Geodesic IVP

Our game here is to find the solution of

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a(s)/ds = \dot{x}^a$ at s = 0.

Algorithm

By successive differentiation of the above equation we can compute

$$\frac{d^n x^a}{ds^n} = -\Gamma^a_{\underline{d}_n} \frac{dx^{\underline{d}_n}}{ds}$$

at s=0 for $n=2,3,4,\ldots$. The $\Gamma^a_{\underline{d}_n}$ are the generalised connections.

We can then construct the Taylor series solution for $x^a(s)$

$$x^{a}(s) = x^{a} + s\dot{x}^{a} - \sum_{k=2}^{\infty} \frac{s^{k}}{k!} \Gamma_{\underline{d}_{k}}^{a} \dot{x}^{\underline{d}_{k}}$$

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w\#}::Indices(position=independent).
\nabla{#}::Derivative.
import cdblib
# change signs to account for - sign in front of the sum for x^a(s), see above preamble
def flip_sign (obj):
   return Ex(0) - obj
sterm21 = flip_sign (cdblib.get ('genGamma01', 'genGamma.json'))
sterm22 = flip_sign (cdblib.get ('genGamma02', 'genGamma.json'))
sterm23 = flip_sign (cdblib.get ('genGamma03', 'genGamma.json'))
sterm24 = flip_sign (cdblib.get ('genGamma04', 'genGamma.json'))
sterm31 = flip_sign (cdblib.get ('genGamma11', 'genGamma.json'))
sterm32 = flip_sign (cdblib.get ('genGamma12', 'genGamma.json'))
sterm33 = flip_sign (cdblib.get ('genGamma13', 'genGamma.json'))
sterm41 = flip_sign (cdblib.get ('genGamma21', 'genGamma.json'))
sterm42 = flip_sign (cdblib.get ('genGamma22', 'genGamma.json'))
sterm51 = flip_sign (cdblib.get ('genGamma31', 'genGamma.json'))
# note: the various ivp21, ivp31 etc. are the pieces of the Taylor series
        for the ivp but *without* the leading 1/n! of the Taylor series
ivp21 := 0(sterm21).
                                                               # cdb (ivp21.000,ivp21)
ivp31 := @(sterm21) + @(sterm22).
                                                               # cdb (ivp31.000,ivp31)
ivp32 := 0(sterm31).
                                                               # cdb (ivp32.000,ivp32)
ivp41 := 0(sterm21) + 0(sterm22) + 0(sterm23).
                                                            # cdb (ivp41.000,ivp41)
ivp42 := 0(sterm31) + 0(sterm32).
                                                              # cdb (ivp42.000,ivp42)
ivp43 := 0(sterm41).
                                                               # cdb (ivp43.000,ivp43)
ivp51 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24). # cdb (ivp51.000,ivp51)
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ivp52 := 0(sterm31) + 0(sterm32) + 0(sterm33).
                                                              # cdb (ivp52.000,ivp52)
ivp53 := @(sterm41) + @(sterm42).
                                                              # cdb (ivp53.000,ivp53)
ivp54 := 0(sterm51).
                                                              # cdb (ivp54.000,ivp54)
factor_out (ivp21,$A^{a?}$)
                                                              # cdb (ivp21.001,ivp21)
factor_out (ivp31,$A^{a?}$)
                                                              # cdb (ivp31.001,ivp31)
                                                              # cdb (ivp32.001,ivp32)
factor_out (ivp32,$A^{a?}$)
factor_out (ivp41,$A^{a?}$)
                                                              # cdb (ivp41.001,ivp41)
factor_out (ivp42,$A^{a?}$)
                                                              # cdb (ivp42.001,ivp42)
factor_out (ivp43,$A^{a?}$)
                                                              # cdb (ivp43.001,ivp43)
factor_out (ivp51,$A^{a?}$)
                                                              # cdb (ivp51.001,ivp51)
                                                              # cdb (ivp52.001,ivp52)
factor_out (ivp52,$A^{a?}$)
factor_out (ivp53,$A^{a?}$)
                                                              # cdb (ivp53.001,ivp53)
factor_out (ivp54,$A^{a?}$)
                                                              # cdb (ivp54.001,ivp54)
v{#}::LaTeXForm("\dot{x}").
substitute (ivp21, $A^{a} -> v^{a}$)
                                                              # cdb (ivp21.002,ivp21)
substitute (ivp31, $A^{a} -> v^{a}$)
                                                              # cdb (ivp31.002,ivp31)
substitute (ivp32, $A^{a} -> v^{a}$)
                                                              # cdb (ivp32.002,ivp32)
substitute (ivp41, $A^{a} -> v^{a}$)
                                                              # cdb (ivp41.002,ivp41)
substitute (ivp42, $A^{a} -> v^{a}$)
                                                              # cdb (ivp42.002,ivp42)
substitute (ivp43, $A^{a} -> v^{a}$)
                                                              # cdb (ivp43.002,ivp43)
substitute (ivp51, $A^{a} -> v^{a}$)
                                                              # cdb (ivp51.002,ivp51)
substitute (ivp52, $A^{a} -> v^{a}$)
                                                              # cdb (ivp52.002,ivp52)
substitute (ivp53, $A^{a} -> v^{a}$)
                                                              # cdb (ivp53.002,ivp53)
substitute (ivp54, $A^{a} -> v^{a}$)
                                                              # cdb (ivp54.002,ivp54)
# build the Taylor series
# note the inclusion of the 1/n! factors
ivp2 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp21).
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ivp3 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp31) + (1/6) (s**3) @(ivp32).
ivp4 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp41) + (1/6) (s**3) @(ivp42) + (1/24) (s**4) @(ivp43).
ivp5 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp51) + (1/6) (s**3) @(ivp52) + (1/24) (s**4) @(ivp53) + (1/120) (s**5) @(ivp54).
# cdb (ivp2.000,ivp2)
# cdb (ivp3.000,ivp3)
# cdb (ivp4.000,ivp4)
# cdb (ivp5.000,ivp5)
# now construct the scaled terms for ivp5
sterm2 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24). # cdb (sterm2.000, sterm2)
sterm3 := 0(sterm31) + 0(sterm32) + 0(sterm33).
                                                              # cdb (sterm3.000,sterm3)
sterm4 := @(sterm41) + @(sterm42).
                                                              # cdb (sterm4.000,sterm4)
sterm5 := @(sterm51).
                                                              # cdb (sterm5.000,sterm5)
factor_out (sterm2,$A^{a?}$)
                                                              # cdb (sterm2.001,sterm2)
factor_out (sterm3,$A^{a?}$)
                                                              # cdb (sterm3.001,sterm3)
factor_out (sterm4,$A^{a?}$)
                                                              # cdb (sterm4.001,sterm4)
factor_out (sterm5,$A^{a?}$)
                                                              # cdb (sterm5.001,sterm5)
sterm2 := 360 @(sterm2).
sterm3 := 360 @(sterm3).
sterm4 := 90 @(sterm4).
sterm5 := 3 @(sterm5).
substitute (sterm2,$A^{a}->1$)
                                                              # cdb (sterm2.002,sterm2)
substitute (sterm3,$A^{a}->1$)
                                                              # cdb (sterm3.002,sterm3)
substitute (sterm4,$A^{a}->1$)
                                                              # cdb (sterm4.002,sterm4)
substitute (sterm5,$A^{a}->1$)
                                                              # cdb (sterm5.002,sterm5)
```

The geodesic ivp

$$x^{a}(s) = x^{a} + s\dot{x}^{a} + \frac{s^{2}}{2!}\dot{x}^{b}\dot{x}^{c}A^{a}_{bc} + \frac{s^{3}}{3!}\dot{x}^{b}\dot{x}^{c}\dot{x}^{d}A^{a}_{bcd} + \frac{s^{4}}{4!}\dot{x}^{b}\dot{x}^{c}\dot{x}^{d}\dot{x}^{e}A^{a}_{bcde} + \frac{s^{5}}{5!}\dot{x}^{b}\dot{x}^{c}\dot{x}^{d}\dot{x}^{e}\dot{x}^{f}A^{a}_{bcdef} + \cdots$$

$$360A_{bc}^{a} = -240x^{d}g^{ae}R_{bdce} - 30x^{d}x^{e}\left(2g^{af}\nabla_{b}R_{cdef} + 4g^{af}\nabla_{d}R_{becf} + g^{af}\nabla_{f}R_{bdce}\right) - x^{d}x^{e}x^{f}\left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\right) \\ - 2x^{d}x^{e}x^{f}x^{g}\left(16g^{ah}g^{ij}R_{bdci}\nabla_{e}R_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_{b}R_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_{f}R_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_{j}R_{bfcg} - 8g^{ah}g^{ij}R_{bhdi}\nabla_{e}R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_{e}R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_{f}R_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_{f}R_{cghj} + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh} + 2g^{ah}\nabla_{deh}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\right) \\ + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_{e}R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_{h}R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_{f}R_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} \\ + g^{ah}\nabla_{deh}R_{bfcg}\right)$$

$$360A_{bcd}^{a} = -180x^{e}g^{af}\nabla_{b}R_{cedf} - 3x^{e}x^{f}\left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg} \\ + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right) \\ - 2x^{e}x^{f}x^{g}\left(32g^{ah}g^{ij}R_{beci}\nabla_{d}R_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_{f}R_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_{f}R_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_{c}R_{dfgj} + 2g^{ah}g^{ij}R_{bieh}\nabla_{c}R_{dfgj} \\ + 22g^{ah}g^{ij}R_{ehfi}\nabla_{b}R_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_{f}R_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_{f}R_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_{j}R_{cfdg} + 5g^{ah}g^{ij}R_{bieh}\nabla_{j}R_{cfdg} \\ - 12g^{ah}g^{ij}R_{bhci}\nabla_{e}R_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_{c}R_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_{c}R_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_{g}R_{chdj} + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bce}R_{dfgh} \\ + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{efh}R_{cgdh} + 6g^{ah}\nabla_{efh}R_{cgdh} + 16g^{ah}g^{ij}R_{behi}\nabla_{c}R_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_{f}R_{cgdj} \\ + 16g^{ah}g^{ij}R_{bcci}\nabla_{h}R_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_{h}R_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_{f}R_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_{c}R_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} +$$

$$90A_{bcde}^{a} = -6x^{f} \left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) - x^{f}x^{g} \left(64g^{ah}g^{ij}R_{bfci}\nabla_{d}R_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_{d}R_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_{g}R_{dhej} \right.$$

$$+ 4g^{ah}g^{ij}R_{bhci}\nabla_{d}R_{efgj} + 44g^{ah}g^{ij}R_{bhfi}\nabla_{c}R_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_{c}R_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_{f}R_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_{j}R_{dfeg}$$

$$- 16g^{ah}g^{ij}R_{bfgi}\nabla_{c}R_{dhej} + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_{c}R_{dgej}$$

$$+ 6g^{ah}g^{ij}R_{bfci}\nabla_{h}R_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_{d}R_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_{j}R_{dgeh} \right)$$

 $+g^{ah}\nabla_{bhe}R_{cfdg}+g^{ah}\nabla_{ehb}R_{cfdg}+g^{ah}\nabla_{beh}R_{cfdg}+g^{ah}\nabla_{ebh}R_{cfdg}-20g^{ah}g^{ij}R_{beci}\nabla_{j}R_{dfgh}+10g^{ah}g^{ij}R_{behi}\nabla_{j}R_{cfdg}$

$$3A_{bcdef}^{a} = -x^{g} \left(3g^{ah}g^{ij}R_{bgci}\nabla_{d}R_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_{d}R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh} \right)$$

```
sterm2short := @(sterm21) + @(sterm22).
                                                    # cdb (sterm2.short.001,sterm2short)
sterm3short := @(sterm31).
                                                    # cdb (sterm3.short.001,sterm3short)
sterm2shortscaled := 12 @(sterm2short).
                                                    # cdb (sterm2.short.scaled.002,sterm2shortscaled)
sterm3shortscaled := 2 @(sterm3short).
                                                    # cdb (sterm3.short.scaled.002,sterm3shortscaled)
substitute (sterm2shortscaled,$A^{a}->1$)
                                                    # cdb (sterm2.short.scaled.003,sterm2shortscaled)
substitute (sterm3shortscaled,$A^{a}->1$)
                                                    # cdb (sterm3.short.scaled.003,sterm3shortscaled)
cdblib.create ('geodesic-ivp.export')
# 4th order ivp terms scaled
cdblib.put ('ivp42',sterm2shortscaled,'geodesic-ivp.export')
cdblib.put ('ivp43',sterm3shortscaled,'geodesic-ivp.export')
# 6th order ivp terms scaled
cdblib.put ('ivp62',sterm2,'geodesic-ivp.export')
cdblib.put ('ivp63',sterm3,'geodesic-ivp.export')
cdblib.put ('ivp64',sterm4,'geodesic-ivp.export')
cdblib.put ('ivp65',sterm5,'geodesic-ivp.export')
checkpoint.append (sterm2shortscaled)
checkpoint.append (sterm3shortscaled)
checkpoint.append (sterm2)
checkpoint.append (sterm3)
checkpoint.append (sterm4)
checkpoint.append (sterm5)
cdblib.create ('geodesic-ivp.json')
cdblib.put ('ivp21',ivp21,'geodesic-ivp.json')
cdblib.put ('ivp31',ivp31,'geodesic-ivp.json')
cdblib.put ('ivp32',ivp32,'geodesic-ivp.json')
cdblib.put ('ivp41',ivp41,'geodesic-ivp.json')
cdblib.put ('ivp42',ivp42,'geodesic-ivp.json')
cdblib.put ('ivp43',ivp43,'geodesic-ivp.json')
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```
cdblib.put ('ivp51',ivp51,'geodesic-ivp.json')
cdblib.put ('ivp52',ivp52,'geodesic-ivp.json')
cdblib.put ('ivp53',ivp53,'geodesic-ivp.json')
cdblib.put ('ivp54',ivp54,'geodesic-ivp.json')

cdblib.put ('ivp2',ivp2,'geodesic-ivp.json')
cdblib.put ('ivp3',ivp3,'geodesic-ivp.json')
cdblib.put ('ivp4',ivp4,'geodesic-ivp.json')
cdblib.put ('ivp5',ivp5,'geodesic-ivp.json')
```

$$\begin{aligned} &\texttt{sterm2.short.001} := -\frac{2}{3} A^b A^c x^d g^{ae} R_{bdce} - \frac{1}{12} A^b A^c x^d x^e \left(2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) \\ &\texttt{sterm3.short.001} := -\frac{1}{2} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} \\ &\texttt{sterm2.short.scaled.002} := -8A^b A^c x^d g^{ae} R_{bdce} - A^b A^c x^d x^e \left(2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) \\ &\texttt{sterm3.short.scaled.002} := -A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} \\ &\texttt{sterm2.short.scaled.003} := -8x^d g^{ae} R_{bdce} - x^d x^e \left(2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) \\ &\texttt{sterm3.short.scaled.003} := -x^e g^{af} \nabla_b R_{cedf} \end{aligned}$$

$$\begin{split} \text{ivp21.002} &:= -\frac{2}{3} \dot{x}^b \dot{x}^c x^d g^{ae} R_{bdce} \\ \text{ivp31.002} &:= \dot{x}^b \dot{x}^c \left(-\frac{2}{3} x^d g^{ae} R_{bdce} - \frac{1}{12} x^d x^e \left(2 g^{af} \nabla_b R_{cdef} + 4 g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) \right) \\ \text{ivp32.002} &:= -\frac{1}{2} \dot{x}^b \dot{x}^c \dot{x}^d x^e g^{af} \nabla_b R_{cedf} \\ \text{ivp41.002} &:= \dot{x}^b \dot{x}^c \left(-\frac{2}{3} x^d g^{ae} R_{bdce} - \frac{1}{12} x^d x^e \left(2 g^{af} \nabla_b R_{cdef} + 4 g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) - \frac{1}{360} x^d x^e x^f \left(64 g^{ag} g^{hi} R_{bdch} R_{egfi} - 32 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18 g^{ag} \nabla_{bd} R_{cefg} + 18 g^{ag} \nabla_{db} R_{cefg} + 36 g^{ag} \nabla_{de} R_{bfcg} - 16 g^{ag} g^{hi} R_{bdeh} R_{cfgi} + 9 g^{ag} \nabla_{gd} R_{becf} + 9 g^{ag} \nabla_{dg} R_{becf} \right) \end{split}$$

$$\mathrm{ivp42.002} := \dot{x}^b \dot{x}^c \dot{x}^d \left(-\frac{1}{2} x^c g^{af} \nabla_b R_{ccdf} - \frac{1}{120} x^c x^f \left(64 g^{ag} g^{hi} R_{bech} R_{dgfi} + 16 g^{ag} g^{hi} R_{bech} R_{difg} - 16 g^{ag} g^{hi} R_{bech} R_{egdi} + 12 g^{ag} \nabla_{bc} R_{cdf} + 3 g^{ag} \nabla_{bc} R_{ccdf} \right) \right) \\ \mathrm{ivp43.002} := -\frac{1}{15} \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^c x^f \left(8 g^{ag} g^{hi} R_{bfch} R_{dgei} + 6 g^{ag} \nabla_{bc} R_{dfeg} \right) \\ \mathrm{ivp51.002} := \dot{x}^b \dot{x}^c \left(-\frac{2}{3} x^d g^{ac} R_{bdcc} - \frac{1}{12} x^d x^c \left(2 g^{af} \nabla_b R_{cdef} + 4 g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) - \frac{1}{360} x^d x^c x^f \left(64 g^{ag} g^{hi} R_{bdch} R_{egfi} - 32 g^{ag} g^{hi} R_{bdch} R_{egfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 18 g^{ag} \nabla_{bd} R_{ccfg} + 18 g^{ag} \nabla_{db} R_{ccfg} + 36 g^{ag} \nabla_{dc} R_{bfeg} - 16 g^{ag} g^{hi} R_{bdch} R_{efgi} + 9 g^{ag} \nabla_{gd} R_{bcef} + 9 g^{ag} \nabla_{dg} R_{bcef} \right) \\ - \frac{1}{180} g^{d} x^c x^f x^g \left(16 g^{ah} g^{if} R_{bdch} \nabla_c R_{ffgj} + 9 g^{ag} \nabla_{dc} R_{cfgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{fgj} + 9 g^{ag} \nabla_{gd} R_{bcef} \right) \\ - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{cfgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{cfgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{fgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{fgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{fgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{fgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{fgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 8 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 4 g^{ah} g^{if} R_{bdch} \nabla_c R_{efgj} - 8 g^{ah} g^$$

 $+g^{ah}\nabla_{ehb}R_{cfdg}+g^{ah}\nabla_{beh}R_{cfdg}+g^{ah}\nabla_{ebh}R_{cfdg}-20g^{ah}g^{ij}R_{beci}\nabla_{j}R_{dfgh}+10g^{ah}g^{ij}R_{behi}\nabla_{j}R_{cfdg}$

$$\begin{split} \text{ivp53.002} \coloneqq \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \left(-\frac{1}{15} x^f \left(8g^{ag} g^{hi} R_{bfch} R_{dgei} + 6g^{ag} \nabla_{bc} R_{dfeg} \right) - \frac{1}{90} x^f x^g \left(64g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} \right. \\ \left. + 24g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + 4g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + 44g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} \right. \\ \left. + 10g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} - 16g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + 6g^{ah} \nabla_{bcd} R_{efgh} + 8g^{ah} \nabla_{bcf} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{fbc} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + g^{ah} \nabla_{bfc} R_{dfeg} + g^{ah}$$

$$\text{ivp54.002} := -\frac{1}{3}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^fx^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

$$\begin{split} \text{ivp2.000} &:= x^a + s\dot{x}^a - \frac{1}{3}s^2\dot{x}^b\dot{x}^cx^dg^{ae}R_{bdce} \\ \text{ivp3.000} &:= x^a + s\dot{x}^a + \frac{1}{2}s^2\dot{x}^b\dot{x}^c\left(-\frac{2}{3}x^dg^{ae}R_{bdce} - \frac{1}{12}x^dx^e\left(2g^{af}\nabla_bR_{cdef} + 4g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right)\right) - \frac{1}{12}s^3\dot{x}^b\dot{x}^c\dot{x}^dx^eg^{af}\nabla_bR_{cedf} \\ \text{ivp4.000} &:= x^a + s\dot{x}^a + \frac{1}{2}s^2\dot{x}^b\dot{x}^c\left(-\frac{2}{3}x^dg^{ae}R_{bdce} - \frac{1}{12}x^dx^e\left(2g^{af}\nabla_bR_{cdef} + 4g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right) - \frac{1}{360}x^dx^ex^f\left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{gd}R_{becf}\right) \\ &+ 9g^{ag}\nabla_{dg}R_{becf}\right) \\ &+ \frac{1}{6}s^3\dot{x}^b\dot{x}^c\dot{x}^d\left(-\frac{1}{2}x^eg^{af}\nabla_bR_{cedf} - \frac{1}{120}x^ex^f\left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg} + 18g^{ag}\nabla_{cb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right) \\ &- \frac{1}{360}s^4\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^ex^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) \end{aligned}$$