

Geodesic IVP

Our game here is to find the solution of

$$0 = \frac{d^2 x^a}{ds^2} + \Gamma_{bc}^a(x) \frac{dx^b}{ds} \frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a(s)/ds = \dot{x}^a$ at $s = 0$.

Algorithm

By successive differentiation of the above equation we can compute

$$\frac{d^n x^a}{ds^n} = -\Gamma_{\underline{d}_n}^a \frac{dx^{\underline{d}_n}}{ds}$$

at $s = 0$ for $n = 2, 3, 4, \dots$. The $\Gamma_{\underline{d}_n}^a$ are the *generalised connections*.

We can then construct the Taylor series solution for $x^a(s)$

$$x^a(s) = x^a + s\dot{x}^a - \sum_{k=2}^{\infty} \frac{s^k}{k!} \Gamma_{\underline{d}_k}^a \dot{x}^{\underline{d}_k}$$

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

import cdblib

# change signs to account for - sign in front of the sum for x^a(s), see above preamble

def flip_sign (obj):
    return Ex(0) - obj

sterm21 = flip_sign (cdblib.get ('genGamma01','genGamma.json'))
sterm22 = flip_sign (cdblib.get ('genGamma02','genGamma.json'))
sterm23 = flip_sign (cdblib.get ('genGamma03','genGamma.json'))
sterm24 = flip_sign (cdblib.get ('genGamma04','genGamma.json'))

sterm31 = flip_sign (cdblib.get ('genGamma11','genGamma.json'))
sterm32 = flip_sign (cdblib.get ('genGamma12','genGamma.json'))
sterm33 = flip_sign (cdblib.get ('genGamma13','genGamma.json'))

sterm41 = flip_sign (cdblib.get ('genGamma21','genGamma.json'))
sterm42 = flip_sign (cdblib.get ('genGamma22','genGamma.json'))

sterm51 = flip_sign (cdblib.get ('genGamma31','genGamma.json'))

sterm2 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24). # cdb (sterm2.000,sterm2)
sterm3 := @(sterm31) + @(sterm32) + @(sterm33).           # cdb (sterm3.000,sterm3)
sterm4 := @(sterm41) + @(sterm42).                         # cdb (sterm4.000,sterm4)
sterm5 := @(sterm51).                                     # cdb (sterm5.000,sterm5)

factor_out (sterm2,$A^{a?}$)                             # cdb (sterm2.001,sterm2)
factor_out (sterm3,$A^{a?}$)                             # cdb (sterm3.001,sterm3)
factor_out (sterm4,$A^{a?}$)                             # cdb (sterm4.001,sterm4)
factor_out (sterm5,$A^{a?}$)                             # cdb (sterm5.001,sterm5)

sterm2 := 360 @(sterm2).
sterm3 := 360 @(sterm3).
sterm4 := 90  @(sterm4).

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sterm5 := 3 @(sterm5).
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substitute (sterm2,$A^{a}->1$)      # cdb (sterm2.002,sterm2)  
substitute (sterm3,$A^{a}->1$)      # cdb (sterm3.002,sterm3)  
substitute (sterm4,$A^{a}->1$)      # cdb (sterm4.002,sterm4)  
substitute (sterm5,$A^{a}->1$)      # cdb (sterm5.002,sterm5)
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The geodesic ivp

$$x^a(s) = x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A_{bc}^a + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A_{bcd}^a + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A_{bcde}^a + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A_{bcdef}^a + \dots$$

$$\begin{aligned} 360A_{bc}^a = & -240x^d g^{ae} R_{bdce} - 30x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce}) - x^d x^e x^f (64g^{ag} g^{hi} R_{bdch} R_{egfi} - 32g^{ag} g^{hi} R_{bdeh} R_{cgfi} \\ & - 16g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18g^{ag} \nabla_{bd} R_{cef g} + 18g^{ag} \nabla_{db} R_{cef g} + 36g^{ag} \nabla_{de} R_{bfcg} - 16g^{ag} g^{hi} R_{bdeh} R_{cf gi} + 9g^{ag} \nabla_{gd} R_{becf} + 9g^{ag} \nabla_{dg} R_{becf}) \\ & - 2x^d x^e x^f x^g (16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah} g^{ij} R_{dhei} \nabla_b R_{cf gj} + 16g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 8g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cf gj} \\ & - 4g^{ah} g^{ij} R_{bidh} \nabla_e R_{cf gj} - 4g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} - 8g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 2g^{ah} \nabla_{bde} R_{cf gh} + 2g^{ah} \nabla_{dbe} R_{cf gh} \\ & + 2g^{ah} \nabla_{deb} R_{cf gh} + 4g^{ah} \nabla_{def} R_{bgch} - 4g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cf gj} - 4g^{ah} g^{ij} R_{bdei} \nabla_h R_{cf gj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + g^{ah} \nabla_{hde} R_{bfcg} + g^{ah} \nabla_{dhe} R_{bfcg} \\ & + g^{ah} \nabla_{deh} R_{bfcg}) \end{aligned}$$

$$\begin{aligned} 360A_{bcd}^a = & -180x^e g^{af} \nabla_b R_{cedf} - 3x^e x^f (64g^{ag} g^{hi} R_{bech} R_{d gfi} + 16g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{befh} R_{cgdi} + 12g^{ag} \nabla_{bc} R_{defg} + 18g^{ag} \nabla_{be} R_{cf dg} \\ & + 18g^{ag} \nabla_{eb} R_{cf dg} + 48g^{ag} g^{hi} R_{bech} R_{df gi} + 3g^{ag} \nabla_{gb} R_{cedf} + 3g^{ag} \nabla_{bg} R_{cedf}) \\ & - 2x^e x^f x^g (32g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 48g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + 18g^{ah} g^{ij} R_{bhei} \nabla_c R_{df gj} + 2g^{ah} g^{ij} R_{bieh} \nabla_c R_{df gj} \\ & + 22g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15g^{ah} g^{ij} R_{bhei} \nabla_j R_{cf dg} + 5g^{ah} g^{ij} R_{bieh} \nabla_j R_{cf dg} \\ & - 12g^{ah} g^{ij} R_{bhci} \nabla_e R_{df gj} - 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - 8g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - 12g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 4g^{ah} \nabla_{bce} R_{df gh} + 4g^{ah} \nabla_{bec} R_{df gh} \\ & + 6g^{ah} \nabla_{bef} R_{cgdh} + 4g^{ah} \nabla_{ebc} R_{df gh} + 6g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{efb} R_{cgdh} + 16g^{ah} g^{ij} R_{behi} \nabla_c R_{df gj} + 36g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\ & + 16g^{ah} g^{ij} R_{beci} \nabla_h R_{df gj} - 4g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 36g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - 4g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + g^{ah} \nabla_{hbe} R_{cf dg} + g^{ah} \nabla_{heb} R_{cf dg} \\ & + g^{ah} \nabla_{bhe} R_{cf dg} + g^{ah} \nabla_{ehb} R_{cf dg} + g^{ah} \nabla_{beh} R_{cf dg} + g^{ah} \nabla_{ebh} R_{cf dg} - 20g^{ah} g^{ij} R_{beci} \nabla_j R_{df gh} + 10g^{ah} g^{ij} R_{behi} \nabla_j R_{cf dg}) \end{aligned}$$

$$\begin{aligned} 90A_{bcde}^a = & -6x^f (8g^{ag} g^{hi} R_{bfch} R_{dgei} + 6g^{ag} \nabla_{bc} R_{df eg}) - x^f x^g (64g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} \\ & + 4g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + 44g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 10g^{ah} g^{ij} R_{bhci} \nabla_j R_{df eg} \\ & - 16g^{ah} g^{ij} R_{bf gi} \nabla_c R_{dhej} + 6g^{ah} \nabla_{bcd} R_{ef gh} + 8g^{ah} \nabla_{bcf} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{fbc} R_{dgeh} + 26g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} \\ & + 6g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + 46g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah} \nabla_{hbc} R_{df eg} + g^{ah} \nabla_{bhc} R_{df eg} + g^{ah} \nabla_{bch} R_{df eg} - 40g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh}) \end{aligned}$$

$$3A_{bcdef}^a = -x^g (3g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah} \nabla_{bcd} R_{egfh})$$

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sterm2short := @(sterm21) + @(sterm22).          # cdb (sterm2.short.001,sterm2short)
sterm3short := @(sterm31).                        # cdb (sterm3.short.001,sterm3short)
sterm2shortscaled := 12 @(sterm2short).           # cdb (sterm2.short.scaled.002,sterm2shortscaled)
sterm3shortscaled := 2 @(sterm3short).            # cdb (sterm3.short.scaled.002,sterm3shortscaled)

substitute (sterm2shortscaled,$A^{a}->1$)        # cdb (sterm2.short.scaled.003,sterm2shortscaled)
substitute (sterm3shortscaled,$A^{a}->1$)        # cdb (sterm3.short.scaled.003,sterm3shortscaled)

cdblib.create ('geodesic-ivp.export')

# 4th order ivp terms scaled
cdblib.put ('ivp42',sterm2shortscaled,'geodesic-ivp.export')
cdblib.put ('ivp43',sterm3shortscaled,'geodesic-ivp.export')

# 6th order ivp terms scaled
cdblib.put ('ivp62',sterm2,'geodesic-ivp.export')
cdblib.put ('ivp63',sterm3,'geodesic-ivp.export')
cdblib.put ('ivp64',sterm4,'geodesic-ivp.export')
cdblib.put ('ivp65',sterm5,'geodesic-ivp.export')

checkpoint.append (sterm2shortscaled)
checkpoint.append (sterm3shortscaled)

checkpoint.append (sterm2)
checkpoint.append (sterm3)
checkpoint.append (sterm4)
checkpoint.append (sterm5)

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$$\text{sterm2.short.001} := -\frac{2}{3}A^b A^c x^d g^{ae} R_{bdce} - \frac{1}{12}A^b A^c x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce})$$

$$\text{sterm3.short.001} := -\frac{1}{2}A^b A^c A^d x^e g^{af} \nabla_b R_{cedf}$$

$$\text{sterm2.short.scaled.002} := -8A^b A^c x^d g^{ae} R_{bdce} - A^b A^c x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce})$$

$$\text{sterm3.short.scaled.002} := -A^b A^c A^d x^e g^{af} \nabla_b R_{cedf}$$

$$\text{sterm2.short.scaled.003} := -8x^d g^{ae} R_{bdce} - x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce})$$

$$\text{sterm3.short.scaled.003} := -x^e g^{af} \nabla_b R_{cedf}$$