Exercise 3.8 Symmetry of R_{ab}

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
                                                    # cdb (dgab.000,dgab)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                                             + \partial_{c}{g_{b e}}
                                             - \partial_{e}{g_{b c}}).
                                                    # cdb (Gamma.000,Gamma)

Rabcd := R^{a}_{b c d} ->
         \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
       - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                    # cdb (Rabcd.000,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                    # cdb (Rab.000,Rab)

expr := 4 (R_{a b} - R_{b a}).                      # cdb (ex-0308.100,expr)

substitute   (expr, Rab)                            # cdb (ex-0308.101,expr)
substitute   (expr, Rabcd)                          # cdb (ex-0308.102,expr)
substitute   (expr, Gamma)                          # cdb (ex-0308.103,expr)

distribute   (expr)                                 # cdb (ex-0308.104,expr)
product_rule (expr)                                 # cdb (ex-0308.105,expr)
canonicalise (expr)                                 # cdb (ex-0308.106,expr)

substitute   (expr, dgab)                           # cdb (ex-0308.107,expr)
canonicalise (expr)                                 # cdb (ex-0308.108,expr)
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$$4R_{ab} - 4R_{ba} = 4R^{c}{}_{acb} - 4R^{c}{}_{bca} \tag{ex-0308.101}$$

$$= 4\partial_c\Gamma^{c}{}_{ab} + 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ab} - 4\partial_b\Gamma^{c}{}_{ac} - 4\Gamma^{c}{}_{eb}\Gamma^{e}{}_{ac} - 4\partial_c\Gamma^{c}{}_{ba} - 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ba} + 4\partial_a\Gamma^{c}{}_{bc} + 4\Gamma^{c}{}_{ea}\Gamma^{e}{}_{bc} \tag{ex-0308.102}$$

$$\begin{aligned}
&= 2\partial_c\left(g^{ce}\left(\partial_a g_{eb} + \partial_b g_{ae} - \partial_e g_{ab}\right)\right) + g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_a g_{fb} + \partial_b g_{af} - \partial_f g_{ab}\right) - 2\partial_b\left(g^{ce}\left(\partial_a g_{ec} + \partial_c g_{ae} - \partial_e g_{ac}\right)\right) \\
&\quad - g^{cd}\left(\partial_e g_{db} + \partial_b g_{ed} - \partial_d g_{eb}\right) g^{ef}\left(\partial_a g_{fc} + \partial_c g_{af} - \partial_f g_{ac}\right) - 2\partial_c\left(g^{ce}\left(\partial_b g_{ea} + \partial_a g_{be} - \partial_e g_{ba}\right)\right) \\
&\quad - g^{cd}\left(\partial_e g_{dc} + \partial_c g_{ed} - \partial_d g_{ec}\right) g^{ef}\left(\partial_b g_{fa} + \partial_a g_{bf} - \partial_f g_{ba}\right) + 2\partial_a\left(g^{ce}\left(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}\right)\right) \\
&\quad + g^{cd}\left(\partial_e g_{da} + \partial_a g_{ed} - \partial_d g_{ea}\right) g^{ef}\left(\partial_b g_{fc} + \partial_c g_{bf} - \partial_f g_{bc}\right)
\end{aligned} \tag{ex-0308.103}$$

$$\begin{aligned}
&= 2\partial_c\left(g^{ce}\partial_a g_{eb}\right) + 2\partial_c\left(g^{ce}\partial_b g_{ae}\right) - 2\partial_c\left(g^{ce}\partial_e g_{ab}\right) + g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ab} + g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{fb} \\
&\quad + g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ab} - 2\partial_b\left(g^{ce}\partial_a g_{ec}\right) - 2\partial_b\left(g^{ce}\partial_c g_{ae}\right) \\
&\quad + 2\partial_b\left(g^{ce}\partial_e g_{ac}\right) - g^{cd}\partial_e g_{db} g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db} g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db} g^{ef}\partial_f g_{ac} - g^{cd}\partial_b g_{ed} g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed} g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed} g^{ef}\partial_f g_{ac} \\
&\quad + g^{cd}\partial_d g_{eb} g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb} g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb} g^{ef}\partial_f g_{ac} - 2\partial_c\left(g^{ce}\partial_b g_{ea}\right) - 2\partial_c\left(g^{ce}\partial_a g_{be}\right) + 2\partial_c\left(g^{ce}\partial_e g_{ba}\right) - g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{fa} \\
&\quad - g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{bf} + g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ba} - g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{fa} - g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{bf} + g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ba} + g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{fa} + g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{bf} \\
&\quad - g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ba} + 2\partial_a\left(g^{ce}\partial_b g_{ec}\right) + 2\partial_a\left(g^{ce}\partial_c g_{be}\right) - 2\partial_a\left(g^{ce}\partial_e g_{bc}\right) + g^{cd}\partial_e g_{da} g^{ef}\partial_b g_{fc} + g^{cd}\partial_e g_{da} g^{ef}\partial_c g_{bf} - g^{cd}\partial_e g_{da} g^{ef}\partial_f g_{bc} \\
&\quad + g^{cd}\partial_a g_{ed} g^{ef}\partial_b g_{fc} + g^{cd}\partial_a g_{ed} g^{ef}\partial_c g_{bf} - g^{cd}\partial_a g_{ed} g^{ef}\partial_f g_{bc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_b g_{fc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_c g_{bf} + g^{cd}\partial_d g_{ea} g^{ef}\partial_f g_{bc}
\end{aligned} \tag{ex-0308.104}$$

$$\begin{aligned}
&= 2\partial_c g^{ce}\partial_a g_{eb} + 2g^{ce}\partial_{ca} g_{eb} + 2\partial_c g^{ce}\partial_b g_{ae} + 2g^{ce}\partial_{cb} g_{ae} - 2\partial_c g^{ce}\partial_e g_{ab} - 2g^{ce}\partial_{ce} g_{ab} + g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{fb} + g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{af} - g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ab} \\
&\quad + g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{fb} + g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{af} - g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ab} - g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{fb} - g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{af} + g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ab} - 2\partial_b g^{ce}\partial_a g_{ec} \\
&\quad - 2g^{ce}\partial_{ba} g_{ec} - 2\partial_b g^{ce}\partial_c g_{ae} - 2g^{ce}\partial_{bc} g_{ae} + 2\partial_b g^{ce}\partial_e g_{ac} + 2g^{ce}\partial_{be} g_{ac} - g^{cd}\partial_e g_{db} g^{ef}\partial_a g_{fc} - g^{cd}\partial_e g_{db} g^{ef}\partial_c g_{af} + g^{cd}\partial_e g_{db} g^{ef}\partial_f g_{ac} \\
&\quad - g^{cd}\partial_b g_{ed} g^{ef}\partial_a g_{fc} - g^{cd}\partial_b g_{ed} g^{ef}\partial_c g_{af} + g^{cd}\partial_b g_{ed} g^{ef}\partial_f g_{ac} + g^{cd}\partial_d g_{eb} g^{ef}\partial_a g_{fc} + g^{cd}\partial_d g_{eb} g^{ef}\partial_c g_{af} - g^{cd}\partial_d g_{eb} g^{ef}\partial_f g_{ac} - 2\partial_c g^{ce}\partial_b g_{ea} \\
&\quad - 2g^{ce}\partial_{cb} g_{ea} - 2\partial_c g^{ce}\partial_a g_{be} - 2g^{ce}\partial_{ca} g_{be} + 2\partial_c g^{ce}\partial_e g_{ba} + 2g^{ce}\partial_{ce} g_{ba} - g^{cd}\partial_e g_{dc} g^{ef}\partial_b g_{fa} - g^{cd}\partial_e g_{dc} g^{ef}\partial_a g_{bf} + g^{cd}\partial_e g_{dc} g^{ef}\partial_f g_{ba} \\
&\quad - g^{cd}\partial_c g_{ed} g^{ef}\partial_b g_{fa} - g^{cd}\partial_c g_{ed} g^{ef}\partial_a g_{bf} + g^{cd}\partial_c g_{ed} g^{ef}\partial_f g_{ba} + g^{cd}\partial_d g_{ec} g^{ef}\partial_b g_{fa} + g^{cd}\partial_d g_{ec} g^{ef}\partial_a g_{bf} - g^{cd}\partial_d g_{ec} g^{ef}\partial_f g_{ba} + 2\partial_a g^{ce}\partial_b g_{ec} \\
&\quad + 2g^{ce}\partial_{ab} g_{ec} + 2\partial_a g^{ce}\partial_c g_{be} + 2g^{ce}\partial_{ac} g_{be} - 2\partial_a g^{ce}\partial_e g_{bc} - 2g^{ce}\partial_{ae} g_{bc} + g^{cd}\partial_e g_{da} g^{ef}\partial_b g_{fc} + g^{cd}\partial_e g_{da} g^{ef}\partial_c g_{bf} - g^{cd}\partial_e g_{da} g^{ef}\partial_f g_{bc} \\
&\quad + g^{cd}\partial_a g_{ed} g^{ef}\partial_b g_{fc} + g^{cd}\partial_a g_{ed} g^{ef}\partial_c g_{bf} - g^{cd}\partial_a g_{ed} g^{ef}\partial_f g_{bc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_b g_{fc} - g^{cd}\partial_d g_{ea} g^{ef}\partial_c g_{bf} + g^{cd}\partial_d g_{ea} g^{ef}\partial_f g_{bc}
\end{aligned} \tag{ex-0308.105}$$

$$= -2\partial_b g^{ce}\partial_a g_{ce} + 2\partial_a g^{ce}\partial_b g_{ce} \tag{ex-0308.106}$$

$$= 2g^{cd}g^{ef}\partial_b g_{df}\partial_a g_{ce} - 2g^{cd}g^{ef}\partial_a g_{df}\partial_b g_{ce} \tag{ex-0308.107}$$

$$= 0 \tag{ex-0308.108}$$

Exercise 3.8 Symmetry of R_{ab} alternative solution

This differs from the previous code by the inclusion of a call to `canonicalise` immediately after the first two substitutions and a declaration that $\Gamma^a{}_{bc}$ is symmetric in bc. This pair of changes produces a more compact set of results than given above. Incidently, this also shows that $\partial_a\Gamma^c{}_{bc} = \partial_b\Gamma^c{}_{ac}$.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Metric;
g^{a b}::InverseMetric;

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}.
                                                    # cdb (dgab.000,dgab)

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                                             + \partial_{c}{g_{b e}}
                                             - \partial_{e}{g_{b c}}).
                                                    # cdb (Gamma.000,Gamma)

Rabcd := R^{a}_{b c d} ->
          \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
        - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                    # cdb (Rabcd.000,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                    # cdb (Rab.000,Rab)

expr := 4 (R_{a b} - R_{b a}).                      # cdb (ex-0308.200,expr)

substitute   (expr, Rab)                            # cdb (ex-0308.201,expr)
substitute   (expr, Rabcd)                          # cdb (ex-0308.202,expr)
canonicalise (expr)                                 # cdb (ex-0308.203,expr)
substitute   (expr, Gamma)                          # cdb (ex-0308.204,expr)

distribute   (expr)                                 # cdb (ex-0308.205,expr)
```

```
product_rule (expr)                                  # cdb (ex-0308.206,expr)
canonicalise (expr)                                  # cdb (ex-0308.207,expr)

substitute   (expr, dgab)                            # cdb (ex-0308.208,expr)
canonicalise (expr)                                  # cdb (ex-0308.209,expr)
```

$$
\begin{aligned}
4R_{ab} - 4R_{ba} &= 4R^{c}{}_{acb} - 4R^{c}{}_{bca} && \texttt{(ex-0308.201)}\\
&= 4\partial_{c}\Gamma^{c}{}_{ab} + 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ab} - 4\partial_{b}\Gamma^{c}{}_{ac} - 4\Gamma^{c}{}_{eb}\Gamma^{e}{}_{ac} - 4\partial_{c}\Gamma^{c}{}_{ba} - 4\Gamma^{c}{}_{ec}\Gamma^{e}{}_{ba} + 4\partial_{a}\Gamma^{c}{}_{bc} + 4\Gamma^{c}{}_{ea}\Gamma^{e}{}_{bc} && \texttt{(ex-0308.202)}\\
&= -4\partial_{b}\Gamma^{c}{}_{ac} + 4\partial_{a}\Gamma^{c}{}_{bc} && \texttt{(ex-0308.203)}\\
&= -2\partial_{b}\left(g^{ce}\left(\partial_{a}g_{ec} + \partial_{c}g_{ae} - \partial_{e}g_{ac}\right)\right) + 2\partial_{a}\left(g^{ce}\left(\partial_{b}g_{ec} + \partial_{c}g_{be} - \partial_{e}g_{bc}\right)\right) && \texttt{(ex-0308.204)}\\
&= -2\partial_{b}\left(g^{ce}\partial_{a}g_{ec}\right) - 2\partial_{b}\left(g^{ce}\partial_{c}g_{ae}\right) + 2\partial_{b}\left(g^{ce}\partial_{e}g_{ac}\right) + 2\partial_{a}\left(g^{ce}\partial_{b}g_{ec}\right) + 2\partial_{a}\left(g^{ce}\partial_{c}g_{be}\right) - 2\partial_{a}\left(g^{ce}\partial_{e}g_{bc}\right) && \texttt{(ex-0308.205)}\\
&= -2\partial_{b}g^{ce}\partial_{a}g_{ec} - 2g^{ce}\partial_{ba}g_{ec} - 2\partial_{b}g^{ce}\partial_{c}g_{ae} - 2g^{ce}\partial_{bc}g_{ae} + 2\partial_{b}g^{ce}\partial_{e}g_{ac} + 2g^{ce}\partial_{be}g_{ac} + 2\partial_{a}g^{ce}\partial_{b}g_{ec} + 2g^{ce}\partial_{ab}g_{ec} + 2\partial_{a}g^{ce}\partial_{c}g_{be}\\
&\quad + 2g^{ce}\partial_{ac}g_{be} - 2\partial_{a}g^{ce}\partial_{e}g_{bc} - 2g^{ce}\partial_{ae}g_{bc} && \texttt{(ex-0308.206)}\\
&= -2\partial_{b}g^{ce}\partial_{a}g_{ce} + 2\partial_{a}g^{ce}\partial_{b}g_{ce} && \texttt{(ex-0308.207)}\\
&= 2g^{cd}g^{ef}\partial_{b}g_{df}\partial_{a}g_{ce} - 2g^{cd}g^{ef}\partial_{a}g_{df}\partial_{b}g_{ce} && \texttt{(ex-0308.208)}\\
&= 0 && \texttt{(ex-0308.209)}
\end{aligned}
$$