The inverse metric tensor in Riemann normal coordinates

Here we calculate the Riemann normal expansion of the inverse metric, g^{ab}, by developing the recursive sequences

$$g^{ab}{}_{,d\underline{e}} = -\left(g^{cb}\Gamma^{a}{}_{c(d}\right){}_{,\underline{e})} - \left(g^{ac}\Gamma^{b}{}_{c(d}\right){}_{,\underline{e})} \tag{1}$$

$$(n+3)\Gamma^{a}{}_{d(b,c\underline{e})} = (n+1)\left(R^{a}{}_{(bc\dot{d},\underline{e})} - \left(\Gamma^{a}{}_{f(c}\Gamma^{f}{}_{b\dot{d}}\right){}_{,\underline{e})}\right) \tag{2}$$

for $n = 1, 2, 3, \cdots$. Note in these equations that the (extended) index $\underline{e}$ contains n normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g^{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^{c} + \frac{1}{2!}g^{ab}{}_{,cd}x^{c}x^{d} + \frac{1}{3!}g^{ab}{}_{,cde}x^{c}x^{d}x^{e} + \cdots \\ &= g^{ab} + \sum_{n=1}^{\infty}\frac{1}{n!}\,g^{ab}{}_{,\underline{c}}\,x^{\underline{c}} \end{aligned}$$

Stage 1: Symmetrised partial derivatives of g^{ab}

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g^{ab}{}_{,c}A^{c} &= 0 \\ g^{ab}{}_{,cd}A^{c}A^{d} &= -g^{cb}\partial_{e}\Gamma^{a}{}_{cd}A^{d}A^{e} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}A^{d}A^{e} \\ g^{ab}{}_{,cde}A^{c}A^{d}A^{e} &= -g^{cb}\partial_{fe}\Gamma^{a}{}_{cd}A^{d}A^{e}A^{f} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}A^{d}A^{e}A^{f} \end{aligned}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

Now we use the results from `dGamma` to replace derivatives of Γ with partial derivatives of R. These were computed in `dGamma` using equation (2) above.

$$g^{ab}{}_{,c}A^c = 0$$

$$g^{ab}{}_{,cd}A^cA^d = -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$g^{ab}{}_{,cde}A^cA^dA^e = -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

Stage 3: Replace partial derivs of R with covariant derivs of R

Next we use the results from `dRabcd` to replace the partial derivatives of R with covariant deriavtives.

$$g^{ab}{}_{,c}A^c = 0$$

$$g^{ab}{}_{,cd}A^cA^d = -\frac{1}{3}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}A^cA^dR^b{}_{cd}{}^a$$

$$g^{ab}{}_{,cde}A^cA^dA^e = -\frac{1}{2}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g^{ag} - \frac{1}{2}g^{ac}A^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

Each of the above expressions constitutues one term in the Taylor series for the metric. We also make the trivial change $A \to x$. Then we do some trivial reformatting.

$$\begin{aligned} g_{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^c + \frac{1}{2!}g^{ab}{}_{,cd}x^cx^d + \frac{1}{3!}g^{ab}{}_{,cde}x^cx^dx^e + \mathcal{O}\left(\epsilon^4\right) \\ &= g^{ab} + \frac{1}{3}x^cx^dR_{cedf}g^{ae}g^{bf} + \frac{1}{6}x^cx^dx^e\nabla_cR_{dfeg}g^{af}g^{bg} + \mathcal{O}\left(\epsilon^4\right) \end{aligned}$$

Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d}   -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d}   -> g^{d e} R_{a b c e}$)
    unwrap     (obj)
    sort_product   (obj)
    rename_dummies (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah  = Ex("numx = " + str(n))
    keep_weight (foo,bah)
```

```
    return foo

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                            -> A001^{a}                $)
    substitute (obj,$ x^{a}                            -> A002^{a}                $)
    substitute (obj,$ g_{a b}                          -> A003_{a b}              $)
    substitute (obj,$ g^{a b}                          -> A004^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}    -> A010_{a b c d e f g h}  $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}      -> A009_{a b c d e f g}    $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}        -> A008_{a b c d e f}      $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}          -> A007_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}{R_{a b c d}}  -> A014_{a b c d e f g h}  $)
    substitute (obj,$ \partial_{e f g}{R_{a b c d}}    -> A013_{a b c d e f g}    $)
    substitute (obj,$ \partial_{e f}{R_{a b c d}}      -> A012_{a b c d e f}      $)
    substitute (obj,$ \partial_{e}{R_{a b c d}}        -> A011_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}{R^{a}_{b c d}} -> A018^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}{R^{a}_{b c d}}  -> A017^{a}_{b c d e f g}  $)
    substitute (obj,$ \partial_{e f}{R^{a}_{b c d}}    -> A016^{a}_{b c d e f}    $)
    substitute (obj,$ \partial_{e}{R^{a}_{b c d}}      -> A015^{a}_{b c d e}      $)
    substitute (obj,$ R_{a b c d}                      -> A005_{a b c d}          $)
    substitute (obj,$ R^{a}_{b c d}                    -> A006^{a}_{b c d}        $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                 -> A^{a}                            $)
    substitute (obj,$ A002^{a}                 -> x^{a}                            $)
    substitute (obj,$ A003_{a b}               -> g_{a b}                          $)
    substitute (obj,$ A004^{a b}               -> g^{a b}                          $)
    substitute (obj,$ A005_{a b c d}           -> R_{a b c d}                      $)
    substitute (obj,$ A006^{a}_{b c d}         -> R^{a}_{b c d}                    $)
    substitute (obj,$ A007_{a b c d e}         -> \nabla_{e}{R_{a b c d}}          $)
    substitute (obj,$ A008_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}}        $)
    substitute (obj,$ A009_{a b c d e f g}     -> \nabla_{e f g}{R_{a b c d}}      $)
    substitute (obj,$ A010_{a b c d e f g h}   -> \nabla_{e f g h}{R_{a b c d}}    $)
    substitute (obj,$ A011_{a b c d e}         -> \partial_{e}{R_{a b c d}}        $)
    substitute (obj,$ A012_{a b c d e f}       -> \partial_{e f}{R_{a b c d}}      $)
    substitute (obj,$ A013_{a b c d e f g}     -> \partial_{e f g}{R_{a b c d}}    $)
```

```
    substitute (obj,$ A014_{a b c d e f g h}    -> \partial_{e f g h}{R_{a b c d}}   $)
    substitute (obj,$ A015^{a}_{b c d e}        -> \partial_{e}{R^{a}_{b c d}}       $)
    substitute (obj,$ A016^{a}_{b c d e f}      -> \partial_{e f}{R^{a}_{b c d}}     $)
    substitute (obj,$ A017^{a}_{b c d e f g}    -> \partial_{e f g}{R^{a}_{b c d}}   $)
    substitute (obj,$ A018^{a}_{b c d e f g h}  -> \partial_{e f g h}{R^{a}_{b c d}} $)

    return obj

def reformat_xterm (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute     (bah)
   bah = product_sort (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$x^{a?}$)
   ans := @(bah) / @(foo).
   return ans

def rescale_xterm (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute  (bah)
   factor_out  (bah,$x^{a?}$)
   return bah

def add_tags (obj,tag):
   n = 0
   ans = Ex('0')
   for i in obj.top().terms():
      foo = obj[i]
      bah = Ex(tag+'_{'+str(n)+'}')
      ans := @(ans) + @(bah) @(foo).
      n = n + 1
   return ans

def clear_tags (obj,tag):
   ans := @(obj).
```

```
    foo  = Ex(tag+'_{a?} -> 1')
    substitute (ans,foo)
    return ans

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of g^{ab}

```
beg_stage_1 = time.time()

# symmetrised partial derivatives of g^{ab}

gab00:=g^{a b}.                                               # cdb (gab00.101,gab00)

gab01:= - g^{c b}\Gamma^{a}_{c d} - g^{a c}\Gamma^{b}_{c d}. # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @(gab01) }.                               # cdb (gab02.101,gab02)
distribute   (gab02)                                           # cdb (gab02.102,gab02)
product_rule (gab02)                                           # cdb (gab02.103,gab02)
substitute   (gab02, $\partial_{d}{g^{a b}} -> @(gab01)$)      # cdb (gab02.104,gab02)
distribute   (gab02)                                           # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @(gab02) }.                               # cdb (gab03.101,gab03)
distribute   (gab03)                                           # cdb (gab03.102,gab03)
product_rule (gab03)                                           # cdb (gab03.103,gab03)
substitute   (gab03, $\partial_{d}{g^{a b}} -> @(gab01)$)      # cdb (gab03.104,gab03)
distribute   (gab03)                                           # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @(gab03) }.                               # cdb (gab04.101,gab04)
distribute   (gab04)                                           # cdb (gab04.102,gab04)
product_rule (gab04)                                           # cdb (gab04.103,gab04)
substitute   (gab04, $\partial_{d}{g^{a b}} -> @(gab01)$)      # cdb (gab04.104,gab04)
distribute   (gab04)                                           # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @(gab04) }.                               # cdb (gab05.101,gab05)
distribute   (gab05)                                           # cdb (gab05.102,gab05)
product_rule (gab05)                                           # cdb (gab05.103,gab05)
substitute   (gab05, $\partial_{d}{g^{a b}} -> @(gab01)$)      # cdb (gab05.104,gab05)
distribute   (gab05)                                           # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00)   # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01)   # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02)   # cdb (gab02.106,gab02)
```

```
gab03 = impose_rnc (gab03)   # cdb (gab03.106,gab03)
gab04 = impose_rnc (gab04)   # cdb (gab04.106,gab04)
gab05 = impose_rnc (gab05)   # cdb (gab05.106,gab05)
```

$$\texttt{gab00.101} := g^{ab}$$

$$\texttt{gab00.102} := g^{ab}$$

$$\texttt{gab01.101} := -g^{cb}\Gamma^{a}{}_{cd} - g^{ac}\Gamma^{b}{}_{cd}$$

$$\texttt{gab01.102} := 0$$

$$\texttt{gab02.101} := \partial_{e}\left(-g^{cb}\Gamma^{a}{}_{cd} - g^{ac}\Gamma^{b}{}_{cd}\right)$$

$$\texttt{gab02.102} := -\partial_{e}\left(g^{cb}\Gamma^{a}{}_{cd}\right) - \partial_{e}\left(g^{ac}\Gamma^{b}{}_{cd}\right)$$

$$\texttt{gab02.103} := -\partial_{e}g^{cb}\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - \partial_{e}g^{ac}\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab02.104} := -\left(-g^{fb}\Gamma^{c}{}_{fe} - g^{cf}\Gamma^{b}{}_{fe}\right)\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - \left(-g^{fc}\Gamma^{a}{}_{fe} - g^{af}\Gamma^{c}{}_{fe}\right)\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab02.105} := g^{fb}\Gamma^{c}{}_{fe}\Gamma^{a}{}_{cd} + g^{cf}\Gamma^{b}{}_{fe}\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} + g^{fc}\Gamma^{a}{}_{fe}\Gamma^{b}{}_{cd} + g^{af}\Gamma^{c}{}_{fe}\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab02.106} := -g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab03.101} := \partial_{f}\left(g^{gb}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} + g^{gc}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}\right)$$

$$\texttt{gab03.102} := \partial_{f}\left(g^{gb}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd}\right) + \partial_{f}\left(g^{cg}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd}\right) - \partial_{f}\left(g^{cb}\partial_{e}\Gamma^{a}{}_{cd}\right) + \partial_{f}\left(g^{gc}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd}\right) + \partial_{f}\left(g^{ag}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd}\right) - \partial_{f}\left(g^{ac}\partial_{e}\Gamma^{b}{}_{cd}\right)$$

$$\begin{aligned}\texttt{gab03.103} := {} & \partial_{f}g^{gb}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} + \partial_{f}g^{cg}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\partial_{f}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\Gamma^{b}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} - \partial_{f}g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} \\ & + \partial_{f}g^{gc}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\partial_{f}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\Gamma^{a}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} + \partial_{f}g^{ag}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} - \partial_{f}g^{ac}\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.104} := {} & \left(-g^{hb}\Gamma^{g}{}_{hf} - g^{gh}\Gamma^{b}{}_{hf}\right)\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} + \left(-g^{hg}\Gamma^{c}{}_{hf} - g^{ch}\Gamma^{g}{}_{hf}\right)\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\partial_{f}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} \\ & + g^{cg}\Gamma^{b}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} - \left(-g^{gb}\Gamma^{c}{}_{gf} - g^{cg}\Gamma^{b}{}_{gf}\right)\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} + \left(-g^{hc}\Gamma^{g}{}_{hf} - g^{gh}\Gamma^{c}{}_{hf}\right)\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\partial_{f}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} \\ & + g^{gc}\Gamma^{a}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} + \left(-g^{hg}\Gamma^{a}{}_{hf} - g^{ah}\Gamma^{g}{}_{hf}\right)\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} - \left(-g^{gc}\Gamma^{a}{}_{gf} - g^{ag}\Gamma^{c}{}_{gf}\right)\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.105} := {} & -g^{hb}\Gamma^{g}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} - g^{gh}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} - g^{hg}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{g}{}_{hf}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\partial_{f}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} \\ & + g^{cg}\Gamma^{b}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{gf}\partial_{e}\Gamma^{a}{}_{cd} + g^{cg}\Gamma^{b}{}_{gf}\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{hc}\Gamma^{g}{}_{hf}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} - g^{gh}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\partial_{f}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\Gamma^{a}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} \\ & - g^{hg}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{g}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} + g^{gc}\Gamma^{a}{}_{gf}\partial_{e}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{gf}\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\end{aligned}$$

$$\texttt{gab03.106} := -g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}$$

$$
\texttt{gab04.101} := \partial_g \big(-g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_f\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_f\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}
+ g^{ci}\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\partial_f\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd}
- g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_f\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd} \big)
$$

$$
\texttt{gab04.102} := -\partial_g \left(g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} \right) - \partial_g \left(g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} \right) + \partial_g \left(g^{ib}\partial_f\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} \right) + \partial_g \left(g^{ib}\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd} \right) - \partial_g \left(g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \right)
- \partial_g \left(g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \right) + \partial_g \left(g^{ci}\partial_f\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \right) + \partial_g \left(g^{ci}\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd} \right) + \partial_g \left(g^{ib}\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd} \right) + \partial_g \left(g^{ci}\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd} \right) - \partial_g \left(g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} \right)
- \partial_g \left(g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} \right) - \partial_g \left(g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} \right) + \partial_g \left(g^{ic}\partial_f\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} \right) + \partial_g \left(g^{ic}\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd} \right) - \partial_g \left(g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} \right)
- \partial_g \left(g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} \right) + \partial_g \left(g^{ai}\partial_f\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} \right) + \partial_g \left(g^{ai}\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd} \right) + \partial_g \left(g^{ic}\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd} \right) + \partial_g \left(g^{ai}\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd} \right) - \partial_g \left(g^{ac}\partial_{fe}\Gamma^{b}{}_{cd} \right)
$$

$$
\texttt{gab04.103} := -\partial_g g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\partial_g\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd} - \partial_g g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\partial_g\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}
- g^{ih}\Gamma^{b}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd} + \partial_g g^{ib}\partial_f\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd} + \partial_g g^{ib}\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ib}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd}
+ g^{ib}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} - \partial_g g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\partial_g\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\partial_g\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd} - \partial_g g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}
- g^{ch}\partial_g\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\partial_g\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd} + \partial_g g^{ci}\partial_f\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_{gf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_f\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd}
+ \partial_g g^{ci}\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ci}\partial_g\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} + \partial_g g^{ib}\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ib}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} + \partial_g g^{ci}\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd}
+ g^{ci}\partial_g\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} - \partial_g g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd} - \partial_g g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\partial_g\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hf}\partial_g\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}
- g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd} - \partial_g g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\partial_g\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\partial_g\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd} + \partial_g g^{ic}\partial_f\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}
+ g^{ic}\partial_{gf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\partial_f\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd} + \partial_g g^{ic}\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ic}\partial_g\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} - \partial_g g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\partial_g\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}
- g^{hi}\Gamma^{a}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd} - \partial_g g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\partial_g\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd}
+ \partial_g g^{ai}\partial_f\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd} + \partial_g g^{ai}\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ai}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} + \partial_g g^{ic}\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd}
+ g^{ic}\partial_g\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} + \partial_g g^{ai}\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ai}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} - \partial_g g^{ac}\partial_{fe}\Gamma^{b}{}_{cd} - g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}
$$

$$
\begin{aligned}
\texttt{gab04.104} := {} & - \left(-g^{jb}\Gamma^{h}{}_{jg} - g^{hj}\Gamma^{b}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} - \left(-g^{jh}\Gamma^{i}{}_{jg} - g^{ij}\Gamma^{h}{}_{jg}\right)\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} \\
& - g^{ih}\partial_{g}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + \left(-g^{hb}\Gamma^{i}{}_{hg} - g^{ih}\Gamma^{b}{}_{hg}\right)\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} \\
& + g^{ib}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + \left(-g^{hb}\Gamma^{i}{}_{hg} - g^{ih}\Gamma^{b}{}_{hg}\right)\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} - \left(-g^{ji}\Gamma^{h}{}_{jg} - g^{hj}\Gamma^{i}{}_{jg}\right)\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \\
& - g^{hi}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} - \left(-g^{jh}\Gamma^{c}{}_{jg} - g^{cj}\Gamma^{h}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \\
& - g^{ch}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + \left(-g^{hi}\Gamma^{c}{}_{hg} - g^{ch}\Gamma^{i}{}_{hg}\right)\partial_{f}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_{gf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_{f}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} \\
& + \left(-g^{hi}\Gamma^{c}{}_{hg} - g^{ch}\Gamma^{i}{}_{hg}\right)\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} + \left(-g^{hb}\Gamma^{i}{}_{hg} - g^{ih}\Gamma^{b}{}_{hg}\right)\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} \\
& + g^{ib}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} + \left(-g^{hi}\Gamma^{c}{}_{hg} - g^{ch}\Gamma^{i}{}_{hg}\right)\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} - \left(-g^{hb}\Gamma^{c}{}_{hg} - g^{ch}\Gamma^{b}{}_{hg}\right)\partial_{fe}\Gamma^{a}{}_{cd} - g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd} \\
& - \left(-g^{jc}\Gamma^{h}{}_{jg} - g^{hj}\Gamma^{c}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} - \left(-g^{jh}\Gamma^{i}{}_{jg} - g^{ij}\Gamma^{h}{}_{jg}\right)\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} \\
& - g^{ih}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + \left(-g^{hc}\Gamma^{i}{}_{hg} - g^{ih}\Gamma^{c}{}_{hg}\right)\partial_{f}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\partial_{gf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} \\
& + g^{ic}\partial_{f}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + \left(-g^{hc}\Gamma^{i}{}_{hg} - g^{ih}\Gamma^{c}{}_{hg}\right)\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} - \left(-g^{ji}\Gamma^{h}{}_{jg} - g^{hj}\Gamma^{i}{}_{jg}\right)\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} \\
& - g^{hi}\partial_{g}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\Gamma^{a}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} - \left(-g^{jh}\Gamma^{a}{}_{jg} - g^{aj}\Gamma^{h}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} \\
& - g^{ah}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + \left(-g^{hi}\Gamma^{a}{}_{hg} - g^{ah}\Gamma^{i}{}_{hg}\right)\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} \\
& + \left(-g^{hi}\Gamma^{a}{}_{hg} - g^{ah}\Gamma^{i}{}_{hg}\right)\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} + \left(-g^{hc}\Gamma^{i}{}_{hg} - g^{ih}\Gamma^{c}{}_{hg}\right)\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} \\
& + g^{ic}\Gamma^{a}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} + \left(-g^{hi}\Gamma^{a}{}_{hg} - g^{ah}\Gamma^{i}{}_{hg}\right)\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} - \left(-g^{hc}\Gamma^{a}{}_{hg} - g^{ah}\Gamma^{c}{}_{hg}\right)\partial_{fe}\Gamma^{b}{}_{cd} - g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.105} := {} & g^{jb}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{hj}\Gamma^{b}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + g^{jh}\Gamma^{i}{}_{jg}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} \\
& + g^{ij}\Gamma^{h}{}_{jg}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\partial_{g}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} \\
& + g^{ib}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} + g^{ji}\Gamma^{h}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \\
& + g^{hj}\Gamma^{i}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + g^{jh}\Gamma^{c}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{cj}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \\
& - g^{ch}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hg}\partial_{f}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_{gf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} \\
& + g^{ci}\partial_{f}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hg}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hg}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} \\
& - g^{ih}\Gamma^{b}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hg}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hg}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} \\
& + g^{ci}\Gamma^{b}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} + g^{hb}\Gamma^{c}{}_{hg}\partial_{fe}\Gamma^{a}{}_{cd} + g^{ch}\Gamma^{b}{}_{hg}\partial_{fe}\Gamma^{a}{}_{cd} - g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd} + g^{jc}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{hj}\Gamma^{c}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} \\
& - g^{hc}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + g^{jh}\Gamma^{i}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ij}\Gamma^{h}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} \\
& - g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hg}\partial_{f}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\partial_{gf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\partial_{f}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hg}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} \\
& - g^{ih}\Gamma^{c}{}_{hg}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} + g^{ji}\Gamma^{h}{}_{jg}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{hj}\Gamma^{i}{}_{jg}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\partial_{g}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} \\
& - g^{hi}\Gamma^{a}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + g^{jh}\Gamma^{a}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{aj}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} \\
& - g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} - g^{hi}\Gamma^{a}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} - g^{hi}\Gamma^{a}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} \\
& - g^{ah}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hg}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hg}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} \\
& - g^{hi}\Gamma^{a}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} + g^{hc}\Gamma^{a}{}_{hg}\partial_{fe}\Gamma^{b}{}_{cd} + g^{ah}\Gamma^{c}{}_{hg}\partial_{fe}\Gamma^{b}{}_{cd} - g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}
\end{aligned}
$$

gab04.106 $:= g^{ib}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ci}\partial_{f}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd} + g^{ic}\partial_{f}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ai}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}$

```
# prepare first six terms in the Taylor series expansion of g^{ab}(x)

term0:= @(gab00).
distribute (term0)                                    # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                                    # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                                    # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                                    # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                                    # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                                    # cdb(term5.200,term5)

end_stage_1 = time.time()
```

term0.200 $:= g^{ab}$

term1.200 $:= 0$

term2.200 $:= -g^{cb}\partial_{e}\Gamma^{a}{}_{cd}A^{d}A^{e} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}A^{d}A^{e}$

term3.200 $:= -g^{cb}\partial_{fe}\Gamma^{a}{}_{cd}A^{d}A^{e}A^{f} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}A^{d}A^{e}A^{f}$

Stage 2: Replace derivatives of Γ with partial derivs of R

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json')  # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json')  # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json')  # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json')  # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json')  # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term2.301,term2)
substitute (term2,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term2.302,term2)
distribute (term2)                                                                                          # cdb(term2.303,term2)

substitute (term3,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term3.301,term3)
substitute (term3,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term3.302,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term3.303,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term3.304,term3)
distribute (term3)                                                                                          # cdb(term3.305,term3)

substitute (term4,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term4.301,term4)
substitute (term4,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term4.302,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term4.303,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term4.304,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term4.305,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term4.306,term4)
distribute (term4)                                                                                          # cdb(term4.307,term4)

substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.301,term5)
substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.302,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term5.303,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term5.304,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term5.305,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term5.306,term5)
```

```
substitute (term5,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)               # cdb(term5.307,term5)
substitute (term5,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)               # cdb(term5.308,term5)
distribute (term5)                                                                                     # cdb(term5.309,term5)

# ----------------------------------------------------------------------------------
# this block only produces formatted output, it is not part of the main computation
# ----------------------------------------------------------------------------------

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
      + (1/1) @(term1)
      + (1/2) @(term2)
      + (1/6) @(term3)
      + (1/24) @(term4)
      + (1/120) @(term5).  # cdb(metric.301,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1).     # zero
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{a} -> x^{a}$)
substitute (Xterm1,$A^{a} -> x^{a}$)
substitute (Xterm2,$A^{a} -> x^{a}$)
substitute (Xterm3,$A^{a} -> x^{a}$)
substitute (Xterm4,$A^{a} -> x^{a}$)
substitute (Xterm5,$A^{a} -> x^{a}$)

# Manipulating these expressions is hampered by the presence of the partial derivative on Rabcd.
# Thus we can't freely rasie/lower indices on the dRabcd terms. But we can do so on the first
# derivatives (since these are evaluated at x=0 where the connection vanishes).
```

```
substitute      (Xterm2,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm2.301,Xterm2)
substitute      (Xterm3,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm3.301,Xterm3)
substitute      (Xterm4,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm4.301,Xterm4)
substitute      (Xterm5,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm5.301,Xterm5)

substitute      (Xterm2,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm2.302,Xterm2)
substitute      (Xterm3,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm3.302,Xterm3)
substitute      (Xterm4,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm4.302,Xterm4)
substitute      (Xterm5,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm5.302,Xterm5)

substitute      (Xterm2,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm2.303,Xterm2)
substitute      (Xterm3,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm3.303,Xterm3)
substitute      (Xterm4,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm4.303,Xterm4)
substitute      (Xterm5,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm5.303,Xterm5)

substitute      (Xterm2,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm2.304,Xterm2)
substitute      (Xterm3,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm3.304,Xterm3)
substitute      (Xterm4,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm4.304,Xterm4)
substitute      (Xterm5,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm5.304,Xterm5)

sort_product    (Xterm2)  # cdb(Xterm2.305,Xterm2)
sort_product    (Xterm3)  # cdb(Xterm3.305,Xterm3)
sort_product    (Xterm4)  # cdb(Xterm4.305,Xterm4)
sort_product    (Xterm5)  # cdb(Xterm5.305,Xterm5)

rename_dummies  (Xterm2)  # cdb(Xterm2.306,Xterm2)
rename_dummies  (Xterm3)  # cdb(Xterm3.306,Xterm3)
rename_dummies  (Xterm4)  # cdb(Xterm4.306,Xterm4)
rename_dummies  (Xterm5)  # cdb(Xterm5.306,Xterm5)

canonicalise    (Xterm2)  # cdb(Xterm2.307,Xterm2)
canonicalise    (Xterm3)  # cdb(Xterm3.307,Xterm3)
canonicalise    (Xterm4)  # cdb(Xterm4.307,Xterm4)
canonicalise    (Xterm5)  # cdb(Xterm5.307,Xterm5)

# We can simplify Xterm2 and Xterm3 by careful juggling of the indices (swapping free indices on selected terms)
```

```
tmp = add_tags (Xterm2,'\\mu')        # cdb (tmp.001,tmp)
zoom (tmp, $\mu_{1} Q??$)             # cdb (tmp.002,tmp)
substitute (tmp, $R^{b}_{c d}^{a} x^{c} x^{d} -> R^{a}_{c d}^{b} x^{c} x^{d}$)  # cdb (tmp.003,tmp)
unzoom (tmp)
Xterm2 = clear_tags (tmp,'\\mu')      # cdb (Xterm2.401,Xterm2)

tmp = add_tags (Xterm3,'\\mu')        # cdb (tmp.011,tmp)
zoom (tmp, $\mu_{1} Q??$)             # cdb (tmp.012,tmp)
substitute (tmp, $\partial_{c}{R^{b}_{d e}^{a}} x^{c} x^{d} x^{e} -> \partial_{c}{R^{a}_{d e}^{b}} x^{c} x^{d} x^{e}$)  # cdb (tmp.013,
unzoom (tmp)
Xterm3 = clear_tags (tmp,'\\mu')      # cdb (Xterm3.401,Xterm3)

Xterm0 = reformat_xterm (Xterm0,  1)    # cdb(Xterm0.308,Xterm0)
Xterm2 = reformat_xterm (Xterm2,  3)    # cdb(Xterm2.308,Xterm2)
Xterm3 = reformat_xterm (Xterm3,  6)    # cdb(Xterm3.308,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)    # cdb(Xterm4.308,Xterm4)
Xterm5 = reformat_xterm (Xterm5,360)    # cdb(Xterm5.308,Xterm5)

# metric to 4th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).                                       # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3).                           # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).               # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).   # cdb (Metric6.301,Metric6)

# ---------------------------------------------------------------------------------
# end of format block
# ---------------------------------------------------------------------------------

end_stage_2 = time.time()
```

$$\texttt{term2.301} := -g^{cb}\partial_e\Gamma^a{}_{cd}A^dA^e - g^{ac}\partial_e\Gamma^b{}_{cd}A^dA^e$$

$$\texttt{term2.302} := -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$\texttt{term2.303} := -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$\texttt{term3.301} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.302} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.303} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.304} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.305} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\begin{aligned}\texttt{term4.301} := {} & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)\end{aligned}$$

$$\begin{aligned}\texttt{term4.302} := {} & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)\end{aligned}$$

$$
\begin{aligned}
\texttt{term4.303} := {} & g^{ib}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd}A^dA^eA^fA^g \\
& + g^{ci}\partial_g\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd}A^dA^eA^fA^g \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{a}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{geh}R^{h}{}_{dfc}\right) + g^{ic}\partial_f\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd}A^dA^eA^fA^g \\
& + g^{ic}\partial_g\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd}A^dA^eA^fA^g \\
& + g^{ai}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{b}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{geh}R^{h}{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.304} := {} & g^{ib}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd}A^dA^eA^fA^g \\
& + g^{ci}\partial_g\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd}A^dA^eA^fA^g \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{a}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{geh}R^{h}{}_{dfc}\right) + g^{ic}\partial_f\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd}A^dA^eA^fA^g \\
& + g^{ic}\partial_g\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd}A^dA^eA^fA^g \\
& + g^{ai}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{b}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{geh}R^{h}{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.305} := {} & g^{ib}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd}A^dA^eA^fA^g \\
& + g^{ci}\partial_g\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd}A^dA^eA^fA^g \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{a}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{geh}R^{h}{}_{dfc}\right) + g^{ic}\partial_f\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd}A^dA^eA^fA^g \\
& + g^{ic}\partial_g\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd}A^dA^eA^fA^g \\
& + g^{ai}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{b}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{geh}R^{h}{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.306} := {} & \frac{1}{9}g^{ib}A^eA^fR^{c}{}_{efi}A^dA^gR^{a}{}_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^{c}{}_{egi}A^dA^fR^{a}{}_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^{b}{}_{efi}A^dA^gR^{a}{}_{dgc} \\
& + \frac{1}{9}g^{ci}A^eA^gR^{b}{}_{egi}A^dA^fR^{a}{}_{dfc} + \frac{1}{9}g^{ib}A^fA^gR^{c}{}_{fgi}A^dA^eR^{a}{}_{dec} + \frac{1}{9}g^{ci}A^fA^gR^{b}{}_{fgi}A^dA^eR^{a}{}_{dec} \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{a}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{a}{}_{geh}R^{h}{}_{dfc}\right) + \frac{1}{9}g^{ic}A^eA^fR^{a}{}_{efi}A^dA^gR^{b}{}_{dgc} \\
& + \frac{1}{9}g^{ic}A^eA^gR^{a}{}_{egi}A^dA^fR^{b}{}_{dfc} + \frac{1}{9}g^{ai}A^eA^fR^{c}{}_{efi}A^dA^gR^{b}{}_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^{c}{}_{egi}A^dA^fR^{b}{}_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^{a}{}_{fgi}A^dA^eR^{b}{}_{dec} \\
& + \frac{1}{9}g^{ai}A^fA^gR^{c}{}_{fgi}A^dA^eR^{b}{}_{dec} - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^{b}{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{gfh}R^{h}{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^{b}{}_{geh}R^{h}{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.307} := {} & \frac{1}{9} g^{ib} A^e A^f R^c{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c{}_{egi} A^d A^f R^a{}_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ci} A^e A^g R^b{}_{egi} A^d A^f R^a{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^f A^g R^c{}_{fgi} A^d A^e R^a{}_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b{}_{fgi} A^d A^e R^a{}_{dec} - \frac{3}{5} g^{cb} A^d A^g A^f A^e \partial_{ef} R^a{}_{dgc} + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} \\
& + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9} g^{ic} A^e A^f R^a{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ic} A^e A^g R^a{}_{egi} A^d A^f R^b{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^e A^f R^c{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c{}_{egi} A^d A^f R^b{}_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a{}_{fgi} A^d A^e R^b{}_{dec} + \frac{1}{9} g^{ai} A^f A^g R^c{}_{fgi} A^d A^e R^b{}_{dec} \\
& - \frac{3}{5} g^{ac} A^d A^g A^f A^e \partial_{ef} R^b{}_{dgc} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc}
\end{aligned}
$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f \left(7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg}\right)$$

$$\begin{aligned} g^{ab}(x) = g^{ab} &- \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f \left(7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg}\right) \\ &+ \frac{1}{360}x^c x^d x^e x^f x^g \left(3R^a{}_{cdh}\partial_e R^h{}_{fg}{}^b + 4\partial_c R^a{}_{deh}R^h{}_{fg}{}^b + 5\partial_c R^b{}_{deh}R^a{}_{fg}{}^h + 5\partial_c R^a{}_{deh}R^b{}_{fg}{}^h - 2g^{bh}\partial_{cde}R^a{}_{fgh} + 3R^b{}_{cdh}\partial_e R^h{}_{fg}{}^a + 4\partial_c R^b{}_{deh}R^h{}_{fg}{}^a \right. \\ &\left. - 2g^{ah}\partial_{cde}R^b{}_{fgh}\right) \end{aligned}$$

Stage 3: Replace partial derivs of R with covariant derivs of R

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json')  # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json')  # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json')  # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)   # cdb(term1.401,term1)
sort_product     (term1)   # cdb(term1.402,term1)
rename_dummies   (term1)   # cdb(term1.403,term1)
canonicalise     (term1)   # cdb(term1.404,term1)

eliminate_metric (term2)   # cdb(term2.401,term2)
sort_product     (term2)   # cdb(term2.402,term2)
rename_dummies   (term2)   # cdb(term2.403,term2)
canonicalise     (term2)   # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)         # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)         # cdb(term3.402,term3)
distribute (term3)                                                                                # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)  # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)  # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)         # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)         # cdb(term4.404,term4)
distribute (term4)                                                                                # cdb(term4.405,term4)
```

```
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c d b}} ->   @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c b d}} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()
```

$$\texttt{tmp01.403} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\texttt{tmp02.403} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\texttt{tmp03.403} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}$$

$$\texttt{term1.401} := 0$$
$$\texttt{term1.402} := 0$$
$$\texttt{term1.403} := 0$$
$$\texttt{term1.404} := 0$$

$$\texttt{term2.401} := -\frac{1}{3}A^d A^e R^a{}_{de}{}^b - \frac{1}{3}A^d A^e R^b{}_{de}{}^a$$
$$\texttt{term2.402} := -\frac{1}{3}A^d A^e R^a{}_{de}{}^b - \frac{1}{3}A^d A^e R^b{}_{de}{}^a$$
$$\texttt{term2.403} := -\frac{1}{3}A^c A^d R^a{}_{cd}{}^b - \frac{1}{3}A^c A^d R^b{}_{cd}{}^a$$
$$\texttt{term2.404} := -\frac{1}{3}A^c A^d R^a{}_{cd}{}^b - \frac{1}{3}A^c A^d R^b{}_{cd}{}^a$$

$$\texttt{term3.401} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfeg} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfeg} g^{bg}$$
$$\texttt{term3.402} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfeg} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfeg} g^{bg}$$
$$\texttt{term3.403} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfeg} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfeg} g^{bg}$$

$$\begin{aligned}\texttt{term4.401} := &\frac{1}{9}g^{ib}A^e A^f R^c{}_{efi}A^d A^g R^a{}_{dgc} + \frac{1}{9}g^{ib}A^e A^g R^c{}_{egi}A^d A^f R^a{}_{dfc} + \frac{1}{9}g^{ci}A^e A^f R^b{}_{efi}A^d A^g R^a{}_{dgc} + \frac{1}{9}g^{ci}A^e A^g R^b{}_{egi}A^d A^f R^a{}_{dfc}\\
&+ \frac{1}{9}g^{ib}A^f A^g R^c{}_{fgi}A^d A^e R^a{}_{dec} + \frac{1}{9}g^{ci}A^f A^g R^b{}_{fgi}A^d A^e R^a{}_{dec} - \frac{3}{5}g^{cb}A^d A^g A^e A^f \nabla_{dg}R_{cefh}g^{ah}\\
&+ \frac{1}{15}g^{cb}A^d A^g A^f A^e R^a{}_{gfh}R^h{}_{dec} + \frac{1}{15}g^{cb}A^d A^g A^f A^e R^a{}_{geh}R^h{}_{dfc} + \frac{1}{9}g^{ic}A^e A^f R^a{}_{efi}A^d A^g R^b{}_{dgc} + \frac{1}{9}g^{ic}A^e A^g R^a{}_{egi}A^d A^f R^b{}_{dfc}\\
&+ \frac{1}{9}g^{ai}A^e A^f R^c{}_{efi}A^d A^g R^b{}_{dgc} + \frac{1}{9}g^{ai}A^e A^g R^c{}_{egi}A^d A^f R^b{}_{dfc} + \frac{1}{9}g^{ic}A^f A^g R^a{}_{fgi}A^d A^e R^b{}_{dec} + \frac{1}{9}g^{ai}A^f A^g R^c{}_{fgi}A^d A^e R^b{}_{dec}\\
&- \frac{3}{5}g^{ac}A^d A^g A^e A^f \nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g^{ac}A^d A^g A^f A^e R^b{}_{gfh}R^h{}_{dec} + \frac{1}{15}g^{ac}A^d A^g A^f A^e R^b{}_{geh}R^h{}_{dfc}\end{aligned}$$

$$
\begin{aligned}
\texttt{term4.402} :=\ & \frac{1}{9} g^{ib} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} + \frac{1}{9} g^{ib} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} + \frac{1}{9} g^{ci} A^{e} A^{f} R^{b}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} + \frac{1}{9} g^{ci} A^{e} A^{g} R^{b}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} + \frac{1}{9} g^{ci} A^{f} A^{g} R^{b}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} - \frac{3}{5} g^{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{geh} R^{h}{}_{dfc} + \frac{1}{9} g^{ic} A^{e} A^{f} R^{a}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g^{ic} A^{e} A^{g} R^{a}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g^{ai} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g^{ic} A^{f} A^{g} R^{a}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} + \frac{1}{9} g^{ai} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} \\
& - \frac{3}{5} g^{ac} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{geh} R^{h}{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.403} :=\ & \frac{1}{9} g^{ib} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} + \frac{1}{9} g^{ib} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} + \frac{1}{9} g^{ci} A^{e} A^{f} R^{b}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} + \frac{1}{9} g^{ci} A^{e} A^{g} R^{b}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} + \frac{1}{9} g^{ci} A^{f} A^{g} R^{b}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} - \frac{3}{5} g^{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{geh} R^{h}{}_{dfc} + \frac{1}{9} g^{ic} A^{e} A^{f} R^{a}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g^{ic} A^{e} A^{g} R^{a}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g^{ai} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g^{ic} A^{f} A^{g} R^{a}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} + \frac{1}{9} g^{ai} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} \\
& - \frac{3}{5} g^{ac} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{geh} R^{h}{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.404} :=\ & \frac{1}{9} g^{ib} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} + \frac{1}{9} g^{ib} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} + \frac{1}{9} g^{ci} A^{e} A^{f} R^{b}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} + \frac{1}{9} g^{ci} A^{e} A^{g} R^{b}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} + \frac{1}{9} g^{ci} A^{f} A^{g} R^{b}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} - \frac{3}{5} g^{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{geh} R^{h}{}_{dfc} + \frac{1}{9} g^{ic} A^{e} A^{f} R^{a}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g^{ic} A^{e} A^{g} R^{a}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g^{ai} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g^{ic} A^{f} A^{g} R^{a}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} + \frac{1}{9} g^{ai} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} \\
& - \frac{3}{5} g^{ac} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{geh} R^{h}{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.405} := {}& \frac{1}{9} g^{ib} A^e A^f R^c{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c{}_{egi} A^d A^f R^a{}_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ci} A^e A^g R^b{}_{egi} A^d A^f R^a{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^f A^g R^c{}_{fgi} A^d A^e R^a{}_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b{}_{fgi} A^d A^e R^a{}_{dec} - \frac{3}{5} g^{cb} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9} g^{ic} A^e A^f R^a{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ic} A^e A^g R^a{}_{egi} A^d A^f R^b{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^e A^f R^c{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c{}_{egi} A^d A^f R^b{}_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a{}_{fgi} A^d A^e R^b{}_{dec} + \frac{1}{9} g^{ai} A^f A^g R^c{}_{fgi} A^d A^e R^b{}_{dec} \\
& - \frac{3}{5} g^{ac} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc}
\end{aligned}
$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

```
beg_stage_4 = time.time()

# final housekeeping

# lower the ^{ab} indices to _{uv}

tmp0 := g_{a u} g_{b v} @(term0).
tmp1 := g_{a u} g_{b v} @(term1).
tmp2 := g_{a u} g_{b v} @(term2).
tmp3 := g_{a u} g_{b v} @(term3).
tmp4 := g_{a u} g_{b v} @(term4).
tmp5 := g_{a u} g_{b v} @(term5).

distribute           (tmp1)  # cdb(tmp1.501,tmp1)
eliminate_metric     (tmp1)  # cdb(tmp1.502,tmp1)
eliminate_kronecker  (tmp1)  # cdb(tmp1.503,tmp1)
tmp1 = flatten_Rabcd (tmp1)
canonicalise         (tmp1)  # cdb(tmp1.506,tmp1)

distribute           (tmp2)  # cdb(tmp2.501,tmp2)
eliminate_metric     (tmp2)  # cdb(tmp2.502,tmp2)
eliminate_kronecker  (tmp2)  # cdb(tmp2.503,tmp2)
tmp2 = flatten_Rabcd (tmp2)
canonicalise         (tmp2)  # cdb(tmp2.506,tmp2)

distribute           (tmp3)  # cdb(tmp3.501,tmp3)
eliminate_metric     (tmp3)  # cdb(tmp3.502,tmp3)
eliminate_kronecker  (tmp3)  # cdb(tmp3.503,tmp3)
tmp3 = flatten_Rabcd (tmp3)
canonicalise         (tmp3)  # cdb(tmp3.506,tmp3)

distribute           (tmp4)  # cdb(tmp4.501,tmp4)
eliminate_metric     (tmp4)  # cdb(tmp4.502,tmp4)
eliminate_kronecker  (tmp4)  # cdb(tmp4.503,tmp4)
tmp4 = flatten_Rabcd (tmp4)
canonicalise         (tmp4)  # cdb(tmp4.506,tmp4)
```

```
distribute           (tmp5)  # cdb(tmp5.501,tmp5)
eliminate_metric     (tmp5)  # cdb(tmp5.502,tmp5)
eliminate_kronecker  (tmp5)  # cdb(tmp5.503,tmp5)
tmp5 = flatten_Rabcd (tmp5)
canonicalise         (tmp5)  # cdb(tmp5.506,tmp5)

# this is out final answer

# raise the _{uv} indices to ^{ab}

metric:= g^{a u} g^{b v} (  @(tmp0)
                           + (1/1) @(tmp1)
                           + (1/2) @(tmp2)
                           + (1/6) @(tmp3)
                           + (1/24) @(tmp4)
                           + (1/120) @(tmp5) ).  # cdb(metric.500,metric)

distribute             (metric)   # cdb(metric.501,metric)
eliminate_metric       (metric)   # cdb(metric.502,metric)
eliminate_kronecker    (metric)   # cdb(metric.503,metric)
metric = flatten_Rabcd (metric)   # cdb(metric.504,metric)
canonicalise           (metric)   # cdb(metric.505,metric)

substitute          (metric,$g_{a b} g^{b c} -> g_{a}^{c}$)
substitute          (metric,$g_{b a} g^{b c} -> g_{a}^{c}$)
substitute          (metric,$g_{b a} g^{c b} -> g_{a}^{c}$)
substitute          (metric,$g_{a b} g^{c b} -> g_{a}^{c}$)
eliminate_kronecker (metric)   # cdb(metric.506,metric)
canonicalise        (metric)   # cdb(metric.507,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.508,metric)

cdblib.create ('metric-inv.json')

cdblib.put ('g^ab',metric,'metric-inv.json')

# extract the terms of the metric in powers of x
```

```
term0 = get_xterm (metric,0)   # cdb(term0.501,term0)
term1 = get_xterm (metric,1)   # cdb(term1.501,term1)
term2 = get_xterm (metric,2)   # cdb(term2.501,term2)
term3 = get_xterm (metric,3)   # cdb(term3.501,term3)
term4 = get_xterm (metric,4)   # cdb(term4.501,term4)
term5 = get_xterm (metric,5)   # cdb(term5.501,term5)

cdblib.put ('g^ab_0',term0,'metric-inv.json')
cdblib.put ('g^ab_1',term1,'metric-inv.json')
cdblib.put ('g^ab_2',term2,'metric-inv.json')
cdblib.put ('g^ab_3',term3,'metric-inv.json')
cdblib.put ('g^ab_4',term4,'metric-inv.json')
cdblib.put ('g^ab_5',term5,'metric-inv.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3).  # cdb(metric4.501,metric4)

# these versions of "metric" are created just to add to the metric.json library
# note: term1 = 0, I could have used this fact above but ...

metric2:=@(term0) + @(term2).
metric3:=@(term0) + @(term2) + @(term3).
metric4:=@(term0) + @(term2) + @(term3) + @(term4).
metric5:=@(term0) + @(term2) + @(term3) + @(term4) + @(term5).

cdblib.put ('g^ab2',metric2,'metric-inv.json')
cdblib.put ('g^ab3',metric3,'metric-inv.json')
cdblib.put ('g^ab4',metric4,'metric-inv.json')
cdblib.put ('g^ab5',metric5,'metric-inv.json')
```

$\texttt{term0.501} := g^{ab}$

$\texttt{term1.501} := 0$

$\texttt{term2.501} := \frac{1}{3}x^c x^d R_{cedf} g^{ae} g^{bf}$

$\texttt{term3.501} := \frac{1}{6}x^c x^d x^e \nabla_c R_{dfeg} g^{af} g^{bg}$

$\texttt{term4.501} := \frac{1}{15}x^c x^d x^e x^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{egfh} g^{ag} g^{bh}$

$\texttt{term5.501} := \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}$

$$\texttt{tmp2.501} := -\frac{1}{3} g_{au} g_{bv} A^c A^d R^a{}_{cd}{}^b - \frac{1}{3} g_{au} g_{bv} A^c A^d R^b{}_{cd}{}^a$$

$$\texttt{tmp2.502} := -\frac{1}{3} g_{bv} A^c A^d R_{ucd}{}^b - \frac{1}{3} g_{bv} A^c A^d R^b{}_{cdu}$$

$$\texttt{tmp2.503} := -\frac{1}{3} g_{bv} A^c A^d R_{ucd}{}^b - \frac{1}{3} g_{bv} A^c A^d R^b{}_{cdu}$$

$$\texttt{tmp2.506} := -\frac{2}{3} A^a A^b R_{uabc} g_{vd} g^{cd}$$

$$\texttt{tmp3.501} := -\frac{1}{2} g_{au} g_{bv} g^{cb} A^d A^f A^e \nabla_d R_{cfeg} g^{ag} - \frac{1}{2} g_{au} g_{bv} g^{ac} A^d A^f A^e \nabla_d R_{cfeg} g^{bg}$$

$$\texttt{tmp3.502} := -\frac{1}{2} g_{bv} g^{cb} A^d A^f A^e \nabla_d R_{cfeg} g_u{}^g - \frac{1}{2} g_{bv} g_u{}^c A^d A^f A^e \nabla_d R_{cfeg} g^{bg}$$

$$\texttt{tmp3.503} := -\frac{1}{2} g_{bv} g^{cb} A^d A^f A^e \nabla_d R_{cfeu} - \frac{1}{2} g_{bv} A^d A^f A^e \nabla_d R_{ufeg} g^{bg}$$

$$\texttt{tmp3.506} := -A^a A^b A^c \nabla_a R_{ubcd} g_{ve} g^{de}$$

$$\texttt{tmp4.501} := \frac{1}{9}g_{au}g_{bv}g^{ib}A^eA^fR^c{}_{efi}A^dA^gR^a{}_{dgc} + \frac{1}{9}g_{au}g_{bv}g^{ib}A^eA^gR^c{}_{egi}A^dA^fR^a{}_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ci}A^eA^fR^b{}_{efi}A^dA^gR^a{}_{dgc}$$
$$+ \frac{1}{9}g_{au}g_{bv}g^{ci}A^eA^gR^b{}_{egi}A^dA^fR^a{}_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ib}A^fA^gR^c{}_{fgi}A^dA^eR^a{}_{dec} + \frac{1}{9}g_{au}g_{bv}g^{ci}A^fA^gR^b{}_{fgi}A^dA^eR^a{}_{dec}$$
$$- \frac{3}{5}g_{au}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{ah} + \frac{1}{15}g_{au}g_{bv}g^{cb}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} + \frac{1}{15}g_{au}g_{bv}g^{cb}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}$$
$$+ \frac{1}{9}g_{au}g_{bv}g^{ic}A^eA^fR^a{}_{efi}A^dA^gR^b{}_{dgc} + \frac{1}{9}g_{au}g_{bv}g^{ic}A^eA^gR^a{}_{egi}A^dA^fR^b{}_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ai}A^eA^fR^c{}_{efi}A^dA^gR^b{}_{dgc}$$
$$+ \frac{1}{9}g_{au}g_{bv}g^{ai}A^eA^gR^c{}_{egi}A^dA^fR^b{}_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ic}A^fA^gR^a{}_{fgi}A^dA^eR^b{}_{dec} + \frac{1}{9}g_{au}g_{bv}g^{ai}A^fA^gR^c{}_{fgi}A^dA^eR^b{}_{dec}$$
$$- \frac{3}{5}g_{au}g_{bv}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g_{au}g_{bv}g^{ac}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} + \frac{1}{15}g_{au}g_{bv}g^{ac}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}$$

$$\texttt{tmp4.502} := \frac{1}{9}g_{bv}g^{ib}A^eA^fR^c{}_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ib}A^eA^gR^c{}_{egi}A^dA^fR_{udfc} + \frac{1}{9}g_{bv}g^{ci}A^eA^fR^b{}_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ci}A^eA^gR^b{}_{egi}A^dA^fR_{udfc}$$
$$+ \frac{1}{9}g_{bv}g^{ib}A^fA^gR^c{}_{fgi}A^dA^eR_{udec} + \frac{1}{9}g_{bv}g^{ci}A^fA^gR^b{}_{fgi}A^dA^eR_{udec} - \frac{3}{5}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g_u{}^h$$
$$+ \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugfh}R^h{}_{dec} + \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugeh}R^h{}_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^eA^fR_{uefi}A^dA^gR^b{}_{dgc} + \frac{1}{9}g_{bv}g^{ic}A^eA^gR_{uegi}A^dA^fR^b{}_{dfc}$$
$$+ \frac{1}{9}g_{bv}g_u{}^iA^eA^fR^c{}_{efi}A^dA^gR^b{}_{dgc} + \frac{1}{9}g_{bv}g_u{}^iA^eA^gR^c{}_{egi}A^dA^fR^b{}_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^fA^gR_{ufgi}A^dA^eR^b{}_{dec} + \frac{1}{9}g_{bv}g_u{}^iA^fA^gR^c{}_{fgi}A^dA^eR^b{}_{dec}$$
$$- \frac{3}{5}g_{bv}g_u{}^cA^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g_{bv}g_u{}^cA^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} + \frac{1}{15}g_{bv}g_u{}^cA^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}$$

$$\texttt{tmp4.503} := \frac{1}{9}g_{bv}g^{ib}A^eA^fR^c{}_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ib}A^eA^gR^c{}_{egi}A^dA^fR_{udfc} + \frac{1}{9}g_{bv}g^{ci}A^eA^fR^b{}_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ci}A^eA^gR^b{}_{egi}A^dA^fR_{udfc}$$
$$+ \frac{1}{9}g_{bv}g^{ib}A^fA^gR^c{}_{fgi}A^dA^eR_{udec} + \frac{1}{9}g_{bv}g^{ci}A^fA^gR^b{}_{fgi}A^dA^eR_{udec} - \frac{3}{5}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefu}$$
$$+ \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugfh}R^h{}_{dec} + \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugeh}R^h{}_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^eA^fR_{uefi}A^dA^gR^b{}_{dgc}$$
$$+ \frac{1}{9}g_{bv}g^{ic}A^eA^gR_{uegi}A^dA^fR^b{}_{dfc} + \frac{1}{9}g_{bv}A^eA^fR^c{}_{efu}A^dA^gR^b{}_{dgc} + \frac{1}{9}g_{bv}A^eA^gR^c{}_{egu}A^dA^fR^b{}_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^fA^gR_{ufgi}A^dA^eR^b{}_{dec}$$
$$+ \frac{1}{9}g_{bv}A^fA^gR^c{}_{fgu}A^dA^eR^b{}_{dec} - \frac{3}{5}g_{bv}A^dA^gA^eA^f\nabla_{dg}R_{uefh}g^{bh} + \frac{1}{15}g_{bv}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{deu} + \frac{1}{15}g_{bv}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfu}$$

$$\texttt{tmp4.506} := -\frac{8}{5}A^aA^bA^cA^dR_{uabe}R_{cfdg}g_{vh}g^{ef}g^{gh} - \frac{6}{5}A^aA^bA^cA^d\nabla_{ab}R_{ucde}g_{vf}g^{ef}$$

$$\texttt{tmp5.506} := -4A^aA^bA^cA^dA^eR_{uabf}\nabla_cR_{dgeh}g_{vi}g^{fg}g^{hi} - 4A^aA^bA^cA^dA^eR_{afbg}\nabla_cR_{udeh}g_{vi}g^{fh}g^{gi} - \frac{4}{3}A^aA^bA^cA^dA^e\nabla_{abc}R_{udef}g_{vg}g^{fg}$$

$$\texttt{metric.500} := g^{au}g^{bv}\Big(g_{cu}g_{dv}g^{cd} - \frac{1}{3}A^eA^fR_{uefc}g_{vd}g^{cd} - \frac{1}{6}A^fA^gA^c\nabla_f R_{ugcd}g_{ve}g^{de} - \frac{1}{15}A^iA^jA^cA^dR_{uije}R_{cfdg}g_{vh}g^{ef}g^{gh} - \frac{1}{20}A^iA^jA^cA^d\nabla_{ij}R_{ucde}g_{vf}g^{ef} - \frac{1}{30}A^jA^kA^cA^dA^eR_{ujkf}\nabla_c R_{dgeh}g_{vi}g^{fg}g^{hi} - \frac{1}{30}A^jA^kA^cA^dA^eR_{jfkg}\nabla_c R_{udeh}g_{vi}g^{fh}g^{gi} - \frac{1}{90}A^jA^kA^cA^dA^e\nabla_{jkc}R_{udef}g_{vg}g^{fg}\Big)$$

$$\texttt{metric.501} := g^{au}g^{bv}g_{cu}g_{dv}g^{cd} - \frac{1}{3}g^{au}g^{bv}A^eA^fR_{uefc}g_{vd}g^{cd} - \frac{1}{6}g^{au}g^{bv}A^fA^gA^c\nabla_f R_{ugcd}g_{ve}g^{de} - \frac{1}{15}g^{au}g^{bv}A^iA^jA^cA^dR_{uije}R_{cfdg}g_{vh}g^{ef}g^{gh} - \frac{1}{20}g^{au}g^{bv}A^iA^jA^cA^d\nabla_{ij}R_{ucde}g_{vf}g^{ef} - \frac{1}{30}g^{au}g^{bv}A^jA^kA^cA^dA^eR_{ujkf}\nabla_c R_{dgeh}g_{vi}g^{fg}g^{hi} - \frac{1}{30}g^{au}g^{bv}A^jA^kA^cA^dA^eR_{jfkg}\nabla_c R_{udeh}g_{vi}g^{fh}g^{gi} - \frac{1}{90}g^{au}g^{bv}A^jA^kA^cA^dA^e\nabla_{jkc}R_{udef}g_{vg}g^{fg}$$

$$\texttt{metric.502} := g^{bv}g_c{}^ag_{dv}g^{cd} - \frac{1}{3}g^{bv}A^eA^fR^a{}_{efc}g_{vd}g^{cd} - \frac{1}{6}g^{bv}A^fA^gA^c\nabla_f R^a{}_{gcd}g_{ve}g^{de} - \frac{1}{15}g^{bv}A^iA^jA^cA^dR^a{}_{ije}R_{cfdg}g_{vh}g^{ef}g^{gh} - \frac{1}{20}g^{bv}A^iA^jA^cA^d\nabla_{ij}R^a{}_{cde}g_{vf}g^{ef} - \frac{1}{30}g^{bv}A^jA^kA^cA^dA^eR^a{}_{jkf}\nabla_c R_{dgeh}g_{vi}g^{fg}g^{hi} - \frac{1}{30}g^{bv}A^jA^kA^cA^dA^eR_{jfkg}\nabla_c R^a{}_{deh}g_{vi}g^{fh}g^{gi} - \frac{1}{90}g^{bv}A^jA^kA^cA^dA^e\nabla_{jkc}R^a{}_{def}g_{vg}g^{fg}$$

$$\texttt{metric.503} := g^{bv}g_{dv}g^{ad} - \frac{1}{3}g^{bv}A^eA^fR^a{}_{efc}g_{vd}g^{cd} - \frac{1}{6}g^{bv}A^fA^gA^c\nabla_f R^a{}_{gcd}g_{ve}g^{de} - \frac{1}{15}g^{bv}A^iA^jA^cA^dR^a{}_{ije}R_{cfdg}g_{vh}g^{ef}g^{gh} - \frac{1}{20}g^{bv}A^iA^jA^cA^d\nabla_{ij}R^a{}_{cde}g_{vf}g^{ef} - \frac{1}{30}g^{bv}A^jA^kA^cA^dA^eR^a{}_{jkf}\nabla_c R_{dgeh}g_{vi}g^{fg}g^{hi} - \frac{1}{30}g^{bv}A^jA^kA^cA^dA^eR_{jfkg}\nabla_c R^a{}_{deh}g_{vi}g^{fh}g^{gi} - \frac{1}{90}g^{bv}A^jA^kA^cA^dA^e\nabla_{jkc}R^a{}_{def}g_{vg}g^{fg}$$

$$\texttt{metric.504} := g_{cd}g^{ac}g^{bd} - \frac{1}{3}A^cA^dR_{ecdf}g_{gh}g^{ae}g^{bg}g^{fh} - \frac{1}{6}A^cA^dA^e\nabla_d R_{fecg}g_{hi}g^{af}g^{bh}g^{gi} - \frac{1}{15}A^cA^dA^eA^fR_{cgdh}R_{iefj}g_{kl}g^{ai}g^{bk}g^{jg}g^{hl} - \frac{1}{20}A^cA^dA^eA^f\nabla_{ef}R_{gcdh}g_{ij}g^{ag}g^{bi}g^{hj} - \frac{1}{30}A^cA^dA^eA^fA^gR_{hfgi}\nabla_c R_{djek}g_{lm}g^{ah}g^{bl}g^{ij}g^{km} - \frac{1}{30}A^cA^dA^eA^fA^gR_{fhgi}\nabla_c R_{jdek}g_{lm}g^{aj}g^{bl}g^{hk}g^{im} - \frac{1}{90}A^cA^dA^eA^fA^g\nabla_{fgc}R_{hdei}g_{jk}g^{ah}g^{bj}g^{ik}$$

$$\texttt{metric.505} := g_{cd}g^{ac}g^{bd} + \frac{1}{3}A^cA^dR_{cedf}g_{gh}g^{ae}g^{bg}g^{fh} + \frac{1}{6}A^cA^dA^e\nabla_c R_{dfeg}g_{hi}g^{af}g^{bh}g^{gi} + \frac{1}{15}A^cA^dA^eA^fR_{cgdh}R_{eifj}g_{kl}g^{ag}g^{bk}g^{hi}g^{jl} + \frac{1}{20}A^cA^dA^eA^f\nabla_{cd}R_{egfh}g_{ij}g^{ag}g^{bi}g^{hj} + \frac{1}{30}A^cA^dA^eA^fA^gR_{chdi}\nabla_e R_{fjgk}g_{lm}g^{ah}g^{bl}g^{ij}g^{km} + \frac{1}{30}A^cA^dA^eA^fA^gR_{chdi}\nabla_e R_{fjgk}g_{lm}g^{aj}g^{bl}g^{hk}g^{im} + \frac{1}{90}A^cA^dA^eA^fA^g\nabla_{cde}R_{fhgi}g_{jk}g^{ah}g^{bj}g^{ik}$$

$$\texttt{metric.506} := g^{ba} + \frac{1}{3}A^cA^dR_{cedf}g^{ae}g^{fb} + \frac{1}{6}A^cA^dA^e\nabla_cR_{dfeg}g^{af}g^{gb} + \frac{1}{15}A^cA^dA^eA^fR_{cgdh}R_{eifj}g^{ag}g^{hi}g^{jb} + \frac{1}{20}A^cA^dA^eA^f\nabla_{cd}R_{egfh}g^{ag}g^{hb} + \frac{1}{30}A^cA^dA^eA^fA^gR_{chdi}\nabla_eR_{fjgk}g^{ah}g^{ij}g^{kb} + \frac{1}{30}A^cA^dA^eA^fA^gR_{chdi}\nabla_eR_{fjgk}g^{aj}g^{hk}g^{ib} + \frac{1}{90}A^cA^dA^eA^fA^g\nabla_{cde}R_{fhgi}g^{ah}g^{ib}$$

$$\texttt{metric.507} := g^{ab} + \frac{1}{3}A^cA^dR_{cedf}g^{ae}g^{bf} + \frac{1}{6}A^cA^dA^e\nabla_cR_{dfeg}g^{af}g^{bg} + \frac{1}{15}A^cA^dA^eA^fR_{cgdh}R_{eifj}g^{ag}g^{bi}g^{hj} + \frac{1}{20}A^cA^dA^eA^f\nabla_{cd}R_{egfh}g^{ag}g^{bh} + \frac{1}{30}A^cA^dA^eA^fA^gR_{chdi}\nabla_eR_{fjgk}g^{ah}g^{bj}g^{ik} + \frac{1}{30}A^cA^dA^eA^fA^gR_{chdi}\nabla_eR_{fjgk}g^{aj}g^{bh}g^{ik} + \frac{1}{90}A^cA^dA^eA^fA^g\nabla_{cde}R_{fhgi}g^{ah}g^{bi}$$

$$\texttt{metric.508} := g^{ab} + \frac{1}{3}x^cx^dR_{cedf}g^{ae}g^{bf} + \frac{1}{6}x^cx^dx^e\nabla_cR_{dfeg}g^{af}g^{bg} + \frac{1}{15}x^cx^dx^ex^fR_{cgdh}R_{eifj}g^{ag}g^{bi}g^{hj} + \frac{1}{20}x^cx^dx^ex^f\nabla_{cd}R_{egfh}g^{ag}g^{bh} + \frac{1}{30}x^cx^dx^ex^fx^gR_{chdi}\nabla_eR_{fjgk}g^{ah}g^{bj}g^{ik} + \frac{1}{30}x^cx^dx^ex^fx^gR_{chdi}\nabla_eR_{fjgk}g^{aj}g^{bh}g^{ik} + \frac{1}{90}x^cx^dx^ex^fx^g\nabla_{cde}R_{fhgi}g^{ah}g^{bi}$$

```
Rterm0 := @(term0).
Rterm1 := @(term1).  # zero
Rterm2 := @(term2).
Rterm3 := @(term3).
Rterm4 := @(term4).
Rterm5 := @(term5).

Rterm0 = reformat_xterm (Rterm0,  1)    # cdb(Rterm0.601,Rterm0)
Rterm2 = reformat_xterm (Rterm2,  3)    # cdb(Rterm2.601,Rterm2)
Rterm3 = reformat_xterm (Rterm3,  6)    # cdb(Rterm3.601,Rterm3)
Rterm4 = reformat_xterm (Rterm4, 60)    # cdb(Rterm4.601,Rterm4)
Rterm5 = reformat_xterm (Rterm5, 90)    # cdb(Rterm5.601,Rterm5)

Metric := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Rterm0,  1)    # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Rterm2,  3)    # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Rterm3,  6)    # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Rterm4, 60)    # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Rterm5, 90)    # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()
```

The inverse metric in Riemann normal coordinates

$$g^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \frac{1}{60}x^c x^d x^e x^f \left(4 g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3 g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right)$$
$$+ \frac{1}{90}x^c x^d x^e x^f x^g \left(3 g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3 g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right) + \mathcal{O}\left(\epsilon^6\right)$$

Curvature expansion of the inverse metric

$$g^{ab}(x) = \overset{0}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} + \overset{5}{g}{}^{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned}
\overset{0}{g}{}^{ab} &= g^{ab} \\
3\overset{2}{g}{}^{ab} &= x^c x^d g^{ae} g^{bf} R_{cedf} \\
6\overset{3}{g}{}^{ab} &= x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \\
60\overset{4}{g}{}^{ab} &= x^c x^d x^e x^f \left(4 g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3 g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right) \\
90\overset{5}{g}{}^{ab} &= x^c x^d x^e x^f x^g \left(3 g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3 g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right)
\end{aligned}$$

```
cdblib.create ('metric-inv.export')

cdblib.put ('g^ab_3',Metric3,'metric-inv.export')  # R and \partial R
cdblib.put ('g^ab_4',Metric4,'metric-inv.export')
cdblib.put ('g^ab_5',Metric5,'metric-inv.export')
cdblib.put ('g^ab_6',Metric6,'metric-inv.export')

cdblib.put ('g^ab',  Metric, 'metric-inv.export')  # R and \nabla R

cdblib.put ('g^ab_scaled0',scaled0,'metric-inv.export')
cdblib.put ('g^ab_scaled2',scaled2,'metric-inv.export')
cdblib.put ('g^ab_scaled3',scaled3,'metric-inv.export')
cdblib.put ('g^ab_scaled4',scaled4,'metric-inv.export')
cdblib.put ('g^ab_scaled5',scaled5,'metric-inv.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)
```

Timing

Stage 1: 2.4 secs
Stage 2: 3.6 secs
Stage 3: 54.9 secs
Stage 4: 3.8 secs