

## Exercise 3.7 Commutation of $\nabla$ on the Riemann tensor – direct computation

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 ::Symbol;
4
5 \partial{#}::PartialDerivative.
6
7 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
8
9 RabcdD := \partial_c{\Gamma_{a b d}}
10          - \partial_d{\Gamma_{a b c}}
11          + \Gamma_{e a d} \Gamma^e_{b c}
12          - \Gamma_{e a c} \Gamma^e_{b d} -> R_{a b c d}.          # cdb(Rabcd.010,RabcdD)
13
14 RabcdU := \partial_c{\Gamma^a_{b d}}
15          - \partial_d{\Gamma^a_{b c}}
16          + \Gamma^e_{b d} \Gamma^a_{c e}
17          - \Gamma^e_{b c} \Gamma^a_{d e} -> R^a_{b c d}.          # cdb(Rabcd.000,RabcdU)
18
19 d1Rabcd := R_{a b c d ; e} -> \partial_e{R_{a b c d}}
20          - \Gamma^f_{a e} R_{f b c d}
21          - \Gamma^f_{b e} R_{a f c d}
22          - \Gamma^f_{c e} R_{a b f d}
23          - \Gamma^f_{d e} R_{a b c f}.          # cdb(d1Rabcd.000,d1Rabcd)
24
25 d2Rabcd := R_{a b c d ; e ; f} -> \partial_f{R_{a b c d ; e}}
26          - \Gamma^g_{a f} R_{g b c d ; e}
27          - \Gamma^g_{b f} R_{a g c d ; e}
28          - \Gamma^g_{c f} R_{a b g d ; e}
29          - \Gamma^g_{d f} R_{a b c g ; e}
30          - \Gamma^g_{e f} R_{a b c d ; g}.          # cdb(d2Rabcd.000,d2Rabcd)
31
32 substitute (d2Rabcd,d1Rabcd)          # cdb (d2Rabcd.001, d2Rabcd)
33
34 expr := R_{a b c d ; e ; f} - R_{a b c d ; f ; e}.          # cdb (ex-0307.100, expr)
35
36 substitute (expr,d2Rabcd)          # cdb (ex-0307.101, expr)

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37
38 distribute      (expr)                      # cdb (ex-0307.102, expr)
39 product_rule    (expr)                      # cdb (ex-0307.103, expr)
40
41 sort_product     (expr)                      # cdb (ex-0307.104, expr)
42 rename_dummies  (expr)                      # cdb (ex-0307.105, expr)
43 canonicalise     (expr)                      # cdb (ex-0307.106, expr)
44 factor_out       (expr,$R_{a? b? c? d?}$)    # cdb (ex-0307.107, expr)
45
46 substitute       (expr,RabcdU)              # cdb (ex-0307.108, expr)
47 substitute       (expr,$R^{a}_{b c d} -> -R^{a}_{b d c}$) # cdb (ex-0307.109, expr)

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$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_f(\partial_e R_{abcd} - \Gamma_{ae}^g R_{gbcd} - \Gamma_{be}^g R_{agcd} - \Gamma_{ce}^g R_{abgd} - \Gamma_{de}^g R_{abcg}) - \Gamma_{af}^g (\partial_e R_{gbcd} - \Gamma_{ge}^h R_{hbcd} - \Gamma_{be}^h R_{ghcd} - \Gamma_{ce}^h R_{gbhd} - \Gamma_{de}^h R_{gbch}) \\
& - \Gamma_{bf}^g (\partial_e R_{agcd} - \Gamma_{ae}^h R_{hgcd} - \Gamma_{ge}^h R_{ahcd} - \Gamma_{ce}^h R_{aghd} - \Gamma_{de}^h R_{agch}) \\
& - \Gamma_{cf}^g (\partial_e R_{abgd} - \Gamma_{ae}^h R_{hbgd} - \Gamma_{be}^h R_{ahgd} - \Gamma_{ge}^h R_{abhd} - \Gamma_{de}^h R_{abgh}) \\
& - \Gamma_{df}^g (\partial_e R_{abcg} - \Gamma_{ae}^h R_{hbcg} - \Gamma_{be}^h R_{ahcg} - \Gamma_{ce}^h R_{abhg} - \Gamma_{ge}^h R_{abch}) \\
& - \Gamma_{ef}^g (\partial_g R_{abcd} - \Gamma_{ag}^h R_{hbcd} - \Gamma_{bg}^h R_{ahcd} - \Gamma_{cg}^h R_{abhd} - \Gamma_{dg}^h R_{abch}) \\
& - \partial_e(\partial_f R_{abcd} - \Gamma_{af}^g R_{gbcd} - \Gamma_{bf}^g R_{agcd} - \Gamma_{cf}^g R_{abgd} - \Gamma_{df}^g R_{abcg}) + \Gamma_{ae}^g (\partial_f R_{gbcd} - \Gamma_{gf}^h R_{hbcd} - \Gamma_{bf}^h R_{ghcd} - \Gamma_{cf}^h R_{gbhd} - \Gamma_{df}^h R_{gbch}) \\
& + \Gamma_{be}^g (\partial_f R_{agcd} - \Gamma_{af}^h R_{hgcd} - \Gamma_{gf}^h R_{ahcd} - \Gamma_{cf}^h R_{aghd} - \Gamma_{df}^h R_{agch}) \\
& + \Gamma_{ce}^g (\partial_f R_{abgd} - \Gamma_{af}^h R_{hbgd} - \Gamma_{bf}^h R_{ahgd} - \Gamma_{gf}^h R_{abhd} - \Gamma_{df}^h R_{abgh}) \\
& + \Gamma_{de}^g (\partial_f R_{abcg} - \Gamma_{af}^h R_{hbcg} - \Gamma_{bf}^h R_{ahcg} - \Gamma_{cf}^h R_{abhg} - \Gamma_{gf}^h R_{abch}) \\
& + \Gamma_{fe}^g (\partial_g R_{abcd} - \Gamma_{ag}^h R_{hbcd} - \Gamma_{bg}^h R_{ahcd} - \Gamma_{cg}^h R_{abhd} - \Gamma_{dg}^h R_{abch}) \quad (\text{ex-0307.101})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - \partial_f(\Gamma_{ae}^g R_{gbcd}) - \partial_f(\Gamma_{be}^g R_{agcd}) - \partial_f(\Gamma_{ce}^g R_{abgd}) - \partial_f(\Gamma_{de}^g R_{abcg}) - \Gamma_{af}^g \partial_e R_{gbcd} + \Gamma_{af}^g \Gamma_{ge}^h R_{hbcd} + \Gamma_{af}^g \Gamma_{be}^h R_{ghcd} \\
& + \Gamma_{af}^g \Gamma_{ce}^h R_{gbhd} + \Gamma_{af}^g \Gamma_{de}^h R_{gbch} - \Gamma_{bf}^g \partial_e R_{agcd} + \Gamma_{bf}^g \Gamma_{ae}^h R_{hgcd} + \Gamma_{bf}^g \Gamma_{ge}^h R_{ahcd} + \Gamma_{bf}^g \Gamma_{ce}^h R_{aghd} + \Gamma_{bf}^g \Gamma_{de}^h R_{agch} - \Gamma_{cf}^g \partial_e R_{abgd} \\
& + \Gamma_{cf}^g \Gamma_{ae}^h R_{hbgd} + \Gamma_{cf}^g \Gamma_{be}^h R_{ahgd} + \Gamma_{cf}^g \Gamma_{ge}^h R_{abhd} + \Gamma_{cf}^g \Gamma_{de}^h R_{abgh} - \Gamma_{df}^g \partial_e R_{abcg} + \Gamma_{df}^g \Gamma_{ae}^h R_{hbcg} + \Gamma_{df}^g \Gamma_{be}^h R_{ahcg} + \Gamma_{df}^g \Gamma_{ce}^h R_{abhg} \\
& + \Gamma_{df}^g \Gamma_{ge}^h R_{abch} - \Gamma_{ef}^g \partial_g R_{abcd} + \Gamma_{ef}^g \Gamma_{ag}^h R_{hbcd} + \Gamma_{ef}^g \Gamma_{bg}^h R_{ahcd} + \Gamma_{ef}^g \Gamma_{cg}^h R_{abhd} + \Gamma_{ef}^g \Gamma_{dg}^h R_{abch} - \partial_{ef} R_{abcd} + \partial_e(\Gamma_{af}^g R_{gbcd}) \\
& + \partial_e(\Gamma_{bf}^g R_{agcd}) + \partial_e(\Gamma_{cf}^g R_{abgd}) + \partial_e(\Gamma_{df}^g R_{abcg}) + \Gamma_{ae}^g \partial_f R_{gbcd} - \Gamma_{ae}^g \Gamma_{gf}^h R_{hbcd} - \Gamma_{ae}^g \Gamma_{bf}^h R_{ghcd} - \Gamma_{ae}^g \Gamma_{cf}^h R_{gbhd} - \Gamma_{ae}^g \Gamma_{df}^h R_{gbch} \\
& + \Gamma_{be}^g \partial_f R_{agcd} - \Gamma_{be}^g \Gamma_{af}^h R_{hgcd} - \Gamma_{be}^g \Gamma_{gf}^h R_{ahcd} - \Gamma_{be}^g \Gamma_{cf}^h R_{aghd} - \Gamma_{be}^g \Gamma_{df}^h R_{agch} + \Gamma_{ce}^g \partial_f R_{abgd} - \Gamma_{ce}^g \Gamma_{af}^h R_{hbgd} - \Gamma_{ce}^g \Gamma_{bf}^h R_{ahgd} \\
& - \Gamma_{ce}^g \Gamma_{gf}^h R_{abhd} - \Gamma_{ce}^g \Gamma_{df}^h R_{abgh} + \Gamma_{de}^g \partial_f R_{abcg} - \Gamma_{de}^g \Gamma_{af}^h R_{hbcg} - \Gamma_{de}^g \Gamma_{bf}^h R_{ahcg} - \Gamma_{de}^g \Gamma_{cf}^h R_{abhg} - \Gamma_{de}^g \Gamma_{gf}^h R_{abch} + \Gamma_{fe}^g \partial_g R_{abcd} \\
& - \Gamma_{fe}^g \Gamma_{ag}^h R_{hbcd} - \Gamma_{fe}^g \Gamma_{bg}^h R_{ahcd} - \Gamma_{fe}^g \Gamma_{cg}^h R_{abhd} - \Gamma_{fe}^g \Gamma_{dg}^h R_{abch} \quad (\text{ex-0307.102})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - \partial_f \Gamma_{ae}^g R_{gbcd} - \partial_f \Gamma_{be}^g R_{agcd} - \partial_f \Gamma_{ce}^g R_{abgd} - \partial_f \Gamma_{de}^g R_{abcg} + \Gamma_{af}^g \Gamma_{ge}^h R_{hbcd} + \Gamma_{af}^g \Gamma_{be}^h R_{ghcd} + \Gamma_{af}^g \Gamma_{ce}^h R_{gbhd} + \Gamma_{af}^g \Gamma_{de}^h R_{gbch} \\
& + \Gamma_{bf}^g \Gamma_{ae}^h R_{hgcd} + \Gamma_{bf}^g \Gamma_{ge}^h R_{ahcd} + \Gamma_{bf}^g \Gamma_{ce}^h R_{aghd} + \Gamma_{bf}^g \Gamma_{de}^h R_{agch} + \Gamma_{cf}^g \Gamma_{ae}^h R_{hbgd} + \Gamma_{cf}^g \Gamma_{be}^h R_{ahgd} + \Gamma_{cf}^g \Gamma_{ge}^h R_{abhd} + \Gamma_{cf}^g \Gamma_{de}^h R_{abgh} \\
& + \Gamma_{df}^g \Gamma_{ae}^h R_{hbcg} + \Gamma_{df}^g \Gamma_{be}^h R_{ahcg} + \Gamma_{df}^g \Gamma_{ce}^h R_{abhg} + \Gamma_{df}^g \Gamma_{ge}^h R_{abch} - \Gamma_{ef}^g \partial_g R_{abcd} + \Gamma_{ef}^g \Gamma_{ag}^h R_{hbcd} + \Gamma_{ef}^g \Gamma_{bg}^h R_{ahcd} + \Gamma_{ef}^g \Gamma_{cg}^h R_{abhd} \\
& + \Gamma_{ef}^g \Gamma_{dg}^h R_{abch} - \partial_{ef} R_{abcd} + \partial_e \Gamma_{af}^g R_{gbcd} + \partial_e \Gamma_{bf}^g R_{agcd} + \partial_e \Gamma_{cf}^g R_{abgd} + \partial_e \Gamma_{df}^g R_{abcg} - \Gamma_{ae}^g \Gamma_{gf}^h R_{hbcd} - \Gamma_{ae}^g \Gamma_{bf}^h R_{ghcd} \\
& - \Gamma_{ae}^g \Gamma_{cf}^h R_{gbhd} - \Gamma_{ae}^g \Gamma_{df}^h R_{gbch} - \Gamma_{be}^g \Gamma_{af}^h R_{hgcd} - \Gamma_{be}^g \Gamma_{gf}^h R_{ahcd} - \Gamma_{be}^g \Gamma_{cf}^h R_{aghd} - \Gamma_{be}^g \Gamma_{df}^h R_{agch} - \Gamma_{ce}^g \Gamma_{af}^h R_{hbgd} \\
& - \Gamma_{ce}^g \Gamma_{bf}^h R_{ahgd} - \Gamma_{ce}^g \Gamma_{gf}^h R_{abhd} - \Gamma_{ce}^g \Gamma_{df}^h R_{abgh} - \Gamma_{de}^g \Gamma_{af}^h R_{hbcg} - \Gamma_{de}^g \Gamma_{bf}^h R_{ahcg} - \Gamma_{de}^g \Gamma_{cf}^h R_{abhg} - \Gamma_{de}^g \Gamma_{gf}^h R_{abch} \\
& + \Gamma_{fe}^g \partial_g R_{abcd} - \Gamma_{fe}^g \Gamma_{ag}^h R_{hbcd} - \Gamma_{fe}^g \Gamma_{bg}^h R_{ahcd} - \Gamma_{fe}^g \Gamma_{cg}^h R_{abhd} - \Gamma_{fe}^g \Gamma_{dg}^h R_{abch} \quad (\text{ex-0307.103})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma_{ae}^g - R_{agcd} \partial_f \Gamma_{be}^g - R_{abgd} \partial_f \Gamma_{ce}^g - R_{abcg} \partial_f \Gamma_{de}^g + R_{hbcd} \Gamma_{af}^g \Gamma_{ge}^h + R_{ghcd} \Gamma_{af}^g \Gamma_{be}^h + R_{gbhd} \Gamma_{af}^g \Gamma_{ce}^h + R_{gbch} \Gamma_{af}^g \Gamma_{de}^h \\
& + R_{hgcd} \Gamma_{bf}^g \Gamma_{ae}^h + R_{ahcd} \Gamma_{bf}^g \Gamma_{ge}^h + R_{aghd} \Gamma_{bf}^g \Gamma_{ce}^h + R_{agch} \Gamma_{bf}^g \Gamma_{de}^h + R_{hbgd} \Gamma_{cf}^g \Gamma_{ae}^h + R_{ahgd} \Gamma_{cf}^g \Gamma_{be}^h + R_{abhd} \Gamma_{cf}^g \Gamma_{ge}^h + R_{abgh} \Gamma_{cf}^g \Gamma_{de}^h \\
& + R_{hbcg} \Gamma_{df}^g \Gamma_{ae}^h + R_{ahcg} \Gamma_{df}^g \Gamma_{be}^h + R_{abhg} \Gamma_{df}^g \Gamma_{ce}^h + R_{abch} \Gamma_{df}^g \Gamma_{ge}^h - \Gamma_{ef}^g \partial_g R_{abcd} + R_{hbcd} \Gamma_{ef}^g \Gamma_{ag}^h + R_{ahcd} \Gamma_{ef}^g \Gamma_{bg}^h + R_{abhd} \Gamma_{ef}^g \Gamma_{cg}^h \\
& + R_{abch} \Gamma_{ef}^g \Gamma_{dg}^h - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma_{af}^g + R_{agcd} \partial_e \Gamma_{bf}^g + R_{abgd} \partial_e \Gamma_{cf}^g + R_{abcg} \partial_e \Gamma_{df}^g - R_{hbcd} \Gamma_{ae}^g \Gamma_{gf}^h - R_{ghcd} \Gamma_{ae}^g \Gamma_{bf}^h \\
& - R_{gbhd} \Gamma_{ae}^g \Gamma_{cf}^h - R_{gbch} \Gamma_{ae}^g \Gamma_{df}^h - R_{hgcd} \Gamma_{be}^g \Gamma_{af}^h - R_{ahcd} \Gamma_{be}^g \Gamma_{gf}^h - R_{aghd} \Gamma_{be}^g \Gamma_{cf}^h - R_{agch} \Gamma_{be}^g \Gamma_{df}^h - R_{hbgd} \Gamma_{ce}^g \Gamma_{af}^h \\
& - R_{ahgd} \Gamma_{ce}^g \Gamma_{bf}^h - R_{abhd} \Gamma_{ce}^g \Gamma_{gf}^h - R_{abgh} \Gamma_{ce}^g \Gamma_{df}^h - R_{hbcg} \Gamma_{de}^g \Gamma_{af}^h - R_{ahcg} \Gamma_{de}^g \Gamma_{bf}^h - R_{abhg} \Gamma_{de}^g \Gamma_{cf}^h - R_{abch} \Gamma_{de}^g \Gamma_{gf}^h \\
& + \Gamma_{fe}^g \partial_g R_{abcd} - R_{hbcd} \Gamma_{fe}^g \Gamma_{ag}^h - R_{ahcd} \Gamma_{fe}^g \Gamma_{bg}^h - R_{abhd} \Gamma_{fe}^g \Gamma_{cg}^h - R_{abch} \Gamma_{fe}^g \Gamma_{dg}^h \quad (\text{ex-0307.104})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & \partial_{fe} R_{abcd} - R_{gbcd} \partial_f \Gamma_{ae}^g - R_{agcd} \partial_f \Gamma_{be}^g - R_{abgd} \partial_f \Gamma_{ce}^g - R_{abcg} \partial_f \Gamma_{de}^g + R_{gbcd} \Gamma_{af}^h \Gamma_{he}^g + R_{ghcd} \Gamma_{af}^h \Gamma_{be}^g + R_{gbhd} \Gamma_{af}^h \Gamma_{ce}^g + R_{gbch} \Gamma_{af}^h \Gamma_{de}^g \\
& + R_{ghcd} \Gamma_{bf}^h \Gamma_{ae}^g + R_{agcd} \Gamma_{bf}^h \Gamma_{he}^g + R_{aghd} \Gamma_{bf}^h \Gamma_{ce}^g + R_{agch} \Gamma_{bf}^h \Gamma_{de}^g + R_{gbhd} \Gamma_{cf}^h \Gamma_{ae}^g + R_{aghd} \Gamma_{cf}^h \Gamma_{be}^g + R_{abgd} \Gamma_{cf}^h \Gamma_{he}^g + R_{abgh} \Gamma_{cf}^h \Gamma_{de}^g \\
& + R_{gbch} \Gamma_{df}^h \Gamma_{ae}^g + R_{agch} \Gamma_{df}^h \Gamma_{be}^g + R_{abgh} \Gamma_{df}^h \Gamma_{ce}^g + R_{abcg} \Gamma_{df}^h \Gamma_{he}^g - \Gamma_{ef}^g \partial_g R_{abcd} + R_{gbcd} \Gamma_{ef}^h \Gamma_{ah}^g + R_{agcd} \Gamma_{ef}^h \Gamma_{bh}^g + R_{abgd} \Gamma_{ef}^h \Gamma_{ch}^g \\
& + R_{abcg} \Gamma_{ef}^h \Gamma_{dh}^g - \partial_{ef} R_{abcd} + R_{gbcd} \partial_e \Gamma_{af}^g + R_{agcd} \partial_e \Gamma_{bf}^g + R_{abgd} \partial_e \Gamma_{cf}^g + R_{abcg} \partial_e \Gamma_{df}^g - R_{gbcd} \Gamma_{ae}^h \Gamma_{hf}^g - R_{ghcd} \Gamma_{ae}^h \Gamma_{bf}^g \\
& - R_{gbhd} \Gamma_{ae}^h \Gamma_{cf}^g - R_{gbch} \Gamma_{ae}^h \Gamma_{df}^g - R_{ghcd} \Gamma_{be}^h \Gamma_{af}^g - R_{agcd} \Gamma_{be}^h \Gamma_{hf}^g - R_{aghd} \Gamma_{be}^h \Gamma_{cf}^g - R_{agch} \Gamma_{be}^h \Gamma_{df}^g - R_{gbhd} \Gamma_{ce}^h \Gamma_{af}^g \\
& - R_{aghd} \Gamma_{ce}^h \Gamma_{bf}^g - R_{abgd} \Gamma_{ce}^h \Gamma_{hf}^g - R_{abgh} \Gamma_{ce}^h \Gamma_{df}^g - R_{gbch} \Gamma_{de}^h \Gamma_{af}^g - R_{agch} \Gamma_{de}^h \Gamma_{bf}^g - R_{abgh} \Gamma_{de}^h \Gamma_{cf}^g - R_{abcg} \Gamma_{de}^h \Gamma_{hf}^g \\
& + \Gamma_{fe}^g \partial_g R_{abcd} - R_{gbcd} \Gamma_{fe}^h \Gamma_{ah}^g - R_{agcd} \Gamma_{fe}^h \Gamma_{bh}^g - R_{abgd} \Gamma_{fe}^h \Gamma_{ch}^g - R_{abcg} \Gamma_{fe}^h \Gamma_{dh}^g \quad (\text{ex-0307.105})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & -R_{gbcd} \partial_f \Gamma_{ae}^g - R_{agcd} \partial_f \Gamma_{be}^g - R_{abgd} \partial_f \Gamma_{ce}^g - R_{abcg} \partial_f \Gamma_{de}^g + R_{gbcd} \Gamma_{af}^h \Gamma_{eh}^g + R_{agcd} \Gamma_{bf}^h \Gamma_{eh}^g + R_{abgd} \Gamma_{cf}^h \Gamma_{eh}^g + R_{abcg} \Gamma_{df}^h \Gamma_{eh}^g \\
& + R_{gbcd} \partial_e \Gamma_{af}^g + R_{agcd} \partial_e \Gamma_{bf}^g + R_{abgd} \partial_e \Gamma_{cf}^g + R_{abcg} \partial_e \Gamma_{df}^g - R_{gbcd} \Gamma_{ae}^h \Gamma_{fh}^g - R_{agcd} \Gamma_{be}^h \Gamma_{fh}^g - R_{abgd} \Gamma_{ce}^h \Gamma_{fh}^g \\
& - R_{abcg} \Gamma_{de}^h \Gamma_{fh}^g \quad (\text{ex-0307.106})
\end{aligned}$$

$$\begin{aligned}
R_{abcd;e;f} - R_{abcd;f;e} = & R_{gbcd} (-\partial_f \Gamma_{ae}^g + \Gamma_{af}^h \Gamma_{eh}^g + \partial_e \Gamma_{af}^g - \Gamma_{ae}^h \Gamma_{fh}^g) + R_{agcd} (-\partial_f \Gamma_{be}^g + \Gamma_{bf}^h \Gamma_{eh}^g + \partial_e \Gamma_{bf}^g - \Gamma_{be}^h \Gamma_{fh}^g) \\
& + R_{abgd} (-\partial_f \Gamma_{ce}^g + \Gamma_{cf}^h \Gamma_{eh}^g + \partial_e \Gamma_{cf}^g - \Gamma_{ce}^h \Gamma_{fh}^g) + R_{abcg} (-\partial_f \Gamma_{de}^g + \Gamma_{df}^h \Gamma_{eh}^g + \partial_e \Gamma_{df}^g - \Gamma_{de}^h \Gamma_{fh}^g) \quad (\text{ex-0307.107})
\end{aligned}$$

$$R_{abcd;e;f} - R_{abcd;f;e} = -R_{gbcd} R_{afe}^g - R_{agcd} R_{bfe}^g - R_{abgd} R_{cfe}^g - R_{abcg} R_{dfe}^g \quad (\text{ex-0307.108})$$

$$R_{abcd;e;f} - R_{abcd;f;e} = R_{gbcd} R_{aef}^g + R_{agcd} R_{bef}^g + R_{abgd} R_{cef}^g + R_{abcg} R_{def}^g \quad (\text{ex-0307.109})$$