Exercise 2.1 Using Cadabra's own product rule

```
\{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u\#\}::Indices(position=independent).
    \nabla{#}::Derivative.
    \partial{#}::PartialDerivative.
    # templates for covariant derivatives
    + \Gamma^{b}_{c a} A?^{c}.
10
    11
                              - \Gamma^{c}_{b a} A?_{c}.
12
13
    # create an object
14
15
    uv := \nabla_{a}{v_{b} u^{b}}
16
       - \partial_{a}{v_{b} u^{b}}.
                                   # cdb (ex-0201.101,uv)
17
18
    # apply the rules, then simplify
19
20
    product_rule
                 (uv)
                                     # cdb (ex-0201.102,uv)
21
                 (uv,deriv1)
                                     # cdb (ex-0201.103,uv)
    substitute
22
                 (uv,deriv2)
                                     # cdb (ex-0201.104,uv)
    substitute
                                     # cdb (ex-0201.105,uv)
    distribute
                 (uv)
                 (uv)
    sort_product
                                     # cdb (ex-0201.106,uv)
25
    rename_dummies (uv)
                                     # cdb (ex-0201.107,uv)
```

$$\nabla_{a} (v_{b}u^{b}) - \partial_{a} (v_{b}u^{b}) = \nabla_{a}v_{b}u^{b} + v_{b}\nabla_{a}u^{b} - \partial_{a}v_{b}u^{b} - v_{b}\partial_{a}u^{b}$$

$$= \nabla_{a}v_{b}u^{b} + v_{b} (\partial_{a}u^{b} + \Gamma^{b}{}_{ca}u^{c}) - \partial_{a}v_{b}u^{b} - v_{b}\partial_{a}u^{b}$$

$$= (\partial_{a}v_{b} - \Gamma^{c}{}_{ba}v_{c}) u^{b} + v_{b} (\partial_{a}u^{b} + \Gamma^{b}{}_{ca}u^{c}) - \partial_{a}v_{b}u^{b} - v_{b}\partial_{a}u^{b}$$

$$= -\Gamma^{c}{}_{ba}v_{c}u^{b} + v_{b}\Gamma^{b}{}_{ca}u^{c}$$

$$= -\Gamma^{c}{}_{ba}u^{b}v_{c} + \Gamma^{b}{}_{ca}u^{c}v_{b}$$

$$= 0$$

$$(ex-0201.102)$$

$$(ex-0201.103)$$

$$(ex-0201.105)$$

$$= -\Gamma^{c}{}_{ba}u^{b}v_{c} + \Gamma^{b}{}_{ca}u^{c}v_{b}$$

$$= 0$$

$$(ex-0201.107)$$

Exercise 2.1 Using hand crafted product rules

```
\{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u\#\}::Indices(position=independent).
    \nabla{#}::Derivative.
    \partial{#}::PartialDerivative.
    # templates for covariant derivatives
    + \Gamma^{b}_{c a} A?^{c}.
10
    11
                                - \Gamma^{c}_{b a} A?_{c}.
12
13
    # tempaltes for product rules
14
15
    deriv3 := \frac{a}{A?_{b} B?^{c}} -> B?^{c} \ln_{a}{A?_{b}}
16
                                       + A?_{b} \nabla_{a}{B?^{c}}.
17
18
    deriv4 := \frac{a}{A?_{b} B?^{c}} -> B?^{c} \operatorname{a}_{a}^{A?_{b}}
19
                                        + A?_{b} \partial_{a}{B?^{c}}.
20
21
    # create an object
22
23
    uv := \nabla_{a}{v_{b} u^{b}}
24
        - \partial_{a}{v_{b} u^{b}}.
                                     # cdb (ex-0201.201,uv)
25
26
    # apply the rules, then simplify
27
28
                  (uv,deriv3)
                                        # cdb (ex-0201.202,uv)
    substitute
29
                  (uv,deriv4)
                                        # cdb (ex-0201.203,uv)
    substitute
                  (uv,deriv1)
                                        # cdb (ex-0201.204,uv)
    substitute
31
                                        # cdb (ex-0201.205,uv)
                  (uv,deriv2)
    substitute
32
                                        # cdb (ex-0201.206,uv)
    distribute
                   (uv)
33
    sort_product
                                        # cdb (ex-0201.207,uv)
                   (uv)
34
    rename_dummies (uv)
                                        # cdb (ex-0201.208,uv)
```

$$\nabla_{a} (v_{b}u^{b}) - \partial_{a} (v_{b}u^{b}) = u^{b} \nabla_{a} v_{b} + v_{b} \nabla_{a} u^{b} - \partial_{a} (v_{b}u^{b})$$

$$= u^{b} \nabla_{a} v_{b} + v_{b} \nabla_{a} u^{b} - u^{b} \partial_{a} v_{b} - v_{b} \partial_{a} u^{b}$$

$$= u^{b} \nabla_{a} v_{b} + v_{b} (\partial_{a} u^{b} + \Gamma^{b}{}_{ca} u^{c}) - u^{b} \partial_{a} v_{b} - v_{b} \partial_{a} u^{b}$$

$$= u^{b} (\partial_{a} v_{b} - \Gamma^{c}{}_{ba} v_{c}) + v_{b} (\partial_{a} u^{b} + \Gamma^{b}{}_{ca} u^{c}) - u^{b} \partial_{a} v_{b} - v_{b} \partial_{a} u^{b}$$

$$= u^{b} (\partial_{a} v_{b} - \Gamma^{c}{}_{ba} v_{c}) + v_{b} (\partial_{a} u^{b} + \Gamma^{b}{}_{ca} u^{c}) - u^{b} \partial_{a} v_{b} - v_{b} \partial_{a} u^{b}$$

$$= -u^{b} \Gamma^{c}{}_{ba} v_{c} + v_{b} \Gamma^{b}{}_{ca} u^{c}$$

$$= -\Gamma^{c}{}_{ba} u^{b} v_{c} + \Gamma^{b}{}_{ca} u^{c} v_{b}$$

$$= 0$$

$$(ex-0201.202)$$

$$(ex-0201.205)$$

$$= -\Gamma^{c}{}_{ba} u^{b} v_{c} + \Gamma^{b}{}_{ca} u^{c} v_{b}$$

$$= 0$$

$$(ex-0201.208)$$