

# Symmetrized partial derivatives of the connection

Here we calculate the recursive sequences

$$(n+3)\Gamma^a_{d(b,c\bar{e}_n)} = (n+1) \left( R^a_{(bcd,\bar{e}_n)} - (\Gamma^a_{fc}\Gamma^f_{bd})_{,\bar{e}_n} \right)$$

for  $n = 1, 2, 3, \dots$ . Note that the (extended) index  $\bar{e}_n$  contains  $n$  normal indices.

The result will be expressions for the  $\Gamma^a_{d(b,c\bar{e}_n)}$  in terms of the Riemann tensor and its partial derivatives.

## Stage 1: Compute symmetrised derivatives

In the first stage we simply apply the above recursive equation using a simple trick to impose the symmetries. Start with the original equation and dot out the symmetric indices with  $A^a$  then factor out the partial derivatives. This leads to

$$(n+3)\Gamma^a_{db,c\bar{e}_n} A^b A^c A^{\bar{e}_n} = (n+1) \left( R^a_{bcd} - \Gamma^a_{fc}\Gamma^f_{bd} \right)_{,\bar{e}_n} A^b A^c A^{\bar{e}_n} \quad (1)$$

Thus we also have (for the next iteration)

$$(n+4)\Gamma^a_{db,c\bar{e}_{n+1}} A^b A^c A^{\bar{e}_{n+1}} = (n+2) \left( R^a_{bcd} - \Gamma^a_{fc}\Gamma^f_{bd} \right)_{,\bar{e}_{n+1}} A^b A^c A^{\bar{e}_{n+1}} \quad (2)$$

The  $A^a$  can be freely chosen so choose  $A^a$  to be a constant (i.e., zero derivative). Now define  $P_n$  by

$$P_n = \Gamma^a_{db,c\bar{e}_n} A^b A^c A^{\bar{e}_n} \quad (3)$$

then the above pair of equations can be combined to give

$$P_{n+1} = \frac{(n+2)(n+3)}{(n+4)(n+1)} A^f \partial_f (P_n) \quad (4)$$

This is a very easy equation to compute as it just requires successive rounds of differentiation.

The first term in the sequence is  $P_0$  given by

$$P_0 = \frac{1}{3} A^b A^c (R^a_{bcd} - \Gamma^a_{ce}\Gamma^e_{bd}) \quad (5)$$

The first few results are

$$\begin{aligned}
P_0 &= A^b A^c \Gamma^a_{d(b,c)} = \frac{1}{3} A^b A^c (R^a_{bcd} - \Gamma^a_{ce} \Gamma^e_{bd}) \\
P_1 &= A^b A^c A^e \Gamma^a_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} \\
P_2 &= A^b A^c A^e A^f \Gamma^a_{d(b,cef)} = \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} \\
&\quad - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd}
\end{aligned}$$

## Stage 2: Impose Riemann normal coordinates

Here we impose the RNC condition by setting the  $\Gamma^a_{bc}$  to zero (but not their derivatives).

$$\begin{aligned}
A^b A^c \Gamma^a_{d(b,c)} &= \frac{1}{3} A^b A^c R^a_{bcd} \\
A^b A^c A^e \Gamma^a_{d(b,ce)} &= \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} \\
A^b A^c A^e A^f \Gamma^a_{d(b,cef)} &= \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd}
\end{aligned}$$

## Stage 3: Replace partial derivatives of $\Gamma$ with partial derivatives of $R$

The key point to note is that the partial derivatives of  $\Gamma$  on the right hand side are all symmetrized in exactly the same manner as the partial derivatives on the left hand side. Thus results from the lower order equations can be fed into the later equations to completely eliminate the partial derivatives of  $\Gamma$ .

$$\begin{aligned}
A^b A^c \Gamma^a_{d(b,c)} &= \frac{1}{3} A^b A^c R^a_{bcd} \\
A^b A^c A^e \Gamma^a_{d(b,ce)} &= \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} \\
A^b A^c A^e A^f \Gamma^a_{d(b,cef)} &= \frac{3}{5} A^b A^c A^e A^f \partial_{fe} R^a_{bcd} - \frac{1}{15} A^b A^c A^e A^f R^a_{ceg} R^g_{bfd} - \frac{1}{15} A^b A^c A^e A^f R^a_{cfg} R^g_{bed}
\end{aligned}$$

## Stage 4: Reformatting

This is just simple reformatting.

$$3A^b A^c \Gamma^a_{d(b,c)} = A^b A^c R^a_{bcd}$$

$$6A^b A^c A^e \Gamma^a_{d(b,ce)} = 3A^b A^c A^e \partial_e R^a_{bcd}$$

$$15A^b A^c A^e A^f \Gamma^a_{d(b,cef)} = A^b A^c A^e A^f (9\partial_{fe} R^a_{bcd} - R^a_{ceg} R^g_{bfd} - R^a_{cfg} R^g_{bed})$$

## Stage 1: Compute symmetrised derivatives

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

# symmetrized partial derivatives of \Gamma

dGamma01:= (1/3) A^{b} A^{c} ( R^{a}_{b c d} - \Gamma^{a}_{c e} \Gamma^{e}_{b d} ).
# cdb (dGamma01.101,dGamma01)

dGamma02:= (6/4) A^{a} \partial_{a} { @ (dGamma01) }. # cdb (dGamma02.101,dGamma02)
distribute (dGamma02) # cdb (dGamma02.102,dGamma02)
product_rule (dGamma02) # cdb (dGamma02.103,dGamma02)
unwrap (dGamma02) # cdb (dGamma02.104,dGamma02)
distribute (dGamma02) # cdb (dGamma02.105,dGamma02)

dGamma03:= (12/10) A^{a} \partial_{a} { @ (dGamma02) }. # cdb (dGamma03.101,dGamma03)
distribute (dGamma03) # cdb (dGamma03.102,dGamma03)
product_rule (dGamma03) # cdb (dGamma03.103,dGamma03)
unwrap (dGamma03) # cdb (dGamma03.104,dGamma03)

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distribute      (dGamma03)                      # cdb (dGamma03.105,dGamma03)

dGamma04:= (20/18) A^{a}\partial_{a}{ @(dGamma03) }. # cdb (dGamma04.101,dGamma04)
distribute      (dGamma04)                      # cdb (dGamma04.102,dGamma04)
product_rule    (dGamma04)                      # cdb (dGamma04.103,dGamma04)
unwrap         (dGamma04)                      # cdb (dGamma04.104,dGamma04)
distribute      (dGamma04)                      # cdb (dGamma04.105,dGamma04)

dGamma05:= (30/28) A^{a}\partial_{a}{ @(dGamma04) }. # cdb (dGamma05.101,dGamma05)
distribute      (dGamma05)                      # cdb (dGamma05.102,dGamma05)
product_rule    (dGamma05)                      # cdb (dGamma05.103,dGamma05)
unwrap         (dGamma05)                      # cdb (dGamma05.104,dGamma05)
distribute      (dGamma05)                      # cdb (dGamma05.105,dGamma05)

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$$\text{dGamma01.101} := \frac{1}{3} A^b A^c (R^a_{bcd} - \Gamma^a_{ce} \Gamma^e_{bd})$$

$$\text{dGamma02.101} := \frac{1}{2} A^f \partial_f (A^b A^c (R^a_{bcd} - \Gamma^a_{ce} \Gamma^e_{bd}))$$

$$\text{dGamma02.102} := \frac{1}{2} A^f \partial_f (A^b A^c R^a_{bcd}) - \frac{1}{2} A^f \partial_f (A^b A^c \Gamma^a_{ce} \Gamma^e_{bd})$$

$$\text{dGamma02.103} := \frac{1}{2} A^f (\partial_f A^b A^c R^a_{bcd} + A^b \partial_f A^c R^a_{bcd} + A^b A^c \partial_f R^a_{bcd}) - \frac{1}{2} A^f (\partial_f A^b A^c \Gamma^a_{ce} \Gamma^e_{bd} + A^b \partial_f A^c \Gamma^a_{ce} \Gamma^e_{bd} + A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd})$$

$$\text{dGamma02.104} := \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f (A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd})$$

$$\text{dGamma02.105} := \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd}$$

$$\text{dGamma03.101} := \frac{6}{5} A^g \partial_g \left( \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} \right)$$

$$\text{dGamma03.102} := \frac{3}{5} A^g \partial_g (A^f A^b A^c \partial_f R^a_{bcd}) - \frac{3}{5} A^g \partial_g (A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd}) - \frac{3}{5} A^g \partial_g (A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd})$$

$$\begin{aligned} \text{dGamma03.103} := & \frac{3}{5} A^g (\partial_g A^f A^b A^c \partial_f R^a_{bcd} + A^f \partial_g A^b A^c \partial_f R^a_{bcd} + A^f A^b \partial_g A^c \partial_f R^a_{bcd} + A^f A^b A^c \partial_{gf} R^a_{bcd}) \\ & - \frac{3}{5} A^g (\partial_g A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^f \partial_g A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b \partial_g A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd}) \\ & - \frac{3}{5} A^g (\partial_g A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f \partial_g A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b \partial_g A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd}) \end{aligned}$$

$$\text{dGamma03.104} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g (A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd}) - \frac{3}{5} A^g (A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd})$$

$$\text{dGamma03.105} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd}$$

$$\mathbf{dGamma04.101} := \frac{10}{9} A^h \partial_h \left( \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \right)$$

$$\begin{aligned} \mathbf{dGamma04.102} := & \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd}) - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd}) - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd}) \\ & - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd}) - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd}) \end{aligned}$$

$$\begin{aligned} \text{dGamma04.103} := & \frac{2}{3} A^h \left( \partial_h A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} + A^g \partial_h A^f A^b A^c \partial_{gf} R^a{}_{bcd} + A^g A^f \partial_h A^b A^c \partial_{gf} R^a{}_{bcd} + A^g A^f A^b \partial_h A^c \partial_{gf} R^a{}_{bcd} + A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} \right) \\ & - \frac{2}{3} A^h \left( \partial_h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f \partial_h A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} \right. \\ & \quad \left. + A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \right) - \frac{2}{3} A^h \left( \partial_h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} \right. \\ & \quad \left. + A^g A^f \partial_h A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} \right) \\ & - \frac{2}{3} A^h \left( \partial_h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f \partial_h A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \right. \\ & \quad \left. + A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} \right) - \frac{2}{3} A^h \left( \partial_h A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \right. \\ & \quad \left. + A^g A^f \partial_h A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd} \right) \end{aligned}$$

$$\begin{aligned} \text{dGamma04.104} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h \left( A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \right) \\ & - \frac{2}{3} A^h \left( A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{fhg} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \right) \\ & - \frac{2}{3} A^h \left( A^g A^f A^b A^c \partial_{hgh} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{ghf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \right) - \frac{2}{3} A^h \left( A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gfh} \Gamma^e{}_{bd} + A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd} \right) \end{aligned}$$

$$\begin{aligned} \text{dGamma04.105} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \\ & - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \\ & - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd} \end{aligned}$$

## Stage 2: Impose Riemann normal coordinates

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{\{d\}}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}}_{\{b\}c}\$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{\{a\}}_{\{b\}c} \rightarrow 0\$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{\{d\}}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e\{f\}}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e\{f\}g}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    return obj

# switch to RNC

dGamma01 = impose_rnc (dGamma01)    # cdb (dGamma01.201,dGamma01)
dGamma02 = impose_rnc (dGamma02)    # cdb (dGamma02.202,dGamma02)
dGamma03 = impose_rnc (dGamma03)    # cdb (dGamma03.203,dGamma03)
dGamma04 = impose_rnc (dGamma04)    # cdb (dGamma04.204,dGamma04)
dGamma05 = impose_rnc (dGamma05)    # cdb (dGamma05.205,dGamma05)
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$$d\Gamma_{01}.201 := \frac{1}{3} A^b A^c R^a_{bcd}$$

$$d\Gamma_{02}.202 := \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd}$$

$$d\Gamma_{03}.203 := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd}$$



$$\begin{aligned}
\text{dGamma04.204} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{fhg} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \\
& - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgh} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{ghf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.205} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igfh} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihfh} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ifh} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{fhg} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgh} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igfh} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ghf} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihh} \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgh} \Gamma^a{}_{ce} \partial_{igf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihg} \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd}
\end{aligned}$$

## Stage 3: Replace partial derivatives of $\Gamma$ with partial derivatives of $R$

```
# use lower equations to eliminate partial derivs of Gamma from rhs

# this produces expressions for the partial derivs of the Gamma's in terms of the Rabcd and its partial derivs

substitute (dGamma03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma03.301,dGamma03.302)
substitute (dGamma03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma03.302,dGamma03.303)
distribute (dGamma03)                                                                # cdb(dGamma03.303,dGamma03.304)

substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True)  # cdb(dGamma04.301,dGamma04.302)
substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True)  # cdb(dGamma04.302,dGamma04.303)
substitute (dGamma04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma04.303,dGamma04.304)
substitute (dGamma04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma04.304,dGamma04.305)
distribute (dGamma04)                                                                # cdb(dGamma04.305,dGamma04.306)

substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma03)$,repeat=True)  # cdb(dGamma05.301,dGamma05.302)
substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True)  # cdb(dGamma05.302,dGamma05.303)
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True)      # cdb(dGamma05.303,dGamma05.304)
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True)      # cdb(dGamma05.304,dGamma05.305)
substitute (dGamma05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)          # cdb(dGamma05.305,dGamma05.306)
substitute (dGamma05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)          # cdb(dGamma05.306,dGamma05.307)
distribute (dGamma05)                                                                # cdb(dGamma05.307,dGamma05.308)
```

$$\text{dGamma03.301} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\text{dGamma03.302} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\text{dGamma03.303} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\begin{aligned} \text{dGamma04.301} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} \\ & - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.302} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{3} A^h A^f A^c A^g \partial_f R^a{}_{cge} A^b \partial_h \Gamma^e{}_{bd} - \frac{1}{3} A^f A^c A^h \partial_f R^a{}_{che} A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{1}{3} A^g A^b A^h \partial_g R^e{}_{bhd} A^f A^c \partial_f \Gamma^a{}_{ce} \\ & - \frac{1}{3} A^g A^c A^h \partial_g R^a{}_{che} A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{1}{3} A^f A^b A^h \partial_f R^e{}_{bhd} A^g A^c \partial_g \Gamma^a{}_{ce} - \frac{1}{3} A^h A^f A^b A^g \partial_f R^e{}_{bgd} A^c \partial_h \Gamma^a{}_{ce} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.303} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.304} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.305} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd} \end{aligned}$$

$$\begin{aligned}
\text{dGamma05.301} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ifh} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{fh} \Gamma^a{}_{ce} \partial_{ihg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgh} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gh} \Gamma^a{}_{ce} \partial_{ihf} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gfh} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{igf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_i \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd} \\
\text{dGamma05.302} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jfh} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cjh} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjh} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jfh} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cjh} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jfh} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cjh} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ifh} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jgh} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jgh} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cjh} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgh} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hfh} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jfh} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gfh} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jfh} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jfh} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjh} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.303} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hfg} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gff} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hff} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{iff} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hfg} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hff} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gff} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hfg} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.304} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hfg} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hfg} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.305} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{21} A^b A^i R^e{}_{bid} \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hfg} A^c A^f R^g{}_{cfe} \right) \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^b A^h R^e{}_{bhd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^b A^g R^e{}_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^c A^f R^a{}_{cfe} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^b A^f R^e{}_{bfd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^c A^g R^a{}_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^c A^h R^a{}_{che} \\
& - \frac{5}{21} A^c A^i R^a{}_{cie} \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hfg} A^b A^f R^g{}_{bfd} \right)
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.306} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{21} A^b A^i R^e{}_{bid} \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hfg} A^c A^f R^g{}_{cfe} \right) \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^b A^h R^e{}_{bhd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^b A^g R^e{}_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^c A^f R^a{}_{cfe} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^b A^f R^e{}_{bfd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^c A^g R^a{}_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^c A^h R^a{}_{che} \\
& - \frac{5}{21} A^c A^i R^a{}_{cie} \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hfg} A^b A^f R^g{}_{bfd} \right)
\end{aligned}$$



$$\begin{aligned}
\text{dGamma05.307} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{1}{7} A^b A^i R^e{}_{bid} A^j A^f A^c A^h \partial_{j f} R^a{}_{che} + \frac{1}{63} A^b A^i R^e{}_{bid} A^c A^j R^g{}_{c j e} A^h A^f R^a{}_{h f g} \\
& + \frac{1}{63} A^b A^i R^e{}_{bid} A^h A^j R^a{}_{h j g} A^c A^f R^g{}_{c f e} - \frac{1}{7} A^j A^f A^c A^i \partial_{j f} R^a{}_{c i e} A^b A^h R^e{}_{b h d} + \frac{1}{63} A^c A^j R^g{}_{c j e} A^i A^f R^a{}_{i f g} A^b A^h R^e{}_{b h d} \\
& + \frac{1}{63} A^i A^j R^a{}_{i j g} A^c A^f R^g{}_{c f e} A^b A^h R^e{}_{b h d} - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{c g e} - \frac{1}{7} A^j A^f A^c A^i \partial_{j f} R^a{}_{c i e} A^b A^g R^e{}_{b g d} \\
& + \frac{1}{63} A^c A^j R^h{}_{c j e} A^i A^f R^a{}_{i f h} A^b A^g R^e{}_{b g d} + \frac{1}{63} A^i A^j R^a{}_{i j h} A^c A^f R^h{}_{c f e} A^b A^g R^e{}_{b g d} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{c i e} A^g A^b A^h \partial_g R^e{}_{b h d} - \frac{1}{7} A^j A^g A^b A^i \partial_{j g} R^e{}_{bid} A^c A^f R^a{}_{c f e} \\
& + \frac{1}{63} A^b A^j R^h{}_{b j d} A^i A^g R^e{}_{i g h} A^c A^f R^a{}_{c f e} + \frac{1}{63} A^i A^j R^e{}_{i j h} A^b A^g R^h{}_{b g d} A^c A^f R^a{}_{c f e} - \frac{1}{7} A^j A^g A^c A^i \partial_{j g} R^a{}_{c i e} A^b A^f R^e{}_{b f d} \\
& + \frac{1}{63} A^c A^j R^h{}_{c j e} A^i A^g R^a{}_{i g h} A^b A^f R^e{}_{b f d} + \frac{1}{63} A^i A^j R^a{}_{i j h} A^c A^g R^h{}_{c g e} A^b A^f R^e{}_{b f d} - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} \\
& - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{c i e} A^f A^b A^h \partial_f R^e{}_{b h d} - \frac{1}{7} A^j A^f A^b A^i \partial_{j f} R^e{}_{bid} A^c A^g R^a{}_{c g e} + \frac{1}{63} A^b A^j R^h{}_{b j d} A^i A^f R^e{}_{i f h} A^c A^g R^a{}_{c g e} \\
& + \frac{1}{63} A^i A^j R^e{}_{i j h} A^b A^f R^h{}_{b f d} A^c A^g R^a{}_{c g e} - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{c i e} A^f A^b A^g \partial_f R^e{}_{b g d} - \frac{1}{7} A^j A^f A^b A^i \partial_{j f} R^e{}_{bid} A^c A^h R^a{}_{che} \\
& + \frac{1}{63} A^b A^j R^g{}_{b j d} A^i A^f R^e{}_{i f g} A^c A^h R^a{}_{che} + \frac{1}{63} A^i A^j R^e{}_{i j g} A^b A^f R^g{}_{b f d} A^c A^h R^a{}_{che} - \frac{1}{7} A^c A^i R^a{}_{c i e} A^j A^f A^b A^h \partial_{j f} R^e{}_{b h d} \\
& + \frac{1}{63} A^c A^i R^a{}_{c i e} A^b A^j R^g{}_{b j d} A^h A^f R^e{}_{h f g} + \frac{1}{63} A^c A^i R^a{}_{c i e} A^h A^j R^e{}_{h j g} A^b A^f R^g{}_{b f d}
\end{aligned}$$

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# note:
# canonicalise must not be used here because it may make changes like
#    $R^{\{a\}_{\{b\}c}d} \rightarrow -R_{\{b\}}^{\{a\}_{\{c\}d}}$ 
# these changes can not be applied inside a \partial, must defer use
# of canonicalise until we have \nabla acting on curvatures

sort_product    (dGamma03) # cdb(dGamma03.401,dGamma03)
rename_dummies  (dGamma03) # cdb(dGamma03.402,dGamma03)
# canonicalise   (dGamma03) # cdb(dGamma03.403,dGamma03)

sort_product    (dGamma04) # cdb(dGamma04.401,dGamma04)
rename_dummies  (dGamma04) # cdb(dGamma04.402,dGamma04)
# canonicalise   (dGamma04) # cdb(dGamma04.403,dGamma04)

sort_product    (dGamma05) # cdb(dGamma05.401,dGamma05)
rename_dummies  (dGamma05) # cdb(dGamma05.402,dGamma05)
# canonicalise   (dGamma05) # cdb(dGamma05.403,dGamma05)

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$$\text{dGamma03.401} := \frac{3}{5} A^b A^c A^f A^g \partial_{gf} R^a_{bcd} - \frac{1}{15} A^b A^c A^f A^g R^a_{cfe} R^e_{bgd} - \frac{1}{15} A^b A^c A^f A^g R^a_{cge} R^e_{bfd}$$

$$\text{dGamma03.402} := \frac{3}{5} A^b A^c A^e A^f \partial_{fe} R^a_{bcd} - \frac{1}{15} A^b A^c A^e A^f R^a_{ceg} R^g_{bfd} - \frac{1}{15} A^b A^c A^e A^f R^a_{cfg} R^g_{bed}$$

$$\begin{aligned} \text{dGamma04.401} := & \frac{2}{3} A^b A^c A^f A^g A^h \partial_{hgf} R^a_{bcd} - \frac{1}{9} A^b A^c A^f A^g A^h R^e_{bhd} \partial_f R^a_{cge} - \frac{1}{9} A^b A^c A^f A^g A^h R^e_{bgd} \partial_f R^a_{che} - \frac{1}{9} A^b A^c A^f A^g A^h R^a_{cfe} \partial_g R^e_{bhd} \\ & - \frac{1}{9} A^b A^c A^f A^g A^h R^e_{bfd} \partial_g R^a_{che} - \frac{1}{9} A^b A^c A^f A^g A^h R^a_{cge} \partial_f R^e_{bhd} - \frac{1}{9} A^b A^c A^f A^g A^h R^a_{che} \partial_f R^e_{bgd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.402} := & \frac{2}{3} A^b A^c A^e A^f A^g \partial_{gfe} R^a_{bcd} - \frac{1}{9} A^b A^c A^e A^f A^g R^h_{bgd} \partial_e R^a_{cfh} - \frac{1}{9} A^b A^c A^e A^f A^g R^h_{bfd} \partial_e R^a_{cgh} - \frac{1}{9} A^b A^c A^e A^f A^g R^a_{ceh} \partial_f R^h_{bgd} \\ & - \frac{1}{9} A^b A^c A^e A^f A^g R^h_{bed} \partial_f R^a_{cgh} - \frac{1}{9} A^b A^c A^e A^f A^g R^a_{cfh} \partial_e R^h_{bgd} - \frac{1}{9} A^b A^c A^e A^f A^g R^a_{cgh} \partial_e R^h_{bfd} \end{aligned}$$

$$\begin{aligned} \text{dGamma05.401} := & \frac{5}{7} A^b A^c A^f A^g A^h A^i \partial_{ihgf} R^a_{bcd} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^e_{bid} \partial_{jf} R^a_{che} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{hfg} R^e_{bid} R^g_{cje} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{hfg} R^e_{bid} R^g_{cfe} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^e_{bhd} \partial_{jf} R^a_{cie} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{ifg} R^e_{bhd} R^g_{cje} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{ijg} R^e_{bhd} R^g_{cfe} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a_{cge} \partial_h R^e_{bid} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^e_{bgd} \partial_{jf} R^a_{cie} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{ifh} R^e_{bgd} R^h_{cje} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{ijh} R^e_{bgd} R^h_{cfe} \\ & - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a_{che} \partial_g R^e_{bid} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a_{cie} \partial_g R^e_{bhd} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^a_{cfe} \partial_{jg} R^e_{bid} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cfe} R^e_{igh} R^h_{bjd} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cfe} R^e_{ijh} R^h_{bgd} \\ & - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^e_{bfd} \partial_{jg} R^a_{cie} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{igh} R^e_{bfd} R^h_{cje} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{ijh} R^e_{bfd} R^h_{cge} \\ & - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e_{bid} \partial_g R^a_{che} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e_{bhd} \partial_g R^a_{cie} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^a_{cge} \partial_{jf} R^e_{bid} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cge} R^e_{ifh} R^h_{bjd} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cge} R^e_{ijh} R^h_{bfd} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e_{bgd} \partial_h R^a_{cie} \\ & - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^a_{che} \partial_{jf} R^e_{bid} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{che} R^e_{ifg} R^g_{bjd} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{che} R^e_{ijg} R^g_{bfd} \\ & - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^a_{cie} \partial_{jf} R^e_{bhd} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{cie} R^e_{hfg} R^g_{bjd} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{cie} R^e_{hfg} R^g_{bfd} \end{aligned}$$

$$\begin{aligned}
\text{dGamma05.402} := & \frac{5}{7} A^b A^c A^e A^f A^g A^h \partial_{hgfe} R^a_{bcd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i_{bgd} \partial_{he} R^a_{cfi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{fei} R^j_{bgd} R^i_{chj} \\
& + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{fhi} R^j_{bgd} R^i_{cej} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^i_{bfd} \partial_{he} R^a_{cgi} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{gei} R^j_{bfd} R^i_{chj} \\
& + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{ghi} R^j_{bfd} R^i_{cej} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a_{cfi} \partial_g R^i_{bhd} \\
& - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a_{cgi} \partial_f R^i_{bhd} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a_{chi} \partial_f R^i_{bgd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a_{cei} \partial_{hf} R^i_{bgd} \\
& + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cei} R^i_{gfj} R^j_{bhd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cei} R^i_{ghj} R^j_{bfd} \\
& - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i_{bed} \partial_{hf} R^a_{cgi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{gfi} R^j_{bed} R^i_{chj} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{ghi} R^j_{bed} R^i_{cfj} \\
& - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i_{bhd} \partial_f R^a_{cgi} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i_{bgd} \partial_f R^a_{chi} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^a_{cfi} \partial_{he} R^i_{bgd} \\
& + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{cfi} R^i_{gej} R^j_{bhd} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{cfi} R^i_{ghj} R^j_{bed} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i_{bfd} \partial_g R^a_{chi} \\
& - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a_{cgi} \partial_{he} R^i_{bfd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cgi} R^i_{fej} R^j_{bhd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cgi} R^i_{fhj} R^j_{bed}
\end{aligned}$$

```
import cdblib

cdblib.create ('dGamma.json')

cdblib.put ('dGamma01',dGamma01,'dGamma.json')
cdblib.put ('dGamma02',dGamma02,'dGamma.json')
cdblib.put ('dGamma03',dGamma03,'dGamma.json')
cdblib.put ('dGamma04',dGamma04,'dGamma.json')
cdblib.put ('dGamma05',dGamma05,'dGamma.json')
```

## Stage 4: Reformatting

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                                -> A001^{a}                                $)
    substitute (obj,$ x^{a}                                -> A002^{a}                                $)
    substitute (obj,$ g^{a b}                              -> A003^{a b}                              $)
    substitute (obj,$ \partial_{e f g h}\{R^{a}_{b c d}\}    -> A008^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}\{R^{a}_{b c d}\}       -> A007^{a}_{b c d e f g}  $)
    substitute (obj,$ \partial_{e f}\{R^{a}_{b c d}\}         -> A006^{a}_{b c d e f}   $)
    substitute (obj,$ \partial_e\{R^{a}_{b c d}\}            -> A005^{a}_{b c d e}    $)
    substitute (obj,$ R^{a}_{b c d}                        -> A004^{a}_{b c d}      $)
    sort_product      (obj)
    rename_dummies    (obj)
    substitute (obj,$ A001^{a}                                -> A^{a}                                $)
    substitute (obj,$ A002^{a}                                -> x^{a}                                $)
    substitute (obj,$ A003^{a b}                              -> g^{a b}                              $)
    substitute (obj,$ A004^{a}_{b c d}                        -> R^{a}_{b c d}            $)
    substitute (obj,$ A005^{a}_{b c d e}                      -> \partial_e\{R^{a}_{b c d}\}        $)
    substitute (obj,$ A006^{a}_{b c d e f}                    -> \partial_{e f}\{R^{a}_{b c d}\}     $)
    substitute (obj,$ A007^{a}_{b c d e f g}                  -> \partial_{e f g}\{R^{a}_{b c d}\}    $)
    substitute (obj,$ A008^{a}_{b c d e f g h}                -> \partial_{e f g h}\{R^{a}_{b c d}\}  $)

    return obj

def reformat (obj,scale):
    bah = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute      (tmp)
    tmp = product_sort (tmp)
    rename_dummies  (tmp)
    factor_out      (tmp,$A^{a?}$)
    return tmp

def get_term (obj,n):

    A^{a}::Weight(label=numA).
```

```

foo := @(obj).
bah = Ex("numA = " + str(n))
distribute (foo)
keep_weight (foo, bah)

return foo

Gterm01 := @(dGamma01).
Gterm02 := @(dGamma02).
Gterm03 := @(dGamma03).
Gterm04 := @(dGamma04).
Gterm05 := @(dGamma05).

scaled1 = reformat (Gterm01, 3) # cdb (scaled1.002,scaled1)
scaled2 = reformat (Gterm02, 6) # cdb (scaled2.002,scaled2)
scaled3 = reformat (Gterm03, 15) # cdb (scaled3.002,scaled3)
scaled4 = reformat (Gterm04, 9) # cdb (scaled4.002,scaled4)
scaled5 = reformat (Gterm05, 252) # cdb (scaled5.002,scaled5)

```

## Symmetrised partial derivatives of the connection

$$3A^b A^c \Gamma^a_{d(b,c)} = A^b A^c R^a_{bcd}$$

$$6A^b A^c A^e \Gamma^a_{d(b,ce)} = 3A^b A^c A^e \partial_e R^a_{bcd}$$

$$15A^b A^c A^e A^f \Gamma^a_{d(b,cef)} = A^b A^c A^e A^f (9\partial_{fe} R^a_{bcd} - R^a_{ceg} R^g_{bfd} - R^a_{cfg} R^g_{bed})$$

$$9A^b A^c A^e A^f A^g \Gamma^a_{d(b,cefg)} = A^b A^c A^e A^f A^g (6\partial_{gfe} R^a_{bcd} - R^h_{bgd} \partial_e R^a_{cfh} - R^h_{bfd} \partial_e R^a_{cgh} - R^a_{ceh} \partial_f R^h_{bgd} - R^h_{bed} \partial_f R^a_{cgh} - R^a_{cfh} \partial_e R^h_{bgd} - R^a_{cgh} \partial_e R^h_{bfd})$$

$$\begin{aligned} 252A^b A^c A^e A^f A^g A^h \Gamma^a_{d(b,cefg h)} = & A^b A^c A^e A^f A^g A^h (180\partial_{hgfe} R^a_{bcd} - 36R^i_{bgd} \partial_{he} R^a_{cfi} + 4R^a_{fei} R^i_{chj} R^j_{bgd} + 4R^a_{fhi} R^i_{cej} R^j_{bgd} - 72R^i_{bfd} \partial_{he} R^a_{cgi} \\ & + 8R^a_{gei} R^i_{chj} R^j_{bfd} + 8R^a_{ghi} R^i_{cej} R^j_{bfd} - 45\partial_e R^a_{cfi} \partial_g R^i_{bhd} - 45\partial_e R^a_{cgi} \partial_f R^i_{bhd} - 45\partial_e R^a_{chi} \partial_f R^i_{bgd} \\ & - 36R^a_{cei} \partial_{hf} R^i_{bgd} + 4R^a_{cei} R^i_{g f j} R^j_{bhd} + 4R^a_{cei} R^i_{ghj} R^j_{bfd} - 36R^i_{bed} \partial_{hf} R^a_{cgi} + 4R^a_{gfi} R^i_{chj} R^j_{bed} \\ & + 4R^a_{ghi} R^i_{cfj} R^j_{bed} - 45\partial_f R^a_{cgi} \partial_e R^i_{bhd} - 45\partial_f R^a_{chi} \partial_e R^i_{bgd} - 72R^a_{cfi} \partial_{he} R^i_{bgd} + 8R^a_{cfi} R^i_{gej} R^j_{bhd} \\ & + 8R^a_{cfi} R^i_{ghj} R^j_{bed} - 45\partial_g R^a_{chi} \partial_e R^i_{bfd} - 36R^a_{cgi} \partial_{he} R^i_{bfd} + 4R^a_{cgi} R^i_{fej} R^j_{bhd} + 4R^a_{cgi} R^i_{fhj} R^j_{bed}) \end{aligned}$$



```

substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dGamma.export')

# 6th order dGamma, scaled
cdblib.put ('dGamma61scaled',scaled1,'dGamma.export')
cdblib.put ('dGamma62scaled',scaled2,'dGamma.export')
cdblib.put ('dGamma63scaled',scaled3,'dGamma.export')
cdblib.put ('dGamma64scaled',scaled4,'dGamma.export')
cdblib.put ('dGamma65scaled',scaled5,'dGamma.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

```