The metric connection in Riemann normal coordinates

In local Riemann normal coordinates, the metric components can always be expanded as a power series in the Riemann curvatures and its derivatives (provided the curvatives are finite at the expansion point). In particular

$$
\begin{aligned}
g_{ab}(x) &= g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \cdots \\
g^{ab}(x) &= g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \cdots
\end{aligned}
$$

where g_{ab} and g^{ab} are independent of the coordinates x^a and where ∇ is the metric compatable derivative operator (i.e., $\nabla(g) = 0$). In applications in General Relativity the g_{ab} are often chosen to be $g_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

Here we will use the standard metric compatible connection

$$
\Gamma^d_{ab}(x) = \frac{1}{2} g^{dc} \left(g_{cb,a} + g_{ac,b} - g_{ab,c} \right)
$$

to compute $\Gamma^d_{ab}(x)$ to terms linear in R_{abcd} and $\nabla_e R_{abcd}$.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::PartialDerivative.
\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

\delta{#}::KroneckerDelta.

R_{a b c d}::RiemannTensor.

x^{a}::Depends(D{#}).
x^{a}::Weight(label=num,value=1).

R_{a b c d}::Depends(\nabla{#}).

DxaDxb := D_{a}{x^{b}}->\delta^{b}_{a}.

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```
# can chose lower order approximations by truncating the following pair

gab := g_{a b} - 1/3 x^{c} x^{d} R_{a c b d}
               - 1/6 x^{c} x^{d} x^{e} \nabla_{c}{R_{a d b e}}.                    # cdb(gab.000,gab)

iab := g^{a b} + 1/3 x^{c} x^{d} g^{a e} g^{b f} R_{c e d f}
               + 1/6 x^{c} x^{d} x^{e} g^{a f} g^{b g} \nabla_{c}{R_{d f e g}}.    # cdb(iab.000,iab)

gab := g_{a b} -> @(gab).
iab := g^{a b} -> @(iab).

gam := 1/2 g^{d c} (D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}).             # cdb(gam.001,gam)

substitute   (gam,gab)
substitute   (gam,iab)
distribute   (gam)              # cdb(gam.002,gam)
unwrap       (gam)              # cdb(gam.003,gam)
product_rule (gam)              # cdb(gam.004,gam)
distribute   (gam)              # cdb(gam.005,gam)
substitute   (gam,DxaDxb)       # cdb(gam.006,gam)
eliminate_kronecker (gam)       # cdb(gam.007,gam)
sort_product   (gam)            # cdb(gam.008,gam)
rename_dummies (gam)            # cdb(gam.009,gam)
canonicalise   (gam)            # cdb(gam.010,gam)

def truncate (obj,n):

    ans = Ex(0)  # create a Cadabra object with value zero

    for i in range (0,n+1):
       foo := @(obj).
       bah  = Ex("num = " + str(i))
       distribute  (foo)
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

```

```
gam = truncate (gam,2)   # cdb (gam.101,gam)  # allow up to 2nd order in x^a

# ==========================================================================
# the remaining code is just for pretty printing

{x^{a},g^{a b},R_{a b c d},\nabla_{e}{R_{a b c d}}}::SortOrder.

def get_term (obj,n):

    foo := @(obj).
    bah  = Ex("num = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

def reformat (obj,scale):

   foo  = Ex(str(scale))
   bah := @(foo) @(obj).

   distribute     (bah)
   sort_product   (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$x^{a?},g^{b? c?}$)
   ans := @(bah) / @(foo).

   return ans

gam1 = get_term (gam,1)       # cdb (gam1.301,gam1)
gam2 = get_term (gam,2)       # cdb (gam2.301,gam2)

gam1 = reformat (gam1,  3)    # cdb (gam1.301,gam1)
gam2 = reformat (gam2, 12)    # cdb (gam2.301,gam2)

Gamma  := @(gam1) + @(gam2).  # cdb (Gamma.301,Gamma)
Scaled := 12 @(Gamma).        # cdb (Scaled.301,Scaled)
```

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$$\Gamma^{d}_{ab} = \frac{1}{3}x^{c}g^{de}\left(R_{aebc} + R_{acbe}\right) + \frac{1}{12}x^{c}x^{e}g^{df}\left(\nabla_{a}R_{bcef} + 2\nabla_{c}R_{afbe} + \nabla_{b}R_{acef} + 2\nabla_{c}R_{aebf} + \nabla_{f}R_{acbe}\right)$$

$$12\Gamma^{d}_{ab} = 4x^{c}g^{de}\left(R_{aebc} + R_{acbe}\right) + x^{c}x^{e}g^{df}\left(\nabla_{a}R_{bcef} + 2\nabla_{c}R_{afbe} + \nabla_{b}R_{acef} + 2\nabla_{c}R_{aebf} + \nabla_{f}R_{acbe}\right)$$

```
\begin{dgroup*}
   \begin{dmath*} \Gamma^{d}_{a b} = \cdb{Gamma.301} \end{dmath*}
\end{dgroup*}

\begin{dgroup*}
   \begin{dmath*} 12 \Gamma^{d}_{a b} = \cdb{Scaled.301} \end{dmath*}
\end{dgroup*}
```