Symmetry of the Ricci tensor

This simple example shows that, for the metric connection, the Ricci tensor is symmetric, that is $R_{ab} = R_{ba}$.

```
{a,b,c,d,e,f,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial{#}::PartialDerivative;

g_{a b}::Symmetric;
g^{a b}::Symmetric;

g_{a b}::Depends(\partial{#});
g^{a b}::Depends(\partial{#});

dgab := \partial_{c}{g^{a b}} -> - g^{a e} g^{b f} \partial_{c}{g_{e f}}. # cdb (dgab,dgab)

Gamma := \Gamma^{a}_{b c} ->
         (1/2) g^{a e} (   \partial_{b}{g_{e c}}
                         + \partial_{c}{g_{b e}}
                         - \partial_{e}{g_{b c}}).     # cdb (Chr,Gamma)

Rabcd := R^{a}_{b c d} ->
        \partial_{c}{\Gamma^{a}_{b d}} + \Gamma^{a}_{e c} \Gamma^{e}_{b d}
      - \partial_{d}{\Gamma^{a}_{b c}} - \Gamma^{a}_{e d} \Gamma^{e}_{b c}.
                                                       # cdb (Rabcd,Rabcd)

Rab := R_{a b} -> R^{c}_{a c b}.                       # cdb (Rab,Rab)

eqn := 2 (R_{a b} - R_{b a}).

substitute (eqn, Rab)
substitute (eqn, Rabcd)
substitute (eqn, Gamma)

distribute   (eqn)
product_rule (eqn)
canonicalise (eqn)                                     # cdb (final1,eqn)
```

```
substitute   (eqn,dgab)
canonicalise (eqn)                                          # cdb (final2,eqn)
```

Symmetry of the Ricci tensor

$$
\begin{aligned}
g^{ab}{}_{,c} &:= \partial_{c}g^{ab} \rightarrow -g^{ae} g^{bf} \partial_{c}g_{ef}\\
\Gamma^{a}_{bc} &:= \Gamma^{a}{}_{bc} \rightarrow \frac{1}{2}g^{ae} \left(\partial_{b}g_{ec} + \partial_{c}g_{be} - \partial_{e}g_{bc}\right)\\
R^{a}{}_{bcd} &:= R^{a}{}_{bcd} \rightarrow \partial_{c}\Gamma^{a}{}_{bd} + \Gamma^{a}{}_{ec}\Gamma^{e}{}_{bd} - \partial_{d}\Gamma^{a}{}_{bc} - \Gamma^{a}{}_{ed}\Gamma^{e}{}_{bc}\\
R_{ab} &:= R_{ab} \rightarrow R^{c}{}_{acb}\\[10pt]
2(R_{ab} - R_{ba}) &= -\partial_{b}g^{ce}\partial_{a}g_{ce} + \partial_{a}g^{ce}\partial_{b}g_{ce}\\
&= 0
\end{aligned}
$$

```
\begin{align*}
   g^{a b}{}_{,c}    &:=\cdb{dgab}\\
   \Gamma^{a}_{bc}  &:=\cdb{Chr}\\
   R^a{}_{bcd}      &:=\cdb{Rabcd}\\
   R_{ab}           &:=\cdb{Rab}\\[10pt]
   2(R_{ab}-R_{ba}) &= \cdb{final1}\\
                    &= \cdb{final2}
\end{align*}
```