

## Geodesic mid-point for arc-length

This code uses the results of `geodesic-lsq` and `metric` to show that the 2nd and 3rd order estimates for  $L_{PQ}^2$  can be recovered using a mid-point estimate. For the 3rd order estimate we have

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adb e} + \mathcal{O}(\epsilon^4)$$
$$L_{PQ}^2 = g_{ab} D x^a D x^b - \frac{1}{3}x^a x^b D x^c D x^d R_{acbd} - \frac{1}{12}x^a x^b D x^c D x^d D x^e \nabla_c R_{adb e} - \frac{1}{6}x^a x^b x^c D x^d D x^e \nabla_a R_{bdce} + \mathcal{O}(\epsilon^4)$$

The code below verifies that

$$L_{PQ}^2 = g_{ab}(\bar{x}) D x^a D x^b + \mathcal{O}(\epsilon^4)$$

where  $\bar{x}$  is the *coordinate* midpoint of the geodesic

$$\bar{x}^a = \frac{1}{2} (x_P^a + x_Q^a)$$

This result holds true only for the 2nd and 3rd order estimates. Note that the *coordinate* midpoint is not the *geometric* midpoint of the geodesic.

It might be interesting to see if the higher order estimates could be recovered by sampling the metric at points other than the mid point.

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.

import cdblib

gab = cdblib.get('g_ab','metric.json')

lsq2 = cdblib.get('lsq2','geodesic-lsq.json')
lsq3 = cdblib.get('lsq3','geodesic-lsq.json')
lsq4 = cdblib.get('lsq4','geodesic-lsq.json')
lsq5 = cdblib.get('lsq5','geodesic-lsq.json')

substitute (gab,$x^{a}->(p^{a}+q^{a})/2$)  # evaluate rnc gab at mid-point
distribute (gab)

defgab := g_{a b} -> @(gab).

mid := g_{a b} (q^{a}-p^{a}) (q^{b}-p^{b}).

substitute      (mid,defgab)
distribute      (mid)
sort_product    (mid)
rename_dummies  (mid)
canonicalise     (mid)

tst2 := @(lsq2) - @(mid).                # cdb (tst2.201,tst2)
tst3 := @(lsq3) - @(mid).                # cdb (tst3.201,tst3)
tst4 := @(lsq4) - @(mid).                # cdb (tst4.201,tst4)
tst5 := @(lsq5) - @(mid).                # cdb (tst5.201,tst5)

substitute      (tst2,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst2,$x^{a} -> p^{a}$)

```

```

distribute      (tst2)
sort_product    (tst2)
rename_dummies  (tst2)
canonicalise    (tst2)                                # cdb (tst2.202,tst2)

substitute      (tst3,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst3,$x^{a} -> p^{a}$)
distribute      (tst3)
sort_product    (tst3)
rename_dummies  (tst3)
canonicalise    (tst3)                                # cdb (tst3.202,tst3)

substitute      (tst4,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst4,$x^{a} -> p^{a}$)
distribute      (tst4)
sort_product    (tst4)
rename_dummies  (tst4)
canonicalise    (tst4)                                # cdb (tst4.202,tst4)

substitute      (tst5,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst5,$x^{a} -> p^{a}$)
distribute      (tst5)
sort_product    (tst5)
rename_dummies  (tst5)
canonicalise    (tst5)                                # cdb (tst5.202,tst5)

```

# Reformatting

```
def truncateR (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    substitute (tmp, $\nabla_{e f g}\{R_{a b c d}\} \rightarrow Q_{a b c d e f g}\$)
    substitute (tmp, $\nabla_{e f}\{R_{a b c d}\} \rightarrow Q_{a b c d e f}\$)
    substitute (tmp, $\nabla_e\{R_{a b c d}\} \rightarrow Q_{a b c d e}\$)
    substitute (tmp, $R_{a b c d} \rightarrow Q_{a b c d}\$)

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(tmp).
        bah = Ex("numR = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    substitute (ans, $Q_{a b c d e f g} \rightarrow \nabla_{e f g}\{R_{a b c d}\}\$)
    substitute (ans, $Q_{a b c d e f} \rightarrow \nabla_{e f}\{R_{a b c d}\}\$)
    substitute (ans, $Q_{a b c d e} \rightarrow \nabla_e\{R_{a b c d}\}\$)
    substitute (ans, $Q_{a b c d} \rightarrow R_{a b c d}\$)

    return ans

tst2 = truncateR (tst2,2)  # cdb (tst2.301,tst2)
tst3 = truncateR (tst3,3)  # cdb (tst3.301,tst3)
tst4 = truncateR (tst4,4)  # cdb (tst4.301,tst4)
```

```
tst5 = truncateR (tst5,5)  # cdb (tst5.301,tst5)
```

## Errors is mid-point estimates for $L_{PQ}^2$

$$(L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_2 = 0$$

$$(L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_3 = 0$$

$$\begin{aligned} (L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_4 = & -\frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^cp^gp^bq^dq^fq^h + \frac{1}{15}R_{abcd}R_{efgh}g^{ae}p^bp^cp^gq^dq^fq^h - \frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^bp^cp^fp^gq^dq^h \\ & + \frac{1}{240}\nabla_{ab}R_{cdef}p^bp^cp^eq^aq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^bp^cp^eq^dq^f + \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^cp^eq^bq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^cp^eq^aq^bq^dq^f \end{aligned}$$

$$\begin{aligned} (L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_5 = & -\frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^cp^gp^bq^dq^fq^h + \frac{1}{15}R_{abcd}R_{efgh}g^{ae}p^bp^cp^gq^dq^fq^h - \frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^bp^cp^fp^gq^dq^h \\ & + \frac{1}{240}\nabla_{ab}R_{cdef}p^bp^cp^eq^aq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^bp^cp^eq^dq^f + \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^cp^eq^bq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^cp^eq^aq^bq^dq^f \\ & + \frac{1}{135}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^gp^hqbq^dq^eq^i + \frac{7}{270}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^ep^gp^hqbq^dq^i - \frac{1}{90}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^gp^hqbq^dq^eq^i \\ & - \frac{1}{45}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^ep^gp^hqbq^dq^i - \frac{1}{90}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^ep^hqbq^dq^gq^i + \frac{1}{135}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^ep^hqbq^dq^gq^i \\ & - \frac{1}{108}R_{abcd}\nabla_eR_{fghi}g^{ae}p^cp^fp^hqbq^dq^gq^i + \frac{1}{108}R_{abcd}\nabla_eR_{fghi}g^{ae}p^bp^cp^fp^hqbq^dq^gq^i - \frac{1}{45}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^hqbq^dq^eq^gq^i \\ & + \frac{7}{270}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^hqbq^dq^eq^gq^i + \frac{1}{2160}\nabla_{abc}R_{defg}p^bp^cp^dp^fp^aq^eq^g - \frac{1}{720}\nabla_{abc}R_{defg}p^ap^bp^cp^dp^fp^aq^eq^g \\ & + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^cp^dp^fp^bq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^bp^dp^fp^cq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^cp^dp^fp^aq^bq^eq^g \\ & + \frac{1}{2160}\nabla_{abc}R_{defg}p^bp^dp^fp^aq^cq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^dp^fp^bq^cq^eq^g - \frac{1}{720}\nabla_{abc}R_{defg}p^dp^fp^aq^bq^cq^eq^g \end{aligned}$$