Exercise 3.3 Computing R_{abcd}

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1    {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3    \partial{#}::PartialDerivative.
4
5    \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
6    \Gamma_{a b c}::TableauSymmetry(shape={2}, indices={1,2}).
7
8    dgab := \partial_{c}{g_{a b}} ->   \Gamma^{d}_{a c} g_{d b}
9                                     + \Gamma^{d}_{b c} g_{a d}.        # cdb(dgab.000,dgab)
10
11   RabcdU := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
12                              - \partial_{d}{\Gamma^{a}_{b c}}
13                              + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
14                              - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.    # cdb(Rabcd.000,RabcdU)
15
16   GammaD := {g_{a e} \Gamma^{e}_{b c} -> \Gamma_{a b c},
17             g_{e a} \Gamma^{e}_{b c} -> \Gamma_{a b c}}.             # cdb(Gamma.010,GammaD)
18
19   RabcdD := R_{a b c d} -> g_{a e} R^{e}_{b c d}.                    # cdb(Rabcd.010,RabcdD)
20
21   gabDGamma := g_{a e} \partial_{c}{\Gamma^{e}_{b d}} ->
22                   \partial_{c}{g_{a e} \Gamma^{e}_{b d}}
23                 - \Gamma^{e}_{b d} \partial_{c}{g_{a e}}.           # cdb(gabDGamma.000,gabDGamma)
24
25   # this pair of rules needed to sort \Gamma_{a b c} to the very left
26   # this helps canonicalise spot the terms that cancel
27   bah := \Gamma_{a b c} -> A_{a b c}.
28   foo := A_{a b c} -> \Gamma_{a b c}.
29
30   expr := R_{a b c d}.                                              # cdb(ex-0303.101,expr)
31
32   substitute      (expr, RabcdD)                                    # cdb(ex-0303.102,expr)
33   substitute      (expr, RabcdU)                                    # cdb(ex-0303.103,expr)
34   distribute      (expr)                                            # cdb(ex-0303.104,expr)
35   substitute      (expr, gabDGamma)                                 # cdb(ex-0303.105,expr)
36   substitute      (expr, dgab)                                      # cdb(ex-0303.106,expr)
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37   substitute      (expr, GammaD)                                    # cdb(ex-0303.107,expr)
38   distribute      (expr)                                            # cdb(ex-0303.109,expr)
39   substitute      (expr, bah)                                       # cdb(ex-0303.110,expr)
40   sort_product    (expr)                                            # cdb(ex-0303.111,expr)
41   rename_dummies  (expr)                                            # cdb(ex-0303.112,expr)
42   substitute      (expr, foo)                                       # cdb(ex-0303.113,expr)
43   canonicalise    (expr)                                            # cdb(ex-0303.114,expr)
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$$R_{abcd} = g_{ae} R^e{}_{bcd} \tag{ex-0303.102}$$

$$= g_{ae} \left(\partial_c \Gamma^e{}_{bd} - \partial_d \Gamma^e{}_{bc} + \Gamma^f{}_{bd} \Gamma^e{}_{cf} - \Gamma^f{}_{bc} \Gamma^e{}_{df} \right) \tag{ex-0303.103}$$

$$= g_{ae} \partial_c \Gamma^e{}_{bd} - g_{ae} \partial_d \Gamma^e{}_{bc} + g_{ae} \Gamma^f{}_{bd} \Gamma^e{}_{cf} - g_{ae} \Gamma^f{}_{bc} \Gamma^e{}_{df} \tag{ex-0303.104}$$

$$= \partial_c \left(g_{ae} \Gamma^e{}_{bd} \right) - \Gamma^e{}_{bd} \partial_c g_{ae} - \partial_d \left(g_{ae} \Gamma^e{}_{bc} \right) + \Gamma^e{}_{bc} \partial_d g_{ae} + g_{ae} \Gamma^f{}_{bd} \Gamma^e{}_{cf} - g_{ae} \Gamma^f{}_{bc} \Gamma^e{}_{df} \tag{ex-0303.105}$$

$$= \partial_c \left(g_{ae} \Gamma^e{}_{bd} \right) - \Gamma^e{}_{bd} \left(\Gamma^f{}_{ac} g_{fe} + \Gamma^f{}_{ec} g_{af} \right) - \partial_d \left(g_{ae} \Gamma^e{}_{bc} \right) + \Gamma^e{}_{bc} \left(\Gamma^f{}_{ad} g_{fe} + \Gamma^f{}_{ed} g_{af} \right) + g_{ae} \Gamma^f{}_{bd} \Gamma^e{}_{cf} - g_{ae} \Gamma^f{}_{bc} \Gamma^e{}_{df} \tag{ex-0303.106}$$

$$= \partial_c \Gamma_{abd} - \Gamma^e{}_{bd} \left(\Gamma_{eac} + \Gamma_{aec} \right) - \partial_d \Gamma_{abc} + \Gamma^e{}_{bc} \left(\Gamma_{ead} + \Gamma_{aed} \right) + \Gamma_{acf} \Gamma^f{}_{bd} - \Gamma_{adf} \Gamma^f{}_{bc} \tag{ex-0303.107}$$

$$= \partial_c \Gamma_{abd} - \Gamma^e{}_{bd} \Gamma_{eac} - \Gamma^e{}_{bd} \Gamma_{aec} - \partial_d \Gamma_{abc} + \Gamma^e{}_{bc} \Gamma_{ead} + \Gamma^e{}_{bc} \Gamma_{aed} + \Gamma_{acf} \Gamma^f{}_{bd} - \Gamma_{adf} \Gamma^f{}_{bc} \tag{ex-0303.109}$$

$$= \partial_c A_{abd} - \Gamma^e{}_{bd} A_{eac} - \Gamma^e{}_{bd} A_{aec} - \partial_d A_{abc} + \Gamma^e{}_{bc} A_{ead} + \Gamma^e{}_{bc} A_{aed} + A_{acf} \Gamma^f{}_{bd} - A_{adf} \Gamma^f{}_{bc} \tag{ex-0303.110}$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e{}_{bd} - A_{aec} \Gamma^e{}_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e{}_{bc} + A_{aed} \Gamma^e{}_{bc} + A_{acf} \Gamma^f{}_{bd} - A_{adf} \Gamma^f{}_{bc} \tag{ex-0303.111}$$

$$= \partial_c A_{abd} - A_{eac} \Gamma^e{}_{bd} - A_{aec} \Gamma^e{}_{bd} - \partial_d A_{abc} + A_{ead} \Gamma^e{}_{bc} + A_{aed} \Gamma^e{}_{bc} + A_{ace} \Gamma^e{}_{bd} - A_{ade} \Gamma^e{}_{bc} \tag{ex-0303.112}$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e{}_{bd} - \Gamma_{aec} \Gamma^e{}_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e{}_{bc} + \Gamma_{aed} \Gamma^e{}_{bc} + \Gamma_{ace} \Gamma^e{}_{bd} - \Gamma_{ade} \Gamma^e{}_{bc} \tag{ex-0303.113}$$

$$= \partial_c \Gamma_{abd} - \Gamma_{eac} \Gamma^e{}_{bd} - \partial_d \Gamma_{abc} + \Gamma_{ead} \Gamma^e{}_{bc} \tag{ex-0303.114}$$