Geodesic BVP

Consider a geodesic that connects two points P_i and P_j with RNC coordinates x^a_i and x^a_j. Our aim is to construct a solution $x^a(s)$ of the geodesic equation such that $x^a(0) = x^a_i$ and $x^a(1) = x^a_j$.

We will do this in two stages. First we will solve

$$x^a_j = x^a_i + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\cdot\underline{b}_k} \tag{1}$$

for y^a as an explicit polynomial in x^a_i and x^a_j. The functions $\Gamma^a_{\underline{b}_k}$ are the generalised connections for the RNC frame evaluated at $x^a = x^a_i$.

In the second stage, we will substitute our expression for y^a into

$$x^a(s) = x^a_i + sy^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\cdot\underline{b}_k} s^k \tag{2}$$

to obtain the desired solution to the two point boundary value problem.

Stage 1: The fixed point iteration scheme

First we rewrite the main equation (1) in the suggestive form

$$y^a = \Delta x^a + \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\underline{b}_k}$$

where $\Delta x^a = x^a_j - x^a_i$. Our approximate solution for y^a will be taken to be the partial sums for the infinite series. Thus we will solve

$$\overset{n}{y}{}^a = \Delta x^a + \sum_{k=2}^{n} \frac{1}{k!} \Gamma^a_{\underline{b}_k} \overset{n}{y}{}^{\underline{b}_k}$$

for $\overset{n}{y}{}^a$. Note that in the last term of the sum, the $\Gamma^a_{\underline{b}_n}$ will contain curvature terms of order $\mathcal{O}\left(\epsilon^n\right)$. Thus in truncating the series at this point we will loose contributions to the curvature terms of order $\mathcal{O}\left(\epsilon^{n+1}\right)$ and higher. So to be consistent we must truncate all terms of the partial sum to order $\mathcal{O}\left(\epsilon^n\right)$ (i.e., exclude any contributions from terms $\mathcal{O}\left(\epsilon^{n+1}\right)$ and higher, these are the terms that would couple with the terms that we excluded when truncating the original infinite series). Let $\overset{k}{T}$ the operator that truncates its argument to contain terms no higher than $\mathcal{O}\left(\epsilon^n\right)$. Then we have the following modified version of the equation for $\overset{n}{y}{}^a$

$$\overset{n}{y}{}^a = \Delta x^a + \sum_{k=2}^{n} \frac{1}{k!} \overset{k}{T}\left(\Gamma^a_{\underline{b}_k} \overset{n}{y}{}^{\underline{b}_k}\right)$$

Finally we note that since $\Gamma^a_{\underline{b}_k} = \mathcal{O}\left(\epsilon^k\right)$, we can use lower order estimates for the $\overset{k}{y}{}^a$ in the right hand side of the sum. This allows us to compute $\overset{n}{y}{}^a$ by successive approximations such as

$$\begin{aligned}
\overset{0}{y}{}^a &= \Delta x^a \\
\overset{2}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{2}{T}\left(\Gamma^a_{bc}\, \overset{0}{y}{}^b \overset{0}{y}{}^c\right) \\
\overset{3}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{3}{T}\left(\Gamma^a_{bc}\, \overset{2}{y}{}^b \overset{2}{y}{}^c\right) + \frac{1}{3!}\overset{3}{T}\left(\Gamma^a_{bcd}\, \overset{0}{y}{}^b \overset{0}{y}{}^c \overset{0}{y}{}^d\right) \\
\overset{4}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{4}{T}\left(\Gamma^a_{bc}\, \overset{3}{y}{}^b \overset{3}{y}{}^c\right) + \frac{1}{3!}\overset{4}{T}\left(\Gamma^a_{bcd}\, \overset{2}{y}{}^b \overset{2}{y}{}^c \overset{2}{y}{}^d\right) + \frac{1}{4!}\overset{4}{T}\left(\Gamma^a_{bcde}\, \overset{0}{y}{}^b \overset{0}{y}{}^c \overset{0}{y}{}^d \overset{0}{y}{}^e\right) \\
\overset{5}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{5}{T}\left(\Gamma^a_{bc}\, \overset{4}{y}{}^b \overset{4}{y}{}^c\right) + \frac{1}{3!}\overset{5}{T}\left(\Gamma^a_{bcd}\, \overset{3}{y}{}^b \overset{3}{y}{}^c \overset{3}{y}{}^d\right) + \frac{1}{4!}\overset{5}{T}\left(\Gamma^a_{bcde}\, \overset{2}{y}{}^b \overset{2}{y}{}^c \overset{2}{y}{}^d \overset{2}{y}{}^e\right) + \frac{1}{5!}\overset{5}{T}\left(\Gamma^a_{bcdef}\, \overset{0}{y}{}^b \overset{0}{y}{}^c \overset{0}{y}{}^d \overset{0}{y}{}^e \overset{0}{y}{}^f\right)
\end{aligned}$$

and so on. Note that there are no $\overset{1}{y}{}^a$ terms.

Stage 2: Introduce the generalised connections

This is the final stage – it introduces the generalised connecstion after the completion of the fixed point scheme.

All that needs be done is to substitute our expression for y^a into

$$x^a(s) = x^a_i + s y^a - \sum_{k=2}^{\infty} \frac{1}{k!}\, \Gamma^a{}_{\underline{b}_k} y^{\cdot \underline{b}_k} s^k \tag{3}$$

to obtain the desired solution to the two point boundary value problem.

The generalised connections $\Gamma^a{}_{\underline{b}_k}$ are taken from the results of the `genGamma` code.

Stage 1: The fixed point iteration scheme

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.
R_{a b c d}::Depends(\nabla{#}).

{Gam22^{a}_{b c},Gam23^{a}_{b c},Gam24^{a}_{b c},Gam25^{a}_{b c}}::TableauSymmetry(shape={2}, indices={1,2}).
{Gam33^{a}_{b c d},Gam34^{a}_{b c d},Gam35^{a}_{b c d}}::TableauSymmetry(shape={3}, indices={1,2,3}).
{Gam44^{a}_{b c d e},Gam45^{a}_{b c d e}}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
{Gam55^{a}_{b c d e f}}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).

{Gam22^{a}_{b c}}::Weight(label=eps,value=2).
{Gam23^{a}_{b c},Gam33^{a}_{b c d}}::Weight(label=eps,value=3).
{Gam24^{a}_{b c},Gam34^{a}_{b c d},Gam44^{a}_{b c d e}}::Weight(label=eps,value=4).
{Gam25^{a}_{b c},Gam35^{a}_{b c d},Gam45^{a}_{b c d e},Gam55^{a}_{b c d e f}}::Weight(label=eps,value=5).

{Dx^{a}}::Weight(label=eps,value=0).

{y00^{a},y20^{a},y30^{a},y40^{a},y50^{a}}::Weight(label=eps,value=0).
{y22^{a},y32^{a},y42^{a},y52^{a}}::Weight(label=eps,value=2).
{y33^{a},y43^{a},y53^{a}}::Weight(label=eps,value=3).
{y44^{a},y54^{a}}::Weight(label=eps,value=4).
{y55^{a}}::Weight(label=eps,value=5).

# Dx{#}::LaTeXForm{"{\Dx}"}.  # LCB: currently causes a bug, it kills ::KeepWeight for Dx

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}                          -> A001^{a}             $)
    substitute (obj,$ Dx^{a}                         -> A002^{a}             $)
```

```
    substitute (obj,$ g^{a b}                          -> A003^{a b}             $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}    -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}      -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}        -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}          -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                      -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> x^{a}                           $)
    substitute (obj,$ A002^{a}              -> Dx^{a}                          $)
    substitute (obj,$ A003^{a b}            -> g^{a b}                         $)
    substitute (obj,$ A004_{a b c d}        -> R_{a b c d}                     $)
    substitute (obj,$ A005_{a b c d e}      -> \nabla_{e}{R_{a b c d}}         $)
    substitute (obj,$ A006_{a b c d e f}    -> \nabla_{e f}{R_{a b c d}}       $)
    substitute (obj,$ A007_{a b c d e f g}  -> \nabla_{e f g}{R_{a b c d}}     $)
    substitute (obj,$ A008_{a b c d e f g h} -> \nabla_{e f g h}{R_{a b c d}} $)

    return obj

def get_term (obj,n):

    tmp := @(obj).
    foo = Ex("eps = " + str(n))
    distribute  (tmp)
    keep_weight (tmp, foo)

    return tmp

def truncate (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah = Ex("eps = " + str(i))
       distribute  (foo)
       keep_weight (foo, bah)
       ans = ans + foo
```

```
    return ans

def substitute_eps (obj):
    substitute     (obj,epsy0)
    substitute     (obj,epsy2)
    substitute     (obj,epsy3)
    substitute     (obj,epsy4)
    substitute     (obj,epsy5)
    substitute     (obj,epsGam2)
    substitute     (obj,epsGam3)
    substitute     (obj,epsGam4)
    substitute     (obj,epsGam5)
    distribute     (obj)
    obj = truncate     (obj,5)
    obj = product_sort (obj)
    rename_dummies (obj)
    canonicalise   (obj)

    return obj

beg_stage_1 = time.time()

# yn = y expanded to terms upto and including O(eps^n)

y0 := Dx^{a}.
y2 := Dx^{a} +   (1/2) Gam^{a}_{b c} y0^{b} y0^{c}.
y3 := Dx^{a} +   (1/2) Gam^{a}_{b c} y2^{b} y2^{c}
             +   (1/6) Gam^{a}_{b c d} y0^{b} y0^{c} y0^{d}.
y4 := Dx^{a} +   (1/2) Gam^{a}_{b c} y3^{b} y3^{c}
             +   (1/6) Gam^{a}_{b c d} y2^{b} y2^{c} y2^{d}
             +  (1/24) Gam^{a}_{b c d e} y0^{b} y0^{c} y0^{d} y0^{e}.
y5 := Dx^{a} +   (1/2) Gam^{a}_{b c} y4^{b} y4^{c}
             +   (1/6) Gam^{a}_{b c d} y3^{b} y3^{c} y3^{d}
             +  (1/24) Gam^{a}_{b c d e} y2^{b} y2^{c} y2^{d} y2^{e}
             + (1/120) Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

# epsyN = y expanded to terms upto and including O(eps^N)
```

```
# yPQ = O(eps^Q) term of epsyP

# expand to O(eps^5)

epsy0 := y0^{a} -> y00^{a}.
epsy2 := y2^{a} -> y20^{a}+y22^{a}.
epsy3 := y3^{a} -> y30^{a}+y32^{a}+y33^{a}.
epsy4 := y4^{a} -> y40^{a}+y42^{a}+y43^{a}+y44^{a}.
epsy5 := y5^{a} -> y50^{a}+y52^{a}+y53^{a}+y54^{a}+y55^{a}.

# epsGamN = gen. gamma with N lower indices (epsGam2 = the connection)
# epsGamPQ = O(eps^Q) term of epsGamP

epsGam2 := Gam^{a}_{b c} -> Gam22^{a}_{b c}+Gam23^{a}_{b c}+Gam24^{a}_{b c}+Gam25^{a}_{b c}.
epsGam3 := Gam^{a}_{b c d} -> Gam33^{a}_{b c d}+Gam34^{a}_{b c d}+Gam35^{a}_{b c d}.
epsGam4 := Gam^{a}_{b c d e} -> Gam44^{a}_{b c d e}+Gam45^{a}_{b c d e}.
epsGam5 := Gam^{a}_{b c d e f} -> Gam55^{a}_{b c d e f}.

y0 = substitute_eps (y0)   # cdb (y0.001,y0)
y2 = substitute_eps (y2)   # cdb (y2.001,y2)
y3 = substitute_eps (y3)   # cdb (y3.001,y3)
y4 = substitute_eps (y4)   # cdb (y4.001,y4)
y5 = substitute_eps (y5)   # cdb (y5.001,y5)

y0 = truncate (y0,1)       # cdb (y0.002,y0)
y2 = truncate (y2,2)       # cdb (y2.002,y2)
y3 = truncate (y3,3)       # cdb (y3.002,y3)
y4 = truncate (y4,4)       # cdb (y4.002,y4)
y5 = truncate (y5,5)       # cdb (y5.002,y5)

defy0 := y0^{a} -> @(y0).
defy2 := y2^{a} -> @(y2).
defy3 := y3^{a} -> @(y3).
defy4 := y4^{a} -> @(y4).
defy5 := y5^{a} -> @(y5).

# -----------------------------------
def tidy (obj):
```

```
    obj = product_sort (obj)
    rename_dummies     (obj)
    canonicalise       (obj)
    return obj

# ----------------------------------
# y0

y00 := @(y0).           # cdb (y00.101,y00)

defy00 := y00^{a} -> @(y00).

# ----------------------------------
# y2

substitute (y2,defy00)

distribute (y2)

y20 = get_term (y2,0)   # cdb (y20.101,y20)
y22 = get_term (y2,2)   # cdb (y22.101,y22)

y20 = tidy (y20)        # cdb (y20.201,y20)
y22 = tidy (y22)        # cdb (y22.201,y22)

defy20 := y20^{a} -> @(y20).
defy22 := y22^{a} -> @(y22).

# ----------------------------------
# y3

substitute (y3,defy00)

substitute (y3,defy20)
substitute (y3,defy22)

distribute (y3)
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```
y30 = get_term (y3,0)   # cdb (y30.101,y30)
y32 = get_term (y3,2)   # cdb (y32.101,y32)
y33 = get_term (y3,3)   # cdb (y33.101,y33)

y30 = tidy (y30)        # cdb (y30.201,y30)
y32 = tidy (y32)        # cdb (y32.201,y32)
y33 = tidy (y33)        # cdb (y33.201,y33)

defy30 := y30^{a} -> @(y30).
defy32 := y32^{a} -> @(y32).
defy33 := y33^{a} -> @(y33).

# -----------------------------------
# y4

substitute (y4,defy00)

substitute (y4,defy20)
substitute (y4,defy22)

substitute (y4,defy30)
substitute (y4,defy32)
substitute (y4,defy33)

distribute (y4)

y40 = get_term (y4,0)   # cdb (y40.101,y40)
y42 = get_term (y4,2)   # cdb (y42.101,y42)
y43 = get_term (y4,3)   # cdb (y43.101,y43)
y44 = get_term (y4,4)   # cdb (y44.101,y44)

y40 = tidy (y40)        # cdb (y40.201,y40)
y42 = tidy (y42)        # cdb (y42.201,y42)
y43 = tidy (y43)        # cdb (y43.201,y43)
y44 = tidy (y44)        # cdb (y44.201,y44)

defy40 := y40^{a} -> @(y40).
defy42 := y42^{a} -> @(y42).
```

```
defy43 := y43^{a} -> @(y43).
defy44 := y44^{a} -> @(y44).

# ----------------------------------
# y5

substitute (y5,defy00)

substitute (y5,defy20)
substitute (y5,defy22)

substitute (y5,defy30)
substitute (y5,defy32)
substitute (y5,defy33)

substitute (y5,defy40)
substitute (y5,defy42)
substitute (y5,defy43)
substitute (y5,defy44)

distribute (y5)

y50 = get_term (y5,0)   # cdb (y50.101,y50)
y52 = get_term (y5,2)   # cdb (y52.101,y52)
y53 = get_term (y5,3)   # cdb (y53.101,y53)
y54 = get_term (y5,4)   # cdb (y54.101,y54)
y55 = get_term (y5,5)   # cdb (y55.101,y55)

y50 = tidy (y50)        # cdb (y50.201,y50)
y52 = tidy (y52)        # cdb (y52.201,y52)
y53 = tidy (y53)        # cdb (y53.201,y53)
y54 = tidy (y54)        # cdb (y54.201,y54)
y55 = tidy (y55)        # cdb (y55.201,y55)

defy50 := y50^{a} -> @(y50).
defy52 := y52^{a} -> @(y52).
defy53 := y53^{a} -> @(y53).
defy54 := y54^{a} -> @(y54).
```

```
defy55 := y55^{a} -> @(y55).

end_stage_1 = time.time()
```

$$\texttt{y0.001} := Dx^a$$

$$\texttt{y2.001} := Dx^a + \frac{1}{2}Gam22^a{}_{bc}y00^b y00^c + \frac{1}{2}Gam23^a{}_{bc}y00^b y00^c + \frac{1}{2}Gam24^a{}_{bc}y00^b y00^c + \frac{1}{2}Gam25^a{}_{bc}y00^b y00^c$$

$$\begin{aligned}\texttt{y3.001} := {} & Dx^a + \frac{1}{2}Gam22^a{}_{bc}y20^b y20^c + \frac{1}{2}Gam23^a{}_{bc}y20^b y20^c + \frac{1}{6}Gam33^a{}_{bcd}y00^b y00^c y00^d + Gam22^a{}_{bc}y20^b y22^c + \frac{1}{2}Gam24^a{}_{bc}y20^b y20^c \\ & + \frac{1}{6}Gam34^a{}_{bcd}y00^b y00^c y00^d + Gam23^a{}_{bc}y20^b y22^c + \frac{1}{2}Gam25^a{}_{bc}y20^b y20^c + \frac{1}{6}Gam35^a{}_{bcd}y00^b y00^c y00^d\end{aligned}$$

$$\begin{aligned}\texttt{y4.001} := {} & Dx^a + \frac{1}{2}Gam22^a{}_{bc}y30^b y30^c + \frac{1}{2}Gam23^a{}_{bc}y30^b y30^c + \frac{1}{6}Gam33^a{}_{bcd}y20^b y20^c y20^d + Gam22^a{}_{bc}y30^b y32^c + \frac{1}{2}Gam24^a{}_{bc}y30^b y30^c \\ & + \frac{1}{6}Gam34^a{}_{bcd}y20^b y20^c y20^d + \frac{1}{24}Gam44^a{}_{bcde}y00^b y00^c y00^d y00^e + Gam22^a{}_{bc}y30^b y33^c + Gam23^a{}_{bc}y30^b y32^c \\ & + \frac{1}{2}Gam25^a{}_{bc}y30^b y30^c + \frac{1}{2}Gam33^a{}_{bcd}y20^b y20^c y22^d + \frac{1}{6}Gam35^a{}_{bcd}y20^b y20^c y20^d + \frac{1}{24}Gam45^a{}_{bcde}y00^b y00^c y00^d y00^e\end{aligned}$$

$$\begin{aligned}\texttt{y5.001} := {} & Dx^a + \frac{1}{2}Gam22^a{}_{bc}y40^b y40^c + \frac{1}{2}Gam23^a{}_{bc}y40^b y40^c + \frac{1}{6}Gam33^a{}_{bcd}y30^b y30^c y30^d + Gam22^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam24^a{}_{bc}y40^b y40^c \\ & + \frac{1}{6}Gam34^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam44^a{}_{bcde}y20^b y20^c y20^d y20^e + Gam22^a{}_{bc}y40^b y43^c + Gam23^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam25^a{}_{bc}y40^b y40^c \\ & + \frac{1}{2}Gam33^a{}_{bcd}y30^b y30^c y32^d + \frac{1}{6}Gam35^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam45^a{}_{bcde}y20^b y20^c y20^d y20^e + \frac{1}{120}Gam55^a{}_{bcdef}y00^b y00^c y00^d y00^e y00^f\end{aligned}$$

$$\texttt{y0.002} := Dx^a$$

$$\texttt{y2.002} := Dx^a + \frac{1}{2}Gam22^a{}_{bc}y00^b y00^c$$

$$\texttt{y3.002} := Dx^a + \frac{1}{2}Gam22^a{}_{bc}y20^b y20^c + \frac{1}{2}Gam23^a{}_{bc}y20^b y20^c + \frac{1}{6}Gam33^a{}_{bcd}y00^b y00^c y00^d$$

$$\begin{aligned}\texttt{y4.002} := Dx^a &+ \frac{1}{2}Gam22^a{}_{bc}y30^b y30^c + \frac{1}{2}Gam23^a{}_{bc}y30^b y30^c + \frac{1}{6}Gam33^a{}_{bcd}y20^b y20^c y20^d + Gam22^a{}_{bc}y30^b y32^c \\ &+ \frac{1}{2}Gam24^a{}_{bc}y30^b y30^c + \frac{1}{6}Gam34^a{}_{bcd}y20^b y20^c y20^d + \frac{1}{24}Gam44^a{}_{bcde}y00^b y00^c y00^d y00^e\end{aligned}$$

$$\begin{aligned}\texttt{y5.002} := Dx^a &+ \frac{1}{2}Gam22^a{}_{bc}y40^b y40^c + \frac{1}{2}Gam23^a{}_{bc}y40^b y40^c + \frac{1}{6}Gam33^a{}_{bcd}y30^b y30^c y30^d + Gam22^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam24^a{}_{bc}y40^b y40^c \\ &+ \frac{1}{6}Gam34^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam44^a{}_{bcde}y20^b y20^c y20^d y20^e + Gam22^a{}_{bc}y40^b y43^c + Gam23^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam25^a{}_{bc}y40^b y40^c \\ &+ \frac{1}{2}Gam33^a{}_{bcd}y30^b y30^c y32^d + \frac{1}{6}Gam35^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam45^a{}_{bcde}y20^b y20^c y20^d y20^e + \frac{1}{120}Gam55^a{}_{bcdef}y00^b y00^c y00^d y00^e y00^f\end{aligned}$$

$$\texttt{y00.101} := Dx^a$$

$$\texttt{y20.201} := Dx^a$$

$$\texttt{y22.201} := \frac{1}{2} Dx^b Dx^c Gam22^a{}_{bc}$$

$$\texttt{y30.201} := Dx^a$$

$$\texttt{y32.201} := \frac{1}{2} Dx^b Dx^c Gam22^a{}_{bc}$$

$$\texttt{y33.201} := \frac{1}{2} Dx^b Dx^c Gam23^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a{}_{bcd}$$

$$\texttt{y40.201} := Dx^a$$

$$\texttt{y42.201} := \frac{1}{2} Dx^b Dx^c Gam22^a{}_{bc}$$

$$\texttt{y43.201} := \frac{1}{2} Dx^b Dx^c Gam23^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a{}_{bcd}$$

$$\texttt{y44.201} := \frac{1}{2} Dx^b Dx^c Dx^d Gam22^a{}_{be} Gam22^e{}_{cd} + \frac{1}{2} Dx^b Dx^c Gam24^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam34^a{}_{bcd} + \frac{1}{24} Dx^b Dx^c Dx^d Dx^e Gam44^a{}_{bcde}$$

$$\texttt{y50.201} := Dx^{a}$$

$$\texttt{y52.201} := \frac{1}{2}Dx^{b}Dx^{c}Gam22^{a}{}_{bc}$$

$$\texttt{y53.201} := \frac{1}{2}Dx^{b}Dx^{c}Gam23^{a}{}_{bc} + \frac{1}{6}Dx^{b}Dx^{c}Dx^{d}Gam33^{a}{}_{bcd}$$

$$\texttt{y54.201} := \frac{1}{2}Dx^{b}Dx^{c}Dx^{d}Gam22^{a}{}_{be}Gam22^{e}{}_{cd} + \frac{1}{2}Dx^{b}Dx^{c}Gam24^{a}{}_{bc} + \frac{1}{6}Dx^{b}Dx^{c}Dx^{d}Gam34^{a}{}_{bcd} + \frac{1}{24}Dx^{b}Dx^{c}Dx^{d}Dx^{e}Gam44^{a}{}_{bcde}$$

$$\begin{aligned}\texttt{y55.201} := {}& \frac{1}{2}Dx^{b}Dx^{c}Dx^{d}Gam22^{a}{}_{be}Gam23^{e}{}_{cd} + \frac{1}{6}Dx^{b}Dx^{c}Dx^{d}Dx^{e}Gam22^{a}{}_{bf}Gam33^{f}{}_{cde} + \frac{1}{2}Dx^{b}Dx^{c}Dx^{d}Gam22^{e}{}_{bc}Gam23^{a}{}_{de} \\ & + \frac{1}{2}Dx^{b}Dx^{c}Gam25^{a}{}_{bc} + \frac{1}{4}Dx^{b}Dx^{c}Dx^{d}Dx^{e}Gam22^{f}{}_{bc}Gam33^{a}{}_{def} + \frac{1}{6}Dx^{b}Dx^{c}Dx^{d}Gam35^{a}{}_{bcd} \\ & + \frac{1}{24}Dx^{b}Dx^{c}Dx^{d}Dx^{e}Gam45^{a}{}_{bcde} + \frac{1}{120}Dx^{b}Dx^{c}Dx^{d}Dx^{e}Dx^{f}Gam55^{a}{}_{bcdef}\end{aligned}$$

Stage 2a: Introduce the generalised connections, build terms of y^a

```
def substitute_gam (obj):

    substitute     (obj,defGam22)
    substitute     (obj,defGam23)
    substitute     (obj,defGam24)
    substitute     (obj,defGam25)

    substitute     (obj,defGam33)
    substitute     (obj,defGam34)
    substitute     (obj,defGam35)

    substitute     (obj,defGam44)
    substitute     (obj,defGam45)

    substitute     (obj,defGam55)

    distribute     (obj)
    return obj

import cdblib

beg_stage_2a = time.time()

Gam22 = cdblib.get ('genGamma01','genGamma.json')
Gam23 = cdblib.get ('genGamma02','genGamma.json')
Gam24 = cdblib.get ('genGamma03','genGamma.json')
Gam25 = cdblib.get ('genGamma04','genGamma.json')

Gam33 = cdblib.get ('genGamma11','genGamma.json')
Gam34 = cdblib.get ('genGamma12','genGamma.json')
Gam35 = cdblib.get ('genGamma13','genGamma.json')

Gam44 = cdblib.get ('genGamma21','genGamma.json')
Gam45 = cdblib.get ('genGamma22','genGamma.json')

Gam55 = cdblib.get ('genGamma31','genGamma.json')
```

```
# peel off the A^{a}, must then symmetrise over revealed indices

substitute (Gam22,$A^{a}->1$)
substitute (Gam23,$A^{a}->1$)
substitute (Gam24,$A^{a}->1$)
substitute (Gam25,$A^{a}->1$)

substitute (Gam33,$A^{a}->1$)
substitute (Gam34,$A^{a}->1$)
substitute (Gam35,$A^{a}->1$)

substitute (Gam44,$A^{a}->1$)
substitute (Gam45,$A^{a}->1$)

substitute (Gam55,$A^{a}->1$)

# now symmetrise

sym (Gam22,$_{b},_{c}$)
sym (Gam23,$_{b},_{c}$)
sym (Gam24,$_{b},_{c}$)
sym (Gam25,$_{b},_{c}$)

sym (Gam33,$_{b},_{c},_{d}$)
sym (Gam34,$_{b},_{c},_{d}$)
sym (Gam35,$_{b},_{c},_{d}$)

sym (Gam44,$_{b},_{c},_{d},_{e}$)
sym (Gam45,$_{b},_{c},_{d},_{e}$)

sym (Gam55,$_{b},_{c},_{d},_{e},_{f}$)

defGam22 := Gam22^{a}_{b c} -> @(Gam22).
defGam23 := Gam23^{a}_{b c} -> @(Gam23).
defGam24 := Gam24^{a}_{b c} -> @(Gam24).
defGam25 := Gam25^{a}_{b c} -> @(Gam25).
```

```
defGam33 := Gam33^{a}_{b c d} -> @(Gam33).
defGam34 := Gam34^{a}_{b c d} -> @(Gam34).
defGam35 := Gam35^{a}_{b c d} -> @(Gam35).

defGam44 := Gam44^{a}_{b c d e} -> @(Gam44).
defGam45 := Gam45^{a}_{b c d e} -> @(Gam45).

defGam55 := Gam55^{a}_{b c d e f} -> @(Gam55).

# ---------------------------------------------------
# y2

y22 = substitute_gam (y22)

y22 = tidy (y22)                                        # cdb (y22.301,y22)

y2 := @(y20) + @(y22).                                  # cdb (y2.301,y2)

# ---------------------------------------------------
# y3

y32 = substitute_gam (y32)
y33 = substitute_gam (y33)

y32 = tidy (y32)                                        # cdb (y32.301,y32)
y33 = tidy (y33)                                        # cdb (y33.301,y33)

y3 := @(y30) + @(y32) + @(y33).                         # cdb (y3.301,y3)

# ---------------------------------------------------
# y4

y42 = substitute_gam (y42)
y43 = substitute_gam (y43)
y44 = substitute_gam (y44)

y42 = tidy (y42)                                        # cdb (y42.301,y42)
y43 = tidy (y43)                                        # cdb (y43.301,y43)
```

```
y44 = tidy (y44)                                        # cdb (y44.301,y44)

y4 := @(y40) + @(y42) + @(y43) + @(y44).                # cdb (y4.301,y4)

# ---------------------------------------------------
# y5

y52 = substitute_gam (y52)
y53 = substitute_gam (y53)
y54 = substitute_gam (y54)
y55 = substitute_gam (y55)

y52 = tidy (y52)                                        # cdb (y52.301,y52)
y53 = tidy (y53)                                        # cdb (y53.301,y53)
y54 = tidy (y54)                                        # cdb (y54.301,y54)
y55 = tidy (y55)                                        # cdb (y55.301,y55)

y5 := @(y50) + @(y52) + @(y53) + @(y54) + @(y55).       # cdb (y5.301,y5)

# ---------------------------------------------------
cdblib.create ('geodesic-bvp.json')

cdblib.put ('y2',y2,'geodesic-bvp.json')
cdblib.put ('y3',y3,'geodesic-bvp.json')
cdblib.put ('y4',y4,'geodesic-bvp.json')
cdblib.put ('y5',y5,'geodesic-bvp.json')

cdblib.put ('y20',y20,'geodesic-bvp.json')
cdblib.put ('y22',y22,'geodesic-bvp.json')

cdblib.put ('y30',y30,'geodesic-bvp.json')
cdblib.put ('y32',y32,'geodesic-bvp.json')
cdblib.put ('y33',y33,'geodesic-bvp.json')

cdblib.put ('y40',y40,'geodesic-bvp.json')
cdblib.put ('y42',y42,'geodesic-bvp.json')
cdblib.put ('y43',y43,'geodesic-bvp.json')
cdblib.put ('y44',y44,'geodesic-bvp.json')
```

```
cdblib.put ('y50',y50,'geodesic-bvp.json')
cdblib.put ('y52',y52,'geodesic-bvp.json')
cdblib.put ('y53',y53,'geodesic-bvp.json')
cdblib.put ('y54',y54,'geodesic-bvp.json')
cdblib.put ('y55',y55,'geodesic-bvp.json')

end_stage_2a = time.time()
```

$\texttt{y50.201} := Dx^a$

$\texttt{y52.301} := -\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$

$\texttt{y53.301} := -\frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} + \frac{1}{24}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$

$$
\begin{aligned}
\texttt{y54.301} := &-\frac{2}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{4}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{befh} R_{cgdi} \\
&+ \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dgfi} - \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{eb} R_{cfdg} \\
&- \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bc} R_{defg} - \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dfgi} \\
&+ \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bg} R_{cedf} - \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cgfi} \\
&+ \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{60}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{de} R_{bfcg} - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{db} R_{cefg} \\
&- \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{gd} R_{becf} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{dg} R_{becf} \\
&- \frac{1}{45}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{1}{60}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} \nabla_{cd} R_{befg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{y55.301} :=& -\frac{7}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} - \frac{1}{45}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} \\
&+ \frac{1}{216}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} \\
&- \frac{17}{1080}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} + \frac{1}{135}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} \\
&- \frac{1}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{108}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} \\
&- \frac{1}{45}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - \frac{7}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} \\
&- \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} - \frac{1}{60}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} - \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} \\
&+ \frac{1}{72}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} \\
&+ \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} \\
&- \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{fbc} R_{dgeh} - \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bfc} R_{dgeh} - \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcf} R_{dgeh} \\
&- \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcd} R_{efgh} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} \\
&- \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + \frac{1}{360}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{hbc} R_{dfeg} + \frac{1}{360}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bhc} R_{dfeg} \\
&+ \frac{1}{360}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bch} R_{dfeg} - \frac{7}{1080}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - \frac{1}{540}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} \\
&+ \frac{1}{108}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh} - \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} \\
&- \frac{1}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} - \frac{11}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
&+ \frac{1}{216}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} \\
&+ \frac{1}{135}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{efb} R_{cgdh} \\
&- \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ebf} R_{cgdh} - \frac{1}{180}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ebc} R_{dfgh} - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bef} R_{cgdh} \\
&- \frac{1}{180}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bec} R_{dfgh} - \frac{1}{180}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bce} R_{dfgh} - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} \\
&- \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{heb} R_{cfdg} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{hbe} R_{cfdg} \\
&+ \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ehb} R_{cfdg} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bhe} R_{cfdg} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ebh} R_{cfdg} \\
&+ \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{beh} R_{cfdg} - \frac{1}{120}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - \frac{1}{90}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj}
\end{aligned}
$$

Stage 2b: Building the terms of $x^a(s)$

```
def substitute_y (obj):
    substitute (obj,defy00)
    substitute (obj,defy20)
    substitute (obj,defy30)
    substitute (obj,defy32)
    substitute (obj,defy40)
    substitute (obj,defy42)
    substitute (obj,defy43)
    distribute (obj)
    return obj

beg_stage_2b = time.time()

term2 := Gam^{a}_{b c} y4^{b} y4^{c}.
term3 := Gam^{a}_{b c d} y3^{b} y3^{c} y3^{d}.
term4 := Gam^{a}_{b c d e} y2^{b} y2^{c} y2^{d} y2^{e}.
term5 := Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

term2 = substitute_eps (term2)   # cdb (term2.401,term2)
term3 = substitute_eps (term3)   # cdb (term3.401,term3)
term4 = substitute_eps (term4)   # cdb (term4.401,term4)
term5 = substitute_eps (term5)   # cdb (term5.401,term5)

term2 = substitute_y (term2)
term3 = substitute_y (term3)
term4 = substitute_y (term4)
term5 = substitute_y (term5)

term2 = substitute_gam (term2)
term3 = substitute_gam (term3)
term4 = substitute_gam (term4)
term5 = substitute_gam (term5)

term2 = tidy (term2)   # cdb (term2.501,term2)
term3 = tidy (term3)   # cdb (term3.501,term3)
term4 = tidy (term4)   # cdb (term4.501,term4)
```

```
term5 = tidy (term5)    # cdb (term5.501,term5)
```

$$\texttt{term2.401} := Gam22^a{}_{bc} y40^b y40^c + Gam23^a{}_{bc} y40^b y40^c + 2Gam22^a{}_{bc} y40^b y42^c + Gam24^a{}_{bc} y40^b y40^c$$
$$+ 2Gam22^a{}_{bc} y40^b y43^c + 2Gam23^a{}_{bc} y40^b y42^c + Gam25^a{}_{bc} y40^b y40^c$$

$$\texttt{term3.401} := Gam33^a{}_{bcd} y30^b y30^c y30^d + Gam34^a{}_{bcd} y30^b y30^c y30^d + 3Gam33^a{}_{bcd} y30^b y30^c y32^d + Gam35^a{}_{bcd} y30^b y30^c y30^d$$

$$\texttt{term4.401} := Gam44^a{}_{bcde} y20^b y20^c y20^d y20^e + Gam45^a{}_{bcde} y20^b y20^c y20^d y20^e$$

$$\texttt{term5.401} := Gam55^a{}_{bcdef} y00^b y00^c y00^d y00^e y00^f$$

$$
\begin{aligned}
\texttt{term2.501} := & -\frac{2}{3}x^bDx^cDx^dg^{ae}R_{bcde} - \frac{1}{6}x^bx^cDx^dDx^eg^{af}\nabla_dR_{becf} - \frac{1}{3}x^bx^cDx^dDx^eg^{af}\nabla_bR_{cdef} + \frac{1}{12}x^bx^cDx^dDx^eg^{af}\nabla_fR_{bdce} \\
& - \frac{2}{9}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cfgi} + \frac{2}{9}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cifg} - \frac{8}{45}x^bx^cx^dDx^eDx^fg^{ag}g^{hi}R_{befh}R_{cgdi} \\
& + \frac{4}{45}x^bx^cx^dDx^eDx^fg^{ag}g^{hi}R_{bech}R_{difg} + \frac{2}{45}x^bx^cx^dDx^eDx^fg^{ag}g^{hi}R_{bech}R_{dgfi} - \frac{1}{20}x^bx^cx^dDx^eDx^fg^{ag}\nabla_{eb}R_{cfdg} \\
& - \frac{1}{20}x^bx^cx^dDx^eDx^fg^{ag}\nabla_{be}R_{cfdg} - \frac{1}{10}x^bx^cx^dDx^eDx^fg^{ag}\nabla_{bc}R_{defg} - \frac{2}{45}x^bx^cx^dDx^eDx^fg^{ag}g^{hi}R_{bech}R_{dfgi} \\
& + \frac{1}{40}x^bx^cx^dDx^eDx^fg^{ag}\nabla_{gb}R_{cedf} + \frac{1}{40}x^bx^cx^dDx^eDx^fg^{ag}\nabla_{bg}R_{cedf} - \frac{1}{18}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} \\
& - \frac{1}{9}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj} + \frac{1}{36}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg} + \frac{1}{18}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} \\
& + \frac{1}{9}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} - \frac{1}{36}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} \\
& - \frac{1}{18}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} + \frac{1}{18}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} \\
& + \frac{1}{18}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_gR_{chdj} + \frac{1}{18}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_jR_{cgdh} - \frac{1}{9}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} \\
& + \frac{1}{9}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - \frac{1}{18}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} - \frac{4}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej} \\
& - \frac{1}{30}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} - \frac{4}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj} + \frac{1}{36}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg} \\
& + \frac{2}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + \frac{1}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + \frac{1}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} \\
& + \frac{2}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + \frac{1}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} - \frac{1}{90}x^bx^cx^dx^eDx^fDx^gg^{ah}\nabla_{fbc}R_{dgeh} \\
& - \frac{1}{90}x^bx^cx^dx^eDx^fDx^gg^{ah}\nabla_{bfc}R_{dgeh} - \frac{1}{90}x^bx^cx^dx^eDx^fDx^gg^{ah}\nabla_{bcf}R_{dgeh} - \frac{1}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}\nabla_{bcd}R_{efgh} \\
& - \frac{1}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej} - \frac{1}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej} - \frac{1}{45}x^bx^cx^dx^eDx^fDx^gg^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} \\
& + \frac{1}{180}x^bx^cx^dx^eDx^fDx^gg^{ah}\nabla_{hbc}R_{dfeg} + \frac{1}{180}x^bx^cx^dx^eDx^fDx^gg^{ah}\nabla_{bhc}R_{dfeg} + \frac{1}{180}x^bx^cx^dx^eDx^fDx^gg^{ah}\nabla_{bch}R_{dfeg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term3.501} := {} & -\frac{1}{2}x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef}-\frac{8}{15}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cifg}-\frac{2}{15}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cgfi}\\
& +\frac{2}{15}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdch}R_{egfi}-\frac{1}{10}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{de}R_{bfcg}-\frac{3}{20}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{db}R_{cefg}\\
& -\frac{3}{20}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{bd}R_{cefg}+\frac{2}{5}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cfgi}+\frac{1}{40}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{gd}R_{becf}\\
& +\frac{1}{40}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{dg}R_{becf}-\frac{1}{6}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj}+\frac{1}{6}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh}\\
& +\frac{1}{6}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_jR_{cfgh}-\frac{8}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_gR_{chdj}\\
& -\frac{4}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{djgh}-\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj}-\frac{1}{10}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj}\\
& -\frac{1}{90}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj}-\frac{11}{90}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj}-\frac{4}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj}\\
& -\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj}+\frac{1}{12}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg}+\frac{1}{36}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg}\\
& +\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj}+\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_fR_{djgh}+\frac{2}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj}\\
& +\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj}-\frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{efb}R_{cgdh}-\frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ebf}R_{cgdh}\\
& -\frac{1}{30}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ebc}R_{dfgh}-\frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bef}R_{cgdh}-\frac{1}{30}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bec}R_{dfgh}\\
& -\frac{1}{30}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bce}R_{dfgh}+\frac{4}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj}+\frac{1}{5}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj}\\
& +\frac{4}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj}-\frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj}+\frac{1}{5}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{dghj}\\
& -\frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_fR_{dghj}+\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{heb}R_{cfdg}+\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{hbe}R_{cfdg}\\
& +\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ehb}R_{cfdg}+\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bhe}R_{cfdg}+\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ebh}R_{cfdg}\\
& +\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{beh}R_{cfdg}-\frac{1}{9}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_jR_{cgdh}-\frac{1}{18}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.501} := &-\frac{8}{15}x^bDx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bcdh}R_{egfi} - \frac{2}{5}x^bDx^cDx^dDx^eDx^fg^{ag}\nabla_{cd}R_{befg} - \frac{32}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh} \\
&- \frac{1}{5}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - \frac{4}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} \\
&- \frac{2}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} - \frac{22}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} \\
&- \frac{1}{5}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - \frac{4}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} \\
&+ \frac{1}{9}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg} + \frac{8}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} - \frac{1}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{def}R_{bgch} \\
&- \frac{4}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{deb}R_{cfgh} - \frac{4}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{dbe}R_{cfgh} - \frac{4}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{bde}R_{cfgh} \\
&+ \frac{13}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} + \frac{1}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj} \\
&+ \frac{23}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + \frac{1}{90}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{hde}R_{bfcg} \\
&+ \frac{1}{90}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{dhe}R_{bfcg} + \frac{1}{90}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{deh}R_{bfcg} - \frac{4}{9}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_jR_{cfgh}
\end{aligned}
$$

$$
\texttt{term5.501} := -x^bDx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bcdi}\nabla_eR_{fhgj} - x^bDx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{chdi}\nabla_eR_{bfgj} - \frac{1}{3}x^bDx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{cde}R_{bfgh}
$$

```
# Check:
#    x^{a} at s=1 should equal x^{a} + Dx^{a}
#    but x^{a}(s) = x^{a} + s y^{a} - \sum (1/n!) @(termn) s^n
#    thus foo should equal Dx^{a} and it does (yeah)

foo := @(y5)
     -   (1/2) @(term2)
     -   (1/6) @(term3)
     -  (1/24) @(term4)
     - (1/120) @(term5).

distribute          (foo)
obj = product_sort  (foo)
rename_dummies      (foo)
canonicalise        (foo)     # cdb (foo.001,foo)

term2 :=   (1/2) @(term2).   # cdb(term2.502,term2)
term3 :=   (1/6) @(term3).   # cdb(term3.502,term3)
term4 :=  (1/24) @(term4).   # cdb(term4.502,term4)
term5 := (1/120) @(term5).   # cdb(term5.502,term5)

end_stage_2b = time.time()
```

$$\texttt{foo.001} := Dx^{a}$$

$$\texttt{y2.301} := Dx^{a} - \frac{1}{3}x^{b}Dx^{c}Dx^{d}g^{ae}R_{bcde}$$

$$\texttt{y3.301} := Dx^{a} - \frac{1}{3}x^{b}Dx^{c}Dx^{d}g^{ae}R_{bcde} - \frac{1}{12}x^{b}x^{c}Dx^{d}Dx^{e}g^{af}\nabla_{d}R_{becf} - \frac{1}{6}x^{b}x^{c}Dx^{d}Dx^{e}g^{af}\nabla_{b}R_{cdef} + \frac{1}{24}x^{b}x^{c}Dx^{d}Dx^{e}g^{af}\nabla_{f}R_{bdce} - \frac{1}{12}x^{b}Dx^{c}Dx^{d}Dx^{e}g^{af}\nabla_{c}R_{bdef}$$

$$
\begin{aligned}
\texttt{y4.301} := Dx^a &- \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} + \frac{1}{24}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} \\
&- \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} - \frac{2}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cifg} \\
&- \frac{4}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{befh} R_{cgdi} + \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dgfi} \\
&- \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{eb} R_{cfdg} - \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bc} R_{defg} \\
&- \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dfgi} + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bg} R_{cedf} \\
&- \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cgfi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{60}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{de} R_{bfcg} \\
&- \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{db} R_{cefg} - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{gd} R_{becf} \\
&+ \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{dg} R_{becf} - \frac{1}{45}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{1}{60}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} \nabla_{cd} R_{befg}
\end{aligned}
$$

Stage 3: Reformatting and output

```
def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    distribute (tmp)

    substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
    substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
    substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d} -> R_{a b c d}$)

    return foo

def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
```

```
    rename_dummies (bah)
    canonicalise   (bah)
    substitute     (bah,$Dx^{b}->zzz^{b}$)
    factor_out     (bah,$x^{a?},zzz^{b?}$)
    substitute     (bah,$zzz^{b}->Dx^{b}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute  (bah)
    substitute  (bah,$Dx^{b}->zzz^{b}$)
    factor_out  (bah,$x^{a?},zzz^{b?}$)
    substitute  (bah,$zzz^{b}->Dx^{b}$)
    return bah

beg_stage_3 = time.time()

Rterm22 = get_Rterm (term2,2)                           # cdb(Rterm22.101,Rterm22)
Rterm23 = get_Rterm (term2,3)                           # cdb(Rterm23.101,Rterm23)
Rterm24 = get_Rterm (term2,4)                           # cdb(Rterm24.101,Rterm24)
Rterm25 = get_Rterm (term2,5)                           # cdb(Rterm25.101,Rterm25)

Rterm32 = get_Rterm (term3,2)                           # cdb(Rterm32.101,Rterm32)  # zero
Rterm33 = get_Rterm (term3,3)                           # cdb(Rterm33.101,Rterm33)
Rterm34 = get_Rterm (term3,4)                           # cdb(Rterm34.101,Rterm34)
Rterm35 = get_Rterm (term3,5)                           # cdb(Rterm35.101,Rterm35)

Rterm42 = get_Rterm (term4,2)                           # cdb(Rterm42.101,Rterm42)  # zero
Rterm43 = get_Rterm (term4,3)                           # cdb(Rterm43.101,Rterm43)  # zero
Rterm44 = get_Rterm (term4,4)                           # cdb(Rterm44.101,Rterm44)
Rterm45 = get_Rterm (term4,5)                           # cdb(Rterm45.101,Rterm45)

Rterm52 = get_Rterm (term5,2)                           # cdb(Rterm52.101,Rterm52)  # zero
Rterm53 = get_Rterm (term5,3)                           # cdb(Rterm53.101,Rterm53)  # zero
Rterm54 = get_Rterm (term5,4)                           # cdb(Rterm54.101,Rterm54)  # zero
Rterm55 = get_Rterm (term5,5)                           # cdb(Rterm55.101,Rterm55)
```

```
Rterm22 = rescale ( reformat (Rterm22,   -3),    -3 )   # cdb(Rterm22.102,Rterm22)
Rterm23 = rescale ( reformat (Rterm23,  -24),   -24 )   # cdb(Rterm23.102,Rterm23)
Rterm24 = rescale ( reformat (Rterm24, -720),  -720 )   # cdb(Rterm24.102,Rterm24)
Rterm25 = rescale ( reformat (Rterm25, -360),  -360 )   # cdb(Rterm25.102,Rterm25)

Rterm33 = rescale ( reformat (Rterm33,  -12),   -12 )   # cdb(Rterm33.102,Rterm33)
Rterm34 = rescale ( reformat (Rterm34, -720),  -720 )   # cdb(Rterm34.102,Rterm34)
Rterm35 = rescale ( reformat (Rterm35,-1080), -1080 )   # cdb(Rterm35.102,Rterm35)

Rterm44 = rescale ( reformat (Rterm44, -180),  -180 )   # cdb(Rterm44.102,Rterm44)
Rterm45 = rescale ( reformat (Rterm45,-2160), -2160 )   # cdb(Rterm45.102,Rterm45)

Rterm55 = rescale ( reformat (Rterm55, -360),  -360 )   # cdb(Rterm55.102,Rterm55)
```

```
# ----------------------------------------------------------------
# bvp to terms linear in R

tmp2 := -(1/3) @(Rterm22).

bvp2 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2).                                  # cdb(bvp.601,bvp2)

cdblib.put ('bvp2',bvp2,'geodesic-bvp.json')
cdblib.put ('bvp22',tmp2,'geodesic-bvp.json')

y2 := Dx^{a} + @(tmp2).                                   # cdb(y2.600,y2)

# ----------------------------------------------------------------
# bvp to terms linear in dR

tmp2 :=  -(1/3) @(Rterm22) - (1/24) @(Rterm23).
tmp3 := -(1/12) @(Rterm33).

bvp3 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2)
     + (s-s**3) @(tmp3).                                  # cdb(bvp.602,bvp3)

cdblib.put ('bvp3',bvp3,'geodesic-bvp.json')
cdblib.put ('bvp32',tmp2,'geodesic-bvp.json')
cdblib.put ('bvp33',tmp3,'geodesic-bvp.json')

y3 := Dx^{a} + @(tmp2) + @(tmp3).                         # cdb(y3.600,y3)

# ----------------------------------------------------------------
# bvp to terms linear in d^2 R

tmp2 :=   -(1/3) @(Rterm22) -  (1/24) @(Rterm23) - (1/720) @(Rterm24).
tmp3 :=  -(1/12) @(Rterm33) - (1/720) @(Rterm34).
tmp4 := -(1/180) @(Rterm44).
```

```
bvp4 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2)
     + (s-s**3) @(tmp3)
     + (s-s**4) @(tmp4).                                # cdb(bvp.603,bvp4)

cdblib.put ('bvp4',bvp4,'geodesic-bvp.json')
cdblib.put ('bvp42',tmp2,'geodesic-bvp.json')
cdblib.put ('bvp43',tmp3,'geodesic-bvp.json')
cdblib.put ('bvp44',tmp4,'geodesic-bvp.json')

y4 := Dx^{a} + @(tmp2) + @(tmp3) + @(tmp4).             # cdb(y4.600,y4)

# ----------------------------------------------------------------
# bvp to terms linear in d^3 R

tmp2 := @(term2).
tmp3 := @(term3).
tmp4 := @(term4).
tmp5 := @(term5).

bvp5 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2)
     + (s-s**3) @(tmp3)
     + (s-s**4) @(tmp4)
     + (s-s**5) @(tmp5).                                # cdb(bvp.604,bvp5)

cdblib.put ('bvp5',bvp5,'geodesic-bvp.json')
cdblib.put ('bvp52',term2,'geodesic-bvp.json')
cdblib.put ('bvp53',term3,'geodesic-bvp.json')
cdblib.put ('bvp54',term4,'geodesic-bvp.json')
cdblib.put ('bvp55',term5,'geodesic-bvp.json')

y5 := Dx^{a} + @(tmp2) + @(tmp3) + @(tmp4) + @(tmp5).   # cdb(y5.600,y5)

end_stage_3 = time.time()
```

```
# cdbBeg (timing)
print ("Stage 1:  {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2a: {:7.1f} secs\\hfill\\break".format(end_stage_2a-beg_stage_2a))
print ("Stage 2b: {:7.1f} secs\\hfill\\break".format(end_stage_2b-beg_stage_2b))
print ("Stage 3:  {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
# cdbEnd (timing)
```

Non-unit tangent vectors at P

These are not unit vectors, their length is the geodesic distance from P to Q

$$\texttt{y2.600} := Dx^{a} - \frac{1}{3}x^{b}Dx^{c}Dx^{d}g^{ae}R_{bcde}$$

$$\texttt{y3.600} := Dx^{a} - \frac{1}{3}x^{b}Dx^{c}Dx^{d}g^{ae}R_{bcde} - \frac{1}{24}x^{b}x^{c}Dx^{d}Dx^{e}\left(2g^{af}\nabla_{d}R_{becf} + 4g^{af}\nabla_{b}R_{cdef} - g^{af}\nabla_{f}R_{bdce}\right) - \frac{1}{12}x^{b}Dx^{c}Dx^{d}Dx^{e}g^{af}\nabla_{c}R_{bdef}$$

$$\begin{aligned}\texttt{y4.600} := Dx^{a} &- \frac{1}{3}x^{b}Dx^{c}Dx^{d}g^{ae}R_{bcde} - \frac{1}{24}x^{b}x^{c}Dx^{d}Dx^{e}\left(2g^{af}\nabla_{d}R_{becf} + 4g^{af}\nabla_{b}R_{cdef} - g^{af}\nabla_{f}R_{bdce}\right)\\
&- \frac{1}{720}x^{b}x^{c}Dx^{d}Dx^{e}Dx^{f}\left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right)\\
&- \frac{1}{720}x^{b}x^{c}x^{d}Dx^{e}Dx^{f}\left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg}\right.\\
&\qquad\left. + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right) - \frac{1}{12}x^{b}Dx^{c}Dx^{d}Dx^{e}g^{af}\nabla_{c}R_{bdef}\\
&- \frac{1}{720}x^{b}x^{c}Dx^{d}Dx^{e}Dx^{f}\left(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} + 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg}\right.\\
&\left. + 18g^{ag}\nabla_{bd}R_{cefg} - 48g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\right) - \frac{1}{180}x^{b}Dx^{c}Dx^{d}Dx^{e}Dx^{f}\left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right)\end{aligned}$$

Geodesic boundary value problem to terms linear in R

$$x^a(s) = x^a + sDx^a - \frac{1}{3}\left(s - s^2\right) x^b Dx^c Dx^d g^{ae} R_{bcde} + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2) x^a_2 + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \mathcal{O}\left(\epsilon^3\right)$$

$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

Geodesic boundary value problem to terms linear in ∇R

$$x^a(s) = x^a + sDx^a + \left(s - s^2\right)\left(-\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\right)$$
$$- \frac{1}{12}\left(s - s^3\right) x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} + \mathcal{O}\left(s^4, \epsilon^4\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2)x^a_2 + (s - s^3)x^a_3 + \mathcal{O}\left(s^4, \epsilon^4\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \mathcal{O}\left(\epsilon^4\right)$$
$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$
$$-24\overset{3}{x}{}^a_2 = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)$$

$$x^a_3 = \overset{3}{x}{}^a_3 + \mathcal{O}\left(\epsilon^4\right)$$
$$-12\overset{3}{x}{}^a_3 = x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef}$$

Geodesic boundary value problem to terms linear in $\nabla^2 R$

$$
\begin{aligned}
x^a(s) = x^a + sDx^a + \left(s - s^2\right) \Big(&-\frac{1}{3} x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24} x^b x^c Dx^d Dx^e \left(2g^{af} \nabla_d R_{becf} + 4g^{af} \nabla_b R_{cdef} - g^{af} \nabla_f R_{bdce}\right) \\
&- \frac{1}{720} x^b x^c Dx^d Dx^e Dx^f \left(80 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 80 g^{ag} g^{hi} R_{bdeh} R_{cifg}\right) - \frac{1}{720} x^b x^c x^d Dx^e Dx^f \big(64 g^{ag} g^{hi} R_{befh} R_{cgdi} - 32 g^{ag} g^{hi} R_{bech} R_{difg} \\
&- 16 g^{ag} g^{hi} R_{bech} R_{dgfi} + 18 g^{ag} \nabla_{eb} R_{cfdg} + 18 g^{ag} \nabla_{be} R_{cfdg} + 36 g^{ag} \nabla_{bc} R_{defg} + 16 g^{ag} g^{hi} R_{bech} R_{dfgi} - 9 g^{ag} \nabla_{gb} R_{cedf} - 9 g^{ag} \nabla_{bg} R_{cedf}\big)\Big) \\
+ \left(s - s^3\right) \Big(&-\frac{1}{12} x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} - \frac{1}{720} x^b x^c Dx^d Dx^e Dx^f \big(64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} \\
&+ 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg} - 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\big)\Big) \\
- \frac{1}{180} \left(s - s^4\right) &x^b Dx^c Dx^d Dx^e Dx^f \left(4 g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3 g^{ag} \nabla_{cd} R_{befg}\right) + \mathcal{O}\left(s^5, \epsilon^5\right)
\end{aligned}
$$

$$
x^a(s) = x^a + sDx^a + (s - s^2) x^a_2 + (s - s^3) x^a_3 + (s - s^4) x^a_4 + \mathcal{O}\left(s^5, \epsilon^5\right)
$$

$$
\begin{aligned}
x^a_2 &= \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \overset{4}{x}{}^a_2 + \mathcal{O}\left(\epsilon^5\right) \\
-3 \overset{2}{x}{}^a_2 &= x^b Dx^c Dx^d g^{ae} R_{bcde} \\
-24 \overset{3}{x}{}^a_2 &= x^b x^c Dx^d Dx^e \left(2 g^{af} \nabla_d R_{becf} + 4 g^{af} \nabla_b R_{cdef} - g^{af} \nabla_f R_{bdce}\right) \\
-720 \overset{4}{x}{}^a_2 &= x^b x^c Dx^d Dx^e Dx^f \left(80 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 80 g^{ag} g^{hi} R_{bdeh} R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \big(64 g^{ag} g^{hi} R_{befh} R_{cgdi} - 32 g^{ag} g^{hi} R_{bech} R_{difg} \\
&\quad - 16 g^{ag} g^{hi} R_{bech} R_{dgfi} + 18 g^{ag} \nabla_{eb} R_{cfdg} + 18 g^{ag} \nabla_{be} R_{cfdg} + 36 g^{ag} \nabla_{bc} R_{defg} + 16 g^{ag} g^{hi} R_{bech} R_{dfgi} - 9 g^{ag} \nabla_{gb} R_{cedf} - 9 g^{ag} \nabla_{bg} R_{cedf}\big)
\end{aligned}
$$

$$
\begin{aligned}
x^a_3 &= \overset{3}{x}{}^a_3 + \overset{4}{x}{}^a_3 + \mathcal{O}\left(\epsilon^5\right) \\
-12 \overset{3}{x}{}^a_3 &= x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} \\
-720 \overset{4}{x}{}^a_3 &= x^b x^c Dx^d Dx^e Dx^f \big(64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg} \\
&\quad - 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\big)
\end{aligned}
$$

$$x_4^a = \overset{4}{x}{}_4^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right)$$

Geodesic boundary value problem to terms linear in $\nabla^3 R$

The geodesic that connects the points with RNC coordinates x^a and $x^a + Dx^a$ is described, for $0 \le s \le 1$, by

$$x^a(s) = x^a + sDx^a + (s-s^2)x^a_2 + (s-s^3)x^a_3 + (s-s^4)x^a_4 + (s-s^5)x^a_5 + \mathcal{O}\left(s^6, \epsilon^6\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \overset{4}{x}{}^a_2 + \overset{5}{x}{}^a_2 + \mathcal{O}\left(\epsilon^6\right)$$

$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$-24\overset{3}{x}{}^a_2 = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)$$

$$\begin{aligned}
-720\overset{4}{x}{}^a_2 = {} & x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right. \\
& \left. - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right)
\end{aligned}$$

$$\begin{aligned}
-360\overset{5}{x}{}^a_2 = {} & x^b x^c x^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 20g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj}\right. \\
& - 20g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 20g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} \\
& \left. - 20g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 10g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj}\right) + x^b x^c Dx^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 10g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj}\right) \\
& + x^b x^c x^d x^e Dx^f Dx^g \left(16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg}\right. \\
& - 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 2g^{ah}\nabla_{fbc}R_{dgeh} \\
& + 2g^{ah}\nabla_{bfc}R_{dgeh} + 2g^{ah}\nabla_{bcf}R_{dgeh} + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} \\
& \left. - g^{ah}\nabla_{hbc}R_{dfeg} - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\right)
\end{aligned}$$

$$
\begin{aligned}
x^a_3 &= \overset{3}{x}{}^a_3 + \overset{4}{x}{}^a_3 + \overset{5}{x}{}^a_3 + \mathcal{O}\left(\epsilon^6\right)\\
-12\overset{3}{x}{}^a_3 &= x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}\\
-720\overset{4}{x}{}^a_3 &= x^b x^c Dx^d Dx^e Dx^f \big(64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg}\\
&\qquad - 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\big)\\
-1080\overset{5}{x}{}^a_3 &= x^b x^c Dx^d Dx^e Dx^f Dx^g \big(30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - 30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} - 30 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\big)\\
&\quad + x^b x^c x^d Dx^e Dx^f Dx^g \big(32 g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 48 g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 12 g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 18 g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj}\\
&\qquad + 2 g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22 g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 48 g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 12 g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} - 15 g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg}\\
&\qquad - 5 g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} - 12 g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12 g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8 g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12 g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj}\\
&\qquad + 4 g^{ah} \nabla_{efb} R_{cgdh} + 4 g^{ah} \nabla_{ebf} R_{cgdh} + 6 g^{ah} \nabla_{ebc} R_{dfgh} + 4 g^{ah} \nabla_{bef} R_{cgdh} + 6 g^{ah} \nabla_{bec} R_{dfgh} + 6 g^{ah} \nabla_{bce} R_{dfgh} - 16 g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj}\\
&\qquad - 36 g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 16 g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 4 g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 36 g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + 4 g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg}\\
&\qquad - g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg} + 20 g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 10 g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}\big)
\end{aligned}
$$

$$
\begin{aligned}
x^a_4 &= \overset{4}{x}{}^a_4 + \overset{5}{x}{}^a_4 + \mathcal{O}\left(\epsilon^6\right)\\
-180\overset{4}{x}{}^a_4 &= x^b Dx^c Dx^d Dx^e Dx^f \left(4 g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3 g^{ag} \nabla_{cd} R_{befg}\right)\\
-2160\overset{5}{x}{}^a_4 &= x^b x^c Dx^d Dx^e Dx^f Dx^g \big(64 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 18 g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24 g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4 g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj}\\
&\qquad + 44 g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 18 g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} + 24 g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10 g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 16 g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj}\\
&\qquad + 6 g^{ah} \nabla_{def} R_{bgch} + 8 g^{ah} \nabla_{deb} R_{cfgh} + 8 g^{ah} \nabla_{dbe} R_{cfgh} + 8 g^{ah} \nabla_{bde} R_{cfgh} - 26 g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 6 g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj}\\
&\qquad - 46 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - g^{ah} \nabla_{hde} R_{bfcg} - g^{ah} \nabla_{dhe} R_{bfcg} - g^{ah} \nabla_{deh} R_{bfcg} + 40 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\big)
\end{aligned}
$$

$$
\begin{aligned}
x^a_5 &= \overset{5}{x}{}^a_5 + \mathcal{O}\left(\epsilon^6\right)\\
-360\overset{5}{x}{}^a_5 &= x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(3 g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3 g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh}\right)
\end{aligned}
$$

```
tmp2 := 8 @(Rterm22) + @(Rterm23).
tmp3 := @(Rterm33).

factor_out     (tmp2,$Dx^{a?}$) # cdb(tmp2.001,tmp2)
rename_dummies (tmp2)
factor_out     (tmp2,$Dx^{a?}$) # cdb(tmp2.002,tmp2)

bvp4 := x^{a}
     + \lam Dx^{a}
     - (1/24) (\lam-\lam**2) @(tmp2)
     - (1/12) (\lam-\lam**3) @(tmp3).     # cdb(bvp4,bvp4)

cdblib.create ('geodesic-bvp.export')

# 4th order bvp
cdblib.put ('bvp4',bvp4,'geodesic-bvp.export')

# 6th order bvp terms, scaled
cdblib.put ('bvp622',Rterm22,'geodesic-bvp.export')
cdblib.put ('bvp623',Rterm23,'geodesic-bvp.export')
cdblib.put ('bvp624',Rterm24,'geodesic-bvp.export')
cdblib.put ('bvp625',Rterm25,'geodesic-bvp.export')

cdblib.put ('bvp633',Rterm33,'geodesic-bvp.export')
cdblib.put ('bvp634',Rterm34,'geodesic-bvp.export')
cdblib.put ('bvp635',Rterm35,'geodesic-bvp.export')

cdblib.put ('bvp644',Rterm44,'geodesic-bvp.export')
cdblib.put ('bvp645',Rterm45,'geodesic-bvp.export')

cdblib.put ('bvp655',Rterm55,'geodesic-bvp.export')

checkpoint.append (bvp4)

checkpoint.append (Rterm22)
checkpoint.append (Rterm23)
checkpoint.append (Rterm24)
checkpoint.append (Rterm25)
```

```
checkpoint.append (Rterm33)
checkpoint.append (Rterm34)
checkpoint.append (Rterm35)

checkpoint.append (Rterm44)
checkpoint.append (Rterm45)

checkpoint.append (Rterm55)
```

Timing

Stage 1: 5.7 secs
Stage 2a: 68.9 secs
Stage 2b: 67.8 secs
Stage 3: 13.6 secs