Exercise 6.5 Schwarzschild spacetime in isotropic coordinates

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{t, r, \theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

Rab := R_{a b} -> R^{c}_{a c b}.

gab := { g_{t t}             = -((2*r-m)/(2*r+m))**2,
         g_{r r}             = (1+m/(2*r))**4,
         g_{\theta\theta}    = r**2 (1+m/(2*r))**4,
         g_{\varphi\varphi} = r**2 \sin(\theta)**2 (1+m/(2*r))**4}.  # cdb(ex-0605.101,gab)

complete   (gab, $g^{a b}$)                                  # cdb(ex-0605.102,gab)

substitute (Rabcd, Gamma)
substitute (Rab, Rabcd)

evaluate   (Gamma, gab, rhsonly=True)                        # cdb(ex-0605.103,Gamma)
evaluate   (Rabcd, gab, rhsonly=True)                        # cdb(ex-0605.104,Rabcd)
evaluate   (Rab,   gab, rhsonly=True)                        # cdb(ex-0605.105,Rab)
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$$\left[g_{tt} = -\left((2r-m)\,(2r+m)^{-1}\right)^2, g_{rr} = \left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\theta\theta} = r^2\left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1+\frac{1}{2}mr^{-1}\right)^4\right] \tag{ex-0605.101}$$

$$\left[g_{tt} = -\left((2r-m)\,(2r+m)^{-1}\right)^2, g_{rr} = \left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\theta\theta} = r^2\left(1+\frac{1}{2}mr^{-1}\right)^4, g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1+\frac{1}{2}mr^{-1}\right)^4, g^{tt} = \left(-m^2-4mr -4r^2\right)\left(m^2-4mr+4r^2\right)^{-1}, g^{rr} = 16r^4\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-1}, g^{\theta\theta} = 16r^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-1}, g^{\varphi\varphi} = 16r^2\left(m^4(\sin\theta)^2+8m^3r(\sin\theta)^2+24m^2r^2(\sin\theta)^2+32mr^3(\sin\theta)^2+16r^4(\sin\theta)^2\right)^{-1}\right] \tag{ex-0605.102}$$

$$\Gamma^a{}_{bc} \to \square_{cb}{}^a \begin{cases} \square_{\varphi r}{}^{\varphi} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{\varphi\theta}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\theta r}{}^{\theta} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{rr}{}^{r} = -2m(r\,(m+2r))^{-1} \\ \square_{tr}{}^{t} = 4m\left(-m^2+4r^2\right)^{-1} \\ \square_{r\varphi}{}^{\varphi} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{\theta\varphi}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{r\theta}{}^{\theta} = (-m+2r)\,(r\,(m+2r))^{-1} \\ \square_{rt}{}^{t} = 4m\left(-m^2+4r^2\right)^{-1} \\ \square_{\varphi\varphi}{}^{r} = r\,(m-2r)\,(\sin\theta)^2(m+2r)^{-1} \\ \square_{\varphi\varphi}{}^{\theta} = -\frac{1}{2}\sin(2\theta) \\ \square_{\theta\theta}{}^{r} = r\,(m-2r)\,(m+2r)^{-1} \\ \square_{tt}{}^{r} = -64mr^4\,(m-2r)\left((m+2r)^3\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)\right)^{-1} \end{cases} \tag{ex-0605.103}$$

$$R^{a}{}_{bcd}$$

$$\to \Box_{db}{}^{a}{}_{c} \begin{cases} \Box_{tt}{}^{r}{}_{r} = 128mr^{3}\left(-m^{2}+4mr-4r^{2}\right)\left(m^{8}+16m^{7}r+112m^{6}r^{2}+448m^{5}r^{3}+1120m^{4}r^{4}+1792m^{3}r^{5}+1792m^{2}r^{6}+1024mr^{7}+256r^{8}\right)^{-1} \\ \Box_{\theta\theta}{}^{r}{}_{r} = -4mr\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{\varphi\varphi}{}^{\theta}{}_{\theta} = 8mr(\sin\theta)^{2}\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{\varphi\varphi}{}^{r}{}_{r} = -4mr(\sin\theta)^{2}\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{tr}{}^{t}{}_{r} = -8m\left(r\left(m^{2}+4mr+4r^{2}\right)\right)^{-1} \\ \Box_{\theta r}{}^{\theta}{}_{r} = 4m\left(r\left(m^{2}+4mr+4r^{2}\right)\right)^{-1} \\ \Box_{\varphi\theta}{}^{\varphi}{}_{\theta} = -8mr\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{\varphi r}{}^{\varphi}{}_{r} = 4m\left(r\left(m^{2}+4mr+4r^{2}\right)\right)^{-1} \\ \Box_{rt}{}^{r}{}_{t} = 128mr^{3}\left(m^{2}-4mr+4r^{2}\right)\left(m^{8}+16m^{7}r+112m^{6}r^{2}+448m^{5}r^{3}+1120m^{4}r^{4}+1792m^{3}r^{5}+1792m^{2}r^{6}+1024mr^{7}+256r^{8}\right)^{-1} \\ \Box_{r\theta}{}^{r}{}_{\theta} = 4mr\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{\theta\varphi}{}^{\theta}{}_{\varphi} = -8mr(\sin\theta)^{2}\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{r\varphi}{}^{r}{}_{\varphi} = 4mr(\sin\theta)^{2}\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{rr}{}^{t}{}_{t} = 8m\left(r\left(m^{2}+4mr+4r^{2}\right)\right)^{-1} \\ \Box_{rr}{}^{\theta}{}_{\theta} = -4m\left(r\left(m^{2}+4mr+4r^{2}\right)\right)^{-1} \\ \Box_{\theta\theta}{}^{\varphi}{}_{\varphi} = 8mr\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{rr}{}^{\varphi}{}_{\varphi} = -4m\left(r\left(m^{2}+4mr+4r^{2}\right)\right)^{-1} \\ \Box_{\varphi\varphi}{}^{t}{}_{t} = -4mr(\sin\theta)^{2}\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{\theta\theta}{}^{t}{}_{t} = -4mr\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{tt}{}^{\varphi}{}_{\varphi} = 64mr^{3}(m-2r)^{2}\left(m^{4}+8m^{3}r+24m^{2}r^{2}+32mr^{3}+16r^{4}\right)^{-2} \\ \Box_{tt}{}^{\theta}{}_{\theta} = 64mr^{3}(m-2r)^{2}\left(m^{4}+8m^{3}r+24m^{2}r^{2}+32mr^{3}+16r^{4}\right)^{-2} \\ \Box_{t\varphi}{}^{t}{}_{\varphi} = 4mr(\sin\theta)^{2}\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{t\theta}{}^{t}{}_{\theta} = 4mr\left(m^{2}+4mr+4r^{2}\right)^{-1} \\ \Box_{\varphi t}{}^{\varphi}{}_{t} = -64mr^{3}(m-2r)^{2}\left(m^{4}+8m^{3}r+24m^{2}r^{2}+32mr^{3}+16r^{4}\right)^{-2} \\ \Box_{\theta t}{}^{\theta}{}_{t} = -64mr^{3}(m-2r)^{2}\left(m^{4}+8m^{3}r+24m^{2}r^{2}+32mr^{3}+16r^{4}\right)^{-2} \end{cases} \tag{ex-0605.104}$$

$$R_{ab} \to 0 \tag{ex-0605.105}$$