

The metric connection in Riemann normal coordinates

In local Riemann normal coordinates, the metric components can always be expanded as a power series in the Riemann curvatures and its derivatives (provided the curvatures are finite at the expansion point). In particular

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \dots$$
$$g^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \dots$$

where g_{ab} and g^{ab} are independent of the coordinates x^a and where ∇ is the metric compatible derivative operator (i.e., $\nabla(g) = 0$). In applications in General Relativity the g_{ab} are often chosen to be $g_{ab} = \text{diag}(-1, 1, 1, 1)$.

Here we will use the standard metric compatible connection

$$\Gamma_{ab}^d(x) = \frac{1}{2}g^{dc}(g_{cb,a} + g_{ac,b} - g_{ab,c})$$

to compute $\Gamma_{ab}^d(x)$ to terms linear in R_{abcd} and $\nabla_e R_{abcd}$.

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1 {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).
2
3 D{#}::PartialDerivative.
4 \nabla{#}::Derivative.
5
6 g_{a b}::Metric.
7 g^{a b}::InverseMetric.
8
9 \delta{#}::KroneckerDelta.
10
11 R_{a b c d}::RiemannTensor.
12
13 x^{a}::Depends(D{#}).
14 x^{a}::Weight(label=num,value=1).
15
16 R_{a b c d}::Depends(\nabla{#}).
17
18 DxaDxb := D_{a}{x^{b}}->\delta^{b}_{a}.
19
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20 # can chose lower order approximations by truncating the following pair
21
22 gab := g_{a b} - 1/3 x^{c} x^{d} R_{a c b d}
23         - 1/6 x^{c} x^{d} x^{e} \nabla_{c}\{R_{a d b e}\}. # cdb(gab.000,gab)
24
25 iab := g^{a b} + 1/3 x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
26         + 1/6 x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}\{R_{d f e g}\}. # cdb(iab.000,iab)
27
28 gab := g_{a b} -> @(gab).
29 iab := g^{a b} -> @(iab).
30
31 gam := 1/2 g^{d c} (D_{a}\{g_{c b}\} + D_{b}\{g_{a c}\} - D_{c}\{g_{a b}\}). # cdb(gam.001,gam)
32
33 substitute (gam,gab)
34 substitute (gam,iab)
35 distribute (gam) # cdb(gam.002,gam)
36 unwrap (gam) # cdb(gam.003,gam)
37 product_rule (gam) # cdb(gam.004,gam)
38 distribute (gam) # cdb(gam.005,gam)
39 substitute (gam,DxaDxb) # cdb(gam.006,gam)
40 eliminate_kronecker (gam) # cdb(gam.007,gam)
41 sort_product (gam) # cdb(gam.008,gam)
42 rename_dummies (gam) # cdb(gam.009,gam)
43 canonicalise (gam) # cdb(gam.010,gam)
44
45 def truncate (obj,n):
46
47     ans = Ex(0) # create a Cadabra object with value zero
48
49     for i in range (0,n+1):
50         foo := @(obj).
51         bah = Ex("num = " + str(i))
52         distribute (foo)
53         keep_weight (foo, bah)
54         ans = ans + foo
55
56     return ans
57

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58 gam = truncate (gam,2)      # cdb (gam.101,gam) # allow up to 2nd order in x^a
59
60 # =====
61 # the remaining code is just for pretty printing
62
63 {x^{a},g^{a b},R_{a b c d},\nabla_{e}\{R_{a b c d}\}}::SortOrder.
64
65 def get_term (obj,n):
66
67     foo := @(obj).
68     bah  = Ex("num = " + str(n))
69     distribute    (foo)
70     keep_weight   (foo, bah)
71
72     return foo
73
74 def reformat (obj,scale):
75
76     foo  = Ex(str(scale))
77     bah := @(foo) @(obj).
78
79     distribute    (bah)
80     sort_product  (bah)
81     rename_dummies (bah)
82     canonicalise   (bah)
83     factor_out     (bah,$x^{a?},g^{b? c?}$)
84     ans := @(bah) / @(foo).
85
86     return ans
87
88 gam1 = get_term (gam,1)      # cdb (gam1.301,gam1)
89 gam2 = get_term (gam,2)      # cdb (gam2.301,gam2)
90
91 gam1 = reformat (gam1, 3)    # cdb (gam1.301,gam1)
92 gam2 = reformat (gam2, 12)   # cdb (gam2.301,gam2)
93
94 Gamma := @(gam1) + @(gam2).  # cdb (Gamma.301,Gamma)
95 Scaled := 12 @(Gamma).       # cdb (Scaled.301,Scaled)

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$$\Gamma_{ab}^d = \frac{1}{3} x^c g^{de} (R_{aebc} + R_{acbe}) + \frac{1}{12} x^c x^e g^{df} (\nabla_a R_{bcef} + 2 \nabla_c R_{afbe} + \nabla_b R_{acef} + 2 \nabla_e R_{aebf} + \nabla_f R_{acbe})$$

$$12\Gamma_{ab}^d = 4 x^c g^{de} (R_{aebc} + R_{acbe}) + x^c x^e g^{df} (\nabla_a R_{bcef} + 2 \nabla_c R_{afbe} + \nabla_b R_{acef} + 2 \nabla_e R_{aebf} + \nabla_f R_{acbe})$$

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\begin{dgroup*}
  \begin{dmath*} \Gamma^{\mathrm{d}}_{\mathrm{a} \mathrm{b}} = \mathrm{cdb}\{\mathrm{Gamma}.301\} \end{dmath*}
\end{dgroup*}
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\begin{dgroup*}
  \begin{dmath*} 12 \Gamma^{\mathrm{d}}_{\mathrm{a} \mathrm{b}} = \mathrm{cdb}\{\mathrm{Scaled}.301\} \end{dmath*}
\end{dgroup*}
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