

# The inverse metric tensor in Riemann normal coordinates

Here we calculate the Riemann normal expansion of the inverse metric,  $g^{ab}$ , by developing the recursive sequences

$$g^{ab}{}_{,d\underline{e}} = -\left(g^{cb}\Gamma^a{}_{c(d),\underline{e}}\right) - \left(g^{ac}\Gamma^b{}_{c(d),\underline{e}}\right) \quad (1)$$

$$(n+3)\Gamma^a{}_{d(b,c\underline{e})} = (n+1)\left(R^a{}_{(bcd,\underline{e})} - \left(\Gamma^a{}_{f(c}\Gamma^f{}_{bd),\underline{e}}\right)\right) \quad (2)$$

for  $n = 1, 2, 3, \dots$ . Note in these equations that the (extended) index  $\underline{e}$  contains  $n$  normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g^{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^c + \frac{1}{2!}g^{ab}{}_{,cd}x^cx^d + \frac{1}{3!}g^{ab}{}_{,cde}x^cx^dx^e + \dots \\ &= g^{ab} + \sum_{n=1}^{\infty} \frac{1}{n!} g^{ab}{}_{,\underline{c}} x^{\underline{c}} \end{aligned}$$

## Stage 1: Symmetrised partial derivatives of $g^{ab}$

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g^{ab}{}_{,c}A^c &= 0 \\ g^{ab}{}_{,cd}A^cA^d &= -g^{cb}\partial_e\Gamma^a{}_{cd}A^dA^e - g^{ac}\partial_e\Gamma^b{}_{cd}A^dA^e \\ g^{ab}{}_{,cde}A^cA^dA^e &= -g^{cb}\partial_{fe}\Gamma^a{}_{cd}A^dA^eA^f - g^{ac}\partial_{fe}\Gamma^b{}_{cd}A^dA^eA^f \end{aligned}$$

## Stage 2: Replace derivatives of $\Gamma$ with partial derivs of $R$

Now we use the results from `dGamma` to replace derivatives of  $\Gamma$  with partial derivatives of  $R$ . These were computed in `dGamma` using equation (2) above.

$$\begin{aligned}
g^{ab}{}_{,c}A^c &= 0 \\
g^{ab}{}_{,cd}A^cA^d &= -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec} \\
g^{ab}{}_{,cde}A^cA^dA^e &= -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}
\end{aligned}$$

### Stage 3: Replace partial derivs of $R$ with covariant derivs of $R$

Next we use the results from `dRabcd` to replace the partial derivatives of  $R$  with covariant derivatives.

$$\begin{aligned}
g^{ab}{}_{,c}A^c &= 0 \\
g^{ab}{}_{,cd}A^cA^d &= -\frac{1}{3}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}A^cA^dR^b{}_{cd}{}^a \\
g^{ab}{}_{,cde}A^cA^dA^e &= -\frac{1}{2}g^{cb}A^dA^fA^e\nabla_dR_{cfe}g^{ag} - \frac{1}{2}g^{ac}A^dA^fA^e\nabla_dR_{cfe}g^{bg}
\end{aligned}$$

### Stage 4: Build the Taylor series for $g_{ab}$ , reformatting and output

Each of the above expressions constitutes one term in the Taylor series for the metric. We also make the trivial change  $A \rightarrow x$ . Then we do some trivial reformatting.

$$\begin{aligned}
g_{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^c + \frac{1}{2!}g^{ab}{}_{,cd}x^cx^d + \frac{1}{3!}g^{ab}{}_{,cde}x^cx^dx^e + \mathcal{O}(\epsilon^4) \\
&= g^{ab} + \frac{1}{3}x^cx^dR_{cedf}g^{ae}g^{bf} + \frac{1}{6}x^cx^dx^e\nabla_cR_{dfe}g^{af}g^{bg} + \mathcal{O}(\epsilon^4)
\end{aligned}$$

# Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R_{a}_{b c d} -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d} -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    sort_product (obj)
    rename_dummies (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}\{\Gamma^{a}_{b c}\} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e}\{\Gamma^{a}_{b c}\} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f}\{\Gamma^{a}_{b c}\} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g h}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}\{\Gamma^{a}_{b c}\}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @ (obj).
    bah = Ex("numx = " + str(n))
    keep_weight (foo,bah)
```

```

return foo

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                                -> A001^{a}                                $)
    substitute (obj,$ x^{a}                                -> A002^{a}                                $)
    substitute (obj,$ g_{a b}                              -> A003_{a b}                              $)
    substitute (obj,$ g^{a b}                              -> A004^{a b}                              $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\}       -> A010_{a b c d e f g h}                 $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}          -> A009_{a b c d e f g}                   $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}             -> A008_{a b c d e f}                     $)
    substitute (obj,$ \nabla_{e}\{R_{a b c d}\}               -> A007_{a b c d e}                       $)
    substitute (obj,$ \partial_{e f g h}\{R_{a b c d}\}      -> A014_{a b c d e f g h}                 $)
    substitute (obj,$ \partial_{e f g}\{R_{a b c d}\}         -> A013_{a b c d e f g}                   $)
    substitute (obj,$ \partial_{e f}\{R_{a b c d}\}           -> A012_{a b c d e f}                     $)
    substitute (obj,$ \partial_{e}\{R_{a b c d}\}              -> A011_{a b c d e}                       $)
    substitute (obj,$ \partial_{e f g h}\{R^{a}_{a}_{b c d}\}  -> A018^{a}_{a}_{b c d e f g h}           $)
    substitute (obj,$ \partial_{e f g}\{R^{a}_{a}_{b c d}\}     -> A017^{a}_{a}_{b c d e f g}             $)
    substitute (obj,$ \partial_{e f}\{R^{a}_{a}_{b c d}\}       -> A016^{a}_{a}_{b c d e f}               $)
    substitute (obj,$ \partial_{e}\{R^{a}_{a}_{b c d}\}         -> A015^{a}_{a}_{b c d e}                 $)
    substitute (obj,$ R_{a b c d}                           -> A005_{a b c d}                         $)
    substitute (obj,$ R^{a}_{a}_{b c d}                     -> A006^{a}_{a}_{b c d}                   $)
    sort_product      (obj)
    rename_dummies    (obj)
    substitute (obj,$ A001^{a}                              -> A^{a}                                $)
    substitute (obj,$ A002^{a}                              -> x^{a}                                $)
    substitute (obj,$ A003_{a b}                            -> g_{a b}                              $)
    substitute (obj,$ A004^{a b}                            -> g^{a b}                              $)
    substitute (obj,$ A005_{a b c d}                        -> R_{a b c d}                          $)
    substitute (obj,$ A006^{a}_{a}_{b c d}                  -> R^{a}_{a}_{b c d}                     $)
    substitute (obj,$ A007_{a b c d e}                      -> \nabla_{e}\{R_{a b c d}\}                $)
    substitute (obj,$ A008_{a b c d e f}                   -> \nabla_{e f}\{R_{a b c d}\}               $)
    substitute (obj,$ A009_{a b c d e f g}                  -> \nabla_{e f g}\{R_{a b c d}\}             $)
    substitute (obj,$ A010_{a b c d e f g h}                -> \nabla_{e f g h}\{R_{a b c d}\}           $)
    substitute (obj,$ A011_{a b c d e}                     -> \partial_{e}\{R_{a b c d}\}                $)
    substitute (obj,$ A012_{a b c d e f}                    -> \partial_{e f}\{R_{a b c d}\}               $)
    substitute (obj,$ A013_{a b c d e f g}                  -> \partial_{e f g}\{R_{a b c d}\}             $)

```

```

substitute (obj,$ A014_{a b c d e f g h} -> \partial_{e f g h}\{R_{a b c d}\} $)
substitute (obj,$ A015^{\{a\}}_{\{b c d e\}} -> \partial_{e}\{R^{\{a\}}_{\{b c d\}}\} $)
substitute (obj,$ A016^{\{a\}}_{\{b c d e f\}} -> \partial_{e f}\{R^{\{a\}}_{\{b c d\}}\} $)
substitute (obj,$ A017^{\{a\}}_{\{b c d e f g\}} -> \partial_{e f g}\{R^{\{a\}}_{\{b c d\}}\} $)
substitute (obj,$ A018^{\{a\}}_{\{b c d e f g h\}} -> \partial_{e f g h}\{R^{\{a\}}_{\{b c d\}}\} $)

return obj

def reformat_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    factor_out (bah,$x^{\{a?\}}$)
    ans := @(bah) / @(foo).
    return ans

def rescale_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    factor_out (bah,$x^{\{a?\}}$)
    return bah

def add_tags (obj,tag):
    n = 0
    ans = Ex('0')
    for i in obj.top().terms():
        foo = obj[i]
        bah = Ex(tag+'_{'+str(n)+'}')
        ans := @(ans) + @(bah) @(foo).
        n = n + 1
    return ans

def clear_tags (obj,tag):
    ans := @(obj).

```

```

foo = Ex(tag+'_{a?} -> 1')
substitute (ans,foo)
return ans

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

```

## Stage 1: Symmetrised partial derivatives of $g^{ab}$

```

beg_stage_1 = time.time()

# symmetrised partial derivatives of  $g^{ab}$ 

gab00:=g^{a b}. # cdb (gab00.101,gab00)

gab01:= - g^{c b}\Gamma^{a}_{c d} - g^{a c}\Gamma^{b}_{c d}. # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @(gab01) }. # cdb (gab02.101,gab02)
distribute (gab02) # cdb (gab02.102,gab02)
product_rule (gab02) # cdb (gab02.103,gab02)
substitute (gab02, $\partial_{d}{g^{a b}} \rightarrow @(gab01)$) # cdb (gab02.104,gab02)
distribute (gab02) # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @(gab02) }. # cdb (gab03.101,gab03)
distribute (gab03) # cdb (gab03.102,gab03)
product_rule (gab03) # cdb (gab03.103,gab03)
substitute (gab03, $\partial_{d}{g^{a b}} \rightarrow @(gab01)$) # cdb (gab03.104,gab03)
distribute (gab03) # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @(gab03) }. # cdb (gab04.101,gab04)
distribute (gab04) # cdb (gab04.102,gab04)
product_rule (gab04) # cdb (gab04.103,gab04)
substitute (gab04, $\partial_{d}{g^{a b}} \rightarrow @(gab01)$) # cdb (gab04.104,gab04)
distribute (gab04) # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @(gab04) }. # cdb (gab05.101,gab05)
distribute (gab05) # cdb (gab05.102,gab05)
product_rule (gab05) # cdb (gab05.103,gab05)
substitute (gab05, $\partial_{d}{g^{a b}} \rightarrow @(gab01)$) # cdb (gab05.104,gab05)
distribute (gab05) # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00) # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01) # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02) # cdb (gab02.106,gab02)

```

```
gab03 = impose_rnc (gab03)  # cdb (gab03.106,gab03)  
gab04 = impose_rnc (gab04)  # cdb (gab04.106,gab04)  
gab05 = impose_rnc (gab05)  # cdb (gab05.106,gab05)
```



$$\text{gab00.101} := g^{ab}$$

$$\text{gab00.102} := g^{ab}$$

$$\text{gab01.101} := -g^{cb}\Gamma_{cd}^a - g^{ac}\Gamma_{cd}^b$$

$$\text{gab01.102} := 0$$

$$\text{gab02.101} := \partial_e (-g^{cb}\Gamma_{cd}^a - g^{ac}\Gamma_{cd}^b)$$

$$\text{gab02.102} := -\partial_e (g^{cb}\Gamma_{cd}^a) - \partial_e (g^{ac}\Gamma_{cd}^b)$$

$$\text{gab02.103} := -\partial_e g^{cb}\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a - \partial_e g^{ac}\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab02.104} := -(-g^{fb}\Gamma_{fe}^c - g^{cf}\Gamma_{fe}^b)\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a - (-g^{fc}\Gamma_{fe}^a - g^{af}\Gamma_{fe}^c)\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab02.105} := g^{fb}\Gamma_{fe}^c\Gamma_{cd}^a + g^{cf}\Gamma_{fe}^b\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a + g^{fc}\Gamma_{fe}^a\Gamma_{cd}^b + g^{af}\Gamma_{fe}^c\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab02.106} := -g^{cb}\partial_e \Gamma_{cd}^a - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab03.101} := \partial_f (g^{gb}\Gamma_{ge}^c\Gamma_{cd}^a + g^{cg}\Gamma_{ge}^b\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a + g^{gc}\Gamma_{ge}^a\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b)$$

$$\text{gab03.102} := \partial_f (g^{gb}\Gamma_{ge}^c\Gamma_{cd}^a) + \partial_f (g^{cg}\Gamma_{ge}^b\Gamma_{cd}^a) - \partial_f (g^{cb}\partial_e \Gamma_{cd}^a) + \partial_f (g^{gc}\Gamma_{ge}^a\Gamma_{cd}^b) + \partial_f (g^{ag}\Gamma_{ge}^c\Gamma_{cd}^b) - \partial_f (g^{ac}\partial_e \Gamma_{cd}^b)$$

$$\begin{aligned} \text{gab03.103} := & \partial_f g^{gb}\Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\partial_f \Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\Gamma_{ge}^c\partial_f \Gamma_{cd}^a + \partial_f g^{cg}\Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\partial_f \Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\Gamma_{ge}^b\partial_f \Gamma_{cd}^a - \partial_f g^{cb}\partial_e \Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a \\ & + \partial_f g^{gc}\Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\partial_f \Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\Gamma_{ge}^a\partial_f \Gamma_{cd}^b + \partial_f g^{ag}\Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\partial_f \Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\partial_f \Gamma_{cd}^b - \partial_f g^{ac}\partial_e \Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \end{aligned}$$

$$\begin{aligned} \text{gab03.104} := & (-g^{hb}\Gamma_{hf}^g - g^{gh}\Gamma_{hf}^b)\Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\partial_f \Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\Gamma_{ge}^c\partial_f \Gamma_{cd}^a + (-g^{hg}\Gamma_{hf}^c - g^{ch}\Gamma_{hf}^g)\Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\partial_f \Gamma_{ge}^b\Gamma_{cd}^a \\ & + g^{cg}\Gamma_{ge}^b\partial_f \Gamma_{cd}^a - (-g^{gb}\Gamma_{gf}^c - g^{cg}\Gamma_{gf}^b)\partial_e \Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a + (-g^{hc}\Gamma_{hf}^g - g^{gh}\Gamma_{hf}^c)\Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\partial_f \Gamma_{ge}^a\Gamma_{cd}^b \\ & + g^{gc}\Gamma_{ge}^a\partial_f \Gamma_{cd}^b + (-g^{hg}\Gamma_{hf}^a - g^{ah}\Gamma_{hf}^g)\Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\partial_f \Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\partial_f \Gamma_{cd}^b - (-g^{gc}\Gamma_{gf}^a - g^{ag}\Gamma_{gf}^c)\partial_e \Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \end{aligned}$$

$$\begin{aligned} \text{gab03.105} := & -g^{hb}\Gamma_{hf}^g\Gamma_{ge}^c\Gamma_{cd}^a - g^{gh}\Gamma_{hf}^b\Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\partial_f \Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\Gamma_{ge}^c\partial_f \Gamma_{cd}^a - g^{hg}\Gamma_{hf}^c\Gamma_{ge}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^g\Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\partial_f \Gamma_{ge}^b\Gamma_{cd}^a \\ & + g^{cg}\Gamma_{ge}^b\partial_f \Gamma_{cd}^a + g^{gb}\Gamma_{gf}^c\partial_e \Gamma_{cd}^a + g^{cg}\Gamma_{gf}^b\partial_e \Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{hc}\Gamma_{hf}^g\Gamma_{ge}^a\Gamma_{cd}^b - g^{gh}\Gamma_{hf}^c\Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\partial_f \Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\Gamma_{ge}^a\partial_f \Gamma_{cd}^b \\ & - g^{hg}\Gamma_{hf}^a\Gamma_{ge}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^g\Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\partial_f \Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\partial_f \Gamma_{cd}^b + g^{gc}\Gamma_{gf}^a\partial_e \Gamma_{cd}^b + g^{ag}\Gamma_{gf}^c\partial_e \Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \end{aligned}$$

$$\text{gab03.106} := -g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{ac}\partial_{fe}\Gamma_{cd}^b$$

$$\begin{aligned} \text{gab04.101} := & \partial_g \left( -g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\partial_f\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\Gamma_{ie}^c\partial_f\Gamma_{cd}^a - g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a + g^{ci}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \right. \\ & + g^{ci}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ib}\Gamma_{if}^c\partial_e\Gamma_{cd}^a + g^{ci}\Gamma_{if}^b\partial_e\Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a + g^{ic}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a + g^{ic}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a \\ & \left. - g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\partial_f\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\Gamma_{ie}^c\partial_f\Gamma_{cd}^b + g^{ic}\Gamma_{if}^a\partial_e\Gamma_{cd}^b + g^{ai}\Gamma_{if}^c\partial_e\Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \right) \end{aligned}$$

$$\begin{aligned} \text{gab04.102} := & -\partial_g \left( g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a \right) - \partial_g \left( g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a \right) + \partial_g \left( g^{ib}\partial_f\Gamma_{ie}^c\Gamma_{cd}^a \right) + \partial_g \left( g^{ib}\Gamma_{ie}^c\partial_f\Gamma_{cd}^a \right) - \partial_g \left( g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a \right) \\ & - \partial_g \left( g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ci}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ci}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a \right) + \partial_g \left( g^{ib}\Gamma_{if}^c\partial_e\Gamma_{cd}^a \right) + \partial_g \left( g^{ci}\Gamma_{if}^b\partial_e\Gamma_{cd}^a \right) - \partial_g \left( g^{cb}\partial_{fe}\Gamma_{cd}^a \right) \\ & - \partial_g \left( g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a \right) - \partial_g \left( g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ic}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ic}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a \right) - \partial_g \left( g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b \right) \\ & - \partial_g \left( g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b \right) + \partial_g \left( g^{ai}\partial_f\Gamma_{ie}^c\Gamma_{cd}^b \right) + \partial_g \left( g^{ai}\Gamma_{ie}^c\partial_f\Gamma_{cd}^b \right) + \partial_g \left( g^{ic}\Gamma_{if}^a\partial_e\Gamma_{cd}^b \right) + \partial_g \left( g^{ai}\Gamma_{if}^c\partial_e\Gamma_{cd}^b \right) - \partial_g \left( g^{ac}\partial_{fe}\Gamma_{cd}^b \right) \end{aligned}$$

$$\begin{aligned} \text{gab04.103} := & -\partial_g g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a - g^{hb}\partial_g\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a - g^{hb}\Gamma_{hf}^i\partial_g\Gamma_{ie}^c\Gamma_{cd}^a - g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\partial_g\Gamma_{cd}^a - \partial_g g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a - g^{ih}\partial_g\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a \\ & - g^{ih}\Gamma_{hf}^b\partial_g\Gamma_{ie}^c\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\partial_g\Gamma_{cd}^a + \partial_g g^{ib}\partial_f\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\partial_{gf}\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\partial_f\Gamma_{ie}^c\partial_g\Gamma_{cd}^a + \partial_g g^{ib}\Gamma_{ie}^c\partial_f\Gamma_{cd}^a + g^{ib}\partial_g\Gamma_{ie}^c\partial_f\Gamma_{cd}^a \\ & + g^{ib}\Gamma_{ie}^c\partial_{gf}\Gamma_{cd}^a - \partial_g g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{hi}\partial_g\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{hi}\Gamma_{hf}^c\partial_g\Gamma_{ie}^b\Gamma_{cd}^a - g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\partial_g\Gamma_{cd}^a - \partial_g g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a \\ & - g^{ch}\partial_g\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^i\partial_g\Gamma_{ie}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\partial_g\Gamma_{cd}^a + \partial_g g^{ci}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a + g^{ci}\partial_{gf}\Gamma_{ie}^b\Gamma_{cd}^a + g^{ci}\partial_f\Gamma_{ie}^b\partial_g\Gamma_{cd}^a \\ & + \partial_g g^{ci}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ci}\partial_g\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ci}\Gamma_{ie}^b\partial_{gf}\Gamma_{cd}^a + \partial_g g^{ib}\Gamma_{if}^c\partial_e\Gamma_{cd}^a + g^{ib}\partial_g\Gamma_{if}^c\partial_e\Gamma_{cd}^a + g^{ib}\Gamma_{if}^c\partial_{ge}\Gamma_{cd}^a + \partial_g g^{ci}\Gamma_{if}^b\partial_e\Gamma_{cd}^a \\ & + g^{ci}\partial_g\Gamma_{if}^b\partial_e\Gamma_{cd}^a + g^{ci}\Gamma_{if}^b\partial_{ge}\Gamma_{cd}^a - \partial_g g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{cb}\partial_{gfe}\Gamma_{cd}^a - \partial_g g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{hc}\partial_g\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{hc}\Gamma_{hf}^i\partial_g\Gamma_{ie}^b\Gamma_{cd}^a \\ & - g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\partial_g\Gamma_{cd}^a - \partial_g g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\partial_g\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^c\partial_g\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\partial_g\Gamma_{cd}^a + \partial_g g^{ic}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \\ & + g^{ic}\partial_{gf}\Gamma_{ie}^b\Gamma_{cd}^a + g^{ic}\partial_f\Gamma_{ie}^b\partial_g\Gamma_{cd}^a + \partial_g g^{ic}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ic}\partial_g\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ic}\Gamma_{ie}^b\partial_{gf}\Gamma_{cd}^a - \partial_g g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b - g^{hi}\partial_g\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b \\ & - g^{hi}\Gamma_{hf}^a\partial_g\Gamma_{ie}^c\Gamma_{cd}^b - g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\partial_g\Gamma_{cd}^b - \partial_g g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\partial_g\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^i\partial_g\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\partial_g\Gamma_{cd}^b \\ & + \partial_g g^{ai}\partial_f\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\partial_{gf}\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\partial_f\Gamma_{ie}^c\partial_g\Gamma_{cd}^b + \partial_g g^{ai}\Gamma_{ie}^c\partial_f\Gamma_{cd}^b + g^{ai}\partial_g\Gamma_{ie}^c\partial_f\Gamma_{cd}^b + g^{ai}\Gamma_{ie}^c\partial_{gf}\Gamma_{cd}^b + \partial_g g^{ic}\Gamma_{if}^a\partial_e\Gamma_{cd}^b \\ & + g^{ic}\partial_g\Gamma_{if}^a\partial_e\Gamma_{cd}^b + g^{ic}\Gamma_{if}^a\partial_{ge}\Gamma_{cd}^b + \partial_g g^{ai}\Gamma_{if}^c\partial_e\Gamma_{cd}^b + g^{ai}\partial_g\Gamma_{if}^c\partial_e\Gamma_{cd}^b + g^{ai}\Gamma_{if}^c\partial_{ge}\Gamma_{cd}^b - \partial_g g^{ac}\partial_{fe}\Gamma_{cd}^b - g^{ac}\partial_{gfe}\Gamma_{cd}^b \end{aligned}$$



$$\begin{aligned} \text{gab04.106} := & g^{ib} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^a + g^{ib} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^a + g^{ci} \partial_f \Gamma_{ie}^b \partial_g \Gamma_{cd}^a + g^{ci} \partial_g \Gamma_{ie}^b \partial_f \Gamma_{cd}^a + g^{ib} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^a + g^{ci} \partial_g \Gamma_{if}^b \partial_e \Gamma_{cd}^a - g^{cb} \partial_{gfe} \Gamma_{cd}^a \\ & + g^{ic} \partial_f \Gamma_{ie}^a \partial_g \Gamma_{cd}^b + g^{ic} \partial_g \Gamma_{ie}^a \partial_f \Gamma_{cd}^b + g^{ai} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^b + g^{ai} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^b + g^{ic} \partial_g \Gamma_{if}^a \partial_e \Gamma_{cd}^b + g^{ai} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^b - g^{ac} \partial_{gfe} \Gamma_{cd}^b \end{aligned}$$

```
# prepare first six terms in the Taylor series expansion of g^{ab}(x)

term0:= @(gab00).
distribute (term0)                                # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                                # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                                # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                                # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                                # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                                # cdb(term5.200,term5)

end_stage_1 = time.time()
```

$$\begin{aligned} \text{term0.200} &:= g^{ab} \\ \text{term1.200} &:= 0 \\ \text{term2.200} &:= -g^{cb} \partial_e \Gamma_{cd}^a A^d A^e - g^{ac} \partial_e \Gamma_{cd}^b A^d A^e \\ \text{term3.200} &:= -g^{cb} \partial_{fe} \Gamma_{cd}^a A^d A^e A^f - g^{ac} \partial_{fe} \Gamma_{cd}^b A^d A^e A^f \end{aligned}$$

## Stage 2: Replace derivatives of $\Gamma$ with partial derivs of $R$

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json') # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json') # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json') # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json') # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json') # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.301,term2)
substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.302,term2)
distribute (term2) # cdb(term2.303,term2)

substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.301,term3)
substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.302,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.303,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.304,term3)
distribute (term3) # cdb(term3.305,term3)

substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.301,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.302,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.303,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.304,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.305,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.306,term4)
distribute (term4) # cdb(term4.307,term4)

substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.301,term5)
substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.302,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.303,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.304,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.305,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.306,term5)
```

```

substitute (term5,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term5.307,term5)
substitute (term5,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term5.308,term5)
distribute (term5) # cdb(term5.309,term5)

# -----
# this block only produces formatted output, it is not part of the main computation
# -----

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
+ (1/1) @(term1)
+ (1/2) @(term2)
+ (1/6) @(term3)
+ (1/24) @(term4)
+ (1/120) @(term5). # cdb(metric.301,metric)

substitute (metric,$A^{\{a\}} \rightarrow x^{\{a\}}$) # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1). # zero
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm1,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm2,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm3,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm4,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm5,$A^{\{a\}} \rightarrow x^{\{a\}}$)

# Manipulating these expressions is hampered by the presence of the partial derivative on Rabcd.
# Thus we can't freely raise/lower indices on the dRabcd terms. But we can do so on the first
# derivatives (since these are evaluated at x=0 where the connection vanishes).

```

```

substitute      (Xterm2,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm2.301,Xterm2)
substitute      (Xterm3,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm3.301,Xterm3)
substitute      (Xterm4,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm4.301,Xterm4)
substitute      (Xterm5,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm5.301,Xterm5)

substitute      (Xterm2,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm2.302,Xterm2)
substitute      (Xterm3,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm3.302,Xterm3)
substitute      (Xterm4,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm4.302,Xterm4)
substitute      (Xterm5,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm5.302,Xterm5)

substitute      (Xterm2,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm2.303,Xterm2)
substitute      (Xterm3,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm3.303,Xterm3)
substitute      (Xterm4,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm4.303,Xterm4)
substitute      (Xterm5,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm5.303,Xterm5)

substitute      (Xterm2,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm2.304,Xterm2)
substitute      (Xterm3,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm3.304,Xterm3)
substitute      (Xterm4,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm4.304,Xterm4)
substitute      (Xterm5,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm5.304,Xterm5)

sort_product    (Xterm2) # cdb(Xterm2.305,Xterm2)
sort_product    (Xterm3) # cdb(Xterm3.305,Xterm3)
sort_product    (Xterm4) # cdb(Xterm4.305,Xterm4)
sort_product    (Xterm5) # cdb(Xterm5.305,Xterm5)

rename_dummies  (Xterm2) # cdb(Xterm2.306,Xterm2)
rename_dummies  (Xterm3) # cdb(Xterm3.306,Xterm3)
rename_dummies  (Xterm4) # cdb(Xterm4.306,Xterm4)
rename_dummies  (Xterm5) # cdb(Xterm5.306,Xterm5)

canonicalise    (Xterm2) # cdb(Xterm2.307,Xterm2)
canonicalise    (Xterm3) # cdb(Xterm3.307,Xterm3)
canonicalise    (Xterm4) # cdb(Xterm4.307,Xterm4)
canonicalise    (Xterm5) # cdb(Xterm5.307,Xterm5)

```

# We can simplify Xterm2 and Xterm3 by careful juggling of the indices (swapping free indices on selected terms)

```

tmp = add_tags (Xterm2, '\\mu')          # cdb (tmp.001,tmp)
zoom (tmp, $\\mu_{1} Q??)$              # cdb (tmp.002,tmp)
substitute (tmp, $R^{b}_{c d}^{a} x^{c} x^{d} \to R^{a}_{c d}^{b} x^{c} x^{d}$) # cdb (tmp.003,tmp)
unzoom (tmp)
Xterm2 = clear_tags (tmp, '\\mu')        # cdb (Xterm2.401,Xterm2)

tmp = add_tags (Xterm3, '\\mu')          # cdb (tmp.011,tmp)
zoom (tmp, $\\mu_{1} Q??)$              # cdb (tmp.012,tmp)
substitute (tmp, $\\partial_{c} \{R^{b}_{d e}^{a}\} x^{c} x^{d} x^{e} \to \\partial_{c} \{R^{a}_{d e}^{b}\} x^{c} x^{d} x^{e}$) # cdb (tmp.013,
unzoom (tmp)
Xterm3 = clear_tags (tmp, '\\mu')        # cdb (Xterm3.401,Xterm3)

Xterm0 = reformat_xterm (Xterm0, 1)      # cdb(Xterm0.308,Xterm0)
Xterm2 = reformat_xterm (Xterm2, 3)      # cdb(Xterm2.308,Xterm2)
Xterm3 = reformat_xterm (Xterm3, 6)      # cdb(Xterm3.308,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)     # cdb(Xterm4.308,Xterm4)
Xterm5 = reformat_xterm (Xterm5,360)     # cdb(Xterm5.308,Xterm5)

# metric to 4th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).        # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3). # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4). # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5). # cdb (Metric6.301,Metric6)

# -----
# end of format block
# -----

end_stage_2 = time.time()

```



$$\text{term2.301} := -g^{cb}\partial_e\Gamma^a{}_{cd}A^dA^e - g^{ac}\partial_e\Gamma^b{}_{cd}A^dA^e$$

$$\text{term2.302} := -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$\text{term2.303} := -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$\text{term3.301} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\text{term3.302} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\text{term3.303} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\text{term3.304} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\text{term3.305} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\begin{aligned} \text{term4.301} := & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.302} := & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.303} := & g^{ib} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^a A^d A^e A^f A^g + g^{ib} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^a A^d A^e A^f A^g + g^{ci} \partial_f \Gamma_{ie}^b \partial_g \Gamma_{cd}^a A^d A^e A^f A^g \\ & + g^{ci} \partial_g \Gamma_{ie}^b \partial_f \Gamma_{cd}^a A^d A^e A^f A^g + g^{ib} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^a A^d A^e A^f A^g + g^{ci} \partial_g \Gamma_{if}^b \partial_e \Gamma_{cd}^a A^d A^e A^f A^g \\ & - g^{cb} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^a_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^a_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^a_{geh} R^h_{dfc} \right) + g^{ic} \partial_f \Gamma_{ie}^a \partial_g \Gamma_{cd}^b A^d A^e A^f A^g \\ & + g^{ic} \partial_g \Gamma_{ie}^a \partial_f \Gamma_{cd}^b A^d A^e A^f A^g + g^{ai} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^b A^d A^e A^f A^g + g^{ai} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^b A^d A^e A^f A^g + g^{ic} \partial_g \Gamma_{if}^a \partial_e \Gamma_{cd}^b A^d A^e A^f A^g \\ & + g^{ai} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^b A^d A^e A^f A^g - g^{ac} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^b_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^b_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^b_{geh} R^h_{dfc} \right) \end{aligned}$$

$$\begin{aligned} \text{term4.304} := & g^{ib} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^a A^d A^e A^f A^g + g^{ib} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^a A^d A^e A^f A^g + g^{ci} \partial_f \Gamma_{ie}^b \partial_g \Gamma_{cd}^a A^d A^e A^f A^g \\ & + g^{ci} \partial_g \Gamma_{ie}^b \partial_f \Gamma_{cd}^a A^d A^e A^f A^g + g^{ib} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^a A^d A^e A^f A^g + g^{ci} \partial_g \Gamma_{if}^b \partial_e \Gamma_{cd}^a A^d A^e A^f A^g \\ & - g^{cb} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^a{}_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} - \frac{1}{15} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} \right) + g^{ic} \partial_f \Gamma_{ie}^a \partial_g \Gamma_{cd}^b A^d A^e A^f A^g \\ & + g^{ic} \partial_g \Gamma_{ie}^a \partial_f \Gamma_{cd}^b A^d A^e A^f A^g + g^{ai} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^b A^d A^e A^f A^g + g^{ai} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^b A^d A^e A^f A^g + g^{ic} \partial_g \Gamma_{if}^a \partial_e \Gamma_{cd}^b A^d A^e A^f A^g \\ & + g^{ai} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^b A^d A^e A^f A^g - g^{ac} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^b{}_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} - \frac{1}{15} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc} \right) \end{aligned}$$

$$\begin{aligned} \text{term4.305} := & g^{ib} \partial_f \Gamma^c_{ie} \partial_g \Gamma^a_{cd} A^d A^e A^f A^g + g^{ib} \partial_g \Gamma^c_{ie} \partial_f \Gamma^a_{cd} A^d A^e A^f A^g + g^{ci} \partial_f \Gamma^b_{ie} \partial_g \Gamma^a_{cd} A^d A^e A^f A^g \\ & + g^{ci} \partial_g \Gamma^b_{ie} \partial_f \Gamma^a_{cd} A^d A^e A^f A^g + g^{ib} \partial_g \Gamma^c_{if} \partial_e \Gamma^a_{cd} A^d A^e A^f A^g + g^{ci} \partial_g \Gamma^b_{if} \partial_e \Gamma^a_{cd} A^d A^e A^f A^g \\ & - g^{cb} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^a_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^a_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^a_{geh} R^h_{dfc} \right) + g^{ic} \partial_f \Gamma^a_{ie} \partial_g \Gamma^b_{cd} A^d A^e A^f A^g \\ & + g^{ic} \partial_g \Gamma^a_{ie} \partial_f \Gamma^b_{cd} A^d A^e A^f A^g + g^{ai} \partial_f \Gamma^c_{ie} \partial_g \Gamma^b_{cd} A^d A^e A^f A^g + g^{ai} \partial_g \Gamma^c_{ie} \partial_f \Gamma^b_{cd} A^d A^e A^f A^g + g^{ic} \partial_g \Gamma^a_{if} \partial_e \Gamma^b_{cd} A^d A^e A^f A^g \\ & + g^{ai} \partial_g \Gamma^c_{if} \partial_e \Gamma^b_{cd} A^d A^e A^f A^g - g^{ac} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^b_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^b_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^b_{geh} R^h_{dfc} \right) \end{aligned}$$

$$\begin{aligned} \text{term4.306} := & \frac{1}{9} g^{ib} A^e A^f R^c_{efi} A^d A^g R^a_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c_{egi} A^d A^f R^a_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b_{efi} A^d A^g R^a_{dgc} \\ & + \frac{1}{9} g^{ci} A^e A^g R^b_{egi} A^d A^f R^a_{dfc} + \frac{1}{9} g^{ib} A^f A^g R^c_{fgi} A^d A^e R^a_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b_{fgi} A^d A^e R^a_{dec} \\ & - g^{cb} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^a_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^a_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^a_{geh} R^h_{dfc} \right) + \frac{1}{9} g^{ic} A^e A^f R^a_{efi} A^d A^g R^b_{dgc} \\ & + \frac{1}{9} g^{ic} A^e A^g R^a_{egi} A^d A^f R^b_{dfc} + \frac{1}{9} g^{ai} A^e A^f R^c_{efi} A^d A^g R^b_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c_{egi} A^d A^f R^b_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a_{fgi} A^d A^e R^b_{dec} \\ & + \frac{1}{9} g^{ai} A^f A^g R^c_{fgi} A^d A^e R^b_{dec} - g^{ac} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^b_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^b_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^b_{geh} R^h_{dfc} \right) \end{aligned}$$

$$\begin{aligned}
\text{term4.307} := & \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
& + \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^fA^e\partial_{ef}R^a_{dgc} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} \\
& + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
& + \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
& - \frac{3}{5}g^{ac}A^dA^gA^fA^e\partial_{ef}R^b_{dgc} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}
\end{aligned}$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f (7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg})$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f (7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg})$$

$$+ \frac{1}{360}x^c x^d x^e x^f x^g (3R^a{}_{cdh}\partial_e R^h{}_{fg}{}^b + 4\partial_c R^a{}_{deh}R^h{}_{fg}{}^b + 5\partial_c R^b{}_{deh}R^a{}_{fg}{}^h + 5\partial_c R^a{}_{deh}R^b{}_{fg}{}^h - 2g^{bh}\partial_{cde}R^a{}_{fgh} + 3R^b{}_{cdh}\partial_e R^h{}_{fg}{}^a + 4\partial_c R^b{}_{deh}R^h{}_{fg}{}^a$$

$$- 2g^{ah}\partial_{cde}R^b{}_{fgh})$$

## Stage 3: Replace partial derivs of $R$ with covariant derivs of $R$

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json') # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json') # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json') # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)    # cdb(term1.401,term1)
sort_product      (term1)    # cdb(term1.402,term1)
rename_dummies    (term1)    # cdb(term1.403,term1)
canonicalise      (term1)    # cdb(term1.404,term1)

eliminate_metric (term2)    # cdb(term2.401,term2)
sort_product      (term2)    # cdb(term2.402,term2)
rename_dummies    (term2)    # cdb(term2.403,term2)
canonicalise      (term2)    # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term3.402,term3)
distribute (term3)                                                # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c d b}\} ->  @(tmp02)$,repeat=True) # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True) # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term4.404,term4)
distribute (term4)                                                # cdb(term4.405,term4)
```

```

substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c d b}\} -> @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c b d}\} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{ef}\{R^{a}_{c d b}\} -> @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{ef}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()

```

$$\text{tmp01.403} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\text{tmp02.403} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\text{tmp03.403} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}$$

$$\text{term1.401} := 0$$

$$\text{term1.402} := 0$$

$$\text{term1.403} := 0$$

$$\text{term1.404} := 0$$

$$\text{term2.401} := -\frac{1}{3}A^d A^e R^a{}_{de}{}^b - \frac{1}{3}A^d A^e R^b{}_{de}{}^a$$

$$\text{term2.402} := -\frac{1}{3}A^d A^e R^a{}_{de}{}^b - \frac{1}{3}A^d A^e R^b{}_{de}{}^a$$

$$\text{term2.403} := -\frac{1}{3}A^c A^d R^a{}_{cd}{}^b - \frac{1}{3}A^c A^d R^b{}_{cd}{}^a$$

$$\text{term2.404} := -\frac{1}{3}A^c A^d R^a{}_{cd}{}^b - \frac{1}{3}A^c A^d R^b{}_{cd}{}^a$$

$$\text{term3.401} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfe} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfe} g^{bg}$$

$$\text{term3.402} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfe} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfe} g^{bg}$$

$$\text{term3.403} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfe} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfe} g^{bg}$$

$$\begin{aligned} \text{term4.401} := & \frac{1}{9}g^{ib}A^e A^f R^c{}_{efi}A^d A^g R^a{}_{dgc} + \frac{1}{9}g^{ib}A^e A^g R^c{}_{egi}A^d A^f R^a{}_{dfc} + \frac{1}{9}g^{ci}A^e A^f R^b{}_{efi}A^d A^g R^a{}_{dgc} + \frac{1}{9}g^{ci}A^e A^g R^b{}_{egi}A^d A^f R^a{}_{dfc} \\ & + \frac{1}{9}g^{ib}A^f A^g R^c{}_{fgi}A^d A^e R^a{}_{dec} + \frac{1}{9}g^{ci}A^f A^g R^b{}_{fgi}A^d A^e R^a{}_{dec} - \frac{3}{5}g^{cb}A^d A^g A^e A^f \nabla_{dg} R_{cef} g^{ah} \\ & + \frac{1}{15}g^{cb}A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} + \frac{1}{15}g^{cb}A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9}g^{ic}A^e A^f R^a{}_{efi}A^d A^g R^b{}_{dgc} + \frac{1}{9}g^{ic}A^e A^g R^a{}_{egi}A^d A^f R^b{}_{dfc} \\ & + \frac{1}{9}g^{ai}A^e A^f R^c{}_{efi}A^d A^g R^b{}_{dgc} + \frac{1}{9}g^{ai}A^e A^g R^c{}_{egi}A^d A^f R^b{}_{dfc} + \frac{1}{9}g^{ic}A^f A^g R^a{}_{fgi}A^d A^e R^b{}_{dec} + \frac{1}{9}g^{ai}A^f A^g R^c{}_{fgi}A^d A^e R^b{}_{dec} \\ & - \frac{3}{5}g^{ac}A^d A^g A^e A^f \nabla_{dg} R_{cef} g^{bh} + \frac{1}{15}g^{ac}A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15}g^{ac}A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc} \end{aligned}$$

$$\begin{aligned}
\text{term4.402} &:= \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
&+ \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cef}hg^{ah} \\
&+ \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
&+ \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
&- \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cef}hg^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{term4.403} &:= \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
&+ \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cef}hg^{ah} \\
&+ \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
&+ \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
&- \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cef}hg^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{term4.404} &:= \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
&+ \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cef}hg^{ah} \\
&+ \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
&+ \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
&- \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cef}hg^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}
\end{aligned}$$



$$\begin{aligned}
\text{term4.405} := & \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
& + \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{ah} \\
& + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
& + \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
& - \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}
\end{aligned}$$

## Stage 4: Build the Taylor series for $g_{ab}$ , reformatting and output

```
beg_stage_4 = time.time()

# final housekeeping

# lower the  $\{ab\}$  indices to  $\{uv\}$ 

tmp0 := g_{a u} g_{b v} @(term0).
tmp1 := g_{a u} g_{b v} @(term1).
tmp2 := g_{a u} g_{b v} @(term2).
tmp3 := g_{a u} g_{b v} @(term3).
tmp4 := g_{a u} g_{b v} @(term4).
tmp5 := g_{a u} g_{b v} @(term5).

distribute      (tmp1) # cdb(tmp1.501,tmp1)
eliminate_metric (tmp1) # cdb(tmp1.502,tmp1)
eliminate_kronecker (tmp1) # cdb(tmp1.503,tmp1)
tmp1 = flatten_Rabcd (tmp1)
canonicalise     (tmp1) # cdb(tmp1.506,tmp1)

distribute      (tmp2) # cdb(tmp2.501,tmp2)
eliminate_metric (tmp2) # cdb(tmp2.502,tmp2)
eliminate_kronecker (tmp2) # cdb(tmp2.503,tmp2)
tmp2 = flatten_Rabcd (tmp2)
canonicalise     (tmp2) # cdb(tmp2.506,tmp2)

distribute      (tmp3) # cdb(tmp3.501,tmp3)
eliminate_metric (tmp3) # cdb(tmp3.502,tmp3)
eliminate_kronecker (tmp3) # cdb(tmp3.503,tmp3)
tmp3 = flatten_Rabcd (tmp3)
canonicalise     (tmp3) # cdb(tmp3.506,tmp3)

distribute      (tmp4) # cdb(tmp4.501,tmp4)
eliminate_metric (tmp4) # cdb(tmp4.502,tmp4)
eliminate_kronecker (tmp4) # cdb(tmp4.503,tmp4)
tmp4 = flatten_Rabcd (tmp4)
canonicalise     (tmp4) # cdb(tmp4.506,tmp4)
```

```

distribute      (tmp5) # cdb(tmp5.501,tmp5)
eliminate_metric (tmp5) # cdb(tmp5.502,tmp5)
eliminate_kronecker (tmp5) # cdb(tmp5.503,tmp5)
tmp5 = flatten_Rabcd (tmp5)
canonicalise     (tmp5) # cdb(tmp5.506,tmp5)

# this is out final answer

# raise the  $_{uv}$  indices to  $^{ab}$ 

metric:=  $g^{\{a\} u} g^{\{b\} v} ($  @ (tmp0)
      + (1/1) @ (tmp1)
      + (1/2) @ (tmp2)
      + (1/6) @ (tmp3)
      + (1/24) @ (tmp4)
      + (1/120) @ (tmp5) ). # cdb(metric.500,metric)

distribute      (metric) # cdb(metric.501,metric)
eliminate_metric (metric) # cdb(metric.502,metric)
eliminate_kronecker (metric) # cdb(metric.503,metric)
metric = flatten_Rabcd (metric) # cdb(metric.504,metric)
canonicalise     (metric) # cdb(metric.505,metric)

substitute      (metric,$g_{\{a\} b} g^{\{b\} c} \rightarrow g_{\{a\}^{\{c\}}}$)
substitute      (metric,$g_{\{b\} a} g^{\{b\} c} \rightarrow g_{\{a\}^{\{c\}}}$)
substitute      (metric,$g_{\{b\} a} g^{\{c\} b} \rightarrow g_{\{a\}^{\{c\}}}$)
substitute      (metric,$g_{\{a\} b} g^{\{c\} b} \rightarrow g_{\{a\}^{\{c\}}}$)
eliminate_kronecker (metric) # cdb(metric.506,metric)
canonicalise     (metric) # cdb(metric.507,metric)

substitute (metric,$A^{\{a\}} \rightarrow x^{\{a\}}$) # cdb (metric.508,metric)

cdblib.create ('metric-inv.json')

cdblib.put (' $g^{ab}$ ',metric,'metric-inv.json')

# extract the terms of the metric in powers of x

```

```

term0 = get_xterm (metric,0)    # cdb(term0.501,term0)
term1 = get_xterm (metric,1)    # cdb(term1.501,term1)
term2 = get_xterm (metric,2)    # cdb(term2.501,term2)
term3 = get_xterm (metric,3)    # cdb(term3.501,term3)
term4 = get_xterm (metric,4)    # cdb(term4.501,term4)
term5 = get_xterm (metric,5)    # cdb(term5.501,term5)

cdblib.put ('g^ab_0',term0,'metric-inv.json')
cdblib.put ('g^ab_1',term1,'metric-inv.json')
cdblib.put ('g^ab_2',term2,'metric-inv.json')
cdblib.put ('g^ab_3',term3,'metric-inv.json')
cdblib.put ('g^ab_4',term4,'metric-inv.json')
cdblib.put ('g^ab_5',term5,'metric-inv.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3).  # cdb(metric4.501,metric4)

```

$$\text{term0.501} := g^{ab}$$

$$\text{term1.501} := 0$$

$$\text{term2.501} := \frac{1}{3}x^c x^d R_{cedf} g^{ae} g^{bf}$$

$$\text{term3.501} := \frac{1}{6}x^c x^d x^e \nabla_c R_{dfe} g^{af} g^{bg}$$

$$\text{term4.501} := \frac{1}{15}x^c x^d x^e x^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{egfh} g^{ag} g^{bh}$$

$$\text{term5.501} := \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}$$

$$\text{tmp2.501} := -\frac{1}{3}g_{au}g_{bv}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}g_{au}g_{bv}A^cA^dR^b{}_{cd}{}^a$$

$$\text{tmp2.502} := -\frac{1}{3}g_{bv}A^cA^dR_{ucd}{}^b - \frac{1}{3}g_{bv}A^cA^dR^b{}_{cd}{}^u$$

$$\text{tmp2.503} := -\frac{1}{3}g_{bv}A^cA^dR_{ucd}{}^b - \frac{1}{3}g_{bv}A^cA^dR^b{}_{cd}{}^u$$

$$\text{tmp2.506} := -\frac{2}{3}A^aA^bR_{uabc}g_{vd}g^{cd}$$

$$\text{tmp3.501} := -\frac{1}{2}g_{au}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfe}g^{ag} - \frac{1}{2}g_{au}g_{bv}g^{ac}A^dA^fA^e\nabla_dR_{cfe}g^{bg}$$

$$\text{tmp3.502} := -\frac{1}{2}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfe}g_u{}^g - \frac{1}{2}g_{bv}g_u{}^cA^dA^fA^e\nabla_dR_{cfe}g^{bg}$$

$$\text{tmp3.503} := -\frac{1}{2}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfe} - \frac{1}{2}g_{bv}A^dA^fA^e\nabla_dR_{ufe}g^{bg}$$

$$\text{tmp3.506} := -A^aA^bA^c\nabla_aR_{ubcd}g_{ve}g^{de}$$

$$\begin{aligned}
\text{tmp4.501} &:= \frac{1}{9}g_{au}g_{bv}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g_{au}g_{bv}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} \\
&+ \frac{1}{9}g_{au}g_{bv}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g_{au}g_{bv}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} \\
&- \frac{3}{5}g_{au}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{ah} + \frac{1}{15}g_{au}g_{bv}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g_{au}g_{bv}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} \\
&+ \frac{1}{9}g_{au}g_{bv}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g_{au}g_{bv}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} \\
&+ \frac{1}{9}g_{au}g_{bv}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g_{au}g_{bv}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
&- \frac{3}{5}g_{au}g_{bv}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g_{au}g_{bv}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g_{au}g_{bv}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{tmp4.502} &:= \frac{1}{9}g_{bv}g^{ib}A^eA^fR^c_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ib}A^eA^gR^c_{egi}A^dA^fR_{udfc} + \frac{1}{9}g_{bv}g^{ci}A^eA^fR^b_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ci}A^eA^gR^b_{egi}A^dA^fR_{udfc} \\
&+ \frac{1}{9}g_{bv}g^{ib}A^fA^gR^c_{fgi}A^dA^eR_{udec} + \frac{1}{9}g_{bv}g^{ci}A^fA^gR^b_{fgi}A^dA^eR_{udec} - \frac{3}{5}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g_u^h \\
&+ \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugfh}R^h_{dec} + \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugeh}R^h_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^eA^fR_{uefi}A^dA^gR^b_{dgc} + \frac{1}{9}g_{bv}g^{ic}A^eA^gR_{uegi}A^dA^fR^b_{dfc} \\
&+ \frac{1}{9}g_{bv}g_u^iA^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g_{bv}g_u^iA^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^fA^gR_{ufgi}A^dA^eR^b_{dec} + \frac{1}{9}g_{bv}g_u^iA^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
&- \frac{3}{5}g_{bv}g_u^cA^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g_{bv}g_u^cA^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g_{bv}g_u^cA^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{tmp4.503} &:= \frac{1}{9}g_{bv}g^{ib}A^eA^fR^c_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ib}A^eA^gR^c_{egi}A^dA^fR_{udfc} + \frac{1}{9}g_{bv}g^{ci}A^eA^fR^b_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ci}A^eA^gR^b_{egi}A^dA^fR_{udfc} \\
&+ \frac{1}{9}g_{bv}g^{ib}A^fA^gR^c_{fgi}A^dA^eR_{udec} + \frac{1}{9}g_{bv}g^{ci}A^fA^gR^b_{fgi}A^dA^eR_{udec} - \frac{3}{5}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefu} \\
&+ \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugfh}R^h_{dec} + \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugeh}R^h_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^eA^fR_{uefi}A^dA^gR^b_{dgc} \\
&+ \frac{1}{9}g_{bv}g^{ic}A^eA^gR_{uegi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{bv}A^eA^fR^c_{efu}A^dA^gR^b_{dgc} + \frac{1}{9}g_{bv}A^eA^gR^c_{egu}A^dA^fR^b_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^fA^gR_{ufgi}A^dA^eR^b_{dec} \\
&+ \frac{1}{9}g_{bv}A^fA^gR^c_{fgu}A^dA^eR^b_{dec} - \frac{3}{5}g_{bv}A^dA^gA^eA^f\nabla_{dg}R_{uefh}g^{bh} + \frac{1}{15}g_{bv}A^dA^gA^fA^eR^b_{gfh}R^h_{deu} + \frac{1}{15}g_{bv}A^dA^gA^fA^eR^b_{geh}R^h_{dfu} \\
\text{tmp4.506} &:= -\frac{8}{5}A^aA^bA^cA^dR_{uabe}R_{cfdg}g_{vh}g^{ef}g^{gh} - \frac{6}{5}A^aA^bA^cA^d\nabla_{ab}R_{ucde}g_{vf}g^{ef}
\end{aligned}$$

$$\text{tmp5.506} := -4A^a A^b A^c A^d A^e R_{uabf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} - 4A^a A^b A^c A^d A^e R_{afbg} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{4}{3} A^a A^b A^c A^d A^e \nabla_{abc} R_{udef} g_{vg} g^{fg}$$



$$\begin{aligned} \text{metric.500} := & g^{au} g^{bv} \left( g_{cu} g_{dv} g^{cd} - \frac{1}{3} A^e A^f R_{uefc} g_{vd} g^{cd} - \frac{1}{6} A^f A^g A^c \nabla_f R_{ugcd} g_{ve} g^{de} - \frac{1}{15} A^i A^j A^c A^d R_{uije} R_{cfdg} g_{vh} g^{ef} g^{gh} - \frac{1}{20} A^i A^j A^c A^d \nabla_{ij} R_{ucde} g_{vf} g^{ef} \right. \\ & \left. - \frac{1}{30} A^j A^k A^c A^d A^e R_{ujkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} - \frac{1}{30} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} A^j A^k A^c A^d A^e \nabla_{jkc} R_{udef} g_{vg} g^{fg} \right) \end{aligned}$$

$$\begin{aligned} \text{metric.501} := & g^{au} g^{bv} g_{cu} g_{dv} g^{cd} - \frac{1}{3} g^{au} g^{bv} A^e A^f R_{uefc} g_{vd} g^{cd} - \frac{1}{6} g^{au} g^{bv} A^f A^g A^c \nabla_f R_{ugcd} g_{ve} g^{de} - \frac{1}{15} g^{au} g^{bv} A^i A^j A^c A^d R_{uije} R_{cfdg} g_{vh} g^{ef} g^{gh} \\ & - \frac{1}{20} g^{au} g^{bv} A^i A^j A^c A^d \nabla_{ij} R_{ucde} g_{vf} g^{ef} - \frac{1}{30} g^{au} g^{bv} A^j A^k A^c A^d A^e R_{ujkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\ & - \frac{1}{30} g^{au} g^{bv} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{au} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R_{udef} g_{vg} g^{fg} \end{aligned}$$

$$\begin{aligned} \text{metric.502} := & g^{bv} g_c^a g_{dv} g^{cd} - \frac{1}{3} g^{bv} A^e A^f R_{efc}^a g_{vd} g^{cd} - \frac{1}{6} g^{bv} A^f A^g A^c \nabla_f R_{gcd}^a g_{ve} g^{de} - \frac{1}{15} g^{bv} A^i A^j A^c A^d R_{ije}^a R_{cfdg} g_{vh} g^{ef} g^{gh} \\ & - \frac{1}{20} g^{bv} A^i A^j A^c A^d \nabla_{ij} R_{cde}^a g_{vf} g^{ef} - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jkf}^a \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\ & - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{deh}^a g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R_{def}^a g_{vg} g^{fg} \end{aligned}$$

$$\begin{aligned} \text{metric.503} := & g^{bv} g_{dv} g^{ad} - \frac{1}{3} g^{bv} A^e A^f R_{efc}^a g_{vd} g^{cd} - \frac{1}{6} g^{bv} A^f A^g A^c \nabla_f R_{gcd}^a g_{ve} g^{de} - \frac{1}{15} g^{bv} A^i A^j A^c A^d R_{ije}^a R_{cfdg} g_{vh} g^{ef} g^{gh} \\ & - \frac{1}{20} g^{bv} A^i A^j A^c A^d \nabla_{ij} R_{cde}^a g_{vf} g^{ef} - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jkf}^a \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\ & - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{deh}^a g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R_{def}^a g_{vg} g^{fg} \end{aligned}$$

$$\begin{aligned} \text{metric.504} := & g^{bc} g_{dc} g^{ad} - \frac{1}{3} A^c A^d R_{ecdf} g^{ae} g^{bg} g_{gh} g^{fh} - \frac{1}{6} A^c A^d A^e \nabla_d R_{fecg} g^{af} g^{bh} g_{hi} g^{gi} - \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{iefj} g^{ai} g^{bk} g_{kl} g^{jg} g^{hl} \\ & - \frac{1}{20} A^c A^d A^e A^f \nabla_{ef} R_{gcdh} g^{ag} g^{bi} g_{ij} g^{hj} - \frac{1}{30} A^c A^d A^e A^f A^g R_{hfgi} \nabla_c R_{djek} g^{ah} g^{bl} g_{lm} g^{ij} g^{km} \\ & - \frac{1}{30} A^c A^d A^e A^f A^g R_{fhgi} \nabla_c R_{jdek} g^{aj} g^{bl} g_{lm} g^{hk} g^{im} - \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{fgc} R_{hdei} g^{ah} g^{bj} g_{jk} g^{ik} \end{aligned}$$

$$\begin{aligned} \text{metric.505} := & g^{ac} g_{cd} g^{bd} + \frac{1}{3} A^c A^d R_{cedf} g^{ae} g^{bg} g_{gh} g^{fh} + \frac{1}{6} A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{bh} g_{hi} g^{gi} + \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{iefj} g^{ag} g^{bk} g_{kl} g^{hi} g^{jl} \\ & + \frac{1}{20} A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{bi} g_{ij} g^{hj} + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bl} g_{lm} g^{ij} g^{km} \\ & + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bl} g_{lm} g^{hk} g^{im} + \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{bj} g_{jk} g^{ik} \end{aligned}$$

$$\begin{aligned}
\text{metric.506} &:= g^{ba} + \frac{1}{3} A^c A^d R_{cedf} g^{ae} g^{fb} + \frac{1}{6} A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{gb} + \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{eifj} g^{ag} g^{hi} g^{jb} + \frac{1}{20} A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{hb} \\
&\quad + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{ij} g^{kb} + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{hk} g^{ib} + \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{ib} \\
\text{metric.507} &:= g^{ab} + \frac{1}{3} A^c A^d R_{cedf} g^{ae} g^{bf} + \frac{1}{6} A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{bg} + \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20} A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{bh} \\
&\quad + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi} \\
\text{metric.508} &:= g^{ab} + \frac{1}{3} x^c x^d R_{cedf} g^{ae} g^{bf} + \frac{1}{6} x^c x^d x^e \nabla_c R_{dfeg} g^{af} g^{bg} + \frac{1}{15} x^c x^d x^e x^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20} x^c x^d x^e x^f \nabla_{cd} R_{egfh} g^{ag} g^{bh} \\
&\quad + \frac{1}{30} x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30} x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90} x^c x^d x^e x^f x^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}
\end{aligned}$$

```

Rterm0 := @(term0).
Rterm1 := @(term1).  # zero
Rterm2 := @(term2).
Rterm3 := @(term3).
Rterm4 := @(term4).
Rterm5 := @(term5).

Rterm0 = reformat_xterm (Rterm0, 1)      # cdb(Rterm0.601,Rterm0)
Rterm2 = reformat_xterm (Rterm2, 3)      # cdb(Rterm2.601,Rterm2)
Rterm3 = reformat_xterm (Rterm3, 6)      # cdb(Rterm3.601,Rterm3)
Rterm4 = reformat_xterm (Rterm4, 60)     # cdb(Rterm4.601,Rterm4)
Rterm5 = reformat_xterm (Rterm5, 90)     # cdb(Rterm5.601,Rterm5)

Metric := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Rterm0, 1)      # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Rterm2, 3)      # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Rterm3, 6)      # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Rterm4, 60)     # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Rterm5, 90)     # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()

```

## The inverse metric in Riemann normal coordinates

$$\begin{aligned} g^{ab}(x) = & g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \frac{1}{60}x^c x^d x^e x^f (4g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3g^{ag} g^{bh} \nabla_{cd} R_{egfh}) \\ & + \frac{1}{90}x^c x^d x^e x^f x^g (3g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

## Curvature expansion of the inverse metric

$$g^{ab}(x) = \overset{0}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} + \overset{5}{g}{}^{ab} + \mathcal{O}(\epsilon^6)$$

$$\overset{0}{g}{}^{ab} = g^{ab}$$

$$3\overset{2}{g}{}^{ab} = x^c x^d g^{ae} g^{bf} R_{cedf}$$

$$6\overset{3}{g}{}^{ab} = x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg}$$

$$60\overset{4}{g}{}^{ab} = x^c x^d x^e x^f (4g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3g^{ag} g^{bh} \nabla_{cd} R_{egfh})$$

$$90\overset{5}{g}{}^{ab} = x^c x^d x^e x^f x^g (3g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi})$$

```

cdblib.create ('metric-inv.export')

cdblib.put ('g^ab_3',Metric3,'metric-inv.export')  # R and \partial R
cdblib.put ('g^ab_4',Metric4,'metric-inv.export')
cdblib.put ('g^ab_5',Metric5,'metric-inv.export')
cdblib.put ('g^ab_6',Metric6,'metric-inv.export')

cdblib.put ('g^ab', Metric, 'metric-inv.export')  # R and \nabla R

cdblib.put ('g^ab_scaled0',scaled0,'metric-inv.export')
cdblib.put ('g^ab_scaled2',scaled2,'metric-inv.export')
cdblib.put ('g^ab_scaled3',scaled3,'metric-inv.export')
cdblib.put ('g^ab_scaled4',scaled4,'metric-inv.export')
cdblib.put ('g^ab_scaled5',scaled5,'metric-inv.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)

```

# Timing

Stage 1: 3.3 secs

Stage 2: 5.8 secs

Stage 3: 97.9 secs

Stage 4: 7.2 secs