## Exercise 2.6 Cummutation of $\nabla$ on a scalar

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\{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u\#\}::Indices(position=independent).
     \nabla{#}::Derivative.
     \partial{#}::PartialDerivative.
     # covariant derivative of \phi
     dphi := \nabla_{a}{\phi} -> \partial_{a}{\phi}.
     # rules to hide and reveal \partial\phi
10
11
            := \partial_{a}{\phi} -> w_{a}.
12
     reveal := w_{a} \rightarrow \beta_{a}.
14
     # template for covariant derivative of a dual-vector
15
16
     deriv := \nabla_{a}_{A?_{b}} - \nabla_{a}_{A?_{b}} - \nabla_{a}_{A?_{b}} - \nabla_{a}_{A?_{b}}.
17
18
     # create an object
19
     expr := \nabla_{a}{\nabla_{b}{\phi}}
21
             - \nabla_{b}{\nabla_{a}{\phi}}.
                                                 # cdb (ex-0206.101,expr)
22
23
     # apply the rules, then simplify
                     (expr,dphi)
                                                 # cdb (ex-0206.102,expr)
     substitute
26
                     (expr, hide)
                                                 # cdb (ex-0206.103,expr)
     substitute
27
                     (expr,deriv)
                                                 # cdb (ex-0206.104,expr)
     substitute
28
     substitute
                     (expr,reveal)
                                                 # cdb (ex-0206.105,expr)
29
                     (expr)
                                                 # cdb (ex-0206.106,expr)
     canonicalise
```

$$\nabla_{a} (\nabla_{b} \phi) - \nabla_{b} (\nabla_{a} \phi) = \nabla_{a} (\partial_{b} \phi) - \nabla_{b} (\partial_{a} \phi)$$

$$= \nabla_{a} w_{b} - \nabla_{b} w_{a}$$

$$= \partial_{a} w_{b} - \Gamma^{c}_{ba} w_{c} - \partial_{b} w_{a} + \Gamma^{c}_{ab} w_{c}$$

$$= \partial_{ab} \phi - \Gamma^{c}_{ba} \partial_{c} \phi - \partial_{ba} \phi + \Gamma^{c}_{ab} \partial_{c} \phi$$

$$= -\Gamma^{c}_{ba} \partial_{c} \phi + \Gamma^{c}_{ab} \partial_{c} \phi$$

$$= -\Gamma^{c}_{ba} \partial_{c} \phi + \Gamma^{c}_{ab} \partial_{c} \phi$$

$$(ex-0206.105)$$

$$= -\Gamma^{c}_{ba} \partial_{c} \phi + \Gamma^{c}_{ab} \partial_{c} \phi$$

$$(ex-0206.106)$$