19 Jan 2024

Notes

The convention for the curvature used in these notes conforms to that of Misner-Thorne-Wheeler (MTW, eq. 11.12) , namely

$$V^a{}_{;bc} - V^a{}_{;cb} = -R^a{}_{dbc}V^d$$

Also, note the following shorthand for mixed covariant derivatives

$$\begin{aligned}\nabla_a\left(\nabla_b\right) &= \nabla_{ab}\\ \nabla_a\left(\nabla_b\left(\nabla_c\right)\right) &= \nabla_{abc}\\ \nabla_a\left(\nabla_b\left(\nabla_c\left(\nabla_d\right)\right)\right) &= \nabla_{abcd}\end{aligned}$$

and so on.

See for example the Python function `combine_nabla` in `cadabra/dRabcd.tex`.

In terms of ∇ the above MTW definition of $R^a{}_{bcd}$ can written as

$$\left(\nabla_{cb} - \nabla_{bc}\right)V^a = -R^a{}_{dbc}V^d$$

Symmetrisation

In the following pages there will be frequent constructions of the form

$$\begin{aligned}3A^bA^c\Gamma^a{}_{d(b,c)} &= A^bA^cR^a{}_{bcd}\\ 6A^bA^cA^e\Gamma^a{}_{d(b,ce)} &= 3\,A^bA^cA^e\partial_e R^a{}_{bcd}\\ 15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} &= A^bA^cA^eA^f\left(9\,\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right)\end{aligned}$$

The vector A^a has no special meaning. Its purpose is to indicate that the associciated tensor is symmetric over a selection of its indices. If the A^a were not included then the right hand side would either need to be spelt out in full or some other device would be needed to denote the symmetries. The symmetrisation brackets are included on the left hand side though they are redundant (in the presence of the A^a).

The metric in RNC

$$g_{ab}(x) = g_{ab} - \frac{1}{3}\, x^c x^d R_{acbd} - \frac{1}{6}\, x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}\, x^c x^d x^e x^f \left(8\, g^{gh} R_{acdg} R_{befh} - 9\, \nabla_{cd} R_{aebf}\right)$$
$$+ \frac{1}{90}\, x^c x^d x^e x^f x^g \left(2\, g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2\, g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right) + \mathcal{O}\left(\epsilon^6\right)$$

Curvature expansion of the metric

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{g}_{ab} = g_{ab}$$

$$3\overset{2}{g}_{ab} = -\, x^c x^d R_{acbd}$$

$$6\overset{3}{g}_{ab} = -\, x^c x^d x^e \nabla_c R_{adbe}$$

$$180\overset{4}{g}_{ab} = x^c x^d x^e x^f \left(8\, g^{gh} R_{acdg} R_{befh} - 9\, \nabla_{cd} R_{aebf}\right)$$

$$90\overset{5}{g}_{ab} = x^c x^d x^e x^f x^g \left(2\, g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2\, g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right)$$

The inverse metric in RNC

$$g^{ab}(x) = g^{ab} + \frac{1}{3}\, x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}\, x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \frac{1}{60}\, x^c x^d x^e x^f \left(4\, g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3\, g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right)$$
$$+ \frac{1}{90}\, x^c x^d x^e x^f x^g \left(3\, g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3\, g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right) + \mathcal{O}\left(\epsilon^6\right)$$

Curvature expansion of the inverse metric

$$g^{ab}(x) = \overset{0}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} + \overset{5}{g}{}^{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{g}{}^{ab} = g^{ab}$$

$$3\overset{2}{g}{}^{ab} = x^c x^d g^{ae} g^{bf} R_{cedf}$$

$$6\overset{3}{g}{}^{ab} = x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg}$$

$$60\overset{4}{g}{}^{ab} = x^c x^d x^e x^f \left(4\, g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3\, g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right)$$

$$90\overset{5}{g}{}^{ab} = x^c x^d x^e x^f x^g \left(3\, g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3\, g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right)$$

The metric determinant in RNC

$$
\begin{aligned}
-\det g(x) = 1 - \frac{1}{3}\,x^a x^b R_{ab} - \frac{1}{6}\,x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}\,x^a x^b x^c x^d \left(-9\,\nabla_{ab} R_{cd} + 10\,R_{ab} R_{cd} - 2\,g^{ef} g^{gh} R_{aebg} R_{cfdh}\right) \\
+ \frac{1}{90}\,x^a x^b x^c x^d x^e \left(-\,\nabla_{abc} R_{de} + 5\,R_{ab} \nabla_c R_{de} -\; g^{fg} g^{hi} R_{afbh} \nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

The metric Jacobian in RNC

$$
\begin{aligned}
\sqrt{-\det g(x)} = 1 - \frac{1}{6}\,x^a x^b R_{ab} - \frac{1}{12}\,x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}\,x^a x^b x^c x^d \left(-9\,\nabla_{ab} R_{cd} + 5\,R_{ab} R_{cd} - 2\,g^{ef} g^{gh} R_{aebg} R_{cfdh}\right) \\
+ \frac{1}{360}\,x^a x^b x^c x^d x^e \left(-2\,\nabla_{abc} R_{de} + 5\,R_{ab} \nabla_c R_{de} - 2\,g^{fg} g^{hi} R_{afbh} \nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

The log of detg in RNC

$$
\begin{aligned}
\log\left(-\det g(x)\right) = -\frac{1}{3}\,x^a x^b R_{ab} - \frac{1}{6}\,x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}\,x^a x^b x^c x^d \left(-9\,\nabla_{ab} R_{cd} - 2\,g^{ef} g^{gh} R_{aebg} R_{cfdh}\right) \\
+ \frac{1}{90}\,x^a x^b x^c x^d x^e \left(-\,\nabla_{abc} R_{de} -\; g^{fg} g^{hi} R_{afbh} \nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

The connection in RNC

$$
\begin{aligned}
A^aA^b\Gamma^d_{ab} = {} & \frac{2}{3}\,A^aA^bx^cg^{de}R_{acbe} + \frac{1}{12}\,A^aA^bx^cx^e\left(2\,g^{df}\nabla_{a}R_{bcef} + 4\,g^{df}\nabla_{c}R_{aebf} + g^{df}\nabla_{f}R_{acbe}\right)\\
& + \frac{1}{360}\,A^aA^bx^cx^ex^f\left(64\,g^{dg}g^{hi}R_{acbh}R_{egfi} - 32\,g^{dg}g^{hi}R_{aceh}R_{bgfi} - 16\,g^{dg}g^{hi}R_{aceh}R_{bifg} + 18\,g^{dg}\nabla_{ac}R_{befg}\right.\\
& \left. + 18\,g^{dg}\nabla_{ca}R_{befg} + 36\,g^{dg}\nabla_{ce}R_{afbg} - 16\,g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9\,g^{dg}\nabla_{gc}R_{aebf} + 9\,g^{dg}\nabla_{cg}R_{aebf}\right) + \frac{1}{180}\,A^aA^bx^cx^ex^fx^g\left(16\,g^{dh}g^{ij}R_{acbi}\nabla_{e}R_{fhgj}\right.\\
& + 6\,g^{dh}g^{ij}R_{chei}\nabla_{a}R_{bfgj} + 16\,g^{dh}g^{ij}R_{chei}\nabla_{f}R_{agbj} + 5\,g^{dh}g^{ij}R_{chei}\nabla_{j}R_{afbg} - 8\,g^{dh}g^{ij}R_{ahci}\nabla_{e}R_{bfgj} - 4\,g^{dh}g^{ij}R_{aich}\nabla_{e}R_{bfgj}\\
& - 4\,g^{dh}g^{ij}R_{acei}\nabla_{b}R_{fhgj} - 8\,g^{dh}g^{ij}R_{acei}\nabla_{f}R_{bhgj} - 4\,g^{dh}g^{ij}R_{acei}\nabla_{f}R_{bjgh} + 2\,g^{dh}\nabla_{ace}R_{bfgh} + 2\,g^{dh}\nabla_{cae}R_{bfgh} + 2\,g^{dh}\nabla_{cea}R_{bfgh}\\
& \left. + 4\,g^{dh}\nabla_{cef}R_{agbh} - 4\,g^{dh}g^{ij}R_{achi}\nabla_{e}R_{bfgj} - 4\,g^{dh}g^{ij}R_{acei}\nabla_{h}R_{bfgj} - 4\,g^{dh}g^{ij}R_{acei}\nabla_{f}R_{bghj} + g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + g^{dh}\nabla_{ceh}R_{afbg}\right)
\end{aligned}
$$

$$
\begin{aligned}
360A^aA^b\Gamma^d_{ab} = {} & 240\,A^aA^bx^cg^{de}R_{acbe} + 30\,A^aA^bx^cx^e\left(2\,g^{df}\nabla_{a}R_{bcef} + 4\,g^{df}\nabla_{c}R_{aebf} + g^{df}\nabla_{f}R_{acbe}\right)\\
& + A^aA^bx^cx^ex^f\left(64\,g^{dg}g^{hi}R_{acbh}R_{egfi} - 32\,g^{dg}g^{hi}R_{aceh}R_{bgfi} - 16\,g^{dg}g^{hi}R_{aceh}R_{bifg} + 18\,g^{dg}\nabla_{ac}R_{befg} + 18\,g^{dg}\nabla_{ca}R_{befg} + 36\,g^{dg}\nabla_{ce}R_{afbg}\right.\\
& \left. - 16\,g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9\,g^{dg}\nabla_{gc}R_{aebf} + 9\,g^{dg}\nabla_{cg}R_{aebf}\right) + 2\,A^aA^bx^cx^ex^fx^g\left(16\,g^{dh}g^{ij}R_{acbi}\nabla_{e}R_{fhgj} + 6\,g^{dh}g^{ij}R_{chei}\nabla_{a}R_{bfgj}\right.\\
& + 16\,g^{dh}g^{ij}R_{chei}\nabla_{f}R_{agbj} + 5\,g^{dh}g^{ij}R_{chei}\nabla_{j}R_{afbg} - 8\,g^{dh}g^{ij}R_{ahci}\nabla_{e}R_{bfgj} - 4\,g^{dh}g^{ij}R_{aich}\nabla_{e}R_{bfgj} - 4\,g^{dh}g^{ij}R_{acei}\nabla_{b}R_{fhgj}\\
& - 8\,g^{dh}g^{ij}R_{acei}\nabla_{f}R_{bhgj} - 4\,g^{dh}g^{ij}R_{acei}\nabla_{f}R_{bjgh} + 2\,g^{dh}\nabla_{ace}R_{bfgh} + 2\,g^{dh}\nabla_{cae}R_{bfgh} + 2\,g^{dh}\nabla_{cea}R_{bfgh} + 4\,g^{dh}\nabla_{cef}R_{agbh}\\
& \left. - 4\,g^{dh}g^{ij}R_{achi}\nabla_{e}R_{bfgj} - 4\,g^{dh}g^{ij}R_{acei}\nabla_{h}R_{bfgj} - 4\,g^{dh}g^{ij}R_{acei}\nabla_{f}R_{bghj} + g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + g^{dh}\nabla_{ceh}R_{afbg}\right)
\end{aligned}
$$

Curvature expansion of the connection

$$A^a A^b \Gamma^d_{ab} = A^a A^b \overset{2}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{3}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{4}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{5}{\Gamma}{}^d{}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$3A^a A^b \overset{2}{\Gamma}{}^d_{ab} = 2\, A^a A^b x^c g^{de} R_{acbe}$$

$$12A^a A^b \overset{3}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e \left(2\, g^{df} \nabla_a R_{bcef} + 4\, g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}\right)$$

$$\begin{aligned}360A^a A^b \overset{4}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f \big(&64\, g^{dg} g^{hi} R_{acbh} R_{egfi} - 32\, g^{dg} g^{hi} R_{aceh} R_{bgfi} - 16\, g^{dg} g^{hi} R_{aceh} R_{bifg} + 18\, g^{dg} \nabla_{ac} R_{befg} + 18\, g^{dg} \nabla_{ca} R_{befg} + 36\, g^{dg} \nabla_{ce} R_{afbg} \\ &- 16\, g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9\, g^{dg} \nabla_{gc} R_{aebf} + 9\, g^{dg} \nabla_{cg} R_{aebf}\big)\end{aligned}$$

$$\begin{aligned}180A^a A^b \overset{5}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f x^g \big(&16\, g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6\, g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16\, g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5\, g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} - 8\, g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} \\ &- 4\, g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4\, g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8\, g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4\, g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2\, g^{dh} \nabla_{ace} R_{bfgh} + 2\, g^{dh} \nabla_{cae} R_{bfgh} \\ &+ 2\, g^{dh} \nabla_{cea} R_{bfgh} + 4\, g^{dh} \nabla_{cef} R_{agbh} - 4\, g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4\, g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} - 4\, g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} + g^{dh} \nabla_{hce} R_{afbg} \\ &+ g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg}\big)\end{aligned}$$

Symmetrised partial derivatives of the connection

$$
\begin{aligned}
3A^bA^c\Gamma^a{}_{d(b,c)} &= A^bA^cR^a{}_{bcd} \\
6A^bA^cA^e\Gamma^a{}_{d(b,ce)} &= 3\,A^bA^cA^e\partial_e R^a{}_{bcd} \\
15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} &= A^bA^cA^eA^f\left(9\,\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right) \\
9A^bA^cA^eA^fA^g\Gamma^a{}_{d(b,cefg)} &= A^bA^cA^eA^fA^g\big(6\,\partial_{gfe}R^a{}_{bcd} - R^h{}_{bgd}\partial_e R^a{}_{cfh} - R^h{}_{bfd}\partial_e R^a{}_{cgh} - R^a{}_{ceh}\partial_f R^h{}_{bgd} - R^h{}_{bed}\partial_f R^a{}_{cgh} - R^a{}_{cfh}\partial_e R^h{}_{bgd} - R^a{}_{cgh}\partial_e R^h{}_{bfd}\big) \\
252A^bA^cA^eA^fA^gA^h\Gamma^a{}_{d(b,cefgh)} &= A^bA^cA^eA^fA^gA^h\big(180\,\partial_{hgfe}R^a{}_{bcd} - 36\,R^i{}_{bgd}\partial_{he}R^a{}_{cfi} + 4\,R^a{}_{fei}R^i{}_{chj}R^j{}_{bgd} + 4\,R^a{}_{fhi}R^i{}_{cej}R^j{}_{bgd} - 72\,R^i{}_{bfd}\partial_{he}R^a{}_{cgi} \\
&\quad + 8\,R^a{}_{gei}R^i{}_{chj}R^j{}_{bfd} + 8\,R^a{}_{ghi}R^i{}_{cej}R^j{}_{bfd} - 45\,\partial_e R^a{}_{cfi}\partial_g R^i{}_{bhd} - 45\,\partial_e R^a{}_{cgi}\partial_f R^i{}_{bhd} - 45\,\partial_e R^a{}_{chi}\partial_f R^i{}_{bgd} - 36\,R^a{}_{cei}\partial_{hf}R^i{}_{bgd} \\
&\quad + 4\,R^a{}_{cei}R^i{}_{gfj}R^j{}_{bhd} + 4\,R^a{}_{cei}R^i{}_{ghj}R^j{}_{bfd} - 36\,R^i{}_{bed}\partial_{hf}R^a{}_{cgi} + 4\,R^a{}_{gfi}R^i{}_{chj}R^j{}_{bed} + 4\,R^a{}_{ghi}R^i{}_{cfj}R^j{}_{bed} - 45\,\partial_f R^a{}_{cgi}\partial_e R^i{}_{bhd} \\
&\quad - 45\,\partial_f R^a{}_{chi}\partial_e R^i{}_{bgd} - 72\,R^a{}_{cfi}\partial_{he}R^i{}_{bgd} + 8\,R^a{}_{cfi}R^i{}_{gej}R^j{}_{bhd} + 8\,R^a{}_{cfi}R^i{}_{ghj}R^j{}_{bed} - 45\,\partial_g R^a{}_{chi}\partial_e R^i{}_{bfd} - 36\,R^a{}_{cgi}\partial_{he}R^i{}_{bfd} \\
&\quad + 4\,R^a{}_{cgi}R^i{}_{fej}R^j{}_{bhd} + 4\,R^a{}_{cgi}R^i{}_{fhj}R^j{}_{bed}\big)
\end{aligned}
$$

Symmetrised partial derivatives of $R^{a}{}_{bcd}$

$$\begin{aligned}
A^c A^d A^e R^a{}_{cdb,e} &= A^c A^d A^e g^{af} \nabla_{c} R_{bdef} \\
A^c A^d A^e A^f R^a{}_{cdb,ef} &= A^c A^d A^e A^f g^{ag} \nabla_{cd} R_{befg} \\
-2 A^c A^d A^e A^f A^g R^a{}_{cdb,efg} &= A^c A^d A^e A^f A^g \left(g^{ah} g^{ij} R_{bcdi} \nabla_{e} R_{fhgj} - g^{ah} g^{ij} R_{chdi} \nabla_{e} R_{bfgj} - 2\, g^{ah} \nabla_{cde} R_{bfgh} \right) \\
-5 A^c A^d A^e A^f A^g A^h R^a{}_{cdb,efgh} &= A^c A^d A^e A^f A^g A^h \left(7\, g^{ai} g^{jk} R_{bcdj} \nabla_{ef} R_{gihk} - 7\, g^{ai} g^{jk} R_{cidj} \nabla_{ef} R_{bghk} - 5\, g^{ai} \nabla_{cdef} R_{bghi} \right) \\
-3 A^c A^d A^e A^f A^g A^h A^i R^a{}_{cdb,efghi} &= A^c A^d A^e A^f A^g A^h A^i \left(6\, g^{aj} g^{kl} \nabla_{c} R_{bdek} \nabla_{fg} R_{hjil} - 6\, g^{aj} g^{kl} \nabla_{c} R_{djek} \nabla_{fg} R_{bhil} + 8\, g^{aj} g^{kl} R_{bcdk} \nabla_{efg} R_{hjil} \right. \\
&\quad - 8\, g^{aj} g^{kl} R_{cjdk} \nabla_{efg} R_{bhil} - g^{aj} g^{kl} g^{mn} R_{bcdk} R_{elfm} \nabla_{g} R_{hjin} - 3\, g^{aj} g^{kl} g^{mn} R_{cjdk} R_{elfm} \nabla_{g} R_{bhin} \\
&\quad \left. + 4\, g^{aj} g^{kl} g^{mn} R_{bcdk} R_{ejfm} \nabla_{g} R_{hlin} - 3\, g^{aj} \nabla_{cdefg} R_{bhij} \right)
\end{aligned}$$

The generalised connection in RNC

$$A^bA^c\Gamma^a_{bc} = \frac{2}{3}\,A^bA^cx^dg^{ae}R_{bdce} + \frac{1}{12}\,A^bA^cx^dx^e\left(2\,g^{af}\nabla_bR_{cdef} + 4\,g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right)$$
$$+\frac{1}{360}\,A^bA^cx^dx^ex^f\left(64\,g^{ag}g^{hi}R_{bdch}R_{egfi} - 32\,g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16\,g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18\,g^{ag}\nabla_{bd}R_{cefg}\right.$$
$$\left.+18\,g^{ag}\nabla_{db}R_{cefg} + 36\,g^{ag}\nabla_{de}R_{bfcg} - 16\,g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9\,g^{ag}\nabla_{gd}R_{becf} + 9\,g^{ag}\nabla_{dg}R_{becf}\right) + \frac{1}{180}\,A^bA^cx^dx^ex^fx^g\left(16\,g^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj}\right.$$
$$+6\,g^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} + 16\,g^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} + 5\,g^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg} - 8\,g^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - 4\,g^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj}$$
$$-4\,g^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} - 8\,g^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - 4\,g^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh} + 2\,g^{ah}\nabla_{bde}R_{cfgh} + 2\,g^{ah}\nabla_{dbe}R_{cfgh} + 2\,g^{ah}\nabla_{deb}R_{cfgh}$$
$$\left.+4\,g^{ah}\nabla_{def}R_{bgch} - 4\,g^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} - 4\,g^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj} - 4\,g^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\right)$$

$$A^bA^cA^d\Gamma^a_{bcd} = \frac{1}{2}\,A^bA^cA^dx^eg^{af}\nabla_bR_{cedf} + \frac{1}{120}\,A^bA^cA^dx^ex^f\left(64\,g^{ag}g^{hi}R_{bech}R_{dgfi} + 16\,g^{ag}g^{hi}R_{bech}R_{difg} - 16\,g^{ag}g^{hi}R_{befh}R_{cgdi} + 12\,g^{ag}\nabla_{bc}R_{defg}\right.$$
$$\left.+18\,g^{ag}\nabla_{be}R_{cfdg} + 18\,g^{ag}\nabla_{eb}R_{cfdg} + 48\,g^{ag}g^{hi}R_{bech}R_{dfgi} + 3\,g^{ag}\nabla_{gb}R_{cedf} + 3\,g^{ag}\nabla_{bg}R_{cedf}\right)$$
$$+\frac{1}{180}\,A^bA^cA^dx^ex^fx^g\left(32\,g^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} + 48\,g^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj} + 12\,g^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + 18\,g^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj}\right.$$
$$+2\,g^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} + 22\,g^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + 48\,g^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} + 12\,g^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} + 15\,g^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg}$$
$$+5\,g^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} - 12\,g^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - 12\,g^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - 8\,g^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - 12\,g^{ah}g^{ij}R_{befi}\nabla_gR_{chdj}$$
$$+4\,g^{ah}\nabla_{bce}R_{dfgh} + 4\,g^{ah}\nabla_{bec}R_{dfgh} + 6\,g^{ah}\nabla_{bef}R_{cgdh} + 4\,g^{ah}\nabla_{ebc}R_{dfgh} + 6\,g^{ah}\nabla_{ebf}R_{cgdh} + 6\,g^{ah}\nabla_{efb}R_{cgdh} + 16\,g^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj}$$
$$+36\,g^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + 16\,g^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} - 4\,g^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} + 36\,g^{ah}g^{ij}R_{beci}\nabla_fR_{dghj} - 4\,g^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg}$$
$$\left.+g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20\,g^{ah}g^{ij}R_{beci}\nabla_jR_{dfgh} + 10\,g^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}\right)$$

$$A^bA^cA^dA^e\Gamma^a_{bcde} = \frac{1}{15}\,A^bA^cA^dA^ex^f\left(8\,g^{ag}g^{hi}R_{bfch}R_{dgei} + 6\,g^{ag}\nabla_{bc}R_{dfeg}\right)$$
$$+\frac{1}{90}\,A^bA^cA^dA^ex^fx^g\left(64\,g^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + 18\,g^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + 24\,g^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} + 4\,g^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj}\right.$$
$$+44\,g^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + 18\,g^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + 24\,g^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + 10\,g^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg} - 16\,g^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej}$$
$$+6\,g^{ah}\nabla_{bcd}R_{efgh} + 8\,g^{ah}\nabla_{bcf}R_{dgeh} + 8\,g^{ah}\nabla_{bfc}R_{dgeh} + 8\,g^{ah}\nabla_{fbc}R_{dgeh} + 26\,g^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej} + 6\,g^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej}$$
$$\left.+46\,g^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40\,g^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\right)$$

$$A^bA^cA^dA^eA^f\Gamma^a_{bcdef} = \frac{1}{3}\,A^bA^cA^dA^eA^fx^g\left(3\,g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3\,g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

The generalised connection in RNC

This is the same as the previous page but with a small change in the format to avoid fractions.

$$
\begin{aligned}
360A^bA^c\Gamma^a_{bc} = {} & 240\, A^bA^cx^dg^{ae}R_{bdce} + 30\, A^bA^cx^dx^e\left(2\, g^{af}\nabla_bR_{cdef} + 4\, g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right)\\
& + A^bA^cx^dx^ex^f\big(64\, g^{ag}g^{hi}R_{bdch}R_{egfi} - 32\, g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16\, g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18\, g^{ag}\nabla_{bd}R_{cefg}\\
& \qquad + 18\, g^{ag}\nabla_{db}R_{cefg} + 36\, g^{ag}\nabla_{de}R_{bfcg} - 16\, g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9\, g^{ag}\nabla_{gd}R_{becf} + 9\, g^{ag}\nabla_{dg}R_{becf}\big)\\
& + 2\, A^bA^cx^dx^ex^fx^g\big(16\, g^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} + 6\, g^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} + 16\, g^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} + 5\, g^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg}\\
& \qquad - 8\, g^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - 4\, g^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} - 4\, g^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} - 8\, g^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - 4\, g^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh}\\
& \qquad + 2\, g^{ah}\nabla_{bde}R_{cfgh} + 2\, g^{ah}\nabla_{dbe}R_{cfgh} + 2\, g^{ah}\nabla_{deb}R_{cfgh} + 4\, g^{ah}\nabla_{def}R_{bgch} - 4\, g^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} - 4\, g^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj}\\
& \qquad - 4\, g^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\big)
\end{aligned}
$$

$$
\begin{aligned}
360A^bA^cA^d\Gamma^a_{bcd} = {} & 180\, A^bA^cA^dx^eg^{af}\nabla_bR_{cedf} + 3\, A^bA^cA^dx^ex^f\big(64\, g^{ag}g^{hi}R_{bech}R_{dgfi} + 16\, g^{ag}g^{hi}R_{bech}R_{difg} - 16\, g^{ag}g^{hi}R_{befh}R_{cgdi} + 12\, g^{ag}\nabla_{bc}R_{defg}\\
& \qquad + 18\, g^{ag}\nabla_{be}R_{cfdg} + 18\, g^{ag}\nabla_{eb}R_{cfdg} + 48\, g^{ag}g^{hi}R_{bech}R_{dfgi} + 3\, g^{ag}\nabla_{gb}R_{cedf} + 3\, g^{ag}\nabla_{bg}R_{cedf}\big)\\
& + 2\, A^bA^cA^dx^ex^fx^g\big(32\, g^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} + 48\, g^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj} + 12\, g^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + 18\, g^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj}\\
& \qquad + 2\, g^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} + 22\, g^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + 48\, g^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} + 12\, g^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} + 15\, g^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg}\\
& \qquad + 5\, g^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} - 12\, g^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - 12\, g^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - 8\, g^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - 12\, g^{ah}g^{ij}R_{befi}\nabla_gR_{chdj}\\
& \qquad + 4\, g^{ah}\nabla_{bce}R_{dfgh} + 4\, g^{ah}\nabla_{bec}R_{dfgh} + 6\, g^{ah}\nabla_{bef}R_{cgdh} + 4\, g^{ah}\nabla_{ebc}R_{dfgh} + 6\, g^{ah}\nabla_{ebf}R_{cgdh} + 6\, g^{ah}\nabla_{efb}R_{cgdh}\\
& \qquad + 16\, g^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj} + 36\, g^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + 16\, g^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} - 4\, g^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} + 36\, g^{ah}g^{ij}R_{beci}\nabla_fR_{dghj}\\
& \qquad - 4\, g^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg}\\
& \qquad - 20\, g^{ah}g^{ij}R_{beci}\nabla_jR_{dfgh} + 10\, g^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}\big)
\end{aligned}
$$

$$
\begin{aligned}
90A^bA^cA^dA^e\Gamma^a_{bcde} = {} & 6\, A^bA^cA^dA^ex^f\left(8\, g^{ag}g^{hi}R_{bfch}R_{dgei} + 6\, g^{ag}\nabla_{bc}R_{dfeg}\right)\\
& + A^bA^cA^dA^ex^fx^g\big(64\, g^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + 18\, g^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + 24\, g^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} + 4\, g^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj}\\
& \qquad + 44\, g^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + 18\, g^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + 24\, g^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + 10\, g^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg} - 16\, g^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej}\\
& \qquad + 6\, g^{ah}\nabla_{bcd}R_{efgh} + 8\, g^{ah}\nabla_{bcf}R_{dgeh} + 8\, g^{ah}\nabla_{bfc}R_{dgeh} + 8\, g^{ah}\nabla_{fbc}R_{dgeh} + 26\, g^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej} + 6\, g^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej}\\
& \qquad + 46\, g^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40\, g^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\big)
\end{aligned}
$$

$$
3A^bA^cA^dA^eA^f\Gamma^a_{bcdef} = A^bA^cA^dA^eA^fx^g\left(3\, g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3\, g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)
$$

Convert from generic (x) to local RNC coords (y)

$$y^a = \overset{0}{y}{}^a + \overset{1}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a$$

$$\begin{aligned}
\overset{0}{y}{}^a &= x^a \\
2\overset{1}{y}{}^a &= x^b x^c \Gamma^a{}_{bc} \\
6\overset{2}{y}{}^a &= x^b x^c x^d \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} + \partial_b\Gamma^a{}_{cd}\right) \\
24\overset{3}{y}{}^a &= x^b x^c x^d x^e \left(2\,\Gamma^a{}_{bf}\partial_c\Gamma^f{}_{de} + \Gamma^a{}_{fg}\Gamma^f{}_{bc}\Gamma^g{}_{de} + \Gamma^f{}_{bc}\partial_f\Gamma^a{}_{de} + \partial_{bc}\Gamma^a{}_{de}\right) \\
360\overset{4}{y}{}^a &= x^b x^c x^d x^e x^f \Big(-4\,\Gamma^a{}_{bg}\Gamma^g{}_{ch}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 2\,\Gamma^a{}_{bg}\Gamma^g{}_{ch}\partial_d\Gamma^h{}_{ef} + 3\,\Gamma^a{}_{bg}\Gamma^g{}_{hi}\Gamma^h{}_{cd}\Gamma^i{}_{ef} - 6\,\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_e\Gamma^g{}_{fh} + 6\,\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_h\Gamma^g{}_{ef} + 9\,\Gamma^a{}_{bg}\partial_{cd}\Gamma^g{}_{ef} \\
&\quad + 4\,\Gamma^a{}_{gh}\Gamma^g{}_{bc}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 13\,\Gamma^a{}_{gh}\Gamma^g{}_{bc}\partial_d\Gamma^h{}_{ef} - 4\,\Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_e\Gamma^a{}_{fh} + \Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_h\Gamma^a{}_{ef} + 2\,\partial_b\Gamma^a{}_{cg}\partial_d\Gamma^g{}_{ef} + 7\,\partial_g\Gamma^a{}_{bc}\partial_d\Gamma^g{}_{ef} + 3\,\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_f\Gamma^a{}_{gh} \\
&\quad + 3\,\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_g\Gamma^a{}_{fh} - 3\,\Gamma^g{}_{bc}\partial_{de}\Gamma^a{}_{fg} + 6\,\Gamma^g{}_{bc}\partial_{dg}\Gamma^a{}_{ef} + 3\,\partial_{bcd}\Gamma^a{}_{ef}\Big)
\end{aligned}$$

The geodesic ivp

$$x^a(s) = x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A^a_{bc} + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A^a_{bcd} + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A^a_{bcde} + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A^a_{bcdef} + \cdots$$

$$\begin{aligned}
360A^a_{bc} = {}& -240\,x^d g^{ae} R_{bdce} - 30\,x^d x^e \left(2\,g^{af}\nabla_b R_{cdef} + 4\,g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) - x^d x^e x^f \big(64\,g^{ag}g^{hi}R_{bdch}R_{egfi} - 32\,g^{ag}g^{hi}R_{bdeh}R_{cgfi} \\
& - 16\,g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18\,g^{ag}\nabla_{bd}R_{cefg} + 18\,g^{ag}\nabla_{db}R_{cefg} + 36\,g^{ag}\nabla_{de}R_{bfcg} - 16\,g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9\,g^{ag}\nabla_{gd}R_{becf} + 9\,g^{ag}\nabla_{dg}R_{becf}\big) \\
& - 2\,x^d x^e x^f x^g \big(16\,g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6\,g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} + 16\,g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 5\,g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 8\,g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj} \\
& - 4\,g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} - 4\,g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} - 8\,g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} - 4\,g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} + 2\,g^{ah}\nabla_{bde}R_{cfgh} + 2\,g^{ah}\nabla_{dbe}R_{cfgh} \\
& + 2\,g^{ah}\nabla_{deb}R_{cfgh} + 4\,g^{ah}\nabla_{def}R_{bgch} - 4\,g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 4\,g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - 4\,g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} \\
& + g^{ah}\nabla_{deh}R_{bfcg}\big)
\end{aligned}$$

$$\begin{aligned}
360A^a_{bcd} = {}& -180\,x^e g^{af}\nabla_b R_{cedf} - 3\,x^e x^f \big(64\,g^{ag}g^{hi}R_{bech}R_{dgfi} + 16\,g^{ag}g^{hi}R_{bech}R_{difg} - 16\,g^{ag}g^{hi}R_{befh}R_{cgdi} + 12\,g^{ag}\nabla_{bc}R_{defg} + 18\,g^{ag}\nabla_{be}R_{cfdg} \\
& + 18\,g^{ag}\nabla_{eb}R_{cfdg} + 48\,g^{ag}g^{hi}R_{bech}R_{dfgi} + 3\,g^{ag}\nabla_{gb}R_{cedf} + 3\,g^{ag}\nabla_{bg}R_{cedf}\big) \\
& - 2\,x^e x^f x^g \big(32\,g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 48\,g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} + 12\,g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} + 18\,g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} + 2\,g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} \\
& + 22\,g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} + 48\,g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 12\,g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} + 15\,g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} + 5\,g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} \\
& - 12\,g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} - 12\,g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} - 8\,g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} - 12\,g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} + 4\,g^{ah}\nabla_{bce}R_{dfgh} + 4\,g^{ah}\nabla_{bec}R_{dfgh} \\
& + 6\,g^{ah}\nabla_{bef}R_{cgdh} + 4\,g^{ah}\nabla_{ebc}R_{dfgh} + 6\,g^{ah}\nabla_{ebf}R_{cgdh} + 6\,g^{ah}\nabla_{efb}R_{cgdh} + 16\,g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} + 36\,g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} \\
& + 16\,g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} - 4\,g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 36\,g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - 4\,g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg} \\
& + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20\,g^{ah}g^{ij}R_{beci}\nabla_j R_{dfgh} + 10\,g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg}\big)
\end{aligned}$$

$$\begin{aligned}
90A^a_{bcde} = {}& -6\,x^f \left(8\,g^{ag}g^{hi}R_{bfch}R_{dgei} + 6\,g^{ag}\nabla_{bc}R_{dfeg}\right) - x^f x^g \big(64\,g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 18\,g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + 24\,g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} \\
& + 4\,g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} + 44\,g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} + 18\,g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} + 24\,g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 10\,g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} \\
& - 16\,g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6\,g^{ah}\nabla_{bcd}R_{efgh} + 8\,g^{ah}\nabla_{bcf}R_{dgeh} + 8\,g^{ah}\nabla_{bfc}R_{dgeh} + 8\,g^{ah}\nabla_{fbc}R_{dgeh} + 26\,g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} \\
& + 6\,g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 46\,g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40\,g^{ah}g^{ij}R_{bfci}\nabla_j R_{dgeh}\big)
\end{aligned}$$

$$3A^a_{bcdef} = -\,x^g \left(3\,g^{ah}g^{ij}R_{bgci}\nabla_d R_{ehfj} + 3\,g^{ah}g^{ij}R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

Geodesic boundary value problem to terms linear in R

$$x^a(s) = x^a + sDx^a - \frac{1}{3}\left(s - s^2\right) x^b Dx^c Dx^d g^{ae} R_{bcde} + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2) x^a_2 + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \mathcal{O}\left(\epsilon^3\right)$$

$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

Geodesic boundary value problem to terms linear in ∇R

$$
\begin{aligned}
x^a(s) = x^a + sDx^a + \left(s - \ s^2\right)\left(-\frac{1}{3}\, x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}\, x^b x^c Dx^d Dx^e \left(2\, g^{af} \nabla_d R_{becf} + 4\, g^{af} \nabla_b R_{cdef} - \ g^{af} \nabla_f R_{bdce}\right)\right) \\
- \frac{1}{12}\, \left(s - \ s^3\right) x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} + \mathcal{O}\left(s^4, \epsilon^4\right)
\end{aligned}
$$

$$
x^a(s) = x^a + sDx^a + (s - s^2) x_2^a + (s - s^3) x_3^a + \mathcal{O}\left(s^4, \epsilon^4\right)
$$

$$
\begin{aligned}
x_2^a &= \overset{2}{x}{}_2^a + \overset{3}{x}{}_2^a + \mathcal{O}\left(\epsilon^4\right) \\
-3 \overset{2}{x}{}_2^a &= x^b Dx^c Dx^d g^{ae} R_{bcde} \\
-24 \overset{3}{x}{}_2^a &= x^b x^c Dx^d Dx^e \left(2\, g^{af} \nabla_d R_{becf} + 4\, g^{af} \nabla_b R_{cdef} - \ g^{af} \nabla_f R_{bdce}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_3^a &= \overset{3}{x}{}_3^a + \mathcal{O}\left(\epsilon^4\right) \\
-12 \overset{3}{x}{}_3^a &= x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}
\end{aligned}
$$

Geodesic boundary value problem to terms linear in $\nabla^2 R$

$$
\begin{aligned}
x^a(s) = x^a + sDx^a + \left(s - s^2\right)\Bigg(&-\frac{1}{3}\,x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}\,x^b x^c Dx^d Dx^e \left(2\,g^{af}\nabla_d R_{becf} + 4\,g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\\
&- \frac{1}{720}\,x^b x^c Dx^d Dx^e Dx^f \left(80\,g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80\,g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) - \frac{1}{720}\,x^b x^c x^d Dx^e Dx^f \left(64\,g^{ag}g^{hi}R_{befh}R_{cgdi} - 32\,g^{ag}g^{hi}R_{bech}R_{difg}\right.\\
&\left. - 16\,g^{ag}g^{hi}R_{bech}R_{dgfi} + 18\,g^{ag}\nabla_{eb}R_{cfdg} + 18\,g^{ag}\nabla_{be}R_{cfdg} + 36\,g^{ag}\nabla_{bc}R_{defg} + 16\,g^{ag}g^{hi}R_{bech}R_{dfgi} - 9\,g^{ag}\nabla_{gb}R_{cedf} - 9\,g^{ag}\nabla_{bg}R_{cedf}\right)\Bigg)\\
+ \left(s - s^3\right)\Bigg(&-\frac{1}{12}\,x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} - \frac{1}{720}\,x^b x^c Dx^d Dx^e Dx^f \left(64\,g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16\,g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16\,g^{ag}g^{hi}R_{bdch}R_{egfi}\right.\\
&\left. + 12\,g^{ag}\nabla_{de}R_{bfcg} + 18\,g^{ag}\nabla_{db}R_{cefg} + 18\,g^{ag}\nabla_{bd}R_{cefg} - 48\,g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 3\,g^{ag}\nabla_{gd}R_{becf} - 3\,g^{ag}\nabla_{dg}R_{becf}\right)\Bigg)\\
- \frac{1}{180}&\left(s - s^4\right) x^b Dx^c Dx^d Dx^e Dx^f \left(4\,g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3\,g^{ag}\nabla_{cd}R_{befg}\right) + \mathcal{O}\left(s^5, \epsilon^5\right)
\end{aligned}
$$

$$
x^a(s) = x^a + sDx^a + (s - s^2)x^a_2 + (s - s^3)x^a_3 + (s - s^4)x^a_4 + \mathcal{O}\left(s^5, \epsilon^5\right)
$$

$$
\begin{aligned}
x^a_2 &= \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \overset{4}{x}{}^a_2 + \mathcal{O}\left(\epsilon^5\right)\\
-3\overset{2}{x}{}^a_2 &= x^b Dx^c Dx^d g^{ae} R_{bcde}\\
-24\overset{3}{x}{}^a_2 &= x^b x^c Dx^d Dx^e \left(2\,g^{af}\nabla_d R_{becf} + 4\,g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\\
-720\overset{4}{x}{}^a_2 &= x^b x^c Dx^d Dx^e Dx^f \left(80\,g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80\,g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \left(64\,g^{ag}g^{hi}R_{befh}R_{cgdi} - 32\,g^{ag}g^{hi}R_{bech}R_{difg}\right.\\
&\quad\left. - 16\,g^{ag}g^{hi}R_{bech}R_{dgfi} + 18\,g^{ag}\nabla_{eb}R_{cfdg} + 18\,g^{ag}\nabla_{be}R_{cfdg} + 36\,g^{ag}\nabla_{bc}R_{defg} + 16\,g^{ag}g^{hi}R_{bech}R_{dfgi} - 9\,g^{ag}\nabla_{gb}R_{cedf} - 9\,g^{ag}\nabla_{bg}R_{cedf}\right)
\end{aligned}
$$

$$x_3^a = \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-12\overset{3}{x}{}_3^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned}-720\overset{4}{x}{}_3^a = x^b x^c Dx^d Dx^e Dx^f \Big(&64\, g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16\, g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16\, g^{ag} g^{hi} R_{bdch} R_{egfi} + 12\, g^{ag} \nabla_{de} R_{bfcg} + 18\, g^{ag} \nabla_{db} R_{cefg} + 18\, g^{ag} \nabla_{bd} R_{cefg} \\ &- 48\, g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3\, g^{ag} \nabla_{gd} R_{becf} - 3\, g^{ag} \nabla_{dg} R_{becf}\Big)\end{aligned}$$

$$x_4^a = \overset{4}{x}{}_4^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4\, g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3\, g^{ag} \nabla_{cd} R_{befg}\right)$$

Geodesic boundary value problem to terms linear in $\nabla^3 R$

The geodesic that connects the points with RNC coordinates x^a and $x^a + Dx^a$ is described, for $0 \leq s \leq 1$, by

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + (s - s^4)x_4^a + (s - s^5)x_5^a + \mathcal{O}\left(s^6, \epsilon^6\right)$$

$$\begin{aligned}
x_2^a &= \overset{2}{x}{}_2^a + \overset{3}{x}{}_2^a + \overset{4}{x}{}_2^a + \overset{5}{x}{}_2^a + \mathcal{O}\left(\epsilon^6\right)\\
-3\overset{2}{x}{}_2^a &= x^b Dx^c Dx^d g^{ae} R_{bcde}\\
-24\overset{3}{x}{}_2^a &= x^b x^c Dx^d Dx^e \left(2\, g^{af} \nabla_{d} R_{becf} + 4\, g^{af} \nabla_{b} R_{cdef} -\, g^{af} \nabla_{f} R_{bdce}\right)\\
-720\overset{4}{x}{}_2^a &= x^b x^c Dx^d Dx^e Dx^f \left(80\, g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 80\, g^{ag} g^{hi} R_{bdeh} R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \left(64\, g^{ag} g^{hi} R_{befh} R_{cgdi} - 32\, g^{ag} g^{hi} R_{bech} R_{difg}\right.\\
&\qquad \left. - 16\, g^{ag} g^{hi} R_{bech} R_{dgfi} + 18\, g^{ag} \nabla_{eb} R_{cfdg} + 18\, g^{ag} \nabla_{be} R_{cfdg} + 36\, g^{ag} \nabla_{bc} R_{defg} + 16\, g^{ag} g^{hi} R_{bech} R_{dfgi} - 9\, g^{ag} \nabla_{gb} R_{cedf} - 9\, g^{ag} \nabla_{bg} R_{cedf}\right)\\
-360\overset{5}{x}{}_2^a &= x^b x^c x^d Dx^e Dx^f Dx^g \left(10\, g^{ah} g^{ij} R_{behi} \nabla_{f} R_{cgdj} + 20\, g^{ah} g^{ij} R_{behi} \nabla_{c} R_{dfgj} - 5\, g^{ah} g^{ij} R_{behi} \nabla_{j} R_{cfdg} - 10\, g^{ah} g^{ij} R_{bieh} \nabla_{f} R_{cgdj}\right.\\
&\qquad - 20\, g^{ah} g^{ij} R_{bieh} \nabla_{c} R_{dfgj} + 5\, g^{ah} g^{ij} R_{bieh} \nabla_{j} R_{cfdg} - 10\, g^{ah} g^{ij} R_{befi} \nabla_{g} R_{chdj} - 10\, g^{ah} g^{ij} R_{befi} \nabla_{j} R_{cgdh} + 20\, g^{ah} g^{ij} R_{befi} \nabla_{c} R_{dghj}\\
&\qquad \left. - 20\, g^{ah} g^{ij} R_{befi} \nabla_{c} R_{djgh} + 10\, g^{ah} g^{ij} R_{befi} \nabla_{h} R_{cgdj}\right) + x^b x^c Dx^d Dx^e Dx^f Dx^g \left(10\, g^{ah} g^{ij} R_{bdhi} \nabla_{e} R_{cfgj} - 10\, g^{ah} g^{ij} R_{bidh} \nabla_{e} R_{cfgj}\right)\\
&\qquad + x^b x^c x^d x^e Dx^f Dx^g \left(16\, g^{ah} g^{ij} R_{bfgi} \nabla_{c} R_{dhej} + 6\, g^{ah} g^{ij} R_{bhci} \nabla_{f} R_{dgej} + 16\, g^{ah} g^{ij} R_{bhci} \nabla_{d} R_{efgj} - 5\, g^{ah} g^{ij} R_{bhci} \nabla_{j} R_{dfeg}\right.\\
&\qquad - 8\, g^{ah} g^{ij} R_{bifh} \nabla_{c} R_{dgej} - 4\, g^{ah} g^{ij} R_{bhfi} \nabla_{c} R_{dgej} - 4\, g^{ah} g^{ij} R_{bfci} \nabla_{g} R_{dhej} - 8\, g^{ah} g^{ij} R_{bfci} \nabla_{d} R_{ejgh} - 4\, g^{ah} g^{ij} R_{bfci} \nabla_{d} R_{ehgj} + 2\, g^{ah} \nabla_{fbc} R_{dgeh}\\
&\qquad + 2\, g^{ah} \nabla_{bfc} R_{dgeh} + 2\, g^{ah} \nabla_{bcf} R_{dgeh} + 4\, g^{ah} \nabla_{bcd} R_{efgh} + 4\, g^{ah} g^{ij} R_{bfhi} \nabla_{c} R_{dgej} + 4\, g^{ah} g^{ij} R_{bfci} \nabla_{h} R_{dgej} + 4\, g^{ah} g^{ij} R_{bfci} \nabla_{d} R_{eghj}\\
&\qquad \left. -\, g^{ah} \nabla_{hbc} R_{dfeg} -\, g^{ah} \nabla_{bhc} R_{dfeg} -\, g^{ah} \nabla_{bch} R_{dfeg}\right)
\end{aligned}$$

$$x_3^a = \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \overset{5}{x}{}_3^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-12\overset{3}{x}{}_3^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned} -720\overset{4}{x}{}_3^a = x^b x^c Dx^d Dx^e Dx^f \big(&64\, g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16\, g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16\, g^{ag} g^{hi} R_{bdch} R_{egfi} + 12\, g^{ag} \nabla_{de} R_{bfcg} + 18\, g^{ag} \nabla_{db} R_{cefg} + 18\, g^{ag} \nabla_{bd} R_{cefg} \\ &- 48\, g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3\, g^{ag} \nabla_{gd} R_{becf} - 3\, g^{ag} \nabla_{dg} R_{becf}\big) \end{aligned}$$

$$\begin{aligned} -1080\overset{5}{x}{}_3^a = {} & x^b x^c Dx^d Dx^e Dx^f Dx^g \left(30\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - 30\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} - 30\, g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\right) \\ & + x^b x^c x^d Dx^e Dx^f Dx^g \big(32\, g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 48\, g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 12\, g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 18\, g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\ & + 2\, g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22\, g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 48\, g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 12\, g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} - 15\, g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\ & - 5\, g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} - 12\, g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12\, g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8\, g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12\, g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\ & + 4\, g^{ah} \nabla_{efb} R_{cgdh} + 4\, g^{ah} \nabla_{ebf} R_{cgdh} + 6\, g^{ah} \nabla_{ebc} R_{dfgh} + 4\, g^{ah} \nabla_{bef} R_{cgdh} + 6\, g^{ah} \nabla_{bec} R_{dfgh} + 6\, g^{ah} \nabla_{bce} R_{dfgh} - 16\, g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\ & - 36\, g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 16\, g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 4\, g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 36\, g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + 4\, g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg} \\ & - g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg} + 20\, g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 10\, g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}\big) \end{aligned}$$

$$x_4^a = \overset{4}{x}{}_4^a + \overset{5}{x}{}_4^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4\, g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3\, g^{ag} \nabla_{cd} R_{befg}\right)$$

$$\begin{aligned} -2160\overset{5}{x}{}_4^a = x^b x^c Dx^d Dx^e Dx^f Dx^g \big(&64\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 18\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24\, g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4\, g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} \\ & + 44\, g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 18\, g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} + 24\, g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10\, g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 16\, g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} \\ & + 6\, g^{ah} \nabla_{def} R_{bgch} + 8\, g^{ah} \nabla_{deb} R_{cfgh} + 8\, g^{ah} \nabla_{dbe} R_{cfgh} + 8\, g^{ah} \nabla_{bde} R_{cfgh} - 26\, g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 6\, g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} \\ & - 46\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - g^{ah} \nabla_{hde} R_{bfcg} - g^{ah} \nabla_{dhe} R_{bfcg} - g^{ah} \nabla_{deh} R_{bfcg} + 40\, g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\big) \end{aligned}$$

$$x_5^a = \overset{5}{x}{}_5^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-360\overset{5}{x}{}_5^a = x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(3\, g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3\, g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh}\right)$$

Geodesic arc-length

$$
\begin{aligned}
(\Delta s)^2 = {} & g_{ab} Dx^a Dx^b - \frac{1}{3}\, x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12}\, x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6}\, x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \\
& + \frac{1}{360}\, x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8\, g^{gh} R_{acdg} R_{befh} - 6\, \nabla_{cd} R_{aebf}\right) + \frac{1}{360}\, x^a x^b x^c Dx^d Dx^e Dx^f \left(16\, g^{gh} R_{adbg} R_{cefh} - 9\, \nabla_{da} R_{becf} - 9\, \nabla_{ad} R_{becf}\right) \\
& + \frac{1}{360}\, x^a x^b x^c x^d Dx^e Dx^f \left(16\, g^{gh} R_{aebg} R_{cfdh} - 18\, \nabla_{ab} R_{cedf}\right) + \frac{1}{1080}\, x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \Big(-4\, g^{hi} R_{adeh} \nabla_f R_{bgci} - 24\, g^{hi} R_{adeh} \nabla_b R_{cfgi} \\
& \qquad + 10\, g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16\, g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4\, \nabla_{dea} R_{bfcg} - 4\, \nabla_{dae} R_{bfcg} - 4\, \nabla_{ade} R_{bfcg}\Big) \\
& + \frac{1}{1080}\, x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18\, g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3\, \nabla_{cde} R_{afbg}\right) \\
& + \frac{1}{1080}\, x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24\, g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24\, g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24\, g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6\, \nabla_{eab} R_{cfdg} - 6\, \nabla_{aeb} R_{cfdg} - 6\, \nabla_{abe} R_{cfdg}\right) \\
& + \frac{1}{1080}\, x^a x^b x^c x^d x^e Dx^f Dx^g \left(48\, g^{hi} R_{afbh} \nabla_c R_{dgei} - 12\, \nabla_{abc} R_{dfeg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

Geodesic arc-length curvature expansion

$$(\Delta s)^2 = \overset{0}{\Delta} + \overset{2}{\Delta} + \overset{3}{\Delta} + \overset{4}{\Delta} + \overset{5}{\Delta} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{\Delta} = g_{ab} Dx^a Dx^b$$

$$3\overset{2}{\Delta} = -\, x^a x^b Dx^c Dx^d R_{acbd}$$

$$12\overset{3}{\Delta} = -\, x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - 2\, x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce}$$

$$\begin{aligned} 360\overset{4}{\Delta} = {} & x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8\, g^{gh} R_{acdg} R_{befh} - 6\, \nabla_{cd} R_{aebf}\right) + x^a x^b x^c Dx^d Dx^e Dx^f \left(16\, g^{gh} R_{adbg} R_{cefh} - 9\, \nabla_{da} R_{becf} - 9\, \nabla_{ad} R_{becf}\right) \\ & + x^a x^b x^c x^d Dx^e Dx^f \left(16\, g^{gh} R_{aebg} R_{cfdh} - 18\, \nabla_{ab} R_{cedf}\right) \end{aligned}$$

$$\begin{aligned} 1080\overset{5}{\Delta} = {} & x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4\, g^{hi} R_{adeh} \nabla_f R_{bgci} - 24\, g^{hi} R_{adeh} \nabla_b R_{cfgi} + 10\, g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16\, g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4\, \nabla_{dea} R_{bfcg} - 4\, \nabla_{dae} R_{bfcg} \right. \\ & \left. - 4\, \nabla_{ade} R_{bfcg}\right) + x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18\, g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3\, \nabla_{cde} R_{afbg}\right) \\ & + x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24\, g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24\, g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24\, g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6\, \nabla_{eab} R_{cfdg} - 6\, \nabla_{aeb} R_{cfdg} - 6\, \nabla_{abe} R_{cfdg}\right) \\ & + x^a x^b x^c x^d x^e Dx^f Dx^g \left(48\, g^{hi} R_{afbh} \nabla_c R_{dgei} - 12\, \nabla_{abc} R_{dfeg}\right) \end{aligned}$$

Tranformation between two RNC frames

$$y^a = \overset{0}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a + \overset{5}{y}{}^a + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{y}{}^a = Dx^a$$

$$\overset{2}{y}{}^a = \overset{2}{y}{}^a_1$$

$$3\overset{2}{y}{}^a_1 = -\,x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\overset{3}{y}{}^a = \overset{3}{y}{}^a_1 + \overset{3}{y}{}^a_2$$

$$-12\overset{3}{y}{}^a_1 = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$-24\overset{3}{y}{}^a_2 = x^b x^c Dx^d Dx^e \left(2\, g^{af} \nabla_d R_{becf} + 4\, g^{af} \nabla_b R_{cdef} -\, g^{af} \nabla_f R_{bdce}\right)$$

$$\overset{4}{y}{}^a = \overset{4}{y}{}^a_1 + \overset{4}{y}{}^a_2 + \overset{4}{y}{}^a_3$$

$$-180\overset{4}{y}{}^a_1 = x^b Dx^c Dx^d Dx^e Dx^f \left(4\, g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3\, g^{ag} \nabla_{cd} R_{befg}\right)$$

$$-720\overset{4}{y}{}^a_2 = x^b x^c Dx^d Dx^e Dx^f \left(32\, g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 16\, g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16\, g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16\, g^{ag} g^{hi} R_{bdch} R_{egfi} + 12\, g^{ag} \nabla_{de} R_{bfcg} \right.$$
$$\left. + 18\, g^{ag} \nabla_{db} R_{cefg} + 18\, g^{ag} \nabla_{bd} R_{cefg} - 3\, g^{ag} \nabla_{gd} R_{becf} - 3\, g^{ag} \nabla_{dg} R_{becf}\right)$$

$$-720\overset{4}{y}{}^a_3 = x^b x^c x^d Dx^e Dx^f \left(64\, g^{ag} g^{hi} R_{befh} R_{cgdi} - 32\, g^{ag} g^{hi} R_{bech} R_{difg} - 16\, g^{ag} g^{hi} R_{bech} R_{dgfi} + 18\, g^{ag} \nabla_{eb} R_{cfdg} + 18\, g^{ag} \nabla_{be} R_{cfdg} + 36\, g^{ag} \nabla_{bc} R_{defg} \right.$$
$$\left. + 16\, g^{ag} g^{hi} R_{bech} R_{dfgi} - 9\, g^{ag} \nabla_{gb} R_{cedf} - 9\, g^{ag} \nabla_{bg} R_{cedf}\right)$$

$$
\begin{aligned}
\overset{5}{y}{}^a &= \overset{5}{y}{}^a_1 + \overset{5}{y}{}^a_2 + \overset{5}{y}{}^a_3 + \overset{5}{y}{}^a_4 \\
-360\overset{5}{y}{}^a_1 &= x^b Dx^c Dx^d Dx^e Dx^f Dx^g \big(3\, g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3\, g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh}\big) \\
-2160\overset{5}{y}{}^a_2 &= x^b x^c Dx^d Dx^e Dx^f Dx^g \big(34\, g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 16\, g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 14\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + 4\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} \\
&\quad - 20\, g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh} + 18\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24\, g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4\, g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 18\, g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} \\
&\quad + 24\, g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10\, g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 16\, g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6\, g^{ah} \nabla_{def} R_{bgch} + 8\, g^{ah} \nabla_{deb} R_{cfgh} + 8\, g^{ah} \nabla_{dbe} R_{cfgh} \\
&\quad + 8\, g^{ah} \nabla_{bde} R_{cfgh} - 6\, g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} - g^{ah} \nabla_{hde} R_{bfcg} - g^{ah} \nabla_{dhe} R_{bfcg} - g^{ah} \nabla_{deh} R_{bfcg}\big) \\
-1080\overset{5}{y}{}^a_3 &= x^b x^c x^d Dx^e Dx^f Dx^g \big(14\, g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + 24\, g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 5\, g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} - 12\, g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} - 12\, g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} \\
&\quad + 2\, g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} - 10\, g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 24\, g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} - 12\, g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 14\, g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} \\
&\quad + 12\, g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 2\, g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22\, g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 12\, g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} - 5\, g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} \\
&\quad - 12\, g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12\, g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8\, g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12\, g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 4\, g^{ah} \nabla_{efb} R_{cgdh} + 4\, g^{ah} \nabla_{ebf} R_{cgdh} \\
&\quad + 6\, g^{ah} \nabla_{ebc} R_{dfgh} + 4\, g^{ah} \nabla_{bef} R_{cgdh} + 6\, g^{ah} \nabla_{bec} R_{dfgh} + 6\, g^{ah} \nabla_{bce} R_{dfgh} + 4\, g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} + 4\, g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg} \\
&\quad - g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg}\big) \\
-360\overset{5}{y}{}^a_4 &= x^b x^c x^d x^e Dx^f Dx^g \big(16\, g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + 6\, g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 16\, g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} - 5\, g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} - 8\, g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} \\
&\quad - 4\, g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} - 4\, g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} - 8\, g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} - 4\, g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 2\, g^{ah} \nabla_{fbc} R_{dgeh} + 2\, g^{ah} \nabla_{bfc} R_{dgeh} \\
&\quad + 2\, g^{ah} \nabla_{bcf} R_{dgeh} + 4\, g^{ah} \nabla_{bcd} R_{efgh} + 4\, g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} + 4\, g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + 4\, g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} - g^{ah} \nabla_{hbc} R_{dfeg} \\
&\quad - g^{ah} \nabla_{bhc} R_{dfeg} - g^{ah} \nabla_{bch} R_{dfeg}\big)
\end{aligned}
$$