Example 1 The metric connection

```
# Define some properties

{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

g_{a b}::Metric.
g_{a}^{b}::KroneckerDelta.

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# Define rules for covariant derivative and the Christoffel symbol

nabla := \nabla_{c}{g_{a b}} -> \partial_{c}{g_{a b}} - g_{a d}\Gamma^{d}_{b c}
                                                      - g_{d b}\Gamma^{d}_{a c}.   # cdb (nabla.100,nabla)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
                                            + \partial_{c}{g_{b d}}
                                            - \partial_{d}{g_{b c}} ).   # cdb (Gamma.100,Gamma)

# Start with a simple expression

cderiv := \nabla_{c}{g_{a b}}.                                            # cdb (ex-01.100,cderiv)

# Do the computations

substitute          (cderiv, nabla)                                       # cdb (ex-01.101,cderiv)
substitute          (cderiv, Gamma)                                       # cdb (ex-01.102,cderiv)
distribute          (cderiv)                                              # cdb (ex-01.103,cderiv)
eliminate_metric    (cderiv)                                              # cdb (ex-01.104,cderiv)
eliminate_kronecker (cderiv)                                              # cdb (ex-01.105,cderiv)
canonicalise        (cderiv)                                              # cdb (ex-01.106,cderiv)

checkpoint.append (cderiv)
```

$$\nabla_c g_{ab} \to \partial_c g_{ab} - g_{ad}\Gamma^d{}_{bc} - g_{db}\Gamma^d{}_{ac} \tag{nabla.100}$$

$$\Gamma^a{}_{bc} \to \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right) \tag{Gamma.100}$$

$$\begin{aligned}
\nabla_c g_{ab} &= \partial_c g_{ab} - g_{ad}\Gamma^d{}_{bc} - g_{db}\Gamma^d{}_{ac} && \text{(ex-01.101)}\\
&= \partial_c g_{ab} - \frac{1}{2}g_{ad}g^{de}\left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right) - \frac{1}{2}g_{db}g^{de}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right) && \text{(ex-01.102)}\\
&= \partial_c g_{ab} - \frac{1}{2}g_{ad}g^{de}\partial_b g_{ec} - \frac{1}{2}g_{ad}g^{de}\partial_c g_{be} + \frac{1}{2}g_{ad}g^{de}\partial_e g_{bc} - \frac{1}{2}g_{db}g^{de}\partial_a g_{ec} - \frac{1}{2}g_{db}g^{de}\partial_c g_{ae} + \frac{1}{2}g_{db}g^{de}\partial_e g_{ac} && \text{(ex-01.103)}\\
&= \partial_c g_{ab} - \frac{1}{2}g_a{}^e\partial_b g_{ec} - \frac{1}{2}g_a{}^e\partial_c g_{be} + \frac{1}{2}g_a{}^e\partial_e g_{bc} - \frac{1}{2}g_b{}^e\partial_a g_{ec} - \frac{1}{2}g_b{}^e\partial_c g_{ae} + \frac{1}{2}g_b{}^e\partial_e g_{ac} && \text{(ex-01.104)}\\
&= \frac{1}{2}\partial_c g_{ab} - \frac{1}{2}\partial_c g_{ba} && \text{(ex-01.105)}\\
&= 0 && \text{(ex-01.106)}
\end{aligned}$$

Example 2 Covariant derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# rule for covariant derivative of v^{a}

deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.

# create an expression

foo := \nabla_{a}{v^{b}}.                         # cdb (ex-02.101,foo)

# apply the rule, then simplify

substitute   (foo,deriv)                          # cdb (ex-02.102,foo)
canonicalise (foo)                                # cdb (ex-02.103,foo)

checkpoint.append (foo)
```

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ca} v^c \qquad (\texttt{ex-02.102})$$

$$= \partial_a v^b + \Gamma^{bc}{}_a v_c \qquad (\texttt{ex-02.103})$$

Example 2 Covariant derivatives using "position=independent"

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# rule for covariant derivative of v^{a}

deriv := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}} + \Gamma^{b}_{c a} v^{c}.

# create an expression

foo := \nabla_{a}{v^{b}}.                          # cdb (ex-02.201,foo)

# apply the rule, then simplify

substitute   (foo,deriv)                           # cdb (ex-02.202,foo)
canonicalise (foo)                                 # cdb (ex-02.203,foo)

checkpoint.append (foo)
```

$$\begin{aligned} \nabla_{a}v^{b} &= \partial_{a}v^{b} + \Gamma^{b}{}_{c a} v^{c} &\quad (\texttt{ex-02.202}) \\ &= \partial_{a}v^{b} + \Gamma^{b}{}_{c a} v^{c} &\quad (\texttt{ex-02.203}) \end{aligned}$$

Example 2 Covariant derivatives using generic rule for deriv

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

# template for covariant derivative of a vector

deriv := \nabla_{a}{A?^{b}} -> \partial_{a}{A?^{b}} + \Gamma^{b}_{c a} A?^{c}.

# create an expression

foo := \nabla_{a}{u^{b}} + \nabla_{a}{v^{b}}. # cdb (ex-02.301,foo)

# apply the rule, then simplify

substitute   (foo,deriv)                     # cdb (ex-02.302,foo)
canonicalise (foo)                           # cdb (ex-02.303,foo)

checkpoint.append (foo)
```

$$\begin{aligned} \nabla_{a}u^{b} + \nabla_{a}v^{b} &= \partial_{a}u^{b} + \Gamma^{b}{}_{ca}u^{c} + \partial_{a}v^{b} + \Gamma^{b}{}_{ca}v^{c} && \texttt{(ex-02.302)} \\ &= \partial_{a}u^{b} + \Gamma^{b}{}_{ca}u^{c} + \partial_{a}v^{b} + \Gamma^{b}{}_{ca}v^{c} && \texttt{(ex-02.303)} \end{aligned}$$

Example 3a The Riemann curvature tensor

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});

;::Symbol;  # Suggsted by Kasper as a way to make use of ; legal
            # see https://cadabra.science/qa/473/is-this-legal-syntax
            # this code works with and without this trick

# rules for the first two covariant derivs of V^a

deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
                            + \Gamma^{a}_{c b} V^{c}.        # cdb (ex-03.101,deriv1)

deriv2 := V^{a}_{; b ; c}  -> \partial_{c}{V^{a}_{; b}}
                            + \Gamma^{a}_{d c} V^{d}_{; b}
                            - \Gamma^{d}_{b c} V^{a}_{; d}.  # cdb (ex-03.102,deriv2)

substitute (deriv2,deriv1)                  # cdb (ex-03.103, deriv2)

Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}.   # cdb (ex-03.104, Vabc)

substitute (Vabc,deriv2)                    # cdb (ex-03.105, Vabc)

distribute     (Vabc)                       # cdb (ex-03.106, Vabc)
product_rule   (Vabc)                       # cdb (ex-03.107, Vabc)

sort_product   (Vabc)                       # cdb (ex-03.108, Vabc)
rename_dummies (Vabc)                       # cdb (ex-03.109, Vabc)
canonicalise   (Vabc)                       # cdb (ex-03.110, Vabc)

sort_sum       (Vabc)                       # cdb (ex-03.111, Vabc)
factor_out     (Vabc,$V^{a?}$)              # cdb (ex-03.112, Vabc)

checkpoint.append (Vabc)
```

```

# create rule for Riemann, export later (for use by lib/dgeom)

substitute (Vabc,$V^{a} -> -1$)                # cdb (ex-03.113, Vabc)
                                               # note use of -1 to get correct
                                               # signs when coupled with the rule
                                               # for Rabcd (next statement)

Rabcd := R^{a}_{d b c} -> @(Vabc).             # cdb (ex-03.114, Rabcd) #

foo   := R^{a}_{b c d}.                        # cdb (ex-03.115, foo)
substitute (foo, Rabcd)                        # cdb (ex-03.116, foo)

# update rule to use nice indices

Rabcd := R^{a}_{b c d} -> @(foo).

checkpoint.append (Rabcd)
```

$$V^{a}{}_{;b} \rightarrow \partial_{b}V^{a} + \Gamma^{a}{}_{cb}V^{c} \tag{ex-03.101}$$

$$V^{a}{}_{;b;c} \rightarrow \partial_{c}V^{a}{}_{;b} + \Gamma^{a}{}_{dc}V^{d}{}_{;b} - \Gamma^{d}{}_{bc}V^{a}{}_{;d} \tag{ex-03.102}$$

$$V^{a}{}_{;b;c} \rightarrow \partial_{c}\left(\partial_{b}V^{a} + \Gamma^{a}{}_{db}V^{d}\right) + \Gamma^{a}{}_{dc}\left(\partial_{b}V^{d} + \Gamma^{d}{}_{eb}V^{e}\right) - \Gamma^{d}{}_{bc}\left(\partial_{d}V^{a} + \Gamma^{a}{}_{ed}V^{e}\right) \tag{ex-03.103}$$

$$
\begin{aligned}
V^a{}_{;b;c} - V^a{}_{;c;b} &= \partial_c \left(\partial_b V^a + \Gamma^a{}_{db} V^d\right) + \Gamma^a{}_{dc} \left(\partial_b V^d + \Gamma^d{}_{eb} V^e\right) - \Gamma^d{}_{bc} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e\right) - \partial_b \left(\partial_c V^a + \Gamma^a{}_{dc} V^d\right) - \Gamma^a{}_{db} \left(\partial_c V^d + \Gamma^d{}_{ec} V^e\right) \\
&\quad + \Gamma^d{}_{cb} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e\right) && \texttt{(ex-03.105)} \\
&= \partial_{cb} V^a + \partial_c \left(\Gamma^a{}_{db} V^d\right) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \left(\Gamma^a{}_{dc} V^d\right) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e \\
&\quad + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e && \texttt{(ex-03.106)} \\
&= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e && \texttt{(ex-03.107)} \\
&= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^e \Gamma^a{}_{dc} \Gamma^d{}_{eb} - \Gamma^d{}_{bc} \partial_d V^a - V^e \Gamma^a{}_{ed} \Gamma^d{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^e \Gamma^a{}_{db} \Gamma^d{}_{ec} + \Gamma^d{}_{cb} \partial_d V^a + V^e \Gamma^a{}_{ed} \Gamma^d{}_{cb} && \texttt{(ex-03.108)} \\
&= \partial_{cb} V^a + V^d \partial_c \Gamma^a{}_{db} + V^d \Gamma^a{}_{ec} \Gamma^e{}_{db} - \Gamma^d{}_{bc} \partial_d V^a - V^d \Gamma^a{}_{de} \Gamma^e{}_{bc} - \partial_{bc} V^a - V^d \partial_b \Gamma^a{}_{dc} - V^d \Gamma^a{}_{eb} \Gamma^e{}_{dc} + \Gamma^d{}_{cb} \partial_d V^a + V^d \Gamma^a{}_{de} \Gamma^e{}_{cb} && \texttt{(ex-03.109)} \\
&= V^d \partial_c \Gamma^a{}_{bd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} && \texttt{(ex-03.110)} \\
&= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} + V^d \Gamma^a{}_{ce} \Gamma^e{}_{bd} && \texttt{(ex-03.111)} \\
&= V^d \left(\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} - \Gamma^a{}_{be} \Gamma^e{}_{cd} + \Gamma^a{}_{ce} \Gamma^e{}_{bd}\right) && \texttt{(ex-03.112)} \\
&= -R^a{}_{dbc} V^d \quad \text{(MTW)}
\end{aligned}
$$

$$
R^a{}_{bcd} = -\partial_d \Gamma^a{}_{cb} + \partial_c \Gamma^a{}_{db} + \Gamma^a{}_{ce} \Gamma^e{}_{db} - \Gamma^a{}_{de} \Gamma^e{}_{cb}
$$

Example 3b The Riemann curvature tensor

This differs from the above by not using the `::TableauSymmetry` property. It gives the same results as above but it does require a little bit more housekeeping.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative.

;::Symbol;  # Suggsted by Kasper as a way to make use of ; legal
            # see https://cadabra.science/qa/473/is-this-legal-syntax
            # this code works with and without this trick

# rules for the first two covariant derivs of V^a

deriv1 := V^{a}_{; b}      -> \partial_{b}{V^{a}}
                            + \Gamma^{a}_{c b} V^{c}.        # cdb (ex-03.301,deriv1)

deriv2 := V^{a}_{; b ; c}  -> \partial_{c}{V^{a}_{; b}}
                            + \Gamma^{a}_{d c} V^{d}_{; b}
                            - \Gamma^{d}_{b c} V^{a}_{; d}.  # cdb (ex-03.302,deriv2)

substitute (deriv2,deriv1)                       # cdb (ex-03.303, deriv2)

Vabc := V^{a}_{; b ; c} - V^{a}_{; c ; b}.       # cdb (ex-03.304, Vabc)

substitute (Vabc,deriv2)                         # cdb (ex-03.305, Vabc)

distribute     (Vabc)                            # cdb (ex-03.306, Vabc)
product_rule   (Vabc)                            # cdb (ex-03.307, Vabc)

# ---------------------------------------------------------
# trick to obtain a symmetric connection

G_{a b}::Symmetric.

substitute     (Vabc,$\Gamma^{a}_{b c} ->  G^{a} G_{b c}$)
sort_product   (Vabc)                            # cdb (ex-03.308, Vabc)
```

```
rename_dummies (Vabc)                          # cdb (ex-03.309, Vabc)
canonicalise   (Vabc)                          # cdb (ex-03.310, Vabc)
substitute     (Vabc,$G^{a} G_{b c} -> \Gamma^{a}_{b c}$,repeat=True)
# -----------------------------------------------------

sort_product   (Vabc)
rename_dummies (Vabc)
canonicalise   (Vabc)

sort_sum       (Vabc)                          # cdb (ex-03.311, Vabc)
factor_out     (Vabc,$V^{a?}$)                 # cdb (ex-03.312, Vabc)

checkpoint.append (Vabc)
```

$$V^{a}{}_{;b} \rightarrow \partial_{b}V^{a} + \Gamma^{a}{}_{cb}V^{c} \tag{ex-03.301}$$

$$V^{a}{}_{;b;c} \rightarrow \partial_{c}V^{a}{}_{;b} + \Gamma^{a}{}_{dc}V^{d}{}_{;b} - \Gamma^{d}{}_{bc}V^{a}{}_{;d} \tag{ex-03.302}$$

$$V^{a}{}_{;b;c} \rightarrow \partial_{c}\left(\partial_{b}V^{a} + \Gamma^{a}{}_{db}V^{d}\right) + \Gamma^{a}{}_{dc}\left(\partial_{b}V^{d} + \Gamma^{d}{}_{eb}V^{e}\right) - \Gamma^{d}{}_{bc}\left(\partial_{d}V^{a} + \Gamma^{a}{}_{ed}V^{e}\right) \tag{ex-03.303}$$

$$
\begin{aligned}
V^a{}_{;b;c} - V^a{}_{;c;b} &= \partial_c \left(\partial_b V^a + \Gamma^a{}_{db} V^d\right) + \Gamma^a{}_{dc} \left(\partial_b V^d + \Gamma^d{}_{eb} V^e\right) - \Gamma^d{}_{bc} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e\right) - \partial_b \left(\partial_c V^a + \Gamma^a{}_{dc} V^d\right) - \Gamma^a{}_{db} \left(\partial_c V^d + \Gamma^d{}_{ec} V^e\right) \\
&\quad + \Gamma^d{}_{cb} \left(\partial_d V^a + \Gamma^a{}_{ed} V^e\right) && \texttt{(ex-03.305)} \\
&= \partial_{cb} V^a + \partial_c \left(\Gamma^a{}_{db} V^d\right) + \Gamma^a{}_{dc} \partial_b V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \left(\Gamma^a{}_{dc} V^d\right) - \Gamma^a{}_{db} \partial_c V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e \\
&\quad + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e && \texttt{(ex-03.306)} \\
&= \partial_{cb} V^a + \partial_c \Gamma^a{}_{db} V^d + \Gamma^a{}_{dc} \Gamma^d{}_{eb} V^e - \Gamma^d{}_{bc} \partial_d V^a - \Gamma^d{}_{bc} \Gamma^a{}_{ed} V^e - \partial_{bc} V^a - \partial_b \Gamma^a{}_{dc} V^d - \Gamma^a{}_{db} \Gamma^d{}_{ec} V^e + \Gamma^d{}_{cb} \partial_d V^a + \Gamma^d{}_{cb} \Gamma^a{}_{ed} V^e && \texttt{(ex-03.307)} \\
&= \partial_{cb} V^a + V^d \partial_c \left(G^a G_{db}\right) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b \left(G^a G_{dc}\right) - G^a G^d G_{db} G_{ec} V^e \\
&\quad + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e && \texttt{(ex-03.308)} \\
&= \partial_{cb} V^a + V^d \partial_c \left(G^a G_{db}\right) + G^a G^d G_{dc} G_{eb} V^e - G^d G_{bc} \partial_d V^a - G^a G^d G_{bc} G_{ed} V^e - \partial_{bc} V^a - V^d \partial_b \left(G^a G_{dc}\right) - G^a G^d G_{db} G_{ec} V^e \\
&\quad + G^d G_{cb} \partial_d V^a + G^a G^d G_{cb} G_{ed} V^e && \texttt{(ex-03.309)} \\
&= V^d \partial_c \left(G^a G_{bd}\right) + G^a G^d G_{be} G_{cd} V^e - V^d \partial_b \left(G^a G_{cd}\right) - G^a G^d G_{bd} G_{ce} V^e && \texttt{(ex-03.310)} \\
&= V^d \partial_c \Gamma^a{}_{bd} - V^d \partial_b \Gamma^a{}_{cd} + V^d \Gamma^a{}_{bd} \Gamma^e{}_{ce} - V^d \Gamma^a{}_{be} \Gamma^e{}_{cd} && \texttt{(ex-03.311)} \\
&= V^d \left(\partial_c \Gamma^a{}_{bd} - \partial_b \Gamma^a{}_{cd} + \Gamma^a{}_{bd} \Gamma^e{}_{ce} - \Gamma^a{}_{be} \Gamma^e{}_{cd}\right) && \texttt{(ex-03.312)} \\
&= -R^a{}_{dbc} V^d \quad \text{(MTW)}
\end{aligned}
$$

Example 4 Python functions

```
{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

def truncate (poly,n):

    # define the weight and give it a label
    x^{a}::Weight(label=\epsilon).

    # start with an empty espression
    ans = Ex("0")

    # loop over selected terms in the source
    for i in range (0,n+1):

       foo := @(poly).
       bah  = Ex("\epsilon = " + str(i))

       # extract a single term
       keep_weight (foo, bah)

       # update the running sum
       ans = ans + foo

    # all done, return final answer
    return ans

Quartic := c^{a}
         + c^{a}_{b} x^b
         + c^{a}_{b c} x^b x^c
         + c^{a}_{b c d} x^b x^c x^d
         + c^{a}_{b c d e} x^b x^c x^d x^e.   # cdb (ex-04.100,Quartic)

Cubic = truncate (Quartic,3)                  # cdb (ex-04.101,Cubic)

checkpoint.append (Cubic)
```

$$p(x) = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c + c^a{}_{bcd} x^b x^c x^d + c^a{}_{bcde} x^b x^c x^d x^e \tag{ex-04.100}$$

$$q(x) = c^a + c^a{}_b x^b + c^a{}_{bc} x^b x^c + c^a{}_{bcd} x^b x^c x^d \tag{ex-04.101}$$

Example 5a Keeping focused

```
{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

expr := A_{a} v^{a} + B_{a} v^{a} + C_{a} v^{a}.  # cdb (ex-05.100,expr)

zoom (expr,$B_{a} Q??$)                          # cdb (ex-05.101,expr)
substitute (expr, $v^{a} -> w^{a}$)              # cdb (ex-05.102,expr)
unzoom (expr)                                    # cdb (ex-05.103,expr)

checkpoint.append (expr)
```

$$
\begin{aligned}
A_a v^a + B_a v^a + C_a v^a &= \ldots + B_a v^a + \ldots && (\texttt{ex-05.101})\\
&= \ldots + B_a w^a + \ldots && (\texttt{ex-05.102})\\
&= A_a v^a + B_a w^a + C_a v^a && (\texttt{ex-05.103})
\end{aligned}
$$

Example 5b Tags

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

def add_tags (obj,tag):
   n = 0
   ans = Ex('0')
   for i in obj.top().terms():
      foo = obj[i]
      bah = Ex(tag+'_{'+str(n)+'}')
      ans := @(ans) + @(bah) @(foo).
      n = n + 1
   return ans

def clear_tags (obj,tag):
   ans := @(obj).
   foo  = Ex(tag+'_{a?} -> 1')
   substitute (ans,foo)
   return ans

expr := 2 V_{p q} - 3 V_{q p}.                        # cdb (ex-05.200,expr)

expr = add_tags (expr,'\\mu')                          # cdb (ex-05.201,expr)

zoom       (expr, $\mu_{1} Q??$)                       # cdb (ex-05.202,expr)
substitute (expr, $V_{a b} -> - V_{b a}$)              # cdb (ex-05.203,expr)
unzoom     (expr)                                      # cdb (ex-05.204,expr)

expr = clear_tags (expr,'\\mu')                        # cdb (ex-05.205,expr)

checkpoint.append (expr)
```

$$
\begin{aligned}
2V_{pq} - 3V_{qp} &= 2\mu_0 V_{pq} - 3\mu_1 V_{qp} && \text{(ex-05.201)} \\
&= \ldots - 3\mu_1 V_{qp} && \text{(ex-05.202)} \\
&= \ldots + 3\mu_1 V_{pq} && \text{(ex-05.203)} \\
&= 2\mu_0 V_{pq} + 3\mu_1 V_{pq} && \text{(ex-05.204)} \\
&= 5V_{pq} && \text{(ex-05.205)}
\end{aligned}
$$

Example 6-01 Evaluating components

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

V  := { V_{\theta} = \varphi, V_{\varphi} = \sin(\theta) }.   # cdb(ex-06.100,V)
dV := \partial_{b}{V_{a}} - \partial_{a}{V_{b}}.              # cdb(ex-06.101,dV)

evaluate (dV, V)   # cdb(ex-06.102,dV)
```

$$V_a = [V_\theta = \varphi,\ V_\varphi = \sin\theta] \tag{ex-06.100}$$

$$\partial_b V_a - \partial_a V_b = \square_{ab} \begin{cases} \square_{\varphi\theta} = \cos\theta - 1 \\ \square_{\theta\varphi} = 1 - \cos\theta \end{cases} \tag{ex-06.102}$$

Example 6-02 Riemann tensor of a 2-sphere

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

gab := { g_{\theta\theta}   = r**2,
         g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.      # cdb(ex-06.201,gab)

iab := { g^{\theta\theta}   = 1/r**2,
         g^{\varphi\varphi} = 1/(r**2 \sin(\theta)**2) }.  # cdb(ex-06.202,iab)

substitute (Rabcd, Gamma)                                  # cdb(ex-06.203,Gamma)

evaluate   (Gamma, join (gab,iab), rhsonly=True)           # cdb(ex-06.204,Gamma)
evaluate   (Rabcd, join (gab,iab), rhsonly=True)           # cdb(ex-06.205,Rabcd)

# convert from a rule to a simple expression
Riem := R^{a}_{b c d}.
substitute (Riem, Rabcd)                                   # cdb(ex-06.206,Riem)

from cdb.core.component import *

RiemCompt = get_component (Riem, $\theta, \varphi, \theta, \varphi$)   # cdb(ex-06.207,RiemCompt)
```

$$\left[g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-06.201}$$

$$\left[g^{\theta\theta} = r^{-2},\ g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\right] \tag{ex-06.202}$$

$$\Gamma^a{}_{bc} \to \square_{cb}{}^a \begin{cases} \square_{\varphi\theta}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\theta\varphi}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\varphi\varphi}{}^{\theta} = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-06.204}$$

$$R^a{}_{bcd} \to \square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^{\theta}{}_{\theta} = (\sin\theta)^2 \\ \square_{\varphi\theta}{}^{\varphi}{}_{\theta} = -1 \\ \square_{\theta\varphi}{}^{\theta}{}_{\varphi} = -(\sin\theta)^2 \\ \square_{\theta\theta}{}^{\varphi}{}_{\varphi} = 1 \end{cases} \tag{ex-06.205}$$

$$\square_{db}{}^a{}_c \begin{cases} \square_{\varphi\varphi}{}^{\theta}{}_{\theta} = (\sin\theta)^2 \\ \square_{\varphi\theta}{}^{\varphi}{}_{\theta} = -1 \\ \square_{\theta\varphi}{}^{\theta}{}_{\varphi} = -(\sin\theta)^2 \\ \square_{\theta\theta}{}^{\varphi}{}_{\varphi} = 1 \end{cases} \tag{ex-06.206}$$

$$R^{\theta}{}_{\varphi\varphi\theta} = -(\sin\theta)^2 \tag{ex-06.207}$$

Example 6-03 Using `complete` to compute the inverse metric

This version uses `complete` to compute the inverse metric.

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

gab := { g_{\theta\theta}   = r**2,
         g_{\varphi\varphi} = r**2 \sin(\theta)**2 }.      # cdb(ex-06.301,gab)

complete   (gab, $g^{a b}$)                                # cdb(ex-06.302,gab)

substitute (Rabcd, Gamma)

evaluate   (Gamma, gab, rhsonly=True)                      # cdb(ex-06.303,Gamma)
evaluate   (Rabcd, gab, rhsonly=True)                      # cdb(ex-06.304,Rabcd)
```

$$\left[g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2 (\sin\theta)^2\right] \tag{ex-06.301}$$

$$\left[g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2 (\sin\theta)^2,\ g^{\theta\theta} = r^{-2},\ g^{\varphi\varphi} = \left(r^2 (\sin\theta)^2\right)^{-1}\right] \tag{ex-06.302}$$

$$\Gamma^a{}_{bc} \to \Box_{cb}{}^a \begin{cases} \Box_{\varphi\theta}{}^{\varphi} = (\tan\theta)^{-1} \\ \Box_{\theta\varphi}{}^{\varphi} = (\tan\theta)^{-1} \\ \Box_{\varphi\varphi}{}^{\theta} = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-06.303}$$

$$R^a{}_{bcd} \to \Box_{db}{}^a{}_c \begin{cases} \Box_{\varphi\varphi}{}^{\theta}{}_{\theta} = (\sin\theta)^2 \\ \Box_{\varphi\theta}{}^{\varphi}{}_{\theta} = -1 \\ \Box_{\theta\varphi}{}^{\theta}{}_{\varphi} = -(\sin\theta)^2 \\ \Box_{\theta\theta}{}^{\varphi}{}_{\varphi} = 1 \end{cases} \tag{ex-06.304}$$

Example 6-04 Components by scalar projection

This example shows how one component of the Riemann tensor can be computed using a scalar projection.

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

theta{#}::LaTeXForm{"\theta"}.
varphi{#}::LaTeXForm{"\varphi"}.

# usual definitions for the connection and Riemann tensor

Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                           + \partial_{c}{g_{b d}}
                                           - \partial_{d}{g_{b c}}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.

gab := { g_{\theta \theta}   = r**2,
         g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.   # cdb(ex-06.400,gab)

iab := { g^{\theta\theta}   = 1/r**2,
         g^{\varphi\varphi} = 1/(r**2 \sin(\theta)**2) }.

substitute (Rabcd, Gamma)
evaluate   (Rabcd, join (gab,iab), rhsonly=True)

# above code just to compute Rabcd
# following code is all that is needed for the scalar projection method

# define the basis for vectors and dual vectors

basis := {theta^{\theta} = 1, varphi^{\varphi} = 1}.
dual  := {theta_{\theta} = 1, varphi_{\varphi} = 1}.

```

```
# obtain components by contracting with basis

compt := R^{a}_{b c d} theta_{a} varphi^{b} theta^{c} varphi^{d}.   # cdb(ex-06.401,compt)
substitute (compt,Rabcd)

evaluate (compt, join (basis,dual))                                 # cdb(ex-06.402,compt)

compt_sympy = compt._sympy_()

# cdbBeg(print.ex-06.04)
print ('type compt   = ' + str(type(compt)))         # shows that compt is a Cadabra object
print ('type gphiphi = ' + str(type(compt_sympy)))   # shows that gphiphi is a Python object
print ('     compt   = ' + str(compt))               # will contain LaTeX markup
print ('     gphiphi = ' + str(compt_sympy))         # will be pure Python/SymPy
# cdbEnd(print.ex-06.04)

checkpoint.append (compt)
```

$$
\begin{aligned}
R^{\theta}{}_{\varphi\theta\varphi} &= R^{a}{}_{bcd}\theta_{a}\varphi^{b}\theta^{c}\varphi^{d} && \text{(ex-06.401)}\\
&= (\sin\theta)^{2} && \text{(ex-06.402)}
\end{aligned}
$$

```
type compt   = <class 'cadabra2.Ex'>
type gphiphi = <class 'sympy.core.power.Pow'>
     compt   = (\sin(\theta))**2
     gphiphi = sin(theta)**2
```

Example 6-05 Components by selection

This example shows how one component of the metric tensor can be computed by indexing the result of a call to `evaluate`.

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

theta{#}::LaTeXForm{"\theta"}.
varphi{#}::LaTeXForm{"\varphi"}.

gab := { g_{\theta \theta}   = r**2,
         g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.   # cdb(ex-06.500,gab)

metric := g_{a b}.

evaluate (metric,gab)

indcs = metric[2][1][0]                                  # cdb(ex-06.501,indcs)
compt = metric[2][1][1]                                  # cdb(ex-06.502,compt)

# cdbBeg(print.ex-06.05)
print ('metric = ' + str(metric.input_form())+'\n')  # reveals Cadabra's internal structure for storing metric

print ('metric[0] = ' + str(metric[0]))
print ('metric[1] = ' + str(metric[1]))
print ('metric[2] = ' + str(metric[2])+'\n')

print ('metric[2][1] = '+ str(metric[2][1]))
print ('metric[2][1][0] = '+ str(metric[2][1][0]))
print ('metric[2][1][1] = '+ str(metric[2][1][1]))
# cdbEnd(print.ex-06.05)

checkpoint.append (indcs)
checkpoint.append (compt)
```

$$g_{\varphi\varphi} = g_{[\varphi,\ \varphi]} \tag{ex-06.501}$$
$$= r^2(\sin\theta)^2 \tag{ex-06.502}$$

```
metric = \components_{a b}({{\theta, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2})

metric[0] = a
metric[1] = b
metric[2] = {{\theta, \theta} = (r)**2, {\varphi, \varphi} = (r)**2 (\sin(\theta))**2}

metric[2][1] = {\varphi, \varphi} = (r)**2 (\sin(\theta))**2
metric[2][1][0] = {\varphi, \varphi}
metric[2][1][1] = (r)**2 (\sin(\theta))**2
```

Example 7 Export to C-code

```
def write_code (obj,name,filename,rank):

   import os

   from sympy.printing.c import C99CodePrinter as printer
   from sympy.codegen.ast import Assignment

   idx=[]  # indices in the form [{x, x}, {x, y} ...]
   lst=[]  # corresponding terms [termxx, termxy, ...]

   for i in range( len(obj[rank]) ):                 # rank = number of free indices
       idx.append( str(obj[rank][i][0]._sympy_()) )  # indices for this term
       lst.append( str(obj[rank][i][1]._sympy_()) )  # the matching term

   mat = sympy.Matrix([lst])                         # row vector of terms
   sub_exprs, simplified_rhs = sympy.cse(mat)        # optimise code

   with open(os.getcwd() + '/' + filename, 'w') as out:

      for lhs, rhs in sub_exprs:
         out.write(printer().doprint(Assignment(lhs, rhs))+'\n')

      for index, rhs in enumerate (simplified_rhs[0]):
         lhs = sympy.Symbol(name+' '+(idx[index]).replace(', ',']['))
         out.write(printer().doprint(Assignment(lhs, rhs))+'\n')
```

```
{\theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={\theta, \varphi}, position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} (   \partial_{g}{g_{b f}}
                                           + \partial_{f}{g_{b g}}
                                           - \partial_{b}{g_{f g}} ).

Rabcd := R^{d}_{e f g} ->   \partial_{f}{\Gamma^{d}_{e g}}
                          - \partial_{g}{\Gamma^{d}_{e f}}
                          + \Gamma^{d}_{b f} \Gamma^{b}_{e g}
                          - \Gamma^{d}_{b g} \Gamma^{b}_{e f}.

Rab := R_{a b} -> R^{c}_{a c b}.

gab := { g_{\theta \theta}   = r**2,
         g_{\varphi \varphi} = r**2 \sin(\theta)**2 }.   # cdb(ex-07.101,gab)

complete (gab, $g^{a b}$)                                 # cdb(ex-07.102,gab)

substitute (Rabcd, Gamma)
substitute (Rab, Rabcd)

evaluate   (Gamma, gab, rhsonly=True)                     # cdb(ex-07.103,Gamma)
evaluate   (Rabcd, gab, rhsonly=True)                     # cdb(ex-07.104,Rabcd)
evaluate   (Rab,   gab, rhsonly=True)                     # cdb(ex-07.105,Rab)

write_code (Gamma[1],'myGamma','example-07-gamma.c',3)
write_code (Rabcd[1],'myRabcd','example-07-rabcd.c',4)
write_code (Rab[1],  'myRab',  'example-07-rab.c',2)
```

$$\left[g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2(\sin\theta)^2\right] \tag{ex-07.101}$$

$$\left[g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2(\sin\theta)^2,\ g^{\theta\theta} = r^{-2},\ g^{\varphi\varphi} = \left(r^2(\sin\theta)^2\right)^{-1}\right] \tag{ex-07.102}$$

$$\Gamma^{a}{}_{fg} \rightarrow \square_{fg}{}^{a} \begin{cases} \square_{\varphi\theta}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\theta\varphi}{}^{\varphi} = (\tan\theta)^{-1} \\ \square_{\varphi\varphi}{}^{\theta} = -\dfrac{1}{2}\sin(2\theta) \end{cases} \tag{ex-07.103}$$

$$R^{d}{}_{efg} \rightarrow \square_{eg}{}^{d}{}_{f} \begin{cases} \square_{\varphi\varphi}{}^{\theta}{}_{\theta} = (\sin\theta)^2 \\ \square_{\theta\varphi}{}^{\varphi}{}_{\theta} = -1 \\ \square_{\varphi\theta}{}^{\theta}{}_{\varphi} = -(\sin\theta)^2 \\ \square_{\theta\theta}{}^{\varphi}{}_{\varphi} = 1 \end{cases} \tag{ex-07.104}$$

$$R_{ab} \rightarrow \square_{ab} \begin{cases} \square_{\varphi\varphi} = (\sin\theta)^2 \\ \square_{\theta\theta} = 1 \end{cases} \tag{ex-07.105}$$

Example 8 Importing and exporting Cadabra expressions

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

def create (file_name):
    import json, io, os, errno

    try:
        os.remove(file_name)                # delete the file if it exsists
        with open(file_name, 'w'): pass     # create an empty file
    except OSError as e:
        if e.errno == errno.ENOENT:         # errno.ENOENT = no such file or directory
           with open(file_name, 'w'): pass  # create an empty file
        else:
            raise                           # report an exception

    # Create and save an empty dict
    data_out = {}
    with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
        out_file.write(json.dumps(data_out,
                                  indent=2,
                                  sort_keys=True,
                                  separators=(',', ': '),
                                  ensure_ascii=False)+'\n')

def put (key_name,object,file_name):
    import json, io, os

    # Read the current dict
    with io.open(os.getcwd() + '/' + file_name) as inp_file:
        data_out = json.load(inp_file)

    # Add a new entry to the dict
    data_out[key_name] = object.input_form()

    # Save the updated dict
    with io.open(os.getcwd() + '/' + file_name, 'w', encoding='utf-8') as out_file:
        out_file.write(json.dumps(data_out,
```

```
                                    indent=2,
                                    sort_keys=True,
                                    separators=(',', ': '),
                                    ensure_ascii=False)+'\n')

def get (key_name,file_name):
    import json, io, os

    # Read the current dict
    with io.open(os.getcwd() + '/' + file_name) as inp_file:
        data_inp = json.load(inp_file)

    # Return one entry from the dict
    return Ex (data_inp[key_name])

lib_name = 'example-08.json'

create (lib_name)

\nabla{#}::Derivative.

gab := g_{a b} - 1/3 x^{c} x^{d} R_{a c b d}
               - 1/6 x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}.                   # cdb (ex-08-02.101,gab)

iab := g^{a b} + 1/3 x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
               + 1/6 x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}.   # cdb (ex-08-02.102,iab)

put ('g_ab',gab,lib_name)
put ('g^ab',iab,lib_name)

foo = get ('g_ab',lib_name)                            # cdb (ex-08-02.103,foo)
bah = get ('g^ab',lib_name)                            # cdb (ex-08-02.104,bah)

tmp := @(gab) - @(foo).                                # cdb (ex-08-02.105,tmp)
tmp := @(iab) - @(bah).                                # cdb (ex-08-02.106,tmp)
```

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} \tag{ex-08-02.101}$$

$$g^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \tag{ex-08-02.102}$$

$$\bar{g}_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} \tag{ex-08-02.103}$$

$$\bar{g}^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \tag{ex-08-02.104}$$

$$g_{ab}(x) - \bar{g}_{ab}(x) = 0 \tag{ex-08-02.105}$$

$$g^{ab}(x) - \bar{g}^{ab}(x) = 0 \tag{ex-08-02.106}$$

Example 9 The Gauss equation

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla{#}::Derivative.

K_{a b}::Symmetric.
g^{a}_{b}::KroneckerDelta.

# define the projection operator

hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}.

# 3-covariant derivative obtained by projection on 4-covariant derivative

vpq:=v_{p q} -> h^{a}_{p} h^{b}_{q} \nabla_{b}{v_{a}}.

# compute 3-curvature by commutation of covariant derivatives

vpqr:= h^{a}_{p} h^{b}_{q} h^{c}_{r} ( \nabla_{c}{v_{a b}} - \nabla_{b}{v_{a c}} ).

substitute (vpq,hab)
substitute (vpqr,vpq)

distribute   (vpqr)
product_rule (vpqr)
distribute   (vpqr)
eliminate_kronecker (vpqr)

# standard substitutions

substitute (vpqr,$h^{a}_{b} n^{b} -> 0$)
substitute (vpqr,$h^{a}_{b} n_{a} -> 0$)
substitute (vpqr,$\nabla_{a}{g^{b}_{c}} -> 0$)
substitute (vpqr,$n^{a} \nabla_{b}{v_{a}} -> -v_{a} \nabla_{b}{n^{a}}$)
substitute (vpqr,$v_{a} \nabla_{b}{n^{a}} -> v_{p} h^{p}_{a}\nabla_{b}{n^{a}}$)
substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n_{q}} -> K_{a b}$)
substitute (vpqr,$h^{p}_{a} h^{q}_{b} \nabla_{p}{n^{b}} -> K_{a}^{q}$)  # cdb(ex-09.095,vpqr)
```

```
# tidy up

{v_{a},h^{a}_{b},K_{a}^{b},K_{a b},R^{a}_{b c d},\nabla_{a}{v_{b}}}::SortOrder.

sort_product   (vpqr)                                          # cdb(ex-09.096,vpqr)
rename_dummies (vpqr)                                          # cdb(ex-09.097,vpqr)
canonicalise   (vpqr)                                          # cdb(ex-09.098,vpqr)
factor_out     (vpqr,$h^{a?}_{b?}$)                            # cdb(ex-09.099,vpqr)
factor_out     (vpqr,$v_{a?}$)                                 # cdb(ex-09.101,vpqr)

checkpoint.append (vpqr)
```

$$
\begin{aligned}
(D_r D_q - D_q D_r) v_p &= h^{e}{}_{p} h^{d}{}_{q} h^{c}{}_{r} \nabla_c \left(\nabla_d v_e\right) - h^{e}{}_{p} K_{rq} n^{d} \nabla_d v_e + K_{q}{}^{b} K_{rp} v_b - h^{d}{}_{p} h^{b}{}_{q} h^{e}{}_{r} \nabla_b \left(\nabla_e v_d\right) + h^{d}{}_{p} K_{qr} n^{e} \nabla_e v_d - K_{qp} K_{r}{}^{c} v_c \tag{ex-09.095}\\
&= h^{c}{}_{r} h^{d}{}_{q} h^{e}{}_{p} \nabla_c \left(\nabla_d v_e\right) - h^{e}{}_{p} K_{rq} \nabla_d v_e n^{d} + v_b K_{q}{}^{b} K_{rp} - h^{b}{}_{q} h^{d}{}_{p} h^{e}{}_{r} \nabla_b \left(\nabla_e v_d\right) + h^{d}{}_{p} K_{qr} \nabla_e v_d n^{e} - v_c K_{r}{}^{c} K_{qp} \tag{ex-09.096}\\
&= h^{a}{}_{r} h^{b}{}_{q} h^{c}{}_{p} \nabla_a \left(\nabla_b v_c\right) - h^{b}{}_{p} K_{rq} \nabla_a v_b n^{a} + v_a K_{q}{}^{a} K_{rp} - h^{a}{}_{q} h^{c}{}_{p} h^{b}{}_{r} \nabla_a \left(\nabla_b v_c\right) + h^{b}{}_{p} K_{qr} \nabla_a v_b n^{a} - v_a K_{r}{}^{a} K_{qp} \tag{ex-09.097}\\
&= h^{a}{}_{p} h^{b}{}_{q} h^{c}{}_{r} \nabla_c \left(\nabla_b v_a\right) + v_a K_{q}{}^{a} K_{pr} - h^{a}{}_{p} h^{b}{}_{q} h^{c}{}_{r} \nabla_b \left(\nabla_c v_a\right) - v_a K_{r}{}^{a} K_{pq} \tag{ex-09.098}\\
&= v_a K_{q}{}^{a} K_{pr} - v_a K_{r}{}^{a} K_{pq} + h^{a}{}_{p} h^{b}{}_{q} h^{c}{}_{r} \left(\nabla_c \left(\nabla_b v_a\right) - \nabla_b \left(\nabla_c v_a\right)\right) \tag{ex-09.099}\\
&= h^{a}{}_{p} h^{b}{}_{q} h^{c}{}_{r} \left(\nabla_c \left(\nabla_b v_a\right) - \nabla_b \left(\nabla_c v_a\right)\right) + v_a \left(K_{q}{}^{a} K_{pr} - K_{r}{}^{a} K_{pq}\right) \tag{ex-09.101}
\end{aligned}
$$

```
R{#}::LaTeXForm("{{\strut}^g R}").

gRabcd := \nabla_{c}{\nabla_{b}{v_{a}}}
         -\nabla_{b}{\nabla_{c}{v_{a}}} -> R^{d}_{a b c} v_{d}.

substitute     (vpqr,gRabcd)                          # cdb(ex-09.102,vpqr)
distribute     (vpqr)                                 # cdb(ex-09.103,vpqr)
substitute     (vpqr,$v_{a} -> h^{b}_{a} v_{b}$)      # cdb(ex-09.104,vpqr)
substitute     (vpqr,$h^{b}_{a} K_{c}^{a} -> K_{c}^{b}$) # cdb(ex-09.105,vpqr)
sort_product   (vpqr)                                 # cdb(ex-09.106,vpqr)
rename_dummies (vpqr)                                 # cdb(ex-09.107,vpqr)
canonicalise   (vpqr)                                 # cdb(ex-09.108,vpqr)
factor_out     (vpqr,$v_{a?}$)                        # cdb(ex-09.109,vpqr)
substitute     (vpqr,$v_{a}->1$)                      # cdb(ex-09.110,vpqr)
sort_product   (vpqr)                                 # cdb(ex-09.111,vpqr)

checkpoint.append (vpqr)
```

$$
\begin{aligned}
(D_r D_q - D_q D_r) v_p &= h^a{}_p h^b{}_q h^c{}_r \left(\nabla_c \left(\nabla_b v_a\right) - \nabla_b \left(\nabla_c v_a\right)\right) + v_a \left(K_q{}^a K_{pr} - K_r{}^a K_{pq}\right) && \text{(ex-09.101)}\\
&= h^a{}_p h^b{}_q h^c{}_r {}^g R^d{}_{abc} v_d + v_a \left(K_q{}^a K_{pr} - K_r{}^a K_{pq}\right) && \text{(ex-09.102)}\\
&= h^a{}_p h^b{}_q h^c{}_r {}^g R^d{}_{abc} v_d + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} && \text{(ex-09.103)}\\
&= h^a{}_p h^b{}_q h^c{}_r {}^g R^d{}_{abc} h^e{}_d v_e + h^b{}_a v_b K_q{}^a K_{pr} - h^b{}_a v_b K_r{}^a K_{pq} && \text{(ex-09.104)}\\
&= h^a{}_p h^b{}_q h^c{}_r {}^g R^d{}_{abc} h^e{}_d v_e + K_q{}^b v_b K_{pr} - K_r{}^b v_b K_{pq} && \text{(ex-09.105)}\\
&= v_e h^a{}_p h^b{}_q h^c{}_r h^e{}_d {}^g R^d{}_{abc} + v_b K_q{}^b K_{pr} - v_b K_r{}^b K_{pq} && \text{(ex-09.106)}\\
&= v_e h^b{}_p h^c{}_q h^d{}_r h^e{}_a {}^g R^a{}_{bcd} + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} && \text{(ex-09.107)}\\
&= v_a h^b{}_p h^c{}_q h^d{}_r h^a{}_e {}^g R^e{}_{bcd} + v_a K_q{}^a K_{pr} - v_a K_r{}^a K_{pq} && \text{(ex-09.108)}\\
&= v_a \left(h^b{}_p h^c{}_q h^d{}_r h^a{}_e {}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq}\right) && \text{(ex-09.109)}
\end{aligned}
$$

$$
{}^h R^a{}_{pqr} = h^b{}_p h^c{}_q h^d{}_r h^a{}_e {}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq} \qquad \text{(ex-09.110)}
$$

$$
{}^h R^a{}_{pqr} = h^a{}_e h^b{}_p h^c{}_q h^d{}_r {}^g R^e{}_{bcd} + K_q{}^a K_{pr} - K_r{}^a K_{pq} \qquad \text{(ex-09.111)}
$$

Example 10 The determinant of the metric

Our game here is to compute (the leading terms) in $\det g$ of the metric in RNC form

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f \left(8g^{gh}R_{acdg}R_{befh} - 9\nabla_{cd}R_{aebf}\right) + \cdots$$

For the sake of simplicity let's assume that we are working in 3-dimensions. The following analysis is easily generalsied to other dimensions (and the final answers for $\det g$ and friends are unchanged).

Define ϵ^{abc}_{ijk} by

$$\epsilon^{abc}_{ijk} = \delta^a_i\delta^b_j\delta^c_k - \delta^b_i\delta^a_j\delta^c_k + \delta^c_i\delta^a_j\delta^b_k - \delta^c_i\delta^b_j\delta^a_k + \delta^b_i\delta^c_j\delta^a_k - \delta^a_i\delta^c_j\delta^b_k \tag{1}$$

It is easy to see that ϵ^{abc}_{ijk} is anti-symmetric in both its upper and lower indices. A trivial computation shows that for any 3×3 square matrix M_{ab},

$$\epsilon^{abc}_{123}M_{1a}M_{2b}M_{3c} = \left(\delta^a_1\delta^b_2\delta^c_3 - \delta^b_1\delta^a_2\delta^c_3 + \delta^c_1\delta^a_2\delta^b_3 - \delta^c_1\delta^b_2\delta^a_3 + \delta^b_1\delta^c_2\delta^a_3 - \delta^a_1\delta^c_2\delta^b_3\right) M_{1a}M_{2b}M_{3c} = \det M \tag{2}$$

This can be easily generalised to

$$\epsilon^{abc}_{ijk}M_{pa}M_{qb}M_{rc} = \begin{cases} \pm\det M & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The $\pm$ sign in the above depends on the particular permutations of (ijk) and (pqr). If both permutations are even or both odd then the sign is $+1$ otherwise the sign is -1. The same arguments can also be applied to a matrix inverse N^{-1} leading to

$$\epsilon^{ijk}_{uvw}N^{pu}N^{qv}M^{rw} = \begin{cases} \pm\det N^{-1} & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Note that the $\pm$ in this case will match exactly that for the case of $\det M$. Thus, multiplying both expressions and summing over all choices for (ijk) and (pqr) leads to

$$\sum_{\substack{(ijk)\\(pqr)}} \left(\det N^{-1}\right)\det M = \epsilon^{ijk}_{uvw}N^{pu}N^{qv}M^{rw}\epsilon^{abc}_{ijk}M_{pa}M_{qb}M_{rc} \tag{5}$$

where the sum on the left hand side includes just those (ijk) and (prq) that are permutations of (123). There are 3! choices for (ijk) and 3! choices for (pqr) and thus the left hand side is easily reduced to $(3!\,)^2 \det M/\det N$ where $\det N = 1/\det(N^{-1})$. For the right hand side notice that

$$\epsilon^{ijk}_{uvw}\epsilon^{abc}_{ijk} = 3!\,\epsilon^{abc}_{uvw} \tag{6}$$

which leads to

$$\det M = \frac{1}{3!}\det N \epsilon^{abc}_{uvw} M_{pa} M_{qb} M_{rc} N^{pu} N^{qv} N^{rw} \tag{7}$$

For our RNC metric we will set $N^{ab} = g^{ab}$ and $M_{ij} = g_{ij}(x)$. Since g^{ab} is of the form $\mathrm{diag}(-1,1,1,1)$ we have $\det g = -1$ and thus

$$\det g(x) = -\frac{1}{3!}\epsilon^{abc}_{ijk}\, g_{pa}(x)\, g_{qb}(x)\, g_{rc}(x)\, g^{ip} g^{jq} g^{kr} \tag{8}$$

The ϵ^{abc}_{ijk} can be constructed in Cadabra by applying the `asym` algorithm to the upper indices of $\delta^a_i \delta^b_j \delta^c_k$. Note that `asym` will include the 1/3! coeffcient as part of its output.

The following code computes $-\det g$ rather than $\det g$.

Note that Calzetta etal. use an opposite sign for R_{abcd} so when comparing the following results against Calzetta do take note of this flipped sign in R_{abcd}.

The determinant of the metric

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Integer(1..3).

\nabla{#}::Derivative.

d{#}::KroneckerDelta.

g^{a b}::Symmetric.
g_{a b}::Symmetric.

R_{a b c d}::RiemannTensor.

x^{a}::Weight(label=num,value=1).

def truncate (obj,n):

    ans = Ex("0")  # create a Cadabra object with value zero

    for i in range (0,n+1):
       foo := @(obj).
       bah  = Ex("num = " + str(i))
       distribute  (foo)
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

gab := g_{a b}
       - (1/3)   x^{c} x^{d} R_{a c b d}
       - (1/6)   x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
       + (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
                                   -9 \nabla_{c d}{R_{a e b f}} ).          # cdb (ex-10.gab.000,gab)

iab := g^{a b}
       + (1/3)  x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
```

```
       + (1/6)  x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}
       + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
                              ( 4 g^{i j} R_{c g d i} R_{e h f j}
                               +3 \nabla_{c d}{R_{e g f h}} ).                    # cdb(ex-10.iab.000,iab)

distribute (gab)
distribute (iab)

gxab := gx_{a b} -> @(gab).

eps := d^{a}_{i} d^{b}_{j} d^{c}_{k}.   # cdb (ex-10.eps.001,eps) # includes a factor of 1/3!
asym (eps,$^{a},^{b},^{c}$)             # cdb (ex-10.eps.002,eps)

# compute negative detg rather than det g, note 1/3! included in eps
Ndetg := @(eps) gx_{p a} gx_{q b} gx_{r c} g^{i p} g^{j q} g^{k r}.   # cdb (ex-10.Ndetg.001,Ndetg)

substitute       (Ndetg,gxab)                                         # cdb (ex-10.Ndetg.002,Ndetg)
distribute       (Ndetg)                                              # cdb (ex-10.Ndetg.003,Ndetg)
Ndetg = truncate (Ndetg,4)                                            # cdb (ex-10.Ndetg.004,Ndetg)
substitute       (Ndetg,$g^{a b} g_{b c} -> d^{a}_{c}$,repeat=True)   # cdb (ex-10.Ndetg.005,Ndetg)
eliminate_kronecker (Ndetg)                                           # cdb (ex-10.Ndetg.006,Ndetg)
sort_product     (Ndetg)                                              # cdb (ex-10.Ndetg.007,Ndetg)
rename_dummies   (Ndetg)                                              # cdb (ex-10.Ndetg.008,Ndetg)
canonicalise     (Ndetg)                                              # cdb (ex-10.Ndetg.009,Ndetg)

# introduce the Ricci tensor

substitute (Ndetg,$R_{a b c d} g^{a c}               -> R_{b d}$,repeat=True)              # cdb (ex-10.Ndetg.101,Ndetg)
substitute (Ndetg,$\nabla_{a}{R_{b c d e}} g^{b d}   -> \nabla_{a}{R_{c e}}$,repeat=True)   # cdb (ex-10.Ndetg.102,Ndetg)
substitute (Ndetg,$\nabla_{a b}{R_{c d e f}} g^{c e} -> \nabla_{a b}{R_{d f}}$,repeat=True) # cdb (ex-10.Ndetg.103,Ndetg)

# the following was based on sqrt-Ndetg.tex

sqrtNdetg := 1/2 + (1/2) @(Ndetg)
            - (1/8) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
            - (1/4) (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
                                                                      # cdb (ex-10.sqrtNdetg.001,sqrtNdetg)

```

```
sort_product   (sqrtNdetg)                                          # cdb (ex-10.sqrtNdetg.002,sqrtNdetg)
rename_dummies (sqrtNdetg)                                          # cdb (ex-10.sqrtNdetg.003,sqrtNdetg)
canonicalise   (sqrtNdetg)                                          # cdb (ex-10.sqrtNdetg.004,sqrtNdetg)

logNdetg := -1 + @(Ndetg)
           - (1/2) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
           - (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
                                                                    # cdb (ex-10.logNdetg.001,logNdetg)

sort_product   (logNdetg)                                           # cdb (ex-10.logNdetg.002,logNdetg)
rename_dummies (logNdetg)                                           # cdb (ex-10.logNdetg.003,logNdetg)
canonicalise   (logNdetg)                                           # cdb (ex-10.logNdetg.004,logNdetg)

# =================================================================================
# the remaining code is just for pretty printing

def product_sort (obj):
    substitute (obj,$ x^{a}                         -> A000^{a}              $)
    substitute (obj,$ g^{a b}                       -> A001^{a b}            $)
    substitute (obj,$ \nabla_{c}{R_{a b}}           -> A004_{a b c}          $)
    substitute (obj,$ \nabla_{c d}{R_{a b}}         -> A005_{a b c d}        $)
    substitute (obj,$ \nabla_{c d e}{R_{a b}}       -> A006_{a b c d e}      $)
    substitute (obj,$ \nabla_{c d e f}{R_{a b}}     -> A007_{a b c d e f}    $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}       -> A008_{a b c d e}      $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}     -> A009_{a b c d e f}    $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}   -> A010_{a b c d e f g}  $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}} -> A011_{a b c d e f g h} $)
    substitute (obj,$ R_{a b}                       -> A002_{a b}            $)
    substitute (obj,$ R_{a b c d}                   -> A003_{a b c d}        $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A000^{a}              -> x^{a}                         $)
    substitute (obj,$ A001^{a b}            -> g^{a b}                       $)
    substitute (obj,$ A002_{a b}            -> R_{a b}                       $)
    substitute (obj,$ A003_{a b c d}        -> R_{a b c d}                   $)
    substitute (obj,$ A004_{a b c}          -> \nabla_{c}{R_{a b}}           $)
    substitute (obj,$ A005_{a b c d}        -> \nabla_{c d}{R_{a b}}         $)
    substitute (obj,$ A006_{a b c d e}      -> \nabla_{c d e}{R_{a b}}       $)
```

```
    substitute (obj,$ A007_{a b c d e f}       -> \nabla_{c d e f}{R_{a b}}      $)
    substitute (obj,$ A008_{a b c d e}         -> \nabla_{e}{R_{a b c d}}        $)
    substitute (obj,$ A009_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}}      $)
    substitute (obj,$ A010_{a b c d e f g}     -> \nabla_{e f g}{R_{a b c d}}    $)
    substitute (obj,$ A011_{a b c d e f g h}   -> \nabla_{e f g h}{R_{a b c d}}  $)

def get_term (obj,n):

    x^{a}::Weight(label=xnum).

    foo := @(obj).
    bah  = Ex("xnum = " + str(n))
    keep_weight (foo,bah)

    return foo

def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    product_sort   (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    sort_sum       (bah)
    factor_out     (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute  (bah)
    factor_out  (bah,$x^{a?}$)
    return bah

# ---------------------------------------------------------------
# reformat Ndetg

```

```
Rterm0 = get_term (Ndetg,0)       # cdb (ex-10.Rterm0.701,Rterm0)
Rterm1 = get_term (Ndetg,1)       # cdb (ex-10.Rterm1.701,Rterm1)
Rterm2 = get_term (Ndetg,2)       # cdb (ex-10.Rterm2.701,Rterm2)
Rterm3 = get_term (Ndetg,3)       # cdb (ex-10.Rterm3.701,Rterm3)
Rterm4 = get_term (Ndetg,4)       # cdb (ex-10.Rterm4.701,Rterm4)

Rterm0 = reformat (Rterm0,  1)    # cdb (ex-10.Rterm0.702,Rterm0)
Rterm1 = reformat (Rterm1,  1)    # cdb (ex-10.Rterm1.702,Rterm1)
Rterm2 = reformat (Rterm2,  3)    # cdb (ex-10.Rterm2.702,Rterm2)
Rterm3 = reformat (Rterm3,  6)    # cdb (ex-10.Rterm3.702,Rterm3)
Rterm4 = reformat (Rterm4,180)    # cdb (ex-10.Rterm4.702,Rterm4)

Ndetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.Ndetg.701,Ndetg)

# ---------------------------------------------------------------
# reformat sqrtNdetg

Rterm0 = get_term (sqrtNdetg,0)   # cdb (ex-10.Rterm0.801,Rterm0)
Rterm1 = get_term (sqrtNdetg,1)   # cdb (ex-10.Rterm1.801,Rterm1)
Rterm2 = get_term (sqrtNdetg,2)   # cdb (ex-10.Rterm2.801,Rterm2)
Rterm3 = get_term (sqrtNdetg,3)   # cdb (ex-10.Rterm3.801,Rterm3)
Rterm4 = get_term (sqrtNdetg,4)   # cdb (ex-10.Rterm4.801,Rterm4)

Rterm0 = reformat (Rterm0,  1)    # cdb (ex-10.Rterm0.802,Rterm0)
Rterm1 = reformat (Rterm1,  1)    # cdb (ex-10.Rterm1.802,Rterm1)
Rterm2 = reformat (Rterm2,  6)    # cdb (ex-10.Rterm2.802,Rterm2)
Rterm3 = reformat (Rterm3, 12)    # cdb (ex-10.Rterm3.802,Rterm3)
Rterm4 = reformat (Rterm4,360)    # cdb (ex-10.Rterm4.802,Rterm4)

sqrtNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.sqrtNdetg.801,sqrtNdetg)

# ---------------------------------------------------------------
# reformat logNdetg

Rterm0 = get_term (logNdetg,0)    # cdb (ex-10.Rterm0.801,Rterm0)
Rterm1 = get_term (logNdetg,1)    # cdb (ex-10.Rterm1.801,Rterm1)
Rterm2 = get_term (logNdetg,2)    # cdb (ex-10.Rterm2.801,Rterm2)
Rterm3 = get_term (logNdetg,3)    # cdb (ex-10.Rterm3.801,Rterm3)
```

```
Rterm4 = get_term (logNdetg,4)    # cdb (ex-10.Rterm4.801,Rterm4)

Rterm0 = reformat (Rterm0,  1)    # cdb (ex-10.Rterm0.802,Rterm0)
Rterm1 = reformat (Rterm1,  1)    # cdb (ex-10.Rterm1.802,Rterm1)
Rterm2 = reformat (Rterm2,  3)    # cdb (ex-10.Rterm2.802,Rterm2)
Rterm3 = reformat (Rterm3,  6)    # cdb (ex-10.Rterm3.802,Rterm3)
Rterm4 = reformat (Rterm4,180)    # cdb (ex-10.Rterm4.802,Rterm4)

logNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4).  # cdb (ex-10.logNdetg.901,logNdetg)

checkpoint.append (Ndetg)
checkpoint.append (sqrtNdetg)
checkpoint.append (logNdetg)
```

The metric determinant in Riemann normal coordinates

$$-\det g(x) = 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) + \cdots$$

The volume element in RNC

If $-\det g(x)$ is non-negative then we also have

$$\sqrt{-\det g(x)} = 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) + \cdots$$

The log of -detg in RNC

$$\log\left(-\det g(x)\right) = -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) + \cdots$$

Apart from the signs, this matches exactly the expression given by Calzetta etal. (eq. A14)

Example 11 The RNC connection.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent).

D{#}::PartialDerivative.
\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g^{a b}::Weight(label=gnum,value=1).

\delta{#}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R_{a b c d}::Depends(\nabla{#}).

x^{a}::Depends(D{#}).
x^{a}::Weight(label=xnum,value=1).

Dx := D_{a}{x^{b}} -> \delta^{b}_{a}.   # cdb (ex-11.000,Dx)

gab := g_{a b} -> g_{a b}
               -  (1/3)   x^{c} x^{d} R_{a c b d}
               -  (1/6)   x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}
               +  (1/180) x^{c} x^{d} x^{e} x^{f} ( 8 g^{g h} R_{a c d g} R_{b e f h}
                                                   -9 \nabla_{c d}{R_{a e b f}} ).      # cdb (ex-11.001,gab)

iab := g^{a b} -> g^{a b}
               + (1/3)  x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
               + (1/6)  x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}
               + (1/60) x^{c} x^{d} x^{e} x^{f} g^{a g} g^{b h}
                                     ( 4 g^{i j} R_{c g d i} R_{e h f j}
                                      +3 \nabla_{c d}{R_{e g f h}} ).                  # cdb(ex-11.002,iab)

distribute (gab)
distribute (iab)

ChrSym := \Gamma^{a}_{b c} -> 1/2 g^{a d} (  D_{b}{g_{d c}}
```

```
                                              + D_{c}{g_{b d}}
                                              - D_{d}{g_{b c}} ).   # cdb (ex-11.003,ChrSym)

Gamma := \Gamma^{a}_{b c}.                # cdb (ex-11.100,Gamma)

substitute     (Gamma,ChrSym)             # cdb (ex-11.101,Gamma)
substitute     (Gamma,gab)                # cdb (ex-11.102,Gamma)
substitute     (Gamma,iab)                # cdb (ex-11.103,Gamma)
distribute     (Gamma)                    # cdb (ex-11.104,Gamma)
unwrap         (Gamma)                    # cdb (ex-11.105,Gamma)
product_rule   (Gamma)                    # cdb (ex-11.106,Gamma)
distribute     (Gamma)                    # cdb (ex-11.107,Gamma)
substitute     (Gamma,Dx)                 # cdb (ex-11.108,Gamma)
eliminate_kronecker (Gamma)               # cdb (ex-11.109,Gamma)

def truncate (obj,n):

    ans = Ex("0")  # create a Cadabra object with value zero

    for i in range (0,n+1):
       foo := @(obj).
       bah  = Ex("xnum = " + str(i))
       distribute  (foo)
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

checkpoint.append (Gamma)

# sort_product   (Gamma)  # 52.3 sec, 49 Mbyte
# rename_dummies (Gamma)  # 58.6 sec, 51 Mbyte
# canonicalise   (Gamma)  # killed after 20 mins and over 500 Mbyte

Gamma = truncate (Gamma,3)   # cdb (ex-11.110,Gamma)  # allow up to 3rd order in x^a

sort_product   (Gamma)
rename_dummies (Gamma)
```

```
canonicalise   (Gamma)

checkpoint.append (Gamma)

# =========================================================================
# the remaining code is just for pretty printing

def product_sort (obj):
    substitute (obj,$ g^{a b}                  -> A001^{a b}              $)
    substitute (obj,$ x^{a}                    -> A002^{a}                $)
    substitute (obj,$ z^{a}                    -> A003^{a}                $)
    substitute (obj,$ R_{a b c d}              -> A004_{a b c d}          $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}   -> A005_{a b c d e}        $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}} -> A006_{a b c d e f}      $)
    sort_sum       (obj)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a b}               -> g^{a b}                 $)
    substitute (obj,$ A002^{a}                 -> x^{a}                   $)
    substitute (obj,$ A003^{a}                 -> z^{a}                   $)
    substitute (obj,$ A004_{a b c d}           -> R_{a b c d}             $)
    substitute (obj,$ A005_{a b c d e}         -> \nabla_{e}{R_{a b c d}}   $)
    substitute (obj,$ A006_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}} $)

def get_xterm (obj,n):

    foo := @(obj).
    bah  = Ex("xnum = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

def get_gterm (obj,n):

    foo := @(obj).
    bah  = Ex("gnum = " + str(n))
    distribute  (foo)
```

```
     keep_weight (foo, bah)

     return foo

def reformat (obj,scale):

   foo  = Ex(str(scale))
   bah := @(foo) @(obj).

   distribute     (bah)
   product_sort   (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$x^{a?},g^{b? c?}$)
   ans := @(bah) / @(foo).

   return ans

gam1  = get_xterm (Gamma, 1)                            # cdb (ex-11.200,gam1)
gam2  = get_xterm (Gamma, 2)                            # cdb (ex-11.201,gam2)
gam3  = get_xterm (Gamma, 3)                            # cdb (ex-11.202,gam3)

gam31 = get_gterm (gam3,  1)                            # cdb (ex-11.210,gam31)
gam32 = get_gterm (gam3,  2)                            # cdb (ex-11.211,gam31)

gam1  = reformat (gam1,   3)                            # cdb (ex-11.220,gam1)
gam2  = reformat (gam2,  12)                            # cdb (ex-11.221,gam2)

gam31 = reformat (gam31, 40)                            # cdb (ex-11.222,gam31)
gam32 = reformat (gam32, 45)                            # cdb (ex-11.223,gam32)

Gamma  := @(gam1) + @(gam2) + @(gam31) + @(gam32).    # cdb (ex-11.230,Gamma)
Scaled := 360 @(Gamma).                                 # cdb (ex-11.231,Scaled)

checkpoint.append (Gamma)
```

$$\Gamma^a{}_{bc}(x) = \frac{1}{3}g^{ad}x^e\left(R_{bdce} + R_{becd}\right) + \frac{1}{12}g^{ad}x^ex^f\left(-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}\right)$$
$$+ \frac{1}{40}g^{ad}x^ex^fx^g\left(-\nabla_{ce}R_{bfdg} - \nabla_{ec}R_{bfdg} + \nabla_{de}R_{bfcg} + \nabla_{ed}R_{bfcg} + 2\nabla_{ef}R_{bdcg} + 2\nabla_{ef}R_{bgcd} - \nabla_{be}R_{cfdg} - \nabla_{eb}R_{cfdg}\right)$$
$$+ \frac{1}{45}g^{ad}g^{ef}x^gx^hx^i\left(4R_{becg}R_{dhfi} + 4R_{bgce}R_{dhfi} - 2R_{bdeg}R_{chfi} - R_{bedg}R_{chfi} + R_{bgde}R_{chfi} - 2R_{bgeh}R_{cdfi} - R_{bgeh}R_{cfdi} + R_{bgeh}R_{cidf}\right) \qquad (\texttt{ex-11.230})$$

$$360\Gamma^a{}_{bc}(x) = 120g^{ad}x^e\left(R_{bdce} + R_{becd}\right) + 30g^{ad}x^ex^f\left(-\nabla_c R_{bedf} + \nabla_d R_{becf} + 2\nabla_e R_{bdcf} + 2\nabla_e R_{bfcd} - \nabla_b R_{cedf}\right)$$
$$+ 9g^{ad}x^ex^fx^g\left(-\nabla_{ce}R_{bfdg} - \nabla_{ec}R_{bfdg} + \nabla_{de}R_{bfcg} + \nabla_{ed}R_{bfcg} + 2\nabla_{ef}R_{bdcg} + 2\nabla_{ef}R_{bgcd} - \nabla_{be}R_{cfdg} - \nabla_{eb}R_{cfdg}\right)$$
$$+ 8g^{ad}g^{ef}x^gx^hx^i\left(4R_{becg}R_{dhfi} + 4R_{bgce}R_{dhfi} - 2R_{bdeg}R_{chfi} - R_{bedg}R_{chfi} + R_{bgde}R_{chfi} - 2R_{bgeh}R_{cdfi} - R_{bgeh}R_{cfdi} + R_{bgeh}R_{cidf}\right) \qquad (\texttt{ex-11.231})$$

Save $\Gamma^a{}_{bc}$ for later use in Example 12.

```
jsonfile = 'example-11.json'
cdblib.create (jsonfile)
cdblib.put ('Gamma',Gamma,jsonfile)
```

Example 12 Checking the 2nd and 3rd order terms of Calzetta etal.

The following calculations show that my results for the RNC connection agree with those of Calzetta etal. to third order terms.

Note that I take ∇_{ab} to be $\nabla_a(\nabla_b)$.

Note also that $(LCB)\ R_{abcd} = -(Calzetta)\ R_{abcd}$. Consequently, I replace R_{abcd} with $-R_{abcd}$ in the Calzetta expressions (done as a Cadabra substitution rule).

This is relatively straightforward. We just apply a few carefully chosen applications of the first and second Bianchi identities.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,u,v#}::Indices("latin",position=independent).
{\mu,\nu,\rho,\sigma,\tau,\lambda,\xi#}::Indices("greek",position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g^{a b}::Weight(label=gnum,value=1).

\delta{#}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R_{a b c d}::Depends(\nabla{#}).

x^{a}::Weight(label=xnum,value=1).

def add_tags (obj,tag):

   n = 0
   ans = Ex('0')

   for i in obj.top().terms():
      foo = obj[i]
      bah = Ex(tag+'_{'+str(n)+'}')
      ans := @(ans) + @(bah) @(foo).
      n = n + 1

   return ans

def clear_tags (obj,tag):

   ans := @(obj).
   foo  = Ex(tag+'_{a?} -> 1')
   substitute (ans,foo)

   return ans

def get_xterm (obj,n):
```

```

    foo := @(obj).
    bah  = Ex("xnum = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

def get_gterm (obj,n):

    foo := @(obj).
    bah  = Ex("gnum = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

def product_sort (obj):
    substitute (obj,$ g^{a b}                   -> A001^{a b}               $)
    substitute (obj,$ x^{a}                     -> A002^{a}                 $)
    substitute (obj,$ z^{a}                     -> A003^{a}                 $)
    substitute (obj,$ R_{a b c d}               -> A004_{a b c d}           $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}   -> A005_{a b c d e}         $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}} -> A006_{a b c d e f}       $)
    sort_sum       (obj)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a b}                -> g^{a b}                   $)
    substitute (obj,$ A002^{a}                  -> x^{a}                     $)
    substitute (obj,$ A003^{a}                  -> z^{a}                     $)
    substitute (obj,$ A004_{a b c d}            -> R_{a b c d}               $)
    substitute (obj,$ A005_{a b c d e}          -> \nabla_{e}{R_{a b c d}}   $)
    substitute (obj,$ A006_{a b c d e f}        -> \nabla_{e f}{R_{a b c d}} $)

def reformat (obj,scaleA,scaleB):

   foo  = Ex(str(scaleA))
   moo  = Ex(str(scaleB))
```

```
   bah := @(foo) @(obj) / @(moo).

   distribute     (bah)
   product_sort   (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$g^{c? d?}$)
   factor_out     (bah,$x^{a?},z^{b?}$)
   ans := @(moo) @(bah) / @(foo).

   return ans

# ==========================================================================
# LCB

import cdblib
Gamma  = cdblib.get ('Gamma','example-11.json')                # cdb(ex-12.100,Gamma)

# note that the next two lines require careful inspection of the free indices on Gamma
# expecting Gamma = \Gamma^{a}_{bc}
Gamma := z^{b} z^{c} @(Gamma).

# lower index ^{a} to _{v}

Gamma := g_{v a} @(Gamma).

distribute (Gamma)
substitute (Gamma, $g_{a d} g^{d b} -> \delta_{a}^{b}$)
eliminate_kronecker (Gamma)                                   # cdb(ex-12.101,Gamma)

# change free index _{v} to _{a}

foo := tmp_{v} -> @(Gamma).                                   # cdb(ex-12.191,foo)
bah := tmp_{a}.                                               # cdb(ex-12.192,bah)
substitute (bah, foo)                                         # cdb(ex-12.193,bah)

Gamma := @(bah).                                              # cdb(ex-12.102,Gamma)
```

```
product_sort (Gamma)                                          # cdb(ex-12.103,Gamma)

checkpoint.append (Gamma)

gam1  = get_xterm (Gamma,1)                                   # cdb(ex-12.200,gam1)
gam2  = get_xterm (Gamma,2)                                   # cdb(ex-12.201,gam2)
gam3  = get_xterm (Gamma,3)                                   # cdb(ex-12.202,gam3)

gam30 = get_gterm (gam3,0)                                    # cdb(ex-12.203,gam30)
gam31 = get_gterm (gam3,1)                                    # cdb(ex-12.204,gam31)

gam1  = reformat (gam1, 3,1)                                  # cdb(ex-12.300,gam1)
gam2  = reformat (gam2,12,1)                                  # cdb(ex-12.301,gam2)

gam30 = reformat (gam30,40,1)                                 # cdb(ex-12.302,gam30)
gam31 = reformat (gam31,45,2)                                 # cdb(ex-12.303,gam31)

gam3 := @(gam30) + @(gam31).                                  # cdb(ex-12.304,gam3)

Gamma := @(gam1) + @(gam2) + @(gam3).                         # cdb(ex-12.305,Gamma)

checkpoint.append (Gamma)
```

$$\texttt{ex-12.100} := \frac{1}{3}g^{ad}x^{e}\left(R_{bdce}+R_{becd}\right)+\frac{1}{12}g^{ad}x^{e}x^{f}\left(-\nabla_{c}R_{bedf}+\nabla_{d}R_{becf}+2\nabla_{e}R_{bdcf}+2\nabla_{e}R_{bfcd}-\nabla_{b}R_{cedf}\right)$$
$$+\frac{1}{40}g^{ad}x^{e}x^{f}x^{g}\left(-\nabla_{ce}R_{bfdg}-\nabla_{ec}R_{bfdg}+\nabla_{de}R_{bfcg}+\nabla_{ed}R_{bfcg}+2\nabla_{ef}R_{bdcg}+2\nabla_{ef}R_{bgcd}-\nabla_{be}R_{cfdg}-\nabla_{eb}R_{cfdg}\right)$$
$$+\frac{1}{45}g^{ad}g^{ef}x^{g}x^{h}x^{i}\left(4R_{becg}R_{dhfi}+4R_{bgce}R_{dhfi}-2R_{bdeg}R_{chfi}-R_{bedg}R_{chfi}+R_{bgde}R_{chfi}-2R_{bgeh}R_{cdfi}-R_{bgeh}R_{cfdi}+R_{bgeh}R_{cidf}\right)$$

$$\texttt{ex-12.191} := tmp_{v}$$
$$\rightarrow \frac{1}{3}z^{b}z^{c}x^{e}R_{bvce}+\frac{1}{3}z^{b}z^{c}x^{e}R_{becv}-\frac{1}{12}z^{b}z^{c}x^{e}x^{f}\nabla_{c}R_{bevf}+\frac{1}{12}z^{b}z^{c}x^{e}x^{f}\nabla_{v}R_{becf}+\frac{1}{6}z^{b}z^{c}x^{e}x^{f}\nabla_{e}R_{bvcf}+\frac{1}{6}z^{b}z^{c}x^{e}x^{f}\nabla_{e}R_{bfcv}$$
$$-\frac{1}{12}z^{b}z^{c}x^{e}x^{f}\nabla_{b}R_{cevf}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ce}R_{bfvg}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ec}R_{bfvg}+\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ve}R_{bfcg}+\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ev}R_{bfcg}$$
$$+\frac{1}{20}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ef}R_{bvcg}+\frac{1}{20}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ef}R_{bgcv}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{be}R_{cfvg}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{eb}R_{cfvg}$$
$$+\frac{4}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{becg}R_{vhfi}+\frac{4}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgce}R_{vhfi}-\frac{2}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bveg}R_{chfi}-\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bevg}R_{chfi}$$
$$+\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgve}R_{chfi}-\frac{2}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgeh}R_{cvfi}-\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgeh}R_{cfvi}+\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgeh}R_{civf}$$

$$\texttt{ex-12.192} := tmp_{a}$$

$$\texttt{ex-12.193} := \frac{1}{3}z^{b}z^{c}x^{e}R_{bace}+\frac{1}{3}z^{b}z^{c}x^{e}R_{beca}-\frac{1}{12}z^{b}z^{c}x^{e}x^{f}\nabla_{c}R_{beaf}+\frac{1}{12}z^{b}z^{c}x^{e}x^{f}\nabla_{a}R_{becf}+\frac{1}{6}z^{b}z^{c}x^{e}x^{f}\nabla_{e}R_{bacf}+\frac{1}{6}z^{b}z^{c}x^{e}x^{f}\nabla_{e}R_{bfca}$$
$$-\frac{1}{12}z^{b}z^{c}x^{e}x^{f}\nabla_{b}R_{ceaf}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ce}R_{bfag}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ec}R_{bfag}+\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ae}R_{bfcg}+\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ea}R_{bfcg}$$
$$+\frac{1}{20}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ef}R_{bacg}+\frac{1}{20}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{ef}R_{bgca}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{be}R_{cfag}-\frac{1}{40}z^{b}z^{c}x^{e}x^{f}x^{g}\nabla_{eb}R_{cfag}$$
$$+\frac{4}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{becg}R_{ahfi}+\frac{4}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgce}R_{ahfi}-\frac{2}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{baeg}R_{chfi}-\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{beag}R_{chfi}$$
$$+\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgae}R_{chfi}-\frac{2}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgeh}R_{cafi}-\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgeh}R_{cfai}+\frac{1}{45}z^{b}z^{c}g^{ef}x^{g}x^{h}x^{i}R_{bgeh}R_{ciaf}$$

$$\texttt{ex-12.101} := \frac{1}{3}z^b z^c x^e R_{bvce} + \frac{1}{3}z^b z^c x^e R_{becv} - \frac{1}{12}z^b z^c x^e x^f \nabla_c R_{bevf} + \frac{1}{12}z^b z^c x^e x^f \nabla_v R_{becf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bvcf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bfcv}$$
$$- \frac{1}{12}z^b z^c x^e x^f \nabla_b R_{cevf} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ce} R_{bfvg} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ec} R_{bfvg} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ve} R_{bfcg} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ev} R_{bfcg}$$
$$+ \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bvcg} + \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bgcv} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{be} R_{cfvg} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{eb} R_{cfvg}$$
$$+ \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{becg} R_{vhfi} + \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgce} R_{vhfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{bveg} R_{chfi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bevg} R_{chfi}$$
$$+ \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgve} R_{chfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cvfi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cfvi} + \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{civf}$$

$$\texttt{ex-12.102} := \frac{1}{3}z^b z^c x^e R_{bace} + \frac{1}{3}z^b z^c x^e R_{beca} - \frac{1}{12}z^b z^c x^e x^f \nabla_c R_{beaf} + \frac{1}{12}z^b z^c x^e x^f \nabla_a R_{becf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bacf} + \frac{1}{6}z^b z^c x^e x^f \nabla_e R_{bfca}$$
$$- \frac{1}{12}z^b z^c x^e x^f \nabla_b R_{ceaf} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ce} R_{bfag} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ec} R_{bfag} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ae} R_{bfcg} + \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{ea} R_{bfcg}$$
$$+ \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bacg} + \frac{1}{20}z^b z^c x^e x^f x^g \nabla_{ef} R_{bgca} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{be} R_{cfag} - \frac{1}{40}z^b z^c x^e x^f x^g \nabla_{eb} R_{cfag}$$
$$+ \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{becg} R_{ahfi} + \frac{4}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgce} R_{ahfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{baeg} R_{chfi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{beag} R_{chfi}$$
$$+ \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgae} R_{chfi} - \frac{2}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cafi} - \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{cfai} + \frac{1}{45}z^b z^c g^{ef} x^g x^h x^i R_{bgeh} R_{ciaf}$$

$$\texttt{ex-12.103} := \frac{1}{3}x^b z^c z^d R_{cadb} + \frac{1}{3}x^b z^c z^d R_{cbda} + \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{daec} - \frac{1}{12}x^b x^c z^d z^e \nabla_e R_{dbac} + \frac{1}{12}x^b x^c z^d z^e \nabla_a R_{dbec} + \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{dcea}$$
$$- \frac{1}{12}x^b x^c z^d z^e \nabla_d R_{ebac} + \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{eafd} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fb} R_{ecad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{bf} R_{ecad} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{ecfd}$$
$$+ \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{ecfd} + \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{edfa} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{fcad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{fcad}$$
$$- \frac{2}{45}g^{bc} x^d x^e x^f z^g z^h R_{gabd} R_{hecf} - \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gbad} R_{hecf} + \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{aecf} R_{gbhd} + \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdab} R_{hecf}$$
$$+ \frac{4}{45}g^{bc} x^d x^e x^f z^g z^h R_{aecf} R_{gdhb} - \frac{2}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdbe} R_{hacf} - \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdbe} R_{hcaf} + \frac{1}{45}g^{bc} x^d x^e x^f z^g z^h R_{gdbe} R_{hfac}$$

$$\texttt{ex-12.200} := \frac{1}{3}x^b z^c z^d R_{cadb} + \frac{1}{3}x^b z^c z^d R_{cbda}$$

$$\texttt{ex-12.201} := \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{daec} - \frac{1}{12}x^b x^c z^d z^e \nabla_e R_{dbac} + \frac{1}{12}x^b x^c z^d z^e \nabla_a R_{dbec} + \frac{1}{6}x^b x^c z^d z^e \nabla_b R_{dcea} - \frac{1}{12}x^b x^c z^d z^e \nabla_d R_{ebac}$$

$$\texttt{ex-12.202} := \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{eafd} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fb} R_{ecad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{bf} R_{ecad} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{ecfd}$$
$$+ \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{ecfd} + \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{edfa} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{fcad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{fcad}$$
$$- \frac{2}{45}g^{bc}x^d x^e x^f z^g z^h R_{gabd} R_{hecf} - \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gbad} R_{hecf} + \frac{4}{45}g^{bc}x^d x^e x^f z^g z^h R_{aecf} R_{gbhd} + \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdab} R_{hecf}$$
$$+ \frac{4}{45}g^{bc}x^d x^e x^f z^g z^h R_{aecf} R_{gdhb} - \frac{2}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdbe} R_{hacf} - \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdbe} R_{hcaf} + \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdbe} R_{hfac}$$

$$\texttt{ex-12.203} := \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{eafd} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fb} R_{ecad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{bf} R_{ecad} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{ecfd}$$
$$+ \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{ecfd} + \frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{edfa} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{fcad} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{fcad}$$

$$\texttt{ex-12.204} := -\frac{2}{45}g^{bc}x^d x^e x^f z^g z^h R_{gabd} R_{hecf} - \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gbad} R_{hecf} + \frac{4}{45}g^{bc}x^d x^e x^f z^g z^h R_{aecf} R_{gbhd} + \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdab} R_{hecf}$$
$$+ \frac{4}{45}g^{bc}x^d x^e x^f z^g z^h R_{aecf} R_{gdhb} - \frac{2}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdbe} R_{hacf} - \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdbe} R_{hcaf} + \frac{1}{45}g^{bc}x^d x^e x^f z^g z^h R_{gdbe} R_{hfac}$$

$$\texttt{ex-12.300} := \frac{2}{3}x^b z^c z^d R_{acbd}$$

$$\texttt{ex-12.301} := \frac{1}{12}x^b x^c z^d z^e \left(4\nabla_b R_{adce} + 2\nabla_d R_{abce} + \nabla_a R_{bdce}\right)$$

$$\texttt{ex-12.302} := \frac{1}{40}x^b x^c x^d z^e z^f \left(4\nabla_{bc} R_{aedf} + 2\nabla_{be} R_{acdf} + 2\nabla_{eb} R_{acdf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right)$$

$$\texttt{ex-12.303} := \frac{2}{45}g^{bc}x^d x^e x^f z^g z^h \left(4R_{adbe} R_{cgfh} - 2R_{agbd} R_{cefh} - R_{adbg} R_{cefh} + R_{abdg} R_{cefh}\right)$$

$$\texttt{ex-12.304} := \frac{1}{40}x^b x^c x^d z^e z^f \left(4\nabla_{bc} R_{aedf} + 2\nabla_{be} R_{acdf} + 2\nabla_{eb} R_{acdf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right)$$
$$+ \frac{2}{45}g^{bc}x^d x^e x^f z^g z^h \left(4R_{adbe} R_{cgfh} - 2R_{agbd} R_{cefh} - R_{adbg} R_{cefh} + R_{abdg} R_{cefh}\right)$$

$$\texttt{ex-12.305} := \frac{2}{3}x^b z^c z^d R_{acbd} + \frac{1}{12}x^b x^c z^d z^e \left(4\nabla_b R_{adce} + 2\nabla_d R_{abce} + \nabla_a R_{bdce}\right)$$
$$+ \frac{1}{40}x^b x^c x^d z^e z^f \left(4\nabla_{bc} R_{aedf} + 2\nabla_{be} R_{acdf} + 2\nabla_{eb} R_{acdf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right)$$
$$+ \frac{2}{45}g^{bc}x^d x^e x^f z^g z^h \left(4R_{adbe} R_{cgfh} - 2R_{agbd} R_{cefh} - R_{adbg} R_{cefh} + R_{abdg} R_{cefh}\right)$$

```
# ==========================================================================
# Calzetta
# note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}

GammaBar := z^{\nu} z^{\rho} (
              (2/3) R^{\mu}_{\nu\rho\sigma} x^{\sigma}
            + (1/12) (5 \nabla_{\lambda}{R^{\mu}_{\nu\rho\sigma}}
                      + \nabla_{\rho}{R^{\mu}_{\sigma\nu\lambda}}) x^{\sigma} x^{\lambda}
            + (1/6) (  (9/10) \nabla_{\tau\lambda}{R^{\mu}_{\rho\nu\sigma}}
                     + (3/20) (  \nabla_{\tau\rho}{R^{\mu}_{\sigma\nu\lambda}}
                               + \nabla_{\rho\tau}{R^{\mu}_{\sigma\nu\lambda}} )
                     + (1/60) (  21 R^{\mu}_{\lambda\xi\rho} R^{\xi}_{\sigma\nu\tau}
                               + 48 R^{\mu}_{\xi\rho\lambda} R^{\xi}_{\sigma\nu\tau}
                               - 37 R^{\mu}_{\sigma\xi\lambda} R^{\xi}_{\nu\rho\tau} ) ) x^{\sigma} x^{\lambda} x^{\tau} ).
                                                                     # cdb(ex-12.400,GammaBar)

# convert from Greek to Latin indices

distribute (GammaBar)
rename_dummies (GammaBar,"greek","latin")                            # cdb(ex-12.401,GammaBar)

# lower the \mu index

GammaBar := \delta_{a \mu} @(GammaBar).                              # cdb(ex-12.402,GammaBar)
distribute (GammaBar)                                                # cdb(ex-12.403,GammaBar)
eliminate_kronecker (GammaBar)                                       # cdb(ex-12.404,GammaBar)

# sort products

product_sort (GammaBar)                                              # cdb(ex-12.405,GammaBar)

checkpoint.append (GammaBar)

# Replace R with - R (Calzetta uses the non-MTW convention for Riemann)

substitute (GammaBar, $R_{a b c d} -> - R_{a b c d}$)                # cdb(ex-12.406,GammaBar)
substitute (GammaBar, $R^{a}_{b c d} -> - R^{a}_{b c d}$)            # cdb(ex-12.407,GammaBar)
```

```
substitute (GammaBar, $R^{a}_{b c d} -> g^{a e} R_{e b c d}$)  # cdb(ex-12.408,GammaBar)

cal1 = get_xterm (GammaBar,1)                                    # cdb(ex-12.500,cal1)
cal2 = get_xterm (GammaBar,2)                                    # cdb(ex-12.501,cal2)
cal3 = get_xterm (GammaBar,3)                                    # cdb(ex-12.502,cal3)

cal1 = reformat (cal1,3,1)                                       # cdb(ex-12.600,cal1)
cal2 = reformat (cal2,12,1)                                      # cdb(ex-12.601,cal2)
# cal3 = reformat (cal3,360,1)                                     # cdb(ex-12.602,cal3)

cal30 = get_gterm (cal3,0)                                       # cdb(ex-12.602,cal30)
cal31 = get_gterm (cal3,1)                                       # cdb(ex-12.603,cal31)

cal1  = reformat (cal1, 3,1)                                     # cdb(ex-12.604,cal1)
cal2  = reformat (cal2,12,1)                                     # cdb(ex-12.605,cal2)

cal30 = reformat (cal30,40,1)                                    # cdb(ex-12.606,cal30)
cal31 = reformat (cal31,360,1)                                   # cdb(ex-12.607,cal31)

cal3 := @(cal30) + @(cal31).                                     # cdb(ex-12.608,cal3)

GammaBar := @(cal1) + @(cal2) + @(cal3).                         # cdb(ex-12.409,GammaBar)

checkpoint.append (GammaBar)
```

$$\texttt{ex-12.400} := z^{\nu}z^{\rho}\left(\frac{2}{3}R^{\mu}{}_{\nu\rho\sigma}x^{\sigma} + \frac{1}{12}\left(5\nabla_{\lambda}R^{\mu}{}_{\nu\rho\sigma} + \nabla_{\rho}R^{\mu}{}_{\sigma\nu\lambda}\right)x^{\sigma}x^{\lambda} + \frac{1}{6}\left(\frac{9}{10}\nabla_{\tau\lambda}R^{\mu}{}_{\rho\nu\sigma} + \frac{3}{20}\nabla_{\tau\rho}R^{\mu}{}_{\sigma\nu\lambda} + \frac{3}{20}\nabla_{\rho\tau}R^{\mu}{}_{\sigma\nu\lambda} + \frac{7}{20}R^{\mu}{}_{\lambda\xi\rho}R^{\xi}{}_{\sigma\nu\tau} + \frac{4}{5}R^{\mu}{}_{\xi\rho\lambda}R^{\xi}{}_{\sigma\nu\tau} - \frac{37}{60}R^{\mu}{}_{\sigma\xi\lambda}R^{\xi}{}_{\nu\rho\tau}\right)x^{\sigma}x^{\lambda}x^{\tau}\right)$$

$$\texttt{ex-12.401} := \frac{2}{3}z^{a}z^{b}R^{\mu}{}_{abc}x^{c} + \frac{5}{12}z^{a}z^{b}\nabla_{d}R^{\mu}{}_{abc}x^{c}x^{d} + \frac{1}{12}z^{b}z^{d}\nabla_{d}R^{\mu}{}_{abc}x^{a}x^{c} + \frac{3}{20}z^{b}z^{a}\nabla_{de}R^{\mu}{}_{abc}x^{c}x^{e}x^{d} + \frac{1}{40}z^{b}z^{e}\nabla_{de}R^{\mu}{}_{abc}x^{a}x^{c}x^{d} + \frac{1}{40}z^{b}z^{d}\nabla_{de}R^{\mu}{}_{abc}x^{a}x^{c}x^{e} + \frac{7}{120}z^{e}z^{c}R^{\mu}{}_{abc}R^{b}{}_{def}x^{d}x^{a}x^{f} + \frac{2}{15}z^{e}z^{b}R^{\mu}{}_{abc}R^{a}{}_{def}x^{d}x^{c}x^{f} - \frac{37}{360}z^{d}z^{e}R^{\mu}{}_{abc}R^{b}{}_{def}x^{a}x^{c}x^{f}$$

$$\texttt{ex-12.402} := \delta_{a\mu}\left(\frac{2}{3}z^{g}z^{b}R^{\mu}{}_{gbc}x^{c} + \frac{5}{12}z^{g}z^{b}\nabla_{d}R^{\mu}{}_{gbc}x^{c}x^{d} + \frac{1}{12}z^{b}z^{d}\nabla_{d}R^{\mu}{}_{gbc}x^{g}x^{c} + \frac{3}{20}z^{b}z^{g}\nabla_{de}R^{\mu}{}_{gbc}x^{c}x^{e}x^{d} + \frac{1}{40}z^{b}z^{e}\nabla_{de}R^{\mu}{}_{gbc}x^{g}x^{c}x^{d} + \frac{1}{40}z^{b}z^{d}\nabla_{de}R^{\mu}{}_{gbc}x^{g}x^{c}x^{e} + \frac{7}{120}z^{e}z^{c}R^{\mu}{}_{gbc}R^{b}{}_{def}x^{d}x^{g}x^{f} + \frac{2}{15}z^{e}z^{b}R^{\mu}{}_{gbc}R^{g}{}_{def}x^{d}x^{c}x^{f} - \frac{37}{360}z^{d}z^{e}R^{\mu}{}_{gbc}R^{b}{}_{def}x^{g}x^{c}x^{f}\right)$$

$$\texttt{ex-12.403} := \frac{2}{3}\delta_{a\mu}z^{g}z^{b}R^{\mu}{}_{gbc}x^{c} + \frac{5}{12}\delta_{a\mu}z^{g}z^{b}\nabla_{d}R^{\mu}{}_{gbc}x^{c}x^{d} + \frac{1}{12}\delta_{a\mu}z^{b}z^{d}\nabla_{d}R^{\mu}{}_{gbc}x^{g}x^{c} + \frac{3}{20}\delta_{a\mu}z^{b}z^{g}\nabla_{de}R^{\mu}{}_{gbc}x^{c}x^{e}x^{d} + \frac{1}{40}\delta_{a\mu}z^{b}z^{e}\nabla_{de}R^{\mu}{}_{gbc}x^{g}x^{c}x^{d} + \frac{1}{40}\delta_{a\mu}z^{b}z^{d}\nabla_{de}R^{\mu}{}_{gbc}x^{g}x^{c}x^{e} + \frac{7}{120}\delta_{a\mu}z^{e}z^{c}R^{\mu}{}_{gbc}R^{b}{}_{def}x^{d}x^{g}x^{f} + \frac{2}{15}\delta_{a\mu}z^{e}z^{b}R^{\mu}{}_{gbc}R^{g}{}_{def}x^{d}x^{c}x^{f} - \frac{37}{360}\delta_{a\mu}z^{d}z^{e}R^{\mu}{}_{gbc}R^{b}{}_{def}x^{g}x^{c}x^{f}$$

$$\texttt{ex-12.404} := \frac{2}{3}z^{g}z^{b}R_{agbc}x^{c} + \frac{5}{12}z^{g}z^{b}\nabla_{d}R_{agbc}x^{c}x^{d} + \frac{1}{12}z^{b}z^{d}\nabla_{d}R_{agbc}x^{g}x^{c} + \frac{3}{20}z^{b}z^{g}\nabla_{de}R_{agbc}x^{c}x^{e}x^{d} + \frac{1}{40}z^{b}z^{e}\nabla_{de}R_{agbc}x^{g}x^{c}x^{d} + \frac{1}{40}z^{b}z^{d}\nabla_{de}R_{agbc}x^{g}x^{c}x^{e} + \frac{7}{120}z^{e}z^{c}R_{agbc}R^{b}{}_{def}x^{d}x^{g}x^{f} + \frac{2}{15}z^{e}z^{b}R_{agbc}R^{g}{}_{def}x^{d}x^{c}x^{f} - \frac{37}{360}z^{d}z^{e}R_{agbc}R^{b}{}_{def}x^{g}x^{c}x^{f}$$

$$\texttt{ex-12.405} := \frac{2}{3}x^{b}z^{c}z^{d}R_{adcb} + \frac{1}{12}x^{b}x^{c}z^{d}z^{e}\nabla_{e}R_{acdb} + \frac{5}{12}x^{b}x^{c}z^{d}z^{e}\nabla_{c}R_{aedb} + \frac{1}{40}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{fc}R_{adeb} + \frac{1}{40}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{cf}R_{adeb} + \frac{3}{20}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{cd}R_{afeb} - \frac{37}{360}x^{b}x^{c}x^{d}z^{e}z^{f}R_{adgb}R^{g}{}_{efc} + \frac{2}{15}x^{b}x^{c}x^{d}z^{e}z^{f}R_{ageb}R^{g}{}_{cfd} + \frac{7}{120}x^{b}x^{c}x^{d}z^{e}z^{f}R_{adge}R^{g}{}_{bfc}$$

$$\texttt{ex-12.406} := -\frac{2}{3}x^{b}z^{c}z^{d}R_{adcb} - \frac{1}{12}x^{b}x^{c}z^{d}z^{e}\nabla_{e}R_{acdb} - \frac{5}{12}x^{b}x^{c}z^{d}z^{e}\nabla_{c}R_{aedb} - \frac{1}{40}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{fc}R_{adeb} - \frac{1}{40}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{cf}R_{adeb} - \frac{3}{20}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{cd}R_{afeb} + \frac{37}{360}x^{b}x^{c}x^{d}z^{e}z^{f}R_{adgb}R^{g}{}_{efc} - \frac{2}{15}x^{b}x^{c}x^{d}z^{e}z^{f}R_{ageb}R^{g}{}_{cfd} - \frac{7}{120}x^{b}x^{c}x^{d}z^{e}z^{f}R_{adge}R^{g}{}_{bfc}$$

$$\texttt{ex-12.407} := -\frac{2}{3}x^{b}z^{c}z^{d}R_{adcb} - \frac{1}{12}x^{b}x^{c}z^{d}z^{e}\nabla_{e}R_{acdb} - \frac{5}{12}x^{b}x^{c}z^{d}z^{e}\nabla_{c}R_{aedb} - \frac{1}{40}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{fc}R_{adeb} - \frac{1}{40}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{cf}R_{adeb} - \frac{3}{20}x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{cd}R_{afeb} - \frac{37}{360}x^{b}x^{c}x^{d}z^{e}z^{f}R_{adgb}R^{g}{}_{efc} + \frac{2}{15}x^{b}x^{c}x^{d}z^{e}z^{f}R_{ageb}R^{g}{}_{cfd} + \frac{7}{120}x^{b}x^{c}x^{d}z^{e}z^{f}R_{adge}R^{g}{}_{bfc}$$

$$\texttt{ex-12.408} := -\frac{2}{3}x^b z^c z^d R_{adcb} - \frac{1}{12}x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12}x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb}$$
$$- \frac{3}{20}x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360}x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15}x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcfd} + \frac{7}{120}x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}$$

$$\texttt{ex-12.500} := -\frac{2}{3}x^b z^c z^d R_{adcb}$$

$$\texttt{ex-12.501} := -\frac{1}{12}x^b x^c z^d z^e \nabla_e R_{acdb} - \frac{5}{12}x^b x^c z^d z^e \nabla_c R_{aedb}$$

$$\begin{aligned}\texttt{ex-12.502} := &-\frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20}x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb}\\ &- \frac{37}{360}x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15}x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcfd} + \frac{7}{120}x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}\end{aligned}$$

$\texttt{ex-12.600} := \frac{2}{3}x^b z^c z^d R_{acbd}$

$\texttt{ex-12.601} := \frac{1}{12}x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5\nabla_b R_{adce}\right)$

$\texttt{ex-12.602} := -\frac{1}{40}x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} - \frac{3}{20}x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb}$

$\texttt{ex-12.603} := -\frac{37}{360}x^b x^c x^d z^e z^f R_{adgb} g^{gh} R_{hefc} + \frac{2}{15}x^b x^c x^d z^e z^f R_{ageb} g^{gh} R_{hcfd} + \frac{7}{120}x^b x^c x^d z^e z^f R_{adge} g^{gh} R_{hbfc}$

$\texttt{ex-12.604} := \frac{2}{3}x^b z^c z^d R_{acbd}$

$\texttt{ex-12.605} := \frac{1}{12}x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5\nabla_b R_{adce}\right)$

$\texttt{ex-12.606} := \frac{1}{40}x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}\right)$

$\texttt{ex-12.607} := \frac{1}{360}g^{bc} x^d x^e x^f z^g z^h \left(37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh}\right)$

$\texttt{ex-12.608} := \frac{1}{40}x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}\right) + \frac{1}{360}g^{bc} x^d x^e x^f z^g z^h \left(37 R_{adbe} R_{cgfh} - 21 R_{adbg} R_{cefh} + 48 R_{abdg} R_{cefh}\right)$

$$\texttt{ex-12.409} := \frac{2}{3}x^b z^c z^d R_{acbd} + \frac{1}{12}x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5\nabla_b R_{adce}\right) + \frac{1}{40}x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}\right)$$
$$+ \frac{1}{360}g^{bc} x^d x^e x^f z^g z^h \left(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh}\right)$$

The fun begins $\Gamma - \bar{\Gamma}$

It's now time to compute the difference $\Gamma - \bar{\Gamma}$. Here it is.

```
def reformat_diff (obj):

    distribute (obj)

    obj1  = get_xterm (obj,1)
    obj2  = get_xterm (obj,2)
    obj3  = get_xterm (obj,3)

    obj30 = get_gterm (obj3,0)
    obj31 = get_gterm (obj3,1)

    obj1  = reformat (obj1, 3,1)
    obj2  = reformat (obj2,12,1)

    obj30 = reformat (obj30,40,1)
    obj31 = reformat (obj31,360,1)

    obj3 := @(obj30) + @(obj31).

    ans := @(obj1) + @(obj2) + @(obj3).

    return ans

# We could use reformat_diff here but instead we'll do it one step at a time so that
# we can see exactly what's going on. Later on we will use reformat_diff to do the job.

diff := @(Gamma) - @(GammaBar).                                 # cdb(ex-12.diff.100,diff)
distribute (diff)

diff1  = get_xterm (diff,1)                                     # cdb(ex-12.diff.200,diff1)
diff2  = get_xterm (diff,2)                                     # cdb(ex-12.diff.201,diff2)
diff3  = get_xterm (diff,3)                                     # cdb(ex-12.diff.202,diff3)

diff30 = get_gterm (diff3,0)                                    # cdb(ex-12.diff.203,diff30)
```

```
diff31 = get_gterm (diff3,1)                                         # cdb(ex-12.diff.204,diff31)

diff1  = reformat (diff1, 3,1)                                       # cdb(ex-12.diff.300,diff1)
diff2  = reformat (diff2,12,1)                                       # cdb(ex-12.diff.301,diff2)

diff30 = reformat (diff30,40,1)                                      # cdb(ex-12.diff.302,diff30)
diff31 = reformat (diff31,360,1)                                     # cdb(ex-12.diff.303,diff31)

diff3 := @(diff30) + @(diff31).                                      # cdb(ex-12.diff.304,diff3)

diff := @(diff1) + @(diff2) + @(diff3).                              # cdb(ex-12.diff.305,diff)
```

$$\texttt{ex-12.diff.100} := \frac{1}{12}x^b x^c z^d z^e \left(4\nabla_b R_{adce} + 2\nabla_d R_{abce} + \nabla_a R_{bdce}\right) + \frac{1}{40}x^b x^c x^d z^e z^f \left(4\nabla_{bc} R_{aedf} + 2\nabla_{be} R_{acdf} + 2\nabla_{eb} R_{acdf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right) + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h \left(4R_{adbe}R_{cgfh} - 2R_{agbd}R_{cefh} - R_{adbg}R_{cefh} + R_{abdg}R_{cefh}\right) - \frac{1}{12}x^b x^c z^d z^e \left(\nabla_d R_{abce} + 5\nabla_b R_{adce}\right) - \frac{1}{40}x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}\right) - \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh}\right)$$

$$\texttt{ex-12.diff.200} := 0$$

$$\texttt{ex-12.diff.201} := -\frac{1}{12}x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12}x^b x^c z^d z^e \nabla_d R_{abce} + \frac{1}{12}x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\texttt{ex-12.diff.202} := -\frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh}$$

$$\texttt{ex-12.diff.203} := -\frac{1}{20}x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40}x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$

$$\texttt{ex-12.diff.204} := \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh}$$

$$\texttt{ex-12.diff.300} := 0$$

$$\texttt{ex-12.diff.301} := \frac{1}{12} x^b x^c z^d z^e \left(\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce}\right)$$

$$\texttt{ex-12.diff.302} := \frac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right)$$

$$\texttt{ex-12.diff.303} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}\right)$$

$$\begin{aligned}\texttt{ex-12.diff.304} := & \frac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}\right)\end{aligned}$$

$$\begin{aligned}\texttt{ex-12.diff.305} := & \frac{1}{12} x^b x^c z^d z^e \left(\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce}\right) + \frac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}\right)\end{aligned}$$

Second order terms

```
diff2  = get_xterm (diff,2)
diff2 := 12 @(diff2).                                                     # cdb (ex-12.701,diff2)
distribute (diff2)                                                        # cdb (ex-12.702,diff2)

diff2 = add_tags (diff2,'\\mu')                                           # cdb (ex-12.711,diff2)

# swap indices on middle term, then apply 2nd Bianchi identity

zoom       (diff2, $\mu_{1} Q??$)                                         # cdb (ex-12.712,diff2)
substitute (diff2, $\nabla_{b}{R_{a d c e}} -> - \nabla_{b}{R_{d a c e}}$)   # cdb (ex-12.713,diff2)
unzoom     (diff2)

substitute (diff2, $\mu_{1} -> \mu_{0}, \mu_{2} -> \mu_{0}$)              # cdb (ex-12.714,diff2)
substitute (diff2, $\mu_{0} -> 0$)                                        # cdb (ex-12.715,diff2)

diff2 = clear_tags (diff2,'\\mu')                                         # cdb (ex-12.716,diff2)

diff2 := @(diff2) / 12 .

diff := @(diff1) + @(diff2) + @(diff3).

diff  = reformat_diff (diff)                                              # cdb(ex-12.diff.306,diff)
```

$$\texttt{ex-12.701} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$
$$\texttt{ex-12.702} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\texttt{ex-12.711} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_2 x^b x^c z^d z^e \nabla_a R_{bdce}$$
$$\texttt{ex-12.712} := \ldots - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \ldots$$
$$\texttt{ex-12.713} := \ldots + \mu_1 x^b x^c z^d z^e \nabla_b R_{dace} + \ldots$$
$$\texttt{ex-12.714} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} + \mu_0 x^b x^c z^d z^e \nabla_b R_{dace} + \mu_0 x^b x^c z^d z^e \nabla_a R_{bdce}$$
$$\texttt{ex-12.715} := 0$$
$$\texttt{ex-12.716} := 0$$

$$\begin{aligned}\texttt{ex-12.diff.306} := {}& \frac{1}{40} x^b x^c x^d z^e z^f \left(\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}\right) \\ &+ \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h \left(-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}\right)\end{aligned}$$

Third order terms, commute $\nabla\nabla R$ terms

```
diff3  = get_xterm (diff,3)
diff3 := 360 @(diff3).                                  # cdb (ex-12.801,diff3)
distribute (diff3)                                      # cdb (ex-12.802,diff3)

# commutation rule for covariant derivs on Rabcd, see exrecise 3.6
# note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}
CommuteNablaRiemann := \nabla_{f e}(R_{a b c d}) -> \nabla_{e f}(R_{a b c d})
                                                  + g^{u v} R_{u a e f} R_{v b c d}
                                                  + g^{u v} R_{u b e f} R_{a v c d}
                                                  + g^{u v} R_{u c e f} R_{a b v d}
                                                  + g^{u v} R_{u d e f} R_{a b c v}.

diff3 = add_tags (diff3,'\\mu')                         # cdb (ex-12.901,diff3)

# commute derivs on Rabcd so that each double deriv is of the form \nabla_{b*}

substitute (diff3, $\mu_{3} -> \mu_{1}$)                # cdb (ex-12.902,diff3)

zoom       (diff3, $\mu_{1} Q??$)                       # cdb (ex-12.903,diff3)
substitute (diff3, CommuteNablaRiemann)                 # cdb (ex-12.904,diff3)
unzoom     (diff3)

diff3 = clear_tags (diff3,'\\mu')
diff3 := @(diff3) / 360 .

distribute   (diff3)
canonicalise (diff3)                                    # cdb (ex-12.905,diff3)

diff := @(diff1) + @(diff2) + @(diff3).

diff  = reformat_diff (diff)                            # cdb(ex-12.diff.307,diff)
```

$$\texttt{ex-12.801} := 9x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} + 9x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} - 18x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + 9x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + 9x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf} - 32g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + 27g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} + 5g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - 32g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}$$

$$\texttt{ex-12.802} := 9x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} + 9x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} - 18x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + 9x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + 9x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf} - 32g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + 27g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} + 5g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - 32g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}$$

$$\texttt{ex-12.901} := 9\mu_0x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} + 9\mu_1x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} - 18\mu_2x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + 9\mu_3x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + 9\mu_4x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf} - 32\mu_5g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + 27\mu_6g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} + 5\mu_7g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - 32\mu_8g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}$$

$$\texttt{ex-12.902} := 9\mu_0x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} + 9\mu_1x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} - 18\mu_2x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + 9\mu_1x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + 9\mu_4x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf} - 32\mu_5g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + 27\mu_6g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} + 5\mu_7g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - 32\mu_8g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}$$

$$\texttt{ex-12.903} := \ldots + 9\mu_1x^bx^cx^dz^ez^f\nabla_{eb}R_{acdf} + \ldots + 9\mu_1x^bx^cx^dz^ez^f\nabla_{ab}R_{cedf} + \ldots$$

$$\texttt{ex-12.904} := \ldots + 9\mu_1x^bx^cx^dz^ez^f\left(\nabla_{be}R_{acdf} + g^{uv}R_{uabe}R_{vcdf} + g^{uv}R_{ucbe}R_{avdf} + g^{uv}R_{udbe}R_{acvf} + g^{uv}R_{ufbe}R_{acdv}\right) + \ldots + 9\mu_1x^bx^cx^dz^ez^f\left(\nabla_{ba}R_{cedf} + g^{uv}R_{ucba}R_{vedf} + g^{uv}R_{ueba}R_{cvdf} + g^{uv}R_{udba}R_{cevf} + g^{uv}R_{ufba}R_{cedv}\right) + \ldots$$

$$\texttt{ex-12.905} := \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{be}R_{acdf} + \frac{3}{40}x^bx^cx^dz^ez^fg^{uv}R_{abeu}R_{cfdv} - \frac{3}{40}x^bx^cx^dz^ez^fg^{uv}R_{abcu}R_{defv} - \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{aedf} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{ba}R_{cedf} - \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{abdg}R_{cefh} + \frac{3}{40}g^{bc}x^dx^ex^fz^gz^hR_{adbe}R_{cgfh} + \frac{1}{72}g^{bc}x^dx^ex^fz^gz^hR_{adbg}R_{cefh} - \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{agbd}R_{cefh}$$

$$\texttt{ex-12.diff.307} := \frac{1}{40}x^bx^cx^dz^ez^f\left(2\nabla_{be}R_{acdf} - 2\nabla_{bc}R_{aedf} + 2\nabla_{ba}R_{cedf}\right) + \frac{1}{360}g^{bc}x^dx^ex^fz^gz^h\left(-32R_{abdg}R_{cefh} + 32R_{adbg}R_{cefh} - 32R_{agbd}R_{cefh}\right)$$

Third order terms, use 2nd Bianchi identity on $\nabla\nabla R$ terms

```
diff3  = get_xterm (diff,3)
diff3 := 360 @(diff3).                                                                  # cdb (ex-12.910,diff3)
distribute (diff3)                                                                      # cdb (ex-12.911,diff3)

diff3 = add_tags (diff3,'\\mu')                                                         # cdb (ex-12.912,diff3)

# swap indices on middle second deriv term, then apply 2nd Bianchi identity

zoom       (diff3, $\mu_{1} Q??$)                                                       # cdb (ex-12.913,diff3)
substitute (diff3, $\nabla_{b c}{R_{a e d f}} -> - \nabla_{b c}{R_{e a d f}}$)   # cdb (ex-12.914,diff3)
unzoom     (diff3)

substitute (diff3, $\mu_{1} -> \mu_{0}, \mu_{2} -> \mu_{0}$)                            # cdb (ex-12.915,diff3)
substitute (diff3, $\mu_{0} -> 0$)                                                      # cdb (ex-12.916,diff3)

diff3 = clear_tags (diff3,'\\mu')
diff3 := @(diff3) / 360 .

distribute   (diff3)
canonicalise (diff3)                                                                    # cdb (ex-12.917,diff3)

diff := @(diff1) + @(diff2) + @(diff3).

diff  = reformat_diff (diff)                                                            # cdb(ex-12.diff.308,diff)
```

$$\texttt{ex-12.910} := 18x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{be}R_{acdf} - 18x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{bc}R_{aedf} + 18x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{ba}R_{cedf} - 32g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{abdg}R_{cefh} + 32g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{adbg}R_{cefh} - 32g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{agbd}R_{cefh}$$

$$\texttt{ex-12.911} := 18x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{be}R_{acdf} - 18x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{bc}R_{aedf} + 18x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{ba}R_{cedf} - 32g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{abdg}R_{cefh} + 32g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{adbg}R_{cefh} - 32g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{agbd}R_{cefh}$$

$$\texttt{ex-12.912} := 18\mu_0x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{be}R_{acdf} - 18\mu_1x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{bc}R_{aedf} + 18\mu_2x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{ba}R_{cedf} - 32\mu_3g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{abdg}R_{cefh} + 32\mu_4g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{adbg}R_{cefh} - 32\mu_5g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{agbd}R_{cefh}$$

$$\texttt{ex-12.913} := \ldots - 18\mu_1x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{bc}R_{aedf} + \ldots$$

$$\texttt{ex-12.914} := \ldots + 18\mu_1x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{bc}R_{eadf} + \ldots$$

$$\texttt{ex-12.915} := 18\mu_0x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{be}R_{acdf} + 18\mu_0x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{bc}R_{eadf} + 18\mu_0x^{b}x^{c}x^{d}z^{e}z^{f}\nabla_{ba}R_{cedf} - 32\mu_3g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{abdg}R_{cefh} + 32\mu_4g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{adbg}R_{cefh} - 32\mu_5g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{agbd}R_{cefh}$$

$$\texttt{ex-12.916} := -32\mu_3g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{abdg}R_{cefh} + 32\mu_4g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{adbg}R_{cefh} - 32\mu_5g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{agbd}R_{cefh}$$

$$\texttt{ex-12.917} := -\frac{4}{45}g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{abdg}R_{cefh} + \frac{4}{45}g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{adbg}R_{cefh} - \frac{4}{45}g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}R_{agbd}R_{cefh}$$

$$\texttt{ex-12.diff.308} := \frac{1}{360}g^{bc}x^{d}x^{e}x^{f}z^{g}z^{h}\left(-32R_{abdg}R_{cefh} + 32R_{adbg}R_{cefh} - 32R_{agbd}R_{cefh}\right)$$

Third order terms, use 1st Bianchi identity on RR terms

```
diff3  = get_xterm (diff,3)
diff3 := 360 @(diff3).
distribute (diff3)

diff3 = add_tags (diff3,'\\mu')                                         # cdb (ex-12.921,diff3)

# swap indices on middle term, then apply 1st Bianchi identity

zoom       (diff3, $\mu_{1} Q??$)                                        # cdb (ex-12.922,diff3)
substitute (diff3, $R_{a d b g} R_{c e f h} -> - R_{a d g b} R_{c e f h}$)   # cdb (ex-12.923,diff3)
unzoom     (diff3)

substitute (diff3, $\mu_{1} -> \mu_{0}, \mu_{2} -> \mu_{0}$)             # cdb (ex-12.924,diff3)
substitute (diff3, $\mu_{0} -> 0$)                                       # cdb (ex-12.925,diff3)

diff3 = clear_tags (diff3,'\\mu')                                       # cdb (ex-12.926,diff3)

diff := @(diff1) + @(diff2) + @(diff3).

diff  = reformat_diff (diff)                                            # cdb(ex-12.diff.309,diff)
```

$\texttt{ex-12.921} := -32\mu_0 g^{bc}x^d x^e x^f z^g z^h R_{abdg}R_{cefh} + 32\mu_1 g^{bc}x^d x^e x^f z^g z^h R_{adbg}R_{cefh} - 32\mu_2 g^{bc}x^d x^e x^f z^g z^h R_{agbd}R_{cefh}$

$\texttt{ex-12.922} := \ldots + 32\mu_1 g^{bc}x^d x^e x^f z^g z^h R_{adbg}R_{cefh} + \ldots$

$\texttt{ex-12.923} := \ldots - 32\mu_1 g^{bc}x^d x^e x^f z^g z^h R_{adgb}R_{cefh} + \ldots$

$\texttt{ex-12.924} := -32\mu_0 g^{bc}x^d x^e x^f z^g z^h R_{abdg}R_{cefh} - 32\mu_0 g^{bc}x^d x^e x^f z^g z^h R_{adgb}R_{cefh} - 32\mu_0 g^{bc}x^d x^e x^f z^g z^h R_{agbd}R_{cefh}$

$\texttt{ex-12.925} := 0$

$\texttt{ex-12.926} := 0$

$$\texttt{ex-12.diff.309} := 0$$

Example 13a The Weyl tensor vanishes in 3d – direct proof

```
{x,y,z}::Coordinate.
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices (values={x,y,z},position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

{\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.

GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                             + \partial_{c}{g_{b d}}
                                             - \partial_{d}{g_{b c}}). # cdb(Gamma.000,GammaU)

GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
                                   + \partial_{c}{g_{b a}}
                                   - \partial_{a}{g_{b c}}).           # cdb(Gamma.010,GammaD)

Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
                        - \partial_{d}{\Gamma_{a b c}}
                        + \Gamma_{e a d} \Gamma^{e}_{b c}
                        - \Gamma_{e a c} \Gamma^{e}_{b d}.             # cdb (Rabcd.000,Rabcd)

Rab     := R_{a b} -> g^{c d} R_{a c b d}.                             # cdb (Rab.000,Rab)

Rscalar := R -> g^{a b} R_{a b}.                                       # cdb (R.000,Rscalar)

# Weyl in 3-dimensions

Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
                     - (g_{a c} R_{b d} - g_{a d} R_{b c})
               + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).            # cdb (ex-13a.100,Cabcd)

# Use 8 Cabcd to clear the fractions

EightCabcd := 8 @(Cabcd).                                              # cdb (ex-13a.110,EightCabcd)
```

```
substitute     (Cabcd,Rscalar)
substitute     (Cabcd,Rab)
substitute     (Cabcd,Rabcd)
substitute     (Cabcd,GammaU)
substitute     (Cabcd,GammaD)

distribute     (Cabcd)

sort_product   (Cabcd)
rename_dummies (Cabcd)
canonicalise   (Cabcd)                                               # cdb (ex-13a.101,Cabcd)

EightCabcd := 8 @(Cabcd).                                            # cdb (ex-13a.111,EightCabcd)

gab := {g_{x x} = gxx, g_{x y} = gxy, g_{x z} = gxz,
        g_{y x} = gxy, g_{y y} = gyy, g_{y z} = gyz,
        g_{z x} = gxz, g_{z y} = gyz, g_{z z} = gzz}.

complete  (gab, $g^{a b}$)
evaluate  (Cabcd,gab)                                                # cdb (ex-13a.102,Cabcd)
evaluate  (EightCabcd,gab)                                           # cdb (ex-13a.112,EightCabcd)
```

$$
\begin{aligned}
8C_{abcd} &= 8R_{abcd} - 8R_{ac}g_{bd} + 8R_{ad}g_{bc} - 8g_{ac}R_{bd} + 8g_{ad}R_{bc} + 4R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) && \texttt{(ex-13a.110)} \\
&= 4\partial_{bc}g_{ad} - 4\partial_{ac}g_{bd} - 4\partial_{bd}g_{ac} + 4\partial_{ad}g_{bc} + 2\partial_{a}g_{de}\partial_{b}g_{cf}g^{ef} + 2\partial_{a}g_{de}\partial_{c}g_{bf}g^{ef} - 2\partial_{a}g_{de}\partial_{f}g_{bc}g^{ef} + 2\partial_{b}g_{ce}\partial_{d}g_{af}g^{ef} + 2\partial_{c}g_{be}\partial_{d}g_{af}g^{ef} \\
&\quad - 2\partial_{d}g_{ae}\partial_{f}g_{bc}g^{ef} - 2\partial_{b}g_{ce}\partial_{f}g_{ad}g^{ef} - 2\partial_{c}g_{be}\partial_{f}g_{ad}g^{ef} + 2\partial_{e}g_{ad}\partial_{f}g_{bc}g^{ef} - 2\partial_{a}g_{ce}\partial_{b}g_{df}g^{ef} - 2\partial_{a}g_{ce}\partial_{d}g_{bf}g^{ef} + 2\partial_{a}g_{ce}\partial_{f}g_{bd}g^{ef} - 2\partial_{b}g_{de}\partial_{c}g_{af}g^{ef} \\
&\quad - 2\partial_{c}g_{ae}\partial_{d}g_{bf}g^{ef} + 2\partial_{c}g_{ae}\partial_{f}g_{bd}g^{ef} + 2\partial_{b}g_{de}\partial_{f}g_{ac}g^{ef} + 2\partial_{d}g_{be}\partial_{f}g_{ac}g^{ef} - 2\partial_{e}g_{ac}\partial_{f}g_{bd}g^{ef} - 4\partial_{ce}g_{af}g_{bd}g^{ef} + 4\partial_{ac}g_{ef}g_{bd}g^{ef} + 4\partial_{ef}g_{ac}g_{bd}g^{ef} \\
&\quad - 4\partial_{ae}g_{cf}g_{bd}g^{ef} - 2\partial_{a}g_{ef}\partial_{c}g_{gh}g_{bd}g^{eg}g^{fh} - 4\partial_{e}g_{af}\partial_{g}g_{ch}g_{bd}g^{eg}g^{fh} + 4\partial_{e}g_{af}\partial_{g}g_{ch}g_{bd}g^{eh}g^{fg} + 4\partial_{a}g_{ce}\partial_{f}g_{gh}g_{bd}g^{eg}g^{fh} - 2\partial_{a}g_{ce}\partial_{f}g_{gh}g_{bd}g^{ef}g^{gh} \\
&\quad + 4\partial_{c}g_{ae}\partial_{f}g_{gh}g_{bd}g^{eg}g^{fh} - 2\partial_{c}g_{ae}\partial_{f}g_{gh}g_{bd}g^{ef}g^{gh} - 4\partial_{e}g_{ac}\partial_{f}g_{gh}g_{bd}g^{eg}g^{fh} + 2\partial_{e}g_{ac}\partial_{f}g_{gh}g_{bd}g^{ef}g^{gh} + 4\partial_{de}g_{af}g_{bc}g^{ef} - 4\partial_{ad}g_{ef}g_{bc}g^{ef} \\
&\quad - 4\partial_{ef}g_{ad}g_{bc}g^{ef} + 4\partial_{ae}g_{df}g_{bc}g^{ef} + 2\partial_{a}g_{ef}\partial_{d}g_{gh}g_{bc}g^{eg}g^{fh} + 4\partial_{e}g_{af}\partial_{g}g_{dh}g_{bc}g^{eg}g^{fh} - 4\partial_{e}g_{af}\partial_{g}g_{dh}g_{bc}g^{eh}g^{fg} - 4\partial_{a}g_{de}\partial_{f}g_{gh}g_{bc}g^{eg}g^{fh} \\
&\quad + 2\partial_{a}g_{de}\partial_{f}g_{gh}g_{bc}g^{ef}g^{gh} - 4\partial_{d}g_{ae}\partial_{f}g_{gh}g_{bc}g^{eg}g^{fh} + 2\partial_{d}g_{ae}\partial_{f}g_{gh}g_{bc}g^{ef}g^{gh} + 4\partial_{e}g_{ad}\partial_{f}g_{gh}g_{bc}g^{eg}g^{fh} - 2\partial_{e}g_{ad}\partial_{f}g_{gh}g_{bc}g^{ef}g^{gh} - 4\partial_{de}g_{bf}g_{ac}g^{ef} \\
&\quad + 4\partial_{bd}g_{ef}g_{ac}g^{ef} + 4\partial_{ef}g_{bd}g_{ac}g^{ef} - 4\partial_{be}g_{df}g_{ac}g^{ef} - 2\partial_{b}g_{ef}\partial_{d}g_{gh}g_{ac}g^{eg}g^{fh} - 4\partial_{e}g_{bf}\partial_{g}g_{dh}g_{ac}g^{eg}g^{fh} + 4\partial_{e}g_{bf}\partial_{g}g_{dh}g_{ac}g^{eh}g^{fg} + 4\partial_{b}g_{de}\partial_{f}g_{gh}g_{ac}g^{eg}g^{fh} \\
&\quad - 2\partial_{b}g_{de}\partial_{f}g_{gh}g_{ac}g^{ef}g^{gh} + 4\partial_{d}g_{be}\partial_{f}g_{gh}g_{ac}g^{eg}g^{fh} - 2\partial_{d}g_{be}\partial_{f}g_{gh}g_{ac}g^{ef}g^{gh} - 4\partial_{e}g_{bd}\partial_{f}g_{gh}g_{ac}g^{eg}g^{fh} + 2\partial_{e}g_{bd}\partial_{f}g_{gh}g_{ac}g^{ef}g^{gh} + 4\partial_{ce}g_{bf}g_{ad}g^{ef} \\
&\quad - 4\partial_{bc}g_{ef}g_{ad}g^{ef} - 4\partial_{ef}g_{bc}g_{ad}g^{ef} + 4\partial_{be}g_{cf}g_{ad}g^{ef} + 2\partial_{b}g_{ef}\partial_{c}g_{gh}g_{ad}g^{eg}g^{fh} + 4\partial_{e}g_{bf}\partial_{g}g_{ch}g_{ad}g^{eg}g^{fh} - 4\partial_{e}g_{bf}\partial_{g}g_{ch}g_{ad}g^{eh}g^{fg} - 4\partial_{b}g_{ce}\partial_{f}g_{gh}g_{ad}g^{eg}g^{fh} \\
&\quad + 2\partial_{b}g_{ce}\partial_{f}g_{gh}g_{ad}g^{ef}g^{gh} - 4\partial_{c}g_{be}\partial_{f}g_{gh}g_{ad}g^{eg}g^{fh} + 2\partial_{c}g_{be}\partial_{f}g_{gh}g_{ad}g^{ef}g^{gh} + 4\partial_{e}g_{bc}\partial_{f}g_{gh}g_{ad}g^{eg}g^{fh} - 2\partial_{e}g_{bc}\partial_{f}g_{gh}g_{ad}g^{ef}g^{gh} + 4\partial_{ef}g_{gh}g_{ac}g_{bd}g^{eg}g^{fh} \\
&\quad - 4\partial_{ef}g_{gh}g_{ad}g_{bc}g^{eg}g^{fh} - 4\partial_{ef}g_{gh}g_{ac}g_{bd}g^{ef}g^{gh} + 4\partial_{ef}g_{gh}g_{ad}g_{bc}g^{ef}g^{gh} - 2\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{ei}g^{fh}g^{gj} + 2\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{ei}g^{fh}g^{gj} \\
&\quad + 3\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{eh}g^{fi}g^{gj} - 3\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{eh}g^{fi}g^{gj} - 4\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{ef}g^{gi}g^{hj} + 4\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{ef}g^{gi}g^{hj} \\
&\quad + 4\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{ef}g^{gh}g^{ij} - 4\partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{ef}g^{gh}g^{ij} - \partial_{e}g_{fg}\partial_{h}g_{ij}g_{ac}g_{bd}g^{eh}g^{fg}g^{ij} + \partial_{e}g_{fg}\partial_{h}g_{ij}g_{ad}g_{bc}g^{eh}g^{fg}g^{ij} && \texttt{(ex-13a.111)} \\
&= 0 && \texttt{(ex-13a.112)}
\end{aligned}
$$

Example 13b The Weyl tensor vanishes in 3d – orthonormal basis

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.

ex{#}::LaTeXForm{"e_x"}.
ey{#}::LaTeXForm{"e_y"}.
ez{#}::LaTeXForm{"e_z"}.

{R_{a b c d}, g_{a b}, g^{a b}}::SortOrder.

Rab     := R_{a b} -> g^{c d} R_{a c b d}.

Rscalar := R -> g^{a b} R_{a b}.

gab := g^{a b} -> ex^{a} ex^{b} + ey^{a} ey^{b} + ez^{a} ez^{b}.

ortho := {ex^{a} ex^{b} g_{a b} -> 1, ey^{a} ey^{b} g_{a b} -> 1, ez^{a} ez^{b} g_{a b} -> 1,
          ex^{a} ey^{b} g_{a b} -> 0, ex^{a} ez^{b} g_{a b} -> 0,
          ey^{a} ex^{b} g_{a b} -> 0, ey^{a} ez^{b} g_{a b} -> 0,
          ez^{a} ex^{b} g_{a b} -> 0, ez^{a} ey^{b} g_{a b} -> 0}.

# Weyl in 3-dimensions

Cabcd := R_{a b c d} - (R_{a c} g_{b d} - R_{a d} g_{b c})
                     - (g_{a c} R_{b d} - g_{a d} R_{b c})
               + 1/2 R (g_{a c} g_{b d} - g_{a d} g_{b c}).   # cdb (ex-13b.100,Cabcd)


substitute     (Cabcd, Rscalar)                               # cdb(ex-13b.101,Cabcd)
substitute     (Cabcd, Rab)                                   # cdb(ex-13b.102,Cabcd)
distribute     (Cabcd)                                        # cdb(ex-13b.103,Cabcd)

Cabcd := C_{a b c d} -> @(Cabcd).
```

```

Cxyxy := C_{a b c d} ex^{a} ey^{b} ex^{c} ey^{d}.                 # cdb(ex-13b.104,Cxyxy)

substitute     (Cxyxy,Cabcd)                                     # cdb(ex-13b.105,Cxyxy)
distribute     (Cxyxy)                                           # cdb(ex-13b.106,Cxyxy)

substitute     (Cxyxy, ortho, repeat=True)                       # cdb(ex-13b.107,Cxyxy)

substitute     (Cxyxy, gab)                                      # cdb(ex-13b.108,Cxyxy)
distribute     (Cxyxy)                                           # cdb(ex-13b.109,Cxyxy)

sort_product   (Cxyxy)                                           # cdb(ex-13b.110,Cxyxy)
rename_dummies (Cxyxy)                                           # cdb(ex-13b.111,Cxyxy)
canonicalise   (Cxyxy)                                           # cdb(ex-13b.112,Cxyxy)
```

$\texttt{ex-13b.101} := R_{abcd} - R_{ac}g_{bd} + R_{ad}g_{bc} - g_{ac}R_{bd} + g_{ad}R_{bc} + \frac{1}{2}g^{ef}R_{ef}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$

$\texttt{ex-13b.102} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$

$\texttt{ex-13b.103} := R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}$

$$C_{abcd}e_x^a e_y^b e_x^c e_y^d = \left(R_{abcd} - g^{ef} R_{aecf} g_{bd} + g^{fe} R_{afde} g_{bc} - g_{ac} g^{fe} R_{bfde} + g_{ad} g^{ef} R_{becf} + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} \right) e_x^a e_y^b e_x^c e_y^d \tag{ex-13b.105}$$

$$\begin{split} &= R_{abcd} e_x^a e_y^b e_x^c e_y^d - g^{ef} R_{aecf} g_{bd} e_x^a e_y^b e_x^c e_y^d + g^{fe} R_{afde} g_{bc} e_x^a e_y^b e_x^c e_y^d - g_{ac} g^{fe} R_{bfde} e_x^a e_y^b e_x^c e_y^d + g_{ad} g^{ef} R_{becf} e_x^a e_y^b e_x^c e_y^d \\ &\quad + \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ac} g_{bd} e_x^a e_y^b e_x^c e_y^d - \frac{1}{2} g^{ef} g^{gh} R_{egfh} g_{ad} g_{bc} e_x^a e_y^b e_x^c e_y^d \end{split} \tag{ex-13b.106}$$

$$= R_{abcd} e_x^a e_y^b e_x^c e_y^d - g^{ef} R_{aecf} e_x^a e_x^c - g^{fe} R_{bfde} e_y^b e_y^d + \frac{1}{2} g^{ef} g^{gh} R_{egfh} \tag{ex-13b.107}$$

$$\begin{split} &= R_{abcd} e_x^a e_y^b e_x^c e_y^d - \left(e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f \right) R_{aecf} e_x^a e_x^c - \left(e_x^f e_x^e + e_y^f e_y^e + e_z^f e_z^e \right) R_{bfde} e_y^b e_y^d \\ &\quad + \frac{1}{2} \left(e_x^e e_x^f + e_y^e e_y^f + e_z^e e_z^f \right) \left(e_x^g e_x^h + e_y^g e_y^h + e_z^g e_z^h \right) R_{egfh} \end{split} \tag{ex-13b.108}$$

$$\begin{split} &= R_{abcd} e_x^a e_y^b e_x^c e_y^d - e_x^e e_x^f R_{aecf} e_x^a e_x^c - e_y^e e_y^f R_{aecf} e_x^a e_x^c - e_z^e e_z^f R_{aecf} e_x^a e_x^c - e_x^f e_x^e R_{bfde} e_y^b e_y^d - e_y^f e_y^e R_{bfde} e_y^b e_y^d - e_z^f e_z^e R_{bfde} e_y^b e_y^d \\ &\quad + \frac{1}{2} e_x^e e_x^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_x^e e_x^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_x^e e_x^f e_z^g e_z^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_x^g e_x^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_y^e e_y^f e_z^g e_z^h R_{egfh} + \frac{1}{2} e_z^e e_z^f e_x^g e_x^h R_{egfh} \\ &\quad + \frac{1}{2} e_z^e e_z^f e_y^g e_y^h R_{egfh} + \frac{1}{2} e_z^e e_z^f e_z^g e_z^h R_{egfh} \end{split} \tag{ex-13b.109}$$

$$\begin{split} &= R_{abcd} e_x^a e_x^c e_y^b e_y^d - R_{aecf} e_x^a e_x^c e_x^e e_x^f - R_{aecf} e_x^a e_x^c e_y^e e_y^f - R_{aecf} e_x^a e_x^c e_z^e e_z^f - R_{bfde} e_x^e e_x^f e_y^b e_y^d - R_{bfde} e_y^b e_y^d e_y^e e_y^f - R_{bfde} e_y^b e_y^d e_z^e e_z^f \\ &\quad + \frac{1}{2} R_{egfh} e_x^e e_x^f e_x^g e_x^h + \frac{1}{2} R_{egfh} e_x^e e_x^f e_y^g e_y^h + \frac{1}{2} R_{egfh} e_x^e e_x^f e_z^g e_z^h + \frac{1}{2} R_{egfh} e_x^g e_x^h e_y^e e_y^f + \frac{1}{2} R_{egfh} e_y^e e_y^f e_y^g e_y^h + \frac{1}{2} R_{egfh} e_y^e e_y^f e_z^g e_z^h + \frac{1}{2} R_{egfh} e_x^g e_x^h e_z^e e_z^f \\ &\quad + \frac{1}{2} R_{egfh} e_y^g e_y^h e_z^e e_z^f + \frac{1}{2} R_{egfh} e_z^e e_z^f e_z^g e_z^h \end{split} \tag{ex-13b.110}$$

$$\begin{split} &= \frac{1}{2} R_{abcd} e_x^a e_x^c e_y^b e_y^d - \frac{1}{2} R_{abcd} e_x^a e_x^c e_x^b e_x^d - \frac{1}{2} R_{abcd} e_x^a e_x^c e_z^b e_z^d - R_{abcd} e_x^d e_x^b e_y^a e_y^c - R_{abcd} e_y^a e_y^c e_y^d e_y^b - R_{abcd} e_y^a e_y^c e_z^d e_z^b + \frac{1}{2} R_{abcd} e_x^b e_x^d e_y^a e_y^c \\ &\quad + \frac{1}{2} R_{abcd} e_y^a e_y^c e_y^b e_y^d + \frac{1}{2} R_{abcd} e_y^a e_y^c e_z^b e_z^d + \frac{1}{2} R_{abcd} e_x^b e_x^d e_z^a e_z^c + \frac{1}{2} R_{abcd} e_y^b e_y^d e_z^a e_z^c + \frac{1}{2} R_{abcd} e_z^a e_z^c e_z^b e_z^d \end{split} \tag{ex-13b.111}$$

$$= 0 \tag{ex-13b.112}$$

Example 13c The Weyl tensor vanishes in 3d – orthonormal basis

```
Cxyxz := C_{a b c d} ex^{a} ey^{b} ex^{c} ez^{d}.            # cdb(ex-13c.101,Cxyxz)

substitute     (Cxyxz,Cabcd)                                  # cdb(ex-13c.102,Cxyxz)

distribute     (Cxyxz)                                        # cdb(ex-13c.103,Cxyxz)

substitute     (Cxyxz, ortho, repeat=True)                    # cdb(ex-13c.104,Cxyxz)

substitute     (Cxyxz, gab)                                   # cdb(ex-13c.105,Cxyxz)
distribute     (Cxyxz)                                        # cdb(ex-13c.106,Cxyxz)

sort_product   (Cxyxz)                                        # cdb(ex-13c.107,Cxyxz)
rename_dummies (Cxyxz)                                        # cdb(ex-13c.108,Cxyxz)
canonicalise   (Cxyxz)                                        # cdb(ex-13c.109,Cxyxz)
```

$$C_{abcd}e_x^a e_y^b e_x^c e_z^d = \left(R_{abcd} - g^{ef}R_{aecf}g_{bd} + g^{fe}R_{afde}g_{bc} - g_{ac}g^{fe}R_{bfde} + g_{ad}g^{ef}R_{becf} + \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd} - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}\right) e_x^a e_y^b e_x^c e_z^d \quad \texttt{(ex-13c.102)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_z^d - g^{ef}R_{aecf}g_{bd}e_x^a e_y^b e_x^c e_z^d + g^{fe}R_{afde}g_{bc}e_x^a e_y^b e_x^c e_z^d - g_{ac}g^{fe}R_{bfde}e_x^a e_y^b e_x^c e_z^d + g_{ad}g^{ef}R_{becf}e_x^a e_y^b e_x^c e_z^d$$

$$+ \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ac}g_{bd}e_x^a e_y^b e_x^c e_z^d - \frac{1}{2}g^{ef}g^{gh}R_{egfh}g_{ad}g_{bc}e_x^a e_y^b e_x^c e_z^d \quad \texttt{(ex-13c.103)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_z^d - g^{fe}R_{bfde}e_y^b e_z^d \quad \texttt{(ex-13c.104)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_z^d - \left(e_x^f e_x^e + e_y^f e_y^e + e_z^f e_z^e\right) R_{bfde}e_y^b e_z^d \quad \texttt{(ex-13c.105)}$$

$$= R_{abcd}e_x^a e_y^b e_x^c e_z^d - e_x^f e_x^e R_{bfde}e_y^b e_z^d - e_y^f e_y^e R_{bfde}e_y^b e_z^d - e_z^f e_z^e R_{bfde}e_y^b e_z^d \quad \texttt{(ex-13c.106)}$$

$$= R_{abcd}e_x^a e_x^c e_y^b e_z^d - R_{bfde}e_x^e e_x^f e_y^b e_z^d - R_{bfde}e_y^b e_y^e e_y^f e_z^d - R_{bfde}e_y^b e_z^d e_z^e e_z^f \quad \texttt{(ex-13c.107)}$$

$$= R_{abcd}e_x^a e_x^c e_y^b e_z^d - R_{abcd}e_x^d e_x^b e_y^a e_z^c - R_{abcd}e_y^a e_y^d e_y^b e_z^c - R_{abcd}e_y^a e_z^c e_z^d e_z^b \quad \texttt{(ex-13c.108)}$$

$$= 0 \quad \texttt{(ex-13c.109)}$$

Example 14 The Weyl tensor is conformally invariant

This example shows that the Weyl tensor is conformally invariant. That is, for a pair of metrics g and $\overline{g}$ related by a conformal transformation, $\overline{g}_{ab} = \phi g_{ab}$ then $\overline{C}^{a}_{bcd} = C^{a}_{bcd}$ or equally $\overline{C}_{abcd} = \phi C_{abcd}$.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w#}::Indices (position=independent).

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.

GammaU := \Gamma^{a}_{b c} ->  1/2 g^{a d} (   \partial_{b}{g_{d c}}
                                             + \partial_{c}{g_{b d}}
                                             - \partial_{d}{g_{b c}}).

GammaD := \Gamma_{a b c} ->  1/2 (   \partial_{b}{g_{a c}}
                                   + \partial_{c}{g_{b a}}
                                   - \partial_{a}{g_{b c}}).

Rabcd := R_{a b c d} ->   \partial_{c}{\Gamma_{a b d}}
                        - \partial_{d}{\Gamma_{a b c}}
                        + \Gamma_{e a d} \Gamma^{e}_{b c}
                        - \Gamma_{e a c} \Gamma^{e}_{b d}.

Rab     := R_{a b} -> g^{c d} R_{a c b d}.

Rscalar := R -> g^{a b} R_{a b}.

# Weyl in 4-dimensions

Cabcd := R_{a b c d} - (1/2) (R_{a c} g_{b d} - R_{a d} g_{b c})
                     - (1/2) (g_{a c} R_{b d} - g_{a d} R_{b c})
                     + (R/6) (g_{a c} g_{b d} - g_{a d} g_{b c}).

{\partial_{a b}{\phi},\partial_{a}{\phi},\phi}::SortOrder.
{\partial_{a b}{g_{c d}},\partial_{a}{g_{b c}},g_{a b},g^{a b}}::SortOrder.
```

```

substitute (Cabcd,Rscalar)
substitute (Cabcd,Rab)
substitute (Cabcd,Rabcd)
substitute (Cabcd,GammaU)
substitute (Cabcd,GammaD)

distribute     (Cabcd)

sort_product   (Cabcd)
rename_dummies (Cabcd)
canonicalise   (Cabcd)

# this is the Weyl tensor on the base metric
baseC := @(Cabcd).

conformal := {g_{a b} -> \phi g_{a b}, g^{a b} -> (1/phi) g^{a b}}.

substitute     (Cabcd, conformal)
product_rule   (Cabcd)
distribute     (Cabcd)
product_rule   (Cabcd)
distribute     (Cabcd)

map_sympy      (Cabcd, "simplify")

sort_product   (Cabcd)
rename_dummies (Cabcd)
canonicalise   (Cabcd)

# this is the Weyl tensor on the conformal metric
confC := @(Cabcd).

# their difference, should be zero
diff := @(confC) - \phi @(baseC).  # cdb (ex-14.diff.100,diff)

distribute     (diff)
sort_product   (diff)
```

```
rename_dummies (diff)
canonicalise   (diff)  # cdb (ex-14.diff.101,diff)

# this trick is not essential but it does reduce the number of terms in diff
substitute     (diff,$\partial_{a}{\partial_{b}{g_{c d}}} -> g_{c d b a}$)
substitute     (diff,$\partial_{a}{g_{b c}} -> 0$)
substitute     (diff,$g_{c d b a} -> \partial_{a}{\partial_{b}{g_{c d}}}$)  # cdb (ex-14.diff.102,diff)

# standard expressions in 4-d
substitute     (diff,$g_{a b} g^{a b} -> 4$,repeat=True)           # cdb (ex-14.diff.201,diff)
substitute     (diff,$g_{a b} g^{c b} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.202,diff)
substitute     (diff,$g_{b a} g^{b c} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.203,diff)
substitute     (diff,$g_{a}^{a} -> 4$,repeat=True)                 # cdb (ex-14.diff.204,diff)
substitute     (diff,$g^{a}_{a} -> 4$,repeat=True)                 # cdb (ex-14.diff.205,diff)
eliminate_kronecker (diff)                                         # cdb (ex-14.diff.206,diff)

# need a second round since the above block introduces new terms that match those just eliminated
substitute     (diff,$g_{a b} g^{a b} -> 4$,repeat=True)           # cdb (ex-14.diff.301,diff)
substitute     (diff,$g_{a b} g^{c b} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.302,diff)
substitute     (diff,$g_{b a} g^{b c} -> g_{a}^{c}$,repeat=True)  # cdb (ex-14.diff.303,diff)
substitute     (diff,$g_{a}^{a} -> 4$,repeat=True)                 # cdb (ex-14.diff.304,diff)
substitute     (diff,$g^{a}_{a} -> 4$,repeat=True)                 # cdb (ex-14.diff.305,diff)
eliminate_kronecker (diff)                                         # cdb (ex-14.diff.306,diff)

sort_product   (diff)
rename_dummies (diff)
canonicalise   (diff)  # cdb (ex-14.diff.400,diff)

checkpoint.append (baseC)
checkpoint.append (confC)
```

$$
\begin{aligned}
\Delta = {} & \frac{1}{2}\partial_{bc}\phi g_{ad} - \frac{1}{2}\partial_{ac}\phi g_{bd} - \frac{1}{2}\partial_{bd}\phi g_{ac} + \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{be}g_{df}g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ae}g_{cf}g^{ef} \\
& - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{be}g_{cf}g^{ef} \\
& + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g^{ef} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ae}g_{df}g^{ef} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g^{ef} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g^{ef} \\
& - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} + \frac{1}{4}\partial_{ac}\phi g_{bd}g_{ef}g^{ef} + \frac{1}{2}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd}g_{cf}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bd}g_{gh}g^{ef}g^{gh} - \frac{1}{2}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} \\
& + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} - \frac{1}{4}\partial_{ad}\phi g_{bc}g_{ef}g^{ef} - \frac{1}{2}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} \\
& - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc}g_{df}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af}g_{bc}g_{gh}g^{ef}g^{gh} + \frac{1}{2}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} + \frac{1}{4}\partial_{bd}\phi g_{ac}g_{ef}g^{ef} \\
& - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{8}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{df}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} - \frac{1}{8}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} - \frac{1}{4}\partial_{bc}\phi g_{ad}g_{ef}g^{ef} \\
& + \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
& + \frac{1}{8}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{cf}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{8}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad}g_{bf}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} \\
& - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{ef}g^{gh} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{ef}g^{gh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
& + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} + \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} - \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fh}g^{ij} \\
& - \frac{1}{24}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij} + \frac{1}{24}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gh}g^{ij}
\end{aligned}
\tag{ex-14.diff.102}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1}g_{be}g_{df}g^{ef} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{df}g^{ef} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ae}g_{cf}g^{ef} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bc}g^{ef} \\
&+ \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{cf}g^{ef} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{bf}g^{ef} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{be}g_{cf}g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{cf}g^{ef} \\
&- \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ae}g_{df}g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bd}g^{ef} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{df}g^{ef} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{bf}g^{ef} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{af}g_{bd}g^{ef} \\
&- \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{cf}g^{ef} - \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{ef}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{ef}g^{gh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bd}g_{ch}g^{eg}g^{fh} \\
&+ \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{cf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{de}\phi g_{af}g_{bc}g^{ef} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} \\
&+ \frac{1}{4}\partial_{ae}\phi g_{bc}g_{df}g^{ef} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{ef}g^{gh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bc}g_{dh}g^{eg}g^{fh} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{df}g_{gh}g^{eg}g^{fh} \\
&- \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{bf}g^{ef} - \frac{1}{4}\partial_{be}\phi g_{ac}g_{df}g^{ef} - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{ef}g_{gh}g^{eg}g^{fh} \\
&- \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{ef}g^{gh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bg}g_{dh}g^{eg}g^{fh} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{df}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{bf}g^{ef} \\
&+ \frac{1}{4}\partial_{be}\phi g_{ad}g_{cf}g^{ef} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{ef}g_{gh}g^{eg}g^{fh} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{ef}g^{gh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bg}g_{ch}g^{eg}g^{fh} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{cf}g_{gh}g^{eg}g^{fh} \\
&- \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{bf}g_{gh}g^{eg}g^{fh} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{gh}g^{eg}g^{fh} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{gh}g^{eg}g^{fh} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} \\
&+ \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{eg}g^{fi}g^{hj} + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{ij}g^{ef}g^{gi}g^{hj}
\end{aligned}
\tag{\texttt{ex-14.diff.201}}
$$

$$
\begin{aligned}
\Delta = {} & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1}g_{be}g_{d}{}^{e} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{d}{}^{e} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ae}g_{c}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{a}{}^{e}g_{bc} \\
& + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{b}{}^{e} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{be}g_{c}{}^{e} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{c}{}^{e} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ae}g_{d}{}^{e} \\
& - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{a}{}^{e}g_{bd} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{b}{}^{e} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{a}{}^{e}g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd}g_{c}{}^{e} \\
& - \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1}g_{bd}g_{ef}g_{g}{}^{f}g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bd}g_{c}{}^{g}g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{a}{}^{e}g_{bd}g_{c}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{cf}g_{g}{}^{f}g^{eg} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bd}g_{g}{}^{f}g^{eg} \\
& + \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}{}^{f}g^{eg} + \frac{1}{4}\partial_{de}\phi g_{a}{}^{e}g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc}g_{d}{}^{e} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{bc}g_{ef}g_{g}{}^{f}g^{eg} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ag}g_{bc}g_{d}{}^{g}g^{ef} \\
& - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{a}{}^{e}g_{bc}g_{d}{}^{f} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{df}g_{g}{}^{f}g^{eg} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{af}g_{bc}g_{g}{}^{f}g^{eg} - \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}{}^{f}g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac}g_{b}{}^{e} - \frac{1}{4}\partial_{be}\phi g_{ac}g_{d}{}^{e} \\
& - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac}g_{ef}g_{g}{}^{f}g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bg}g_{d}{}^{g}g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{b}{}^{e}g_{d}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{df}g_{g}{}^{f}g^{eg} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{bf}g_{g}{}^{f}g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad}g_{b}{}^{e} + \frac{1}{4}\partial_{be}\phi g_{ad}g_{c}{}^{e} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}g_{ef}g_{g}{}^{f}g^{eg} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bg}g_{c}{}^{g}g^{ef} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{b}{}^{e}g_{c}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{cf}g_{g}{}^{f}g^{eg} \\
& - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{bf}g_{g}{}^{f}g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g_{g}{}^{f}g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g_{g}{}^{f}g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{i}{}^{h}g^{eg}g^{fi} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{i}{}^{h}g^{eg}g^{fi} \\
& + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{gh}g_{i}{}^{h}g^{ef}g^{gi} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{gh}g_{i}{}^{h}g^{ef}g^{gi} \qquad (\texttt{ex-14.diff.202})
\end{aligned}
$$

$$
\begin{aligned}
\Delta = & -\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{be} g_{d}{}^{e} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{d}{}^{e} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ae} g_{c}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bc} \\
& + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{b}{}^{e} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{be} g_{c}{}^{e} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{c}{}^{e} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ae} g_{d}{}^{e} \\
& - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bd} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{b}{}^{e} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{a}{}^{e} g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd} g_{c}{}^{e} \\
& - \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}{}^{g} g_{g}{}^{f} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bd} g_{c}{}^{g} g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{a}{}^{e} g_{bd} g_{c}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{cf} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{af} g_{bd} g_{g}{}^{f} g^{eg} \\
& + \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{de}\phi g_{a}{}^{e} g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc} g_{d}{}^{e} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}{}^{g} g_{g}{}^{f} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bc} g_{d}{}^{g} g^{ef} \\
& - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{a}{}^{e} g_{bc} g_{d}{}^{f} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{df} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac} g_{b}{}^{e} - \frac{1}{4}\partial_{be}\phi g_{ac} g_{d}{}^{e} \\
& - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}{}^{g} g_{g}{}^{f} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bg} g_{d}{}^{g} g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{b}{}^{e} g_{d}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{df} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{bf} g_{g}{}^{f} g^{eg} \\
& + \frac{1}{4}\partial_{ce}\phi g_{ad} g_{b}{}^{e} + \frac{1}{4}\partial_{be}\phi g_{ad} g_{c}{}^{e} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}{}^{g} g_{g}{}^{f} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bg} g_{c}{}^{g} g^{ef} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{b}{}^{e} g_{c}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{cf} g_{g}{}^{f} g^{eg} \\
& - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{bf} g_{g}{}^{f} g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g_{g}{}^{f} g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{gh} g_{i}{}^{h} g^{eg} g^{fi} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{gh} g_{i}{}^{h} g^{eg} g^{fi} \\
& + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}{}^{i} g_{i}{}^{h} g^{ef} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}{}^{i} g_{i}{}^{h} g^{ef}
\end{aligned}
\tag{ex-14.diff.203}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{be} g_{d}{}^{e} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{d}{}^{e} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ae} g_{c}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bc} \\
&+ \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{b}{}^{e} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{be} g_{c}{}^{e} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{c}{}^{e} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ae} g_{d}{}^{e} \\
&- \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bd} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{b}{}^{e} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{a}{}^{e} g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd} g_{c}{}^{e} \\
&- \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}{}^{g} g_{g}{}^{f} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bd} g_{c}{}^{g} g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{a}{}^{e} g_{bd} g_{c}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{cf} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{af} g_{bd} g_{g}{}^{f} g^{eg} \\
&+ \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{de}\phi g_{a}{}^{e} g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc} g_{d}{}^{e} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}{}^{g} g_{g}{}^{f} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bc} g_{d}{}^{g} g^{ef} \\
&- \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{a}{}^{e} g_{bc} g_{d}{}^{f} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{df} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac} g_{b}{}^{e} - \frac{1}{4}\partial_{be}\phi g_{ac} g_{d}{}^{e} \\
&- \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}{}^{g} g_{g}{}^{f} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bg} g_{d}{}^{g} g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{b}{}^{e} g_{d}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{df} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{bf} g_{g}{}^{f} g^{eg} \\
&+ \frac{1}{4}\partial_{ce}\phi g_{ad} g_{b}{}^{e} + \frac{1}{4}\partial_{be}\phi g_{ad} g_{c}{}^{e} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}{}^{g} g_{g}{}^{f} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bg} g_{c}{}^{g} g^{ef} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{b}{}^{e} g_{c}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{cf} g_{g}{}^{f} g^{eg} \\
&- \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{bf} g_{g}{}^{f} g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g_{g}{}^{f} g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{gh} g_{i}{}^{h} g^{eg} g^{fi} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{gh} g_{i}{}^{h} g^{eg} g^{fi} \\
&+ \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}{}^{i} g_{i}{}^{h} g^{ef} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}{}^{i} g_{i}{}^{h} g^{ef}
\end{aligned}
\tag{\texttt{ex-14.diff.204}}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{2}\partial_{bc}\phi g_{ad} + \frac{1}{2}\partial_{ac}\phi g_{bd} + \frac{1}{2}\partial_{bd}\phi g_{ac} - \frac{1}{2}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{be} g_{d}{}^{e} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{d}{}^{e} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ae} g_{c}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bc} \\
&+ \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{b}{}^{e} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{be} g_{c}{}^{e} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{c}{}^{e} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ae} g_{d}{}^{e} \\
&- \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bd} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{b}{}^{e} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ce}\phi g_{a}{}^{e} g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ae}\phi g_{bd} g_{c}{}^{e} \\
&- \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}{}^{g} g_{g}{}^{f} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bd} g_{c}{}^{g} g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{a}{}^{e} g_{bd} g_{c}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{cf} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{af} g_{bd} g_{g}{}^{f} g^{eg} \\
&+ \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{de}\phi g_{a}{}^{e} g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{4}\partial_{ae}\phi g_{bc} g_{d}{}^{e} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}{}^{g} g_{g}{}^{f} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ag} g_{bc} g_{d}{}^{g} g^{ef} \\
&- \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{a}{}^{e} g_{bc} g_{d}{}^{f} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{df} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{af} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{6}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{de}\phi g_{ac} g_{b}{}^{e} - \frac{1}{4}\partial_{be}\phi g_{ac} g_{d}{}^{e} \\
&- \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}{}^{g} g_{g}{}^{f} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bg} g_{d}{}^{g} g^{ef} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{b}{}^{e} g_{d}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{df} g_{g}{}^{f} g^{eg} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{bf} g_{g}{}^{f} g^{eg} \\
&+ \frac{1}{4}\partial_{ce}\phi g_{ad} g_{b}{}^{e} + \frac{1}{4}\partial_{be}\phi g_{ad} g_{c}{}^{e} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}{}^{g} g_{g}{}^{f} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bg} g_{c}{}^{g} g^{ef} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{b}{}^{e} g_{c}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{cf} g_{g}{}^{f} g^{eg} \\
&- \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{bf} g_{g}{}^{f} g^{eg} + \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g_{g}{}^{f} g^{eg} - \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g_{g}{}^{f} g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{gh} g_{i}{}^{h} g^{eg} g^{fi} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{gh} g_{i}{}^{h} g^{eg} g^{fi} \\
&+ \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}{}^{i} g_{i}{}^{h} g^{ef} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}{}^{i} g_{i}{}^{h} g^{ef}
\end{aligned}
\tag{\texttt{ex-14.diff.205}}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} + \frac{1}{4}\partial_{d}\phi\partial_{a}\phi\phi^{-1} g_{bc} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} \\
&+ \frac{1}{4}\partial_{c}\phi\partial_{b}\phi\phi^{-1} g_{ad} + \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{a}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{d}\phi\partial_{b}\phi\phi^{-1} g_{ac} - \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g^{ef} \\
&- \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{cg} g^{eg} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ag} g_{bd} g^{eg} + \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ac} g_{bd} g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}{}^{f} \\
&- \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{dg} g^{eg} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ag} g_{bc} g^{eg} - \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ad} g_{bc} g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{dg} g^{eg} \\
&+ \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{bg} g^{eg} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{cg} g^{eg} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{bg} g^{eg} + \frac{1}{6}\partial_{eg}\phi g_{ac} g_{bd} g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad} g_{bc} g^{eg} \\
&- \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{gi} g^{eg} g^{fi} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{gi} g^{eg} g^{fi} + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}{}^{h} g^{ef} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}{}^{h} g^{ef}
\end{aligned}
\tag{\texttt{ex-14.diff.206}}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} + \frac{1}{4}\partial_{d}\phi\partial_{a}\phi\phi^{-1} g_{bc} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} \\
&+ \frac{1}{4}\partial_{c}\phi\partial_{b}\phi\phi^{-1} g_{ad} + \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{a}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{d}\phi\partial_{b}\phi\phi^{-1} g_{ac} - \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g^{ef} \\
&- \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{cg} g^{eg} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ag} g_{bd} g^{eg} + \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ac} g_{bd} g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}{}^{f} \\
&- \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{dg} g^{eg} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ag} g_{bc} g^{eg} - \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ad} g_{bc} g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{dg} g^{eg} \\
&+ \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{bg} g^{eg} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{cg} g^{eg} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{bg} g^{eg} + \frac{1}{6}\partial_{eg}\phi g_{ac} g_{bd} g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad} g_{bc} g^{eg} \\
&- \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{gi} g^{eg} g^{fi} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{gi} g^{eg} g^{fi} + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}{}^{h} g^{ef} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}{}^{h} g^{ef}
\end{aligned}
\tag{ex-14.diff.301}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} + \frac{1}{4}\partial_{d}\phi\partial_{a}\phi\phi^{-1} g_{bc} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} \\
&+ \frac{1}{4}\partial_{c}\phi\partial_{b}\phi\phi^{-1} g_{ad} + \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{a}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{d}\phi\partial_{b}\phi\phi^{-1} g_{ac} - \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g^{ef} \\
&- \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bd} + \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ac} g_{bd} g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}{}^{f} \\
&- \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bc} - \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ad} g_{bc} g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{d}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{b}{}^{e} \\
&+ \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{c}{}^{e} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{b}{}^{e} + \frac{1}{6}\partial_{eg}\phi g_{ac} g_{bd} g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad} g_{bc} g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{g}{}^{f} g^{eg} \\
&+ \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{g}{}^{f} g^{eg} + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}{}^{h} g^{ef} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}{}^{h} g^{ef}
\end{aligned}
\tag{ex-14.diff.302}
$$

$$
\begin{aligned}
\Delta = &-\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} + \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} + \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} + \frac{1}{4}\partial_{d}\phi\partial_{a}\phi\phi^{-1} g_{bc} - \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} \\
&+ \frac{1}{4}\partial_{c}\phi\partial_{b}\phi\phi^{-1} g_{ad} + \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{a}\phi\phi^{-1} g_{bd} - \frac{1}{4}\partial_{d}\phi\partial_{b}\phi\phi^{-1} g_{ac} - \frac{5}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac} g_{bd} g^{ef} \\
&- \frac{1}{8}\partial_{a}\phi\partial_{c}\phi\phi^{-1} g_{bd} g_{f}{}^{f} + \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bd} g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bd} + \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ac} g_{bd} g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad} g_{bc} g^{ef} + \frac{1}{8}\partial_{a}\phi\partial_{d}\phi\phi^{-1} g_{bc} g_{f}{}^{f} \\
&- \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1} g_{bc} g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{a}{}^{e} g_{bc} - \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1} g_{ad} g_{bc} g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} - \frac{1}{8}\partial_{b}\phi\partial_{d}\phi\phi^{-1} g_{ac} g_{f}{}^{f} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{d}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1} g_{ac} g_{b}{}^{e} \\
&+ \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{8}\partial_{b}\phi\partial_{c}\phi\phi^{-1} g_{ad} g_{f}{}^{f} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{c}{}^{e} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1} g_{ad} g_{b}{}^{e} + \frac{1}{6}\partial_{eg}\phi g_{ac} g_{bd} g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad} g_{bc} g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{g}{}^{f} g^{eg} \\
&+ \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{g}{}^{f} g^{eg} + \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ac} g_{bd} g_{h}{}^{h} g^{ef} - \frac{1}{8}\partial_{e}\phi\partial_{f}\phi\phi^{-1} g_{ad} g_{bc} g_{h}{}^{h} g^{ef}
\end{aligned}
\tag{ex-14.diff.303}
$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_{d}\phi\partial_{a}\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\partial_{c}\phi\partial_{b}\phi\phi^{-1}g_{ad} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{a}\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_{d}\phi\partial_{b}\phi\phi^{-1}g_{ac} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$+ \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{a}{}^{e}g_{bd} + \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{a}{}^{e}g_{bc}$$
$$- \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{d}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{b}{}^{e} + \frac{1}{4}\partial_{cb}\phi g_{ad} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{c}{}^{e} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{b}{}^{e}$$
$$+ \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}{}^{f}g^{eg} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}{}^{f}g^{eg} \qquad (\texttt{ex-14.diff.304})$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{4}\partial_{a}\phi\partial_{c}\phi\phi^{-1}g_{bd} + \frac{1}{4}\partial_{a}\phi\partial_{d}\phi\phi^{-1}g_{bc} - \frac{1}{4}\partial_{b}\phi\partial_{d}\phi\phi^{-1}g_{ac} + \frac{1}{4}\partial_{d}\phi\partial_{a}\phi\phi^{-1}g_{bc} + \frac{1}{4}\partial_{b}\phi\partial_{c}\phi\phi^{-1}g_{ad}$$
$$+ \frac{1}{4}\partial_{c}\phi\partial_{b}\phi\phi^{-1}g_{ad} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{c}\phi\partial_{a}\phi\phi^{-1}g_{bd} - \frac{1}{4}\partial_{d}\phi\partial_{b}\phi\phi^{-1}g_{ac} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$+ \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bd}g_{c}{}^{e} + \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{a}{}^{e}g_{bd} + \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} - \frac{1}{4}\partial_{a}\phi\partial_{e}\phi\phi^{-1}g_{bc}g_{d}{}^{e} - \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{a}{}^{e}g_{bc}$$
$$- \frac{1}{6}\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{d}{}^{e} + \frac{1}{4}\partial_{d}\phi\partial_{e}\phi\phi^{-1}g_{ac}g_{b}{}^{e} + \frac{1}{4}\partial_{cb}\phi g_{ad} - \frac{1}{4}\partial_{b}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{c}{}^{e} - \frac{1}{4}\partial_{c}\phi\partial_{e}\phi\phi^{-1}g_{ad}g_{b}{}^{e}$$
$$+ \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg} - \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g_{g}{}^{f}g^{eg} + \frac{1}{4}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g_{g}{}^{f}g^{eg} \qquad (\texttt{ex-14.diff.305})$$

$$\Delta = -\frac{1}{4}\partial_{bc}\phi g_{ad} + \frac{1}{4}\partial_{ac}\phi g_{bd} + \frac{1}{4}\partial_{bd}\phi g_{ac} - \frac{1}{4}\partial_{ad}\phi g_{bc} - \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_{e}\phi\partial_{f}\phi\phi^{-1}g_{ac}g_{bd}g^{ef} - \frac{1}{4}\partial_{ca}\phi g_{bd} - \frac{1}{6}\partial_{ef}\phi g_{ac}g_{bd}g^{ef}$$
$$- \frac{1}{12}\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ac}g_{bd}g^{eg} + \frac{1}{4}\partial_{da}\phi g_{bc} + \frac{1}{6}\partial_{ef}\phi g_{ad}g_{bc}g^{ef} + \frac{1}{12}\partial_{e}\phi\partial_{g}\phi\phi^{-1}g_{ad}g_{bc}g^{eg} - \frac{1}{4}\partial_{db}\phi g_{ac} + \frac{1}{4}\partial_{cb}\phi g_{ad} + \frac{1}{6}\partial_{eg}\phi g_{ac}g_{bd}g^{eg}$$
$$- \frac{1}{6}\partial_{eg}\phi g_{ad}g_{bc}g^{eg} \qquad (\texttt{ex-14.diff.306})$$

$$\Delta = 0 \qquad (\texttt{ex-14.diff.400})$$

Example 15 Verifying the BSSN equations

This is short example verifies two of the main equations in the Phys Rev D paper by Miguel Alcubierre, Bernd Brugmann etal. (Phys.Rev.D. (62) 044034 (2000)).

The code for the full set of BSSN equations can be found at `https://github.com/leo-brewin/adm-bssn-equations`

```
{a,b,c,d,e,f,i,j,k,l,m,n,o,p,q,r,s,u#}::Indices(position=independent,values={t,x,y,z}).
{t,x,y,z}::Coordinate.

\partial{#}::PartialDerivative.
D{#}::Derivative.
DBar{#}::Derivative.

N::Depends(t,x,y,z).

g_{a b}::Symmetric.
g^{a b}::Symmetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

g_{a b}::Depends(t,x,y,z).
g^{a b}::Depends(t,x,y,z).

gBar_{a b}::Symmetric.
gBar^{a b}::Symmetric.
gBar_{a}^{b}::KroneckerDelta.
gBar^{a}_{b}::KroneckerDelta.

gBar_{a b}::Depends(t,x,y,z).
gBar^{a b}::Depends(t,x,y,z).

trK::LaTeXForm("K").
detg::LaTeXForm("g").
ABar{#}::LaTeXForm("{\bar{A}}").
DBar{#}::LaTeXForm("{\bar{D}}").
```

15.1 Evolution equation for ϕ

```
phi      := \phi -> (1/12) \log(detg).
gdotK    := g^{i j} K_{i j} -> trK.
DdetgDt  := \partial_{t}{detg} -> detg g^{i j} \partial_{t}{g_{i j}}.

DgijDt   := \partial_{t}{g_{i j}} -> -2 N K_{i j}.

dlog     := \partial_{a?}{\log(A?)} -> (1/A?)\partial_{a?}{A?}.
dexp     := \partial_{a?}{\exp(A?)} -> \exp(A?)\partial_{a?}{A?}.

dotphi   := \partial_{t}{\phi}.

substitute (dotphi, phi)                # cdb (ex-15-02.101,dotphi)
substitute (dotphi, dlog)               # cdb (ex-15-02.102,dotphi)
substitute (dotphi, DdetgDt)            # cdb (ex-15-02.103,dotphi)
substitute (dotphi, DgijDt)             # cdb (ex-15-02.104,dotphi)
substitute (dotphi, gdotK)              # cdb (ex-15-02.105,dotphi)
map_sympy  (dotphi, "simplify")         # cdb (ex-15-02.106,dotphi)

DphiDt := \partial_{t}{\phi} -> @(dotphi).

checkpoint.append (dotphi)
```

$$\frac{d\phi}{dt} = \frac{1}{12}\partial_t \left(\log\left(g\right)\right) \tag{ex-15-02.101}$$

$$= \frac{1}{12}g^{-1}\partial_t g \tag{ex-15-02.102}$$

$$= \frac{1}{12}g^{-1}gg^{ij}\partial_t g_{ij} \tag{ex-15-02.103}$$

$$= -\frac{1}{6}g^{-1}gg^{ij}NK_{ij} \tag{ex-15-02.104}$$

$$= -\frac{1}{6}g^{-1}gKN \tag{ex-15-02.105}$$

$$= -\frac{1}{6}KN \tag{ex-15-02.106}$$

15.2 Evolution equation for $\bar{g}_{ij}$

```
gBarij := gBar_{i j} -> \exp(-4\phi) g_{i j}.
Kij    := K_{i j} -> A_{i j} + (1/3) g_{i j} trK.
A2ABar := \exp(-4\phi) A_{i j} -> ABar_{i j}.
ABar2A := ABar_{i j} -> \exp(-4\phi) A_{i j}.

dotgBarij := \partial_{t}{gBar_{i j}}.

substitute   (dotgBarij, gBarij)        # cdb (ex-15-03.101,dotgBarij)
product_rule (dotgBarij)                # cdb (ex-15-03.102,dotgBarij)
substitute   (dotgBarij, dexp)          # cdb (ex-15-03.103,dotgBarij)
substitute   (dotgBarij, DgijDt)        # cdb (ex-15-03.104,dotgBarij)
substitute   (dotgBarij, DphiDt)        # cdb (ex-15-03.105,dotgBarij)
substitute   (dotgBarij, Kij)           # cdb (ex-15-03.106,dotgBarij)
distribute   (dotgBarij)                # cdb (ex-15-03.107,dotgBarij)
map_sympy    (dotgBarij, "simplify")    # cdb (ex-15-03.108,dotgBarij)
substitute   (dotgBarij, A2ABar)        # cdb (ex-15-03.109,dotgBarij)

DgBarijDt := \partial_{t}{gBar_{i j}} -> @(dotgBarij).

checkpoint.append (dotgBarij)
```

$$
\begin{aligned}
\frac{d\bar{g}_{ij}}{dt} &= \partial_t \left(\exp\left(-4\phi\right) g_{ij}\right) && \texttt{(ex-15-03.101)}\\
&= \partial_t \left(\exp\left(-4\phi\right)\right) g_{ij} + \exp\left(-4\phi\right) \partial_t g_{ij} && \texttt{(ex-15-03.102)}\\
&= -4\exp\left(-4\phi\right) \partial_t \phi g_{ij} + \exp\left(-4\phi\right) \partial_t g_{ij} && \texttt{(ex-15-03.103)}\\
&= -4\exp\left(-4\phi\right) \partial_t \phi g_{ij} - 2\exp\left(-4\phi\right) N K_{ij} && \texttt{(ex-15-03.104)}\\
&= \frac{2}{3}\exp\left(-4\phi\right) K N g_{ij} - 2\exp\left(-4\phi\right) N K_{ij} && \texttt{(ex-15-03.105)}\\
&= \frac{2}{3}\exp\left(-4\phi\right) K N g_{ij} - 2\exp\left(-4\phi\right) N \left(A_{ij} + \frac{1}{3} g_{ij} K\right) && \texttt{(ex-15-03.106)}\\
&= \frac{2}{3}\exp\left(-4\phi\right) K N g_{ij} - 2\exp\left(-4\phi\right) N A_{ij} - \frac{2}{3}\exp\left(-4\phi\right) N g_{ij} K && \texttt{(ex-15-03.107)}\\
&= -2N\exp\left(-4\phi\right) A_{ij} && \texttt{(ex-15-03.108)}\\
&= -2N\bar{A}_{ij} && \texttt{(ex-15-03.109)}
\end{aligned}
$$