Geodesic IVP

Our game here is to find the solution of

$$0 = \frac{d^2 x^a}{ds^2} + \Gamma^a_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a(s)/ds = \dot{x}^a$ at $s = 0$.

Algorithm

By successive differentiation of the above equation we can compute

$$\frac{d^n x^a}{ds^n} = -\Gamma^a_{\underline{d}_n}\frac{dx^{\underline{d}_n}}{ds}$$

at $s = 0$ for $n = 2, 3, 4, \ldots$. The $\Gamma^a_{\underline{d}_n}$ are the *generalised connections*.

We can then construct the Taylor series solution for $x^a(s)$

$$x^a(s) = x^a + s\dot{x}^a - \sum_{k=2}^{\infty}\frac{s^k}{k!}\Gamma^a_{\underline{d}_k}\dot{x}^{\underline{d}_k}$$

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

import cdblib

# change signs to account for - sign in front of the sum for x^a(s), see above preamble

def flip_sign (obj):
    return Ex(0) - obj

sterm21 = flip_sign (cdblib.get ('genGamma01','genGamma.json'))
sterm22 = flip_sign (cdblib.get ('genGamma02','genGamma.json'))
sterm23 = flip_sign (cdblib.get ('genGamma03','genGamma.json'))
sterm24 = flip_sign (cdblib.get ('genGamma04','genGamma.json'))


sterm31 = flip_sign (cdblib.get ('genGamma11','genGamma.json'))
sterm32 = flip_sign (cdblib.get ('genGamma12','genGamma.json'))
sterm33 = flip_sign (cdblib.get ('genGamma13','genGamma.json'))


sterm41 = flip_sign (cdblib.get ('genGamma21','genGamma.json'))
sterm42 = flip_sign (cdblib.get ('genGamma22','genGamma.json'))


sterm51 = flip_sign (cdblib.get ('genGamma31','genGamma.json'))

# note: the various ivp21, ivp31  etc. are the pieces of the Taylor series
#       for the ivp but *without* the leading 1/n! of the Taylor series

ivp21 := @(sterm21).                                         # cdb (ivp21.000,ivp21)

ivp31 := @(sterm21) + @(sterm22).                           # cdb (ivp31.000,ivp31)
ivp32 := @(sterm31).                                         # cdb (ivp32.000,ivp32)

ivp41 := @(sterm21) + @(sterm22) + @(sterm23).              # cdb (ivp41.000,ivp41)
ivp42 := @(sterm31) + @(sterm32).                           # cdb (ivp42.000,ivp42)
ivp43 := @(sterm41).                                         # cdb (ivp43.000,ivp43)

ivp51 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24).   # cdb (ivp51.000,ivp51)
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ivp52 := @(sterm31) + @(sterm32) + @(sterm33).                    # cdb (ivp52.000,ivp52)
ivp53 := @(sterm41) + @(sterm42).                                 # cdb (ivp53.000,ivp53)
ivp54 := @(sterm51).                                              # cdb (ivp54.000,ivp54)


factor_out (ivp21,$A^{a?}$)                                       # cdb (ivp21.001,ivp21)


factor_out (ivp31,$A^{a?}$)                                       # cdb (ivp31.001,ivp31)
factor_out (ivp32,$A^{a?}$)                                       # cdb (ivp32.001,ivp32)


factor_out (ivp41,$A^{a?}$)                                       # cdb (ivp41.001,ivp41)
factor_out (ivp42,$A^{a?}$)                                       # cdb (ivp42.001,ivp42)
factor_out (ivp43,$A^{a?}$)                                       # cdb (ivp43.001,ivp43)


factor_out (ivp51,$A^{a?}$)                                       # cdb (ivp51.001,ivp51)
factor_out (ivp52,$A^{a?}$)                                       # cdb (ivp52.001,ivp52)
factor_out (ivp53,$A^{a?}$)                                       # cdb (ivp53.001,ivp53)
factor_out (ivp54,$A^{a?}$)                                       # cdb (ivp54.001,ivp54)


v{#}::LaTeXForm("\dot{x}").


substitute (ivp21, $A^{a} -> v^{a}$)                             # cdb (ivp21.002,ivp21)


substitute (ivp31, $A^{a} -> v^{a}$)                             # cdb (ivp31.002,ivp31)
substitute (ivp32, $A^{a} -> v^{a}$)                             # cdb (ivp32.002,ivp32)


substitute (ivp41, $A^{a} -> v^{a}$)                             # cdb (ivp41.002,ivp41)
substitute (ivp42, $A^{a} -> v^{a}$)                             # cdb (ivp42.002,ivp42)
substitute (ivp43, $A^{a} -> v^{a}$)                             # cdb (ivp43.002,ivp43)


substitute (ivp51, $A^{a} -> v^{a}$)                             # cdb (ivp51.002,ivp51)
substitute (ivp52, $A^{a} -> v^{a}$)                             # cdb (ivp52.002,ivp52)
substitute (ivp53, $A^{a} -> v^{a}$)                             # cdb (ivp53.002,ivp53)
substitute (ivp54, $A^{a} -> v^{a}$)                             # cdb (ivp54.002,ivp54)


# build the Taylor series
# note the inclusion of the 1/n! factors


ivp2 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp21).
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ivp3 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp31) + (1/6) (s**3) @(ivp32).
ivp4 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp41) + (1/6) (s**3) @(ivp42) + (1/24) (s**4) @(ivp43).
ivp5 := x^{a} + s v^{a} + (1/2) (s**2) @(ivp51) + (1/6) (s**3) @(ivp52) + (1/24) (s**4) @(ivp53) + (1/120) (s**5) @(ivp54).


# cdb (ivp2.000,ivp2)
# cdb (ivp3.000,ivp3)
# cdb (ivp4.000,ivp4)
# cdb (ivp5.000,ivp5)


# now construct the scaled terms for ivp5

sterm2 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24).  # cdb (sterm2.000,sterm2)
sterm3 := @(sterm31) + @(sterm32) + @(sterm33).              # cdb (sterm3.000,sterm3)
sterm4 := @(sterm41) + @(sterm42).                          # cdb (sterm4.000,sterm4)
sterm5 := @(sterm51).                                       # cdb (sterm5.000,sterm5)


factor_out (sterm2,$A^{a?}$)                                # cdb (sterm2.001,sterm2)
factor_out (sterm3,$A^{a?}$)                                # cdb (sterm3.001,sterm3)
factor_out (sterm4,$A^{a?}$)                                # cdb (sterm4.001,sterm4)
factor_out (sterm5,$A^{a?}$)                                # cdb (sterm5.001,sterm5)


sterm2 := 360 @(sterm2).
sterm3 := 360 @(sterm3).
sterm4 :=  90 @(sterm4).
sterm5 :=   3 @(sterm5).


substitute (sterm2,$A^{a}->1$)                              # cdb (sterm2.002,sterm2)
substitute (sterm3,$A^{a}->1$)                              # cdb (sterm3.002,sterm3)
substitute (sterm4,$A^{a}->1$)                              # cdb (sterm4.002,sterm4)
substitute (sterm5,$A^{a}->1$)                              # cdb (sterm5.002,sterm5)
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The geodesic ivp

$$x^a(s) = x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A^a_{bc} + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A^a_{bcd} + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A^a_{bcde} + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A^a_{bcdef} + \cdots$$

$$
\begin{aligned}
360 A^a_{bc} = & -240\, x^d g^{ae} R_{bdce} - 30\, x^d x^e \left(2\, g^{af}\nabla_b R_{cdef} + 4\, g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce} \right) - x^d x^e x^f \left(64\, g^{ag}g^{hi} R_{bdch} R_{egfi} - 32\, g^{ag}g^{hi} R_{bdeh} R_{cgfi} \right.\\
& - 16\, g^{ag}g^{hi} R_{bdeh} R_{cifg} + 18\, g^{ag}\nabla_{bd} R_{cefg} + 18\, g^{ag}\nabla_{db} R_{cefg} + 36\, g^{ag}\nabla_{de} R_{bfcg} - 16\, g^{ag}g^{hi} R_{bdeh} R_{cfgi} + 9\, g^{ag}\nabla_{gd} R_{becf} + 9\, g^{ag}\nabla_{dg} R_{becf} \big)\\
& - 2\, x^d x^e x^f x^g \left(16\, g^{ah}g^{ij} R_{bdci}\nabla_e R_{fhgj} + 6\, g^{ah}g^{ij} R_{dhei}\nabla_b R_{cfgj} + 16\, g^{ah}g^{ij} R_{dhei}\nabla_f R_{bgcj} + 5\, g^{ah}g^{ij} R_{dhei}\nabla_j R_{bfcg} - 8\, g^{ah}g^{ij} R_{bhdi}\nabla_e R_{cfgj} \right.\\
& - 4\, g^{ah}g^{ij} R_{bidh}\nabla_e R_{cfgj} - 4\, g^{ah}g^{ij} R_{bdei}\nabla_c R_{fhgj} - 8\, g^{ah}g^{ij} R_{bdei}\nabla_f R_{chgj} - 4\, g^{ah}g^{ij} R_{bdei}\nabla_f R_{cjgh} + 2\, g^{ah}\nabla_{bde} R_{cfgh} + 2\, g^{ah}\nabla_{dbe} R_{cfgh}\\
& + 2\, g^{ah}\nabla_{deb} R_{cfgh} + 4\, g^{ah}\nabla_{def} R_{bgch} - 4\, g^{ah}g^{ij} R_{bdhi}\nabla_e R_{cfgj} - 4\, g^{ah}g^{ij} R_{bdei}\nabla_h R_{cfgj} - 4\, g^{ah}g^{ij} R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde} R_{bfcg} + g^{ah}\nabla_{dhe} R_{bfcg}\\
& \left. + g^{ah}\nabla_{deh} R_{bfcg} \right)
\end{aligned}
$$

$$
\begin{aligned}
360 A^a_{bcd} = & -180\, x^e g^{af}\nabla_b R_{cedf} - 3\, x^e x^f \left(64\, g^{ag}g^{hi} R_{bech} R_{dgfi} + 16\, g^{ag}g^{hi} R_{bech} R_{difg} - 16\, g^{ag}g^{hi} R_{befh} R_{cgdi} + 12\, g^{ag}\nabla_{bc} R_{defg} + 18\, g^{ag}\nabla_{be} R_{cfdg} \right.\\
& \left. + 18\, g^{ag}\nabla_{eb} R_{cfdg} + 48\, g^{ag}g^{hi} R_{bech} R_{dfgi} + 3\, g^{ag}\nabla_{gb} R_{cedf} + 3\, g^{ag}\nabla_{bg} R_{cedf} \right)\\
& - 2\, x^e x^f x^g \left(32\, g^{ah}g^{ij} R_{beci}\nabla_d R_{fhgj} + 48\, g^{ah}g^{ij} R_{beci}\nabla_f R_{dhgj} + 12\, g^{ah}g^{ij} R_{beci}\nabla_f R_{djgh} + 18\, g^{ah}g^{ij} R_{bhei}\nabla_c R_{dfgj} + 2\, g^{ah}g^{ij} R_{bieh}\nabla_c R_{dfgj} \right.\\
& + 22\, g^{ah}g^{ij} R_{ehfi}\nabla_b R_{cgdj} + 48\, g^{ah}g^{ij} R_{bhei}\nabla_f R_{cgdj} + 12\, g^{ah}g^{ij} R_{bieh}\nabla_f R_{cgdj} + 15\, g^{ah}g^{ij} R_{bhei}\nabla_j R_{cfdg} + 5\, g^{ah}g^{ij} R_{bieh}\nabla_j R_{cfdg}\\
& - 12\, g^{ah}g^{ij} R_{bhci}\nabla_e R_{dfgj} - 12\, g^{ah}g^{ij} R_{befi}\nabla_c R_{dhgj} - 8\, g^{ah}g^{ij} R_{befi}\nabla_c R_{djgh} - 12\, g^{ah}g^{ij} R_{befi}\nabla_g R_{chdj} + 4\, g^{ah}\nabla_{bce} R_{dfgh} + 4\, g^{ah}\nabla_{bec} R_{dfgh}\\
& + 6\, g^{ah}\nabla_{bef} R_{cgdh} + 4\, g^{ah}\nabla_{ebc} R_{dfgh} + 6\, g^{ah}\nabla_{ebf} R_{cgdh} + 6\, g^{ah}\nabla_{efb} R_{cgdh} + 16\, g^{ah}g^{ij} R_{behi}\nabla_c R_{dfgj} + 36\, g^{ah}g^{ij} R_{behi}\nabla_f R_{cgdj}\\
& + 16\, g^{ah}g^{ij} R_{beci}\nabla_h R_{dfgj} - 4\, g^{ah}g^{ij} R_{befi}\nabla_h R_{cgdj} + 36\, g^{ah}g^{ij} R_{beci}\nabla_f R_{dghj} - 4\, g^{ah}g^{ij} R_{befi}\nabla_c R_{dghj} + g^{ah}\nabla_{hbe} R_{cfdg} + g^{ah}\nabla_{heb} R_{cfdg}\\
& \left. + g^{ah}\nabla_{bhe} R_{cfdg} + g^{ah}\nabla_{ehb} R_{cfdg} + g^{ah}\nabla_{beh} R_{cfdg} + g^{ah}\nabla_{ebh} R_{cfdg} - 20\, g^{ah}g^{ij} R_{beci}\nabla_j R_{dfgh} + 10\, g^{ah}g^{ij} R_{behi}\nabla_j R_{cfdg} \right)
\end{aligned}
$$

$$
\begin{aligned}
90 A^a_{bcde} = & -6\, x^f \left(8\, g^{ag}g^{hi} R_{bfch} R_{dgei} + 6\, g^{ag}\nabla_{bc} R_{dfeg} \right) - x^f x^g \left(64\, g^{ah}g^{ij} R_{bfci}\nabla_d R_{ehgj} + 18\, g^{ah}g^{ij} R_{bfci}\nabla_d R_{ejgh} + 24\, g^{ah}g^{ij} R_{bfci}\nabla_g R_{dhej} \right.\\
& + 4\, g^{ah}g^{ij} R_{bhci}\nabla_d R_{efgj} + 44\, g^{ah}g^{ij} R_{bhfi}\nabla_c R_{dgej} + 18\, g^{ah}g^{ij} R_{bifh}\nabla_c R_{dgej} + 24\, g^{ah}g^{ij} R_{bhci}\nabla_f R_{dgej} + 10\, g^{ah}g^{ij} R_{bhci}\nabla_j R_{dfeg}\\
& - 16\, g^{ah}g^{ij} R_{bfgi}\nabla_c R_{dhej} + 6\, g^{ah}\nabla_{bcd} R_{efgh} + 8\, g^{ah}\nabla_{bcf} R_{dgeh} + 8\, g^{ah}\nabla_{bfc} R_{dgeh} + 8\, g^{ah}\nabla_{fbc} R_{dgeh} + 26\, g^{ah}g^{ij} R_{bfhi}\nabla_c R_{dgej}\\
& \left. + 6\, g^{ah}g^{ij} R_{bfci}\nabla_h R_{dgej} + 46\, g^{ah}g^{ij} R_{bfci}\nabla_d R_{eghj} + g^{ah}\nabla_{hbc} R_{dfeg} + g^{ah}\nabla_{bhc} R_{dfeg} + g^{ah}\nabla_{bch} R_{dfeg} - 40\, g^{ah}g^{ij} R_{bfci}\nabla_j R_{dgeh} \right)
\end{aligned}
$$

$$3 A^a_{bcdef} = -x^g \left(3\, g^{ah}g^{ij} R_{bgci}\nabla_d R_{ehfj} + 3\, g^{ah}g^{ij} R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd} R_{egfh} \right)$$

```
sterm2short := @(sterm21) + @(sterm22).              # cdb (sterm2.short.001,sterm2short)
sterm3short := @(sterm31).                           # cdb (sterm3.short.001,sterm3short)
sterm2shortscaled := 12 @(sterm2short).             # cdb (sterm2.short.scaled.002,sterm2shortscaled)
sterm3shortscaled :=  2 @(sterm3short).             # cdb (sterm3.short.scaled.002,sterm3shortscaled)


substitute (sterm2shortscaled,$A^{a}->1$)           # cdb (sterm2.short.scaled.003,sterm2shortscaled)
substitute (sterm3shortscaled,$A^{a}->1$)           # cdb (sterm3.short.scaled.003,sterm3shortscaled)


cdblib.create ('geodesic-ivp.export')


# 4th order ivp terms scaled
cdblib.put ('ivp42',sterm2shortscaled,'geodesic-ivp.export')
cdblib.put ('ivp43',sterm3shortscaled,'geodesic-ivp.export')


# 6th order ivp terms scaled
cdblib.put ('ivp62',sterm2,'geodesic-ivp.export')
cdblib.put ('ivp63',sterm3,'geodesic-ivp.export')
cdblib.put ('ivp64',sterm4,'geodesic-ivp.export')
cdblib.put ('ivp65',sterm5,'geodesic-ivp.export')


checkpoint.append (sterm2shortscaled)
checkpoint.append (sterm3shortscaled)


checkpoint.append (sterm2)
checkpoint.append (sterm3)
checkpoint.append (sterm4)
checkpoint.append (sterm5)


cdblib.create ('geodesic-ivp.json')


cdblib.put ('ivp21',ivp21,'geodesic-ivp.json')


cdblib.put ('ivp31',ivp31,'geodesic-ivp.json')
cdblib.put ('ivp32',ivp32,'geodesic-ivp.json')


cdblib.put ('ivp41',ivp41,'geodesic-ivp.json')
cdblib.put ('ivp42',ivp42,'geodesic-ivp.json')
cdblib.put ('ivp43',ivp43,'geodesic-ivp.json')
```

```
cdblib.put ('ivp51',ivp51,'geodesic-ivp.json')
cdblib.put ('ivp52',ivp52,'geodesic-ivp.json')
cdblib.put ('ivp53',ivp53,'geodesic-ivp.json')
cdblib.put ('ivp54',ivp54,'geodesic-ivp.json')

cdblib.put ('ivp2',ivp2,'geodesic-ivp.json')
cdblib.put ('ivp3',ivp3,'geodesic-ivp.json')
cdblib.put ('ivp4',ivp4,'geodesic-ivp.json')
cdblib.put ('ivp5',ivp5,'geodesic-ivp.json')
```

$$\texttt{sterm2.short.001} := -\frac{2}{3}\,A^b A^c x^d g^{ae} R_{bdce} - \frac{1}{12}\,A^b A^c x^d x^e \left(2\,g^{af}\nabla_b R_{cdef} + 4\,g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$$

$$\texttt{sterm3.short.001} := -\frac{1}{2}\,A^b A^c A^d x^e g^{af}\nabla_b R_{cedf}$$

$$\texttt{sterm2.short.scaled.002} := -8\,A^b A^c x^d g^{ae} R_{bdce} - A^b A^c x^d x^e \left(2\,g^{af}\nabla_b R_{cdef} + 4\,g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$$

$$\texttt{sterm3.short.scaled.002} := -A^b A^c A^d x^e g^{af}\nabla_b R_{cedf}$$

$$\texttt{sterm2.short.scaled.003} := -8\,x^d g^{ae} R_{bdce} - x^d x^e \left(2\,g^{af}\nabla_b R_{cdef} + 4\,g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$$

$$\texttt{sterm3.short.scaled.003} := -x^e g^{af}\nabla_b R_{cedf}$$

$$\texttt{ivp21.002} := -\frac{2}{3}\,\dot{x}^b \dot{x}^c x^d g^{ae} R_{bdce}$$

$$\texttt{ivp31.002} := \dot{x}^b \dot{x}^c \left(-\frac{2}{3}\,x^d g^{ae} R_{bdce} - \frac{1}{12}\,x^d x^e \left(2\,g^{af}\nabla_b R_{cdef} + 4\,g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)\right)$$

$$\texttt{ivp32.002} := -\frac{1}{2}\,\dot{x}^b \dot{x}^c \dot{x}^d x^e g^{af}\nabla_b R_{cedf}$$

$$\texttt{ivp41.002} := \dot{x}^b \dot{x}^c \left(-\frac{2}{3}\,x^d g^{ae} R_{bdce} - \frac{1}{12}\,x^d x^e \left(2\,g^{af}\nabla_b R_{cdef} + 4\,g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) - \frac{1}{360}\,x^d x^e x^f \left(64\,g^{ag} g^{hi} R_{bdch} R_{egfi} - 32\,g^{ag} g^{hi} R_{bdeh} R_{cgfi}\right.\right.$$
$$\left.\left. - 16\,g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18\,g^{ag}\nabla_{bd} R_{cefg} + 18\,g^{ag}\nabla_{db} R_{cefg} + 36\,g^{ag}\nabla_{de} R_{bfcg} - 16\,g^{ag} g^{hi} R_{bdeh} R_{cfgi} + 9\,g^{ag}\nabla_{gd} R_{becf} + 9\,g^{ag}\nabla_{dg} R_{becf}\right)\right)$$

$$\texttt{ivp42.002} := \dot{x}^b \dot{x}^c \dot{x}^d \left(-\frac{1}{2} x^e g^{af} \nabla_b R_{cedf} - \frac{1}{120} x^e x^f \left(64\, g^{ag} g^{hi} R_{bech} R_{dgfi} + 16\, g^{ag} g^{hi} R_{bech} R_{difg} - 16\, g^{ag} g^{hi} R_{befh} R_{cgdi} + 12\, g^{ag} \nabla_{bc} R_{defg} \right. \right.$$

$$\left. \left. + 18\, g^{ag} \nabla_{be} R_{cfdg} + 18\, g^{ag} \nabla_{eb} R_{cfdg} + 48\, g^{ag} g^{hi} R_{bech} R_{dfgi} + 3\, g^{ag} \nabla_{gb} R_{cedf} + 3\, g^{ag} \nabla_{bg} R_{cedf} \right) \right)$$

$$\texttt{ivp43.002} := -\frac{1}{15} \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f \left(8\, g^{ag} g^{hi} R_{bfch} R_{dgei} + 6\, g^{ag} \nabla_{bc} R_{dfeg} \right)$$

$$\texttt{ivp51.002} := \dot{x}^b \dot{x}^c \left(-\frac{2}{3} x^d g^{ae} R_{bdce} - \frac{1}{12} x^d x^e \left(2\, g^{af} \nabla_b R_{cdef} + 4\, g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) - \frac{1}{360} x^d x^e x^f \left(64\, g^{ag} g^{hi} R_{bdch} R_{egfi} - 32\, g^{ag} g^{hi} R_{bdeh} R_{cgfi} \right. \right.$$

$$\left. - 16\, g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18\, g^{ag} \nabla_{bd} R_{cefg} + 18\, g^{ag} \nabla_{db} R_{cefg} + 36\, g^{ag} \nabla_{de} R_{bfcg} - 16\, g^{ag} g^{hi} R_{bdeh} R_{cfgi} + 9\, g^{ag} \nabla_{gd} R_{becf} + 9\, g^{ag} \nabla_{dg} R_{becf} \right)$$

$$- \frac{1}{180} x^d x^e x^f x^g \left(16\, g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6\, g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} + 16\, g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5\, g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} \right.$$

$$- 8\, g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} - 4\, g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} - 4\, g^{ah} g^{ij} R_{bdei} \nabla_e R_{fhgj} - 8\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - 4\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh}$$

$$+ 2\, g^{ah} \nabla_{bde} R_{cfgh} + 2\, g^{ah} \nabla_{dbe} R_{cfgh} + 2\, g^{ah} \nabla_{deb} R_{cfgh} + 4\, g^{ah} \nabla_{def} R_{bgch} - 4\, g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 4\, g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj}$$

$$\left. \left. - 4\, g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + g^{ah} \nabla_{hde} R_{bfcg} + g^{ah} \nabla_{dhe} R_{bfcg} + g^{ah} \nabla_{deh} R_{bfcg} \right) \right)$$

$$\texttt{ivp52.002} := \dot{x}^b \dot{x}^c \dot{x}^d \left(-\frac{1}{2} x^e g^{af} \nabla_b R_{cedf} - \frac{1}{120} x^e x^f \left(64\, g^{ag} g^{hi} R_{bech} R_{dgfi} + 16\, g^{ag} g^{hi} R_{bech} R_{difg} - 16\, g^{ag} g^{hi} R_{befh} R_{cgdi} + 12\, g^{ag} \nabla_{bc} R_{defg} \right. \right.$$

$$\left. + 18\, g^{ag} \nabla_{be} R_{cfdg} + 18\, g^{ag} \nabla_{eb} R_{cfdg} + 48\, g^{ag} g^{hi} R_{bech} R_{dfgi} + 3\, g^{ag} \nabla_{gb} R_{cedf} + 3\, g^{ag} \nabla_{bg} R_{cedf} \right)$$

$$- \frac{1}{180} x^e x^f x^g \left(32\, g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 48\, g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + 12\, g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + 18\, g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} + 2\, g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} \right.$$

$$+ 22\, g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48\, g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12\, g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15\, g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} + 5\, g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg}$$

$$- 12\, g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - 12\, g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - 8\, g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - 12\, g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 4\, g^{ah} \nabla_{bce} R_{dfgh} + 4\, g^{ah} \nabla_{bec} R_{dfgh}$$

$$+ 6\, g^{ah} \nabla_{bef} R_{cgdh} + 4\, g^{ah} \nabla_{ebc} R_{dfgh} + 6\, g^{ah} \nabla_{ebf} R_{cgdh} + 6\, g^{ah} \nabla_{efb} R_{cgdh} + 16\, g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + 36\, g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj}$$

$$+ 16\, g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 4\, g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 36\, g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - 4\, g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + g^{ah} \nabla_{hbe} R_{cfdg} + g^{ah} \nabla_{heb} R_{cfdg}$$

$$\left. \left. + g^{ah} \nabla_{bhe} R_{cfdg} + g^{ah} \nabla_{ehb} R_{cfdg} + g^{ah} \nabla_{beh} R_{cfdg} + g^{ah} \nabla_{ebh} R_{cfdg} - 20\, g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} + 10\, g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} \right) \right)$$

$$\texttt{ivp53.002} := \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \left(-\frac{1}{15} x^f \left(8\, g^{ag} g^{hi} R_{bfch} R_{dgei} + 6\, g^{ag} \nabla_{bc} R_{dfeg} \right) \right.$$

$$- \frac{1}{90} x^f x^g \left(64\, g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18\, g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24\, g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + 4\, g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + 44\, g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} \right.$$

$$+ 18\, g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24\, g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 10\, g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} - 16\, g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + 6\, g^{ah} \nabla_{bcd} R_{efgh} + 8\, g^{ah} \nabla_{bcf} R_{dgeh}$$

$$+ 8\, g^{ah} \nabla_{bfc} R_{dgeh} + 8\, g^{ah} \nabla_{fbc} R_{dgeh} + 26\, g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} + 6\, g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + 46\, g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah} \nabla_{hbc} R_{dfeg}$$

$$\left. \left. + g^{ah} \nabla_{bhc} R_{dfeg} + g^{ah} \nabla_{bch} R_{dfeg} - 40\, g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh} \right) \right)$$

$$\texttt{ivp54.002} := -\frac{1}{3} \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \dot{x}^f x^g \left(3\, g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3\, g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah} \nabla_{bcd} R_{egfh} \right)$$

$$\texttt{ivp2.000} := x^a + s \dot{x}^a - \frac{1}{3} s^2 \dot{x}^b \dot{x}^c x^d g^{ae} R_{bdce}$$

$$\texttt{ivp3.000} := x^a + s \dot{x}^a + \frac{1}{2} s^2 \dot{x}^b \dot{x}^c \left(-\frac{2}{3} x^d g^{ae} R_{bdce} - \frac{1}{12} x^d x^e \left(2\, g^{af} \nabla_b R_{cdef} + 4\, g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) \right) - \frac{1}{12} s^3 \dot{x}^b \dot{x}^c \dot{x}^d x^e g^{af} \nabla_b R_{cedf}$$

$$\texttt{ivp4.000} := x^a + s \dot{x}^a + \frac{1}{2} s^2 \dot{x}^b \dot{x}^c \left(-\frac{2}{3} x^d g^{ae} R_{bdce} - \frac{1}{12} x^d x^e \left(2\, g^{af} \nabla_b R_{cdef} + 4\, g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right) \right.$$

$$- \frac{1}{360} x^d x^e x^f \left(64\, g^{ag} g^{hi} R_{bdch} R_{egfi} - 32\, g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16\, g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18\, g^{ag} \nabla_{bd} R_{cefg} + 18\, g^{ag} \nabla_{db} R_{cefg} + 36\, g^{ag} \nabla_{de} R_{bfcg} \right.$$

$$\left. \left. - 16\, g^{ag} g^{hi} R_{bdeh} R_{cfgi} + 9\, g^{ag} \nabla_{gd} R_{becf} + 9\, g^{ag} \nabla_{dg} R_{becf} \right) \right)$$

$$+ \frac{1}{6} s^3 \dot{x}^b \dot{x}^c \dot{x}^d \left(-\frac{1}{2} x^e g^{af} \nabla_b R_{cedf} - \frac{1}{120} x^e x^f \left(64\, g^{ag} g^{hi} R_{bech} R_{dgfi} + 16\, g^{ag} g^{hi} R_{bech} R_{difg} - 16\, g^{ag} g^{hi} R_{befh} R_{cgdi} + 12\, g^{ag} \nabla_{bc} R_{defg} \right.$$

$$\left. \left. + 18\, g^{ag} \nabla_{be} R_{cfdg} + 18\, g^{ag} \nabla_{eb} R_{cfdg} + 48\, g^{ag} g^{hi} R_{bech} R_{dfgi} + 3\, g^{ag} \nabla_{gb} R_{cedf} + 3\, g^{ag} \nabla_{bg} R_{cedf} \right) \right)$$

$$- \frac{1}{360} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e x^f \left(8\, g^{ag} g^{hi} R_{bfch} R_{dgei} + 6\, g^{ag} \nabla_{bc} R_{dfeg} \right)$$