

Exercise 2.6 Commutation of ∇ on a scalar

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # covariant derivative of \phi
7
8 dphi := \nabla_{a}{\phi} -> \partial_{a}{\phi}.
9
10 # rules to hide and reveal \partial\phi
11
12 hide := \partial_{a}{\phi} -> w_{a}.
13 reveal := w_{a} -> \partial_{a}{\phi}.
14
15 # template for covariant derivative of a dual-vector
16
17 deriv := \nabla_{a}{A?_{b}} -> \partial_{a}{A?_{b}} - \Gamma^{c}_{b a} A?_{c}.
18
19 # create an object
20
21 expr := \nabla_{a}{\nabla_{b}{\phi}}
22         - \nabla_{b}{\nabla_{a}{\phi}}. # cdb (ex-0206.101,expr)
23
24 # apply the rules, then simplify
25
26 substitute (expr,dphi) # cdb (ex-0206.102,expr)
27 substitute (expr,hide) # cdb (ex-0206.103,expr)
28 substitute (expr,deriv) # cdb (ex-0206.104,expr)
29 substitute (expr,reveal) # cdb (ex-0206.105,expr)
30 canonicalise (expr) # cdb (ex-0206.106,expr)
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$$\nabla_a (\nabla_b \phi) - \nabla_b (\nabla_a \phi) = \nabla_a (\partial_b \phi) - \nabla_b (\partial_a \phi) \quad (\text{ex-0206.102})$$

$$= \nabla_a w_b - \nabla_b w_a \quad (\text{ex-0206.103})$$

$$= \partial_a w_b - \Gamma_{ba}^c w_c - \partial_b w_a + \Gamma_{ab}^c w_c \quad (\text{ex-0206.104})$$

$$= \partial_{ab} \phi - \Gamma_{ba}^c \partial_c \phi - \partial_{ba} \phi + \Gamma_{ab}^c \partial_c \phi \quad (\text{ex-0206.105})$$

$$= -\Gamma_{ba}^c \partial_c \phi + \Gamma_{ab}^c \partial_c \phi \quad (\text{ex-0206.106})$$