

Exercise 3.9 Ricci in terms of the metric and its derivatives

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1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \partial{#}::PartialDerivative;
4
5  g_{a b}::Metric;
6  g^{a b}::InverseMetric;
7
8  dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.      # cdb (ex-0309.dgab,dgab)
9
10 Gamma := \Gamma^{a}_{b c} ->
11         (1/2) g^{a e} ( \partial_{b}{g_{e c}}
12                        + \partial_{c}{g_{b e}}
13                        - \partial_{e}{g_{b c}}).      # cdb (ex-0309.Gamma,Gamma)
14
15 Rabcd := R^{a}_{b c d} ->
16         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
17         - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.      # cdb (ex-0309.Rabcd,Rabcd)
18
19 FourRab := 4 R^{c}_{a c b}.      # cdb (ex-0309.101,FourRab)
20
21 substitute      (FourRab, Rabcd)      # cdb (ex-0309.102,FourRab)
22 substitute      (FourRab, Gamma)      # cdb (ex-0309.103,FourRab)
23
24 product_rule    (FourRab)      # cdb (ex-0309.104,FourRab)
25 distribute      (FourRab)      # cdb (ex-0309.105,FourRab)
26
27 substitute      (FourRab, dgab)      # cdb (ex-0309.106,FourRab)
28
29 sort_product    (FourRab)      # cdb (ex-0309.107,FourRab)
30 rename_dummies  (FourRab)      # cdb (ex-0309.108,FourRab)
31 canonicalise    (FourRab)      # cdb (ex-0309.109,FourRab)
32
33 # sort so that g to appeares before dg
34
35 substitute      (FourRab, $g^{a b} -> A^{a b}$)
36 sort_product    (FourRab)

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37 rename_dummies (FourRab)
38 substitute      (FourRab, $A^{a b} -> g^{a b}$)    # cdb (ex-0309.110,FourRab)
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$$\begin{aligned}
4R_{ab} &= 4R_{acb} & (\text{ex-0309.101}) \\
&= 4\partial_c \Gamma_{ab}^c + 4\Gamma_{ec}^c \Gamma_{ab}^e - 4\partial_b \Gamma_{ac}^c - 4\Gamma_{eb}^c \Gamma_{ac}^e & (\text{ex-0309.102}) \\
&= 2\partial_c (g^{ce} (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab})) + g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}) - 2\partial_b (g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac})) \\
&\quad - g^{cd} (\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) & (\text{ex-0309.103}) \\
&= 2\partial_c g^{ce} (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}) + 2g^{ce} \partial_c (\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}) + g^{cd} (\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}) g^{ef} (\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}) \\
&\quad - 2\partial_b g^{ce} (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}) - 2g^{ce} \partial_b (\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}) - g^{cd} (\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}) g^{ef} (\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}) & (\text{ex-0309.104}) \\
&= 2\partial_c g^{ce} \partial_a g_{eb} + 2\partial_c g^{ce} \partial_b g_{ae} - 2\partial_c g^{ce} \partial_e g_{ab} + 2g^{ce} \partial_{ca} g_{eb} + 2g^{ce} \partial_{cb} g_{ae} - 2g^{ce} \partial_{ce} g_{ab} + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} \\
&\quad + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} - 2\partial_b g^{ce} \partial_a g_{ec} \\
&\quad - 2\partial_b g^{ce} \partial_c g_{ae} + 2\partial_b g^{ce} \partial_e g_{ac} - 2g^{ce} \partial_{ba} g_{ec} - 2g^{ce} \partial_{bc} g_{ae} + 2g^{ce} \partial_{be} g_{ac} - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} \\
&\quad - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} & (\text{ex-0309.105}) \\
&= -2g^{cd} g^{ef} \partial_c g_{df} \partial_a g_{eb} - 2g^{cd} g^{ef} \partial_c g_{df} \partial_b g_{ae} + 2g^{cd} g^{ef} \partial_c g_{df} \partial_e g_{ab} + 2g^{ce} \partial_{ca} g_{eb} + 2g^{ce} \partial_{cb} g_{ae} - 2g^{ce} \partial_{ce} g_{ab} + g^{cd} \partial_e g_{dc} g^{ef} \partial_a g_{fb} + g^{cd} \partial_e g_{dc} g^{ef} \partial_b g_{af} \\
&\quad - g^{cd} \partial_e g_{dc} g^{ef} \partial_f g_{ab} + g^{cd} \partial_c g_{ed} g^{ef} \partial_a g_{fb} + g^{cd} \partial_c g_{ed} g^{ef} \partial_b g_{af} - g^{cd} \partial_c g_{ed} g^{ef} \partial_f g_{ab} - g^{cd} \partial_d g_{ec} g^{ef} \partial_a g_{fb} - g^{cd} \partial_d g_{ec} g^{ef} \partial_b g_{af} + g^{cd} \partial_d g_{ec} g^{ef} \partial_f g_{ab} \\
&\quad + 2g^{cd} g^{ef} \partial_b g_{df} \partial_a g_{ec} + 2g^{cd} g^{ef} \partial_b g_{df} \partial_c g_{ae} - 2g^{cd} g^{ef} \partial_b g_{df} \partial_e g_{ac} - 2g^{ce} \partial_{ba} g_{ec} - 2g^{ce} \partial_{bc} g_{ae} + 2g^{ce} \partial_{be} g_{ac} - g^{cd} \partial_e g_{db} g^{ef} \partial_a g_{fc} \\
&\quad - g^{cd} \partial_e g_{db} g^{ef} \partial_c g_{af} + g^{cd} \partial_e g_{db} g^{ef} \partial_f g_{ac} - g^{cd} \partial_b g_{ed} g^{ef} \partial_a g_{fc} - g^{cd} \partial_b g_{ed} g^{ef} \partial_c g_{af} + g^{cd} \partial_b g_{ed} g^{ef} \partial_f g_{ac} + g^{cd} \partial_d g_{eb} g^{ef} \partial_a g_{fc} + g^{cd} \partial_d g_{eb} g^{ef} \partial_c g_{af} \\
&\quad - g^{cd} \partial_d g_{eb} g^{ef} \partial_f g_{ac} & (\text{ex-0309.106}) \\
&= -2\partial_a g_{eb} \partial_c g_{df} g^{cd} g^{ef} - 2\partial_b g_{ae} \partial_c g_{df} g^{cd} g^{ef} + 2\partial_c g_{df} \partial_e g_{ab} g^{cd} g^{ef} + 2\partial_{ca} g_{eb} g^{ce} + 2\partial_{cb} g_{ae} g^{ce} - 2\partial_{ce} g_{ab} g^{ce} + \partial_a g_{fb} \partial_e g_{dc} g^{cd} g^{ef} + \partial_b g_{af} \partial_e g_{dc} g^{cd} g^{ef} \\
&\quad - \partial_e g_{dc} \partial_f g_{ab} g^{cd} g^{ef} + \partial_a g_{fb} \partial_c g_{ed} g^{cd} g^{ef} + \partial_b g_{af} \partial_c g_{ed} g^{cd} g^{ef} - \partial_c g_{ed} \partial_f g_{ab} g^{cd} g^{ef} - \partial_a g_{fb} \partial_d g_{ec} g^{cd} g^{ef} - \partial_b g_{af} \partial_d g_{ec} g^{cd} g^{ef} + \partial_d g_{ec} \partial_f g_{ab} g^{cd} g^{ef} \\
&\quad + 2\partial_a g_{ec} \partial_b g_{df} g^{cd} g^{ef} + 2\partial_b g_{df} \partial_c g_{ae} g^{cd} g^{ef} - 2\partial_b g_{df} \partial_e g_{ac} g^{cd} g^{ef} - 2\partial_{ba} g_{ec} g^{ce} - 2\partial_{bc} g_{ae} g^{ce} + 2\partial_{be} g_{ac} g^{ce} - \partial_a g_{fc} \partial_e g_{db} g^{cd} g^{ef} \\
&\quad - \partial_c g_{af} \partial_e g_{db} g^{cd} g^{ef} + \partial_e g_{db} \partial_f g_{ac} g^{cd} g^{ef} - \partial_a g_{fc} \partial_b g_{ed} g^{cd} g^{ef} - \partial_b g_{ed} \partial_c g_{af} g^{cd} g^{ef} + \partial_b g_{ed} \partial_f g_{ac} g^{cd} g^{ef} + \partial_a g_{fc} \partial_d g_{eb} g^{cd} g^{ef} + \partial_c g_{af} \partial_d g_{eb} g^{cd} g^{ef} \\
&\quad - \partial_d g_{eb} \partial_f g_{ac} g^{cd} g^{ef} & (\text{ex-0309.107}) \\
&= -2\partial_a g_{db} \partial_c g_{ef} g^{ce} g^{df} - 2\partial_b g_{ad} \partial_c g_{ef} g^{ce} g^{df} + 2\partial_c g_{ef} \partial_d g_{ab} g^{ce} g^{df} + 2\partial_{ca} g_{db} g^{cd} + 2\partial_{cb} g_{ad} g^{cd} - 2\partial_{cd} g_{ab} g^{cd} + \partial_a g_{db} \partial_c g_{ef} g^{fe} g^{cd} + \partial_b g_{ad} \partial_c g_{ef} g^{fe} g^{cd} \\
&\quad - \partial_c g_{ef} \partial_d g_{ab} g^{fe} g^{cd} + \partial_a g_{db} \partial_c g_{ef} g^{cf} g^{ed} + \partial_b g_{ad} \partial_c g_{ef} g^{cf} g^{ed} - \partial_c g_{ef} \partial_d g_{ab} g^{cf} g^{ed} - \partial_a g_{db} \partial_c g_{ef} g^{fc} g^{ed} - \partial_b g_{ad} \partial_c g_{ef} g^{fc} g^{ed} + \partial_c g_{ef} \partial_d g_{ab} g^{fc} g^{ed} \\
&\quad + 2\partial_a g_{cd} \partial_b g_{ef} g^{de} g^{cf} + 2\partial_b g_{de} \partial_c g_{af} g^{cd} g^{fe} - 2\partial_b g_{de} \partial_c g_{af} g^{fd} g^{ce} - 2\partial_{ba} g_{cd} g^{dc} - 2\partial_{bc} g_{ad} g^{cd} + 2\partial_{bc} g_{ad} g^{dc} - \partial_a g_{de} \partial_c g_{fb} g^{ef} g^{cd} \\
&\quad - \partial_c g_{ae} \partial_d g_{fb} g^{cf} g^{de} + \partial_c g_{eb} \partial_d g_{af} g^{fe} g^{cd} - \partial_a g_{cd} \partial_b g_{ef} g^{df} g^{ec} - \partial_b g_{de} \partial_c g_{af} g^{ce} g^{df} + \partial_b g_{de} \partial_c g_{af} g^{fe} g^{dc} + \partial_a g_{de} \partial_c g_{fb} g^{ec} g^{fd} + \partial_c g_{ae} \partial_d g_{fb} g^{cd} g^{fe} \\
&\quad - \partial_c g_{eb} \partial_d g_{af} g^{fc} g^{ed} & (\text{ex-0309.108}) \\
&= -2\partial_a g_{bc} \partial_d g_{ef} g^{ce} g^{df} - 2\partial_b g_{ac} \partial_d g_{ef} g^{ce} g^{df} + 2\partial_c g_{ab} \partial_d g_{ef} g^{ce} g^{df} + 2\partial_{ac} g_{bd} g^{cd} + 2\partial_{bc} g_{ad} g^{cd} - 2\partial_{cd} g_{ab} g^{cd} + \partial_a g_{bc} \partial_d g_{ef} g^{cd} g^{ef} + \partial_b g_{ac} \partial_d g_{ef} g^{cd} g^{ef} \\
&\quad - \partial_c g_{ab} \partial_d g_{ef} g^{cd} g^{ef} + \partial_a g_{cd} \partial_b g_{ef} g^{ce} g^{df} - 2\partial_{ab} g_{cd} g^{cd} - 2\partial_c g_{ad} \partial_e g_{bf} g^{cf} g^{de} + 2\partial_c g_{ad} \partial_e g_{bf} g^{ce} g^{df} & (\text{ex-0309.109}) \\
&= -2g^{cd} g^{ef} \partial_a g_{bc} \partial_e g_{df} - 2g^{cd} g^{ef} \partial_b g_{ac} \partial_e g_{df} + 2g^{cd} g^{ef} \partial_c g_{ab} \partial_e g_{df} + 2g^{cd} \partial_{ac} g_{bd} + 2g^{cd} \partial_{bc} g_{ad} - 2g^{cd} \partial_{cd} g_{ab} + g^{cd} g^{ef} \partial_a g_{bc} \partial_d g_{ef} + g^{cd} g^{ef} \partial_b g_{ac} \partial_d g_{ef} \\
&\quad - g^{cd} g^{ef} \partial_c g_{ab} \partial_d g_{ef} + g^{cd} g^{ef} \partial_a g_{ce} \partial_b g_{df} - 2g^{cd} \partial_{ab} g_{cd} - 2g^{cd} g^{ef} \partial_c g_{ae} \partial_f g_{bd} + 2g^{cd} g^{ef} \partial_c g_{ae} \partial_d g_{bf} & (\text{ex-0309.110})
\end{aligned}$$