Exercise 6.5 Schwarzschild spacetime in isotropic coordinates

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1    {t, r, \theta, \varphi}::Coordinate.
2    {a,b,c,d,e,f,g,h#}::Indices(values={t, r, \theta, \varphi}, position=independent).
3
4    \partial{#}::PartialDerivative.
5
6    g^{a b}::InverseMetric.  # essential when using complete (gab, $g^{a b}$)
7
8    Gamma := \Gamma^{a}_{b c} -> 1/2 g^{a d} (   \partial_{b}{g_{d c}}
9                                              + \partial_{c}{g_{b d}}
10                                             - \partial_{d}{g_{b c}}).
11
12   Rabcd := R^{a}_{b c d} ->   \partial_{c}{\Gamma^{a}_{b d}}
13                            - \partial_{d}{\Gamma^{a}_{b c}}
14                            + \Gamma^{e}_{b d} \Gamma^{a}_{c e}
15                            - \Gamma^{e}_{b c} \Gamma^{a}_{d e}.
16
17   Rab := R_{a b} -> R^{c}_{a c b}.
18
19   gab := { g_{t t}             = -((2*r-m)/(2*r+m))**2,
20            g_{r r}             = (1+m/(2*r))**4,
21            g_{\theta\theta}    = r**2 (1+m/(2*r))**4,
22            g_{\varphi\varphi} = r**2 \sin(\theta)**2 (1+m/(2*r))**4}.  # cdb(ex-0605.101,gab)
23
24   complete   (gab, $g^{a b}$)                                         # cdb(ex-0605.102,gab)
25
26   substitute (Rabcd, Gamma)
27   substitute (Rab, Rabcd)
28
29   evaluate   (Gamma, gab, rhsonly=True)                               # cdb(ex-0605.103,Gamma)
30   evaluate   (Rabcd, gab, rhsonly=True)                               # cdb(ex-0605.104,Rabcd)
31   evaluate   (Rab,   gab, rhsonly=True)                               # cdb(ex-0605.105,Rab)
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$$\left[g_{tt} = -\left((2r - m)\,(2r + m)^{-1}\right)^2, \; g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, \; g_{\theta\theta} = r^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \; g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1 + \frac{1}{2}mr^{-1}\right)^4 \right]$$

(ex-0605.101)

$$\left[g_{tt} = -\left((2r - m)\,(2r + m)^{-1}\right)^2, \; g_{rr} = \left(1 + \frac{1}{2}mr^{-1}\right)^4, \; g_{\theta\theta} = r^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \right.$$

$$g_{\varphi\varphi} = r^2(\sin\theta)^2\left(1 + \frac{1}{2}mr^{-1}\right)^4, \; g^{tt} = \left(-m^2 - 4mr - 4r^2\right)\left(m^2 - 4mr + 4r^2\right)^{-1},$$

$$g^{rr} = 16r^4\left(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4\right)^{-1}, \; g^{\theta\theta} = 16r^2\left(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4\right)^{-1},$$

$$\left. g^{\varphi\varphi} = 16r^2\left(m^4(\sin\theta)^2 + 8m^3r(\sin\theta)^2 + 24m^2r^2(\sin\theta)^2 + 32mr^3(\sin\theta)^2 + 16r^4(\sin\theta)^2\right)^{-1} \right]$$

(ex-0605.102)

$$\Gamma^a{}_{bc} \to \Box_{cb}{}^a \begin{cases} \Box_{\varphi r}{}^\varphi = (-m + 2r)\,(r\,(m + 2r))^{-1} \\[4pt] \Box_{\varphi\theta}{}^\varphi = (\tan\theta)^{-1} \\[4pt] \Box_{\theta r}{}^\theta = (-m + 2r)\,(r\,(m + 2r))^{-1} \\[4pt] \Box_{rr}{}^r = -2m(r\,(m + 2r))^{-1} \\[4pt] \Box_{tr}{}^t = 4m\left(-m^2 + 4r^2\right)^{-1} \\[4pt] \Box_{r\varphi}{}^\varphi = (-m + 2r)\,(r\,(m + 2r))^{-1} \\[4pt] \Box_{\theta\varphi}{}^\varphi = (\tan\theta)^{-1} \\[4pt] \Box_{r\theta}{}^\theta = (-m + 2r)\,(r\,(m + 2r))^{-1} \\[4pt] \Box_{rt}{}^t = 4m\left(-m^2 + 4r^2\right)^{-1} \\[4pt] \Box_{\varphi\varphi}{}^r = r\,(m - 2r)\,(\sin\theta)^2(m + 2r)^{-1} \\[4pt] \Box_{\varphi\varphi}{}^\theta = -\frac{1}{2}\sin(2\theta) \\[4pt] \Box_{\theta\theta}{}^r = r\,(m - 2r)\,(m + 2r)^{-1} \\[4pt] \Box_{tt}{}^r = -64mr^4\,(m - 2r)\left((m + 2r)^3\left(m^4 + 8m^3r + 24m^2r^2 + 32mr^3 + 16r^4\right)\right)^{-1} \end{cases}$$

(ex-0605.103)

$R^a{}_{bcd}$

$\rightarrow \Box_{db}{}^a{}_c$
$\begin{cases}
\Box_{tt}{}^r{}_r = 128mr^3\left(-m^2+4mr-4r^2\right)\left(m^8+16m^7r+112m^6r^2+448m^5r^3+1120m^4r^4+1792m^3r^5+1792m^2r^6+1024mr^7+256r^8\right)^{-1} \\[4pt]
\Box_{\theta\theta}{}^r{}_r = -4mr\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{\varphi\varphi}{}^\theta{}_\theta = 8mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{\varphi\varphi}{}^r{}_r = -4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{tr}{}^t{}_r = -8m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\[4pt]
\Box_{\theta r}{}^\theta{}_r = 4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\[4pt]
\Box_{\varphi\theta}{}^\varphi{}_\theta = -8mr\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{\varphi r}{}^\varphi{}_r = 4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\[4pt]
\Box_{rt}{}^r{}_t = 128mr^3\left(m^2-4mr+4r^2\right)\left(m^8+16m^7r+112m^6r^2+448m^5r^3+1120m^4r^4+1792m^3r^5+1792m^2r^6+1024mr^7+256r^8\right)^{-1} \\[4pt]
\Box_{r\theta}{}^r{}_\theta = 4mr\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{\theta\varphi}{}^\theta{}_\varphi = -8mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{r\varphi}{}^r{}_\varphi = 4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{rr}{}^t{}_t = 8m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\[4pt]
\Box_{rr}{}^\theta{}_\theta = -4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\[4pt]
\Box_{\theta\theta}{}^\varphi{}_\varphi = 8mr\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{rr}{}^\varphi{}_\varphi = -4m\left(r\left(m^2+4mr+4r^2\right)\right)^{-1} \\[4pt]
\Box_{\varphi\varphi}{}^t{}_t = -4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{\theta\theta}{}^t{}_t = -4mr\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{tt}{}^\varphi{}_\varphi = 64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2} \\[4pt]
\Box_{tt}{}^\theta{}_\theta = 64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2} \\[4pt]
\Box_{t\varphi}{}^t{}_\varphi = 4mr(\sin\theta)^2\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{t\theta}{}^t{}_\theta = 4mr\left(m^2+4mr+4r^2\right)^{-1} \\[4pt]
\Box_{\varphi t}{}^\varphi{}_t = -64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2} \\[4pt]
\Box_{\theta t}{}^\theta{}_t = -64mr^3(m-2r)^2\left(m^4+8m^3r+24m^2r^2+32mr^3+16r^4\right)^{-2}
\end{cases}$

(ex-0605.104)

$R_{ab} \rightarrow 0$

(ex-0605.105)