Converting from generic to rnc coordinates

The following is based on the approach used in the `geodesic-bvp.tex` code. The main difference here is that this time we will *not* be assuming that the coordinates are in Riemann normal form. This will be apparent in the expression for the generalised connections – they will be expressed in terms of the partial derivatives of the connection rather the covariant derivatives of the Riemann tensor. There will also be a change in the way the Taylor series are developed. In this case the expansion parameter ϵ will be associated with the connection and its derivatives rather than the Riemann tensor. We will use

$$\Gamma^a{}_{bc} = \mathcal{O}\left(\epsilon\right)\ , \qquad \Gamma^a{}_{bc,d} = \mathcal{O}\left(\epsilon^2\right)\ , \qquad \Gamma^a{}_{bc,de} = \mathcal{O}\left(\epsilon^3\right)\ , \qquad \text{etc.}$$

The generalised connections are defined recursively by

$$\Gamma^a{}_{b\underline{c}d} = \Gamma^a{}_{(b\underline{c},d)} - (n+1)\Gamma^a{}_{p(\underline{c}}\Gamma^p{}_{bd)} \tag{1}$$

where $\underline{c}$ contains $n > 0$ indices. It is easy to see from this equation that the generalised connections will behave much the same as the connection, that is

$$\Gamma^a{}_{bc} = \mathcal{O}\left(\epsilon\right)\ , \qquad \Gamma^a{}_{bcd} = \mathcal{O}\left(\epsilon^2\right)\ , \qquad \Gamma^a{}_{bcde} = \mathcal{O}\left(\epsilon^3\right)\ , \qquad \text{etc.}$$

This allows us to represent each generalised connection by a single expression (typically `GamNN`).

The situation is slighly different in `geodesic-bvp.tex`. In that code the connection and the generalised connection are expanded as a series in the Riemann tensor and its derivatives. Thus each connection is written in the form

$$\bar{\Gamma}^a{}_{\underline{c}_n} = \overset{0}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} + \overset{1}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} + \overset{2}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} + \cdots + \overset{m}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} \tag{2}$$

where $\underline{c}_n$ denotes a set of indices such as $c_1 c_2 c_3 \ldots c_n$. The terms of the RHS are each of a different weight in ϵ.

Stage 1: The generalised connections

The generalised connections $\Gamma^a{}_{\underline{c}_n}$ could be computed directly by successive application of equation (1). But a more effiecent method exists and its basis lies in the original definition of the generalised connections. Recall that the generalised connecstions arose when buidling a formal power series solution of the geodesic equation

$$0 = \frac{d^2 x^a}{ds^s} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} \tag{3}$$

The key idea was that the coeffcients c_n in the formal power series

$$x^a = c^a_0 + s c^a_1 + s^2 c^a_2 + \cdots \tag{4}$$

could be computed using

$$c^a_n = \frac{1}{n!} \left. \frac{d^n x^a}{ds^n} \right|_{s=0} \tag{5}$$

with the second, third and higher derivatives of x^a found by successive differentiation of the geodesic equation. The generalised connections were introduced as part of this algorithm, leading to

$$c^a_n = - \left. \Gamma^a{}_{\underline{c}_n} A^{\cdot \underline{c}_n} \right|_{s=0} \qquad n = 2, 3, 4 \cdots \tag{6}$$

and

$$\Gamma^a{}_{\underline{c}_{n+1}} A^{\cdot \underline{c}_{n+1}} = \frac{d}{ds} \left(\Gamma^a{}_{\underline{c}_n} A^{\cdot \underline{c}_n} \right) \tag{7}$$

with $d/ds = A^a \partial_a$, $A^a = dx^a/ds$ and $dA^a/ds = -\Gamma^a{}_{bc} A^b A^c$.

The upshot is that computing the $\Gamma^a{}_{\underline{c}_n} A^{\cdot \underline{c}_n}$ requires little more than successive rounds of differentiation (and a few substitutions for the derivaties of A^a).

Note that the coefficients c_0 and c_1 must be determined from the initial conditions. Suppose that $x^a = x_i$ at $s = 0$ then $c_0 = x^a_i$ while $c_1 = A^a$.

The Riemann normal coordinates of the point j (where $s = 1$) are introduced by setting

$$y^a = A^a \tag{8}$$

This leads to

$$x^a_j = x^a_i + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a{}_{\underline{b}_k} y^{\cdot \underline{b}_k} \tag{9}$$

Note that given two points i and j, the y^a would be found as a root of this non-linear equation for y^a.

Stage 2: The fixed point scheme for y^a

This second stage is almost exactly the same as the corresponding stage in `geodesic-bvp`. The difference here is that the generalised connections involve partial derivatives of the connection. In contsrat, the `geodesic-bvp` code is specfic to RNC and thus uses the generalised connections based on covariant derivatives of the Riemann tensor.

We begin this second stage by rewriting the equation (9) in the suggestive form

$$y^a = x^a_j - x^a_i + \sum_{k=2}^{\infty} \frac{1}{k!}\,\Gamma^a{}_{\underline{b}_k} y^{\underline{b}_k}$$

and then use this as the basis of a fixed point iteration scheme.

Start with the first approximation $y^a_1 = x^a_j - x^a_i = \Delta x^a$, then compute the successive approximations

$$
\begin{aligned}
y^a_1 &= \Delta x^a \\
y^a_2 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_1 y^c_1 \\
y^a_3 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_2 y^c_2 + \frac{1}{3!}\Gamma^a{}_{bcd}\, y^b_1 y^c_1 y^d_1 \\
y^a_4 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_3 y^c_3 + \frac{1}{3!}\Gamma^a{}_{bcd}\, y^b_2 y^c_2 y^d_2 + \frac{1}{4!}\Gamma^a{}_{bcde}\, y^b_1 y^c_1 y^d_1 y^e_1 \\
y^a_5 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_4 y^c_4 + \frac{1}{3!}\Gamma^a{}_{bcd}\, y^b_3 y^c_3 y^d_3 + \frac{1}{4!}\Gamma^a{}_{bcde}\, y^b_2 y^c_2 y^d_2 y^e_2 + \frac{1}{5!}\Gamma^a{}_{bcdef}\, y^b_1 y^c_1 y^d_1 y^e_1 y^f_1
\end{aligned}
$$

and so on. Not that the $\Gamma^a{}_{bc}$, $\Gamma^a{}_{bcd}$, $\Gamma^a{}_{bcde}$ etc. will all depend on the original coordinates x^a at the initial point (i.e., $P = x^a_i$).

Stage3: Introduce the generalised connections from Stage 1

This is the final stage – it introduces the generalised connecstion after the completion of the fixed point scheme.

The result will be an equation for the y^a in terms of the original coordinates x^a and the connections (and its derivatives) at a chosen point $s = 0$ (aka i).

The y^a define an RNC frame in the neighbourhood of the chosen point i.

Stage 1: The generalised connections

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

A^{a}::Depends(\partial{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).

\Gamma^{a}_{b c}::Depends(\partial{#}).

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
\Gamma^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
\Gamma^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
\Gamma^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

\Gamma^{p}_{a b}::Weight(label=numG,value=1).
\Gamma^{p}_{a b c}::Weight(label=numG,value=2).
\Gamma^{p}_{a b c d}::Weight(label=numG,value=3).
```

```
\Gamma^{p}_{a b c d e}::Weight(label=numG,value=4).
\Gamma^{p}_{a b c d e f}::Weight(label=numG,value=5).

def product_sort (obj):
    substitute (obj,$ A^{a}                    -> A001^{a}                $)
    substitute (obj,$ x^{a}                    -> A002^{a}                $)
    substitute (obj,$ g^{a b}                  -> A003^{a b}              $)
    substitute (obj,$ \Gamma^{p}_{a b}         -> A004^{p}_{a b}          $)
    substitute (obj,$ \Gamma^{p}_{a b c}       -> A005^{p}_{a b c}        $)
    substitute (obj,$ \Gamma^{p}_{a b c d}     -> A006^{p}_{a b c d}      $)
    substitute (obj,$ \Gamma^{p}_{a b c d e}   -> A007^{p}_{a b c d e}    $)
    substitute (obj,$ \Gamma^{p}_{a b c d e f} -> A008^{p}_{a b c d e f}  $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                 -> A^{a}                   $)
    substitute (obj,$ A002^{a}                 -> x^{a}                   $)
    substitute (obj,$ A003^{a b}               -> g^{a b}                 $)
    substitute (obj,$ A004^{p}_{a b}           -> \Gamma^{p}_{a b}        $)
    substitute (obj,$ A005^{p}_{a b c}         -> \Gamma^{p}_{a b c}      $)
    substitute (obj,$ A006^{p}_{a b c d}       -> \Gamma^{p}_{a b c d}    $)
    substitute (obj,$ A007^{p}_{a b c d e}     -> \Gamma^{p}_{a b c d e}  $)
    substitute (obj,$ A008^{p}_{a b c d e f}   -> \Gamma^{p}_{a b c d e f} $)

    return obj

def truncateGam (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah  = Ex("numG = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

beg_stage_1 = time.time()
```

```
# note that we use A^{a} in place of dx^a/ds

Gamma := \Gamma^{d}_{a b} A^{a} A^{b}.

# the geodesic equation

dAds  := A^{c} \partial_{c}{A^{d}} -> - @(Gamma).

# eq0, eq1, eq2 ... are the the successive derivates of Gamma
# thus they are the generalised gamma's dotted into (multiple copies of) A^{a} = dx^a/ds

# =============================================================
eq0 := @(Gamma).                       # cdb (eq0.000,eq0)

# =============================================================
eq1 := A^{c} \partial_{c}{@(eq0)}.     # cdb (eq1.000,eq1)

distribute      (eq1)                  # cdb (eq1.001,eq1)
unwrap          (eq1)                  # cdb (eq1.002,eq1)
product_rule    (eq1)                  # cdb (eq1.003,eq1)
distribute      (eq1)                  # cdb (eq1.004,eq1)
substitute      (eq1,dAds)             # cdb (eq1.005,eq1)
distribute      (eq1)                  # cdb (eq1.006,eq1)
eq1 = truncateGam (eq1,5)              # cdb (eq1.007,eq1)
sort_product    (eq1)                  # cdb (eq1.008,eq1)
rename_dummies  (eq1)                  # cdb (eq1.009,eq1)
canonicalise    (eq1)                  # cdb (eq1.010,eq1)

# =============================================================
eq2 := A^{c} \partial_{c}{@(eq1)}.     # cdb (eq2.000,eq2)

distribute      (eq2)                  # cdb (eq2.001,eq2)
unwrap          (eq2)                  # cdb (eq2.002,eq2)
product_rule    (eq2)                  # cdb (eq2.003,eq2)
distribute      (eq2)                  # cdb (eq2.004,eq2)
substitute      (eq2,dAds)             # cdb (eq2.005,eq2)
distribute      (eq2)                  # cdb (eq2.006,eq2)
```

```
eq2 = truncateGam (eq2,5)              # cdb (eq2.007,eq2)
sort_product     (eq2)                 # cdb (eq2.008,eq2)
rename_dummies   (eq2)                 # cdb (eq2.009,eq2)
canonicalise     (eq2)                 # cdb (eq2.010,eq2)

# ==============================================================
eq3 := A^{c} \partial_{c}{@(eq2)}.     # cdb (eq3.000,eq3)

distribute       (eq3)                 # cdb (eq3.001,eq3)
unwrap           (eq3)                 # cdb (eq3.002,eq3)
product_rule     (eq3)                 # cdb (eq3.003,eq3)
distribute       (eq3)                 # cdb (eq3.004,eq3)
substitute       (eq3,dAds)            # cdb (eq3.005,eq3)
distribute       (eq3)                 # cdb (eq3.006,eq3)
eq3 = truncateGam (eq3,5)              # cdb (eq3.007,eq3)
sort_product     (eq3)                 # cdb (eq3.008,eq3)
rename_dummies   (eq3)                 # cdb (eq3.009,eq3)
canonicalise     (eq3)                 # cdb (eq3.010,eq3)

# ==============================================================
eq4 := A^{c} \partial_{c}{@(eq3)}.     # cdb (eq4.000,eq4)

distribute       (eq4)                 # cdb (eq4.001,eq4)
unwrap           (eq4)                 # cdb (eq4.002,eq4)
product_rule     (eq4)                 # cdb (eq4.003,eq4)
distribute       (eq4)                 # cdb (eq4.004,eq4)
substitute       (eq4,dAds)            # cdb (eq4.005,eq4)
distribute       (eq4)                 # cdb (eq4.006,eq4)
eq4 = truncateGam (eq4,5)              # cdb (eq4.007,eq4)
sort_product     (eq4)                 # cdb (eq4.008,eq4)
rename_dummies   (eq4)                 # cdb (eq4.009,eq4)
canonicalise     (eq4)                 # cdb (eq4.010,eq4)

end_stage_1 = time.time()
```

$$\texttt{eq0.000} := \Gamma^{d}{}_{ab} A^{a} A^{b}$$

$\texttt{eq1.000} := A^c\partial_c\left(\Gamma^d{}_{ab}A^aA^b\right)$

$\texttt{eq1.001} := A^c\partial_c\left(\Gamma^d{}_{ab}A^aA^b\right)$

$\texttt{eq1.002} := A^c\partial_c\left(\Gamma^d{}_{ab}A^aA^b\right)$

$\texttt{eq1.003} := A^c\left(\partial_c\Gamma^d{}_{ab}A^aA^b + \Gamma^d{}_{ab}\partial_cA^aA^b + \Gamma^d{}_{ab}A^a\partial_cA^b\right)$

$\texttt{eq1.004} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b + A^c\Gamma^d{}_{ab}\partial_cA^aA^b + A^c\Gamma^d{}_{ab}A^a\partial_cA^b$

$\texttt{eq1.005} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b - \Gamma^a{}_{ce}A^cA^e\Gamma^d{}_{ab}A^b - \Gamma^b{}_{ec}A^eA^c\Gamma^d{}_{ab}A^a$

$\texttt{eq1.006} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b - \Gamma^a{}_{ce}A^cA^e\Gamma^d{}_{ab}A^b - \Gamma^b{}_{ec}A^eA^c\Gamma^d{}_{ab}A^a$

$\texttt{eq1.007} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b - \Gamma^a{}_{ce}A^cA^e\Gamma^d{}_{ab}A^b - \Gamma^b{}_{ec}A^eA^c\Gamma^d{}_{ab}A^a$

$\texttt{eq1.008} := A^aA^bA^c\partial_c\Gamma^d{}_{ab} - A^bA^cA^e\Gamma^a{}_{ce}\Gamma^d{}_{ab} - A^aA^cA^e\Gamma^b{}_{ec}\Gamma^d{}_{ab}$

$\texttt{eq1.009} := A^aA^bA^c\partial_c\Gamma^d{}_{ab} - A^aA^bA^c\Gamma^e{}_{bc}\Gamma^d{}_{ea} - A^aA^bA^c\Gamma^e{}_{cb}\Gamma^d{}_{ae}$

$\texttt{eq1.010} := A^aA^bA^c\partial_a\Gamma^d{}_{bc} - 2A^aA^bA^c\Gamma^d{}_{ae}\Gamma^e{}_{bc}$

$$\texttt{eq2.000} := A^c\partial_c\left(A^aA^bA^f\partial_a\Gamma^d{}_{bf} - 2A^aA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf}\right)$$

$$\texttt{eq2.001} := A^c\partial_c\left(A^aA^bA^f\partial_a\Gamma^d{}_{bf}\right) - 2A^c\partial_c\left(A^aA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf}\right)$$

$$\texttt{eq2.002} := A^c\partial_c\left(A^aA^bA^f\partial_a\Gamma^d{}_{bf}\right) - 2A^c\partial_c\left(A^aA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf}\right)$$

$$\begin{aligned}\texttt{eq2.003} := {}& A^c\left(\partial_cA^aA^bA^f\partial_a\Gamma^d{}_{bf} + A^a\partial_cA^bA^f\partial_a\Gamma^d{}_{bf} + A^aA^b\partial_cA^f\partial_a\Gamma^d{}_{bf} + A^aA^bA^f\partial_{ca}\Gamma^d{}_{bf}\right)\\ &- 2A^c\left(\partial_cA^aA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} + A^a\partial_cA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} + A^aA^b\partial_cA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} + A^aA^bA^f\partial_c\Gamma^d{}_{ae}\Gamma^e{}_{bf} + A^aA^bA^f\Gamma^d{}_{ae}\partial_c\Gamma^e{}_{bf}\right)\end{aligned}$$

$$\begin{aligned}\texttt{eq2.004} := {}& A^c\partial_cA^aA^bA^f\partial_a\Gamma^d{}_{bf} + A^cA^a\partial_cA^bA^f\partial_a\Gamma^d{}_{bf} + A^cA^aA^b\partial_cA^f\partial_a\Gamma^d{}_{bf} + A^cA^aA^bA^f\partial_{ca}\Gamma^d{}_{bf} - 2A^c\partial_cA^aA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf}\\ &- 2A^cA^a\partial_cA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} - 2A^cA^aA^b\partial_cA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} - 2A^cA^aA^bA^f\partial_c\Gamma^d{}_{ae}\Gamma^e{}_{bf} - 2A^cA^aA^bA^f\Gamma^d{}_{ae}\partial_c\Gamma^e{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.005} := {}& -\Gamma^a{}_{ce}A^cA^eA^bA^f\partial_a\Gamma^d{}_{bf} - \Gamma^b{}_{ec}A^eA^cA^aA^f\partial_a\Gamma^d{}_{bf} - \Gamma^f{}_{ce}A^cA^eA^aA^b\partial_a\Gamma^d{}_{bf} + A^cA^aA^bA^f\partial_{ca}\Gamma^d{}_{bf} + 2\Gamma^a{}_{cg}A^cA^gA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf}\\ &+ 2\Gamma^b{}_{gc}A^gA^cA^aA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} + 2\Gamma^f{}_{cg}A^cA^gA^aA^b\Gamma^d{}_{ae}\Gamma^e{}_{bf} - 2A^cA^aA^bA^f\partial_c\Gamma^d{}_{ae}\Gamma^e{}_{bf} - 2A^cA^aA^bA^f\Gamma^d{}_{ae}\partial_c\Gamma^e{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.006} := {}& -\Gamma^a{}_{ce}A^cA^eA^bA^f\partial_a\Gamma^d{}_{bf} - \Gamma^b{}_{ec}A^eA^cA^aA^f\partial_a\Gamma^d{}_{bf} - \Gamma^f{}_{ce}A^cA^eA^aA^b\partial_a\Gamma^d{}_{bf} + A^cA^aA^bA^f\partial_{ca}\Gamma^d{}_{bf} + 2\Gamma^a{}_{cg}A^cA^gA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf}\\ &+ 2\Gamma^b{}_{gc}A^gA^cA^aA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} + 2\Gamma^f{}_{cg}A^cA^gA^aA^b\Gamma^d{}_{ae}\Gamma^e{}_{bf} - 2A^cA^aA^bA^f\partial_c\Gamma^d{}_{ae}\Gamma^e{}_{bf} - 2A^cA^aA^bA^f\Gamma^d{}_{ae}\partial_c\Gamma^e{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.007} := {}& A^cA^aA^bA^f\partial_{ca}\Gamma^d{}_{bf} - \Gamma^a{}_{ce}A^cA^eA^bA^f\partial_a\Gamma^d{}_{bf} - \Gamma^b{}_{ec}A^eA^cA^aA^f\partial_a\Gamma^d{}_{bf} - \Gamma^f{}_{ce}A^cA^eA^aA^b\partial_a\Gamma^d{}_{bf} - 2A^cA^aA^bA^f\partial_c\Gamma^d{}_{ae}\Gamma^e{}_{bf}\\ &- 2A^cA^aA^bA^f\Gamma^d{}_{ae}\partial_c\Gamma^e{}_{bf} + 2\Gamma^a{}_{cg}A^cA^gA^bA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} + 2\Gamma^b{}_{gc}A^gA^cA^aA^f\Gamma^d{}_{ae}\Gamma^e{}_{bf} + 2\Gamma^f{}_{cg}A^cA^gA^aA^b\Gamma^d{}_{ae}\Gamma^e{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.008} := {}& A^aA^bA^cA^f\partial_{ca}\Gamma^d{}_{bf} - A^bA^cA^eA^f\Gamma^a{}_{ce}\partial_a\Gamma^d{}_{bf} - A^aA^cA^eA^f\Gamma^b{}_{ec}\partial_a\Gamma^d{}_{bf} - A^aA^bA^cA^e\Gamma^f{}_{ce}\partial_a\Gamma^d{}_{bf} - 2A^aA^bA^cA^f\Gamma^e{}_{bf}\partial_c\Gamma^d{}_{ae}\\ &- 2A^aA^bA^cA^f\Gamma^d{}_{ae}\partial_c\Gamma^e{}_{bf} + 2A^bA^cA^fA^g\Gamma^a{}_{cg}\Gamma^d{}_{ae}\Gamma^e{}_{bf} + 2A^aA^cA^fA^g\Gamma^b{}_{gc}\Gamma^d{}_{ae}\Gamma^e{}_{bf} + 2A^aA^bA^cA^g\Gamma^d{}_{ae}\Gamma^e{}_{bf}\Gamma^f{}_{cg}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.009} := {}& A^aA^bA^cA^e\partial_{ca}\Gamma^d{}_{be} - A^aA^bA^cA^e\Gamma^f{}_{bc}\partial_f\Gamma^d{}_{ae} - A^aA^bA^cA^e\Gamma^f{}_{cb}\partial_a\Gamma^d{}_{fe} - A^aA^bA^cA^e\Gamma^f{}_{ce}\partial_a\Gamma^d{}_{bf} - 2A^aA^bA^cA^e\Gamma^f{}_{be}\partial_c\Gamma^d{}_{af}\\ &- 2A^aA^bA^cA^e\Gamma^d{}_{af}\partial_c\Gamma^f{}_{be} + 2A^aA^bA^cA^e\Gamma^f{}_{be}\Gamma^d{}_{fg}\Gamma^g{}_{ac} + 2A^aA^bA^cA^e\Gamma^f{}_{eb}\Gamma^d{}_{ag}\Gamma^g{}_{fc} + 2A^aA^bA^cA^e\Gamma^d{}_{af}\Gamma^f{}_{bg}\Gamma^g{}_{ce}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.010} := {}& A^aA^bA^cA^e\partial_{ab}\Gamma^d{}_{ce} - A^aA^bA^cA^e\Gamma^f{}_{ab}\partial_f\Gamma^d{}_{ce} - 4A^aA^bA^cA^e\Gamma^f{}_{ab}\partial_c\Gamma^d{}_{ef}\\ &- 2A^aA^bA^cA^e\Gamma^d{}_{af}\partial_b\Gamma^f{}_{ce} + 2A^aA^bA^cA^e\Gamma^d{}_{fg}\Gamma^f{}_{ab}\Gamma^g{}_{ce} + 4A^aA^bA^cA^e\Gamma^d{}_{af}\Gamma^f{}_{bg}\Gamma^g{}_{ce}\end{aligned}$$

$$
\begin{aligned}
\texttt{eq3.010} := {} & A^aA^bA^cA^eA^f\partial_{abc}\Gamma^d{}_{ef} - A^aA^bA^cA^eA^f\partial_g\Gamma^d{}_{ab}\partial_c\Gamma^g{}_{ef} - 6A^aA^bA^cA^eA^f\partial_a\Gamma^d{}_{bg}\partial_c\Gamma^g{}_{ef} - 3A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\partial_{cg}\Gamma^d{}_{ef} \\
& - 6A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\partial_{ce}\Gamma^d{}_{fg} - 2A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\partial_{bc}\Gamma^g{}_{ef} + 2A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{cg}\partial_h\Gamma^d{}_{ef} + 6A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{ce}\partial_g\Gamma^d{}_{fh} \\
& + 12A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{cg}\partial_e\Gamma^d{}_{fh} + 6A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{ce}\partial_f\Gamma^d{}_{gh} + 6A^aA^bA^cA^eA^f\Gamma^d{}_{gh}\Gamma^g{}_{ab}\partial_c\Gamma^h{}_{ef} \\
& + 2A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^h{}_{bc}\partial_h\Gamma^g{}_{ef} + 8A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^h{}_{bc}\partial_e\Gamma^g{}_{fh} + 4A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^g{}_{bh}\partial_c\Gamma^h{}_{ef} \\
& - 12A^aA^bA^cA^eA^f\Gamma^d{}_{gh}\Gamma^g{}_{ab}\Gamma^h{}_{ci}\Gamma^i{}_{ef} - 4A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^g{}_{hi}\Gamma^h{}_{bc}\Gamma^i{}_{ef} - 8A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^g{}_{bh}\Gamma^h{}_{ci}\Gamma^i{}_{ef}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{eq4.010} := {} & A^aA^bA^cA^eA^fA^g\partial_{abce}\Gamma^{d}{}_{fg} - 4A^aA^bA^cA^eA^fA^g\partial_{a}\Gamma^{h}{}_{bc}\partial_{eh}\Gamma^{d}{}_{fg} - A^aA^bA^cA^eA^fA^g\partial_{h}\Gamma^{d}{}_{ab}\partial_{ce}\Gamma^{h}{}_{fg} - 12A^aA^bA^cA^eA^fA^g\partial_{a}\Gamma^{h}{}_{bc}\partial_{ef}\Gamma^{d}{}_{gh} \\
& - 8A^aA^bA^cA^eA^fA^g\partial_{a}\Gamma^{d}{}_{bh}\partial_{ce}\Gamma^{h}{}_{fg} - 6A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{ceh}\Gamma^{d}{}_{fg} - 8A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{cef}\Gamma^{d}{}_{gh} \\
& + 8A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{i}\Gamma^{d}{}_{ch}\partial_{e}\Gamma^{i}{}_{fg} + A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{i}\Gamma^{d}{}_{ce}\partial_{h}\Gamma^{i}{}_{fg} + 4A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{i}\Gamma^{d}{}_{ce}\partial_{f}\Gamma^{i}{}_{gh} \\
& + 12A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{h}\Gamma^{d}{}_{ci}\partial_{e}\Gamma^{i}{}_{fg} + 24A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{c}\Gamma^{d}{}_{hi}\partial_{e}\Gamma^{i}{}_{fg} + 8A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{c}\Gamma^{d}{}_{ei}\partial_{h}\Gamma^{i}{}_{fg} \\
& + 32A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\partial_{c}\Gamma^{d}{}_{ei}\partial_{f}\Gamma^{i}{}_{gh} - 2A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\partial_{bce}\Gamma^{h}{}_{fg} + 2A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ai}\partial_{h}\Gamma^{d}{}_{bc}\partial_{e}\Gamma^{i}{}_{fg} \\
& + 16A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ai}\partial_{b}\Gamma^{d}{}_{ch}\partial_{e}\Gamma^{i}{}_{fg} + 6A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\partial_{a}\Gamma^{h}{}_{bc}\partial_{e}\Gamma^{i}{}_{fg} + 2A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\partial_{b}\Gamma^{i}{}_{ce}\partial_{i}\Gamma^{h}{}_{fg} \\
& + 12A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\partial_{b}\Gamma^{h}{}_{ci}\partial_{e}\Gamma^{i}{}_{fg} + 8A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\partial_{ei}\Gamma^{d}{}_{fg} + 3A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\partial_{hi}\Gamma^{d}{}_{fg} \\
& + 24A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\partial_{fh}\Gamma^{d}{}_{gi} + 24A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\partial_{ef}\Gamma^{d}{}_{gi} + 12A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\partial_{fg}\Gamma^{d}{}_{hi} \\
& + 8A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\partial_{ce}\Gamma^{i}{}_{fg} + 6A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\partial_{ei}\Gamma^{h}{}_{fg} + 12A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\partial_{ef}\Gamma^{h}{}_{gi} \\
& + 4A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\partial_{ce}\Gamma^{i}{}_{fg} - 4A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\Gamma^{j}{}_{ei}\partial_{j}\Gamma^{d}{}_{fg} - 2A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{hi}\partial_{j}\Gamma^{d}{}_{fg} \\
& - 16A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fh}\partial_{j}\Gamma^{d}{}_{gi} - 24A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fh}\partial_{i}\Gamma^{d}{}_{gj} - 12A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fg}\partial_{h}\Gamma^{d}{}_{ij} \\
& - 32A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\Gamma^{j}{}_{ei}\partial_{f}\Gamma^{d}{}_{gj} - 16A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{hi}\partial_{f}\Gamma^{d}{}_{gj} - 48A^aA^bA^cA^eA^fA^g\Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fh}\partial_{g}\Gamma^{d}{}_{ij} \\
& - 24A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{aj}\Gamma^{j}{}_{bc}\partial_{e}\Gamma^{i}{}_{fg} - 8A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{j}{}_{ce}\partial_{j}\Gamma^{i}{}_{fg} - 32A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{j}{}_{ce}\partial_{f}\Gamma^{i}{}_{gj} \\
& - 4A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ei}\partial_{j}\Gamma^{h}{}_{fg} - 12A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ef}\partial_{i}\Gamma^{h}{}_{gj} - 24A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ei}\partial_{f}\Gamma^{h}{}_{gj} \\
& - 12A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ef}\partial_{g}\Gamma^{h}{}_{ij} - 16A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{i}{}_{cj}\partial_{e}\Gamma^{j}{}_{fg} - 12A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{ij}\Gamma^{i}{}_{bc}\partial_{e}\Gamma^{j}{}_{fg} \\
& - 4A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{j}{}_{ce}\partial_{j}\Gamma^{i}{}_{fg} - 16A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{j}{}_{ce}\partial_{f}\Gamma^{i}{}_{gj} - 8A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{i}{}_{cj}\partial_{e}\Gamma^{j}{}_{fg} \\
& + 24A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{aj}\Gamma^{i}{}_{bk}\Gamma^{j}{}_{ce}\Gamma^{k}{}_{fg} + 16A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{i}{}_{jk}\Gamma^{j}{}_{ce}\Gamma^{k}{}_{fg} + 32A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{i}{}_{cj}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg} \\
& + 24A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{ij}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg} + 8A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{i}{}_{jk}\Gamma^{j}{}_{ce}\Gamma^{k}{}_{fg} + 16A^aA^bA^cA^eA^fA^g\Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{i}{}_{cj}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg}
\end{aligned}
$$

Stage 2: The fixed point scheme for y^a

```
{x^{a}}::Weight(label=eps,value=0).

{y00^{a},y10^{a},y20^{a},y30^{a},y40^{a}}::Weight(label=eps,value=0).
{y11^{a},y21^{a},y31^{a},y41^{a}}::Weight(label=eps,value=1).
{y22^{a},y32^{a},y42^{a}}::Weight(label=eps,value=2).
{y33^{a},y43^{a}}::Weight(label=eps,value=3).
{y44^{a}}::Weight(label=eps,value=4).

{Gam11^{a}_{b c}}::Weight(label=eps,value=1).
{Gam22^{a}_{b c d}}::Weight(label=eps,value=2).
{Gam33^{a}_{b c d e}}::Weight(label=eps,value=3).
{Gam44^{a}_{b c d e f}}::Weight(label=eps,value=4).
{Gam55^{a}_{b c d e f g}}::Weight(label=eps,value=5).

Gam11^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
Gam22^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
Gam33^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
Gam44^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
Gam55^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

y00{#}::LaTeXForm ("\ny{00}").
y10{#}::LaTeXForm ("\ny{10}").
y20{#}::LaTeXForm ("\ny{20}").
y30{#}::LaTeXForm ("\ny{30}").
y40{#}::LaTeXForm ("\ny{40}").
y11{#}::LaTeXForm ("\ny{11}").
y21{#}::LaTeXForm ("\ny{21}").
y31{#}::LaTeXForm ("\ny{31}").
y41{#}::LaTeXForm ("\ny{41}").
y22{#}::LaTeXForm ("\ny{22}").
y32{#}::LaTeXForm ("\ny{32}").
y42{#}::LaTeXForm ("\ny{42}").
y33{#}::LaTeXForm ("\ny{33}").
y43{#}::LaTeXForm ("\ny{43}").
y44{#}::LaTeXForm ("\ny{44}").
```

```
Gam11{#}::LaTeXForm ("\nGamma{11}").
Gam22{#}::LaTeXForm ("\nGamma{22}").
Gam33{#}::LaTeXForm ("\nGamma{33}").
Gam44{#}::LaTeXForm ("\nGamma{44}").
Gam55{#}::LaTeXForm ("\nGamma{55}").

def get_term (obj,n):

    foo := @(obj).
    bah = Ex("eps = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

def truncateEps (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah = Ex("eps = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

def substitute_eps (obj):
    substitute         (obj,epsy0)
    substitute         (obj,epsy1)
    substitute         (obj,epsy2)
    substitute         (obj,epsy3)
    substitute         (obj,epsy4)
    substitute         (obj,epsGam1)
    substitute         (obj,epsGam2)
    substitute         (obj,epsGam3)
    substitute         (obj,epsGam4)
    substitute         (obj,epsGam5)
```

```
    distribute          (obj)
    obj = truncateEps  (obj,4)
    obj = product_sort (obj)
    rename_dummies     (obj)
    canonicalise       (obj)

    return obj

def tidy (obj):
    obj = product_sort (obj)
    rename_dummies (obj)
    canonicalise   (obj)
    return obj

beg_stage_2 = time.time()

y0 := x^{a}.
y1 := x^{a} +   (1/2) Gam^{a}_{b c} y0^{b} y0^{c}.
y2 := x^{a} +   (1/2) Gam^{a}_{b c} y1^{b} y1^{c}
            +   (1/6) Gam^{a}_{b c d} y0^{b} y0^{c} y0^{d}.
y3 := x^{a} +   (1/2) Gam^{a}_{b c} y2^{b} y2^{c}
            +   (1/6) Gam^{a}_{b c d} y1^{b} y1^{c} y1^{d}
            +  (1/24) Gam^{a}_{b c d e} y0^{b} y0^{c} y0^{d} y0^{e}.
y4 := x^{a} +   (1/2) Gam^{a}_{b c} y3^{b} y3^{c}
            +   (1/6) Gam^{a}_{b c d} y2^{b} y2^{c} y2^{d}
            +  (1/24) Gam^{a}_{b c d e} y1^{b} y1^{c} y1^{d} y1^{e}
            + (1/120) Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

# note that:
#   y00 = y10 = y20 = y30 = y40
#   y11 = y21 = y31 = y41
#   y22 = y32 = y42
#   y33 = y43
#   y44

# expand each y in powers of eps

epsy0 := y0^{a} -> y00^{a}.
```

```
epsy1 := y1^{a} -> y10^{a}+y11^{a}.
epsy2 := y2^{a} -> y20^{a}+y21^{a}+y22^{a}.
epsy3 := y3^{a} -> y30^{a}+y31^{a}+y32^{a}+y33^{a}.
epsy4 := y4^{a} -> y40^{a}+y41^{a}+y42^{a}+y43^{a}+y44^{a}.

epsGam1 := Gam^{a}_{b c} -> Gam11^{a}_{b c}.
epsGam2 := Gam^{a}_{b c d} -> Gam22^{a}_{b c d}.
epsGam3 := Gam^{a}_{b c d e} -> Gam33^{a}_{b c d e}.
epsGam4 := Gam^{a}_{b c d e f} -> Gam44^{a}_{b c d e f}.
epsGam5 := Gam^{a}_{b c d e f g} -> Gam55^{a}_{b c d e f g}.

y0 = substitute_eps (y0)   # cdb (y0.001,y0)
y1 = substitute_eps (y1)   # cdb (y1.001,y1)
y2 = substitute_eps (y2)   # cdb (y2.001,y2)
y3 = substitute_eps (y3)   # cdb (y3.001,y3)
y4 = substitute_eps (y4)   # cdb (y4.001,y4)

defy0 := y0^{a} -> @(y0).
defy1 := y1^{a} -> @(y1).
defy2 := y2^{a} -> @(y2).
defy3 := y3^{a} -> @(y3).
defy4 := y4^{a} -> @(y4).

# -----------------------------------
# y0

y00 := @(y0).            # cdb (y00.101,y00)

defy00 := y00^{a} -> @(y00).

# -----------------------------------
# y1

substitute (y1,defy00)

distribute (y1)

y10 = get_term (y1,0)   # cdb (y10.101,y10)
```

```
y11 = get_term (y1,1)   # cdb (y11.101,y11)

defy10 := y10^{a} -> @(y10).
defy11 := y11^{a} -> @(y11).

# -----------------------------------
# y2

substitute (y2,defy00)

substitute (y2,defy10)
substitute (y2,defy11)

distribute (y2)

y20 = get_term (y2,0)   # cdb (y20.101,y20)
y21 = get_term (y2,1)   # cdb (y21.101,y21)
y22 = get_term (y2,2)   # cdb (y22.101,y22)

y20 = tidy (y20)   # cdb (y20.201,y20)
y21 = tidy (y21)   # cdb (y21.201,y21)
y22 = tidy (y22)   # cdb (y22.201,y22)

defy20 := y20^{a} -> @(y20).
defy21 := y21^{a} -> @(y21).
defy22 := y22^{a} -> @(y22).

# -----------------------------------
# y3

substitute (y3,defy00)

substitute (y3,defy10)
substitute (y3,defy11)

substitute (y3,defy20)
substitute (y3,defy21)
substitute (y3,defy22)
```

```
distribute (y3)

y30 = get_term (y3,0)   # cdb (y30.101,y30)
y31 = get_term (y3,1)   # cdb (y31.101,y31)
y32 = get_term (y3,2)   # cdb (y32.101,y32)
y33 = get_term (y3,3)   # cdb (y33.101,y33)

y30 = tidy (y30)   # cdb (y30.201,y30)
y31 = tidy (y31)   # cdb (y31.201,y31)
y32 = tidy (y32)   # cdb (y32.201,y32)
y33 = tidy (y33)   # cdb (y33.201,y33)

defy30 := y30^{a} -> @(y30).
defy31 := y31^{a} -> @(y31).
defy32 := y32^{a} -> @(y32).
defy33 := y33^{a} -> @(y33).

# -----------------------------------
# y4

substitute (y4,defy00)

substitute (y4,defy10)
substitute (y4,defy11)

substitute (y4,defy20)
substitute (y4,defy21)
substitute (y4,defy22)

substitute (y4,defy30)
substitute (y4,defy31)
substitute (y4,defy32)
substitute (y4,defy33)

distribute (y4)

y40 = get_term (y4,0)   # cdb (y40.101,y40)
```

```
y41 = get_term (y4,1)   # cdb (y41.101,y41)
y42 = get_term (y4,2)   # cdb (y42.101,y42)
y43 = get_term (y4,3)   # cdb (y43.101,y43)
y44 = get_term (y4,4)   # cdb (y44.101,y44)

y40 = tidy (y40)   # cdb (y40.201,y40)
y41 = tidy (y41)   # cdb (y41.201,y41)
y42 = tidy (y42)   # cdb (y42.201,y42)
y43 = tidy (y43)   # cdb (y43.201,y43)
y44 = tidy (y44)   # cdb (y44.201,y44)

defy40 := y40^{a} -> @(y40).
defy41 := y41^{a} -> @(y41).
defy42 := y42^{a} -> @(y42).
defy43 := y43^{a} -> @(y43).
defy44 := y44^{a} -> @(y44).

end_stage_2 = time.time()
```

$$\texttt{y1.001} := x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}$$

$$\texttt{y2.001} := x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{10}{y}{}^{b}\overset{11}{y}{}^{c} + \frac{1}{6}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}\overset{00}{y}{}^{d} + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{11}{y}{}^{b}\overset{11}{y}{}^{c}$$

$$\texttt{y3.001} := x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{20}{y}{}^{b}\overset{21}{y}{}^{c} + \frac{1}{6}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{10}{y}{}^{d} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{20}{y}{}^{b}\overset{22}{y}{}^{c} + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{21}{y}{}^{b}\overset{21}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{11}{y}{}^{d} + \frac{1}{24}\overset{33}{\Gamma}{}^{a}{}_{bcde}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}\overset{00}{y}{}^{d}\overset{00}{y}{}^{e} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{21}{y}{}^{b}\overset{22}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{10}{y}{}^{b}\overset{11}{y}{}^{c}\overset{11}{y}{}^{d}$$

$$\begin{aligned}\texttt{y4.001} := {}& x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{30}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{31}{y}{}^{c} + \frac{1}{6}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c}\overset{20}{y}{}^{d} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{32}{y}{}^{c} + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{31}{y}{}^{b}\overset{31}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c}\overset{21}{y}{}^{d} + \frac{1}{24}\overset{33}{\Gamma}{}^{a}{}_{bcde}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{10}{y}{}^{d}\overset{10}{y}{}^{e} \\ &+ \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{33}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{31}{y}{}^{b}\overset{32}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c}\overset{22}{y}{}^{d} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{21}{y}{}^{c}\overset{21}{y}{}^{d} + \frac{1}{6}\overset{33}{\Gamma}{}^{a}{}_{bcde}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{10}{y}{}^{d}\overset{11}{y}{}^{e} + \frac{1}{120}\overset{44}{\Gamma}{}^{a}{}_{bcdef}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}\overset{00}{y}{}^{d}\overset{00}{y}{}^{e}\overset{00}{y}{}^{f}\end{aligned}$$

$$\texttt{y10.101} := x^a$$

$$\texttt{y11.101} := \frac{1}{2}\overset{11}{\Gamma}{}^a{}_{bc}x^b x^c$$

$$\texttt{y20.201} := x^a$$

$$\texttt{y21.201} := \frac{1}{2}x^b x^c \overset{11}{\Gamma}{}^a{}_{bc}$$

$$\texttt{y22.201} := \frac{1}{2}x^b x^c x^d \overset{11}{\Gamma}{}^a{}_{be}\overset{11}{\Gamma}{}^e{}_{cd} + \frac{1}{6}x^b x^c x^d \overset{22}{\Gamma}{}^a{}_{bcd}$$

$$\texttt{y30.201} := x^a$$

$$\texttt{y31.201} := \frac{1}{2}x^b x^c \overset{11}{\Gamma}{}^a{}_{bc}$$

$$\texttt{y32.201} := \frac{1}{2}x^b x^c x^d \overset{11}{\Gamma}{}^a{}_{be}\overset{11}{\Gamma}{}^e{}_{cd} + \frac{1}{6}x^b x^c x^d \overset{22}{\Gamma}{}^a{}_{bcd}$$

$$\texttt{y33.201} := \frac{1}{2}x^b x^c x^d x^e \overset{11}{\Gamma}{}^a{}_{bf}\overset{11}{\Gamma}{}^f{}_{cg}\overset{11}{\Gamma}{}^g{}_{de} + \frac{1}{6}x^b x^c x^d x^e \overset{11}{\Gamma}{}^a{}_{bf}\overset{22}{\Gamma}{}^f{}_{cde} + \frac{1}{8}x^b x^c x^d x^e \overset{11}{\Gamma}{}^a{}_{fg}\overset{11}{\Gamma}{}^f{}_{bc}\overset{11}{\Gamma}{}^g{}_{de} + \frac{1}{4}x^b x^c x^d x^e \overset{11}{\Gamma}{}^f{}_{bc}\overset{22}{\Gamma}{}^a{}_{def} + \frac{1}{24}x^b x^c x^d x^e \overset{33}{\Gamma}{}^a{}_{bcde}$$

$$\texttt{y40.201} := x^{a}$$

$$\texttt{y41.201} := \frac{1}{2} x^{b} x^{c} \overset{11}{\Gamma}{}^{a}{}_{bc}$$

$$\texttt{y42.201} := \frac{1}{2} x^{b} x^{c} x^{d} \overset{11}{\Gamma}{}^{a}{}_{be} \overset{11}{\Gamma}{}^{e}{}_{cd} + \frac{1}{6} x^{b} x^{c} x^{d} \overset{22}{\Gamma}{}^{a}{}_{bcd}$$

$$\texttt{y43.201} := \frac{1}{2} x^{b} x^{c} x^{d} x^{e} \overset{11}{\Gamma}{}^{a}{}_{bf} \overset{11}{\Gamma}{}^{f}{}_{cg} \overset{11}{\Gamma}{}^{g}{}_{de} + \frac{1}{6} x^{b} x^{c} x^{d} x^{e} \overset{11}{\Gamma}{}^{a}{}_{bf} \overset{22}{\Gamma}{}^{f}{}_{cde} + \frac{1}{8} x^{b} x^{c} x^{d} x^{e} \overset{11}{\Gamma}{}^{a}{}_{fg} \overset{11}{\Gamma}{}^{f}{}_{bc} \overset{11}{\Gamma}{}^{g}{}_{de} + \frac{1}{4} x^{b} x^{c} x^{d} x^{e} \overset{11}{\Gamma}{}^{f}{}_{bc} \overset{22}{\Gamma}{}^{a}{}_{def} + \frac{1}{24} x^{b} x^{c} x^{d} x^{e} \overset{33}{\Gamma}{}^{a}{}_{bcde}$$

$$\begin{aligned}\texttt{y44.201} := {} & \frac{1}{2} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{a}{}_{bg} \overset{11}{\Gamma}{}^{g}{}_{ch} \overset{11}{\Gamma}{}^{h}{}_{di} \overset{11}{\Gamma}{}^{i}{}_{ef} + \frac{1}{6} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{a}{}_{bg} \overset{11}{\Gamma}{}^{g}{}_{ch} \overset{22}{\Gamma}{}^{h}{}_{def} + \frac{1}{8} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{a}{}_{bg} \overset{11}{\Gamma}{}^{g}{}_{hi} \overset{11}{\Gamma}{}^{h}{}_{cd} \overset{11}{\Gamma}{}^{i}{}_{ef} + \frac{1}{4} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{a}{}_{bg} \overset{11}{\Gamma}{}^{h}{}_{cd} \overset{22}{\Gamma}{}^{g}{}_{efh} \\ & + \frac{1}{24} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{a}{}_{bg} \overset{33}{\Gamma}{}^{g}{}_{cdef} + \frac{1}{4} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{a}{}_{gh} \overset{11}{\Gamma}{}^{g}{}_{bc} \overset{11}{\Gamma}{}^{h}{}_{di} \overset{11}{\Gamma}{}^{i}{}_{ef} + \frac{1}{12} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{a}{}_{gh} \overset{11}{\Gamma}{}^{g}{}_{bc} \overset{22}{\Gamma}{}^{h}{}_{def} + \frac{1}{4} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{g}{}_{bc} \overset{11}{\Gamma}{}^{h}{}_{dg} \overset{22}{\Gamma}{}^{a}{}_{efh} \\ & + \frac{1}{12} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{22}{\Gamma}{}^{a}{}_{bcg} \overset{22}{\Gamma}{}^{g}{}_{def} + \frac{1}{8} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{g}{}_{bc} \overset{11}{\Gamma}{}^{h}{}_{de} \overset{22}{\Gamma}{}^{a}{}_{fgh} + \frac{1}{12} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{11}{\Gamma}{}^{g}{}_{bc} \overset{33}{\Gamma}{}^{a}{}_{defg} + \frac{1}{120} x^{b} x^{c} x^{d} x^{e} x^{f} \overset{44}{\Gamma}{}^{a}{}_{bcdef}\end{aligned}$$

Stage3: Introduce the generalised connections from Stage 1

```
def substitute_gam (obj):
    substitute     (obj,defGam11)
    substitute     (obj,defGam22)
    substitute     (obj,defGam33)
    substitute     (obj,defGam44)
    substitute     (obj,defGam55)
    distribute     (obj)

    return obj

beg_stage_3 = time.time()

Gam11 := @(eq0).
Gam22 := @(eq1).
Gam33 := @(eq2).
Gam44 := @(eq3).
Gam55 := @(eq4).

# peel off the A^{a}, must then symmetrise over revealed indices

substitute (Gam11,$A^{a}->1$)
substitute (Gam22,$A^{a}->1$)
substitute (Gam33,$A^{a}->1$)
substitute (Gam44,$A^{a}->1$)
substitute (Gam55,$A^{a}->1$)

# now symmetrise

sym (Gam11,$_{a},_{b}$)
sym (Gam22,$_{a},_{b},_{c}$)
sym (Gam33,$_{a},_{b},_{c},_{e}$)
sym (Gam44,$_{a},_{b},_{c},_{e},_{f}$)
sym (Gam55,$_{a},_{b},_{c},_{e},_{f},_{g}$)

defGam11 := Gam11^{d}_{a b} -> @(Gam11).
defGam22 := Gam22^{d}_{a b c} -> @(Gam22).
```

```
defGam33 := Gam33^{d}_{a b c e} -> @(Gam33).
defGam44 := Gam44^{d}_{a b c e f} -> @(Gam44).
defGam55 := Gam55^{d}_{a b c e f g} -> @(Gam55).

y31 = substitute_gam (y31)
y32 = substitute_gam (y32)
y33 = substitute_gam (y33)

y31 = tidy (y31)   # cdb (y31.301,y31)
y32 = tidy (y32)   # cdb (y32.301,y32)
y33 = tidy (y33)   # cdb (y33.301,y33)

y3 := @(y30) + @(y31) + @(y32) + @(y33).

y41 = substitute_gam (y41)
y42 = substitute_gam (y42)
y43 = substitute_gam (y43)
y44 = substitute_gam (y44)

y41 = tidy (y41)   # cdb (y41.301,y41)
y42 = tidy (y42)   # cdb (y42.301,y42)
y43 = tidy (y43)   # cdb (y43.301,y43)
y44 = tidy (y44)   # cdb (y44.301,y44)

y4 := @(y40) + @(y41) + @(y42) + @(y43) + @(y44).

end_stage_3 = time.time()
```

$\texttt{y30.201} := x^a$

$\texttt{y31.301} := \frac{1}{2} x^b x^c \Gamma^a{}_{bc}$

$\texttt{y32.301} := \frac{1}{6} x^b x^c x^d \Gamma^a{}_{be} \Gamma^e{}_{cd} + \frac{1}{6} x^b x^c x^d \partial_b \Gamma^a{}_{cd}$

$\texttt{y33.301} := \frac{1}{12} x^b x^c x^d x^e \Gamma^a{}_{bf} \partial_c \Gamma^f{}_{de} + \frac{1}{24} x^b x^c x^d x^e \Gamma^a{}_{fg} \Gamma^f{}_{bc} \Gamma^g{}_{de} + \frac{1}{24} x^b x^c x^d x^e \Gamma^f{}_{bc} \partial_f \Gamma^a{}_{de} + \frac{1}{24} x^b x^c x^d x^e \partial_{bc} \Gamma^a{}_{de}$

$$\texttt{y40.201} := x^a$$

$$\texttt{y41.301} := \frac{1}{2}x^b x^c \Gamma^a{}_{bc}$$

$$\texttt{y42.301} := \frac{1}{6}x^b x^c x^d \Gamma^a{}_{be}\Gamma^e{}_{cd} + \frac{1}{6}x^b x^c x^d \partial_b \Gamma^a{}_{cd}$$

$$\texttt{y43.301} := \frac{1}{12}x^b x^c x^d x^e \Gamma^a{}_{bf}\partial_c\Gamma^f{}_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^a{}_{fg}\Gamma^f{}_{bc}\Gamma^g{}_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^f{}_{bc}\partial_f\Gamma^a{}_{de} + \frac{1}{24}x^b x^c x^d x^e \partial_{bc}\Gamma^a{}_{de}$$

$$\begin{aligned}\texttt{y44.301} := & -\frac{1}{90}x^b x^c x^d x^e x^f \Gamma^a{}_{bg}\Gamma^g{}_{ch}\Gamma^h{}_{di}\Gamma^i{}_{ef} + \frac{1}{180}x^b x^c x^d x^e x^f \Gamma^a{}_{bg}\Gamma^g{}_{ch}\partial_d\Gamma^h{}_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^a{}_{bg}\Gamma^g{}_{hi}\Gamma^h{}_{cd}\Gamma^i{}_{ef} \\ & - \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_e\Gamma^g{}_{fh} + \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_h\Gamma^g{}_{ef} + \frac{1}{40}x^b x^c x^d x^e x^f \Gamma^a{}_{bg}\partial_{cd}\Gamma^g{}_{ef} + \frac{1}{90}x^b x^c x^d x^e x^f \Gamma^a{}_{gh}\Gamma^g{}_{bc}\Gamma^h{}_{di}\Gamma^i{}_{ef} \\ & + \frac{13}{360}x^b x^c x^d x^e x^f \Gamma^a{}_{gh}\Gamma^g{}_{bc}\partial_d\Gamma^h{}_{ef} - \frac{1}{90}x^b x^c x^d x^e x^f \Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_e\Gamma^a{}_{fh} + \frac{1}{360}x^b x^c x^d x^e x^f \Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_h\Gamma^a{}_{ef} \\ & + \frac{1}{180}x^b x^c x^d x^e x^f \partial_b\Gamma^a{}_{cg}\partial_d\Gamma^g{}_{ef} + \frac{7}{360}x^b x^c x^d x^e x^f \partial_g\Gamma^a{}_{bc}\partial_d\Gamma^g{}_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_f\Gamma^a{}_{gh} \\ & + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_g\Gamma^a{}_{fh} - \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g{}_{bc}\partial_{de}\Gamma^a{}_{fg} + \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^g{}_{bc}\partial_{dg}\Gamma^a{}_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \partial_{bcd}\Gamma^a{}_{ef}\end{aligned}$$

Stage4: Reformatting and output

```
{x^{a}}::Weight(label=numx).
\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

def reformat (obj,scale):

    bah  = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute         (tmp)
    tmp = product_sort (tmp)
    rename_dummies     (tmp)
    canonicalise       (tmp)
    factor_out         (tmp,$x^{a?}$)

    return tmp

def get_term (obj,n):

    tmp := @(obj).
    foo = Ex("numx = " + str(n))
    distribute  (tmp)
    keep_weight (tmp, foo)

    return tmp

beg_stage_4 = time.time()

rnc := x^{a}
     + @(y41)
     + @(y42)
     + @(y43)
     + @(y44).

# substitute (rnc,$A^{a}->x^{a}$)

rnc1 = get_term (rnc,1)          # cdb (rnc1.001,rnc1)
rnc2 = get_term (rnc,2)          # cdb (rnc2.001,rnc2)
```

```
rnc3 = get_term (rnc,3)          # cdb (rnc3.001,rnc3)
rnc4 = get_term (rnc,4)          # cdb (rnc4.001,rnc4)
rnc5 = get_term (rnc,5)          # cdb (rnc5.001,rnc5)

scaled1 = reformat (rnc1,   1)   # cdb (scaled1.002,scaled1)
scaled2 = reformat (rnc2,   2)   # cdb (scaled2.002,scaled2)
scaled3 = reformat (rnc3,   6)   # cdb (scaled3.002,scaled3)
scaled4 = reformat (rnc4,  24)   # cdb (scaled4.002,scaled4)
scaled5 = reformat (rnc5, 360)   # cdb (scaled5.002,scaled5)

import cdblib

cdblib.create ('gen2rnc.json')

cdblib.put ('rnc',rnc,'gen2rnc.json')

cdblib.put ('rnc1',rnc1,'gen2rnc.json')
cdblib.put ('rnc2',rnc2,'gen2rnc.json')
cdblib.put ('rnc3',rnc3,'gen2rnc.json')
cdblib.put ('rnc4',rnc4,'gen2rnc.json')
cdblib.put ('rnc5',rnc5,'gen2rnc.json')

end_stage_4 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)
```

Timing

Stage 1: 6.4 secs
Stage 2: 1.7 secs
Stage 3: 262.8 secs
Stage 4: 0.9 secs

Convert from generic (x) to local RNC coords (y)

$$y^a = \overset{0}{y}{}^a + \overset{1}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a$$

$$\begin{aligned}
\overset{0}{y}{}^a &= x^a \\
2\overset{1}{y}{}^a &= x^b x^c \Gamma^a{}_{bc} \\
6\overset{2}{y}{}^a &= x^b x^c x^d \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} + \partial_b \Gamma^a{}_{cd}\right) \\
24\overset{3}{y}{}^a &= x^b x^c x^d x^e \left(2\Gamma^a{}_{bf}\partial_c\Gamma^f{}_{de} + \Gamma^a{}_{fg}\Gamma^f{}_{bc}\Gamma^g{}_{de} + \Gamma^f{}_{bc}\partial_f\Gamma^a{}_{de} + \partial_{bc}\Gamma^a{}_{de}\right) \\
360\overset{4}{y}{}^a &= x^b x^c x^d x^e x^f \Big(-4\Gamma^a{}_{bg}\Gamma^g{}_{ch}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 2\Gamma^a{}_{bg}\Gamma^g{}_{ch}\partial_d\Gamma^h{}_{ef} + 3\Gamma^a{}_{bg}\Gamma^g{}_{hi}\Gamma^h{}_{cd}\Gamma^i{}_{ef} - 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_e\Gamma^g{}_{fh} + 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_h\Gamma^g{}_{ef} + 9\Gamma^a{}_{bg}\partial_{cd}\Gamma^g{}_{ef} \\
&\qquad + 4\Gamma^a{}_{gh}\Gamma^g{}_{bc}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 13\Gamma^a{}_{gh}\Gamma^g{}_{bc}\partial_d\Gamma^h{}_{ef} - 4\Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_e\Gamma^a{}_{fh} + \Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_h\Gamma^a{}_{ef} + 2\partial_b\Gamma^a{}_{cg}\partial_d\Gamma^g{}_{ef} + 7\partial_g\Gamma^a{}_{bc}\partial_d\Gamma^g{}_{ef} + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_f\Gamma^a{}_{gh} \\
&\qquad + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_g\Gamma^a{}_{fh} - 3\Gamma^g{}_{bc}\partial_{de}\Gamma^a{}_{fg} + 6\Gamma^g{}_{bc}\partial_{dg}\Gamma^a{}_{ef} + 3\partial_{bcd}\Gamma^a{}_{ef}\Big)
\end{aligned}$$

```
cdblib.create ('gen2rnc.export')

# 6th order terms, scaled
cdblib.put ('rnc61scaled',scaled1,'gen2rnc.export')
cdblib.put ('rnc62scaled',scaled2,'gen2rnc.export')
cdblib.put ('rnc63scaled',scaled3,'gen2rnc.export')
cdblib.put ('rnc64scaled',scaled4,'gen2rnc.export')
cdblib.put ('rnc65scaled',scaled5,'gen2rnc.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```