Example 14 The Weyl tensor is conformally invariant

This example shows that the Weyl tensor is conformally invariant. That is, for a pair of metrics g and $\overline{g}$ related by a conformal transformation, $\overline{g}_{ab} = \phi g_{ab}$ then $\overline{C}^{a}_{bcd} = C^{a}_{bcd}$ or equally $\overline{C}_{abcd} = \phi C_{abcd}$.

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices(position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.

GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                            + \partial_{c}{g_{b d}}
                                            - \partial_{d}{g_{b c}}).

GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
                                   + \partial_{c}{g_{b a}}
                                   - \partial_{a}{g_{b c}}).

Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
                        - \partial_{d}{\Gamma_{a b c}}
                        + \Gamma_{e a d} \Gamma^{e}_{b c}
                        - \Gamma_{e a c} \Gamma^{e}_{b d}.

Rab     := R_{a b} -> g^{c d} R_{a c b d}.

Rscalar := R -> g^{a b} R_{a b}.

# Weyl in 4-dimensions

Cabcd := R_{a b c d} - (1/2) (R_{a c} g_{b d} - R_{a d} g_{b c})
                     - (1/2) (g_{a c} R_{b d} - g_{a d} R_{b c})
                     + (R/6) (g_{a c} g_{b d} - g_{a d} g_{b c}).

{\partial_{a b}{\phi},\partial_{a}{\phi},\phi}::SortOrder.
{\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.
```

```
substitute (Cabcd,Rscalar)
substitute (Cabcd,Rab)
substitute (Cabcd,Rabcd)
substitute (Cabcd,GammaU)
substitute (Cabcd,GammaD)

distribute     (Cabcd)

sort_product   (Cabcd)
rename_dummies (Cabcd)
canonicalise   (Cabcd)

# this is the Weyl tensor on the base metric
baseC := @(Cabcd).

conformal := {g_{a b} -> \phi g_{a b}, g^{a b} -> (1/phi) g^{a b}}.

substitute     (Cabcd, conformal)
product_rule   (Cabcd)
distribute     (Cabcd)
product_rule   (Cabcd)
distribute     (Cabcd)

map_sympy      (Cabcd, "simplify")

sort_product   (Cabcd)
rename_dummies (Cabcd)
canonicalise   (Cabcd)

# this is the Weyl tensor on the conformal metric
confC := @(Cabcd).

# their difference, should be zero
diff := @(confC) - \phi @(baseC).  # cdb (ex-14.diff.100,diff)

distribute     (diff)
sort_product   (diff)
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```
rename_dummies (diff)
canonicalise   (diff)  # cdb (ex-14.diff.101,diff)

# this trick is not essential but it does reduce the number of terms in diff
substitute     (diff,$\partial_{a}{\partial_{b}{g_{c d}}} -> g_{c d b a}$)
substitute     (diff,$\partial_{a}{g_{b c}} -> 0$)
substitute     (diff,$g_{c d b a} -> \partial_{a}{\partial_{b}{g_{c d}}}$)  # cdb (ex-14.diff.102,diff)

# standard expressions in 4-d
substitute     (diff,$g_{a b} g^{a b} -> 4$,repeat=True)          # cdb (ex-14.diff.201,diff)
substitute     (diff,$g_{a b} g^{c b} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.202,diff)
substitute     (diff,$g_{b a} g^{b c} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.203,diff)
substitute     (diff,$g_{a}^{a} -> 4$,repeat=True)                # cdb (ex-14.diff.204,diff)
substitute     (diff,$g^{a}_{a} -> 4$,repeat=True)                # cdb (ex-14.diff.205,diff)
eliminate_kronecker (diff)                                        # cdb (ex-14.diff.206,diff)

# need a second round since the above block introduces new terms that match those just eliminated
substitute     (diff,$g_{a b} g^{a b} -> 4$,repeat=True)          # cdb (ex-14.diff.301,diff)
substitute     (diff,$g_{a b} g^{c b} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.302,diff)
substitute     (diff,$g_{b a} g^{b c} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.303,diff)
substitute     (diff,$g_{a}^{a} -> 4$,repeat=True)                # cdb (ex-14.diff.304,diff)
substitute     (diff,$g^{a}_{a} -> 4$,repeat=True)                # cdb (ex-14.diff.305,diff)
eliminate_kronecker (diff)                                        # cdb (ex-14.diff.306,diff)

sort_product   (diff)
rename_dummies (diff)
canonicalise   (diff)  # cdb (ex-14.diff.400,diff)

checkpoint.append (baseC)
checkpoint.append (confC)
```

$$
\begin{aligned}
\Delta = {} & \frac{1}{2}\,\partial_{bc}\phi g_{ad} - \frac{1}{2}\,\partial_{ac}\phi g_{bd} - \frac{1}{2}\,\partial_{bd}\phi g_{ac} + \frac{1}{2}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{be}g_{df}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g^{ef} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ae}g_{cf}g^{ef} \\
& - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g^{ef} - \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g^{ef} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{be}g_{cf}g^{ef} \\
& + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g^{ef} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ae}g_{df}g^{ef} + \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g^{ef} + \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g^{ef} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g^{ef} \\
& - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{ce}\phi g_{af}g_{bd}g^{ef} + \frac{1}{4}\,\partial_{ae}\phi g_{bd}g_{ef}g^{ef} + \frac{1}{2}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g_{gh}g^{ef}g^{gh} - \frac{1}{2}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\,\partial_{de}\phi g_{af}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{ad}\phi g_{bc}g_{ef}g^{ef} - \frac{1}{2}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\,\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} \\
& - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g_{gh}g^{ef}g^{gh} + \frac{1}{2}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\,\partial_{de}\phi g_{ac}g_{bf}g^{ef} + \frac{1}{4}\,\partial_{bd}\phi g_{ac}g_{ef}g^{ef} \\
& - \frac{1}{4}\,\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{8}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\,\partial_{ce}\phi g_{ad}g_{bf}g^{ef} - \frac{1}{4}\,\partial_{bc}\phi g_{ad}g_{ef}g^{ef} \\
& + \frac{1}{4}\,\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{8}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} + \frac{1}{6}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} - \frac{1}{6}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} \\
& - \frac{1}{24}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij} + \frac{1}{24}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij}
\end{aligned}
\tag{ex-14.diff.102}
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$$
\begin{aligned}
\Delta = &-\frac{1}{2}\,\partial_{bc}\phi g_{ad} + \frac{1}{2}\,\partial_{ac}\phi g_{bd} + \frac{1}{2}\,\partial_{bd}\phi g_{ac} - \frac{1}{2}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{be}g_{df}g^{ef} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g^{ef} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ae}g_{cf}g^{ef} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g^{ef} \\
&+ \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g^{ef} + \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g^{ef} - \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{be}g_{cf}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g^{ef} \\
&- \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ae}g_{df}g^{ef} - \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g^{ef} + \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{ce}\phi g_{af}g_{bd}g^{ef} \\
&- \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} \\
&+ \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{de}\phi g_{af}g_{bc}g^{ef} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} \\
&+ \frac{1}{4}\,\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g_{gh}g^{eg}g^{fh} \\
&- \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{de}\phi g_{ac}g_{bf}g^{ef} - \frac{1}{4}\,\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} \\
&- \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{ce}\phi g_{ad}g_{bf}g^{ef} \\
&+ \frac{1}{4}\,\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
&- \frac{1}{4}\,\partial_{e}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
&+ \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{8}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj}
\end{aligned}
\tag{ex-14.diff.201}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\,\partial_{b}\partial_{c}\phi g_{ad} + \frac{1}{2}\,\partial_{a}\partial_{c}\phi g_{bd} + \frac{1}{2}\,\partial_{b}\partial_{d}\phi g_{ac} - \frac{1}{2}\,\partial_{a}\partial_{d}\phi g_{bc} + \frac{1}{4}\,\partial_{a}\phi\partial_{c}\phi\phi^{-1}g_{be}g_{d}^{e} + \frac{1}{4}\,\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{d}^{e} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ae}g_{c}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{a}^{e}g_{bc} \\
&+ \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{c}^{e} + \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{b}^{e} - \frac{1}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{be}g_{c}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{c}^{e} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ae}g_{d}^{e} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{a}^{e}g_{bd} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{d}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{b}^{e} + \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{c}\phi g_{a}^{e}g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{a}\phi g_{bd}g_{c}^{e} \\
&- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{ef}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bd}g_{c}^{g}g^{ef} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{a}^{e}g_{bd}g_{c}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{cf}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bd}g_{g}^{f}g^{eg} \\
&+ \frac{1}{6}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{de}\phi g_{a}^{e}g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\,\partial_{ac}\phi g_{bc}g_{d}^{e} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{ef}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bc}g_{d}^{g}g^{ef} \\
&- \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{a}^{e}g_{bc}g_{d}^{f} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{df}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{6}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{de}\phi g_{ac}g_{b}^{e} - \frac{1}{4}\,\partial_{be}\phi g_{ac}g_{d}^{e} \\
&- \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{ef}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bg}g_{d}^{g}g^{ef} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{b}^{e}g_{d}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{df}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{bf}g_{g}^{f}g^{eg} \\
&+ \frac{1}{4}\,\partial_{ce}\phi g_{ad}g_{b}^{e} + \frac{1}{4}\,\partial_{be}\phi g_{ad}g_{c}^{e} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{ef}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bg}g_{c}^{g}g^{ef} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{b}^{e}g_{c}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{cf}g_{g}^{f}g^{eg} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{bf}g_{g}^{f}g^{eg} + \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g_{g}^{f}g^{eg} - \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{i}^{h}g^{eg}g^{fi} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{i}^{h}g^{eg}g^{fi} \\
&+ \frac{1}{8}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{i}^{h}g^{ef}g^{gi} - \frac{1}{8}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{i}^{h}g^{ef}g^{gi}
\end{aligned}
\tag{ex-14.diff.202}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\,\partial_{bc}\phi g_{ad} + \frac{1}{2}\,\partial_{ac}\phi g_{bd} + \frac{1}{2}\,\partial_{bd}\phi g_{ac} - \frac{1}{2}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{be}g_{d}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bc}g_{d}^{e} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ae}g_{c}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{a}^{e}g_{bc} \\
&+ \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{c}^{e} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{b}^{e} - \frac{1}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{be}g_{c}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{c}^{e} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ae}g_{d}^{e} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{a}^{e}g_{bd} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{d}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{b}^{e} + \frac{1}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{c}\phi g_{a}^{e}g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{ac}\phi g_{bd}g_{c}^{e} \\
&- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{f}^{g}g_{g}^{f} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bd}g_{c}^{g}g^{ef} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{a}^{e}g_{bd}g_{c}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{cf}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{af}g_{bd}g_{g}^{f}g^{eg} \\
&+ \frac{1}{6}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{dc}\phi g_{a}^{e}g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\,\partial_{ac}\phi g_{bc}g_{d}^{e} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{f}^{g}g_{g}^{f} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bc}g_{d}^{g}g^{ef} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{a}^{e}g_{bc}g_{d}^{f} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bc}g_{df}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{af}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{6}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{dc}\phi g_{ac}g_{b}^{e} - \frac{1}{4}\,\partial_{bc}\phi g_{ac}g_{d}^{e} \\
&- \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{f}^{g}g_{g}^{f} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bg}g_{d}^{g}g^{ef} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{b}^{e}g_{d}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{df}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{bf}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{cc}\phi g_{ad}g_{b}^{e} \\
&+ \frac{1}{4}\,\partial_{bc}\phi g_{ad}g_{c}^{e} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{f}^{g}g_{g}^{f} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bg}g_{c}^{g}g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{b}^{e}g_{c}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{cf}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{bf}g_{g}^{f}g^{eg} \\
&+ \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g_{g}^{f}g^{eg} - \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{i}^{h}g^{eg}g^{fi} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{i}^{h}g^{eg}g^{fi} + \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{h}^{i}g_{i}^{h}g^{ef} \\
&- \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{h}^{i}g_{i}^{h}g^{ef}
\end{aligned}
\tag{ex-14.diff.203}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\,\partial_{bc}\phi g_{ad} + \frac{1}{2}\,\partial_{ac}\phi g_{bd} + \frac{1}{2}\,\partial_{bd}\phi g_{ac} - \frac{1}{2}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{be}g_{d}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bc}g_{d}^{e} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ae}g_{c}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{a}^{e}g_{bc} \\
&+ \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{c}^{e} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{b}^{e} - \frac{1}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{be}g_{c}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{c}^{e} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ae}g_{d}^{e} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{a}^{e}g_{bd} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{d}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{b}^{e} + \frac{1}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{c}\phi g_{a}^{e}g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{ac}\phi g_{bd}g_{c}^{e} \\
&- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{f}^{g}g_{g}^{f} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bd}g_{c}^{g}g^{ef} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{a}^{e}g_{bd}g_{c}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{cf}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{af}g_{bd}g_{g}^{f}g^{eg} \\
&+ \frac{1}{6}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{dc}\phi g_{a}^{e}g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\,\partial_{ac}\phi g_{bc}g_{d}^{e} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{f}^{g}g_{g}^{f} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bc}g_{d}^{g}g^{ef} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{a}^{e}g_{bc}g_{d}^{f} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bc}g_{df}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{af}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{6}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{dc}\phi g_{ac}g_{b}^{e} - \frac{1}{4}\,\partial_{bc}\phi g_{ac}g_{d}^{e} \\
&- \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{f}^{g}g_{g}^{f} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bg}g_{d}^{g}g^{ef} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{b}^{e}g_{d}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{df}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{bf}g_{g}^{f}g^{eg} + \frac{1}{4}\,\partial_{cc}\phi g_{ad}g_{b}^{e} \\
&+ \frac{1}{4}\,\partial_{bc}\phi g_{ad}g_{c}^{e} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{f}^{g}g_{g}^{f} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bg}g_{c}^{g}g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{b}^{e}g_{c}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{cf}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{bf}g_{g}^{f}g^{eg} \\
&+ \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g_{g}^{f}g^{eg} - \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g_{g}^{f}g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{i}^{h}g^{eg}g^{fi} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{i}^{h}g^{eg}g^{fi} + \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{h}^{i}g_{i}^{h}g^{ef} \\
&- \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{h}^{i}g_{i}^{h}g^{ef}
\end{aligned}
\qquad \texttt{(ex-14.diff.204)}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\,\partial_{bc}\phi g_{ad} + \frac{1}{2}\,\partial_{ac}\phi g_{bd} + \frac{1}{2}\,\partial_{bd}\phi g_{ac} - \frac{1}{2}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{be} g_{d}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bc} g_{d}^{e} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ae} g_{c}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{a}^{e} g_{bc} \\
&+ \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{c}^{e} + \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{b}^{e} - \frac{1}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{be} g_{c}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{c}^{e} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ae} g_{d}^{e} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1} g_{a}^{e} g_{bd} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ac} g_{d}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{ac} g_{b}^{e} + \frac{1}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\,\partial_{c}\phi g_{a}^{e} g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac} g_{bd} g^{ef} - \frac{1}{4}\,\partial_{ac}\phi g_{bd} g_{c}^{e} \\
&- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}^{g} g_{g}^{f} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bd} g_{c}^{g} g^{ef} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{a}^{e} g_{bd} g_{c}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{cf} g_{g}^{f} g^{eg} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1} g_{af} g_{bd} g_{g}^{f} g^{eg} \\
&+ \frac{1}{6}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{g}^{f} g^{eg} + \frac{1}{4}\,\partial_{dc}\phi g_{a}^{e} g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{4}\,\partial_{ac}\phi g_{bc} g_{d}^{e} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}^{g} g_{g}^{f} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bc} g_{d}^{g} g^{ef} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{a}^{e} g_{bc} g_{d}^{f} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bc} g_{df} g_{g}^{f} g^{eg} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{af} g_{bc} g_{g}^{f} g^{eg} - \frac{1}{6}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{g}^{f} g^{eg} - \frac{1}{4}\,\partial_{dc}\phi g_{ac} g_{b}^{e} - \frac{1}{4}\,\partial_{bc}\phi g_{ac} g_{d}^{e} \\
&- \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}^{g} g_{g}^{f} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bg} g_{d}^{g} g^{ef} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{b}^{e} g_{d}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ac} g_{df} g_{g}^{f} g^{eg} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{ac} g_{bf} g_{g}^{f} g^{eg} + \frac{1}{4}\,\partial_{cc}\phi g_{ad} g_{b}^{e} \\
&+ \frac{1}{4}\,\partial_{bc}\phi g_{ad} g_{c}^{e} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}^{g} g_{g}^{f} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bg} g_{c}^{g} g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{b}^{e} g_{c}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{cf} g_{g}^{f} g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{bf} g_{g}^{f} g^{eg} \\
&+ \frac{1}{6}\,\partial_{ef}\phi g_{ac} g_{bd} g_{g}^{f} g^{eg} - \frac{1}{6}\,\partial_{ef}\phi g_{ad} g_{bc} g_{g}^{f} g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{gh} g_{i}^{h} g^{eg} g^{fi} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{gh} g_{i}^{h} g^{eg} g^{fi} + \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}^{i} g_{i}^{h} g^{ef} \\
&- \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}^{i} g_{i}^{h} g^{ef}
\end{aligned}
\tag{ex-14.diff.205}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{4}\,\partial_{bc}\phi g_{ad} + \frac{1}{4}\,\partial_{ac}\phi g_{bd} + \frac{1}{4}\,\partial_{bd}\phi g_{ac} - \frac{1}{4}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{bc} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{bc} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} \\
&+ \frac{1}{4}\,\partial_{c}\phi\partial_{b}\phi\phi^{-1} g_{ad} + \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{d}\phi\phi^{-1} g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{b}\phi\phi^{-1} g_{ac} - \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\,\partial_{cd}\phi g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac} g_{bd} g^{ef} \\
&- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{cg} g^{eg} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1} g_{ag} g_{bd} g^{eg} + \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1} g_{ac} g_{bd} g^{eg} + \frac{1}{4}\,\partial_{dd}\phi g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}^{f} \\
&- \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{bc} g_{dg} g^{eg} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{ag} g_{bc} g^{eg} - \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1} g_{ad} g_{bc} g^{eg} - \frac{1}{4}\,\partial_{db}\phi g_{ac} - \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ac} g_{dg} g^{eg} \\
&+ \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1} g_{ac} g_{bg} g^{eg} + \frac{1}{4}\,\partial_{cb}\phi g_{ad} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{cg} g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{bg} g^{eg} + \frac{1}{6}\,\partial_{eg}\phi g_{ac} g_{bd} g^{eg} - \frac{1}{6}\,\partial_{eg}\phi g_{ad} g_{bc} g^{eg} \\
&- \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{gi} g^{eg} g^{fi} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{gi} g^{eg} g^{fi} + \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}^{h} g^{ef} - \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}^{h} g^{ef}
\end{aligned}
\tag{ex-14.diff.206}
$$

$$\Delta = -\frac{1}{4}\,\partial_{bc}\phi g_{ad} + \frac{1}{4}\,\partial_{ac}\phi g_{bd} + \frac{1}{4}\,\partial_{bd}\phi g_{ac} - \frac{1}{4}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\,\partial_{c}\phi\partial_{b}\phi\phi^{-1}g_{ad} + \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{d}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{b}\phi\phi^{-1}g_{ac} - \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{cd}\phi g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{f}{}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{cg}g^{eg} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ag}g_{bd}g^{eg} + \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\,\partial_{dd}\phi g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{f}{}^{f}$$
$$- \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bc}g_{dg}g^{eg} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ag}g_{bc}g^{eg} - \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{db}\phi g_{ac} - \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{f}{}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{dg}g^{eg}$$
$$+ \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{bg}g^{eg} + \frac{1}{4}\,\partial_{cb}\phi g_{ad} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{f}{}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{cg}g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{bg}g^{eg} + \frac{1}{6}\,\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\,\partial_{eg}\phi g_{ad}g_{bc}g^{eg}$$
$$- \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gi}g^{eg}g^{fi} + \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gi}g^{eg}g^{fi} + \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{h}{}^{h}g^{ef} - \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{h}{}^{h}g^{ef} \qquad \texttt{(ex-14.diff.301)}$$

$$\Delta = -\frac{1}{4}\,\partial_{bc}\phi g_{ad} + \frac{1}{4}\,\partial_{ac}\phi g_{bd} + \frac{1}{4}\,\partial_{bd}\phi g_{ac} - \frac{1}{4}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\,\partial_{c}\phi\partial_{b}\phi\phi^{-1}g_{ad} + \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{d}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{b}\phi\phi^{-1}g_{ac} - \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{cd}\phi g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{f}{}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{c}{}^{e} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{a}{}^{e}g_{bd} + \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\,\partial_{dd}\phi g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{f}{}^{f}$$
$$- \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bc}g_{d}{}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{a}{}^{e}g_{bc} - \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{db}\phi g_{ac} - \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{f}{}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{d}{}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{b}{}^{e}$$
$$+ \frac{1}{4}\,\partial_{cb}\phi g_{ad} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{f}{}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{c}{}^{e} - \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{b}{}^{e} + \frac{1}{6}\,\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\,\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}{}^{f}g^{eg}$$
$$+ \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}{}^{f}g^{eg} + \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{h}{}^{h}g^{ef} - \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{h}{}^{h}g^{ef} \qquad \texttt{(ex-14.diff.302)}$$

$$\Delta = -\frac{1}{4}\,\partial_{bc}\phi g_{ad} + \frac{1}{4}\,\partial_{ac}\phi g_{bd} + \frac{1}{4}\,\partial_{bd}\phi g_{ac} - \frac{1}{4}\,\partial_{ad}\phi g_{bc} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} + \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\,\partial_{c}\phi\partial_{b}\phi\phi^{-1}g_{ad} + \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{d}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{b}\phi\phi^{-1}g_{ac} - \frac{5}{12}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{cd}\phi g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{8}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{f}{}^{f} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{c}{}^{e} + \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{a}{}^{e}g_{bd} + \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\,\partial_{dd}\phi g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{8}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{f}{}^{f}$$
$$- \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bc}g_{d}{}^{e} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{a}{}^{e}g_{bc} - \frac{1}{6}\,\partial_{c}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{db}\phi g_{ac} - \frac{1}{8}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{f}{}^{f} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{d}{}^{e} + \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{ac}g_{b}{}^{e}$$
$$+ \frac{1}{4}\,\partial_{cb}\phi g_{ad} + \frac{1}{8}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{f}{}^{f} - \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{c}{}^{e} - \frac{1}{4}\,\partial_{c}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{b}{}^{e} + \frac{1}{6}\,\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\,\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}{}^{f}g^{eg}$$
$$+ \frac{1}{4}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}{}^{f}g^{eg} + \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{h}{}^{h}g^{ef} - \frac{1}{8}\,\partial_{c}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{h}{}^{h}g^{ef} \qquad \texttt{(ex-14.diff.303)}$$

$$\begin{aligned}\Delta = &-\frac{1}{4}\,\partial_{bc}\phi g_{ad} + \frac{1}{4}\,\partial_{ac}\phi g_{bd} + \frac{1}{4}\,\partial_{bd}\phi g_{ac} - \frac{1}{4}\,\partial_{ad}\phi g_{bc} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} - \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad} \\ &+ \frac{1}{4}\,\partial_{c}\phi\partial_{b}\phi\phi^{-1}g_{ad} - \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{d}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{b}\phi\phi^{-1}g_{ac} + \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{cd}\phi g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ &+ \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{c}^{\ e} + \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{a}^{\ e}g_{bd} + \frac{1}{6}\,\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\,\partial_{da}\phi g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{d}^{\ e} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{a}^{\ e}g_{bc} \\ &- \frac{1}{6}\,\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{db}\phi g_{ac} + \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{d}^{\ e} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{b}^{\ e} + \frac{1}{4}\,\partial_{cb}\phi g_{ad} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{c}^{\ e} - \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{b}^{\ e} \\ &+ \frac{1}{6}\,\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\,\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}^{\ f}g^{eg} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}^{\ f}g^{eg}\end{aligned} \tag{ex-14.diff.304}$$

$$\begin{aligned}\Delta = &-\frac{1}{4}\,\partial_{bc}\phi g_{ad} + \frac{1}{4}\,\partial_{ac}\phi g_{bd} + \frac{1}{4}\,\partial_{bd}\phi g_{ac} - \frac{1}{4}\,\partial_{ad}\phi g_{bc} - \frac{1}{4}\,\partial_{d}\phi\partial_{c}\phi\phi^{-1}g_{bd} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} - \frac{1}{4}\,\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac} + \frac{1}{4}\,\partial_{d}\phi\partial_{d}\phi\phi^{-1}g_{bc} + \frac{1}{4}\,\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad} \\ &+ \frac{1}{4}\,\partial_{c}\phi\partial_{b}\phi\phi^{-1}g_{ad} - \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{c}\phi\partial_{d}\phi\phi^{-1}g_{bd} - \frac{1}{4}\,\partial_{d}\phi\partial_{b}\phi\phi^{-1}g_{ac} + \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{cd}\phi g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ &+ \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{c}^{\ e} + \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{a}^{\ e}g_{bd} + \frac{1}{6}\,\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\,\partial_{da}\phi g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{d}^{\ e} - \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{a}^{\ e}g_{bc} \\ &- \frac{1}{6}\,\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{db}\phi g_{ac} + \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{d}^{\ e} + \frac{1}{4}\,\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{b}^{\ e} + \frac{1}{4}\,\partial_{cb}\phi g_{ad} - \frac{1}{4}\,\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{c}^{\ e} - \frac{1}{4}\,\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{b}^{\ e} \\ &+ \frac{1}{6}\,\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\,\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}^{\ f}g^{eg} + \frac{1}{4}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}^{\ f}g^{eg}\end{aligned} \tag{ex-14.diff.305}$$

$$\begin{aligned}\Delta = &-\frac{1}{4}\,\partial_{bc}\phi g_{ad} + \frac{1}{4}\,\partial_{ac}\phi g_{bd} + \frac{1}{4}\,\partial_{bd}\phi g_{ac} - \frac{1}{4}\,\partial_{ad}\phi g_{bc} - \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} + \frac{1}{12}\,\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\,\partial_{cd}\phi g_{bd} - \frac{1}{6}\,\partial_{ef}\phi g_{ac}g_{bd}g^{ef} \\ &- \frac{1}{12}\,\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\,\partial_{da}\phi g_{bc} + \frac{1}{6}\,\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{12}\,\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\,\partial_{db}\phi g_{ac} + \frac{1}{4}\,\partial_{cb}\phi g_{ad} + \frac{1}{6}\,\partial_{eg}\phi g_{ac}g_{bd}g^{eg} \\ &- \frac{1}{6}\,\partial_{eg}\phi g_{ad}g_{bc}g^{eg}\end{aligned} \tag{ex-14.diff.306}$$

$$\Delta = 0 \tag{ex-14.diff.400}$$