Exercise 3.1 Some symmetries of Riemann

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

;::Symbol;

\partial{#}::PartialDerivative.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
                          - \partial_{d}{\Gamma^{a}_{b c}}
                          + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
                          - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.      # cdb(Rabcd.000,Rabcd)

dRabcd := R^{a}_{b c d ; e} -> \partial_{e}{R^{a}_{b c d}}
                          + \Gamma^{a}_{f e} R^{f}_{b c d}
                          - \Gamma^{f}_{b e} R^{a}_{f c d}
                          - \Gamma^{f}_{c e} R^{a}_{b f d}
                          - \Gamma^{f}_{d e} R^{a}_{b c f}.      # cdb(dRabcd.000,dRabcd)
```

Exercise 3.1 Antisymmetry on last pair of indices

```
1   expr := R^{a}_{b c d} + R^{a}_{b d c}.                    # cdb(ex-0301.101,expr)
2
3   substitute (expr, Rabcd)                                  # cdb(ex-0301.102,expr)
```

$$R^a{}_{bcd} + R^a{}_{bdc} = 0$$

Exercise 3.1 First Bianchi identity

```
1   expr := R^{a}_{b c d} + R^{a}_{d b c} + R^{a}_{c d b}.        # cdb(ex-0301.201,expr)
2
3   substitute   (expr, Rabcd)                                    # cdb(ex-0301.202,expr)
4   canonicalise (expr)                                           # cdb(ex-0301.203,expr)
```

$$R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de} + \partial_b\Gamma^a{}_{dc} - \partial_c\Gamma^a{}_{db} + \Gamma^e{}_{dc}\Gamma^a{}_{be} - \Gamma^e{}_{db}\Gamma^a{}_{ce} + \partial_d\Gamma^a{}_{cb} - \partial_b\Gamma^a{}_{cd} + \Gamma^e{}_{cb}\Gamma^a{}_{de}$$
$$- \Gamma^e{}_{cd}\Gamma^a{}_{be} \tag{ex-0301.202}$$
$$= 0 \tag{ex-0301.203}$$

Exercise 3.1 Second Bianchi identity

```
1   expr := R^{a}_{b c d ; e} + R^{a}_{b e c ; d} + R^{a}_{b d e ; c}.   # cdb(ex-0301.301,expr)
2
3   substitute      (expr, dRabcd)                                        # cdb(ex-0301.302,expr)
4   substitute      (expr,  Rabcd)                                        # cdb(ex-0301.303,expr)
5   distribute      (expr)                                                # cdb(ex-0301.304,expr)
6   product_rule    (expr)                                                # cdb(ex-0301.305,expr)
7   sort_product    (expr)                                                # cdb(ex-0301.306,expr)
8   rename_dummies  (expr)                                                # cdb(ex-0301.307,expr)
9   canonicalise    (expr)                                                # cdb(ex-0301.308,expr)
```

$$\begin{aligned}
R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} &= \partial_e R^a{}_{bcd} + \Gamma^a{}_{fe} R^f{}_{bcd} - \Gamma^f{}_{be} R^a{}_{fcd} - \Gamma^f{}_{ce} R^a{}_{bfd} - \Gamma^f{}_{de} R^a{}_{bcf} + \partial_d R^a{}_{bec} + \Gamma^a{}_{fd} R^f{}_{bec} - \Gamma^f{}_{bd} R^a{}_{fec} - \Gamma^f{}_{ed} R^a{}_{bfc} - \Gamma^f{}_{cd} R^a{}_{bef} \\
&\quad + \partial_c R^a{}_{bde} + \Gamma^a{}_{fc} R^f{}_{bde} - \Gamma^f{}_{bc} R^a{}_{fde} - \Gamma^f{}_{dc} R^a{}_{bfe} - \Gamma^f{}_{ec} R^a{}_{bdf} \qquad\qquad\text{(ex-0301.302)}
\end{aligned}$$

$$\begin{aligned}
&= \partial_e\left(\partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^f{}_{bd}\Gamma^a{}_{cf} - \Gamma^f{}_{bc}\Gamma^a{}_{df}\right) + \Gamma^a{}_{fe}\left(\partial_c\Gamma^f{}_{bd} - \partial_d\Gamma^f{}_{bc} + \Gamma^g{}_{bd}\Gamma^f{}_{cg} - \Gamma^g{}_{bc}\Gamma^f{}_{dg}\right) \\
&\quad - \Gamma^f{}_{be}\left(\partial_c\Gamma^a{}_{fd} - \partial_d\Gamma^a{}_{fc} + \Gamma^g{}_{fd}\Gamma^a{}_{cg} - \Gamma^g{}_{fc}\Gamma^a{}_{dg}\right) - \Gamma^f{}_{ce}\left(\partial_f\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bf} + \Gamma^g{}_{bd}\Gamma^a{}_{fg} - \Gamma^g{}_{bf}\Gamma^a{}_{dg}\right) \\
&\quad - \Gamma^f{}_{de}\left(\partial_c\Gamma^a{}_{bf} - \partial_f\Gamma^a{}_{bc} + \Gamma^g{}_{bf}\Gamma^a{}_{cg} - \Gamma^g{}_{bc}\Gamma^a{}_{fg}\right) + \partial_d\left(\partial_e\Gamma^a{}_{bc} - \partial_c\Gamma^a{}_{be} + \Gamma^f{}_{bc}\Gamma^a{}_{ef} - \Gamma^f{}_{be}\Gamma^a{}_{cf}\right) \\
&\quad + \Gamma^a{}_{fd}\left(\partial_e\Gamma^f{}_{bc} - \partial_c\Gamma^f{}_{be} + \Gamma^g{}_{bc}\Gamma^f{}_{eg} - \Gamma^g{}_{be}\Gamma^f{}_{cg}\right) - \Gamma^f{}_{bd}\left(\partial_e\Gamma^a{}_{fc} - \partial_c\Gamma^a{}_{fe} + \Gamma^g{}_{fc}\Gamma^a{}_{eg} - \Gamma^g{}_{fe}\Gamma^a{}_{cg}\right) \\
&\quad - \Gamma^f{}_{ed}\left(\partial_f\Gamma^a{}_{bc} - \partial_c\Gamma^a{}_{bf} + \Gamma^g{}_{bc}\Gamma^a{}_{fg} - \Gamma^g{}_{bf}\Gamma^a{}_{cg}\right) - \Gamma^f{}_{cd}\left(\partial_e\Gamma^a{}_{bf} - \partial_f\Gamma^a{}_{be} + \Gamma^g{}_{bf}\Gamma^a{}_{eg} - \Gamma^g{}_{be}\Gamma^a{}_{fg}\right) \\
&\quad + \partial_c\left(\partial_d\Gamma^a{}_{be} - \partial_e\Gamma^a{}_{bd} + \Gamma^f{}_{be}\Gamma^a{}_{df} - \Gamma^f{}_{bd}\Gamma^a{}_{ef}\right) + \Gamma^a{}_{fc}\left(\partial_d\Gamma^f{}_{be} - \partial_e\Gamma^f{}_{bd} + \Gamma^g{}_{be}\Gamma^f{}_{dg} - \Gamma^g{}_{bd}\Gamma^f{}_{eg}\right) \\
&\quad - \Gamma^f{}_{bc}\left(\partial_d\Gamma^a{}_{fe} - \partial_e\Gamma^a{}_{fd} + \Gamma^g{}_{fe}\Gamma^a{}_{dg} - \Gamma^g{}_{fd}\Gamma^a{}_{eg}\right) - \Gamma^f{}_{dc}\left(\partial_f\Gamma^a{}_{be} - \partial_e\Gamma^a{}_{bf} + \Gamma^g{}_{be}\Gamma^a{}_{fg} - \Gamma^g{}_{bf}\Gamma^a{}_{eg}\right) \\
&\quad - \Gamma^f{}_{ec}\left(\partial_d\Gamma^a{}_{bf} - \partial_f\Gamma^a{}_{bd} + \Gamma^g{}_{bf}\Gamma^a{}_{dg} - \Gamma^g{}_{bd}\Gamma^a{}_{fg}\right) \qquad\qquad\text{(ex-0301.303)}
\end{aligned}$$

$$\begin{aligned}
&= \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \partial_e\left(\Gamma^f{}_{bd}\Gamma^a{}_{cf}\right) - \partial_e\left(\Gamma^f{}_{bc}\Gamma^a{}_{df}\right) + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^g{}_{bd}\Gamma^f{}_{cg} - \Gamma^a{}_{fe}\Gamma^g{}_{bc}\Gamma^f{}_{dg} \\
&\quad - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^f{}_{be}\Gamma^g{}_{fd}\Gamma^a{}_{cg} + \Gamma^f{}_{be}\Gamma^g{}_{fc}\Gamma^a{}_{dg} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^f{}_{ce}\Gamma^g{}_{bd}\Gamma^a{}_{fg} + \Gamma^f{}_{ce}\Gamma^g{}_{bf}\Gamma^a{}_{dg} \\
&\quad - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^f{}_{de}\Gamma^g{}_{bf}\Gamma^a{}_{cg} + \Gamma^f{}_{de}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \partial_d\left(\Gamma^f{}_{bc}\Gamma^a{}_{ef}\right) - \partial_d\left(\Gamma^f{}_{be}\Gamma^a{}_{cf}\right) \\
&\quad + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^g{}_{bc}\Gamma^f{}_{eg} - \Gamma^a{}_{fd}\Gamma^g{}_{be}\Gamma^f{}_{cg} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} - \Gamma^f{}_{bd}\Gamma^g{}_{fc}\Gamma^a{}_{eg} + \Gamma^f{}_{bd}\Gamma^g{}_{fe}\Gamma^a{}_{cg} \\
&\quad - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^f{}_{ed}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \Gamma^f{}_{ed}\Gamma^g{}_{bf}\Gamma^a{}_{cg} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} - \Gamma^f{}_{cd}\Gamma^g{}_{bf}\Gamma^a{}_{eg} + \Gamma^f{}_{cd}\Gamma^g{}_{be}\Gamma^a{}_{fg} \\
&\quad + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \partial_c\left(\Gamma^f{}_{be}\Gamma^a{}_{df}\right) - \partial_c\left(\Gamma^f{}_{bd}\Gamma^a{}_{ef}\right) + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^g{}_{be}\Gamma^f{}_{dg} - \Gamma^a{}_{fc}\Gamma^g{}_{bd}\Gamma^f{}_{eg} \\
&\quad - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^f{}_{bc}\Gamma^g{}_{fe}\Gamma^a{}_{dg} + \Gamma^f{}_{bc}\Gamma^g{}_{fd}\Gamma^a{}_{eg} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^f{}_{dc}\Gamma^g{}_{be}\Gamma^a{}_{fg} + \Gamma^f{}_{dc}\Gamma^g{}_{bf}\Gamma^a{}_{eg} \\
&\quad - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^f{}_{ec}\Gamma^g{}_{bf}\Gamma^a{}_{dg} + \Gamma^f{}_{ec}\Gamma^g{}_{bd}\Gamma^a{}_{fg} \qquad\qquad\text{(ex-0301.304)}
\end{aligned}$$

$$
\begin{aligned}
R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} ={}& \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \partial_e\Gamma^f{}_{bd}\Gamma^a{}_{cf} + \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{cf} - \partial_e\Gamma^f{}_{bc}\Gamma^a{}_{df} - \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{df} + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^g{}_{bd}\Gamma^f{}_{cg} \\
& - \Gamma^a{}_{fe}\Gamma^g{}_{bc}\Gamma^f{}_{dg} - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^f{}_{be}\Gamma^g{}_{fd}\Gamma^a{}_{cg} + \Gamma^f{}_{be}\Gamma^g{}_{fc}\Gamma^a{}_{dg} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^f{}_{ce}\Gamma^g{}_{bd}\Gamma^a{}_{fg} \\
& + \Gamma^f{}_{ce}\Gamma^g{}_{bf}\Gamma^a{}_{dg} - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^f{}_{de}\Gamma^g{}_{bf}\Gamma^a{}_{cg} + \Gamma^f{}_{de}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \partial_d\Gamma^f{}_{bc}\Gamma^a{}_{ef} \\
& + \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{ef} - \partial_d\Gamma^f{}_{be}\Gamma^a{}_{cf} - \Gamma^f{}_{be}\partial_d\Gamma^a{}_{cf} + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^g{}_{bc}\Gamma^f{}_{eg} - \Gamma^a{}_{fd}\Gamma^g{}_{be}\Gamma^f{}_{cg} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} \\
& + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} - \Gamma^f{}_{bd}\Gamma^g{}_{fc}\Gamma^a{}_{eg} + \Gamma^f{}_{bd}\Gamma^g{}_{fe}\Gamma^a{}_{cg} - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^f{}_{ed}\Gamma^g{}_{bc}\Gamma^a{}_{fg} + \Gamma^f{}_{ed}\Gamma^g{}_{bf}\Gamma^a{}_{cg} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} \\
& + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} - \Gamma^f{}_{cd}\Gamma^g{}_{bf}\Gamma^a{}_{eg} + \Gamma^f{}_{cd}\Gamma^g{}_{be}\Gamma^a{}_{fg} + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \partial_c\Gamma^f{}_{be}\Gamma^a{}_{df} + \Gamma^f{}_{be}\partial_c\Gamma^a{}_{df} - \partial_c\Gamma^f{}_{bd}\Gamma^a{}_{ef} \\
& - \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{ef} + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^g{}_{be}\Gamma^f{}_{dg} - \Gamma^a{}_{fc}\Gamma^g{}_{bd}\Gamma^f{}_{eg} - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^f{}_{bc}\Gamma^g{}_{fe}\Gamma^a{}_{dg} \\
& + \Gamma^f{}_{bc}\Gamma^g{}_{fd}\Gamma^a{}_{eg} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^f{}_{dc}\Gamma^g{}_{be}\Gamma^a{}_{fg} + \Gamma^f{}_{dc}\Gamma^g{}_{bf}\Gamma^a{}_{eg} - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^f{}_{ec}\Gamma^g{}_{bf}\Gamma^a{}_{dg} \\
& + \Gamma^f{}_{ec}\Gamma^g{}_{bd}\Gamma^a{}_{fg} \hspace{9cm} \text{(ex-0301.305)} \\[4pt]
={}& \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \Gamma^a{}_{cf}\partial_e\Gamma^f{}_{bd} + \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{cf} - \Gamma^a{}_{df}\partial_e\Gamma^f{}_{bc} - \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{df} + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^f{}_{cg}\Gamma^g{}_{bd} \\
& - \Gamma^a{}_{fe}\Gamma^f{}_{dg}\Gamma^g{}_{bc} - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^a{}_{cg}\Gamma^f{}_{be}\Gamma^g{}_{fd} + \Gamma^a{}_{dg}\Gamma^f{}_{be}\Gamma^g{}_{fc} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ce}\Gamma^g{}_{bd} \\
& + \Gamma^a{}_{dg}\Gamma^f{}_{ce}\Gamma^g{}_{bf} - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^a{}_{cg}\Gamma^f{}_{de}\Gamma^g{}_{bf} + \Gamma^a{}_{fg}\Gamma^f{}_{de}\Gamma^g{}_{bc} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \Gamma^a{}_{ef}\partial_d\Gamma^f{}_{bc} \\
& + \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{ef} - \Gamma^a{}_{cf}\partial_d\Gamma^f{}_{be} - \Gamma^f{}_{be}\partial_d\Gamma^a{}_{cf} + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^f{}_{eg}\Gamma^g{}_{bc} - \Gamma^a{}_{fd}\Gamma^f{}_{cg}\Gamma^g{}_{be} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} \\
& + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe} - \Gamma^a{}_{eg}\Gamma^f{}_{bd}\Gamma^g{}_{fc} + \Gamma^a{}_{cg}\Gamma^f{}_{bd}\Gamma^g{}_{fe} - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ed}\Gamma^g{}_{bc} + \Gamma^a{}_{cg}\Gamma^f{}_{ed}\Gamma^g{}_{bf} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} \\
& + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be} - \Gamma^a{}_{eg}\Gamma^f{}_{cd}\Gamma^g{}_{bf} + \Gamma^a{}_{fg}\Gamma^f{}_{cd}\Gamma^g{}_{be} + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \Gamma^a{}_{df}\partial_c\Gamma^f{}_{be} + \Gamma^f{}_{be}\partial_c\Gamma^a{}_{df} - \Gamma^a{}_{ef}\partial_c\Gamma^f{}_{bd} \\
& - \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{ef} + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be} - \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^f{}_{dg}\Gamma^g{}_{be} - \Gamma^a{}_{fc}\Gamma^f{}_{eg}\Gamma^g{}_{bd} - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^a{}_{dg}\Gamma^f{}_{bc}\Gamma^g{}_{fe} \\
& + \Gamma^a{}_{eg}\Gamma^f{}_{bc}\Gamma^g{}_{fd} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be} + \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{dc}\Gamma^g{}_{be} + \Gamma^a{}_{eg}\Gamma^f{}_{dc}\Gamma^g{}_{bf} - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^a{}_{dg}\Gamma^f{}_{ec}\Gamma^g{}_{bf} \\
& + \Gamma^a{}_{fg}\Gamma^f{}_{ec}\Gamma^g{}_{bd} \hspace{9cm} \text{(ex-0301.306)}
\end{aligned}
$$

$$R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = \partial_{ec}\Gamma^a{}_{bd} - \partial_{ed}\Gamma^a{}_{bc} + \Gamma^a{}_{cf}\partial_e\Gamma^f{}_{bd} + \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{cf} - \Gamma^a{}_{df}\partial_e\Gamma^f{}_{bc} - \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{df} + \Gamma^a{}_{fe}\partial_c\Gamma^f{}_{bd} - \Gamma^a{}_{fe}\partial_d\Gamma^f{}_{bc} + \Gamma^a{}_{fe}\Gamma^f{}_{cg}\Gamma^g{}_{bd}$$

$$- \Gamma^a{}_{fe}\Gamma^f{}_{dg}\Gamma^g{}_{bc} - \Gamma^f{}_{be}\partial_c\Gamma^a{}_{fd} + \Gamma^f{}_{be}\partial_d\Gamma^a{}_{fc} - \Gamma^a{}_{cf}\Gamma^g{}_{be}\Gamma^f{}_{gd} + \Gamma^a{}_{df}\Gamma^g{}_{be}\Gamma^f{}_{gc} - \Gamma^f{}_{ce}\partial_f\Gamma^a{}_{bd} + \Gamma^f{}_{ce}\partial_d\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ce}\Gamma^g{}_{bd}$$

$$+ \Gamma^a{}_{df}\Gamma^g{}_{ce}\Gamma^f{}_{bg} - \Gamma^f{}_{de}\partial_c\Gamma^a{}_{bf} + \Gamma^f{}_{de}\partial_f\Gamma^a{}_{bc} - \Gamma^a{}_{cf}\Gamma^g{}_{de}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{de}\Gamma^g{}_{bc} + \partial_{de}\Gamma^a{}_{bc} - \partial_{dc}\Gamma^a{}_{be} + \Gamma^a{}_{ef}\partial_d\Gamma^f{}_{bc} + \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{ef}$$

$$- \Gamma^a{}_{cf}\partial_d\Gamma^f{}_{be} - \Gamma^f{}_{be}\partial_d\Gamma^a{}_{cf} + \Gamma^a{}_{fd}\partial_e\Gamma^f{}_{bc} - \Gamma^a{}_{fd}\partial_c\Gamma^f{}_{be} + \Gamma^a{}_{fd}\Gamma^f{}_{eg}\Gamma^g{}_{bc} - \Gamma^a{}_{fd}\Gamma^f{}_{cg}\Gamma^g{}_{be} - \Gamma^f{}_{bd}\partial_e\Gamma^a{}_{fc} + \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{fe}$$

$$- \Gamma^a{}_{ef}\Gamma^g{}_{bd}\Gamma^f{}_{gc} + \Gamma^a{}_{cf}\Gamma^g{}_{bd}\Gamma^f{}_{ge} - \Gamma^f{}_{ed}\partial_f\Gamma^a{}_{bc} + \Gamma^f{}_{ed}\partial_c\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{ed}\Gamma^g{}_{bc} + \Gamma^a{}_{cf}\Gamma^g{}_{ed}\Gamma^f{}_{bg} - \Gamma^f{}_{cd}\partial_e\Gamma^a{}_{bf} + \Gamma^f{}_{cd}\partial_f\Gamma^a{}_{be}$$

$$- \Gamma^a{}_{ef}\Gamma^g{}_{cd}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{cd}\Gamma^g{}_{be} + \partial_{cd}\Gamma^a{}_{be} - \partial_{ce}\Gamma^a{}_{bd} + \Gamma^a{}_{df}\partial_c\Gamma^f{}_{be} + \Gamma^f{}_{be}\partial_c\Gamma^a{}_{df} - \Gamma^a{}_{ef}\partial_c\Gamma^f{}_{bd} - \Gamma^f{}_{bd}\partial_c\Gamma^a{}_{ef} + \Gamma^a{}_{fc}\partial_d\Gamma^f{}_{be}$$

$$- \Gamma^a{}_{fc}\partial_e\Gamma^f{}_{bd} + \Gamma^a{}_{fc}\Gamma^f{}_{dg}\Gamma^g{}_{be} - \Gamma^a{}_{fc}\Gamma^f{}_{eg}\Gamma^g{}_{bd} - \Gamma^f{}_{bc}\partial_d\Gamma^a{}_{fe} + \Gamma^f{}_{bc}\partial_e\Gamma^a{}_{fd} - \Gamma^a{}_{df}\Gamma^g{}_{bc}\Gamma^f{}_{ge} + \Gamma^a{}_{ef}\Gamma^g{}_{bc}\Gamma^f{}_{gd} - \Gamma^f{}_{dc}\partial_f\Gamma^a{}_{be}$$

$$+ \Gamma^f{}_{dc}\partial_e\Gamma^a{}_{bf} - \Gamma^a{}_{fg}\Gamma^f{}_{dc}\Gamma^g{}_{be} + \Gamma^a{}_{ef}\Gamma^g{}_{dc}\Gamma^f{}_{bg} - \Gamma^f{}_{ec}\partial_d\Gamma^a{}_{bf} + \Gamma^f{}_{ec}\partial_f\Gamma^a{}_{bd} - \Gamma^a{}_{df}\Gamma^g{}_{ec}\Gamma^f{}_{bg} + \Gamma^a{}_{fg}\Gamma^f{}_{ec}\Gamma^g{}_{bd} \qquad (\text{ex-0301.307})$$

$$= 0 \qquad\qquad (\text{ex-0301.308})$$