

Exercise 2.1 Using Cadabra's own product rule

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # templates for covariant derivatives
7
8 deriv1 := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}}
9         + \Gamma^{b}_{c a} A^{c}.
10
11 deriv2 := \nabla_{a}{A_{b}} -> \partial_{a}{A_{b}}
12        - \Gamma^{c}_{b a} A_{c}.
13
14 # create an object
15
16 uv := \nabla_{a}{v_{b} u^{b}}
17      - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.101,uv)
18
19 # apply the rules, then simplify
20
21 product_rule (uv)                  # cdb (ex-0201.102,uv)
22 substitute (uv,deriv1)             # cdb (ex-0201.103,uv)
23 substitute (uv,deriv2)             # cdb (ex-0201.104,uv)
24 distribute (uv)                   # cdb (ex-0201.105,uv)
25 sort_product (uv)                 # cdb (ex-0201.106,uv)
26 rename_dummies (uv)               # cdb (ex-0201.107,uv)
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$$\nabla_a (v_b u^b) - \partial_a (v_b u^b) = \nabla_a v_b u^b + v_b \nabla_a u^b - \partial_a v_b u^b - v_b \partial_a u^b \quad (\text{ex-0201.102})$$

$$= \nabla_a v_b u^b + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - \partial_a v_b u^b - v_b \partial_a u^b \quad (\text{ex-0201.103})$$

$$= (\partial_a v_b - \Gamma^c_{ba} v_c) u^b + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - \partial_a v_b u^b - v_b \partial_a u^b \quad (\text{ex-0201.104})$$

$$= -\Gamma^c_{ba} v_c u^b + v_b \Gamma^b_{ca} u^c \quad (\text{ex-0201.105})$$

$$= -\Gamma^c_{ba} u^b v_c + \Gamma^b_{ca} u^c v_b \quad (\text{ex-0201.106})$$

$$= 0 \quad (\text{ex-0201.107})$$

Exercise 2.1 Using hand crafted product rules

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 # templates for covariant derivatives
7
8 deriv1 := \nabla_{a}{A^{b}} -> \partial_{a}{A^{b}}
9         + \Gamma^{b}_{c a} A^{c}.
10
11 deriv2 := \nabla_{a}{A_{b}} -> \partial_{a}{A_{b}}
12         - \Gamma^{c}_{b a} A_{c}.
13
14 # templates for product rules
15
16 deriv3 := \nabla_{a}{A_{b} B^{c}} -> B^{c} \nabla_{a}{A_{b}}
17         + A_{b} \nabla_{a}{B^{c}}.
18
19 deriv4 := \partial_{a}{A_{b} B^{c}} -> B^{c} \partial_{a}{A_{b}}
20         + A_{b} \partial_{a}{B^{c}}.
21
22 # create an object
23
24 uv := \nabla_{a}{v_{b} u^{b}}
25      - \partial_{a}{v_{b} u^{b}}.      # cdb (ex-0201.201,uv)
26
27 # apply the rules, then simplify
28
29 substitute (uv,deriv3)      # cdb (ex-0201.202,uv)
30 substitute (uv,deriv4)      # cdb (ex-0201.203,uv)
31 substitute (uv,deriv1)      # cdb (ex-0201.204,uv)
32 substitute (uv,deriv2)      # cdb (ex-0201.205,uv)
33 distribute (uv)             # cdb (ex-0201.206,uv)
34 sort_product (uv)           # cdb (ex-0201.207,uv)
35 rename_dummies (uv)         # cdb (ex-0201.208,uv)

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$$\nabla_a (v_b u^b) - \partial_a (v_b u^b) = u^b \nabla_a v_b + v_b \nabla_a u^b - \partial_a (v_b u^b) \quad (\text{ex-0201.202})$$

$$= u^b \nabla_a v_b + v_b \nabla_a u^b - u^b \partial_a v_b - v_b \partial_a u^b \quad (\text{ex-0201.203})$$

$$= u^b \nabla_a v_b + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - u^b \partial_a v_b - v_b \partial_a u^b \quad (\text{ex-0201.204})$$

$$= u^b (\partial_a v_b - \Gamma^c_{ba} v_c) + v_b (\partial_a u^b + \Gamma^b_{ca} u^c) - u^b \partial_a v_b - v_b \partial_a u^b \quad (\text{ex-0201.205})$$

$$= -u^b \Gamma^c_{ba} v_c + v_b \Gamma^b_{ca} u^c \quad (\text{ex-0201.206})$$

$$= -\Gamma^c_{ba} u^b v_c + \Gamma^b_{ca} u^c v_b \quad (\text{ex-0201.207})$$

$$= 0 \quad (\text{ex-0201.208})$$