

## Exercise 3.2 Riemann tensor from commutation of $\nabla$

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).
2
3 \nabla{#}::Derivative.
4 \partial{#}::PartialDerivative.
5
6 \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2});
7
8 # rules for the first two covariant derivs of V^a
9
10 deriv1 := \nabla_{a}{V^{b}} -> \partial_{a}{V^{b}}
11          + \Gamma^{b}_{d a} V^{d}.          # cdb (ex-0302.101,deriv1)
12
13 deriv2 := \nabla_{a}{\nabla_{b}{V^{c}}} -> \partial_{a}{\nabla_{b}{V^{c}}}
14          + \Gamma^{c}_{d a} \nabla_{b}{V^{d}}
15          - \Gamma^{d}_{b a} \nabla_{d}{V^{c}}.
16          # cdb (ex-0302.102,deriv2)
17
18 Vabc := \nabla_{c}{\nabla_{b}{V^{a}}}
19         - \nabla_{b}{\nabla_{c}{V^{a}}}.          # cdb (ex-0302.103, Vabc)
20
21 substitute (Vabc,deriv2)          # cdb (ex-0302.104, Vabc)
22 substitute (Vabc,deriv1)          # cdb (ex-0302.105, Vabc)
23
24 distribute      (Vabc)          # cdb (ex-0302.106, Vabc)
25 product_rule    (Vabc)          # cdb (ex-0302.107, Vabc)
26
27 sort_product    (Vabc)          # cdb (ex-0302.108, Vabc)
28 rename_dummies  (Vabc)          # cdb (ex-0302.109, Vabc)
29 canonicalise    (Vabc)          # cdb (ex-0302.110, Vabc)
30 factor_out      (Vabc,$V^{a?}$) # cdb (ex-0302.111, Vabc)
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$$\nabla_c (\nabla_b V^a) - \nabla_b (\nabla_c V^a) = \partial_c (\nabla_b V^a) + \Gamma_{dc}^a \nabla_b V^d - \Gamma_{bc}^d \nabla_d V^a - \partial_b (\nabla_c V^a) - \Gamma_{db}^a \nabla_c V^d + \Gamma_{cb}^d \nabla_d V^a \quad (\text{ex-0302.104})$$

$$\begin{aligned} &= \partial_c (\partial_b V^a + \Gamma_{db}^a V^d) + \Gamma_{dc}^a (\partial_b V^d + \Gamma_{eb}^d V^e) - \Gamma_{bc}^d (\partial_d V^a + \Gamma_{ed}^a V^e) - \partial_b (\partial_c V^a + \Gamma_{dc}^a V^d) - \Gamma_{db}^a (\partial_c V^d + \Gamma_{ec}^d V^e) \\ &\quad + \Gamma_{cb}^d (\partial_d V^a + \Gamma_{ed}^a V^e) \end{aligned} \quad (\text{ex-0302.105})$$

$$\begin{aligned} &= \partial_{cb} V^a + \partial_c (\Gamma_{db}^a V^d) + \Gamma_{dc}^a \partial_b V^d + \Gamma_{dc}^a \Gamma_{eb}^d V^e - \Gamma_{bc}^d \partial_d V^a - \Gamma_{bc}^d \Gamma_{ed}^a V^e - \partial_{bc} V^a - \partial_b (\Gamma_{dc}^a V^d) - \Gamma_{db}^a \partial_c V^d \\ &\quad - \Gamma_{db}^a \Gamma_{ec}^d V^e + \Gamma_{cb}^d \partial_d V^a + \Gamma_{cb}^d \Gamma_{ed}^a V^e \end{aligned} \quad (\text{ex-0302.106})$$

$$\begin{aligned} &= \partial_{cb} V^a + \partial_c \Gamma_{db}^a V^d + \Gamma_{dc}^a \Gamma_{eb}^d V^e - \Gamma_{bc}^d \partial_d V^a - \Gamma_{bc}^d \Gamma_{ed}^a V^e - \partial_{bc} V^a - \partial_b \Gamma_{dc}^a V^d - \Gamma_{db}^a \Gamma_{ec}^d V^e + \Gamma_{cb}^d \partial_d V^a \\ &\quad + \Gamma_{cb}^d \Gamma_{ed}^a V^e \end{aligned} \quad (\text{ex-0302.107})$$

$$\begin{aligned} &= \partial_{cb} V^a + V^d \partial_c \Gamma_{db}^a + V^e \Gamma_{dc}^a \Gamma_{eb}^d - \Gamma_{bc}^d \partial_d V^a - V^e \Gamma_{ed}^a \Gamma_{bc}^d - \partial_{bc} V^a - V^d \partial_b \Gamma_{dc}^a - V^e \Gamma_{db}^a \Gamma_{ec}^d + \Gamma_{cb}^d \partial_d V^a \\ &\quad + V^e \Gamma_{ed}^a \Gamma_{cb}^d \end{aligned} \quad (\text{ex-0302.108})$$

$$\begin{aligned} &= \partial_{cb} V^a + V^d \partial_c \Gamma_{db}^a + V^d \Gamma_{ec}^a \Gamma_{db}^e - \Gamma_{bc}^d \partial_d V^a - V^d \Gamma_{de}^a \Gamma_{bc}^e - \partial_{bc} V^a - V^d \partial_b \Gamma_{dc}^a - V^d \Gamma_{eb}^a \Gamma_{dc}^e + \Gamma_{cb}^d \partial_d V^a \\ &\quad + V^d \Gamma_{de}^a \Gamma_{cb}^e \end{aligned} \quad (\text{ex-0302.109})$$

$$= V^d \partial_c \Gamma_{bd}^a + V^d \Gamma_{ce}^a \Gamma_{bd}^e - V^d \partial_b \Gamma_{cd}^a - V^d \Gamma_{be}^a \Gamma_{cd}^e \quad (\text{ex-0302.110})$$

$$= V^d (\partial_c \Gamma_{bd}^a + \Gamma_{ce}^a \Gamma_{bd}^e - \partial_b \Gamma_{cd}^a - \Gamma_{be}^a \Gamma_{cd}^e) \quad (\text{ex-0302.111})$$

$$= -R_{dbc}^a V^d$$

This result agrees with Misner, Thorne and Wheeler. pg. 266.