

Exercise 6.5 Schwarzschild spacetime in isotropic coordinates

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1 {t, r, \theta, \varphi}::Coordinate.
2 {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4 \partial{#}::PartialDerivative.
5
6 g^{a b}::InverseMetric. # essential when using complete (gab, $g^{a b}$)
7
8 Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} ( \partial_{b}{g_{d c}}
9                                     + \partial_{c}{g_{b d}}
10                                    - \partial_{d}{g_{b c}}).
11
12 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
13                        - \partial_{d}{\Gamma^{a}_{b c}}
14                        + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                        - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17 Rab := R_{a b} -> R^{c}_{c a b}.
18
19 gab := { g_{t t}          = -((2*r-m)/(2*r+m))**2,
20         g_{r r}          = (1+m/(2*r))**4,
21         g_{\theta\theta}   = r**2 (1+m/(2*r))**4,
22         g_{\varphi\varphi} = r**2 \sin(\theta)**2 (1+m/(2*r))**4}. # cdb(ex-0605.101,gab)
23
24 complete (gab, $g^{a b}$) # cdb(ex-0605.102,gab)
25
26 substitute (Rabcd, Gamma)
27 substitute (Rab, Rabcd)
28
29 evaluate (Gamma, gab, rhsonly=True) # cdb(ex-0605.103,Gamma)
30 evaluate (Rabcd, gab, rhsonly=True) # cdb(ex-0605.104,Rabcd)
31 evaluate (Rab, gab, rhsonly=True) # cdb(ex-0605.105,Rab)

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$$\left[g_{tt} = -((2r - m)(2r + m)^{-1})^2, \quad g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g_{\theta\theta} = r^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1 + \frac{1}{2}mr^{-1}\right)^4 \right] \quad (\text{ex-0605.101})$$

$$\begin{aligned} & \left[g_{tt} = -((2r - m)(2r + m)^{-1})^2, \quad g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g_{\theta\theta} = r^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \right. \\ & g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \quad g^{tt} = (-m^2 - 4mr - 4r^2)(m^2 - 4mr + 4r^2)^{-1}, \\ & g^{rr} = 16r^4(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-1}, \quad g^{\theta\theta} = 16r^2(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-1}, \\ & \left. g^{\varphi\varphi} = 16r^2(m^4(\sin\theta)^2 + 8m^3r(\sin\theta)^2 + 24m^2r^2(\sin\theta)^2 + 32mr^3(\sin\theta)^2 + 16r^4(\sin\theta)^2)^{-1} \right] \quad (\text{ex-0605.102}) \end{aligned}$$

$$\Gamma^a_{bc} \rightarrow \square_{cb}^a \left\{ \begin{array}{l} \square_{\varphi r}^{\varphi} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{\varphi\theta}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\theta r}^{\theta} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{rr}^r = -2m(r(m + 2r))^{-1} \\ \square_{tr}^t = 4m(-m^2 + 4r^2)^{-1} \\ \square_{r\varphi}^{\varphi} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{\theta\varphi}^{\varphi} = (\tan\theta)^{-1} \\ \square_{r\theta}^{\theta} = (-m + 2r)(r(m + 2r))^{-1} \\ \square_{rt}^t = 4m(-m^2 + 4r^2)^{-1} \\ \square_{\varphi\varphi}^r = r(m - 2r)(\sin\theta)^2(m + 2r)^{-1} \\ \square_{\varphi\varphi}^{\theta} = -\frac{1}{2}\sin(2\theta) \\ \square_{\theta\theta}^r = r(m - 2r)(m + 2r)^{-1} \\ \square_{tt}^r = -64mr^4(m - 2r)((m + 2r)^3(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4))^{-1} \end{array} \right. \quad (\text{ex-0605.103})$$

$R^a{}_{bcd}$

$$\rightarrow \square_{db}{}^a{}_c \left\{ \begin{array}{l} \square_{tt}{}^r{}_r = 128mr^3(-m^2 + 4mr - 4r^2)(m^8 + 16m^7r + 112m^6r^2 + 448m^5r^3 + 1120m^4r^4 + 1792m^3r^5 + 1792m^2r^6 + 1024mr^7 + 256r^8)^{-1} \\ \square_{\theta\theta}{}^r{}_r = -4mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\varphi\varphi}{}^\theta{}_\theta = 8mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\varphi\varphi}{}^r{}_r = -4mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{tr}{}^t{}_r = -8m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\theta r}{}^\theta{}_r = 4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\varphi\theta}{}^\varphi{}_\theta = -8mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\varphi r}{}^\varphi{}_r = 4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{rt}{}^r{}_t = 128mr^3(m^2 - 4mr + 4r^2)(m^8 + 16m^7r + 112m^6r^2 + 448m^5r^3 + 1120m^4r^4 + 1792m^3r^5 + 1792m^2r^6 + 1024mr^7 + 256r^8)^{-1} \\ \square_{r\theta}{}^r{}_\theta = 4mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\theta\varphi}{}^\theta{}_\varphi = -8mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{r\varphi}{}^r{}_\varphi = 4mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{rr}{}^t{}_t = 8m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{rr}{}^\theta{}_\theta = -4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\theta\theta}{}^\varphi{}_\varphi = 8mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{rr}{}^\varphi{}_\varphi = -4m(r(m^2 + 4mr + 4r^2))^{-1} \\ \square_{\varphi\varphi}{}^t{}_t = -4mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\theta\theta}{}^t{}_t = -4mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{tt}{}^\varphi{}_\varphi = 64mr^3(m - 2r)^2(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-2} \\ \square_{tt}{}^\theta{}_\theta = 64mr^3(m - 2r)^2(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-2} \\ \square_{t\varphi}{}^t{}_\varphi = 4mr(\sin\theta)^2(m^2 + 4mr + 4r^2)^{-1} \\ \square_{t\theta}{}^t{}_\theta = 4mr(m^2 + 4mr + 4r^2)^{-1} \\ \square_{\varphi t}{}^\varphi{}_t = -64mr^3(m - 2r)^2(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-2} \\ \square_{\theta t}{}^\theta{}_t = -64mr^3(m - 2r)^2(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4)^{-2} \end{array} \right. \quad (\text{ex-0605.104})$$

 $R_{ab} \rightarrow 0$
 (ex-0605.105)