

Example 12 Checking the 2nd and 3rd order terms of Calzetta etal.

The following calculations show that my results for the RNC connection agree with those of Calzetta etal. to third order terms.

Note that I take ∇_{ab} to be $\nabla_a(\nabla_b)$.

Note also that $(LCB) R_{abcd} = -(Calzetta) R_{abcd}$. Consequently, I replace R_{abcd} with $-R_{abcd}$ in the Calzetta expressions (done as a Cadabra substitution rule).

This is relatively straightforward. We just apply a few carefully chosen applications of the first and second Bianchi identities.

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,u,v#}::Indices("latin",position=independent).
2 {\mu,\nu,\rho,\sigma,\tau,\lambda,\xi#}::Indices("greek",position=independent).
3
4 \nabla{#}::Derivative.
5
6 g_{a b}::Metric.
7 g^{a b}::InverseMetric.
8 g^{a b}::Weight(label=gnum,value=1).
9
10 \delta{#}::KroneckerDelta.
11
12 R_{a b c d}::RiemannTensor.
13 R_{a b c d}::Depends(\nabla{#}).
14
15 x^{a}::Weight(label=xnum,value=1).
16
17 def add_tags (obj,tag):
18
19     n = 0
20     ans = Ex('0')
21
22     for i in obj.top().terms():
23         foo = obj[i]
24         bah = Ex(tag+'_{'+str(n)+'}')
25         ans := @(ans) + @(bah) @(foo).
26         n = n + 1
27
28     return ans
29
30 def clear_tags (obj,tag):
31
32     ans := @(obj).
33     foo = Ex(tag+'_{a?} -> 1')
34     substitute (ans,foo)
35
36     return ans
37
38 def get_xterm (obj,n):

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39
40     foo := @(obj).
41     bah  = Ex("xnum = " + str(n))
42     distribute (foo)
43     keep_weight (foo, bah)
44
45     return foo
46
47 def get_gterm (obj,n):
48
49     foo := @(obj).
50     bah  = Ex("gnum = " + str(n))
51     distribute (foo)
52     keep_weight (foo, bah)
53
54     return foo
55
56 def product_sort (obj):
57     substitute (obj,$ g^{a b}                -> A001^{a b}                $)
58     substitute (obj,$ x^{a}                  -> A002^{a}                  $)
59     substitute (obj,$ z^{a}                  -> A003^{a}                  $)
60     substitute (obj,$ R_{a b c d}            -> A004_{a b c d}            $)
61     substitute (obj,$ \nabla_{e}\{R_{a b c d}\} -> A005_{a b c d e}        $)
62     substitute (obj,$ \nabla_{e f}\{R_{a b c d}\} -> A006_{a b c d e f}      $)
63     sort_sum      (obj)
64     sort_product  (obj)
65     rename_dummies (obj)
66     substitute (obj,$ A001^{a b}              -> g^{a b}              $)
67     substitute (obj,$ A002^{a}                -> x^{a}                $)
68     substitute (obj,$ A003^{a}                -> z^{a}                $)
69     substitute (obj,$ A004_{a b c d}          -> R_{a b c d}          $)
70     substitute (obj,$ A005_{a b c d e}        -> \nabla_{e}\{R_{a b c d}\} $)
71     substitute (obj,$ A006_{a b c d e f}      -> \nabla_{e f}\{R_{a b c d}\} $)
72
73 def reformat (obj,scaleA,scaleB):
74
75     foo  = Ex(str(scaleA))
76     moo  = Ex(str(scaleB))

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77     bah := @(foo) @(obj) / @(moo).
78
79     distribute      (bah)
80     product_sort    (bah)
81     rename_dummies  (bah)
82     canonicalise     (bah)
83     factor_out       (bah,$g^{c? d?}$)
84     factor_out       (bah,$x^{a?},z^{b?}$)
85     ans := @(moo) @(bah) / @(foo).
86
87     return ans
88
89     # =====
90     # LCB
91
92     import cdblib
93     Gamma = cdblib.get ('Gamma','example-11.json')           # cdb(ex-12.100,Gamma)
94
95     # -----
96     # note: in versions prior to 24 may 2024, the first \Gamma^a shown in example 12
97     #       of the tutorial was actually the *downstairs* version
98     #       this block added to create reformed version of \Gamma^a for the tutorial
99
100    # note that the next line requires careful inspection of the free indices on Gamma
101    # expecting Gamma = \Gamma^a_{bc}
102
103    GammaUp := z^{b} z^{c} @(Gamma).                          # cdb(ex-12.110,GammaUp)
104
105    product_sort (GammaUp)                                     # cdb(ex-12.113,GammaUp)
106
107    checkpoint.append (GammaUp)
108
109    gam1 = get_xterm (GammaUp,1)                               # cdb(ex-12.210,gam1)
110    gam2 = get_xterm (GammaUp,2)                               # cdb(ex-12.211,gam2)
111    gam3 = get_xterm (GammaUp,3)                               # cdb(ex-12.212,gam3)
112
113    gam30 = get_gterm (gam3,1)                                 # cdb(ex-12.213,gam30)
114    gam31 = get_gterm (gam3,2)                                 # cdb(ex-12.214,gam31)

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115
116 gam1 = reformat (gam1, 3,1) # cdb(ex-12.310,gam1)
117 gam2 = reformat (gam2,12,1) # cdb(ex-12.311,gam2)
118
119 gam30 = reformat (gam30,40,1) # cdb(ex-12.312,gam30)
120 gam31 = reformat (gam31,45,2) # cdb(ex-12.313,gam31)
121
122 gam3 := @(gam30) + @(gam31). # cdb(ex-12.314,gam3)
123
124 GammaUp := @(gam1) + @(gam2) + @(gam3). # cdb(ex-12.315,GammaUp)
125
126 checkpoint.append (GammaUp)
127 # -----
128
129 # lower index  $\{a\}$  to  $\{v\}$ 
130
131 Gamma := g_ $\{v\}$  a) @(GammaUp).
132
133 distribute (Gamma)
134 substitute (Gamma, $g_ $\{a\}$  d} g $\{d\}$  b} -> \delta_ $\{a\}$  $\{b\}$  $)
135 eliminate_kronecker (Gamma) # cdb(ex-12.101,Gamma)
136
137 # change free index  $\{v\}$  to  $\{a\}$ 
138
139 foo := tmp_ $\{v\}$  -> @(Gamma). # cdb(ex-12.191,foo)
140 bah := tmp_ $\{a\}$ . # cdb(ex-12.192,bah)
141 substitute (bah, foo) # cdb(ex-12.193,bah)
142
143 Gamma := @(bah). # cdb(ex-12.102,Gamma)
144
145 product_sort (Gamma) # cdb(ex-12.103,Gamma)
146
147 checkpoint.append (Gamma)
148
149 gam1 = get_xterm (Gamma,1) # cdb(ex-12.200,gam1)
150 gam2 = get_xterm (Gamma,2) # cdb(ex-12.201,gam2)
151 gam3 = get_xterm (Gamma,3) # cdb(ex-12.202,gam3)
152

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153 gam30 = get_gterm (gam3,0)           # cdb(ex-12.203,gam30)
154 gam31 = get_gterm (gam3,1)           # cdb(ex-12.204,gam31)
155
156 gam1  = reformat (gam1, 3,1)           # cdb(ex-12.300,gam1)
157 gam2  = reformat (gam2,12,1)           # cdb(ex-12.301,gam2)
158
159 gam30 = reformat (gam30,40,1)           # cdb(ex-12.302,gam30)
160 gam31 = reformat (gam31,45,2)           # cdb(ex-12.303,gam31)
161
162 gam3  := @(gam30) + @(gam31).           # cdb(ex-12.304,gam3)
163
164 Gamma := @(gam1) + @(gam2) + @(gam3).   # cdb(ex-12.305,Gamma)
165
166 checkpoint.append (Gamma)

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$$\begin{aligned}
\text{ex-12.100} := & \frac{1}{3} g^{ad} x^e (R_{bdce} + R_{becd}) + \frac{1}{12} g^{ad} x^e x^f (-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}) \\
& + \frac{1}{40} g^{ad} x^e x^f x^g (-\nabla_{ce} R_{bfdg} - \nabla_{ec} R_{bfdg} + \nabla_{de} R_{bfcg} + \nabla_{ed} R_{bfcg} + 2\nabla_{ef} R_{bdcg} + 2\nabla_{ef} R_{bgcd} - \nabla_{be} R_{cfdg} - \nabla_{eb} R_{cfdg}) \\
& + \frac{1}{45} g^{ad} g^{ef} x^g x^h x^i (4 R_{becg} R_{dhfi} + 4 R_{bgce} R_{dhfi} - 2 R_{bdeg} R_{chfi} - R_{bedg} R_{chfi} + R_{bgde} R_{chfi} - 2 R_{bgeh} R_{cdfi} - R_{bgeh} R_{cfdi} + R_{bgeh} R_{cidf})
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.191} := & tmp_v \\
\rightarrow & \frac{2}{3} x^c z^d z^e R_{vdce} + \frac{1}{3} x^c x^d z^e z^f \nabla_c R_{vedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{vcdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_v R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{vfeg} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{vdeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{vdeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{vc} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{cv} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{vhce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{vecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vceh} R_{dfgi}
\end{aligned}$$

$$\text{ex-12.192} := tmp_a$$

$$\begin{aligned}
\text{ex-12.193} := & \frac{2}{3} x^c z^d z^e R_{adce} + \frac{1}{3} x^c x^d z^e z^f \nabla_c R_{aedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{acdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_a R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{afeg} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{adeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{adeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ac} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ca} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{ahce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{aecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aceh} R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.101} := & \frac{2}{3} x^c z^d z^e R_{vdce} + \frac{1}{3} x^c x^d z^e z^f \nabla_c R_{vedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{vcdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_v R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{vfeg} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{vdeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{vdeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{vc} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{cv} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{vhce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{vecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{vceh} R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.102} := & \frac{2}{3} x^c z^d z^e R_{adce} + \frac{1}{3} x^c x^d z^e z^f \nabla_c R_{aedf} + \frac{1}{6} x^c x^d z^e z^f \nabla_e R_{acdf} + \frac{1}{12} x^c x^d z^e z^f \nabla_a R_{cedf} + \frac{1}{10} x^c x^d x^e z^f z^g \nabla_{cd} R_{afeg} \\
& + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{cf} R_{adeg} + \frac{1}{20} x^c x^d x^e z^f z^g \nabla_{fe} R_{adeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ac} R_{dfeg} + \frac{1}{40} x^c x^d x^e z^f z^g \nabla_{ca} R_{dfeg} \\
& - \frac{4}{45} g^{cd} x^e x^f x^g z^h z^i R_{ahce} R_{dfgi} - \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aech} R_{dfgi} + \frac{8}{45} g^{cd} x^e x^f x^g z^h z^i R_{aecf} R_{dhgi} + \frac{2}{45} g^{cd} x^e x^f x^g z^h z^i R_{aceh} R_{dfgi}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.103} := & \frac{2}{3} x^b z^c z^d R_{acbd} + \frac{1}{6} x^b x^c z^d z^e \nabla_d R_{abce} + \frac{1}{3} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} \\
& + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{10} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{8}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}
\end{aligned}$$

$$\text{ex-12.200} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.201} := \frac{1}{6} x^b x^c z^d z^e \nabla_d R_{abce} + \frac{1}{3} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\begin{aligned}
\text{ex-12.202} := & \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{10} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{8}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}
\end{aligned}$$

$$\text{ex-12.203} := \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{10} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$

$$\text{ex-12.204} := \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{8}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.300} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.301} := \frac{1}{12} x^b x^c z^d z^e (2 \nabla_d R_{abce} + 4 \nabla_b R_{adce} + \nabla_a R_{bdce})$$

$$\text{ex-12.302} := \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf})$$

$$\text{ex-12.303} := \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2 R_{agbd} R_{cefh})$$

$$\begin{aligned}
\text{ex-12.304} := & \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2 R_{agbd} R_{cefh})
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.305} := & \frac{2}{3} x^b z^c z^d R_{acbd} + \frac{1}{12} x^b x^c z^d z^e (2 \nabla_d R_{abce} + 4 \nabla_b R_{adce} + \nabla_a R_{bdce}) \\
& + \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2 R_{agbd} R_{cefh})
\end{aligned}$$

$$\begin{aligned} \text{ex-12.110} := & z^b z^c \left(\frac{1}{3} g^{ad} x^e (R_{bdce} + R_{becd}) + \frac{1}{12} g^{ad} x^e x^f (-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}) \right. \\ & + \frac{1}{40} g^{ad} x^e x^f x^g (-\nabla_{ce} R_{bf dg} - \nabla_{ec} R_{bf dg} + \nabla_{de} R_{bf cg} + \nabla_{ed} R_{bf cg} + 2\nabla_{ef} R_{bd cg} + 2\nabla_{ef} R_{bg cd} - \nabla_{be} R_{cf dg} - \nabla_{eb} R_{cf dg}) \\ & \left. + \frac{1}{45} g^{ad} g^{ef} x^g x^h x^i (4 R_{becg} R_{dh fi} + 4 R_{bg ce} R_{dh fi} - 2 R_{bdeg} R_{ch fi} - R_{bedg} R_{ch fi} + R_{bg de} R_{ch fi} - 2 R_{bge h} R_{cdf i} - R_{bge h} R_{cf di} + R_{bge h} R_{cid f}) \right) \end{aligned}$$

$$\begin{aligned} \text{ex-12.113} := & z^h z^i \left(\frac{1}{3} g^{ab} x^c (R_{hbic} + R_{hcib}) + \frac{1}{12} g^{ab} x^c x^d (2\nabla_c R_{hbid} + \nabla_b R_{hcid} - \nabla_i R_{hcbd} + 2\nabla_c R_{hdib} - \nabla_h R_{icbd}) \right. \\ & + \frac{1}{40} g^{ab} x^c x^d x^e (2\nabla_{cd} R_{hb ie} + \nabla_{bc} R_{hd ie} + \nabla_{cb} R_{hd ie} - \nabla_{ic} R_{hd be} - \nabla_{ci} R_{hd be} + 2\nabla_{cd} R_{he ib} - \nabla_{hc} R_{id be} - \nabla_{ch} R_{id be}) \\ & \left. + \frac{1}{45} g^{ab} g^{cd} x^e x^f x^g (-2 R_{hbce} R_{if dg} + 4 R_{hc ie} R_{bf dg} - R_{hc be} R_{if dg} + 4 R_{he ic} R_{bf dg} + R_{he bc} R_{if dg} - 2 R_{he cf} R_{ib dg} - R_{he cf} R_{id bg} + R_{he cf} R_{ig bd}) \right) \end{aligned}$$

$$\text{ex-12.210} := \frac{1}{3} z^h z^i g^{ab} x^c R_{hbic} + \frac{1}{3} z^h z^i g^{ab} x^c R_{hcib}$$

$$\text{ex-12.211} := \frac{1}{6} z^h z^i g^{ab} x^c x^d \nabla_c R_{hbid} + \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_b R_{hcid} - \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_i R_{hcbd} + \frac{1}{6} z^h z^i g^{ab} x^c x^d \nabla_c R_{hdib} - \frac{1}{12} z^h z^i g^{ab} x^c x^d \nabla_h R_{icbd}$$

$$\begin{aligned} \text{ex-12.212} := & \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{hb ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{bc} R_{hd ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{cb} R_{hd ie} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ic} R_{hd be} \\ & - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ci} R_{hd be} + \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{he ib} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{hc} R_{id be} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ch} R_{id be} \\ & - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hbce} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc ie} R_{bf dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc be} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he ic} R_{bf dg} \\ & + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he bc} R_{if dg} - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ib dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{id bg} + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ig bd} \end{aligned}$$

$$\begin{aligned} \text{ex-12.213} := & \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{hb ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{bc} R_{hd ie} + \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{cb} R_{hd ie} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ic} R_{hd be} \\ & - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ci} R_{hd be} + \frac{1}{20} z^h z^i g^{ab} x^c x^d x^e \nabla_{cd} R_{he ib} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{hc} R_{id be} - \frac{1}{40} z^h z^i g^{ab} x^c x^d x^e \nabla_{ch} R_{id be} \end{aligned}$$

$$\begin{aligned} \text{ex-12.214} := & -\frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hbce} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc ie} R_{bf dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{hc be} R_{if dg} + \frac{4}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he ic} R_{bf dg} \\ & + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he bc} R_{if dg} - \frac{2}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ib dg} - \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{id bg} + \frac{1}{45} z^h z^i g^{ab} g^{cd} x^e x^f x^g R_{he cf} R_{ig bd} \end{aligned}$$

$$\begin{aligned}
\text{ex-12.310} &:= \frac{2}{3} g^{ab} x^c z^d z^e R_{bdce} \\
\text{ex-12.311} &:= \frac{1}{12} g^{ab} x^c x^d z^e z^f (4 \nabla_c R_{bedf} + 2 \nabla_e R_{bcd f} + \nabla_b R_{cedf}) \\
\text{ex-12.312} &:= \frac{1}{40} g^{ab} x^c x^d x^e z^f z^g (4 \nabla_{cd} R_{bfeg} + 2 \nabla_{cf} R_{bdeg} + 2 \nabla_{fc} R_{bdeg} + \nabla_{bc} R_{dfeg} + \nabla_{cb} R_{dfeg}) \\
\text{ex-12.313} &:= \frac{2}{45} g^{ab} g^{cd} x^e x^f x^g z^h z^i (-2 R_{bhce} R_{dfgi} - R_{bech} R_{dfgi} + 4 R_{becf} R_{dhgi} + R_{bceh} R_{dfgi}) \\
\text{ex-12.314} &:= \frac{1}{40} g^{ab} x^c x^d x^e z^f z^g (4 \nabla_{cd} R_{bfeg} + 2 \nabla_{cf} R_{bdeg} + 2 \nabla_{fc} R_{bdeg} + \nabla_{bc} R_{dfeg} + \nabla_{cb} R_{dfeg}) \\
&\quad + \frac{2}{45} g^{ab} g^{cd} x^e x^f x^g z^h z^i (-2 R_{bhce} R_{dfgi} - R_{bech} R_{dfgi} + 4 R_{becf} R_{dhgi} + R_{bceh} R_{dfgi}) \\
\text{ex-12.315} &:= \frac{2}{3} g^{ab} x^c z^d z^e R_{bdce} + \frac{1}{12} g^{ab} x^c x^d z^e z^f (4 \nabla_c R_{bedf} + 2 \nabla_e R_{bcd f} + \nabla_b R_{cedf}) \\
&\quad + \frac{1}{40} g^{ab} x^c x^d x^e z^f z^g (4 \nabla_{cd} R_{bfeg} + 2 \nabla_{cf} R_{bdeg} + 2 \nabla_{fc} R_{bdeg} + \nabla_{bc} R_{dfeg} + \nabla_{cb} R_{dfeg}) \\
&\quad + \frac{2}{45} g^{ab} g^{cd} x^e x^f x^g z^h z^i (-2 R_{bhce} R_{dfgi} - R_{bech} R_{dfgi} + 4 R_{becf} R_{dhgi} + R_{bceh} R_{dfgi})
\end{aligned}$$

```

1  # =====
2  # Calzetta
3  # note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}
4
5  GammaBar := z^{\nu} z^{\rho} (
6      (2/3) R^{\mu}_{\nu\rho\sigma} x^{\sigma}
7      + (1/12) (5 \nabla_{\lambda}\{R^{\mu}_{\nu\rho\sigma}\}
8          + \nabla_{\rho}\{R^{\mu}_{\sigma\nu\lambda}\}) x^{\sigma} x^{\lambda}
9      + (1/6) ( (9/10) \nabla_{\tau\lambda}\{R^{\mu}_{\rho\nu\sigma}\}
10         + (3/20) ( \nabla_{\tau\rho}\{R^{\mu}_{\sigma\nu\lambda}\}
11             + \nabla_{\rho\tau}\{R^{\mu}_{\sigma\nu\lambda}\} )
12         + (1/60) ( 21 R^{\mu}_{\lambda\xi\rho} R^{\xi}_{\sigma\nu\tau}
13             + 48 R^{\mu}_{\xi\rho\lambda} R^{\xi}_{\sigma\nu\tau}
14             - 37 R^{\mu}_{\sigma\xi\lambda} R^{\xi}_{\nu\rho\tau} ) ) x^{\sigma} x^{\lambda} x^{\tau} ).
15      # cdb(ex-12.400,GammaBar)
16
17  # convert from Greek to Latin indices
18
19  distribute (GammaBar)
20  rename_dummies (GammaBar,"greek","latin") # cdb(ex-12.401,GammaBar)
21
22  # lower the \mu index
23
24  GammaBar := \delta_{a \mu} @(GammaBar). # cdb(ex-12.402,GammaBar)
25  distribute (GammaBar) # cdb(ex-12.403,GammaBar)
26  eliminate_kronecker (GammaBar) # cdb(ex-12.404,GammaBar)
27
28  # sort products
29
30  product_sort (GammaBar) # cdb(ex-12.405,GammaBar)
31
32  checkpoint.append (GammaBar)
33
34  # Replace R with - R (Calzetta uses the non-MTW convention for Riemann)
35
36  substitute (GammaBar, $R_{a b c d} -> - R_{a b c d}$) # cdb(ex-12.406,GammaBar)
37  substitute (GammaBar, $R^{\{a}_{b c d} -> - R^{\{a}_{b c d}$) # cdb(ex-12.407,GammaBar)
38

```

```

39  substitute (GammaBar, $R^{a}_{b c d} -> g^{a e} R_{e b c d}$) # cdb(ex-12.408,GammaBar)
40
41  cal1 = get_xterm (GammaBar,1) # cdb(ex-12.500,cal1)
42  cal2 = get_xterm (GammaBar,2) # cdb(ex-12.501,cal2)
43  cal3 = get_xterm (GammaBar,3) # cdb(ex-12.502,cal3)
44
45  cal1 = reformat (cal1,3,1) # cdb(ex-12.600,cal1)
46  cal2 = reformat (cal2,12,1) # cdb(ex-12.601,cal2)
47  # cal3 = reformat (cal3,360,1) # cdb(ex-12.602,cal3)
48
49  cal30 = get_gterm (cal3,0) # cdb(ex-12.602,cal30)
50  cal31 = get_gterm (cal3,1) # cdb(ex-12.603,cal31)
51
52  cal1 = reformat (cal1, 3,1) # cdb(ex-12.604,cal1)
53  cal2 = reformat (cal2,12,1) # cdb(ex-12.605,cal2)
54
55  cal30 = reformat (cal30,40,1) # cdb(ex-12.606,cal30)
56  cal31 = reformat (cal31,360,1) # cdb(ex-12.607,cal31)
57
58  cal3 := @(cal30) + @(cal31). # cdb(ex-12.608,cal3)
59
60  GammaBar := @(cal1) + @(cal2) + @(cal3). # cdb(ex-12.409,GammaBar)
61
62  checkpoint.append (GammaBar)

```

$$\begin{aligned} \text{ex-12.400} := & z^\nu z^\rho \left(\frac{2}{3} R^\mu_{\nu\rho\sigma} x^\sigma + \frac{1}{12} (5 \nabla_\lambda R^\mu_{\nu\rho\sigma} + \nabla_\rho R^\mu_{\sigma\nu\lambda}) x^\sigma x^\lambda \right. \\ & \left. + \frac{1}{6} \left(\frac{9}{10} \nabla_\tau R^\mu_{\rho\nu\sigma} + \frac{3}{20} \nabla_\tau R^\mu_{\sigma\nu\lambda} + \frac{3}{20} \nabla_\rho R^\mu_{\sigma\nu\lambda} + \frac{7}{20} R^\mu_{\lambda\xi\rho} R^\xi_{\sigma\nu\tau} + \frac{4}{5} R^\mu_{\xi\rho\lambda} R^\xi_{\sigma\nu\tau} - \frac{37}{60} R^\mu_{\sigma\xi\lambda} R^\xi_{\nu\rho\tau} \right) x^\sigma x^\lambda x^\tau \right) \end{aligned}$$

$$\begin{aligned} \text{ex-12.401} := & \frac{2}{3} z^a z^b R^\mu_{abc} x^c + \frac{5}{12} z^a z^b \nabla_d R^\mu_{abc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu_{abc} x^a x^c + \frac{3}{20} z^b z^a \nabla_{de} R^\mu_{abc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu_{abc} x^a x^c x^d \\ & + \frac{1}{40} z^b z^d \nabla_{de} R^\mu_{abc} x^a x^c x^e + \frac{7}{120} z^e z^c R^\mu_{abc} R^b_{def} x^d x^a x^f + \frac{2}{15} z^e z^b R^\mu_{abc} R^a_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu_{abc} R^b_{def} x^a x^c x^f \end{aligned}$$

$$\begin{aligned} \text{ex-12.402} := & \delta_{a\mu} \left(\frac{2}{3} z^g z^b R^\mu_{gbc} x^c + \frac{5}{12} z^g z^b \nabla_d R^\mu_{gbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu_{gbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R^\mu_{gbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu_{gbc} x^g x^c x^d \right. \\ & \left. + \frac{1}{40} z^b z^d \nabla_{de} R^\mu_{gbc} x^g x^c x^e + \frac{7}{120} z^e z^c R^\mu_{gbc} R^b_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R^\mu_{gbc} R^g_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu_{gbc} R^b_{def} x^g x^c x^f \right) \end{aligned}$$

$$\begin{aligned} \text{ex-12.403} := & \frac{2}{3} \delta_{a\mu} z^g z^b R^\mu_{gbc} x^c + \frac{5}{12} \delta_{a\mu} z^g z^b \nabla_d R^\mu_{gbc} x^c x^d + \frac{1}{12} \delta_{a\mu} z^b z^d \nabla_d R^\mu_{gbc} x^g x^c + \frac{3}{20} \delta_{a\mu} z^b z^g \nabla_{de} R^\mu_{gbc} x^c x^e x^d + \frac{1}{40} \delta_{a\mu} z^b z^e \nabla_{de} R^\mu_{gbc} x^g x^c x^d \\ & + \frac{1}{40} \delta_{a\mu} z^b z^d \nabla_{de} R^\mu_{gbc} x^g x^c x^e + \frac{7}{120} \delta_{a\mu} z^e z^c R^\mu_{gbc} R^b_{def} x^d x^g x^f + \frac{2}{15} \delta_{a\mu} z^e z^b R^\mu_{gbc} R^g_{def} x^d x^c x^f - \frac{37}{360} \delta_{a\mu} z^d z^e R^\mu_{gbc} R^b_{def} x^g x^c x^f \end{aligned}$$

$$\begin{aligned} \text{ex-12.404} := & \frac{2}{3} z^g z^b R_{agbc} x^c + \frac{5}{12} z^g z^b \nabla_d R_{agbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R_{agbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R_{agbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R_{agbc} x^g x^c x^d \\ & + \frac{1}{40} z^b z^d \nabla_{de} R_{agbc} x^g x^c x^e + \frac{7}{120} z^e z^c R_{agbc} R^b_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R_{agbc} R^g_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R_{agbc} R^b_{def} x^g x^c x^f \end{aligned}$$

$$\begin{aligned} \text{ex-12.405} := & \frac{2}{3} x^b z^c z^d R_{adcb} + \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} + \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\ & + \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g_{bfc} \end{aligned}$$

$$\begin{aligned} \text{ex-12.406} := & -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} - \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\ & - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} + \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g_{efc} - \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g_{cfd} - \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g_{bfc} \end{aligned}$$

$$\begin{aligned} \text{ex-12.407} := & -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} - \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\ & - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g_{bfc} \end{aligned}$$

$$\begin{aligned}
\text{ex-12.408} := & -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_f R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
& - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcf d} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}
\end{aligned}$$

$$\text{ex-12.500} := -\frac{2}{3} x^b z^c z^d R_{adcb}$$

$$\text{ex-12.501} := -\frac{1}{12} x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12} x^b x^c z^d z^e \nabla_e R_{aedb}$$

$$\begin{aligned} \text{ex-12.502} := & -\frac{1}{40} x^b x^c x^d z^e z^f \nabla_f R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} \\ & - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcf d} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc} \end{aligned}$$

$$\text{ex-12.600} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.601} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce})$$

$$\text{ex-12.602} := -\frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb}$$

$$\text{ex-12.603} := -\frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcf d} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}$$

$$\text{ex-12.604} := \frac{2}{3} x^b z^c z^d R_{acbd}$$

$$\text{ex-12.605} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce})$$

$$\text{ex-12.606} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf})$$

$$\text{ex-12.607} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adb g} R_{cefh} + 48 R_{abdg} R_{cefh})$$

$$\text{ex-12.608} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf}) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adb g} R_{cefh} + 48 R_{abdg} R_{cefh})$$

$$\begin{aligned}
\mathbf{ex-12.409} := & \frac{2}{3} x^b z^c z^d R_{acbd} + \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce}) + \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf}) \\
& + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh})
\end{aligned}$$

The fun begins $\Gamma - \bar{\Gamma}$

It's now time to compute the difference $\Gamma - \bar{\Gamma}$. Here it is.

```
1  def reformat_diff (obj):
2
3      distribute (obj)
4
5      obj1  = get_xterm (obj,1)
6      obj2  = get_xterm (obj,2)
7      obj3  = get_xterm (obj,3)
8
9      obj30 = get_gterm (obj3,0)
10     obj31 = get_gterm (obj3,1)
11
12     obj1  = reformat (obj1, 3,1)
13     obj2  = reformat (obj2,12,1)
14
15     obj30 = reformat (obj30,40,1)
16     obj31 = reformat (obj31,360,1)
17
18     obj3  := @(obj30) + @(obj31).
19
20     ans  := @(obj1) + @(obj2) + @(obj3).
21
22     return ans
23
24     # We could use reformat_diff here but instead we'll do it one step at a time so that
25     # we can see exactly what's going on. Later on we will use reformat_diff to do the job.
26
27     diff := @(Gamma) - @(GammaBar).                # cdb(ex-12.diff.100,diff)
28     distribute (diff)
29
30     diff1  = get_xterm (diff,1)                    # cdb(ex-12.diff.200,diff1)
31     diff2  = get_xterm (diff,2)                    # cdb(ex-12.diff.201,diff2)
32     diff3  = get_xterm (diff,3)                    # cdb(ex-12.diff.202,diff3)
33
34     diff30 = get_gterm (diff3,0)                    # cdb(ex-12.diff.203,diff30)
```

```

35 diff31 = get_gterm (diff3,1) # cdb(ex-12.diff.204,diff31)
36
37 diff1 = reformat (diff1, 3,1) # cdb(ex-12.diff.300,diff1)
38 diff2 = reformat (diff2,12,1) # cdb(ex-12.diff.301,diff2)
39
40 diff30 = reformat (diff30,40,1) # cdb(ex-12.diff.302,diff30)
41 diff31 = reformat (diff31,360,1) # cdb(ex-12.diff.303,diff31)
42
43 diff3 := @(diff30) + @(diff31). # cdb(ex-12.diff.304,diff3)
44
45 diff := @(diff1) + @(diff2) + @(diff3). # cdb(ex-12.diff.305,diff)

```

$$\begin{aligned}
\text{ex-12.diff.100} := & \frac{1}{12} x^b x^c z^d z^e (2 \nabla_d R_{abce} + 4 \nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (2 \nabla_{be} R_{acdf} + 2 \nabla_{eb} R_{acdf} + 4 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (R_{abdg} R_{cefh} + 4 R_{adbe} R_{cgfh} - R_{adbg} R_{cefh} - 2 R_{agbd} R_{cefh}) - \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5 \nabla_b R_{adce}) \\
& - \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6 \nabla_{bc} R_{aedf}) - \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh})
\end{aligned}$$

$$\text{ex-12.diff.200} := 0$$

$$\text{ex-12.diff.201} := \frac{1}{12} x^b x^c z^d z^e \nabla_d R_{abce} - \frac{1}{12} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\begin{aligned}
\text{ex-12.diff.202} := & \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}
\end{aligned}$$

$$\text{ex-12.diff.203} := \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$

$$\text{ex-12.diff.204} := -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.diff.300} := 0$$

$$\text{ex-12.diff.301} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce})$$

$$\text{ex-12.diff.302} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf})$$

$$\text{ex-12.diff.303} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh})$$

$$\begin{aligned} \text{ex-12.diff.304} := & \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

$$\begin{aligned} \text{ex-12.diff.305} := & \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

Second order terms

```
1 diff2 = get_xterm (diff,2)
2 diff2 := 12 @(diff2). # cdb (ex-12.701,diff2)
3 distribute (diff2) # cdb (ex-12.702,diff2)
4
5 diff2 = add_tags (diff2,'\\mu') # cdb (ex-12.711,diff2)
6
7 # swap indices on middle term, then apply 2nd Bianchi identity
8
9 zoom (diff2, $\\mu_{1} Q??$) # cdb (ex-12.712,diff2)
10 substitute (diff2, $\\nabla_{b}\\{R_{a d c e}\\} -> - \\nabla_{b}\\{R_{d a c e}\\}$) # cdb (ex-12.713,diff2)
11 unzoom (diff2)
12
13 substitute (diff2, $\\mu_{1} -> \\mu_{0}, \\mu_{2} -> \\mu_{0}$) # cdb (ex-12.714,diff2)
14 substitute (diff2, $\\mu_{0} -> 0$) # cdb (ex-12.715,diff2)
15
16 diff2 = clear_tags (diff2,'\\mu') # cdb (ex-12.716,diff2)
17
18 diff2 := @(diff2) / 12 .
19
20 diff := @(diff1) + @(diff2) + @(diff3).
21
22 diff = reformat_diff (diff) # cdb(ex-12.diff.306,diff)
```

$$\text{ex-12.701} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.702} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.711} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_2 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.712} := \dots - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.713} := \dots + \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.714} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} + \mu_0 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_0 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.715} := 0$$

$$\text{ex-12.716} := 0$$

$$\begin{aligned} \text{ex-12.diff.306} := & \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2 \nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

Third order terms, commute $\nabla\nabla R$ terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3).                                # cdb (ex-12.801,diff3)
3  distribute (diff3)                                    # cdb (ex-12.802,diff3)
4
5  # commutation rule for covariant derivs on Rabcd, see exrecise 3.6
6  # note: \nabla_{a b} defined as \nabla_a \nabla_b
7  CommuteNablaRiemann := \nabla_{f e}(R_{a b c d}) -> \nabla_{e f}(R_{a b c d})
8                                     + g^{u v} R_{u a e f} R_{v b c d}
9                                     + g^{u v} R_{u b e f} R_{a v c d}
10                                    + g^{u v} R_{u c e f} R_{a b v d}
11                                    + g^{u v} R_{u d e f} R_{a b c v}.
12
13 diff3 = add_tags (diff3, '\\mu')                        # cdb (ex-12.901,diff3)
14
15 # commute derivs on Rabcd so that each double deriv is of the form \nabla_{b*}
16
17 substitute (diff3, $\\mu_{3} -> \\mu_{1}$)              # cdb (ex-12.902,diff3)
18
19 zoom (diff3, $\\mu_{1} Q??$)                            # cdb (ex-12.903,diff3)
20 substitute (diff3, CommuteNablaRiemann)                 # cdb (ex-12.904,diff3)
21 unzoom (diff3)
22
23 diff3 = clear_tags (diff3, '\\mu')
24 diff3 := @(diff3) / 360 .
25
26 distribute (diff3)
27 canonicalise (diff3)                                    # cdb (ex-12.905,diff3)
28
29 diff := @(diff1) + @(diff2) + @(diff3).
30
31 diff = reformat_diff (diff)                             # cdb(ex-12.diff.307,diff)

```


$$\begin{aligned} \text{ex-12.801} := & 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27g^{bc} x^d x^e x^f z^g z^h R_{adbeg} R_{cgfh} + 5g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.802} := & 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27g^{bc} x^d x^e x^f z^g z^h R_{adbeg} R_{cgfh} + 5g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.901} := & 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_3 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbeg} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.902} := & 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbeg} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.903} := \dots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \dots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \dots$$

$$\begin{aligned} \text{ex-12.904} := & \dots + 9\mu_1 x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + g^{uv} R_{uabe} R_{vcdf} + g^{uv} R_{ucbe} R_{avdf} + g^{uv} R_{udbe} R_{acvf} + g^{uv} R_{ufbe} R_{acdv}) + \dots \\ & + 9\mu_1 x^b x^c x^d z^e z^f (\nabla_{ba} R_{cedf} + g^{uv} R_{ucba} R_{vedf} + g^{uv} R_{ueba} R_{cvdf} + g^{uv} R_{udba} R_{cevf} + g^{uv} R_{ufba} R_{cedv}) + \dots \end{aligned}$$

$$\begin{aligned} \text{ex-12.905} := & \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abeu} R_{cfdv} - \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abcu} R_{defv} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} \\ & + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbeg} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} \\ & - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.diff.307} := \frac{1}{40} x^b x^c x^d z^e z^f (2\nabla_{be} R_{acdf} - 2\nabla_{bc} R_{aedf} + 2\nabla_{ba} R_{cedf}) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32R_{abdg} R_{cefh} + 32R_{adbg} R_{cefh} - 32R_{agbd} R_{cefh})$$

Third order terms, use 2nd Bianchi identity on $\nabla\nabla R$ terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3).                                     # cdb (ex-12.910,diff3)
3  distribute (diff3)                                         # cdb (ex-12.911,diff3)
4
5  diff3 = add_tags (diff3,'\\mu')                             # cdb (ex-12.912,diff3)
6
7  # swap indices on middle second deriv term, then apply 2nd Bianchi identity
8
9  zoom      (diff3, $\mu_{1} Q??$)                           # cdb (ex-12.913,diff3)
10 substitute (diff3, $\nabla_{b c}\{R_{a e d f}\} \rightarrow - \nabla_{a c}\{R_{b e d f}\}$) # cdb (ex-12.914,diff3)
11 unzoom    (diff3)
12
13 substitute (diff3, $\mu_{1} \rightarrow \mu_{0}, \mu_{2} \rightarrow \mu_{0}$) # cdb (ex-12.915,diff3)
14 substitute (diff3, $\mu_{0} \rightarrow 0$)                  # cdb (ex-12.916,diff3)
15
16 diff3 = clear_tags (diff3,'\\mu')
17 diff3 := @(diff3) / 360 .
18
19 distribute      (diff3)
20 canonicalise    (diff3)                                     # cdb (ex-12.917,diff3)
21
22 diff := @(diff1) + @(diff2) + @(diff3).
23
24 diff = reformat_diff (diff)                                 # cdb(ex-12.diff.308,diff)

```

$$\begin{aligned}
\text{ex-12.910} &:= 18 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.911} &:= 18 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.912} &:= 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18 \mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18 \mu_2 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 \mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.913} &:= \dots - 18 \mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \dots \\
\text{ex-12.914} &:= \dots + 18 \mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + \dots \\
\text{ex-12.915} &:= 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + 18 \mu_0 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32 \mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.916} &:= -32 \mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.917} &:= -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.diff.308} &:= \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 32 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh})
\end{aligned}$$

Third order terms, use 1st Bianchi identity on RR terms

```

1  diff3 = get_xterm (diff,3)
2  diff3 := 360 @(diff3).
3  distribute (diff3)
4
5  diff3 = add_tags (diff3,'\mu') # cdb (ex-12.921,diff3)
6
7  # swap indices on middle term, then apply 1st Bianchi identity
8
9  zoom (diff3, $\mu_{1}$ Q??$) # cdb (ex-12.922,diff3)
10 substitute (diff3, $R_{a d b g} R_{c e f h} \rightarrow - R_{a d g b} R_{c e f h}$) # cdb (ex-12.923,diff3)
11 unzoom (diff3)
12
13 substitute (diff3, $\mu_{1} \rightarrow \mu_{0}$, $\mu_{2} \rightarrow \mu_{0}$) # cdb (ex-12.924,diff3)
14 substitute (diff3, $\mu_{0} \rightarrow 0$) # cdb (ex-12.925,diff3)
15
16 diff3 = clear_tags (diff3,'\mu') # cdb (ex-12.926,diff3)
17
18 diff := @(diff1) + @(diff2) + @(diff3).
19
20 diff = reformat_diff (diff) # cdb(ex-12.diff.309,diff)

```

$$\text{ex-12.921} := -32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32 \mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32 \mu_2 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.922} := \dots + 32 \mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} + \dots$$

$$\text{ex-12.923} := \dots - 32 \mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} + \dots$$

$$\text{ex-12.924} := -32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} - 32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} - 32 \mu_0 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.925} := 0$$

$$\text{ex-12.926} := 0$$

$$\text{ex-12.diff.309} := 0$$