

Exercise 3.1 Some symmetries of Riemann

```
1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 ::Symbol;
4
5 \partial{#}::PartialDerivative.
6
7 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
8
9 Rabcd := R^{a}_{b c d} -> \partial_{c}{\Gamma^{a}_{b d}}
10                      - \partial_{d}{\Gamma^{a}_{b c}}
11                      + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
12                      - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.      # cdb(Rabcd.000,Rabcd)
13
14 dRabcd := R^{a}_{b c d ; e} -> \partial_{e}{R^{a}_{b c d}}
15                      + \Gamma^{a}_{f e} R^{f}_{b c d}
16                      - \Gamma^{f}_{b e} R^{a}_{f c d}
17                      - \Gamma^{f}_{c e} R^{a}_{b f d}
18                      - \Gamma^{f}_{d e} R^{a}_{b c f}.      # cdb(dRabcd.000,dRabcd)
```

Exercise 3.1 Antisymmetry on last pair of indices

```
1  expr := R^{a}_{b c d} + R^{a}_{b d c}.                # cdb(ex-0301.101,expr)
2
3  substitute (expr, Rabcd)                             # cdb(ex-0301.102,expr)
```

$$R^a_{bcd} + R^a_{bdc} = 0 \qquad (\text{ex-0301.102})$$

Exercise 3.1 First Bianchi identity

```

1  expr := R^{a}_{b c d} + R^{a}_{d b c} + R^{a}_{c d b}.           # cdb(ex-0301.201,expr)
2
3  substitute (expr, Rabcd)                                     # cdb(ex-0301.202,expr)
4  canonicalise (expr)                                         # cdb(ex-0301.203,expr)

```

$$\begin{aligned}
 R^a_{bcd} + R^a_{dbc} + R^a_{cdb} &= \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ce} - \Gamma^e_{bc} \Gamma^a_{de} + \partial_b \Gamma^a_{dc} - \partial_c \Gamma^a_{db} + \Gamma^e_{dc} \Gamma^a_{be} - \Gamma^e_{db} \Gamma^a_{ce} + \partial_d \Gamma^a_{cb} - \partial_b \Gamma^a_{cd} + \Gamma^e_{cb} \Gamma^a_{de} \\
 &\quad - \Gamma^e_{cd} \Gamma^a_{be} \qquad \qquad \qquad \text{(ex-0301.202)} \\
 &= 0 \qquad \qquad \qquad \text{(ex-0301.203)}
 \end{aligned}$$

Exercise 3.1 Second Bianchi identity

```
1  expr := R^{a}_{b c d ; e} + R^{a}_{b e c ; d} + R^{a}_{b d e ; c}.    # cdb(ex-0301.301,expr)
2
3  substitute      (expr, dRabcd)      # cdb(ex-0301.302,expr)
4  substitute      (expr, Rabcd)       # cdb(ex-0301.303,expr)
5  distribute      (expr)              # cdb(ex-0301.304,expr)
6  product_rule    (expr)              # cdb(ex-0301.305,expr)
7  sort_product    (expr)              # cdb(ex-0301.306,expr)
8  rename_dummies  (expr)              # cdb(ex-0301.307,expr)
9  canonicalise    (expr)              # cdb(ex-0301.308,expr)
```

$$\begin{aligned}
R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} &= \partial_e R^a_{bcd} + \Gamma^a_{fe} R^f_{bcd} - \Gamma^f_{be} R^a_{fcd} - \Gamma^f_{ce} R^a_{bfd} - \Gamma^f_{de} R^a_{bcf} + \partial_d R^a_{bec} + \Gamma^a_{fd} R^f_{bec} - \Gamma^f_{bd} R^a_{fec} - \Gamma^f_{ed} R^a_{bfc} - \Gamma^f_{cd} R^a_{bef} \\
&\quad + \partial_c R^a_{bde} + \Gamma^a_{fc} R^f_{bde} - \Gamma^f_{bc} R^a_{fde} - \Gamma^f_{dc} R^a_{bfe} - \Gamma^f_{ec} R^a_{bdf} \tag{ex-0301.302}
\end{aligned}$$

$$\begin{aligned}
&= \partial_e (\partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^f_{bd} \Gamma^a_{cf} - \Gamma^f_{bc} \Gamma^a_{df}) + \Gamma^a_{fe} (\partial_c \Gamma^f_{bd} - \partial_d \Gamma^f_{bc} + \Gamma^g_{bd} \Gamma^f_{cg} - \Gamma^g_{bc} \Gamma^f_{dg}) \\
&\quad - \Gamma^f_{be} (\partial_c \Gamma^a_{fd} - \partial_d \Gamma^a_{fc} + \Gamma^g_{fd} \Gamma^a_{cg} - \Gamma^g_{fc} \Gamma^a_{dg}) - \Gamma^f_{ce} (\partial_f \Gamma^a_{bd} - \partial_d \Gamma^a_{bf} + \Gamma^g_{bd} \Gamma^a_{fg} - \Gamma^g_{bf} \Gamma^a_{dg}) \\
&\quad - \Gamma^f_{de} (\partial_c \Gamma^a_{bf} - \partial_f \Gamma^a_{bc} + \Gamma^g_{bf} \Gamma^a_{cg} - \Gamma^g_{bc} \Gamma^a_{fg}) + \partial_d (\partial_e \Gamma^a_{bc} - \partial_c \Gamma^a_{be} + \Gamma^f_{bc} \Gamma^a_{ef} - \Gamma^f_{be} \Gamma^a_{cf}) \\
&\quad + \Gamma^a_{fd} (\partial_e \Gamma^f_{bc} - \partial_c \Gamma^f_{be} + \Gamma^g_{bc} \Gamma^f_{eg} - \Gamma^g_{be} \Gamma^f_{cg}) - \Gamma^f_{bd} (\partial_e \Gamma^a_{fc} - \partial_c \Gamma^a_{fe} + \Gamma^g_{fc} \Gamma^a_{eg} - \Gamma^g_{fe} \Gamma^a_{cg}) \\
&\quad - \Gamma^f_{ed} (\partial_f \Gamma^a_{bc} - \partial_c \Gamma^a_{bf} + \Gamma^g_{bc} \Gamma^a_{fg} - \Gamma^g_{bf} \Gamma^a_{cg}) - \Gamma^f_{cd} (\partial_e \Gamma^a_{bf} - \partial_f \Gamma^a_{be} + \Gamma^g_{bf} \Gamma^a_{eg} - \Gamma^g_{be} \Gamma^a_{fg}) \\
&\quad + \partial_c (\partial_d \Gamma^a_{be} - \partial_e \Gamma^a_{bd} + \Gamma^f_{be} \Gamma^a_{df} - \Gamma^f_{bd} \Gamma^a_{ef}) + \Gamma^a_{fc} (\partial_d \Gamma^f_{be} - \partial_e \Gamma^f_{bd} + \Gamma^g_{be} \Gamma^f_{dg} - \Gamma^g_{bd} \Gamma^f_{eg}) \\
&\quad - \Gamma^f_{bc} (\partial_d \Gamma^a_{fe} - \partial_e \Gamma^a_{fd} + \Gamma^g_{fe} \Gamma^a_{dg} - \Gamma^g_{fd} \Gamma^a_{eg}) - \Gamma^f_{dc} (\partial_f \Gamma^a_{be} - \partial_e \Gamma^a_{bf} + \Gamma^g_{be} \Gamma^a_{fg} - \Gamma^g_{bf} \Gamma^a_{eg}) \\
&\quad - \Gamma^f_{ec} (\partial_d \Gamma^a_{bf} - \partial_f \Gamma^a_{bd} + \Gamma^g_{bf} \Gamma^a_{dg} - \Gamma^g_{bd} \Gamma^a_{fg}) \tag{ex-0301.303}
\end{aligned}$$

$$\begin{aligned}
&= \partial_{ec} \Gamma^a_{bd} - \partial_{ed} \Gamma^a_{bc} + \partial_e (\Gamma^f_{bd} \Gamma^a_{cf}) - \partial_e (\Gamma^f_{bc} \Gamma^a_{df}) + \Gamma^a_{fe} \partial_c \Gamma^f_{bd} - \Gamma^a_{fe} \partial_d \Gamma^f_{bc} + \Gamma^a_{fe} \Gamma^g_{bd} \Gamma^f_{cg} - \Gamma^a_{fe} \Gamma^g_{bc} \Gamma^f_{dg} \\
&\quad - \Gamma^f_{be} \partial_c \Gamma^a_{fd} + \Gamma^f_{be} \partial_d \Gamma^a_{fc} - \Gamma^f_{be} \Gamma^g_{fd} \Gamma^a_{cg} + \Gamma^f_{be} \Gamma^g_{fc} \Gamma^a_{dg} - \Gamma^f_{ce} \partial_f \Gamma^a_{bd} + \Gamma^f_{ce} \partial_d \Gamma^a_{bf} - \Gamma^f_{ce} \Gamma^g_{bd} \Gamma^a_{fg} + \Gamma^f_{ce} \Gamma^g_{bf} \Gamma^a_{dg} \\
&\quad - \Gamma^f_{de} \partial_c \Gamma^a_{bf} + \Gamma^f_{de} \partial_f \Gamma^a_{bc} - \Gamma^f_{de} \Gamma^g_{bf} \Gamma^a_{cg} + \Gamma^f_{de} \Gamma^g_{bc} \Gamma^a_{fg} + \partial_{de} \Gamma^a_{bc} - \partial_{dc} \Gamma^a_{be} + \partial_d (\Gamma^f_{bc} \Gamma^a_{ef}) - \partial_d (\Gamma^f_{be} \Gamma^a_{cf}) \\
&\quad + \Gamma^a_{fd} \partial_e \Gamma^f_{bc} - \Gamma^a_{fd} \partial_c \Gamma^f_{be} + \Gamma^a_{fd} \Gamma^g_{bc} \Gamma^f_{eg} - \Gamma^a_{fd} \Gamma^g_{be} \Gamma^f_{cg} - \Gamma^f_{bd} \partial_e \Gamma^a_{fc} + \Gamma^f_{bd} \partial_c \Gamma^a_{fe} - \Gamma^f_{bd} \Gamma^g_{fc} \Gamma^a_{eg} + \Gamma^f_{bd} \Gamma^g_{fe} \Gamma^a_{cg} \\
&\quad - \Gamma^f_{ed} \partial_f \Gamma^a_{bc} + \Gamma^f_{ed} \partial_c \Gamma^a_{bf} - \Gamma^f_{ed} \Gamma^g_{bc} \Gamma^a_{fg} + \Gamma^f_{ed} \Gamma^g_{bf} \Gamma^a_{cg} - \Gamma^f_{cd} \partial_e \Gamma^a_{bf} + \Gamma^f_{cd} \partial_f \Gamma^a_{be} - \Gamma^f_{cd} \Gamma^g_{bf} \Gamma^a_{eg} + \Gamma^f_{cd} \Gamma^g_{be} \Gamma^a_{fg} \\
&\quad + \partial_{cd} \Gamma^a_{be} - \partial_{ce} \Gamma^a_{bd} + \partial_c (\Gamma^f_{be} \Gamma^a_{df}) - \partial_c (\Gamma^f_{bd} \Gamma^a_{ef}) + \Gamma^a_{fc} \partial_d \Gamma^f_{be} - \Gamma^a_{fc} \partial_e \Gamma^f_{bd} + \Gamma^a_{fc} \Gamma^g_{be} \Gamma^f_{dg} - \Gamma^a_{fc} \Gamma^g_{bd} \Gamma^f_{eg} \\
&\quad - \Gamma^f_{bc} \partial_d \Gamma^a_{fe} + \Gamma^f_{bc} \partial_e \Gamma^a_{fd} - \Gamma^f_{bc} \Gamma^g_{fe} \Gamma^a_{dg} + \Gamma^f_{bc} \Gamma^g_{fd} \Gamma^a_{eg} - \Gamma^f_{dc} \partial_f \Gamma^a_{be} + \Gamma^f_{dc} \partial_e \Gamma^a_{bf} - \Gamma^f_{dc} \Gamma^g_{be} \Gamma^a_{fg} + \Gamma^f_{dc} \Gamma^g_{bf} \Gamma^a_{eg} \\
&\quad - \Gamma^f_{ec} \partial_d \Gamma^a_{bf} + \Gamma^f_{ec} \partial_f \Gamma^a_{bd} - \Gamma^f_{ec} \Gamma^g_{bf} \Gamma^a_{dg} + \Gamma^f_{ec} \Gamma^g_{bd} \Gamma^a_{fg} \tag{ex-0301.304}
\end{aligned}$$

$$\begin{aligned}
R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} = & \partial_{ec}\Gamma^a_{bd} - \partial_{ed}\Gamma^a_{bc} + \partial_e\Gamma^f_{bd}\Gamma^a_{cf} + \Gamma^f_{bd}\partial_e\Gamma^a_{cf} - \partial_e\Gamma^f_{bc}\Gamma^a_{df} - \Gamma^f_{bc}\partial_e\Gamma^a_{df} + \Gamma^a_{fe}\partial_c\Gamma^f_{bd} - \Gamma^a_{fe}\partial_d\Gamma^f_{bc} + \Gamma^a_{fe}\Gamma^g_{bd}\Gamma^f_{cg} \\
& - \Gamma^a_{fe}\Gamma^g_{bc}\Gamma^f_{dg} - \Gamma^f_{be}\partial_c\Gamma^a_{fd} + \Gamma^f_{be}\partial_d\Gamma^a_{fc} - \Gamma^f_{be}\Gamma^g_{fd}\Gamma^a_{cg} + \Gamma^f_{be}\Gamma^g_{fc}\Gamma^a_{dg} - \Gamma^f_{ce}\partial_f\Gamma^a_{bd} + \Gamma^f_{ce}\partial_d\Gamma^a_{bf} - \Gamma^f_{ce}\Gamma^g_{bd}\Gamma^a_{fg} \\
& + \Gamma^f_{ce}\Gamma^g_{bf}\Gamma^a_{dg} - \Gamma^f_{de}\partial_c\Gamma^a_{bf} + \Gamma^f_{de}\partial_f\Gamma^a_{bc} - \Gamma^f_{de}\Gamma^g_{bf}\Gamma^a_{cg} + \Gamma^f_{de}\Gamma^g_{bc}\Gamma^a_{fg} + \partial_{de}\Gamma^a_{bc} - \partial_{dc}\Gamma^a_{be} + \partial_d\Gamma^f_{bc}\Gamma^a_{ef} \\
& + \Gamma^f_{bc}\partial_d\Gamma^a_{ef} - \partial_d\Gamma^f_{be}\Gamma^a_{cf} - \Gamma^f_{be}\partial_d\Gamma^a_{cf} + \Gamma^a_{fd}\partial_e\Gamma^f_{bc} - \Gamma^a_{fd}\partial_c\Gamma^f_{be} + \Gamma^a_{fd}\Gamma^g_{bc}\Gamma^f_{eg} - \Gamma^a_{fd}\Gamma^g_{be}\Gamma^f_{cg} - \Gamma^f_{bd}\partial_e\Gamma^a_{fc} \\
& + \Gamma^f_{bd}\partial_c\Gamma^a_{fe} - \Gamma^f_{bd}\Gamma^g_{fc}\Gamma^a_{eg} + \Gamma^f_{bd}\Gamma^g_{fe}\Gamma^a_{cg} - \Gamma^f_{ed}\partial_f\Gamma^a_{bc} + \Gamma^f_{ed}\partial_c\Gamma^a_{bf} - \Gamma^f_{ed}\Gamma^g_{bc}\Gamma^a_{fg} + \Gamma^f_{ed}\Gamma^g_{bf}\Gamma^a_{cg} - \Gamma^f_{cd}\partial_e\Gamma^a_{bf} \\
& + \Gamma^f_{cd}\partial_f\Gamma^a_{be} - \Gamma^f_{cd}\Gamma^g_{bf}\Gamma^a_{eg} + \Gamma^f_{cd}\Gamma^g_{be}\Gamma^a_{fg} + \partial_{cd}\Gamma^a_{be} - \partial_{ce}\Gamma^a_{bd} + \partial_c\Gamma^f_{be}\Gamma^a_{df} + \Gamma^f_{be}\partial_c\Gamma^a_{df} - \partial_c\Gamma^f_{bd}\Gamma^a_{ef} \\
& - \Gamma^f_{bd}\partial_c\Gamma^a_{ef} + \Gamma^a_{fc}\partial_d\Gamma^f_{be} - \Gamma^a_{fc}\partial_e\Gamma^f_{bd} + \Gamma^a_{fc}\Gamma^g_{be}\Gamma^f_{dg} - \Gamma^a_{fc}\Gamma^g_{bd}\Gamma^f_{eg} - \Gamma^f_{bc}\partial_d\Gamma^a_{fe} + \Gamma^f_{bc}\partial_e\Gamma^a_{fd} - \Gamma^f_{bc}\Gamma^g_{fe}\Gamma^a_{dg} \\
& + \Gamma^f_{bc}\Gamma^g_{fd}\Gamma^a_{eg} - \Gamma^f_{dc}\partial_f\Gamma^a_{be} + \Gamma^f_{dc}\partial_e\Gamma^a_{bf} - \Gamma^f_{dc}\Gamma^g_{be}\Gamma^a_{fg} + \Gamma^f_{dc}\Gamma^g_{bf}\Gamma^a_{eg} - \Gamma^f_{ec}\partial_d\Gamma^a_{bf} + \Gamma^f_{ec}\partial_f\Gamma^a_{bd} - \Gamma^f_{ec}\Gamma^g_{bf}\Gamma^a_{dg} \\
& + \Gamma^f_{ec}\Gamma^g_{bd}\Gamma^a_{fg} \quad (\text{ex-0301.305}) \\
= & \partial_{ec}\Gamma^a_{bd} - \partial_{ed}\Gamma^a_{bc} + \Gamma^a_{cf}\partial_e\Gamma^f_{bd} + \Gamma^f_{bd}\partial_e\Gamma^a_{cf} - \Gamma^a_{df}\partial_e\Gamma^f_{bc} - \Gamma^f_{bc}\partial_e\Gamma^a_{df} + \Gamma^a_{fe}\partial_c\Gamma^f_{bd} - \Gamma^a_{fe}\partial_d\Gamma^f_{bc} + \Gamma^a_{fe}\Gamma^f_{cg}\Gamma^g_{bd} \\
& - \Gamma^a_{fe}\Gamma^f_{dg}\Gamma^g_{bc} - \Gamma^f_{be}\partial_c\Gamma^a_{fd} + \Gamma^f_{be}\partial_d\Gamma^a_{fc} - \Gamma^a_{cg}\Gamma^f_{be}\Gamma^g_{fd} + \Gamma^a_{dg}\Gamma^f_{be}\Gamma^g_{fc} - \Gamma^f_{ce}\partial_f\Gamma^a_{bd} + \Gamma^f_{ce}\partial_d\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ce}\Gamma^g_{bd} \\
& + \Gamma^a_{dg}\Gamma^f_{ce}\Gamma^g_{bf} - \Gamma^f_{de}\partial_c\Gamma^a_{bf} + \Gamma^f_{de}\partial_f\Gamma^a_{bc} - \Gamma^a_{cg}\Gamma^f_{de}\Gamma^g_{bf} + \Gamma^a_{fg}\Gamma^f_{de}\Gamma^g_{bc} + \partial_{de}\Gamma^a_{bc} - \partial_{dc}\Gamma^a_{be} + \Gamma^a_{ef}\partial_d\Gamma^f_{bc} \\
& + \Gamma^f_{bc}\partial_d\Gamma^a_{ef} - \Gamma^a_{cf}\partial_d\Gamma^f_{be} - \Gamma^f_{be}\partial_d\Gamma^a_{cf} + \Gamma^a_{fd}\partial_e\Gamma^f_{bc} - \Gamma^a_{fd}\partial_c\Gamma^f_{be} + \Gamma^a_{fd}\Gamma^f_{eg}\Gamma^g_{bc} - \Gamma^a_{fd}\Gamma^f_{cg}\Gamma^g_{be} - \Gamma^f_{bd}\partial_e\Gamma^a_{fc} \\
& + \Gamma^f_{bd}\partial_c\Gamma^a_{fe} - \Gamma^a_{eg}\Gamma^f_{bd}\Gamma^g_{fc} + \Gamma^a_{cg}\Gamma^f_{bd}\Gamma^g_{fe} - \Gamma^f_{ed}\partial_f\Gamma^a_{bc} + \Gamma^f_{ed}\partial_c\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ed}\Gamma^g_{bc} + \Gamma^a_{cg}\Gamma^f_{ed}\Gamma^g_{bf} - \Gamma^f_{cd}\partial_e\Gamma^a_{bf} \\
& + \Gamma^f_{cd}\partial_f\Gamma^a_{be} - \Gamma^a_{eg}\Gamma^f_{cd}\Gamma^g_{bf} + \Gamma^a_{fg}\Gamma^f_{cd}\Gamma^g_{be} + \partial_{cd}\Gamma^a_{be} - \partial_{ce}\Gamma^a_{bd} + \Gamma^a_{df}\partial_c\Gamma^f_{be} + \Gamma^f_{be}\partial_c\Gamma^a_{df} - \Gamma^a_{ef}\partial_c\Gamma^f_{bd} \\
& - \Gamma^f_{bd}\partial_c\Gamma^a_{ef} + \Gamma^a_{fc}\partial_d\Gamma^f_{be} - \Gamma^a_{fc}\partial_e\Gamma^f_{bd} + \Gamma^a_{fc}\Gamma^f_{dg}\Gamma^g_{be} - \Gamma^a_{fc}\Gamma^f_{eg}\Gamma^g_{bd} - \Gamma^f_{bc}\partial_d\Gamma^a_{fe} + \Gamma^f_{bc}\partial_e\Gamma^a_{fd} - \Gamma^a_{dg}\Gamma^f_{bc}\Gamma^g_{fe} \\
& + \Gamma^a_{eg}\Gamma^f_{bc}\Gamma^g_{fd} - \Gamma^f_{dc}\partial_f\Gamma^a_{be} + \Gamma^f_{dc}\partial_e\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{dc}\Gamma^g_{be} + \Gamma^a_{eg}\Gamma^f_{dc}\Gamma^g_{bf} - \Gamma^f_{ec}\partial_d\Gamma^a_{bf} + \Gamma^f_{ec}\partial_f\Gamma^a_{bd} - \Gamma^a_{dg}\Gamma^f_{ec}\Gamma^g_{bf} \\
& + \Gamma^a_{fg}\Gamma^f_{ec}\Gamma^g_{bd} \quad (\text{ex-0301.306})
\end{aligned}$$

$$\begin{aligned}
R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} = & \partial_{ec}\Gamma^a_{bd} - \partial_{ed}\Gamma^a_{bc} + \Gamma^a_{cf}\partial_e\Gamma^f_{bd} + \Gamma^f_{bd}\partial_e\Gamma^a_{cf} - \Gamma^a_{df}\partial_e\Gamma^f_{bc} - \Gamma^f_{bc}\partial_e\Gamma^a_{df} + \Gamma^a_{fe}\partial_c\Gamma^f_{bd} - \Gamma^a_{fe}\partial_d\Gamma^f_{bc} + \Gamma^a_{fe}\Gamma^f_{cg}\Gamma^g_{bd} \\
& - \Gamma^a_{fe}\Gamma^f_{dg}\Gamma^g_{bc} - \Gamma^f_{be}\partial_c\Gamma^a_{fd} + \Gamma^f_{be}\partial_d\Gamma^a_{fc} - \Gamma^a_{cf}\Gamma^g_{be}\Gamma^f_{gd} + \Gamma^a_{df}\Gamma^g_{be}\Gamma^f_{gc} - \Gamma^f_{ce}\partial_f\Gamma^a_{bd} + \Gamma^f_{ce}\partial_d\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ce}\Gamma^g_{bd} \\
& + \Gamma^a_{df}\Gamma^g_{ce}\Gamma^f_{bg} - \Gamma^f_{de}\partial_c\Gamma^a_{bf} + \Gamma^f_{de}\partial_f\Gamma^a_{bc} - \Gamma^a_{cf}\Gamma^g_{de}\Gamma^f_{bg} + \Gamma^a_{fg}\Gamma^f_{de}\Gamma^g_{bc} + \partial_{de}\Gamma^a_{bc} - \partial_{dc}\Gamma^a_{be} + \Gamma^a_{ef}\partial_d\Gamma^f_{bc} + \Gamma^f_{bc}\partial_d\Gamma^a_{ef} \\
& - \Gamma^a_{cf}\partial_d\Gamma^f_{be} - \Gamma^f_{be}\partial_d\Gamma^a_{cf} + \Gamma^a_{fd}\partial_e\Gamma^f_{bc} - \Gamma^a_{fd}\partial_c\Gamma^f_{be} + \Gamma^a_{fd}\Gamma^f_{eg}\Gamma^g_{bc} - \Gamma^a_{fd}\Gamma^f_{cg}\Gamma^g_{be} - \Gamma^f_{bd}\partial_e\Gamma^a_{fc} + \Gamma^f_{bd}\partial_c\Gamma^a_{fe} \\
& - \Gamma^a_{ef}\Gamma^g_{bd}\Gamma^f_{gc} + \Gamma^a_{cf}\Gamma^g_{bd}\Gamma^f_{ge} - \Gamma^f_{ed}\partial_f\Gamma^a_{bc} + \Gamma^f_{ed}\partial_c\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{ed}\Gamma^g_{bc} + \Gamma^a_{cf}\Gamma^g_{ed}\Gamma^f_{bg} - \Gamma^f_{cd}\partial_e\Gamma^a_{bf} + \Gamma^f_{cd}\partial_f\Gamma^a_{be} \\
& - \Gamma^a_{ef}\Gamma^g_{cd}\Gamma^f_{bg} + \Gamma^a_{fg}\Gamma^f_{cd}\Gamma^g_{be} + \partial_{cd}\Gamma^a_{be} - \partial_{ce}\Gamma^a_{bd} + \Gamma^a_{df}\partial_c\Gamma^f_{be} + \Gamma^f_{be}\partial_c\Gamma^a_{df} - \Gamma^a_{ef}\partial_c\Gamma^f_{bd} - \Gamma^f_{bd}\partial_c\Gamma^a_{ef} + \Gamma^a_{fc}\partial_d\Gamma^f_{be} \\
& - \Gamma^a_{fc}\partial_e\Gamma^f_{bd} + \Gamma^a_{fc}\Gamma^f_{dg}\Gamma^g_{be} - \Gamma^a_{fc}\Gamma^f_{eg}\Gamma^g_{bd} - \Gamma^f_{bc}\partial_d\Gamma^a_{fe} + \Gamma^f_{bc}\partial_e\Gamma^a_{fd} - \Gamma^a_{df}\Gamma^g_{bc}\Gamma^f_{ge} + \Gamma^a_{ef}\Gamma^g_{bc}\Gamma^f_{gd} - \Gamma^f_{dc}\partial_f\Gamma^a_{be} \\
& + \Gamma^f_{dc}\partial_e\Gamma^a_{bf} - \Gamma^a_{fg}\Gamma^f_{dc}\Gamma^g_{be} + \Gamma^a_{ef}\Gamma^g_{dc}\Gamma^f_{bg} - \Gamma^f_{ec}\partial_d\Gamma^a_{bf} + \Gamma^f_{ec}\partial_f\Gamma^a_{bd} - \Gamma^a_{df}\Gamma^g_{ec}\Gamma^f_{bg} + \Gamma^a_{fg}\Gamma^f_{ec}\Gamma^g_{bd} \quad (\text{ex-0301.307}) \\
= 0 & \quad (\text{ex-0301.308})
\end{aligned}$$