.
5
6   p1::LaTeXForm("p_1").
7   p2::LaTeXForm("p_2").
8   p3::LaTeXForm("p_3").
9
10  g^{a b}::InverseMetric.   # essential when using complete (gab, $g^{a b}$)
11
12  Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
13                                           + \partial_{c}{g_{b d}}
14                                           - \partial_{d}{g_{b c}}).
15
16  Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
17                            - \partial_{d}{\Gamma^{a}_{b c}}
18                            + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
19                            - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
20
21  Rab := R_{a b} -> R^{c}_{a c b}.
22
23  gab := { g_{t t} = -1,
24           g_{x x} = t**(2*p1),
25           g_{y y} = t**(2*p2),
26           g_{z z} = t**(2*p3)}.                            # cdb(ex-0606.101,gab)
27
28  complete   (gab, $g^{a b}$)                              # cdb(ex-0606.102,gab)
29
30  substitute (Rabcd, Gamma)
31  substitute (Rab, Rabcd)
32
33  evaluate   (Gamma, gab, rhsonly=True)                    # cdb(ex-0606.103,Gamma)
34  evaluate   (Rabcd, gab, rhsonly=True)                    # cdb(ex-0606.104,Rabcd)
35  evaluate   (Rab,   gab, rhsonly=True)                    # cdb(ex-0606.105,Rab)
```

$$\left[g_{tt} = -1, \; g_{xx} = t^{2p_1}, \; g_{yy} = t^{2p_2}, \; g_{zz} = t^{2p_3} \right] \tag{ex-0606.101}$$

$$\left[g_{tt} = -1, \; g_{xx} = t^{2p_1}, \; g_{yy} = t^{2p_2}, \; g_{zz} = t^{2p_3}, \; g^{tt} = -1, \; g^{xx} = t^{-2p_1}, \; g^{yy} = t^{-2p_2}, \; g^{zz} = t^{-2p_3} \right] \tag{ex-0606.102}$$

$$\Gamma^a{}_{bc} \to \Box_{cb}{}^a \begin{cases} \Box_{zt}{}^z = p_3 t^{-1} \\ \Box_{yt}{}^y = p_2 t^{-1} \\ \Box_{xt}{}^x = p_1 t^{-1} \\ \Box_{tz}{}^z = p_3 t^{-1} \\ \Box_{ty}{}^y = p_2 t^{-1} \\ \Box_{tx}{}^x = p_1 t^{-1} \\ \Box_{zz}{}^t = p_3 t^{(2p_3-1)} \\ \Box_{yy}{}^t = p_2 t^{(2p_2-1)} \\ \Box_{xx}{}^t = p_1 t^{(2p_1-1)} \end{cases} \tag{ex-0606.103}$$

$$R^a{}_{bcd} \to \Box_{db}{}^a{}_c \begin{cases} \Box_{xx}{}^t{}_t = p_1 t^{(2p_1-2)}(p_1-1) \\ \Box_{yy}{}^t{}_t = p_2 t^{(2p_2-2)}(p_2-1) \\ \Box_{zz}{}^t{}_t = p_3 t^{(2p_3-2)}(p_3-1) \\ \Box_{xt}{}^x{}_t = p_1(p_1-1)t^{-2} \\ \Box_{yt}{}^y{}_t = p_2(p_2-1)t^{-2} \\ \Box_{zt}{}^z{}_t = p_3(p_3-1)t^{-2} \\ \Box_{tx}{}^t{}_x = -p_1 t^{(2p_1-2)}(p_1-1) \\ \Box_{ty}{}^t{}_y = -p_2 t^{(2p_2-2)}(p_2-1) \\ \Box_{tz}{}^t{}_z = -p_3 t^{(2p_3-2)}(p_3-1) \\ \Box_{tt}{}^x{}_x = p_1(-p_1+1)t^{-2} \\ \Box_{tt}{}^y{}_y = p_2(-p_2+1)t^{-2} \\ \Box_{tt}{}^z{}_z = p_3(-p_3+1)t^{-2} \\ \Box_{zz}{}^y{}_y = p_2 p_3 t^{(2p_3-2)} \\ \Box_{zz}{}^x{}_x = p_1 p_3 t^{(2p_3-2)} \\ \Box_{yy}{}^z{}_z = p_2 p_3 t^{(2p_2-2)} \\ \Box_{yy}{}^x{}_x = p_1 p_2 t^{(2p_2-2)} \\ \Box_{xx}{}^z{}_z = p_1 p_3 t^{(2p_1-2)} \\ \Box_{xx}{}^y{}_y = p_1 p_2 t^{(2p_1-2)} \\ \Box_{yz}{}^y{}_z = -p_2 p_3 t^{(2p_3-2)} \\ \Box_{xz}{}^x{}_z = -p_1 p_3 t^{(2p_3-2)} \\ \Box_{zy}{}^z{}_y = -p_2 p_3 t^{(2p_2-2)} \\ \Box_{xy}{}^x{}_y = -p_1 p_2 t^{(2p_2-2)} \\ \Box_{zx}{}^z{}_x = -p_1 p_3 t^{(2p_1-2)} \\ \Box_{yx}{}^y{}_x = -p_1 p_2 t^{(2p_1-2)} \end{cases} \qquad (\text{ex-0606.104})$$

$$R_{ab} \to \Box_{ba} \begin{cases} \Box_{xx} = p_1 t^{(2p_1-2)}(p_1+p_2+p_3-1) \\ \Box_{yy} = p_2 t^{(2p_2-2)}(p_1+p_2+p_3-1) \\ \Box_{zz} = p_3 t^{(2p_3-2)}(p_1+p_2+p_3-1) \\ \Box_{tt} = \left(-p_1{}^2+p_1-p_2{}^2+p_2-p_3{}^2+p_3\right)t^{-2} \end{cases} \qquad (\text{ex-0606.105})$$

Exercise 6.7 Killing vectors of the Schwarzschild spacetime

```
1    {t, r, \theta, \varphi}::Coordinate.
2    {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4    ;::Symbol.
5
6    \partial{#}::PartialDerivative.
7
8    g_{a b}::Metric.
9    g^{a b}::InverseMetric.   # essential when using complete (gab, $g^{a b}$)
10
11   Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} (    \partial_{g}{g_{b f}}
12                                             + \partial_{f}{g_{b g}}
13                                             - \partial_{b}{g_{f g}} ).
14
15   deriv := \xi_{a ; b} -> \partial_{b}{\xi_{a}} - \Gamma^{c}_{a b} \xi_{c}.
16   lower := \xi_{a} -> g_{a b} \xi^{b}.
17
18   expr  := \xi_{a ; b} + \xi_{b ; a}.                      # cdb(ex-0607.100,expr)
19
20   substitute   (expr, deriv)                               # cdb(ex-0607.101,expr)
21   substitute   (expr, lower)                               # cdb(ex-0607.102,expr)
22   substitute   (expr, Gamma)                               # cdb(ex-0607.103,expr)
23   distribute   (expr)                                      # cdb(ex-0607.104,expr)
24   product_rule (expr)                                      # cdb(ex-0607.105,expr)
25   canonicalise (expr)                                      # cdb(ex-0607.106,expr)
26
27   # choose a vector
28
29   # Kvect := {\xi^{t} = 1}.
30   # Kvect := {\xi^{\varphi} = 1}.
31   Kvect := {\xi^{\theta} = \sin(\varphi), \xi^{\varphi} = \cos(\theta)/\sin(\theta) \cos(\varphi)}.
32   # Kvect := {\xi^{\theta} = \cos(\varphi), \xi^{\varphi} = - \cos(\theta)/\sin(\theta) \sin(\varphi)}.
33                                                 # cdb(ex-0607.107,Kvect)
34
35   gab := { g_{t t}              = -(1-2*m/r),
36            g_{r r}              = 1/(1-(2*m/r)),
```

```
37          g_{\theta\theta}    = r**2,
38          g_{\varphi\varphi} = r**2 \sin(\theta)**2}.  # cdb(ex-0607.108,gab)
39
40   complete   (gab, $g^{a b}$)                          # cdb(ex-0607.109,gab)
41
42   evaluate   (expr, gab+Kvect)                         # cdb(ex-0607.110,expr)
```

$$[\xi^a] = \left[\xi^\theta = \sin(\varphi),\ \xi^\varphi = \cos\theta(\sin\theta)^{-1}\cos(\varphi)\right] \tag{ex-0607.107}$$

$$[g_{ab}] = \left[g_{tt} = -1 + 2mr^{-1},\ g_{rr} = \left(1 - 2mr^{-1}\right)^{-1},\ g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-0607.108}$$

$$[g_{ab}, g^{ab}] = \left[g_{tt} = -1 + 2mr^{-1},\ g_{rr} = \left(1 - 2mr^{-1}\right)^{-1},\ g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2(\sin\theta)^2,\ g^{tt} = \left(2mr^{-1} - 1\right)^{-1},\ g^{rr} = -2mr^{-1} + 1,\ g^{\theta\theta} = r^{-2},\right.$$
$$\left.g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\right] \tag{ex-0607.109}$$

$$\xi_{a;b} + \xi_{b;a} = \partial_b\xi_a - \Gamma^c{}_{ab}\xi_c + \partial_a\xi_b - \Gamma^c{}_{ba}\xi_c \tag{ex-0607.101}$$
$$= \partial_b\left(g_{ac}\xi^c\right) - \Gamma^c{}_{ab}g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) - \Gamma^c{}_{ba}g_{cd}\xi^d \tag{ex-0607.102}$$
$$= \partial_b\left(g_{ac}\xi^c\right) - \frac{1}{2}g^{ce}\left(\partial_b g_{ea} + \partial_a g_{eb} - \partial_e g_{ab}\right)g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) - \frac{1}{2}g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ba}\right)g_{cd}\xi^d \tag{ex-0607.103}$$
$$= \partial_b\left(g_{ac}\xi^c\right) - g^{ce}\partial_b g_{ea}g_{cd}\xi^d - g^{ce}\partial_a g_{eb}g_{cd}\xi^d + \frac{1}{2}g^{ce}\partial_e g_{ab}g_{cd}\xi^d + \partial_a\left(g_{bc}\xi^c\right) + \frac{1}{2}g^{ce}\partial_e g_{ba}g_{cd}\xi^d \tag{ex-0607.104}$$
$$= \partial_b g_{ac}\xi^c + g_{ac}\partial_b\xi^c - g^{ce}\partial_b g_{ea}g_{cd}\xi^d - g^{ce}\partial_a g_{eb}g_{cd}\xi^d + \frac{1}{2}g^{ce}\partial_e g_{ab}g_{cd}\xi^d + \partial_a g_{bc}\xi^c + g_{bc}\partial_a\xi^c + \frac{1}{2}g^{ce}\partial_e g_{ba}g_{cd}\xi^d \tag{ex-0607.105}$$
$$= \partial_b g_{ac}\xi^c + g_{ac}\partial_b\xi^c - g^{cd}\partial_b g_{ac}g_{de}\xi^e - g^{cd}\partial_a g_{bc}g_{de}\xi^e + g^{cd}\partial_c g_{ab}g_{de}\xi^e + \partial_a g_{bc}\xi^c + g_{bc}\partial_a\xi^c \tag{ex-0607.106}$$
$$= 0 \tag{ex-0607.110}$$

Exercise 6.08a A problem with `evaluate`

```
1   {\theta, \varphi}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4   \partial{#}::PartialDerivative.
5
6   V_{a}::Depends(\theta,\varphi,\partial{#}).
7
8   dVrule := { \partial_{\theta}{V_{\varphi}} = \sin(\theta),
9            \partial_{\varphi}{V_{\theta}} = \cos(\theta)}.    # cdb(ex-0608.101,dVrule)
10  dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.          # cdb(ex-0608.102,dV)
11
12  evaluate (dV, dVrule)                                      # cdb(ex-0608.103,dV)
```

```
Traceback (most recent call last):
  File "/usr/local/bin/cadabra2", line 248, in <module>
    exec(cmp)
  File "ex-0608.py", line 27, in <module>
    evaluate (dV, dVrule)
RuntimeError: Dependencies on derivatives are not yet handled in the SymPy bridge
```

Exercise 6.08b A work around

```
1   {\theta, \varphi}::Coordinate.
2   {a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).
3
4   \partial{#}::PartialDerivative.
5
6   V_{a}::Depends(\theta,\varphi,\partial{#}).
7
8   hide := \partial_{a}{V_{b}} -> dV_{a b}.
9
10  dVrule := { dV_{\theta\varphi} = \sin(\theta),
11              dV_{\varphi\theta} = \cos(\theta)}.          # cdb(ex-0608.201,dVrule)
12  dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.        # cdb(ex-0608.202,dV)
13
14  substitute (dV, hide)                                   # cdb(ex-0608.212,dV)
15  evaluate (dV, dVrule)                                   # cdb(ex-0608.203,dV)
```

The workaround here is to to hide the derivatives before calling `evaluate`.

$$dV_{ba} - dV_{ab} \tag{ex-0608.212}$$

$$dV_{ab} = \partial_b V_a - \partial_a V_b \tag{ex-0608.202}$$

$$= \Box_{ab} \begin{cases} \Box_{\varphi\theta} = \sin\theta - \cos\theta \\ \Box_{\theta\varphi} = -\sin\theta + \cos\theta \end{cases} \tag{ex-0608.203}$$

Exercise 7.1 C-code for a R_{ab} for a generic metric

```
1    {x,y,z}::Coordinate.
2    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(values={x,y,z},position=independent).
3
4    \partial{#}::PartialDerivative.
5
6    g_{a b}::Metric.
7    g^{a b}::InverseMetric.
8
9    import cdblib
10
11   FourRab = cdblib.get ('FourRab','ex-0309.json')
12
13   Rab := 1/4 @(FourRab).
14
15   substitute (Rab, $ \partial_{a b}{g_{c d}} -> dg_{c d a b} $)
16   substitute (Rab, $ \partial_{a}{g_{b c}}   -> dg_{b c a}   $)
17
18   # ------------------------------------------------------------
19   # build rules to export Cadabra expressions to Python
20   # use known symmetries for g_{a b}, dg_{ab,c,d} etc.
21   # note: replacements must not contain underscores (reserved for subscripts),
22   #       so g_{x x}-> g_xx is not allowed
23
24   gabRule := {g_{x x} -> gxx, g_{x y} -> gxy, g_{x z} -> gxz,
25               g_{y x} -> gxy, g_{y y} -> gyy, g_{y z} -> gyz,
26               g_{z x} -> gxz, g_{z y} -> gyz, g_{z z} -> gzz}.
27
28   iabRule := {g^{x x} -> ixx, g^{x y} -> ixy, g^{x z} -> ixz,
29               g^{y x} -> ixy, g^{y y} -> iyy, g^{y z} -> iyz,
30               g^{z x} -> ixz, g^{z y} -> iyz, g^{z z} -> izz}.
31
32   d1gabRule := {dg_{x x x} -> dgxxx, dg_{x y x} -> dgxyx, dg_{x z x} -> dgxzx,
33                 dg_{y x x} -> dgxyx, dg_{y y x} -> dgyyx, dg_{y z x} -> dgyzx,
34                 dg_{z x x} -> dgxzx, dg_{z y x} -> dgyzx, dg_{z z x} -> dgzzx,
35
36                 dg_{x x y} -> dgxxy, dg_{x y y} -> dgxyy, dg_{x z y} -> dgxzy,
```

```
              dg_{y x y} -> dgxyy, dg_{y y y} -> dgyyy, dg_{y z y} -> dgyzy,
              dg_{z x y} -> dgxzy, dg_{z y y} -> dgyzy, dg_{z z y} -> dgzzy,

              dg_{x x z} -> dgxxz, dg_{x y z} -> dgxyz, dg_{x z z} -> dgxzz,
              dg_{y x z} -> dgxyz, dg_{y y z} -> dgyyz, dg_{y z z} -> dgyzz,
              dg_{z x z} -> dgxzz, dg_{z y z} -> dgyzz, dg_{z z z} -> dgzzz}.

   d2gabRule := {dg_{x x x x} -> dgxxxx, dg_{x y x x} -> dgxyxx, dg_{x z x x} -> dgxzxx,
              dg_{y x x x} -> dgxyxx, dg_{y y x x} -> dgyyxx, dg_{y z x x} -> dgyzxx,
              dg_{z x x x} -> dgxzxx, dg_{z y x x} -> dgyzxx, dg_{z z x x} -> dgzzxx,
              dg_{x x y x} -> dgxxxy, dg_{x y y x} -> dgxyxy, dg_{x z y x} -> dgxzxy,
              dg_{y x y x} -> dgxyxy, dg_{y y y x} -> dgyyxy, dg_{y z y x} -> dgyzxy,
              dg_{z x y x} -> dgxzxy, dg_{z y y x} -> dgyzxy, dg_{z z y x} -> dgzzxy,
              dg_{x x z x} -> dgxxxz, dg_{x y z x} -> dgxyxz, dg_{x z z x} -> dgxzxz,
              dg_{y x z x} -> dgxyxz, dg_{y y z x} -> dgyyxz, dg_{y z z x} -> dgyzxz,
              dg_{z x z x} -> dgxzxz, dg_{z y z x} -> dgyzxz, dg_{z z z x} -> dgzzxz,

              dg_{x x x y} -> dgxxxy, dg_{x y x y} -> dgxyxy, dg_{x z x y} -> dgxzxy,
              dg_{y x x y} -> dgxyxy, dg_{y y x y} -> dgyyxy, dg_{y z x y} -> dgyzxy,
              dg_{z x x y} -> dgxzxy, dg_{z y x y} -> dgyzxy, dg_{z z x y} -> dgzzxy,
              dg_{x x y y} -> dgxxyy, dg_{x y y y} -> dgxyyy, dg_{x z y y} -> dgxzyy,
              dg_{y x y y} -> dgxyyy, dg_{y y y y} -> dgyyyy, dg_{y z y y} -> dgyzyy,
              dg_{z x y y} -> dgxzyy, dg_{z y y y} -> dgyzyy, dg_{z z y y} -> dgzzyy,
              dg_{x x z y} -> dgxxyz, dg_{x y z y} -> dgxyyz, dg_{x z z y} -> dgxzyz,
              dg_{y x z y} -> dgxyyz, dg_{y y z y} -> dgyyyz, dg_{y z z y} -> dgyzyz,
              dg_{z x z y} -> dgxzyz, dg_{z y z y} -> dgyzyz, dg_{z z z y} -> dgzzyz,

              dg_{x x x z} -> dgxxxz, dg_{x y x z} -> dgxyxz, dg_{x z x z} -> dgxzxz,
              dg_{y x x z} -> dgxyxz, dg_{y y x z} -> dgyyxz, dg_{y z x z} -> dgyzxz,
              dg_{z x x z} -> dgxzxz, dg_{z y x z} -> dgyzxz, dg_{z z x z} -> dgzzxz,
              dg_{x x y z} -> dgxxyz, dg_{x y y z} -> dgxyyz, dg_{x z y z} -> dgxzyz,
              dg_{y x y z} -> dgxyyz, dg_{y y y z} -> dgyyyz, dg_{y z y z} -> dgyzyz,
              dg_{z x y z} -> dgxzyz, dg_{z y y z} -> dgyzyz, dg_{z z y z} -> dgzzyz,
              dg_{x x z z} -> dgxxzz, dg_{x y z z} -> dgxyzz, dg_{x z z z} -> dgxzzz,
              dg_{y x z z} -> dgxyzz, dg_{y y z z} -> dgyyzz, dg_{y z z z} -> dgyzzz,
              dg_{z x z z} -> dgxzzz, dg_{z y z z} -> dgyzzz, dg_{z z z z} -> dgzzzz}.

   def write_code (obj,name,filename,rank):
```

```python
    import os

    from sympy.printing.ccode import C99CodePrinter as printer
    from sympy.printing.codeprinter import Assignment

    idx=[]   # indices in the form [{x, x}, {x, y} ...]
    lst=[]   # corresponding terms [termxx, termxy, ...]

    for i in range( len(obj[rank]) ):                    # rank = number of free indices
        idx.append( str(obj[rank][i][0]._sympy_()) )  # indices for this term
        lst.append( str(obj[rank][i][1]._sympy_()) )  # the matching term

    mat = sympy.Matrix([lst])                            # row vector of terms
    sub_exprs, simplified_rhs = sympy.cse(mat)         # optimise code

    with open(os.getcwd() + '/' + filename, 'w') as out:

        for lhs, rhs in sub_exprs:
            out.write(printer().doprint(Assignment(lhs, rhs))+'\n')

        for index, rhs in enumerate (simplified_rhs[0]):
            lhs = sympy.Symbol(name+' '+(idx[index]).replace(', ',',']['))
            out.write(printer().doprint(Assignment(lhs, rhs))+'\n')

evaluate    (Rab,  gabRule+d1gabRule+d2gabRule+iabRule, simplify=False)

write_code (Rab, 'Rab', 'ex-0701-rab.c',2)
```

The code for R_{ab} can be found in the file `ex-0701-rab.c`. It is long and it would require more work to turn it into something useful in a numerical code. For example, functions would be needed to compute the first and second partial derivatives of the metric. But that is not a Cadabra issue.