

The metric tensor in Riemann normal coordinates

In this notebook we compute the recursive sequences

$$g_{ab,d\underline{e}} = (g_{cb}\Gamma^c_{a(d),\underline{e}}) + (g_{ac}\Gamma^c_{b(d),\underline{e}}) \quad (1)$$

$$(n+3)\Gamma^a_{d(b,c\underline{e})} = (n+1) \left(R^a_{(bcd,\underline{e})} - (\Gamma^a_{f(c}\Gamma^f_{bd),\underline{e}}) \right) \quad (2)$$

for $n = 1, 2, 3, \dots$. Note in these equations that the (extended) index \underline{e} contains n normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g_{ab}(x) &= g_{ab} + g_{ab,c}x^c + \frac{1}{2!}g_{ab,cd}x^cx^d + \frac{1}{3!}g_{ab,cde}x^cx^dx^e + \dots \\ &= g_{ab} + \sum_{n=1}^{\infty} \frac{1}{n!} g_{ab,\underline{c}} x^{\underline{c}} \end{aligned}$$

Stage 1: Symmetrised partial derivatives of g_{ab}

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g_{ab,c}A^c &= 0 \\ g_{ab,cd}A^cA^d &= g_{cb}\partial_e\Gamma^c_{ad}A^dA^e + g_{ac}\partial_e\Gamma^c_{bd}A^dA^e \\ g_{ab,cde}A^cA^dA^e &= g_{cb}\partial_{fe}\Gamma^c_{ad}A^dA^eA^f + g_{ac}\partial_{fe}\Gamma^c_{bd}A^dA^eA^f \end{aligned}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

Now we use the results from `dGamma` to replace derivatives of Γ with partial derivatives of R . These were computed in `dGamma` using equation (2) above.

$$\begin{aligned}
g_{ab,c}A^c &= 0 \\
g_{ab,cd}A^cA^d &= \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb} \\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}
\end{aligned}$$

Stage 3: Replace partial derivs of R with covariant derivs of R

Next we use the results from `dRabcd` to replace the partial derivatives of R with covariant derivatives.

$$\begin{aligned}
g_{ab,c}A^c &= 0 \\
g_{ab,cd}A^cA^d &= -\frac{2}{3}A^cA^dR_{acbd} \\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}
\end{aligned}$$

Stage 4: Build the Taylor series for g_{ab} , reformatting and output

Each of the above expressions constitutes one term in the Taylor series for the metric. We also make the trivial change $A \rightarrow x$. Then we do some trivial reformatting.

$$\begin{aligned}
g_{ab}(x) &= g_{ab} + g_{ab,c}x^c + \frac{1}{2!}g_{ab,cd}x^cx^d + \frac{1}{3!}g_{ab,cde}x^cx^dx^e + \mathcal{O}(\epsilon^4) \\
&= g_{ab} - \frac{1}{3}x^cx^dR_{acbd} - \frac{1}{6}x^cx^dx^e\nabla_cR_{adbe} + \mathcal{O}(\epsilon^4)
\end{aligned}$$

Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d} -> g^{a e} R_{e b c d}$)
    substitute (obj,$R^{a}_{b}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R^{a b}_{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R^{a b c}_{d} -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}\{\Gamma^{a}_{b c}\} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e}\{\Gamma^{a}_{b c}\} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f}\{\Gamma^{a}_{b c}\} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g h}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}\{\Gamma^{a}_{b c}\}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah = Ex("numx = " + str(n))
    keep_weight (foo,bah)

    return foo
```

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ g_{a b}              -> A003_{a b}              $)
    substitute (obj,$ g^{a b}              -> A004^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\} -> A010_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}   -> A009_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}     -> A008_{a b c d e f}    $)
    substitute (obj,$ \nabla_e\{R_{a b c d}\}        -> A007_{a b c d e}      $)
    substitute (obj,$ \partial_{e f g h}\{R_{a b c d}\} -> A014_{a b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}\{R_{a b c d}\}   -> A013_{a b c d e f g}   $)
    substitute (obj,$ \partial_{e f}\{R_{a b c d}\}      -> A012_{a b c d e f}    $)
    substitute (obj,$ \partial_e\{R_{a b c d}\}          -> A011_{a b c d e}      $)
    substitute (obj,$ \partial_{e f g h}\{R^{a}_{a}_{b c d}\} -> A018^{a}_{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}\{R^{a}_{a}_{b c d}\}   -> A017^{a}_{a}_{b c d e f g}   $)
    substitute (obj,$ \partial_{e f}\{R^{a}_{a}_{b c d}\}     -> A016^{a}_{a}_{b c d e f}    $)
    substitute (obj,$ \partial_e\{R^{a}_{a}_{b c d}\}        -> A015^{a}_{a}_{b c d e}      $)
    substitute (obj,$ R_{a b c d}            -> A005_{a b c d}        $)
    substitute (obj,$ R^{a}_{a}_{b c d}          -> A006^{a}_{a}_{b c d}      $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> A^{a}                $)
    substitute (obj,$ A002^{a}              -> x^{a}                $)
    substitute (obj,$ A003_{a b}            -> g_{a b}              $)
    substitute (obj,$ A004^{a b}            -> g^{a b}              $)
    substitute (obj,$ A005_{a b c d}        -> R_{a b c d}          $)
    substitute (obj,$ A006^{a}_{a}_{b c d}   -> R^{a}_{a}_{b c d}      $)
    substitute (obj,$ A007_{a b c d e}      -> \nabla_e\{R_{a b c d}\}   $)
    substitute (obj,$ A008_{a b c d e f}    -> \nabla_{e f}\{R_{a b c d}\} $)
    substitute (obj,$ A009_{a b c d e f g}  -> \nabla_{e f g}\{R_{a b c d}\} $)
    substitute (obj,$ A010_{a b c d e f g h} -> \nabla_{e f g h}\{R_{a b c d}\} $)
    substitute (obj,$ A011_{a b c d e}      -> \partial_e\{R_{a b c d}\}   $)
    substitute (obj,$ A012_{a b c d e f}    -> \partial_{e f}\{R_{a b c d}\} $)
    substitute (obj,$ A013_{a b c d e f g}  -> \partial_{e f g}\{R_{a b c d}\} $)
    substitute (obj,$ A014_{a b c d e f g h} -> \partial_{e f g h}\{R_{a b c d}\} $)
    substitute (obj,$ A015^{a}_{a}_{b c d e} -> \partial_e\{R^{a}_{a}_{b c d}\} $)

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substitute (obj,$ A016^{a}_{b c d e f}      -> \partial_{e f}{R^{a}_{b c d}}      $)
substitute (obj,$ A017^{a}_{b c d e f g}    -> \partial_{e f g}{R^{a}_{b c d}}    $)
substitute (obj,$ A018^{a}_{b c d e f g h}  -> \partial_{e f g h}{R^{a}_{b c d}}  $)

return obj

def reformat_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute      (bah)
    bah = product_sort (bah)
    rename_dummies  (bah)
    canonicalise    (bah)
    factor_out      (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute      (bah)
    factor_out      (bah,$x^{a?}$)
    return bah

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

```

```
\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
```

```
g_{a b}::Depends(\partial{#}).
```

```
R_{a b c d}::Depends(\partial{#}).
```

```
R^{a}_{b c d}::Depends(\partial{#}).
```

```
\Gamma^{a}_{b c}::Depends(\partial{#}).
```

```
R_{a b c d}::Depends(\nabla{#}).
```

```
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of g_{ab}

```
beg_stage_1 = time.time()

# symmetrised partial derivatives of g_{ab}

gab00:=g_{a b}. # cdb (gab00.101,gab00)

gab01:=g_{c b}\Gamma^{c}_{a d} + g_{a c}\Gamma^{c}_{b d}. # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @(gab01) }. # cdb (gab02.101,gab02)
distribute (gab02) # cdb (gab02.102,gab02)
product_rule (gab02) # cdb (gab02.103,gab02)
substitute (gab02, $\partial_{d}{g_{a b}} \rightarrow @(gab01)$) # cdb (gab02.104,gab02)
distribute (gab02) # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @(gab02) }. # cdb (gab03.101,gab03)
distribute (gab03) # cdb (gab03.102,gab03)
product_rule (gab03) # cdb (gab03.103,gab03)
substitute (gab03, $\partial_{d}{g_{a b}} \rightarrow @(gab01)$) # cdb (gab03.104,gab03)
distribute (gab03) # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @(gab03) }. # cdb (gab04.101,gab04)
distribute (gab04) # cdb (gab04.102,gab04)
product_rule (gab04) # cdb (gab04.103,gab04)
substitute (gab04, $\partial_{d}{g_{a b}} \rightarrow @(gab01)$) # cdb (gab04.104,gab04)
distribute (gab04) # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @(gab04) }. # cdb (gab05.101,gab05)
distribute (gab05) # cdb (gab05.102,gab05)
product_rule (gab05) # cdb (gab05.103,gab05)
substitute (gab05, $\partial_{d}{g_{a b}} \rightarrow @(gab01)$) # cdb (gab05.104,gab05)
distribute (gab05) # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00) # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01) # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02) # cdb (gab02.106,gab02)
gab03 = impose_rnc (gab03) # cdb (gab03.106,gab03)
```

```
gab04 = impose_rnc (gab04)    # cdb (gab04.106,gab04)  
gab05 = impose_rnc (gab05)    # cdb (gab05.106,gab05)
```


$$\begin{aligned}
\text{gab00.101} &:= g_{ab} \\
\text{gab00.102} &:= g_{ab} \\
\text{gab01.101} &:= g_{cb}\Gamma^c_{ad} + g_{ac}\Gamma^c_{bd} \\
\text{gab01.102} &:= 0
\end{aligned}$$

$$\begin{aligned}
\text{gab02.101} &:= \partial_e (g_{cb}\Gamma^c_{ad} + g_{ac}\Gamma^c_{bd}) \\
\text{gab02.102} &:= \partial_e (g_{cb}\Gamma^c_{ad}) + \partial_e (g_{ac}\Gamma^c_{bd}) \\
\text{gab02.103} &:= \partial_e g_{cb}\Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + \partial_e g_{ac}\Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd} \\
\text{gab02.104} &:= (g_{fb}\Gamma^f_{ce} + g_{cf}\Gamma^f_{be}) \Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + (g_{fc}\Gamma^f_{ae} + g_{af}\Gamma^f_{ce}) \Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd} \\
\text{gab02.105} &:= g_{fb}\Gamma^f_{ce}\Gamma^c_{ad} + g_{cf}\Gamma^f_{be}\Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + g_{fc}\Gamma^f_{ae}\Gamma^c_{bd} + g_{af}\Gamma^f_{ce}\Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd} \\
\text{gab02.106} &:= g_{cb}\partial_e \Gamma^c_{ad} + g_{ac}\partial_e \Gamma^c_{bd}
\end{aligned}$$

$$\begin{aligned}
\text{gab03.101} &:= \partial_f (g_{gb}\Gamma^g_{ce}\Gamma^c_{ad} + g_{cg}\Gamma^g_{be}\Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + g_{gc}\Gamma^g_{ae}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd}) \\
\text{gab03.102} &:= \partial_f (g_{gb}\Gamma^g_{ce}\Gamma^c_{ad}) + \partial_f (g_{cg}\Gamma^g_{be}\Gamma^c_{ad}) + \partial_f (g_{cb}\partial_e \Gamma^c_{ad}) + \partial_f (g_{gc}\Gamma^g_{ae}\Gamma^c_{bd}) + \partial_f (g_{ag}\Gamma^g_{ce}\Gamma^c_{bd}) + \partial_f (g_{ac}\partial_e \Gamma^c_{bd}) \\
\text{gab03.103} &:= \partial_f g_{gb}\Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\partial_f \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\Gamma^g_{ce}\partial_f \Gamma^c_{ad} + \partial_f g_{cg}\Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\partial_f \Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\Gamma^g_{be}\partial_f \Gamma^c_{ad} + \partial_f g_{cb}\partial_e \Gamma^c_{ad} + g_{cb}\partial_{fe}\Gamma^c_{ad} \\
&\quad + \partial_f g_{gc}\Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\partial_f \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\Gamma^g_{ae}\partial_f \Gamma^c_{bd} + \partial_f g_{ag}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\partial_f \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\partial_f \Gamma^c_{bd} + \partial_f g_{ac}\partial_e \Gamma^c_{bd} + g_{ac}\partial_{fe}\Gamma^c_{bd} \\
\text{gab03.104} &:= (g_{hb}\Gamma^h_{gf} + g_{gh}\Gamma^h_{bf}) \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\partial_f \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\Gamma^g_{ce}\partial_f \Gamma^c_{ad} + (g_{hg}\Gamma^h_{cf} + g_{ch}\Gamma^h_{gf}) \Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\partial_f \Gamma^g_{be}\Gamma^c_{ad} \\
&\quad + g_{cg}\Gamma^g_{be}\partial_f \Gamma^c_{ad} + (g_{gb}\Gamma^g_{cf} + g_{cg}\Gamma^g_{bf}) \partial_e \Gamma^c_{ad} + g_{cb}\partial_{fe}\Gamma^c_{ad} + (g_{hc}\Gamma^h_{gf} + g_{gh}\Gamma^h_{cf}) \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\partial_f \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\Gamma^g_{ae}\partial_f \Gamma^c_{bd} \\
&\quad + (g_{hg}\Gamma^h_{af} + g_{ah}\Gamma^h_{gf}) \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\partial_f \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\partial_f \Gamma^c_{bd} + (g_{gc}\Gamma^g_{af} + g_{ag}\Gamma^g_{cf}) \partial_e \Gamma^c_{bd} + g_{ac}\partial_{fe}\Gamma^c_{bd} \\
\text{gab03.105} &:= g_{hb}\Gamma^h_{gf}\Gamma^g_{ce}\Gamma^c_{ad} + g_{gh}\Gamma^h_{bf}\Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\partial_f \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\Gamma^g_{ce}\partial_f \Gamma^c_{ad} + g_{hg}\Gamma^h_{cf}\Gamma^g_{be}\Gamma^c_{ad} + g_{ch}\Gamma^h_{gf}\Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\partial_f \Gamma^g_{be}\Gamma^c_{ad} \\
&\quad + g_{cg}\Gamma^g_{be}\partial_f \Gamma^c_{ad} + g_{gb}\Gamma^g_{cf}\partial_e \Gamma^c_{ad} + g_{cg}\Gamma^g_{bf}\partial_e \Gamma^c_{ad} + g_{cb}\partial_{fe}\Gamma^c_{ad} + g_{hc}\Gamma^h_{gf}\Gamma^g_{ae}\Gamma^c_{bd} + g_{gh}\Gamma^h_{cf}\Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\partial_f \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\Gamma^g_{ae}\partial_f \Gamma^c_{bd} \\
&\quad + g_{hg}\Gamma^h_{af}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ah}\Gamma^h_{gf}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\partial_f \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\partial_f \Gamma^c_{bd} + g_{gc}\Gamma^g_{af}\partial_e \Gamma^c_{bd} + g_{ag}\Gamma^g_{cf}\partial_e \Gamma^c_{bd} + g_{ac}\partial_{fe}\Gamma^c_{bd} \\
\text{gab03.106} &:= g_{cb}\partial_{fe}\Gamma^c_{ad} + g_{ac}\partial_{fe}\Gamma^c_{bd}
\end{aligned}$$

$$\begin{aligned} \text{gab04.101} := & \partial_g (g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\partial_f\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\Gamma_{ce}^i\partial_f\Gamma_{ad}^c + g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{ch}\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\partial_f\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\Gamma_{be}^i\partial_f\Gamma_{ad}^c \\ & + g_{ib}\Gamma_{cf}^i\partial_e\Gamma_{ad}^c + g_{ci}\Gamma_{bf}^i\partial_e\Gamma_{ad}^c + g_{cb}\partial_{fe}\Gamma_{ad}^c + g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ic}\partial_f\Gamma_{ae}^i\Gamma_{bd}^c + g_{ic}\Gamma_{ae}^i\partial_f\Gamma_{bd}^c + g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c \\ & + g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\partial_f\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\Gamma_{ce}^i\partial_f\Gamma_{bd}^c + g_{ic}\Gamma_{af}^i\partial_e\Gamma_{bd}^c + g_{ai}\Gamma_{cf}^i\partial_e\Gamma_{bd}^c + g_{ac}\partial_{fe}\Gamma_{bd}^c) \end{aligned}$$

$$\begin{aligned} \text{gab04.102} := & \partial_g (g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c) + \partial_g (g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c) + \partial_g (g_{ib}\partial_f\Gamma_{ce}^i\Gamma_{ad}^c) + \partial_g (g_{ib}\Gamma_{ce}^i\partial_f\Gamma_{ad}^c) + \partial_g (g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c) \\ & + \partial_g (g_{ch}\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c) + \partial_g (g_{ci}\partial_f\Gamma_{be}^i\Gamma_{ad}^c) + \partial_g (g_{ci}\Gamma_{be}^i\partial_f\Gamma_{ad}^c) + \partial_g (g_{ib}\Gamma_{cf}^i\partial_e\Gamma_{ad}^c) + \partial_g (g_{ci}\Gamma_{bf}^i\partial_e\Gamma_{ad}^c) + \partial_g (g_{cb}\partial_{fe}\Gamma_{ad}^c) \\ & + \partial_g (g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c) + \partial_g (g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c) + \partial_g (g_{ic}\partial_f\Gamma_{ae}^i\Gamma_{bd}^c) + \partial_g (g_{ic}\Gamma_{ae}^i\partial_f\Gamma_{bd}^c) + \partial_g (g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c) \\ & + \partial_g (g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c) + \partial_g (g_{ai}\partial_f\Gamma_{ce}^i\Gamma_{bd}^c) + \partial_g (g_{ai}\Gamma_{ce}^i\partial_f\Gamma_{bd}^c) + \partial_g (g_{ic}\Gamma_{af}^i\partial_e\Gamma_{bd}^c) + \partial_g (g_{ai}\Gamma_{cf}^i\partial_e\Gamma_{bd}^c) + \partial_g (g_{ac}\partial_{fe}\Gamma_{bd}^c) \end{aligned}$$

$$\begin{aligned} \text{gab04.103} := & \partial_g g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{hb}\partial_g\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{hb}\Gamma_{if}^h\partial_g\Gamma_{ce}^i\Gamma_{ad}^c + g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{ih}\partial_g\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c \\ & + g_{ih}\Gamma_{bf}^h\partial_g\Gamma_{ce}^i\Gamma_{ad}^c + g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ib}\partial_f\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\partial_{gf}\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\partial_f\Gamma_{ce}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ib}\Gamma_{ce}^i\partial_f\Gamma_{ad}^c + g_{ib}\partial_g\Gamma_{ce}^i\partial_f\Gamma_{ad}^c \\ & + g_{ib}\Gamma_{ce}^i\partial_{gf}\Gamma_{ad}^c + \partial_g g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{hi}\partial_g\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{hi}\Gamma_{cf}^h\partial_g\Gamma_{be}^i\Gamma_{ad}^c + g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ch}\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c \\ & + g_{ch}\partial_g\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{ch}\Gamma_{if}^h\partial_g\Gamma_{be}^i\Gamma_{ad}^c + g_{ch}\Gamma_{if}^h\Gamma_{be}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ci}\partial_f\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\partial_{gf}\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\partial_f\Gamma_{be}^i\partial_g\Gamma_{ad}^c \\ & + \partial_g g_{ci}\Gamma_{be}^i\partial_f\Gamma_{ad}^c + g_{ci}\partial_g\Gamma_{be}^i\partial_f\Gamma_{ad}^c + g_{ci}\Gamma_{be}^i\partial_{gf}\Gamma_{ad}^c + \partial_g g_{ib}\Gamma_{cf}^i\partial_e\Gamma_{ad}^c + g_{ib}\partial_g\Gamma_{cf}^i\partial_e\Gamma_{ad}^c + g_{ib}\Gamma_{cf}^i\partial_{ge}\Gamma_{ad}^c + \partial_g g_{ci}\Gamma_{bf}^i\partial_e\Gamma_{ad}^c \\ & + g_{ci}\partial_g\Gamma_{bf}^i\partial_e\Gamma_{ad}^c + g_{ci}\Gamma_{bf}^i\partial_{ge}\Gamma_{ad}^c + \partial_g g_{cb}\partial_{fe}\Gamma_{ad}^c + g_{cb}\partial_{ge}\Gamma_{ad}^c + \partial_g g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{hc}\partial_g\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{hc}\Gamma_{if}^h\partial_g\Gamma_{ae}^i\Gamma_{bd}^c \\ & + g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\partial_g\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\Gamma_{cf}^h\partial_g\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ic}\partial_f\Gamma_{ae}^i\Gamma_{bd}^c \\ & + g_{ic}\partial_{gf}\Gamma_{ae}^i\Gamma_{bd}^c + g_{ic}\partial_f\Gamma_{ae}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ic}\Gamma_{ae}^i\partial_f\Gamma_{bd}^c + g_{ic}\partial_g\Gamma_{ae}^i\partial_f\Gamma_{bd}^c + g_{ic}\Gamma_{ae}^i\partial_{gf}\Gamma_{bd}^c + \partial_g g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{hi}\partial_g\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c \\ & + g_{hi}\Gamma_{af}^h\partial_g\Gamma_{ce}^i\Gamma_{bd}^c + g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{ah}\partial_g\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{ah}\Gamma_{if}^h\partial_g\Gamma_{ce}^i\Gamma_{bd}^c + g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\partial_g\Gamma_{bd}^c \\ & + \partial_g g_{ai}\partial_f\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\partial_{gf}\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\partial_f\Gamma_{ce}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ai}\Gamma_{ce}^i\partial_f\Gamma_{bd}^c + g_{ai}\partial_g\Gamma_{ce}^i\partial_f\Gamma_{bd}^c + g_{ai}\Gamma_{ce}^i\partial_{gf}\Gamma_{bd}^c + \partial_g g_{ic}\Gamma_{af}^i\partial_e\Gamma_{bd}^c \\ & + g_{ic}\partial_g\Gamma_{af}^i\partial_e\Gamma_{bd}^c + g_{ic}\Gamma_{af}^i\partial_{ge}\Gamma_{bd}^c + \partial_g g_{ai}\Gamma_{cf}^i\partial_e\Gamma_{bd}^c + g_{ai}\partial_g\Gamma_{cf}^i\partial_e\Gamma_{bd}^c + g_{ai}\Gamma_{cf}^i\partial_{ge}\Gamma_{bd}^c + \partial_g g_{ac}\partial_{fe}\Gamma_{bd}^c + g_{ac}\partial_{gfe}\Gamma_{bd}^c \end{aligned}$$

$$\begin{aligned} \text{gab04.104} := & (g_{jb}\Gamma^j_{hg} + g_{hj}\Gamma^j_{bg})\Gamma^h_{if}\Gamma^i_{ce}\Gamma^c_{ad} + g_{hb}\partial_g\Gamma^h_{if}\Gamma^i_{ce}\Gamma^c_{ad} + g_{hb}\Gamma^h_{if}\partial_g\Gamma^i_{ce}\Gamma^c_{ad} + g_{hb}\Gamma^h_{if}\Gamma^i_{ce}\partial_g\Gamma^c_{ad} + (g_{jh}\Gamma^j_{ig} + g_{ij}\Gamma^j_{hg})\Gamma^h_{bf}\Gamma^i_{ce}\Gamma^c_{ad} \\ & + g_{ih}\partial_g\Gamma^h_{bf}\Gamma^i_{ce}\Gamma^c_{ad} + g_{ih}\Gamma^h_{bf}\partial_g\Gamma^i_{ce}\Gamma^c_{ad} + g_{ih}\Gamma^h_{bf}\Gamma^i_{ce}\partial_g\Gamma^c_{ad} + (g_{hb}\Gamma^h_{ig} + g_{ih}\Gamma^h_{bg})\partial_f\Gamma^i_{ce}\Gamma^c_{ad} + g_{ib}\partial_g\Gamma^i_{ce}\Gamma^c_{ad} + g_{ib}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{ad} \\ & + (g_{hb}\Gamma^h_{ig} + g_{ih}\Gamma^h_{bg})\Gamma^i_{ce}\partial_f\Gamma^c_{ad} + g_{ib}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{ad} + g_{ib}\Gamma^i_{ce}\partial_g\Gamma^c_{ad} + (g_{ji}\Gamma^j_{hg} + g_{hj}\Gamma^j_{ig})\Gamma^h_{cf}\Gamma^i_{be}\Gamma^c_{ad} + g_{hi}\partial_g\Gamma^h_{cf}\Gamma^i_{be}\Gamma^c_{ad} \\ & + g_{hi}\Gamma^h_{cf}\partial_g\Gamma^i_{be}\Gamma^c_{ad} + g_{hi}\Gamma^h_{cf}\Gamma^i_{be}\partial_g\Gamma^c_{ad} + (g_{jh}\Gamma^j_{cg} + g_{cj}\Gamma^j_{hg})\Gamma^h_{if}\Gamma^i_{be}\Gamma^c_{ad} + g_{ch}\partial_g\Gamma^h_{if}\Gamma^i_{be}\Gamma^c_{ad} + g_{ch}\Gamma^h_{if}\partial_g\Gamma^i_{be}\Gamma^c_{ad} \\ & + g_{ch}\Gamma^h_{if}\Gamma^i_{be}\partial_g\Gamma^c_{ad} + (g_{hi}\Gamma^h_{cg} + g_{ch}\Gamma^h_{ig})\partial_f\Gamma^i_{be}\Gamma^c_{ad} + g_{ci}\partial_g\Gamma^i_{be}\Gamma^c_{ad} + g_{ci}\partial_f\Gamma^i_{be}\partial_g\Gamma^c_{ad} + (g_{hi}\Gamma^h_{cg} + g_{ch}\Gamma^h_{ig})\Gamma^i_{be}\partial_f\Gamma^c_{ad} \\ & + g_{ci}\partial_g\Gamma^i_{be}\partial_f\Gamma^c_{ad} + g_{ci}\Gamma^i_{be}\partial_g\Gamma^c_{ad} + (g_{hb}\Gamma^h_{ig} + g_{ih}\Gamma^h_{bg})\Gamma^i_{cf}\partial_e\Gamma^c_{ad} + g_{ib}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{ad} + g_{ib}\Gamma^i_{cf}\partial_g\Gamma^c_{ad} + (g_{hi}\Gamma^h_{cg} + g_{ch}\Gamma^h_{ig})\Gamma^i_{bf}\partial_e\Gamma^c_{ad} \\ & + g_{ci}\partial_g\Gamma^i_{bf}\partial_e\Gamma^c_{ad} + g_{ci}\Gamma^i_{bf}\partial_g\Gamma^c_{ad} + (g_{hb}\Gamma^h_{cg} + g_{ch}\Gamma^h_{bg})\partial_{fe}\Gamma^c_{ad} + g_{cb}\partial_g\Gamma^c_{ad} + (g_{jc}\Gamma^j_{hg} + g_{hj}\Gamma^j_{cg})\Gamma^h_{if}\Gamma^i_{ae}\Gamma^c_{bd} + g_{hc}\partial_g\Gamma^h_{if}\Gamma^i_{ae}\Gamma^c_{bd} \\ & + g_{hc}\Gamma^h_{if}\partial_g\Gamma^i_{ae}\Gamma^c_{bd} + g_{hc}\Gamma^h_{if}\Gamma^i_{ae}\partial_g\Gamma^c_{bd} + (g_{jh}\Gamma^j_{ig} + g_{ij}\Gamma^j_{hg})\Gamma^h_{cf}\Gamma^i_{ae}\Gamma^c_{bd} + g_{ih}\partial_g\Gamma^h_{cf}\Gamma^i_{ae}\Gamma^c_{bd} + g_{ih}\Gamma^h_{cf}\partial_g\Gamma^i_{ae}\Gamma^c_{bd} + g_{ih}\Gamma^h_{cf}\Gamma^i_{ae}\partial_g\Gamma^c_{bd} \\ & + (g_{hc}\Gamma^h_{ig} + g_{ih}\Gamma^h_{cg})\partial_f\Gamma^i_{ae}\Gamma^c_{bd} + g_{ic}\partial_g\Gamma^i_{ae}\Gamma^c_{bd} + g_{ic}\partial_f\Gamma^i_{ae}\partial_g\Gamma^c_{bd} + (g_{hc}\Gamma^h_{ig} + g_{ih}\Gamma^h_{cg})\Gamma^i_{ae}\partial_f\Gamma^c_{bd} + g_{ic}\partial_g\Gamma^i_{ae}\partial_f\Gamma^c_{bd} + g_{ic}\Gamma^i_{ae}\partial_g\Gamma^c_{bd} \\ & + (g_{ji}\Gamma^j_{hg} + g_{hj}\Gamma^j_{ig})\Gamma^h_{af}\Gamma^i_{ce}\Gamma^c_{bd} + g_{hi}\partial_g\Gamma^h_{af}\Gamma^i_{ce}\Gamma^c_{bd} + g_{hi}\Gamma^h_{af}\partial_g\Gamma^i_{ce}\Gamma^c_{bd} + g_{hi}\Gamma^h_{af}\Gamma^i_{ce}\partial_g\Gamma^c_{bd} + (g_{jh}\Gamma^j_{ag} + g_{aj}\Gamma^j_{hg})\Gamma^h_{if}\Gamma^i_{ce}\Gamma^c_{bd} \\ & + g_{ah}\partial_g\Gamma^h_{if}\Gamma^i_{ce}\Gamma^c_{bd} + g_{ah}\Gamma^h_{if}\partial_g\Gamma^i_{ce}\Gamma^c_{bd} + g_{ah}\Gamma^h_{if}\Gamma^i_{ce}\partial_g\Gamma^c_{bd} + (g_{hi}\Gamma^h_{ag} + g_{ah}\Gamma^h_{ig})\partial_f\Gamma^i_{ce}\Gamma^c_{bd} + g_{ai}\partial_g\Gamma^i_{ce}\Gamma^c_{bd} + g_{ai}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{bd} \\ & + (g_{hi}\Gamma^h_{ag} + g_{ah}\Gamma^h_{ig})\Gamma^i_{ce}\partial_f\Gamma^c_{bd} + g_{ai}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{bd} + g_{ai}\Gamma^i_{ce}\partial_g\Gamma^c_{bd} + (g_{hc}\Gamma^h_{ig} + g_{ih}\Gamma^h_{cg})\Gamma^i_{af}\partial_e\Gamma^c_{bd} + g_{ic}\partial_g\Gamma^i_{af}\partial_e\Gamma^c_{bd} \\ & + g_{ic}\Gamma^i_{af}\partial_g\Gamma^c_{bd} + (g_{hi}\Gamma^h_{ag} + g_{ah}\Gamma^h_{ig})\Gamma^i_{cf}\partial_e\Gamma^c_{bd} + g_{ai}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{bd} + g_{ai}\Gamma^i_{cf}\partial_g\Gamma^c_{bd} + (g_{hc}\Gamma^h_{ag} + g_{ah}\Gamma^h_{cg})\partial_{fe}\Gamma^c_{bd} + g_{ac}\partial_g\Gamma^c_{bd} \end{aligned}$$

$$\begin{aligned} \text{gab04.106} := & g_{ib}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{ad} + g_{ib}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{ad} + g_{ci}\partial_f\Gamma^i_{be}\partial_g\Gamma^c_{ad} + g_{ci}\partial_g\Gamma^i_{be}\partial_f\Gamma^c_{ad} + g_{ib}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{ad} + g_{ci}\partial_g\Gamma^i_{bf}\partial_e\Gamma^c_{ad} + g_{cb}\partial_{gfe}\Gamma^c_{ad} \\ & + g_{ic}\partial_f\Gamma^i_{ae}\partial_g\Gamma^c_{bd} + g_{ic}\partial_g\Gamma^i_{ae}\partial_f\Gamma^c_{bd} + g_{ai}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{bd} + g_{ai}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{bd} + g_{ic}\partial_g\Gamma^i_{af}\partial_e\Gamma^c_{bd} + g_{ai}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{bd} + g_{ac}\partial_{gfe}\Gamma^c_{bd} \end{aligned}$$

```

# prepare first six terms in the Taylor series expansion of g_{ab}(x)

term0:= @(gab00).
distribute (term0)                # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                # cdb(term5.200,term5)

end_stage_1 = time.time()

```

$$\text{term0.200} := g_{ab}$$

$$\text{term1.200} := 0$$

$$\text{term2.200} := g_{cb} \partial_e \Gamma^c_{ad} A^d A^e + g_{ac} \partial_e \Gamma^c_{bd} A^d A^e$$

$$\text{term3.200} := g_{cb} \partial_{fe} \Gamma^c_{ad} A^d A^e A^f + g_{ac} \partial_{fe} \Gamma^c_{bd} A^d A^e A^f$$

Stage 2: Replace derivatives of Γ with partial derivs of R

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json') # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json') # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json') # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json') # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json') # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.301,term2)
substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.302,term2)
distribute (term2) # cdb(term2.303,term2)

substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.301,term3)
substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.302,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.303,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.304,term3)
distribute (term3) # cdb(term3.305,term3)

substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.301,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.302,term4)
substitute (term4,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.303,term4)
substitute (term4,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.304,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.305,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.306,term4)
distribute (term4) # cdb(term4.307,term4)

substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.301,term5)
substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.302,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.303,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.304,term5)
substitute (term5,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.305,term5)
substitute (term5,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.306,term5)
```

```

substitute (term5,$\partial_{c}\{\Gamma^{a}_{b\ d}\}A^{c}A^{b} \rightarrow @(\mathrm{d}\Gamma_{01})$,repeat=True) # cdb(term5.307,term5)
substitute (term5,$\partial_{c}\{\Gamma^{a}_{d\ b}\}A^{c}A^{b} \rightarrow @(\mathrm{d}\Gamma_{01})$,repeat=True) # cdb(term5.308,term5)
distribute (term5) # cdb(term5.309,term5)

end_stage_2 = time.time()

# -----
# this block of Xterms only produces formatted output, it's not part of the main computation
# -----

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
+ (1/1) @(term1) # zero
+ (1/2) @(term2)
+ (1/6) @(term3)
+ (1/24) @(term4)
+ (1/120) @(term5). # cdb(metric.301,metric)

substitute (metric,$A^{a} \rightarrow x^{a}$) # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1).
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{a} \rightarrow x^{a}$)
substitute (Xterm1,$A^{a} \rightarrow x^{a}$)
substitute (Xterm2,$A^{a} \rightarrow x^{a}$)
substitute (Xterm3,$A^{a} \rightarrow x^{a}$)
substitute (Xterm4,$A^{a} \rightarrow x^{a}$)
substitute (Xterm5,$A^{a} \rightarrow x^{a}$)

substitute (Xterm2,$g_{a\ b} \ \partial_{c}\{R^{b}_{d\ e\ f}\} \rightarrow \partial_{c}\{R_{a\ d\ e\ f}\}$) # cdb(Xterm2.301,Xterm2)

```

```

substitute (Xterm3,$g_{a b} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{a b} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{a b} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm5.301,Xterm5)

substitute (Xterm2,$g_{\{b a\}} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm2.301,Xterm2)
substitute (Xterm3,$g_{\{b a\}} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{\{b a\}} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{\{b a\}} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm5.301,Xterm5)

eliminate_metric (Xterm2) # cdb(Xterm2.302,Xterm2)
eliminate_metric (Xterm3) # cdb(Xterm3.302,Xterm3)
eliminate_metric (Xterm4) # cdb(Xterm4.302,Xterm4)
eliminate_metric (Xterm5) # cdb(Xterm5.302,Xterm5)

sort_product (Xterm2) # cdb(Xterm2.303,Xterm2)
sort_product (Xterm3) # cdb(Xterm3.303,Xterm3)
sort_product (Xterm4) # cdb(Xterm4.303,Xterm4)
sort_product (Xterm5) # cdb(Xterm5.303,Xterm5)

rename_dummies (Xterm2) # cdb(Xterm2.304,Xterm2)
rename_dummies (Xterm3) # cdb(Xterm3.304,Xterm3)
rename_dummies (Xterm4) # cdb(Xterm4.304,Xterm4)
rename_dummies (Xterm5) # cdb(Xterm5.304,Xterm5)

canonicalise (Xterm2) # cdb(Xterm2.305,Xterm2)
canonicalise (Xterm3) # cdb(Xterm3.305,Xterm3)
canonicalise (Xterm4) # cdb(Xterm4.305,Xterm4)
canonicalise (Xterm5) # cdb(Xterm5.305,Xterm5)

# push upper index to the left
def tidy_Rabcd (obj):
    substitute (obj,$R_{\{a b c\}}^{\{d\}} \rightarrow - R^{\{d\}}_{\{c a b\}})
    substitute (obj,$R_{\{a b\}}^{\{c\}}_{\{d\}} \rightarrow R^{\{c\}}_{\{d a b\}})
    substitute (obj,$R_{\{a\}}^{\{b\}}_{\{c d\}} \rightarrow - R^{\{b\}}_{\{a c d\}})
    return obj

Xterm0 = tidy_Rabcd (Xterm0) # cdb(Xterm0.666,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2) # cdb(Xterm2.666,Xterm2)

```

```

Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.666,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.666,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.666,Xterm5)

Xterm0 = reformat_xterm (Xterm0, 1)    # cdb(Xterm0.301,Xterm0)
Xterm2 = reformat_xterm (Xterm2, 3)    # cdb(Xterm2.301,Xterm2)
Xterm3 = reformat_xterm (Xterm3, 6)    # cdb(Xterm3.301,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)   # cdb(Xterm4.301,Xterm4)
Xterm5 = reformat_xterm (Xterm5,180)   # cdb(Xterm5.301,Xterm5)

# canonicalise from reformat_xterm will slide upper index from left hand side
# so now we slide the upper index back to the left

Xterm0 = tidy_Rabcd (Xterm0)  # cdb(Xterm0.667,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2)  # cdb(Xterm2.667,Xterm2)
Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.667,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.667,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.667,Xterm5)

# metric to 3rd, 4th, 5th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).      # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3).  # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).  # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (Metric6.301,Metric6)

# -----
# end of format block
# -----

```


$$\text{term2.301} := g_{cb}\partial_e\Gamma^c_{ad}A^dA^e + g_{ac}\partial_e\Gamma^c_{bd}A^dA^e$$

$$\text{term2.302} := \frac{1}{3}g_{cb}A^dA^eR^c_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c_{deb}$$

$$\text{term2.303} := \frac{1}{3}g_{cb}A^dA^eR^c_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c_{deb}$$

$$\text{term3.301} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.302} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.303} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.304} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.305} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\begin{aligned} \text{term4.301} := & g_{ib}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i_{be}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{ci}\partial_g\Gamma^i_{be}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i_{bf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfa}\right) + g_{ic}\partial_f\Gamma^i_{ae}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ic}\partial_g\Gamma^i_{ae}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i_{af}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ai}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.302} := & g_{ib}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i_{be}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{ci}\partial_g\Gamma^i_{be}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i_{bf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfa}\right) + g_{ic}\partial_f\Gamma^i_{ae}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ic}\partial_g\Gamma^i_{ae}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i_{af}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ai}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.303} := & g_{ib}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i_{be}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{ci}\partial_g\Gamma^i_{be}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i_{bf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfa}\right) + g_{ic}\partial_f\Gamma^i_{ae}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ic}\partial_g\Gamma^i_{ae}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i_{af}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ai}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.304} := & g_{ib} \partial_f \Gamma_{ce}^i \partial_g \Gamma_{ad}^c A^d A^e A^f A^g + g_{ib} \partial_g \Gamma_{ce}^i \partial_f \Gamma_{ad}^c A^d A^e A^f A^g + g_{ci} \partial_f \Gamma_{be}^i \partial_g \Gamma_{ad}^c A^d A^e A^f A^g \\ & + g_{ci} \partial_g \Gamma_{be}^i \partial_f \Gamma_{ad}^c A^d A^e A^f A^g + g_{ib} \partial_g \Gamma_{cf}^i \partial_e \Gamma_{ad}^c A^d A^e A^f A^g + g_{ci} \partial_g \Gamma_{bf}^i \partial_e \Gamma_{ad}^c A^d A^e A^f A^g \\ & + g_{cb} \left(\frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^c{}_{dga} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{ghf} R^h{}_{dea} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} \right) + g_{ic} \partial_f \Gamma_{ae}^i \partial_g \Gamma_{bd}^c A^d A^e A^f A^g \\ & + g_{ic} \partial_g \Gamma_{ae}^i \partial_f \Gamma_{bd}^c A^d A^e A^f A^g + g_{ai} \partial_f \Gamma_{ce}^i \partial_g \Gamma_{bd}^c A^d A^e A^f A^g + g_{ai} \partial_g \Gamma_{ce}^i \partial_f \Gamma_{bd}^c A^d A^e A^f A^g + g_{ic} \partial_g \Gamma_{af}^i \partial_e \Gamma_{bd}^c A^d A^e A^f A^g \\ & + g_{ai} \partial_g \Gamma_{cf}^i \partial_e \Gamma_{bd}^c A^d A^e A^f A^g + g_{ac} \left(\frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^c{}_{dgb} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{ghf} R^h{}_{deb} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb} \right) \end{aligned}$$

$$\begin{aligned} \text{term4.305} := & g_{ib} \partial_f \Gamma_{ce}^i \partial_g \Gamma_{ad}^c A^d A^e A^f A^g + g_{ib} \partial_g \Gamma_{ce}^i \partial_f \Gamma_{ad}^c A^d A^e A^f A^g + g_{ci} \partial_f \Gamma_{be}^i \partial_g \Gamma_{ad}^c A^d A^e A^f A^g \\ & + g_{ci} \partial_g \Gamma_{be}^i \partial_f \Gamma_{ad}^c A^d A^e A^f A^g + g_{ib} \partial_g \Gamma_{cf}^i \partial_e \Gamma_{ad}^c A^d A^e A^f A^g + g_{ci} \partial_g \Gamma_{bf}^i \partial_e \Gamma_{ad}^c A^d A^e A^f A^g \\ & + g_{cb} \left(\frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^c{}_{dga} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{gh} R^h{}_{dea} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} \right) + g_{ic} \partial_f \Gamma_{ae}^i \partial_g \Gamma_{bd}^c A^d A^e A^f A^g \\ & + g_{ic} \partial_g \Gamma_{ae}^i \partial_f \Gamma_{bd}^c A^d A^e A^f A^g + g_{ai} \partial_f \Gamma_{ce}^i \partial_g \Gamma_{bd}^c A^d A^e A^f A^g + g_{ai} \partial_g \Gamma_{ce}^i \partial_f \Gamma_{bd}^c A^d A^e A^f A^g + g_{ic} \partial_g \Gamma_{af}^i \partial_e \Gamma_{bd}^c A^d A^e A^f A^g \\ & + g_{ai} \partial_g \Gamma_{cf}^i \partial_e \Gamma_{bd}^c A^d A^e A^f A^g + g_{ac} \left(\frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^c{}_{dgb} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{gh} R^h{}_{deb} - \frac{1}{15} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb} \right) \end{aligned}$$

$$\begin{aligned} \text{term4.306} := & \frac{1}{9}g_{ib}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i{}_{efb}A^dA^gR^c{}_{dga} \\ & + \frac{1}{9}g_{ci}A^eA^gR^i{}_{egb}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ib}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i{}_{fgb}A^dA^eR^c{}_{dea} \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + \frac{1}{9}g_{ic}A^eA^fR^i{}_{efa}A^dA^gR^c{}_{dgb} \\ & + \frac{1}{9}g_{ic}A^eA^gR^i{}_{ega}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ai}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i{}_{fga}A^dA^eR^c{}_{deb} \\ & + \frac{1}{9}g_{ai}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{deb} + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right) \end{aligned}$$

$$\begin{aligned}
\text{term4.307} := & \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
& + \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^fA^e\partial_{ef}R^c_{dga} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} \\
& - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
& + \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
& + \frac{3}{5}g_{ac}A^dA^gA^fA^e\partial_{ef}R^c_{dgb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}
\end{aligned}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f (-3R_{bcdg}R^g_{fae} - 13R_{acdg}R^g_{fbe} - 9g_{bg}\partial_{cd}R^g_{fae} - 9g_{ag}\partial_{cd}R^g_{fbe})$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f (-3R_{bcdg}R^g_{fae} - 13R_{acdg}R^g_{fbe} - 9g_{bg}\partial_{cd}R^g_{fae} - 9g_{ag}\partial_{cd}R^g_{fbe}) \\ + \frac{1}{180}x^c x^d x^e x^f x^g (-3R^h_{dac}\partial_e R_{bfg h} - R_{bcdh}\partial_e R^h_{gaf} - 3R^h_{dbc}\partial_e R_{afgh} - g_{bh}\partial_{cde}R^h_{gaf} - R_{acd h}\partial_e R^h_{gbf} - g_{ah}\partial_{cde}R^h_{gbf})$$

Stage 3: Replace partial derivs of R with covariant derivs of R

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json') # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json') # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json') # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)    # cdb(term1.401,term1)
sort_product      (term1)    # cdb(term1.402,term1)
rename_dummies    (term1)    # cdb(term1.403,term1)
canonicalise      (term1)    # cdb(term1.404,term1)

eliminate_metric (term2)    # cdb(term2.401,term2)
sort_product      (term2)    # cdb(term2.402,term2)
rename_dummies    (term2)    # cdb(term2.403,term2)
canonicalise      (term2)    # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term3.402,term3)
distribute (term3)                                                # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c d b}\} ->  @(tmp02)$,repeat=True) # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True) # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term4.404,term4)
distribute (term4)                                                # cdb(term4.405,term4)
```

```

substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c d b}\} -> @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c b d}\} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{ef}\{R^{a}_{c d b}\} -> @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{ef}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()

```

$$\text{tmp01.403} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\text{tmp02.403} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\text{tmp03.403} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bf g j} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}$$

$$\text{term1.401} := 0$$

$$\text{term1.402} := 0$$

$$\text{term1.403} := 0$$

$$\text{term1.404} := 0$$

$$\text{term2.401} := \frac{1}{3}A^d A^e R_{bdea} + \frac{1}{3}A^d A^e R_{adeb}$$

$$\text{term2.402} := \frac{1}{3}A^d A^e R_{bdea} + \frac{1}{3}A^d A^e R_{adeb}$$

$$\text{term2.403} := \frac{1}{3}A^c A^d R_{bcd a} + \frac{1}{3}A^c A^d R_{acdb}$$

$$\text{term2.404} := -\frac{2}{3}A^c A^d R_{acbd}$$

$$\text{term3.401} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\text{term3.402} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\text{term3.403} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\begin{aligned} \text{term4.401} := & \frac{1}{9}g_{ib}A^e A^f R^i_{efc} A^d A^g R^c_{dga} + \frac{1}{9}g_{ib}A^e A^g R^i_{egc} A^d A^f R^c_{dfa} + \frac{1}{9}g_{ci}A^e A^f R^i_{efb} A^d A^g R^c_{dga} + \frac{1}{9}g_{ci}A^e A^g R^i_{egb} A^d A^f R^c_{dfa} \\ & + \frac{1}{9}g_{ib}A^f A^g R^i_{fgc} A^d A^e R^c_{dea} + \frac{1}{9}g_{ci}A^f A^g R^i_{fgb} A^d A^e R^c_{dea} + \frac{3}{5}g_{cb}A^d A^g A^e A^f \nabla_{dg} R_{acfh} g^{ch} \\ & - \frac{1}{15}g_{cb}A^d A^g A^f A^e R^c_{gfh} R^h_{dea} - \frac{1}{15}g_{cb}A^d A^g A^f A^e R^c_{geh} R^h_{dfa} + \frac{1}{9}g_{ic}A^e A^f R^i_{efa} A^d A^g R^c_{dgb} + \frac{1}{9}g_{ic}A^e A^g R^i_{ega} A^d A^f R^c_{dfb} \\ & + \frac{1}{9}g_{ai}A^e A^f R^i_{efc} A^d A^g R^c_{dgb} + \frac{1}{9}g_{ai}A^e A^g R^i_{egc} A^d A^f R^c_{dfb} + \frac{1}{9}g_{ic}A^f A^g R^i_{fga} A^d A^e R^c_{deb} + \frac{1}{9}g_{ai}A^f A^g R^i_{fgc} A^d A^e R^c_{deb} \\ & + \frac{3}{5}g_{ac}A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15}g_{ac}A^d A^g A^f A^e R^c_{gfh} R^h_{deb} - \frac{1}{15}g_{ac}A^d A^g A^f A^e R^c_{geh} R^h_{dfb} \end{aligned}$$

$$\begin{aligned}
\text{term4.402} &:= \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
&+ \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
&- \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
&+ \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
&+ \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb} \\
\text{term4.403} &:= \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
&+ \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
&- \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
&+ \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
&+ \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb} \\
\text{term4.404} &:= \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
&+ \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
&- \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
&+ \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
&+ \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}
\end{aligned}$$

$$\begin{aligned}
\text{term4.405} := & \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
& + \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
& - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
& + \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
& + \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}
\end{aligned}$$

Stage 4: Build the Taylor series for g_{ab} , reformatting and output

```
beg_stage_4 = time.time()
# final housekeeping

term1 = flatten_Rabcd (term1)          # cdb(term1.501,term1)
term2 = flatten_Rabcd (term2)          # cdb(term2.501,term2)
term3 = flatten_Rabcd (term3)          # cdb(term3.501,term3)
term4 = flatten_Rabcd (term4)          # cdb(term4.501,term4)
term5 = flatten_Rabcd (term5)          # cdb(term5.501,term5)

eliminate_metric (term1)
eliminate_metric (term2)
eliminate_metric (term3)
eliminate_metric (term4)
eliminate_metric (term5)

eliminate_kronecker (term1)
eliminate_kronecker (term2)
eliminate_kronecker (term3)
eliminate_kronecker (term4)
eliminate_kronecker (term5)

sort_product (term1)
sort_product (term2)
sort_product (term3)
sort_product (term4)
sort_product (term5)

rename_dummies (term1)
rename_dummies (term2)
rename_dummies (term3)
rename_dummies (term4)
rename_dummies (term5)

canonicalise (term1)                   # cdb(term1.502,term1)
canonicalise (term2)                   # cdb(term2.502,term2)
canonicalise (term3)                   # cdb(term3.502,term3)
```

```

canonicalise (term4)          # cdb(term4.502,term4)
canonicalise (term5)          # cdb(term5.502,term5)

# this is out final answer

metric:=@(term0)
      + (1/1) @(term1)
      + (1/2) @(term2)
      + (1/6) @(term3)
      + (1/24) @(term4)
      + (1/120) @(term5).      # cdb(metric.501,metric)

substitute (metric,$A^{a} -> x^{a}$) # cdb (metric.502,metric)

cdblib.create ('metric.json')

cdblib.put ('g_ab',metric,'metric.json')

# extract the terms of the metric in powers of x

term0 = get_xterm (metric,0)      # cdb(term0.503,term0)
term1 = get_xterm (metric,1)      # cdb(term1.503,term1)
term2 = get_xterm (metric,2)      # cdb(term2.503,term2)
term3 = get_xterm (metric,3)      # cdb(term3.503,term3)
term4 = get_xterm (metric,4)      # cdb(term4.503,term4)
term5 = get_xterm (metric,5)      # cdb(term5.503,term5)

cdblib.put ('g_ab_0',term0,'metric.json')
cdblib.put ('g_ab_1',term1,'metric.json')
cdblib.put ('g_ab_2',term2,'metric.json')
cdblib.put ('g_ab_3',term3,'metric.json')
cdblib.put ('g_ab_4',term4,'metric.json')
cdblib.put ('g_ab_5',term5,'metric.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3). # cdb(metric4.501,metric4)

```

$$\text{term2.501} := -\frac{2}{3}A^c A^d R_{acbd}$$

$$\text{term2.502} := -\frac{2}{3}A^c A^d R_{acbd}$$

$$\text{term3.501} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\text{term3.502} := -A^c A^d A^e \nabla_c R_{adbe}$$

$$\begin{aligned} \text{term4.501} := & \frac{1}{9}g_{ib}A^e A^f g^{ih} R_{hefc}A^d A^g g^{cj} R_{jdga} + \frac{1}{9}g_{ib}A^e A^g g^{ih} R_{hegc}A^d A^f g^{cj} R_{jdfa} + \frac{1}{9}g_{ci}A^e A^f g^{ih} R_{hefb}A^d A^g g^{cj} R_{jdga} \\ & + \frac{1}{9}g_{ci}A^e A^g g^{ih} R_{hegb}A^d A^f g^{cj} R_{jdfa} + \frac{1}{9}g_{ib}A^f A^g g^{ih} R_{hfgc}A^d A^e g^{cj} R_{jdea} + \frac{1}{9}g_{ci}A^f A^g g^{ih} R_{hfgb}A^d A^e g^{cj} R_{jdea} \\ & + \frac{3}{5}g_{cb}A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} - \frac{1}{15}g_{cb}A^d A^g A^f A^e g^{ci} R_{igfh} g^{hj} R_{jdea} - \frac{1}{15}g_{cb}A^d A^g A^f A^e g^{ci} R_{igeh} g^{hj} R_{jdfa} \\ & + \frac{1}{9}g_{ic}A^e A^f g^{ih} R_{hefa}A^d A^g g^{cj} R_{jdgb} + \frac{1}{9}g_{ic}A^e A^g g^{ih} R_{hega}A^d A^f g^{cj} R_{jdfb} + \frac{1}{9}g_{ai}A^e A^f g^{ih} R_{hefc}A^d A^g g^{cj} R_{jdgb} \\ & + \frac{1}{9}g_{ai}A^e A^g g^{ih} R_{hegc}A^d A^f g^{cj} R_{jdfb} + \frac{1}{9}g_{ic}A^f A^g g^{ih} R_{hfga}A^d A^e g^{cj} R_{jdeb} + \frac{1}{9}g_{ai}A^f A^g g^{ih} R_{hfgc}A^d A^e g^{cj} R_{jdeb} \\ & + \frac{3}{5}g_{ac}A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15}g_{ac}A^d A^g A^f A^e g^{ci} R_{igfh} g^{hj} R_{jdeb} - \frac{1}{15}g_{ac}A^d A^g A^f A^e g^{ci} R_{igeh} g^{hj} R_{jdfb} \end{aligned}$$

$$\text{term4.502} := \frac{16}{15}A^c A^d A^e A^f R_{acd g} R_{befh} g^{gh} - \frac{6}{5}A^c A^d A^e A^f \nabla_{cd} R_{aebf}$$

$$\begin{aligned}
\text{term5.501} := & \frac{1}{6}g_{ib}A^eA^gA^f\nabla_eR_{cgfj}g^{ij}A^dA^hg^{ck}R_{kdha} + \frac{1}{6}g_{ib}A^eA^hA^f\nabla_eR_{chfj}g^{ij}A^dA^gA^cR_{kdga} + \frac{1}{6}g_{ib}A^eA^fA^gR_{kefc}A^dA^hA^g\nabla_dR_{ahgj}g^{cj} \\
& + \frac{1}{6}g_{ib}A^eA^hA^g\nabla_eR_{chgj}g^{ij}A^dA^fA^cR_{kdfa} + \frac{1}{6}g_{ib}A^eA^gA^hR_{kegc}A^dA^hA^f\nabla_dR_{ahfj}g^{cj} + \frac{1}{6}g_{ib}A^eA^hA^gR_{kehc}A^dA^gA^f\nabla_dR_{agfj}g^{cj} \\
& + \frac{1}{6}g_{ci}A^eA^gA^f\nabla_eR_{bgfj}g^{ij}A^dA^hA^cR_{kdha} + \frac{1}{6}g_{ci}A^eA^hA^f\nabla_eR_{bhfj}g^{ij}A^dA^gA^cR_{kdga} + \frac{1}{6}g_{ci}A^eA^fA^gR_{kefb}A^dA^hA^g\nabla_dR_{ahgj}g^{cj} \\
& + \frac{1}{6}g_{ci}A^eA^hA^g\nabla_eR_{bhgj}g^{ij}A^dA^fA^cR_{kdfa} + \frac{1}{6}g_{ci}A^eA^gA^hR_{kegb}A^dA^hA^f\nabla_dR_{ahfj}g^{cj} + \frac{1}{6}g_{ci}A^eA^hA^gR_{kehb}A^dA^gA^f\nabla_dR_{agfj}g^{cj} \\
& + \frac{1}{6}g_{ib}A^fA^hA^g\nabla_fR_{chgj}g^{ij}A^dA^eA^cR_{kdea} + \frac{1}{6}g_{ib}A^fA^gA^hR_{kfgc}A^dA^hA^e\nabla_dR_{ahej}g^{cj} + \frac{1}{6}g_{ib}A^fA^hA^gR_{kfhc}A^dA^gA^e\nabla_dR_{agej}g^{cj} \\
& + \frac{1}{6}g_{ci}A^fA^hA^g\nabla_fR_{bhgj}g^{ij}A^dA^eA^cR_{kdea} + \frac{1}{6}g_{ci}A^fA^gA^hR_{kfgb}A^dA^hA^e\nabla_dR_{ahej}g^{cj} + \frac{1}{6}g_{ci}A^fA^hA^gR_{kfhb}A^dA^gA^e\nabla_dR_{agej}g^{cj} \\
& + \frac{1}{6}g_{kb}A^gA^hA^cR_{jghc}A^dA^fA^e\nabla_dR_{afei}g^{ci} + \frac{1}{6}g_{ck}A^gA^hA^cR_{jghb}A^dA^fA^e\nabla_dR_{afei}g^{ci} - \frac{1}{3}g_{cb}A^dA^hA^eA^fA^gR_{adhk}\nabla_eR_{figj}g^{ci}g^{kj} \\
& + \frac{1}{3}g_{cb}A^dA^hA^eA^fA^gR_{dkhi}\nabla_eR_{afgj}g^{ck}g^{ij} + \frac{2}{3}g_{cb}A^dA^hA^eA^fA^g\nabla_{dhe}R_{afgk}g^{ck} - \frac{1}{9}g_{cb}A^dA^hA^eA^fA^g\nabla_hR_{ifgj}g^{cj}A^eA^gR_{kdea} \\
& - \frac{1}{9}g_{cb}A^dA^hA^eA^g\nabla_hR_{iegj}g^{cj}A^fA^gR_{kdfa} - \frac{1}{9}g_{cb}A^dA^eA^f\nabla_dR_{aefj}g^{ij}A^hA^gA^cR_{khgi} - \frac{1}{9}g_{cb}A^dA^hA^eA^f\nabla_hR_{iefj}g^{cj}A^gA^hR_{kdga} \\
& - \frac{1}{9}g_{cb}A^dA^eA^g\nabla_dR_{aegj}g^{ij}A^hA^fA^cR_{khfi} - \frac{1}{9}g_{cb}A^dA^fA^g\nabla_dR_{afgj}g^{ij}A^hA^eA^cR_{khei} + \frac{1}{6}g_{ic}A^eA^gA^f\nabla_eR_{agfj}g^{ij}A^dA^hA^cR_{kdhb} \\
& + \frac{1}{6}g_{ic}A^eA^hA^f\nabla_eR_{ahfj}g^{ij}A^dA^gA^cR_{kdgb} + \frac{1}{6}g_{ic}A^eA^fA^gR_{kefa}A^dA^hA^g\nabla_dR_{bhgj}g^{cj} + \frac{1}{6}g_{ic}A^eA^hA^g\nabla_eR_{ahgj}g^{ij}A^dA^fA^cR_{kdfb} \\
& + \frac{1}{6}g_{ic}A^eA^gA^hR_{kega}A^dA^hA^f\nabla_dR_{bhfj}g^{cj} + \frac{1}{6}g_{ic}A^eA^hA^gR_{keha}A^dA^gA^f\nabla_dR_{bgfj}g^{cj} + \frac{1}{6}g_{ai}A^eA^gA^f\nabla_eR_{cgfj}g^{ij}A^dA^hA^cR_{kdhb} \\
& + \frac{1}{6}g_{ai}A^eA^hA^f\nabla_eR_{chfj}g^{ij}A^dA^gA^cR_{kdgb} + \frac{1}{6}g_{ai}A^eA^fA^gR_{kefc}A^dA^hA^g\nabla_dR_{bhgj}g^{cj} + \frac{1}{6}g_{ai}A^eA^hA^g\nabla_eR_{chgj}g^{ij}A^dA^fA^cR_{kdfb} \\
& + \frac{1}{6}g_{ai}A^eA^gA^hR_{kegc}A^dA^hA^f\nabla_dR_{bhfj}g^{cj} + \frac{1}{6}g_{ai}A^eA^hA^gR_{kehc}A^dA^gA^f\nabla_dR_{bgfj}g^{cj} + \frac{1}{6}g_{ic}A^fA^hA^g\nabla_fR_{ahgj}g^{ij}A^dA^eA^cR_{kdeb} \\
& + \frac{1}{6}g_{ic}A^fA^gA^hR_{kfga}A^dA^hA^e\nabla_dR_{bhej}g^{cj} + \frac{1}{6}g_{ic}A^fA^hA^gR_{kfha}A^dA^gA^e\nabla_dR_{bgej}g^{cj} + \frac{1}{6}g_{ai}A^fA^hA^g\nabla_fR_{chgj}g^{ij}A^dA^eA^cR_{kdeb} \\
& + \frac{1}{6}g_{ai}A^fA^gA^hR_{kfgc}A^dA^hA^e\nabla_dR_{bhej}g^{cj} + \frac{1}{6}g_{ai}A^fA^hA^gR_{kfhc}A^dA^gA^e\nabla_dR_{bgej}g^{cj} + \frac{1}{6}g_{kc}A^gA^hA^cR_{jgha}A^dA^fA^e\nabla_dR_{bfci}g^{ci} \\
& + \frac{1}{6}g_{ak}A^gA^hA^cR_{jghc}A^dA^fA^e\nabla_dR_{bfci}g^{ci} - \frac{1}{3}g_{ac}A^dA^hA^eA^fA^gR_{bdhk}\nabla_eR_{figj}g^{ci}g^{kj} + \frac{1}{3}g_{ac}A^dA^hA^eA^fA^gR_{dkhi}\nabla_eR_{bfgj}g^{ck}g^{ij} \\
& + \frac{2}{3}g_{ac}A^dA^hA^eA^fA^g\nabla_{dhe}R_{bfgk}g^{ck} - \frac{1}{9}g_{ac}A^dA^hA^eA^fA^g\nabla_hR_{ifgj}g^{cj}A^eA^gR_{kdeb} - \frac{1}{9}g_{ac}A^dA^hA^eA^g\nabla_hR_{iegj}g^{cj}A^fA^gR_{kdfb} \\
& - \frac{1}{9}g_{ac}A^dA^eA^f\nabla_dR_{befj}g^{ij}A^hA^gA^cR_{khgi} - \frac{1}{9}g_{ac}A^dA^hA^eA^f\nabla_hR_{iefj}g^{cj}A^gA^hR_{kdgb} \\
& - \frac{1}{9}g_{ac}A^dA^eA^g\nabla_dR_{begj}g^{ij}A^hA^fA^cR_{khfi} - \frac{1}{9}g_{ac}A^dA^fA^g\nabla_dR_{bfgj}g^{ij}A^hA^eA^cR_{khei}
\end{aligned}$$

$$\text{term5.502} := \frac{8}{3} A^c A^d A^e A^f A^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{8}{3} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{4}{3} A^c A^d A^e A^f A^g \nabla_{cde} R_{afbg}$$

$$\begin{aligned}
\text{metric.501} &:= g_{ab} - \frac{1}{3}A^cA^dR_{acbd} - \frac{1}{6}A^cA^dA^e\nabla_cR_{adbe} + \frac{2}{45}A^cA^dA^eA^fR_{acd g}R_{befh}g^{gh} - \frac{1}{20}A^cA^dA^eA^f\nabla_{cd}R_{aebf} \\
&\quad + \frac{1}{45}A^cA^dA^eA^fA^gR_{acd h}\nabla_eR_{bfgi}g^{hi} + \frac{1}{45}A^cA^dA^eA^fA^gR_{bcd h}\nabla_eR_{afgi}g^{hi} - \frac{1}{90}A^cA^dA^eA^fA^g\nabla_{cde}R_{afbg} \\
\text{metric.502} &:= g_{ab} - \frac{1}{3}x^cx^dR_{acbd} - \frac{1}{6}x^cx^dx^e\nabla_cR_{adbe} + \frac{2}{45}x^cx^dx^ex^fR_{acd g}R_{befh}g^{gh} - \frac{1}{20}x^cx^dx^ex^f\nabla_{cd}R_{aebf} \\
&\quad + \frac{1}{45}x^cx^dx^ex^fx^gR_{acd h}\nabla_eR_{bfgi}g^{hi} + \frac{1}{45}x^cx^dx^ex^fx^gR_{bcd h}\nabla_eR_{afgi}g^{hi} - \frac{1}{90}x^cx^dx^ex^fx^g\nabla_{cde}R_{afbg}
\end{aligned}$$

$$\text{term0.503} := g_{ab}$$

$$\text{term1.503} := 0$$

$$\text{term2.503} := -\frac{1}{3}x^c x^d R_{acbd}$$

$$\text{term3.503} := -\frac{1}{6}x^c x^d x^e \nabla_c R_{adb e}$$

$$\text{term4.503} := \frac{2}{45}x^c x^d x^e x^f R_{acd g} R_{bef h} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aeb f}$$

$$\text{term5.503} := \frac{1}{45}x^c x^d x^e x^f x^g R_{acd h} \nabla_e R_{bfg i} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcd h} \nabla_e R_{afg i} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg}$$


```

Xterm0 := @(term0).
Xterm1 := @(term1). # zero
Xterm2 := @(term2).
Xterm3 := @(term3).
Xterm4 := @(term4).
Xterm5 := @(term5).

Xterm0 = reformat_xterm (Xterm0, 1) # cdb(Xterm0.601,Xterm0)
Xterm2 = reformat_xterm (Xterm2, 3) # cdb(Xterm2.601,Xterm2)
Xterm3 = reformat_xterm (Xterm3, 6) # cdb(Xterm3.601,Xterm3)
Xterm4 = reformat_xterm (Xterm4,180) # cdb(Xterm4.601,Xterm4)
Xterm5 = reformat_xterm (Xterm5, 90) # cdb(Xterm5.601,Xterm5)

gab3 := @(Xterm0) + @(Xterm2). # cdb (gab3.601,gab3)
gab4 := @(Xterm0) + @(Xterm2) + @(Xterm3). # cdb (gab4.601,gab4)
gab5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4). # cdb (gab5.601,gab5)
gab6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5). # cdb (gab6.601,gab6)

Metric := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5). # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Xterm0, 1) # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Xterm2, 3) # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Xterm3, 6) # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Xterm4,180) # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Xterm5, 90) # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()

```

The metric in Riemann normal coordinates

$$\begin{aligned} g_{ab}(x) = & g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f (8g^{gh} R_{acd g} R_{b e f h} - 9\nabla_{cd} R_{a e b f}) \\ & + \frac{1}{90}x^c x^d x^e x^f x^g (2g^{hi} R_{acd h} \nabla_e R_{b f g i} + 2g^{hi} R_{bcd h} \nabla_e R_{a f g i} - \nabla_{cde} R_{a f b g}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

Curvature expansion of the metric

$$g_{ab}(x) = g_{ab}^0 + g_{ab}^2 + g_{ab}^3 + g_{ab}^4 + g_{ab}^5 + \mathcal{O}(\epsilon^6)$$

$$g_{ab}^0 = g_{ab}$$

$$3g_{ab}^2 = -x^c x^d R_{acbd}$$

$$6g_{ab}^3 = -x^c x^d x^e \nabla_c R_{adbe}$$

$$180g_{ab}^4 = x^c x^d x^e x^f (8g^{gh} R_{acd} R_{befh} - 9\nabla_{cd} R_{aebf})$$

$$90g_{ab}^5 = x^c x^d x^e x^f x^g (2g^{hi} R_{acd} \nabla_e R_{bfgi} + 2g^{hi} R_{bcd} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg})$$

```

cdblib.create ('metric.export')

cdblib.put ('g_ab_3',Metric3,'metric.export')  # R and \partial R
cdblib.put ('g_ab_4',Metric4,'metric.export')
cdblib.put ('g_ab_5',Metric5,'metric.export')
cdblib.put ('g_ab_6',Metric6,'metric.export')

cdblib.put ('g_ab', Metric, 'metric.export')  # R and \nabla R

cdblib.put ('g_ab_scaled0',scaled0,'metric.export')
cdblib.put ('g_ab_scaled2',scaled2,'metric.export')
cdblib.put ('g_ab_scaled3',scaled3,'metric.export')
cdblib.put ('g_ab_scaled4',scaled4,'metric.export')
cdblib.put ('g_ab_scaled5',scaled5,'metric.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)

```

Timing

Stage 1: 3.3 secs

Stage 2: 1.2 secs

Stage 3: 97.4 secs

Stage 4: 3.5 secs