Symmetrized partial derivatives of the connection

Here we calculate the recursive sequences

$$(n+3)\Gamma^a{}_{d(b,c\underline{e}_n)} = (n+1)\left(R^a{}_{(bcd,\underline{e}_n)} - \left(\Gamma^a{}_{f(c}\Gamma^f{}_{bd)}\right)_{,\underline{e}_n)}\right)$$

for $n = 1, 2, 3, \cdots$. Note that the (extended) index $\underline{e}_n$ contains n normal indices.

The result will be expressions for the $\Gamma^a{}_{d(b,c\underline{e}_n)}$ in terms of the Riemann tensor and its partial derivatives.

Stage 1: Compute symmetrised derivatives

In the first stage we simply apply the above recursive equation using a simple trick to impose the symmetries. Start with the original equation and dot out the symmetric indices with A^a then factor out the partial derivatives. This leads to

$$(n+3)\Gamma^a{}_{db,c\underline{e}_n}A^bA^cA^{\cdot\underline{e}_n} = (n+1)\left(R^a{}_{bcd} - \Gamma^a{}_{fc}\Gamma^f{}_{bd}\right)_{,\underline{e}_n}A^bA^cA^{\cdot\underline{e}_n} \tag{1}$$

Thus we also have (for the next iteration)

$$(n+4)\Gamma^a{}_{db,c\underline{e}_{n+1}}A^bA^cA^{\cdot\underline{e}_{n+1}} = (n+2)\left(R^a{}_{bcd} - \Gamma^a{}_{fc}\Gamma^f{}_{bd}\right)_{,\underline{e}_{n+1}}A^bA^cA^{\cdot\underline{e}_{n+1}} \tag{2}$$

The A^a can be freely chosen so choose A^a to be a constant (i.e., zero derivative). Now define P_n by

$$P_n = \Gamma^a{}_{db,c\underline{e}_n}A^bA^cA^{\cdot\underline{e}_n} \tag{3}$$

then the above pair of equations can be combinded to give

$$P_{n+1} = \frac{(n+2)(n+3)}{(n+4)(n+1)}A^f\partial_f\left(P_n\right) \tag{4}$$

This is a very easy equation to compute as it just requires successive rounds of differentiation.

The first term in the sequence is P_0 given by

$$P_0 = \frac{1}{3}A^bA^c\left(R^a{}_{bcd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right) \tag{5}$$

The first few results are

$$P_0 = A^b A^c \Gamma^a{}_{d(b,c)} = \frac{1}{3} A^b A^c \left(R^a{}_{bcd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right)$$

$$P_1 = A^b A^c A^e \Gamma^a{}_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a{}_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd}$$

$$\begin{aligned} P_2 = A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} = {} & \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce}\partial_g \Gamma^e{}_{bd} \\ & - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a{}_{ce}\partial_{gf} \Gamma^e{}_{bd} \end{aligned}$$

Stage 2: Impose Riemann normal coordinates

Here we impose the RNC condition by setting the $\Gamma^a{}_{bc}$ to zero (but not their derivatives).

$$A^b A^c \Gamma^a{}_{d(b,c)} = \frac{1}{3} A^b A^c R^a{}_{bcd}$$

$$A^b A^c A^e \Gamma^a{}_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a{}_{bcd}$$

$$A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} = \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce}\partial_g \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd}$$

Stage 3: Replace partial derivatives of Γ with partial derivatives of R

The key point to note is that the partial derivatives of Γ on the right hand side are all symmetrized in exactly the same manner as the partial derivatives on the left hand side. Thus results from the lower order equations can be fed into the later equations to completely eliminate the partial derivatives of Γ.

$$A^b A^c \Gamma^a{}_{d(b,c)} = \frac{1}{3} A^b A^c R^a{}_{bcd}$$

$$A^b A^c A^e \Gamma^a{}_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a{}_{bcd}$$

$$A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} = \frac{3}{5} A^b A^c A^e A^f \partial_{fe} R^a{}_{bcd} - \frac{1}{15} A^b A^c A^e A^f R^a{}_{ceg} R^g{}_{bfd} - \frac{1}{15} A^b A^c A^e A^f R^a{}_{cfg} R^g{}_{bed}$$

Stage 4: Reformatting

This is just simple reformatting.

$$
\begin{aligned}
3A^b A^c \Gamma^a{}_{d(b,c)} &= A^b A^c R^a{}_{bcd} \\
6A^b A^c A^e \Gamma^a{}_{d(b,ce)} &= 3\, A^b A^c A^e \partial_e R^a{}_{bcd} \\
15A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} &= A^b A^c A^e A^f \left(9\, \partial_{fe} R^a{}_{bcd} - R^a{}_{ceg} R^g{}_{bfd} - R^a{}_{cfg} R^g{}_{bed}\right)
\end{aligned}
$$

Stage 1: Compute symmetrised derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

# symmetrized partial derivatives of \Gamma

dGamma01:= (1/3) A^{b} A^{c} ( R^{a}_{b c d} - \Gamma^{a}_{c e}\Gamma^{e}_{b d} ).
                                                    # cdb (dGamma01.101,dGamma01)

dGamma02:= (6/4) A^{a}\partial_{a}{ @(dGamma01) }.   # cdb (dGamma02.101,dGamma02)
distribute   (dGamma02)                              # cdb (dGamma02.102,dGamma02)
product_rule (dGamma02)                              # cdb (dGamma02.103,dGamma02)
unwrap       (dGamma02)                              # cdb (dGamma02.104,dGamma02)
distribute   (dGamma02)                              # cdb (dGamma02.105,dGamma02)

dGamma03:= (12/10) A^{a}\partial_{a}{ @(dGamma02) }. # cdb (dGamma03.101,dGamma03)
distribute   (dGamma03)                              # cdb (dGamma03.102,dGamma03)
product_rule (dGamma03)                              # cdb (dGamma03.103,dGamma03)
unwrap       (dGamma03)                              # cdb (dGamma03.104,dGamma03)
```

```
distribute   (dGamma03)                                  # cdb (dGamma03.105,dGamma03)

dGamma04:= (20/18) A^{a}\partial_{a}{ @(dGamma03) }. # cdb (dGamma04.101,dGamma04)
distribute   (dGamma04)                                  # cdb (dGamma04.102,dGamma04)
product_rule (dGamma04)                                  # cdb (dGamma04.103,dGamma04)
unwrap       (dGamma04)                                  # cdb (dGamma04.104,dGamma04)
distribute   (dGamma04)                                  # cdb (dGamma04.105,dGamma04)

dGamma05:= (30/28) A^{a}\partial_{a}{ @(dGamma04) }. # cdb (dGamma05.101,dGamma05)
distribute   (dGamma05)                                  # cdb (dGamma05.102,dGamma05)
product_rule (dGamma05)                                  # cdb (dGamma05.103,dGamma05)
unwrap       (dGamma05)                                  # cdb (dGamma05.104,dGamma05)
distribute   (dGamma05)                                  # cdb (dGamma05.105,dGamma05)
```

$$\texttt{dGamma01.101} := \frac{1}{3}\, A^b A^c \left(R^a{}_{bcd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.101} := \frac{1}{2}\, A^f \partial_f\left(A^b A^c \left(R^a{}_{bcd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right)\right)$$

$$\texttt{dGamma02.102} := \frac{1}{2}\, A^f \partial_f\left(A^b A^c R^a{}_{bcd}\right) - \frac{1}{2}\, A^f \partial_f\left(A^b A^c \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.103} := \frac{1}{2}\, A^f \left(\partial_f A^b A^c R^a{}_{bcd} + A^b \partial_f A^c R^a{}_{bcd} + A^b A^c \partial_f R^a{}_{bcd}\right) - \frac{1}{2}\, A^f \left(\partial_f A^b A^c \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^b \partial_f A^c \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.104} := \frac{1}{2}\, A^f A^b A^c \partial_f R^a{}_{bcd} - \frac{1}{2}\, A^f \left(A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.105} := \frac{1}{2}\, A^f A^b A^c \partial_f R^a{}_{bcd} - \frac{1}{2}\, A^f A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{1}{2}\, A^f A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd}$$

$$\texttt{dGamma03.101} := \frac{6}{5}\, A^g \partial_g\left(\frac{1}{2}\, A^f A^b A^c \partial_f R^a{}_{bcd} - \frac{1}{2}\, A^f A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{1}{2}\, A^f A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma03.102} := \frac{3}{5}\, A^g \partial_g\left(A^f A^b A^c \partial_f R^a{}_{bcd}\right) - \frac{3}{5}\, A^g \partial_g\left(A^f A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right) - \frac{3}{5}\, A^g \partial_g\left(A^f A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd}\right)$$

$$\begin{aligned}\texttt{dGamma03.103} := {} & \frac{3}{5}\, A^g \left(\partial_g A^f A^b A^c \partial_f R^a{}_{bcd} + A^f \partial_g A^b A^c \partial_f R^a{}_{bcd} + A^f A^b \partial_g A^c \partial_f R^a{}_{bcd} + A^f A^b A^c \partial_{gf} R^a{}_{bcd}\right) \\ & - \frac{3}{5}\, A^g \left(\partial_g A^f A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^f \partial_g A^b A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^f A^b \partial_g A^c \partial_f \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^f A^b A^c \partial_f \Gamma^a{}_{ce}\partial_g \Gamma^e{}_{bd}\right) \\ & - \frac{3}{5}\, A^g \left(\partial_g A^f A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd} + A^f \partial_g A^b A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd} + A^f A^b \partial_g A^c \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd} + A^f A^b A^c \partial_g \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd} + A^f A^b A^c \Gamma^a{}_{ce}\partial_{gf} \Gamma^e{}_{bd}\right)\end{aligned}$$

$$\texttt{dGamma03.104} := \frac{3}{5}\, A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5}\, A^g \left(A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^f A^b A^c \partial_f \Gamma^a{}_{ce}\partial_g \Gamma^e{}_{bd}\right) - \frac{3}{5}\, A^g \left(A^f A^b A^c \partial_g \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd} + A^f A^b A^c \Gamma^a{}_{ce}\partial_{gf} \Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma03.105} := \frac{3}{5}\, A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5}\, A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{3}{5}\, A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce}\partial_g \Gamma^e{}_{bd} - \frac{3}{5}\, A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce}\partial_f \Gamma^e{}_{bd} - \frac{3}{5}\, A^g A^f A^b A^c \Gamma^a{}_{ce}\partial_{gf} \Gamma^e{}_{bd}$$

$$\texttt{dGamma04.101} := \frac{10}{9} A^h \partial_h \Big(\frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \Big)$$

$$\begin{aligned}\texttt{dGamma04.102} := {} & \frac{2}{3} A^h \partial_h \big(A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} \big) - \frac{2}{3} A^h \partial_h \big(A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} \big) - \frac{2}{3} A^h \partial_h \big(A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} \big) \\ & - \frac{2}{3} A^h \partial_h \big(A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \big) - \frac{2}{3} A^h \partial_h \big(A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \big)\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.103} := {} & \frac{2}{3} A^h \big(\partial_h A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} + A^g \partial_h A^f A^b A^c \partial_{gf} R^a{}_{bcd} + A^g A^f \partial_h A^b A^c \partial_{gf} R^a{}_{bcd} + A^g A^f A^b \partial_h A^c \partial_{gf} R^a{}_{bcd} + A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} \big) \\ & - \frac{2}{3} A^h \big(\partial_h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f \partial_h A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} \\ & \quad + A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \big) - \frac{2}{3} A^h \big(\partial_h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} \\ & \quad + A^g A^f \partial_h A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} \big) \\ & - \frac{2}{3} A^h \big(\partial_h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f \partial_h A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \\ & \quad + A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} \big) - \frac{2}{3} A^h \big(\partial_h A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g \partial_h A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \\ & \quad + A^g A^f \partial_h A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g A^f A^b \partial_h A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd} \big)\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.104} := {} & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h \big(A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \big) \\ & - \frac{2}{3} A^h \big(A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} \big) \\ & - \frac{2}{3} A^h \big(A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} + A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} \big) - \frac{2}{3} A^h \big(A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} + A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd} \big)\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.105} := {} & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \\ & - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \\ & - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd}\end{aligned}$$

Stage 2: Impose Riemann normal coordinates

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

# switch to RNC

dGamma01 = impose_rnc (dGamma01)   # cdb (dGamma01.201,dGamma01)
dGamma02 = impose_rnc (dGamma02)   # cdb (dGamma02.202,dGamma02)
dGamma03 = impose_rnc (dGamma03)   # cdb (dGamma03.203,dGamma03)
dGamma04 = impose_rnc (dGamma04)   # cdb (dGamma04.204,dGamma04)
dGamma05 = impose_rnc (dGamma05)   # cdb (dGamma05.205,dGamma05)
```

$$\texttt{dGamma01.201} := \frac{1}{3} A^{b} A^{c} R^{a}{}_{bcd}$$

$$\texttt{dGamma02.202} := \frac{1}{2} A^{f} A^{b} A^{c} \partial_{f} R^{a}{}_{bcd}$$

$$\texttt{dGamma03.203} := \frac{3}{5} A^{g} A^{f} A^{b} A^{c} \partial_{gf} R^{a}{}_{bcd} - \frac{3}{5} A^{g} A^{f} A^{b} A^{c} \partial_{f} \Gamma^{a}{}_{ce} \partial_{g} \Gamma^{e}{}_{bd} - \frac{3}{5} A^{g} A^{f} A^{b} A^{c} \partial_{g} \Gamma^{a}{}_{ce} \partial_{f} \Gamma^{e}{}_{bd}$$

$$\texttt{dGamma04.204} := \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd}$$
$$- \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd}$$

$$\texttt{dGamma05.205} := \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{if} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{ihg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{ihf} \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{igf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_i \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd}$$

Stage 3: Replace partial derivatives of Γ with partial derivatives of R

```
# use lower equations to eliminate partial derivs of Gamma from rhs

# this produces experssions for the partial derivs of the Gamma's in terms of the Rabcd and its partial derivs

substitute (dGamma03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)                # cdb(dGamma03.301,dGamma03
substitute (dGamma03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)                # cdb(dGamma03.302,dGamma03
distribute (dGamma03)                                                                                      # cdb(dGamma03.303,dGamma03

substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)        # cdb(dGamma04.301,dGamma04
substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)        # cdb(dGamma04.302,dGamma04
substitute (dGamma04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)               # cdb(dGamma04.303,dGamma04
substitute (dGamma04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)               # cdb(dGamma04.304,dGamma04
distribute (dGamma04)                                                                                      # cdb(dGamma04.305,dGamma04

substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{d b}} -> @(dGamma03)$,repeat=True)  # cdb(dGamma05.301,dGamma05
substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{b d}} -> @(dGamma03)$,repeat=True)  # cdb(dGamma05.302,dGamma05
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)        # cdb(dGamma05.303,dGamma05
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)        # cdb(dGamma05.304,dGamma05
substitute (dGamma05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)               # cdb(dGamma05.305,dGamma05
substitute (dGamma05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)               # cdb(dGamma05.306,dGamma05
distribute (dGamma05)                                                                                      # cdb(dGamma05.307,dGamma05
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$$\texttt{dGamma03.301} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\texttt{dGamma03.302} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\texttt{dGamma03.303} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\begin{aligned}\texttt{dGamma04.301} := {} & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} \\ & - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.302} := {} & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{3} A^h A^f A^c A^g \partial_f R^a{}_{cge} A^b \partial_h \Gamma^e{}_{bd} - \frac{1}{3} A^f A^c A^h \partial_f R^a{}_{che} A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{1}{3} A^g A^b A^h \partial_g R^e{}_{bhd} A^f A^c \partial_f \Gamma^a{}_{ce} \\ & - \frac{1}{3} A^g A^c A^h \partial_g R^a{}_{che} A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{1}{3} A^f A^b A^h \partial_f R^e{}_{bhd} A^g A^c \partial_g \Gamma^a{}_{ce} - \frac{1}{3} A^h A^f A^b A^g \partial_f R^e{}_{bgd} A^c \partial_h \Gamma^a{}_{ce}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.303} := {} & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.304} := {} & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.305} := {} & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd}\end{aligned}$$

$$\texttt{dGamma05.301} := \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{if} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{ihg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{ihf} \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{igf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_i \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd}$$

$$\texttt{dGamma05.302} := \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left(\frac{3}{5} A^j A^f A^c A^h \partial_{jf} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjg} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{if} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} \left(\frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{5}{7} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce}$$
$$- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} - \frac{5}{7} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce}$$
$$- \frac{5}{7} A^i \left(\frac{3}{5} A^j A^f A^b A^h \partial_{jf} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjg} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}$$

$$\texttt{dGamma05.303} := \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{ihgf} R^{a}{}_{bcd} - \frac{5}{7} A^{i} \left(\frac{3}{5} A^{j} A^{f} A^{c} A^{h} \partial_{jf} R^{a}{}_{che} - \frac{1}{15} A^{c} A^{j} R^{g}{}_{cje} A^{h} A^{f} R^{a}{}_{hfg} - \frac{1}{15} A^{h} A^{j} R^{a}{}_{hjg} A^{c} A^{f} R^{g}{}_{cfe} \right) A^{b} \partial_{i} \Gamma^{e}{}_{bd}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{c} A^{i} \partial_{jf} R^{a}{}_{cie} - \frac{1}{15} A^{c} A^{j} R^{g}{}_{cje} A^{i} A^{f} R^{a}{}_{ifg} - \frac{1}{15} A^{i} A^{j} R^{a}{}_{ijg} A^{c} A^{f} R^{g}{}_{cfe} \right) A^{h} A^{b} \partial_{h} \Gamma^{e}{}_{bd} - \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{gf} \Gamma^{a}{}_{ce} \partial_{ih} \Gamma^{e}{}_{bd}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{c} A^{i} \partial_{jf} R^{a}{}_{cie} - \frac{1}{15} A^{c} A^{j} R^{h}{}_{cje} A^{i} A^{f} R^{a}{}_{ifh} - \frac{1}{15} A^{i} A^{j} R^{a}{}_{ijh} A^{c} A^{f} R^{h}{}_{cfe} \right) A^{g} A^{b} \partial_{g} \Gamma^{e}{}_{bd} - \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{hf} \Gamma^{a}{}_{ce} \partial_{ig} \Gamma^{e}{}_{bd}$$
$$- \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{if} \Gamma^{a}{}_{ce} \partial_{hg} \Gamma^{e}{}_{bd} - \frac{5}{7} \left(\frac{3}{5} A^{j} A^{g} A^{b} A^{i} \partial_{jg} R^{e}{}_{bid} - \frac{1}{15} A^{b} A^{j} R^{h}{}_{bjd} A^{i} A^{g} R^{e}{}_{igh} - \frac{1}{15} A^{i} A^{j} R^{e}{}_{ijh} A^{b} A^{g} R^{h}{}_{bgd} \right) A^{f} A^{c} \partial_{f} \Gamma^{a}{}_{ce}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^{j} A^{g} A^{c} A^{i} \partial_{jg} R^{a}{}_{cie} - \frac{1}{15} A^{c} A^{j} R^{h}{}_{cje} A^{i} A^{g} R^{a}{}_{igh} - \frac{1}{15} A^{i} A^{j} R^{a}{}_{ijh} A^{c} A^{g} R^{h}{}_{cge} \right) A^{f} A^{b} \partial_{f} \Gamma^{e}{}_{bd} - \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{hg} \Gamma^{a}{}_{ce} \partial_{if} \Gamma^{e}{}_{bd}$$
$$- \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{ig} \Gamma^{a}{}_{ce} \partial_{hf} \Gamma^{e}{}_{bd} - \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{b} A^{i} \partial_{jf} R^{e}{}_{bid} - \frac{1}{15} A^{b} A^{j} R^{h}{}_{bjd} A^{i} A^{f} R^{e}{}_{ifh} - \frac{1}{15} A^{i} A^{j} R^{e}{}_{ijh} A^{b} A^{f} R^{h}{}_{bfd} \right) A^{g} A^{c} \partial_{g} \Gamma^{a}{}_{ce}$$
$$- \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{ih} \Gamma^{a}{}_{ce} \partial_{gf} \Gamma^{e}{}_{bd} - \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{b} A^{i} \partial_{jf} R^{e}{}_{bid} - \frac{1}{15} A^{b} A^{j} R^{g}{}_{bjd} A^{i} A^{f} R^{e}{}_{ifg} - \frac{1}{15} A^{i} A^{j} R^{e}{}_{ijg} A^{b} A^{f} R^{g}{}_{bfd} \right) A^{h} A^{c} \partial_{h} \Gamma^{a}{}_{ce}$$
$$- \frac{5}{7} A^{i} \left(\frac{3}{5} A^{j} A^{f} A^{b} A^{h} \partial_{jf} R^{e}{}_{bhd} - \frac{1}{15} A^{b} A^{j} R^{g}{}_{bjd} A^{h} A^{f} R^{e}{}_{hfg} - \frac{1}{15} A^{h} A^{j} R^{e}{}_{hjg} A^{b} A^{f} R^{g}{}_{bfd} \right) A^{c} \partial_{i} \Gamma^{a}{}_{ce}$$

$$\texttt{dGamma05.304} := \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{ihgf} R^{a}{}_{bcd} - \frac{5}{7} A^{i} \left(\frac{3}{5} A^{j} A^{f} A^{c} A^{h} \partial_{jf} R^{a}{}_{che} - \frac{1}{15} A^{c} A^{j} R^{g}{}_{cje} A^{h} A^{f} R^{a}{}_{hfg} - \frac{1}{15} A^{h} A^{j} R^{a}{}_{hjg} A^{c} A^{f} R^{g}{}_{cfe} \right) A^{b} \partial_{i} \Gamma^{e}{}_{bd}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{c} A^{i} \partial_{jf} R^{a}{}_{cie} - \frac{1}{15} A^{c} A^{j} R^{g}{}_{cje} A^{i} A^{f} R^{a}{}_{ifg} - \frac{1}{15} A^{i} A^{j} R^{a}{}_{ijg} A^{c} A^{f} R^{g}{}_{cfe} \right) A^{h} A^{b} \partial_{h} \Gamma^{e}{}_{bd} - \frac{5}{28} A^{h} A^{b} A^{i} \partial_{h} R^{e}{}_{bid} A^{f} A^{c} A^{g} \partial_{f} R^{a}{}_{cge}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{c} A^{i} \partial_{jf} R^{a}{}_{cie} - \frac{1}{15} A^{c} A^{j} R^{h}{}_{cje} A^{i} A^{f} R^{a}{}_{ifh} - \frac{1}{15} A^{i} A^{j} R^{a}{}_{ijh} A^{c} A^{f} R^{h}{}_{cfe} \right) A^{g} A^{b} \partial_{g} \Gamma^{e}{}_{bd} - \frac{5}{28} A^{g} A^{b} A^{i} \partial_{g} R^{e}{}_{bid} A^{f} A^{c} A^{h} \partial_{f} R^{a}{}_{che}$$
$$- \frac{5}{28} A^{f} A^{c} A^{i} \partial_{f} R^{a}{}_{cie} A^{g} A^{b} A^{h} \partial_{g} R^{e}{}_{bhd} - \frac{5}{7} \left(\frac{3}{5} A^{j} A^{g} A^{b} A^{i} \partial_{jg} R^{e}{}_{bid} - \frac{1}{15} A^{b} A^{j} R^{h}{}_{bjd} A^{i} A^{g} R^{e}{}_{igh} - \frac{1}{15} A^{i} A^{j} R^{e}{}_{ijh} A^{b} A^{g} R^{h}{}_{bgd} \right) A^{f} A^{c} \partial_{f} \Gamma^{a}{}_{ce}$$
$$- \frac{5}{7} \left(\frac{3}{5} A^{j} A^{g} A^{c} A^{i} \partial_{jg} R^{a}{}_{cie} - \frac{1}{15} A^{c} A^{j} R^{h}{}_{cje} A^{i} A^{g} R^{a}{}_{igh} - \frac{1}{15} A^{i} A^{j} R^{a}{}_{ijh} A^{c} A^{g} R^{h}{}_{cge} \right) A^{f} A^{b} \partial_{f} \Gamma^{e}{}_{bd} - \frac{5}{28} A^{f} A^{b} A^{i} \partial_{f} R^{e}{}_{bid} A^{g} A^{c} A^{h} \partial_{g} R^{a}{}_{che}$$
$$- \frac{5}{28} A^{g} A^{c} A^{i} \partial_{g} R^{a}{}_{cie} A^{f} A^{b} A^{h} \partial_{f} R^{e}{}_{bhd} - \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{b} A^{i} \partial_{jf} R^{e}{}_{bid} - \frac{1}{15} A^{b} A^{j} R^{h}{}_{bjd} A^{i} A^{f} R^{e}{}_{ifh} - \frac{1}{15} A^{i} A^{j} R^{e}{}_{ijh} A^{b} A^{f} R^{h}{}_{bfd} \right) A^{g} A^{c} \partial_{g} \Gamma^{a}{}_{ce}$$
$$- \frac{5}{28} A^{h} A^{c} A^{i} \partial_{h} R^{a}{}_{cie} A^{f} A^{b} A^{g} \partial_{f} R^{e}{}_{bgd} - \frac{5}{7} \left(\frac{3}{5} A^{j} A^{f} A^{b} A^{i} \partial_{jf} R^{e}{}_{bid} - \frac{1}{15} A^{b} A^{j} R^{g}{}_{bjd} A^{i} A^{f} R^{e}{}_{ifg} - \frac{1}{15} A^{i} A^{j} R^{e}{}_{ijg} A^{b} A^{f} R^{g}{}_{bfd} \right) A^{h} A^{c} \partial_{h} \Gamma^{a}{}_{ce}$$
$$- \frac{5}{7} A^{i} \left(\frac{3}{5} A^{j} A^{f} A^{b} A^{h} \partial_{jf} R^{e}{}_{bhd} - \frac{1}{15} A^{b} A^{j} R^{g}{}_{bjd} A^{h} A^{f} R^{e}{}_{hfg} - \frac{1}{15} A^{h} A^{j} R^{e}{}_{hjg} A^{b} A^{f} R^{g}{}_{bfd} \right) A^{c} \partial_{i} \Gamma^{a}{}_{ce}$$

$$
\begin{aligned}
\texttt{dGamma05.305} :=\ & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{21} A^b A^i R^e{}_{bid} \left(\frac{3}{5} A^j A^f A^c A^h \partial_{jf} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjg} A^c A^f R^g{}_{cfe} \right) \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^b A^h R^e{}_{bhd} - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^b A^g R^e{}_{bgd} - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} \\
& - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^c A^f R^a{}_{cfe} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^b A^f R^e{}_{bfd} - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} \\
& - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^c A^g R^a{}_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^c A^h R^a{}_{che} \\
& - \frac{5}{21} A^c A^i R^a{}_{cie} \left(\frac{3}{5} A^j A^f A^b A^h \partial_{jf} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjg} A^b A^f R^g{}_{bfd} \right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dGamma05.306} :=\ & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{21} A^b A^i R^e{}_{bid} \left(\frac{3}{5} A^j A^f A^c A^h \partial_{jf} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjg} A^c A^f R^g{}_{cfe} \right) \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^b A^h R^e{}_{bhd} - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^b A^g R^e{}_{bgd} - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} \\
& - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^c A^f R^a{}_{cfe} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^b A^f R^e{}_{bfd} - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} \\
& - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^c A^g R^a{}_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^c A^h R^a{}_{che} \\
& - \frac{5}{21} A^c A^i R^a{}_{cie} \left(\frac{3}{5} A^j A^f A^b A^h \partial_{jf} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjg} A^b A^f R^g{}_{bfd} \right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dGamma05.307} := {} & \frac{5}{7} A^{i} A^{h} A^{g} A^{f} A^{b} A^{c} \partial_{ihgf} R^{a}{}_{bcd} - \frac{1}{7} A^{b} A^{i} R^{e}{}_{bid} A^{j} A^{f} A^{c} A^{h} \partial_{jf} R^{a}{}_{che} + \frac{1}{63} A^{b} A^{i} R^{e}{}_{bid} A^{c} A^{j} R^{g}{}_{cje} A^{h} A^{f} R^{a}{}_{hfg} \\
& + \frac{1}{63} A^{b} A^{i} R^{e}{}_{bid} A^{h} A^{j} R^{a}{}_{hjg} A^{c} A^{f} R^{g}{}_{cfe} - \frac{1}{7} A^{j} A^{f} A^{c} A^{i} \partial_{jf} R^{a}{}_{cie} A^{b} A^{h} R^{e}{}_{bhd} + \frac{1}{63} A^{c} A^{j} R^{g}{}_{cje} A^{i} A^{f} R^{a}{}_{ifg} A^{b} A^{h} R^{e}{}_{bhd} \\
& + \frac{1}{63} A^{i} A^{j} R^{a}{}_{ijg} A^{c} A^{f} R^{g}{}_{cfe} A^{b} A^{h} R^{e}{}_{bhd} - \frac{5}{28} A^{h} A^{b} A^{i} \partial_{h} R^{e}{}_{bid} A^{f} A^{c} A^{g} \partial_{f} R^{a}{}_{cge} - \frac{1}{7} A^{j} A^{f} A^{c} A^{i} \partial_{jf} R^{a}{}_{cie} A^{b} A^{g} R^{e}{}_{bgd} \\
& + \frac{1}{63} A^{c} A^{j} R^{h}{}_{cje} A^{i} A^{f} R^{a}{}_{ifh} A^{b} A^{g} R^{e}{}_{bgd} + \frac{1}{63} A^{i} A^{j} R^{a}{}_{ijh} A^{c} A^{f} R^{h}{}_{cfe} A^{b} A^{g} R^{e}{}_{bgd} - \frac{5}{28} A^{g} A^{b} A^{i} \partial_{g} R^{e}{}_{bid} A^{f} A^{c} A^{h} \partial_{f} R^{a}{}_{che} \\
& - \frac{5}{28} A^{f} A^{c} A^{i} \partial_{f} R^{a}{}_{cie} A^{g} A^{b} A^{h} \partial_{g} R^{e}{}_{bhd} - \frac{1}{7} A^{j} A^{g} A^{b} A^{i} \partial_{jg} R^{e}{}_{bid} A^{c} A^{f} R^{a}{}_{cfe} + \frac{1}{63} A^{b} A^{j} R^{h}{}_{bjd} A^{i} A^{g} R^{e}{}_{igh} A^{c} A^{f} R^{a}{}_{cfe} \\
& + \frac{1}{63} A^{i} A^{j} R^{e}{}_{ijh} A^{b} A^{g} R^{h}{}_{bgd} A^{c} A^{f} R^{a}{}_{cfe} - \frac{1}{7} A^{j} A^{g} A^{c} A^{i} \partial_{jg} R^{a}{}_{cie} A^{b} A^{f} R^{e}{}_{bfd} + \frac{1}{63} A^{c} A^{j} R^{h}{}_{cje} A^{i} A^{g} R^{a}{}_{igh} A^{b} A^{f} R^{e}{}_{bfd} \\
& + \frac{1}{63} A^{i} A^{j} R^{a}{}_{ijh} A^{c} A^{g} R^{h}{}_{cge} A^{b} A^{f} R^{e}{}_{bfd} - \frac{5}{28} A^{f} A^{b} A^{i} \partial_{f} R^{e}{}_{bid} A^{g} A^{c} A^{h} \partial_{g} R^{a}{}_{che} - \frac{5}{28} A^{g} A^{c} A^{i} \partial_{g} R^{a}{}_{cie} A^{f} A^{b} A^{h} \partial_{f} R^{e}{}_{bhd} \\
& - \frac{1}{7} A^{j} A^{f} A^{b} A^{i} \partial_{jf} R^{e}{}_{bid} A^{c} A^{g} R^{a}{}_{cge} + \frac{1}{63} A^{b} A^{j} R^{h}{}_{bjd} A^{i} A^{f} R^{e}{}_{ifh} A^{c} A^{g} R^{a}{}_{cge} + \frac{1}{63} A^{i} A^{j} R^{e}{}_{ijh} A^{b} A^{f} R^{h}{}_{bfd} A^{c} A^{g} R^{a}{}_{cge} \\
& - \frac{5}{28} A^{h} A^{c} A^{i} \partial_{h} R^{a}{}_{cie} A^{f} A^{b} A^{g} \partial_{f} R^{e}{}_{bgd} - \frac{1}{7} A^{j} A^{f} A^{b} A^{i} \partial_{jf} R^{e}{}_{bid} A^{c} A^{h} R^{a}{}_{che} + \frac{1}{63} A^{b} A^{j} R^{g}{}_{bjd} A^{i} A^{f} R^{e}{}_{ifg} A^{c} A^{h} R^{a}{}_{che} \\
& + \frac{1}{63} A^{i} A^{j} R^{e}{}_{ijg} A^{b} A^{f} R^{g}{}_{bfd} A^{c} A^{h} R^{a}{}_{che} - \frac{1}{7} A^{c} A^{i} R^{a}{}_{cie} A^{j} A^{f} A^{b} A^{h} \partial_{jf} R^{e}{}_{bhd} \\
& + \frac{1}{63} A^{c} A^{i} R^{a}{}_{cie} A^{b} A^{j} R^{g}{}_{bjd} A^{h} A^{f} R^{e}{}_{hfg} + \frac{1}{63} A^{c} A^{i} R^{a}{}_{cie} A^{h} A^{j} R^{e}{}_{hjg} A^{b} A^{f} R^{g}{}_{bfd}
\end{aligned}
$$

```
# note:
# canonicalise must not be used here because it may make changes like
#    R^{a}_{b c d} -> - R_{b}^{a}_{c d}
# these changes can not be applied inside a \partial, must defer use
# of canocialise until we have \nabla acting on curvatures

sort_product   (dGamma03) # cdb(dGamma03.401,dGamma03)
rename_dummies (dGamma03) # cdb(dGamma03.402,dGamma03)
# canonicalise   (dGamma03) # cdb(dGamma03.403,dGamma03)

sort_product   (dGamma04) # cdb(dGamma04.401,dGamma04)
rename_dummies (dGamma04) # cdb(dGamma04.402,dGamma04)
# canonicalise   (dGamma04) # cdb(dGamma04.403,dGamma04)

sort_product   (dGamma05) # cdb(dGamma05.401,dGamma05)
rename_dummies (dGamma05) # cdb(dGamma05.402,dGamma05)
# canonicalise   (dGamma05) # cdb(dGamma05.403,dGamma05)
```

$$\texttt{dGamma03.401} := \frac{3}{5} A^b A^c A^f A^g \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^c A^f A^g R^a{}_{cfe} R^e{}_{bgd} - \frac{1}{15} A^b A^c A^f A^g R^a{}_{cge} R^e{}_{bfd}$$

$$\texttt{dGamma03.402} := \frac{3}{5} A^b A^c A^e A^f \partial_{fe} R^a{}_{bcd} - \frac{1}{15} A^b A^c A^e A^f R^a{}_{ceg} R^g{}_{bfd} - \frac{1}{15} A^b A^c A^e A^f R^a{}_{cfg} R^g{}_{bed}$$

$$\begin{aligned}\texttt{dGamma04.401} := {} & \frac{2}{3} A^b A^c A^f A^g A^h \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^c A^f A^g A^h R^e{}_{bhd} \partial_f R^a{}_{cge} - \frac{1}{9} A^b A^c A^f A^g A^h R^e{}_{bgd} \partial_f R^a{}_{che} - \frac{1}{9} A^b A^c A^f A^g A^h R^a{}_{cfe} \partial_g R^e{}_{bhd} \\ & - \frac{1}{9} A^b A^c A^f A^g A^h R^e{}_{bfd} \partial_g R^a{}_{che} - \frac{1}{9} A^b A^c A^f A^g A^h R^a{}_{cge} \partial_f R^e{}_{bhd} - \frac{1}{9} A^b A^c A^f A^g A^h R^a{}_{che} \partial_f R^e{}_{bgd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.402} := {} & \frac{2}{3} A^b A^c A^e A^f A^g \partial_{gfe} R^a{}_{bcd} - \frac{1}{9} A^b A^c A^e A^f A^g R^h{}_{bgd} \partial_e R^a{}_{cfh} - \frac{1}{9} A^b A^c A^e A^f A^g R^h{}_{bfd} \partial_e R^a{}_{cgh} - \frac{1}{9} A^b A^c A^e A^f A^g R^a{}_{ceh} \partial_f R^h{}_{bgd} \\ & - \frac{1}{9} A^b A^c A^e A^f A^g R^h{}_{bed} \partial_f R^a{}_{cgh} - \frac{1}{9} A^b A^c A^e A^f A^g R^a{}_{cfh} \partial_e R^h{}_{bgd} - \frac{1}{9} A^b A^c A^e A^f A^g R^a{}_{cgh} \partial_e R^h{}_{bfd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma05.401} := {} & \frac{5}{7} A^b A^c A^f A^g A^h A^i \partial_{ihgf} R^a{}_{bcd} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^e{}_{bid} \partial_{jf} R^a{}_{che} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{hfg} R^e{}_{bid} R^g{}_{cje} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{hjg} R^e{}_{bid} R^g{}_{cfe} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^e{}_{bhd} \partial_{jf} R^a{}_{cie} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{ifg} R^e{}_{bhd} R^g{}_{cje} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{ijg} R^e{}_{bhd} R^g{}_{cfe} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a{}_{cge} \partial_h R^e{}_{bid} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^e{}_{bgd} \partial_{jf} R^a{}_{cie} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{ifh} R^e{}_{bgd} R^h{}_{cje} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{ijh} R^e{}_{bgd} R^h{}_{cfe} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a{}_{che} \partial_g R^e{}_{bid} \\ & - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a{}_{cie} \partial_g R^e{}_{bhd} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^a{}_{cfe} \partial_{jg} R^e{}_{bid} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{cfe} R^e{}_{igh} R^h{}_{bjd} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{cfe} R^e{}_{ijh} R^h{}_{bgd} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^e{}_{bfd} \partial_{jg} R^a{}_{cie} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{igh} R^e{}_{bfd} R^h{}_{cje} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{ijh} R^e{}_{bfd} R^h{}_{cge} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e{}_{bid} \partial_g R^a{}_{che} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e{}_{bhd} \partial_g R^a{}_{cie} \\ & - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^a{}_{cge} \partial_{jf} R^e{}_{bid} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{cge} R^e{}_{ifh} R^h{}_{bjd} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a{}_{cge} R^e{}_{ijh} R^h{}_{bfd} \\ & - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e{}_{bgd} \partial_h R^a{}_{cie} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^a{}_{che} \partial_{jf} R^e{}_{bid} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{che} R^e{}_{ifg} R^g{}_{bjd} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{che} R^e{}_{ijg} R^g{}_{bfd} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^a{}_{cie} \partial_{jf} R^e{}_{bhd} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{cie} R^e{}_{hfg} R^g{}_{bjd} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a{}_{cie} R^e{}_{hjg} R^g{}_{bfd}\end{aligned}$$

$$\texttt{dGamma05.402} := \frac{5}{7} A^b A^c A^e A^f A^g A^h \partial_{hgf} R^a{}_{bcd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i{}_{bgd} \partial_{he} R^a{}_{cfi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{fei} R^j{}_{bgd} R^i{}_{chj}$$
$$+ \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{fhi} R^j{}_{bgd} R^i{}_{cej} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^i{}_{bfd} \partial_{he} R^a{}_{cgi} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{gei} R^j{}_{bfd} R^i{}_{chj}$$
$$+ \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{ghi} R^j{}_{bfd} R^i{}_{cej} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a{}_{cfi} \partial_g R^i{}_{bhd} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a{}_{cgi} \partial_f R^i{}_{bhd}$$
$$- \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a{}_{chi} \partial_f R^i{}_{bgd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a{}_{cei} \partial_{hf} R^i{}_{bgd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cei} R^i{}_{gfj} R^j{}_{bhd}$$
$$+ \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cei} R^i{}_{ghj} R^j{}_{bfd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i{}_{bed} \partial_{hf} R^a{}_{cgi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{gfi} R^j{}_{bed} R^i{}_{chj}$$
$$+ \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{ghi} R^j{}_{bed} R^i{}_{cfj} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i{}_{bhd} \partial_f R^a{}_{cgi}$$
$$- \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i{}_{bgd} \partial_f R^a{}_{chi} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^a{}_{cfi} \partial_{he} R^i{}_{bgd} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cfi} R^i{}_{gej} R^j{}_{bhd}$$
$$+ \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cfi} R^i{}_{ghj} R^j{}_{bed} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i{}_{bfd} \partial_g R^a{}_{chi} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a{}_{cgi} \partial_{he} R^i{}_{bfd}$$
$$+ \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cgi} R^i{}_{fej} R^j{}_{bhd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cgi} R^i{}_{fhj} R^j{}_{bed}$$

```
import cdblib

cdblib.create ('dGamma.json')

cdblib.put ('dGamma01',dGamma01,'dGamma.json')
cdblib.put ('dGamma02',dGamma02,'dGamma.json')
cdblib.put ('dGamma03',dGamma03,'dGamma.json')
cdblib.put ('dGamma04',dGamma04,'dGamma.json')
cdblib.put ('dGamma05',dGamma05,'dGamma.json')
```

Stage 4: Reformatting

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                                -> A001^{a}                $)
    substitute (obj,$ x^{a}                                -> A002^{a}                $)
    substitute (obj,$ g^{a b}                              -> A003^{a b}              $)
    substitute (obj,$ \partial_{e f g h}{R^{a}_{b c d}}  -> A008^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}{R^{a}_{b c d}}    -> A007^{a}_{b c d e f g}   $)
    substitute (obj,$ \partial_{e f}{R^{a}_{b c d}}      -> A006^{a}_{b c d e f}     $)
    substitute (obj,$ \partial_{e}{R^{a}_{b c d}}        -> A005^{a}_{b c d e}       $)
    substitute (obj,$ R^{a}_{b c d}                        -> A004^{a}_{b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                  -> A^{a}                           $)
    substitute (obj,$ A002^{a}                  -> x^{a}                           $)
    substitute (obj,$ A003^{a b}                -> g^{a b}                         $)
    substitute (obj,$ A004^{a}_{b c d}          -> R^{a}_{b c d}                   $)
    substitute (obj,$ A005^{a}_{b c d e}        -> \partial_{e}{R^{a}_{b c d}}       $)
    substitute (obj,$ A006^{a}_{b c d e f}      -> \partial_{e f}{R^{a}_{b c d}}     $)
    substitute (obj,$ A007^{a}_{b c d e f g}    -> \partial_{e f g}{R^{a}_{b c d}}   $)
    substitute (obj,$ A008^{a}_{b c d e f g h}  -> \partial_{e f g h}{R^{a}_{b c d}} $)

    return obj

def reformat (obj,scale):
    bah  = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute     (tmp)
    tmp = product_sort (tmp)
    rename_dummies (tmp)
    factor_out     (tmp,$A^{a?}$)
    return tmp

def get_term (obj,n):

    A^{a}::Weight(label=numA).
```

```
    foo := @(obj).
    bah  = Ex("numA = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

Gterm01 := @(dGamma01).
Gterm02 := @(dGamma02).
Gterm03 := @(dGamma03).
Gterm04 := @(dGamma04).
Gterm05 := @(dGamma05).

scaled1 = reformat (Gterm01,   3)   # cdb (scaled1.002,scaled1)
scaled2 = reformat (Gterm02,   6)   # cdb (scaled2.002,scaled2)
scaled3 = reformat (Gterm03,  15)   # cdb (scaled3.002,scaled3)
scaled4 = reformat (Gterm04,   9)   # cdb (scaled4.002,scaled4)
scaled5 = reformat (Gterm05, 252)   # cdb (scaled5.002,scaled5)
```

Symmetrised partial derivatives of the connection

$$3A^bA^c\Gamma^a{}_{d(b,c)} = A^bA^cR^a{}_{bcd}$$

$$6A^bA^cA^e\Gamma^a{}_{d(b,ce)} = 3\,A^bA^cA^e\partial_eR^a{}_{bcd}$$

$$15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} = A^bA^cA^eA^f\left(9\,\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right)$$

$$9A^bA^cA^eA^fA^g\Gamma^a{}_{d(b,cefg)} = A^bA^cA^eA^fA^g\left(6\,\partial_{gfe}R^a{}_{bcd} - R^h{}_{bgd}\partial_eR^a{}_{cfh} - R^h{}_{bfd}\partial_eR^a{}_{cgh} - R^a{}_{ceh}\partial_fR^h{}_{bgd} - R^h{}_{bed}\partial_fR^a{}_{cgh} - R^a{}_{cfh}\partial_eR^h{}_{bgd} - R^a{}_{cgh}\partial_eR^h{}_{bfd}\right)$$

$$\begin{aligned}252A^bA^cA^eA^fA^gA^h\Gamma^a{}_{d(b,cefgh)} = A^bA^cA^eA^fA^gA^h\big(&180\,\partial_{hgfe}R^a{}_{bcd} - 36\,R^i{}_{bgd}\partial_{he}R^a{}_{cfi} + 4\,R^a{}_{fei}R^i{}_{chj}R^j{}_{bgd} + 4\,R^a{}_{fhi}R^i{}_{cej}R^j{}_{bgd} - 72\,R^i{}_{bfd}\partial_{he}R^a{}_{cgi}\\ &+ 8\,R^a{}_{gei}R^i{}_{chj}R^j{}_{bfd} + 8\,R^a{}_{ghi}R^i{}_{cej}R^j{}_{bfd} - 45\,\partial_eR^a{}_{cfi}\partial_gR^i{}_{bhd} - 45\,\partial_eR^a{}_{cgi}\partial_fR^i{}_{bhd} - 45\,\partial_eR^a{}_{chi}\partial_fR^i{}_{bgd} - 36\,R^a{}_{cei}\partial_{hf}R^i{}_{bgd}\\ &+ 4\,R^a{}_{cei}R^i{}_{gfj}R^j{}_{bhd} + 4\,R^a{}_{cei}R^i{}_{ghj}R^j{}_{bfd} - 36\,R^i{}_{bed}\partial_{hf}R^a{}_{cgi} + 4\,R^a{}_{gfi}R^i{}_{chj}R^j{}_{bed} + 4\,R^a{}_{ghi}R^i{}_{cfj}R^j{}_{bed} - 45\,\partial_fR^a{}_{cgi}\partial_eR^i{}_{bhd}\\ &- 45\,\partial_fR^a{}_{chi}\partial_eR^i{}_{bgd} - 72\,R^a{}_{cfi}\partial_{he}R^i{}_{bgd} + 8\,R^a{}_{cfi}R^i{}_{gej}R^j{}_{bhd} + 8\,R^a{}_{cfi}R^i{}_{ghj}R^j{}_{bed} - 45\,\partial_gR^a{}_{chi}\partial_eR^i{}_{bfd} - 36\,R^a{}_{cgi}\partial_{he}R^i{}_{bfd}\\ &+ 4\,R^a{}_{cgi}R^i{}_{fej}R^j{}_{bhd} + 4\,R^a{}_{cgi}R^i{}_{fhj}R^j{}_{bed}\big)\end{aligned}$$

```
substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dGamma.export')

# 6th order dGamma, scaled
cdblib.put ('dGamma61scaled',scaled1,'dGamma.export')
cdblib.put ('dGamma62scaled',scaled2,'dGamma.export')
cdblib.put ('dGamma63scaled',scaled3,'dGamma.export')
cdblib.put ('dGamma64scaled',scaled4,'dGamma.export')
cdblib.put ('dGamma65scaled',scaled5,'dGamma.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```