

Symmetrised partial derivatives of the Riemann tensor

Here we compute the symmetrised partial derivatives $R^a_{(bcd;\underline{e})}$ in terms of the symmetrised covariant derivatives $R^a_{(bcd;\underline{e})}$. Note that the dot over an index indicates that that index does not take part in the symmetrisation.

We will use the algorithm described in section (10.3) of my lcb09-03 paper. Here we will make one small change of notation – the symbol D^a will be replaced with A^a .

We have lots of space (and no annoying editors to appease with brevity) so I will take the liberty to expand slightly on what I wrote in the lcb0-03 paper.

Our starting point is the simple identity

$$(R^a_{cdb}B^b_{a}A^cA^d)_{;e}A^e = (R^a_{cdb}B^b_{a}A^cA^d)_{;e}A^e \quad (1)$$

This is true in all frames since the quantity inside the brackets is a scalar. We are free to make any choice we like for A^a and B^a_b so let's choose A^a to be the tangent vector to any geodesic through the origin and choose the B^a_b to be constants (i.e, all partial derivatives are zero). We will also use local Riemann normal coordinates and as a consequence, the A^a will also be constant along the integral curves of A (the geodesics in an RNC are always of the form $x^a(s) = sA^a$ for some affine parameter s on the geodesic). Let df/ds be the directional derivative of the function f along the geodesics defined by A^a and assume that s is the proper length along the geodesic (although any affine parameter would be sufficient).

Thus at the origin we have, by choice,

$$\begin{aligned} 0 &= B^a_{b,c} = B^a_{b,cd} = B^a_{b,cde} = \dots \\ 0 &= dA^a/ds = d^2A^a/ds^2 = d^3A^a/ds^3 = \dots \\ 0 &= A^a_{;b}A^b = (A^a_{;b}A^b)_{;c}A^c = \left((A^a_{;b}A^b)_{;c}A^c\right)_{;d}A^d \\ 0 &= A^a_{;b}A^b = (A^a_{;b}A^b)_{;c}A^c = \left((A^a_{;b}A^b)_{;c}A^c\right)_{;d}A^d \\ df/ds &= f_{,a}A^a = f_{;a}A^a \\ d^2f/ds^2 &= (f_{,a}A^a)_{;b}A^b = (f_{;a}A^a)_{;b}A^b \\ d^3f/ds^3 &= \left((f_{,a}A^a)_{;b}A^b\right)_{;c}A^c = \left((f_{;a}A^a)_{;b}A^b\right)_{;c}A^c \end{aligned}$$

I admit I've gone overboard here in writing out more than I need to but it's handy to have all of these equations laid bare in one convenient place.

Now put $f = R^p_{abq} B^q_p A^a A^b$. Then upon taking successive derivatives, while taking full advantage of the assumptions just noted, we can easily see that

$$(R^a_{cdb} B^b_a)_{;e} A^c A^d A^e = (R^a_{cdb})_{;e} B^b_a A^c A^d A^e \quad (2)$$

This is the equation that will be computed by the following Cadabra code. All of the computations will be carried out on the left hand side (in the first version of the paper I swapped the left and right hand sides).

We will need the successive covariant derivatives of B . The first covariant derivative is just

$$B^a_{b;c} A^c = \Gamma^a_{dc} B^d_b A^c - \Gamma^d_{bc} B^a_d A^c$$

The quantities on the left hand side are the components of a tensor so further covariant derivatives of the right hand side can be computed (despite the presence of the Γ 's) by application of the usual rule for a covariant derivative of a mixed tensor.

Stage 1: Symmetrised partial derivatives of R

The first stage involves the expansion of the left side of (2). This leads to expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and B^a_b .

$$\begin{aligned} (R^a_{cdb})_{;e} B^b_a A^c A^d A^e &= -A^a A^b A^c B^d_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d_e g^{ef} \\ (R^a_{cdb})_{;ef} B^b_a A^c A^d A^e A^f &= -2A^a A^b A^c A^d \nabla_a B^e_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\ (R^a_{cdb})_{;efg} B^b_a A^c A^d A^e A^f A^g &= -3A^a A^b A^c A^d A^e \nabla_a R_{bfch} \nabla_d (\nabla_e B^f_g) g^{hg} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeh}) g^{gh} \\ &\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeh})) g^{gh} - A^a A^b A^c A^d A^e R_{afbh} \nabla_c (\nabla_d (\nabla_e B^f_g)) g^{hg} \end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

In this stage the symmetrized covariant derivatives of B^a_b are computed in terms of its partial derivatives (which by choice are all zero) and the connection and its partial derivatives (which in general are not zero).

$$\begin{aligned}
A^c \nabla_c (B^a_b) &= \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q \\
A^d A^c \nabla_d (\nabla_c (B^a_b)) &= A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{cd} \Gamma^c_{pq} B^p_b A^d A^q - 2 \Gamma^a_{cd} \Gamma^p_{bq} B^c_p A^d A^q + \Gamma^c_{bd} \Gamma^p_{cq} B^a_p A^d A^q \\
A^e A^d A^c \nabla_e (\nabla_d (\nabla_c (B^a_b))) &= A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q + A^c \partial_c \Gamma^a_{de} \Gamma^d_{pq} B^p_b A^e A^q + A^c \Gamma^a_{cd} \partial_e \Gamma^d_{pq} B^p_b A^e A^q \\
&\quad - 2 A^c \partial_c \Gamma^a_{de} \Gamma^p_{bq} B^d_p A^e A^q - 2 A^c \Gamma^a_{cd} \partial_e \Gamma^p_{bq} B^d_p A^e A^q + A^c \partial_c \Gamma^d_{be} \Gamma^p_{dq} B^a_p A^e A^q + A^c \Gamma^d_{bc} \partial_e \Gamma^p_{dq} B^a_p A^e A^q \\
&\quad + \Gamma^a_{ce} A^c \partial_f \Gamma^e_{pq} B^p_b A^f A^q - \Gamma^a_{ce} A^c \partial_f \Gamma^p_{bq} B^e_p A^f A^q + \Gamma^a_{cd} \Gamma^e_{cf} \Gamma^e_{pq} B^p_b A^d A^f A^q - 3 \Gamma^a_{cd} \Gamma^e_{bf} \Gamma^c_{pq} B^p_e A^d A^f A^q \\
&\quad + 3 \Gamma^a_{cd} \Gamma^e_{bf} \Gamma^p_{eq} B^c_p A^d A^f A^q - \Gamma^c_{be} A^e \partial_f \Gamma^a_{pq} B^p_c A^f A^q + \Gamma^c_{be} A^e \partial_f \Gamma^p_{cq} B^a_p A^f A^q - \Gamma^c_{bd} \Gamma^e_{cf} \Gamma^p_{eq} B^a_p A^d A^f A^q
\end{aligned}$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

Here we impose the RNC condition (that $\Gamma = 0$ while $\partial\Gamma \neq 0$).

$$\begin{aligned}
A^c \nabla_c (B^a_b) &= 0 \\
A^d A^c \nabla_d (\nabla_c (B^a_b)) &= A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q \\
A^e A^d A^c \nabla_e (\nabla_d (\nabla_c (B^a_b))) &= A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q
\end{aligned}$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

This stage uses the results from the second stage to eliminate the ∇B terms from the results of the first stage. This produces expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and the partial derivatives of the connection. In this stage we also set the B^a_b to equal 1.

$$\begin{aligned}
(R^a_{cdb})_{,e} A^c A^d A^e &= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
(R^a_{cdb})_{,ef} A^c A^d A^e A^f &= A^c A^d A^e A^f (-\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma^g_{bf} g^{ha} + R_{cbdg} \partial_e \Gamma^a_{hf} g^{gh}) \\
(R^a_{cdb})_{,efg} A^c A^d A^e A^f A^g &= A^c A^d A^e A^f A^g (-3 \nabla_c R_{dhef} \partial_f \Gamma^h_{bg} g^{ia} + 3 \nabla_c R_{dbeh} \partial_f \Gamma^a_{ig} g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_{ef} \Gamma^h_{bg} g^{ia} + R_{cbdh} \partial_{ef} \Gamma^a_{ig} g^{hi})
\end{aligned}$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

The fifth stage draws in results from `dGamma.tex` to replace the partial derivatives of Γ with partial derivatives of R_{abcd} .

$$\begin{aligned}
(R^a_{\text{cdb}})_{,e} A^c A^d A^e &= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
(R^a_{\text{cdb}})_{,ef} A^c A^d A^e A^f &= -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R^g_{\text{feb}} R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R^a_{\text{feg}} R_{cbdh} g^{hg} \\
(R^a_{\text{cdb}})_{,efg} A^c A^d A^e A^f A^g &= -A^c A^d A^e A^f A^g R^h_{\text{gfb}} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{\text{gfh}} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{\text{geb}} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{\text{gei}} g^{hi}
\end{aligned}$$

Stage 6: Replace partial derivs of R with covariant derivs of R

The final stage is to eliminate the ∂R by using earlier results. For example, in the equation for $\partial^3 R$ we see terms involving ∂R . These first order partial derivatives can be replaced with the expression previously computed for ∂R in terms of ∇R .

$$\begin{aligned}
(R^a_{\text{cdb}})_{,e} A^c A^d A^e &= A^c A^d A^e \nabla_c R_{bdef} g^{af} \\
(R^a_{\text{cdb}})_{,ef} A^c A^d A^e A^f &= A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag} \\
(R^a_{\text{cdb}})_{,efg} A^c A^d A^e A^f A^g &= -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}
\end{aligned}$$

The end result are expressions for the symmetrized partial derivatives of R_{abcd} solely in terms of the symmetrized covariant derivatives of R_{abcd} .

Shared properties

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

B^{a}_{b}::Depends(\nabla{#}).
R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of R

```
def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d} -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{d} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d} -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    sort_product (obj)
    rename_dummies (obj)
    return obj

# compute the symmetric covariant derivatives of  $R^{a}_{bcd} B^{d}_{a} A^{b} A^{c}$ 

beg_stage_1 = time.time()

dRabcd00:= $R^{a}_{b c d} B^{d}_{a} A^{b} A^{c}$ .          # cdb(dRabcd00.101,dRabcd00)

dRabcd01:= $A^{a} \nabla_{a} \{ @ (dRabcd00) \}$ .      # cdb(dRabcd01.101,dRabcd01)
distribute      (dRabcd01)                    # cdb(dRabcd01.102,dRabcd01)
product_rule    (dRabcd01)                    # cdb(dRabcd01.103,dRabcd01)
distribute      (dRabcd01)                    # cdb(dRabcd01.104,dRabcd01)
substitute      (dRabcd01,$\nabla_{a} \{ A^{b} \} -> 0$) # cdb(dRabcd01.105,dRabcd01)
substitute      (dRabcd01,$\nabla_{a} \{ g^{b c} \} -> 0$) # cdb(dRabcd01.106,dRabcd01)

sort_product    (dRabcd01)
rename_dummies  (dRabcd01)
canonicalise    (dRabcd01)                    # cdb(dRabcd01.107,dRabcd01)
dRabcd01 = flatten_Rabcd (dRabcd01)           # cdb(dRabcd01.108,dRabcd01)

dRabcd02:= $A^{a} \nabla_{a} \{ @ (dRabcd01) \}$ .      # cdb(dRabcd02.101,dRabcd02)
distribute      (dRabcd02)                    # cdb(dRabcd02.102,dRabcd02)
product_rule    (dRabcd02)                    # cdb(dRabcd02.103,dRabcd02)
distribute      (dRabcd02)                    # cdb(dRabcd02.104,dRabcd02)
substitute      (dRabcd02,$\nabla_{a} \{ A^{b} \} -> 0$) # cdb(dRabcd02.105,dRabcd02)
substitute      (dRabcd02,$\nabla_{a} \{ g^{b c} \} -> 0$) # cdb(dRabcd02.106,dRabcd02)

sort_product    (dRabcd02)
```

```

rename_dummies (dRabcd02)
canonicalise    (dRabcd02)                # cdb(dRabcd02.107,dRabcd02)
dRabcd02 = flatten_Rabcd (dRabcd02)      # cdb(dRabcd02.108,dRabcd02)

dRabcd03:=A^{a}\nabla_{a}{ @ (dRabcd02) }.      # cdb(dRabcd03.101,dRabcd03)
distribute      (dRabcd03)                # cdb(dRabcd03.102,dRabcd03)
product_rule     (dRabcd03)                # cdb(dRabcd03.103,dRabcd03)
distribute      (dRabcd03)                # cdb(dRabcd03.104,dRabcd03)
substitute       (dRabcd03,$\nabla_{a}{A^{b}} -> 0$) # cdb(dRabcd03.105,dRabcd03)
substitute       (dRabcd03,$\nabla_{a}{g^{b c}} -> 0$) # cdb(dRabcd03.106,dRabcd03)

sort_product     (dRabcd03)
rename_dummies   (dRabcd03)
canonicalise     (dRabcd03)                # cdb(dRabcd03.107,dRabcd03)
dRabcd03 = flatten_Rabcd (dRabcd03)      # cdb(dRabcd03.108,dRabcd03)

dRabcd04:=A^{a}\nabla_{a}{ @ (dRabcd03) }.
distribute      (dRabcd04)
product_rule     (dRabcd04)
distribute      (dRabcd04)
substitute       (dRabcd04,$\nabla_{a}{A^{b}} -> 0$)
substitute       (dRabcd04,$\nabla_{a}{g^{b c}} -> 0$)

sort_product     (dRabcd04)
rename_dummies   (dRabcd04)
canonicalise     (dRabcd04)
dRabcd04 = flatten_Rabcd (dRabcd04)

dRabcd05:=A^{a}\nabla_{a}{ @ (dRabcd04) }.
distribute      (dRabcd05)
product_rule     (dRabcd05)
distribute      (dRabcd05)
substitute       (dRabcd05,$\nabla_{a}{A^{b}} -> 0$)
substitute       (dRabcd05,$\nabla_{a}{g^{b c}} -> 0$)

sort_product     (dRabcd05)
rename_dummies   (dRabcd05)
canonicalise     (dRabcd05)

```

```

dRabcd05 = flatten_Rabcd (dRabcd05)

def combine_nabla (obj):
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{\nabla_{s}\{\nabla_{t}\{A??}\}}\}}\}\rightarrow\nabla_{p\ q\ r\ s\ t}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{\nabla_{s}\{A??\}\}}\}\rightarrow\nabla_{p\ q\ r\ s}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{A??\}\}}\rightarrow\nabla_{p\ q\ r}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{A??\}\}\rightarrow\nabla_{p\ q}\{A??\}\$,repeat=True)
    return obj

dRabcd01 = combine_nabla (dRabcd01)
dRabcd02 = combine_nabla (dRabcd02)
dRabcd03 = combine_nabla (dRabcd03)
dRabcd04 = combine_nabla (dRabcd04)
dRabcd05 = combine_nabla (dRabcd05)

end_stage_1 = time.time()

```


$$\text{dRabcd00.101} := R^a{}_{bcd} B^d{}_e A^b A^c$$

$$\text{dRabcd01.101} := A^a \nabla_a (R^e{}_{bcd} B^d{}_e A^b A^c)$$

$$\text{dRabcd01.102} := A^a \nabla_a (R^e{}_{bcd} B^d{}_e A^b A^c)$$

$$\text{dRabcd01.103} := A^a (\nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c)$$

$$\text{dRabcd01.104} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + A^a R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + A^a R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c$$

$$\text{dRabcd01.105} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\text{dRabcd01.106} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\text{dRabcd01.107} := -A^a A^b A^c B^d{}_e \nabla_a R^e{}_{bcd} - A^a A^b A^c R^d{}_{be} \nabla_c B^e{}_d$$

$$\text{dRabcd01.108} := -A^a A^b A^c B^d{}_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d{}_e g^{ef}$$

$$\text{dRabcd02.101} := A^a \nabla_a (-A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef})$$

$$\text{dRabcd02.102} := -A^a \nabla_a (A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef}) - A^a \nabla_a (A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef})$$

$$\begin{aligned} \text{dRabcd02.103} := & -A^a (\nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ & + A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} + A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef}) - A^a (\nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ & + A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} + A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef}) \end{aligned}$$

$$\begin{aligned} \text{dRabcd02.104} := & -A^a \nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ & - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} - A^a \nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ & - A^a A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef} \end{aligned}$$

$$\begin{aligned} \text{dRabcd02.105} := & -A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} \\ & - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef} \end{aligned}$$

$$\text{dRabcd02.106} := -A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef}$$

$$\text{dRabcd02.107} := -2A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebf} \nabla_c (\nabla_d B^e{}_g) g^{fg}$$

$$\text{dRabcd02.108} := -2A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c (\nabla_d B^e{}_f) g^{fg}$$

$$\begin{aligned}
\text{dRabcd03.101} &:= A^a \nabla_a \left(-2A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right) \\
\text{dRabcd03.102} &:= -2A^a \nabla_a \left(A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \right) - A^a \nabla_a \left(A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \right) - A^a \nabla_a \left(A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right) \\
\text{dRabcd03.103} &:= -2A^a \left(\nabla_a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h \nabla_a A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h A^b \nabla_a A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \right. \\
&\quad + A^h A^b A^c \nabla_a A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} + A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} \\
&\quad + A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} \left. \right) - A^a \left(\nabla_a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h \nabla_a A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \right. \\
&\quad + A^h A^b \nabla_a A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h A^b A^c \nabla_a A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad + A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} + A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} \left. \right) - A^a \left(\nabla_a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right. \\
&\quad + A^h \nabla_a A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b \nabla_a A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b A^c \nabla_a A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad + A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} + A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \left. \right) \\
\text{dRabcd03.104} &:= -2A^a \nabla_a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h \nabla_a A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b \nabla_a A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \\
&\quad - 2A^a A^h A^b A^c \nabla_a A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} \\
&\quad - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} - A^a \nabla_a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h \nabla_a A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b \nabla_a A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c \nabla_a A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} - A^a \nabla_a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad - A^a A^h \nabla_a A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b \nabla_a A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c \nabla_a A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \\
\text{dRabcd03.105} &:= -2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \\
\text{dRabcd03.106} &:= -2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} \\
\text{dRabcd03.107} &:= -3A^a A^b A^c A^d A^e \nabla_a R_{bfeg} \nabla_d (\nabla_e B^f_h) g^{gh} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeg}) g^{gh} \\
&\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeg})) g^{gh} - A^a A^b A^c A^d A^e R_{afbg} \nabla_c (\nabla_d (\nabla_e B^f_h)) g^{gh} \\
\text{dRabcd03.108} &:= -3A^a A^b A^c A^d A^e \nabla_a R_{bfeg} \nabla_d (\nabla_e B^f_g) g^{hg} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeg}) g^{hg} \\
&\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeg})) g^{hg} - A^a A^b A^c A^d A^e R_{afbh} \nabla_c (\nabla_d (\nabla_e B^f_g)) g^{hg}
\end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

```

# compute the covariant derivatives of  $B^{\{a\}_{\{b\}}$ , note  $B^{\{a\}_{\{b,c\}}$  is zero, by choice
# this method of computing covariant derivatives does not use auxillary fields

beg_stage_2 = time.time()

dBab00:= $B^{\{a\}_{\{b\}}$ .      # cdb(dBab00.201,dBab00)

dBab01:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab00}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab01.201,dBab01)
distribute      (dBab01)                # cdb(dBab01.202,dBab01)
product_rule    (dBab01)                # cdb(dBab01.203,dBab01)
distribute      (dBab01)                # cdb(dBab01.204,dBab01)
substitute      (dBab01, $\backslash\text{partial}_{\{a\}}\{A^{\{b\}}\} \rightarrow 0\$$ )  # cdb(dBab01.205,dBab01)
substitute      (dBab01, $\backslash\text{partial}_{\{a\}}\{B^{\{b\}}_{\{c\}}\} \rightarrow 0\$$ )  # cdb(dBab01.206,dBab01)
substitute      (dBab01, $\backslash W^{\{a\}_{\{b\}}} \rightarrow @(\text{dBab00})\$$ )  # cdb(dBab01.207,dBab01)
distribute      (dBab01)                # cdb(dBab01.208,dBab01)
canonicalise    (dBab01)                # cdb(dBab01.209,dBab01)

dBab02:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab01}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab02.201,dBab02)
distribute      (dBab02)                # cdb(dBab02.202,dBab02)
product_rule    (dBab02)                # cdb(dBab02.203,dBab02)
distribute      (dBab02)                # cdb(dBab02.204,dBab02)
substitute      (dBab02, $\backslash\text{partial}_{\{a\}}\{A^{\{b\}}\} \rightarrow 0\$$ )  # cdb(dBab02.205,dBab02)
substitute      (dBab02, $\backslash\text{partial}_{\{a\}}\{B^{\{b\}}_{\{c\}}\} \rightarrow 0\$$ )  # cdb(dBab02.206,dBab02)
substitute      (dBab02, $\backslash W^{\{a\}_{\{b\}}} \rightarrow @(\text{dBab01})\$$ )  # cdb(dBab02.207,dBab02)
distribute      (dBab02)                # cdb(dBab02.208,dBab02)
canonicalise    (dBab02)                # cdb(dBab02.209,dBab02)

dBab03:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab02}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab03.201,dBab03)
distribute      (dBab03)                # cdb(dBab03.202,dBab03)
product_rule    (dBab03)                # cdb(dBab03.203,dBab03)

```

```

distribute      (dBab03)                                # cdb(dBab03.204,dBab03)
substitute      (dBab03,$\partial_{a}\{A^{b}\} \rightarrow 0$) # cdb(dBab03.205,dBab03)
substitute      (dBab03,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$) # cdb(dBab03.206,dBab03)
substitute      (dBab03,$W^{a}_{b} \rightarrow @(dBab02)$) # cdb(dBab03.207,dBab03)
distribute      (dBab03)                                # cdb(dBab03.208,dBab03)
canonicalise    (dBab03)                                # cdb(dBab03.209,dBab03)

dBab04:=A^{c}\partial_{c}\{ @(dBab03) \} + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
              - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.

distribute      (dBab04)
product_rule    (dBab04)
distribute      (dBab04)
substitute      (dBab04,$\partial_{a}\{A^{b}\} \rightarrow 0$)
substitute      (dBab04,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$)
substitute      (dBab04,$W^{a}_{b} \rightarrow @(dBab03)$)
distribute      (dBab04)
canonicalise    (dBab04)

dBab05:=A^{c}\partial_{c}\{ @(dBab04) \} + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
              - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.

distribute      (dBab05)
product_rule    (dBab05)
distribute      (dBab05)
substitute      (dBab05,$\partial_{a}\{A^{b}\} \rightarrow 0$)
substitute      (dBab05,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$)
substitute      (dBab05,$W^{a}_{b} \rightarrow @(dBab04)$)
distribute      (dBab05)
canonicalise    (dBab05)

end_stage_2 = time.time()

```

$$\text{dBab00.201} := B^a_b$$

$$\text{dBab01.201} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.202} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.203} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.204} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.205} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.206} := \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.207} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab01.208} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab01.209} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab02.201} := A^c \partial_c (\Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.202} := A^c \partial_c (\Gamma^a_{pq} B^p_b A^q) - A^c \partial_c (\Gamma^p_{bq} B^a_p A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.203} := A^c (\partial_c \Gamma^a_{pq} B^p_b A^q + \Gamma^a_{pq} \partial_c B^p_b A^q + \Gamma^a_{pq} B^p_b \partial_c A^q) - A^c (\partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^p_{bq} \partial_c B^a_p A^q + \Gamma^p_{bq} B^a_p \partial_c A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.204} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q + A^c \Gamma^a_{pq} \partial_c B^p_b A^q + A^c \Gamma^a_{pq} B^p_b \partial_c A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q - A^c \Gamma^p_{bq} \partial_c B^a_p A^q - A^c \Gamma^p_{bq} B^a_p \partial_c A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.205} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q + A^c \Gamma^a_{pq} \partial_c B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q - A^c \Gamma^p_{bq} \partial_c B^a_p A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.206} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.207} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} (\Gamma^p_{dc} B^d_b A^c - \Gamma^d_{bc} B^p_d A^c) A^q - \Gamma^p_{bq} (\Gamma^a_{dc} B^d_p A^c - \Gamma^d_{pc} B^a_d A^c) A^q$$

$$\text{dBab02.208} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} \Gamma^p_{dc} B^d_b A^c A^q - \Gamma^a_{pq} \Gamma^d_{bc} B^p_d A^c A^q - \Gamma^p_{bq} \Gamma^a_{dc} B^d_p A^c A^q + \Gamma^p_{bq} \Gamma^d_{pc} B^a_d A^c A^q$$

$$\text{dBab02.209} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{cd} \Gamma^c_{pq} B^p_b A^d A^q - 2 \Gamma^a_{cd} \Gamma^p_{bq} B^c_p A^d A^q + \Gamma^c_{bd} \Gamma^p_{cq} B^a_p A^d A^q$$

$$\begin{aligned}
\text{dBab03.209} := & A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q + A^c \partial_c \Gamma^a_{de} \Gamma^d_{pq} B^p_b A^e A^q + A^c \Gamma^a_{cd} \partial_e \Gamma^d_{pq} B^p_b A^e A^q \\
& - 2A^c \partial_c \Gamma^a_{de} \Gamma^p_{bq} B^d_p A^e A^q - 2A^c \Gamma^a_{cd} \partial_e \Gamma^p_{bq} B^d_p A^e A^q + A^c \partial_c \Gamma^d_{be} \Gamma^p_{dq} B^a_p A^e A^q + A^c \Gamma^d_{bc} \partial_e \Gamma^p_{dq} B^a_p A^e A^q \\
& + \Gamma^a_{ce} A^c \partial_f \Gamma^e_{pq} B^p_b A^f A^q - \Gamma^a_{ce} A^c \partial_f \Gamma^p_{bq} B^e_p A^f A^q + \Gamma^a_{cd} \Gamma^c_{ef} \Gamma^e_{pq} B^p_b A^d A^f A^q - 3\Gamma^a_{cd} \Gamma^e_{bf} \Gamma^c_{pq} B^p_e A^d A^f A^q \\
& + 3\Gamma^a_{cd} \Gamma^e_{bf} \Gamma^p_{eq} B^c_p A^d A^f A^q - \Gamma^c_{be} A^e \partial_f \Gamma^a_{pq} B^p_c A^f A^q + \Gamma^c_{be} A^e \partial_f \Gamma^p_{cq} B^a_p A^f A^q - \Gamma^c_{bd} \Gamma^e_{cf} \Gamma^p_{eq} B^a_p A^d A^f A^q
\end{aligned}$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{\{d\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{\{b\}c} \rightarrow 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{\{d\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}g}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    return obj

# switch to RNC

beg_stage_3 = time.time()

dBab01 = impose_rnc (dBab01)    # cdb (dBab01.301,dBab01)
dBab02 = impose_rnc (dBab02)    # cdb (dBab02.301,dBab02)
dBab03 = impose_rnc (dBab03)    # cdb (dBab03.301,dBab03)
dBab04 = impose_rnc (dBab04)    # cdb (dBab04.301,dBab04)
dBab05 = impose_rnc (dBab05)    # cdb (dBab05.301,dBab05)

end_stage_3 = time.time()
```

$$dBab01.301 := 0$$

$$dBab02.301 := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q$$

$$dBab03.301 := A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q$$

$$\begin{aligned} \text{dBab04.301} := & A^c A^e A^g \partial_{ceg} \Gamma_{pq}^a B^p_b A^q - A^c A^e A^g \partial_{ceg} \Gamma_{bq}^p B^a_p A^q + 2A^c A^d \partial_c \Gamma_{de}^a \partial_g \Gamma_{pq}^e B^p_b A^g A^q - 4A^c A^d \partial_c \Gamma_{de}^a \partial_g \Gamma_{bq}^p B^e_p A^g A^q \\ & + 2A^c A^d \partial_c \Gamma_{bd}^e \partial_g \Gamma_{eq}^p B^a_p A^g A^q + A^c \partial_c \Gamma_{ef}^a A^e \partial_g \Gamma_{pq}^f B^p_b A^g A^q - 2A^c \partial_c \Gamma_{ef}^a A^e \partial_g \Gamma_{bq}^p B^f_p A^g A^q + A^c \partial_c \Gamma_{bf}^e A^f \partial_g \Gamma_{eq}^p B^a_p A^g A^q \end{aligned}$$

$$\begin{aligned} \text{dBab05.301} := & A^c A^e A^g A^i \partial_{cegi} \Gamma_{pq}^a B^p_b A^q - A^c A^e A^g A^i \partial_{cegi} \Gamma_{bq}^p B^a_p A^q + 3A^c A^d A^e \partial_{cd} \Gamma_{eg}^a \partial_i \Gamma_{pq}^g B^p_b A^i A^q + 3A^c A^d A^e \partial_{cd} \Gamma_{dg}^a \partial_{ei} \Gamma_{pq}^g B^p_b A^i A^q \\ & - 6A^c A^d A^e \partial_{cd} \Gamma_{eg}^a \partial_i \Gamma_{bq}^p B^g_p A^i A^q - 6A^c A^d A^e \partial_{cd} \Gamma_{dg}^a \partial_{ei} \Gamma_{bq}^p B^g_p A^i A^q + 3A^c A^d A^e \partial_{cd} \Gamma_{be}^g \partial_i \Gamma_{gq}^p B^a_p A^i A^q \\ & + 3A^c A^d A^e \partial_c \Gamma_{bd}^g \partial_{ei} \Gamma_{gq}^p B^a_p A^i A^q + A^c A^e \partial_{ce} \Gamma_{fg}^a A^f \partial_i \Gamma_{pq}^g B^p_b A^i A^q + 2A^c A^e \partial_c \Gamma_{ef}^a A^g \partial_{gi} \Gamma_{pq}^f B^p_b A^i A^q \\ & - 2A^c A^e \partial_{ce} \Gamma_{fg}^a A^f \partial_i \Gamma_{bq}^p B^g_p A^i A^q - 3A^c A^e \partial_c \Gamma_{ef}^a A^g \partial_{gi} \Gamma_{bq}^p B^f_p A^i A^q - A^c A^e \partial_c \Gamma_{be}^f A^g \partial_{gi} \Gamma_{pq}^a B^p_f A^i A^q \\ & + A^c A^e \partial_{ce} \Gamma_{bg}^f A^g \partial_i \Gamma_{fq}^p B^a_p A^i A^q + 2A^c A^e \partial_c \Gamma_{be}^f A^g \partial_{gi} \Gamma_{fq}^p B^a_p A^i A^q + A^c \partial_c \Gamma_{eg}^a A^e A^h \partial_{hi} \Gamma_{pq}^g B^p_b A^i A^q \\ & - A^c \partial_c \Gamma_{eg}^a A^e A^h \partial_{hi} \Gamma_{bq}^p B^g_p A^i A^q - A^c \partial_c \Gamma_{bg}^e A^g A^h \partial_{hi} \Gamma_{pq}^a B^p_e A^i A^q + A^c \partial_c \Gamma_{bg}^e A^g A^h \partial_{hi} \Gamma_{eq}^p B^a_p A^i A^q \end{aligned}$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

```
# substitute covariant derivs of  $B^{\{a\}_{\{b\}}$  into covariant derivs of  $R^{\{a\}_{\{bcd\}}B^{\{d\}_{\{a\}}$ 
# this produces expressions for the partial derivs of Rabcd its covariant derivs and partial derivs of Gamma
# the partial derivs of Gamma will be eliminated later by using results imported from dGamma.json

beg_stage_4 = time.time()

substitute (dRabcd01,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd01)
substitute (dRabcd02,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd02)
substitute (dRabcd03,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd02,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd02)
substitute (dRabcd03,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd03,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab03)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab03)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab03)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab04)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab04)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}A^{\{g\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab05)$,repeat=True); distribute (dRabcd05)

# no longer need B, so let's get rid of it

# two subtle tricks are used here
# 1) rename A and B as A002 and A001 before sort_product,
#    this ensures B will be to left of A after the sort
# 2) indices on B changed from  $B^{\{a\}_{\{b\}}$  to  $B_{\{b\}}^{\{a\}}$ ,
#    this ensures that after factor_out B will have dummy indices  $B_{\{a\}}^{\{b\}}$ 

def remove_Bab (obj):
    foo := @(obj).
```

```

substitute      (foo,$A^{a}->A002^{a},B^{a}_{b}->A001_{b}^{a}$)  # need this to sort B to the left of A
sort_product    (foo)
rename_dummies  (foo)
factor_out      (foo,$A001^{a?}_{b?},A002^{c?}$)
substitute      (foo,$A001_{a}^{b}->1,A002^{a}->A^{a}$)  # recover A and set B = 1, free indices now ^{a}_{b}
return foo

dRabcd01 = remove_Bab (dRabcd01)    # cdb(dRabcd01.401,dRabcd01)
dRabcd02 = remove_Bab (dRabcd02)    # cdb(dRabcd02.401,dRabcd02)
dRabcd03 = remove_Bab (dRabcd03)    # cdb(dRabcd03.401,dRabcd03)
dRabcd04 = remove_Bab (dRabcd04)    # cdb(dRabcd04.401,dRabcd04)
dRabcd05 = remove_Bab (dRabcd05)    # cdb(dRabcd05.401,dRabcd05)

end_stage_4 = time.time()

```

$$\begin{aligned}
\text{dRabcd01.401} &:= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
\text{dRabcd02.401} &:= A^c A^d A^e A^f \left(-\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma_{bf}^g g^{ha} + R_{cbd g} \partial_e \Gamma_{hf}^a g^{gh} \right) \\
\text{dRabcd03.401} &:= A^c A^d A^e A^f A^g \left(-3\nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3\nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_{ef} \Gamma_{bg}^h g^{ia} + R_{cbdh} \partial_{ef} \Gamma_{ig}^a g^{hi} \right) \\
\text{dRabcd04.401} &:= A^c A^d A^e A^f A^g A^h \left(-6\nabla_{de} R_{figj} \partial_c \Gamma_{bh}^i g^{aj} + 6\nabla_{de} R_{fbgi} \partial_c \Gamma_{jh}^a g^{ji} - 4\nabla_c R_{diej} \partial_{fg} \Gamma_{bh}^i g^{ja} + 4\nabla_c R_{dbei} \partial_{fg} \Gamma_{jh}^a g^{ij} - \nabla_{cdef} R_{gbhi} g^{ai} \right. \\
&\quad \left. - R_{cidj} \partial_{efg} \Gamma_{bh}^i g^{ja} + R_{cbdi} \partial_{efg} \Gamma_{jh}^a g^{ij} - 3R_{cidj} \partial_e \Gamma_{fk}^i \partial_g \Gamma_{bh}^k g^{ja} + 6R_{cidj} \partial_e \Gamma_{fb}^i \partial_g \Gamma_{kh}^a g^{jk} - 3R_{cbdi} \partial_e \Gamma_{kf}^j \partial_g \Gamma_{jh}^a g^{ik} \right) \\
\text{dRabcd05.401} &:= A^c A^d A^e A^f A^g A^h A^i \left(-10\nabla_{cd} R_{ejfk} \partial_{gh} \Gamma_{bi}^j g^{ka} + 10\nabla_{cd} R_{ebfj} \partial_{gh} \Gamma_{ki}^a g^{jk} - 10\nabla_{def} R_{gjhk} \partial_c \Gamma_{bi}^j g^{ak} + 10\nabla_{def} R_{gbhj} \partial_c \Gamma_{ki}^a g^{kj} \right. \\
&\quad - 5\nabla_c R_{djek} \partial_{fgh} \Gamma_{bi}^j g^{ka} + 5\nabla_c R_{dbej} \partial_{fgh} \Gamma_{ki}^a g^{jk} - 15\nabla_c R_{djek} \partial_f \Gamma_{gl}^j \partial_h \Gamma_{bi}^l g^{ka} + 30\nabla_c R_{djek} \partial_f \Gamma_{gb}^j \partial_h \Gamma_{li}^a g^{kl} - 15\nabla_c R_{dbej} \partial_f \Gamma_{lg}^k \partial_h \Gamma_{ki}^a g^{jl} \\
&\quad - \nabla_{cdefg} R_{hbij} g^{aj} - R_{cjdk} \partial_{efgh} \Gamma_{bi}^j g^{ka} + R_{cbdj} \partial_{efgh} \Gamma_{ki}^a g^{jk} - 4R_{cjdk} \partial_h \Gamma_{bi}^l \partial_{ef} \Gamma_{gl}^j g^{ka} - 6R_{cjdk} \partial_e \Gamma_{fl}^j \partial_{gh} \Gamma_{bi}^l g^{ka} \\
&\quad \left. + 8R_{cjdk} \partial_h \Gamma_{li}^a \partial_{ef} \Gamma_{gb}^j g^{kl} + 10R_{cjdk} \partial_e \Gamma_{fb}^j \partial_{gh} \Gamma_{li}^a g^{kl} - 4R_{cbdj} \partial_h \Gamma_{ki}^a \partial_{ef} \Gamma_{lg}^k g^{jl} - 6R_{cbdj} \partial_e \Gamma_{lf}^k \partial_{gh} \Gamma_{ki}^a g^{jl} + 2R_{cjdk} \partial_e \Gamma_{lf}^a \partial_{gh} \Gamma_{bi}^j g^{kl} \right)
\end{aligned}$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

```
import cdblib

beg_stage_5 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json') # cdb(dGamma01.500,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json') # cdb(dGamma02.500,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json') # cdb(dGamma03.500,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json') # cdb(dGamma04.500,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json') # cdb(dGamma05.500,dGamma05)

distribute (dRabcd01) # cdb(dRabcd01.500,dRabcd01)
distribute (dRabcd02) # cdb(dRabcd02.500,dRabcd02)
distribute (dRabcd03) # cdb(dRabcd03.500,dRabcd03)
distribute (dRabcd04) # cdb(dRabcd04.500,dRabcd04)
distribute (dRabcd05) # cdb(dRabcd05.500,dRabcd05)

# use dGamma to eliminate the partial derivs of Gamma
# this will introduces some lower order partial dervis of Rabcd on the rhs
# these extra partial derivs of Rabcd will be eliminated (later) by substiting lower order dRabcd into the higher order dRabcd

substitute (dRabcd02,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd02.501,dRabcd02)
substitute (dRabcd02,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd02.502,dRabcd02)
distribute (dRabcd02) # cdb(dRabcd02.503,dRabcd02)
sort_product (dRabcd02) # cdb(dRabcd02.504,dRabcd02)
rename_dummies (dRabcd02) # cdb(dRabcd02.505,dRabcd02)

substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(dGamma02)$,repeat=True) # cdb(dRabcd03.501,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(dGamma02)$,repeat=True) # cdb(dRabcd03.502,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd03.503,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd03.504,dRabcd03)
distribute (dRabcd03) # cdb(dRabcd03.505,dRabcd03)
sort_product (dRabcd03) # cdb(dRabcd03.506,dRabcd03)
rename_dummies (dRabcd03) # cdb(dRabcd03.507,dRabcd03)

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(dGamma03)$,repeat=True) # cdb(dRabcd04.501,dRabcd04)
```

```

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True) # cdb(dRabcd04.502,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True) # cdb(dRabcd04.503,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True) # cdb(dRabcd04.504,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True) # cdb(dRabcd04.505,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True) # cdb(dRabcd04.506,dRabcd04)
distribute (dRabcd04) # cdb(dRabcd04.507,dRabcd04)
sort_product (dRabcd04) # cdb(dRabcd04.508,dRabcd04)
rename_dummies (dRabcd04) # cdb(dRabcd04.509,dRabcd04)

substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)
distribute (dRabcd05)
sort_product (dRabcd05)
rename_dummies (dRabcd05)

end_stage_5 = time.time()

```

$$\text{dRabcd01.500} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\text{dRabcd02.500} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - A^c A^d A^e A^f R_{cgdh} \partial_e \Gamma_{bf}^g g^{ha} + A^c A^d A^e A^f R_{cbdg} \partial_e \Gamma_{hf}^a g^{gh}$$

$$\text{dRabcd02.501} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.502} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.503} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.504} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.505} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{feh}^a R_{cbdh} g^{hg}$$

$$\begin{aligned} \text{dRabcd03.500} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - A^c A^d A^e A^f A^g R_{chdi} \partial_{ef} \Gamma_{bg}^h g^{ia} + A^c A^d A^e A^f A^g R_{cbdh} \partial_{ef} \Gamma_{ig}^a g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.501} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.502} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.503} := & -A^c A^d A^e A^g A^f R_{gfb}^h \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R_{gfi}^a \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.504} := & -A^c A^d A^e A^g A^f R_{gfb}^h \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R_{gfi}^a \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned}
\text{dRabcd03.505} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^h_{geb} R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^a_{gei} R_{cbdh} g^{hi} \\
\text{dRabcd03.506} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{gei} g^{hi} \\
\text{dRabcd03.507} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{gei} g^{hi}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.500} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} \\
&\quad + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{efg} \Gamma^i_{bh} g^{ja} \\
&\quad + A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{efg} \Gamma^a_{jh} g^{ij} - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} \\
&\quad + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.501} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
&\quad - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
&\quad - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
&\quad + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{h fj} \right) R_{cbdi} g^{ij} \\
&\quad - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.502} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
&\quad - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
&\quad - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
&\quad + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{h fj} \right) R_{cbdi} g^{ij} \\
&\quad - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.503} := & -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
& - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik} \\
\text{dRabcd04.504} := & -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
& - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik} \\
\text{dRabcd04.505} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcb} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - A^c A^d A^e A^f A^h A^g R^k_{hgb} R_{cidj} \partial_e \Gamma^i_{fk} g^{ja} + 2A^c A^d A^e A^f A^h A^g R^a_{hgb} R_{cidj} \partial_e \Gamma^i_{fb} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik} \\
\text{dRabcd04.506} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcb} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^f A^e R^i_{fek} A^h A^g R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^f A^e R^i_{feb} A^h A^g R^a_{hgb} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.507} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcb} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{hfb} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^h A^e A^f A^g \partial_{gf} R^i_{heb} R_{cidj} g^{ja} \\
& + \frac{1}{15} A^c A^d A^h A^e A^f A^g R^i_{efk} R^k_{hgb} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^h A^e A^f A^g R^i_{egk} R^k_{hfb} R_{cidj} g^{ja} + \frac{3}{5} A^c A^d A^h A^e A^f A^g \partial_{gf} R^a_{hej} R_{cbdi} g^{ij} \\
& - \frac{1}{15} A^c A^d A^h A^e A^f A^g R^a_{efk} R^k_{hgj} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^h A^e A^f A^g R^a_{egk} R^k_{hfb} R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^f A^e R^i_{fek} A^h A^g R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^f A^e R^i_{feb} A^h A^g R^a_{hgk} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.508} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hcb} \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} \\
& + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{hfb} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{gf} R^i_{heb} g^{ja} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efk} R^k_{hgb} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egk} R^k_{hfb} R_{cidj} g^{ja} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{gf} R^a_{hej} g^{ij} \\
& - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efk} R^k_{hgj} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egk} R^k_{hfb} R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fek} R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgk} R^i_{feb} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgj} R^j_{fek} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.509} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hcb} \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} \\
& + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{hfb} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{gf} R^i_{heb} g^{ja} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efj} R^j_{hgb} R_{cidk} g^{ka} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egj} R^j_{hfb} R_{cidk} g^{ka} \\
& + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{gf} R^a_{hej} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efi} R^i_{hgj} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egi} R^i_{hfb} R_{cbdk} g^{kj} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fej} R^j_{hgb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^j_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^i_{fej} R_{cbdk} g^{kj}
\end{aligned}$$

Stage 6: Replace partial derivs of R with covariant derivs of R

```
# now eliminate remaining partial derivs of Rabcd by substitution from the lower order dRabcd

# note that
#   dRabcd01 = R^a_{cdb,e} A^c A^d A^e
#   dRabcd02 = R^a_{cdb,ef} A^c A^d A^e A^f
#   dRabcd03 = R^a_{cdb,efg} A^c A^d A^e A^f A^g

# thus we can use
#   dRabcd01 to eliminate 1st partial derivs of R in dRabcd03, dRabcd04, etc.
#   dRabcd02 to eliminate 2nd partial derivs of R in dRabcd04, dRabcd05, etc.
#   dRabcd03 to eliminate 3rd partial derivs of R in dRabcd05, dRabcd06, etc.

beg_stage_6 = time.time()

substitute (dRabcd03,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(dRabcd01)$,repeat=True)      # cdb(dRabcd03.601,dRabcd03)
distribute (dRabcd03)                                                                    # cdb(dRabcd03.602,dRabcd03)

# note: dRabcd04 and dRabcd05 unused in this code (or any other code)

substitute (dRabcd04,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(dRabcd02)$,repeat=True) # cdb(dRabcd04.601,dRabcd04)
substitute (dRabcd04,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(dRabcd01)$,repeat=True)      # cdb(dRabcd04.602,dRabcd04)
distribute (dRabcd04)                                                                    # cdb(dRabcd04.603,dRabcd04)

substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c d b}\} -> @(dRabcd03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(dRabcd02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(dRabcd01)$,repeat=True)
distribute (dRabcd05)

end_stage_6 = time.time()
```

$$\begin{aligned}
\text{dRabcd03.601} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad + \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfb} g^{hj} R_{chdi} g^{ia} - \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfi} g^{aj} R_{cbdh} g^{hi} \\
\text{dRabcd03.602} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad + \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfb} g^{hj} R_{chdi} g^{ia} - \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfi} g^{aj} R_{cbdh} g^{hi}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.601} &:= -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hci} \nabla_{de} R_{fbgj} g^{ij} \\
&\quad - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{h fj} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cde} R_{gbhi} g^{ai} \\
&\quad - \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} - \frac{1}{3} A^h A^e A^g A^f R^l_{f gb} R_{hle k} g^{ki} + \frac{1}{3} A^h A^e A^g A^f R^i_{f gl} R_{hbe k} g^{kl} \right) R_{cidj} g^{ja} \\
&\quad + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efj} R^j_{h gb} R_{cidk} g^{ka} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egj} R^j_{h fb} R_{cidk} g^{ka} \\
&\quad + \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gjfl} g^{al} - \frac{1}{3} A^h A^e A^g A^f R^l_{f gj} R_{hle k} g^{ka} + \frac{1}{3} A^h A^e A^g A^f R^a_{f gl} R_{hje k} g^{kl} \right) R_{cbdi} g^{ij} \\
&\quad - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efi} R^i_{h gj} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egi} R^i_{h fj} R_{cbdk} g^{kj} \\
&\quad - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fej} R^j_{h gb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^j_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^i_{fej} R_{cbdk} g^{kj} \\
\text{dRabcd04.602} &:= -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hci} \nabla_{de} R_{fbgj} g^{ij} + 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fk gb} g^{ik} \nabla_c R_{diej} g^{ja} \\
&\quad - 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fk gj} g^{ak} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cde} R_{gbhi} g^{ai} \\
&\quad - \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} - \frac{1}{3} A^h A^e A^g A^f R^l_{f gb} R_{hle k} g^{ki} + \frac{1}{3} A^h A^e A^g A^f R^i_{f gl} R_{hbe k} g^{kl} \right) R_{cidj} g^{ja} \\
&\quad + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efj} R^j_{h gb} R_{cidk} g^{ka} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egj} R^j_{h fb} R_{cidk} g^{ka} \\
&\quad + \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gjfl} g^{al} - \frac{1}{3} A^h A^e A^g A^f R^l_{f gj} R_{hle k} g^{ka} + \frac{1}{3} A^h A^e A^g A^f R^a_{f gl} R_{hje k} g^{kl} \right) R_{cbdi} g^{ij} \\
&\quad - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efi} R^i_{h gj} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egi} R^i_{h fj} R_{cbdk} g^{kj} \\
&\quad - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fej} R^j_{h gb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^j_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^i_{fej} R_{cbdk} g^{kj}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.603} := & -2A^c A^d A^e A^f A^g A^h R^i{}_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a{}_{hci} \nabla_{de} R_{fbgj} g^{ij} + 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fkgb} g^{ik} \nabla_c R_{diej} g^{ja} \\
& - 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fkgj} g^{ak} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} R_{cidj} g^{ja} \\
& + \frac{1}{5} A^c A^d A^h A^e A^g A^f R^l{}_{fgb} R_{hle k} g^{ki} R_{cidj} g^{ja} - \frac{1}{5} A^c A^d A^h A^e A^g A^f R^i{}_{fgl} R_{hbek} g^{kl} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i{}_{efj} R^j{}_{hgb} R_{cidk} g^{ka} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i{}_{egj} R^j{}_{hfb} R_{cidk} g^{ka} - \frac{3}{5} A^c A^d A^h A^e A^g A^f \nabla_{he} R_{g j f l} g^{al} R_{cbdi} g^{ij} - \frac{1}{5} A^c A^d A^h A^e A^g A^f R^l{}_{fgj} R_{hle k} g^{ka} R_{cbdi} g^{ij} \\
& + \frac{1}{5} A^c A^d A^h A^e A^g A^f R^a{}_{fgl} R_{hjek} g^{kl} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a{}_{efi} R^i{}_{hgj} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a{}_{egi} R^i{}_{hfj} R_{cbdk} g^{kj} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i{}_{fej} R^j{}_{hgb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a{}_{hgi} R^j{}_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a{}_{hgi} R^i{}_{fej} R_{cbdk} g^{kj}
\end{aligned}$$

Stage 7: Reformatting

```
beg_stage_7 = time.time()

dRabcd01 = flatten_Rabcd (dRabcd01) # cdb(dRabcd01.701,dRabcd01)
dRabcd02 = flatten_Rabcd (dRabcd02) # cdb(dRabcd02.701,dRabcd02)
dRabcd03 = flatten_Rabcd (dRabcd03) # cdb(dRabcd03.701,dRabcd03)
dRabcd04 = flatten_Rabcd (dRabcd04) # cdb(dRabcd04.701,dRabcd04)
dRabcd05 = flatten_Rabcd (dRabcd05) # cdb(dRabcd05.701,dRabcd05)

canonicalise (dRabcd01) # cdb(dRabcd01.702,dRabcd01)
canonicalise (dRabcd02) # cdb(dRabcd02.702,dRabcd02)
canonicalise (dRabcd03) # cdb(dRabcd03.702,dRabcd03)
canonicalise (dRabcd04) # cdb(dRabcd04.702,dRabcd04)
canonicalise (dRabcd05) # cdb(dRabcd05.702,dRabcd05)

end_stage_7 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.1f} secs\\hfill\\break".format(end_stage_4-beg_stage_4))
print ("Stage 5: {:.1f} secs\\hfill\\break".format(end_stage_5-beg_stage_5))
print ("Stage 6: {:.1f} secs\\hfill\\break".format(end_stage_6-beg_stage_6))
print ("Stage 7: {:.1f} secs".format(end_stage_7-beg_stage_7))
# cdbEnd (timing)
```

$$\text{dRabcd01.701} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\text{dRabcd02.701} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R_{ifeb} g^{gi} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R_{hfei} g^{ah} g^{gi}$$

$$\begin{aligned} \text{dRabcd03.701} := & -A^c A^d A^e A^f A^g R_{hgfb} \nabla_c R_{diej} g^{ih} g^{ja} + A^c A^d A^e A^f A^g R_{hgfi} \nabla_c R_{dbej} g^{ah} g^{ji} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_g R_{ejfb} g^{hj} g^{ia} - \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \nabla_g R_{eifj} g^{ai} g^{hj} \end{aligned}$$

$$\begin{aligned} \text{dRabcd04.701} := & -2A^c A^d A^e A^f A^g A^h R_{ihcb} \nabla_{de} R_{fjgk} g^{ak} g^{ji} + 2A^c A^d A^e A^f A^g A^h R_{ihcj} \nabla_{de} R_{fbgk} g^{ai} g^{jk} + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \nabla_h R_{fkgb} g^{ik} g^{ja} \\ & - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \nabla_h R_{fjgk} g^{aj} g^{ik} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{he} R_{gbfk} g^{ik} g^{ja} \\ & + \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hkel} R_{mfgb} g^{ja} g^{li} g^{km} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hbek} R_{lfgm} g^{il} g^{ja} g^{km} \\ & + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kefl} R_{mhgb} g^{ik} g^{lm} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kegl} R_{mhfb} g^{ik} g^{lm} g^{ja} \\ & - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \nabla_{he} R_{gjfk} g^{ak} g^{ij} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{im} g^{ka} g^{jl} \\ & + \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{al} g^{ij} g^{km} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jefk} R_{lhgm} g^{aj} g^{kl} g^{im} \\ & - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jegk} R_{lhfm} g^{aj} g^{kl} g^{im} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kfel} R_{mhgb} g^{ik} g^{lm} g^{ja} \\ & + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{khgl} R_{mfeb} g^{ak} g^{im} g^{jl} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jhkg} R_{lfem} g^{aj} g^{kl} g^{im} \end{aligned}$$

$$\begin{aligned}
\text{dRabcd05.701} := & 5A^c A^d A^e A^f A^g A^h A^i \nabla_i R_{gjjh} \nabla_{cd} R_{ekfl} g^{kj} g^{la} - 5A^c A^d A^e A^f A^g A^h A^i \nabla_i R_{gjjh} \nabla_{cd} R_{ebfl} g^{aj} g^{lk} \\
& - \frac{10}{3} A^c A^d A^e A^f A^g A^h A^i R_{jicb} \nabla_{def} R_{gkhl} g^{al} g^{kj} + \frac{10}{3} A^c A^d A^e A^f A^g A^h A^i R_{jick} \nabla_{def} R_{gbhl} g^{aj} g^{kl} \\
& + 3A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{djek} \nabla_{if} R_{hbgl} g^{jl} g^{ka} + A^c A^d A^e A^f A^g A^h A^i R_{ijfk} R_{lghb} \nabla_c R_{dmen} g^{na} g^{km} g^{jl} \\
& - A^c A^d A^e A^f A^g A^h A^i R_{ibfj} R_{kghl} \nabla_c R_{dmen} g^{mk} g^{na} g^{jl} + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{jfgk} R_{lihb} \nabla_c R_{dmen} g^{mj} g^{kl} g^{na} \\
& + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{jfhk} R_{ligb} \nabla_c R_{dmen} g^{mj} g^{kl} g^{na} - 3A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{dbej} \nabla_{if} R_{hkgf} g^{al} g^{jk} \\
& - A^c A^d A^e A^f A^g A^h A^i R_{ijfk} R_{lghm} \nabla_c R_{dben} g^{nm} g^{ka} g^{jl} + A^c A^d A^e A^f A^g A^h A^i R_{ijfk} R_{lghm} \nabla_c R_{dben} g^{al} g^{nj} g^{km} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{jfgk} R_{lihm} \nabla_c R_{dben} g^{aj} g^{kl} g^{nm} - \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{jfhk} R_{ligm} \nabla_c R_{dben} g^{aj} g^{kl} g^{nm} \\
& - \frac{5}{3} A^c A^d A^e A^f A^g A^h A^i R_{jgfk} R_{lihb} \nabla_c R_{dmen} g^{mj} g^{kl} g^{na} + \frac{10}{3} A^c A^d A^e A^f A^g A^h A^i R_{jihk} R_{lgfb} \nabla_c R_{dmen} g^{aj} g^{ml} g^{nk} \\
& - \frac{5}{3} A^c A^d A^e A^f A^g A^h A^i R_{jihk} R_{lgfm} \nabla_c R_{dben} g^{aj} g^{kl} g^{nm} - A^c A^d A^e A^f A^g A^h A^i \nabla_{cdefg} R_{hbi} g^{aj} \\
& + \frac{2}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lfgb} \nabla_i R_{emhn} g^{ka} g^{nj} g^{ml} - \frac{2}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lfgm} \nabla_i R_{ebhn} g^{jl} g^{ka} g^{nm} \\
& + \frac{2}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} \nabla_{ieh} R_{gbfl} g^{jl} g^{ka} - \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{ilem} \nabla_f R_{hngb} g^{ka} g^{mj} g^{ln} \\
& + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{ibel} \nabla_f R_{hmgn} g^{jm} g^{ka} g^{ln} - \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lihb} \nabla_e R_{gmfn} g^{nl} g^{jm} g^{ka} \\
& - \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{ligb} \nabla_e R_{hmfn} g^{nl} g^{jm} g^{ka} - \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lef m} \nabla_i R_{hngb} g^{jl} g^{mn} g^{ka} \\
& - \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lifb} \nabla_e R_{hmgn} g^{nl} g^{jm} g^{ka} - \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{legm} \nabla_i R_{hnfb} g^{jl} g^{mn} g^{ka} \\
& - \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lehm} \nabla_i R_{gnfb} g^{jl} g^{mn} g^{ka} - \frac{2}{3} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{kfgl} \nabla_i R_{emhn} g^{jl} g^{na} g^{mk} \\
& + \frac{2}{3} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{kfgl} \nabla_i R_{emhn} g^{ak} g^{jm} g^{nl} - \frac{2}{3} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} \nabla_{ieh} R_{gkfl} g^{al} g^{jk} \\
& + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{ikel} \nabla_f R_{hmgn} g^{jn} g^{la} g^{km} - \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{ikel} \nabla_f R_{hmgn} g^{am} g^{jk} g^{ln} \\
& + \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{kihl} \nabla_e R_{gmfn} g^{am} g^{nk} g^{jl} + \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{ki gl} \nabla_e R_{hmfn} g^{am} g^{nk} g^{jl} \\
& + \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{kefl} \nabla_i R_{hmgn} g^{ak} g^{lm} g^{jn} + \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{kifl} \nabla_e R_{hmgn} g^{am} g^{nk} g^{jl} \\
& + \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{kegl} \nabla_i R_{hmfn} g^{ak} g^{lm} g^{jn} + \frac{1}{9} A^c A^d A^e A^f A^g A^h A^i R_{cbdj} R_{kehl} \nabla_i R_{gmfn} g^{ak} g^{lm} g^{jn} \\
& + \frac{2}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lihb} \nabla_g R_{emfn} g^{nl} g^{jm} g^{ka} + A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lfem} \nabla_i R_{gnhb} g^{jl} g^{mn} g^{ka} \\
& - \frac{4}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lihm} \nabla_g R_{enfb} g^{al} g^{jn} g^{km} - \frac{5}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{lf eb} \nabla_i R_{gmhn} g^{am} g^{jl} g^{kn}
\end{aligned}$$

$$\text{dRabcd01.702} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\text{dRabcd02.702} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\text{dRabcd03.702} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfg h} g^{ah}$$

$$\text{dRabcd04.702} := -\frac{7}{5} A^c A^d A^e A^f A^g A^h R_{bcdi} \nabla_{ef} R_{gjhk} g^{aj} g^{ik} + \frac{7}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{ef} R_{bghk} g^{ai} g^{jk} + A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{bghi} g^{ai}$$

$$\begin{aligned} \text{dRabcd05.702} := & -2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{bdej} \nabla_{fg} R_{hkil} g^{ak} g^{jl} + 2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{djek} \nabla_{fg} R_{bhil} g^{aj} g^{kl} \\ & - \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} \nabla_{efg} R_{hkil} g^{ak} g^{jl} + \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} \nabla_{efg} R_{bhil} g^{aj} g^{kl} \\ & + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{am} g^{jk} g^{ln} + A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{elfm} \nabla_g R_{bhin} g^{aj} g^{kl} g^{mn} \\ & - \frac{4}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{ak} g^{jm} g^{ln} + A^c A^d A^e A^f A^g A^h A^i \nabla_{cdefg} R_{bhij} g^{aj} \end{aligned}$$

```
cdblib.create ('dRabcd.json')
```

```
cdblib.put ('dRabcd01',dRabcd01,'dRabcd.json')
```

```
cdblib.put ('dRabcd02',dRabcd02,'dRabcd.json')
```

```
cdblib.put ('dRabcd03',dRabcd03,'dRabcd.json')
```

```
cdblib.put ('dRabcd04',dRabcd04,'dRabcd.json')
```

```
cdblib.put ('dRabcd05',dRabcd05,'dRabcd.json')
```

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ g^{a b}              -> A003^{a b}              $)
    substitute (obj,$ \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} -> A008_{\{a b c d e f g h\}} $)
    substitute (obj,$ \nabla_{\{e f g\}}\{R_{\{a b c d\}}\}      -> A007_{\{a b c d e f g\}}  $)
    substitute (obj,$ \nabla_{\{e f\}}\{R_{\{a b c d\}}\}         -> A006_{\{a b c d e f\}}   $)
    substitute (obj,$ \nabla_{\{e\}}\{R_{\{a b c d\}}\}           -> A005_{\{a b c d e\}}     $)
    substitute (obj,$ R_{\{a b c d\}}        -> A004_{\{a b c d\}}        $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> A^{a}              $)
    substitute (obj,$ A002^{a}              -> x^{a}              $)
    substitute (obj,$ A003^{a b}            -> g^{a b}            $)
    substitute (obj,$ A004_{\{a b c d\}}     -> R_{\{a b c d\}}     $)
    substitute (obj,$ A005_{\{a b c d e\}}    -> \nabla_{\{e\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A006_{\{a b c d e f\}} -> \nabla_{\{e f\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A007_{\{a b c d e f g\}} -> \nabla_{\{e f g\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A008_{\{a b c d e f g h\}} -> \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} $)

    return obj

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    factor_out (bah,$A^{a?}$)
    ans := @(bah).
    return ans

scaled1 = reformat (dRabcd01, 1) # cdb(scaled1.601,scaled1)
scaled2 = reformat (dRabcd02, 1) # cdb(scaled2.601,scaled2)
scaled3 = reformat (dRabcd03,-2) # cdb(scaled3.601,scaled3)
scaled4 = reformat (dRabcd04,-5) # cdb(scaled4.601,scaled4)

```

```
scaled5 = reformat (dRabcd05,-3)    # cdb(scaled5.601,scaled5)
```

Symmetrised partial derivatives of $R^a{}_{bcd}$

$$\begin{aligned}
A^c A^d A^e R^a{}_{cdb,e} &= A^c A^d A^e g^{af} \nabla_c R_{bdef} \\
A^c A^d A^e A^f R^a{}_{cdb,ef} &= A^c A^d A^e A^f g^{ag} \nabla_{cd} R_{befg} \\
-2A^c A^d A^e A^f A^g R^a{}_{cdb,efg} &= A^c A^d A^e A^f A^g (g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - 2g^{ah} \nabla_{cde} R_{bfgh}) \\
-5A^c A^d A^e A^f A^g A^h R^a{}_{cdb,efgh} &= A^c A^d A^e A^f A^g A^h (7g^{ai} g^{jk} R_{bcdj} \nabla_{ef} R_{gihk} - 7g^{ai} g^{jk} R_{cidi} \nabla_{ef} R_{bghk} - 5g^{ai} \nabla_{cdef} R_{bghi}) \\
-3A^c A^d A^e A^f A^g A^h A^i R^a{}_{cdb,efghi} &= A^c A^d A^e A^f A^g A^h A^i (6g^{aj} g^{kl} \nabla_c R_{bdek} \nabla_{fg} R_{hjil} - 6g^{aj} g^{kl} \nabla_c R_{djek} \nabla_{fg} R_{bhil} + 8g^{aj} g^{kl} R_{bcdk} \nabla_{efg} R_{hjil} \\
&\quad - 8g^{aj} g^{kl} R_{cjdk} \nabla_{efg} R_{bhil} - g^{aj} g^{kl} g^{mn} R_{bcdk} R_{elfm} \nabla_g R_{hjin} - 3g^{aj} g^{kl} g^{mn} R_{cjdk} R_{elfm} \nabla_g R_{bhin} \\
&\quad + 4g^{aj} g^{kl} g^{mn} R_{bcdk} R_{ejfm} \nabla_g R_{hlin} - 3g^{aj} \nabla_{cdefg} R_{bhij})
\end{aligned}$$

```

substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dRabcd.export')

# 6th order dRabcd, scaled
cdblib.put ('dRabcd61scaled',scaled1,'dRabcd.export')
cdblib.put ('dRabcd62scaled',scaled2,'dRabcd.export')
cdblib.put ('dRabcd63scaled',scaled3,'dRabcd.export')
cdblib.put ('dRabcd64scaled',scaled4,'dRabcd.export')
cdblib.put ('dRabcd65scaled',scaled5,'dRabcd.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

```

Timing

Stage 1: 2.4 secs

Stage 2: 6.3 secs

Stage 3: 0.5 secs

Stage 4: 179.3 secs

Stage 5: 194.8 secs

Stage 6: 232.1 secs

Stage 7: 7.9 secs