Exercise 6.9 Killing vectors of the Schwarzschild spacetime

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{t, r, \theta, \varphi}::Coordinate.
{a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).

;::Symbol.

\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)

Gamma := \Gamma^{a}_{f g} -> 1/2 g^{a b} (   \partial_{g}{g_{b f}}
                                           + \partial_{f}{g_{b g}}
                                           - \partial_{b}{g_{f g}} ).

deriv := \xi_{a ; b} -> \partial_{b}{\xi_{a}} - \Gamma^{c}_{a b} \xi_{c}.
lower := \xi_{a} -> g_{a b} \xi^{b}.

expr  := \xi_{a ; b} + \xi_{b ; a}.                  # cdb(ex-0609.100,expr)

substitute   (expr, deriv)                           # cdb(ex-0609.101,expr)
substitute   (expr, lower)                           # cdb(ex-0609.102,expr)
substitute   (expr, Gamma)                           # cdb(ex-0609.103,expr)
distribute   (expr)                                  # cdb(ex-0609.104,expr)
product_rule (expr)                                  # cdb(ex-0609.105,expr)
canonicalise (expr)                                  # cdb(ex-0609.106,expr)

# choose a vector

# Kvect := {\xi^{t} = 1}.
# Kvect := {\xi^{\varphi} = 1}.
Kvect := {\xi^{\theta} = \sin(\varphi), \xi^{\varphi} = \cos(\theta)/\sin(\theta) \cos(\varphi)}.
# Kvect := {\xi^{\theta} = \cos(\varphi), \xi^{\varphi} = - \cos(\theta)/\sin(\theta) \sin(\varphi)}.
                                                      # cdb(ex-0609.107,Kvect)

gab := { g_{t t}            = -(1-2*m/r),
         g_{r r}            = 1/(1-(2*m/r)),
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             g_{\theta\theta}   = r**2,
             g_{\varphi\varphi} = r**2 \sin(\theta)**2}.  # cdb(ex-0609.108,gab)

complete   (gab, $g^{a b}$)                            # cdb(ex-0609.109,gab)

evaluate   (expr, gab+Kvect)                           # cdb(ex-0609.110,expr)
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$$[\xi^{a}] = \left[\xi^{\theta} = \sin\left(\varphi\right),\ \xi^{\varphi} = \cos\theta(\sin\theta)^{-1}\cos\left(\varphi\right)\right] \tag{ex-0609.107}$$

$$[g_{ab}] = \left[g_{tt} = -1 + 2mr^{-1},\ g_{rr} = \left(1 - 2mr^{-1}\right)^{-1},\ g_{\theta\theta} = r^{2},\ g_{\varphi\varphi} = r^{2}(\sin\theta)^{2}\right] \tag{ex-0609.108}$$

$$[g_{ab}, g^{ab}] = \left[g_{tt} = -1 + 2mr^{-1},\ g_{rr} = \left(1 - 2mr^{-1}\right)^{-1},\ g_{\theta\theta} = r^{2},\ g_{\varphi\varphi} = r^{2}(\sin\theta)^{2},\ g^{tt} = \left(2mr^{-1} - 1\right)^{-1},\ g^{rr} = -2mr^{-1} + 1,\ g^{\theta\theta} = r^{-2},\ g^{\varphi\varphi} = \left(r^{2}(\sin\theta)^{2}\right)^{-1}\right] \tag{ex-0609.109}$$

$$\begin{aligned}
\xi_{a;b} + \xi_{b;a} &= \partial_{b}\xi_{a} - \Gamma^{c}{}_{ab}\xi_{c} + \partial_{a}\xi_{b} - \Gamma^{c}{}_{ba}\xi_{c} && \text{(ex-0609.101)}\\
&= \partial_{b}\left(g_{ac}\xi^{c}\right) - \Gamma^{c}{}_{ab}g_{cd}\xi^{d} + \partial_{a}\left(g_{bc}\xi^{c}\right) - \Gamma^{c}{}_{ba}g_{cd}\xi^{d} && \text{(ex-0609.102)}\\
&= \partial_{b}\left(g_{ac}\xi^{c}\right) - \frac{1}{2}g^{ce}\left(\partial_{b}g_{ea} + \partial_{a}g_{eb} - \partial_{e}g_{ab}\right)g_{cd}\xi^{d} + \partial_{a}\left(g_{bc}\xi^{c}\right) - \frac{1}{2}g^{ce}\left(\partial_{a}g_{eb} + \partial_{b}g_{ea} - \partial_{e}g_{ba}\right)g_{cd}\xi^{d} && \text{(ex-0609.103)}\\
&= \partial_{b}\left(g_{ac}\xi^{c}\right) - g^{ce}\partial_{b}g_{ea}g_{cd}\xi^{d} - g^{ce}\partial_{a}g_{eb}g_{cd}\xi^{d} + \frac{1}{2}g^{ce}\partial_{e}g_{ab}g_{cd}\xi^{d} + \partial_{a}\left(g_{bc}\xi^{c}\right) + \frac{1}{2}g^{ce}\partial_{e}g_{ba}g_{cd}\xi^{d} && \text{(ex-0609.104)}\\
&= \partial_{b}g_{ac}\xi^{c} + g_{ac}\partial_{b}\xi^{c} - g^{ce}\partial_{b}g_{ea}g_{cd}\xi^{d} - g^{ce}\partial_{a}g_{eb}g_{cd}\xi^{d} + \frac{1}{2}g^{ce}\partial_{e}g_{ab}g_{cd}\xi^{d} + \partial_{a}g_{bc}\xi^{c} + g_{bc}\partial_{a}\xi^{c} + \frac{1}{2}g^{ce}\partial_{e}g_{ba}g_{cd}\xi^{d} && \text{(ex-0609.105)}\\
&= \partial_{b}g_{ac}\xi^{c} + g_{ac}\partial_{b}\xi^{c} - g^{cd}\partial_{b}g_{ac}g_{de}\xi^{e} - g^{cd}\partial_{a}g_{bc}g_{de}\xi^{e} + g^{cd}\partial_{c}g_{ab}g_{de}\xi^{e} + \partial_{a}g_{bc}\xi^{c} + g_{bc}\partial_{a}\xi^{c} && \text{(ex-0609.106)}\\
&= 0 && \text{(ex-0609.110)}
\end{aligned}$$