

Symmetry of the Ricci tensor

This simple example shows that, for the metric connection, the Ricci tensor is symmetric, that is $R_{ab} = R_{ba}$.

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{a,b,c,d,e,f,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

g_{a b}::Symmetric;
g^{a b}::Symmetric;

g_{a b}::Depends(\partial{#});
g^{a b}::Depends(\partial{#});

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}. # cdb (dgab,dgab)

Gamma := \Gamma^{a}_{b c} ->
(1/2) g^{a e} ( \partial_{b}{g_{e c}}
+ \partial_{c}{g_{b e}}
- \partial_{e}{g_{b c}}). # cdb (Chr,Gamma)

Rabcd := R^{a}_{b c d} ->
\partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
- \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
# cdb (Rabcd,Rabcd)

Rab := R_{a b} -> R^{c}_{c a b}. # cdb (Rab,Rab)

eqn := 2 (R_{a b} - R_{b a}).

substitute (eqn, Rab)
substitute (eqn, Rabcd)
substitute (eqn, Gamma)

distribute (eqn)
product_rule (eqn)
canonicalise (eqn) # cdb (final1,eqn)
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substitute (eqn,dgab)
canonicalise (eqn)                                     # cdb (final2,eqn)

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$$\begin{aligned}
 g^{ab}{}_{,c} &:= \partial_c g^{ab} \rightarrow -g^{ae} g^{bf} \partial_c g_{ef} \\
 \Gamma^a_{bc} &:= \Gamma^a{}_{bc} \rightarrow \frac{1}{2} g^{ae} (\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) \\
 R^a{}_{bcd} &:= R^a{}_{bcd} \rightarrow \partial_c \Gamma^a{}_{bd} + \Gamma^a{}_{ec} \Gamma^e{}_{bd} - \partial_d \Gamma^a{}_{bc} - \Gamma^a{}_{ed} \Gamma^e{}_{bc} \\
 R_{ab} &:= R_{ab} \rightarrow R^c{}_{acb} \\
 2(R_{ab} - R_{ba}) &= -\partial_b g^{ce} \partial_a g_{ce} + \partial_a g^{ce} \partial_b g_{ce} \\
 &= 0
 \end{aligned}$$

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\begin{align*}
g^{\{a\}{}_b\}{}_{,c} & \&:= \backslash cdb{dgab} \\
\Gamma^{\{a\}{}_b\}{}_c & \&:= \backslash cdb{Chr} \\
R^{\{a\}{}_b\}{}_c{}_d & \&:= \backslash cdb{Rabcd} \\
R_{\{ab\}} & \&:= \backslash cdb{Rab} \\
2(R_{\{ab\}} - R_{\{ba\}}) & \&= \backslash cdb{final1} \\
& \&= \backslash cdb{final2}
\end{align*}

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