Exercise 2.5 Combining rules – a solution

```
1  {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3  \nabla{#}::Derivative.
4  \partial{#}::PartialDerivative.
5
6  # rules for covariant derivatives of v
7
8  deriv1 := \nabla_{a}{v^{b}} -> \partial_{a}{v^{b}}
9                              + \Gamma^{b}_{d a} v^{d}.
10
11 deriv2 := \nabla_{a}{\nabla_{b}{v^{c}}} -> \partial_{a}{\nabla_{b}{v^{c}}}
12                                          + \Gamma^{c}_{d a} \nabla_{b}{v^{d}}
13                                          - \Gamma^{d}_{b a} \nabla_{d}{v^{c}}.
14
15 # second covariant derivative of v
16
17 expr := v^{c}_{b a} -> \nabla_{a}{\nabla_{b}{v^{c}}}.  # cdb (ex-0205.101,expr)
18 save := @(expr).
19
20 # apply the rules, then simplify
21
22 substitute     (expr,deriv2)       # cdb (ex-0205.102,expr)
23 substitute     (expr,deriv1)       # cdb (ex-0205.103,expr)
24 distribute     (expr)              # cdb (ex-0205.104,expr)
25 product_rule   (expr)              # cdb (ex-0205.105,expr)
26 canonicalise   (expr)              # cdb (ex-0205.107,expr)
27 substitute     (expr,save)         # cdb (ex-0205.108,expr)
```

The trick here is to introduce in line 17 a dummy left hand side, `v^{c}{}_{b a}`, that is invisible with respect to the substitution rules of lines 8 and 11. Thus lines 22 and 23 will only target the right hand side of `expr`.

Notice how a copy of the initial expression is made in 18. This is used later in line 27 to replace the dummy object `v^{c}{}_{b a}` with `\nabla_{a}{\nabla_{b}{v^{c}}}` but this time acting on the left hand side of the rule. The result is a rule for second covariant deriavtives.

$$v^{c}{}_{ba} \to \nabla_a(\nabla_b v^c) \tag{ex-0205.101}$$

$$v^{c}{}_{ba} \to \partial_a(\nabla_b v^c) + \Gamma^{c}{}_{da}\nabla_b v^d - \Gamma^{d}{}_{ba}\nabla_d v^c \tag{ex-0205.102}$$

$$v^{c}{}_{ba} \to \partial_a\left(\partial_b v^c + \Gamma^{c}{}_{db}v^d\right) + \Gamma^{c}{}_{da}\left(\partial_b v^d + \Gamma^{d}{}_{eb}v^e\right) - \Gamma^{d}{}_{ba}\left(\partial_d v^c + \Gamma^{c}{}_{ed}v^e\right) \tag{ex-0205.103}$$

$$v^{c}{}_{ba} \to \partial_a\partial_b v^c + \partial_a\left(\Gamma^{c}{}_{db}v^d\right) + \Gamma^{c}{}_{da}\partial_b v^d + \Gamma^{c}{}_{da}\Gamma^{d}{}_{eb}v^e - \Gamma^{d}{}_{ba}\partial_d v^c - \Gamma^{d}{}_{ba}\Gamma^{c}{}_{ed}v^e \tag{ex-0205.104}$$

$$v^{c}{}_{ba} \to \partial_a\partial_b v^c + \partial_a\Gamma^{c}{}_{db}v^d + \Gamma^{c}{}_{db}\partial_a v^d + \Gamma^{c}{}_{da}\partial_b v^d + \Gamma^{c}{}_{da}\Gamma^{d}{}_{eb}v^e - \Gamma^{d}{}_{ba}\partial_d v^c - \Gamma^{d}{}_{ba}\Gamma^{c}{}_{ed}v^e \tag{ex-0205.105}$$

$$v^{c}{}_{ba} \to \partial_a\partial_b v^c + \partial_a\Gamma^{c}{}_{db}v^d + \Gamma^{c}{}_{db}\partial_a v^d + \Gamma^{c}{}_{da}\partial_b v^d + \Gamma^{c}{}_{da}\Gamma^{d}{}_{eb}v^e - \Gamma^{d}{}_{ba}\partial_d v^c - \Gamma^{c}{}_{de}\Gamma^{e}{}_{ba}v^d \tag{ex-0205.107}$$

$$\nabla_a(\nabla_b v^c) \to \partial_a\partial_b v^c + \partial_a\Gamma^{c}{}_{db}v^d + \Gamma^{c}{}_{db}\partial_a v^d + \Gamma^{c}{}_{da}\partial_b v^d + \Gamma^{c}{}_{da}\Gamma^{d}{}_{eb}v^e - \Gamma^{d}{}_{ba}\partial_d v^c - \Gamma^{c}{}_{de}\Gamma^{e}{}_{ba}v^d \tag{ex-0205.108}$$