The metric tensor in Riemann normal coordinates

In this notebook we compute the recursive sequences

$$g_{ab,d\underline{e}} = \left(g_{cb}\Gamma^{c}{}_{a(d}\right)_{,\underline{e})} + \left(g_{ac}\Gamma^{c}{}_{b(d}\right)_{,\underline{e})} \tag{1}$$

$$(n+3)\Gamma^{a}{}_{d(b,c\underline{e})} = (n+1)\left(R^{a}{}_{(bcd,\underline{e})} - \left(\Gamma^{a}{}_{f(c}\Gamma^{f}{}_{bd}\right)_{,\underline{e})}\right) \tag{2}$$

for $n = 1, 2, 3, \cdots$. Note in these equations that the (extended) index $\underline{e}$ contains n normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g_{ab}(x) &= g_{ab} + g_{ab,c}x^{c} + \frac{1}{2!}g_{ab,cd}x^{c}x^{d} + \frac{1}{3!}g_{ab,cde}x^{c}x^{d}x^{e} + \cdots \\ &= g_{ab} + \sum_{n=1}^{\infty} \frac{1}{n!}\, g_{ab,\underline{c}}\, x^{\cdot\underline{c}} \end{aligned}$$

Stage 1: Symmetrised partial derivatives of g_{ab}

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g_{ab,c}A^{c} &= 0 \\ g_{ab,cd}A^{c}A^{d} &= g_{cb}\partial_{e}\Gamma^{c}{}_{ad}A^{d}A^{e} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}A^{d}A^{e} \\ g_{ab,cde}A^{c}A^{d}A^{e} &= g_{cb}\partial_{fe}\Gamma^{c}{}_{ad}A^{d}A^{e}A^{f} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}A^{d}A^{e}A^{f} \end{aligned}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

Now we use the results from `dGamma` to replace derivatives of Γ with partial derivatives of R. These were computed in `dGamma` using equation (2) above.

$$
\begin{aligned}
g_{ab,c}A^c &= 0\\
g_{ab,cd}A^cA^d &= \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb}\\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}
\end{aligned}
$$

Stage 3: Replace partial derivs of R with covariant derivs of R

Next we use the results from `dRabcd` to replace the partial derivatives of R with covariant deriavtives.

$$
\begin{aligned}
g_{ab,c}A^c &= 0\\
g_{ab,cd}A^cA^d &= -\frac{2}{3}A^cA^dR_{acbd}\\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}
\end{aligned}
$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

Each of the above expressions constitutues one term in the Taylor series for the metric. We also make the trivial change $A \to x$. Then we do some trivial reformatting.

$$
\begin{aligned}
g_{ab}(x) &= g_{ab} + g_{ab,c}x^c + \frac{1}{2!}g_{ab,cd}x^cx^d + \frac{1}{3!}g_{ab,cde}x^cx^dx^e + \mathcal{O}\left(\epsilon^4\right)\\
&= g_{ab} - \frac{1}{3}x^cx^dR_{acbd} - \frac{1}{6}x^cx^dx^e\nabla_cR_{adbe} + \mathcal{O}\left(\epsilon^4\right)
\end{aligned}
$$

Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d}   -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d}   -> g^{d e} R_{a b c e}$)
    unwrap     (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah  = Ex("numx = " + str(n))
    keep_weight (foo,bah)

    return foo
```

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                              -> A001^{a}                $)
    substitute (obj,$ x^{a}                              -> A002^{a}                $)
    substitute (obj,$ g_{a b}                            -> A003_{a b}              $)
    substitute (obj,$ g^{a b}                            -> A004^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A010_{a b c d e f g h}  $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A009_{a b c d e f g}    $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A008_{a b c d e f}      $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A007_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}{R_{a b c d}}    -> A014_{a b c d e f g h}  $)
    substitute (obj,$ \partial_{e f g}{R_{a b c d}}      -> A013_{a b c d e f g}    $)
    substitute (obj,$ \partial_{e f}{R_{a b c d}}        -> A012_{a b c d e f}      $)
    substitute (obj,$ \partial_{e}{R_{a b c d}}          -> A011_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}{R^{a}_{b c d}} -> A018^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}{R^{a}_{b c d}}   -> A017^{a}_{b c d e f g}  $)
    substitute (obj,$ \partial_{e f}{R^{a}_{b c d}}     -> A016^{a}_{b c d e f}    $)
    substitute (obj,$ \partial_{e}{R^{a}_{b c d}}       -> A015^{a}_{b c d e}      $)
    substitute (obj,$ R_{a b c d}                        -> A005_{a b c d}          $)
    substitute (obj,$ R^{a}_{b c d}                      -> A006^{a}_{b c d}        $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                   -> A^{a}                            $)
    substitute (obj,$ A002^{a}                   -> x^{a}                            $)
    substitute (obj,$ A003_{a b}                 -> g_{a b}                          $)
    substitute (obj,$ A004^{a b}                 -> g^{a b}                          $)
    substitute (obj,$ A005_{a b c d}             -> R_{a b c d}                      $)
    substitute (obj,$ A006^{a}_{b c d}           -> R^{a}_{b c d}                    $)
    substitute (obj,$ A007_{a b c d e}           -> \nabla_{e}{R_{a b c d}}          $)
    substitute (obj,$ A008_{a b c d e f}         -> \nabla_{e f}{R_{a b c d}}        $)
    substitute (obj,$ A009_{a b c d e f g}       -> \nabla_{e f g}{R_{a b c d}}      $)
    substitute (obj,$ A010_{a b c d e f g h}     -> \nabla_{e f g h}{R_{a b c d}}    $)
    substitute (obj,$ A011_{a b c d e}           -> \partial_{e}{R_{a b c d}}        $)
    substitute (obj,$ A012_{a b c d e f}         -> \partial_{e f}{R_{a b c d}}      $)
    substitute (obj,$ A013_{a b c d e f g}       -> \partial_{e f g}{R_{a b c d}}    $)
    substitute (obj,$ A014_{a b c d e f g h}     -> \partial_{e f g h}{R_{a b c d}}  $)
    substitute (obj,$ A015^{a}_{b c d e}         -> \partial_{e}{R^{a}_{b c d}}      $)
```

```
    substitute (obj,$ A016^{a}_{b c d e f}      -> \partial_{e f}{R^{a}_{b c d}}     $)
    substitute (obj,$ A017^{a}_{b c d e f g}    -> \partial_{e f g}{R^{a}_{b c d}}   $)
    substitute (obj,$ A018^{a}_{b c d e f g h}  -> \partial_{e f g h}{R^{a}_{b c d}} $)

    return obj

def reformat_xterm (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    factor_out     (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale_xterm (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute  (bah)
    factor_out  (bah,$x^{a?}$)
    return bah

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.
```

```
\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of g_{ab}

```
beg_stage_1 = time.time()

# symmetrised partial derivatives of g_{ab}

gab00:=g_{a b}.                                               # cdb (gab00.101,gab00)

gab01:=g_{c b}\Gamma^{c}_{a d} + g_{a c}\Gamma^{c}_{b d}.     # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @(gab01) }.                              # cdb (gab02.101,gab02)
distribute   (gab02)                                          # cdb (gab02.102,gab02)
product_rule (gab02)                                          # cdb (gab02.103,gab02)
substitute   (gab02, $\partial_{d}{g_{a b}} -> @(gab01)$)     # cdb (gab02.104,gab02)
distribute   (gab02)                                          # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @(gab02) }.                              # cdb (gab03.101,gab03)
distribute   (gab03)                                          # cdb (gab03.102,gab03)
product_rule (gab03)                                          # cdb (gab03.103,gab03)
substitute   (gab03, $\partial_{d}{g_{a b}} -> @(gab01)$)     # cdb (gab03.104,gab03)
distribute   (gab03)                                          # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @(gab03) }.                              # cdb (gab04.101,gab04)
distribute   (gab04)                                          # cdb (gab04.102,gab04)
product_rule (gab04)                                          # cdb (gab04.103,gab04)
substitute   (gab04, $\partial_{d}{g_{a b}} -> @(gab01)$)     # cdb (gab04.104,gab04)
distribute   (gab04)                                          # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @(gab04) }.                              # cdb (gab05.101,gab05)
distribute   (gab05)                                          # cdb (gab05.102,gab05)
product_rule (gab05)                                          # cdb (gab05.103,gab05)
substitute   (gab05, $\partial_{d}{g_{a b}} -> @(gab01)$)     # cdb (gab05.104,gab05)
distribute   (gab05)                                          # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00)   # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01)   # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02)   # cdb (gab02.106,gab02)
gab03 = impose_rnc (gab03)   # cdb (gab03.106,gab03)
```

```
gab04 = impose_rnc (gab04)   # cdb (gab04.106,gab04)
gab05 = impose_rnc (gab05)   # cdb (gab05.106,gab05)
```

$$\texttt{gab00.101} := g_{ab}$$

$$\texttt{gab00.102} := g_{ab}$$

$$\texttt{gab01.101} := g_{cb}\Gamma^{c}{}_{ad} + g_{ac}\Gamma^{c}{}_{bd}$$

$$\texttt{gab01.102} := 0$$

$$\texttt{gab02.101} := \partial_{e}\left(g_{cb}\Gamma^{c}{}_{ad} + g_{ac}\Gamma^{c}{}_{bd}\right)$$

$$\texttt{gab02.102} := \partial_{e}\left(g_{cb}\Gamma^{c}{}_{ad}\right) + \partial_{e}\left(g_{ac}\Gamma^{c}{}_{bd}\right)$$

$$\texttt{gab02.103} := \partial_{e}g_{cb}\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + \partial_{e}g_{ac}\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab02.104} := \left(g_{fb}\Gamma^{f}{}_{ce} + g_{cf}\Gamma^{f}{}_{be}\right)\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + \left(g_{fc}\Gamma^{f}{}_{ae} + g_{af}\Gamma^{f}{}_{ce}\right)\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab02.105} := g_{fb}\Gamma^{f}{}_{ce}\Gamma^{c}{}_{ad} + g_{cf}\Gamma^{f}{}_{be}\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{fc}\Gamma^{f}{}_{ae}\Gamma^{c}{}_{bd} + g_{af}\Gamma^{f}{}_{ce}\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab02.106} := g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab03.101} := \partial_{f}\left(g_{gb}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{cg}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{gc}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}\right)$$

$$\texttt{gab03.102} := \partial_{f}\left(g_{gb}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad}\right) + \partial_{f}\left(g_{cg}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad}\right) + \partial_{f}\left(g_{cb}\partial_{e}\Gamma^{c}{}_{ad}\right) + \partial_{f}\left(g_{gc}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd}\right) + \partial_{f}\left(g_{ag}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd}\right) + \partial_{f}\left(g_{ac}\partial_{e}\Gamma^{c}{}_{bd}\right)$$

$$\begin{aligned}\texttt{gab03.103} := {} & \partial_{f}g_{gb}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + \partial_{f}g_{cg}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\partial_{f}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\Gamma^{g}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + \partial_{f}g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{fe}\Gamma^{c}{}_{ad}\\ & + \partial_{f}g_{gc}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\partial_{f}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + \partial_{f}g_{ag}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + \partial_{f}g_{ac}\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.104} := {} & \left(g_{hb}\Gamma^{h}{}_{gf} + g_{gh}\Gamma^{h}{}_{bf}\right)\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + \left(g_{hg}\Gamma^{h}{}_{cf} + g_{ch}\Gamma^{h}{}_{gf}\right)\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\partial_{f}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad}\\ & + g_{cg}\Gamma^{g}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + \left(g_{gb}\Gamma^{g}{}_{cf} + g_{cg}\Gamma^{g}{}_{bf}\right)\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + \left(g_{hc}\Gamma^{h}{}_{gf} + g_{gh}\Gamma^{h}{}_{cf}\right)\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\partial_{f}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd}\\ & + \left(g_{hg}\Gamma^{h}{}_{af} + g_{ah}\Gamma^{h}{}_{gf}\right)\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + \left(g_{gc}\Gamma^{g}{}_{af} + g_{ag}\Gamma^{g}{}_{cf}\right)\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.105} := {} & g_{hb}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gh}\Gamma^{h}{}_{bf}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{hg}\Gamma^{h}{}_{cf}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\partial_{f}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad}\\ & + g_{cg}\Gamma^{g}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{cg}\Gamma^{g}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + g_{hc}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gh}\Gamma^{h}{}_{cf}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\partial_{f}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd}\\ & + g_{hg}\Gamma^{h}{}_{af}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}\end{aligned}$$

$$\texttt{gab03.106} := g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}$$

$$
\begin{aligned}
\texttt{gab04.101} := \partial_{g} \Big(& g_{hb}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ib}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ib}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ci}\partial_{f}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ci}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} \\
& + g_{ib}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ci}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + g_{hc}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ic}\partial_{f}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ic}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{hi}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} \\
& + g_{ah}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd} \Big)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.102} := & \partial_{g}\left(g_{hb}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{ih}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{ib}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{ib}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{hi}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad}\right) \\
& + \partial_{g}\left(g_{ch}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{ci}\partial_{f}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{ci}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{ib}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{ci}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad}\right) + \partial_{g}\left(g_{cb}\partial_{fe}\Gamma^{c}{}_{ad}\right) \\
& + \partial_{g}\left(g_{hc}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ih}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ic}\partial_{f}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ic}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{hi}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd}\right) \\
& + \partial_{g}\left(g_{ah}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ai}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ai}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ic}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ai}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd}\right) + \partial_{g}\left(g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.103} := & \partial_{g}g_{hb}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + \partial_{g}g_{ih}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\partial_{g}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} \\
& + g_{ih}\Gamma^{h}{}_{bf}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + \partial_{g}g_{ib}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ib}\partial_{gf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ib}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + \partial_{g}g_{ib}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} \\
& + g_{ib}\Gamma^{i}{}_{ce}\partial_{gf}\Gamma^{c}{}_{ad} + \partial_{g}g_{hi}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\partial_{g}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cf}\partial_{g}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + \partial_{g}g_{ch}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} \\
& + g_{ch}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + \partial_{g}g_{ci}\partial_{f}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ci}\partial_{gf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ci}\partial_{f}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} \\
& + \partial_{g}g_{ci}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{ci}\partial_{g}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{ci}\Gamma^{i}{}_{be}\partial_{gf}\Gamma^{c}{}_{ad} + \partial_{g}g_{ib}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ib}\Gamma^{i}{}_{cf}\partial_{ge}\Gamma^{c}{}_{ad} + \partial_{g}g_{ci}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} \\
& + g_{ci}\partial_{g}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ci}\Gamma^{i}{}_{bf}\partial_{ge}\Gamma^{c}{}_{ad} + \partial_{g}g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + g_{cb}\partial_{gfe}\Gamma^{c}{}_{ad} + \partial_{g}g_{hc}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{hc}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{hc}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} \\
& + g_{hc}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + \partial_{g}g_{ih}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\partial_{g}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cf}\partial_{g}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + \partial_{g}g_{ic}\partial_{f}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} \\
& + g_{ic}\partial_{gf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ic}\partial_{f}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + \partial_{g}g_{ic}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\partial_{g}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\Gamma^{i}{}_{ae}\partial_{gf}\Gamma^{c}{}_{bd} + \partial_{g}g_{hi}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hi}\partial_{g}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} \\
& + g_{hi}\Gamma^{h}{}_{af}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hi}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + \partial_{g}g_{ah}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} \\
& + \partial_{g}g_{ai}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\partial_{gf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + \partial_{g}g_{ai}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{ce}\partial_{gf}\Gamma^{c}{}_{bd} + \partial_{g}g_{ic}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} \\
& + g_{ic}\partial_{g}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ic}\Gamma^{i}{}_{af}\partial_{ge}\Gamma^{c}{}_{bd} + \partial_{g}g_{ai}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{cf}\partial_{ge}\Gamma^{c}{}_{bd} + \partial_{g}g_{ac}\partial_{fe}\Gamma^{c}{}_{bd} + g_{ac}\partial_{gfe}\Gamma^{c}{}_{bd}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.104} := {} & \left(g_{jb}\Gamma^j{}_{hg} + g_{hj}\Gamma^j{}_{bg}\right)\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{if}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + \left(g_{jh}\Gamma^j{}_{ig} + g_{ij}\Gamma^j{}_{hg}\right)\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} \\
& + g_{ih}\partial_g\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bf}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + \left(g_{hb}\Gamma^h{}_{ig} + g_{ih}\Gamma^h{}_{bg}\right)\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ib}\partial_{gf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} \\
& + \left(g_{hb}\Gamma^h{}_{ig} + g_{ih}\Gamma^h{}_{bg}\right)\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{ce}\partial_{gf}\Gamma^c{}_{ad} + \left(g_{ji}\Gamma^j{}_{hg} + g_{hj}\Gamma^j{}_{ig}\right)\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\partial_g\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} \\
& + g_{hi}\Gamma^h{}_{cf}\partial_g\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cf}\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + \left(g_{jh}\Gamma^j{}_{cg} + g_{cj}\Gamma^j{}_{hg}\right)\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{be}\Gamma^c{}_{ad} \\
& + g_{ch}\Gamma^h{}_{if}\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + \left(g_{hi}\Gamma^h{}_{cg} + g_{ch}\Gamma^h{}_{ig}\right)\partial_f\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ci}\partial_{gf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + \left(g_{hi}\Gamma^h{}_{cg} + g_{ch}\Gamma^h{}_{ig}\right)\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} \\
& + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ci}\Gamma^i{}_{be}\partial_{gf}\Gamma^c{}_{ad} + \left(g_{hb}\Gamma^h{}_{ig} + g_{ih}\Gamma^h{}_{bg}\right)\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{cf}\partial_{ge}\Gamma^c{}_{ad} + \left(g_{hi}\Gamma^h{}_{cg} + g_{ch}\Gamma^h{}_{ig}\right)\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} \\
& + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} + g_{ci}\Gamma^i{}_{bf}\partial_{ge}\Gamma^c{}_{ad} + \left(g_{hb}\Gamma^h{}_{cg} + g_{ch}\Gamma^h{}_{bg}\right)\partial_{fe}\Gamma^c{}_{ad} + g_{cb}\partial_{gfe}\Gamma^c{}_{ad} + \left(g_{jc}\Gamma^j{}_{hg} + g_{hj}\Gamma^j{}_{cg}\right)\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hc}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} \\
& + g_{hc}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{if}\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + \left(g_{jh}\Gamma^j{}_{ig} + g_{ij}\Gamma^j{}_{hg}\right)\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\partial_g\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cf}\partial_g\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} \\
& + \left(g_{hc}\Gamma^h{}_{ig} + g_{ih}\Gamma^h{}_{cg}\right)\partial_f\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\partial_{gf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + \left(g_{hc}\Gamma^h{}_{ig} + g_{ih}\Gamma^h{}_{cg}\right)\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{ic}\Gamma^i{}_{ae}\partial_{gf}\Gamma^c{}_{bd} \\
& + \left(g_{ji}\Gamma^j{}_{hg} + g_{hj}\Gamma^j{}_{ig}\right)\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\partial_g\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{af}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{af}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + \left(g_{jh}\Gamma^j{}_{ag} + g_{aj}\Gamma^j{}_{hg}\right)\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} \\
& + g_{ah}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{if}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + \left(g_{hi}\Gamma^h{}_{ag} + g_{ah}\Gamma^h{}_{ig}\right)\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\partial_{gf}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} \\
& + \left(g_{hi}\Gamma^h{}_{ag} + g_{ah}\Gamma^h{}_{ig}\right)\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{ce}\partial_{gf}\Gamma^c{}_{bd} + \left(g_{hc}\Gamma^h{}_{ig} + g_{ih}\Gamma^h{}_{cg}\right)\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} \\
& + g_{ic}\Gamma^i{}_{af}\partial_{ge}\Gamma^c{}_{bd} + \left(g_{hi}\Gamma^h{}_{ag} + g_{ah}\Gamma^h{}_{ig}\right)\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{cf}\partial_{ge}\Gamma^c{}_{bd} + \left(g_{hc}\Gamma^h{}_{ag} + g_{ah}\Gamma^h{}_{cg}\right)\partial_{fe}\Gamma^c{}_{bd} + g_{ac}\partial_{gfe}\Gamma^c{}_{bd}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.105} := {} & g_{jb}\Gamma^j{}_{hg}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hj}\Gamma^j{}_{bg}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{if}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + g_{jh}\Gamma^j{}_{ig}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} \\
& + g_{ij}\Gamma^j{}_{hg}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\partial_g\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bf}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{ig}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bg}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{ad} \\
& + g_{ib}\partial_{gf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{ig}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bg}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{ce}\partial_{gf}\Gamma^c{}_{ad} + g_{ji}\Gamma^j{}_{hg}\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} \\
& + g_{hj}\Gamma^j{}_{ig}\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\partial_g\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cf}\partial_g\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cf}\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + g_{jh}\Gamma^j{}_{cg}\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{cj}\Gamma^j{}_{hg}\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} \\
& + g_{ch}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{if}\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cg}\partial_f\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{ig}\partial_f\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ci}\partial_{gf}\Gamma^i{}_{be}\Gamma^c{}_{ad} \\
& + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cg}\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{ig}\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ci}\Gamma^i{}_{be}\partial_{gf}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{ig}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} \\
& + g_{ih}\Gamma^h{}_{bg}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{cf}\partial_{ge}\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cg}\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{ig}\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} \\
& + g_{ci}\Gamma^i{}_{bf}\partial_{ge}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{cg}\partial_{fe}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{bg}\partial_{fe}\Gamma^c{}_{ad} + g_{cb}\partial_{gfe}\Gamma^c{}_{ad} + g_{jc}\Gamma^j{}_{hg}\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hj}\Gamma^j{}_{cg}\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hc}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} \\
& + g_{hc}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{if}\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + g_{jh}\Gamma^j{}_{ig}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ij}\Gamma^j{}_{hg}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\partial_g\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cf}\partial_g\Gamma^i{}_{ae}\Gamma^c{}_{bd} \\
& + g_{ih}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{ig}\partial_f\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cg}\partial_f\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\partial_{gf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{ig}\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} \\
& + g_{ih}\Gamma^h{}_{cg}\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{ic}\Gamma^i{}_{ae}\partial_{gf}\Gamma^c{}_{bd} + g_{ji}\Gamma^j{}_{hg}\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hj}\Gamma^j{}_{ig}\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\partial_g\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} \\
& + g_{hi}\Gamma^h{}_{af}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{af}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + g_{jh}\Gamma^j{}_{ag}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{aj}\Gamma^j{}_{hg}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{bd} \\
& + g_{ah}\Gamma^h{}_{if}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{ag}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{ig}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\partial_{gf}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{ag}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} \\
& + g_{ah}\Gamma^h{}_{ig}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{ce}\partial_{gf}\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{ig}\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cg}\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ic}\Gamma^i{}_{af}\partial_{ge}\Gamma^c{}_{bd} \\
& + g_{hi}\Gamma^h{}_{ag}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{ig}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{cf}\partial_{ge}\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{ag}\partial_{fe}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{cg}\partial_{fe}\Gamma^c{}_{bd} + g_{ac}\partial_{gfe}\Gamma^c{}_{bd}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.106} := {} & g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} + g_{cb}\partial_{gfe}\Gamma^c{}_{ad} \\
& + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ac}\partial_{gfe}\Gamma^c{}_{bd}
\end{aligned}
$$

```
# prepare first six terms in the Taylor series expansion of g_{ab}(x)

term0:= @(gab00).
distribute (term0)                              # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                              # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                              # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                              # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                              # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                              # cdb(term5.200,term5)

end_stage_1 = time.time()
```

$$\texttt{term0.200} := g_{ab}$$

$$\texttt{term1.200} := 0$$

$$\texttt{term2.200} := g_{cb}\partial_{e}\Gamma^{c}{}_{ad}A^{d}A^{e} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}A^{d}A^{e}$$

$$\texttt{term3.200} := g_{cb}\partial_{fe}\Gamma^{c}{}_{ad}A^{d}A^{e}A^{f} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}A^{d}A^{e}A^{f}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json')  # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json')  # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json')  # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json')  # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json')  # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                # cdb(term2.301,term2)
substitute (term2,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                # cdb(term2.302,term2)
distribute (term2)                                                                                       # cdb(term2.303,term2)

substitute (term3,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)         # cdb(term3.301,term3)
substitute (term3,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)         # cdb(term3.302,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                # cdb(term3.303,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                # cdb(term3.304,term3)
distribute (term3)                                                                                       # cdb(term3.305,term3)

substitute (term4,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)  # cdb(term4.301,term4)
substitute (term4,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)  # cdb(term4.302,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)         # cdb(term4.303,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)         # cdb(term4.304,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                # cdb(term4.305,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                # cdb(term4.306,term4)
distribute (term4)                                                                                       # cdb(term4.307,term4)

substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.301,term5)
substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.302,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)  # cdb(term5.303,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)  # cdb(term5.304,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)         # cdb(term5.305,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)         # cdb(term5.306,term5)
```

```
substitute (term5,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term5.307,term5)
substitute (term5,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term5.308,term5)
distribute (term5)                                                                                           # cdb(term5.309,term5)

end_stage_2 = time.time()

# ---------------------------------------------------------------------------------------------
# this block of Xterms only produces formatted output, it's not part of the main computation
# ---------------------------------------------------------------------------------------------

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
      + (1/1) @(term1)  # zero
      + (1/2) @(term2)
      + (1/6) @(term3)
      + (1/24) @(term4)
      + (1/120) @(term5).  # cdb(metric.301,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1).
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{a} -> x^{a}$)
substitute (Xterm1,$A^{a} -> x^{a}$)
substitute (Xterm2,$A^{a} -> x^{a}$)
substitute (Xterm3,$A^{a} -> x^{a}$)
substitute (Xterm4,$A^{a} -> x^{a}$)
substitute (Xterm5,$A^{a} -> x^{a}$)

substitute (Xterm2,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm2.301,Xterm2)
```

```
substitute (Xterm3,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm5.301,Xterm5)

substitute (Xterm2,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm2.301,Xterm2)
substitute (Xterm3,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm5.301,Xterm5)

eliminate_metric (Xterm2)  # cdb(Xterm2.302,Xterm2)
eliminate_metric (Xterm3)  # cdb(Xterm3.302,Xterm3)
eliminate_metric (Xterm4)  # cdb(Xterm4.302,Xterm4)
eliminate_metric (Xterm5)  # cdb(Xterm5.302,Xterm5)

sort_product     (Xterm2)  # cdb(Xterm2.303,Xterm2)
sort_product     (Xterm3)  # cdb(Xterm3.303,Xterm3)
sort_product     (Xterm4)  # cdb(Xterm4.303,Xterm4)
sort_product     (Xterm5)  # cdb(Xterm5.303,Xterm5)

rename_dummies   (Xterm2)  # cdb(Xterm2.304,Xterm2)
rename_dummies   (Xterm3)  # cdb(Xterm3.304,Xterm3)
rename_dummies   (Xterm4)  # cdb(Xterm4.304,Xterm4)
rename_dummies   (Xterm5)  # cdb(Xterm5.304,Xterm5)

canonicalise     (Xterm2)  # cdb(Xterm2.305,Xterm2)
canonicalise     (Xterm3)  # cdb(Xterm3.305,Xterm3)
canonicalise     (Xterm4)  # cdb(Xterm4.305,Xterm4)
canonicalise     (Xterm5)  # cdb(Xterm5.305,Xterm5)

# push upper index to the left
def tidy_Rabcd (obj):
    substitute (obj,$R_{a b c}^{d} -> - R^{d}_{c a b}$)
    substitute (obj,$R_{a b}^{c}_{d} -> R^{c}_{d a b}$)
    substitute (obj,$R_{a}^{b}_{c d} -> - R^{b}_{a c d}$)
    return obj

Xterm0 = tidy_Rabcd (Xterm0)  # cdb(Xterm0.666,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2)  # cdb(Xterm2.666,Xterm2)
```

```
Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.666,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.666,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.666,Xterm5)

Xterm0 = reformat_xterm (Xterm0,  1)    # cdb(Xterm0.301,Xterm0)
Xterm2 = reformat_xterm (Xterm2,  3)    # cdb(Xterm2.301,Xterm2)
Xterm3 = reformat_xterm (Xterm3,  6)    # cdb(Xterm3.301,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)    # cdb(Xterm4.301,Xterm4)
Xterm5 = reformat_xterm (Xterm5,180)    # cdb(Xterm5.301,Xterm5)

# canonicalise from reformat_xterm will slide upper index from left hand side
# so now we slide the upper index back to the left

Xterm0 = tidy_Rabcd (Xterm0)  # cdb(Xterm0.667,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2)  # cdb(Xterm2.667,Xterm2)
Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.667,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.667,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.667,Xterm5)

# metric to 3rd, 4th, 5th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).                                   # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3).                       # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).           # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (Metric6.301,Metric6)

# -----------------------------------------------------------------------------------
# end of format block
# -----------------------------------------------------------------------------------
```

$$\texttt{term2.301} := g_{cb}\partial_e\Gamma^c{}_{ad}A^dA^e + g_{ac}\partial_e\Gamma^c{}_{bd}A^dA^e$$

$$\texttt{term2.302} := \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb}$$

$$\texttt{term2.303} := \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb}$$

$$\texttt{term3.301} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.302} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.303} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.304} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.305} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\begin{aligned}\texttt{term4.301} := {} & g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g \\ & + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g \\ & + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g \\ & + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)\end{aligned}$$

$$\begin{aligned}\texttt{term4.302} := {} & g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g \\ & + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g \\ & + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g \\ & + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)\end{aligned}$$

$$\texttt{term4.303} := g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)$$

$$\texttt{term4.304} := g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)$$

$$\texttt{term4.305} := g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)$$

$$\texttt{term4.306} := \frac{1}{9}g_{ib}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i{}_{efb}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i{}_{egb}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ib}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i{}_{fgb}A^dA^eR^c{}_{dea} + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + \frac{1}{9}g_{ic}A^eA^fR^i{}_{efa}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i{}_{ega}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ai}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i{}_{fga}A^dA^eR^c{}_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{deb} + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)$$

$$\texttt{term4.307} := \frac{1}{9} g_{ib} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ib} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfa} + \frac{1}{9} g_{ci} A^e A^f R^i{}_{efb} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ci} A^e A^g R^i{}_{egb} A^d A^f R^c{}_{dfa}$$
$$+ \frac{1}{9} g_{ib} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{dea} + \frac{1}{9} g_{ci} A^f A^g R^i{}_{fgb} A^d A^e R^c{}_{dea} + \frac{3}{5} g_{cb} A^d A^g A^f A^e \partial_{ef} R^c{}_{dga} - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{dea}$$
$$- \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} + \frac{1}{9} g_{ic} A^e A^f R^i{}_{efa} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ic} A^e A^g R^i{}_{ega} A^d A^f R^c{}_{dfb}$$
$$+ \frac{1}{9} g_{ai} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ai} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfb} + \frac{1}{9} g_{ic} A^f A^g R^i{}_{fga} A^d A^e R^c{}_{deb} + \frac{1}{9} g_{ai} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{deb}$$
$$+ \frac{3}{5} g_{ac} A^d A^g A^f A^e \partial_{ef} R^c{}_{dgb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{deb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f \left(-3R_{bcdg}R^g{}_{fae} - 13R_{acdg}R^g{}_{fbe} - 9g_{bg}\partial_{cd}R^g{}_{fae} - 9g_{ag}\partial_{cd}R^g{}_{fbe}\right)$$

$$\begin{aligned} g_{ab}(x) = {} & g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f \left(-3R_{bcdg}R^g{}_{fae} - 13R_{acdg}R^g{}_{fbe} - 9g_{bg}\partial_{cd}R^g{}_{fae} - 9g_{ag}\partial_{cd}R^g{}_{fbe}\right) \\ & + \frac{1}{180}x^c x^d x^e x^f x^g \left(-3R^h{}_{dac}\partial_e R_{bfgh} - R_{bcdh}\partial_e R^h{}_{gaf} - 3R^h{}_{dbc}\partial_e R_{afgh} - g_{bh}\partial_{cde}R^h{}_{gaf} - R_{acdh}\partial_e R^h{}_{gbf} - g_{ah}\partial_{cde}R^h{}_{gbf}\right) \end{aligned}$$

Stage 3: Replace partial derivs of R with covariant derivs of R

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json')  # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json')  # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json')  # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)   # cdb(term1.401,term1)
sort_product     (term1)   # cdb(term1.402,term1)
rename_dummies   (term1)   # cdb(term1.403,term1)
canonicalise     (term1)   # cdb(term1.404,term1)

eliminate_metric (term2)   # cdb(term2.401,term2)
sort_product     (term2)   # cdb(term2.402,term2)
rename_dummies   (term2)   # cdb(term2.403,term2)
canonicalise     (term2)   # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)         # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)         # cdb(term3.402,term3)
distribute (term3)                                                                                # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)  # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)  # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)         # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)         # cdb(term4.404,term4)
distribute (term4)                                                                                # cdb(term4.405,term4)
```

```
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c d b}} ->   @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c b d}} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()
```

$$\texttt{tmp01.403} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\texttt{tmp02.403} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\texttt{tmp03.403} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}$$

$$\texttt{term1.401} := 0$$
$$\texttt{term1.402} := 0$$
$$\texttt{term1.403} := 0$$
$$\texttt{term1.404} := 0$$

$$\texttt{term2.401} := \frac{1}{3}A^dA^eR_{bdea} + \frac{1}{3}A^dA^eR_{adeb}$$
$$\texttt{term2.402} := \frac{1}{3}A^dA^eR_{bdea} + \frac{1}{3}A^dA^eR_{adeb}$$
$$\texttt{term2.403} := \frac{1}{3}A^cA^dR_{bcda} + \frac{1}{3}A^cA^dR_{acdb}$$
$$\texttt{term2.404} := -\frac{2}{3}A^cA^dR_{acbd}$$

$$\texttt{term3.401} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$
$$\texttt{term3.402} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$
$$\texttt{term3.403} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$

$$\begin{aligned}\texttt{term4.401} := {} & \frac{1}{9}g_{ib}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i{}_{efb}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i{}_{egb}A^dA^fR^c{}_{dfa} \\ & + \frac{1}{9}g_{ib}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i{}_{fgb}A^dA^eR^c{}_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\ & - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i{}_{efa}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i{}_{ega}A^dA^fR^c{}_{dfb} \\ & + \frac{1}{9}g_{ai}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i{}_{fga}A^dA^eR^c{}_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{deb} \\ & + \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\end{aligned}$$

$$
\begin{aligned}
\texttt{term4.402} := {} & \frac{1}{9} g_{ib} A^{e} A^{f} R^{i}{}_{efc} A^{d} A^{g} R^{c}{}_{dga} + \frac{1}{9} g_{ib} A^{e} A^{g} R^{i}{}_{egc} A^{d} A^{f} R^{c}{}_{dfa} + \frac{1}{9} g_{ci} A^{e} A^{f} R^{i}{}_{efb} A^{d} A^{g} R^{c}{}_{dga} + \frac{1}{9} g_{ci} A^{e} A^{g} R^{i}{}_{egb} A^{d} A^{f} R^{c}{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^{f} A^{g} R^{i}{}_{fgc} A^{d} A^{e} R^{c}{}_{dea} + \frac{1}{9} g_{ci} A^{f} A^{g} R^{i}{}_{fgb} A^{d} A^{e} R^{c}{}_{dea} + \frac{3}{5} g_{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{gfh} R^{h}{}_{dea} - \frac{1}{15} g_{cb} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{geh} R^{h}{}_{dfa} + \frac{1}{9} g_{ic} A^{e} A^{f} R^{i}{}_{efa} A^{d} A^{g} R^{c}{}_{dgb} + \frac{1}{9} g_{ic} A^{e} A^{g} R^{i}{}_{ega} A^{d} A^{f} R^{c}{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^{e} A^{f} R^{i}{}_{efc} A^{d} A^{g} R^{c}{}_{dgb} + \frac{1}{9} g_{ai} A^{e} A^{g} R^{i}{}_{egc} A^{d} A^{f} R^{c}{}_{dfb} + \frac{1}{9} g_{ic} A^{f} A^{g} R^{i}{}_{fga} A^{d} A^{e} R^{c}{}_{deb} + \frac{1}{9} g_{ai} A^{f} A^{g} R^{i}{}_{fgc} A^{d} A^{e} R^{c}{}_{deb} \\
& + \frac{3}{5} g_{ac} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{gfh} R^{h}{}_{deb} - \frac{1}{15} g_{ac} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{geh} R^{h}{}_{dfb}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.403} := {} & \frac{1}{9} g_{ib} A^{e} A^{f} R^{i}{}_{efc} A^{d} A^{g} R^{c}{}_{dga} + \frac{1}{9} g_{ib} A^{e} A^{g} R^{i}{}_{egc} A^{d} A^{f} R^{c}{}_{dfa} + \frac{1}{9} g_{ci} A^{e} A^{f} R^{i}{}_{efb} A^{d} A^{g} R^{c}{}_{dga} + \frac{1}{9} g_{ci} A^{e} A^{g} R^{i}{}_{egb} A^{d} A^{f} R^{c}{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^{f} A^{g} R^{i}{}_{fgc} A^{d} A^{e} R^{c}{}_{dea} + \frac{1}{9} g_{ci} A^{f} A^{g} R^{i}{}_{fgb} A^{d} A^{e} R^{c}{}_{dea} + \frac{3}{5} g_{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{gfh} R^{h}{}_{dea} - \frac{1}{15} g_{cb} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{geh} R^{h}{}_{dfa} + \frac{1}{9} g_{ic} A^{e} A^{f} R^{i}{}_{efa} A^{d} A^{g} R^{c}{}_{dgb} + \frac{1}{9} g_{ic} A^{e} A^{g} R^{i}{}_{ega} A^{d} A^{f} R^{c}{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^{e} A^{f} R^{i}{}_{efc} A^{d} A^{g} R^{c}{}_{dgb} + \frac{1}{9} g_{ai} A^{e} A^{g} R^{i}{}_{egc} A^{d} A^{f} R^{c}{}_{dfb} + \frac{1}{9} g_{ic} A^{f} A^{g} R^{i}{}_{fga} A^{d} A^{e} R^{c}{}_{deb} + \frac{1}{9} g_{ai} A^{f} A^{g} R^{i}{}_{fgc} A^{d} A^{e} R^{c}{}_{deb} \\
& + \frac{3}{5} g_{ac} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{gfh} R^{h}{}_{deb} - \frac{1}{15} g_{ac} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{geh} R^{h}{}_{dfb}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.404} := {} & \frac{1}{9} g_{ib} A^{e} A^{f} R^{i}{}_{efc} A^{d} A^{g} R^{c}{}_{dga} + \frac{1}{9} g_{ib} A^{e} A^{g} R^{i}{}_{egc} A^{d} A^{f} R^{c}{}_{dfa} + \frac{1}{9} g_{ci} A^{e} A^{f} R^{i}{}_{efb} A^{d} A^{g} R^{c}{}_{dga} + \frac{1}{9} g_{ci} A^{e} A^{g} R^{i}{}_{egb} A^{d} A^{f} R^{c}{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^{f} A^{g} R^{i}{}_{fgc} A^{d} A^{e} R^{c}{}_{dea} + \frac{1}{9} g_{ci} A^{f} A^{g} R^{i}{}_{fgb} A^{d} A^{e} R^{c}{}_{dea} + \frac{3}{5} g_{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{gfh} R^{h}{}_{dea} - \frac{1}{15} g_{cb} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{geh} R^{h}{}_{dfa} + \frac{1}{9} g_{ic} A^{e} A^{f} R^{i}{}_{efa} A^{d} A^{g} R^{c}{}_{dgb} + \frac{1}{9} g_{ic} A^{e} A^{g} R^{i}{}_{ega} A^{d} A^{f} R^{c}{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^{e} A^{f} R^{i}{}_{efc} A^{d} A^{g} R^{c}{}_{dgb} + \frac{1}{9} g_{ai} A^{e} A^{g} R^{i}{}_{egc} A^{d} A^{f} R^{c}{}_{dfb} + \frac{1}{9} g_{ic} A^{f} A^{g} R^{i}{}_{fga} A^{d} A^{e} R^{c}{}_{deb} + \frac{1}{9} g_{ai} A^{f} A^{g} R^{i}{}_{fgc} A^{d} A^{e} R^{c}{}_{deb} \\
& + \frac{3}{5} g_{ac} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{gfh} R^{h}{}_{deb} - \frac{1}{15} g_{ac} A^{d} A^{g} A^{f} A^{e} R^{c}{}_{geh} R^{h}{}_{dfb}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.405} := {} & \frac{1}{9} g_{ib} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ib} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfa} + \frac{1}{9} g_{ci} A^e A^f R^i{}_{efb} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ci} A^e A^g R^i{}_{egb} A^d A^f R^c{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{dea} + \frac{1}{9} g_{ci} A^f A^g R^i{}_{fgb} A^d A^e R^c{}_{dea} + \frac{3}{5} g_{cb} A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{dea} - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} + \frac{1}{9} g_{ic} A^e A^f R^i{}_{efa} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ic} A^e A^g R^i{}_{ega} A^d A^f R^c{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ai} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfb} + \frac{1}{9} g_{ic} A^f A^g R^i{}_{fga} A^d A^e R^c{}_{deb} + \frac{1}{9} g_{ai} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{deb} \\
& + \frac{3}{5} g_{ac} A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{deb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb}
\end{aligned}
$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

```
beg_stage_4 = time.time()
# final housekeeping

term1 = flatten_Rabcd (term1)          # cdb(term1.501,term1)
term2 = flatten_Rabcd (term2)          # cdb(term2.501,term2)
term3 = flatten_Rabcd (term3)          # cdb(term3.501,term3)
term4 = flatten_Rabcd (term4)          # cdb(term4.501,term4)
term5 = flatten_Rabcd (term5)          # cdb(term5.501,term5)

eliminate_metric (term1)
eliminate_metric (term2)
eliminate_metric (term3)
eliminate_metric (term4)
eliminate_metric (term5)

eliminate_kronecker (term1)
eliminate_kronecker (term2)
eliminate_kronecker (term3)
eliminate_kronecker (term4)
eliminate_kronecker (term5)

sort_product (term1)
sort_product (term2)
sort_product (term3)
sort_product (term4)
sort_product (term5)

rename_dummies (term1)
rename_dummies (term2)
rename_dummies (term3)
rename_dummies (term4)
rename_dummies (term5)

canonicalise (term1)                   # cdb(term1.502,term1)
canonicalise (term2)                   # cdb(term2.502,term2)
canonicalise (term3)                   # cdb(term3.502,term3)
```

```
canonicalise (term4)                    # cdb(term4.502,term4)
canonicalise (term5)                    # cdb(term5.502,term5)

# this is out final answer

metric:=@(term0)
      + (1/1) @(term1)
      + (1/2) @(term2)
      + (1/6) @(term3)
      + (1/24) @(term4)
      + (1/120) @(term5).               # cdb(metric.501,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.502,metric)

cdblib.create ('metric.json')

cdblib.put ('g_ab',metric,'metric.json')

# extract the terms of the metric in powers of x

term0 = get_xterm (metric,0)            # cdb(term0.503,term0)
term1 = get_xterm (metric,1)            # cdb(term1.503,term1)
term2 = get_xterm (metric,2)            # cdb(term2.503,term2)
term3 = get_xterm (metric,3)            # cdb(term3.503,term3)
term4 = get_xterm (metric,4)            # cdb(term4.503,term4)
term5 = get_xterm (metric,5)            # cdb(term5.503,term5)

cdblib.put ('g_ab_0',term0,'metric.json')
cdblib.put ('g_ab_1',term1,'metric.json')
cdblib.put ('g_ab_2',term2,'metric.json')
cdblib.put ('g_ab_3',term3,'metric.json')
cdblib.put ('g_ab_4',term4,'metric.json')
cdblib.put ('g_ab_5',term5,'metric.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3).  # cdb(metric4.501,metric4)
```

$$\texttt{term2.501} := -\frac{2}{3}A^cA^dR_{acbd}$$

$$\texttt{term2.502} := -\frac{2}{3}A^cA^dR_{acbd}$$

$$\texttt{term3.501} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$

$$\texttt{term3.502} := -A^cA^dA^e\nabla_cR_{adbe}$$

$$\begin{aligned}\texttt{term4.501} := &\frac{1}{9}g_{ib}A^eA^fg^{ih}R_{hefc}A^dA^gg^{cj}R_{jdga} + \frac{1}{9}g_{ib}A^eA^gg^{ih}R_{hegc}A^dA^fg^{cj}R_{jdfa} + \frac{1}{9}g_{ci}A^eA^fg^{ih}R_{hefb}A^dA^gg^{cj}R_{jdga}\\ &+ \frac{1}{9}g_{ci}A^eA^gg^{ih}R_{hegb}A^dA^fg^{cj}R_{jdfa} + \frac{1}{9}g_{ib}A^fA^gg^{ih}R_{hfgc}A^dA^eg^{cj}R_{jdea} + \frac{1}{9}g_{ci}A^fA^gg^{ih}R_{hfgb}A^dA^eg^{cj}R_{jdea}\\ &+ \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} - \frac{1}{15}g_{cb}A^dA^gA^fA^eg^{ci}R_{igfh}g^{hj}R_{jdea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eg^{ci}R_{igeh}g^{hj}R_{jdfa}\\ &+ \frac{1}{9}g_{ic}A^eA^fg^{ih}R_{hefa}A^dA^gg^{cj}R_{jdgb} + \frac{1}{9}g_{ic}A^eA^gg^{ih}R_{hega}A^dA^fg^{cj}R_{jdfb} + \frac{1}{9}g_{ai}A^eA^fg^{ih}R_{hefc}A^dA^gg^{cj}R_{jdgb}\\ &+ \frac{1}{9}g_{ai}A^eA^gg^{ih}R_{hegc}A^dA^fg^{cj}R_{jdfb} + \frac{1}{9}g_{ic}A^fA^gg^{ih}R_{hfga}A^dA^eg^{cj}R_{jdeb} + \frac{1}{9}g_{ai}A^fA^gg^{ih}R_{hfgc}A^dA^eg^{cj}R_{jdeb}\\ &+ \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eg^{ci}R_{igfh}g^{hj}R_{jdeb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eg^{ci}R_{igeh}g^{hj}R_{jdfb}\end{aligned}$$

$$\texttt{term4.502} := \frac{16}{15}A^cA^dA^eA^fR_{acdg}R_{befh}g^{gh} - \frac{6}{5}A^cA^dA^eA^f\nabla_{cd}R_{aebf}$$

$$
\begin{aligned}
\texttt{term5.501} := {} & \frac{1}{6} g_{ib} A^e A^g A^f \nabla_e R_{cgfj} g^{ij} A^d A^h g^{ck} R_{kdha} + \frac{1}{6} g_{ib} A^e A^h A^f \nabla_e R_{chfj} g^{ij} A^d A^g g^{ck} R_{kdga} + \frac{1}{6} g_{ib} A^e A^f g^{ik} R_{kefc} A^d A^h A^g \nabla_d R_{ahgj} g^{cj} \\
& + \frac{1}{6} g_{ib} A^e A^h A^g \nabla_e R_{chgj} g^{ij} A^d A^f g^{ck} R_{kdfa} + \frac{1}{6} g_{ib} A^e A^g g^{ik} R_{kegc} A^d A^h A^f \nabla_d R_{ahfj} g^{cj} + \frac{1}{6} g_{ib} A^e A^h g^{ik} R_{kehc} A^d A^g A^f \nabla_d R_{agfj} g^{cj} \\
& + \frac{1}{6} g_{ci} A^e A^g A^f \nabla_e R_{bgfj} g^{ij} A^d A^h g^{ck} R_{kdha} + \frac{1}{6} g_{ci} A^e A^h A^f \nabla_e R_{bhfj} g^{ij} A^d A^g g^{ck} R_{kdga} + \frac{1}{6} g_{ci} A^e A^f g^{ik} R_{kefb} A^d A^h A^g \nabla_d R_{ahgj} g^{cj} \\
& + \frac{1}{6} g_{ci} A^e A^h A^g \nabla_e R_{bhgj} g^{ij} A^d A^f g^{ck} R_{kdfa} + \frac{1}{6} g_{ci} A^e A^g g^{ik} R_{kegb} A^d A^h A^f \nabla_d R_{ahfj} g^{cj} + \frac{1}{6} g_{ci} A^e A^h g^{ik} R_{kehb} A^d A^g A^f \nabla_d R_{agfj} g^{cj} \\
& + \frac{1}{6} g_{ib} A^f A^h A^g \nabla_f R_{chgj} g^{ij} A^d A^e g^{ck} R_{kdea} + \frac{1}{6} g_{ib} A^f A^g g^{ik} R_{kfgc} A^d A^h A^e \nabla_d R_{ahej} g^{cj} + \frac{1}{6} g_{ib} A^f A^h g^{ik} R_{kfhc} A^d A^g A^e \nabla_d R_{agej} g^{cj} \\
& + \frac{1}{6} g_{ci} A^f A^h A^g \nabla_f R_{bhgj} g^{ij} A^d A^e g^{ck} R_{kdea} + \frac{1}{6} g_{ci} A^f A^g g^{ik} R_{kfgb} A^d A^h A^e \nabla_d R_{ahej} g^{cj} + \frac{1}{6} g_{ci} A^f A^h g^{ik} R_{kfhb} A^d A^g A^e \nabla_d R_{agej} g^{cj} \\
& + \frac{1}{6} g_{kb} A^g A^h g^{kj} R_{jghc} A^d A^f A^e \nabla_d R_{afei} g^{ci} + \frac{1}{6} g_{ck} A^g A^h g^{kj} R_{jghb} A^d A^f A^e \nabla_d R_{afei} g^{ci} - \frac{1}{3} g_{cb} A^d A^h A^e A^f A^g R_{adhk} \nabla_e R_{figj} g^{ci} g^{kj} \\
& + \frac{1}{3} g_{cb} A^d A^h A^e A^f A^g R_{dkhi} \nabla_e R_{afgj} g^{ck} g^{ij} + \frac{2}{3} g_{cb} A^d A^h A^e A^f A^g \nabla_{dhe} R_{afgk} g^{ck} - \frac{1}{9} g_{cb} A^d A^h A^f A^g \nabla_h R_{ifgj} g^{cj} A^e g^{ik} R_{kdea} \\
& - \frac{1}{9} g_{cb} A^d A^h A^e A^g \nabla_h R_{iegj} g^{cj} A^f g^{ik} R_{kdfa} - \frac{1}{9} g_{cb} A^d A^e A^f \nabla_d R_{aefj} g^{ij} A^h A^g g^{ck} R_{khgi} - \frac{1}{9} g_{cb} A^d A^h A^e A^f \nabla_h R_{iefj} g^{cj} A^g g^{ik} R_{kdga} \\
& - \frac{1}{9} g_{cb} A^d A^e A^g \nabla_d R_{aegj} g^{ij} A^h A^f g^{ck} R_{khfi} - \frac{1}{9} g_{cb} A^d A^f A^g \nabla_d R_{afgj} g^{ij} A^h A^e g^{ck} R_{khei} + \frac{1}{6} g_{ic} A^e A^g A^f \nabla_e R_{agfj} g^{ij} A^d A^h g^{ck} R_{kdhb} \\
& + \frac{1}{6} g_{ic} A^e A^h A^f \nabla_e R_{ahfj} g^{ij} A^d A^g g^{ck} R_{kdgb} + \frac{1}{6} g_{ic} A^e A^f g^{ik} R_{kefa} A^d A^h A^g \nabla_d R_{bhgj} g^{cj} + \frac{1}{6} g_{ic} A^e A^h A^g \nabla_e R_{ahgj} g^{ij} A^d A^f g^{ck} R_{kdfb} \\
& + \frac{1}{6} g_{ic} A^e A^g g^{ik} R_{kega} A^d A^h A^f \nabla_d R_{bhfj} g^{cj} + \frac{1}{6} g_{ic} A^e A^h g^{ik} R_{keha} A^d A^g A^f \nabla_d R_{bgfj} g^{cj} + \frac{1}{6} g_{ai} A^e A^g A^f \nabla_e R_{cgfj} g^{ij} A^d A^h g^{ck} R_{kdhb} \\
& + \frac{1}{6} g_{ai} A^e A^h A^f \nabla_e R_{chfj} g^{ij} A^d A^g g^{ck} R_{kdgb} + \frac{1}{6} g_{ai} A^e A^f g^{ik} R_{kefc} A^d A^h A^g \nabla_d R_{bhgj} g^{cj} + \frac{1}{6} g_{ai} A^e A^h A^g \nabla_e R_{chgj} g^{ij} A^d A^f g^{ck} R_{kdfb} \\
& + \frac{1}{6} g_{ai} A^e A^g g^{ik} R_{kegc} A^d A^h A^f \nabla_d R_{bhfj} g^{cj} + \frac{1}{6} g_{ai} A^e A^h g^{ik} R_{kehc} A^d A^g A^f \nabla_d R_{bgfj} g^{cj} + \frac{1}{6} g_{ic} A^f A^h A^g \nabla_f R_{ahgj} g^{ij} A^d A^e g^{ck} R_{kdeb} \\
& + \frac{1}{6} g_{ic} A^f A^g g^{ik} R_{kfga} A^d A^h A^e \nabla_d R_{bhej} g^{cj} + \frac{1}{6} g_{ic} A^f A^h g^{ik} R_{kfha} A^d A^g A^e \nabla_d R_{bgej} g^{cj} + \frac{1}{6} g_{ai} A^f A^h A^g \nabla_f R_{chgj} g^{ij} A^d A^e g^{ck} R_{kdeb} \\
& + \frac{1}{6} g_{ai} A^f A^g g^{ik} R_{kfgc} A^d A^h A^e \nabla_d R_{bhej} g^{cj} + \frac{1}{6} g_{ai} A^f A^h g^{ik} R_{kfhc} A^d A^g A^e \nabla_d R_{bgej} g^{cj} + \frac{1}{6} g_{kc} A^g A^h g^{kj} R_{jgha} A^d A^f A^e \nabla_d R_{bfei} g^{ci} \\
& + \frac{1}{6} g_{ak} A^g A^h g^{kj} R_{jghc} A^d A^f A^e \nabla_d R_{bfei} g^{ci} - \frac{1}{3} g_{ac} A^d A^h A^e A^f A^g R_{bdhk} \nabla_e R_{figj} g^{ci} g^{kj} + \frac{1}{3} g_{ac} A^d A^h A^e A^f A^g R_{dkhi} \nabla_e R_{bfgj} g^{ck} g^{ij} \\
& + \frac{2}{3} g_{ac} A^d A^h A^e A^f A^g \nabla_{dhe} R_{bfgk} g^{ck} - \frac{1}{9} g_{ac} A^d A^h A^f A^g \nabla_h R_{ifgj} g^{cj} A^e g^{ik} R_{kdeb} - \frac{1}{9} g_{ac} A^d A^h A^e A^g \nabla_h R_{iegj} g^{cj} A^f g^{ik} R_{kdfb} \\
& - \frac{1}{9} g_{ac} A^d A^e A^f \nabla_d R_{befj} g^{ij} A^h A^g g^{ck} R_{khgi} - \frac{1}{9} g_{ac} A^d A^h A^e A^f \nabla_h R_{iefj} g^{cj} A^g g^{ik} R_{kdgb} \\
& - \frac{1}{9} g_{ac} A^d A^e A^g \nabla_d R_{begj} g^{ij} A^h A^f g^{ck} R_{khfi} - \frac{1}{9} g_{ac} A^d A^f A^g \nabla_d R_{bfgj} g^{ij} A^h A^e g^{ck} R_{khei}
\end{aligned}
$$

$$\texttt{term5.502} := \frac{8}{3}A^cA^dA^eA^fA^gR_{acdh}\nabla_eR_{bfgi}g^{hi} + \frac{8}{3}A^cA^dA^eA^fA^gR_{bcdh}\nabla_eR_{afgi}g^{hi} - \frac{4}{3}A^cA^dA^eA^fA^g\nabla_{cde}R_{afbg}$$

$$\texttt{metric.501} \coloneqq g_{ab} - \frac{1}{3}A^{c}A^{d}R_{acbd} - \frac{1}{6}A^{c}A^{d}A^{e}\nabla_{c}R_{adbe} + \frac{2}{45}A^{c}A^{d}A^{e}A^{f}R_{acdg}R_{befh}g^{gh} - \frac{1}{20}A^{c}A^{d}A^{e}A^{f}\nabla_{cd}R_{aebf}$$
$$+ \frac{1}{45}A^{c}A^{d}A^{e}A^{f}A^{g}R_{acdh}\nabla_{e}R_{bfgi}g^{hi} + \frac{1}{45}A^{c}A^{d}A^{e}A^{f}A^{g}R_{bcdh}\nabla_{e}R_{afgi}g^{hi} - \frac{1}{90}A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{cde}R_{afbg}$$

$$\texttt{metric.502} \coloneqq g_{ab} - \frac{1}{3}x^{c}x^{d}R_{acbd} - \frac{1}{6}x^{c}x^{d}x^{e}\nabla_{c}R_{adbe} + \frac{2}{45}x^{c}x^{d}x^{e}x^{f}R_{acdg}R_{befh}g^{gh} - \frac{1}{20}x^{c}x^{d}x^{e}x^{f}\nabla_{cd}R_{aebf}$$
$$+ \frac{1}{45}x^{c}x^{d}x^{e}x^{f}x^{g}R_{acdh}\nabla_{e}R_{bfgi}g^{hi} + \frac{1}{45}x^{c}x^{d}x^{e}x^{f}x^{g}R_{bcdh}\nabla_{e}R_{afgi}g^{hi} - \frac{1}{90}x^{c}x^{d}x^{e}x^{f}x^{g}\nabla_{cde}R_{afbg}$$

$\texttt{term0.503} := g_{ab}$

$\texttt{term1.503} := 0$

$$\texttt{term2.503} := -\frac{1}{3}x^c x^d R_{acbd}$$

$$\texttt{term3.503} := -\frac{1}{6}x^c x^d x^e \nabla_c R_{adbe}$$

$$\texttt{term4.503} := \frac{2}{45}x^c x^d x^e x^f R_{acdg} R_{befh} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf}$$

$$\texttt{term5.503} := \frac{1}{45}x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg}$$

```
Xterm0 := @(term0).
Xterm1 := @(term1).  # zero
Xterm2 := @(term2).
Xterm3 := @(term3).
Xterm4 := @(term4).
Xterm5 := @(term5).

Xterm0 = reformat_xterm (Xterm0,  1)    # cdb(Xterm0.601,Xterm0)
Xterm2 = reformat_xterm (Xterm2,  3)    # cdb(Xterm2.601,Xterm2)
Xterm3 = reformat_xterm (Xterm3,  6)    # cdb(Xterm3.601,Xterm3)
Xterm4 = reformat_xterm (Xterm4,180)    # cdb(Xterm4.601,Xterm4)
Xterm5 = reformat_xterm (Xterm5, 90)    # cdb(Xterm5.601,Xterm5)

gab3   := @(Xterm0) + @(Xterm2).                                  # cdb (gab3.601,gab3)
gab4   := @(Xterm0) + @(Xterm2) + @(Xterm3).                      # cdb (gab4.601,gab4)
gab5   := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).          # cdb (gab5.601,gab5)
gab6   := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (gab6.601,gab6)

Metric := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Xterm0,  1)    # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Xterm2,  3)    # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Xterm3,  6)    # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Xterm4,180)    # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Xterm5, 90)    # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()
```

The metric in Riemann normal coordinates

$$
\begin{aligned}
g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f \left(8g^{gh} R_{acdg} R_{befh} - 9\nabla_{cd} R_{aebf}\right) \\
+ \frac{1}{90}x^c x^d x^e x^f x^g \left(2g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

Curvature expansion of the metric

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned}
\overset{0}{g}_{ab} &= g_{ab} \\
3\overset{2}{g}_{ab} &= -x^c x^d R_{acbd} \\
6\overset{3}{g}_{ab} &= -x^c x^d x^e \nabla_c R_{adbe} \\
180\overset{4}{g}_{ab} &= x^c x^d x^e x^f \left(8g^{gh} R_{acdg} R_{befh} - 9\nabla_{cd} R_{aebf}\right) \\
90\overset{5}{g}_{ab} &= x^c x^d x^e x^f x^g \left(2g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right)
\end{aligned}$$

```
cdblib.create ('metric.export')

cdblib.put ('g_ab_3',Metric3,'metric.export')  # R and \partial R
cdblib.put ('g_ab_4',Metric4,'metric.export')
cdblib.put ('g_ab_5',Metric5,'metric.export')
cdblib.put ('g_ab_6',Metric6,'metric.export')

cdblib.put ('g_ab',  Metric, 'metric.export')  # R and \nabla R

cdblib.put ('g_ab_scaled0',scaled0,'metric.export')
cdblib.put ('g_ab_scaled2',scaled2,'metric.export')
cdblib.put ('g_ab_scaled3',scaled3,'metric.export')
cdblib.put ('g_ab_scaled4',scaled4,'metric.export')
cdblib.put ('g_ab_scaled5',scaled5,'metric.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)
```

Timing

Stage 1: 3.1 secs
Stage 2: 1.2 secs
Stage 3: 97.6 secs
Stage 4: 3.6 secs

The inverse metric tensor in Riemann normal coordinates

Here we calculate the Riemann normal expansion of the inverse metric, g^{ab}, by developing the recursive sequences

$$g^{ab}{}_{,d\underline{e}} = -\left(g^{cb}\Gamma^{a}{}_{c(d}\right)_{,\underline{e})} - \left(g^{ac}\Gamma^{b}{}_{c(d}\right)_{,\underline{e})} \tag{1}$$

$$(n+3)\Gamma^{a}{}_{d(b,c\underline{e})} = (n+1)\left(R^{a}{}_{(bcd,\underline{e})} - \left(\Gamma^{a}{}_{f(c}\Gamma^{f}{}_{bd}\right)_{,\underline{e})}\right) \tag{2}$$

for $n = 1, 2, 3, \cdots$. Note in these equations that the (extended) index $\underline{e}$ contains n normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g^{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^{c} + \frac{1}{2!}g^{ab}{}_{,cd}x^{c}x^{d} + \frac{1}{3!}g^{ab}{}_{,cde}x^{c}x^{d}x^{e} + \cdots \\ &= g^{ab} + \sum_{n=1}^{\infty}\frac{1}{n!}\,g^{ab}{}_{,\underline{c}}\,x^{\underline{c}} \end{aligned}$$

Stage 1: Symmetrised partial derivatives of g^{ab}

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g^{ab}{}_{,c}A^{c} &= 0 \\ g^{ab}{}_{,cd}A^{c}A^{d} &= -g^{cb}\partial_{e}\Gamma^{a}{}_{cd}A^{d}A^{e} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}A^{d}A^{e} \\ g^{ab}{}_{,cde}A^{c}A^{d}A^{e} &= -g^{cb}\partial_{fe}\Gamma^{a}{}_{cd}A^{d}A^{e}A^{f} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}A^{d}A^{e}A^{f} \end{aligned}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

Now we use the results from `dGamma` to replace derivatives of Γ with partial derivatives of R. These were computed in `dGamma` using equation (2) above.

$$g^{ab}{}_{,c}A^c = 0$$

$$g^{ab}{}_{,cd}A^cA^d = -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$g^{ab}{}_{,cde}A^cA^dA^e = -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

Stage 3: Replace partial derivs of R with covariant derivs of R

Next we use the results from `dRabcd` to replace the partial derivatives of R with covariant deriavtives.

$$g^{ab}{}_{,c}A^c = 0$$

$$g^{ab}{}_{,cd}A^cA^d = -\frac{1}{3}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}A^cA^dR^b{}_{cd}{}^a$$

$$g^{ab}{}_{,cde}A^cA^dA^e = -\frac{1}{2}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g^{ag} - \frac{1}{2}g^{ac}A^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

Each of the above expressions constitutues one term in the Taylor series for the metric. We also make the trivial change $A \to x$. Then we do some trivial reformatting.

$$\begin{aligned} g_{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^c + \frac{1}{2!}g^{ab}{}_{,cd}x^cx^d + \frac{1}{3!}g^{ab}{}_{,cde}x^cx^dx^e + \mathcal{O}\left(\epsilon^4\right) \\ &= g^{ab} + \frac{1}{3}x^cx^dR_{cedf}g^{ae}g^{bf} + \frac{1}{6}x^cx^dx^e\nabla_cR_{dfeg}g^{af}g^{bg} + \mathcal{O}\left(\epsilon^4\right) \end{aligned}$$

Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d}   -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d}   -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    sort_product    (obj)
    rename_dummies (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah  = Ex("numx = " + str(n))
    keep_weight (foo,bah)
```

```
    return foo

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                              -> A001^{a}                $)
    substitute (obj,$ x^{a}                              -> A002^{a}                $)
    substitute (obj,$ g_{a b}                            -> A003_{a b}              $)
    substitute (obj,$ g^{a b}                            -> A004^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A010_{a b c d e f g h}  $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A009_{a b c d e f g}    $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A008_{a b c d e f}      $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A007_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}{R_{a b c d}}    -> A014_{a b c d e f g h}  $)
    substitute (obj,$ \partial_{e f g}{R_{a b c d}}      -> A013_{a b c d e f g}    $)
    substitute (obj,$ \partial_{e f}{R_{a b c d}}        -> A012_{a b c d e f}      $)
    substitute (obj,$ \partial_{e}{R_{a b c d}}          -> A011_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}{R^{a}_{b c d}} -> A018^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}{R^{a}_{b c d}}   -> A017^{a}_{b c d e f g}  $)
    substitute (obj,$ \partial_{e f}{R^{a}_{b c d}}     -> A016^{a}_{b c d e f}    $)
    substitute (obj,$ \partial_{e}{R^{a}_{b c d}}       -> A015^{a}_{b c d e}      $)
    substitute (obj,$ R_{a b c d}                        -> A005_{a b c d}          $)
    substitute (obj,$ R^{a}_{b c d}                      -> A006^{a}_{b c d}        $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                   -> A^{a}                            $)
    substitute (obj,$ A002^{a}                   -> x^{a}                            $)
    substitute (obj,$ A003_{a b}                 -> g_{a b}                          $)
    substitute (obj,$ A004^{a b}                 -> g^{a b}                          $)
    substitute (obj,$ A005_{a b c d}             -> R_{a b c d}                      $)
    substitute (obj,$ A006^{a}_{b c d}           -> R^{a}_{b c d}                    $)
    substitute (obj,$ A007_{a b c d e}           -> \nabla_{e}{R_{a b c d}}          $)
    substitute (obj,$ A008_{a b c d e f}         -> \nabla_{e f}{R_{a b c d}}        $)
    substitute (obj,$ A009_{a b c d e f g}       -> \nabla_{e f g}{R_{a b c d}}      $)
    substitute (obj,$ A010_{a b c d e f g h}     -> \nabla_{e f g h}{R_{a b c d}}    $)
    substitute (obj,$ A011_{a b c d e}           -> \partial_{e}{R_{a b c d}}        $)
    substitute (obj,$ A012_{a b c d e f}         -> \partial_{e f}{R_{a b c d}}      $)
    substitute (obj,$ A013_{a b c d e f g}       -> \partial_{e f g}{R_{a b c d}}    $)
```

```
    substitute (obj,$ A014_{a b c d e f g h}    -> \partial_{e f g h}{R_{a b c d}}   $)
    substitute (obj,$ A015^{a}_{b c d e}        -> \partial_{e}{R^{a}_{b c d}}       $)
    substitute (obj,$ A016^{a}_{b c d e f}      -> \partial_{e f}{R^{a}_{b c d}}     $)
    substitute (obj,$ A017^{a}_{b c d e f g}    -> \partial_{e f g}{R^{a}_{b c d}}   $)
    substitute (obj,$ A018^{a}_{b c d e f g h}  -> \partial_{e f g h}{R^{a}_{b c d}} $)

    return obj

def reformat_xterm (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute     (bah)
   bah = product_sort (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$x^{a?}$)
   ans := @(bah) / @(foo).
   return ans

def rescale_xterm (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute  (bah)
   factor_out  (bah,$x^{a?}$)
   return bah

def add_tags (obj,tag):
   n = 0
   ans = Ex('0')
   for i in obj.top().terms():
      foo = obj[i]
      bah = Ex(tag+'_{'+str(n)+'}')
      ans := @(ans) + @(bah) @(foo).
      n = n + 1
   return ans

def clear_tags (obj,tag):
   ans := @(obj).
```

```
    foo  = Ex(tag+'_{a?} -> 1')
    substitute (ans,foo)
    return ans

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of g^{ab}

```
beg_stage_1 = time.time()

# symmetrised partial derivatives of g^{ab}

gab00:=g^{a b}.                                               # cdb (gab00.101,gab00)

gab01:= - g^{c b}\Gamma^{a}_{c d} - g^{a c}\Gamma^{b}_{c d}. # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @(gab01) }.                              # cdb (gab02.101,gab02)
distribute   (gab02)                                          # cdb (gab02.102,gab02)
product_rule (gab02)                                          # cdb (gab02.103,gab02)
substitute   (gab02, $\partial_{d}{g^{a b}} -> @(gab01)$)     # cdb (gab02.104,gab02)
distribute   (gab02)                                          # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @(gab02) }.                              # cdb (gab03.101,gab03)
distribute   (gab03)                                          # cdb (gab03.102,gab03)
product_rule (gab03)                                          # cdb (gab03.103,gab03)
substitute   (gab03, $\partial_{d}{g^{a b}} -> @(gab01)$)     # cdb (gab03.104,gab03)
distribute   (gab03)                                          # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @(gab03) }.                              # cdb (gab04.101,gab04)
distribute   (gab04)                                          # cdb (gab04.102,gab04)
product_rule (gab04)                                          # cdb (gab04.103,gab04)
substitute   (gab04, $\partial_{d}{g^{a b}} -> @(gab01)$)     # cdb (gab04.104,gab04)
distribute   (gab04)                                          # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @(gab04) }.                              # cdb (gab05.101,gab05)
distribute   (gab05)                                          # cdb (gab05.102,gab05)
product_rule (gab05)                                          # cdb (gab05.103,gab05)
substitute   (gab05, $\partial_{d}{g^{a b}} -> @(gab01)$)     # cdb (gab05.104,gab05)
distribute   (gab05)                                          # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00)   # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01)   # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02)   # cdb (gab02.106,gab02)
```

```
gab03 = impose_rnc (gab03)   # cdb (gab03.106,gab03)
gab04 = impose_rnc (gab04)   # cdb (gab04.106,gab04)
gab05 = impose_rnc (gab05)   # cdb (gab05.106,gab05)
```

$$\texttt{gab00.101} := g^{ab}$$

$$\texttt{gab00.102} := g^{ab}$$

$$\texttt{gab01.101} := -g^{cb}\Gamma^{a}{}_{cd} - g^{ac}\Gamma^{b}{}_{cd}$$

$$\texttt{gab01.102} := 0$$

$$\texttt{gab02.101} := \partial_{e}\left(-g^{cb}\Gamma^{a}{}_{cd} - g^{ac}\Gamma^{b}{}_{cd}\right)$$

$$\texttt{gab02.102} := -\partial_{e}\left(g^{cb}\Gamma^{a}{}_{cd}\right) - \partial_{e}\left(g^{ac}\Gamma^{b}{}_{cd}\right)$$

$$\texttt{gab02.103} := -\partial_{e}g^{cb}\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - \partial_{e}g^{ac}\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab02.104} := -\left(-g^{fb}\Gamma^{c}{}_{fe} - g^{cf}\Gamma^{b}{}_{fe}\right)\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - \left(-g^{fc}\Gamma^{a}{}_{fe} - g^{af}\Gamma^{c}{}_{fe}\right)\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab02.105} := g^{fb}\Gamma^{c}{}_{fe}\Gamma^{a}{}_{cd} + g^{cf}\Gamma^{b}{}_{fe}\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} + g^{fc}\Gamma^{a}{}_{fe}\Gamma^{b}{}_{cd} + g^{af}\Gamma^{c}{}_{fe}\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab02.106} := -g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}$$

$$\texttt{gab03.101} := \partial_{f}\left(g^{gb}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} - g^{cb}\partial_{e}\Gamma^{a}{}_{cd} + g^{gc}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}\right)$$

$$\texttt{gab03.102} := \partial_{f}\left(g^{gb}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd}\right) + \partial_{f}\left(g^{cg}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd}\right) - \partial_{f}\left(g^{cb}\partial_{e}\Gamma^{a}{}_{cd}\right) + \partial_{f}\left(g^{gc}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd}\right) + \partial_{f}\left(g^{ag}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd}\right) - \partial_{f}\left(g^{ac}\partial_{e}\Gamma^{b}{}_{cd}\right)$$

$$\begin{aligned}\texttt{gab03.103} := {}& \partial_{f}g^{gb}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} + \partial_{f}g^{cg}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\partial_{f}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\Gamma^{b}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} - \partial_{f}g^{cb}\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd}\\ &+ \partial_{f}g^{gc}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\partial_{f}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\Gamma^{a}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} + \partial_{f}g^{ag}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} - \partial_{f}g^{ac}\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.104} := {}& \left(-g^{hb}\Gamma^{g}{}_{hf} - g^{gh}\Gamma^{b}{}_{hf}\right)\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} + \left(-g^{hg}\Gamma^{c}{}_{hf} - g^{ch}\Gamma^{g}{}_{hf}\right)\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\partial_{f}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd}\\ &+ g^{cg}\Gamma^{b}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} - \left(-g^{gb}\Gamma^{c}{}_{gf} - g^{cg}\Gamma^{b}{}_{gf}\right)\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} + \left(-g^{hc}\Gamma^{g}{}_{hf} - g^{gh}\Gamma^{c}{}_{hf}\right)\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\partial_{f}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd}\\ &+ g^{gc}\Gamma^{a}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} + \left(-g^{hg}\Gamma^{a}{}_{hf} - g^{ah}\Gamma^{g}{}_{hf}\right)\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} - \left(-g^{gc}\Gamma^{a}{}_{gf} - g^{ag}\Gamma^{c}{}_{gf}\right)\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.105} := {}& -g^{hb}\Gamma^{g}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} - g^{gh}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} - g^{hg}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{g}{}_{hf}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd} + g^{cg}\partial_{f}\Gamma^{b}{}_{ge}\Gamma^{a}{}_{cd}\\ &+ g^{cg}\Gamma^{b}{}_{ge}\partial_{f}\Gamma^{a}{}_{cd} + g^{gb}\Gamma^{c}{}_{gf}\partial_{e}\Gamma^{a}{}_{cd} + g^{cg}\Gamma^{b}{}_{gf}\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{hc}\Gamma^{g}{}_{hf}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} - g^{gh}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\partial_{f}\Gamma^{a}{}_{ge}\Gamma^{b}{}_{cd} + g^{gc}\Gamma^{a}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd}\\ &- g^{hg}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{g}{}_{hf}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\partial_{f}\Gamma^{c}{}_{ge}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{ge}\partial_{f}\Gamma^{b}{}_{cd} + g^{gc}\Gamma^{a}{}_{gf}\partial_{e}\Gamma^{b}{}_{cd} + g^{ag}\Gamma^{c}{}_{gf}\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\end{aligned}$$

$$\texttt{gab03.106} := -g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}$$

$$\texttt{gab04.101} := \partial_g \Big(-g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_f\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_f\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}$$
$$+ g^{ci}\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd} - g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\partial_f\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd}$$
$$- g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_f\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\Big)$$

$$\texttt{gab04.102} := -\partial_g\left(g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}\right) - \partial_g\left(g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}\right) + \partial_g\left(g^{ib}\partial_f\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}\right) + \partial_g\left(g^{ib}\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd}\right) - \partial_g\left(g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\right)$$
$$- \partial_g\left(g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\right) + \partial_g\left(g^{ci}\partial_f\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\right) + \partial_g\left(g^{ci}\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd}\right) + \partial_g\left(g^{ib}\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd}\right) + \partial_g\left(g^{ci}\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd}\right) - \partial_g\left(g^{cb}\partial_{fe}\Gamma^{a}{}_{cd}\right)$$
$$- \partial_g\left(g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}\right) - \partial_g\left(g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}\right) + \partial_g\left(g^{ic}\partial_f\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}\right) + \partial_g\left(g^{ic}\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd}\right) - \partial_g\left(g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}\right)$$
$$- \partial_g\left(g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}\right) + \partial_g\left(g^{ai}\partial_f\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}\right) + \partial_g\left(g^{ai}\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd}\right) + \partial_g\left(g^{ic}\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd}\right) + \partial_g\left(g^{ai}\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd}\right) - \partial_g\left(g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}\right)$$

$$\texttt{gab04.103} := -\partial_g g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\partial_g\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd} - \partial_g g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\partial_g\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}$$
$$- g^{ih}\Gamma^{b}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} - g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd} + \partial_g g^{ib}\partial_f\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd} + g^{ib}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{a}{}_{cd} + \partial_g g^{ib}\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ib}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{a}{}_{cd}$$
$$+ g^{ib}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} - \partial_g g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\partial_g\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\partial_g\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd} - \partial_g g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}$$
$$- g^{ch}\partial_g\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\partial_g\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} - g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd} + \partial_g g^{ci}\partial_f\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_{gf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd} + g^{ci}\partial_f\Gamma^{b}{}_{ie}\partial_g\Gamma^{a}{}_{cd}$$
$$+ \partial_g g^{ci}\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ci}\partial_g\Gamma^{b}{}_{ie}\partial_f\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd} + \partial_g g^{ib}\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ib}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ib}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} + \partial_g g^{ci}\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd}$$
$$+ g^{ci}\partial_g\Gamma^{b}{}_{if}\partial_e\Gamma^{a}{}_{cd} + g^{ci}\Gamma^{b}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd} - \partial_g g^{cb}\partial_{fe}\Gamma^{a}{}_{cd} - g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd} - \partial_g g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\partial_g\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{hc}\Gamma^{i}{}_{hf}\partial_g\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}$$
$$- g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd} - \partial_g g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\partial_g\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\partial_g\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} - g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd} + \partial_g g^{ic}\partial_f\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}$$
$$+ g^{ic}\partial_{gf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd} + g^{ic}\partial_f\Gamma^{a}{}_{ie}\partial_g\Gamma^{b}{}_{cd} + \partial_g g^{ic}\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ic}\partial_g\Gamma^{a}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} - \partial_g g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\partial_g\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}$$
$$- g^{hi}\Gamma^{a}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd} - \partial_g g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\partial_g\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\partial_g\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} - g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd}$$
$$+ \partial_g g^{ai}\partial_f\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd} + g^{ai}\partial_f\Gamma^{c}{}_{ie}\partial_g\Gamma^{b}{}_{cd} + \partial_g g^{ai}\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ai}\partial_g\Gamma^{c}{}_{ie}\partial_f\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd} + \partial_g g^{ic}\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd}$$
$$+ g^{ic}\partial_g\Gamma^{a}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ic}\Gamma^{a}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} + \partial_g g^{ai}\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ai}\partial_g\Gamma^{c}{}_{if}\partial_e\Gamma^{b}{}_{cd} + g^{ai}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd} - \partial_g g^{ac}\partial_{fe}\Gamma^{b}{}_{cd} - g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}$$

$$
\begin{aligned}
\texttt{gab04.104} := & -\left(-g^{jb}\Gamma^{h}{}_{jg}-g^{hj}\Gamma^{b}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{hb}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{hb}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}-\left(-g^{jh}\Gamma^{i}{}_{jg}-g^{ij}\Gamma^{h}{}_{jg}\right)\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}\\
& -g^{ih}\partial_{g}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{ih}\Gamma^{b}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}+\left(-g^{hb}\Gamma^{i}{}_{hg}-g^{ih}\Gamma^{b}{}_{hg}\right)\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}+g^{ib}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}\\
& +g^{ib}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}+\left(-g^{hb}\Gamma^{i}{}_{hg}-g^{ih}\Gamma^{b}{}_{hg}\right)\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ib}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ib}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd}-\left(-g^{ji}\Gamma^{h}{}_{jg}-g^{hj}\Gamma^{i}{}_{jg}\right)\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\\
& -g^{hi}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{hi}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}-\left(-g^{jh}\Gamma^{c}{}_{jg}-g^{cj}\Gamma^{h}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{ch}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\\
& -g^{ch}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}+\left(-g^{hi}\Gamma^{c}{}_{hg}-g^{ch}\Gamma^{i}{}_{hg}\right)\partial_{f}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}+g^{ci}\partial_{gf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}+g^{ci}\partial_{f}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}\\
& +\left(-g^{hi}\Gamma^{c}{}_{hg}-g^{ch}\Gamma^{i}{}_{hg}\right)\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ci}\partial_{g}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ci}\Gamma^{b}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd}+\left(-g^{hb}\Gamma^{i}{}_{hg}-g^{ih}\Gamma^{b}{}_{hg}\right)\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}+g^{ib}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}\\
& +g^{ib}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd}+\left(-g^{hi}\Gamma^{c}{}_{hg}-g^{ch}\Gamma^{i}{}_{hg}\right)\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}+g^{ci}\partial_{g}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}+g^{ci}\Gamma^{b}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd}-\left(-g^{hb}\Gamma^{c}{}_{hg}-g^{ch}\Gamma^{b}{}_{hg}\right)\partial_{fe}\Gamma^{a}{}_{cd}-g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd}\\
& -\left(-g^{jc}\Gamma^{h}{}_{jg}-g^{hj}\Gamma^{c}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{hc}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{hc}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}-\left(-g^{jh}\Gamma^{i}{}_{jg}-g^{ij}\Gamma^{h}{}_{jg}\right)\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}\\
& -g^{ih}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{ih}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}+\left(-g^{hc}\Gamma^{i}{}_{hg}-g^{ih}\Gamma^{c}{}_{hg}\right)\partial_{f}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}+g^{ic}\partial_{gf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}\\
& +g^{ic}\partial_{f}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}+\left(-g^{hc}\Gamma^{i}{}_{hg}-g^{ih}\Gamma^{c}{}_{hg}\right)\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ic}\partial_{g}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ic}\Gamma^{a}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd}-\left(-g^{ji}\Gamma^{h}{}_{jg}-g^{hj}\Gamma^{i}{}_{jg}\right)\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}\\
& -g^{hi}\partial_{g}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{hi}\Gamma^{a}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}-\left(-g^{jh}\Gamma^{a}{}_{jg}-g^{aj}\Gamma^{h}{}_{jg}\right)\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{ah}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}\\
& -g^{ah}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}+\left(-g^{hi}\Gamma^{a}{}_{hg}-g^{ah}\Gamma^{i}{}_{hg}\right)\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}+g^{ai}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}+g^{ai}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}\\
& +\left(-g^{hi}\Gamma^{a}{}_{hg}-g^{ah}\Gamma^{i}{}_{hg}\right)\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ai}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ai}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd}+\left(-g^{hc}\Gamma^{i}{}_{hg}-g^{ih}\Gamma^{c}{}_{hg}\right)\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}+g^{ic}\partial_{g}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}\\
& +g^{ic}\Gamma^{a}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd}+\left(-g^{hi}\Gamma^{a}{}_{hg}-g^{ah}\Gamma^{i}{}_{hg}\right)\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}+g^{ai}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}+g^{ai}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd}-\left(-g^{hc}\Gamma^{a}{}_{hg}-g^{ah}\Gamma^{c}{}_{hg}\right)\partial_{fe}\Gamma^{b}{}_{cd}-g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.105} := & \; g^{jb}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}+g^{hj}\Gamma^{b}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{hb}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{hb}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{hb}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}+g^{jh}\Gamma^{i}{}_{jg}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}\\
& +g^{ij}\Gamma^{h}{}_{jg}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{ih}\partial_{g}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{ih}\Gamma^{b}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{ih}\Gamma^{b}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}-g^{hb}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}-g^{ih}\Gamma^{b}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}\\
& +g^{ib}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{a}{}_{cd}+g^{ib}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}-g^{hb}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}-g^{ih}\Gamma^{b}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ib}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ib}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd}+g^{ji}\Gamma^{h}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\\
& +g^{hj}\Gamma^{i}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{hi}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{hi}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{hi}\Gamma^{c}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}+g^{jh}\Gamma^{c}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}+g^{cj}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\\
& -g^{ch}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{ch}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{ch}\Gamma^{i}{}_{hf}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}-g^{hi}\Gamma^{c}{}_{hg}\partial_{f}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}-g^{ch}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}+g^{ci}\partial_{gf}\Gamma^{b}{}_{ie}\Gamma^{a}{}_{cd}\\
& +g^{ci}\partial_{f}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd}-g^{hi}\Gamma^{c}{}_{hg}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}-g^{ch}\Gamma^{i}{}_{hg}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ci}\partial_{g}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd}+g^{ci}\Gamma^{b}{}_{ie}\partial_{gf}\Gamma^{a}{}_{cd}-g^{hb}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}\\
& -g^{ih}\Gamma^{b}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}+g^{ib}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}+g^{ib}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd}-g^{hi}\Gamma^{c}{}_{hg}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}-g^{ch}\Gamma^{i}{}_{hg}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}+g^{ci}\partial_{g}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd}\\
& +g^{ci}\Gamma^{b}{}_{if}\partial_{ge}\Gamma^{a}{}_{cd}+g^{hb}\Gamma^{c}{}_{hg}\partial_{fe}\Gamma^{a}{}_{cd}+g^{ch}\Gamma^{b}{}_{hg}\partial_{fe}\Gamma^{a}{}_{cd}-g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd}+g^{jc}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}+g^{hj}\Gamma^{c}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{hc}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}\\
& -g^{hc}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{hc}\Gamma^{i}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}+g^{jh}\Gamma^{i}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}+g^{ij}\Gamma^{h}{}_{jg}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{ih}\partial_{g}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{ih}\Gamma^{c}{}_{hf}\partial_{g}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}\\
& -g^{ih}\Gamma^{c}{}_{hf}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}-g^{hc}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}-g^{ih}\Gamma^{c}{}_{hg}\partial_{f}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}+g^{ic}\partial_{gf}\Gamma^{a}{}_{ie}\Gamma^{b}{}_{cd}+g^{ic}\partial_{f}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}-g^{hc}\Gamma^{i}{}_{hg}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}\\
& -g^{ih}\Gamma^{c}{}_{hg}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ic}\partial_{g}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ic}\Gamma^{a}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd}+g^{ji}\Gamma^{h}{}_{jg}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}+g^{hj}\Gamma^{i}{}_{jg}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{hi}\partial_{g}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}\\
& -g^{hi}\Gamma^{a}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{hi}\Gamma^{a}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}+g^{jh}\Gamma^{a}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}+g^{aj}\Gamma^{h}{}_{jg}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{ah}\partial_{g}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{ah}\Gamma^{i}{}_{hf}\partial_{g}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}\\
& -g^{ah}\Gamma^{i}{}_{hf}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}-g^{hi}\Gamma^{a}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}-g^{ah}\Gamma^{i}{}_{hg}\partial_{f}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}+g^{ai}\partial_{gf}\Gamma^{c}{}_{ie}\Gamma^{b}{}_{cd}+g^{ai}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd}-g^{hi}\Gamma^{a}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}\\
& -g^{ah}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ai}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd}+g^{ai}\Gamma^{c}{}_{ie}\partial_{gf}\Gamma^{b}{}_{cd}-g^{hc}\Gamma^{i}{}_{hg}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}-g^{ih}\Gamma^{c}{}_{hg}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}+g^{ic}\partial_{g}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}+g^{ic}\Gamma^{a}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd}\\
& -g^{hi}\Gamma^{a}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}-g^{ah}\Gamma^{i}{}_{hg}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}+g^{ai}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd}+g^{ai}\Gamma^{c}{}_{if}\partial_{ge}\Gamma^{b}{}_{cd}+g^{hc}\Gamma^{a}{}_{hg}\partial_{fe}\Gamma^{b}{}_{cd}+g^{ah}\Gamma^{c}{}_{hg}\partial_{fe}\Gamma^{b}{}_{cd}-g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}
\end{aligned}
$$

$$\texttt{gab04.106} := g^{ib}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ci}\partial_{f}\Gamma^{b}{}_{ie}\partial_{g}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{ie}\partial_{f}\Gamma^{a}{}_{cd} + g^{ib}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} + g^{ci}\partial_{g}\Gamma^{b}{}_{if}\partial_{e}\Gamma^{a}{}_{cd} - g^{cb}\partial_{gfe}\Gamma^{a}{}_{cd}$$
$$+ g^{ic}\partial_{f}\Gamma^{a}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ai}\partial_{f}\Gamma^{c}{}_{ie}\partial_{g}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{ie}\partial_{f}\Gamma^{b}{}_{cd} + g^{ic}\partial_{g}\Gamma^{a}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} + g^{ai}\partial_{g}\Gamma^{c}{}_{if}\partial_{e}\Gamma^{b}{}_{cd} - g^{ac}\partial_{gfe}\Gamma^{b}{}_{cd}$$

```
# prepare first six terms in the Taylor series expansion of g^{ab}(x)

term0:= @(gab00).
distribute (term0)                              # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                              # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                              # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                              # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                              # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                              # cdb(term5.200,term5)

end_stage_1 = time.time()
```

$$\texttt{term0.200} := g^{ab}$$

$$\texttt{term1.200} := 0$$

$$\texttt{term2.200} := -g^{cb}\partial_{e}\Gamma^{a}{}_{cd}A^{d}A^{e} - g^{ac}\partial_{e}\Gamma^{b}{}_{cd}A^{d}A^{e}$$

$$\texttt{term3.200} := -g^{cb}\partial_{fe}\Gamma^{a}{}_{cd}A^{d}A^{e}A^{f} - g^{ac}\partial_{fe}\Gamma^{b}{}_{cd}A^{d}A^{e}A^{f}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json')  # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json')  # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json')  # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json')  # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json')  # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                 # cdb(term2.301,term2)
substitute (term2,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                 # cdb(term2.302,term2)
distribute (term2)                                                                                       # cdb(term2.303,term2)

substitute (term3,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)          # cdb(term3.301,term3)
substitute (term3,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)          # cdb(term3.302,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                 # cdb(term3.303,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                 # cdb(term3.304,term3)
distribute (term3)                                                                                       # cdb(term3.305,term3)

substitute (term4,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)   # cdb(term4.301,term4)
substitute (term4,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)   # cdb(term4.302,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)          # cdb(term4.303,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)          # cdb(term4.304,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                 # cdb(term4.305,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                 # cdb(term4.306,term4)
distribute (term4)                                                                                       # cdb(term4.307,term4)

substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.301,term5)
substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.302,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)   # cdb(term5.303,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)   # cdb(term5.304,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)          # cdb(term5.305,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)          # cdb(term5.306,term5)
```

```
substitute (term5,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)          # cdb(term5.307,term5)
substitute (term5,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)          # cdb(term5.308,term5)
distribute (term5)                                                                                  # cdb(term5.309,term5)

# ---------------------------------------------------------------------------------
# this block only produces formatted output, it is not part of the main computation
# ---------------------------------------------------------------------------------

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
      + (1/1) @(term1)
      + (1/2) @(term2)
      + (1/6) @(term3)
      + (1/24) @(term4)
      + (1/120) @(term5).  # cdb(metric.301,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1).     # zero
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{a} -> x^{a}$)
substitute (Xterm1,$A^{a} -> x^{a}$)
substitute (Xterm2,$A^{a} -> x^{a}$)
substitute (Xterm3,$A^{a} -> x^{a}$)
substitute (Xterm4,$A^{a} -> x^{a}$)
substitute (Xterm5,$A^{a} -> x^{a}$)

# Manipulating these expressions is hampered by the presence of the partial derivative on Rabcd.
# Thus we can't freely rasie/lower indices on the dRabcd terms. But we can do so on the first
# derivatives (since these are evaluated at x=0 where the connection vanishes).
```

```
substitute      (Xterm2,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm2.301,Xterm2)
substitute      (Xterm3,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm3.301,Xterm3)
substitute      (Xterm4,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm4.301,Xterm4)
substitute      (Xterm5,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm5.301,Xterm5)

substitute      (Xterm2,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm2.302,Xterm2)
substitute      (Xterm3,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm3.302,Xterm3)
substitute      (Xterm4,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm4.302,Xterm4)
substitute      (Xterm5,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a}}$)  # cdb(Xterm5.302,Xterm5)

substitute      (Xterm2,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm2.303,Xterm2)
substitute      (Xterm3,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm3.303,Xterm3)
substitute      (Xterm4,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm4.303,Xterm4)
substitute      (Xterm5,$g^{a b} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm5.303,Xterm5)

substitute      (Xterm2,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm2.304,Xterm2)
substitute      (Xterm3,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm3.304,Xterm3)
substitute      (Xterm4,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm4.304,Xterm4)
substitute      (Xterm5,$g^{b a} \partial_{c}{R^{d}_{e f b}} -> \partial_{c}{R^{d}_{e f}^{a}$)  # cdb(Xterm5.304,Xterm5)

sort_product    (Xterm2)  # cdb(Xterm2.305,Xterm2)
sort_product    (Xterm3)  # cdb(Xterm3.305,Xterm3)
sort_product    (Xterm4)  # cdb(Xterm4.305,Xterm4)
sort_product    (Xterm5)  # cdb(Xterm5.305,Xterm5)

rename_dummies  (Xterm2)  # cdb(Xterm2.306,Xterm2)
rename_dummies  (Xterm3)  # cdb(Xterm3.306,Xterm3)
rename_dummies  (Xterm4)  # cdb(Xterm4.306,Xterm4)
rename_dummies  (Xterm5)  # cdb(Xterm5.306,Xterm5)

canonicalise    (Xterm2)  # cdb(Xterm2.307,Xterm2)
canonicalise    (Xterm3)  # cdb(Xterm3.307,Xterm3)
canonicalise    (Xterm4)  # cdb(Xterm4.307,Xterm4)
canonicalise    (Xterm5)  # cdb(Xterm5.307,Xterm5)

# We can simplify Xterm2 and Xterm3 by careful juggling of the indices (swapping free indices on selected terms)
```

```
tmp = add_tags (Xterm2,'\\mu')       # cdb (tmp.001,tmp)
zoom (tmp, $\mu_{1} Q??$)            # cdb (tmp.002,tmp)
substitute (tmp, $R^{b}_{c d}^{a} x^{c} x^{d} -> R^{a}_{c d}^{b} x^{c} x^{d}$)  # cdb (tmp.003,tmp)
unzoom (tmp)
Xterm2 = clear_tags (tmp,'\\mu')     # cdb (Xterm2.401,Xterm2)

tmp = add_tags (Xterm3,'\\mu')       # cdb (tmp.011,tmp)
zoom (tmp, $\mu_{1} Q??$)            # cdb (tmp.012,tmp)
substitute (tmp, $\partial_{c}{R^{b}_{d e}^{a}} x^{c} x^{d} x^{e} -> \partial_{c}{R^{a}_{d e}^{b}} x^{c} x^{d} x^{e}$)  # cdb (tmp.013,
unzoom (tmp)
Xterm3 = clear_tags (tmp,'\\mu')     # cdb (Xterm3.401,Xterm3)

Xterm0 = reformat_xterm (Xterm0,  1)    # cdb(Xterm0.308,Xterm0)
Xterm2 = reformat_xterm (Xterm2,  3)    # cdb(Xterm2.308,Xterm2)
Xterm3 = reformat_xterm (Xterm3,  6)    # cdb(Xterm3.308,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)    # cdb(Xterm4.308,Xterm4)
Xterm5 = reformat_xterm (Xterm5,360)    # cdb(Xterm5.308,Xterm5)

# metric to 4th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).                                      # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3).                          # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).              # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (Metric6.301,Metric6)

# -----------------------------------------------------------------------------------
# end of format block
# -----------------------------------------------------------------------------------

end_stage_2 = time.time()
```

$$\texttt{term2.301} := -g^{cb}\partial_e\Gamma^a{}_{cd}A^dA^e - g^{ac}\partial_e\Gamma^b{}_{cd}A^dA^e$$

$$\texttt{term2.302} := -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$\texttt{term2.303} := -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec}$$

$$\texttt{term3.301} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.302} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.303} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.304} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\texttt{term3.305} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}$$

$$\begin{aligned}\texttt{term4.301} := {} & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)\end{aligned}$$

$$\begin{aligned}\texttt{term4.302} := {} & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)\end{aligned}$$

$$
\begin{aligned}
\texttt{term4.303} := {} & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\
& + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\
& + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\
& + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.304} := {} & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\
& + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\
& + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\
& + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.305} := {} & g^{ib}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b{}_{ie}\partial_g\Gamma^a{}_{cd}A^dA^eA^fA^g \\
& + g^{ci}\partial_g\Gamma^b{}_{ie}\partial_f\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b{}_{if}\partial_e\Gamma^a{}_{cd}A^dA^eA^fA^g \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + g^{ic}\partial_f\Gamma^a{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g \\
& + g^{ic}\partial_g\Gamma^a{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c{}_{ie}\partial_g\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c{}_{ie}\partial_f\Gamma^b{}_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g \\
& + g^{ai}\partial_g\Gamma^c{}_{if}\partial_e\Gamma^b{}_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.306} := {} & \frac{1}{9}g^{ib}A^eA^fR^c{}_{efi}A^dA^gR^a{}_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c{}_{egi}A^dA^fR^a{}_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b{}_{efi}A^dA^gR^a{}_{dgc} \\
& + \frac{1}{9}g^{ci}A^eA^gR^b{}_{egi}A^dA^fR^a{}_{dfc} + \frac{1}{9}g^{ib}A^fA^gR^c{}_{fgi}A^dA^eR^a{}_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b{}_{fgi}A^dA^eR^a{}_{dec} \\
& - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc}\right) + \frac{1}{9}g^{ic}A^eA^fR^a{}_{efi}A^dA^gR^b{}_{dgc} \\
& + \frac{1}{9}g^{ic}A^eA^gR^a{}_{egi}A^dA^fR^b{}_{dfc} + \frac{1}{9}g^{ai}A^eA^fR^c{}_{efi}A^dA^gR^b{}_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c{}_{egi}A^dA^fR^b{}_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a{}_{fgi}A^dA^eR^b{}_{dec} \\
& + \frac{1}{9}g^{ai}A^fA^gR^c{}_{fgi}A^dA^eR^b{}_{dec} - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b{}_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.307} :=\ & \frac{1}{9} g^{ib} A^e A^f R^c{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c{}_{egi} A^d A^f R^a{}_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ci} A^e A^g R^b{}_{egi} A^d A^f R^a{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^f A^g R^c{}_{fgi} A^d A^e R^a{}_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b{}_{fgi} A^d A^e R^a{}_{dec} - \frac{3}{5} g^{cb} A^d A^g A^f A^e \partial_{ef} R^a{}_{dgc} + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} \\
& + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9} g^{ic} A^e A^f R^a{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ic} A^e A^g R^a{}_{egi} A^d A^f R^b{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^e A^f R^c{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c{}_{egi} A^d A^f R^b{}_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a{}_{fgi} A^d A^e R^b{}_{dec} + \frac{1}{9} g^{ai} A^f A^g R^c{}_{fgi} A^d A^e R^b{}_{dec} \\
& - \frac{3}{5} g^{ac} A^d A^g A^f A^e \partial_{ef} R^b{}_{dgc} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc}
\end{aligned}
$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f \left(7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg}\right)$$

$$\begin{aligned} g^{ab}(x) = g^{ab} &- \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f \left(7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg}\right) \\ &+ \frac{1}{360}x^c x^d x^e x^f x^g \left(3R^a{}_{cdh}\partial_e R^h{}_{fg}{}^b + 4\partial_c R^a{}_{deh}R^h{}_{fg}{}^b + 5\partial_c R^b{}_{deh}R^a{}_{fg}{}^h + 5\partial_c R^a{}_{deh}R^b{}_{fg}{}^h - 2g^{bh}\partial_{cde}R^a{}_{fgh} + 3R^b{}_{cdh}\partial_e R^h{}_{fg}{}^a + 4\partial_c R^b{}_{deh}R^h{}_{fg}{}^a \right. \\ &\left. - 2g^{ah}\partial_{cde}R^b{}_{fgh}\right) \end{aligned}$$

Stage 3: Replace partial derivs of R with covariant derivs of R

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json')  # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json')  # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json')  # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)   # cdb(term1.401,term1)
sort_product     (term1)   # cdb(term1.402,term1)
rename_dummies   (term1)   # cdb(term1.403,term1)
canonicalise     (term1)   # cdb(term1.404,term1)

eliminate_metric (term2)   # cdb(term2.401,term2)
sort_product     (term2)   # cdb(term2.402,term2)
rename_dummies   (term2)   # cdb(term2.403,term2)
canonicalise     (term2)   # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)         # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)         # cdb(term3.402,term3)
distribute (term3)                                                                                # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)  # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)  # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)         # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)         # cdb(term4.404,term4)
distribute (term4)                                                                                # cdb(term4.405,term4)
```

```
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c d b}} ->   @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c b d}} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()
```

$$\texttt{tmp01.403} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\texttt{tmp02.403} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\texttt{tmp03.403} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}$$

$$\texttt{term1.401} := 0$$
$$\texttt{term1.402} := 0$$
$$\texttt{term1.403} := 0$$
$$\texttt{term1.404} := 0$$

$$\texttt{term2.401} := -\frac{1}{3}A^dA^eR^a{}_{de}{}^b - \frac{1}{3}A^dA^eR^b{}_{de}{}^a$$
$$\texttt{term2.402} := -\frac{1}{3}A^dA^eR^a{}_{de}{}^b - \frac{1}{3}A^dA^eR^b{}_{de}{}^a$$
$$\texttt{term2.403} := -\frac{1}{3}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}A^cA^dR^b{}_{cd}{}^a$$
$$\texttt{term2.404} := -\frac{1}{3}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}A^cA^dR^b{}_{cd}{}^a$$

$$\texttt{term3.401} := -\frac{1}{2}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g^{ag} - \frac{1}{2}g^{ac}A^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$
$$\texttt{term3.402} := -\frac{1}{2}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g^{ag} - \frac{1}{2}g^{ac}A^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$
$$\texttt{term3.403} := -\frac{1}{2}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g^{ag} - \frac{1}{2}g^{ac}A^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$

$$\begin{aligned}\texttt{term4.401} := {} & \frac{1}{9}g^{ib}A^eA^fR^c{}_{efi}A^dA^gR^a{}_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c{}_{egi}A^dA^fR^a{}_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b{}_{efi}A^dA^gR^a{}_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b{}_{egi}A^dA^fR^a{}_{dfc} \\ & + \frac{1}{9}g^{ib}A^fA^gR^c{}_{fgi}A^dA^eR^a{}_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b{}_{fgi}A^dA^eR^a{}_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{ah} \\ & + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a{}_{gfh}R^h{}_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a{}_{geh}R^h{}_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a{}_{efi}A^dA^gR^b{}_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a{}_{egi}A^dA^fR^b{}_{dfc} \\ & + \frac{1}{9}g^{ai}A^eA^fR^c{}_{efi}A^dA^gR^b{}_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c{}_{egi}A^dA^fR^b{}_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a{}_{fgi}A^dA^eR^b{}_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c{}_{fgi}A^dA^eR^b{}_{dec} \\ & - \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b{}_{gfh}R^h{}_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b{}_{geh}R^h{}_{dfc}\end{aligned}$$

$$
\begin{aligned}
\texttt{term4.402} := {} & \frac{1}{9} g^{ib} A^e A^f R^c{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c{}_{egi} A^d A^f R^a{}_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ci} A^e A^g R^b{}_{egi} A^d A^f R^a{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^f A^g R^c{}_{fgi} A^d A^e R^a{}_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b{}_{fgi} A^d A^e R^a{}_{dec} - \frac{3}{5} g^{cb} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9} g^{ic} A^e A^f R^a{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ic} A^e A^g R^a{}_{egi} A^d A^f R^b{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^e A^f R^c{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c{}_{egi} A^d A^f R^b{}_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a{}_{fgi} A^d A^e R^b{}_{dec} + \frac{1}{9} g^{ai} A^f A^g R^c{}_{fgi} A^d A^e R^b{}_{dec} \\
& - \frac{3}{5} g^{ac} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.403} := {} & \frac{1}{9} g^{ib} A^e A^f R^c{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c{}_{egi} A^d A^f R^a{}_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ci} A^e A^g R^b{}_{egi} A^d A^f R^a{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^f A^g R^c{}_{fgi} A^d A^e R^a{}_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b{}_{fgi} A^d A^e R^a{}_{dec} - \frac{3}{5} g^{cb} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9} g^{ic} A^e A^f R^a{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ic} A^e A^g R^a{}_{egi} A^d A^f R^b{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^e A^f R^c{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c{}_{egi} A^d A^f R^b{}_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a{}_{fgi} A^d A^e R^b{}_{dec} + \frac{1}{9} g^{ai} A^f A^g R^c{}_{fgi} A^d A^e R^b{}_{dec} \\
& - \frac{3}{5} g^{ac} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.404} := {} & \frac{1}{9} g^{ib} A^e A^f R^c{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c{}_{egi} A^d A^f R^a{}_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ci} A^e A^g R^b{}_{egi} A^d A^f R^a{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^f A^g R^c{}_{fgi} A^d A^e R^a{}_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b{}_{fgi} A^d A^e R^a{}_{dec} - \frac{3}{5} g^{cb} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9} g^{ic} A^e A^f R^a{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ic} A^e A^g R^a{}_{egi} A^d A^f R^b{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^e A^f R^c{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c{}_{egi} A^d A^f R^b{}_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a{}_{fgi} A^d A^e R^b{}_{dec} + \frac{1}{9} g^{ai} A^f A^g R^c{}_{fgi} A^d A^e R^b{}_{dec} \\
& - \frac{3}{5} g^{ac} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.405} := {} & \frac{1}{9} g^{ib} A^e A^f R^c{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c{}_{egi} A^d A^f R^a{}_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b{}_{efi} A^d A^g R^a{}_{dgc} + \frac{1}{9} g^{ci} A^e A^g R^b{}_{egi} A^d A^f R^a{}_{dfc} \\
& + \frac{1}{9} g^{ib} A^f A^g R^c{}_{fgi} A^d A^e R^a{}_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b{}_{fgi} A^d A^e R^a{}_{dec} - \frac{3}{5} g^{cb} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{ah} \\
& + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{cb} A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9} g^{ic} A^e A^f R^a{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ic} A^e A^g R^a{}_{egi} A^d A^f R^b{}_{dfc} \\
& + \frac{1}{9} g^{ai} A^e A^f R^c{}_{efi} A^d A^g R^b{}_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c{}_{egi} A^d A^f R^b{}_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a{}_{fgi} A^d A^e R^b{}_{dec} + \frac{1}{9} g^{ai} A^f A^g R^c{}_{fgi} A^d A^e R^b{}_{dec} \\
& - \frac{3}{5} g^{ac} A^d A^g A^e A^f \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15} g^{ac} A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc}
\end{aligned}
$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

```
beg_stage_4 = time.time()

# final housekeeping

# lower the ^{ab} indices to _{uv}

tmp0 := g_{a u} g_{b v} @(term0).
tmp1 := g_{a u} g_{b v} @(term1).
tmp2 := g_{a u} g_{b v} @(term2).
tmp3 := g_{a u} g_{b v} @(term3).
tmp4 := g_{a u} g_{b v} @(term4).
tmp5 := g_{a u} g_{b v} @(term5).

distribute          (tmp1)  # cdb(tmp1.501,tmp1)
eliminate_metric    (tmp1)  # cdb(tmp1.502,tmp1)
eliminate_kronecker (tmp1)  # cdb(tmp1.503,tmp1)
tmp1 = flatten_Rabcd (tmp1)
canonicalise        (tmp1)  # cdb(tmp1.506,tmp1)

distribute          (tmp2)  # cdb(tmp2.501,tmp2)
eliminate_metric    (tmp2)  # cdb(tmp2.502,tmp2)
eliminate_kronecker (tmp2)  # cdb(tmp2.503,tmp2)
tmp2 = flatten_Rabcd (tmp2)
canonicalise        (tmp2)  # cdb(tmp2.506,tmp2)

distribute          (tmp3)  # cdb(tmp3.501,tmp3)
eliminate_metric    (tmp3)  # cdb(tmp3.502,tmp3)
eliminate_kronecker (tmp3)  # cdb(tmp3.503,tmp3)
tmp3 = flatten_Rabcd (tmp3)
canonicalise        (tmp3)  # cdb(tmp3.506,tmp3)

distribute          (tmp4)  # cdb(tmp4.501,tmp4)
eliminate_metric    (tmp4)  # cdb(tmp4.502,tmp4)
eliminate_kronecker (tmp4)  # cdb(tmp4.503,tmp4)
tmp4 = flatten_Rabcd (tmp4)
canonicalise        (tmp4)  # cdb(tmp4.506,tmp4)
```

```
distribute           (tmp5)  # cdb(tmp5.501,tmp5)
eliminate_metric     (tmp5)  # cdb(tmp5.502,tmp5)
eliminate_kronecker  (tmp5)  # cdb(tmp5.503,tmp5)
tmp5 = flatten_Rabcd (tmp5)
canonicalise         (tmp5)  # cdb(tmp5.506,tmp5)

# this is out final answer

# raise the _{uv} indices to ^{ab}

metric:= g^{a u} g^{b v} (  @(tmp0)
                          + (1/1) @(tmp1)
                          + (1/2) @(tmp2)
                          + (1/6) @(tmp3)
                          + (1/24) @(tmp4)
                          + (1/120) @(tmp5) ).  # cdb(metric.500,metric)

distribute             (metric)   # cdb(metric.501,metric)
eliminate_metric       (metric)   # cdb(metric.502,metric)
eliminate_kronecker    (metric)   # cdb(metric.503,metric)
metric = flatten_Rabcd (metric)   # cdb(metric.504,metric)
canonicalise           (metric)   # cdb(metric.505,metric)

substitute          (metric,$g_{a b} g^{b c} -> g_{a}^{c}$)
substitute          (metric,$g_{b a} g^{b c} -> g_{a}^{c}$)
substitute          (metric,$g_{b a} g^{c b} -> g_{a}^{c}$)
substitute          (metric,$g_{a b} g^{c b} -> g_{a}^{c}$)
eliminate_kronecker (metric)   # cdb(metric.506,metric)
canonicalise        (metric)   # cdb(metric.507,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.508,metric)

cdblib.create ('metric-inv.json')

cdblib.put ('g^ab',metric,'metric-inv.json')

# extract the terms of the metric in powers of x
```

```
term0 = get_xterm (metric,0)   # cdb(term0.501,term0)
term1 = get_xterm (metric,1)   # cdb(term1.501,term1)
term2 = get_xterm (metric,2)   # cdb(term2.501,term2)
term3 = get_xterm (metric,3)   # cdb(term3.501,term3)
term4 = get_xterm (metric,4)   # cdb(term4.501,term4)
term5 = get_xterm (metric,5)   # cdb(term5.501,term5)

cdblib.put ('g^ab_0',term0,'metric-inv.json')
cdblib.put ('g^ab_1',term1,'metric-inv.json')
cdblib.put ('g^ab_2',term2,'metric-inv.json')
cdblib.put ('g^ab_3',term3,'metric-inv.json')
cdblib.put ('g^ab_4',term4,'metric-inv.json')
cdblib.put ('g^ab_5',term5,'metric-inv.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3).  # cdb(metric4.501,metric4)
```

$\texttt{term0.501} := g^{ab}$

$\texttt{term1.501} := 0$

$\texttt{term2.501} := \frac{1}{3} x^c x^d R_{cedf} g^{ae} g^{bf}$

$\texttt{term3.501} := \frac{1}{6} x^c x^d x^e \nabla_c R_{dfeg} g^{af} g^{bg}$

$\texttt{term4.501} := \frac{1}{15} x^c x^d x^e x^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20} x^c x^d x^e x^f \nabla_{cd} R_{egfh} g^{ag} g^{bh}$

$\texttt{term5.501} := \frac{1}{30} x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30} x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90} x^c x^d x^e x^f x^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}$

$$\texttt{tmp2.501} := -\frac{1}{3}g_{au}g_{bv}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}g_{au}g_{bv}A^cA^dR^b{}_{cd}{}^a$$

$$\texttt{tmp2.502} := -\frac{1}{3}g_{bv}A^cA^dR_{ucd}{}^b - \frac{1}{3}g_{bv}A^cA^dR^b{}_{cdu}$$

$$\texttt{tmp2.503} := -\frac{1}{3}g_{bv}A^cA^dR_{ucd}{}^b - \frac{1}{3}g_{bv}A^cA^dR^b{}_{cdu}$$

$$\texttt{tmp2.506} := -\frac{2}{3}A^aA^bR_{uabc}g_{vd}g^{cd}$$

$$\texttt{tmp3.501} := -\frac{1}{2}g_{au}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g^{ag} - \frac{1}{2}g_{au}g_{bv}g^{ac}A^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$

$$\texttt{tmp3.502} := -\frac{1}{2}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g_u{}^g - \frac{1}{2}g_{bv}g_u{}^cA^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$

$$\texttt{tmp3.503} := -\frac{1}{2}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfeu} - \frac{1}{2}g_{bv}A^dA^fA^e\nabla_dR_{ufeg}g^{bg}$$

$$\texttt{tmp3.506} := -A^aA^bA^c\nabla_aR_{ubcd}g_{ve}g^{de}$$

$$
\begin{aligned}
\texttt{tmp4.501} := {} & \frac{1}{9} g_{au} g_{bv} g^{ib} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} + \frac{1}{9} g_{au} g_{bv} g^{ib} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} + \frac{1}{9} g_{au} g_{bv} g^{ci} A^{e} A^{f} R^{b}{}_{efi} A^{d} A^{g} R^{a}{}_{dgc} \\
& + \frac{1}{9} g_{au} g_{bv} g^{ci} A^{e} A^{g} R^{b}{}_{egi} A^{d} A^{f} R^{a}{}_{dfc} + \frac{1}{9} g_{au} g_{bv} g^{ib} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} + \frac{1}{9} g_{au} g_{bv} g^{ci} A^{f} A^{g} R^{b}{}_{fgi} A^{d} A^{e} R^{a}{}_{dec} \\
& - \frac{3}{5} g_{au} g_{bv} g^{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{ah} + \frac{1}{15} g_{au} g_{bv} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g_{au} g_{bv} g^{cb} A^{d} A^{g} A^{f} A^{e} R^{a}{}_{geh} R^{h}{}_{dfc} \\
& + \frac{1}{9} g_{au} g_{bv} g^{ic} A^{e} A^{f} R^{a}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g_{au} g_{bv} g^{ic} A^{e} A^{g} R^{a}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g_{au} g_{bv} g^{ai} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} \\
& + \frac{1}{9} g_{au} g_{bv} g^{ai} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g_{au} g_{bv} g^{ic} A^{f} A^{g} R^{a}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} + \frac{1}{9} g_{au} g_{bv} g^{ai} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} \\
& - \frac{3}{5} g_{au} g_{bv} g^{ac} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g_{au} g_{bv} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g_{au} g_{bv} g^{ac} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{geh} R^{h}{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{tmp4.502} := {} & \frac{1}{9} g_{bv} g^{ib} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R_{udgc} + \frac{1}{9} g_{bv} g^{ib} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R_{udfc} + \frac{1}{9} g_{bv} g^{ci} A^{e} A^{f} R^{b}{}_{efi} A^{d} A^{g} R_{udgc} + \frac{1}{9} g_{bv} g^{ci} A^{e} A^{g} R^{b}{}_{egi} A^{d} A^{f} R_{udfc} \\
& + \frac{1}{9} g_{bv} g^{ib} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R_{udec} + \frac{1}{9} g_{bv} g^{ci} A^{f} A^{g} R^{b}{}_{fgi} A^{d} A^{e} R_{udec} - \frac{3}{5} g_{bv} g^{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g_{u}{}^{h} \\
& + \frac{1}{15} g_{bv} g^{cb} A^{d} A^{g} A^{f} A^{e} R_{ugfh} R^{h}{}_{dec} + \frac{1}{15} g_{bv} g^{cb} A^{d} A^{g} A^{f} A^{e} R_{ugeh} R^{h}{}_{dfc} + \frac{1}{9} g_{bv} g^{ic} A^{e} A^{f} R_{uefi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g_{bv} g^{ic} A^{e} A^{g} R_{uegi} A^{d} A^{f} R^{b}{}_{dfc} \\
& + \frac{1}{9} g_{bv} g_{u}{}^{i} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g_{bv} g_{u}{}^{i} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g_{bv} g^{ic} A^{f} A^{g} R_{ufgi} A^{d} A^{e} R^{b}{}_{dec} + \frac{1}{9} g_{bv} g_{u}{}^{i} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R^{b}{}_{dec} \\
& - \frac{3}{5} g_{bv} g_{u}{}^{c} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefh} g^{bh} + \frac{1}{15} g_{bv} g_{u}{}^{c} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{gfh} R^{h}{}_{dec} + \frac{1}{15} g_{bv} g_{u}{}^{c} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{geh} R^{h}{}_{dfc}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{tmp4.503} := {} & \frac{1}{9} g_{bv} g^{ib} A^{e} A^{f} R^{c}{}_{efi} A^{d} A^{g} R_{udgc} + \frac{1}{9} g_{bv} g^{ib} A^{e} A^{g} R^{c}{}_{egi} A^{d} A^{f} R_{udfc} + \frac{1}{9} g_{bv} g^{ci} A^{e} A^{f} R^{b}{}_{efi} A^{d} A^{g} R_{udgc} + \frac{1}{9} g_{bv} g^{ci} A^{e} A^{g} R^{b}{}_{egi} A^{d} A^{f} R_{udfc} \\
& + \frac{1}{9} g_{bv} g^{ib} A^{f} A^{g} R^{c}{}_{fgi} A^{d} A^{e} R_{udec} + \frac{1}{9} g_{bv} g^{ci} A^{f} A^{g} R^{b}{}_{fgi} A^{d} A^{e} R_{udec} - \frac{3}{5} g_{bv} g^{cb} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{cefu} \\
& + \frac{1}{15} g_{bv} g^{cb} A^{d} A^{g} A^{f} A^{e} R_{ugfh} R^{h}{}_{dec} + \frac{1}{15} g_{bv} g^{cb} A^{d} A^{g} A^{f} A^{e} R_{ugeh} R^{h}{}_{dfc} + \frac{1}{9} g_{bv} g^{ic} A^{e} A^{f} R_{uefi} A^{d} A^{g} R^{b}{}_{dgc} \\
& + \frac{1}{9} g_{bv} g^{ic} A^{e} A^{g} R_{uegi} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g_{bv} A^{e} A^{f} R^{c}{}_{efu} A^{d} A^{g} R^{b}{}_{dgc} + \frac{1}{9} g_{bv} A^{e} A^{g} R^{c}{}_{egu} A^{d} A^{f} R^{b}{}_{dfc} + \frac{1}{9} g_{bv} g^{ic} A^{f} A^{g} R_{ufgi} A^{d} A^{e} R^{b}{}_{dec} \\
& + \frac{1}{9} g_{bv} A^{f} A^{g} R^{c}{}_{fgu} A^{d} A^{e} R^{b}{}_{dec} - \frac{3}{5} g_{bv} A^{d} A^{g} A^{e} A^{f} \nabla_{dg} R_{uefh} g^{bh} + \frac{1}{15} g_{bv} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{gfh} R^{h}{}_{deu} + \frac{1}{15} g_{bv} A^{d} A^{g} A^{f} A^{e} R^{b}{}_{geh} R^{h}{}_{dfu}
\end{aligned}
$$

$$
\texttt{tmp4.506} := -\frac{8}{5} A^{a} A^{b} A^{c} A^{d} R_{uabe} R_{cfdg} g_{vh} g^{ef} g^{gh} - \frac{6}{5} A^{a} A^{b} A^{c} A^{d} \nabla_{ab} R_{ucde} g_{vf} g^{ef}
$$

$$\texttt{tmp5.506} := -4A^aA^bA^cA^dA^eR_{uabf}\nabla_cR_{dgeh}g_{vi}g^{fg}g^{hi} - 4A^aA^bA^cA^dA^eR_{afbg}\nabla_cR_{udeh}g_{vi}g^{fh}g^{gi} - \frac{4}{3}A^aA^bA^cA^dA^e\nabla_{abc}R_{udef}g_{vg}g^{fg}$$

$$
\begin{aligned}
\texttt{metric.500} := g^{au} g^{bv} \Big(& g_{cu} g_{dv} g^{cd} - \frac{1}{3} A^e A^f R_{uefc} g_{vd} g^{cd} - \frac{1}{6} A^f A^g A^c \nabla_f R_{ugcd} g_{ve} g^{de} - \frac{1}{15} A^i A^j A^c A^d R_{uije} R_{cfdg} g_{vh} g^{ef} g^{gh} - \frac{1}{20} A^i A^j A^c A^d \nabla_{ij} R_{ucde} g_{vf} g^{ef} \\
& - \frac{1}{30} A^j A^k A^c A^d A^e R_{ujkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} - \frac{1}{30} A^j A^k A^c A^d A^e R_{jfkg} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} A^j A^k A^c A^d A^e \nabla_{jkc} R_{udef} g_{vg} g^{fg} \Big)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{metric.501} := {} & g^{au} g^{bv} g_{cu} g_{dv} g^{cd} - \frac{1}{3} g^{au} g^{bv} A^e A^f R_{uefc} g_{vd} g^{cd} - \frac{1}{6} g^{au} g^{bv} A^f A^g A^c \nabla_f R_{ugcd} g_{ve} g^{de} - \frac{1}{15} g^{au} g^{bv} A^i A^j A^c A^d R_{uije} R_{cfdg} g_{vh} g^{ef} g^{gh} \\
& - \frac{1}{20} g^{au} g^{bv} A^i A^j A^c A^d \nabla_{ij} R_{ucde} g_{vf} g^{ef} - \frac{1}{30} g^{au} g^{bv} A^j A^k A^c A^d A^e R_{ujkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\
& - \frac{1}{30} g^{au} g^{bv} A^j A^k A^c A^d A^e R_{jfkg} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{au} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R_{udef} g_{vg} g^{fg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{metric.502} := {} & g^{bv} g_c{}^a g_{dv} g^{cd} - \frac{1}{3} g^{bv} A^e A^f R^a{}_{efc} g_{vd} g^{cd} - \frac{1}{6} g^{bv} A^f A^g A^c \nabla_f R^a{}_{gcd} g_{ve} g^{de} - \frac{1}{15} g^{bv} A^i A^j A^c A^d R^a{}_{ije} R_{cfdg} g_{vh} g^{ef} g^{gh} \\
& - \frac{1}{20} g^{bv} A^i A^j A^c A^d \nabla_{ij} R^a{}_{cde} g_{vf} g^{ef} - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R^a{}_{jkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\
& - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jfkg} \nabla_c R^a{}_{deh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R^a{}_{def} g_{vg} g^{fg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{metric.503} := {} & g^{bv} g_{dv} g^{ad} - \frac{1}{3} g^{bv} A^e A^f R^a{}_{efc} g_{vd} g^{cd} - \frac{1}{6} g^{bv} A^f A^g A^c \nabla_f R^a{}_{gcd} g_{ve} g^{de} - \frac{1}{15} g^{bv} A^i A^j A^c A^d R^a{}_{ije} R_{cfdg} g_{vh} g^{ef} g^{gh} \\
& - \frac{1}{20} g^{bv} A^i A^j A^c A^d \nabla_{ij} R^a{}_{cde} g_{vf} g^{ef} - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R^a{}_{jkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\
& - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jfkg} \nabla_c R^a{}_{deh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R^a{}_{def} g_{vg} g^{fg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{metric.504} := {} & g^{bc} g_{dc} g^{ad} - \frac{1}{3} A^c A^d R_{ecdf} g^{ae} g^{bg} g_{gh} g^{fh} - \frac{1}{6} A^c A^d A^e \nabla_d R_{fecg} g^{af} g^{bh} g_{hi} g^{gi} - \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{iefj} g^{ai} g^{bk} g_{kl} g^{jg} g^{hl} \\
& - \frac{1}{20} A^c A^d A^e A^f \nabla_{ef} R_{gcdh} g^{ag} g^{bi} g_{ij} g^{hj} - \frac{1}{30} A^c A^d A^e A^f A^g R_{hfgi} \nabla_c R_{djek} g^{ah} g^{bl} g_{lm} g^{ij} g^{km} \\
& - \frac{1}{30} A^c A^d A^e A^f A^g R_{fhgi} \nabla_c R_{jdek} g^{aj} g^{bl} g_{lm} g^{hk} g^{im} - \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{fgc} R_{hdei} g^{ah} g^{bj} g_{jk} g^{ik}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{metric.505} := {} & g^{ac} g_{cd} g^{bd} + \frac{1}{3} A^c A^d R_{cedf} g^{ae} g^{bg} g_{gh} g^{fh} + \frac{1}{6} A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{bh} g_{hi} g^{gi} + \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{eifj} g^{ag} g^{bk} g_{kl} g^{hi} g^{jl} \\
& + \frac{1}{20} A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{bi} g_{ij} g^{hj} + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bl} g_{lm} g^{ij} g^{km} \\
& + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bl} g_{lm} g^{hk} g^{im} + \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{bj} g_{jk} g^{ik}
\end{aligned}
$$

$$\texttt{metric.506} := g^{ba} + \frac{1}{3}A^c A^d R_{cedf} g^{ae} g^{fb} + \frac{1}{6}A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{gb} + \frac{1}{15}A^c A^d A^e A^f R_{cgdh} R_{eifj} g^{ag} g^{hi} g^{jb} + \frac{1}{20}A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{hb} + \frac{1}{30}A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{ij} g^{kb} + \frac{1}{30}A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{hk} g^{ib} + \frac{1}{90}A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{ib}$$

$$\texttt{metric.507} := g^{ab} + \frac{1}{3}A^c A^d R_{cedf} g^{ae} g^{bf} + \frac{1}{6}A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{bg} + \frac{1}{15}A^c A^d A^e A^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20}A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{bh} + \frac{1}{30}A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30}A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90}A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}$$

$$\texttt{metric.508} := g^{ab} + \frac{1}{3}x^c x^d R_{cedf} g^{ae} g^{bf} + \frac{1}{6}x^c x^d x^e \nabla_c R_{dfeg} g^{af} g^{bg} + \frac{1}{15}x^c x^d x^e x^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{egfh} g^{ag} g^{bh} + \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}$$

```
Rterm0 := @(term0).
Rterm1 := @(term1).  # zero
Rterm2 := @(term2).
Rterm3 := @(term3).
Rterm4 := @(term4).
Rterm5 := @(term5).

Rterm0 = reformat_xterm (Rterm0,  1)    # cdb(Rterm0.601,Rterm0)
Rterm2 = reformat_xterm (Rterm2,  3)    # cdb(Rterm2.601,Rterm2)
Rterm3 = reformat_xterm (Rterm3,  6)    # cdb(Rterm3.601,Rterm3)
Rterm4 = reformat_xterm (Rterm4, 60)    # cdb(Rterm4.601,Rterm4)
Rterm5 = reformat_xterm (Rterm5, 90)    # cdb(Rterm5.601,Rterm5)

Metric := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Rterm0,  1)    # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Rterm2,  3)    # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Rterm3,  6)    # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Rterm4, 60)    # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Rterm5, 90)    # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()
```

The inverse metric in Riemann normal coordinates

$$g^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \frac{1}{60}x^c x^d x^e x^f \left(4g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right)$$
$$+ \frac{1}{90}x^c x^d x^e x^f x^g \left(3g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right) + \mathcal{O}\left(\epsilon^6\right)$$

Curvature expansion of the inverse metric

$$g^{ab}(x) = \overset{0}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} + \overset{5}{g}{}^{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{g}{}^{ab} = g^{ab}$$

$$3\overset{2}{g}{}^{ab} = x^c x^d g^{ae} g^{bf} R_{cedf}$$

$$6\overset{3}{g}{}^{ab} = x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg}$$

$$60\overset{4}{g}{}^{ab} = x^c x^d x^e x^f \left(4 g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3 g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right)$$

$$90\overset{5}{g}{}^{ab} = x^c x^d x^e x^f x^g \left(3 g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3 g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right)$$

```
cdblib.create ('metric-inv.export')

cdblib.put ('g^ab_3',Metric3,'metric-inv.export')  # R and \partial R
cdblib.put ('g^ab_4',Metric4,'metric-inv.export')
cdblib.put ('g^ab_5',Metric5,'metric-inv.export')
cdblib.put ('g^ab_6',Metric6,'metric-inv.export')

cdblib.put ('g^ab',  Metric, 'metric-inv.export')  # R and \nabla R

cdblib.put ('g^ab_scaled0',scaled0,'metric-inv.export')
cdblib.put ('g^ab_scaled2',scaled2,'metric-inv.export')
cdblib.put ('g^ab_scaled3',scaled3,'metric-inv.export')
cdblib.put ('g^ab_scaled4',scaled4,'metric-inv.export')
cdblib.put ('g^ab_scaled5',scaled5,'metric-inv.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)
```

Timing

Stage 1: 3.4 secs
Stage 2: 5.9 secs
Stage 3: 99.6 secs
Stage 4: 7.3 secs

The connection

Here we use the output from `metric.tex` and `metric-inv.tex` to compute the metric connection Γ^d_{ab}. We use the standard metric compatible connection

$$\Gamma^d_{ab} = \frac{1}{2} g^{dc} \left(g_{cb,a} + g_{ac,b} - g_{ab,c} \right) \tag{1}$$

Since `metric.tex` and `metric-inv.tex` generate truncated expressions for g_{ab} and g^{ab} a similar truncation must be applied to this computation of Γ^d_{ab}. The naive choice is to truncate Γ^d_{ab} *after* it has been fully evaluated on the truncated expersions for g_{ab} and g^{ab}. This will work but it wastes time and memory (big time).

A better approach is to truncate Γ^d_{ab} during its construction. That is, we take careful note of how the terms in the finite series for g_{ab} and g^{ab} combine to produce the terms of a particular order in the expansion of Γ^d_{ab}.

Suppose g_{ab} and g^{ab} are known to say fourth order. We can write each of these as follows

$$g_{ab} = \overset{0}{g}_{ab} + \overset{1}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} \tag{2}$$

$$g^{ab} = \overset{0}{g}{}^{ab} + \overset{1}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} \tag{3}$$

where $\overset{n}{g}$ denotes a term of order $\mathcal{O}\left(\epsilon^n\right)$. A similar expansion applies for Γ^d_{ab}, that is

$$\Gamma^d_{ab} = \overset{0}{\Gamma}{}^d_{ab} + \overset{1}{\Gamma}{}^d_{ab} + \overset{2}{\Gamma}{}^d_{ab} + \overset{3}{\Gamma}{}^d_{ab} + \overset{4}{\Gamma}{}^d_{ab} \tag{4}$$

After substituting these formal expansions into the equation (1) and then matching corresponing terms we obtain

$$\overset{n}{\Gamma}{}^d_{ab} = \frac{1}{2} \sum_{i=0}^{i=n} \overset{i}{g}{}^{dc} \left(\overset{n-i}{g}{}_{cb,a} + \overset{n-i}{g}{}_{ac,b} - \overset{n-i}{g}{}_{ab,c} \right) \tag{5}$$

We use this equation to compute the successive terms in Γ^d_{ab}.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

x^{a}::Depends(D{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

import cdblib

gab = cdblib.get ('g_ab','metric.json')      # cdb(gab.000,gab)
iab = cdblib.get ('g^ab','metric-inv.json')  # cdb(iab.000,iab)

defgab := g_{a b} -> @(gab).
defiab := g^{a b} -> @(iab).

dgab := D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}.  # cdb(dgab.001,dgab)

substitute   (dgab,defgab)

distribute   (dgab)              # cdb(dgab.002,dgab)
unwrap       (dgab)              # cdb(dgab.003,dgab)
product_rule (dgab)              # cdb(dgab.004,dgab)
distribute   (dgab)              # cdb(dgab.005,dgab)
substitute   (dgab,$D_{a}{x^{b}}->\delta^{b}_{a}$,repeat=True)  # cdb(dgab.006,dgab)
```

```
eliminate_kronecker (dgab)          # cdb(dgab.007,dgab)
sort_product    (dgab)              # cdb(dgab.008,dgab)
rename_dummies  (dgab)              # cdb(dgab.009,dgab)
canonicalise    (dgab)              # cdb(dgab.010,dgab)
```

$$\texttt{gab.000} := g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45}x^c x^d x^e x^f R_{acdg} R_{befh} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf} + \frac{1}{45}x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg}$$

$$\texttt{dgab.001} := D_a g_{cb} + D_b g_{ac} - D_c g_{ab}$$

$$\begin{aligned}\texttt{dgab.002} := {} & D_a g_{cb} - \frac{1}{3}D_a\left(x^j x^d R_{cjbd}\right) - \frac{1}{6}D_a\left(x^j x^d x^e \nabla_j R_{cdbe}\right) + \frac{2}{45}D_a\left(x^j x^d x^e x^f R_{cjdg} R_{befh} g^{gh}\right) - \frac{1}{20}D_a\left(x^j x^d x^e x^f \nabla_{jd} R_{cebf}\right) \\ & + \frac{1}{45}D_a\left(x^j x^d x^e x^f x^g R_{cjdh} \nabla_e R_{bfgi} g^{hi}\right) + \frac{1}{45}D_a\left(x^j x^d x^e x^f x^g R_{bjdh} \nabla_e R_{cfgi} g^{hi}\right) - \frac{1}{90}D_a\left(x^j x^d x^e x^f x^g \nabla_{jde} R_{cfbg}\right) \\ & + D_b g_{ac} - \frac{1}{3}D_b\left(x^j x^d R_{ajcd}\right) - \frac{1}{6}D_b\left(x^j x^d x^e \nabla_j R_{adce}\right) + \frac{2}{45}D_b\left(x^j x^d x^e x^f R_{ajdg} R_{cefh} g^{gh}\right) - \frac{1}{20}D_b\left(x^j x^d x^e x^f \nabla_{jd} R_{aecf}\right) \\ & + \frac{1}{45}D_b\left(x^j x^d x^e x^f x^g R_{ajdh} \nabla_e R_{cfgi} g^{hi}\right) + \frac{1}{45}D_b\left(x^j x^d x^e x^f x^g R_{cjdh} \nabla_e R_{afgi} g^{hi}\right) - \frac{1}{90}D_b\left(x^j x^d x^e x^f x^g \nabla_{jde} R_{afcg}\right) \\ & - D_c g_{ab} + \frac{1}{3}D_c\left(x^j x^d R_{ajbd}\right) + \frac{1}{6}D_c\left(x^j x^d x^e \nabla_j R_{adbe}\right) - \frac{2}{45}D_c\left(x^j x^d x^e x^f R_{ajdg} R_{befh} g^{gh}\right) + \frac{1}{20}D_c\left(x^j x^d x^e x^f \nabla_{jd} R_{aebf}\right) \\ & - \frac{1}{45}D_c\left(x^j x^d x^e x^f x^g R_{ajdh} \nabla_e R_{bfgi} g^{hi}\right) - \frac{1}{45}D_c\left(x^j x^d x^e x^f x^g R_{bjdh} \nabla_e R_{afgi} g^{hi}\right) + \frac{1}{90}D_c\left(x^j x^d x^e x^f x^g \nabla_{jde} R_{afbg}\right)\end{aligned}$$

$$
\begin{aligned}
\texttt{dgab.010} := {} & \frac{2}{3}R_{acbd}x^d - \frac{1}{6}\nabla_a R_{bdce}x^d x^e + \frac{1}{3}\nabla_d R_{acbe}x^d x^e - \frac{4}{45}R_{acde}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adce}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adbe}R_{cfgh}g^{dg}x^e x^f x^h \\
& - \frac{1}{20}\nabla_{ad}R_{becf}x^d x^e x^f - \frac{1}{20}\nabla_{da}R_{becf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{acbf}x^d x^e x^f - \frac{2}{45}R_{acde}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i \\
& + \frac{1}{45}R_{cdef}\nabla_a R_{bghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{bdef}\nabla_a R_{cghi}g^{eh}x^d x^f x^g x^i \\
& - \frac{2}{45}R_{bdef}\nabla_g R_{achi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{eh}x^d x^f x^g x^i - \frac{1}{90}\nabla_{ade}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dae}R_{bfcg}x^d x^e x^f x^g \\
& - \frac{1}{90}\nabla_{dea}R_{bfcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{acbg}x^d x^e x^f x^g + \frac{2}{3}R_{adbc}x^d - \frac{1}{6}\nabla_b R_{adce}x^d x^e + \frac{1}{3}\nabla_d R_{aebc}x^d x^e - \frac{2}{45}R_{adbe}R_{cfgh}g^{eg}x^d x^f x^h \\
& - \frac{4}{45}R_{adef}R_{bcgh}g^{eg}x^d x^f x^h - \frac{2}{45}R_{adef}R_{bgch}g^{eg}x^d x^f x^h - \frac{1}{20}\nabla_{bd}R_{aecf}x^d x^e x^f - \frac{1}{20}\nabla_{db}R_{aecf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{afbc}x^d x^e x^f \\
& - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_b R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{adef}\nabla_g R_{bchi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{eh}x^d x^f x^g x^i \\
& - \frac{2}{45}R_{bcde}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_b R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{ei}x^d x^f x^g x^h \\
& - \frac{1}{90}\nabla_{bde}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dbe}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{deb}R_{afcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{agbc}x^d x^e x^f x^g + \frac{1}{6}\nabla_c R_{adbe}x^d x^e \\
& + \frac{2}{45}R_{adce}R_{bfgh}g^{eg}x^d x^f x^h + \frac{2}{45}R_{adef}R_{bgch}g^{eh}x^d x^f x^g + \frac{1}{20}\nabla_{cd}R_{aebf}x^d x^e x^f + \frac{1}{20}\nabla_{dc}R_{aebf}x^d x^e x^f + \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{eh}x^d x^f x^g x^i \\
& - \frac{1}{45}R_{adef}\nabla_c R_{bghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{ei}x^d x^f x^g x^h + \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_c R_{aghi}g^{eh}x^d x^f x^g x^i \\
& + \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{ei}x^d x^f x^g x^h + \frac{1}{90}\nabla_{cde}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dce}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dec}R_{afbg}x^d x^e x^f x^g
\end{aligned}
$$

```
# Note:
# Computing Gamma directly by (1/2) iab dgab and *then* truncating to lower order
# is not optimal. We only want the leading oder terms (to 4th order in x). But the direct
# calculation would compute *all* terms before the truncation. This does work but it
# is slower than the following code.
#
# The better approach (as adopted in this code) is to extract all of the terms of iab
# and dgab then construct the leading order terms of Gamma (to fifth order) term by term.

def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    distribute (tmp)

    substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
    substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
    substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d} -> R_{a b c d}$)
```

```
    return foo

# terms of the curvature expansion of dg_{ab}

dgab00 = get_Rterm (dgab,0)   # cdb(dgab00.105,dgab00)  # zero
dgab01 = get_Rterm (dgab,1)   # cdb(dgab01.105,dgab01)  # zero
dgab02 = get_Rterm (dgab,2)   # cdb(dgab02.105,dgab02)
dgab03 = get_Rterm (dgab,3)   # cdb(dgab03.105,dgab03)
dgab04 = get_Rterm (dgab,4)   # cdb(dgab04.105,dgab04)
dgab05 = get_Rterm (dgab,5)   # cdb(dgab05.105,dgab05)

# Convert free indices on iab from ^{a b} to ^{d c}
# This ensures we can later build products like @(iab) @(dgab) knowing that the indices are correctly ordered.
# Without this step we would be using free indices ^{a b} and _{a b c}. Thus the product @(iab) @(dgab) would
# have just one free index _{c}. This is clearly wrong.

tmp := @(iab) \delta_{a}^{d} \delta_{b}^{c}.

distribute     (tmp)
eliminate_kronecker (tmp)
sort_product   (tmp)
rename_dummies (tmp)
canonicalise   (tmp)

idc := @(tmp).

# terms of the curvature expansion of g^{ab}

idc00 = get_Rterm (idc,0)   # cdb(idc00.105,idc00)
idc01 = get_Rterm (idc,1)   # cdb(idc01.105,idc01)  # zero
idc02 = get_Rterm (idc,2)   # cdb(idc02.105,idc02)
idc03 = get_Rterm (idc,3)   # cdb(idc03.105,idc03)
idc04 = get_Rterm (idc,4)   # cdb(idc04.105,idc04)
idc05 = get_Rterm (idc,5)   # cdb(idc05.105,idc05)
```

$$\texttt{dgab00.105} := 0$$

$$\texttt{dgab01.105} := 0$$

$$\texttt{dgab02.105} := \frac{2}{3}R_{acbd}x^d + \frac{2}{3}R_{adbc}x^d$$

$$\texttt{dgab03.105} := -\frac{1}{6}\nabla_a R_{bdce}x^d x^e + \frac{1}{3}\nabla_d R_{acbe}x^d x^e - \frac{1}{6}\nabla_b R_{adce}x^d x^e + \frac{1}{3}\nabla_d R_{aebc}x^d x^e + \frac{1}{6}\nabla_c R_{adbe}x^d x^e$$

$$\begin{aligned}\texttt{dgab04.105} := &-\frac{4}{45}R_{acde}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adce}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adbe}R_{cfgh}g^{dg}x^e x^f x^h - \frac{1}{20}\nabla_{ad}R_{becf}x^d x^e x^f \\ &- \frac{1}{20}\nabla_{da}R_{becf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{acbf}x^d x^e x^f - \frac{2}{45}R_{adbe}R_{cfgh}g^{eg}x^d x^f x^h - \frac{4}{45}R_{adef}R_{bcgh}g^{eg}x^d x^f x^h \\ &- \frac{2}{45}R_{adef}R_{bgch}g^{eg}x^d x^f x^h - \frac{1}{20}\nabla_{bd}R_{aecf}x^d x^e x^f - \frac{1}{20}\nabla_{db}R_{aecf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{afbc}x^d x^e x^f \\ &+ \frac{2}{45}R_{adce}R_{bfgh}g^{eg}x^d x^f x^h + \frac{2}{45}R_{adef}R_{bgch}g^{eh}x^d x^f x^g + \frac{1}{20}\nabla_{cd}R_{aebf}x^d x^e x^f + \frac{1}{20}\nabla_{dc}R_{aebf}x^d x^e x^f\end{aligned}$$

$$\begin{aligned}\texttt{dgab05.105} := &-\frac{2}{45}R_{acde}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_a R_{bghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{eh}x^d x^f x^g x^i \\ &- \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{bdef}\nabla_a R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bdef}\nabla_g R_{achi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{eh}x^d x^f x^g x^i \\ &- \frac{1}{90}\nabla_{ade}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dae}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dea}R_{bfcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{acbg}x^d x^e x^f x^g - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{eh}x^d x^f x^g x^i \\ &+ \frac{1}{45}R_{adef}\nabla_b R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{adef}\nabla_g R_{bchi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bcde}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i \\ &- \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_b R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{ei}x^d x^f x^g x^h - \frac{1}{90}\nabla_{bde}R_{afcg}x^d x^e x^f x^g \\ &- \frac{1}{90}\nabla_{dbe}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{deb}R_{afcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{agbc}x^d x^e x^f x^g + \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{eh}x^d x^f x^g x^i \\ &- \frac{1}{45}R_{adef}\nabla_c R_{bghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{ei}x^d x^f x^g x^h + \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_c R_{aghi}g^{eh}x^d x^f x^g x^i \\ &+ \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{ei}x^d x^f x^g x^h + \frac{1}{90}\nabla_{cde}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dce}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dec}R_{afbg}x^d x^e x^f x^g\end{aligned}$$

$\texttt{idc00.105} := g^{cd}$

$\texttt{idc01.105} := 0$

$\texttt{idc02.105} := \frac{1}{3} R_{abef} g^{ca} g^{de} x^b x^f$

$\texttt{idc03.105} := \frac{1}{6} \nabla_a R_{befg} g^{cb} g^{df} x^a x^e x^g$

$\texttt{idc04.105} := \frac{1}{15} R_{abef} R_{ghij} g^{ca} g^{dg} g^{ei} x^b x^f x^h x^j + \frac{1}{20} \nabla_{ab} R_{efgh} g^{ce} g^{dg} x^a x^b x^f x^h$

$\texttt{idc05.105} := \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ch} g^{da} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ca} g^{dh} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{90} \nabla_{abe} R_{fghi} g^{cf} g^{dh} x^a x^b x^e x^g x^i$

```
# idc  = g^{d c}
# dgab = D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}

# terms of the curvature expansion of \Gamma^{d}_{a b}

# term0 := (1/2)  @(idc00) @(dgab00).
# term1 := (1/2) (@(idc01) @(dgab00) + @(idc00) @(dgab01)).
# term2 := (1/2) (@(idc02) @(dgab00) + @(idc01) @(dgab01) + @(idc00) @(dgab02)).
# term3 := (1/2) (@(idc03) @(dgab00) + @(idc02) @(dgab01) + @(idc01) @(dgab02) + @(idc00) @(dgab03)).
# term4 := (1/2) (@(idc04) @(dgab00) + @(idc03) @(dgab01) + @(idc02) @(dgab02) + @(idc01) @(dgab03) + @(idc00) @(dgab04)).
# term5 := (1/2) (@(idc05) @(dgab00) + @(idc04) @(dgab01) + @(idc03) @(dgab02) + @(idc02) @(dgab03) + @(idc01) @(dgab04) + @(idc00) @(d

# simplidied version of the above after noting dgab00 = dgab01 = 0

term0 := 0.
term1 := 0.
term2 := (1/2) (@(idc00) @(dgab02)).
term3 := (1/2) (@(idc01) @(dgab02) + @(idc00) @(dgab03)).
term4 := (1/2) (@(idc02) @(dgab02) + @(idc01) @(dgab03) + @(idc00) @(dgab04)).
term5 := (1/2) (@(idc03) @(dgab02) + @(idc02) @(dgab03) + @(idc01) @(dgab04) + @(idc00) @(dgab05)).

def tidy_terms (obj):
    substitute     (obj,$x^{a}->AA^{a}$,repeat=True)  # will force AA to the left of all terms
    distribute     (obj)
    sort_product   (obj)
    rename_dummies (obj)
    canonicalise   (obj)
    substitute     (obj,$AA^{a}->x^{a}$,repeat=True)  # replace AA with x
    factor_out     (obj,$x^{a?}$)

    return obj

term0 = tidy_terms (term0)   # cdb(term0.201,term0)  # zero
term1 = tidy_terms (term1)   # cdb(term1.201,term1)  # zero
term2 = tidy_terms (term2)   # cdb(term2.201,term2)
term3 = tidy_terms (term3)   # cdb(term3.201,term3)
term4 = tidy_terms (term4)   # cdb(term4.201,term4)
term5 = tidy_terms (term5)   # cdb(term5.201,term5)
```

```
Gamma := @(term0) + @(term1) + @(term2) + @(term3) + @(term4) + @(term5). # cdb(Gamma.200,Gamma)
```

$$\texttt{term0.201} := 0$$

$$\texttt{term1.201} := 0$$

$$\texttt{term2.201} := x^c \left(\frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de}\right)$$

$$\texttt{term3.201} := x^c x^e \left(\frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df}\right)$$

$$\begin{aligned}\texttt{term4.201} := x^c x^e x^f \Big(&\frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \\ &+ \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \\ &- \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg}\Big)\end{aligned}$$

$$\begin{aligned}\texttt{term5.201} := x^c x^e x^f x^g \Big(&\frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} \\ &+ \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} \\ &- \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cea} R_{bfgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} \\ &- \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} \\ &+ \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} \\ &- \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{agij} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} \\ &+ \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh}\Big)\end{aligned}$$

$$
\begin{aligned}
\texttt{Gamma.200} := {} & x^c \left(\frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de}\right) + x^c x^e \left(\frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df}\right) \\
& + x^c x^e x^f \Bigg(\frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \\
& + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \\
& - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg}\Bigg) \\
& + x^c x^e x^f x^g \Bigg(\frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} \\
& + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} \\
& - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgh} g^{dh} \\
& + \frac{1}{180} \nabla_{cea} R_{bfgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} \\
& - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} \\
& - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} \\
& - \frac{1}{90} R_{bceh} \nabla_f R_{agij} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh}\Bigg)
\end{aligned}
$$

```
cdblib.create ('connection.json')

cdblib.put ('Gamma',Gamma,'connection.json')

cdblib.put ('GammaRterm0',term0,'connection.json')
cdblib.put ('GammaRterm1',term1,'connection.json')
cdblib.put ('GammaRterm2',term2,'connection.json')
cdblib.put ('GammaRterm3',term3,'connection.json')
cdblib.put ('GammaRterm4',term4,'connection.json')
cdblib.put ('GammaRterm5',term5,'connection.json')

checkpoint.append (term0)
checkpoint.append (term1)
checkpoint.append (term2)
checkpoint.append (term3)
checkpoint.append (term4)
checkpoint.append (term5)
```

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                            -> A001^{a}               $)
    substitute (obj,$ x^{a}                            -> A002^{a}               $)
    substitute (obj,$ g^{a b}                          -> A003^{a b}             $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}    -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}      -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}        -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}          -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                      -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                  -> A^{a}                           $)
    substitute (obj,$ A002^{a}                  -> x^{a}                           $)
    substitute (obj,$ A003^{a b}                -> g^{a b}                         $)
    substitute (obj,$ A008_{a b c d e f g h}    -> \nabla_{e f g h}{R_{a b c d}} $)
    substitute (obj,$ A007_{a b c d e f g}      -> \nabla_{e f g}{R_{a b c d}}   $)
    substitute (obj,$ A006_{a b c d e f}        -> \nabla_{e f}{R_{a b c d}}     $)
    substitute (obj,$ A005_{a b c d e}          -> \nabla_{e}{R_{a b c d}}       $)
    substitute (obj,$ A004_{a b c d}            -> R_{a b c d}                   $)

    return obj

def reformat (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute     (bah)
   bah = product_sort (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$A^{a?},x^{b?}$)
   ans := @(bah) / @(foo).
   return ans

def rescale (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute  (bah)
```

```
   factor_out  (bah,$A^{a?},x^{b?}$)
   return bah

Rterm2 := @(term2) A^{a} A^{b}.
Rterm3 := @(term3) A^{a} A^{b}.
Rterm4 := @(term4) A^{a} A^{b}.
Rterm5 := @(term5) A^{a} A^{b}.

Rterm2 = reformat (Rterm2,  3)    # cdb(Rterm2.301,Rterm2)
Rterm3 = reformat (Rterm3, 12)    # cdb(Rterm3.301,Rterm3)
Rterm4 = reformat (Rterm4,360)    # cdb(Rterm4.301,Rterm4)
Rterm5 = reformat (Rterm5,180)    # cdb(Rterm5.301,Rterm5)

Gamma  := @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Gamma.301,Gamma)
Scaled := 360 @(Gamma).                                   # cdb (Scaled.301,Scaled)

scaled2 = rescale (Rterm2,   3)   # cdb (scaled2.301,scaled2)
scaled3 = rescale (Rterm3,  12)   # cdb (scaled3.301,scaled3)
scaled4 = rescale (Rterm4, 360)   # cdb (scaled4.301,scaled4)
scaled5 = rescale (Rterm5, 180)   # cdb (scaled5.301,scaled5)
```

The connection in Riemann normal coordinates

$$
\begin{aligned}
A^aA^b\Gamma^d_{ab} = {}& \frac{2}{3}A^aA^bx^cg^{de}R_{acbe} + \frac{1}{12}A^aA^bx^cx^e\left(2g^{df}\nabla_aR_{bcef} + 4g^{df}\nabla_cR_{aebf} + g^{df}\nabla_fR_{acbe}\right) + \frac{1}{360}A^aA^bx^cx^ex^f\left(64g^{dg}g^{hi}R_{acbh}R_{egfi} - 32g^{dg}g^{hi}R_{aceh}R_{bgfi}\right.\\
&\left. - 16g^{dg}g^{hi}R_{aceh}R_{bifg} + 18g^{dg}\nabla_{ac}R_{befg} + 18g^{dg}\nabla_{ca}R_{befg} + 36g^{dg}\nabla_{ce}R_{afbg} - 16g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9g^{dg}\nabla_{gc}R_{aebf} + 9g^{dg}\nabla_{cg}R_{aebf}\right)\\
&+ \frac{1}{180}A^aA^bx^cx^ex^fx^g\left(16g^{dh}g^{ij}R_{acbi}\nabla_eR_{fhgj} + 6g^{dh}g^{ij}R_{chei}\nabla_aR_{bfgj} + 16g^{dh}g^{ij}R_{chei}\nabla_fR_{agbj} + 5g^{dh}g^{ij}R_{chei}\nabla_jR_{afbg}\right.\\
&- 8g^{dh}g^{ij}R_{ahci}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{aich}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_bR_{fhgj} - 8g^{dh}g^{ij}R_{acei}\nabla_fR_{bhgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bjgh} + 2g^{dh}\nabla_{ace}R_{bfgh}\\
&+ 2g^{dh}\nabla_{cae}R_{bfgh} + 2g^{dh}\nabla_{cea}R_{bfgh} + 4g^{dh}\nabla_{cef}R_{agbh} - 4g^{dh}g^{ij}R_{achi}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_hR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bghj}\\
&\left. + g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + g^{dh}\nabla_{ceh}R_{afbg}\right)
\end{aligned}
$$

$$
\begin{aligned}
360A^aA^b\Gamma^d_{ab} = {}& 240A^aA^bx^cg^{de}R_{acbe} + 30A^aA^bx^cx^e\left(2g^{df}\nabla_aR_{bcef} + 4g^{df}\nabla_cR_{aebf} + g^{df}\nabla_fR_{acbe}\right) + A^aA^bx^cx^ex^f\left(64g^{dg}g^{hi}R_{acbh}R_{egfi} - 32g^{dg}g^{hi}R_{aceh}R_{bgfi}\right.\\
&\left. - 16g^{dg}g^{hi}R_{aceh}R_{bifg} + 18g^{dg}\nabla_{ac}R_{befg} + 18g^{dg}\nabla_{ca}R_{befg} + 36g^{dg}\nabla_{ce}R_{afbg} - 16g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9g^{dg}\nabla_{gc}R_{aebf} + 9g^{dg}\nabla_{cg}R_{aebf}\right)\\
&+ 2A^aA^bx^cx^ex^fx^g\left(16g^{dh}g^{ij}R_{acbi}\nabla_eR_{fhgj} + 6g^{dh}g^{ij}R_{chei}\nabla_aR_{bfgj} + 16g^{dh}g^{ij}R_{chei}\nabla_fR_{agbj} + 5g^{dh}g^{ij}R_{chei}\nabla_jR_{afbg}\right.\\
&- 8g^{dh}g^{ij}R_{ahci}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{aich}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_bR_{fhgj} - 8g^{dh}g^{ij}R_{acei}\nabla_fR_{bhgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bjgh}\\
&+ 2g^{dh}\nabla_{ace}R_{bfgh} + 2g^{dh}\nabla_{cae}R_{bfgh} + 2g^{dh}\nabla_{cea}R_{bfgh} + 4g^{dh}\nabla_{cef}R_{agbh} - 4g^{dh}g^{ij}R_{achi}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_hR_{bfgj}\\
&\left. - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bghj} + g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + g^{dh}\nabla_{ceh}R_{afbg}\right)
\end{aligned}
$$

Curvature expansion of the connection

$$A^a A^b \Gamma^d_{ab} = A^a A^b \overset{2}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{3}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{4}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{5}{\Gamma}{}^d{}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$3A^a A^b \overset{2}{\Gamma}{}^d_{ab} = 2A^a A^b x^c g^{de} R_{acbe}$$

$$12A^a A^b \overset{3}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e \left(2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}\right)$$

$$\begin{aligned} 360A^a A^b \overset{4}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f \big(&64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} \\ &- 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}\big) \end{aligned}$$

$$\begin{aligned} 180A^a A^b \overset{5}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f x^g \big(&16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} \\ &- 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bfgh} + 2g^{dh} \nabla_{cae} R_{bfgh} \\ &+ 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} + g^{dh} \nabla_{hce} R_{afbg} \\ &+ g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg}\big) \end{aligned}$$

The generalised connections

The generalised connections may be computed recursively using

$$\Gamma^{a}{}_{b\underline{c}d} = \Gamma^{a}{}_{(b\underline{c},d)} - (n+1)\Gamma^{a}{}_{p(\underline{c}}\Gamma^{p}{}_{bd)} \tag{1}$$

where $\underline{c}$ contains $n > 0$ indices. The sequence begins with the standard metric compatible connection

$$\Gamma^{d}_{ab} = \frac{1}{2} g^{dc} \left(g_{cb,a} + g_{ac,b} - g_{ab,c} \right) \tag{2}$$

Here we will use the results of `metric.tex` and `metric-inv.tex` to compute the metric connection Γ^{d}_{ab}. But since the g_{ab} and g^{ab} provided by those codes are truncated at a particular order in the curvatures (and thus are only approximations to the g_{ab} and g^{ab}) similar truncations will arise in the $\Gamma^{a}{}_{b\underline{c}d}$.

Approximations will be denoted by the addition of an overbar to an object. In this notation the metric g can be written as

$$g = \bar{g} + \mathcal{O}\left(\epsilon^{n}\right) \tag{3}$$

in which $\bar{g}$ is the truncated polynomial approximation to g and $\mathcal{O}\left(\epsilon^{n}\right)$ is the error term (containing terms no smaller than ϵ^{n}). The polynomial structure of $\bar{g}$ can be expressed as

$$\bar{g} = \overset{0}{\bar{g}} + \overset{1}{\bar{g}} + \overset{2}{\bar{g}} + \cdots + \overset{p}{\bar{g}} \tag{4}$$

in which each terms like $\overset{m}{\bar{g}}$ contains only terms of order m. This notation will be applied to other quantities in particular the generalised connections.

The notation $\mathcal{O}\left(\epsilon^{n}\right)$ denotes terms in the curvatures that are of order ϵ^{n}. What does this actually mean? Each term in R is of order ϵ^{2} while each derivative of R carries an extra power of ϵ. Thus $R \cdot R = \mathcal{O}\left(\epsilon^{4}\right)$, $R \cdot R \cdot \nabla R = \mathcal{O}\left(\epsilon^{7}\right)$ and $R \cdot R \cdot \nabla^{2} R = \mathcal{O}\left(\epsilon^{8}\right)$.

We will also adopt the convention that an object is said to be an $\mathcal{O}\left(\epsilon^{m}\right)$ approximation when the corresponding error term is $\mathcal{O}\left(\epsilon^{m+1}\right)$.

Consider the $\mathcal{O}\left(\epsilon^{m}\right)$ approximation of the generalised connection, namely,

$$\bar{\Gamma}^{a}{}_{b\underline{c}_n d} = \overset{0}{\bar{\Gamma}}{}^{a}{}_{b\underline{c}_n d} + \overset{1}{\bar{\Gamma}}{}^{a}{}_{b\underline{c}_n d} + \overset{2}{\bar{\Gamma}}{}^{a}{}_{b\underline{c}_n d} + \cdots + \overset{m}{\bar{\Gamma}}{}^{a}{}_{b\underline{c}_n d} \tag{5}$$

where $\underline{c}_n$ denotes a set of indices such as $c_1 c_2 c_3 \ldots c_n$.

The first thing to note is that

$$0 = \overset{1+n}{\bar{\Gamma}}{}^{a}{}_{(b\underline{c}_n,d)} \tag{6}$$

There are two proofs of this claim. For the first proof, note (by inspection) that the order $\mathcal{O}\left(\epsilon^p\right)$ approximation for $\bar{\Gamma}^a{}_{b\underline{c}_n d}$ is a polynomial in x of degree $p-n-1$. Thus $\overset{1+n}{\bar{\Gamma}}{}^a{}_{(b\underline{c}_n,d)}$ is a polynomial in x of degree zero, i.e., a constant. However, we know that all generalised connections vanish at the origin of the RNC frame. Thus this constant must be zero. The second proof makes explicit use of the first (and second?) Bianchi identity, that is $0 = R_{a(bcd)}$. The term $\overset{1+n}{\bar{\Gamma}}{}^a{}_{(b\underline{c}_n,d)}$ will itself consist of a sum of terms built from combinations of x, R, ∇R etc. The x^a will always appear in a contraction with one of the indices on R_{abcd} or one of its derivatives. Consider any one of these terms, denoted by A, and assume for the moment that $1+n$ is an even number, say $1+n=2p$. The indices $(b\underline{c}_n, d)$ must somehow be assigned to the factors that comprise A. Our aim is to show that at least one R factor in A will receive 3 of these indices and thus by the Bianchi identities will be zero. If there are too many R factors then the Bianchi identities will not come into play. So how many R factors can we expect? Since A is a term in an $\mathcal{O}\left(\epsilon^{(n+1)}\right)$ approximation there can be no more than $(n+1)/2 = p$ Riemann factors. There will be at least one x term contracted with one of the p Riemann factors. However, we have $n+2 = 2p+1$ indices to distribute amongst the x term and p Riemann factors. One of the indices is a derivative index and will have nett effect of transferring that index from x to one of the Riemann factors. The remaining $2p$ indices must be distributed amongst the p Riemann factors. It is not possible to avoid assigning three indices to at least one of the Riemann factors. Thus, by the Bianchi identity, this A term must vanish. Similar arguments can be applied to the other cases where the A terms consists of products of R and its derivatives and in the case where $n+1$ is an odd number. The analysis always comes down to the distribution of the indices $(b\underline{c}_n, d)$ amongst the factors of a typical A term. In all cases the Bianchi identity will enter the play and force A to be zero.

A corollary of the second proof is that for all $m < n+2$

$$0 = \overset{m}{\bar{\Gamma}}{}^a{}_{b\underline{c}_n d} \tag{7}$$

The proof follows exactly that of the second proof given above.

We can use the above results to streamline the computation of the generalised connections. We begin with the formal expression for the $\mathcal{O}\left(\epsilon^m\right)$ approximations

$$\Gamma^a{}_{bc} = \overset{2}{\bar{\Gamma}}{}^a{}_{bc} + \overset{3}{\bar{\Gamma}}{}^a{}_{bc} + \overset{4}{\bar{\Gamma}}{}^a{}_{bc} + \cdots + \overset{m}{\bar{\Gamma}}{}^a{}_{bc} \tag{8}$$

$$\Gamma^a{}_{b\underline{c}} = \overset{n+1}{\bar{\Gamma}}{}^a{}_{b\underline{c}} + \overset{n+2}{\bar{\Gamma}}{}^a{}_{b\underline{c}} + \overset{n+3}{\bar{\Gamma}}{}^a{}_{b\underline{c}} + \cdots + \overset{m}{\bar{\Gamma}}{}^a{}_{b\underline{c}} \tag{9}$$

$$\Gamma^a{}_{b\underline{c}d} = \overset{n+2}{\bar{\Gamma}}{}^a{}_{b\underline{c}d} + \overset{n+3}{\bar{\Gamma}}{}^a{}_{b\underline{c}d} + \overset{n+4}{\bar{\Gamma}}{}^a{}_{b\underline{c}d} + \cdots + \overset{m}{\bar{\Gamma}}{}^a{}_{b\underline{c}d} \tag{10}$$

These can be substituted into equation (1) with the result

$$\Gamma^a{}_{b\underline{c}d} = \overset{n+1}{\bar{\Gamma}}{}^a{}_{(b\underline{c},d)} + \overset{n+2}{\bar{\Gamma}}{}^a{}_{(b\underline{c},d)} + \overset{n+3}{\bar{\Gamma}}{}^a{}_{(b\underline{c},d)} + \cdots + \overset{m}{\bar{\Gamma}}{}^a{}_{(b\underline{c},d)} - (n+1)\left(\overset{n+1}{\bar{\Gamma}}{}^a{}_{p\underline{c}} + \overset{n+2}{\bar{\Gamma}}{}^a{}_{p\underline{c}} + \overset{n+3}{\bar{\Gamma}}{}^a{}_{p\underline{c}} + \cdots + \overset{m}{\bar{\Gamma}}{}^a{}_{p\underline{c}}\right)\left(\overset{2}{\bar{\Gamma}}{}^p{}_{bd} + \overset{3}{\bar{\Gamma}}{}^p{}_{bd} + \overset{4}{\bar{\Gamma}}{}^p{}_{bd} + \cdots + \overset{m}{\bar{\Gamma}}{}^p{}_{bd}\right) \tag{11}$$

where it is understood that in expanding the pair of bracketed terms in the last result the terms should be symmetrised over $b\underline{c}d$ and also truncated to terms of order $\mathcal{O}\left(\epsilon^m\right)$. Note that the first term on the right hand side of this equation vanishes by way of the results described above.

Comparing the order m terms in equation (10) and (11) leads to the following equation

$$\overset{m}{\bar{\Gamma}}{}^{a}{}_{b\underline{c}d} = \overset{m}{\bar{\Gamma}}{}^{a}{}_{(b\underline{c},d)} - (n+1)\left(\overset{m-2}{\bar{\Gamma}}{}^{a}{}_{p(\underline{c}}\overset{2}{\bar{\Gamma}}{}^{p}{}_{bd)} + \overset{m-3}{\bar{\Gamma}}{}^{a}{}_{p(\underline{c}}\overset{3}{\bar{\Gamma}}{}^{p}{}_{bd)} + \overset{m-4}{\bar{\Gamma}}{}^{a}{}_{p(\underline{c}}\overset{4}{\bar{\Gamma}}{}^{p}{}_{bd)} + \cdots + \overset{n+1}{\bar{\Gamma}}{}^{a}{}_{p(\underline{c}}\overset{m-n-1}{\bar{\Gamma}}{}^{p}{}_{bd)} \right) \tag{12}$$

This one equation is all that is needed to compute all of the $\overset{p}{\bar{\Gamma}}{}^{a}{}_{b\underline{c}d}$ for $p = 3, 4, 5, \ldots m$ given just the $\overset{p}{\bar{\Gamma}}{}^{a}{}_{bd}$ for $p = 2, 3, 4, \ldots m$. For example, suppose $m = 5$ and suppose that we are given $\overset{p}{\bar{\Gamma}}{}^{a}{}_{bd}$ for $p = 2, 3, 4, 5$. Then with $n = 1$ we can use equation (12) to compute in turn, $\overset{p}{\bar{\Gamma}}{}^{a}{}_{bc_1d}$ for $p = 3, 4, 5$. Then with $n = 2$ we compute $\overset{p}{\bar{\Gamma}}{}^{a}{}_{bc_1c_2d}$ for $p = 4, 5$ and finally with $n = 3$ we compute $\overset{p}{\bar{\Gamma}}{}^{a}{}_{bc_1c_2c_3d}$ for $p = 5$. There are no terms like $\overset{p}{\bar{\Gamma}}{}^{a}{}_{bc_1c_2c_3c_4d}$ for $p \leq 5$ due to the corollary given earlier.

The explicit computations for $m = 5$ are as follows.

For $n = 1$,

$$\overset{3}{\bar{\Gamma}}{}^{a}{}_{bc_1d} = \overset{3}{\bar{\Gamma}}{}^{a}{}_{(bc_1,d)} \tag{13}$$

$$\overset{4}{\bar{\Gamma}}{}^{a}{}_{bc_1d} = \overset{4}{\bar{\Gamma}}{}^{a}{}_{(bc_1,d)} - 2\overset{2}{\bar{\Gamma}}{}^{a}{}_{p(c_1}\overset{2}{\bar{\Gamma}}{}^{p}{}_{bd)} \tag{14}$$

$$\overset{5}{\bar{\Gamma}}{}^{a}{}_{bc_1d} = \overset{5}{\bar{\Gamma}}{}^{a}{}_{(bc_1,d)} - 2\overset{3}{\bar{\Gamma}}{}^{a}{}_{p(c_1}\overset{2}{\bar{\Gamma}}{}^{p}{}_{bd)} - 2\overset{2}{\bar{\Gamma}}{}^{a}{}_{p(c_1}\overset{3}{\bar{\Gamma}}{}^{p}{}_{bd)} \tag{15}$$

For $n = 2$,

$$\overset{4}{\bar{\Gamma}}{}^{a}{}_{bc_1c_2d} = \overset{4}{\bar{\Gamma}}{}^{a}{}_{(bc_1c_2,d)} \tag{16}$$

$$\overset{5}{\bar{\Gamma}}{}^{a}{}_{bc_1c_2d} = \overset{5}{\bar{\Gamma}}{}^{a}{}_{(bc_1c_2,d)} - 3\overset{2}{\bar{\Gamma}}{}^{a}{}_{p(c_1c_2}\overset{2}{\bar{\Gamma}}{}^{p}{}_{bd)} \tag{17}$$

For $n = 3$,

$$\overset{5}{\bar{\Gamma}}{}^{a}{}_{bc_1c_2c_3d} = \overset{5}{\bar{\Gamma}}{}^{a}{}_{(bc_1c_2c_3,d)} \tag{18}$$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,c1,c2,c3,c4,c5,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

x^{a}::Depends(D{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

import cdblib

term0 = cdblib.get ('GammaRterm0','connection.json')
term1 = cdblib.get ('GammaRterm2','connection.json')
term2 = cdblib.get ('GammaRterm3','connection.json')
term3 = cdblib.get ('GammaRterm4','connection.json')
term4 = cdblib.get ('GammaRterm5','connection.json')

# LCB: these terms were not computed in connection.tex so set them to zero
```

```
#      maybe in the future I will compute down to term6.

term5 := 0.
term6 := 0.

# genGmn : m = eps order of Rabcd terms
#          n = number of c indices

# ---------------------------------------------------------------------------
# rules for building the genGmn

# note: after applying each rule, must symmetrise over (b c1 c2 ... cn d)

# n = 0

genG20 := genG2^{a}_{b d}.
genG30 := genG3^{a}_{b d}.
genG40 := genG4^{a}_{b d}.
genG50 := genG5^{a}_{b d}.

defG20 := genG2^{d}_{a b} -> @(term1).
defG30 := genG3^{d}_{a b} -> @(term2).
defG40 := genG4^{d}_{a b} -> @(term3).
defG50 := genG5^{d}_{a b} -> @(term4).

# LCB: rncGamma in connection.json limited to "term4" (ie. to 4th order in x)
#      so can only compute genG3*, genG4* and genG5* (at this stage)
#      but it doesn't hurt to provide the definitions for genG6*, genG7* etc. we just won't use them (at this atage)

defG60 := genG6^{d}_{a b} -> @(term5).
defG70 := genG7^{d}_{a b} -> @(term6).

# n = 1

defG31 := genG3^{a}_{b c1 d} -> D_{d}{genG3^{a}_{b c1}}.

defG41 := genG4^{a}_{b c1 d} -> D_{d}{genG4^{a}_{b c1}}
                              - 2 genG2^{a}_{p c1} genG2^{p}_{b d}.
```

```
defG51 := genG5^{a}_{b c1 d} -> D_{d}{genG5^{a}_{b c1}}
                                - 2 genG3^{a}_{p c1} genG2^{p}_{b d}
                                - 2 genG2^{a}_{p c1} genG3^{p}_{b d}.

defG61 := genG6^{a}_{b c1 d} -> D_{d}{genG6^{a}_{b c1}}
                                - 2 genG4^{a}_{p c1} genG2^{p}_{b d}
                                - 2 genG3^{a}_{p c1} genG3^{p}_{b d}
                                - 2 genG3^{a}_{p c1} genG4^{p}_{b d}.

defG71 := genG7^{a}_{b c1 d} -> D_{d}{genG7^{a}_{b c1}}
                                - 2 genG5^{a}_{p c1} genG2^{p}_{b d}
                                - 2 genG4^{a}_{p c1} genG3^{p}_{b d}
                                - 2 genG3^{a}_{p c1} genG4^{p}_{b d}
                                - 2 genG2^{a}_{p c1} genG5^{p}_{b d}.

# n = 2

defG42 := genG4^{a}_{b c1 c2 d} -> D_{d}{genG4^{a}_{b c1 c2}}.

defG52 := genG5^{a}_{b c1 c2 d} -> D_{d}{genG5^{a}_{b c1 c2}}
                                   - 3 genG3^{a}_{p c1 c2} genG2^{p}_{b d}.

defG62 := genG6^{a}_{b c1 c2 d} -> D_{d}{genG6^{a}_{b c1 c2}}
                                   - 3 genG4^{a}_{p c1 c2} genG2^{p}_{b d}
                                   - 3 genG3^{a}_{p c1 c2} genG3^{p}_{b d}.

defG72 := genG7^{a}_{b c1 c2 d} -> D_{d}{genG7^{a}_{b c1 c2}}
                                   - 3 genG5^{a}_{p c1 c2} genG2^{p}_{b d}
                                   - 3 genG4^{a}_{p c1 c2} genG3^{p}_{b d}
                                   - 3 genG3^{a}_{p c1 c2} genG4^{p}_{b d}.

# n = 3

defG53 := genG5^{a}_{b c1 c2 c3 d} -> D_{d}{genG5^{a}_{b c1 c2 c3}}.

defG63 := genG6^{a}_{b c1 c2 c3 d} -> D_{d}{genG6^{a}_{b c1 c2 c3}}
                                      - 4 genG3^{a}_{p c1 c2 c3} genG3^{p}_{b d}.
```

```
defG73 := genG7^{a}_{b c1 c2 c3 d} -> D_{d}{genG7^{a}_{b c1 c2 c3}}
                                      - 4 genG4^{a}_{p c1 c2 c3} genG3^{p}_{b d}
                                      - 4 genG3^{a}_{p c1 c2 c3} genG4^{p}_{b d}.

# n = 4

defG64 := genG6^{a}_{b c1 c2 c3 c4 d} -> D_{d}{genG6^{a}_{b c1 c2 c3 c4}}.

defG74 := genG7^{a}_{b c1 c2 c3 c4 d} -> D_{d}{genG7^{a}_{b c1 c2 c3 c4}}
                                         - 5 genG5^{a}_{p c1 c2 c3 c4} genG2^{p}_{b d}.

# n = 5

defG75 := genG7^{a}_{b c1 c2 c3 c4 c5 d} -> D_{d}{genG7^{a}_{b c1 c2 c3 c4 c5}}.

# -------------------------------------------------------------------------
# build the genGmn

# =========================================================================
# n = 1

genG31 := genG3^{a}_{b c1 d}.                             # cdb (genG31.000,genG31)
genG41 := genG4^{a}_{b c1 d}.                             # cdb (genG41.000,genG41)
genG51 := genG5^{a}_{b c1 d}.
# genG61 := genG6^{a}_{b c1 d}.
# genG71 := genG7^{a}_{b c1 d}.

# -------------------------------------------------------------------------
substitute     (genG20,defG20)                            # cdb (genG20.001,genG20)
substitute     (genG30,defG30)                            # cdb (genG30.001,genG30)
substitute     (genG40,defG40)                            # cdb (genG40.001,genG40)
substitute     (genG50,defG50)                            # cdb (genG50.001,genG50)

# -------------------------------------------------------------------------
substitute     (genG31,defG31)                            # cdb (genG31.001,genG31)
substitute     (genG31,defG30)                            # cdb (genG31.002,genG31)
```

```
distribute      (genG31)                                   # cdb (genG31.002,genG31)
unwrap          (genG31)                                   # cdb (genG31.003,genG31)
product_rule    (genG31)                                   # cdb (genG31.004,genG31)
distribute      (genG31)                                   # cdb (genG31.005,genG31)
substitute      (genG31,$D_{a}{x^b}->\delta_{a}^{b}$)      # cdb (genG31.006,genG31)
eliminate_kronecker (genG31)                               # cdb (genG31.007,genG31)
sym             (genG31,$_{b}, _{c1}, _{d}$)
sort_product    (genG31)                                   # cdb (genG31.008,genG31)
rename_dummies  (genG31)                                   # cdb (genG31.009,genG31)
canonicalise    (genG31)                                   # cdb (genG31.010,genG31)

# ---------------------------------------------------------------------------
substitute      (genG41,defG41)                            # cdb (genG41.001,genG41)
substitute      (genG41,defG40)                            # cdb (genG41.002,genG41)
substitute      (genG41,defG20,repeat=True)                # cdb (genG41.003,genG41)

distribute      (genG41)                                   # cdb (genG41.004,genG41)
unwrap          (genG41)                                   # cdb (genG41.005,genG41)
product_rule    (genG41)                                   # cdb (genG41.006,genG41)
distribute      (genG41)                                   # cdb (genG41.007,genG41)
substitute      (genG41,$D_{a}{x^b}->\delta_{a}^{b}$)      # cdb (genG41.008,genG41)
eliminate_kronecker (genG41)                               # cdb (genG41.009,genG41)
sym             (genG41,$_{b}, _{c1}, _{d}$)
sort_product    (genG41)                                   # cdb (genG41.010,genG41)
rename_dummies  (genG41)                                   # cdb (genG41.011,genG41)
canonicalise    (genG41)                                   # cdb (genG41.012,genG41)

# ---------------------------------------------------------------------------
substitute      (genG51,defG51)
substitute      (genG51,defG50)
substitute      (genG51,defG30,repeat=True)
substitute      (genG51,defG20,repeat=True)

distribute      (genG51)
unwrap          (genG51)
product_rule    (genG51)
distribute      (genG51)
substitute      (genG51,$D_{a}{x^b}->\delta_{a}^{b}$)
```

```
eliminate_kronecker (genG51)
sym             (genG51,$_{b}, _{c1}, _{d}$)
sort_product    (genG51)
rename_dummies (genG51)
canonicalise    (genG51)

# update the rules

defG31 := genG3^{a}_{b c1 d} -> @(genG31).
defG41 := genG4^{a}_{b c1 d} -> @(genG41).
defG51 := genG5^{a}_{b c1 d} -> @(genG51).

# ===========================================================================
# n = 2

genG42 := genG4^{a}_{b c1 c2 d}.                          # cdb (genG42.000,genG42)
genG52 := genG5^{a}_{b c1 c2 d}.
# genG62 := genG6^{a}_{b c1 c2 d}.
# genG72 := genG7^{a}_{b c1 c2 d}.

# ---------------------------------------------------------------------------
substitute      (genG42,defG42)                           # cdb (genG42.001,genG42)
substitute      (genG42,defG41)                           # cdb (genG42.002,genG42)

distribute      (genG42)                                  # cdb (genG42.003,genG42)
unwrap          (genG42)                                  # cdb (genG42.004,genG42)
product_rule    (genG42)                                  # cdb (genG42.005,genG42)
distribute      (genG42)                                  # cdb (genG42.006,genG42)
substitute      (genG42,$D_{a}{x^b}->\delta_{a}^{b}$)     # cdb (genG42.007,genG42)
eliminate_kronecker (genG42)                              # cdb (genG42.008,genG42)
sym             (genG42,$_{b}, _{c1}, _{c2}, _{d}$)
sort_product    (genG42)                                  # cdb (genG42.009,genG42)
rename_dummies (genG42)                                   # cdb (genG42.010,genG42)
canonicalise    (genG42)                                  # cdb (genG42.011,genG42)

# ---------------------------------------------------------------------------
substitute      (genG52,defG52)
substitute      (genG52,defG51)
```

```
substitute      (genG52,defG31,repeat=True)
substitute      (genG52,defG20,repeat=True)

distribute      (genG52)
unwrap          (genG52)
product_rule    (genG52)
distribute      (genG52)
substitute      (genG52,$D_{a}{x^b}->\delta_{a}^{b}$)
eliminate_kronecker (genG52)
sym             (genG52,$_{b}, _{c1}, _{c2}, _{d}$)
sort_product    (genG52)
rename_dummies  (genG52)
canonicalise    (genG52)                                    # cdb (genG52.001,genG52)

# update the rules

defG42 := genG4^{a}_{b c1 c2 d} -> @(genG42).
defG52 := genG5^{a}_{b c1 c2 d} -> @(genG52).

# ===========================================================================
# n = 3

genG53 := genG5^{a}_{b c1 c2 c3 d}.
# genG63 := genG6^{a}_{b c1 c2 c3 d}.
# genG73 := genG7^{a}_{b c1 c2 c3 d}.

# ---------------------------------------------------------------------------
substitute      (genG53,defG53)
substitute      (genG53,defG52)

distribute      (genG53)
unwrap          (genG53)
product_rule    (genG53)
distribute      (genG53)
substitute      (genG53,$D_{a}{x^b}->\delta_{a}^{b}$)
eliminate_kronecker (genG53)
sym             (genG53,$_{b}, _{c1}, _{c2}, _{c3}, _{d}$)
sort_product    (genG53)
```

```
rename_dummies (genG53)
canonicalise   (genG53)                                    # cdb (genG53.001,genG53)

# update the rules

defG53 := genG5^{a}_{b c1 c2 c3 d} -> @(genG53).
```

$$\texttt{genG31.000} := genG3^{a}{}_{bc1d}$$

$$\texttt{genG31.001} := D_d\left(genG3^{a}{}_{bc1}\right)$$

$$\texttt{genG31.002} := \frac{1}{12}D_d\left(x^c x^e \nabla_b R_{c1cef} g^{af}\right) + \frac{1}{6}D_d\left(x^c x^e \nabla_c R_{bfc1e} g^{af}\right) + \frac{1}{12}D_d\left(x^c x^e \nabla_{c1} R_{bcef} g^{af}\right) + \frac{1}{6}D_d\left(x^c x^e \nabla_c R_{bec1f} g^{af}\right) + \frac{1}{12}D_d\left(x^c x^e \nabla_f R_{bcc1e} g^{af}\right)$$

$$\texttt{genG31.003} := \frac{1}{12}\nabla_b R_{c1cef} g^{af} D_d\left(x^c x^e\right) + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} D_d\left(x^c x^e\right) + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} D_d\left(x^c x^e\right) + \frac{1}{6}\nabla_c R_{bec1f} g^{af} D_d\left(x^c x^e\right) + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} D_d\left(x^c x^e\right)$$

$$\begin{aligned}\texttt{genG31.004} := {} & \frac{1}{12}\nabla_b R_{c1cef} g^{af}\left(D_d x^c x^e + x^c D_d x^e\right) + \frac{1}{6}\nabla_c R_{bfc1e} g^{af}\left(D_d x^c x^e + x^c D_d x^e\right) + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af}\left(D_d x^c x^e + x^c D_d x^e\right) \\ & + \frac{1}{6}\nabla_c R_{bec1f} g^{af}\left(D_d x^c x^e + x^c D_d x^e\right) + \frac{1}{12}\nabla_f R_{bcc1e} g^{af}\left(D_d x^c x^e + x^c D_d x^e\right)\end{aligned}$$

$$\begin{aligned}\texttt{genG31.005} := {} & \frac{1}{12}\nabla_b R_{c1cef} g^{af} D_d x^c x^e + \frac{1}{12}\nabla_b R_{c1cef} g^{af} x^c D_d x^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} D_d x^c x^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} x^c D_d x^e + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} D_d x^c x^e \\ & + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} x^c D_d x^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} D_d x^c x^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} x^c D_d x^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} D_d x^c x^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} x^c D_d x^e\end{aligned}$$

$$\begin{aligned}\texttt{genG31.006} := {} & \frac{1}{12}\nabla_b R_{c1cef} g^{af} \delta_d{}^c x^e + \frac{1}{12}\nabla_b R_{c1cef} g^{af} x^c \delta_d{}^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} \delta_d{}^c x^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} x^c \delta_d{}^e + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} \delta_d{}^c x^e \\ & + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} x^c \delta_d{}^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} \delta_d{}^c x^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} x^c \delta_d{}^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} \delta_d{}^c x^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} x^c \delta_d{}^e\end{aligned}$$

$$\begin{aligned}\texttt{genG31.007} := {} & \frac{1}{12}\nabla_b R_{c1def} g^{af} x^e + \frac{1}{12}\nabla_b R_{c1cdf} g^{af} x^c + \frac{1}{6}\nabla_d R_{bfc1e} g^{af} x^e + \frac{1}{6}\nabla_c R_{bfc1d} g^{af} x^c + \frac{1}{12}\nabla_{c1} R_{bdef} g^{af} x^e \\ & + \frac{1}{12}\nabla_{c1} R_{bcdf} g^{af} x^c + \frac{1}{6}\nabla_d R_{bec1f} g^{af} x^e + \frac{1}{6}\nabla_c R_{bdc1f} g^{af} x^c + \frac{1}{12}\nabla_f R_{bdc1e} g^{af} x^e + \frac{1}{12}\nabla_f R_{bcc1d} g^{af} x^c\end{aligned}$$

$$\texttt{genG31.008} := \frac{1}{36}\nabla_b R_{c1def} g^{af} x^e + \frac{1}{36}\nabla_b R_{dc1ef} g^{af} x^e + \frac{1}{36}\nabla_{c1} R_{bdef} g^{af} x^e + \frac{1}{36}\nabla_{c1} R_{dbef} g^{af} x^e + \frac{1}{36}\nabla_d R_{bc1ef} g^{af} x^e + \frac{1}{36}\nabla_d R_{c1bef} g^{af} x^e$$
$$+ \frac{1}{36}\nabla_b R_{c1cdf} g^{af} x^c + \frac{1}{36}\nabla_b R_{dcc1f} g^{af} x^c + \frac{1}{36}\nabla_{c1} R_{bcdf} g^{af} x^c + \frac{1}{36}\nabla_{c1} R_{dcbf} g^{af} x^c + \frac{1}{36}\nabla_d R_{bcc1f} g^{af} x^c + \frac{1}{36}\nabla_d R_{c1cbf} g^{af} x^c$$
$$+ \frac{1}{36}\nabla_d R_{bfc1e} g^{af} x^e + \frac{1}{36}\nabla_{c1} R_{bfde} g^{af} x^e + \frac{1}{36}\nabla_d R_{c1fbe} g^{af} x^e + \frac{1}{36}\nabla_b R_{c1fde} g^{af} x^e + \frac{1}{36}\nabla_{c1} R_{dfbe} g^{af} x^e + \frac{1}{36}\nabla_b R_{dfc1e} g^{af} x^e$$
$$+ \frac{1}{6}\nabla_c \left(\frac{1}{6}R_{bfc1d} + \frac{1}{6}R_{bfdc1} + \frac{1}{6}R_{c1fbd} + \frac{1}{6}R_{c1fdb} + \frac{1}{6}R_{dfbc1} + \frac{1}{6}R_{dfc1b}\right) g^{af} x^c + \frac{1}{36}\nabla_d R_{bec1f} g^{af} x^e + \frac{1}{36}\nabla_{c1} R_{bedf} g^{af} x^e + \frac{1}{36}\nabla_d R_{c1ebf} g^{af} x^e$$
$$+ \frac{1}{36}\nabla_b R_{c1edf} g^{af} x^e + \frac{1}{36}\nabla_{c1} R_{debf} g^{af} x^e + \frac{1}{36}\nabla_b R_{dec1f} g^{af} x^e + \frac{1}{6}\nabla_c \left(\frac{1}{6}R_{bdc1f} + \frac{1}{6}R_{bc1df} + \frac{1}{6}R_{c1dbf} + \frac{1}{6}R_{c1bdf} + \frac{1}{6}R_{dc1bf} + \frac{1}{6}R_{dbc1f}\right) g^{af} x^c$$
$$+ \frac{1}{12}\nabla_f \left(\frac{1}{6}R_{bdc1e} + \frac{1}{6}R_{bc1de} + \frac{1}{6}R_{c1dbe} + \frac{1}{6}R_{c1bde} + \frac{1}{6}R_{dc1be} + \frac{1}{6}R_{dbc1e}\right) g^{af} x^e$$
$$+ \frac{1}{12}\nabla_f \left(\frac{1}{6}R_{bcc1d} + \frac{1}{6}R_{bcdc1} + \frac{1}{6}R_{c1cbd} + \frac{1}{6}R_{c1cdb} + \frac{1}{6}R_{dcbc1} + \frac{1}{6}R_{dcc1b}\right) g^{af} x^c$$

$$\texttt{genG31.009} := \frac{1}{36}\nabla_b R_{c1dce} g^{ae} x^c + \frac{1}{36}\nabla_b R_{dc1ce} g^{ae} x^c + \frac{1}{36}\nabla_{c1} R_{bdce} g^{ae} x^c + \frac{1}{36}\nabla_{c1} R_{dbce} g^{ae} x^c + \frac{1}{36}\nabla_d R_{bc1ce} g^{ae} x^c + \frac{1}{36}\nabla_d R_{c1bce} g^{ae} x^c$$
$$+ \frac{1}{18}\nabla_b R_{c1cde} g^{ae} x^c + \frac{1}{18}\nabla_b R_{dcc1e} g^{ae} x^c + \frac{1}{18}\nabla_{c1} R_{bcde} g^{ae} x^c + \frac{1}{18}\nabla_{c1} R_{dcbe} g^{ae} x^c + \frac{1}{18}\nabla_d R_{bcc1e} g^{ae} x^c$$
$$+ \frac{1}{18}\nabla_d R_{c1cbe} g^{ae} x^c + \frac{1}{36}\nabla_d R_{bcc1e} g^{ac} x^e + \frac{1}{36}\nabla_{c1} R_{bcde} g^{ac} x^e + \frac{1}{36}\nabla_d R_{c1cbe} g^{ac} x^e + \frac{1}{36}\nabla_b R_{c1cde} g^{ac} x^e + \frac{1}{36}\nabla_{c1} R_{dcbe} g^{ac} x^e$$
$$+ \frac{1}{36}\nabla_b R_{dcc1e} g^{ac} x^e + \frac{1}{6}\nabla_e \left(\frac{1}{6}R_{bcc1d} + \frac{1}{6}R_{bcdc1} + \frac{1}{6}R_{c1cbd} + \frac{1}{6}R_{c1cdb} + \frac{1}{6}R_{dcbc1} + \frac{1}{6}R_{dcc1b}\right) g^{ac} x^e$$
$$+ \frac{1}{6}\nabla_e \left(\frac{1}{6}R_{bdc1c} + \frac{1}{6}R_{bc1dc} + \frac{1}{6}R_{c1dbc} + \frac{1}{6}R_{c1bdc} + \frac{1}{6}R_{dc1bc} + \frac{1}{6}R_{dbc1c}\right) g^{ac} x^e$$
$$+ \frac{1}{12}\nabla_e \left(\frac{1}{6}R_{bdc1c} + \frac{1}{6}R_{bc1dc} + \frac{1}{6}R_{c1dbc} + \frac{1}{6}R_{c1bdc} + \frac{1}{6}R_{dc1bc} + \frac{1}{6}R_{dbc1c}\right) g^{ae} x^c$$
$$+ \frac{1}{12}\nabla_e \left(\frac{1}{6}R_{bcc1d} + \frac{1}{6}R_{bcdc1} + \frac{1}{6}R_{c1cbd} + \frac{1}{6}R_{c1cdb} + \frac{1}{6}R_{dcbc1} + \frac{1}{6}R_{dcc1b}\right) g^{ae} x^c$$

$$\texttt{genG31.010} := \frac{1}{12}\nabla_b R_{c1cde} g^{ae} x^c + \frac{1}{12}\nabla_b R_{c1cde} g^{ac} x^e + \frac{1}{12}\nabla_{c1} R_{bcde} g^{ae} x^c + \frac{1}{12}\nabla_{c1} R_{bcde} g^{ac} x^e + \frac{1}{12}\nabla_d R_{bcc1e} g^{ae} x^c + \frac{1}{12}\nabla_d R_{bcc1e} g^{ac} x^e$$

$$\texttt{genG41.000} := genG4^{a}{}_{bc1d}$$

$$\texttt{genG41.001} := D_{d}\left(genG4^{a}{}_{bc1}\right) - 2genG2^{a}{}_{pc1}genG2^{p}{}_{bd}$$

$$\begin{aligned}\texttt{genG41.002} := D_{d}\Bigg(x^{c}x^{e}x^{f}\Bigg(&\frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi} + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi} - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi} - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi} + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}\\&+ \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag} + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag} - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi} - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh} + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag} + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}\\&+ \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag} - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi} - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi} + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag} + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}\Bigg)\Bigg) - 2genG2^{a}{}_{pc1}genG2^{p}{}_{bd}\end{aligned}$$

$$\begin{aligned}\texttt{genG41.003} := D_{d}\Bigg(x^{c}x^{e}x^{f}\Bigg(&\frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi} + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi} - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi} - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi} + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}\\&+ \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag} + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag} - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi} - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh} + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag} + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}\\&+ \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag} - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi} - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi} + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag} + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}\Bigg)\Bigg)\\&- 2x^{c}\left(\frac{1}{3}R_{pec1c}g^{ae} + \frac{1}{3}R_{pcc1e}g^{ae}\right)x^{f}\left(\frac{1}{3}R_{bgdf}g^{pg} + \frac{1}{3}R_{bfdg}g^{pg}\right)\end{aligned}$$

$$\begin{aligned}\texttt{genG41.004} := &\frac{4}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bgc1c}R_{ehfi}g^{ah}g^{gi}\right) + \frac{4}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bcc1g}R_{ehfi}g^{ah}g^{gi}\right) - \frac{2}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bgch}R_{c1efi}g^{ag}g^{hi}\right)\\&- \frac{1}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bgch}R_{c1efi}g^{ah}g^{gi}\right) + \frac{1}{40}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{bc}R_{c1efg}g^{ag}\right) + \frac{1}{40}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{cb}R_{c1efg}g^{ag}\right) + \frac{1}{20}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{ce}R_{bgc1f}g^{ag}\right)\\&- \frac{2}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bceg}R_{c1hfi}g^{ah}g^{gi}\right) - \frac{1}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bceg}R_{c1hfi}g^{ai}g^{gh}\right) + \frac{1}{40}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{c1c}R_{befg}g^{ag}\right)\\&+ \frac{1}{40}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{cc1}R_{befg}g^{ag}\right) + \frac{1}{20}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{ce}R_{bfc1g}g^{ag}\right) - \frac{1}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bcgh}R_{c1efi}g^{ag}g^{hi}\right)\\&- \frac{1}{45}D_{d}\left(x^{c}x^{e}x^{f}R_{bceg}R_{c1fhi}g^{ah}g^{gi}\right) + \frac{1}{40}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{gc}R_{bec1f}g^{ag}\right) + \frac{1}{40}D_{d}\left(x^{c}x^{e}x^{f}\nabla_{cg}R_{bec1f}g^{ag}\right)\\&- \frac{2}{9}x^{c}R_{pec1c}g^{ae}x^{f}R_{bgdf}g^{pg} - \frac{2}{9}x^{c}R_{pec1c}g^{ae}x^{f}R_{bfdg}g^{pg} - \frac{2}{9}x^{c}R_{pcc1e}g^{ae}x^{f}R_{bgdf}g^{pg} - \frac{2}{9}x^{c}R_{pcc1e}g^{ae}x^{f}R_{bfdg}g^{pg}\end{aligned}$$

$$
\begin{aligned}
\texttt{genG41.005} :=\ & \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} D_d \left(x^c x^e x^f\right) + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} D_d \left(x^c x^e x^f\right) - \frac{2}{45} R_{bgch} R_{c1efi} g^{ag} g^{hi} D_d \left(x^c x^e x^f\right) \\
& - \frac{1}{45} R_{bgch} R_{c1efi} g^{ah} g^{gi} D_d \left(x^c x^e x^f\right) + \frac{1}{40} \nabla_{bc} R_{c1efg} g^{ag} D_d \left(x^c x^e x^f\right) + \frac{1}{40} \nabla_{cb} R_{c1efg} g^{ag} D_d \left(x^c x^e x^f\right) + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} D_d \left(x^c x^e x^f\right) \\
& - \frac{2}{45} R_{bceg} R_{c1hfi} g^{ah} g^{gi} D_d \left(x^c x^e x^f\right) - \frac{1}{45} R_{bceg} R_{c1hfi} g^{ai} g^{gh} D_d \left(x^c x^e x^f\right) + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} D_d \left(x^c x^e x^f\right) \\
& + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} D_d \left(x^c x^e x^f\right) + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} D_d \left(x^c x^e x^f\right) - \frac{1}{45} R_{bcgh} R_{c1efi} g^{ag} g^{hi} D_d \left(x^c x^e x^f\right) \\
& - \frac{1}{45} R_{bceg} R_{c1fhi} g^{ah} g^{gi} D_d \left(x^c x^e x^f\right) + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} D_d \left(x^c x^e x^f\right) + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} D_d \left(x^c x^e x^f\right) \\
& - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG41.006} :=\ & \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& - \frac{2}{45} R_{bgch} R_{c1efi} g^{ag} g^{hi} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) - \frac{1}{45} R_{bgch} R_{c1efi} g^{ah} g^{gi} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& + \frac{1}{40} \nabla_{bc} R_{c1efg} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) + \frac{1}{40} \nabla_{cb} R_{c1efg} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) - \frac{2}{45} R_{bceg} R_{c1hfi} g^{ah} g^{gi} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& - \frac{1}{45} R_{bceg} R_{c1hfi} g^{ai} g^{gh} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& - \frac{1}{45} R_{bcgh} R_{c1efi} g^{ag} g^{hi} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) - \frac{1}{45} R_{bceg} R_{c1fhi} g^{ah} g^{gi} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} \left(D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f\right) \\
& - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG41.007} := {} & \frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi}D_dx^cx^ex^f + \frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi}x^cD_dx^ex^f + \frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi}x^cx^eD_dx^f + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi}D_dx^cx^ex^f \\
& + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi}x^cD_dx^ex^f + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi}x^cx^eD_dx^f - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi}D_dx^cx^ex^f - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi}x^cD_dx^ex^f \\
& - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi}x^cx^eD_dx^f - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi}D_dx^cx^ex^f - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi}x^cD_dx^ex^f - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi}x^cx^eD_dx^f \\
& + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}D_dx^cx^ex^f + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}x^cD_dx^ex^f + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}x^cx^eD_dx^f + \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag}D_dx^cx^ex^f \\
& + \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag}x^cD_dx^ex^f + \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag}x^cx^eD_dx^f + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag}D_dx^cx^ex^f + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag}x^cD_dx^ex^f \\
& + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag}x^cx^eD_dx^f - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi}D_dx^cx^ex^f - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi}x^cD_dx^ex^f - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi}x^cx^eD_dx^f \\
& - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh}D_dx^cx^ex^f - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh}x^cD_dx^ex^f - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh}x^cx^eD_dx^f + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag}D_dx^cx^ex^f \\
& + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag}x^cD_dx^ex^f + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag}x^cx^eD_dx^f + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}D_dx^cx^ex^f + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}x^cD_dx^ex^f \\
& + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}x^cx^eD_dx^f + \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag}D_dx^cx^ex^f + \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag}x^cD_dx^ex^f + \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag}x^cx^eD_dx^f \\
& - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi}D_dx^cx^ex^f - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi}x^cD_dx^ex^f - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi}x^cx^eD_dx^f - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi}D_dx^cx^ex^f \\
& - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi}x^cD_dx^ex^f - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi}x^cx^eD_dx^f + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag}D_dx^cx^ex^f + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag}x^cD_dx^ex^f \\
& + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag}x^cx^eD_dx^f + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}D_dx^cx^ex^f + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}x^cD_dx^ex^f + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}x^cx^eD_dx^f \\
& - \frac{2}{9}x^cR_{pec1c}g^{ae}x^fR_{bgdf}g^{pg} - \frac{2}{9}x^cR_{pec1c}g^{ae}x^fR_{bfdg}g^{pg} - \frac{2}{9}x^cR_{pcc1e}g^{ae}x^fR_{bgdf}g^{pg} - \frac{2}{9}x^cR_{pcc1e}g^{ae}x^fR_{bfdg}g^{pg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG41.008} := {} & \frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi}\delta_d{}^c x^e x^f + \frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi}x^c\delta_d{}^e x^f + \frac{4}{45}R_{bgc1c}R_{ehfi}g^{ah}g^{gi}x^c x^e\delta_d{}^f + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi}\delta_d{}^c x^e x^f \\
& + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi}x^c\delta_d{}^e x^f + \frac{4}{45}R_{bcc1g}R_{ehfi}g^{ah}g^{gi}x^c x^e\delta_d{}^f - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi}\delta_d{}^c x^e x^f - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi}x^c\delta_d{}^e x^f \\
& - \frac{2}{45}R_{bgch}R_{c1efi}g^{ag}g^{hi}x^c x^e\delta_d{}^f - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi}\delta_d{}^c x^e x^f - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi}x^c\delta_d{}^e x^f - \frac{1}{45}R_{bgch}R_{c1efi}g^{ah}g^{gi}x^c x^e\delta_d{}^f \\
& + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}\delta_d{}^c x^e x^f + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}x^c\delta_d{}^e x^f + \frac{1}{40}\nabla_{bc}R_{c1efg}g^{ag}x^c x^e\delta_d{}^f + \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag}\delta_d{}^c x^e x^f \\
& + \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag}x^c\delta_d{}^e x^f + \frac{1}{40}\nabla_{cb}R_{c1efg}g^{ag}x^c x^e\delta_d{}^f + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag}\delta_d{}^c x^e x^f + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag}x^c\delta_d{}^e x^f + \frac{1}{20}\nabla_{ce}R_{bgc1f}g^{ag}x^c x^e\delta_d{}^f \\
& - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi}\delta_d{}^c x^e x^f - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi}x^c\delta_d{}^e x^f - \frac{2}{45}R_{bceg}R_{c1hfi}g^{ah}g^{gi}x^c x^e\delta_d{}^f - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh}\delta_d{}^c x^e x^f \\
& - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh}x^c\delta_d{}^e x^f - \frac{1}{45}R_{bceg}R_{c1hfi}g^{ai}g^{gh}x^c x^e\delta_d{}^f + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag}\delta_d{}^c x^e x^f + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag}x^c\delta_d{}^e x^f \\
& + \frac{1}{40}\nabla_{c1c}R_{befg}g^{ag}x^c x^e\delta_d{}^f + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}\delta_d{}^c x^e x^f + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}x^c\delta_d{}^e x^f + \frac{1}{40}\nabla_{cc1}R_{befg}g^{ag}x^c x^e\delta_d{}^f + \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag}\delta_d{}^c x^e x^f \\
& + \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag}x^c\delta_d{}^e x^f + \frac{1}{20}\nabla_{ce}R_{bfc1g}g^{ag}x^c x^e\delta_d{}^f - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi}\delta_d{}^c x^e x^f - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi}x^c\delta_d{}^e x^f \\
& - \frac{1}{45}R_{bcgh}R_{c1efi}g^{ag}g^{hi}x^c x^e\delta_d{}^f - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi}\delta_d{}^c x^e x^f - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi}x^c\delta_d{}^e x^f - \frac{1}{45}R_{bceg}R_{c1fhi}g^{ah}g^{gi}x^c x^e\delta_d{}^f \\
& + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag}\delta_d{}^c x^e x^f + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag}x^c\delta_d{}^e x^f + \frac{1}{40}\nabla_{gc}R_{bec1f}g^{ag}x^c x^e\delta_d{}^f + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}\delta_d{}^c x^e x^f + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}x^c\delta_d{}^e x^f \\
& + \frac{1}{40}\nabla_{cg}R_{bec1f}g^{ag}x^c x^e\delta_d{}^f - \frac{2}{9}x^c R_{pec1c}g^{ae}x^f R_{bgdf}g^{pg} - \frac{2}{9}x^c R_{pec1c}g^{ae}x^f R_{bfdg}g^{pg} - \frac{2}{9}x^c R_{pcc1e}g^{ae}x^f R_{bgdf}g^{pg} - \frac{2}{9}x^c R_{pcc1e}g^{ae}x^f R_{bfdg}g^{pg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG41.009} := {} & \frac{4}{45}R_{bgc1d}R_{ehfi}g^{ah}g^{gi}x^{e}x^{f} + \frac{4}{45}R_{bgc1c}R_{dhfi}g^{ah}g^{gi}x^{c}x^{f} + \frac{4}{45}R_{bgc1c}R_{ehdi}g^{ah}g^{gi}x^{c}x^{e} + \frac{4}{45}R_{bdc1g}R_{ehfi}g^{ah}g^{gi}x^{e}x^{f} \\
& + \frac{4}{45}R_{bcc1g}R_{dhfi}g^{ah}g^{gi}x^{c}x^{f} + \frac{4}{45}R_{bcc1g}R_{ehdi}g^{ah}g^{gi}x^{c}x^{e} - \frac{2}{45}R_{bgdh}R_{c1efi}g^{ag}g^{hi}x^{e}x^{f} - \frac{2}{45}R_{bgch}R_{c1dfi}g^{ag}g^{hi}x^{c}x^{f} \\
& - \frac{2}{45}R_{bgch}R_{c1edi}g^{ag}g^{hi}x^{c}x^{e} - \frac{1}{45}R_{bgdh}R_{c1efi}g^{ah}g^{gi}x^{e}x^{f} - \frac{1}{45}R_{bgch}R_{c1dfi}g^{ah}g^{gi}x^{c}x^{f} - \frac{1}{45}R_{bgch}R_{c1edi}g^{ah}g^{gi}x^{c}x^{e} + \frac{1}{40}\nabla_{bd}R_{c1efg}g^{ag}x^{e}x^{f} \\
& + \frac{1}{40}\nabla_{bc}R_{c1dfg}g^{ag}x^{c}x^{f} + \frac{1}{40}\nabla_{bc}R_{c1edg}g^{ag}x^{c}x^{e} + \frac{1}{40}\nabla_{db}R_{c1efg}g^{ag}x^{e}x^{f} + \frac{1}{40}\nabla_{cb}R_{c1dfg}g^{ag}x^{c}x^{f} + \frac{1}{40}\nabla_{cb}R_{c1edg}g^{ag}x^{c}x^{e} \\
& + \frac{1}{20}\nabla_{de}R_{bgc1f}g^{ag}x^{e}x^{f} + \frac{1}{20}\nabla_{cd}R_{bgc1f}g^{ag}x^{c}x^{f} + \frac{1}{20}\nabla_{ce}R_{bgc1d}g^{ag}x^{c}x^{e} - \frac{2}{45}R_{bdeg}R_{c1hfi}g^{ah}g^{gi}x^{e}x^{f} - \frac{2}{45}R_{bcdg}R_{c1hfi}g^{ah}g^{gi}x^{c}x^{f} \\
& - \frac{2}{45}R_{bceg}R_{c1hdi}g^{ah}g^{gi}x^{c}x^{e} - \frac{1}{45}R_{bdeg}R_{c1hfi}g^{ai}g^{gh}x^{e}x^{f} - \frac{1}{45}R_{bcdg}R_{c1hfi}g^{ai}g^{gh}x^{c}x^{f} - \frac{1}{45}R_{bceg}R_{c1hdi}g^{ai}g^{gh}x^{c}x^{e} + \frac{1}{40}\nabla_{c1d}R_{befg}g^{ag}x^{e}x^{f} \\
& + \frac{1}{40}\nabla_{c1c}R_{bdfg}g^{ag}x^{c}x^{f} + \frac{1}{40}\nabla_{c1c}R_{bedg}g^{ag}x^{c}x^{e} + \frac{1}{40}\nabla_{dc1}R_{befg}g^{ag}x^{e}x^{f} + \frac{1}{40}\nabla_{cc1}R_{bdfg}g^{ag}x^{c}x^{f} + \frac{1}{40}\nabla_{cc1}R_{bedg}g^{ag}x^{c}x^{e} \\
& + \frac{1}{20}\nabla_{de}R_{bfc1g}g^{ag}x^{e}x^{f} + \frac{1}{20}\nabla_{cd}R_{bfc1g}g^{ag}x^{c}x^{f} + \frac{1}{20}\nabla_{ce}R_{bdc1g}g^{ag}x^{c}x^{e} - \frac{1}{45}R_{bdgh}R_{c1efi}g^{ag}g^{hi}x^{e}x^{f} - \frac{1}{45}R_{bcgh}R_{c1dfi}g^{ag}g^{hi}x^{c}x^{f} \\
& - \frac{1}{45}R_{bcgh}R_{c1edi}g^{ag}g^{hi}x^{c}x^{e} - \frac{1}{45}R_{bdeg}R_{c1fhi}g^{ah}g^{gi}x^{e}x^{f} - \frac{1}{45}R_{bcdg}R_{c1fhi}g^{ah}g^{gi}x^{c}x^{f} - \frac{1}{45}R_{bceg}R_{c1dhi}g^{ah}g^{gi}x^{c}x^{e} \\
& + \frac{1}{40}\nabla_{gd}R_{bec1f}g^{ag}x^{e}x^{f} + \frac{1}{40}\nabla_{gc}R_{bdc1f}g^{ag}x^{c}x^{f} + \frac{1}{40}\nabla_{gc}R_{bec1d}g^{ag}x^{c}x^{e} + \frac{1}{40}\nabla_{dg}R_{bec1f}g^{ag}x^{e}x^{f} + \frac{1}{40}\nabla_{cg}R_{bdc1f}g^{ag}x^{c}x^{f} \\
& + \frac{1}{40}\nabla_{cg}R_{bec1d}g^{ag}x^{c}x^{e} - \frac{2}{9}x^{c}R_{pec1c}g^{ae}x^{f}R_{bgdf}g^{pg} - \frac{2}{9}x^{c}R_{pec1c}g^{ae}x^{f}R_{bfdg}g^{pg} - \frac{2}{9}x^{c}R_{pcc1e}g^{ae}x^{f}R_{bgdf}g^{pg} - \frac{2}{9}x^{c}R_{pcc1e}g^{ae}x^{f}R_{bfdg}g^{pg}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG41.012} := &-\frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{cg}x^{e}x^{h}-\frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{cg}x^{e}x^{h}-\frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{eg}x^{c}x^{h}-\frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eg}x^{f}x^{h}\\
&-\frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{eg}x^{c}x^{h}-\frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eh}x^{f}x^{g}-\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{cf}x^{e}x^{h}-\frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{cf}x^{e}x^{h}\\
&-\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{ef}x^{c}x^{h}-\frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{cg}x^{f}x^{h}-\frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{ef}x^{c}x^{h}-\frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{ch}x^{f}x^{g}\\
&+\frac{1}{45}R_{bcde}R_{c1fgh}g^{ac}g^{eg}x^{f}x^{h}+\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ac}g^{eg}x^{f}x^{h}+\frac{1}{45}R_{bcef}R_{c1gdh}g^{ag}g^{eh}x^{c}x^{f}+\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ae}g^{cg}x^{f}x^{h}\\
&+\frac{1}{45}R_{bcef}R_{c1gdh}g^{ah}g^{eg}x^{c}x^{f}+\frac{1}{45}R_{bcde}R_{c1fgh}g^{ae}g^{cg}x^{f}x^{h}-\frac{1}{60}\nabla_{bd}R_{c1cef}g^{ae}x^{c}x^{f}-\frac{1}{60}\nabla_{bc1}R_{dcef}g^{ae}x^{c}x^{f}\\
&-\frac{1}{60}\nabla_{c1d}R_{bcef}g^{ae}x^{c}x^{f}-\frac{1}{60}\nabla_{c1b}R_{dcef}g^{ae}x^{c}x^{f}-\frac{1}{60}\nabla_{dc1}R_{bcef}g^{ae}x^{c}x^{f}-\frac{1}{60}\nabla_{db}R_{c1cef}g^{ae}x^{c}x^{f}+\frac{1}{40}\nabla_{bc}R_{c1edf}g^{af}x^{c}x^{e}\\
&+\frac{1}{40}\nabla_{bc}R_{c1edf}g^{ae}x^{c}x^{f}+\frac{1}{40}\nabla_{c1c}R_{bedf}g^{af}x^{c}x^{e}+\frac{1}{40}\nabla_{c1c}R_{bedf}g^{ae}x^{c}x^{f}+\frac{1}{40}\nabla_{dc}R_{bec1f}g^{af}x^{c}x^{e}+\frac{1}{40}\nabla_{dc}R_{bec1f}g^{ae}x^{c}x^{f}\\
&+\frac{1}{40}\nabla_{cb}R_{c1edf}g^{af}x^{c}x^{e}+\frac{1}{40}\nabla_{cb}R_{c1edf}g^{ae}x^{c}x^{f}+\frac{1}{40}\nabla_{cc1}R_{bedf}g^{af}x^{c}x^{e}+\frac{1}{40}\nabla_{cc1}R_{bedf}g^{ae}x^{c}x^{f}+\frac{1}{40}\nabla_{cd}R_{bec1f}g^{af}x^{c}x^{e}\\
&+\frac{1}{40}\nabla_{cd}R_{bec1f}g^{ae}x^{c}x^{f}+\frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fh}x^{c}x^{g}+\frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fg}x^{c}x^{h}+\frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{eh}x^{c}x^{f}\\
&+\frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{ch}x^{e}x^{f}+\frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{eh}x^{c}x^{f}+\frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{ch}x^{e}x^{f}+\frac{1}{120}\nabla_{cd}R_{bec1f}g^{ac}x^{e}x^{f}\\
&+\frac{1}{120}\nabla_{cc1}R_{bedf}g^{ac}x^{e}x^{f}+\frac{1}{120}\nabla_{cb}R_{c1edf}g^{ac}x^{e}x^{f}+\frac{1}{120}\nabla_{dc}R_{bec1f}g^{ac}x^{e}x^{f}+\frac{1}{120}\nabla_{c1c}R_{bedf}g^{ac}x^{e}x^{f}+\frac{1}{120}\nabla_{bc}R_{c1edf}g^{ac}x^{e}x^{f}
\end{aligned}
$$

$$\texttt{genG42.000} := genG4^{a}{}_{bc1c2d}$$

$$\texttt{genG42.001} := D_d\left(genG4^{a}{}_{bc1c2}\right)$$

$$\texttt{genG42.002} := D_d\Big(-\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{cg}x^ex^h-\frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{cg}x^ex^h-\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{eg}x^cx^h-\frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eg}x^fx^h$$
$$-\frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{eg}x^cx^h-\frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eh}x^fx^g-\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{cf}x^ex^h-\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{cf}x^ex^h$$
$$-\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{ef}x^cx^h-\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{cg}x^fx^h-\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{ef}x^cx^h-\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{ch}x^fx^g$$
$$+\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ac}g^{eg}x^fx^h+\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ac}g^{eg}x^fx^h+\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ag}g^{eh}x^cx^f+\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ae}g^{cg}x^fx^h$$
$$+\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ah}g^{eg}x^cx^f+\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ae}g^{cg}x^fx^h-\frac{1}{60}\nabla_{bc2}R_{c1cef}g^{ae}x^cx^f-\frac{1}{60}\nabla_{bc1}R_{c2cef}g^{ae}x^cx^f-\frac{1}{60}\nabla_{c1c2}R_{bcef}g^{ae}x^cx^f$$
$$-\frac{1}{60}\nabla_{c1b}R_{c2cef}g^{ae}x^cx^f-\frac{1}{60}\nabla_{c2c1}R_{bcef}g^{ae}x^cx^f-\frac{1}{60}\nabla_{c2b}R_{c1cef}g^{ae}x^cx^f+\frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{af}x^cx^e+\frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{ae}x^cx^f$$
$$+\frac{1}{40}\nabla_{c1c}R_{bec2f}g^{af}x^cx^e+\frac{1}{40}\nabla_{c1c}R_{bec2f}g^{ae}x^cx^f+\frac{1}{40}\nabla_{c2c}R_{bec1f}g^{af}x^cx^e+\frac{1}{40}\nabla_{c2c}R_{bec1f}g^{ae}x^cx^f+\frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{af}x^cx^e$$
$$+\frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{ae}x^cx^f+\frac{1}{40}\nabla_{cc1}R_{bec2f}g^{af}x^cx^e+\frac{1}{40}\nabla_{cc1}R_{bec2f}g^{ae}x^cx^f+\frac{1}{40}\nabla_{cc2}R_{bec1f}g^{af}x^cx^e+\frac{1}{40}\nabla_{cc2}R_{bec1f}g^{ae}x^cx^f$$
$$+\frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fh}x^cx^g+\frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fg}x^cx^h+\frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{eh}x^cx^f+\frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{ch}x^ex^f$$
$$+\frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{eh}x^cx^f+\frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{ch}x^ex^f+\frac{1}{120}\nabla_{cc2}R_{bec1f}g^{ac}x^ex^f+\frac{1}{120}\nabla_{cc1}R_{bec2f}g^{ac}x^ex^f$$
$$+\frac{1}{120}\nabla_{cb}R_{c1ec2f}g^{ac}x^ex^f+\frac{1}{120}\nabla_{c2c}R_{bec1f}g^{ac}x^ex^f+\frac{1}{120}\nabla_{c1c}R_{bec2f}g^{ac}x^ex^f+\frac{1}{120}\nabla_{bc}R_{c1ec2f}g^{ac}x^ex^f\Big)$$

$$
\begin{aligned}
\texttt{genG42.003} := &-\frac{4}{45}D_d\left(R_{bcc1e}R_{c2fgh}g^{af}g^{cg}x^e x^h\right) - \frac{4}{45}D_d\left(R_{bcc2e}R_{c1fgh}g^{af}g^{cg}x^e x^h\right) - \frac{4}{45}D_d\left(R_{bcc1e}R_{c2fgh}g^{af}g^{eg}x^c x^h\right)\\
&- \frac{4}{45}D_d\left(R_{bcef}R_{c1gc2h}g^{ac}g^{eg}x^f x^h\right) - \frac{4}{45}D_d\left(R_{bcc2e}R_{c1fgh}g^{af}g^{eg}x^c x^h\right) - \frac{4}{45}D_d\left(R_{bcef}R_{c1gc2h}g^{ac}g^{eh}x^f x^g\right)\\
&- \frac{1}{45}D_d\left(R_{bcc1e}R_{c2fgh}g^{ag}g^{cf}x^e x^h\right) - \frac{1}{45}D_d\left(R_{bcc2e}R_{c1fgh}g^{ag}g^{cf}x^e x^h\right) - \frac{1}{45}D_d\left(R_{bcc1e}R_{c2fgh}g^{ag}g^{ef}x^c x^h\right)\\
&- \frac{1}{45}D_d\left(R_{bcef}R_{c1gc2h}g^{ae}g^{cg}x^f x^h\right) - \frac{1}{45}D_d\left(R_{bcc2e}R_{c1fgh}g^{ag}g^{ef}x^c x^h\right) - \frac{1}{45}D_d\left(R_{bcef}R_{c1gc2h}g^{ae}g^{ch}x^f x^g\right)\\
&+ \frac{1}{45}D_d\left(R_{bcc2e}R_{c1fgh}g^{ac}g^{eg}x^f x^h\right) + \frac{1}{45}D_d\left(R_{bcc1e}R_{c2fgh}g^{ac}g^{eg}x^f x^h\right) + \frac{1}{45}D_d\left(R_{bcef}R_{c1gc2h}g^{ag}g^{eh}x^c x^f\right)\\
&+ \frac{1}{45}D_d\left(R_{bcc1e}R_{c2fgh}g^{ae}g^{cg}x^f x^h\right) + \frac{1}{45}D_d\left(R_{bcef}R_{c1gc2h}g^{ah}g^{eg}x^c x^f\right) + \frac{1}{45}D_d\left(R_{bcc2e}R_{c1fgh}g^{ae}g^{cg}x^f x^h\right)\\
&- \frac{1}{60}D_d\left(\nabla_{bc2}R_{c1cef}g^{ae}x^c x^f\right) - \frac{1}{60}D_d\left(\nabla_{bc1}R_{c2cef}g^{ae}x^c x^f\right) - \frac{1}{60}D_d\left(\nabla_{c1c2}R_{bcef}g^{ae}x^c x^f\right) - \frac{1}{60}D_d\left(\nabla_{c1b}R_{c2cef}g^{ae}x^c x^f\right)\\
&- \frac{1}{60}D_d\left(\nabla_{c2c1}R_{bcef}g^{ae}x^c x^f\right) - \frac{1}{60}D_d\left(\nabla_{c2b}R_{c1cef}g^{ae}x^c x^f\right) + \frac{1}{40}D_d\left(\nabla_{bc}R_{c1ec2f}g^{af}x^c x^e\right) + \frac{1}{40}D_d\left(\nabla_{bc}R_{c1ec2f}g^{ae}x^c x^f\right)\\
&+ \frac{1}{40}D_d\left(\nabla_{c1c}R_{bec2f}g^{af}x^c x^e\right) + \frac{1}{40}D_d\left(\nabla_{c1c}R_{bec2f}g^{ae}x^c x^f\right) + \frac{1}{40}D_d\left(\nabla_{c2c}R_{bec1f}g^{af}x^c x^e\right) + \frac{1}{40}D_d\left(\nabla_{c2c}R_{bec1f}g^{ae}x^c x^f\right)\\
&+ \frac{1}{40}D_d\left(\nabla_{cb}R_{c1ec2f}g^{af}x^c x^e\right) + \frac{1}{40}D_d\left(\nabla_{cb}R_{c1ec2f}g^{ae}x^c x^f\right) + \frac{1}{40}D_d\left(\nabla_{cc1}R_{bec2f}g^{af}x^c x^e\right) + \frac{1}{40}D_d\left(\nabla_{cc1}R_{bec2f}g^{ae}x^c x^f\right)\\
&+ \frac{1}{40}D_d\left(\nabla_{cc2}R_{bec1f}g^{af}x^c x^e\right) + \frac{1}{40}D_d\left(\nabla_{cc2}R_{bec1f}g^{ae}x^c x^f\right) + \frac{1}{15}D_d\left(R_{bcef}R_{c1gc2h}g^{ae}g^{fh}x^c x^g\right) + \frac{1}{15}D_d\left(R_{bcef}R_{c1gc2h}g^{ae}g^{fg}x^c x^h\right)\\
&+ \frac{1}{15}D_d\left(R_{bcc2e}R_{c1fgh}g^{ag}g^{eh}x^c x^f\right) + \frac{1}{15}D_d\left(R_{bcc2e}R_{c1fgh}g^{ag}g^{ch}x^e x^f\right) + \frac{1}{15}D_d\left(R_{bcc1e}R_{c2fgh}g^{ag}g^{eh}x^c x^f\right)\\
&+ \frac{1}{15}D_d\left(R_{bcc1e}R_{c2fgh}g^{ag}g^{ch}x^e x^f\right) + \frac{1}{120}D_d\left(\nabla_{cc2}R_{bec1f}g^{ac}x^e x^f\right) + \frac{1}{120}D_d\left(\nabla_{cc1}R_{bec2f}g^{ac}x^e x^f\right)\\
&+ \frac{1}{120}D_d\left(\nabla_{cb}R_{c1ec2f}g^{ac}x^e x^f\right) + \frac{1}{120}D_d\left(\nabla_{c2c}R_{bec1f}g^{ac}x^e x^f\right) + \frac{1}{120}D_d\left(\nabla_{c1c}R_{bec2f}g^{ac}x^e x^f\right) + \frac{1}{120}D_d\left(\nabla_{bc}R_{c1ec2f}g^{ac}x^e x^f\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG42.004} := & -\frac{4}{45} R_{bcc1e} R_{c2fgh} g^{af} g^{cg} D_d \left(x^e x^h\right) - \frac{4}{45} R_{bcc2e} R_{c1fgh} g^{af} g^{cg} D_d \left(x^e x^h\right) - \frac{4}{45} R_{bcc1e} R_{c2fgh} g^{af} g^{eg} D_d \left(x^c x^h\right) \\
& - \frac{4}{45} R_{bcef} R_{c1gc2h} g^{ac} g^{eg} D_d \left(x^f x^h\right) - \frac{4}{45} R_{bcc2e} R_{c1fgh} g^{af} g^{eg} D_d \left(x^c x^h\right) - \frac{4}{45} R_{bcef} R_{c1gc2h} g^{ac} g^{eh} D_d \left(x^f x^g\right) \\
& - \frac{1}{45} R_{bcc1e} R_{c2fgh} g^{ag} g^{cf} D_d \left(x^e x^h\right) - \frac{1}{45} R_{bcc2e} R_{c1fgh} g^{ag} g^{cf} D_d \left(x^e x^h\right) - \frac{1}{45} R_{bcc1e} R_{c2fgh} g^{ag} g^{ef} D_d \left(x^c x^h\right) \\
& - \frac{1}{45} R_{bcef} R_{c1gc2h} g^{ae} g^{cg} D_d \left(x^f x^h\right) - \frac{1}{45} R_{bcc2e} R_{c1fgh} g^{ag} g^{ef} D_d \left(x^c x^h\right) - \frac{1}{45} R_{bcef} R_{c1gc2h} g^{ae} g^{ch} D_d \left(x^f x^g\right) \\
& + \frac{1}{45} R_{bcc2e} R_{c1fgh} g^{ac} g^{eg} D_d \left(x^f x^h\right) + \frac{1}{45} R_{bcc1e} R_{c2fgh} g^{ac} g^{eg} D_d \left(x^f x^h\right) + \frac{1}{45} R_{bcef} R_{c1gc2h} g^{ag} g^{eh} D_d \left(x^c x^f\right) \\
& + \frac{1}{45} R_{bcc1e} R_{c2fgh} g^{ae} g^{cg} D_d \left(x^f x^h\right) + \frac{1}{45} R_{bcef} R_{c1gc2h} g^{ah} g^{eg} D_d \left(x^c x^f\right) + \frac{1}{45} R_{bcc2e} R_{c1fgh} g^{ae} g^{cg} D_d \left(x^f x^h\right) \\
& - \frac{1}{60} \nabla_{bc2} R_{c1cef} g^{ae} D_d \left(x^c x^f\right) - \frac{1}{60} \nabla_{bc1} R_{c2cef} g^{ae} D_d \left(x^c x^f\right) - \frac{1}{60} \nabla_{c1c2} R_{bcef} g^{ae} D_d \left(x^c x^f\right) - \frac{1}{60} \nabla_{c1b} R_{c2cef} g^{ae} D_d \left(x^c x^f\right) \\
& - \frac{1}{60} \nabla_{c2c1} R_{bcef} g^{ae} D_d \left(x^c x^f\right) - \frac{1}{60} \nabla_{c2b} R_{c1cef} g^{ae} D_d \left(x^c x^f\right) + \frac{1}{40} \nabla_{bc} R_{c1ec2f} g^{af} D_d \left(x^c x^e\right) + \frac{1}{40} \nabla_{bc} R_{c1ec2f} g^{ae} D_d \left(x^c x^f\right) \\
& + \frac{1}{40} \nabla_{c1c} R_{bec2f} g^{af} D_d \left(x^c x^e\right) + \frac{1}{40} \nabla_{c1c} R_{bec2f} g^{ae} D_d \left(x^c x^f\right) + \frac{1}{40} \nabla_{c2c} R_{bec1f} g^{af} D_d \left(x^c x^e\right) + \frac{1}{40} \nabla_{c2c} R_{bec1f} g^{ae} D_d \left(x^c x^f\right) \\
& + \frac{1}{40} \nabla_{cb} R_{c1ec2f} g^{af} D_d \left(x^c x^e\right) + \frac{1}{40} \nabla_{cb} R_{c1ec2f} g^{ae} D_d \left(x^c x^f\right) + \frac{1}{40} \nabla_{cc1} R_{bec2f} g^{af} D_d \left(x^c x^e\right) + \frac{1}{40} \nabla_{cc1} R_{bec2f} g^{ae} D_d \left(x^c x^f\right) \\
& + \frac{1}{40} \nabla_{cc2} R_{bec1f} g^{af} D_d \left(x^c x^e\right) + \frac{1}{40} \nabla_{cc2} R_{bec1f} g^{ae} D_d \left(x^c x^f\right) + \frac{1}{15} R_{bcef} R_{c1gc2h} g^{ae} g^{fh} D_d \left(x^c x^g\right) + \frac{1}{15} R_{bcef} R_{c1gc2h} g^{ae} g^{fg} D_d \left(x^c x^h\right) \\
& + \frac{1}{15} R_{bcc2e} R_{c1fgh} g^{ag} g^{eh} D_d \left(x^c x^f\right) + \frac{1}{15} R_{bcc2e} R_{c1fgh} g^{ag} g^{ch} D_d \left(x^e x^f\right) + \frac{1}{15} R_{bcc1e} R_{c2fgh} g^{ag} g^{eh} D_d \left(x^c x^f\right) \\
& + \frac{1}{15} R_{bcc1e} R_{c2fgh} g^{ag} g^{ch} D_d \left(x^e x^f\right) + \frac{1}{120} \nabla_{cc2} R_{bec1f} g^{ac} D_d \left(x^e x^f\right) + \frac{1}{120} \nabla_{cc1} R_{bec2f} g^{ac} D_d \left(x^e x^f\right) \\
& + \frac{1}{120} \nabla_{cb} R_{c1ec2f} g^{ac} D_d \left(x^e x^f\right) + \frac{1}{120} \nabla_{c2c} R_{bec1f} g^{ac} D_d \left(x^e x^f\right) + \frac{1}{120} \nabla_{c1c} R_{bec2f} g^{ac} D_d \left(x^e x^f\right) + \frac{1}{120} \nabla_{bc} R_{c1ec2f} g^{ac} D_d \left(x^e x^f\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG42.005} := &-\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{cg}\left(D_dx^ex^h+x^eD_dx^h\right)-\frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{cg}\left(D_dx^ex^h+x^eD_dx^h\right)-\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{eg}\left(D_dx^cx^h+x^cD_dx^h\right)\\
&-\frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eg}\left(D_dx^fx^h+x^fD_dx^h\right)-\frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{eg}\left(D_dx^cx^h+x^cD_dx^h\right)-\frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eh}\left(D_dx^fx^g+x^fD_dx^g\right)\\
&-\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{cf}\left(D_dx^ex^h+x^eD_dx^h\right)-\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{cf}\left(D_dx^ex^h+x^eD_dx^h\right)-\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{ef}\left(D_dx^cx^h+x^cD_dx^h\right)\\
&-\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{cg}\left(D_dx^fx^h+x^fD_dx^h\right)-\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{ef}\left(D_dx^cx^h+x^cD_dx^h\right)-\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{ch}\left(D_dx^fx^g+x^fD_dx^g\right)\\
&+\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ac}g^{eg}\left(D_dx^fx^h+x^fD_dx^h\right)+\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ac}g^{eg}\left(D_dx^fx^h+x^fD_dx^h\right)+\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ag}g^{eh}\left(D_dx^cx^f+x^cD_dx^f\right)\\
&+\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ae}g^{cg}\left(D_dx^fx^h+x^fD_dx^h\right)+\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ah}g^{eg}\left(D_dx^cx^f+x^cD_dx^f\right)+\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ae}g^{cg}\left(D_dx^fx^h+x^fD_dx^h\right)\\
&-\frac{1}{60}\nabla_{bc2}R_{c1cef}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)-\frac{1}{60}\nabla_{bc1}R_{c2cef}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)-\frac{1}{60}\nabla_{c1c2}R_{bcef}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)\\
&-\frac{1}{60}\nabla_{c1b}R_{c2cef}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)-\frac{1}{60}\nabla_{c2c1}R_{bcef}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)-\frac{1}{60}\nabla_{c2b}R_{c1cef}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)\\
&+\frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{af}\left(D_dx^cx^e+x^cD_dx^e\right)+\frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)+\frac{1}{40}\nabla_{c1c}R_{bec2f}g^{af}\left(D_dx^cx^e+x^cD_dx^e\right)\\
&+\frac{1}{40}\nabla_{c1c}R_{bec2f}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)+\frac{1}{40}\nabla_{c2c}R_{bec1f}g^{af}\left(D_dx^cx^e+x^cD_dx^e\right)+\frac{1}{40}\nabla_{c2c}R_{bec1f}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)\\
&+\frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{af}\left(D_dx^cx^e+x^cD_dx^e\right)+\frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)+\frac{1}{40}\nabla_{cc1}R_{bec2f}g^{af}\left(D_dx^cx^e+x^cD_dx^e\right)\\
&+\frac{1}{40}\nabla_{cc1}R_{bec2f}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)+\frac{1}{40}\nabla_{cc2}R_{bec1f}g^{af}\left(D_dx^cx^e+x^cD_dx^e\right)+\frac{1}{40}\nabla_{cc2}R_{bec1f}g^{ae}\left(D_dx^cx^f+x^cD_dx^f\right)\\
&+\frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fh}\left(D_dx^cx^g+x^cD_dx^g\right)+\frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fg}\left(D_dx^cx^h+x^cD_dx^h\right)+\frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{eh}\left(D_dx^cx^f+x^cD_dx^f\right)\\
&+\frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{ch}\left(D_dx^ex^f+x^eD_dx^f\right)+\frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{eh}\left(D_dx^cx^f+x^cD_dx^f\right)+\frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{ch}\left(D_dx^ex^f+x^eD_dx^f\right)\\
&+\frac{1}{120}\nabla_{cc2}R_{bec1f}g^{ac}\left(D_dx^ex^f+x^eD_dx^f\right)+\frac{1}{120}\nabla_{cc1}R_{bec2f}g^{ac}\left(D_dx^ex^f+x^eD_dx^f\right)+\frac{1}{120}\nabla_{cb}R_{c1ec2f}g^{ac}\left(D_dx^ex^f+x^eD_dx^f\right)\\
&+\frac{1}{120}\nabla_{c2c}R_{bec1f}g^{ac}\left(D_dx^ex^f+x^eD_dx^f\right)+\frac{1}{120}\nabla_{c1c}R_{bec2f}g^{ac}\left(D_dx^ex^f+x^eD_dx^f\right)+\frac{1}{120}\nabla_{bc}R_{c1ec2f}g^{ac}\left(D_dx^ex^f+x^eD_dx^f\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{genG42.011} :=\ & \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{af}g^{cg}x^{e} + \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{ag}g^{cf}x^{e} + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{af}g^{cg}x^{e} + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{ag}g^{cf}x^{e} + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{af}g^{cg}x^{e} \\
& + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{ag}g^{cf}x^{e} + \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{af}g^{eg}x^{c} + \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{ag}g^{ef}x^{c} + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{ac}g^{ef}x^{g} + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{ae}g^{cf}x^{g} \\
& + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{ac}g^{ef}x^{g} + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{ae}g^{cf}x^{g} + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{af}g^{eg}x^{c} + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{ag}g^{ef}x^{c} + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{ac}g^{eg}x^{f} \\
& + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{ae}g^{cg}x^{f} + \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{ac}g^{ef}x^{g} + \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{ae}g^{cf}x^{g} + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{af}g^{eg}x^{c} + \frac{1}{45}R_{bcde}R_{c1fc2g}g^{ag}g^{ef}x^{c} \\
& + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{ac}g^{eg}x^{f} + \frac{1}{45}R_{bcc2e}R_{c1fdg}g^{ae}g^{cg}x^{f} + \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{ac}g^{eg}x^{f} + \frac{1}{45}R_{bcc1e}R_{c2fdg}g^{ae}g^{cg}x^{f} + \frac{1}{60}\nabla_{bc2}R_{c1cde}g^{ae}x^{c} \\
& + \frac{1}{60}\nabla_{bd}R_{c1cc2e}g^{ae}x^{c} + \frac{1}{60}\nabla_{bc1}R_{c2cde}g^{ae}x^{c} + \frac{1}{60}\nabla_{bd}R_{c1cc2e}g^{ac}x^{e} + \frac{1}{60}\nabla_{bc1}R_{c2cde}g^{ac}x^{e} + \frac{1}{60}\nabla_{bc2}R_{c1cde}g^{ac}x^{e} + \frac{1}{60}\nabla_{c1c2}R_{bcde}g^{ae}x^{c} \\
& + \frac{1}{60}\nabla_{c1d}R_{bcc2e}g^{ae}x^{c} + \frac{1}{60}\nabla_{c1b}R_{c2cde}g^{ae}x^{c} + \frac{1}{60}\nabla_{c1d}R_{bcc2e}g^{ac}x^{e} + \frac{1}{60}\nabla_{c1b}R_{c2cde}g^{ac}x^{e} + \frac{1}{60}\nabla_{c1c2}R_{bcde}g^{ac}x^{e} + \frac{1}{60}\nabla_{c2c1}R_{bcde}g^{ae}x^{c} \\
& + \frac{1}{60}\nabla_{c2d}R_{bcc1e}g^{ae}x^{c} + \frac{1}{60}\nabla_{c2b}R_{c1cde}g^{ae}x^{c} + \frac{1}{60}\nabla_{c2d}R_{bcc1e}g^{ac}x^{e} + \frac{1}{60}\nabla_{c2b}R_{c1cde}g^{ac}x^{e} + \frac{1}{60}\nabla_{c2c1}R_{bcde}g^{ac}x^{e} + \frac{1}{60}\nabla_{dc1}R_{bcc2e}g^{ae}x^{c} \\
& + \frac{1}{60}\nabla_{dc2}R_{bcc1e}g^{ae}x^{c} + \frac{1}{60}\nabla_{db}R_{c1cc2e}g^{ae}x^{c} + \frac{1}{60}\nabla_{dc2}R_{bcc1e}g^{ac}x^{e} + \frac{1}{60}\nabla_{db}R_{c1cc2e}g^{ac}x^{e} + \frac{1}{60}\nabla_{dc1}R_{bcc2e}g^{ac}x^{e}
\end{aligned}
$$

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                              -> A001^{a}             $)
    substitute (obj,$ x^{a}                              -> A002^{a}             $)
    substitute (obj,$ g^{a b}                            -> A003^{a b}           $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                        -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                -> A^{a}                           $)
    substitute (obj,$ A002^{a}                -> x^{a}                           $)
    substitute (obj,$ A003^{a b}              -> g^{a b}                         $)
    substitute (obj,$ A004_{a b c d}          -> R_{a b c d}                     $)
    substitute (obj,$ A005_{a b c d e}        -> \nabla_{e}{R_{a b c d}}         $)
    substitute (obj,$ A006_{a b c d e f}      -> \nabla_{e f}{R_{a b c d}}       $)
    substitute (obj,$ A007_{a b c d e f g}    -> \nabla_{e f g}{R_{a b c d}}     $)
    substitute (obj,$ A008_{a b c d e f g h}  -> \nabla_{e f g h}{R_{a b c d}}   $)

    return obj

# ---------------------------------------------------------------------------
symG20 := @(genG20) A^{b} A^{d}.                         # cdb (symG20.100,symG20)

distribute              (symG20)                         # cdb (symG20.101,symG20)
symG20 = product_sort (symG20)                           # cdb (symG20.102,symG20)
rename_dummies          (symG20)                         # cdb (symG20.103,symG20)
canonicalise            (symG20)                         # cdb (symG20.104,symG20)

# ---------------------------------------------------------------------------
symG30 := @(genG30) A^{b} A^{d}.                         # cdb (symG30.100,symG30)

distribute              (symG30)                         # cdb (symG30.101,symG30)
symG30 = product_sort (symG30)                           # cdb (symG30.102,symG30)
rename_dummies          (symG30)                         # cdb (symG30.103,symG30)
canonicalise            (symG30)                         # cdb (symG30.104,symG30)
```

```
# ------------------------------------------------------------------------------
symG40 := @(genG40) A^{b} A^{d}.                           # cdb (symG40.100,symG40)

distribute             (symG40)                            # cdb (symG40.101,symG40)
symG40 = product_sort (symG40)                             # cdb (symG40.102,symG40)
rename_dummies         (symG40)                            # cdb (symG40.103,symG40)
canonicalise           (symG40)                            # cdb (symG40.104,symG40)

# ------------------------------------------------------------------------------
symG50 := @(genG50) A^{b} A^{d}.                           # cdb (symG50.100,symG50)

distribute             (symG50)                            # cdb (symG50.101,symG50)
symG50 = product_sort (symG50)                             # cdb (symG50.102,symG50)
rename_dummies         (symG50)                            # cdb (symG50.103,symG50)
canonicalise           (symG50)                            # cdb (symG50.104,symG50)

# ------------------------------------------------------------------------------
symG31 := @(genG31) A^{b} A^{c1} A^{d}.                    # cdb (symG31.100,symG31)

distribute             (symG31)                            # cdb (symG31.101,symG31)
symG31 = product_sort (symG31)                             # cdb (symG31.102,symG31)
rename_dummies         (symG31)                            # cdb (symG31.103,symG31)
canonicalise           (symG31)                            # cdb (symG31.104,symG31)

# ------------------------------------------------------------------------------
symG41 := @(genG41) A^{b} A^{c1} A^{d}.                    # cdb (symG41.100,symG41)

distribute             (symG41)                            # cdb (symG41.101,symG41)
symG41 = product_sort (symG41)                             # cdb (symG41.102,symG41)
rename_dummies         (symG41)                            # cdb (symG41.103,symG41)
canonicalise           (symG41)                            # cdb (symG41.104,symG41)

# ------------------------------------------------------------------------------
symG51 := @(genG51) A^{b} A^{c1} A^{d}.                    # cdb (symG51.100,symG51)

distribute             (symG51)                            # cdb (symG51.101,symG51)
symG51 = product_sort (symG51)                             # cdb (symG51.102,symG51)
```

```
rename_dummies         (symG51)                        # cdb (symG51.103,symG51)
canonicalise           (symG51)                        # cdb (symG51.104,symG51)

# ---------------------------------------------------------------------------
symG42 := @(genG42) A^{b} A^{c1} A^{c2} A^{d}.         # cdb (symG42.100,symG42)

distribute             (symG42)                        # cdb (symG42.101,symG42)
symG42 = product_sort (symG42)                         # cdb (symG42.102,symG42)
rename_dummies         (symG42)                        # cdb (symG42.103,symG42)
canonicalise           (symG42)                        # cdb (symG42.104,symG42)

# ---------------------------------------------------------------------------
symG52 := @(genG52) A^{b} A^{c1} A^{c2} A^{d}.         # cdb (symG52.100,symG52)

distribute             (symG52)                        # cdb (symG52.101,symG52)
symG52 = product_sort (symG52)                         # cdb (symG52.102,symG52)
rename_dummies         (symG52)                        # cdb (symG52.103,symG52)
canonicalise           (symG52)                        # cdb (symG52.104,symG52)

# ---------------------------------------------------------------------------
symG53 := @(genG53) A^{b} A^{c1} A^{c2} A^{c3} A^{d}.  # cdb (symG53.100,symG53)

distribute             (symG53)                        # cdb (symG53.101,symG53)
symG53 = product_sort (symG53)                         # cdb (symG53.102,symG53)
rename_dummies         (symG53)                        # cdb (symG53.103,symG53)
canonicalise           (symG53)                        # cdb (symG53.104,symG53)
```

$$\texttt{symG31.100} := \left(\frac{1}{12}\nabla_b R_{c1cde} g^{ae} x^c + \frac{1}{12}\nabla_b R_{c1cde} g^{ac} x^e + \frac{1}{12}\nabla_{c1} R_{bcde} g^{ae} x^c + \frac{1}{12}\nabla_{c1} R_{bcde} g^{ac} x^e + \frac{1}{12}\nabla_d R_{bcc1e} g^{ae} x^c + \frac{1}{12}\nabla_d R_{bcc1e} g^{ac} x^e\right) A^b A^{c1} A^d$$

$$\texttt{symG31.101} := \frac{1}{12}\nabla_b R_{c1cde} g^{ae} x^c A^b A^{c1} A^d + \frac{1}{12}\nabla_b R_{c1cde} g^{ac} x^e A^b A^{c1} A^d + \frac{1}{12}\nabla_{c1} R_{bcde} g^{ae} x^c A^b A^{c1} A^d + \frac{1}{12}\nabla_{c1} R_{bcde} g^{ac} x^e A^b A^{c1} A^d + \frac{1}{12}\nabla_d R_{bcc1e} g^{ae} x^c A^b A^{c1} A^d + \frac{1}{12}\nabla_d R_{bcc1e} g^{ac} x^e A^b A^{c1} A^d$$

$$\texttt{symG31.102} := \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_b R_{cfde} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_c R_{bedf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_c R_{bfde} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_d R_{becf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_d R_{bfce}$$

$$\texttt{symG31.103} := \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} + \frac{1}{12} A^b A^c A^d x^f g^{ae} \nabla_b R_{cedf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_c R_{bedf} + \frac{1}{12} A^b A^c A^d x^f g^{ae} \nabla_c R_{bedf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_d R_{becf} + \frac{1}{12} A^b A^c A^d x^f g^{ae} \nabla_d R_{becf}$$

$$\texttt{symG31.104} := \frac{1}{2} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf}$$

$$
\begin{aligned}
\texttt{symG41.100} := \Big(&-\frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{cg}x^e x^h-\frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{cg}x^e x^h-\frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{eg}x^c x^h-\frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eg}x^f x^h\\
&-\frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{eg}x^c x^h-\frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eh}x^f x^g-\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{cf}x^e x^h-\frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{cf}x^e x^h\\
&-\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{ef}x^c x^h-\frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{cg}x^f x^h-\frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{ef}x^c x^h-\frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{ch}x^f x^g\\
&+\frac{1}{45}R_{bcde}R_{c1fgh}g^{ac}g^{eg}x^f x^h+\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ac}g^{eg}x^f x^h+\frac{1}{45}R_{bcef}R_{c1gdh}g^{ag}g^{eh}x^c x^f+\frac{1}{45}R_{bcc1e}R_{dfgh}g^{ae}g^{cg}x^f x^h\\
&+\frac{1}{45}R_{bcef}R_{c1gdh}g^{ah}g^{eg}x^c x^f+\frac{1}{45}R_{bcde}R_{c1fgh}g^{ae}g^{cg}x^f x^h-\frac{1}{60}\nabla_{bd}R_{c1cef}g^{ae}x^c x^f-\frac{1}{60}\nabla_{bc1}R_{dcef}g^{ae}x^c x^f-\frac{1}{60}\nabla_{c1d}R_{bcef}g^{ae}x^c x^f\\
&-\frac{1}{60}\nabla_{c1b}R_{dcef}g^{ae}x^c x^f-\frac{1}{60}\nabla_{dc1}R_{bcef}g^{ae}x^c x^f-\frac{1}{60}\nabla_{db}R_{c1cef}g^{ae}x^c x^f+\frac{1}{40}\nabla_{bc}R_{c1edf}g^{af}x^c x^e+\frac{1}{40}\nabla_{bc}R_{c1edf}g^{ae}x^c x^f\\
&+\frac{1}{40}\nabla_{c1c}R_{bedf}g^{af}x^c x^e+\frac{1}{40}\nabla_{c1c}R_{bedf}g^{ae}x^c x^f+\frac{1}{40}\nabla_{dc}R_{bec1f}g^{af}x^c x^e+\frac{1}{40}\nabla_{dc}R_{bec1f}g^{ae}x^c x^f+\frac{1}{40}\nabla_{cb}R_{c1edf}g^{af}x^c x^e\\
&+\frac{1}{40}\nabla_{cb}R_{c1edf}g^{ae}x^c x^f+\frac{1}{40}\nabla_{cc1}R_{bedf}g^{af}x^c x^e+\frac{1}{40}\nabla_{cc1}R_{bedf}g^{ae}x^c x^f+\frac{1}{40}\nabla_{cd}R_{bec1f}g^{af}x^c x^e+\frac{1}{40}\nabla_{cd}R_{bec1f}g^{ae}x^c x^f\\
&+\frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fh}x^c x^g+\frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fg}x^c x^h+\frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{eh}x^c x^f+\frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{ch}x^e x^f\\
&+\frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{eh}x^c x^f+\frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{ch}x^e x^f+\frac{1}{120}\nabla_{cd}R_{bec1f}g^{ac}x^e x^f+\frac{1}{120}\nabla_{cc1}R_{bedf}g^{ac}x^e x^f+\frac{1}{120}\nabla_{cb}R_{c1edf}g^{ac}x^e x^f\\
&+\frac{1}{120}\nabla_{dc}R_{bec1f}g^{ac}x^e x^f+\frac{1}{120}\nabla_{c1c}R_{bedf}g^{ac}x^e x^f+\frac{1}{120}\nabla_{bc}R_{c1edf}g^{ac}x^e x^f\Big)A^b A^{c1} A^d
\end{aligned}
$$

$$
\begin{aligned}
\texttt{symG41.101} :=& -\frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{cg}x^{e}x^{h}A^{b}A^{c1}A^{d} - \frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{cg}x^{e}x^{h}A^{b}A^{c1}A^{d} - \frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{eg}x^{c}x^{h}A^{b}A^{c1}A^{d} \\
& - \frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eg}x^{f}x^{h}A^{b}A^{c1}A^{d} - \frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{eg}x^{c}x^{h}A^{b}A^{c1}A^{d} - \frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eh}x^{f}x^{g}A^{b}A^{c1}A^{d} \\
& - \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{cf}x^{e}x^{h}A^{b}A^{c1}A^{d} - \frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{cf}x^{e}x^{h}A^{b}A^{c1}A^{d} - \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{ef}x^{c}x^{h}A^{b}A^{c1}A^{d} \\
& - \frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{cg}x^{f}x^{h}A^{b}A^{c1}A^{d} - \frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{ef}x^{c}x^{h}A^{b}A^{c1}A^{d} - \frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{ch}x^{f}x^{g}A^{b}A^{c1}A^{d} \\
& + \frac{1}{45}R_{bcde}R_{c1fgh}g^{ac}g^{eg}x^{f}x^{h}A^{b}A^{c1}A^{d} + \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ac}g^{eg}x^{f}x^{h}A^{b}A^{c1}A^{d} + \frac{1}{45}R_{bcef}R_{c1gdh}g^{ag}g^{eh}x^{c}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ae}g^{cg}x^{f}x^{h}A^{b}A^{c1}A^{d} + \frac{1}{45}R_{bcef}R_{c1gdh}g^{ah}g^{eg}x^{c}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{45}R_{bcde}R_{c1fgh}g^{ae}g^{cg}x^{f}x^{h}A^{b}A^{c1}A^{d} \\
& - \frac{1}{60}\nabla_{bd}R_{c1cef}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} - \frac{1}{60}\nabla_{bc1}R_{dcef}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} - \frac{1}{60}\nabla_{c1d}R_{bcef}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} \\
& - \frac{1}{60}\nabla_{c1b}R_{dcef}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} - \frac{1}{60}\nabla_{dc1}R_{bcef}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} - \frac{1}{60}\nabla_{db}R_{c1cef}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{40}\nabla_{bc}R_{c1edf}g^{af}x^{c}x^{e}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{bc}R_{c1edf}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{c1c}R_{bedf}g^{af}x^{c}x^{e}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{c1c}R_{bedf}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{40}\nabla_{dc}R_{bec1f}g^{af}x^{c}x^{e}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{dc}R_{bec1f}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{cb}R_{c1edf}g^{af}x^{c}x^{e}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{cb}R_{c1edf}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{40}\nabla_{cc1}R_{bedf}g^{af}x^{c}x^{e}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{cc1}R_{bedf}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{cd}R_{bec1f}g^{af}x^{c}x^{e}A^{b}A^{c1}A^{d} + \frac{1}{40}\nabla_{cd}R_{bec1f}g^{ae}x^{c}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fh}x^{c}x^{g}A^{b}A^{c1}A^{d} + \frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fg}x^{c}x^{h}A^{b}A^{c1}A^{d} + \frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{eh}x^{c}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{ch}x^{e}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{eh}x^{c}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{ch}x^{e}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{120}\nabla_{cd}R_{bec1f}g^{ac}x^{e}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{120}\nabla_{cc1}R_{bedf}g^{ac}x^{e}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{120}\nabla_{cb}R_{c1edf}g^{ac}x^{e}x^{f}A^{b}A^{c1}A^{d} \\
& + \frac{1}{120}\nabla_{dc}R_{bec1f}g^{ac}x^{e}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{120}\nabla_{c1c}R_{bedf}g^{ac}x^{e}x^{f}A^{b}A^{c1}A^{d} + \frac{1}{120}\nabla_{bc}R_{c1edf}g^{ac}x^{e}x^{f}A^{b}A^{c1}A^{d}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{symG41.102} := & -\frac{4}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhce}R_{dgif}-\frac{4}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhde}R_{cgif}-\frac{4}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bech}R_{dgif}\\
& -\frac{4}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bghe}R_{cidf}-\frac{4}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bedh}R_{cgif}-\frac{4}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bghe}R_{cfdi}\\
& -\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhce}R_{digf}-\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhde}R_{cigf}-\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bech}R_{digf}\\
& -\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhge}R_{cidf}-\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bedh}R_{cigf}-\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhge}R_{cfdi}\\
& +\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bgdh}R_{ceif}+\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bgch}R_{deif}+\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{behf}R_{cgdi}\\
& +\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhcg}R_{deif}+\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{behf}R_{cidg}+\frac{1}{45}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhdg}R_{ceif}\\
& -\frac{1}{60}A^bA^cA^dx^ex^fg^{ag}\nabla_{bd}R_{cegf}-\frac{1}{60}A^bA^cA^dx^ex^fg^{ag}\nabla_{bc}R_{degf}-\frac{1}{60}A^bA^cA^dx^ex^fg^{ag}\nabla_{cd}R_{begf}-\frac{1}{60}A^bA^cA^dx^ex^fg^{ag}\nabla_{cb}R_{degf}\\
& -\frac{1}{60}A^bA^cA^dx^ex^fg^{ag}\nabla_{dc}R_{begf}-\frac{1}{60}A^bA^cA^dx^ex^fg^{ag}\nabla_{db}R_{cegf}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{be}R_{cfdg}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{be}R_{cgdf}\\
& +\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{ce}R_{bfdg}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{ce}R_{bgdf}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{de}R_{bfcg}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{de}R_{bgcf}\\
& +\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{eb}R_{cfdg}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{eb}R_{cgdf}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{ec}R_{bfdg}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{ec}R_{bgdf}\\
& +\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{ed}R_{bfcg}+\frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{ed}R_{bgcf}+\frac{1}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{begh}R_{cfdi}+\frac{1}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{begh}R_{cidf}\\
& +\frac{1}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bedh}R_{cfgi}+\frac{1}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhde}R_{cfgi}+\frac{1}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bech}R_{dfgi}\\
& +\frac{1}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bhce}R_{dfgi}+\frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{gd}R_{becf}+\frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{gc}R_{bedf}\\
& +\frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{gb}R_{cedf}+\frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{dg}R_{becf}+\frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{cg}R_{bedf}+\frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{bg}R_{cedf}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{symG41.103} :=\; & -\frac{4}{45}A^bA^cA^dx^fx^ig^{ag}g^{eh}R_{becf}R_{dghi} - \frac{4}{45}A^bA^cA^dx^fx^ig^{ag}g^{eh}R_{bedf}R_{cghi} - \frac{4}{45}A^bA^cA^dx^ex^ig^{ag}g^{fh}R_{becf}R_{dghi} \\
& - \frac{4}{45}A^bA^cA^dx^gx^ig^{ae}g^{fh}R_{befg}R_{chdi} - \frac{4}{45}A^bA^cA^dx^ex^ig^{ag}g^{fh}R_{bedf}R_{cghi} - \frac{4}{45}A^bA^cA^dx^gx^hg^{ae}g^{fi}R_{befg}R_{chdi} \\
& - \frac{1}{45}A^bA^cA^dx^fx^ig^{ah}g^{eg}R_{becf}R_{dghi} - \frac{1}{45}A^bA^cA^dx^fx^ig^{ah}g^{eg}R_{bedf}R_{cghi} - \frac{1}{45}A^bA^cA^dx^ex^ig^{ah}g^{fg}R_{becf}R_{dghi} \\
& - \frac{1}{45}A^bA^cA^dx^gx^ig^{af}g^{eh}R_{befg}R_{chdi} - \frac{1}{45}A^bA^cA^dx^ex^ig^{ah}g^{fg}R_{bedf}R_{cghi} - \frac{1}{45}A^bA^cA^dx^gx^hg^{af}g^{ei}R_{befg}R_{chdi} \\
& + \frac{1}{45}A^bA^cA^dx^gx^ig^{ae}g^{fh}R_{bedf}R_{cghi} + \frac{1}{45}A^bA^cA^dx^gx^ig^{ae}g^{fh}R_{becf}R_{dghi} + \frac{1}{45}A^bA^cA^dx^ex^gg^{ah}g^{fi}R_{befg}R_{chdi} \\
& + \frac{1}{45}A^bA^cA^dx^gx^ig^{af}g^{eh}R_{becf}R_{dghi} + \frac{1}{45}A^bA^cA^dx^ex^gg^{ai}g^{fh}R_{befg}R_{chdi} + \frac{1}{45}A^bA^cA^dx^gx^ig^{af}g^{eh}R_{bedf}R_{cghi} \\
& - \frac{1}{60}A^bA^cA^dx^ex^gg^{af}\nabla_{bd}R_{cefg} - \frac{1}{60}A^bA^cA^dx^ex^gg^{af}\nabla_{bc}R_{defg} - \frac{1}{60}A^bA^cA^dx^ex^gg^{af}\nabla_{cd}R_{befg} - \frac{1}{60}A^bA^cA^dx^ex^gg^{af}\nabla_{cb}R_{defg} \\
& - \frac{1}{60}A^bA^cA^dx^ex^gg^{af}\nabla_{dc}R_{befg} - \frac{1}{60}A^bA^cA^dx^ex^gg^{af}\nabla_{db}R_{cefg} + \frac{1}{40}A^bA^cA^dx^gx^eg^{af}\nabla_{bg}R_{cedf} + \frac{1}{40}A^bA^cA^dx^gx^fg^{ae}\nabla_{bg}R_{cedf} \\
& + \frac{1}{40}A^bA^cA^dx^gx^eg^{af}\nabla_{cg}R_{bedf} + \frac{1}{40}A^bA^cA^dx^gx^fg^{ae}\nabla_{cg}R_{bedf} + \frac{1}{40}A^bA^cA^dx^gx^eg^{af}\nabla_{dg}R_{becf} + \frac{1}{40}A^bA^cA^dx^gx^fg^{ae}\nabla_{dg}R_{becf} \\
& + \frac{1}{40}A^bA^cA^dx^gx^eg^{af}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dx^gx^fg^{ae}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dx^gx^eg^{af}\nabla_{gc}R_{bedf} + \frac{1}{40}A^bA^cA^dx^gx^fg^{ae}\nabla_{gc}R_{bedf} \\
& + \frac{1}{40}A^bA^cA^dx^gx^eg^{af}\nabla_{gd}R_{becf} + \frac{1}{40}A^bA^cA^dx^gx^fg^{ae}\nabla_{gd}R_{becf} + \frac{1}{15}A^bA^cA^dx^ex^hg^{af}g^{gi}R_{befg}R_{chdi} + \frac{1}{15}A^bA^cA^dx^ex^ig^{af}g^{gh}R_{befg}R_{chdi} \\
& + \frac{1}{15}A^bA^cA^dx^ex^gg^{ah}g^{fi}R_{bedf}R_{cghi} + \frac{1}{15}A^bA^cA^dx^fx^gg^{ah}g^{ei}R_{bedf}R_{cghi} + \frac{1}{15}A^bA^cA^dx^ex^gg^{ah}g^{fi}R_{becf}R_{dghi} \\
& + \frac{1}{15}A^bA^cA^dx^fx^gg^{ah}g^{ei}R_{becf}R_{dghi} + \frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{gd}R_{becf} + \frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{gc}R_{bedf} \\
& + \frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{gb}R_{cedf} + \frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{dg}R_{becf} + \frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{cg}R_{bedf} + \frac{1}{120}A^bA^cA^dx^ex^fg^{ag}\nabla_{bg}R_{cedf}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{symG41.104} :=\; & \frac{8}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bech}R_{dgfi} + \frac{2}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bech}R_{difg} - \frac{2}{15}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{befh}R_{cgdi} \\
& + \frac{1}{10}A^bA^cA^dx^ex^fg^{ag}\nabla_{bc}R_{defg} + \frac{3}{20}A^bA^cA^dx^ex^fg^{ag}\nabla_{be}R_{cfdg} + \frac{3}{20}A^bA^cA^dx^ex^fg^{ag}\nabla_{eb}R_{cfdg} \\
& + \frac{2}{5}A^bA^cA^dx^ex^fg^{ag}g^{hi}R_{bech}R_{dfgi} + \frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dx^ex^fg^{ag}\nabla_{bg}R_{cedf}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{symG51.104} := {} & \frac{8}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + \frac{4}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} \\
& + \frac{1}{10} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} + \frac{1}{90} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + \frac{11}{90} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} \\
& + \frac{4}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\
& + \frac{1}{12} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} + \frac{1}{36} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\
& - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - \frac{2}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} \\
& - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bce} R_{dfgh} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bec} R_{dfgh} \\
& + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bef} R_{cgdh} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebc} R_{dfgh} + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebf} R_{cgdh} \\
& + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{efb} R_{cgdh} + \frac{4}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + \frac{1}{5} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\
& + \frac{4}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + \frac{1}{5} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} \\
& - \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{hbe} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{heb} R_{cfdg} \\
& + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bhe} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ehb} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{beh} R_{cfdg} \\
& + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebh} R_{cfdg} - \frac{1}{9} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} + \frac{1}{18} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}
\end{aligned}
$$

$$\texttt{symG42.104} := \frac{8}{15}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{bfch}R_{dgei} + \frac{2}{5}A^bA^cA^dA^ex^fg^{ag}\nabla_{bc}R_{dfeg}$$

$$\begin{aligned}\texttt{symG52.104} := &\frac{32}{45}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + \frac{1}{5}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + \frac{4}{15}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej}\\
&+ \frac{2}{45}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj} + \frac{22}{45}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej}\\
&+ \frac{1}{5}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + \frac{4}{15}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + \frac{1}{9}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg}\\
&- \frac{8}{45}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej} + \frac{1}{15}A^bA^cA^dA^ex^fx^gg^{ah}\nabla_{bcd}R_{efgh} + \frac{4}{45}A^bA^cA^dA^ex^fx^gg^{ah}\nabla_{bcf}R_{dgeh}\\
&+ \frac{4}{45}A^bA^cA^dA^ex^fx^gg^{ah}\nabla_{bfc}R_{dgeh} + \frac{4}{45}A^bA^cA^dA^ex^fx^gg^{ah}\nabla_{fbc}R_{dgeh} + \frac{13}{45}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej}\\
&+ \frac{1}{15}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej} + \frac{23}{45}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + \frac{1}{90}A^bA^cA^dA^ex^fx^gg^{ah}\nabla_{hbc}R_{dfeg}\\
&+ \frac{1}{90}A^bA^cA^dA^ex^fx^gg^{ah}\nabla_{bhc}R_{dfeg} + \frac{1}{90}A^bA^cA^dA^ex^fx^gg^{ah}\nabla_{bch}R_{dfeg} - \frac{4}{9}A^bA^cA^dA^ex^fx^gg^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\end{aligned}$$

$$\texttt{symG53.104} := A^b A^c A^d A^e A^f x^g g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + A^b A^c A^d A^e A^f x^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + \frac{1}{3} A^b A^c A^d A^e A^f x^g g^{ah} \nabla_{bcd} R_{egfh}$$

```
def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    factor_out (bah,$A^{a?},x^{b?}$)
    ans := @(bah) / @(foo).
    return ans

fooG20 = reformat (symG20,3)
fooG30 = reformat (symG30,12)
fooG40 = reformat (symG40,360)
fooG50 = reformat (symG50,180)

fooG31 = reformat (symG31,2)
fooG41 = reformat (symG41,120)
fooG51 = reformat (symG51,180)

fooG42 = reformat (symG42,15)
fooG52 = reformat (symG52,90)

fooG53 = reformat (symG53,3)

genGamma0 := @(fooG20) + @(fooG30) + @(fooG40) + @(fooG50).  # cdb (genGamma0.000,genGamma0)
genGamma1 := @(fooG31) + @(fooG41) + @(fooG51).              # cdb (genGamma1.000,genGamma1)
genGamma2 := @(fooG42) + @(fooG52).                          # cdb (genGamma2.000,genGamma2)
genGamma3 := @(fooG53).                                      # cdb (genGamma3.000,genGamma3)

cdblib.create ('genGamma.json')

cdblib.put ('genGamma0',genGamma0,'genGamma.json')
cdblib.put ('genGamma1',genGamma1,'genGamma.json')
cdblib.put ('genGamma2',genGamma2,'genGamma.json')
cdblib.put ('genGamma3',genGamma3,'genGamma.json')

cdblib.put ('genGamma01',fooG20,'genGamma.json')
cdblib.put ('genGamma02',fooG30,'genGamma.json')
cdblib.put ('genGamma03',fooG40,'genGamma.json')
cdblib.put ('genGamma04',fooG50,'genGamma.json')
```

```
cdblib.put ('genGamma11',fooG31,'genGamma.json')
cdblib.put ('genGamma12',fooG41,'genGamma.json')
cdblib.put ('genGamma13',fooG51,'genGamma.json')

cdblib.put ('genGamma21',fooG42,'genGamma.json')
cdblib.put ('genGamma22',fooG52,'genGamma.json')

cdblib.put ('genGamma31',fooG53,'genGamma.json')
```

The generalised connection in Riemann normal coordinates

$$
\begin{aligned}
A^b A^c \Gamma^a_{bc}(x) = {} & \frac{2}{3} A^b A^c x^d g^{ae} R_{bdce} + \frac{1}{12} A^b A^c x^d x^e \left(2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce}\right) \\
& + \frac{1}{360} A^b A^c x^d x^e x^f \left(64g^{ag} g^{hi} R_{bdch} R_{egfi} - 32g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18g^{ag} \nabla_{bd} R_{cefg} + 18g^{ag} \nabla_{db} R_{cefg} + 36g^{ag} \nabla_{de} R_{bfcg}\right. \\
& \left. - 16g^{ag} g^{hi} R_{bdeh} R_{cfgi} + 9g^{ag} \nabla_{gd} R_{becf} + 9g^{ag} \nabla_{dg} R_{becf}\right) + \frac{1}{180} A^b A^c x^d x^e x^f x^g \left(16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj}\right. \\
& + 16g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 8g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} \\
& - 8g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 2g^{ah} \nabla_{bde} R_{cfgh} + 2g^{ah} \nabla_{dbe} R_{cfgh} + 2g^{ah} \nabla_{deb} R_{cfgh} + 4g^{ah} \nabla_{def} R_{bgch} \\
& \left. - 4g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + g^{ah} \nabla_{hde} R_{bfcg} + g^{ah} \nabla_{dhe} R_{bfcg} + g^{ah} \nabla_{deh} R_{bfcg}\right)
\end{aligned}
$$

$$
\begin{aligned}
A^b A^c A^d \Gamma^a_{bcd}(x) = {} & \frac{1}{2} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} + \frac{1}{120} A^b A^c A^d x^e x^f \left(64g^{ag} g^{hi} R_{bech} R_{dgfi} + 16g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{befh} R_{cgdi} + 12g^{ag} \nabla_{bc} R_{defg}\right. \\
& \left. + 18g^{ag} \nabla_{be} R_{cfdg} + 18g^{ag} \nabla_{eb} R_{cfdg} + 48g^{ag} g^{hi} R_{bech} R_{dfgi} + 3g^{ag} \nabla_{gb} R_{cedf} + 3g^{ag} \nabla_{bg} R_{cedf}\right) \\
& + \frac{1}{180} A^b A^c A^d x^e x^f x^g \left(32g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 48g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + 18g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj}\right. \\
& + 2g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 22g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} \\
& + 5g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} - 12g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - 8g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - 12g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} \\
& + 4g^{ah} \nabla_{bce} R_{dfgh} + 4g^{ah} \nabla_{bec} R_{dfgh} + 6g^{ah} \nabla_{bef} R_{cgdh} + 4g^{ah} \nabla_{ebc} R_{dfgh} + 6g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{efb} R_{cgdh} + 16g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} \\
& + 36g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + 16g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 4g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 36g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - 4g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} \\
& + g^{ah} \nabla_{hbe} R_{cfdg} + g^{ah} \nabla_{heb} R_{cfdg} + g^{ah} \nabla_{bhe} R_{cfdg} + g^{ah} \nabla_{ehb} R_{cfdg} + g^{ah} \nabla_{beh} R_{cfdg} + g^{ah} \nabla_{ebh} R_{cfdg} - 20g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} \\
& \left. + 10g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}\right)
\end{aligned}
$$

$$
\begin{aligned}
A^b A^c A^d A^e \Gamma^a_{bcde}(x) = {} & \frac{1}{15} A^b A^c A^d A^e x^f \left(8g^{ag} g^{hi} R_{bfch} R_{dgei} + 6g^{ag} \nabla_{bc} R_{dfeg}\right) \\
& + \frac{1}{90} A^b A^c A^d A^e x^f x^g \left(64g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + 4g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj}\right. \\
& + 44g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 10g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} - 16g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} \\
& + 6g^{ah} \nabla_{bcd} R_{efgh} + 8g^{ah} \nabla_{bcf} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{fbc} R_{dgeh} + 26g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} + 6g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} \\
& \left. + 46g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah} \nabla_{hbc} R_{dfeg} + g^{ah} \nabla_{bhc} R_{dfeg} + g^{ah} \nabla_{bch} R_{dfeg} - 40g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh}\right)
\end{aligned}
$$

$$
A^b A^c A^d A^e A^f \Gamma^a_{bcdef}(x) = \frac{1}{3} A^b A^c A^d A^e A^f x^g \left(3g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah} \nabla_{bcd} R_{egfh}\right)
$$

```
scaledGamma0 := 360 @(genGamma0).  # cdb (scaledGamma0.001,scaledGamma0)
scaledGamma1 := 360 @(genGamma1).  # cdb (scaledGamma1.001,scaledGamma1)
scaledGamma2 :=  90 @(genGamma2).  # cdb (scaledGamma2.001,scaledGamma2)
scaledGamma3 :=   3 @(genGamma3).  # cdb (scaledGamma3.001,scaledGamma3)
```

The generalised connection in Riemann normal coordinates

This is the same as the previous page but with a small change in the format to avoid fractions.

$$
\begin{aligned}
360A^bA^c\Gamma^a_{bc}(x) &= 240A^bA^cx^dg^{ae}R_{bdce} + 30A^bA^cx^dx^e\left(2g^{af}\nabla_bR_{cdef} + 4g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right)\\
&\quad + A^bA^cx^dx^ex^f\big(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg}\\
&\qquad + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\big)\\
&\quad + 2A^bA^cx^dx^ex^fx^g\big(16g^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg}\\
&\qquad - 8g^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh}\\
&\qquad + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh} + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj}\\
&\qquad - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\big)
\end{aligned}
$$

$$
\begin{aligned}
360A^bA^cA^d\Gamma^a_{bcd}(x) &= 180A^bA^cA^dx^eg^{af}\nabla_bR_{cedf} + 3A^bA^cA^dx^ex^f\big(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi}\\
&\qquad + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg} + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\big)\\
&\quad + 2A^bA^cA^dx^ex^fx^g\big(32g^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj}\\
&\qquad + 2g^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} + 22g^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg}\\
&\qquad + 5g^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} - 12g^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_gR_{chdj}\\
&\qquad + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh}\\
&\qquad + 16g^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + 16g^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_fR_{dghj}\\
&\qquad - 4g^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg}\\
&\qquad - 20g^{ah}g^{ij}R_{beci}\nabla_jR_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}\big)
\end{aligned}
$$

$$
\begin{aligned}
90A^bA^cA^dA^e\Gamma^a_{bcde}(x) &= 6A^bA^cA^dA^ex^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right)\\
&\quad + A^bA^cA^dA^ex^fx^g\big(64g^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} + 4g^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj}\\
&\qquad + 44g^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg}\\
&\qquad - 16g^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej} + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej}\\
&\qquad + 6g^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\big)
\end{aligned}
$$

$$
3A^bA^cA^dA^eA^f\Gamma^a_{bcdef}(x) = A^bA^cA^dA^eA^fx^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)
$$

```
tmp0 := @(fooG20) + @(fooG30).
tmp1 := @(fooG31).

alt0 := @(genGamma0).
alt1 := @(genGamma1).
alt2 := @(genGamma2).
alt3 := @(genGamma3).

alt0scaled := @(scaledGamma0).
alt1scaled := @(scaledGamma1).
alt2scaled := @(scaledGamma2).
alt3scaled := @(scaledGamma3).

substitute (tmp0, $A^{a}->1$)
substitute (tmp1, $A^{a}->1$)

substitute (alt0, $A^{a}->1$)
substitute (alt1, $A^{a}->1$)
substitute (alt2, $A^{a}->1$)
substitute (alt3, $A^{a}->1$)

substitute (alt0scaled, $A^{a}->1$)
substitute (alt1scaled, $A^{a}->1$)
substitute (alt2scaled, $A^{a}->1$)
substitute (alt3scaled, $A^{a}->1$)

cdblib.create ('genGamma.export')

# 4th order gen gamma
cdblib.put ('gen_gamma_0_4th',tmp0,'genGamma.export')
cdblib.put ('gen_gamma_1_4th',tmp1,'genGamma.export')

# 6th order gen gamma
cdblib.put ('gen_gamma_0',alt0,'genGamma.export')
cdblib.put ('gen_gamma_1',alt1,'genGamma.export')
cdblib.put ('gen_gamma_2',alt2,'genGamma.export')
cdblib.put ('gen_gamma_3',alt3,'genGamma.export')
```

```
# 6th order gen gamma scaled
cdblib.put ('gen_gamma_0_scaled',alt0scaled,'genGamma.export')
cdblib.put ('gen_gamma_1_scaled',alt1scaled,'genGamma.export')
cdblib.put ('gen_gamma_2_scaled',alt2scaled,'genGamma.export')
cdblib.put ('gen_gamma_3_scaled',alt3scaled,'genGamma.export')

checkpoint.append (tmp0)
checkpoint.append (tmp1)

checkpoint.append (alt0)
checkpoint.append (alt1)
checkpoint.append (alt2)
checkpoint.append (alt3)

checkpoint.append (alt0scaled)
checkpoint.append (alt1scaled)
checkpoint.append (alt2scaled)
checkpoint.append (alt3scaled)
```

Symmetrized partial derivatives of the connection

Here we calculate the recursive sequences

$$(n+3)\Gamma^{a}{}_{d(b,c\underline{e}_n)} = (n+1)\left(R^{a}{}_{(bc\dot{d},\underline{e}_n)} - \left(\Gamma^{a}{}_{f(c}\Gamma^{f}{}_{b\dot{d}}\right)_{,\underline{e}_n)}\right)$$

for $n = 1, 2, 3, \cdots$. Note that the (extended) index $\underline{e}_n$ contains n normal indices.

The result will be expressions for the $\Gamma^{a}{}_{d(b,c\underline{e}_n)}$ in terms of the Riemann tensor and its partial derivatives.

Stage 1: Compute symmetrised derivatives

In the first stage we simply apply the above recursive equation using a simple trick to impose the symmetries. Start with the original equation and dot out the symmetric indices with A^a then factor out the partial derivatives. This leads to

$$(n+3)\Gamma^{a}{}_{db,c\underline{e}_n}A^{b}A^{c}A^{\cdot\underline{e}_n} = (n+1)\left(R^{a}{}_{bcd} - \Gamma^{a}{}_{fc}\Gamma^{f}{}_{bd}\right)_{,\underline{e}_n}A^{b}A^{c}A^{\cdot\underline{e}_n} \tag{1}$$

Thus we also have (for the next iteration)

$$(n+4)\Gamma^{a}{}_{db,c\underline{e}_{n+1}}A^{b}A^{c}A^{\cdot\underline{e}_{n+1}} = (n+2)\left(R^{a}{}_{bcd} - \Gamma^{a}{}_{fc}\Gamma^{f}{}_{bd}\right)_{,\underline{e}_{n+1}}A^{b}A^{c}A^{\cdot\underline{e}_{n+1}} \tag{2}$$

The A^a can be freely chosen so choose A^a to be a constant (i.e., zero derivative). Now define P_n by

$$P_n = \Gamma^{a}{}_{db,c\underline{e}_n}A^{b}A^{c}A^{\cdot\underline{e}_n} \tag{3}$$

then the above pair of equations can be combinded to give

$$P_{n+1} = \frac{(n+2)(n+3)}{(n+4)(n+1)}A^{f}\partial_{f}\left(P_n\right) \tag{4}$$

This is a very easy equation to compute as it just requires successive rounds of differentiation.

The first term in the sequence is P_0 given by

$$P_0 = \frac{1}{3}A^{b}A^{c}\left(R^{a}{}_{bcd} - \Gamma^{a}{}_{ce}\Gamma^{e}{}_{bd}\right) \tag{5}$$

The first few results are

$$P_0 = A^b A^c \Gamma^a{}_{d(b,c)} = \frac{1}{3} A^b A^c \left(R^a{}_{bcd} - \Gamma^a{}_{ce} \Gamma^e{}_{bd} \right)$$

$$P_1 = A^b A^c A^e \Gamma^a{}_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a{}_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd}$$

$$\begin{aligned} P_2 = A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} &= \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} \\ &\quad - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \end{aligned}$$

Stage 2: Impose Riemann normal coordinates

Here we impose the RNC condition by setting the $\Gamma^a{}_{bc}$ to zero (but not their derivatives).

$$A^b A^c \Gamma^a{}_{d(b,c)} = \frac{1}{3} A^b A^c R^a{}_{bcd}$$

$$A^b A^c A^e \Gamma^a{}_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a{}_{bcd}$$

$$A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} = \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd}$$

Stage 3: Replace partial derivatives of Γ with partial derivatives of R

The key point to note is that the partial derivatives of Γ on the right hand side are all symmetrized in exactly the same manner as the partial derivatives on the left hand side. Thus results from the lower order equations can be fed into the later equations to completely eliminate the partial derivatives of Γ.

$$A^b A^c \Gamma^a{}_{d(b,c)} = \frac{1}{3} A^b A^c R^a{}_{bcd}$$

$$A^b A^c A^e \Gamma^a{}_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a{}_{bcd}$$

$$A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} = \frac{3}{5} A^b A^c A^e A^f \partial_{fe} R^a{}_{bcd} - \frac{1}{15} A^b A^c A^e A^f R^a{}_{ceg} R^g{}_{bfd} - \frac{1}{15} A^b A^c A^e A^f R^a{}_{cfg} R^g{}_{bed}$$

Stage 4: Reformatting

This is just simple reformatting.

$$
\begin{aligned}
3A^bA^c\Gamma^a{}_{d(b,c)} &= A^bA^cR^a{}_{bcd}\\
6A^bA^cA^e\Gamma^a{}_{d(b,ce)} &= 3A^bA^cA^e\partial_eR^a{}_{bcd}\\
15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} &= A^bA^cA^eA^f\left(9\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right)
\end{aligned}
$$

Stage 1: Compute symmetrised derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

# symmetrized partial derivatives of \Gamma

dGamma01:= (1/3) A^{b} A^{c} ( R^{a}_{b c d} - \Gamma^{a}_{c e}\Gamma^{e}_{b d} ).
                                                     # cdb (dGamma01.101,dGamma01)

dGamma02:= (6/4) A^{a}\partial_{a}{ @(dGamma01) }.   # cdb (dGamma02.101,dGamma02)
distribute   (dGamma02)                              # cdb (dGamma02.102,dGamma02)
product_rule (dGamma02)                              # cdb (dGamma02.103,dGamma02)
unwrap       (dGamma02)                              # cdb (dGamma02.104,dGamma02)
distribute   (dGamma02)                              # cdb (dGamma02.105,dGamma02)

dGamma03:= (12/10) A^{a}\partial_{a}{ @(dGamma02) }. # cdb (dGamma03.101,dGamma03)
distribute   (dGamma03)                              # cdb (dGamma03.102,dGamma03)
product_rule (dGamma03)                              # cdb (dGamma03.103,dGamma03)
unwrap       (dGamma03)                              # cdb (dGamma03.104,dGamma03)
```

```
distribute   (dGamma03)                             # cdb (dGamma03.105,dGamma03)

dGamma04:= (20/18) A^{a}\partial_{a}{ @(dGamma03) }. # cdb (dGamma04.101,dGamma04)
distribute   (dGamma04)                             # cdb (dGamma04.102,dGamma04)
product_rule (dGamma04)                             # cdb (dGamma04.103,dGamma04)
unwrap       (dGamma04)                             # cdb (dGamma04.104,dGamma04)
distribute   (dGamma04)                             # cdb (dGamma04.105,dGamma04)

dGamma05:= (30/28) A^{a}\partial_{a}{ @(dGamma04) }. # cdb (dGamma05.101,dGamma05)
distribute   (dGamma05)                             # cdb (dGamma05.102,dGamma05)
product_rule (dGamma05)                             # cdb (dGamma05.103,dGamma05)
unwrap       (dGamma05)                             # cdb (dGamma05.104,dGamma05)
distribute   (dGamma05)                             # cdb (dGamma05.105,dGamma05)
```

$$\texttt{dGamma01.101} := \frac{1}{3}A^bA^c\left(R^a{}_{bcd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.101} := \frac{1}{2}A^f\partial_f\left(A^bA^c\left(R^a{}_{bcd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd}\right)\right)$$

$$\texttt{dGamma02.102} := \frac{1}{2}A^f\partial_f\left(A^bA^cR^a{}_{bcd}\right) - \frac{1}{2}A^f\partial_f\left(A^bA^c\Gamma^a{}_{ce}\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.103} := \frac{1}{2}A^f\left(\partial_fA^bA^cR^a{}_{bcd} + A^b\partial_fA^cR^a{}_{bcd} + A^bA^c\partial_fR^a{}_{bcd}\right) - \frac{1}{2}A^f\left(\partial_fA^bA^c\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^b\partial_fA^c\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^bA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^bA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.104} := \frac{1}{2}A^fA^bA^c\partial_fR^a{}_{bcd} - \frac{1}{2}A^f\left(A^bA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^bA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma02.105} := \frac{1}{2}A^fA^bA^c\partial_fR^a{}_{bcd} - \frac{1}{2}A^fA^bA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{1}{2}A^fA^bA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}$$

$$\texttt{dGamma03.101} := \frac{6}{5}A^g\partial_g\left(\frac{1}{2}A^fA^bA^c\partial_fR^a{}_{bcd} - \frac{1}{2}A^fA^bA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{1}{2}A^fA^bA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma03.102} := \frac{3}{5}A^g\partial_g\left(A^fA^bA^c\partial_fR^a{}_{bcd}\right) - \frac{3}{5}A^g\partial_g\left(A^fA^bA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd}\right) - \frac{3}{5}A^g\partial_g\left(A^fA^bA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}\right)$$

$$\begin{aligned}\texttt{dGamma03.103} := {} & \frac{3}{5}A^g\left(\partial_gA^fA^bA^c\partial_fR^a{}_{bcd} + A^f\partial_gA^bA^c\partial_fR^a{}_{bcd} + A^fA^b\partial_gA^c\partial_fR^a{}_{bcd} + A^fA^bA^c\partial_{gf}R^a{}_{bcd}\right)\\ & - \frac{3}{5}A^g\left(\partial_gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^f\partial_gA^bA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^fA^b\partial_gA^c\partial_f\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd}\right)\\ & - \frac{3}{5}A^g\left(\partial_gA^fA^bA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^f\partial_gA^bA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^fA^b\partial_gA^c\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^fA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}\right)\end{aligned}$$

$$\texttt{dGamma03.104} := \frac{3}{5}A^gA^fA^bA^c\partial_{gf}R^a{}_{bcd} - \frac{3}{5}A^g\left(A^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd}\right) - \frac{3}{5}A^g\left(A^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^fA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma03.105} := \frac{3}{5}A^gA^fA^bA^c\partial_{gf}R^a{}_{bcd} - \frac{3}{5}A^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{3}{5}A^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} - \frac{3}{5}A^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} - \frac{3}{5}A^gA^fA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}$$

$$\texttt{dGamma04.101} := \frac{10}{9}A^h\partial_h\left(\frac{3}{5}A^gA^fA^bA^c\partial_{gf}R^a{}_{bcd} - \frac{3}{5}A^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{3}{5}A^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} - \frac{3}{5}A^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} - \frac{3}{5}A^gA^fA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}\right)$$

$$\texttt{dGamma04.102} := \frac{2}{3}A^h\partial_h\left(A^gA^fA^bA^c\partial_{gf}R^a{}_{bcd}\right) - \frac{2}{3}A^h\partial_h\left(A^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd}\right) - \frac{2}{3}A^h\partial_h\left(A^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd}\right) - \frac{2}{3}A^h\partial_h\left(A^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}\right) - \frac{2}{3}A^h\partial_h\left(A^gA^fA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}\right)$$

$$\begin{aligned}\texttt{dGamma04.103} := {}& \frac{2}{3}A^h\left(\partial_hA^gA^fA^bA^c\partial_{gf}R^a{}_{bcd} + A^g\partial_hA^fA^bA^c\partial_{gf}R^a{}_{bcd} + A^gA^f\partial_hA^bA^c\partial_{gf}R^a{}_{bcd} + A^gA^fA^b\partial_hA^c\partial_{gf}R^a{}_{bcd} + A^gA^fA^bA^c\partial_{hgf}R^a{}_{bcd}\right)\\
&- \frac{2}{3}A^h\left(\partial_hA^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^g\partial_hA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^gA^f\partial_hA^bA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^gA^fA^b\partial_hA^c\partial_{gf}\Gamma^a{}_{ce}\Gamma^e{}_{bd}\right.\\
&\left. + A^gA^fA^bA^c\partial_{hgf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_h\Gamma^e{}_{bd}\right) - \frac{2}{3}A^h\left(\partial_hA^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} + A^g\partial_hA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd}\right.\\
&\left. + A^gA^f\partial_hA^bA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} + A^gA^fA^b\partial_hA^c\partial_f\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd}\right)\\
&- \frac{2}{3}A^h\left(\partial_hA^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^g\partial_hA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^gA^f\partial_hA^bA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^gA^fA^b\partial_hA^c\partial_g\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}\right.\\
&\left. + A^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd}\right) - \frac{2}{3}A^h\left(\partial_hA^gA^fA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} + A^g\partial_hA^fA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}\right.\\
&\left. + A^gA^f\partial_hA^bA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} + A^gA^fA^b\partial_hA^c\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_h\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} + A^gA^fA^bA^c\Gamma^a{}_{ce}\partial_{hgf}\Gamma^e{}_{bd}\right)\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.104} := {}& \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hgf}R^a{}_{bcd} - \frac{2}{3}A^h\left(A^gA^fA^bA^c\partial_{hgf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_h\Gamma^e{}_{bd}\right)\\
&- \frac{2}{3}A^h\left(A^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd}\right)\\
&- \frac{2}{3}A^h\left(A^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} + A^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd}\right) - \frac{2}{3}A^h\left(A^gA^fA^bA^c\partial_h\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} + A^gA^fA^bA^c\Gamma^a{}_{ce}\partial_{hgf}\Gamma^e{}_{bd}\right)\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.105} := {}& \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hgf}R^a{}_{bcd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hgf}\Gamma^a{}_{ce}\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_h\Gamma^e{}_{bd}\\
&- \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}\\
&- \frac{2}{3}A^hA^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_h\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\Gamma^a{}_{ce}\partial_{hgf}\Gamma^e{}_{bd}\end{aligned}$$

Stage 2: Impose Riemann normal coordinates

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

# switch to RNC

dGamma01 = impose_rnc (dGamma01)   # cdb (dGamma01.201,dGamma01)
dGamma02 = impose_rnc (dGamma02)   # cdb (dGamma02.202,dGamma02)
dGamma03 = impose_rnc (dGamma03)   # cdb (dGamma03.203,dGamma03)
dGamma04 = impose_rnc (dGamma04)   # cdb (dGamma04.204,dGamma04)
dGamma05 = impose_rnc (dGamma05)   # cdb (dGamma05.205,dGamma05)
```

$$\texttt{dGamma01.201} := \frac{1}{3}A^{b}A^{c}R^{a}{}_{bcd}$$

$$\texttt{dGamma02.202} := \frac{1}{2}A^{f}A^{b}A^{c}\partial_{f}R^{a}{}_{bcd}$$

$$\texttt{dGamma03.203} := \frac{3}{5}A^{g}A^{f}A^{b}A^{c}\partial_{gf}R^{a}{}_{bcd} - \frac{3}{5}A^{g}A^{f}A^{b}A^{c}\partial_{f}\Gamma^{a}{}_{ce}\partial_{g}\Gamma^{e}{}_{bd} - \frac{3}{5}A^{g}A^{f}A^{b}A^{c}\partial_{g}\Gamma^{a}{}_{ce}\partial_{f}\Gamma^{e}{}_{bd}$$

$$\texttt{dGamma04.204} := \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hgf}R^a{}_{bcd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_h\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd}$$
$$- \frac{2}{3}A^hA^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd} - \frac{2}{3}A^hA^gA^fA^bA^c\partial_h\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}$$

$$\texttt{dGamma05.205} := \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihgf}R^a{}_{bcd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hgf}\Gamma^a{}_{ce}\partial_i\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{igf}\Gamma^a{}_{ce}\partial_h\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_{ih}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihf}\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_{ig}\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{if}\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_{ihg}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihg}\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_{if}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ig}\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_{ihf}\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ih}\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_h\Gamma^a{}_{ce}\partial_{igf}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_i\Gamma^a{}_{ce}\partial_{hgf}\Gamma^e{}_{bd}$$

Stage 3: Replace partial derivatives of Γ with partial derivatives of R

```
# use lower equations to eliminate partial derivs of Gamma from rhs

# this produces experssions for the partial derivs of the Gamma's in terms of the Rabcd and its partial derivs

substitute (dGamma03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)             # cdb(dGamma03.301,dGamma03
substitute (dGamma03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)             # cdb(dGamma03.302,dGamma03
distribute (dGamma03)                                                                                  # cdb(dGamma03.303,dGamma03

substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)      # cdb(dGamma04.301,dGamma04
substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)      # cdb(dGamma04.302,dGamma04
substitute (dGamma04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)             # cdb(dGamma04.303,dGamma04
substitute (dGamma04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)             # cdb(dGamma04.304,dGamma04
distribute (dGamma04)                                                                                  # cdb(dGamma04.305,dGamma04

substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{d b}} -> @(dGamma03)$,repeat=True)   # cdb(dGamma05.301,dGamma05
substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{b d}} -> @(dGamma03)$,repeat=True)   # cdb(dGamma05.302,dGamma05
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)      # cdb(dGamma05.303,dGamma05
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)      # cdb(dGamma05.304,dGamma05
substitute (dGamma05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)             # cdb(dGamma05.305,dGamma05
substitute (dGamma05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)             # cdb(dGamma05.306,dGamma05
distribute (dGamma05)                                                                                  # cdb(dGamma05.307,dGamma05
```

$$\texttt{dGamma03.301} := \frac{3}{5}A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15}A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15}A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\texttt{dGamma03.302} := \frac{3}{5}A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15}A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15}A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\texttt{dGamma03.303} := \frac{3}{5}A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15}A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15}A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\begin{aligned}\texttt{dGamma04.301} := {}& \frac{2}{3}A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3}A^h A^g A^f A^b A^c \partial_{gf}\Gamma^a{}_{ce}\partial_h\Gamma^e{}_{bd} - \frac{2}{3}A^h A^g A^f A^b A^c \partial_{hf}\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} - \frac{2}{3}A^h A^g A^f A^b A^c \partial_f\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd} \\ & - \frac{2}{3}A^h A^g A^f A^b A^c \partial_{hg}\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd} - \frac{2}{3}A^h A^g A^f A^b A^c \partial_g\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd} - \frac{2}{3}A^h A^g A^f A^b A^c \partial_h\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.302} := {}& \frac{2}{3}A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{3}A^h A^f A^c A^g \partial_f R^a{}_{cge} A^b \partial_h\Gamma^e{}_{bd} - \frac{1}{3}A^f A^c A^h \partial_f R^a{}_{che} A^g A^b \partial_g\Gamma^e{}_{bd} - \frac{1}{3}A^g A^b A^h \partial_g R^e{}_{bhd} A^f A^c \partial_f\Gamma^a{}_{ce} \\ & - \frac{1}{3}A^g A^c A^h \partial_g R^a{}_{che} A^f A^b \partial_f\Gamma^e{}_{bd} - \frac{1}{3}A^f A^b A^h \partial_f R^e{}_{bhd} A^g A^c \partial_g\Gamma^a{}_{ce} - \frac{1}{3}A^h A^f A^b A^g \partial_f R^e{}_{bgd} A^c \partial_h\Gamma^a{}_{ce}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.303} := {}& \frac{2}{3}A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9}A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9}A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9}A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9}A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9}A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9}A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.304} := {}& \frac{2}{3}A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9}A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9}A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9}A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9}A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9}A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9}A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.305} := {}& \frac{2}{3}A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9}A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9}A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9}A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9}A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9}A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9}A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd}\end{aligned}$$

$$\texttt{dGamma05.301} := \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihgf}R^a{}_{bcd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hgf}\Gamma^a{}_{ce}\partial_i\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{igf}\Gamma^a{}_{ce}\partial_h\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_{ih}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihf}\Gamma^a{}_{ce}\partial_g\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_{ig}\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{if}\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_f\Gamma^a{}_{ce}\partial_{ihg}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihg}\Gamma^a{}_{ce}\partial_f\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_{if}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ig}\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_g\Gamma^a{}_{ce}\partial_{ihf}\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ih}\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_h\Gamma^a{}_{ce}\partial_{igf}\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_i\Gamma^a{}_{ce}\partial_{hgf}\Gamma^e{}_{bd}$$

$$\texttt{dGamma05.302} := \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihgf}R^a{}_{bcd} - \frac{5}{7}A^i\left(\frac{3}{5}A^jA^fA^cA^h\partial_{jf}R^a{}_{che} - \frac{1}{15}A^cA^jR^g{}_{cje}A^hA^fR^a{}_{hfg} - \frac{1}{15}A^hA^jR^a{}_{hjg}A^cA^fR^g{}_{cfe}\right)A^b\partial_i\Gamma^e{}_{bd}$$
$$- \frac{5}{7}\left(\frac{3}{5}A^jA^fA^cA^i\partial_{jf}R^a{}_{cie} - \frac{1}{15}A^cA^jR^g{}_{cje}A^iA^fR^a{}_{ifg} - \frac{1}{15}A^iA^jR^a{}_{ijg}A^cA^fR^g{}_{cfe}\right)A^hA^b\partial_h\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_{ih}\Gamma^e{}_{bd}$$
$$- \frac{5}{7}\left(\frac{3}{5}A^jA^fA^cA^i\partial_{jf}R^a{}_{cie} - \frac{1}{15}A^cA^jR^h{}_{cje}A^iA^fR^a{}_{ifh} - \frac{1}{15}A^iA^jR^a{}_{ijh}A^cA^fR^h{}_{cfe}\right)A^gA^b\partial_g\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_{ig}\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{if}\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd} - \frac{5}{7}\left(\frac{3}{5}A^jA^gA^bA^i\partial_{jg}R^e{}_{bid} - \frac{1}{15}A^bA^jR^h{}_{bjd}A^iA^gR^e{}_{igh} - \frac{1}{15}A^iA^jR^e{}_{ijh}A^bA^gR^h{}_{bgd}\right)A^fA^c\partial_f\Gamma^a{}_{ce}$$
$$- \frac{5}{7}\left(\frac{3}{5}A^jA^gA^cA^i\partial_{jg}R^a{}_{cie} - \frac{1}{15}A^cA^jR^h{}_{cje}A^iA^gR^a{}_{igh} - \frac{1}{15}A^iA^jR^a{}_{ijh}A^cA^gR^h{}_{cge}\right)A^fA^b\partial_f\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_{if}\Gamma^e{}_{bd}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ig}\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd} - \frac{5}{7}\left(\frac{3}{5}A^jA^fA^bA^i\partial_{jf}R^e{}_{bid} - \frac{1}{15}A^bA^jR^h{}_{bjd}A^iA^fR^e{}_{ifh} - \frac{1}{15}A^iA^jR^e{}_{ijh}A^bA^fR^h{}_{bfd}\right)A^gA^c\partial_g\Gamma^a{}_{ce}$$
$$- \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ih}\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} - \frac{5}{7}\left(\frac{3}{5}A^jA^fA^bA^i\partial_{jf}R^e{}_{bid} - \frac{1}{15}A^bA^jR^g{}_{bjd}A^iA^fR^e{}_{ifg} - \frac{1}{15}A^iA^jR^e{}_{ijg}A^bA^fR^g{}_{bfd}\right)A^hA^c\partial_h\Gamma^a{}_{ce}$$
$$- \frac{5}{7}A^i\left(\frac{3}{5}A^jA^fA^bA^h\partial_{jf}R^e{}_{bhd} - \frac{1}{15}A^bA^jR^g{}_{bjd}A^hA^fR^e{}_{hfg} - \frac{1}{15}A^hA^jR^e{}_{hjg}A^bA^fR^g{}_{bfd}\right)A^c\partial_i\Gamma^a{}_{ce}$$

$$
\begin{aligned}
\texttt{dGamma05.303} :=\ & \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ihgf}R^a{}_{bcd} - \frac{5}{7}A^i\left(\frac{3}{5}A^jA^fA^cA^h\partial_{jf}R^a{}_{che} - \frac{1}{15}A^cA^jR^g{}_{cje}A^hA^fR^a{}_{hfg} - \frac{1}{15}A^hA^jR^a{}_{hjg}A^cA^fR^g{}_{cfe}\right)A^b\partial_i\Gamma^e{}_{bd} \\
& - \frac{5}{7}\left(\frac{3}{5}A^jA^fA^cA^i\partial_{jf}R^a{}_{cie} - \frac{1}{15}A^cA^jR^g{}_{cje}A^iA^fR^a{}_{ifg} - \frac{1}{15}A^iA^jR^a{}_{ijg}A^cA^fR^g{}_{cfe}\right)A^hA^b\partial_h\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{gf}\Gamma^a{}_{ce}\partial_{ih}\Gamma^e{}_{bd} \\
& - \frac{5}{7}\left(\frac{3}{5}A^jA^fA^cA^i\partial_{jf}R^a{}_{cie} - \frac{1}{15}A^cA^jR^h{}_{cje}A^iA^fR^a{}_{ifh} - \frac{1}{15}A^iA^jR^a{}_{ijh}A^cA^fR^h{}_{cfe}\right)A^gA^b\partial_g\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hf}\Gamma^a{}_{ce}\partial_{ig}\Gamma^e{}_{bd} \\
& - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{if}\Gamma^a{}_{ce}\partial_{hg}\Gamma^e{}_{bd} - \frac{5}{7}\left(\frac{3}{5}A^jA^gA^bA^i\partial_{jg}R^e{}_{bid} - \frac{1}{15}A^bA^jR^h{}_{bjd}A^iA^gR^e{}_{igh} - \frac{1}{15}A^iA^jR^e{}_{ijh}A^bA^gR^h{}_{bgd}\right)A^fA^c\partial_f\Gamma^a{}_{ce} \\
& - \frac{5}{7}\left(\frac{3}{5}A^jA^gA^cA^i\partial_{jg}R^a{}_{cie} - \frac{1}{15}A^cA^jR^h{}_{cje}A^iA^gR^a{}_{igh} - \frac{1}{15}A^iA^jR^a{}_{ijh}A^cA^gR^h{}_{cge}\right)A^fA^b\partial_f\Gamma^e{}_{bd} - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{hg}\Gamma^a{}_{ce}\partial_{if}\Gamma^e{}_{bd} \\
& - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ig}\Gamma^a{}_{ce}\partial_{hf}\Gamma^e{}_{bd} - \frac{5}{7}\left(\frac{3}{5}A^jA^fA^bA^i\partial_{jf}R^e{}_{bid} - \frac{1}{15}A^bA^jR^h{}_{bjd}A^iA^fR^e{}_{ifh} - \frac{1}{15}A^iA^jR^e{}_{ijh}A^bA^fR^h{}_{bfd}\right)A^gA^c\partial_g\Gamma^a{}_{ce} \\
& - \frac{5}{7}A^iA^hA^gA^fA^bA^c\partial_{ih}\Gamma^a{}_{ce}\partial_{gf}\Gamma^e{}_{bd} - \frac{5}{7}\left(\frac{3}{5}A^jA^fA^bA^i\partial_{jf}R^e{}_{bid} - \frac{1}{15}A^bA^jR^g{}_{bjd}A^iA^fR^e{}_{ifg} - \frac{1}{15}A^iA^jR^e{}_{ijg}A^bA^fR^g{}_{bfd}\right)A^hA^c\partial_h\Gamma^a{}_{ce} \\
& - \frac{5}{7}A^i\left(\frac{3}{5}A^jA^fA^bA^h\partial_{jf}R^e{}_{bhd} - \frac{1}{15}A^bA^jR^g{}_{bjd}A^hA^fR^e{}_{hfg} - \frac{1}{15}A^hA^jR^e{}_{hjg}A^bA^fR^g{}_{bfd}\right)A^c\partial_i\Gamma^a{}_{ce}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dGamma05.304} := {} & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left(\frac{3}{5} A^j A^f A^c A^h \partial_{jf} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjg} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{7} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} \\
& - \frac{5}{7} \left(\frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce} \\
& - \frac{5}{7} \left(\frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} \\
& - \frac{5}{7} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} \\
& - \frac{5}{7} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i \left(\frac{3}{5} A^j A^f A^b A^h \partial_{jf} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjg} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dGamma05.305} := {} & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{21} A^b A^i R^e{}_{bid} \left(\frac{3}{5} A^j A^f A^c A^h \partial_{jf} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjg} A^c A^f R^g{}_{cfe} \right) \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^b A^h R^e{}_{bhd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^b A^g R^e{}_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^c A^f R^a{}_{cfe} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^b A^f R^e{}_{bfd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^c A^g R^a{}_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^c A^h R^a{}_{che} \\
& - \frac{5}{21} A^c A^i R^a{}_{cie} \left(\frac{3}{5} A^j A^f A^b A^h \partial_{jf} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjg} A^b A^f R^g{}_{bfd} \right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dGamma05.306} := {} & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{21} A^b A^i R^e{}_{bid} \left(\frac{3}{5} A^j A^f A^c A^h \partial_{jf} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjg} A^c A^f R^g{}_{cfe} \right) \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^b A^h R^e{}_{bhd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^b A^g R^e{}_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^c A^f R^a{}_{cfe} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^b A^f R^e{}_{bfd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^c A^g R^a{}_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} \\
& - \frac{5}{21} \left(\frac{3}{5} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^c A^h R^a{}_{che} \\
& - \frac{5}{21} A^c A^i R^a{}_{cie} \left(\frac{3}{5} A^j A^f A^b A^h \partial_{jf} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjg} A^b A^f R^g{}_{bfd} \right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dGamma05.307} := {} & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{1}{7} A^b A^i R^e{}_{bid} A^j A^f A^c A^h \partial_{jf} R^a{}_{che} + \frac{1}{63} A^b A^i R^e{}_{bid} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} \\
& + \frac{1}{63} A^b A^i R^e{}_{bid} A^h A^j R^a{}_{hjg} A^c A^f R^g{}_{cfe} - \frac{1}{7} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} A^b A^h R^e{}_{bhd} + \frac{1}{63} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} A^b A^h R^e{}_{bhd} \\
& + \frac{1}{63} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} A^b A^h R^e{}_{bhd} - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{7} A^j A^f A^c A^i \partial_{jf} R^a{}_{cie} A^b A^g R^e{}_{bgd} \\
& + \frac{1}{63} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} A^b A^g R^e{}_{bgd} + \frac{1}{63} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} A^b A^g R^e{}_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} - \frac{1}{7} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} A^c A^f R^a{}_{cfe} \\
& + \frac{1}{63} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} A^c A^f R^a{}_{cfe} + \frac{1}{63} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{7} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} A^b A^f R^e{}_{bfd} \\
& + \frac{1}{63} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} A^b A^f R^e{}_{bfd} + \frac{1}{63} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} A^b A^f R^e{}_{bfd} - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} \\
& - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} - \frac{1}{7} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} A^c A^g R^a{}_{cge} + \frac{1}{63} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} A^c A^g R^a{}_{cge} \\
& + \frac{1}{63} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} A^c A^g R^a{}_{cge} - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} - \frac{1}{7} A^j A^f A^b A^i \partial_{jf} R^e{}_{bid} A^c A^h R^a{}_{che} \\
& + \frac{1}{63} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} A^c A^h R^a{}_{che} + \frac{1}{63} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} A^c A^h R^a{}_{che} - \frac{1}{7} A^c A^i R^a{}_{cie} A^j A^f A^b A^h \partial_{jf} R^e{}_{bhd} \\
& + \frac{1}{63} A^c A^i R^a{}_{cie} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} + \frac{1}{63} A^c A^i R^a{}_{cie} A^h A^j R^e{}_{hjg} A^b A^f R^g{}_{bfd}
\end{aligned}
$$

```
# note:
# canonicalise must not be used here because it may make changes like
#    R^{a}_{b c d} -> - R_{b}^{a}_{c d}
# these changes can not be applied inside a \partial, must defer use
# of canocialise until we have \nabla acting on curvatures

sort_product   (dGamma03) # cdb(dGamma03.401,dGamma03)
rename_dummies (dGamma03) # cdb(dGamma03.402,dGamma03)
# canonicalise   (dGamma03) # cdb(dGamma03.403,dGamma03)

sort_product   (dGamma04) # cdb(dGamma04.401,dGamma04)
rename_dummies (dGamma04) # cdb(dGamma04.402,dGamma04)
# canonicalise   (dGamma04) # cdb(dGamma04.403,dGamma04)

sort_product   (dGamma05) # cdb(dGamma05.401,dGamma05)
rename_dummies (dGamma05) # cdb(dGamma05.402,dGamma05)
# canonicalise   (dGamma05) # cdb(dGamma05.403,dGamma05)
```

$$\texttt{dGamma03.401} := \frac{3}{5}A^bA^cA^fA^g\partial_{gf}R^a{}_{bcd} - \frac{1}{15}A^bA^cA^fA^gR^a{}_{cfe}R^e{}_{bgd} - \frac{1}{15}A^bA^cA^fA^gR^a{}_{cge}R^e{}_{bfd}$$

$$\texttt{dGamma03.402} := \frac{3}{5}A^bA^cA^eA^f\partial_{fe}R^a{}_{bcd} - \frac{1}{15}A^bA^cA^eA^fR^a{}_{ceg}R^g{}_{bfd} - \frac{1}{15}A^bA^cA^eA^fR^a{}_{cfg}R^g{}_{bed}$$

$$\begin{aligned}\texttt{dGamma04.401} := {}& \frac{2}{3}A^bA^cA^fA^gA^h\partial_{hgf}R^a{}_{bcd} - \frac{1}{9}A^bA^cA^fA^gA^hR^e{}_{bhd}\partial_fR^a{}_{cge} - \frac{1}{9}A^bA^cA^fA^gA^hR^e{}_{bgd}\partial_fR^a{}_{che} - \frac{1}{9}A^bA^cA^fA^gA^hR^a{}_{cfe}\partial_gR^e{}_{bhd}\\ & - \frac{1}{9}A^bA^cA^fA^gA^hR^e{}_{bfd}\partial_gR^a{}_{che} - \frac{1}{9}A^bA^cA^fA^gA^hR^a{}_{cge}\partial_fR^e{}_{bhd} - \frac{1}{9}A^bA^cA^fA^gA^hR^a{}_{che}\partial_fR^e{}_{bgd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma04.402} := {}& \frac{2}{3}A^bA^cA^eA^fA^g\partial_{gfe}R^a{}_{bcd} - \frac{1}{9}A^bA^cA^eA^fA^gR^h{}_{bgd}\partial_eR^a{}_{cfh} - \frac{1}{9}A^bA^cA^eA^fA^gR^h{}_{bfd}\partial_eR^a{}_{cgh} - \frac{1}{9}A^bA^cA^eA^fA^gR^a{}_{ceh}\partial_fR^h{}_{bgd}\\ & - \frac{1}{9}A^bA^cA^eA^fA^gR^h{}_{bed}\partial_fR^a{}_{cgh} - \frac{1}{9}A^bA^cA^eA^fA^gR^a{}_{cfh}\partial_eR^h{}_{bgd} - \frac{1}{9}A^bA^cA^eA^fA^gR^a{}_{cgh}\partial_eR^h{}_{bfd}\end{aligned}$$

$$\begin{aligned}\texttt{dGamma05.401} := {}& \frac{5}{7}A^bA^cA^fA^gA^hA^i\partial_{ihgf}R^a{}_{bcd} - \frac{1}{7}A^bA^cA^fA^hA^iA^jR^e{}_{bid}\partial_{jf}R^a{}_{che} + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{hfg}R^e{}_{bid}R^g{}_{cje}\\ & + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{hjg}R^e{}_{bid}R^g{}_{cfe} - \frac{1}{7}A^bA^cA^fA^hA^iA^jR^e{}_{bhd}\partial_{jf}R^a{}_{cie} + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{ifg}R^e{}_{bhd}R^g{}_{cje}\\ & + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{ijg}R^e{}_{bhd}R^g{}_{cfe} - \frac{5}{28}A^bA^cA^fA^gA^hA^i\partial_fR^a{}_{cge}\partial_hR^e{}_{bid} - \frac{1}{7}A^bA^cA^fA^gA^iA^jR^e{}_{bgd}\partial_{jf}R^a{}_{cie}\\ & + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{ifh}R^e{}_{bgd}R^h{}_{cje} + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{ijh}R^e{}_{bgd}R^h{}_{cfe}\\ & - \frac{5}{28}A^bA^cA^fA^gA^hA^i\partial_fR^a{}_{che}\partial_gR^e{}_{bid} - \frac{5}{28}A^bA^cA^fA^gA^hA^i\partial_fR^a{}_{cie}\partial_gR^e{}_{bhd} - \frac{1}{7}A^bA^cA^fA^gA^iA^jR^a{}_{cfe}\partial_{jg}R^e{}_{bid}\\ & + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{cfe}R^e{}_{igh}R^h{}_{bjd} + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{cfe}R^e{}_{ijh}R^h{}_{bgd}\\ & - \frac{1}{7}A^bA^cA^fA^gA^iA^jR^e{}_{bfd}\partial_{jg}R^a{}_{cie} + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{igh}R^e{}_{bfd}R^h{}_{cje} + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{ijh}R^e{}_{bfd}R^h{}_{cge}\\ & - \frac{5}{28}A^bA^cA^fA^gA^hA^i\partial_fR^e{}_{bid}\partial_gR^a{}_{che} - \frac{5}{28}A^bA^cA^fA^gA^hA^i\partial_fR^e{}_{bhd}\partial_gR^a{}_{cie} - \frac{1}{7}A^bA^cA^fA^gA^iA^jR^a{}_{cge}\partial_{jf}R^e{}_{bid}\\ & + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{cge}R^e{}_{ifh}R^h{}_{bjd} + \frac{1}{63}A^bA^cA^fA^gA^iA^jR^a{}_{cge}R^e{}_{ijh}R^h{}_{bfd} - \frac{5}{28}A^bA^cA^fA^gA^hA^i\partial_fR^e{}_{bgd}\partial_hR^a{}_{cie}\\ & - \frac{1}{7}A^bA^cA^fA^hA^iA^jR^a{}_{che}\partial_{jf}R^e{}_{bid} + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{che}R^e{}_{ifg}R^g{}_{bjd} + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{che}R^e{}_{ijg}R^g{}_{bfd}\\ & - \frac{1}{7}A^bA^cA^fA^hA^iA^jR^a{}_{cie}\partial_{jf}R^e{}_{bhd} + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{cie}R^e{}_{hfg}R^g{}_{bjd} + \frac{1}{63}A^bA^cA^fA^hA^iA^jR^a{}_{cie}R^e{}_{hjg}R^g{}_{bfd}\end{aligned}$$

$$
\begin{aligned}
\texttt{dGamma05.402} := {}& \frac{5}{7} A^b A^c A^e A^f A^g A^h \partial_{hgfe} R^a{}_{bcd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i{}_{bgd} \partial_{he} R^a{}_{cfi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{fei} R^j{}_{bgd} R^i{}_{chj} \\
& + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{fhi} R^j{}_{bgd} R^i{}_{cej} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^i{}_{bfd} \partial_{he} R^a{}_{cgi} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{gei} R^j{}_{bfd} R^i{}_{chj} \\
& + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{ghi} R^j{}_{bfd} R^i{}_{cej} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a{}_{cfi} \partial_g R^i{}_{bhd} \\
& - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a{}_{cgi} \partial_f R^i{}_{bhd} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a{}_{chi} \partial_f R^i{}_{bgd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a{}_{cei} \partial_{hf} R^i{}_{bgd} \\
& + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cei} R^i{}_{gfj} R^j{}_{bhd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cei} R^i{}_{ghj} R^j{}_{bfd} \\
& - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i{}_{bed} \partial_{hf} R^a{}_{cgi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{gfi} R^j{}_{bed} R^i{}_{chj} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{ghi} R^j{}_{bed} R^i{}_{cfj} \\
& - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i{}_{bhd} \partial_f R^a{}_{cgi} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i{}_{bgd} \partial_f R^a{}_{chi} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^a{}_{cfi} \partial_{he} R^i{}_{bgd} \\
& + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cfi} R^i{}_{gej} R^j{}_{bhd} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cfi} R^i{}_{ghj} R^j{}_{bed} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i{}_{bfd} \partial_g R^a{}_{chi} \\
& - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a{}_{cgi} \partial_{he} R^i{}_{bfd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cgi} R^i{}_{fej} R^j{}_{bhd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a{}_{cgi} R^i{}_{fhj} R^j{}_{bed}
\end{aligned}
$$

```python
import cdblib

cdblib.create ('dGamma.json')

cdblib.put ('dGamma01',dGamma01,'dGamma.json')
cdblib.put ('dGamma02',dGamma02,'dGamma.json')
cdblib.put ('dGamma03',dGamma03,'dGamma.json')
cdblib.put ('dGamma04',dGamma04,'dGamma.json')
cdblib.put ('dGamma05',dGamma05,'dGamma.json')
```

Stage 4: Reformatting

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                              -> A001^{a}                $)
    substitute (obj,$ x^{a}                              -> A002^{a}                $)
    substitute (obj,$ g^{a b}                            -> A003^{a b}              $)
    substitute (obj,$ \partial_{e f g h}{R^{a}_{b c d}}  -> A008^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}{R^{a}_{b c d}}    -> A007^{a}_{b c d e f g}  $)
    substitute (obj,$ \partial_{e f}{R^{a}_{b c d}}      -> A006^{a}_{b c d e f}    $)
    substitute (obj,$ \partial_{e}{R^{a}_{b c d}}        -> A005^{a}_{b c d e}      $)
    substitute (obj,$ R^{a}_{b c d}                      -> A004^{a}_{b c d}        $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                 -> A^{a}                             $)
    substitute (obj,$ A002^{a}                 -> x^{a}                             $)
    substitute (obj,$ A003^{a b}               -> g^{a b}                           $)
    substitute (obj,$ A004^{a}_{b c d}         -> R^{a}_{b c d}                     $)
    substitute (obj,$ A005^{a}_{b c d e}       -> \partial_{e}{R^{a}_{b c d}}       $)
    substitute (obj,$ A006^{a}_{b c d e f}     -> \partial_{e f}{R^{a}_{b c d}}     $)
    substitute (obj,$ A007^{a}_{b c d e f g}   -> \partial_{e f g}{R^{a}_{b c d}}   $)
    substitute (obj,$ A008^{a}_{b c d e f g h} -> \partial_{e f g h}{R^{a}_{b c d}} $)

    return obj

def reformat (obj,scale):
    bah  = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute     (tmp)
    tmp = product_sort (tmp)
    rename_dummies (tmp)
    factor_out     (tmp,$A^{a?}$)
    return tmp

def get_term (obj,n):

    A^{a}::Weight(label=numA).
```

```
    foo := @(obj).
    bah  = Ex("numA = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

Gterm01 := @(dGamma01).
Gterm02 := @(dGamma02).
Gterm03 := @(dGamma03).
Gterm04 := @(dGamma04).
Gterm05 := @(dGamma05).

scaled1 = reformat (Gterm01,   3)   # cdb (scaled1.002,scaled1)
scaled2 = reformat (Gterm02,   6)   # cdb (scaled2.002,scaled2)
scaled3 = reformat (Gterm03,  15)   # cdb (scaled3.002,scaled3)
scaled4 = reformat (Gterm04,   9)   # cdb (scaled4.002,scaled4)
scaled5 = reformat (Gterm05, 252)   # cdb (scaled5.002,scaled5)
```

Symmetrised partial derivatives of the connection

$$
\begin{aligned}
3A^bA^c\Gamma^a{}_{d(b,c)} &= A^bA^cR^a{}_{bcd} \\
6A^bA^cA^e\Gamma^a{}_{d(b,ce)} &= 3A^bA^cA^e\partial_eR^a{}_{bcd} \\
15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} &= A^bA^cA^eA^f\left(9\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right) \\
9A^bA^cA^eA^fA^g\Gamma^a{}_{d(b,cefg)} &= A^bA^cA^eA^fA^g\left(6\partial_{gfe}R^a{}_{bcd} - R^h{}_{bgd}\partial_eR^a{}_{cfh} - R^h{}_{bfd}\partial_eR^a{}_{cgh} - R^a{}_{ceh}\partial_fR^h{}_{bgd} - R^h{}_{bed}\partial_fR^a{}_{cgh} - R^a{}_{cfh}\partial_eR^h{}_{bgd} - R^a{}_{cgh}\partial_eR^h{}_{bfd}\right) \\
252A^bA^cA^eA^fA^gA^h\Gamma^a{}_{d(b,cefgh)} &= A^bA^cA^eA^fA^gA^h\Big(180\partial_{hgfe}R^a{}_{bcd} - 36R^i{}_{bgd}\partial_{he}R^a{}_{cfi} + 4R^a{}_{fei}R^i{}_{chj}R^j{}_{bgd} + 4R^a{}_{fhi}R^i{}_{cej}R^j{}_{bgd} - 72R^i{}_{bfd}\partial_{he}R^a{}_{cgi} \\
&\quad + 8R^a{}_{gei}R^i{}_{chj}R^j{}_{bfd} + 8R^a{}_{ghi}R^i{}_{cej}R^j{}_{bfd} - 45\partial_eR^a{}_{cfi}\partial_gR^i{}_{bhd} - 45\partial_eR^a{}_{cgi}\partial_fR^i{}_{bhd} - 45\partial_eR^a{}_{chi}\partial_fR^i{}_{bgd} \\
&\quad - 36R^a{}_{cei}\partial_{hf}R^i{}_{bgd} + 4R^a{}_{cei}R^i{}_{gfj}R^j{}_{bhd} + 4R^a{}_{cei}R^i{}_{ghj}R^j{}_{bfd} - 36R^i{}_{bed}\partial_{hf}R^a{}_{cgi} + 4R^a{}_{gfi}R^i{}_{chj}R^j{}_{bed} \\
&\quad + 4R^a{}_{ghi}R^i{}_{cfj}R^j{}_{bed} - 45\partial_fR^a{}_{cgi}\partial_eR^i{}_{bhd} - 45\partial_fR^a{}_{chi}\partial_eR^i{}_{bgd} - 72R^a{}_{cfi}\partial_{he}R^i{}_{bgd} + 8R^a{}_{cfi}R^i{}_{gej}R^j{}_{bhd} \\
&\quad + 8R^a{}_{cfi}R^i{}_{ghj}R^j{}_{bed} - 45\partial_gR^a{}_{chi}\partial_eR^i{}_{bfd} - 36R^a{}_{cgi}\partial_{he}R^i{}_{bfd} + 4R^a{}_{cgi}R^i{}_{fej}R^j{}_{bhd} + 4R^a{}_{cgi}R^i{}_{fhj}R^j{}_{bed}\Big)
\end{aligned}
$$

```
substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dGamma.export')

# 6th order dGamma, scaled
cdblib.put ('dGamma61scaled',scaled1,'dGamma.export')
cdblib.put ('dGamma62scaled',scaled2,'dGamma.export')
cdblib.put ('dGamma63scaled',scaled3,'dGamma.export')
cdblib.put ('dGamma64scaled',scaled4,'dGamma.export')
cdblib.put ('dGamma65scaled',scaled5,'dGamma.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```

Symmetrised partial derivatives of the Riemann tensor

Here we compute the symmetrised partial derivatoves $R^a{}_{(b\dot{c}d,\underline{e})}$ in terms of the symmetrised covariant derivatives $R^a{}_{(b\dot{c}d;\underline{e})}$. Note that the dot over an index indicates that that index does not take part in the symmetrisation.

We will use the algorithm described in section (10.3) of my lcb09-03 paper. Here we will make one small change of notation – the symbol D^a will replaced with A^a.

We have lots of space (and no annoying editors to appease with brevity) so I will take the liberty to expand slightly on what I wrote in the lcb0-03 paper.

Our starting point is the simple identity

$$\left(R^a{}_{cdb}B^b{}_aA^cA^d\right)_{;e}A^e = \left(R^a{}_{cdb}B^b{}_aA^cA^d\right)_{,e}A^e \tag{1}$$

This is true in all frames since the quantity inside the brackets is a scalar. We are free to make any choice we like for A^a and $B^a{}_b$ so let's choose A^a to be the tangent vector to any geodesic through the origin and choose the $B^a{}_b$ to be constants (i.e, all partial derivatives are zero). We will also use local Riemann normal coordinates and as a consequence, the A^a will also be constant along the integral curves of A (the geodesics in an RNC are always of the form $x^a(s) = sA^a$ for some affine parameter s on the geodesic). Let df/ds be the directional derivative of the function f along the geodesics defined by A^a and assume that s is the proper length along the geodesic (although any affine parameter would be sufficient).

Thus at the origin we have, by choice,

$$0 = B^a{}_{b,c} = B^a{}_{b,cd} = B^a{}_{b,cde} = \dots$$

$$0 = dA^a/ds = d^2A^a/ds^2 = d^3A^a/ds^3 = \dots$$

$$0 = A^a{}_{,b}A^b = \left(A^a{}_{,b}A^b\right)_{,c}A^c = \left(\left(A^a{}_{,b}A^b\right)_{,c}A^c\right)_{,d}A^d$$

$$0 = A^a{}_{;b}A^b = \left(A^a{}_{;b}A^b\right)_{;c}A^c = \left(\left(A^a{}_{;b}A^b\right)_{;c}A^c\right)_{;d}A^d$$

$$df/ds = f_{,a}A^a = f_{;a}A^a$$

$$d^2f/ds^2 = \left(f_{,a}A^a\right)_{,b}A^b = \left(f_{;a}A^a\right)_{;b}A^b$$

$$d^3f/ds^3 = \left(\left(f_{,a}A^a\right)_{,b}A^b\right)_{,c}A^c = \left(\left(f_{;a}A^a\right)_{;b}A^b\right)_{;c}A^c$$

I admit I've gone overboard here in writing out more than I need to but it's handy to have all of these equations laid bare in one convenient place.

Now put $f = R^p{}_{abq} B^q{}_p A^a A^b$. Then upon taking successive derivatives, while taking full advantage of the asummptions just noted, we can eaily see that

$$\left(R^a{}_{cdb} B^b{}_a\right)_{;\underline{e}} A^c A^d A^{\underline{e}} = \left(R^a{}_{cdb}\right)_{,\underline{e}} B^b{}_a A^c A^d A^{\underline{e}} \tag{2}$$

This is the equation that will be computed by the following Cadabra code. All of the compuations will be carried out on the left hand side (in the first version of the paper I swapped the left and righ hand sides).

We will need the successive covariant derivatives of B. The first covariant derivative is just

$$B^a{}_{b;c} A^c = \Gamma^a{}_{dc} B^d{}_b A^c - \Gamma^d{}_{bc} B^a{}_d A^c$$

The quantities on the left hand side are the components of a tensor so further covariant derivatives of the right hand side can be computed (despite the presence of the Γ's) by application of the usual rule for a covariant derivative of a mixed tensor.

Stage 1: Symmetrised partial derivatives of R

The first stage involves the expansion of the left side of (2). This leads to expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and $B^a{}_b$.

$$\begin{aligned}
\left(R^a{}_{cdb}\right)_{,e} B^b{}_a A^c A^d A^e &= -A^a A^b A^c B^d{}_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d{}_e g^{ef} \\
\left(R^a{}_{cdb}\right)_{,ef} B^b{}_a A^c A^d A^e A^f &= -2 A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a \left(\nabla_b R_{cedg}\right) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c \left(\nabla_d B^e{}_f\right) g^{gf} \\
\left(R^a{}_{cdb}\right)_{,efg} B^b{}_a A^c A^d A^e A^f A^g &= -3 A^a A^b A^c A^d A^e \nabla_a R_{bfch} \nabla_d \left(\nabla_e B^f{}_g\right) g^{hg} - 3 A^a A^b A^c A^d A^e \nabla_a B^f{}_g \nabla_b \left(\nabla_c R_{dfeh}\right) g^{gh} \\
&\quad - A^a A^b A^c A^d A^e B^f{}_g \nabla_a \left(\nabla_b \left(\nabla_c R_{dfeh}\right)\right) g^{gh} - A^a A^b A^c A^d A^e R_{afbh} \nabla_c \left(\nabla_d \left(\nabla_e B^f{}_g\right)\right) g^{hg}
\end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

In this stage the symmetrized covariant derivatives of $B^a{}_b$ are computed in terms of its partial derivatives (which by choice are all zero) and the connection and its partial derivatives (which in general are not zero).

$$
\begin{aligned}
A^c\nabla_c\left(B^a{}_b\right) &= \Gamma^a{}_{pq}B^p{}_bA^q - \Gamma^p{}_{bq}B^a{}_pA^q\\
A^dA^c\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right) &= A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{cd}\Gamma^c{}_{pq}B^p{}_bA^dA^q - 2\Gamma^a{}_{cd}\Gamma^p{}_{bq}B^c{}_pA^dA^q + \Gamma^c{}_{bd}\Gamma^p{}_{cq}B^a{}_pA^dA^q\\
A^eA^dA^c\nabla_e\left(\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right)\right) &= A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{de}\Gamma^d{}_{pq}B^p{}_bA^eA^q + A^c\Gamma^a{}_{cd}\partial_e\Gamma^d{}_{pq}B^p{}_bA^eA^q\\
&\quad - 2A^c\partial_c\Gamma^a{}_{de}\Gamma^p{}_{bq}B^d{}_pA^eA^q - 2A^c\Gamma^a{}_{cd}\partial_e\Gamma^p{}_{bq}B^d{}_pA^eA^q + A^c\partial_c\Gamma^d{}_{be}\Gamma^p{}_{dq}B^a{}_pA^eA^q + A^c\Gamma^d{}_{bc}\partial_e\Gamma^p{}_{dq}B^a{}_pA^eA^q\\
&\quad + \Gamma^a{}_{ce}A^c\partial_f\Gamma^e{}_{pq}B^p{}_bA^fA^q - \Gamma^a{}_{ce}A^c\partial_f\Gamma^p{}_{bq}B^e{}_pA^fA^q + \Gamma^a{}_{cd}\Gamma^c{}_{ef}\Gamma^e{}_{pq}B^p{}_bA^dA^fA^q - 3\Gamma^a{}_{cd}\Gamma^e{}_{bf}\Gamma^c{}_{pq}B^p{}_eA^dA^fA^q\\
&\quad + 3\Gamma^a{}_{cd}\Gamma^e{}_{bf}\Gamma^p{}_{eq}B^c{}_pA^dA^fA^q - \Gamma^c{}_{be}A^e\partial_f\Gamma^a{}_{pq}B^p{}_cA^fA^q + \Gamma^c{}_{be}A^e\partial_f\Gamma^p{}_{cq}B^a{}_pA^fA^q - \Gamma^c{}_{bd}\Gamma^e{}_{cf}\Gamma^p{}_{eq}B^a{}_pA^dA^fA^q
\end{aligned}
$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

Here we impose the RNC condition (that $\Gamma = 0$ while $\partial\Gamma \neq 0$).

$$
\begin{aligned}
A^c\nabla_c\left(B^a{}_b\right) &= 0\\
A^dA^c\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right) &= A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q\\
A^eA^dA^c\nabla_e\left(\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right)\right) &= A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q
\end{aligned}
$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

This stage uses the results from the second stage to eliminate the ∇B terms from the results of the first stage. This produces expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and the partial derivatives of the connection. In this stage we also set the $B^a{}_b$ to equal 1.

$$
\begin{aligned}
\left(R^a{}_{cdb}\right)_{,e}A^cA^dA^e &= -A^cA^dA^e\nabla_cR_{dfeb}g^{af}\\
\left(R^a{}_{cdb}\right)_{,ef}A^cA^dA^eA^e &= A^cA^dA^eA^f\left(-\nabla_{cd}R_{ebfg}g^{ag} - R_{cgdh}\partial_e\Gamma^g{}_{bf}g^{ha} + R_{cbdg}\partial_e\Gamma^a{}_{hf}g^{gh}\right)\\
\left(R^a{}_{cdb}\right)_{,efg}A^cA^dA^eA^fA^g &= A^cA^dA^eA^fA^g\left(-3\nabla_cR_{dhei}\partial_f\Gamma^h{}_{bg}g^{ia} + 3\nabla_cR_{dbeh}\partial_f\Gamma^a{}_{ig}g^{hi} - \nabla_{cde}R_{fbgh}g^{ah} - R_{chdi}\partial_{ef}\Gamma^h{}_{bg}g^{ia} + R_{cbdh}\partial_{ef}\Gamma^a{}_{ig}g^{hi}\right)
\end{aligned}
$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

The fifth stage draws in results from `dGamma.tex` to replace the partial derivatives of Γ with partial derivatives of R_{abcd}.

$$
\begin{aligned}
{(R^{a}{}_{cdb})}_{,e}\,A^{c}A^{d}A^{e} &= -A^{c}A^{d}A^{e}\nabla_{c}R_{dfeb}g^{af}\\
{(R^{a}{}_{cdb})}_{,ef}\,A^{c}A^{d}A^{e}A^{f} &= -A^{c}A^{d}A^{e}A^{f}\nabla_{cd}R_{ebfg}g^{ag} - \frac{1}{3}A^{c}A^{d}A^{e}A^{f}R^{g}{}_{feb}R_{cgdh}g^{ha} + \frac{1}{3}A^{c}A^{d}A^{e}A^{f}R^{a}{}_{feg}R_{cbdh}g^{hg}\\
{(R^{a}{}_{cdb})}_{,efg}\,A^{c}A^{d}A^{e}A^{f}A^{g} &= -A^{c}A^{d}A^{e}A^{f}A^{g}R^{h}{}_{gfb}\nabla_{c}R_{dhei}g^{ia} + A^{c}A^{d}A^{e}A^{f}A^{g}R^{a}{}_{gfh}\nabla_{c}R_{dbei}g^{ih} - A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{cde}R_{fbgh}g^{ah}\\
&\quad - \frac{1}{2}A^{c}A^{d}A^{e}A^{f}A^{g}R_{chdi}\partial_{f}R^{h}{}_{geb}g^{ia} + \frac{1}{2}A^{c}A^{d}A^{e}A^{f}A^{g}R_{cbdh}\partial_{f}R^{a}{}_{gei}g^{hi}
\end{aligned}
$$

Stage 6: Replace partial derivs of R with covariant derivs of R

The final stage is to eliminate the ∂R by using earlier results. For example, in the equation for $\partial^3 R$ we see terms involving ∂R. These first order partial derivatives can be replaced with the expression previously computed for ∂R in terms of ∇R.

$$
\begin{aligned}
{(R^{a}{}_{cdb})}_{,e}\,A^{c}A^{d}A^{e} &= A^{c}A^{d}A^{e}\nabla_{c}R_{bdef}g^{af}\\
{(R^{a}{}_{cdb})}_{,ef}\,A^{c}A^{d}A^{e}A^{e} &= A^{c}A^{d}A^{e}A^{f}\nabla_{cd}R_{befg}g^{ag}\\
{(R^{a}{}_{cdb})}_{,efg}\,A^{c}A^{d}A^{e}A^{f}A^{g} &= -\frac{1}{2}A^{c}A^{d}A^{e}A^{f}A^{g}R_{bcdh}\nabla_{e}R_{figj}g^{ai}g^{hj} + \frac{1}{2}A^{c}A^{d}A^{e}A^{f}A^{g}R_{chdi}\nabla_{e}R_{bfgj}g^{ah}g^{ij} + A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{cde}R_{bfgh}g^{ah}
\end{aligned}
$$

The end result are expressions for the symmetrized partial derivatives of R_{abcd} solely in terms of the symmetrized covariant derivatives of R_{abcd}.

Shared properties

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

B^{a}_{b::Depends(\nabla{#}).
R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of R

```
def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d}   -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d}   -> g^{d e} R_{a b c e}$)
    unwrap     (obj)
    sort_product   (obj)
    rename_dummies (obj)
    return obj

# compute the symmetric covariant derivatives of R^{a}_{bcd} B^{d}_{a}

beg_stage_1 = time.time()

dRabcd00:=R^{a}_{b c d} B^{d}_{a} A^{b} A^{c}.        # cdb(dRabcd00.101,dRabcd00)

dRabcd01:=A^{a}\nabla_{a}{ @(dRabcd00) }.             # cdb(dRabcd01.101,dRabcd01)
distribute     (dRabcd01)                             # cdb(dRabcd01.102,dRabcd01)
product_rule   (dRabcd01)                             # cdb(dRabcd01.103,dRabcd01)
distribute     (dRabcd01)                             # cdb(dRabcd01.104,dRabcd01)
substitute     (dRabcd01,$\nabla_{a}{A^{b}} -> 0$)    # cdb(dRabcd01.105,dRabcd01)
substitute     (dRabcd01,$\nabla_{a}{g^{b c}} -> 0$)  # cdb(dRabcd01.106,dRabcd01)

sort_product   (dRabcd01)
rename_dummies (dRabcd01)
canonicalise   (dRabcd01)                             # cdb(dRabcd01.107,dRabcd01)
dRabcd01 = flatten_Rabcd (dRabcd01)                   # cdb(dRabcd01.108,dRabcd01)

dRabcd02:=A^{a}\nabla_{a}{ @(dRabcd01) }.             # cdb(dRabcd02.101,dRabcd02)
distribute     (dRabcd02)                             # cdb(dRabcd02.102,dRabcd02)
product_rule   (dRabcd02)                             # cdb(dRabcd02.103,dRabcd02)
distribute     (dRabcd02)                             # cdb(dRabcd02.104,dRabcd02)
substitute     (dRabcd02,$\nabla_{a}{A^{b}} -> 0$)    # cdb(dRabcd02.105,dRabcd02)
substitute     (dRabcd02,$\nabla_{a}{g^{b c}} -> 0$)  # cdb(dRabcd02.106,dRabcd02)

sort_product   (dRabcd02)
```

```
rename_dummies (dRabcd02)
canonicalise   (dRabcd02)                            # cdb(dRabcd02.107,dRabcd02)
dRabcd02 = flatten_Rabcd (dRabcd02)                  # cdb(dRabcd02.108,dRabcd02)

dRabcd03:=A^{a}\nabla_{a}{ @(dRabcd02) }.            # cdb(dRabcd03.101,dRabcd03)
distribute     (dRabcd03)                            # cdb(dRabcd03.102,dRabcd03)
product_rule   (dRabcd03)                            # cdb(dRabcd03.103,dRabcd03)
distribute     (dRabcd03)                            # cdb(dRabcd03.104,dRabcd03)
substitute     (dRabcd03,$\nabla_{a}{A^{b}} -> 0$)    # cdb(dRabcd03.105,dRabcd03)
substitute     (dRabcd03,$\nabla_{a}{g^{b c}} -> 0$)  # cdb(dRabcd03.106,dRabcd03)

sort_product   (dRabcd03)
rename_dummies (dRabcd03)
canonicalise   (dRabcd03)                            # cdb(dRabcd03.107,dRabcd03)
dRabcd03 = flatten_Rabcd (dRabcd03)                  # cdb(dRabcd03.108,dRabcd03)

dRabcd04:=A^{a}\nabla_{a}{ @(dRabcd03) }.
distribute     (dRabcd04)
product_rule   (dRabcd04)
distribute     (dRabcd04)
substitute     (dRabcd04,$\nabla_{a}{A^{b}} -> 0$)
substitute     (dRabcd04,$\nabla_{a}{g^{b c}} -> 0$)

sort_product   (dRabcd04)
rename_dummies (dRabcd04)
canonicalise   (dRabcd04)
dRabcd04 = flatten_Rabcd (dRabcd04)

dRabcd05:=A^{a}\nabla_{a}{ @(dRabcd04) }.
distribute     (dRabcd05)
product_rule   (dRabcd05)
distribute     (dRabcd05)
substitute     (dRabcd05,$\nabla_{a}{A^{b}} -> 0$)
substitute     (dRabcd05,$\nabla_{a}{g^{b c}} -> 0$)

sort_product   (dRabcd05)
rename_dummies (dRabcd05)
canonicalise   (dRabcd05)
```

```
dRabcd05 = flatten_Rabcd (dRabcd05)

def combine_nabla (obj):
    substitute (obj,$\nabla_{p}{\nabla_{q}{\nabla_{r}{\nabla_{s}{\nabla_{t}{A??}}}}}->\nabla_{p q r s t}{A??}$,repeat=True)
    substitute (obj,$\nabla_{p}{\nabla_{q}{\nabla_{r}{\nabla_{s}{A??}}}}->\nabla_{p q r s}{A??}$,repeat=True)
    substitute (obj,$\nabla_{p}{\nabla_{q}{\nabla_{r}{A??}}}->\nabla_{p q r}{A??}$,repeat=True)
    substitute (obj,$\nabla_{p}{\nabla_{q}{A??}}->\nabla_{p q}{A??}$,repeat=True)
    return obj

dRabcd01 = combine_nabla (dRabcd01)
dRabcd02 = combine_nabla (dRabcd02)
dRabcd03 = combine_nabla (dRabcd03)
dRabcd04 = combine_nabla (dRabcd04)
dRabcd05 = combine_nabla (dRabcd05)

end_stage_1 = time.time()
```

$$\texttt{dRabcd00.101} := R^a{}_{bcd} B^d{}_a A^b A^c$$

$$\texttt{dRabcd01.101} := A^a \nabla_a \left(R^e{}_{bcd} B^d{}_e A^b A^c \right)$$

$$\texttt{dRabcd01.102} := A^a \nabla_a \left(R^e{}_{bcd} B^d{}_e A^b A^c \right)$$

$$\texttt{dRabcd01.103} := A^a \left(\nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c \right)$$

$$\texttt{dRabcd01.104} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + A^a R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + A^a R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c$$

$$\texttt{dRabcd01.105} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\texttt{dRabcd01.106} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\texttt{dRabcd01.107} := -A^a A^b A^c B^d{}_e \nabla_a R_b{}^e{}_{cd} - A^a A^b A^c R_a{}^d{}_{be} \nabla_c B^e{}_d$$

$$\texttt{dRabcd01.108} := -A^a A^b A^c B^d{}_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d{}_e g^{ef}$$

$$\texttt{dRabcd02.101} := A^a \nabla_a \left(-A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \right)$$

$$\texttt{dRabcd02.102} := -A^a \nabla_a \left(A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} \right) - A^a \nabla_a \left(A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \right)$$

$$\begin{aligned}\texttt{dRabcd02.103} := -A^a \Big(&\nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ &+ A^g A^b A^c B^d{}_e \nabla_a \left(\nabla_g R_{bfcd} \right) g^{ef} + A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} \Big) - A^a \Big(\nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ &+ A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c R_{gfbd} \nabla_a \left(\nabla_c B^d{}_e \right) g^{ef} + A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef} \Big)\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd02.104} := &-A^a \nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ &- A^a A^g A^b A^c B^d{}_e \nabla_a \left(\nabla_g R_{bfcd} \right) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} - A^a \nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ &- A^a A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a \left(\nabla_c B^d{}_e \right) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd02.105} := &-A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a \left(\nabla_g R_{bfcd} \right) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} \\ &- A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a \left(\nabla_c B^d{}_e \right) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef}\end{aligned}$$

$$\texttt{dRabcd02.106} := -A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a \left(\nabla_g R_{bfcd} \right) g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a \left(\nabla_c B^d{}_e \right) g^{ef}$$

$$\texttt{dRabcd02.107} := -2 A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a \left(\nabla_b R_{cedg} \right) g^{fg} - A^a A^b A^c A^d R_{aebf} \nabla_c \left(\nabla_d B^e{}_g \right) g^{fg}$$

$$\texttt{dRabcd02.108} := -2 A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a \left(\nabla_b R_{cedg} \right) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c \left(\nabla_d B^e{}_f \right) g^{gf}$$

$$\texttt{dRabcd03.101} := A^a\nabla_a\left(-2A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - A^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\right)$$

$$\texttt{dRabcd03.102} := -2A^a\nabla_a\left(A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg}\right) - A^a\nabla_a\left(A^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\right) - A^a\nabla_a\left(A^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\right)$$

$$\begin{aligned}\texttt{dRabcd03.103} := -2A^a\big(&\nabla_aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} + A^h\nabla_aA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} + A^hA^b\nabla_aA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg}\\
&+ A^hA^bA^c\nabla_aA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} + A^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} + A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg}\\
&+ A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}\nabla_ag^{fg}\big) - A^a\big(\nabla_aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} + A^h\nabla_aA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&+ A^hA^b\nabla_aA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} + A^hA^bA^c\nabla_aA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} + A^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&+ A^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} + A^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)\nabla_ag^{fg}\big) - A^a\big(\nabla_aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&+ A^h\nabla_aA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} + A^hA^b\nabla_aA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} + A^hA^bA^c\nabla_aA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&+ A^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} + A^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf} + A^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)\nabla_ag^{gf}\big)\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.104} := &-2A^a\nabla_aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - 2A^aA^h\nabla_aA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^b\nabla_aA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg}\\
&- 2A^aA^hA^bA^c\nabla_aA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}\nabla_ag^{fg} - A^a\nabla_aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^h\nabla_aA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- A^aA^hA^b\nabla_aA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^c\nabla_aA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- A^aA^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} - A^aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)\nabla_ag^{fg} - A^a\nabla_aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&- A^aA^h\nabla_aA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^b\nabla_aA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^c\nabla_aA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&- A^aA^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)\nabla_ag^{gf}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.105} := &-2A^aA^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}\nabla_ag^{fg}\\
&- A^aA^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} - A^aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)\nabla_ag^{fg}\\
&- A^aA^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)\nabla_ag^{gf}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.106} := &-2A^aA^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- A^aA^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} - A^aA^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.107} := &-3A^aA^bA^cA^dA^e\nabla_aR_{bfcg}\nabla_d\left(\nabla_eB^f{}_h\right)g^{gh} - 3A^aA^bA^cA^dA^e\nabla_aB^f{}_g\nabla_b\left(\nabla_cR_{dfeh}\right)g^{gh}\\
&- A^aA^bA^cA^dA^eB^f{}_g\nabla_a\left(\nabla_b\left(\nabla_cR_{dfeh}\right)\right)g^{gh} - A^aA^bA^cA^dA^eR_{afbg}\nabla_c\left(\nabla_d\left(\nabla_eB^f{}_h\right)\right)g^{gh}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.108} := &-3A^aA^bA^cA^dA^e\nabla_aR_{bfch}\nabla_d\left(\nabla_eB^f{}_g\right)g^{hg} - 3A^aA^bA^cA^dA^e\nabla_aB^f{}_g\nabla_b\left(\nabla_cR_{dfeh}\right)g^{gh}\\
&- A^aA^bA^cA^dA^eB^f{}_g\nabla_a\left(\nabla_b\left(\nabla_cR_{dfeh}\right)\right)g^{gh} - A^aA^bA^cA^dA^eR_{afbh}\nabla_c\left(\nabla_d\left(\nabla_eB^f{}_g\right)\right)g^{hg}\end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

```
# compute the covariant derivatives of B^{a}_{b}, note B^{a}_{b,c} is zero, by choice
# this method of computing covariant derivatives does not use auxillary fields

beg_stage_2 = time.time()

dBab00:=B^{a}_{b}.      # cdb(dBab00.201,dBab00)

dBab01:=A^{c}\partial_{c}{ @(dBab00) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
                                                      # cdb(dBab01.201,dBab01)
distribute   (dBab01)                                 # cdb(dBab01.202,dBab01)
product_rule (dBab01)                                 # cdb(dBab01.203,dBab01)
distribute   (dBab01)                                 # cdb(dBab01.204,dBab01)
substitute   (dBab01,$\partial_{a}{A^{b}} -> 0$)      # cdb(dBab01.205,dBab01)
substitute   (dBab01,$\partial_{a}{B^{b}_{c}} -> 0$)  # cdb(dBab01.206,dBab01)
substitute   (dBab01,$W^{a}_{b} -> @(dBab00)$)        # cdb(dBab01.207,dBab01)
distribute   (dBab01)                                 # cdb(dBab01.208,dBab01)
canonicalise (dBab01)                                 # cdb(dBab01.209,dBab01)

dBab02:=A^{c}\partial_{c}{ @(dBab01) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
                                                      # cdb(dBab02.201,dBab02)
distribute   (dBab02)                                 # cdb(dBab02.202,dBab02)
product_rule (dBab02)                                 # cdb(dBab02.203,dBab02)
distribute   (dBab02)                                 # cdb(dBab02.204,dBab02)
substitute   (dBab02,$\partial_{a}{A^{b}} -> 0$)      # cdb(dBab02.205,dBab02)
substitute   (dBab02,$\partial_{a}{B^{b}_{c}} -> 0$)  # cdb(dBab02.206,dBab02)
substitute   (dBab02,$W^{a}_{b} -> @(dBab01)$)        # cdb(dBab02.207,dBab02)
distribute   (dBab02)                                 # cdb(dBab02.208,dBab02)
canonicalise (dBab02)                                 # cdb(dBab02.209,dBab02)

dBab03:=A^{c}\partial_{c}{ @(dBab02) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
                                                      # cdb(dBab03.201,dBab03)
distribute   (dBab03)                                 # cdb(dBab03.202,dBab03)
product_rule (dBab03)                                 # cdb(dBab03.203,dBab03)
```

```
distribute   (dBab03)                              # cdb(dBab03.204,dBab03)
substitute   (dBab03,$\partial_{a}{A^{b}} -> 0$)    # cdb(dBab03.205,dBab03)
substitute   (dBab03,$\partial_{a}{B^{b}_{c}} -> 0$) # cdb(dBab03.206,dBab03)
substitute   (dBab03,$W^{a}_{b} -> @(dBab02)$)      # cdb(dBab03.207,dBab03)
distribute   (dBab03)                              # cdb(dBab03.208,dBab03)
canonicalise (dBab03)                              # cdb(dBab03.209,dBab03)

dBab04:=A^{c}\partial_{c}{ @(dBab03) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
distribute   (dBab04)
product_rule (dBab04)
distribute   (dBab04)
substitute   (dBab04,$\partial_{a}{A^{b}} -> 0$)
substitute   (dBab04,$\partial_{a}{B^{b}_{c}} -> 0$)
substitute   (dBab04,$W^{a}_{b} -> @(dBab03)$)
distribute   (dBab04)
canonicalise (dBab04)

dBab05:=A^{c}\partial_{c}{ @(dBab04) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
distribute   (dBab05)
product_rule (dBab05)
distribute   (dBab05)
substitute   (dBab05,$\partial_{a}{A^{b}} -> 0$)
substitute   (dBab05,$\partial_{a}{B^{b}_{c}} -> 0$)
substitute   (dBab05,$W^{a}_{b} -> @(dBab04)$)
distribute   (dBab05)
canonicalise (dBab05)

end_stage_2 = time.time()
```

$$\texttt{dBab00.201} := B^a{}_b$$

$$\texttt{dBab01.201} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab01.202} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab01.203} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab01.204} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab01.205} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab01.206} := \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab01.207} := \Gamma^a{}_{pq}B^p{}_bA^q - \Gamma^p{}_{bq}B^a{}_pA^q$$

$$\texttt{dBab01.208} := \Gamma^a{}_{pq}B^p{}_bA^q - \Gamma^p{}_{bq}B^a{}_pA^q$$

$$\texttt{dBab01.209} := \Gamma^a{}_{pq}B^p{}_bA^q - \Gamma^p{}_{bq}B^a{}_pA^q$$

$$\texttt{dBab02.201} := A^c\partial_c\left(\Gamma^a{}_{pq}B^p{}_bA^q - \Gamma^p{}_{bq}B^a{}_pA^q\right) + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab02.202} := A^c\partial_c\left(\Gamma^a{}_{pq}B^p{}_bA^q\right) - A^c\partial_c\left(\Gamma^p{}_{bq}B^a{}_pA^q\right) + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab02.203} := A^c\left(\partial_c\Gamma^a{}_{pq}B^p{}_bA^q + \Gamma^a{}_{pq}\partial_cB^p{}_bA^q + \Gamma^a{}_{pq}B^p{}_b\partial_cA^q\right) - A^c\left(\partial_c\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^p{}_{bq}\partial_cB^a{}_pA^q + \Gamma^p{}_{bq}B^a{}_p\partial_cA^q\right) + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab02.204} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q + A^c\Gamma^a{}_{pq}\partial_cB^p{}_bA^q + A^c\Gamma^a{}_{pq}B^p{}_b\partial_cA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q - A^c\Gamma^p{}_{bq}\partial_cB^a{}_pA^q - A^c\Gamma^p{}_{bq}B^a{}_p\partial_cA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab02.205} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q + A^c\Gamma^a{}_{pq}\partial_cB^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q - A^c\Gamma^p{}_{bq}\partial_cB^a{}_pA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab02.206} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\texttt{dBab02.207} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{pq}\left(\Gamma^p{}_{dc}B^d{}_bA^c - \Gamma^d{}_{bc}B^p{}_dA^c\right)A^q - \Gamma^p{}_{bq}\left(\Gamma^a{}_{dc}B^d{}_pA^c - \Gamma^d{}_{pc}B^a{}_dA^c\right)A^q$$

$$\texttt{dBab02.208} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{pq}\Gamma^p{}_{dc}B^d{}_bA^cA^q - \Gamma^a{}_{pq}\Gamma^d{}_{bc}B^p{}_dA^cA^q - \Gamma^p{}_{bq}\Gamma^a{}_{dc}B^d{}_pA^cA^q + \Gamma^p{}_{bq}\Gamma^d{}_{pc}B^a{}_dA^cA^q$$

$$\texttt{dBab02.209} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{cd}\Gamma^c{}_{pq}B^p{}_bA^dA^q - 2\Gamma^a{}_{cd}\Gamma^p{}_{bq}B^c{}_pA^dA^q + \Gamma^c{}_{bd}\Gamma^p{}_{cq}B^a{}_pA^dA^q$$

$$\texttt{dBab03.201} := A^c\partial_c\left(A^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q - A^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q - 2\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q\right) + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q$$

$$\begin{aligned}\texttt{dBab03.202} := {}& A^c\partial_c\left(A^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q\right) - A^c\partial_c\left(A^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q\right) + A^c\partial_c\left(\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q\right)\\ &- 2A^c\partial_c\left(\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q\right) + A^c\partial_c\left(\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q\right) + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab03.203} := {}& A^c\left(\partial_cA^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q + A^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q + A^e\partial_e\Gamma^a{}_{pq}\partial_cB^p{}_bA^q + A^e\partial_e\Gamma^a{}_{pq}B^p{}_b\partial_cA^q\right)\\ &- A^c\left(\partial_cA^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q + A^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^e\partial_e\Gamma^p{}_{bq}\partial_cB^a{}_pA^q + A^e\partial_e\Gamma^p{}_{bq}B^a{}_p\partial_cA^q\right)\\ &+ A^c\left(\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + \Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}\partial_cB^p{}_bA^dA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_b\partial_cA^dA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^d\partial_cA^q\right)\\ &- 2A^c\left(\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q + \Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + \Gamma^a{}_{ed}\Gamma^p{}_{bq}\partial_cB^e{}_pA^dA^q + \Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_p\partial_cA^dA^q + \Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^d\partial_cA^q\right)\\ &+ A^c\left(\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}\partial_cB^a{}_pA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_p\partial_cA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^d\partial_cA^q\right)\\ &+ \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab03.204} := {}& A^c\partial_cA^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q + A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q + A^cA^e\partial_e\Gamma^a{}_{pq}\partial_cB^p{}_bA^q + A^cA^e\partial_e\Gamma^a{}_{pq}B^p{}_b\partial_cA^q - A^c\partial_cA^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q\\ &- A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q - A^cA^e\partial_e\Gamma^p{}_{bq}\partial_cB^a{}_pA^q - A^cA^e\partial_e\Gamma^p{}_{bq}B^a{}_p\partial_cA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q\\ &+ A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}\partial_cB^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_b\partial_cA^dA^q + A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^d\partial_cA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q\\ &- 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}\partial_cB^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_p\partial_cA^dA^q - 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^d\partial_cA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q\\ &+ A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}\partial_cB^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_p\partial_cA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^d\partial_cA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab03.205} := {}& A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q + A^cA^e\partial_e\Gamma^a{}_{pq}\partial_cB^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q - A^cA^e\partial_e\Gamma^p{}_{bq}\partial_cB^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q\\ &+ A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}\partial_cB^p{}_bA^dA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}\partial_cB^e{}_pA^dA^q\\ &+ A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}\partial_cB^a{}_pA^dA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab03.206} := {}& A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q\\ &- 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab03.207} := {}& A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q\\ &- 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q\\ &+ \Gamma^a{}_{pq}\left(A^c\partial_c\Gamma^p{}_{fe}B^f{}_bA^e - A^c\partial_c\Gamma^f{}_{be}B^p{}_fA^e + \Gamma^p{}_{cd}\Gamma^c{}_{fe}B^f{}_bA^dA^e - 2\Gamma^p{}_{cd}\Gamma^f{}_{be}B^c{}_fA^dA^e + \Gamma^c{}_{bd}\Gamma^f{}_{ce}B^p{}_fA^dA^e\right)A^q\\ &- \Gamma^p{}_{bq}\left(A^c\partial_c\Gamma^a{}_{fe}B^f{}_pA^e - A^c\partial_c\Gamma^f{}_{pe}B^a{}_fA^e + \Gamma^a{}_{cd}\Gamma^c{}_{fe}B^f{}_pA^dA^e - 2\Gamma^a{}_{cd}\Gamma^f{}_{pe}B^c{}_fA^dA^e + \Gamma^c{}_{pd}\Gamma^f{}_{ce}B^a{}_fA^dA^e\right)A^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab03.208} := {}& A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q\\ &- 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^a{}_{pq}A^c\partial_c\Gamma^p{}_{fe}B^f{}_bA^eA^q - \Gamma^a{}_{pq}A^c\partial_c\Gamma^f{}_{be}B^p{}_fA^eA^q\\ &+ \Gamma^a{}_{pq}\Gamma^p{}_{cd}\Gamma^c{}_{fe}B^f{}_bA^dA^eA^q - 2\Gamma^a{}_{pq}\Gamma^p{}_{cd}\Gamma^f{}_{be}B^c{}_fA^dA^eA^q + \Gamma^a{}_{pq}\Gamma^c{}_{bd}\Gamma^f{}_{ce}B^p{}_fA^dA^eA^q - \Gamma^p{}_{bq}A^c\partial_c\Gamma^a{}_{fe}B^f{}_pA^eA^q\\ &+ \Gamma^p{}_{bq}A^c\partial_c\Gamma^f{}_{pe}B^a{}_fA^eA^q - \Gamma^p{}_{bq}\Gamma^a{}_{cd}\Gamma^c{}_{fe}B^f{}_pA^dA^eA^q + 2\Gamma^p{}_{bq}\Gamma^a{}_{cd}\Gamma^f{}_{pe}B^c{}_fA^dA^eA^q - \Gamma^p{}_{bq}\Gamma^c{}_{pd}\Gamma^f{}_{ce}B^a{}_fA^dA^eA^q\end{aligned}$$

$$
\begin{aligned}
\texttt{dBab03.209} := {} & A^c A^e \partial_{ce}\Gamma^a{}_{pq} B^p{}_b A^q - A^c A^e \partial_{ce}\Gamma^p{}_{bq} B^a{}_p A^q + A^c \partial_c \Gamma^a{}_{de}\Gamma^d{}_{pq} B^p{}_b A^e A^q + A^c \Gamma^a{}_{cd}\partial_e \Gamma^d{}_{pq} B^p{}_b A^e A^q \\
& - 2A^c \partial_c \Gamma^a{}_{de}\Gamma^p{}_{bq} B^d{}_p A^e A^q - 2A^c \Gamma^a{}_{cd}\partial_e \Gamma^p{}_{bq} B^d{}_p A^e A^q + A^c \partial_c \Gamma^d{}_{be}\Gamma^p{}_{dq} B^a{}_p A^e A^q + A^c \Gamma^d{}_{bc}\partial_e \Gamma^p{}_{dq} B^a{}_p A^e A^q \\
& + \Gamma^a{}_{ce} A^c \partial_f \Gamma^e{}_{pq} B^p{}_b A^f A^q - \Gamma^a{}_{ce} A^c \partial_f \Gamma^p{}_{bq} B^e{}_p A^f A^q + \Gamma^a{}_{cd}\Gamma^c{}_{ef}\Gamma^e{}_{pq} B^p{}_b A^d A^f A^q - 3\Gamma^a{}_{cd}\Gamma^e{}_{bf}\Gamma^c{}_{pq} B^p{}_e A^d A^f A^q \\
& + 3\Gamma^a{}_{cd}\Gamma^e{}_{bf}\Gamma^p{}_{eq} B^c{}_p A^d A^f A^q - \Gamma^c{}_{be} A^e \partial_f \Gamma^a{}_{pq} B^p{}_c A^f A^q + \Gamma^c{}_{be} A^e \partial_f \Gamma^p{}_{cq} B^a{}_p A^f A^q - \Gamma^c{}_{bd}\Gamma^e{}_{cf}\Gamma^p{}_{eq} B^a{}_p A^d A^f A^q
\end{aligned}
$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

# switch to RNC

beg_stage_3 = time.time()

dBab01 = impose_rnc (dBab01)   # cdb (dBab01.301,dBab01)
dBab02 = impose_rnc (dBab02)   # cdb (dBab02.301,dBab02)
dBab03 = impose_rnc (dBab03)   # cdb (dBab03.301,dBab03)
dBab04 = impose_rnc (dBab04)   # cdb (dBab04.301,dBab04)
dBab05 = impose_rnc (dBab05)   # cdb (dBab05.301,dBab05)

end_stage_3 = time.time()
```

$\texttt{dBab01.301} := 0$

$\texttt{dBab02.301} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q$

$\texttt{dBab03.301} := A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q$

$$\begin{aligned}\texttt{dBab04.301} := {} & A^cA^eA^g\partial_{ceg}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^eA^g\partial_{ceg}\Gamma^p{}_{bq}B^a{}_pA^q + 2A^cA^d\partial_c\Gamma^a{}_{de}\partial_g\Gamma^e{}_{pq}B^p{}_bA^gA^q - 4A^cA^d\partial_c\Gamma^a{}_{de}\partial_g\Gamma^p{}_{bq}B^e{}_pA^gA^q \\ & + 2A^cA^d\partial_c\Gamma^e{}_{bd}\partial_g\Gamma^p{}_{eq}B^a{}_pA^gA^q + A^c\partial_c\Gamma^a{}_{ef}A^e\partial_g\Gamma^f{}_{pq}B^p{}_bA^gA^q - 2A^c\partial_c\Gamma^a{}_{ef}A^e\partial_g\Gamma^p{}_{bq}B^f{}_pA^gA^q + A^c\partial_c\Gamma^e{}_{bf}A^f\partial_g\Gamma^p{}_{eq}B^a{}_pA^gA^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab05.301} := {} & A^cA^eA^gA^i\partial_{cegi}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^eA^gA^i\partial_{cegi}\Gamma^p{}_{bq}B^a{}_pA^q + 3A^cA^dA^e\partial_{cd}\Gamma^a{}_{eg}\partial_i\Gamma^g{}_{pq}B^p{}_bA^iA^q + 3A^cA^dA^e\partial_c\Gamma^a{}_{dg}\partial_{ei}\Gamma^g{}_{pq}B^p{}_bA^iA^q \\ & - 6A^cA^dA^e\partial_{cd}\Gamma^a{}_{eg}\partial_i\Gamma^p{}_{bq}B^g{}_pA^iA^q - 6A^cA^dA^e\partial_c\Gamma^a{}_{dg}\partial_{ei}\Gamma^p{}_{bq}B^g{}_pA^iA^q + 3A^cA^dA^e\partial_{cd}\Gamma^g{}_{be}\partial_i\Gamma^p{}_{gq}B^a{}_pA^iA^q \\ & + 3A^cA^dA^e\partial_c\Gamma^g{}_{bd}\partial_{ei}\Gamma^p{}_{gq}B^a{}_pA^iA^q + A^cA^e\partial_{ce}\Gamma^a{}_{fg}A^f\partial_i\Gamma^g{}_{pq}B^p{}_bA^iA^q + 2A^cA^e\partial_c\Gamma^a{}_{ef}A^g\partial_{gi}\Gamma^f{}_{pq}B^p{}_bA^iA^q \\ & - 2A^cA^e\partial_{ce}\Gamma^a{}_{fg}A^f\partial_i\Gamma^p{}_{bq}B^g{}_pA^iA^q - 3A^cA^e\partial_c\Gamma^a{}_{ef}A^g\partial_{gi}\Gamma^p{}_{bq}B^f{}_pA^iA^q - A^cA^e\partial_c\Gamma^f{}_{be}A^g\partial_{gi}\Gamma^a{}_{pq}B^p{}_fA^iA^q \\ & + A^cA^e\partial_{ce}\Gamma^f{}_{bg}A^g\partial_i\Gamma^p{}_{fq}B^a{}_pA^iA^q + 2A^cA^e\partial_c\Gamma^f{}_{be}A^g\partial_{gi}\Gamma^p{}_{fq}B^a{}_pA^iA^q + A^c\partial_c\Gamma^a{}_{eg}A^eA^h\partial_{hi}\Gamma^g{}_{pq}B^p{}_bA^iA^q \\ & - A^c\partial_c\Gamma^a{}_{eg}A^eA^h\partial_{hi}\Gamma^p{}_{bq}B^g{}_pA^iA^q - A^c\partial_c\Gamma^e{}_{bg}A^gA^h\partial_{hi}\Gamma^a{}_{pq}B^p{}_eA^iA^q + A^c\partial_c\Gamma^e{}_{bg}A^gA^h\partial_{hi}\Gamma^p{}_{eq}B^a{}_pA^iA^q\end{aligned}$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

```
# substitute covariant derivs of B^{a}_{b} into covariant derivs of R^{a}_{bcd}B^{d}_{a}
# this produces expressions for the partial derivs of Rabcd its covariant derivs and partial derivs of Gamma
# the partial derivs of Gamma will be eliminted later by using results imported from dGamma.json

beg_stage_4 = time.time()

substitute (dRabcd01,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd01)
substitute (dRabcd02,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd02)
substitute (dRabcd03,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd03)
substitute (dRabcd04,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd04)
substitute (dRabcd05,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd05)

substitute (dRabcd02,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd02)
substitute (dRabcd03,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd03)
substitute (dRabcd04,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd04)
substitute (dRabcd05,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd05)

substitute (dRabcd03,$A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(dBab03)$,repeat=True);   distribute (dRabcd03)
substitute (dRabcd04,$A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(dBab03)$,repeat=True);   distribute (dRabcd04)
substitute (dRabcd05,$A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(dBab03)$,repeat=True);   distribute (dRabcd05)

substitute (dRabcd04,$A^{c}A^{d}A^{e}A^{f}\nabla_{c d e f}{B^{a}_{b}} -> @(dBab04)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}\nabla_{c d e f}{B^{a}_{b}} -> @(dBab04)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{c d e f g}{B^{a}_{b}} -> @(dBab05)$,repeat=True); distribute (dRabcd05)

# no longer need B, so let's get rid of it

# two subtle tricks are used here
# 1) rename A and B as A002 and A001 before sort_product,
#    this ensures B will be to left of A after the sort
# 2) indices on B changed from B^{a}_{b} to B_{b}^{a},
#    this ensures that after factor_out B will have dummy indices B_{a}^{b}

def remove_Bab (obj):
    foo := @(obj).
```

```
    substitute     (foo,$A^{a}->A002^{a},B^{a}_{b}->A001_{b}^{a}$)  # need this to sort B to the left of A
    sort_product   (foo)
    rename_dummies (foo)
    factor_out     (foo,$A001^{a?}_{b?},A002^{c?}$)
    substitute     (foo,$A001_{a}^{b}->1,A002^{a}->A^{a}$)  # recover A and set B = 1, free indices now ^{a}_{b}
    return foo

dRabcd01 = remove_Bab (dRabcd01)   # cdb(dRabcd01.401,dRabcd01)
dRabcd02 = remove_Bab (dRabcd02)   # cdb(dRabcd02.401,dRabcd02)
dRabcd03 = remove_Bab (dRabcd03)   # cdb(dRabcd03.401,dRabcd03)
dRabcd04 = remove_Bab (dRabcd04)   # cdb(dRabcd04.401,dRabcd04)
dRabcd05 = remove_Bab (dRabcd05)   # cdb(dRabcd05.401,dRabcd05)

end_stage_4 = time.time()
```

$$\texttt{dRabcd01.401} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\texttt{dRabcd02.401} := A^c A^d A^e A^f \left(-\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma^g{}_{bf} g^{ha} + R_{cbdg} \partial_e \Gamma^a{}_{hf} g^{gh}\right)$$

$$\texttt{dRabcd03.401} := A^c A^d A^e A^f A^g \left(-3\nabla_c R_{dhei} \partial_f \Gamma^h{}_{bg} g^{ia} + 3\nabla_c R_{dbeh} \partial_f \Gamma^a{}_{ig} g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_{ef} \Gamma^h{}_{bg} g^{ia} + R_{cbdh} \partial_{ef} \Gamma^a{}_{ig} g^{hi}\right)$$

$$\begin{aligned}\texttt{dRabcd04.401} := A^c A^d A^e A^f A^g A^h \Big(&-6\nabla_{de} R_{figj} \partial_c \Gamma^i{}_{bh} g^{aj} + 6\nabla_{de} R_{fbgi} \partial_c \Gamma^a{}_{jh} g^{ji} - 4\nabla_c R_{diej} \partial_{fg} \Gamma^i{}_{bh} g^{ja} + 4\nabla_c R_{dbei} \partial_{fg} \Gamma^a{}_{jh} g^{ij} - \nabla_{cdef} R_{gbhi} g^{ai} \\ &- R_{cidj} \partial_{efg} \Gamma^i{}_{bh} g^{ja} + R_{cbdi} \partial_{efg} \Gamma^a{}_{jh} g^{ij} - 3R_{cidj} \partial_e \Gamma^i{}_{fk} \partial_g \Gamma^k{}_{bh} g^{ja} + 6R_{cidj} \partial_e \Gamma^i{}_{fb} \partial_g \Gamma^a{}_{kh} g^{jk} - 3R_{cbdi} \partial_e \Gamma^j{}_{kf} \partial_g \Gamma^a{}_{jh} g^{ik}\Big)\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd05.401} := A^c A^d A^e A^f A^g A^h A^i \Big(&-10\nabla_{cd} R_{ejfk} \partial_{gh} \Gamma^j{}_{bi} g^{ka} + 10\nabla_{cd} R_{ebfj} \partial_{gh} \Gamma^a{}_{ki} g^{jk} - 10\nabla_{def} R_{gjhk} \partial_c \Gamma^j{}_{bi} g^{ak} + 10\nabla_{def} R_{gbhj} \partial_c \Gamma^a{}_{ki} g^{kj} \\ &-5\nabla_c R_{djek} \partial_{fgh} \Gamma^j{}_{bi} g^{ka} + 5\nabla_c R_{dbej} \partial_{fgh} \Gamma^a{}_{ki} g^{jk} - 15\nabla_c R_{djek} \partial_f \Gamma^j{}_{gl} \partial_h \Gamma^l{}_{bi} g^{ka} + 30\nabla_c R_{djek} \partial_f \Gamma^j{}_{gb} \partial_h \Gamma^a{}_{li} g^{kl} - 15\nabla_c R_{dbej} \partial_f \Gamma^k{}_{lg} \partial_h \Gamma^a{}_{ki} g^{jl} \\ &- \nabla_{cdefg} R_{hbij} g^{aj} - R_{cjdk} \partial_{efgh} \Gamma^j{}_{bi} g^{ka} + R_{cbdj} \partial_{efgh} \Gamma^a{}_{ki} g^{jk} - 4R_{cjdk} \partial_h \Gamma^l{}_{bi} \partial_{ef} \Gamma^j{}_{gl} g^{ka} - 6R_{cjdk} \partial_e \Gamma^j{}_{fl} \partial_{gh} \Gamma^l{}_{bi} g^{ka} \\ &+ 8R_{cjdk} \partial_h \Gamma^a{}_{li} \partial_{ef} \Gamma^j{}_{gb} g^{kl} + 10R_{cjdk} \partial_e \Gamma^j{}_{fb} \partial_{gh} \Gamma^a{}_{li} g^{kl} - 4R_{cbdj} \partial_h \Gamma^a{}_{ki} \partial_{ef} \Gamma^k{}_{lg} g^{jl} - 6R_{cbdj} \partial_e \Gamma^k{}_{lf} \partial_{gh} \Gamma^a{}_{ki} g^{jl} + 2R_{cjdk} \partial_e \Gamma^a{}_{lf} \partial_{gh} \Gamma^j{}_{bi} g^{kl}\Big)\end{aligned}$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

```
import cdblib

beg_stage_5 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json')  # cdb(dGamma01.500,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json')  # cdb(dGamma02.500,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json')  # cdb(dGamma03.500,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json')  # cdb(dGamma04.500,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json')  # cdb(dGamma05.500,dGamma05)

distribute (dRabcd01)   # cdb(dRabcd01.500,dRabcd01)
distribute (dRabcd02)   # cdb(dRabcd02.500,dRabcd02)
distribute (dRabcd03)   # cdb(dRabcd03.500,dRabcd03)
distribute (dRabcd04)   # cdb(dRabcd04.500,dRabcd04)
distribute (dRabcd05)   # cdb(dRabcd05.500,dRabcd05)

# use dGamma to eliminate the partial derivs of Gamma
# this will introduces some lower order partial dervis of Rabcd on the rhs
# these extra partial derivs of Rabcd will be eliminated (later) by substiting lower order dRabcd into the higher order dRabcd

substitute (dRabcd02,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)              # cdb(dRabcd02.501,dRabcd02)
substitute (dRabcd02,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)              # cdb(dRabcd02.502,dRabcd02)
distribute (dRabcd02)                                                                                    # cdb(dRabcd02.503,dRabcd02)
sort_product   (dRabcd02)                                                                                # cdb(dRabcd02.504,dRabcd02)
rename_dummies (dRabcd02)                                                                                # cdb(dRabcd02.505,dRabcd02)


substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)        # cdb(dRabcd03.501,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)        # cdb(dRabcd03.502,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)              # cdb(dRabcd03.503,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)              # cdb(dRabcd03.504,dRabcd03)
distribute (dRabcd03)                                                                                    # cdb(dRabcd03.505,dRabcd03)
sort_product   (dRabcd03)                                                                                # cdb(dRabcd03.506,dRabcd03)
rename_dummies (dRabcd03)                                                                                # cdb(dRabcd03.507,dRabcd03)

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{d b}} -> @(dGamma03)$,repeat=True)  # cdb(dRabcd04.501,dRabcd04)
```

```
substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{b d}} -> @(dGamma03)$,repeat=True)  # cdb(dRabcd04.502,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)         # cdb(dRabcd04.503,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)         # cdb(dRabcd04.504,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd04.505,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd04.506,dRabcd04)
distribute (dRabcd04)                                                                                       # cdb(dRabcd04.507,dRabcd04)
sort_product   (dRabcd04)                                                                                   # cdb(dRabcd04.508,dRabcd04)
rename_dummies (dRabcd04)                                                                                   # cdb(dRabcd04.509,dRabcd04)

substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}{\Gamma^{a}_{d b}} -> @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}{\Gamma^{a}_{b d}} -> @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{d b}} -> @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{b d}} -> @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)
distribute (dRabcd05)
sort_product   (dRabcd05)
rename_dummies (dRabcd05)

end_stage_5 = time.time()
```

$$\texttt{dRabcd01.500} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\texttt{dRabcd02.500} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - A^c A^d A^e A^f R_{cgdh} \partial_e \Gamma^g{}_{bf} g^{ha} + A^c A^d A^e A^f R_{cbdg} \partial_e \Gamma^a{}_{hf} g^{gh}$$

$$\texttt{dRabcd02.501} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R^g{}_{feb} R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R^a{}_{feh} R_{cbdg} g^{gh}$$

$$\texttt{dRabcd02.502} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R^g{}_{feb} R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R^a{}_{feh} R_{cbdg} g^{gh}$$

$$\texttt{dRabcd02.503} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R^g{}_{feb} R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R^a{}_{feh} R_{cbdg} g^{gh}$$

$$\texttt{dRabcd02.504} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R^g{}_{feb} R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R^a{}_{feh} R_{cbdg} g^{gh}$$

$$\texttt{dRabcd02.505} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R^g{}_{feb} R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R^a{}_{feg} R_{cbdh} g^{hg}$$

$$\begin{aligned}\texttt{dRabcd03.500} := &-3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma^h{}_{bg} g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma^a{}_{ig} g^{hi} \\ &- A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - A^c A^d A^e A^f A^g R_{chdi} \partial_{ef} \Gamma^h{}_{bg} g^{ia} + A^c A^d A^e A^f A^g R_{cbdh} \partial_{ef} \Gamma^a{}_{ig} g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.501} := &-3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma^h{}_{bg} g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma^a{}_{ig} g^{hi} \\ &- A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^h{}_{geb} R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^a{}_{gei} R_{cbdh} g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.502} := &-3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma^h{}_{bg} g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma^a{}_{ig} g^{hi} \\ &- A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^h{}_{geb} R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^a{}_{gei} R_{cbdh} g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.503} := &-A^c A^d A^e A^g A^f R^h{}_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R^a{}_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ &- \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^h{}_{geb} R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^a{}_{gei} R_{cbdh} g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.504} := &-A^c A^d A^e A^g A^f R^h{}_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R^a{}_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ &- \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^h{}_{geb} R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^a{}_{gei} R_{cbdh} g^{hi}\end{aligned}$$

$$\texttt{dRabcd03.505} := -A^cA^dA^eA^gA^fR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^gA^fR^a{}_{gfi}\nabla_cR_{dbeh}g^{hi} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^h{}_{geb}R_{chdi}g^{ia} + \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^a{}_{gei}R_{cbdh}g^{hi}$$

$$\texttt{dRabcd03.506} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfi}\nabla_cR_{dbeh}g^{hi} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^eA^fA^gR_{chdi}\partial_fR^h{}_{geb}g^{ia} + \frac{1}{2}A^cA^dA^eA^fA^gR_{cbdh}\partial_fR^a{}_{gei}g^{hi}$$

$$\texttt{dRabcd03.507} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfh}\nabla_cR_{dbei}g^{ih} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^eA^fA^gR_{chdi}\partial_fR^h{}_{geb}g^{ia} + \frac{1}{2}A^cA^dA^eA^fA^gR_{cbdh}\partial_fR^a{}_{gei}g^{hi}$$

$$\texttt{dRabcd04.500} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji} - 4A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_{fg}\Gamma^i{}_{bh}g^{ja} + 4A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_{fg}\Gamma^a{}_{jh}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - A^cA^dA^eA^fA^gA^hR_{cidj}\partial_{efg}\Gamma^i{}_{bh}g^{ja} + A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_{efg}\Gamma^a{}_{jh}g^{ij} - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$\texttt{dRabcd04.501} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji} - 4A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_{fg}\Gamma^i{}_{bh}g^{ja} + 4A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_{fg}\Gamma^a{}_{jh}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja} + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij} - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$\texttt{dRabcd04.502} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji} - 4A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_{fg}\Gamma^i{}_{bh}g^{ja} + 4A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_{fg}\Gamma^a{}_{jh}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja} + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij} - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$\texttt{dRabcd04.503} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji}$$
$$- 2A^cA^dA^eA^gA^hA^f\partial_gR^i{}_{hfb}\nabla_cR_{diej}g^{ja} + 2A^cA^dA^eA^gA^hA^f\partial_gR^a{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}$$
$$- A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja}$$
$$+ A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij}$$
$$- 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$\texttt{dRabcd04.504} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji}$$
$$- 2A^cA^dA^eA^gA^hA^f\partial_gR^i{}_{hfb}\nabla_cR_{diej}g^{ja} + 2A^cA^dA^eA^gA^hA^f\partial_gR^a{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}$$
$$- A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja}$$
$$+ A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij}$$
$$- 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$\texttt{dRabcd04.505} := -2A^hA^cR^i{}_{hcb}A^dA^eA^fA^g\nabla_{de}R_{figj}g^{aj} + 2A^hA^cR^a{}_{hcj}A^dA^eA^fA^g\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^gA^hA^f\partial_gR^i{}_{hfb}\nabla_cR_{diej}g^{ja}$$
$$+ 2A^cA^dA^eA^gA^hA^f\partial_gR^a{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}$$
$$- A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja}$$
$$+ A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij}$$
$$- A^cA^dA^eA^fA^hA^gR^k{}_{hgb}R_{cidj}\partial_e\Gamma^i{}_{fk}g^{ja} + 2A^cA^dA^eA^fA^hA^gR^a{}_{hgk}R_{cidj}\partial_e\Gamma^i{}_{fb}g^{jk} - \frac{1}{3}A^cA^dA^fA^eR^j{}_{fek}A^hA^gR^a{}_{hgj}R_{cbdi}g^{ik}$$

$$\texttt{dRabcd04.506} := -2A^hA^cR^i{}_{hcb}A^dA^eA^fA^g\nabla_{de}R_{figj}g^{aj} + 2A^hA^cR^a{}_{hcj}A^dA^eA^fA^g\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^gA^hA^f\partial_gR^i{}_{hfb}\nabla_cR_{diej}g^{ja}$$
$$+ 2A^cA^dA^eA^gA^hA^f\partial_gR^a{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}$$
$$- A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja}$$
$$+ A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij}$$
$$- \frac{1}{3}A^cA^dA^fA^eR^i{}_{fek}A^hA^gR^k{}_{hgb}R_{cidj}g^{ja} + \frac{2}{3}A^cA^dA^fA^eR^i{}_{feb}A^hA^gR^a{}_{hgk}R_{cidj}g^{jk} - \frac{1}{3}A^cA^dA^fA^eR^j{}_{fek}A^hA^gR^a{}_{hgj}R_{cbdi}g^{ik}$$

$$\texttt{dRabcd04.507} := -2A^hA^cR^i{}_{hcb}A^dA^eA^fA^g\nabla_{de}R_{figj}g^{aj} + 2A^hA^cR^a{}_{hcj}A^dA^eA^fA^g\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^gA^hA^f\partial_gR^i{}_{hfb}\nabla_cR_{diej}g^{ja}$$
$$+ 2A^cA^dA^eA^gA^hA^f\partial_gR^a{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - \frac{3}{5}A^cA^dA^hA^eA^fA^g\partial_{gf}R^i{}_{heb}R_{cidj}g^{ja}$$
$$+ \frac{1}{15}A^cA^dA^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb}R_{cidj}g^{ja} + \frac{1}{15}A^cA^dA^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}R_{cidj}g^{ja} + \frac{3}{5}A^cA^dA^hA^eA^fA^g\partial_{gf}R^a{}_{hej}R_{cbdi}g^{ij}$$
$$- \frac{1}{15}A^cA^dA^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj}R_{cbdi}g^{ij} - \frac{1}{15}A^cA^dA^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}R_{cbdi}g^{ij}$$
$$- \frac{1}{3}A^cA^dA^fA^eR^i{}_{fek}A^hA^gR^k{}_{hgb}R_{cidj}g^{ja} + \frac{2}{3}A^cA^dA^fA^eR^i{}_{feb}A^hA^gR^a{}_{hgk}R_{cidj}g^{jk} - \frac{1}{3}A^cA^dA^fA^eR^j{}_{fek}A^hA^gR^a{}_{hgj}R_{cbdi}g^{ik}$$

$$\texttt{dRabcd04.508} := -2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hcj}\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_gR^i{}_{hfb}g^{ja}$$
$$+ 2A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_gR^a{}_{hfj}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cidj}\partial_{gf}R^i{}_{heb}g^{ja}$$
$$+ \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{efk}R^k{}_{hgb}R_{cidj}g^{ja} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{egk}R^k{}_{hfb}R_{cidj}g^{ja} + \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_{gf}R^a{}_{hej}g^{ij}$$
$$- \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{efk}R^k{}_{hgj}R_{cbdi}g^{ij} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{egk}R^k{}_{hfj}R_{cbdi}g^{ij}$$
$$- \frac{1}{3}A^cA^dA^eA^fA^gA^hR^i{}_{fek}R^k{}_{hgb}R_{cidj}g^{ja} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgk}R^i{}_{feb}R_{cidj}g^{jk} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgj}R^j{}_{fek}R_{cbdi}g^{ik}$$

$$\texttt{dRabcd04.509} := -2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij} - 2A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_gR^i{}_{hfb}g^{ja}$$
$$+ 2A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_gR^a{}_{hfj}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cidj}\partial_{gf}R^i{}_{heb}g^{ja}$$
$$+ \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{efj}R^j{}_{hgb}R_{cidk}g^{ka} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{egj}R^j{}_{hfb}R_{cidk}g^{ka}$$
$$+ \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_{gf}R^a{}_{hej}g^{ij} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{efi}R^i{}_{hgj}R_{cbdk}g^{kj} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{egi}R^i{}_{hfj}R_{cbdk}g^{kj}$$
$$- \frac{1}{3}A^cA^dA^eA^fA^gA^hR^i{}_{fej}R^j{}_{hgb}R_{cidk}g^{ka} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^j{}_{feb}R_{cjdk}g^{ki} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^i{}_{fej}R_{cbdk}g^{kj}$$

Stage 6: Replace partial derivs of R with covariant derivs of R

```
# now eliminate remaining partial derivs of Rabcd by substitution from the lower order dRabcd

# note that
#   dRabcd01 = R^a_{cdb,e} A^c A^d A^e
#   dRabcd02 = R^a_{cdb,ef} A^c A^d A^e A^f
#   dRabcd03 = R^a_{cdb,efg} A^c A^d A^e A^f A^g

# thus we can use
#   dRabcd01 to eliminate 1st partial derivs of R in dRabcd03, dRabcd04, etc.
#   dRabcd02 to eliminate 2nd partial derivs of R in dRabcd04, dRabcd05, etc.
#   dRabcd03 to eliminate 3rd partial derivs of R in dRabcd05, dRabcd06, etc.

beg_stage_6 = time.time()

substitute (dRabcd03,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} -> @(dRabcd01)$,repeat=True)         # cdb(dRabcd03.601,dRabcd03)
distribute (dRabcd03)                                                                                  # cdb(dRabcd03.602,dRabcd03)

# note: dRabcd04 and dRabcd05 unused in this code (or any other code)

substitute (dRabcd04,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} -> @(dRabcd02)$,repeat=True)  # cdb(dRabcd04.601,dRabcd04)
substitute (dRabcd04,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} -> @(dRabcd01)$,repeat=True)         # cdb(dRabcd04.602,dRabcd04)
distribute (dRabcd04)                                                                                  # cdb(dRabcd04.603,dRabcd04)

substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c d b}} -> @(dRabcd03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} -> @(dRabcd02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} -> @(dRabcd01)$,repeat=True)
distribute (dRabcd05)

end_stage_6 = time.time()
```

$$\texttt{dRabcd03.601} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfh}\nabla_cR_{dbei}g^{ih} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah}$$
$$+ \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfb}g^{hj}R_{chdi}g^{ia} - \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfi}g^{aj}R_{cbdh}g^{hi}$$

$$\texttt{dRabcd03.602} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfh}\nabla_cR_{dbei}g^{ih} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah}$$
$$+ \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfb}g^{hj}R_{chdi}g^{ia} - \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfi}g^{aj}R_{cbdh}g^{hi}$$

$$\texttt{dRabcd04.601} := -2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij}$$
$$- 2A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_gR^i{}_{hfb}g^{ja} + 2A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_gR^a{}_{hfj}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}$$
$$- \frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gbfl}g^{il} - \frac{1}{3}A^hA^eA^gA^fR^l{}_{fgb}R_{hlek}g^{ki} + \frac{1}{3}A^hA^eA^gA^fR^i{}_{fgl}R_{hbek}g^{kl}\right)R_{cidj}g^{ja}$$
$$+ \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{efj}R^j{}_{hgb}R_{cidk}g^{ka} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{egj}R^j{}_{hfb}R_{cidk}g^{ka}$$
$$+ \frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gjfl}g^{al} - \frac{1}{3}A^hA^eA^gA^fR^l{}_{fgj}R_{hlek}g^{ka} + \frac{1}{3}A^hA^eA^gA^fR^a{}_{fgl}R_{hjek}g^{kl}\right)R_{cbdi}g^{ij}$$
$$- \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{efi}R^i{}_{hgj}R_{cbdk}g^{kj} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{egi}R^i{}_{hfj}R_{cbdk}g^{kj}$$
$$- \frac{1}{3}A^cA^dA^eA^fA^gA^hR^i{}_{fej}R^j{}_{hgb}R_{cidk}g^{ka} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^j{}_{feb}R_{cjdk}g^{ki} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^i{}_{fej}R_{cbdk}g^{kj}$$

$$\texttt{dRabcd04.602} := -2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij} + 2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgb}g^{ik}\nabla_cR_{diej}g^{ja}$$
$$- 2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgj}g^{ak}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}$$
$$- \frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gbfl}g^{il} - \frac{1}{3}A^hA^eA^gA^fR^l{}_{fgb}R_{hlek}g^{ki} + \frac{1}{3}A^hA^eA^gA^fR^i{}_{fgl}R_{hbek}g^{kl}\right)R_{cidj}g^{ja}$$
$$+ \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{efj}R^j{}_{hgb}R_{cidk}g^{ka} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{egj}R^j{}_{hfb}R_{cidk}g^{ka}$$
$$+ \frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gjfl}g^{al} - \frac{1}{3}A^hA^eA^gA^fR^l{}_{fgj}R_{hlek}g^{ka} + \frac{1}{3}A^hA^eA^gA^fR^a{}_{fgl}R_{hjek}g^{kl}\right)R_{cbdi}g^{ij}$$
$$- \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{efi}R^i{}_{hgj}R_{cbdk}g^{kj} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{egi}R^i{}_{hfj}R_{cbdk}g^{kj}$$
$$- \frac{1}{3}A^cA^dA^eA^fA^gA^hR^i{}_{fej}R^j{}_{hgb}R_{cidk}g^{ka} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^j{}_{feb}R_{cjdk}g^{ki} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^i{}_{fej}R_{cbdk}g^{kj}$$

$$
\begin{aligned}
\texttt{dRabcd04.603} := {} & -2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij} + 2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgb}g^{ik}\nabla_cR_{diej}g^{ja} \\
& - 2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgj}g^{ak}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} + \frac{3}{5}A^cA^dA^hA^eA^gA^f\nabla_{he}R_{gbfl}g^{il}R_{cidj}g^{ja} \\
& + \frac{1}{5}A^cA^dA^hA^eA^gA^fR^l{}_{fgb}R_{hlek}g^{ki}R_{cidj}g^{ja} - \frac{1}{5}A^cA^dA^hA^eA^gA^fR^i{}_{fgl}R_{hbek}g^{kl}R_{cidj}g^{ja} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{efj}R^j{}_{hgb}R_{cidk}g^{ka} \\
& + \frac{1}{15}A^cA^dA^eA^fA^gA^hR^i{}_{egj}R^j{}_{hfb}R_{cidk}g^{ka} - \frac{3}{5}A^cA^dA^hA^eA^gA^f\nabla_{he}R_{gjfl}g^{al}R_{cbdi}g^{ij} - \frac{1}{5}A^cA^dA^hA^eA^gA^fR^l{}_{fgj}R_{hlek}g^{ka}R_{cbdi}g^{ij} \\
& + \frac{1}{5}A^cA^dA^hA^eA^gA^fR^a{}_{fgl}R_{hjek}g^{kl}R_{cbdi}g^{ij} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{efi}R^i{}_{hgj}R_{cbdk}g^{kj} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR^a{}_{egi}R^i{}_{hfj}R_{cbdk}g^{kj} \\
& - \frac{1}{3}A^cA^dA^eA^fA^gA^hR^i{}_{fej}R^j{}_{hgb}R_{cidk}g^{ka} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^j{}_{feb}R_{cjdk}g^{ki} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR^a{}_{hgi}R^i{}_{fej}R_{cbdk}g^{kj}
\end{aligned}
$$

Stage 7: Reformatting

```
beg_stage_7 = time.time()

dRabcd01 = flatten_Rabcd (dRabcd01)  # cdb(dRabcd01.701,dRabcd01)
dRabcd02 = flatten_Rabcd (dRabcd02)  # cdb(dRabcd02.701,dRabcd02)
dRabcd03 = flatten_Rabcd (dRabcd03)  # cdb(dRabcd03.701,dRabcd03)
dRabcd04 = flatten_Rabcd (dRabcd04)  # cdb(dRabcd04.701,dRabcd04)
dRabcd05 = flatten_Rabcd (dRabcd05)  # cdb(dRabcd05.701,dRabcd05)

canonicalise (dRabcd01)   # cdb(dRabcd01.702,dRabcd01)
canonicalise (dRabcd02)   # cdb(dRabcd02.702,dRabcd02)
canonicalise (dRabcd03)   # cdb(dRabcd03.702,dRabcd03)
canonicalise (dRabcd04)   # cdb(dRabcd04.702,dRabcd04)
canonicalise (dRabcd05)   # cdb(dRabcd05.702,dRabcd05)

end_stage_7 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs\\hfill\\break".format(end_stage_4-beg_stage_4))
print ("Stage 5: {:7.1f} secs\\hfill\\break".format(end_stage_5-beg_stage_5))
print ("Stage 6: {:7.1f} secs\\hfill\\break".format(end_stage_6-beg_stage_6))
print ("Stage 7: {:7.1f} secs".format(end_stage_7-beg_stage_7))
# cdbEnd (timing)
```

$$\texttt{dRabcd01.701} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\texttt{dRabcd02.701} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R_{ifeb} g^{gi} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R_{hfei} g^{ah} g^{gi}$$

$$\begin{aligned}\texttt{dRabcd03.701} := &-A^c A^d A^e A^f A^g R_{hgfb} \nabla_c R_{diej} g^{ih} g^{ja} + A^c A^d A^e A^f A^g R_{hgfi} \nabla_c R_{dbej} g^{ah} g^{ji} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ &+ \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_g R_{ejfb} g^{hj} g^{ia} - \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \nabla_g R_{eifj} g^{ai} g^{hj}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd04.701} := &-2 A^c A^d A^e A^f A^g A^h R_{ihcb} \nabla_{de} R_{fjgk} g^{ak} g^{ji} + 2 A^c A^d A^e A^f A^g A^h R_{ihcj} \nabla_{de} R_{fbgk} g^{ai} g^{jk} + 2 A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \nabla_h R_{fkgb} g^{ik} g^{ja} \\ &- 2 A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \nabla_h R_{fjgk} g^{aj} g^{ik} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{he} R_{gbfk} g^{ik} g^{ja} \\ &+ \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hkel} R_{mfgb} g^{ja} g^{li} g^{km} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hbek} R_{lfgm} g^{il} g^{ja} g^{km} \\ &+ \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kefl} R_{mhgb} g^{ik} g^{lm} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kegl} R_{mhfb} g^{ik} g^{lm} g^{ja} \\ &- \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \nabla_{he} R_{gjfk} g^{ak} g^{ij} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{im} g^{ka} g^{jl} \\ &+ \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{al} g^{ij} g^{km} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jefk} R_{lhgm} g^{aj} g^{kl} g^{im} \\ &- \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jegk} R_{lhfm} g^{aj} g^{kl} g^{im} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kfel} R_{mhgb} g^{ik} g^{lm} g^{ja} \\ &+ \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{khgl} R_{mfeb} g^{ak} g^{im} g^{jl} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jhgk} R_{lfem} g^{aj} g^{kl} g^{im}\end{aligned}$$

$$\texttt{dRabcd01.702} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\texttt{dRabcd02.702} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\texttt{dRabcd03.702} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}$$

$$\texttt{dRabcd04.702} := -\frac{7}{5} A^c A^d A^e A^f A^g A^h R_{bcdi} \nabla_{ef} R_{gjhk} g^{aj} g^{ik} + \frac{7}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{ef} R_{bghk} g^{ai} g^{jk} + A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{bghi} g^{ai}$$

$$\begin{aligned}\texttt{dRabcd05.702} := {}& -2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{bdej} \nabla_{fg} R_{hkil} g^{ak} g^{jl} + 2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{djek} \nabla_{fg} R_{bhil} g^{aj} g^{kl} \\ & - \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} \nabla_{efg} R_{hkil} g^{ak} g^{jl} + \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} \nabla_{efg} R_{bhil} g^{aj} g^{kl} \\ & + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{am} g^{jk} g^{ln} + A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{elfm} \nabla_g R_{bhin} g^{aj} g^{kl} g^{mn} \\ & - \frac{4}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{ak} g^{jm} g^{ln} + A^c A^d A^e A^f A^g A^h A^i \nabla_{cdefg} R_{bhij} g^{aj}\end{aligned}$$

```
cdblib.create ('dRabcd.json')

cdblib.put ('dRabcd01',dRabcd01,'dRabcd.json')
cdblib.put ('dRabcd02',dRabcd02,'dRabcd.json')
cdblib.put ('dRabcd03',dRabcd03,'dRabcd.json')
cdblib.put ('dRabcd04',dRabcd04,'dRabcd.json')
cdblib.put ('dRabcd05',dRabcd05,'dRabcd.json')
```

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                            -> A001^{a}               $)
    substitute (obj,$ x^{a}                            -> A002^{a}               $)
    substitute (obj,$ g^{a b}                          -> A003^{a b}             $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}    -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}      -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}        -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}          -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                      -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                  -> A^{a}                          $)
    substitute (obj,$ A002^{a}                  -> x^{a}                          $)
    substitute (obj,$ A003^{a b}                -> g^{a b}                        $)
    substitute (obj,$ A004_{a b c d}            -> R_{a b c d}                    $)
    substitute (obj,$ A005_{a b c d e}          -> \nabla_{e}{R_{a b c d}}        $)
    substitute (obj,$ A006_{a b c d e f}        -> \nabla_{e f}{R_{a b c d}}      $)
    substitute (obj,$ A007_{a b c d e f g}      -> \nabla_{e f g}{R_{a b c d}}    $)
    substitute (obj,$ A008_{a b c d e f g h}    -> \nabla_{e f g h}{R_{a b c d}} $)

    return obj

def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    factor_out     (bah,$A^{a?}$)
    ans := @(bah).
    return ans

scaled1 = reformat (dRabcd01, 1)    # cdb(scaled1.601,scaled1)
scaled2 = reformat (dRabcd02, 1)    # cdb(scaled2.601,scaled2)
scaled3 = reformat (dRabcd03,-2)    # cdb(scaled3.601,scaled3)
scaled4 = reformat (dRabcd04,-5)    # cdb(scaled4.601,scaled4)
```

```
scaled5 = reformat (dRabcd05,-3)    # cdb(scaled5.601,scaled5)
```

Symmetrised partial derivatives of $R^a{}_{bcd}$

$$
\begin{aligned}
A^cA^dA^eR^a{}_{cdb,e} &= A^cA^dA^eg^{af}\nabla_cR_{bdef}\\
A^cA^dA^eA^fR^a{}_{cdb,ef} &= A^cA^dA^eA^fg^{ag}\nabla_{cd}R_{befg}\\
-2A^cA^dA^eA^fA^gR^a{}_{cdb,efg} &= A^cA^dA^eA^fA^g\left(g^{ah}g^{ij}R_{bcdi}\nabla_eR_{fhgj} - g^{ah}g^{ij}R_{chdi}\nabla_eR_{bfgj} - 2g^{ah}\nabla_{cde}R_{bfgh}\right)\\
-5A^cA^dA^eA^fA^gA^hR^a{}_{cdb,efgh} &= A^cA^dA^eA^fA^gA^h\left(7g^{ai}g^{jk}R_{bcdj}\nabla_{ef}R_{gihk} - 7g^{ai}g^{jk}R_{cidj}\nabla_{ef}R_{bghk} - 5g^{ai}\nabla_{cdef}R_{bghi}\right)\\
-3A^cA^dA^eA^fA^gA^hA^iR^a{}_{cdb,efghi} &= A^cA^dA^eA^fA^gA^hA^i\big(6g^{aj}g^{kl}\nabla_cR_{bdek}\nabla_{fg}R_{hjil} - 6g^{aj}g^{kl}\nabla_cR_{djek}\nabla_{fg}R_{bhil} + 8g^{aj}g^{kl}R_{bcdk}\nabla_{efg}R_{hjil}\\
&\quad - 8g^{aj}g^{kl}R_{cjdk}\nabla_{efg}R_{bhil} - g^{aj}g^{kl}g^{mn}R_{bcdk}R_{elfm}\nabla_gR_{hjin} - 3g^{aj}g^{kl}g^{mn}R_{cjdk}R_{elfm}\nabla_gR_{bhin}\\
&\quad + 4g^{aj}g^{kl}g^{mn}R_{bcdk}R_{ejfm}\nabla_gR_{hlin} - 3g^{aj}\nabla_{cdefg}R_{bhij}\big)
\end{aligned}
$$

```
substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dRabcd.export')

# 6th order dRabcd, scaled
cdblib.put ('dRabcd61scaled',scaled1,'dRabcd.export')
cdblib.put ('dRabcd62scaled',scaled2,'dRabcd.export')
cdblib.put ('dRabcd63scaled',scaled3,'dRabcd.export')
cdblib.put ('dRabcd64scaled',scaled4,'dRabcd.export')
cdblib.put ('dRabcd65scaled',scaled5,'dRabcd.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```

Timing

Stage 1: 2.5 secs
Stage 2: 6.8 secs
Stage 3: 0.6 secs
Stage 4: 193.7 secs
Stage 5: 205.1 secs
Stage 6: 245.8 secs
Stage 7: 8.3 secs

Geodesic BVP

Consider a geodesic that connects two points P_i and P_j with RNC coordinates x^a_i and x^a_j. Our aim is to construct a solution $x^a(s)$ of the geodesic equation such that $x^a(0) = x^a_i$ and $x^a(1) = x^a_j$.

We will do this in two stages. First we will solve

$$x^a_j = x^a_i + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\cdot\underline{b}_k} \tag{1}$$

for y^a as an explicit polynomial in x^a_i and x^a_j. The functions $\Gamma^a_{\underline{b}_k}$ are the generalised connections for the RNC frame evaluated at $x^a = x^a_i$.

In the second stage, we will substitute our expression for y^a into

$$x^a(s) = x^a_i + sy^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\cdot\underline{b}_k} s^k \tag{2}$$

to obtain the desired solution to the two point boundary value problem.

Stage 1: The fixed point iteration scheme

First we rewrite the main equation (1) in the suggestive form

$$y^a = \Delta x^a + \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\underline{b}_k}$$

where $\Delta x^a = x^a_j - x^a_i$. Our approximate solution for y^a will be taken to be the partial sums for the infinite series. Thus we will solve

$$\overset{n}{y}{}^a = \Delta x^a + \sum_{k=2}^{n} \frac{1}{k!} \Gamma^a_{\underline{b}_k} \overset{n}{y}{}^{\underline{b}_k}$$

for $\overset{n}{y}{}^a$. Note that in the last term of the sum, the $\Gamma^a_{\underline{b}_n}$ will contain curvature terms of order $\mathcal{O}(\epsilon^n)$. Thus in truncating the series at this point we will loose contributions to the curvature terms of order $\mathcal{O}(\epsilon^{n+1})$ and higher. So to be consistent we must truncate all terms of the partial sum to order $\mathcal{O}(\epsilon^n)$ (i.e., exclude any contributions from terms $\mathcal{O}(\epsilon^{n+1})$ and higher, these are the terms that would couple with the terms that we excluded when truncating the original infinite series). Let $\overset{k}{T}$ the operator that truncates its argument to contain terms no higher than $\mathcal{O}\left(\epsilon^n\right)$. Then we have the following modified version of the equation for $\overset{n}{y}{}^a$

$$\overset{n}{y}{}^a = \Delta x^a + \sum_{k=2}^{n} \frac{1}{k!} \overset{k}{T} \left(\Gamma^a_{\underline{b}_k} \overset{n}{y}{}^{\underline{b}_k} \right)$$

Finally we note that since $\Gamma^a_{\underline{b}_k} = \mathcal{O}\left(\epsilon^k\right)$, we can use lower order estimates for the $\overset{k}{y}{}^a$ in the right hand side of the sum. This allows us to compute $\overset{n}{y}{}^a$ by successive approximations such as

$$\begin{aligned}
\overset{0}{y}{}^a &= \Delta x^a \\
\overset{2}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{2}{T}\left(\Gamma^a_{bc}\,\overset{0}{y}{}^b\overset{0}{y}{}^c\right) \\
\overset{3}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{3}{T}\left(\Gamma^a_{bc}\,\overset{2}{y}{}^b\overset{2}{y}{}^c\right) + \frac{1}{3!}\overset{3}{T}\left(\Gamma^a_{bcd}\,\overset{0}{y}{}^b\overset{0}{y}{}^c\overset{0}{y}{}^d\right) \\
\overset{4}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{4}{T}\left(\Gamma^a_{bc}\,\overset{3}{y}{}^b\overset{3}{y}{}^c\right) + \frac{1}{3!}\overset{4}{T}\left(\Gamma^a_{bcd}\,\overset{2}{y}{}^b\overset{2}{y}{}^c\overset{2}{y}{}^d\right) + \frac{1}{4!}\overset{4}{T}\left(\Gamma^a_{bcde}\,\overset{0}{y}{}^b\overset{0}{y}{}^c\overset{0}{y}{}^d\overset{0}{y}{}^e\right) \\
\overset{5}{y}{}^a &= \overset{0}{y}{}^a + \frac{1}{2!}\overset{5}{T}\left(\Gamma^a_{bc}\,\overset{4}{y}{}^b\overset{4}{y}{}^c\right) + \frac{1}{3!}\overset{5}{T}\left(\Gamma^a_{bcd}\,\overset{3}{y}{}^b\overset{3}{y}{}^c\overset{3}{y}{}^d\right) + \frac{1}{4!}\overset{5}{T}\left(\Gamma^a_{bcde}\,\overset{2}{y}{}^b\overset{2}{y}{}^c\overset{2}{y}{}^d\overset{2}{y}{}^e\right) + \frac{1}{5!}\overset{5}{T}\left(\Gamma^a_{bcdef}\,\overset{0}{y}{}^b\overset{0}{y}{}^c\overset{0}{y}{}^d\overset{0}{y}{}^e\overset{0}{y}{}^f\right)
\end{aligned}$$

and so on. Note that there are no $\overset{1}{y}{}^a$ terms.

Stage 2: Introduce the generalised connections

This is the final stage – it introduces the generalised connecstion after the completion of the fixed point scheme.

All that needs be done is to substitute our expression for y^a into

$$x^a(s) = x^a_i + s y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a{}_{\underline{b}_k} y^{\underline{b}_k} s^k \tag{3}$$

to obtain the desired solution to the two point boundary value problem.

The generalised connections $\Gamma^a{}_{\underline{b}_k}$ are taken from the results of the `genGamma` code.

Stage 1: The fixed point iteration scheme

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.
R_{a b c d}::Depends(\nabla{#}).

{Gam22^{a}_{b c},Gam23^{a}_{b c},Gam24^{a}_{b c},Gam25^{a}_{b c}}::TableauSymmetry(shape={2}, indices={1,2}).
{Gam33^{a}_{b c d},Gam34^{a}_{b c d},Gam35^{a}_{b c d}}::TableauSymmetry(shape={3}, indices={1,2,3}).
{Gam44^{a}_{b c d e},Gam45^{a}_{b c d e}}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
{Gam55^{a}_{b c d e f}}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).

{Gam22^{a}_{b c}}::Weight(label=eps,value=2).
{Gam23^{a}_{b c},Gam33^{a}_{b c d}}::Weight(label=eps,value=3).
{Gam24^{a}_{b c},Gam34^{a}_{b c d},Gam44^{a}_{b c d e}}::Weight(label=eps,value=4).
{Gam25^{a}_{b c},Gam35^{a}_{b c d},Gam45^{a}_{b c d e},Gam55^{a}_{b c d e f}}::Weight(label=eps,value=5).

{Dx^{a}}::Weight(label=eps,value=0).

{y00^{a},y20^{a},y30^{a},y40^{a},y50^{a}}::Weight(label=eps,value=0).
{y22^{a},y32^{a},y42^{a},y52^{a}}::Weight(label=eps,value=2).
{y33^{a},y43^{a},y53^{a}}::Weight(label=eps,value=3).
{y44^{a},y54^{a}}::Weight(label=eps,value=4).
{y55^{a}}::Weight(label=eps,value=5).

# Dx{#}::LaTeXForm{"{\Dx}"}.  # LCB: currently causes a bug, it kills ::KeepWeight for Dx

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}                          -> A001^{a}             $)
    substitute (obj,$ Dx^{a}                         -> A002^{a}             $)
```

```
    substitute (obj,$ g^{a b}                          -> A003^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}    -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}      -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}        -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}          -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                      -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                  -> x^{a}                         $)
    substitute (obj,$ A002^{a}                  -> Dx^{a}                        $)
    substitute (obj,$ A003^{a b}                -> g^{a b}                       $)
    substitute (obj,$ A004_{a b c d}            -> R_{a b c d}                   $)
    substitute (obj,$ A005_{a b c d e}          -> \nabla_{e}{R_{a b c d}}       $)
    substitute (obj,$ A006_{a b c d e f}        -> \nabla_{e f}{R_{a b c d}}     $)
    substitute (obj,$ A007_{a b c d e f g}      -> \nabla_{e f g}{R_{a b c d}}   $)
    substitute (obj,$ A008_{a b c d e f g h}    -> \nabla_{e f g h}{R_{a b c d}} $)

    return obj

def get_term (obj,n):

    tmp := @(obj).
    foo = Ex("eps = " + str(n))
    distribute  (tmp)
    keep_weight (tmp, foo)

    return tmp

def truncate (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah = Ex("eps = " + str(i))
       distribute  (foo)
       keep_weight (foo, bah)
       ans = ans + foo
```

```
    return ans

def substitute_eps (obj):
    substitute     (obj,epsy0)
    substitute     (obj,epsy2)
    substitute     (obj,epsy3)
    substitute     (obj,epsy4)
    substitute     (obj,epsy5)
    substitute     (obj,epsGam2)
    substitute     (obj,epsGam3)
    substitute     (obj,epsGam4)
    substitute     (obj,epsGam5)
    distribute     (obj)
    obj = truncate     (obj,5)
    obj = product_sort (obj)
    rename_dummies (obj)
    canonicalise   (obj)

    return obj

beg_stage_1 = time.time()

# yn = y expanded to terms upto and including O(eps^n)

y0 := Dx^{a}.
y2 := Dx^{a} +   (1/2) Gam^{a}_{b c} y0^{b} y0^{c}.
y3 := Dx^{a} +   (1/2) Gam^{a}_{b c} y2^{b} y2^{c}
             +   (1/6) Gam^{a}_{b c d} y0^{b} y0^{c} y0^{d}.
y4 := Dx^{a} +   (1/2) Gam^{a}_{b c} y3^{b} y3^{c}
             +   (1/6) Gam^{a}_{b c d} y2^{b} y2^{c} y2^{d}
             +  (1/24) Gam^{a}_{b c d e} y0^{b} y0^{c} y0^{d} y0^{e}.
y5 := Dx^{a} +   (1/2) Gam^{a}_{b c} y4^{b} y4^{c}
             +   (1/6) Gam^{a}_{b c d} y3^{b} y3^{c} y3^{d}
             +  (1/24) Gam^{a}_{b c d e} y2^{b} y2^{c} y2^{d} y2^{e}
             + (1/120) Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

# epsyN = y expanded to terms upto and including O(eps^N)
```

```
# yPQ = O(eps^Q) term of epsyP

# expand to O(eps^5)

epsy0 := y0^{a} -> y00^{a}.
epsy2 := y2^{a} -> y20^{a}+y22^{a}.
epsy3 := y3^{a} -> y30^{a}+y32^{a}+y33^{a}.
epsy4 := y4^{a} -> y40^{a}+y42^{a}+y43^{a}+y44^{a}.
epsy5 := y5^{a} -> y50^{a}+y52^{a}+y53^{a}+y54^{a}+y55^{a}.

# epsGamN = gen. gamma with N lower indices (epsGam2 = the connection)
# epsGamPQ = O(eps^Q) term of epsGamP

epsGam2 := Gam^{a}_{b c} -> Gam22^{a}_{b c}+Gam23^{a}_{b c}+Gam24^{a}_{b c}+Gam25^{a}_{b c}.
epsGam3 := Gam^{a}_{b c d} -> Gam33^{a}_{b c d}+Gam34^{a}_{b c d}+Gam35^{a}_{b c d}.
epsGam4 := Gam^{a}_{b c d e} -> Gam44^{a}_{b c d e}+Gam45^{a}_{b c d e}.
epsGam5 := Gam^{a}_{b c d e f} -> Gam55^{a}_{b c d e f}.

y0 = substitute_eps (y0)   # cdb (y0.001,y0)
y2 = substitute_eps (y2)   # cdb (y2.001,y2)
y3 = substitute_eps (y3)   # cdb (y3.001,y3)
y4 = substitute_eps (y4)   # cdb (y4.001,y4)
y5 = substitute_eps (y5)   # cdb (y5.001,y5)

y0 = truncate (y0,1)       # cdb (y0.002,y0)
y2 = truncate (y2,2)       # cdb (y2.002,y2)
y3 = truncate (y3,3)       # cdb (y3.002,y3)
y4 = truncate (y4,4)       # cdb (y4.002,y4)
y5 = truncate (y5,5)       # cdb (y5.002,y5)

defy0 := y0^{a} -> @(y0).
defy2 := y2^{a} -> @(y2).
defy3 := y3^{a} -> @(y3).
defy4 := y4^{a} -> @(y4).
defy5 := y5^{a} -> @(y5).

# -----------------------------------
def tidy (obj):
```

```
    obj = product_sort (obj)
    rename_dummies     (obj)
    canonicalise       (obj)
    return obj

# -----------------------------------
# y0

y00 := @(y0).           # cdb (y00.101,y00)

defy00 := y00^{a} -> @(y00).

# -----------------------------------
# y2

substitute (y2,defy00)

distribute (y2)

y20 = get_term (y2,0)   # cdb (y20.101,y20)
y22 = get_term (y2,2)   # cdb (y22.101,y22)

y20 = tidy (y20)        # cdb (y20.201,y20)
y22 = tidy (y22)        # cdb (y22.201,y22)

defy20 := y20^{a} -> @(y20).
defy22 := y22^{a} -> @(y22).

# -----------------------------------
# y3

substitute (y3,defy00)

substitute (y3,defy20)
substitute (y3,defy22)

distribute (y3)
```

```
y30 = get_term (y3,0)    # cdb (y30.101,y30)
y32 = get_term (y3,2)    # cdb (y32.101,y32)
y33 = get_term (y3,3)    # cdb (y33.101,y33)

y30 = tidy (y30)         # cdb (y30.201,y30)
y32 = tidy (y32)         # cdb (y32.201,y32)
y33 = tidy (y33)         # cdb (y33.201,y33)

defy30 := y30^{a} -> @(y30).
defy32 := y32^{a} -> @(y32).
defy33 := y33^{a} -> @(y33).

# -----------------------------------
# y4

substitute (y4,defy00)

substitute (y4,defy20)
substitute (y4,defy22)

substitute (y4,defy30)
substitute (y4,defy32)
substitute (y4,defy33)

distribute (y4)

y40 = get_term (y4,0)    # cdb (y40.101,y40)
y42 = get_term (y4,2)    # cdb (y42.101,y42)
y43 = get_term (y4,3)    # cdb (y43.101,y43)
y44 = get_term (y4,4)    # cdb (y44.101,y44)

y40 = tidy (y40)         # cdb (y40.201,y40)
y42 = tidy (y42)         # cdb (y42.201,y42)
y43 = tidy (y43)         # cdb (y43.201,y43)
y44 = tidy (y44)         # cdb (y44.201,y44)

defy40 := y40^{a} -> @(y40).
defy42 := y42^{a} -> @(y42).
```

```
defy43 := y43^{a} -> @(y43).
defy44 := y44^{a} -> @(y44).

# -----------------------------------
# y5

substitute (y5,defy00)

substitute (y5,defy20)
substitute (y5,defy22)

substitute (y5,defy30)
substitute (y5,defy32)
substitute (y5,defy33)

substitute (y5,defy40)
substitute (y5,defy42)
substitute (y5,defy43)
substitute (y5,defy44)

distribute (y5)

y50 = get_term (y5,0)   # cdb (y50.101,y50)
y52 = get_term (y5,2)   # cdb (y52.101,y52)
y53 = get_term (y5,3)   # cdb (y53.101,y53)
y54 = get_term (y5,4)   # cdb (y54.101,y54)
y55 = get_term (y5,5)   # cdb (y55.101,y55)

y50 = tidy (y50)        # cdb (y50.201,y50)
y52 = tidy (y52)        # cdb (y52.201,y52)
y53 = tidy (y53)        # cdb (y53.201,y53)
y54 = tidy (y54)        # cdb (y54.201,y54)
y55 = tidy (y55)        # cdb (y55.201,y55)

defy50 := y50^{a} -> @(y50).
defy52 := y52^{a} -> @(y52).
defy53 := y53^{a} -> @(y53).
defy54 := y54^{a} -> @(y54).
```

```
defy55 := y55^{a} -> @(y55).

end_stage_1 = time.time()
```

$$\texttt{y0.001} := Dx^a$$

$$\texttt{y2.001} := Dx^a + \frac{1}{2}Gam22^a{}_{bc}y00^b y00^c + \frac{1}{2}Gam23^a{}_{bc}y00^b y00^c + \frac{1}{2}Gam24^a{}_{bc}y00^b y00^c + \frac{1}{2}Gam25^a{}_{bc}y00^b y00^c$$

$$\begin{aligned}\texttt{y3.001} := Dx^a &+ \frac{1}{2}Gam22^a{}_{bc}y20^b y20^c + \frac{1}{2}Gam23^a{}_{bc}y20^b y20^c + \frac{1}{6}Gam33^a{}_{bcd}y00^b y00^c y00^d + Gam22^a{}_{bc}y20^b y22^c + \frac{1}{2}Gam24^a{}_{bc}y20^b y20^c \\ &+ \frac{1}{6}Gam34^a{}_{bcd}y00^b y00^c y00^d + Gam23^a{}_{bc}y20^b y22^c + \frac{1}{2}Gam25^a{}_{bc}y20^b y20^c + \frac{1}{6}Gam35^a{}_{bcd}y00^b y00^c y00^d\end{aligned}$$

$$\begin{aligned}\texttt{y4.001} := Dx^a &+ \frac{1}{2}Gam22^a{}_{bc}y30^b y30^c + \frac{1}{2}Gam23^a{}_{bc}y30^b y30^c + \frac{1}{6}Gam33^a{}_{bcd}y20^b y20^c y20^d + Gam22^a{}_{bc}y30^b y32^c + \frac{1}{2}Gam24^a{}_{bc}y30^b y30^c \\ &+ \frac{1}{6}Gam34^a{}_{bcd}y20^b y20^c y20^d + \frac{1}{24}Gam44^a{}_{bcde}y00^b y00^c y00^d y00^e + Gam22^a{}_{bc}y30^b y33^c + Gam23^a{}_{bc}y30^b y32^c \\ &+ \frac{1}{2}Gam25^a{}_{bc}y30^b y30^c + \frac{1}{2}Gam33^a{}_{bcd}y20^b y20^c y22^d + \frac{1}{6}Gam35^a{}_{bcd}y20^b y20^c y20^d + \frac{1}{24}Gam45^a{}_{bcde}y00^b y00^c y00^d y00^e\end{aligned}$$

$$\begin{aligned}\texttt{y5.001} := Dx^a &+ \frac{1}{2}Gam22^a{}_{bc}y40^b y40^c + \frac{1}{2}Gam23^a{}_{bc}y40^b y40^c + \frac{1}{6}Gam33^a{}_{bcd}y30^b y30^c y30^d + Gam22^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam24^a{}_{bc}y40^b y40^c \\ &+ \frac{1}{6}Gam34^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam44^a{}_{bcde}y20^b y20^c y20^d y20^e + Gam22^a{}_{bc}y40^b y43^c + Gam23^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam25^a{}_{bc}y40^b y40^c \\ &+ \frac{1}{2}Gam33^a{}_{bcd}y30^b y30^c y32^d + \frac{1}{6}Gam35^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam45^a{}_{bcde}y20^b y20^c y20^d y20^e + \frac{1}{120}Gam55^a{}_{bcdef}y00^b y00^c y00^d y00^e y00^f\end{aligned}$$

$\texttt{y0.002} := Dx^a$

$\texttt{y2.002} := Dx^a + \frac{1}{2}Gam22^a{}_{bc}y00^b y00^c$

$\texttt{y3.002} := Dx^a + \frac{1}{2}Gam22^a{}_{bc}y20^b y20^c + \frac{1}{2}Gam23^a{}_{bc}y20^b y20^c + \frac{1}{6}Gam33^a{}_{bcd}y00^b y00^c y00^d$

$$\begin{aligned}\texttt{y4.002} := Dx^a &+ \frac{1}{2}Gam22^a{}_{bc}y30^b y30^c + \frac{1}{2}Gam23^a{}_{bc}y30^b y30^c + \frac{1}{6}Gam33^a{}_{bcd}y20^b y20^c y20^d + Gam22^a{}_{bc}y30^b y32^c \\ &+ \frac{1}{2}Gam24^a{}_{bc}y30^b y30^c + \frac{1}{6}Gam34^a{}_{bcd}y20^b y20^c y20^d + \frac{1}{24}Gam44^a{}_{bcde}y00^b y00^c y00^d y00^e\end{aligned}$$

$$\begin{aligned}\texttt{y5.002} := Dx^a &+ \frac{1}{2}Gam22^a{}_{bc}y40^b y40^c + \frac{1}{2}Gam23^a{}_{bc}y40^b y40^c + \frac{1}{6}Gam33^a{}_{bcd}y30^b y30^c y30^d + Gam22^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam24^a{}_{bc}y40^b y40^c \\ &+ \frac{1}{6}Gam34^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam44^a{}_{bcde}y20^b y20^c y20^d y20^e + Gam22^a{}_{bc}y40^b y43^c + Gam23^a{}_{bc}y40^b y42^c + \frac{1}{2}Gam25^a{}_{bc}y40^b y40^c \\ &+ \frac{1}{2}Gam33^a{}_{bcd}y30^b y30^c y32^d + \frac{1}{6}Gam35^a{}_{bcd}y30^b y30^c y30^d + \frac{1}{24}Gam45^a{}_{bcde}y20^b y20^c y20^d y20^e + \frac{1}{120}Gam55^a{}_{bcdef}y00^b y00^c y00^d y00^e y00^f\end{aligned}$$

$$\texttt{y00.101} := Dx^a$$

$$\texttt{y20.201} := Dx^a$$

$$\texttt{y22.201} := \frac{1}{2} Dx^b Dx^c Gam22^a{}_{bc}$$

$$\texttt{y30.201} := Dx^a$$

$$\texttt{y32.201} := \frac{1}{2} Dx^b Dx^c Gam22^a{}_{bc}$$

$$\texttt{y33.201} := \frac{1}{2} Dx^b Dx^c Gam23^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a{}_{bcd}$$

$$\texttt{y40.201} := Dx^a$$

$$\texttt{y42.201} := \frac{1}{2} Dx^b Dx^c Gam22^a{}_{bc}$$

$$\texttt{y43.201} := \frac{1}{2} Dx^b Dx^c Gam23^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a{}_{bcd}$$

$$\texttt{y44.201} := \frac{1}{2} Dx^b Dx^c Dx^d Gam22^a{}_{be} Gam22^e{}_{cd} + \frac{1}{2} Dx^b Dx^c Gam24^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam34^a{}_{bcd} + \frac{1}{24} Dx^b Dx^c Dx^d Dx^e Gam44^a{}_{bcde}$$

$\texttt{y50.201} := Dx^a$

$\texttt{y52.201} := \frac{1}{2} Dx^b Dx^c Gam22^a{}_{bc}$

$\texttt{y53.201} := \frac{1}{2} Dx^b Dx^c Gam23^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a{}_{bcd}$

$\texttt{y54.201} := \frac{1}{2} Dx^b Dx^c Dx^d Gam22^a{}_{be} Gam22^e{}_{cd} + \frac{1}{2} Dx^b Dx^c Gam24^a{}_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam34^a{}_{bcd} + \frac{1}{24} Dx^b Dx^c Dx^d Dx^e Gam44^a{}_{bcde}$

$$\begin{aligned}\texttt{y55.201} := {} & \frac{1}{2} Dx^b Dx^c Dx^d Gam22^a{}_{be} Gam23^e{}_{cd} + \frac{1}{6} Dx^b Dx^c Dx^d Dx^e Gam22^a{}_{bf} Gam33^f{}_{cde} + \frac{1}{2} Dx^b Dx^c Dx^d Gam22^e{}_{bc} Gam23^a{}_{de} \\ & + \frac{1}{2} Dx^b Dx^c Gam25^a{}_{bc} + \frac{1}{4} Dx^b Dx^c Dx^d Dx^e Gam22^f{}_{bc} Gam33^a{}_{def} + \frac{1}{6} Dx^b Dx^c Dx^d Gam35^a{}_{bcd} \\ & + \frac{1}{24} Dx^b Dx^c Dx^d Dx^e Gam45^a{}_{bcde} + \frac{1}{120} Dx^b Dx^c Dx^d Dx^e Dx^f Gam55^a{}_{bcdef}\end{aligned}$$

Stage 2a: Introduce the generalised connections, build terms of y^a

```
def substitute_gam (obj):

    substitute     (obj,defGam22)
    substitute     (obj,defGam23)
    substitute     (obj,defGam24)
    substitute     (obj,defGam25)

    substitute     (obj,defGam33)
    substitute     (obj,defGam34)
    substitute     (obj,defGam35)

    substitute     (obj,defGam44)
    substitute     (obj,defGam45)

    substitute     (obj,defGam55)

    distribute     (obj)
    return obj

import cdblib

beg_stage_2a = time.time()

Gam22 = cdblib.get ('genGamma01','genGamma.json')
Gam23 = cdblib.get ('genGamma02','genGamma.json')
Gam24 = cdblib.get ('genGamma03','genGamma.json')
Gam25 = cdblib.get ('genGamma04','genGamma.json')

Gam33 = cdblib.get ('genGamma11','genGamma.json')
Gam34 = cdblib.get ('genGamma12','genGamma.json')
Gam35 = cdblib.get ('genGamma13','genGamma.json')

Gam44 = cdblib.get ('genGamma21','genGamma.json')
Gam45 = cdblib.get ('genGamma22','genGamma.json')

Gam55 = cdblib.get ('genGamma31','genGamma.json')
```

```
# peel off the A^{a}, must then symmetrise over revealed indices

substitute (Gam22,$A^{a}->1$)
substitute (Gam23,$A^{a}->1$)
substitute (Gam24,$A^{a}->1$)
substitute (Gam25,$A^{a}->1$)

substitute (Gam33,$A^{a}->1$)
substitute (Gam34,$A^{a}->1$)
substitute (Gam35,$A^{a}->1$)

substitute (Gam44,$A^{a}->1$)
substitute (Gam45,$A^{a}->1$)

substitute (Gam55,$A^{a}->1$)

# now symmetrise

sym (Gam22,$_{b},_{c}$)
sym (Gam23,$_{b},_{c}$)
sym (Gam24,$_{b},_{c}$)
sym (Gam25,$_{b},_{c}$)

sym (Gam33,$_{b},_{c},_{d}$)
sym (Gam34,$_{b},_{c},_{d}$)
sym (Gam35,$_{b},_{c},_{d}$)

sym (Gam44,$_{b},_{c},_{d},_{e}$)
sym (Gam45,$_{b},_{c},_{d},_{e}$)

sym (Gam55,$_{b},_{c},_{d},_{e},_{f}$)

defGam22 := Gam22^{a}_{b c} -> @(Gam22).
defGam23 := Gam23^{a}_{b c} -> @(Gam23).
defGam24 := Gam24^{a}_{b c} -> @(Gam24).
defGam25 := Gam25^{a}_{b c} -> @(Gam25).
```

```
defGam33 := Gam33^{a}_{b c d} -> @(Gam33).
defGam34 := Gam34^{a}_{b c d} -> @(Gam34).
defGam35 := Gam35^{a}_{b c d} -> @(Gam35).

defGam44 := Gam44^{a}_{b c d e} -> @(Gam44).
defGam45 := Gam45^{a}_{b c d e} -> @(Gam45).

defGam55 := Gam55^{a}_{b c d e f} -> @(Gam55).

# ---------------------------------------------------
# y2

y22 = substitute_gam (y22)

y22 = tidy (y22)                                        # cdb (y22.301,y22)

y2 := @(y20) + @(y22).                                  # cdb (y2.301,y2)

# ---------------------------------------------------
# y3

y32 = substitute_gam (y32)
y33 = substitute_gam (y33)

y32 = tidy (y32)                                        # cdb (y32.301,y32)
y33 = tidy (y33)                                        # cdb (y33.301,y33)

y3 := @(y30) + @(y32) + @(y33).                         # cdb (y3.301,y3)

# ---------------------------------------------------
# y4

y42 = substitute_gam (y42)
y43 = substitute_gam (y43)
y44 = substitute_gam (y44)

y42 = tidy (y42)                                        # cdb (y42.301,y42)
y43 = tidy (y43)                                        # cdb (y43.301,y43)
```

```
y44 = tidy (y44)                                             # cdb (y44.301,y44)

y4 := @(y40) + @(y42) + @(y43) + @(y44).                     # cdb (y4.301,y4)

# ----------------------------------------------------
# y5

y52 = substitute_gam (y52)
y53 = substitute_gam (y53)
y54 = substitute_gam (y54)
y55 = substitute_gam (y55)

y52 = tidy (y52)                                             # cdb (y52.301,y52)
y53 = tidy (y53)                                             # cdb (y53.301,y53)
y54 = tidy (y54)                                             # cdb (y54.301,y54)
y55 = tidy (y55)                                             # cdb (y55.301,y55)

y5 := @(y50) + @(y52) + @(y53) + @(y54) + @(y55).            # cdb (y5.301,y5)

# ----------------------------------------------------
cdblib.create ('geodesic-bvp.json')

cdblib.put ('y2',y2,'geodesic-bvp.json')
cdblib.put ('y3',y3,'geodesic-bvp.json')
cdblib.put ('y4',y4,'geodesic-bvp.json')
cdblib.put ('y5',y5,'geodesic-bvp.json')

cdblib.put ('y20',y20,'geodesic-bvp.json')
cdblib.put ('y22',y22,'geodesic-bvp.json')

cdblib.put ('y30',y30,'geodesic-bvp.json')
cdblib.put ('y32',y32,'geodesic-bvp.json')
cdblib.put ('y33',y33,'geodesic-bvp.json')

cdblib.put ('y40',y40,'geodesic-bvp.json')
cdblib.put ('y42',y42,'geodesic-bvp.json')
cdblib.put ('y43',y43,'geodesic-bvp.json')
cdblib.put ('y44',y44,'geodesic-bvp.json')
```

```
cdblib.put ('y50',y50,'geodesic-bvp.json')
cdblib.put ('y52',y52,'geodesic-bvp.json')
cdblib.put ('y53',y53,'geodesic-bvp.json')
cdblib.put ('y54',y54,'geodesic-bvp.json')
cdblib.put ('y55',y55,'geodesic-bvp.json')

end_stage_2a = time.time()
```

$$\mathtt{y50.201} := Dx^a$$

$$\mathtt{y52.301} := -\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\mathtt{y53.301} := -\frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} + \frac{1}{24}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned}
\mathtt{y54.301} := &-\frac{2}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{4}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{befh} R_{cgdi} \\
&+ \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dgfi} - \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{eb} R_{cfdg} \\
&- \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bc} R_{defg} - \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dfgi} \\
&+ \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bg} R_{cedf} - \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cgfi} \\
&+ \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{60}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{de} R_{bfcg} - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{db} R_{cefg} \\
&- \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{gd} R_{becf} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{dg} R_{becf} \\
&- \frac{1}{45}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{1}{60}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} \nabla_{cd} R_{befg}
\end{aligned}$$

$$
\begin{aligned}
\mathtt{y55.301} := {} & -\frac{7}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} - \frac{1}{45}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} \\
& + \frac{1}{216}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} \\
& - \frac{17}{1080}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} + \frac{1}{135}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} \\
& - \frac{1}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{108}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} \\
& - \frac{1}{45}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - \frac{7}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} \\
& - \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} - \frac{1}{60}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} - \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} \\
& + \frac{1}{72}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} \\
& + \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} \\
& - \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{fbc} R_{dgeh} - \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bfc} R_{dgeh} - \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcf} R_{dgeh} \\
& - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcd} R_{efgh} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} \\
& - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + \frac{1}{360}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{hbc} R_{dfeg} + \frac{1}{360}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bhc} R_{dfeg} \\
& + \frac{1}{360}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bch} R_{dfeg} - \frac{7}{1080}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - \frac{1}{540}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} \\
& + \frac{1}{108}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh} - \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} \\
& - \frac{1}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} - \frac{11}{540}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
& + \frac{1}{216}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} \\
& + \frac{1}{135}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + \frac{1}{90}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{efb} R_{cgdh} \\
& - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ebf} R_{cgdh} - \frac{1}{180}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ebc} R_{dfgh} - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bef} R_{cgdh} \\
& - \frac{1}{180}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bec} R_{dfgh} - \frac{1}{180}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bce} R_{dfgh} - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} \\
& - \frac{1}{270}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{heb} R_{cfdg} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{hbe} R_{cfdg} \\
& + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ehb} R_{cfdg} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{bhe} R_{cfdg} + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{ebh} R_{cfdg} \\
& + \frac{1}{1080}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} \nabla_{beh} R_{cfdg} - \frac{1}{120}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - \frac{1}{90}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj}
\end{aligned}
$$

Stage 2b: Building the terms of $x^a(s)$

```
def substitute_y (obj):
    substitute (obj,defy00)
    substitute (obj,defy20)
    substitute (obj,defy30)
    substitute (obj,defy32)
    substitute (obj,defy40)
    substitute (obj,defy42)
    substitute (obj,defy43)
    distribute (obj)
    return obj

beg_stage_2b = time.time()

term2 := Gam^{a}_{b c} y4^{b} y4^{c}.
term3 := Gam^{a}_{b c d} y3^{b} y3^{c} y3^{d}.
term4 := Gam^{a}_{b c d e} y2^{b} y2^{c} y2^{d} y2^{e}.
term5 := Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

term2 = substitute_eps (term2)   # cdb (term2.401,term2)
term3 = substitute_eps (term3)   # cdb (term3.401,term3)
term4 = substitute_eps (term4)   # cdb (term4.401,term4)
term5 = substitute_eps (term5)   # cdb (term5.401,term5)

term2 = substitute_y (term2)
term3 = substitute_y (term3)
term4 = substitute_y (term4)
term5 = substitute_y (term5)

term2 = substitute_gam (term2)
term3 = substitute_gam (term3)
term4 = substitute_gam (term4)
term5 = substitute_gam (term5)

term2 = tidy (term2)   # cdb (term2.501,term2)
term3 = tidy (term3)   # cdb (term3.501,term3)
term4 = tidy (term4)   # cdb (term4.501,term4)
```

```
term5 = tidy (term5)   # cdb (term5.501,term5)
```

$$\texttt{term2.401} := Gam22^a{}_{bc}y40^b y40^c + Gam23^a{}_{bc}y40^b y40^c + 2Gam22^a{}_{bc}y40^b y42^c + Gam24^a{}_{bc}y40^b y40^c$$
$$+ 2Gam22^a{}_{bc}y40^b y43^c + 2Gam23^a{}_{bc}y40^b y42^c + Gam25^a{}_{bc}y40^b y40^c$$

$$\texttt{term3.401} := Gam33^a{}_{bcd}y30^b y30^c y30^d + Gam34^a{}_{bcd}y30^b y30^c y30^d + 3Gam33^a{}_{bcd}y30^b y30^c y32^d + Gam35^a{}_{bcd}y30^b y30^c y30^d$$

$$\texttt{term4.401} := Gam44^a{}_{bcde}y20^b y20^c y20^d y20^e + Gam45^a{}_{bcde}y20^b y20^c y20^d y20^e$$

$$\texttt{term5.401} := Gam55^a{}_{bcdef}y00^b y00^c y00^d y00^e y00^f$$

$$
\begin{aligned}
\texttt{term2.501} := &-\frac{2}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{3}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} + \frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} \\
&- \frac{2}{9}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{2}{9}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{8}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{befh} R_{cgdi} \\
&+ \frac{4}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{difg} + \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dgfi} - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{eb} R_{cfdg} \\
&- \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{10}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bc} R_{defg} - \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dfgi} \\
&+ \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bg} R_{cedf} - \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\
&- \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + \frac{1}{36}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} + \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\
&+ \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} - \frac{1}{36}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\
&- \frac{1}{18}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} + \frac{1}{18}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} \\
&+ \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} - \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} \\
&+ \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} - \frac{4}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} \\
&- \frac{1}{30}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} - \frac{4}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + \frac{1}{36}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\
&+ \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} \\
&+ \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{fbc} R_{dgeh} \\
&- \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bfc} R_{dgeh} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcf} R_{dgeh} - \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcd} R_{efgh} \\
&- \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} - \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} - \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} \\
&+ \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{hbc} R_{dfeg} + \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bhc} R_{dfeg} + \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bch} R_{dfeg}
\end{aligned}
$$

$$\texttt{term3.501} := -\frac{1}{2}x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef} - \frac{8}{15}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cifg} - \frac{2}{15}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cgfi}$$
$$+\frac{2}{15}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdch}R_{egfi} - \frac{1}{10}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{de}R_{bfcg} - \frac{3}{20}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{db}R_{cefg}$$
$$-\frac{3}{20}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{bd}R_{cefg} + \frac{2}{5}x^bx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bdeh}R_{cfgi} + \frac{1}{40}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{gd}R_{becf}$$
$$+\frac{1}{40}x^bx^cDx^dDx^eDx^fg^{ag}\nabla_{dg}R_{becf} - \frac{1}{6}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + \frac{1}{6}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh}$$
$$+\frac{1}{6}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_jR_{cfgh} - \frac{8}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_gR_{chdj}$$
$$-\frac{4}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - \frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - \frac{1}{10}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj}$$
$$-\frac{1}{90}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} - \frac{11}{90}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - \frac{4}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj}$$
$$-\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj} + \frac{1}{12}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} + \frac{1}{36}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg}$$
$$+\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + \frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + \frac{2}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj}$$
$$+\frac{1}{15}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} - \frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{efb}R_{cgdh} - \frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ebf}R_{cgdh}$$
$$-\frac{1}{30}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ebc}R_{dfgh} - \frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bef}R_{cgdh} - \frac{1}{30}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bec}R_{dfgh}$$
$$-\frac{1}{30}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bce}R_{dfgh} + \frac{4}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + \frac{1}{5}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj}$$
$$+\frac{4}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} - \frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} + \frac{1}{5}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_cR_{dghj}$$
$$-\frac{1}{45}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{beci}\nabla_fR_{dghj} + \frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{heb}R_{cfdg} + \frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{hbe}R_{cfdg}$$
$$+\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ehb}R_{cfdg} + \frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{bhe}R_{cfdg} + \frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{ebh}R_{cfdg}$$
$$+\frac{1}{180}x^bx^cx^dDx^eDx^fDx^gg^{ah}\nabla_{beh}R_{cfdg} - \frac{1}{9}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{befi}\nabla_jR_{cgdh} - \frac{1}{18}x^bx^cx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}$$

$$\texttt{term4.501} := -\frac{8}{15}x^bDx^cDx^dDx^eDx^fg^{ag}g^{hi}R_{bcdh}R_{egfi} - \frac{2}{5}x^bDx^cDx^dDx^eDx^fg^{ag}\nabla_{cd}R_{befg} - \frac{32}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh}$$
$$- \frac{1}{5}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - \frac{4}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj}$$
$$- \frac{2}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} - \frac{22}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj}$$
$$- \frac{1}{5}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - \frac{4}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj}$$
$$+ \frac{1}{9}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg} + \frac{8}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} - \frac{1}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{def}R_{bgch}$$
$$- \frac{4}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{deb}R_{cfgh} - \frac{4}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{dbe}R_{cfgh} - \frac{4}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{bde}R_{cfgh}$$
$$+ \frac{13}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} + \frac{1}{15}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj}$$
$$+ \frac{23}{45}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + \frac{1}{90}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{hde}R_{bfcg}$$
$$+ \frac{1}{90}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{dhe}R_{bfcg} + \frac{1}{90}x^bx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{deh}R_{bfcg} - \frac{4}{9}x^bx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bdei}\nabla_jR_{cfgh}$$

$$\texttt{term5.501} := -x^bDx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{bcdi}\nabla_eR_{fhgj} - x^bDx^cDx^dDx^eDx^fDx^gg^{ah}g^{ij}R_{chdi}\nabla_eR_{bfgj} - \frac{1}{3}x^bDx^cDx^dDx^eDx^fDx^gg^{ah}\nabla_{cde}R_{bfgh}$$

```
# Check:
#    x^{a} at s=1 should equal x^{a} + Dx^{a}
#    but x^{a}(s) = x^{a} + s y^{a} - \sum (1/n!) @(termn) s^n
#    thus foo should equal Dx^{a} and it does (yeah)

foo := @(y5)
     -   (1/2) @(term2)
     -   (1/6) @(term3)
     -  (1/24) @(term4)
     - (1/120) @(term5).

distribute          (foo)
obj = product_sort  (foo)
rename_dummies      (foo)
canonicalise        (foo)     # cdb (foo.001,foo)

term2 :=   (1/2) @(term2).   # cdb(term2.502,term2)
term3 :=   (1/6) @(term3).   # cdb(term3.502,term3)
term4 :=  (1/24) @(term4).   # cdb(term4.502,term4)
term5 := (1/120) @(term5).   # cdb(term5.502,term5)

end_stage_2b = time.time()
```

$$\texttt{foo.001} := Dx^{a}$$

$$\texttt{y2.301} := Dx^{a} - \frac{1}{3}x^{b}Dx^{c}Dx^{d}g^{ae}R_{bcde}$$

$$\texttt{y3.301} := Dx^{a} - \frac{1}{3}x^{b}Dx^{c}Dx^{d}g^{ae}R_{bcde} - \frac{1}{12}x^{b}x^{c}Dx^{d}Dx^{e}g^{af}\nabla_{d}R_{becf} - \frac{1}{6}x^{b}x^{c}Dx^{d}Dx^{e}g^{af}\nabla_{b}R_{cdef} + \frac{1}{24}x^{b}x^{c}Dx^{d}Dx^{e}g^{af}\nabla_{f}R_{bdce} - \frac{1}{12}x^{b}Dx^{c}Dx^{d}Dx^{e}g^{af}\nabla_{c}R_{bdef}$$

$$
\begin{aligned}
\texttt{y4.301} := {} & Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} + \frac{1}{24}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} \\
& - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} - \frac{2}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cifg} \\
& - \frac{4}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{befh} R_{cgdi} + \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dgfi} \\
& - \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{eb} R_{cfdg} - \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bc} R_{defg} \\
& - \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dfgi} + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bg} R_{cedf} \\
& - \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cgfi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{60}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{de} R_{bfcg} \\
& - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{db} R_{cefg} - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{gd} R_{becf} \\
& + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{dg} R_{becf} - \frac{1}{45}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{1}{60}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} \nabla_{cd} R_{befg}
\end{aligned}
$$

Stage 3: Reformatting and output

```
def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    distribute (tmp)

    substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
    substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
    substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d} -> R_{a b c d}$)

    return foo

def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
```

```
    rename_dummies (bah)
    canonicalise   (bah)
    substitute     (bah,$Dx^{b}->zzz^{b}$)
    factor_out     (bah,$x^{a?},zzz^{b?}$)
    substitute     (bah,$zzz^{b}->Dx^{b}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute  (bah)
    substitute  (bah,$Dx^{b}->zzz^{b}$)
    factor_out  (bah,$x^{a?},zzz^{b?}$)
    substitute  (bah,$zzz^{b}->Dx^{b}$)
    return bah

beg_stage_3 = time.time()

Rterm22 = get_Rterm (term2,2)                           # cdb(Rterm22.101,Rterm22)
Rterm23 = get_Rterm (term2,3)                           # cdb(Rterm23.101,Rterm23)
Rterm24 = get_Rterm (term2,4)                           # cdb(Rterm24.101,Rterm24)
Rterm25 = get_Rterm (term2,5)                           # cdb(Rterm25.101,Rterm25)

Rterm32 = get_Rterm (term3,2)                           # cdb(Rterm32.101,Rterm32)  # zero
Rterm33 = get_Rterm (term3,3)                           # cdb(Rterm33.101,Rterm33)
Rterm34 = get_Rterm (term3,4)                           # cdb(Rterm34.101,Rterm34)
Rterm35 = get_Rterm (term3,5)                           # cdb(Rterm35.101,Rterm35)

Rterm42 = get_Rterm (term4,2)                           # cdb(Rterm42.101,Rterm42)  # zero
Rterm43 = get_Rterm (term4,3)                           # cdb(Rterm43.101,Rterm43)  # zero
Rterm44 = get_Rterm (term4,4)                           # cdb(Rterm44.101,Rterm44)
Rterm45 = get_Rterm (term4,5)                           # cdb(Rterm45.101,Rterm45)

Rterm52 = get_Rterm (term5,2)                           # cdb(Rterm52.101,Rterm52)  # zero
Rterm53 = get_Rterm (term5,3)                           # cdb(Rterm53.101,Rterm53)  # zero
Rterm54 = get_Rterm (term5,4)                           # cdb(Rterm54.101,Rterm54)  # zero
Rterm55 = get_Rterm (term5,5)                           # cdb(Rterm55.101,Rterm55)
```

```
Rterm22 = rescale ( reformat (Rterm22,   -3),    -3 )   # cdb(Rterm22.102,Rterm22)
Rterm23 = rescale ( reformat (Rterm23,  -24),   -24 )   # cdb(Rterm23.102,Rterm23)
Rterm24 = rescale ( reformat (Rterm24, -720),  -720 )   # cdb(Rterm24.102,Rterm24)
Rterm25 = rescale ( reformat (Rterm25, -360),  -360 )   # cdb(Rterm25.102,Rterm25)

Rterm33 = rescale ( reformat (Rterm33,  -12),   -12 )   # cdb(Rterm33.102,Rterm33)
Rterm34 = rescale ( reformat (Rterm34, -720),  -720 )   # cdb(Rterm34.102,Rterm34)
Rterm35 = rescale ( reformat (Rterm35,-1080), -1080 )   # cdb(Rterm35.102,Rterm35)

Rterm44 = rescale ( reformat (Rterm44, -180),  -180 )   # cdb(Rterm44.102,Rterm44)
Rterm45 = rescale ( reformat (Rterm45,-2160), -2160 )   # cdb(Rterm45.102,Rterm45)

Rterm55 = rescale ( reformat (Rterm55, -360),  -360 )   # cdb(Rterm55.102,Rterm55)
```

```
# ----------------------------------------------------------------
# bvp to terms linear in R

tmp2 := -(1/3) @(Rterm22).

bvp2 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2).                                # cdb(bvp.601,bvp2)

cdblib.put ('bvp2',bvp2,'geodesic-bvp.json')
cdblib.put ('bvp22',tmp2,'geodesic-bvp.json')

y2 := Dx^{a} + @(tmp2).                                 # cdb(y2.600,y2)

# ----------------------------------------------------------------
# bvp to terms linear in dR

tmp2 :=  -(1/3) @(Rterm22) - (1/24) @(Rterm23).
tmp3 := -(1/12) @(Rterm33).

bvp3 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2)
     + (s-s**3) @(tmp3).                                # cdb(bvp.602,bvp3)

cdblib.put ('bvp3',bvp3,'geodesic-bvp.json')
cdblib.put ('bvp32',tmp2,'geodesic-bvp.json')
cdblib.put ('bvp33',tmp3,'geodesic-bvp.json')

y3 := Dx^{a} + @(tmp2) + @(tmp3).                       # cdb(y3.600,y3)

# ----------------------------------------------------------------
# bvp to terms linear in d^2 R

tmp2 :=   -(1/3) @(Rterm22) -  (1/24) @(Rterm23) - (1/720) @(Rterm24).
tmp3 :=  -(1/12) @(Rterm33) - (1/720) @(Rterm34).
tmp4 := -(1/180) @(Rterm44).
```

```
bvp4 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2)
     + (s-s**3) @(tmp3)
     + (s-s**4) @(tmp4).                                      # cdb(bvp.603,bvp4)

cdblib.put ('bvp4',bvp4,'geodesic-bvp.json')
cdblib.put ('bvp42',tmp2,'geodesic-bvp.json')
cdblib.put ('bvp43',tmp3,'geodesic-bvp.json')
cdblib.put ('bvp44',tmp4,'geodesic-bvp.json')

y4 := Dx^{a} + @(tmp2) + @(tmp3) + @(tmp4).                   # cdb(y4.600,y4)

# ----------------------------------------------------------------
# bvp to terms linear in d^3 R

tmp2 := @(term2).
tmp3 := @(term3).
tmp4 := @(term4).
tmp5 := @(term5).

bvp5 := x^{a}
     + s Dx^{a}
     + (s-s**2) @(tmp2)
     + (s-s**3) @(tmp3)
     + (s-s**4) @(tmp4)
     + (s-s**5) @(tmp5).                                      # cdb(bvp.604,bvp5)

cdblib.put ('bvp5',bvp5,'geodesic-bvp.json')
cdblib.put ('bvp52',term2,'geodesic-bvp.json')
cdblib.put ('bvp53',term3,'geodesic-bvp.json')
cdblib.put ('bvp54',term4,'geodesic-bvp.json')
cdblib.put ('bvp55',term5,'geodesic-bvp.json')

y5 := Dx^{a} + @(tmp2) + @(tmp3) + @(tmp4) + @(tmp5).         # cdb(y5.600,y5)

end_stage_3 = time.time()
```

```python
# cdbBeg (timing)
print ("Stage 1:  {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2a: {:7.1f} secs\\hfill\\break".format(end_stage_2a-beg_stage_2a))
print ("Stage 2b: {:7.1f} secs\\hfill\\break".format(end_stage_2b-beg_stage_2b))
print ("Stage 3:  {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
# cdbEnd (timing)
```

Non-unit tangent vectors at P

These are not unit vectors, their length is the geodesic distance from P to Q

$$\texttt{y2.600} := Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\texttt{y3.600} := Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right) - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef}$$

$$\begin{aligned}\texttt{y4.600} := Dx^a &- \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\\
&- \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right)\\
&- \frac{1}{720}x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg}\right.\\
&\qquad\left. + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right) - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef}\\
&- \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} + 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg}\right.\\
&\left. + 18g^{ag}\nabla_{bd}R_{cefg} - 48g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\right) - \frac{1}{180}x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right)\end{aligned}$$

Geodesic boundary value problem to terms linear in R

$$x^a(s) = x^a + sDx^a - \frac{1}{3}\left(s - s^2\right) x^b Dx^c Dx^d g^{ae} R_{bcde} + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2) x_2^a + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x_2^a = \overset{2}{x}{}_2^a + \mathcal{O}\left(\epsilon^3\right)$$

$$-3\overset{2}{x}{}_2^a = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

Geodesic boundary value problem to terms linear in ∇R

$$
\begin{aligned}
x^a(s) = x^a + sDx^a + \left(s - s^2\right)\left(-\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\right)\\
- \frac{1}{12}\left(s - s^3\right) x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} + \mathcal{O}\left(s^4, \epsilon^4\right)
\end{aligned}
$$

$$
x^a(s) = x^a + sDx^a + (s - s^2)x^a_2 + (s - s^3)x^a_3 + \mathcal{O}\left(s^4, \epsilon^4\right)
$$

$$
\begin{aligned}
x^a_2 &= \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \mathcal{O}\left(\epsilon^4\right)\\
-3\overset{2}{x}{}^a_2 &= x^b Dx^c Dx^d g^{ae} R_{bcde}\\
-24\overset{3}{x}{}^a_2 &= x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)
\end{aligned}
$$

$$
\begin{aligned}
x^a_3 &= \overset{3}{x}{}^a_3 + \mathcal{O}\left(\epsilon^4\right)\\
-12\overset{3}{x}{}^a_3 &= x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef}
\end{aligned}
$$

Geodesic boundary value problem to terms linear in $\nabla^2 R$

$$
\begin{aligned}
x^a(s) = x^a + sDx^a + \left(s - s^2\right)\Bigl(&-\frac{1}{3}x^bDx^cDx^dg^{ae}R_{bcde} - \frac{1}{24}x^bx^cDx^dDx^e\left(2g^{af}\nabla_dR_{becf} + 4g^{af}\nabla_bR_{cdef} - g^{af}\nabla_fR_{bdce}\right)\\
&- \frac{1}{720}x^bx^cDx^dDx^eDx^f\left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) - \frac{1}{720}x^bx^cx^dDx^eDx^f\left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right.\\
&\left.- 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right)\Bigr)\\
&+ \left(s - s^3\right)\Bigl(-\frac{1}{12}x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef} - \frac{1}{720}x^bx^cDx^dDx^eDx^f\left(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi}\right.\\
&\left.+ 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} - 48g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\right)\Bigr)\\
&- \frac{1}{180}\left(s - s^4\right)x^bDx^cDx^dDx^eDx^f\left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right) + \mathcal{O}\left(s^5, \epsilon^5\right)
\end{aligned}
$$

$$
x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + (s - s^4)x_4^a + \mathcal{O}\left(s^5, \epsilon^5\right)
$$

$$
\begin{aligned}
x_2^a &= \overset{2}{x}{}_2^a + \overset{3}{x}{}_2^a + \overset{4}{x}{}_2^a + \mathcal{O}\left(\epsilon^5\right)\\
-3\overset{2}{x}{}_2^a &= x^bDx^cDx^dg^{ae}R_{bcde}\\
-24\overset{3}{x}{}_2^a &= x^bx^cDx^dDx^e\left(2g^{af}\nabla_dR_{becf} + 4g^{af}\nabla_bR_{cdef} - g^{af}\nabla_fR_{bdce}\right)\\
-720\overset{4}{x}{}_2^a &= x^bx^cDx^dDx^eDx^f\left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^bx^cx^dDx^eDx^f\left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right.\\
&\quad\left.- 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_3^a &= \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \mathcal{O}\left(\epsilon^5\right)\\
-12\overset{3}{x}{}_3^a &= x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef}\\
-720\overset{4}{x}{}_3^a &= x^bx^cDx^dDx^eDx^f\left(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} + 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg}\right.\\
&\quad\left.- 48g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\right)
\end{aligned}
$$

$$x_4^a = \overset{4}{x}{}_4^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right)$$

Geodesic boundary value problem to terms linear in $\nabla^3 R$

The geodesic that connects the points with RNC coordinates x^a and $x^a + Dx^a$ is described, for $0 \le s \le 1$, by

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + (s - s^4)x_4^a + (s - s^5)x_5^a + \mathcal{O}\left(s^6, \epsilon^6\right)$$

$$x_2^a = \overset{2}{x}{}_2^a + \overset{3}{x}{}_2^a + \overset{4}{x}{}_2^a + \overset{5}{x}{}_2^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-3\overset{2}{x}{}_2^a = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$-24\overset{3}{x}{}_2^a = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)$$

$$\begin{aligned}-720\overset{4}{x}{}_2^a = {}& x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right.\\ &\left. - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right)\end{aligned}$$

$$\begin{aligned}-360\overset{5}{x}{}_2^a = {}& x^b x^c x^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 20g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj}\right.\\ & - 20g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 20g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj}\\ &\left. - 20g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 10g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj}\right) + x^b x^c Dx^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 10g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj}\right)\\ & + x^b x^c x^d x^e Dx^f Dx^g \left(16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg}\right.\\ & - 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 2g^{ah}\nabla_{fbc}R_{dgeh}\\ & + 2g^{ah}\nabla_{bfc}R_{dgeh} + 2g^{ah}\nabla_{bcf}R_{dgeh} + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj}\\ &\left. - g^{ah}\nabla_{hbc}R_{dfeg} - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\right)\end{aligned}$$

$$x_3^a = \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \overset{5}{x}{}_3^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-12\overset{3}{x}{}_3^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned}-720\overset{4}{x}{}_3^a = x^b x^c Dx^d Dx^e Dx^f \big(&64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg} \\ &- 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\big)\end{aligned}$$

$$\begin{aligned}-1080\overset{5}{x}{}_3^a = {}& x^b x^c Dx^d Dx^e Dx^f Dx^g \left(30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - 30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} - 30 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\right) \\ &+ x^b x^c x^d Dx^e Dx^f Dx^g \big(32 g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 48 g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 12 g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 18 g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\ &+ 2 g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22 g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 48 g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 12 g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} - 15 g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\ &- 5 g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} - 12 g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12 g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8 g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12 g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\ &+ 4 g^{ah} \nabla_{efb} R_{cgdh} + 4 g^{ah} \nabla_{ebf} R_{cgdh} + 6 g^{ah} \nabla_{ebc} R_{dfgh} + 4 g^{ah} \nabla_{bef} R_{cgdh} + 6 g^{ah} \nabla_{bec} R_{dfgh} + 6 g^{ah} \nabla_{bce} R_{dfgh} - 16 g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\ &- 36 g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 16 g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 4 g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 36 g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + 4 g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg} \\ &- g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg} + 20 g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 10 g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}\big)\end{aligned}$$

$$x_4^a = \overset{4}{x}{}_4^a + \overset{5}{x}{}_4^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4 g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3 g^{ag} \nabla_{cd} R_{befg}\right)$$

$$\begin{aligned}-2160\overset{5}{x}{}_4^a = x^b x^c Dx^d Dx^e Dx^f Dx^g \big(&64 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 18 g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24 g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4 g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} \\ &+ 44 g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 18 g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} + 24 g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10 g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 16 g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} \\ &+ 6 g^{ah} \nabla_{def} R_{bgch} + 8 g^{ah} \nabla_{deb} R_{cfgh} + 8 g^{ah} \nabla_{dbe} R_{cfgh} + 8 g^{ah} \nabla_{bde} R_{cfgh} - 26 g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 6 g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} \\ &- 46 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - g^{ah} \nabla_{hde} R_{bfcg} - g^{ah} \nabla_{dhe} R_{bfcg} - g^{ah} \nabla_{deh} R_{bfcg} + 40 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\big)\end{aligned}$$

$$x_5^a = \overset{5}{x}{}_5^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-360\overset{5}{x}{}_5^a = x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(3 g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3 g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh}\right)$$

```
tmp2 := 8 @(Rterm22) + @(Rterm23).
tmp3 := @(Rterm33).

factor_out     (tmp2,$Dx^{a?}$) # cdb(tmp2.001,tmp2)
rename_dummies (tmp2)
factor_out     (tmp2,$Dx^{a?}$) # cdb(tmp2.002,tmp2)

bvp4 := x^{a}
     + \lam Dx^{a}
     - (1/24) (\lam-\lam**2) @(tmp2)
     - (1/12) (\lam-\lam**3) @(tmp3).     # cdb(bvp4,bvp4)

cdblib.create ('geodesic-bvp.export')

# 4th order bvp
cdblib.put ('bvp4',bvp4,'geodesic-bvp.export')

# 6th order bvp terms, scaled
cdblib.put ('bvp622',Rterm22,'geodesic-bvp.export')
cdblib.put ('bvp623',Rterm23,'geodesic-bvp.export')
cdblib.put ('bvp624',Rterm24,'geodesic-bvp.export')
cdblib.put ('bvp625',Rterm25,'geodesic-bvp.export')

cdblib.put ('bvp633',Rterm33,'geodesic-bvp.export')
cdblib.put ('bvp634',Rterm34,'geodesic-bvp.export')
cdblib.put ('bvp635',Rterm35,'geodesic-bvp.export')

cdblib.put ('bvp644',Rterm44,'geodesic-bvp.export')
cdblib.put ('bvp645',Rterm45,'geodesic-bvp.export')

cdblib.put ('bvp655',Rterm55,'geodesic-bvp.export')

checkpoint.append (bvp4)

checkpoint.append (Rterm22)
checkpoint.append (Rterm23)
checkpoint.append (Rterm24)
checkpoint.append (Rterm25)
```

```
checkpoint.append (Rterm33)
checkpoint.append (Rterm34)
checkpoint.append (Rterm35)

checkpoint.append (Rterm44)
checkpoint.append (Rterm45)

checkpoint.append (Rterm55)
```

Timing

Stage 1: 5.7 secs
Stage 2a: 68.9 secs
Stage 2b: 67.8 secs
Stage 3: 13.6 secs

Geodesic IVP

Our game here is to find the solution of

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a(s)/ds = \dot{x}^a$ at $s = 0$.

Algorithm

By successive differentiation of the above equation we can compute

$$\frac{d^nx^a}{ds^n} = -\Gamma^a{}_{\underline{d}_n}\frac{dx^{\underline{d}_n}}{ds}$$

at $s = 0$ for $n = 2, 3, 4, \ldots$. The $\Gamma^a{}_{\underline{d}_n}$ are the *generalised connections.*

We can then construct the Taylor series solution for $x^a(s)$

$$x^a(s) = x^a + s\dot{x}^a - \sum_{k=2}^{\infty}\frac{s^k}{k!}\Gamma^a{}_{\underline{d}_k}\dot{x}^{\underline{d}_k}$$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

import cdblib

sterm21 = cdblib.get ('genGamma01','genGamma.json')
sterm22 = cdblib.get ('genGamma02','genGamma.json')
sterm23 = cdblib.get ('genGamma03','genGamma.json')
sterm24 = cdblib.get ('genGamma04','genGamma.json')

sterm31 = cdblib.get ('genGamma11','genGamma.json')
sterm32 = cdblib.get ('genGamma12','genGamma.json')
sterm33 = cdblib.get ('genGamma13','genGamma.json')

sterm41 = cdblib.get ('genGamma21','genGamma.json')
sterm42 = cdblib.get ('genGamma22','genGamma.json')

sterm51 = cdblib.get ('genGamma31','genGamma.json')

sterm2 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24).  # cdb (sterm2.000,sterm2)
sterm3 := @(sterm31) + @(sterm32) + @(sterm33).               # cdb (sterm3.000,sterm3)
sterm4 := @(sterm41) + @(sterm42).                            # cdb (sterm4.000,sterm4)
sterm5 := @(sterm51).                                         # cdb (sterm5.000,sterm5)

factor_out (sterm2,$A^{a?}$)                                  # cdb (sterm2.001,sterm2)
factor_out (sterm3,$A^{a?}$)                                  # cdb (sterm3.001,sterm3)
factor_out (sterm4,$A^{a?}$)                                  # cdb (sterm4.001,sterm4)
factor_out (sterm5,$A^{a?}$)                                  # cdb (sterm5.001,sterm5)

sterm2 := 360 @(sterm2).
sterm3 := 360 @(sterm3).
sterm4 :=  90 @(sterm4).
sterm5 :=   3 @(sterm5).

substitute (sterm2,$A^{a}->1$)                                # cdb (sterm2.002,sterm2)
substitute (sterm3,$A^{a}->1$)                                # cdb (sterm3.002,sterm3)
substitute (sterm4,$A^{a}->1$)                                # cdb (sterm4.002,sterm4)
```

```
substitute (sterm5,$A^{a}->1$)                                    # cdb (sterm5.002,sterm5)
```

The geodesic ivp

$$x^a(s) = x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A^a_{bc} + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A^a_{bcd} + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A^a_{bcde} + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A^a_{bcdef} + \cdots$$

$$\begin{aligned}
360A^a_{bc} &= 240x^d g^{ae} R_{bdce} + 30x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) + x^d x^e x^f \left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi}\right.\\
&\qquad \left. - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\right)\\
&\quad + 2x^d x^e x^f x^g \left(16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 8g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj}\right.\\
&\qquad - 4g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh}\\
&\qquad + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg}\\
&\qquad \left. + g^{ah}\nabla_{deh}R_{bfcg}\right)
\end{aligned}$$

$$\begin{aligned}
360A^a_{bcd} &= 180x^e g^{af}\nabla_b R_{cedf} + 3x^e x^f \left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg}\right.\\
&\qquad \left. + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right)\\
&\quad + 2x^e x^f x^g \left(32g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} + 2g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj}\right.\\
&\qquad + 22g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg}\\
&\qquad - 12g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh}\\
&\qquad + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} + 16g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj}\\
&\qquad + 16g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg}\\
&\qquad \left. + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20g^{ah}g^{ij}R_{beci}\nabla_j R_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg}\right)
\end{aligned}$$

$$\begin{aligned}
90A^a_{bcde} &= 6x^f \left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) + x^f x^g \left(64g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej}\right.\\
&\qquad + 4g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} + 44g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg}\\
&\qquad - 16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej}\\
&\qquad \left. + 6g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_j R_{dgeh}\right)
\end{aligned}$$

$$3A^a_{bcdef} = x^g \left(3g^{ah}g^{ij}R_{bgci}\nabla_d R_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

```
sterm2short  := @(sterm21) + @(sterm22).            # cdb (sterm2.short.001,sterm2short)
sterm3short  := @(sterm31).                         # cdb (sterm3.short.001,sterm3short)
sterm2shortscaled := 12 @(sterm2short).             # cdb (sterm2.short.scaled.002,sterm2shortscaled)
sterm3shortscaled :=  2 @(sterm3short).             # cdb (sterm3.short.scaled.002,sterm3shortscaled)

substitute (sterm2shortscaled,$A^{a}->1$)           # cdb (sterm2.short.scaled.003,sterm2shortscaled)
substitute (sterm3shortscaled,$A^{a}->1$)           # cdb (sterm3.short.scaled.003,sterm3shortscaled)

cdblib.create ('geodesic-ivp.export')

# 4th order ivp terms scaled
cdblib.put ('ivp42',sterm2shortscaled,'geodesic-ivp.export')
cdblib.put ('ivp43',sterm3shortscaled,'geodesic-ivp.export')

# 6th order ivp terms scaled
cdblib.put ('ivp62',sterm2,'geodesic-ivp.export')
cdblib.put ('ivp63',sterm3,'geodesic-ivp.export')
cdblib.put ('ivp64',sterm4,'geodesic-ivp.export')
cdblib.put ('ivp65',sterm5,'geodesic-ivp.export')

checkpoint.append (sterm2shortscaled)
checkpoint.append (sterm3shortscaled)

checkpoint.append (sterm2)
checkpoint.append (sterm3)
checkpoint.append (sterm4)
checkpoint.append (sterm5)
```

$$\texttt{sterm2.short.001} := \frac{2}{3}A^b A^c x^d g^{ae} R_{bdce} + \frac{1}{12}A^b A^c x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$$

$$\texttt{sterm3.short.001} := \frac{1}{2}A^b A^c A^d x^e g^{af}\nabla_b R_{cedf}$$

$$\texttt{sterm2.short.scaled.002} := 8A^b A^c x^d g^{ae} R_{bdce} + A^b A^c x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$$

$$\texttt{sterm3.short.scaled.002} := A^b A^c A^d x^e g^{af}\nabla_b R_{cedf}$$

`sterm2.short.scaled.003` $:= 8x^d g^{ae} R_{bdce} + x^d x^e \left(2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce}\right)$

`sterm3.short.scaled.003` $:= x^e g^{af} \nabla_b R_{cedf}$

Geodesic arc-length

Give a pair of points P and Q the geodesic arc-length can be computed using

$$L_{PQ} = \int_P^Q \left(g_{ab}(x) \frac{dx^a}{ds} \frac{dx^b}{ds} \right)^{1/2} ds \tag{1}$$

Since the path is a geodesic the integrand is constant and thus

$$L^2_{PQ} = g_{ab}(x) \frac{dx^a}{ds} \frac{dx^b}{ds} \bigg|_P \tag{2}$$

where s is a re-scaled parameter (0 at P and 1 at Q). The point P has RNC coordinates x^a while the point Q has coordinates $x^a + Dx^a$.

The vector dx^a/ds at P is given by the solution of the geodesic boundary value problem. This was found in the previous code (`geodesic-bvp`). That is

$$\frac{dx^b}{ds} \bigg|_P = y^a \tag{3}$$

and thus

$$L^2_{PQ} = g_{ab}(x) y^a y^b \tag{4}$$

It is possible to directly evaluate the right hand side of (4) using the results from the `geodesic-bvp` and `metric` codes. The result would need to be truncated (to an order consistent with the results form those codes). But doing so would be computationaly expensive as at least half of the terms will be thrown away. A better approach is compute just the terms that will survive the truncation. This is done by expanding $g_{ab}(x)$ and y^a as a truncated series in the curvatures and its derivatives.

The $g_{ab}(x)$ and y^a are written in a (truncated) formal power series in the curvature and its derivatives

$$y^a = \overset{0}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a + \overset{5}{y}{}^a + \mathcal{O}\left(\epsilon^6\right) \tag{5}$$

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}\left(\epsilon^6\right) \tag{6}$$

Note that this use of $\overset{i}{y}$ differs from that used in `geodesic-bvp`. Here the index above y^a denotes a particular term in the curvature expansion while in `geodesic-bvp` the index denoted the iteration number (in the fixed point scheme used to solve the BVP for y^a).

Stage 1

The formal curvature expansions are substituted into equation (4), expanded and truncated to retain terms of order $\mathcal{O}(\epsilon^5)$ or less. The expansion to 4th order terms is as follows.

$$L^2_{PQ} = \overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b + 2\overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{2}{y}{}^b + \overset{2}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b + 2\overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{3}{y}{}^b + \overset{3}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b + 2\overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{4}{y}{}^b + \overset{0}{g}_{ab}\overset{2}{y}{}^a\overset{2}{y}{}^b + 2\overset{2}{g}_{ab}\overset{0}{y}{}^a\overset{2}{y}{}^b + \overset{4}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b$$

From `geodesic-bvp` (actually from `rnc2rnc which reformatted the results nicely`) we have

$$\overset{0}{y}{}^a = Dx^a$$

$$\overset{2}{y}{}^a = -\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\overset{3}{y}{}^a = x^b x^c Dx^d Dx^e \left(-\frac{1}{12}g^{af}\nabla_d R_{becf} - \frac{1}{6}g^{af}\nabla_b R_{cdef} + \frac{1}{24}g^{af}\nabla_f R_{bdce}\right) - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef}$$

$$\begin{aligned}\overset{4}{y}{}^a = {} & x^b x^c Dx^d Dx^e Dx^f \left(-\frac{2}{45}g^{ag}g^{hi}R_{bdeh}R_{cfgi} + \frac{1}{45}g^{ag}g^{hi}R_{bdeh}R_{cifg} - \frac{1}{45}g^{ag}g^{hi}R_{bdeh}R_{cgfi} + \frac{1}{45}g^{ag}g^{hi}R_{bdch}R_{egfi} - \frac{1}{60}g^{ag}\nabla_{de}R_{bfcg}\right.\\ & \left. - \frac{1}{40}g^{ag}\nabla_{db}R_{cefg} - \frac{1}{40}g^{ag}\nabla_{bd}R_{cefg} + \frac{1}{240}g^{ag}\nabla_{gd}R_{becf} + \frac{1}{240}g^{ag}\nabla_{dg}R_{becf}\right)\\ & + x^b x^c x^d Dx^e Dx^f \left(-\frac{4}{45}g^{ag}g^{hi}R_{befh}R_{cgdi} + \frac{2}{45}g^{ag}g^{hi}R_{bech}R_{difg} + \frac{1}{45}g^{ag}g^{hi}R_{bech}R_{dgfi} - \frac{1}{40}g^{ag}\nabla_{eb}R_{cfdg} - \frac{1}{40}g^{ag}\nabla_{be}R_{cfdg} - \frac{1}{20}g^{ag}\nabla_{bc}R_{defg}\right.\\ & \left. - \frac{1}{45}g^{ag}g^{hi}R_{bech}R_{dfgi} + \frac{1}{80}g^{ag}\nabla_{gb}R_{cedf} + \frac{1}{80}g^{ag}\nabla_{bg}R_{cedf}\right) + x^b Dx^c Dx^d Dx^e Dx^f \left(-\frac{1}{45}g^{ag}g^{hi}R_{bcdh}R_{egfi} - \frac{1}{60}g^{ag}\nabla_{cd}R_{befg}\right)\end{aligned}$$

and from `metric` we have

$$\overset{0}{g}_{ab} = g_{ab}$$

$$3\overset{2}{g}_{ab} = -x^c x^d R_{acbd}$$

$$6\overset{3}{g}_{ab} = -x^c x^d x^e \nabla_c R_{adbe}$$

$$180\overset{4}{g}_{ab} = x^c x^d x^e x^f \left(8g^{gh}R_{acdg}R_{befh} - 9\nabla_{cd}R_{aebf}\right)$$

Stage 2

The results from the `geodesic-bvp` and `metric` codes are read to provide values for the $\overset{n}{y}{}^a$ and $\overset{m}{g}_{ab}$. These are substituted into the result from Stage 1, et volia, the final answer. To 4th-order terms the result is given by

$$
\begin{aligned}
L^2_{PQ} = {} & g_{ab} Dx^a Dx^b - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \\
& + \frac{1}{360} x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8 g^{gh} R_{acdg} R_{befh} - 6 \nabla_{cd} R_{aebf}\right) + \frac{1}{360} x^a x^b x^c Dx^d Dx^e Dx^f \left(16 g^{gh} R_{adbg} R_{cefh} - 9 \nabla_{da} R_{becf} - 9 \nabla_{ad} R_{becf}\right) \\
& + \frac{1}{360} x^a x^b x^c x^d Dx^e Dx^f \left(16 g^{gh} R_{aebg} R_{cfdh} - 18 \nabla_{ab} R_{cedf}\right) + \frac{1}{1080} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4 g^{hi} R_{adeh} \nabla_f R_{bgci} - 24 g^{hi} R_{adeh} \nabla_b R_{cfgi}\right. \\
& \qquad \left. + 10 g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16 g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4 \nabla_{dea} R_{bfcg} - 4 \nabla_{dae} R_{bfcg} - 4 \nabla_{ade} R_{bfcg}\right) \\
& + \frac{1}{1080} x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18 g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3 \nabla_{cde} R_{afbg}\right) \\
& + \frac{1}{1080} x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24 g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6 \nabla_{eab} R_{cfdg} - 6 \nabla_{aeb} R_{cfdg} - 6 \nabla_{abe} R_{cfdg}\right) \\
& + \frac{1}{1080} x^a x^b x^c x^d x^e Dx^f Dx^g \left(48 g^{hi} R_{afbh} \nabla_c R_{dgei} - 12 \nabla_{abc} R_{dfeg}\right) + \mathcal{O}\left(\epsilon^5\right)
\end{aligned}
$$

Shared properties

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
\Gamma^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
\Gamma^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).

x^{a}::Depends(D{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

g0{#}::LaTeXForm ("\ngab{0}").
g2{#}::LaTeXForm ("\ngab{2}").
g3{#}::LaTeXForm ("\ngab{3}").
g4{#}::LaTeXForm ("\ngab{4}").
```

```
g5{#}::LaTeXForm ("\ngab{5}").

y0{#}::LaTeXForm ("\ny{0}").
y2{#}::LaTeXForm ("\ny{2}").
y3{#}::LaTeXForm ("\ny{3}").
y4{#}::LaTeXForm ("\ny{4}").
y5{#}::LaTeXForm ("\ny{5}").
```

Stage 1: The formal expansion

```
g0_{a b}::Symmetric.
g2_{a b}::Symmetric.
g3_{a b}::Symmetric.
g4_{a b}::Symmetric.
g5_{a b}::Symmetric.

g0_{a b}::Weight(label=num,value=0).
g2_{a b}::Weight(label=num,value=2).
g3_{a b}::Weight(label=num,value=3).
g4_{a b}::Weight(label=num,value=4).
g5_{a b}::Weight(label=num,value=5).

y0^{a}::Weight(label=num,value=0).
y2^{a}::Weight(label=num,value=2).
y3^{a}::Weight(label=num,value=3).
y4^{a}::Weight(label=num,value=4).
y5^{a}::Weight(label=num,value=5).

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                            -> A001^{a}              $)
    substitute (obj,$ x^{a}                            -> A002^{a}              $)
    substitute (obj,$ Dx^{a}                           -> A003^{a}              $)
    substitute (obj,$ g_{a b}                          -> A004_{a b}            $)
    substitute (obj,$ g^{a b}                          -> A005^{a b}            $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}    -> A010_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}      -> A009_{a b c d e f g}  $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}        -> A008_{a b c d e f}    $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}          -> A007_{a b c d e}      $)
    substitute (obj,$ R_{a b c d}                      -> A006_{a b c d}        $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                -> A^{a}                 $)
    substitute (obj,$ A002^{a}                -> x^{a}                 $)
    substitute (obj,$ A003^{a}                -> Dx^{a}                $)
    substitute (obj,$ A004_{a b}              -> g_{a b}               $)
```

```
    substitute (obj,$ A005^{a b}               -> g^{a b}                    $)
    substitute (obj,$ A006_{a b c d}           -> R_{a b c d}                $)
    substitute (obj,$ A007_{a b c d e}         -> \nabla_{e}{R_{a b c d}}    $)
    substitute (obj,$ A008_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}}  $)
    substitute (obj,$ A009_{a b c d e f g}     -> \nabla_{e f g}{R_{a b c d}} $)
    substitute (obj,$ A010_{a b c d e f g h}   -> \nabla_{e f g h}{R_{a b c d}} $)

    return obj

def truncate (obj,n):
    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah = Ex("num = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

# expansions wrt the curvature

defgab := g_{a b} -> g0_{a b} + g2_{a b} + g3_{a b} + g4_{a b} + g5_{a b}.
defy   := y^{a}   -> y0^{a} + y2^{a} + y3^{a} + y4^{a} + y5^{a}.

lsq    := g_{a b} y^{a} y^{b}.

substitute (lsq,defgab)
substitute (lsq,defy)
distribute (lsq)

def tidy (obj):
    foo := @(obj).
    sort_product    (foo)
    rename_dummies  (foo)
    canonicalise    (foo)
    return foo
```

```
lsq0 = tidy ( truncate (lsq,0) )  # cdb (lsq0.002,lsq0)
lsq2 = tidy ( truncate (lsq,2) )  # cdb (lsq2.002,lsq2)
lsq3 = tidy ( truncate (lsq,3) )  # cdb (lsq3.002,lsq3)
lsq4 = tidy ( truncate (lsq,4) )  # cdb (lsq4.002,lsq4)
lsq5 = tidy ( truncate (lsq,5) )  # cdb (lsq5.002,lsq5)

d20 := @(lsq2) - @(lsq0).         # cdb (d20.001,d20)   # check, should contain only O(2) terms
d32 := @(lsq3) - @(lsq2).         # cdb (d32.001,d32)   # check, should contain only O(3) terms
d43 := @(lsq4) - @(lsq3).         # cdb (d43.001,d43)   # check, should contain only O(4) terms
d54 := @(lsq5) - @(lsq4).         # cdb (d54.001,d54)   # check, should contain only O(5) terms

d5 := @(lsq5) - @(lsq).           # cdb (d5.001,d5)
d5  = tidy (d5)                   # cdb (d5.002,d5)  # all higher order terms, should see no O(5) terms
```

$$\texttt{lsq0.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{lsq2.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{lsq3.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{lsq4.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} + \overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\begin{aligned}\texttt{lsq5.002} := {} & \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} + \overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} \\ & + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} + 2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}\end{aligned}$$

$$\texttt{d20.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{d32.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{d43.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} + \overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{d54.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} + 2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\begin{aligned}\texttt{d5.002} := {} & -2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{0}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{0}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{0}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{0}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{0}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{0}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} - 2\overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} \\ & - 2\overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{2}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{2}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{2}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} - 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} \\ & - 2\overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{3}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} \\ & - \overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} - 2\overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{4}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{4}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{4}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{4}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} \\ & - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} - 2\overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{5}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b}\end{aligned}$$

Stage 2: Substution of $\overset{n}{y}{}^{a}$ and $\overset{m}{g}{}_{ab}$

```
import cdblib

g0ab = cdblib.get('g_ab_0','metric.json')
g2ab = cdblib.get('g_ab_2','metric.json')
g3ab = cdblib.get('g_ab_3','metric.json')
g4ab = cdblib.get('g_ab_4','metric.json')
g5ab = cdblib.get('g_ab_5','metric.json')

defg0ab := g0_{a b} -> @(g0ab).
defg2ab := g2_{a b} -> @(g2ab).
defg3ab := g3_{a b} -> @(g3ab).
defg4ab := g4_{a b} -> @(g4ab).
defg5ab := g5_{a b} -> @(g5ab).

y0a = cdblib.get('y50','geodesic-bvp.json')
y2a = cdblib.get('y52','geodesic-bvp.json')
y3a = cdblib.get('y53','geodesic-bvp.json')
y4a = cdblib.get('y54','geodesic-bvp.json')
y5a = cdblib.get('y55','geodesic-bvp.json')

defy0a := y0^{a} -> @(y0a).
defy2a := y2^{a} -> @(y2a).
defy3a := y3^{a} -> @(y3a).
defy4a := y4^{a} -> @(y4a).
defy5a := y5^{a} -> @(y5a).

def substitute_gab_ya (obj):

   foo := @(obj).

   substitute (foo,defg0ab)
   substitute (foo,defg2ab)
   substitute (foo,defg3ab)
   substitute (foo,defg4ab)
   substitute (foo,defg5ab)
```

```
    substitute (foo,defy0a)
    substitute (foo,defy2a)
    substitute (foo,defy3a)
    substitute (foo,defy4a)
    substitute (foo,defy5a)

    distribute     (foo)
    sort_product   (foo)
    rename_dummies (foo)
    canonicalise   (foo)

    substitute     (foo,$g_{a b} g^{c b} -> \delta^{c}_{a}$)
    eliminate_kronecker (foo)
    foo = product_sort  (foo)
    rename_dummies      (foo)
    canonicalise        (foo)

    return foo

def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

     Q_{a b c d}::Weight(label=numR,value=2).
     Q_{a b c d e}::Weight(label=numR,value=3).
     Q_{a b c d e f}::Weight(label=numR,value=4).
     Q_{a b c d e f g}::Weight(label=numR,value=5).

     tmp := @(obj).

     distribute (tmp)

     substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
     substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
     substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
```

```
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d} -> R_{a b c d}$)

    return foo

lsq2 = substitute_gab_ya (lsq2)  # cdb (lsq2.101,lsq2)
lsq3 = substitute_gab_ya (lsq3)  # cdb (lsq3.101,lsq3)
lsq4 = substitute_gab_ya (lsq4)  # cdb (lsq4.101,lsq4)
lsq5 = substitute_gab_ya (lsq5)  # cdb (lsq5.101,lsq5)

lsq50 = get_Rterm (lsq5,0)
lsq52 = get_Rterm (lsq5,2)
lsq53 = get_Rterm (lsq5,3)
lsq54 = get_Rterm (lsq5,4)
lsq55 = get_Rterm (lsq5,5)

cdblib.create ('geodesic-lsq.json')

cdblib.put ('lsq2',lsq2,'geodesic-lsq.json')
cdblib.put ('lsq3',lsq3,'geodesic-lsq.json')
cdblib.put ('lsq4',lsq4,'geodesic-lsq.json')
cdblib.put ('lsq5',lsq5,'geodesic-lsq.json')

cdblib.put ('lsq50',lsq50,'geodesic-lsq.json')
cdblib.put ('lsq52',lsq52,'geodesic-lsq.json')
cdblib.put ('lsq53',lsq53,'geodesic-lsq.json')
cdblib.put ('lsq54',lsq54,'geodesic-lsq.json')
cdblib.put ('lsq55',lsq55,'geodesic-lsq.json')
```

$$\texttt{lsq2.101} := Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd}$$

$$\texttt{lsq3.101} := Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce}$$

$$\begin{aligned}\texttt{lsq4.101} := {} & Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} - \frac{1}{45} x^a x^b Dx^c Dx^d Dx^e Dx^f g^{gh} R_{acdg} R_{befh} \\ & + \frac{2}{45} x^a x^b x^c Dx^d Dx^e Dx^f g^{gh} R_{adbg} R_{cefh} - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{da} R_{becf} - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{ad} R_{becf} \\ & - \frac{1}{60} x^a x^b Dx^c Dx^d Dx^e Dx^f \nabla_{cd} R_{aebf} + \frac{2}{45} x^a x^b x^c x^d Dx^e Dx^f g^{gh} R_{aebg} R_{cfdh} - \frac{1}{20} x^a x^b x^c x^d Dx^e Dx^f \nabla_{ab} R_{cedf}\end{aligned}$$

$$\begin{aligned}\texttt{lsq5.101} := {} & Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \\ & - \frac{1}{45} x^a x^b Dx^c Dx^d Dx^e Dx^f g^{gh} R_{acdg} R_{befh} + \frac{2}{45} x^a x^b x^c Dx^d Dx^e Dx^f g^{gh} R_{adbg} R_{cefh} \\ & - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{da} R_{becf} - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{ad} R_{becf} - \frac{1}{60} x^a x^b Dx^c Dx^d Dx^e Dx^f \nabla_{cd} R_{aebf} \\ & + \frac{2}{45} x^a x^b x^c x^d Dx^e Dx^f g^{gh} R_{aebg} R_{cfdh} - \frac{1}{20} x^a x^b x^c x^d Dx^e Dx^f \nabla_{ab} R_{cedf} - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adeh} \nabla_f R_{bgci} \\ & - \frac{1}{45} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adeh} \nabla_b R_{cfgi} + \frac{1}{108} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adeh} \nabla_i R_{bfcg} \\ & - \frac{1}{60} x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{acdh} \nabla_e R_{bfgi} + \frac{1}{45} x^a x^b x^c x^d Dx^e Dx^f Dx^g g^{hi} R_{aefh} \nabla_b R_{cgdi} \\ & + \frac{1}{45} x^a x^b x^c x^d Dx^e Dx^f Dx^g g^{hi} R_{aebh} \nabla_f R_{cgdi} + \frac{1}{45} x^a x^b x^c x^d Dx^e Dx^f Dx^g g^{hi} R_{aebh} \nabla_c R_{dfgi} - \frac{1}{180} x^a x^b x^c x^d Dx^e Dx^f Dx^g \nabla_{eab} R_{cfdg} \\ & - \frac{1}{180} x^a x^b x^c x^d Dx^e Dx^f Dx^g \nabla_{aeb} R_{cfdg} - \frac{1}{180} x^a x^b x^c x^d Dx^e Dx^f Dx^g \nabla_{abe} R_{cfdg} + \frac{2}{135} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adbh} \nabla_e R_{cfgi} \\ & - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{dea} R_{bfcg} - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{dae} R_{bfcg} - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{ade} R_{bfcg} \\ & - \frac{1}{360} x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \nabla_{cde} R_{afbg} + \frac{2}{45} x^a x^b x^c x^d x^e Dx^f Dx^g g^{hi} R_{afbh} \nabla_c R_{dgei} - \frac{1}{90} x^a x^b x^c x^d x^e Dx^f Dx^g \nabla_{abc} R_{dfeg}\end{aligned}$$

Stage 3: Reformatting

```
def reformat (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute     (bah)
   bah = product_sort (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   substitute     (bah,$Dx^{b}->zzz^{b}$)
   factor_out     (bah,$x^{a?},zzz^{b?}$)
   substitute     (bah,$zzz^{b}->Dx^{b}$)
   ans := @(bah) / @(foo).
   return ans

def rescale (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute  (bah)
   substitute  (bah,$Dx^{b}->zzz^{b}$)
   factor_out  (bah,$x^{a?},zzz^{b?}$)
   substitute  (bah,$zzz^{b}->Dx^{b}$)
   return bah

Rterm0 := @(lsq50).
Rterm2 := @(lsq52).
Rterm3 := @(lsq53).
Rterm4 := @(lsq54).
Rterm5 := @(lsq55).

Rterm0 = reformat (Rterm0,   1)    # cdb(Rterm0.301,Rterm0) # LCB: returns Dx before g, not what I want
Rterm2 = reformat (Rterm2,   3)    # cdb(Rterm2.301,Rterm2)
Rterm3 = reformat (Rterm3,  12)    # cdb(Rterm3.301,Rterm3)
Rterm4 = reformat (Rterm4, 360)    # cdb(Rterm4.301,Rterm4)
Rterm5 = reformat (Rterm5,1080)    # cdb(Rterm5.301,Rterm5)

Rterm0 := g_{a b} Dx^{a} Dx^{b}.   # LCB: fixes the order of terms, g before Dx,
```

```
lsq3 := @(Rterm0) + @(Rterm2).                                       # cdb (lsq4.301,lsq3)
lsq4 := @(Rterm0) + @(Rterm2) + @(Rterm3).                           # cdb (lsq4.301,lsq4)
lsq5 := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4).               # cdb (lsq5.301,lsq5)
lsq6 := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (lsq5.301,lsq6)

lsq  := @(lsq6).                   # cdb (lsq.301,lsq)

scaled0 = rescale (Rterm0,    1)   # cdb (scaled0.301,scaled0) # LCB: returns Dx before g, not what I want
scaled2 = rescale (Rterm2,    3)   # cdb (scaled2.301,scaled2)
scaled3 = rescale (Rterm3,   12)   # cdb (scaled3.301,scaled3)
scaled4 = rescale (Rterm4,  360)   # cdb (scaled4.301,scaled4)
scaled5 = rescale (Rterm5, 1080)   # cdb (scaled5.301,scaled5)

scaled0 := g_{a b} Dx^{a} Dx^{b}.  # cdb (scaled0.301,scaled0) # LCB: fixes the order of terms, g before Dx, good
```

Geodesic arc-length

$$
\begin{aligned}
(\Delta s)^2 = {} & g_{ab}Dx^aDx^b - \frac{1}{3}x^ax^bDx^cDx^dR_{acbd} - \frac{1}{12}x^ax^bDx^cDx^dDx^e\nabla_cR_{adbe} - \frac{1}{6}x^ax^bx^cDx^dDx^e\nabla_aR_{bdce} \\
& + \frac{1}{360}x^ax^bDx^cDx^dDx^eDx^f\left(-8g^{gh}R_{acdg}R_{befh} - 6\nabla_{cd}R_{aebf}\right) + \frac{1}{360}x^ax^bx^cDx^dDx^eDx^f\left(16g^{gh}R_{adbg}R_{cefh} - 9\nabla_{da}R_{becf} - 9\nabla_{ad}R_{becf}\right) \\
& + \frac{1}{360}x^ax^bx^cx^dDx^eDx^f\left(16g^{gh}R_{aebg}R_{cfdh} - 18\nabla_{ab}R_{cedf}\right) + \frac{1}{1080}x^ax^bx^cDx^dDx^eDx^fDx^g\left(-4g^{hi}R_{adeh}\nabla_fR_{bgci} - 24g^{hi}R_{adeh}\nabla_bR_{cfgi}\right. \\
& \qquad \left. + 10g^{hi}R_{adeh}\nabla_iR_{bfcg} + 16g^{hi}R_{adbh}\nabla_eR_{cfgi} - 4\nabla_{dea}R_{bfcg} - 4\nabla_{dae}R_{bfcg} - 4\nabla_{ade}R_{bfcg}\right) \\
& + \frac{1}{1080}x^ax^bDx^cDx^dDx^eDx^fDx^g\left(-18g^{hi}R_{acdh}\nabla_eR_{bfgi} - 3\nabla_{cde}R_{afbg}\right) \\
& + \frac{1}{1080}x^ax^bx^cx^dDx^eDx^fDx^g\left(24g^{hi}R_{aefh}\nabla_bR_{cgdi} + 24g^{hi}R_{aebh}\nabla_fR_{cgdi} + 24g^{hi}R_{aebh}\nabla_cR_{dfgi} - 6\nabla_{eab}R_{cfdg} - 6\nabla_{aeb}R_{cfdg} - 6\nabla_{abe}R_{cfdg}\right) \\
& + \frac{1}{1080}x^ax^bx^cx^dx^eDx^fDx^g\left(48g^{hi}R_{afbh}\nabla_cR_{dgei} - 12\nabla_{abc}R_{dfeg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

Geodesic arc-length curvature expansion

$$(\Delta s)^2 = \overset{0}{\Delta} + \overset{2}{\Delta} + \overset{3}{\Delta} + \overset{4}{\Delta} + \overset{5}{\Delta} + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned}
\overset{0}{\Delta} &= g_{ab} D x^a D x^b \\
3\overset{2}{\Delta} &= -x^a x^b D x^c D x^d R_{acbd} \\
12\overset{3}{\Delta} &= -x^a x^b D x^c D x^d D x^e \nabla_c R_{adbe} - 2 x^a x^b x^c D x^d D x^e \nabla_a R_{bdce} \\
360\overset{4}{\Delta} &= x^a x^b D x^c D x^d D x^e D x^f \left(-8 g^{gh} R_{acdg} R_{befh} - 6 \nabla_{cd} R_{aebf}\right) + x^a x^b x^c D x^d D x^e D x^f \left(16 g^{gh} R_{adbg} R_{cefh} - 9 \nabla_{da} R_{becf} - 9 \nabla_{ad} R_{becf}\right) \\
&\quad + x^a x^b x^c x^d D x^e D x^f \left(16 g^{gh} R_{aebg} R_{cfdh} - 18 \nabla_{ab} R_{cedf}\right) \\
1080\overset{5}{\Delta} &= x^a x^b x^c D x^d D x^e D x^f D x^g \left(-4 g^{hi} R_{adeh} \nabla_f R_{bgci} - 24 g^{hi} R_{adeh} \nabla_b R_{cfgi} + 10 g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16 g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4 \nabla_{dea} R_{bfcg} - 4 \nabla_{dae} R_{bfcg} \right. \\
&\qquad \left. - 4 \nabla_{ade} R_{bfcg}\right) + x^a x^b D x^c D x^d D x^e D x^f D x^g \left(-18 g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3 \nabla_{cde} R_{afbg}\right) \\
&\quad + x^a x^b x^c x^d D x^e D x^f D x^g \left(24 g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6 \nabla_{eab} R_{cfdg} - 6 \nabla_{aeb} R_{cfdg} - 6 \nabla_{abe} R_{cfdg}\right) \\
&\quad + x^a x^b x^c x^d x^e D x^f D x^g \left(48 g^{hi} R_{afbh} \nabla_c R_{dgei} - 12 \nabla_{abc} R_{dfeg}\right)
\end{aligned}$$

```
cdblib.create ('geodesic-lsq.export')

# 3rd to 6th order lsq
cdblib.put ('lsq3',lsq3,'geodesic-lsq.export')
cdblib.put ('lsq4',lsq4,'geodesic-lsq.export')
cdblib.put ('lsq5',lsq5,'geodesic-lsq.export')
cdblib.put ('lsq6',lsq6,'geodesic-lsq.export')

# 6th order lsq terms, scaled
cdblib.put ('lsq60',scaled0,'geodesic-lsq.export')
cdblib.put ('lsq62',scaled2,'geodesic-lsq.export')
cdblib.put ('lsq63',scaled3,'geodesic-lsq.export')
cdblib.put ('lsq64',scaled4,'geodesic-lsq.export')
cdblib.put ('lsq65',scaled5,'geodesic-lsq.export')

checkpoint.append (lsq4)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```

Geodesic mid-point for arc-length

This code uses the results of `geodesic-lsq` and `metric` to show that the 2nd and 3rd order estimates for L^2_{PQ} can be recovered using a mid-point estimate. For the 3rd order estimate we have

$$
\begin{aligned}
g_{ab}(x) &= g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \mathcal{O}\left(\epsilon^4\right)\\
L^2_{PQ} &= g_{ab}Dx^a Dx^b - \frac{1}{3}x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12}x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6}x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} + \mathcal{O}\left(\epsilon^4\right)
\end{aligned}
$$

The code below verifies that

$$L^2_{PQ} = g_{ab}(\bar{x})Dx^a Dx^b + \mathcal{O}\left(\epsilon^4\right)$$

where $\bar{x}$ is the *coordinate* midpoint of the geodesic

$$\bar{x}^a = \frac{1}{2}\left(x^a_P + x^a_Q\right)$$

This result holds true only for the 2nd and 3rd order estimates. Note that the *coordinate* midpoint is not the *geometric* midpoint of the geodesic.

It might be interesting to see if the higher order estimates could be recovered by sampling the metric at points other than the mid point.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.

import cdblib

gab = cdblib.get('g_ab','metric.json')

lsq2 = cdblib.get('lsq2','geodesic-lsq.json')
lsq3 = cdblib.get('lsq3','geodesic-lsq.json')
lsq4 = cdblib.get('lsq4','geodesic-lsq.json')
lsq5 = cdblib.get('lsq5','geodesic-lsq.json')

substitute (gab,$x^{a}->(p^{a}+q^{a})/2$)   # evaluate rnc gab at mid-point
distribute (gab)

defgab := g_{a b} -> @(gab).

mid := g_{a b} (q^{a}-p^{a}) (q^{b}-p^{b}).

substitute     (mid,defgab)
distribute     (mid)
sort_product   (mid)
rename_dummies (mid)
canonicalise   (mid)

tst2 := @(lsq2) - @(mid).                         # cdb (tst2.201,tst2)
tst3 := @(lsq3) - @(mid).                         # cdb (tst3.201,tst3)
tst4 := @(lsq4) - @(mid).                         # cdb (tst4.201,tst4)
tst5 := @(lsq5) - @(mid).                         # cdb (tst5.201,tst5)

substitute     (tst2,$Dx^{a} -> q^{a}-p^{a}$)
substitute     (tst2,$x^{a} -> p^{a}$)
```

```
distribute     (tst2)
sort_product   (tst2)
rename_dummies (tst2)
canonicalise   (tst2)                                  # cdb (tst2.202,tst2)

substitute     (tst3,$Dx^{a} -> q^{a}-p^{a}$)
substitute     (tst3,$x^{a} -> p^{a}$)
distribute     (tst3)
sort_product   (tst3)
rename_dummies (tst3)
canonicalise   (tst3)                                  # cdb (tst3.202,tst3)

substitute     (tst4,$Dx^{a} -> q^{a}-p^{a}$)
substitute     (tst4,$x^{a} -> p^{a}$)
distribute     (tst4)
sort_product   (tst4)
rename_dummies (tst4)
canonicalise   (tst4)                                  # cdb (tst4.202,tst4)

substitute     (tst5,$Dx^{a} -> q^{a}-p^{a}$)
substitute     (tst5,$x^{a} -> p^{a}$)
distribute     (tst5)
sort_product   (tst5)
rename_dummies (tst5)
canonicalise   (tst5)                                  # cdb (tst5.202,tst5)
```

Reformatting

```
def truncateR (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
    substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
    substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(tmp).
       bah = Ex("numR = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    substitute (ans, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (ans, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
    substitute (ans, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (ans, $Q_{a b c d} -> R_{a b c d}$)

    return ans

tst2 = truncateR (tst2,2)  # cdb (tst2.301,tst2)
tst3 = truncateR (tst3,3)  # cdb (tst3.301,tst3)
tst4 = truncateR (tst4,4)  # cdb (tst4.301,tst4)
```

```
tst5 = truncateR (tst5,5)  # cdb (tst5.301,tst5)
```

Errors is mid-point estimates for L^2_{PQ}

$$\left(L^2_{PQ} - g_{ab}(\bar{x})Dx^aDx^b\right)_2 = 0$$

$$\left(L^2_{PQ} - g_{ab}(\bar{x})Dx^aDx^b\right)_3 = 0$$

$$\begin{aligned}\left(L^2_{PQ} - g_{ab}(\bar{x})Dx^aDx^b\right)_4 = &-\frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^cp^gq^bq^dq^fq^h + \frac{1}{15}R_{abcd}R_{efgh}g^{ae}p^bp^cp^gq^dq^fq^h - \frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^bp^cp^fp^gq^dq^h\\
&+\frac{1}{240}\nabla_{ab}R_{cdef}p^bp^cp^eq^aq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^bp^cp^eq^dq^f + \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^cp^eq^bq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^cp^eq^aq^bq^dq^f\end{aligned}$$

$$\begin{aligned}\left(L^2_{PQ} - g_{ab}(\bar{x})Dx^aDx^b\right)_5 = &-\frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^cp^gq^bq^dq^fq^h + \frac{1}{15}R_{abcd}R_{efgh}g^{ae}p^bp^cp^gq^dq^fq^h - \frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^bp^cp^fp^gq^dq^h\\
&+\frac{1}{240}\nabla_{ab}R_{cdef}p^bp^cp^eq^aq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^bp^cp^eq^dq^f + \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^cp^eq^bq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^cp^eq^aq^bq^dq^f\\
&+\frac{1}{135}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^gp^hq^bq^dq^eq^i + \frac{7}{270}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^ep^gp^hq^bq^dq^i - \frac{1}{90}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^gp^hq^dq^eq^i\\
&-\frac{1}{45}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^ep^gp^hq^dq^i - \frac{1}{90}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^ep^hq^bq^dq^gq^i + \frac{1}{135}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^ep^hq^dq^gq^i\\
&-\frac{1}{108}R_{abcd}\nabla_eR_{fghi}g^{ae}p^cp^fp^hq^bq^dq^gq^i + \frac{1}{108}R_{abcd}\nabla_eR_{fghi}g^{ae}p^bp^cp^fp^hq^dq^gq^i - \frac{1}{45}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^hq^bq^dq^eq^gq^i\\
&+\frac{7}{270}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^hq^dq^eq^gq^i + \frac{1}{2160}\nabla_{abc}R_{defg}p^bp^cp^dp^fq^aq^eq^g - \frac{1}{720}\nabla_{abc}R_{defg}p^ap^bp^cp^dp^fq^eq^g\\
&+\frac{1}{2160}\nabla_{abc}R_{defg}p^ap^cp^dp^fq^bq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^bp^dp^fq^cq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^cp^dp^fq^aq^bq^eq^g\\
&+\frac{1}{2160}\nabla_{abc}R_{defg}p^bp^dp^fq^aq^cq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^dp^fq^bq^cq^eq^g - \frac{1}{720}\nabla_{abc}R_{defg}p^dp^fq^aq^bq^cq^eq^g\end{aligned}$$

Converting from generic to rnc coordinates

The following is based on the approach used in the `geodesic-bvp.tex` code. The main difference here is that this time we will *not* be assuming that the coordinates are in Riemann normal form. This will be apparent in the expression for the generalised connections – they will be expressed in terms of the partial derivatives of the connection rather the covariant derivatives of the Riemann tensor. There will also be a change in the way the Taylor series are developed. In this case the expansion parameter ϵ will be associated with the connection and its derivatives rather than the Riemann tensor. We will use

$$\Gamma^a{}_{bc} = \mathcal{O}\left(\epsilon\right) \ , \qquad \Gamma^a{}_{bc,d} = \mathcal{O}\left(\epsilon^2\right) \ , \qquad \Gamma^a{}_{bc,de} = \mathcal{O}\left(\epsilon^3\right) \ , \qquad \text{etc.}$$

The generalised connections are defined recursively by

$$\Gamma^a{}_{b\underline{c}d} = \Gamma^a{}_{(b\underline{c},d)} - (n+1)\Gamma^a{}_{p(\underline{c}}\Gamma^p{}_{bd)} \tag{1}$$

where $\underline{c}$ contains $n > 0$ indices. It is easy to see from this equation that the generalised connections will behave much the same as the connection, that is

$$\Gamma^a{}_{bc} = \mathcal{O}\left(\epsilon\right) \ , \qquad \Gamma^a{}_{bcd} = \mathcal{O}\left(\epsilon^2\right) \ , \qquad \Gamma^a{}_{bcde} = \mathcal{O}\left(\epsilon^3\right) \ , \qquad \text{etc.}$$

This allows us to represent each generalised connection by a single expression (typically `GamNN`).

The situation is slighly different in `geodesic-bvp.tex`. In that code the connection and the generalised connection are expanded as a series in the Riemann tensor and its derivatives. Thus each connection is written in the form

$$\bar{\Gamma}^a{}_{\underline{c}_n} = \overset{0}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} + \overset{1}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} + \overset{2}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} + \cdots + \overset{m}{\bar{\Gamma}}{}^a{}_{\underline{c}_n} \tag{2}$$

where $\underline{c}_n$ denotes a set of indices such as $c_1 c_2 c_3 \ldots c_n$. The terms of the RHS are each of a different weight in ϵ.

Stage 1: The generalised connections

The generalised connections $\Gamma^a{}_{\underline{c}_n}$ could be computed directly by successive application of equation (1). But a more effiecent method exists and its basis lies in the original definition of the generalised connections. Recall that the generalised connecstions arose when buidling a formal power series solution of the geodesic equation

$$0 = \frac{d^2x^a}{ds^s} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} \tag{3}$$

The key idea was that the coeffcients c_n in the formal power series

$$x^a = c^a_0 + s c^a_1 + s^2 c^a_2 + \cdots \tag{4}$$

could be computed using

$$c^a_n = \frac{1}{n!} \left. \frac{d^n x^a}{ds^n} \right|_{s=0} \tag{5}$$

with the second, third and higher derivatives of x^a found by successive differentiation of the geodesic equation. The generalised connections were introduced as part of this algorithm, leading to

$$c^a_n = - \left. \Gamma^a{}_{\underline{c}_n} A^{\cdot \underline{c}_n} \right|_{s=0} \qquad n = 2, 3, 4 \cdots \tag{6}$$

and

$$\Gamma^a{}_{\underline{c}_{n+1}} A^{\cdot \underline{c}_{n+1}} = \frac{d}{ds} \left(\Gamma^a{}_{\underline{c}_n} A^{\cdot \underline{c}_n} \right) \tag{7}$$

with $d/ds = A^a \partial_a$, $A^a = dx^a/ds$ and $dA^a/ds = -\Gamma^a{}_{bc} A^b A^c$.

The upshot is that computing the $\Gamma^a{}_{\underline{c}_n} A^{\cdot \underline{c}_n}$ requires little more than successive rounds of differentiation (and a few substitutions for the derivaties of A^a).

Note that the coefficients c_0 and c_1 must be determined from the initial conditions. Suppose that $x^a = x_i$ at $s = 0$ then $c_0 = x^a_i$ while $c_1 = A^a$.

The Riemann normal coordinates of the point j (where $s = 1$) are introduced by setting

$$y^a = A^a \tag{8}$$

This leads to

$$x^a_j = x^a_i + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a{}_{\underline{b}_k} y^{\cdot \underline{b}_k} \tag{9}$$

Note that given two points i and j, the y^a would be found as a root of this non-linear equation for y^a.

Stage 2: The fixed point scheme for y^a

This second stage is almost exactly the same as the corresponding stage in `geodesic-bvp`. The difference here is that the generalised connections involve partial derivatives of the connection. In contsrat, the `geodesic-bvp` code is specfic to RNC and thus uses the generalised connections based on covariant derivatives of the Riemann tensor.

We begin this second stage by rewriting the equation (9) in the suggestive form

$$y^a = x^a_j - x^a_i + \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a{}_{\underline{b}_k} y^{\cdot \underline{b}_k}$$

and then use this as the basis of a fixed point iteration scheme.

Start with the first approximation $y^a_1 = x^a_j - x^a_i = \Delta x^a$, then compute the successive approximations

$$\begin{aligned}
y^a_1 &= \Delta x^a \\
y^a_2 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_1 y^c_1 \\
y^a_3 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_2 y^c_2 + \frac{1}{3!}\Gamma^a{}_{bcd}\, y^b_1 y^c_1 y^d_1 \\
y^a_4 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_3 y^c_3 + \frac{1}{3!}\Gamma^a{}_{bcd}\, y^b_2 y^c_2 y^d_2 + \frac{1}{4!}\Gamma^a{}_{bcde}\, y^b_1 y^c_1 y^d_1 y^e_1 \\
y^a_5 &= y^a_1 + \frac{1}{2!}\Gamma^a{}_{bc}\, y^b_4 y^c_4 + \frac{1}{3!}\Gamma^a{}_{bcd}\, y^b_3 y^c_3 y^d_3 + \frac{1}{4!}\Gamma^a{}_{bcde}\, y^b_2 y^c_2 y^d_2 y^e_2 + \frac{1}{5!}\Gamma^a{}_{bcdef}\, y^b_1 y^c_1 y^d_1 y^e_1 y^f_1
\end{aligned}$$

and so on. Not that the $\Gamma^a{}_{bc}$, $\Gamma^a{}_{bcd}$, $\Gamma^a{}_{bcde}$ etc. will all depend on the original coordinates x^a at the initial point (i.e., $P = x^a_i$).

Stage3: Introduce the generalised connections from Stage 1

This is the final stage – it introduces the generalised connecstion after the completion of the fixed point scheme.

The result will be an equation for the y^a in terms of the original coordinates x^a and the connections (and its derivatives) at a chosen point $s = 0$ (aka i).

The y^a define an RNC frame in the neighbourhood of the chosen point i.

Stage 1: The generalised connections

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

A^{a}::Depends(\partial{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).

\Gamma^{a}_{b c}::Depends(\partial{#}).

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
\Gamma^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
\Gamma^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
\Gamma^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

\Gamma^{p}_{a b}::Weight(label=numG,value=1).
\Gamma^{p}_{a b c}::Weight(label=numG,value=2).
\Gamma^{p}_{a b c d}::Weight(label=numG,value=3).
```

```
\Gamma^{p}_{a b c d e}::Weight(label=numG,value=4).
\Gamma^{p}_{a b c d e f}::Weight(label=numG,value=5).

def product_sort (obj):
    substitute (obj,$ A^{a}                   -> A001^{a}                $)
    substitute (obj,$ x^{a}                   -> A002^{a}                $)
    substitute (obj,$ g^{a b}                 -> A003^{a b}              $)
    substitute (obj,$ \Gamma^{p}_{a b}        -> A004^{p}_{a b}          $)
    substitute (obj,$ \Gamma^{p}_{a b c}      -> A005^{p}_{a b c}        $)
    substitute (obj,$ \Gamma^{p}_{a b c d}    -> A006^{p}_{a b c d}      $)
    substitute (obj,$ \Gamma^{p}_{a b c d e}  -> A007^{p}_{a b c d e}    $)
    substitute (obj,$ \Gamma^{p}_{a b c d e f} -> A008^{p}_{a b c d e f}  $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                -> A^{a}                   $)
    substitute (obj,$ A002^{a}                -> x^{a}                   $)
    substitute (obj,$ A003^{a b}              -> g^{a b}                 $)
    substitute (obj,$ A004^{p}_{a b}          -> \Gamma^{p}_{a b}        $)
    substitute (obj,$ A005^{p}_{a b c}        -> \Gamma^{p}_{a b c}      $)
    substitute (obj,$ A006^{p}_{a b c d}      -> \Gamma^{p}_{a b c d}    $)
    substitute (obj,$ A007^{p}_{a b c d e}    -> \Gamma^{p}_{a b c d e}  $)
    substitute (obj,$ A008^{p}_{a b c d e f}  -> \Gamma^{p}_{a b c d e f} $)

    return obj

def truncateGam (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah  = Ex("numG = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

beg_stage_1 = time.time()
```

```
# note that we use A^{a} in place of dx^a/ds

Gamma := \Gamma^{d}_{a b} A^{a} A^{b}.

# the geodesic equation

dAds  := A^{c} \partial_{c}{A^{d}} -> - @(Gamma).

# eq0, eq1, eq2 ... are the the successive derivates of Gamma
# thus they are the generalised gamma's dotted into (multiple copies of) A^{a} = dx^a/ds

# =============================================================
eq0 := @(Gamma).                       # cdb (eq0.000,eq0)

# =============================================================
eq1 := A^{c} \partial_{c}{@(eq0)}.     # cdb (eq1.000,eq1)

distribute       (eq1)                 # cdb (eq1.001,eq1)
unwrap           (eq1)                 # cdb (eq1.002,eq1)
product_rule     (eq1)                 # cdb (eq1.003,eq1)
distribute       (eq1)                 # cdb (eq1.004,eq1)
substitute       (eq1,dAds)            # cdb (eq1.005,eq1)
distribute       (eq1)                 # cdb (eq1.006,eq1)
eq1 = truncateGam (eq1,5)              # cdb (eq1.007,eq1)
sort_product     (eq1)                 # cdb (eq1.008,eq1)
rename_dummies   (eq1)                 # cdb (eq1.009,eq1)
canonicalise     (eq1)                 # cdb (eq1.010,eq1)

# =============================================================
eq2 := A^{c} \partial_{c}{@(eq1)}.     # cdb (eq2.000,eq2)

distribute       (eq2)                 # cdb (eq2.001,eq2)
unwrap           (eq2)                 # cdb (eq2.002,eq2)
product_rule     (eq2)                 # cdb (eq2.003,eq2)
distribute       (eq2)                 # cdb (eq2.004,eq2)
substitute       (eq2,dAds)            # cdb (eq2.005,eq2)
distribute       (eq2)                 # cdb (eq2.006,eq2)
```

```
eq2 = truncateGam (eq2,5)            # cdb (eq2.007,eq2)
sort_product     (eq2)               # cdb (eq2.008,eq2)
rename_dummies   (eq2)               # cdb (eq2.009,eq2)
canonicalise     (eq2)               # cdb (eq2.010,eq2)

# ============================================================
eq3 := A^{c} \partial_{c}{@(eq2)}.   # cdb (eq3.000,eq3)

distribute       (eq3)               # cdb (eq3.001,eq3)
unwrap           (eq3)               # cdb (eq3.002,eq3)
product_rule     (eq3)               # cdb (eq3.003,eq3)
distribute       (eq3)               # cdb (eq3.004,eq3)
substitute       (eq3,dAds)          # cdb (eq3.005,eq3)
distribute       (eq3)               # cdb (eq3.006,eq3)
eq3 = truncateGam (eq3,5)            # cdb (eq3.007,eq3)
sort_product     (eq3)               # cdb (eq3.008,eq3)
rename_dummies   (eq3)               # cdb (eq3.009,eq3)
canonicalise     (eq3)               # cdb (eq3.010,eq3)

# ============================================================
eq4 := A^{c} \partial_{c}{@(eq3)}.   # cdb (eq4.000,eq4)

distribute       (eq4)               # cdb (eq4.001,eq4)
unwrap           (eq4)               # cdb (eq4.002,eq4)
product_rule     (eq4)               # cdb (eq4.003,eq4)
distribute       (eq4)               # cdb (eq4.004,eq4)
substitute       (eq4,dAds)          # cdb (eq4.005,eq4)
distribute       (eq4)               # cdb (eq4.006,eq4)
eq4 = truncateGam (eq4,5)            # cdb (eq4.007,eq4)
sort_product     (eq4)               # cdb (eq4.008,eq4)
rename_dummies   (eq4)               # cdb (eq4.009,eq4)
canonicalise     (eq4)               # cdb (eq4.010,eq4)

end_stage_1 = time.time()
```

$$\texttt{eq0.000} := \Gamma^{d}{}_{ab} A^{a} A^{b}$$

$$\texttt{eq1.000} := A^c\partial_c\left(\Gamma^d{}_{ab}A^aA^b\right)$$

$$\texttt{eq1.001} := A^c\partial_c\left(\Gamma^d{}_{ab}A^aA^b\right)$$

$$\texttt{eq1.002} := A^c\partial_c\left(\Gamma^d{}_{ab}A^aA^b\right)$$

$$\texttt{eq1.003} := A^c\left(\partial_c\Gamma^d{}_{ab}A^aA^b + \Gamma^d{}_{ab}\partial_cA^aA^b + \Gamma^d{}_{ab}A^a\partial_cA^b\right)$$

$$\texttt{eq1.004} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b + A^c\Gamma^d{}_{ab}\partial_cA^aA^b + A^c\Gamma^d{}_{ab}A^a\partial_cA^b$$

$$\texttt{eq1.005} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b - \Gamma^a{}_{ce}A^cA^e\Gamma^d{}_{ab}A^b - \Gamma^b{}_{ec}A^eA^c\Gamma^d{}_{ab}A^a$$

$$\texttt{eq1.006} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b - \Gamma^a{}_{ce}A^cA^e\Gamma^d{}_{ab}A^b - \Gamma^b{}_{ec}A^eA^c\Gamma^d{}_{ab}A^a$$

$$\texttt{eq1.007} := A^c\partial_c\Gamma^d{}_{ab}A^aA^b - \Gamma^a{}_{ce}A^cA^e\Gamma^d{}_{ab}A^b - \Gamma^b{}_{ec}A^eA^c\Gamma^d{}_{ab}A^a$$

$$\texttt{eq1.008} := A^aA^bA^c\partial_c\Gamma^d{}_{ab} - A^bA^cA^e\Gamma^a{}_{ce}\Gamma^d{}_{ab} - A^aA^cA^e\Gamma^b{}_{ec}\Gamma^d{}_{ab}$$

$$\texttt{eq1.009} := A^aA^bA^c\partial_c\Gamma^d{}_{ab} - A^aA^bA^c\Gamma^e{}_{bc}\Gamma^d{}_{ea} - A^aA^bA^c\Gamma^e{}_{cb}\Gamma^d{}_{ae}$$

$$\texttt{eq1.010} := A^aA^bA^c\partial_a\Gamma^d{}_{bc} - 2A^aA^bA^c\Gamma^d{}_{ae}\Gamma^e{}_{bc}$$

$$\texttt{eq2.000} := A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} - 2A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\right)$$

$$\texttt{eq2.001} := A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf}\right) - 2A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\right)$$

$$\texttt{eq2.002} := A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf}\right) - 2A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\right)$$

$$\begin{aligned}\texttt{eq2.003} := {} & A^{c}\left(\partial_{c}A^{a}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} + A^{a}\partial_{c}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} + A^{a}A^{b}\partial_{c}A^{f}\partial_{a}\Gamma^{d}{}_{bf} + A^{a}A^{b}A^{f}\partial_{ca}\Gamma^{d}{}_{bf}\right)\\ & - 2A^{c}\left(\partial_{c}A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + A^{a}\partial_{c}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + A^{a}A^{b}\partial_{c}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + A^{a}A^{b}A^{f}\partial_{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\partial_{c}\Gamma^{e}{}_{bf}\right)\end{aligned}$$

$$\begin{aligned}\texttt{eq2.004} := {} & A^{c}\partial_{c}A^{a}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} + A^{c}A^{a}\partial_{c}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} + A^{c}A^{a}A^{b}\partial_{c}A^{f}\partial_{a}\Gamma^{d}{}_{bf} + A^{c}A^{a}A^{b}A^{f}\partial_{ca}\Gamma^{d}{}_{bf} - 2A^{c}\partial_{c}A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\\ & - 2A^{c}A^{a}\partial_{c}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} - 2A^{c}A^{a}A^{b}\partial_{c}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\partial_{c}\Gamma^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.005} := {} & -\Gamma^{a}{}_{ce}A^{c}A^{e}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} - \Gamma^{b}{}_{ec}A^{e}A^{c}A^{a}A^{f}\partial_{a}\Gamma^{d}{}_{bf} - \Gamma^{f}{}_{ce}A^{c}A^{e}A^{a}A^{b}\partial_{a}\Gamma^{d}{}_{bf} + A^{c}A^{a}A^{b}A^{f}\partial_{ca}\Gamma^{d}{}_{bf} + 2\Gamma^{a}{}_{cg}A^{c}A^{g}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\\ & + 2\Gamma^{b}{}_{gc}A^{g}A^{c}A^{a}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + 2\Gamma^{f}{}_{cg}A^{c}A^{g}A^{a}A^{b}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\partial_{c}\Gamma^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.006} := {} & -\Gamma^{a}{}_{ce}A^{c}A^{e}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} - \Gamma^{b}{}_{ec}A^{e}A^{c}A^{a}A^{f}\partial_{a}\Gamma^{d}{}_{bf} - \Gamma^{f}{}_{ce}A^{c}A^{e}A^{a}A^{b}\partial_{a}\Gamma^{d}{}_{bf} + A^{c}A^{a}A^{b}A^{f}\partial_{ca}\Gamma^{d}{}_{bf} + 2\Gamma^{a}{}_{cg}A^{c}A^{g}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\\ & + 2\Gamma^{b}{}_{gc}A^{g}A^{c}A^{a}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + 2\Gamma^{f}{}_{cg}A^{c}A^{g}A^{a}A^{b}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\partial_{c}\Gamma^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.007} := {} & A^{c}A^{a}A^{b}A^{f}\partial_{ca}\Gamma^{d}{}_{bf} - \Gamma^{a}{}_{ce}A^{c}A^{e}A^{b}A^{f}\partial_{a}\Gamma^{d}{}_{bf} - \Gamma^{b}{}_{ec}A^{e}A^{c}A^{a}A^{f}\partial_{a}\Gamma^{d}{}_{bf} - \Gamma^{f}{}_{ce}A^{c}A^{e}A^{a}A^{b}\partial_{a}\Gamma^{d}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\\ & - 2A^{c}A^{a}A^{b}A^{f}\Gamma^{d}{}_{ae}\partial_{c}\Gamma^{e}{}_{bf} + 2\Gamma^{a}{}_{cg}A^{c}A^{g}A^{b}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + 2\Gamma^{b}{}_{gc}A^{g}A^{c}A^{a}A^{f}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + 2\Gamma^{f}{}_{cg}A^{c}A^{g}A^{a}A^{b}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.008} := {} & A^{a}A^{b}A^{c}A^{f}\partial_{ca}\Gamma^{d}{}_{bf} - A^{b}A^{c}A^{e}A^{f}\Gamma^{a}{}_{ce}\partial_{a}\Gamma^{d}{}_{bf} - A^{a}A^{c}A^{e}A^{f}\Gamma^{b}{}_{ec}\partial_{a}\Gamma^{d}{}_{bf} - A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{ce}\partial_{a}\Gamma^{d}{}_{bf} - 2A^{a}A^{b}A^{c}A^{f}\Gamma^{e}{}_{bf}\partial_{c}\Gamma^{d}{}_{ae}\\ & - 2A^{a}A^{b}A^{c}A^{f}\Gamma^{d}{}_{ae}\partial_{c}\Gamma^{e}{}_{bf} + 2A^{b}A^{c}A^{f}A^{g}\Gamma^{a}{}_{cg}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + 2A^{a}A^{c}A^{f}A^{g}\Gamma^{b}{}_{gc}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf} + 2A^{a}A^{b}A^{c}A^{g}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bf}\Gamma^{f}{}_{cg}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.009} := {} & A^{a}A^{b}A^{c}A^{e}\partial_{ca}\Gamma^{d}{}_{be} - A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{bc}\partial_{f}\Gamma^{d}{}_{ae} - A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{cb}\partial_{a}\Gamma^{d}{}_{fe} - A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{ce}\partial_{a}\Gamma^{d}{}_{bf} - 2A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{be}\partial_{c}\Gamma^{d}{}_{af}\\ & - 2A^{a}A^{b}A^{c}A^{e}\Gamma^{d}{}_{af}\partial_{c}\Gamma^{f}{}_{be} + 2A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{be}\Gamma^{d}{}_{fg}\Gamma^{g}{}_{ac} + 2A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{eb}\Gamma^{d}{}_{ag}\Gamma^{g}{}_{fc} + 2A^{a}A^{b}A^{c}A^{e}\Gamma^{d}{}_{af}\Gamma^{f}{}_{bg}\Gamma^{g}{}_{ce}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.010} := {} & A^{a}A^{b}A^{c}A^{e}\partial_{ab}\Gamma^{d}{}_{ce} - A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{ab}\partial_{f}\Gamma^{d}{}_{ce} - 4A^{a}A^{b}A^{c}A^{e}\Gamma^{f}{}_{ab}\partial_{c}\Gamma^{d}{}_{ef}\\ & - 2A^{a}A^{b}A^{c}A^{e}\Gamma^{d}{}_{af}\partial_{b}\Gamma^{f}{}_{ce} + 2A^{a}A^{b}A^{c}A^{e}\Gamma^{d}{}_{fg}\Gamma^{f}{}_{ab}\Gamma^{g}{}_{ce} + 4A^{a}A^{b}A^{c}A^{e}\Gamma^{d}{}_{af}\Gamma^{f}{}_{bg}\Gamma^{g}{}_{ce}\end{aligned}$$

$$
\begin{aligned}
\texttt{eq3.010} := {} & A^aA^bA^cA^eA^f\partial_{abc}\Gamma^d{}_{ef} - A^aA^bA^cA^eA^f\partial_g\Gamma^d{}_{ab}\partial_c\Gamma^g{}_{ef} - 6A^aA^bA^cA^eA^f\partial_a\Gamma^d{}_{bg}\partial_c\Gamma^g{}_{ef} - 3A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\partial_{cg}\Gamma^d{}_{ef} \\
& - 6A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\partial_{ce}\Gamma^d{}_{fg} - 2A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\partial_{bc}\Gamma^g{}_{ef} + 2A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{cg}\partial_h\Gamma^d{}_{ef} + 6A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{ce}\partial_g\Gamma^d{}_{fh} \\
& + 12A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{cg}\partial_e\Gamma^d{}_{fh} + 6A^aA^bA^cA^eA^f\Gamma^g{}_{ab}\Gamma^h{}_{ce}\partial_f\Gamma^d{}_{gh} + 6A^aA^bA^cA^eA^f\Gamma^d{}_{gh}\Gamma^g{}_{ab}\partial_c\Gamma^h{}_{ef} \\
& + 2A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^h{}_{bc}\partial_h\Gamma^g{}_{ef} + 8A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^h{}_{bc}\partial_e\Gamma^g{}_{fh} + 4A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^g{}_{bh}\partial_c\Gamma^h{}_{ef} \\
& - 12A^aA^bA^cA^eA^f\Gamma^d{}_{gh}\Gamma^g{}_{ab}\Gamma^h{}_{ci}\Gamma^i{}_{ef} - 4A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^g{}_{hi}\Gamma^h{}_{bc}\Gamma^i{}_{ef} - 8A^aA^bA^cA^eA^f\Gamma^d{}_{ag}\Gamma^g{}_{bh}\Gamma^h{}_{ci}\Gamma^i{}_{ef}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{eq4.010} := {} & A^a A^b A^c A^e A^f A^g \partial_{abce}\Gamma^{d}{}_{fg} - 4A^a A^b A^c A^e A^f A^g \partial_{a}\Gamma^{h}{}_{bc}\partial_{eh}\Gamma^{d}{}_{fg} - A^a A^b A^c A^e A^f A^g \partial_{h}\Gamma^{d}{}_{ab}\partial_{ce}\Gamma^{h}{}_{fg} - 12A^a A^b A^c A^e A^f A^g \partial_{a}\Gamma^{h}{}_{bc}\partial_{ef}\Gamma^{d}{}_{gh} \\
& - 8A^a A^b A^c A^e A^f A^g \partial_{a}\Gamma^{d}{}_{bh}\partial_{ce}\Gamma^{h}{}_{fg} - 6A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{ceh}\Gamma^{d}{}_{fg} - 8A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{cef}\Gamma^{d}{}_{gh} \\
& + 8A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{i}\Gamma^{d}{}_{ch}\partial_{e}\Gamma^{i}{}_{fg} + A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{i}\Gamma^{d}{}_{ce}\partial_{h}\Gamma^{i}{}_{fg} + 4A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{i}\Gamma^{d}{}_{ce}\partial_{f}\Gamma^{i}{}_{gh} \\
& + 12A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{h}\Gamma^{d}{}_{ci}\partial_{e}\Gamma^{i}{}_{fg} + 24A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{c}\Gamma^{d}{}_{hi}\partial_{e}\Gamma^{i}{}_{fg} + 8A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{c}\Gamma^{d}{}_{ei}\partial_{h}\Gamma^{i}{}_{fg} \\
& + 32A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\partial_{c}\Gamma^{d}{}_{ei}\partial_{f}\Gamma^{i}{}_{gh} - 2A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\partial_{bce}\Gamma^{h}{}_{fg} + 2A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ai}\partial_{h}\Gamma^{d}{}_{bc}\partial_{e}\Gamma^{i}{}_{fg} \\
& + 16A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ai}\partial_{b}\Gamma^{d}{}_{ch}\partial_{e}\Gamma^{i}{}_{fg} + 6A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\partial_{a}\Gamma^{h}{}_{bc}\partial_{e}\Gamma^{i}{}_{fg} + 2A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\partial_{b}\Gamma^{i}{}_{ce}\partial_{i}\Gamma^{h}{}_{fg} \\
& + 12A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\partial_{b}\Gamma^{h}{}_{ci}\partial_{e}\Gamma^{i}{}_{fg} + 8A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\partial_{ei}\Gamma^{d}{}_{fg} + 3A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\partial_{hi}\Gamma^{d}{}_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\partial_{fh}\Gamma^{d}{}_{gi} + 24A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\partial_{ef}\Gamma^{d}{}_{gi} + 12A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\partial_{fg}\Gamma^{d}{}_{hi} \\
& + 8A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\partial_{ce}\Gamma^{i}{}_{fg} + 6A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\partial_{ei}\Gamma^{h}{}_{fg} + 12A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\partial_{ef}\Gamma^{h}{}_{gi} \\
& + 4A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\partial_{ce}\Gamma^{i}{}_{fg} - 4A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\Gamma^{j}{}_{ei}\partial_{j}\Gamma^{d}{}_{fg} - 2A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{hi}\partial_{j}\Gamma^{d}{}_{fg} \\
& - 16A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fh}\partial_{j}\Gamma^{d}{}_{gi} - 24A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fh}\partial_{i}\Gamma^{d}{}_{gj} - 12A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fg}\partial_{h}\Gamma^{d}{}_{ij} \\
& - 32A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ch}\Gamma^{j}{}_{ei}\partial_{f}\Gamma^{d}{}_{gj} - 16A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{hi}\partial_{f}\Gamma^{d}{}_{gj} - 48A^a A^b A^c A^e A^f A^g \Gamma^{h}{}_{ab}\Gamma^{i}{}_{ce}\Gamma^{j}{}_{fh}\partial_{g}\Gamma^{d}{}_{ij} \\
& - 24A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{aj}\Gamma^{j}{}_{bc}\partial_{e}\Gamma^{i}{}_{fg} - 8A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{j}{}_{ce}\partial_{j}\Gamma^{i}{}_{fg} - 32A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{j}{}_{ce}\partial_{f}\Gamma^{i}{}_{gj} \\
& - 4A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ei}\partial_{j}\Gamma^{h}{}_{fg} - 12A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ef}\partial_{i}\Gamma^{h}{}_{gj} - 24A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ei}\partial_{f}\Gamma^{h}{}_{gj} \\
& - 12A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ef}\partial_{g}\Gamma^{h}{}_{ij} - 16A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{i}{}_{cj}\partial_{e}\Gamma^{j}{}_{fg} - 12A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{ij}\Gamma^{i}{}_{bc}\partial_{e}\Gamma^{j}{}_{fg} \\
& - 4A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{j}{}_{ce}\partial_{j}\Gamma^{i}{}_{fg} - 16A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{j}{}_{ce}\partial_{f}\Gamma^{i}{}_{gj} - 8A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{i}{}_{cj}\partial_{e}\Gamma^{j}{}_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{aj}\Gamma^{i}{}_{bk}\Gamma^{j}{}_{ce}\Gamma^{k}{}_{fg} + 16A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{i}{}_{jk}\Gamma^{j}{}_{ce}\Gamma^{k}{}_{fg} + 32A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{hi}\Gamma^{h}{}_{ab}\Gamma^{i}{}_{cj}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{ij}\Gamma^{i}{}_{bc}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg} + 8A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{i}{}_{jk}\Gamma^{j}{}_{ce}\Gamma^{k}{}_{fg} + 16A^a A^b A^c A^e A^f A^g \Gamma^{d}{}_{ah}\Gamma^{h}{}_{bi}\Gamma^{i}{}_{cj}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg}
\end{aligned}
$$

Stage 2: The fixed point scheme for y^a

```
{x^{a}}::Weight(label=eps,value=0).

{y00^{a},y10^{a},y20^{a},y30^{a},y40^{a}}::Weight(label=eps,value=0).
{y11^{a},y21^{a},y31^{a},y41^{a}}::Weight(label=eps,value=1).
{y22^{a},y32^{a},y42^{a}}::Weight(label=eps,value=2).
{y33^{a},y43^{a}}::Weight(label=eps,value=3).
{y44^{a}}::Weight(label=eps,value=4).

{Gam11^{a}_{b c}}::Weight(label=eps,value=1).
{Gam22^{a}_{b c d}}::Weight(label=eps,value=2).
{Gam33^{a}_{b c d e}}::Weight(label=eps,value=3).
{Gam44^{a}_{b c d e f}}::Weight(label=eps,value=4).
{Gam55^{a}_{b c d e f g}}::Weight(label=eps,value=5).

Gam11^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
Gam22^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
Gam33^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
Gam44^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
Gam55^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

y00{#}::LaTeXForm ("\ny{00}").
y10{#}::LaTeXForm ("\ny{10}").
y20{#}::LaTeXForm ("\ny{20}").
y30{#}::LaTeXForm ("\ny{30}").
y40{#}::LaTeXForm ("\ny{40}").
y11{#}::LaTeXForm ("\ny{11}").
y21{#}::LaTeXForm ("\ny{21}").
y31{#}::LaTeXForm ("\ny{31}").
y41{#}::LaTeXForm ("\ny{41}").
y22{#}::LaTeXForm ("\ny{22}").
y32{#}::LaTeXForm ("\ny{32}").
y42{#}::LaTeXForm ("\ny{42}").
y33{#}::LaTeXForm ("\ny{33}").
y43{#}::LaTeXForm ("\ny{43}").
y44{#}::LaTeXForm ("\ny{44}").
```

```
Gam11{#}::LaTeXForm ("\nGamma{11}").
Gam22{#}::LaTeXForm ("\nGamma{22}").
Gam33{#}::LaTeXForm ("\nGamma{33}").
Gam44{#}::LaTeXForm ("\nGamma{44}").
Gam55{#}::LaTeXForm ("\nGamma{55}").

def get_term (obj,n):

    foo := @(obj).
    bah = Ex("eps = " + str(n))
    distribute  (foo)
    keep_weight (foo, bah)

    return foo

def truncateEps (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah = Ex("eps = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

def substitute_eps (obj):
    substitute         (obj,epsy0)
    substitute         (obj,epsy1)
    substitute         (obj,epsy2)
    substitute         (obj,epsy3)
    substitute         (obj,epsy4)
    substitute         (obj,epsGam1)
    substitute         (obj,epsGam2)
    substitute         (obj,epsGam3)
    substitute         (obj,epsGam4)
    substitute         (obj,epsGam5)
```

```
    distribute          (obj)
    obj = truncateEps  (obj,4)
    obj = product_sort (obj)
    rename_dummies     (obj)
    canonicalise       (obj)

    return obj

def tidy (obj):
    obj = product_sort (obj)
    rename_dummies (obj)
    canonicalise   (obj)
    return obj

beg_stage_2 = time.time()

y0 := x^{a}.
y1 := x^{a} +   (1/2) Gam^{a}_{b c} y0^{b} y0^{c}.
y2 := x^{a} +   (1/2) Gam^{a}_{b c} y1^{b} y1^{c}
            +   (1/6) Gam^{a}_{b c d} y0^{b} y0^{c} y0^{d}.
y3 := x^{a} +   (1/2) Gam^{a}_{b c} y2^{b} y2^{c}
            +   (1/6) Gam^{a}_{b c d} y1^{b} y1^{c} y1^{d}
            +  (1/24) Gam^{a}_{b c d e} y0^{b} y0^{c} y0^{d} y0^{e}.
y4 := x^{a} +   (1/2) Gam^{a}_{b c} y3^{b} y3^{c}
            +   (1/6) Gam^{a}_{b c d} y2^{b} y2^{c} y2^{d}
            +  (1/24) Gam^{a}_{b c d e} y1^{b} y1^{c} y1^{d} y1^{e}
            + (1/120) Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

# note that:
#   y00 = y10 = y20 = y30 = y40
#   y11 = y21 = y31 = y41
#   y22 = y32 = y42
#   y33 = y43
#   y44

# expand each y in powers of eps

epsy0 := y0^{a} -> y00^{a}.
```

```
epsy1 := y1^{a} -> y10^{a}+y11^{a}.
epsy2 := y2^{a} -> y20^{a}+y21^{a}+y22^{a}.
epsy3 := y3^{a} -> y30^{a}+y31^{a}+y32^{a}+y33^{a}.
epsy4 := y4^{a} -> y40^{a}+y41^{a}+y42^{a}+y43^{a}+y44^{a}.

epsGam1 := Gam^{a}_{b c} -> Gam11^{a}_{b c}.
epsGam2 := Gam^{a}_{b c d} -> Gam22^{a}_{b c d}.
epsGam3 := Gam^{a}_{b c d e} -> Gam33^{a}_{b c d e}.
epsGam4 := Gam^{a}_{b c d e f} -> Gam44^{a}_{b c d e f}.
epsGam5 := Gam^{a}_{b c d e f g} -> Gam55^{a}_{b c d e f g}.

y0 = substitute_eps (y0)   # cdb (y0.001,y0)
y1 = substitute_eps (y1)   # cdb (y1.001,y1)
y2 = substitute_eps (y2)   # cdb (y2.001,y2)
y3 = substitute_eps (y3)   # cdb (y3.001,y3)
y4 = substitute_eps (y4)   # cdb (y4.001,y4)

defy0 := y0^{a} -> @(y0).
defy1 := y1^{a} -> @(y1).
defy2 := y2^{a} -> @(y2).
defy3 := y3^{a} -> @(y3).
defy4 := y4^{a} -> @(y4).

# -----------------------------------
# y0

y00 := @(y0).           # cdb (y00.101,y00)

defy00 := y00^{a} -> @(y00).

# -----------------------------------
# y1

substitute (y1,defy00)

distribute (y1)

y10 = get_term (y1,0)   # cdb (y10.101,y10)
```

```
y11 = get_term (y1,1)    # cdb (y11.101,y11)

defy10 := y10^{a} -> @(y10).
defy11 := y11^{a} -> @(y11).

# -----------------------------------
# y2

substitute (y2,defy00)

substitute (y2,defy10)
substitute (y2,defy11)

distribute (y2)

y20 = get_term (y2,0)    # cdb (y20.101,y20)
y21 = get_term (y2,1)    # cdb (y21.101,y21)
y22 = get_term (y2,2)    # cdb (y22.101,y22)

y20 = tidy (y20)    # cdb (y20.201,y20)
y21 = tidy (y21)    # cdb (y21.201,y21)
y22 = tidy (y22)    # cdb (y22.201,y22)

defy20 := y20^{a} -> @(y20).
defy21 := y21^{a} -> @(y21).
defy22 := y22^{a} -> @(y22).

# -----------------------------------
# y3

substitute (y3,defy00)

substitute (y3,defy10)
substitute (y3,defy11)

substitute (y3,defy20)
substitute (y3,defy21)
substitute (y3,defy22)
```

```
distribute (y3)

y30 = get_term (y3,0)    # cdb (y30.101,y30)
y31 = get_term (y3,1)    # cdb (y31.101,y31)
y32 = get_term (y3,2)    # cdb (y32.101,y32)
y33 = get_term (y3,3)    # cdb (y33.101,y33)

y30 = tidy (y30)    # cdb (y30.201,y30)
y31 = tidy (y31)    # cdb (y31.201,y31)
y32 = tidy (y32)    # cdb (y32.201,y32)
y33 = tidy (y33)    # cdb (y33.201,y33)

defy30 := y30^{a} -> @(y30).
defy31 := y31^{a} -> @(y31).
defy32 := y32^{a} -> @(y32).
defy33 := y33^{a} -> @(y33).

# -----------------------------------
# y4

substitute (y4,defy00)

substitute (y4,defy10)
substitute (y4,defy11)

substitute (y4,defy20)
substitute (y4,defy21)
substitute (y4,defy22)

substitute (y4,defy30)
substitute (y4,defy31)
substitute (y4,defy32)
substitute (y4,defy33)

distribute (y4)

y40 = get_term (y4,0)    # cdb (y40.101,y40)
```

```
y41 = get_term (y4,1)    # cdb (y41.101,y41)
y42 = get_term (y4,2)    # cdb (y42.101,y42)
y43 = get_term (y4,3)    # cdb (y43.101,y43)
y44 = get_term (y4,4)    # cdb (y44.101,y44)

y40 = tidy (y40)    # cdb (y40.201,y40)
y41 = tidy (y41)    # cdb (y41.201,y41)
y42 = tidy (y42)    # cdb (y42.201,y42)
y43 = tidy (y43)    # cdb (y43.201,y43)
y44 = tidy (y44)    # cdb (y44.201,y44)

defy40 := y40^{a} -> @(y40).
defy41 := y41^{a} -> @(y41).
defy42 := y42^{a} -> @(y42).
defy43 := y43^{a} -> @(y43).
defy44 := y44^{a} -> @(y44).

end_stage_2 = time.time()
```

$$\texttt{y1.001} := x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}$$

$$\texttt{y2.001} := x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{10}{y}{}^{b}\overset{11}{y}{}^{c} + \frac{1}{6}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}\overset{00}{y}{}^{d} + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{11}{y}{}^{b}\overset{11}{y}{}^{c}$$

$$\texttt{y3.001} := x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{20}{y}{}^{b}\overset{21}{y}{}^{c} + \frac{1}{6}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{10}{y}{}^{d} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{20}{y}{}^{b}\overset{22}{y}{}^{c} + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{21}{y}{}^{b}\overset{21}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{11}{y}{}^{d} + \frac{1}{24}\overset{33}{\Gamma}{}^{a}{}_{bcde}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}\overset{00}{y}{}^{d}\overset{00}{y}{}^{e} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{21}{y}{}^{b}\overset{22}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{10}{y}{}^{b}\overset{11}{y}{}^{c}\overset{11}{y}{}^{d}$$

$$\begin{aligned}\texttt{y4.001} := {} & x^a + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{30}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{31}{y}{}^{c} + \frac{1}{6}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c}\overset{20}{y}{}^{d} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{32}{y}{}^{c} + \frac{1}{2}\overset{11}{\Gamma}{}^{a}{}_{bc}\overset{31}{y}{}^{b}\overset{31}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c}\overset{21}{y}{}^{d} + \frac{1}{24}\overset{33}{\Gamma}{}^{a}{}_{bcde}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{10}{y}{}^{d}\overset{10}{y}{}^{e} \\ & + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{30}{y}{}^{b}\overset{33}{y}{}^{c} + \overset{11}{\Gamma}{}^{a}{}_{bc}\overset{31}{y}{}^{b}\overset{32}{y}{}^{c} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{20}{y}{}^{c}\overset{22}{y}{}^{d} + \frac{1}{2}\overset{22}{\Gamma}{}^{a}{}_{bcd}\overset{20}{y}{}^{b}\overset{21}{y}{}^{c}\overset{21}{y}{}^{d} + \frac{1}{6}\overset{33}{\Gamma}{}^{a}{}_{bcde}\overset{10}{y}{}^{b}\overset{10}{y}{}^{c}\overset{10}{y}{}^{d}\overset{11}{y}{}^{e} + \frac{1}{120}\overset{44}{\Gamma}{}^{a}{}_{bcdef}\overset{00}{y}{}^{b}\overset{00}{y}{}^{c}\overset{00}{y}{}^{d}\overset{00}{y}{}^{e}\overset{00}{y}{}^{f}\end{aligned}$$

$$\texttt{y10.101} := x^a$$

$$\texttt{y11.101} := \frac{1}{2}\overset{11}{\Gamma}{}^a{}_{bc}x^b x^c$$

$$\texttt{y20.201} := x^a$$

$$\texttt{y21.201} := \frac{1}{2}x^b x^c \overset{11}{\Gamma}{}^a{}_{bc}$$

$$\texttt{y22.201} := \frac{1}{2}x^b x^c x^d \overset{11}{\Gamma}{}^a{}_{be}\overset{11}{\Gamma}{}^e{}_{cd} + \frac{1}{6}x^b x^c x^d \overset{22}{\Gamma}{}^a{}_{bcd}$$

$$\texttt{y30.201} := x^a$$

$$\texttt{y31.201} := \frac{1}{2}x^b x^c \overset{11}{\Gamma}{}^a{}_{bc}$$

$$\texttt{y32.201} := \frac{1}{2}x^b x^c x^d \overset{11}{\Gamma}{}^a{}_{be}\overset{11}{\Gamma}{}^e{}_{cd} + \frac{1}{6}x^b x^c x^d \overset{22}{\Gamma}{}^a{}_{bcd}$$

$$\texttt{y33.201} := \frac{1}{2}x^b x^c x^d x^e \overset{11}{\Gamma}{}^a{}_{bf}\overset{11}{\Gamma}{}^f{}_{cg}\overset{11}{\Gamma}{}^g{}_{de} + \frac{1}{6}x^b x^c x^d x^e \overset{11}{\Gamma}{}^a{}_{bf}\overset{22}{\Gamma}{}^f{}_{cde} + \frac{1}{8}x^b x^c x^d x^e \overset{11}{\Gamma}{}^a{}_{fg}\overset{11}{\Gamma}{}^f{}_{bc}\overset{11}{\Gamma}{}^g{}_{de} + \frac{1}{4}x^b x^c x^d x^e \overset{11}{\Gamma}{}^f{}_{bc}\overset{22}{\Gamma}{}^a{}_{def} + \frac{1}{24}x^b x^c x^d x^e \overset{33}{\Gamma}{}^a{}_{bcde}$$

$$\texttt{y40.201} := x^{a}$$

$$\texttt{y41.201} := \frac{1}{2}x^{b}x^{c}\overset{11}{\Gamma}{}^{a}{}_{bc}$$

$$\texttt{y42.201} := \frac{1}{2}x^{b}x^{c}x^{d}\overset{11}{\Gamma}{}^{a}{}_{be}\overset{11}{\Gamma}{}^{e}{}_{cd} + \frac{1}{6}x^{b}x^{c}x^{d}\overset{22}{\Gamma}{}^{a}{}_{bcd}$$

$$\texttt{y43.201} := \frac{1}{2}x^{b}x^{c}x^{d}x^{e}\overset{11}{\Gamma}{}^{a}{}_{bf}\overset{11}{\Gamma}{}^{f}{}_{cg}\overset{11}{\Gamma}{}^{g}{}_{de} + \frac{1}{6}x^{b}x^{c}x^{d}x^{e}\overset{11}{\Gamma}{}^{a}{}_{bf}\overset{22}{\Gamma}{}^{f}{}_{cde} + \frac{1}{8}x^{b}x^{c}x^{d}x^{e}\overset{11}{\Gamma}{}^{a}{}_{fg}\overset{11}{\Gamma}{}^{f}{}_{bc}\overset{11}{\Gamma}{}^{g}{}_{de} + \frac{1}{4}x^{b}x^{c}x^{d}x^{e}\overset{11}{\Gamma}{}^{f}{}_{bc}\overset{22}{\Gamma}{}^{a}{}_{def} + \frac{1}{24}x^{b}x^{c}x^{d}x^{e}\overset{33}{\Gamma}{}^{a}{}_{bcde}$$

$$\begin{aligned}\texttt{y44.201} := {} & \frac{1}{2}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{a}{}_{bg}\overset{11}{\Gamma}{}^{g}{}_{ch}\overset{11}{\Gamma}{}^{h}{}_{di}\overset{11}{\Gamma}{}^{i}{}_{ef} + \frac{1}{6}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{a}{}_{bg}\overset{11}{\Gamma}{}^{g}{}_{ch}\overset{22}{\Gamma}{}^{h}{}_{def} + \frac{1}{8}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{a}{}_{bg}\overset{11}{\Gamma}{}^{g}{}_{hi}\overset{11}{\Gamma}{}^{h}{}_{cd}\overset{11}{\Gamma}{}^{i}{}_{ef} + \frac{1}{4}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{a}{}_{bg}\overset{11}{\Gamma}{}^{h}{}_{cd}\overset{22}{\Gamma}{}^{g}{}_{efh} \\ & + \frac{1}{24}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{a}{}_{bg}\overset{33}{\Gamma}{}^{g}{}_{cdef} + \frac{1}{4}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{a}{}_{gh}\overset{11}{\Gamma}{}^{g}{}_{bc}\overset{11}{\Gamma}{}^{h}{}_{di}\overset{11}{\Gamma}{}^{i}{}_{ef} + \frac{1}{12}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{a}{}_{gh}\overset{11}{\Gamma}{}^{g}{}_{bc}\overset{22}{\Gamma}{}^{h}{}_{def} + \frac{1}{4}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{g}{}_{bc}\overset{11}{\Gamma}{}^{h}{}_{dg}\overset{22}{\Gamma}{}^{a}{}_{efh} \\ & + \frac{1}{12}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{22}{\Gamma}{}^{a}{}_{bcg}\overset{22}{\Gamma}{}^{g}{}_{def} + \frac{1}{8}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{g}{}_{bc}\overset{11}{\Gamma}{}^{h}{}_{de}\overset{22}{\Gamma}{}^{a}{}_{fgh} + \frac{1}{12}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{11}{\Gamma}{}^{g}{}_{bc}\overset{33}{\Gamma}{}^{a}{}_{defg} + \frac{1}{120}x^{b}x^{c}x^{d}x^{e}x^{f}\overset{44}{\Gamma}{}^{a}{}_{bcdef}\end{aligned}$$

Stage3: Introduce the generalised connections from Stage 1

```
def substitute_gam (obj):
    substitute     (obj,defGam11)
    substitute     (obj,defGam22)
    substitute     (obj,defGam33)
    substitute     (obj,defGam44)
    substitute     (obj,defGam55)
    distribute     (obj)

    return obj

beg_stage_3 = time.time()

Gam11 := @(eq0).
Gam22 := @(eq1).
Gam33 := @(eq2).
Gam44 := @(eq3).
Gam55 := @(eq4).

# peel off the A^{a}, must then symmetrise over revealed indices

substitute (Gam11,$A^{a}->1$)
substitute (Gam22,$A^{a}->1$)
substitute (Gam33,$A^{a}->1$)
substitute (Gam44,$A^{a}->1$)
substitute (Gam55,$A^{a}->1$)

# now symmetrise

sym (Gam11,$_{a},_{b}$)
sym (Gam22,$_{a},_{b},_{c}$)
sym (Gam33,$_{a},_{b},_{c},_{e}$)
sym (Gam44,$_{a},_{b},_{c},_{e},_{f}$)
sym (Gam55,$_{a},_{b},_{c},_{e},_{f},_{g}$)

defGam11 := Gam11^{d}_{a b} -> @(Gam11).
defGam22 := Gam22^{d}_{a b c} -> @(Gam22).
```

```
defGam33 := Gam33^{d}_{a b c e} -> @(Gam33).
defGam44 := Gam44^{d}_{a b c e f} -> @(Gam44).
defGam55 := Gam55^{d}_{a b c e f g} -> @(Gam55).

y31 = substitute_gam (y31)
y32 = substitute_gam (y32)
y33 = substitute_gam (y33)

y31 = tidy (y31)   # cdb (y31.301,y31)
y32 = tidy (y32)   # cdb (y32.301,y32)
y33 = tidy (y33)   # cdb (y33.301,y33)

y3 := @(y30) + @(y31) + @(y32) + @(y33).

y41 = substitute_gam (y41)
y42 = substitute_gam (y42)
y43 = substitute_gam (y43)
y44 = substitute_gam (y44)

y41 = tidy (y41)   # cdb (y41.301,y41)
y42 = tidy (y42)   # cdb (y42.301,y42)
y43 = tidy (y43)   # cdb (y43.301,y43)
y44 = tidy (y44)   # cdb (y44.301,y44)

y4 := @(y40) + @(y41) + @(y42) + @(y43) + @(y44).

end_stage_3 = time.time()
```

$$\texttt{y30.201} := x^{a}$$

$$\texttt{y31.301} := \frac{1}{2}x^{b}x^{c}\Gamma^{a}{}_{bc}$$

$$\texttt{y32.301} := \frac{1}{6}x^{b}x^{c}x^{d}\Gamma^{a}{}_{be}\Gamma^{e}{}_{cd} + \frac{1}{6}x^{b}x^{c}x^{d}\partial_{b}\Gamma^{a}{}_{cd}$$

$$\texttt{y33.301} := \frac{1}{12}x^{b}x^{c}x^{d}x^{e}\Gamma^{a}{}_{bf}\partial_{c}\Gamma^{f}{}_{de} + \frac{1}{24}x^{b}x^{c}x^{d}x^{e}\Gamma^{a}{}_{fg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{de} + \frac{1}{24}x^{b}x^{c}x^{d}x^{e}\Gamma^{f}{}_{bc}\partial_{f}\Gamma^{a}{}_{de} + \frac{1}{24}x^{b}x^{c}x^{d}x^{e}\partial_{bc}\Gamma^{a}{}_{de}$$

$$\texttt{y40.201} := x^a$$

$$\texttt{y41.301} := \frac{1}{2} x^b x^c \Gamma^a{}_{bc}$$

$$\texttt{y42.301} := \frac{1}{6} x^b x^c x^d \Gamma^a{}_{be} \Gamma^e{}_{cd} + \frac{1}{6} x^b x^c x^d \partial_b \Gamma^a{}_{cd}$$

$$\texttt{y43.301} := \frac{1}{12} x^b x^c x^d x^e \Gamma^a{}_{bf} \partial_c \Gamma^f{}_{de} + \frac{1}{24} x^b x^c x^d x^e \Gamma^a{}_{fg} \Gamma^f{}_{bc} \Gamma^g{}_{de} + \frac{1}{24} x^b x^c x^d x^e \Gamma^f{}_{bc} \partial_f \Gamma^a{}_{de} + \frac{1}{24} x^b x^c x^d x^e \partial_{bc} \Gamma^a{}_{de}$$

$$\begin{aligned}
\texttt{y44.301} := & -\frac{1}{90} x^b x^c x^d x^e x^f \Gamma^a{}_{bg} \Gamma^g{}_{ch} \Gamma^h{}_{di} \Gamma^i{}_{ef} + \frac{1}{180} x^b x^c x^d x^e x^f \Gamma^a{}_{bg} \Gamma^g{}_{ch} \partial_d \Gamma^h{}_{ef} + \frac{1}{120} x^b x^c x^d x^e x^f \Gamma^a{}_{bg} \Gamma^g{}_{hi} \Gamma^h{}_{cd} \Gamma^i{}_{ef} \\
& - \frac{1}{60} x^b x^c x^d x^e x^f \Gamma^a{}_{bg} \Gamma^h{}_{cd} \partial_e \Gamma^g{}_{fh} + \frac{1}{60} x^b x^c x^d x^e x^f \Gamma^a{}_{bg} \Gamma^h{}_{cd} \partial_h \Gamma^g{}_{ef} + \frac{1}{40} x^b x^c x^d x^e x^f \Gamma^a{}_{bg} \partial_{cd} \Gamma^g{}_{ef} + \frac{1}{90} x^b x^c x^d x^e x^f \Gamma^a{}_{gh} \Gamma^g{}_{bc} \Gamma^h{}_{di} \Gamma^i{}_{ef} \\
& + \frac{13}{360} x^b x^c x^d x^e x^f \Gamma^a{}_{gh} \Gamma^g{}_{bc} \partial_d \Gamma^h{}_{ef} - \frac{1}{90} x^b x^c x^d x^e x^f \Gamma^g{}_{bc} \Gamma^h{}_{dg} \partial_e \Gamma^a{}_{fh} + \frac{1}{360} x^b x^c x^d x^e x^f \Gamma^g{}_{bc} \Gamma^h{}_{dg} \partial_h \Gamma^a{}_{ef} \\
& + \frac{1}{180} x^b x^c x^d x^e x^f \partial_b \Gamma^a{}_{cg} \partial_d \Gamma^g{}_{ef} + \frac{7}{360} x^b x^c x^d x^e x^f \partial_g \Gamma^a{}_{bc} \partial_d \Gamma^g{}_{ef} + \frac{1}{120} x^b x^c x^d x^e x^f \Gamma^g{}_{bc} \Gamma^h{}_{de} \partial_f \Gamma^a{}_{gh} \\
& + \frac{1}{120} x^b x^c x^d x^e x^f \Gamma^g{}_{bc} \Gamma^h{}_{de} \partial_g \Gamma^a{}_{fh} - \frac{1}{120} x^b x^c x^d x^e x^f \Gamma^g{}_{bc} \partial_{de} \Gamma^a{}_{fg} + \frac{1}{60} x^b x^c x^d x^e x^f \Gamma^g{}_{bc} \partial_{dg} \Gamma^a{}_{ef} + \frac{1}{120} x^b x^c x^d x^e x^f \partial_{bcd} \Gamma^a{}_{ef}
\end{aligned}$$

Stage4: Reformatting and output

```
{x^{a}}::Weight(label=numx).
\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

def reformat (obj,scale):

    bah  = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute         (tmp)
    tmp = product_sort (tmp)
    rename_dummies     (tmp)
    canonicalise       (tmp)
    factor_out         (tmp,$x^{a?}$)

    return tmp

def get_term (obj,n):

    tmp := @(obj).
    foo = Ex("numx = " + str(n))
    distribute  (tmp)
    keep_weight (tmp, foo)

    return tmp

beg_stage_4 = time.time()

rnc := x^{a}
     + @(y41)
     + @(y42)
     + @(y43)
     + @(y44).

# substitute (rnc,$A^{a}->x^{a}$)

rnc1 = get_term (rnc,1)          # cdb (rnc1.001,rnc1)
rnc2 = get_term (rnc,2)          # cdb (rnc2.001,rnc2)
```

```
rnc3 = get_term (rnc,3)          # cdb (rnc3.001,rnc3)
rnc4 = get_term (rnc,4)          # cdb (rnc4.001,rnc4)
rnc5 = get_term (rnc,5)          # cdb (rnc5.001,rnc5)

scaled1 = reformat (rnc1,   1)   # cdb (scaled1.002,scaled1)
scaled2 = reformat (rnc2,   2)   # cdb (scaled2.002,scaled2)
scaled3 = reformat (rnc3,   6)   # cdb (scaled3.002,scaled3)
scaled4 = reformat (rnc4,  24)   # cdb (scaled4.002,scaled4)
scaled5 = reformat (rnc5, 360)   # cdb (scaled5.002,scaled5)

import cdblib

cdblib.create ('gen2rnc.json')

cdblib.put ('rnc',rnc,'gen2rnc.json')

cdblib.put ('rnc1',rnc1,'gen2rnc.json')
cdblib.put ('rnc2',rnc2,'gen2rnc.json')
cdblib.put ('rnc3',rnc3,'gen2rnc.json')
cdblib.put ('rnc4',rnc4,'gen2rnc.json')
cdblib.put ('rnc5',rnc5,'gen2rnc.json')

end_stage_4 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)
```

Timing

Stage 1: 14.5 secs
Stage 2: 3.2 secs
Stage 3: 453.8 secs
Stage 4: 1.6 secs

Convert from generic (x) to local RNC coords (y)

$$y^a = \overset{0}{y}{}^a + \overset{1}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a$$

$$\begin{aligned}
\overset{0}{y}{}^a &= x^a \\
2\overset{1}{y}{}^a &= x^b x^c \Gamma^a{}_{bc} \\
6\overset{2}{y}{}^a &= x^b x^c x^d \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} + \partial_b \Gamma^a{}_{cd}\right) \\
24\overset{3}{y}{}^a &= x^b x^c x^d x^e \left(2\Gamma^a{}_{bf}\partial_c\Gamma^f{}_{de} + \Gamma^a{}_{fg}\Gamma^f{}_{bc}\Gamma^g{}_{de} + \Gamma^f{}_{bc}\partial_f\Gamma^a{}_{de} + \partial_{bc}\Gamma^a{}_{de}\right) \\
360\overset{4}{y}{}^a &= x^b x^c x^d x^e x^f \big(-4\Gamma^a{}_{bg}\Gamma^g{}_{ch}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 2\Gamma^a{}_{bg}\Gamma^g{}_{ch}\partial_d\Gamma^h{}_{ef} + 3\Gamma^a{}_{bg}\Gamma^g{}_{hi}\Gamma^h{}_{cd}\Gamma^i{}_{ef} - 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_e\Gamma^g{}_{fh} + 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_h\Gamma^g{}_{ef} + 9\Gamma^a{}_{bg}\partial_{cd}\Gamma^g{}_{ef} \\
&\quad + 4\Gamma^a{}_{gh}\Gamma^g{}_{bc}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 13\Gamma^a{}_{gh}\Gamma^g{}_{bc}\partial_d\Gamma^h{}_{ef} - 4\Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_e\Gamma^a{}_{fh} + \Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_h\Gamma^a{}_{ef} + 2\partial_b\Gamma^a{}_{cg}\partial_d\Gamma^g{}_{ef} + 7\partial_g\Gamma^a{}_{bc}\partial_d\Gamma^g{}_{ef} + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_f\Gamma^a{}_{gh} \\
&\quad + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_g\Gamma^a{}_{fh} - 3\Gamma^g{}_{bc}\partial_{de}\Gamma^a{}_{fg} + 6\Gamma^g{}_{bc}\partial_{dg}\Gamma^a{}_{ef} + 3\partial_{bcd}\Gamma^a{}_{ef}\big)
\end{aligned}$$

```
cdblib.create ('gen2rnc.export')

# 6th order terms, scaled
cdblib.put ('rnc61scaled',scaled1,'gen2rnc.export')
cdblib.put ('rnc62scaled',scaled2,'gen2rnc.export')
cdblib.put ('rnc63scaled',scaled3,'gen2rnc.export')
cdblib.put ('rnc64scaled',scaled4,'gen2rnc.export')
cdblib.put ('rnc65scaled',scaled5,'gen2rnc.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```

Convert from rnc to generic coordinates

The following code is based on the `gen2rnc.tex` code.

It is common to do some computations in a local RNC. Doing so makes various parts of the computations much easier to manage than in the original non-RNC coordinates. One simple example is the proof of the second Bianchi identities.

This code develops the inverse transformation, that is from the local RNC coordinates back to generic coordinates. The key equation (drawn form `gen2rnc.tex`) is

$$x_j^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\cdot \underline{b}_k} \tag{1}$$

In `gen2rnc.tex` this equation was solved for the RNC coordinates y given the generic coordinates x_j and x_i. Here we will instead take x_i and y as given and use this equation to compute x_j. The first change we will make is to replace x_j with x (as the subscript j serves no useful purpose).

Thus our job will be to compute

$$x^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\cdot \underline{b}_k} \tag{2}$$

given x_i and y. The generalised connections will be computed recursively by

$$\Gamma^a{}_{b\underline{c}d} = \Gamma^a{}_{(b\underline{c},d)} - (n+1)\Gamma^a{}_{p(\underline{c}}\Gamma^p{}_{bd)} \tag{3}$$

As noted in `gen2rnc.tex`, the generalised connections will scale with the expensions parameter ϵ according to

$$\Gamma^a{}_{bc} = \mathcal{O}(\epsilon) \ , \qquad \Gamma^a{}_{bcd} = \mathcal{O}\left(\epsilon^2\right) \ , \qquad \Gamma^a{}_{bcde} = \mathcal{O}\left(\epsilon^3\right) \ , \qquad \text{etc.}$$

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

A^{a}::Depends(\partial{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).

Q^{a}_{b c}::Depends(\partial{#}).

Q^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
Q^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
Q^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
Q^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
Q^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

Q^{p}_{a b}::Weight(label=numQ,value=1).
Q^{p}_{a b c}::Weight(label=numQ,value=2).
Q^{p}_{a b c d}::Weight(label=numQ,value=3).
Q^{p}_{a b c d e}::Weight(label=numQ,value=4).
Q^{p}_{a b c d e f}::Weight(label=numQ,value=5).

def product_sort (obj):
```

```
    substitute (obj,$ A^{a}                  -> A001^{a}            $)
    substitute (obj,$ x^{a}                  -> A002^{a}            $)
    substitute (obj,$ g^{a b}                -> A003^{a b}          $)
    substitute (obj,$ Q^{p}_{a b}            -> A004^{p}_{a b}      $)
    substitute (obj,$ Q^{p}_{a b c}          -> A005^{p}_{a b c}    $)
    substitute (obj,$ Q^{p}_{a b c d}        -> A006^{p}_{a b c d}  $)
    substitute (obj,$ Q^{p}_{a b c d e}      -> A007^{p}_{a b c d e} $)
    substitute (obj,$ Q^{p}_{a b c d e f}    -> A008^{p}_{a b c d e f} $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}               -> A^{a}               $)
    substitute (obj,$ A002^{a}               -> x^{a}               $)
    substitute (obj,$ A003^{a b}             -> g^{a b}             $)
    substitute (obj,$ A004^{p}_{a b}         -> Q^{p}_{a b}         $)
    substitute (obj,$ A005^{p}_{a b c}       -> Q^{p}_{a b c}       $)
    substitute (obj,$ A006^{p}_{a b c d}     -> Q^{p}_{a b c d}     $)
    substitute (obj,$ A007^{p}_{a b c d e}   -> Q^{p}_{a b c d e}   $)
    substitute (obj,$ A008^{p}_{a b c d e f} -> Q^{p}_{a b c d e f} $)

    return obj

def truncateQ (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah  = Ex("numQ = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

# A^{a} = dx^a/ds

Gamma := Q^{d}_{a b} A^{a} A^{b}.

dAds  := A^{c} \partial_{c}{A^{d}}-> - @(Gamma).
```

```
# ==============================================================
eq0 := @(Gamma).                       # cdb (eq0.000,eq0)

# ==============================================================
eq1 := A^{c} \partial_{c}{@(eq0)}.     # cdb (eq1.000,eq1)

distribute       (eq1)                 # cdb (eq1.001,eq1)
unwrap           (eq1)                 # cdb (eq1.002,eq1)
product_rule     (eq1)                 # cdb (eq1.003,eq1)
distribute       (eq1)                 # cdb (eq1.004,eq1)
substitute       (eq1,dAds)            # cdb (eq1.005,eq1)
distribute       (eq1)                 # cdb (eq1.006,eq1)
eq1 = truncateQ (eq1,5)                # cdb (eq1.007,eq1)
sort_product     (eq1)                 # cdb (eq1.008,eq1)
rename_dummies   (eq1)                 # cdb (eq1.009,eq1)
canonicalise     (eq1)                 # cdb (eq1.010,eq1)

# ==============================================================
eq2 := A^{c} \partial_{c}{@(eq1)}.     # cdb (eq2.000,eq2)

distribute       (eq2)                 # cdb (eq2.001,eq2)
unwrap           (eq2)                 # cdb (eq2.002,eq2)
product_rule     (eq2)                 # cdb (eq2.003,eq2)
distribute       (eq2)                 # cdb (eq2.004,eq2)
substitute       (eq2,dAds)            # cdb (eq2.005,eq2)
distribute       (eq2)                 # cdb (eq2.006,eq2)
eq2 = truncateQ (eq2,5)                # cdb (eq2.007,eq2)
sort_product     (eq2)                 # cdb (eq2.008,eq2)
rename_dummies   (eq2)                 # cdb (eq2.009,eq2)
canonicalise     (eq2)                 # cdb (eq2.010,eq2)

# ==============================================================
eq3 := A^{c} \partial_{c}{@(eq2)}.     # cdb (eq3.000,eq3)

distribute       (eq3)                 # cdb (eq3.001,eq3)
unwrap           (eq3)                 # cdb (eq3.002,eq3)
product_rule     (eq3)                 # cdb (eq3.003,eq3)
```

```
distribute      (eq3)                   # cdb (eq3.004,eq3)
substitute      (eq3,dAds)              # cdb (eq3.005,eq3)
distribute      (eq3)                   # cdb (eq3.006,eq3)
eq3 = truncateQ (eq3,5)                 # cdb (eq3.007,eq3)
sort_product    (eq3)                   # cdb (eq3.008,eq3)
rename_dummies  (eq3)                   # cdb (eq3.009,eq3)
canonicalise    (eq3)                   # cdb (eq3.010,eq3)

# ==============================================================
eq4 := A^{c} \partial_{c}{@(eq3)}.      # cdb (eq4.000,eq4)

distribute      (eq4)                   # cdb (eq4.001,eq4)
unwrap          (eq4)                   # cdb (eq4.002,eq4)
product_rule    (eq4)                   # cdb (eq4.003,eq4)
distribute      (eq4)                   # cdb (eq4.004,eq4)
substitute      (eq4,dAds)              # cdb (eq4.005,eq4)
distribute      (eq4)                   # cdb (eq4.006,eq4)
eq4 = truncateQ (eq4,5)                 # cdb (eq4.007,eq4)
sort_product    (eq4)                   # cdb (eq4.008,eq4)
rename_dummies  (eq4)                   # cdb (eq4.009,eq4)
canonicalise    (eq4)                   # cdb (eq4.010,eq4)
```

$$\texttt{eq0.000} := Q^{d}{}_{ab}A^{a}A^{b}$$

$$\texttt{eq1.000} := A^c\partial_c\left(Q^d{}_{ab}A^aA^b\right)$$

$$\texttt{eq1.001} := A^c\partial_c\left(Q^d{}_{ab}A^aA^b\right)$$

$$\texttt{eq1.002} := A^c\partial_c\left(Q^d{}_{ab}A^aA^b\right)$$

$$\texttt{eq1.003} := A^c\left(\partial_cQ^d{}_{ab}A^aA^b + Q^d{}_{ab}\partial_cA^aA^b + Q^d{}_{ab}A^a\partial_cA^b\right)$$

$$\texttt{eq1.004} := A^c\partial_cQ^d{}_{ab}A^aA^b + A^cQ^d{}_{ab}\partial_cA^aA^b + A^cQ^d{}_{ab}A^a\partial_cA^b$$

$$\texttt{eq1.005} := A^c\partial_cQ^d{}_{ab}A^aA^b - Q^a{}_{ce}A^cA^eQ^d{}_{ab}A^b - Q^b{}_{ec}A^eA^cQ^d{}_{ab}A^a$$

$$\texttt{eq1.006} := A^c\partial_cQ^d{}_{ab}A^aA^b - Q^a{}_{ce}A^cA^eQ^d{}_{ab}A^b - Q^b{}_{ec}A^eA^cQ^d{}_{ab}A^a$$

$$\texttt{eq1.007} := A^c\partial_cQ^d{}_{ab}A^aA^b - Q^a{}_{ce}A^cA^eQ^d{}_{ab}A^b - Q^b{}_{ec}A^eA^cQ^d{}_{ab}A^a$$

$$\texttt{eq1.008} := A^aA^bA^c\partial_cQ^d{}_{ab} - A^bA^cA^eQ^a{}_{ce}Q^d{}_{ab} - A^aA^cA^eQ^b{}_{ec}Q^d{}_{ab}$$

$$\texttt{eq1.009} := A^aA^bA^c\partial_cQ^d{}_{ab} - A^aA^bA^cQ^e{}_{bc}Q^d{}_{ea} - A^aA^bA^cQ^e{}_{cb}Q^d{}_{ae}$$

$$\texttt{eq1.010} := A^aA^bA^c\partial_aQ^d{}_{bc} - 2A^aA^bA^cQ^d{}_{ae}Q^e{}_{bc}$$

$$\texttt{eq2.000} := A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} - 2A^{a}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf}\right)$$

$$\texttt{eq2.001} := A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf}\right) - 2A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf}\right)$$

$$\texttt{eq2.002} := A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf}\right) - 2A^{c}\partial_{c}\left(A^{a}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf}\right)$$

$$\begin{aligned}\texttt{eq2.003} := {} & A^{c}\left(\partial_{c}A^{a}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} + A^{a}\partial_{c}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} + A^{a}A^{b}\partial_{c}A^{f}\partial_{a}Q^{d}{}_{bf} + A^{a}A^{b}A^{f}\partial_{ca}Q^{d}{}_{bf}\right) \\ & - 2A^{c}\left(\partial_{c}A^{a}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} + A^{a}\partial_{c}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} + A^{a}A^{b}\partial_{c}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} + A^{a}A^{b}A^{f}\partial_{c}Q^{d}{}_{ae}Q^{e}{}_{bf} + A^{a}A^{b}A^{f}Q^{d}{}_{ae}\partial_{c}Q^{e}{}_{bf}\right)\end{aligned}$$

$$\begin{aligned}\texttt{eq2.004} := {} & A^{c}\partial_{c}A^{a}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} + A^{c}A^{a}\partial_{c}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} + A^{c}A^{a}A^{b}\partial_{c}A^{f}\partial_{a}Q^{d}{}_{bf} + A^{c}A^{a}A^{b}A^{f}\partial_{ca}Q^{d}{}_{bf} - 2A^{c}\partial_{c}A^{a}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} \\ & - 2A^{c}A^{a}\partial_{c}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} - 2A^{c}A^{a}A^{b}\partial_{c}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}Q^{d}{}_{ae}Q^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}Q^{d}{}_{ae}\partial_{c}Q^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.005} := {} & -Q^{a}{}_{ce}A^{c}A^{e}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} - Q^{b}{}_{ec}A^{e}A^{c}A^{a}A^{f}\partial_{a}Q^{d}{}_{bf} - Q^{f}{}_{ce}A^{c}A^{e}A^{a}A^{b}\partial_{a}Q^{d}{}_{bf} + A^{c}A^{a}A^{b}A^{f}\partial_{ca}Q^{d}{}_{bf} + 2Q^{a}{}_{cg}A^{c}A^{g}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} \\ & + 2Q^{b}{}_{gc}A^{g}A^{c}A^{a}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} + 2Q^{f}{}_{cg}A^{c}A^{g}A^{a}A^{b}Q^{d}{}_{ae}Q^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}Q^{d}{}_{ae}Q^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}Q^{d}{}_{ae}\partial_{c}Q^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.006} := {} & -Q^{a}{}_{ce}A^{c}A^{e}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} - Q^{b}{}_{ec}A^{e}A^{c}A^{a}A^{f}\partial_{a}Q^{d}{}_{bf} - Q^{f}{}_{ce}A^{c}A^{e}A^{a}A^{b}\partial_{a}Q^{d}{}_{bf} + A^{c}A^{a}A^{b}A^{f}\partial_{ca}Q^{d}{}_{bf} + 2Q^{a}{}_{cg}A^{c}A^{g}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} \\ & + 2Q^{b}{}_{gc}A^{g}A^{c}A^{a}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} + 2Q^{f}{}_{cg}A^{c}A^{g}A^{a}A^{b}Q^{d}{}_{ae}Q^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}Q^{d}{}_{ae}Q^{e}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}Q^{d}{}_{ae}\partial_{c}Q^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.007} := {} & A^{c}A^{a}A^{b}A^{f}\partial_{ca}Q^{d}{}_{bf} - Q^{a}{}_{ce}A^{c}A^{e}A^{b}A^{f}\partial_{a}Q^{d}{}_{bf} - Q^{b}{}_{ec}A^{e}A^{c}A^{a}A^{f}\partial_{a}Q^{d}{}_{bf} - Q^{f}{}_{ce}A^{c}A^{e}A^{a}A^{b}\partial_{a}Q^{d}{}_{bf} - 2A^{c}A^{a}A^{b}A^{f}\partial_{c}Q^{d}{}_{ae}Q^{e}{}_{bf} \\ & - 2A^{c}A^{a}A^{b}A^{f}Q^{d}{}_{ae}\partial_{c}Q^{e}{}_{bf} + 2Q^{a}{}_{cg}A^{c}A^{g}A^{b}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} + 2Q^{b}{}_{gc}A^{g}A^{c}A^{a}A^{f}Q^{d}{}_{ae}Q^{e}{}_{bf} + 2Q^{f}{}_{cg}A^{c}A^{g}A^{a}A^{b}Q^{d}{}_{ae}Q^{e}{}_{bf}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.008} := {} & A^{a}A^{b}A^{c}A^{f}\partial_{ca}Q^{d}{}_{bf} - A^{b}A^{c}A^{e}A^{f}Q^{a}{}_{ce}\partial_{a}Q^{d}{}_{bf} - A^{a}A^{c}A^{e}A^{f}Q^{b}{}_{ec}\partial_{a}Q^{d}{}_{bf} - A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{ce}\partial_{a}Q^{d}{}_{bf} - 2A^{a}A^{b}A^{c}A^{f}Q^{e}{}_{bf}\partial_{c}Q^{d}{}_{ae} \\ & - 2A^{a}A^{b}A^{c}A^{f}Q^{d}{}_{ae}\partial_{c}Q^{e}{}_{bf} + 2A^{b}A^{c}A^{f}A^{g}Q^{a}{}_{cg}Q^{d}{}_{ae}Q^{e}{}_{bf} + 2A^{a}A^{c}A^{f}A^{g}Q^{b}{}_{gc}Q^{d}{}_{ae}Q^{e}{}_{bf} + 2A^{a}A^{b}A^{c}A^{g}Q^{d}{}_{ae}Q^{e}{}_{bf}Q^{f}{}_{cg}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.009} := {} & A^{a}A^{b}A^{c}A^{e}\partial_{ca}Q^{d}{}_{be} - A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{bc}\partial_{f}Q^{d}{}_{ae} - A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{cb}\partial_{a}Q^{d}{}_{fe} - A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{ce}\partial_{a}Q^{d}{}_{bf} - 2A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{be}\partial_{c}Q^{d}{}_{af} \\ & - 2A^{a}A^{b}A^{c}A^{e}Q^{d}{}_{af}\partial_{c}Q^{f}{}_{be} + 2A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{be}Q^{d}{}_{fg}Q^{g}{}_{ac} + 2A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{eb}Q^{d}{}_{ag}Q^{g}{}_{fc} + 2A^{a}A^{b}A^{c}A^{e}Q^{d}{}_{af}Q^{f}{}_{bg}Q^{g}{}_{ce}\end{aligned}$$

$$\begin{aligned}\texttt{eq2.010} := {} & A^{a}A^{b}A^{c}A^{e}\partial_{ab}Q^{d}{}_{ce} - A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{ab}\partial_{f}Q^{d}{}_{ce} - 4A^{a}A^{b}A^{c}A^{e}Q^{f}{}_{ab}\partial_{c}Q^{d}{}_{ef} \\ & - 2A^{a}A^{b}A^{c}A^{e}Q^{d}{}_{af}\partial_{b}Q^{f}{}_{ce} + 2A^{a}A^{b}A^{c}A^{e}Q^{d}{}_{fg}Q^{f}{}_{ab}Q^{g}{}_{ce} + 4A^{a}A^{b}A^{c}A^{e}Q^{d}{}_{af}Q^{f}{}_{bg}Q^{g}{}_{ce}\end{aligned}$$

$$
\begin{aligned}
\texttt{eq3.010} := {}& A^aA^bA^cA^eA^f\partial_{abc}Q^d{}_{ef} - A^aA^bA^cA^eA^f\partial_gQ^d{}_{ab}\partial_cQ^g{}_{ef} - 6A^aA^bA^cA^eA^f\partial_aQ^d{}_{bg}\partial_cQ^g{}_{ef} - 3A^aA^bA^cA^eA^fQ^g{}_{ab}\partial_{cg}Q^d{}_{ef}\\
& - 6A^aA^bA^cA^eA^fQ^g{}_{ab}\partial_{ce}Q^d{}_{fg} - 2A^aA^bA^cA^eA^fQ^d{}_{ag}\partial_{bc}Q^g{}_{ef} + 2A^aA^bA^cA^eA^fQ^g{}_{ab}Q^h{}_{cg}\partial_hQ^d{}_{ef} + 6A^aA^bA^cA^eA^fQ^g{}_{ab}Q^h{}_{ce}\partial_gQ^d{}_{fh}\\
& + 12A^aA^bA^cA^eA^fQ^g{}_{ab}Q^h{}_{cg}\partial_eQ^d{}_{fh} + 6A^aA^bA^cA^eA^fQ^g{}_{ab}Q^h{}_{ce}\partial_fQ^d{}_{gh} + 6A^aA^bA^cA^eA^fQ^d{}_{gh}Q^g{}_{ab}\partial_cQ^h{}_{ef}\\
& + 2A^aA^bA^cA^eA^fQ^d{}_{ag}Q^h{}_{bc}\partial_hQ^g{}_{ef} + 8A^aA^bA^cA^eA^fQ^d{}_{ag}Q^h{}_{bc}\partial_eQ^g{}_{fh} + 4A^aA^bA^cA^eA^fQ^d{}_{ag}Q^g{}_{bh}\partial_cQ^h{}_{ef}\\
& - 12A^aA^bA^cA^eA^fQ^d{}_{gh}Q^g{}_{ab}Q^h{}_{ci}Q^i{}_{ef} - 4A^aA^bA^cA^eA^fQ^d{}_{ag}Q^g{}_{hi}Q^h{}_{bc}Q^i{}_{ef} - 8A^aA^bA^cA^eA^fQ^d{}_{ag}Q^g{}_{bh}Q^h{}_{ci}Q^i{}_{ef}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{eq4.010} := {} & A^a A^b A^c A^e A^f A^g \partial_{abce} Q^{d}{}_{fg} - 4 A^a A^b A^c A^e A^f A^g \partial_a Q^{h}{}_{bc} \partial_{eh} Q^{d}{}_{fg} - A^a A^b A^c A^e A^f A^g \partial_h Q^{d}{}_{ab} \partial_{ce} Q^{h}{}_{fg} - 12 A^a A^b A^c A^e A^f A^g \partial_a Q^{h}{}_{bc} \partial_{ef} Q^{d}{}_{gh} \\
& - 8 A^a A^b A^c A^e A^f A^g \partial_a Q^{d}{}_{bh} \partial_{ce} Q^{h}{}_{fg} - 6 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_{ceh} Q^{d}{}_{fg} - 8 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_{cef} Q^{d}{}_{gh} \\
& + 8 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_i Q^{d}{}_{ch} \partial_e Q^{i}{}_{fg} + A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_i Q^{d}{}_{ce} \partial_h Q^{i}{}_{fg} + 4 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_i Q^{d}{}_{ce} \partial_f Q^{i}{}_{gh} \\
& + 12 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_h Q^{d}{}_{ci} \partial_e Q^{i}{}_{fg} + 24 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_c Q^{d}{}_{hi} \partial_e Q^{i}{}_{fg} + 8 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_c Q^{d}{}_{ei} \partial_h Q^{i}{}_{fg} \\
& + 32 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} \partial_c Q^{d}{}_{ei} \partial_f Q^{i}{}_{gh} - 2 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} \partial_{bce} Q^{h}{}_{fg} + 2 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ai} \partial_h Q^{d}{}_{bc} \partial_e Q^{i}{}_{fg} \\
& + 16 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ai} \partial_b Q^{d}{}_{ch} \partial_e Q^{i}{}_{fg} + 6 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} \partial_a Q^{h}{}_{bc} \partial_e Q^{i}{}_{fg} + 2 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} \partial_b Q^{i}{}_{ce} \partial_i Q^{h}{}_{fg} \\
& + 12 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} \partial_b Q^{h}{}_{ci} \partial_e Q^{i}{}_{fg} + 8 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ch} \partial_{ei} Q^{d}{}_{fg} + 3 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} \partial_{hi} Q^{d}{}_{fg} \\
& + 24 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} \partial_{fh} Q^{d}{}_{gi} + 24 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ch} \partial_{ef} Q^{d}{}_{gi} + 12 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} \partial_{fg} Q^{d}{}_{hi} \\
& + 8 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{ab} \partial_{ce} Q^{i}{}_{fg} + 6 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{i}{}_{bc} \partial_{ei} Q^{h}{}_{fg} + 12 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{i}{}_{bc} \partial_{ef} Q^{h}{}_{gi} \\
& + 4 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{bi} \partial_{ce} Q^{i}{}_{fg} - 4 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ch} Q^{j}{}_{ei} \partial_j Q^{d}{}_{fg} - 2 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} Q^{j}{}_{hi} \partial_j Q^{d}{}_{fg} \\
& - 16 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} Q^{j}{}_{fh} \partial_j Q^{d}{}_{gi} - 24 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} Q^{j}{}_{fh} \partial_i Q^{d}{}_{gj} - 12 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} Q^{j}{}_{fg} \partial_h Q^{d}{}_{ij} \\
& - 32 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ch} Q^{j}{}_{ei} \partial_f Q^{d}{}_{gj} - 16 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} Q^{j}{}_{hi} \partial_f Q^{d}{}_{gj} - 48 A^a A^b A^c A^e A^f A^g Q^{h}{}_{ab} Q^{i}{}_{ce} Q^{j}{}_{fh} \partial_g Q^{d}{}_{ij} \\
& - 24 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{aj} Q^{j}{}_{bc} \partial_e Q^{i}{}_{fg} - 8 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{ab} Q^{j}{}_{ce} \partial_j Q^{i}{}_{fg} - 32 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{ab} Q^{j}{}_{ce} \partial_f Q^{i}{}_{gj} \\
& - 4 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{i}{}_{bc} Q^{j}{}_{ei} \partial_j Q^{h}{}_{fg} - 12 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{i}{}_{bc} Q^{j}{}_{ef} \partial_i Q^{h}{}_{gj} - 24 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{i}{}_{bc} Q^{j}{}_{ei} \partial_f Q^{h}{}_{gj} \\
& - 12 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{i}{}_{bc} Q^{j}{}_{ef} \partial_g Q^{h}{}_{ij} - 16 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{ab} Q^{i}{}_{cj} \partial_e Q^{j}{}_{fg} - 12 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{ij} Q^{i}{}_{bc} \partial_e Q^{j}{}_{fg} \\
& - 4 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{bi} Q^{j}{}_{ce} \partial_j Q^{i}{}_{fg} - 16 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{bi} Q^{j}{}_{ce} \partial_f Q^{i}{}_{gj} - 8 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{bi} Q^{i}{}_{cj} \partial_e Q^{j}{}_{fg} \\
& + 24 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{aj} Q^{i}{}_{bk} Q^{j}{}_{ce} Q^{k}{}_{fg} + 16 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{ab} Q^{i}{}_{jk} Q^{j}{}_{ce} Q^{k}{}_{fg} + 32 A^a A^b A^c A^e A^f A^g Q^{d}{}_{hi} Q^{h}{}_{ab} Q^{i}{}_{cj} Q^{j}{}_{ek} Q^{k}{}_{fg} \\
& + 24 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{ij} Q^{i}{}_{bc} Q^{j}{}_{ek} Q^{k}{}_{fg} + 8 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{bi} Q^{i}{}_{jk} Q^{j}{}_{ce} Q^{k}{}_{fg} + 16 A^a A^b A^c A^e A^f A^g Q^{d}{}_{ah} Q^{h}{}_{bi} Q^{i}{}_{cj} Q^{j}{}_{ek} Q^{k}{}_{fg}
\end{aligned}
$$

```
def reformat (obj):
   bah := @(obj).
   distribute     (bah)
   bah = product_sort (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$A^{a?}$)
   substitute     (bah,$A^{a}->y^{a}$)
   substitute     (bah,$Q^{a}_{b c}->\Gamma^{a}_{b c}$)
   ans := @(bah).
   return ans

eq0 = reformat(eq0)  # cdb (eq0.100,eq0)
eq1 = reformat(eq1)  # cdb (eq1.100,eq1)
eq2 = reformat(eq2)  # cdb (eq2.100,eq2)
eq3 = reformat(eq3)  # cdb (eq3.100,eq3)
eq4 = reformat(eq4)  # cdb (eq4.100,eq4)

checkpoint.append (eq0)
checkpoint.append (eq1)
checkpoint.append (eq2)
checkpoint.append (eq3)
checkpoint.append (eq4)
```

Convert from local RNC coords (y) to generic (x)

$$x^a = x^a_i + \overset{0}{x}{}^a - \overset{1}{x}{}^a - \overset{2}{x}{}^a - \overset{3}{x}{}^a - \overset{4}{x}{}^a - \overset{5}{x}{}^a$$

$$\begin{aligned}
\overset{0}{x}{}^a &= y^a \\
2!\,\overset{1}{x}{}^a &= y^a y^b \Gamma^d{}_{ab} \\
3!\,\overset{2}{x}{}^a &= y^a y^b y^c \left(\partial_a \Gamma^d{}_{bc} - 2\Gamma^d{}_{ae}\Gamma^e{}_{bc}\right) \\
4!\,\overset{3}{x}{}^a &= y^a y^b y^c y^e \left(\partial_{ab}\Gamma^d{}_{ce} - \Gamma^f{}_{ab}\partial_f\Gamma^d{}_{ce} - 4\Gamma^f{}_{ab}\partial_c\Gamma^d{}_{ef} - 2\Gamma^d{}_{af}\partial_b\Gamma^f{}_{ce} + 2\Gamma^d{}_{fg}\Gamma^f{}_{ab}\Gamma^g{}_{ce} + 4\Gamma^d{}_{af}\Gamma^f{}_{bg}\Gamma^g{}_{ce}\right) \\
5!\,\overset{4}{x}{}^a &= y^a y^b y^c y^e y^f \big(\partial_{abc}\Gamma^d{}_{ef} - \partial_g\Gamma^d{}_{ab}\partial_c\Gamma^g{}_{ef} - 6\partial_a\Gamma^d{}_{bg}\partial_c\Gamma^g{}_{ef} - 3\Gamma^g{}_{ab}\partial_{cg}\Gamma^d{}_{ef} - 6\Gamma^g{}_{ab}\partial_{ce}\Gamma^d{}_{fg} - 2\Gamma^d{}_{ag}\partial_{bc}\Gamma^g{}_{ef} + 2\Gamma^g{}_{ab}\Gamma^h{}_{cg}\partial_h\Gamma^d{}_{ef} + 6\Gamma^g{}_{ab}\Gamma^h{}_{ce}\partial_g\Gamma^d{}_{fh} \\
&\quad + 12\Gamma^g{}_{ab}\Gamma^h{}_{cg}\partial_e\Gamma^d{}_{fh} + 6\Gamma^g{}_{ab}\Gamma^h{}_{ce}\partial_f\Gamma^d{}_{gh} + 6\Gamma^d{}_{gh}\Gamma^g{}_{ab}\partial_c\Gamma^h{}_{ef} + 2\Gamma^d{}_{ag}\Gamma^h{}_{bc}\partial_h\Gamma^g{}_{ef} + 8\Gamma^d{}_{ag}\Gamma^h{}_{bc}\partial_e\Gamma^g{}_{fh} + 4\Gamma^d{}_{ag}\Gamma^g{}_{bh}\partial_c\Gamma^h{}_{ef} - 12\Gamma^d{}_{gh}\Gamma^g{}_{ab}\Gamma^h{}_{ci}\Gamma^i{}_{ef} \\
&\quad - 4\Gamma^d{}_{ag}\Gamma^g{}_{hi}\Gamma^h{}_{bc}\Gamma^i{}_{ef} - 8\Gamma^d{}_{ag}\Gamma^g{}_{bh}\Gamma^h{}_{ci}\Gamma^i{}_{ef}\big) \\
6!\,\overset{5}{x}{}^a &= y^a y^b y^c y^e y^f y^g \big(\partial_{abce}\Gamma^d{}_{fg} - 4\partial_a\Gamma^h{}_{bc}\partial_{eh}\Gamma^d{}_{fg} - \partial_h\Gamma^d{}_{ab}\partial_{ce}\Gamma^h{}_{fg} - 12\partial_a\Gamma^h{}_{bc}\partial_{ef}\Gamma^d{}_{gh} - 8\partial_a\Gamma^d{}_{bh}\partial_{ce}\Gamma^h{}_{fg} - 6\Gamma^h{}_{ab}\partial_{ceh}\Gamma^d{}_{fg} - 8\Gamma^h{}_{ab}\partial_{cef}\Gamma^d{}_{gh} \\
&\quad + 8\Gamma^h{}_{ab}\partial_i\Gamma^d{}_{ch}\partial_e\Gamma^i{}_{fg} + \Gamma^h{}_{ab}\partial_i\Gamma^d{}_{ce}\partial_h\Gamma^i{}_{fg} + 4\Gamma^h{}_{ab}\partial_i\Gamma^d{}_{ce}\partial_f\Gamma^i{}_{gh} + 12\Gamma^h{}_{ab}\partial_h\Gamma^d{}_{ci}\partial_e\Gamma^i{}_{fg} + 24\Gamma^h{}_{ab}\partial_c\Gamma^d{}_{hi}\partial_e\Gamma^i{}_{fg} + 8\Gamma^h{}_{ab}\partial_c\Gamma^d{}_{ei}\partial_h\Gamma^i{}_{fg} \\
&\quad + 32\Gamma^h{}_{ab}\partial_c\Gamma^d{}_{ei}\partial_f\Gamma^i{}_{gh} - 2\Gamma^d{}_{ah}\partial_{bce}\Gamma^h{}_{fg} + 2\Gamma^h{}_{ai}\partial_h\Gamma^d{}_{bc}\partial_e\Gamma^i{}_{fg} + 16\Gamma^h{}_{ai}\partial_b\Gamma^d{}_{ch}\partial_e\Gamma^i{}_{fg} + 6\Gamma^d{}_{hi}\partial_a\Gamma^h{}_{bc}\partial_e\Gamma^i{}_{fg} + 2\Gamma^d{}_{ah}\partial_b\Gamma^i{}_{ce}\partial_i\Gamma^h{}_{fg} \\
&\quad + 12\Gamma^d{}_{ah}\partial_b\Gamma^h{}_{ci}\partial_e\Gamma^i{}_{fg} + 8\Gamma^h{}_{ab}\Gamma^i{}_{ch}\partial_{ei}\Gamma^d{}_{fg} + 3\Gamma^h{}_{ab}\Gamma^i{}_{ce}\partial_{hi}\Gamma^d{}_{fg} + 24\Gamma^h{}_{ab}\Gamma^i{}_{ce}\partial_{fh}\Gamma^d{}_{gi} + 24\Gamma^h{}_{ab}\Gamma^i{}_{ch}\partial_{ef}\Gamma^d{}_{gi} + 12\Gamma^h{}_{ab}\Gamma^i{}_{ce}\partial_{fg}\Gamma^d{}_{hi} \\
&\quad + 8\Gamma^d{}_{hi}\Gamma^h{}_{ab}\partial_{ce}\Gamma^i{}_{fg} + 6\Gamma^d{}_{ah}\Gamma^i{}_{bc}\partial_{ei}\Gamma^h{}_{fg} + 12\Gamma^d{}_{ah}\Gamma^i{}_{bc}\partial_{ef}\Gamma^h{}_{gi} + 4\Gamma^d{}_{ah}\Gamma^h{}_{bi}\partial_{ce}\Gamma^i{}_{fg} - 4\Gamma^h{}_{ab}\Gamma^i{}_{ch}\Gamma^j{}_{ei}\partial_j\Gamma^d{}_{fg} - 2\Gamma^h{}_{ab}\Gamma^i{}_{ce}\Gamma^j{}_{hi}\partial_j\Gamma^d{}_{fg} \\
&\quad - 16\Gamma^h{}_{ab}\Gamma^i{}_{ce}\Gamma^j{}_{fh}\partial_j\Gamma^d{}_{gi} - 24\Gamma^h{}_{ab}\Gamma^i{}_{ce}\Gamma^j{}_{fh}\partial_i\Gamma^d{}_{gj} - 12\Gamma^h{}_{ab}\Gamma^i{}_{ce}\Gamma^j{}_{fg}\partial_h\Gamma^d{}_{ij} - 32\Gamma^h{}_{ab}\Gamma^i{}_{ch}\Gamma^j{}_{ei}\partial_f\Gamma^d{}_{gj} - 16\Gamma^h{}_{ab}\Gamma^i{}_{ce}\Gamma^j{}_{hi}\partial_f\Gamma^d{}_{gj} \\
&\quad - 48\Gamma^h{}_{ab}\Gamma^i{}_{ce}\Gamma^j{}_{fh}\partial_g\Gamma^d{}_{ij} - 24\Gamma^d{}_{hi}\Gamma^h{}_{aj}\Gamma^j{}_{bc}\partial_e\Gamma^i{}_{fg} - 8\Gamma^d{}_{hi}\Gamma^h{}_{ab}\Gamma^j{}_{ce}\partial_j\Gamma^i{}_{fg} - 32\Gamma^d{}_{hi}\Gamma^h{}_{ab}\Gamma^j{}_{ce}\partial_f\Gamma^i{}_{gj} - 4\Gamma^d{}_{ah}\Gamma^i{}_{bc}\Gamma^j{}_{ei}\partial_j\Gamma^h{}_{fg} \\
&\quad - 12\Gamma^d{}_{ah}\Gamma^i{}_{bc}\Gamma^j{}_{ef}\partial_i\Gamma^h{}_{gj} - 24\Gamma^d{}_{ah}\Gamma^i{}_{bc}\Gamma^j{}_{ei}\partial_f\Gamma^h{}_{gj} - 12\Gamma^d{}_{ah}\Gamma^i{}_{bc}\Gamma^j{}_{ef}\partial_g\Gamma^h{}_{ij} - 16\Gamma^d{}_{hi}\Gamma^h{}_{ab}\Gamma^i{}_{cj}\partial_e\Gamma^j{}_{fg} - 12\Gamma^d{}_{ah}\Gamma^h{}_{ij}\Gamma^i{}_{bc}\partial_e\Gamma^j{}_{fg} \\
&\quad - 4\Gamma^d{}_{ah}\Gamma^h{}_{bi}\Gamma^j{}_{ce}\partial_j\Gamma^i{}_{fg} - 16\Gamma^d{}_{ah}\Gamma^h{}_{bi}\Gamma^j{}_{ce}\partial_f\Gamma^i{}_{gj} - 8\Gamma^d{}_{ah}\Gamma^h{}_{bi}\Gamma^i{}_{cj}\partial_e\Gamma^j{}_{fg} + 24\Gamma^d{}_{hi}\Gamma^h{}_{aj}\Gamma^i{}_{bk}\Gamma^j{}_{ce}\Gamma^k{}_{fg} + 16\Gamma^d{}_{hi}\Gamma^h{}_{ab}\Gamma^i{}_{jk}\Gamma^j{}_{ce}\Gamma^k{}_{fg} \\
&\quad + 32\Gamma^d{}_{hi}\Gamma^h{}_{ab}\Gamma^i{}_{cj}\Gamma^j{}_{ek}\Gamma^k{}_{fg} + 24\Gamma^d{}_{ah}\Gamma^h{}_{ij}\Gamma^i{}_{bc}\Gamma^j{}_{ek}\Gamma^k{}_{fg} + 8\Gamma^d{}_{ah}\Gamma^h{}_{bi}\Gamma^i{}_{jk}\Gamma^j{}_{ce}\Gamma^k{}_{fg} + 16\Gamma^d{}_{ah}\Gamma^h{}_{bi}\Gamma^i{}_{cj}\Gamma^j{}_{ek}\Gamma^k{}_{fg}\big)
\end{aligned}$$

From one RNC to another

Consider an RNC frame with RNC cooridnates x^a.

In the `geodesic-bvp` code the two point boundary value problem (for the geodesic connecting two points) was solved. There is a bonus in that calculation – it can be trivaly adapted to the case of transforming form one RNC into another.

The starting point is the basic equation for the geodesic connecting P (with coordinaties x^a) to Q (with coordinates $x^a + Dx^a$)

$$x^a(s) = x^a_i + sy^a - \sum_{k=2}^{\infty} \frac{1}{k!}\,\Gamma^a{}_{\underline{b}_k} y^{\cdot\underline{b}_k} s^k$$

The affine parameter s varies form 0 (at P) to 1 (at Q).

A new RNC frame, with origin at P, can be defined via the y^a with the coordinates of Q in the new RNC frame defined by y^a (since $s = 1$ at Q). Recall that in an RNC all geodesics through the origin are described by $y^a(s) = sy^a$. Thus the transformation from x^a to y^a satisfies

$$x^a = x^a_i + y^a - \sum_{k=2}^{\infty} \frac{1}{k!}\,\Gamma^a{}_{\underline{b}_k} y^{\cdot\underline{b}_k}$$

where the $\Gamma^a{}_{\underline{b}_k}$ are the generalised connections of the x^a frame evaluated at $x^a = 0$. This equation can be inverted to express y^a in terms of x^a. This computation is done in the `geodesic-bvp` code – we only quote the results here (at the end).

The new y^a frame has origin at P. Its coordinate axes are aligned with those (at P) of the origianl RNC frame. To see this just note that $\partial x^a/\partial y^b = \delta^a_b$ at P. Thus the metric at P in the new frame has values $g_{ab}(x)$ (i.e., exactly those of the original RNC frame). Note that this means that the coordinate axes of the new frame are not ncessarily orthogonal.

The calculations in this code are trivial. It uses the y^a found in `geodesic-bvp` as the basis of the transformation from x^a to y^a. Most of the code involves reformatting the y^a.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

# Dx{#}::LaTeXForm{"{\Dx}"}.  # LCB: currently causes a bug, it kills ::KeepWeight for Dx

import cdblib

Y5 = cdblib.get ('y5','geodesic-bvp.json')

Y50 = cdblib.get ('y50','geodesic-bvp.json')
Y52 = cdblib.get ('y52','geodesic-bvp.json')
Y53 = cdblib.get ('y53','geodesic-bvp.json')
Y54 = cdblib.get ('y54','geodesic-bvp.json')
Y55 = cdblib.get ('y55','geodesic-bvp.json')

# this copies y5* from geodesic-bvp.json to rnc2rnc.json

cdblib.create ('rnc2rnc.json')

cdblib.put ('rnc2rnc',Y5,'rnc2rnc.json')

cdblib.put ('rnc2rnc0',Y50,'rnc2rnc.json')
cdblib.put ('rnc2rnc2',Y52,'rnc2rnc.json')
cdblib.put ('rnc2rnc3',Y53,'rnc2rnc.json')
cdblib.put ('rnc2rnc4',Y54,'rnc2rnc.json')
cdblib.put ('rnc2rnc5',Y55,'rnc2rnc.json')
```

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}                              -> A001^{a}              $)
    substitute (obj,$ Dx^{a}                             -> A002^{a}              $)
    substitute (obj,$ g^{a b}                            -> A003^{a b}            $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                        -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                 -> x^{a}                           $)
    substitute (obj,$ A002^{a}                 -> Dx^{a}                          $)
    substitute (obj,$ A003^{a b}               -> g^{a b}                         $)
    substitute (obj,$ A004_{a b c d}           -> R_{a b c d}                     $)
    substitute (obj,$ A005_{a b c d e}         -> \nabla_{e}{R_{a b c d}}         $)
    substitute (obj,$ A006_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}}       $)
    substitute (obj,$ A007_{a b c d e f g}     -> \nabla_{e f g}{R_{a b c d}}     $)
    substitute (obj,$ A008_{a b c d e f g h}   -> \nabla_{e f g h}{R_{a b c d}}   $)

    return obj

def get_xDxterm (obj,n,m):

    x^{a}::Weight(label=numx,value=1).
    Dx^{a}::Weight(label=numDx,value=1).

    tmp := @(obj).
    distribute  (tmp)

    foo = Ex("numx = " + str(n))
    bah = Ex("numDx = " + str(m))
    keep_weight (tmp, foo)
    keep_weight (tmp, bah)

    return tmp
```

```
def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    substitute     (bah,$Dx^{b}->zzz^{b}$)
    factor_out     (bah,$x^{a?},zzz^{b?}$)
    substitute     (bah,$zzz^{b}->Dx^{b}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute  (bah)
    substitute  (bah,$Dx^{b}->zzz^{b}$)
    factor_out  (bah,$x^{a?},zzz^{b?}$)
    substitute  (bah,$zzz^{b}->Dx^{b}$)
    return bah

term0 := @(Y50).  # cdb (term0.101,term0)
term2 := @(Y52).  # cdb (term2.101,term2)
term3 := @(Y53).  # cdb (term3.101,term3)
term4 := @(Y54).  # cdb (term4.101,term4)
term5 := @(Y55).  # cdb (term5.101,term5)

term0 = reformat (term0,1)  # cdb (term0.102,term0)
term2 = reformat (term2,1)  # cdb (term2.102,term2)
term3 = reformat (term3,1)  # cdb (term3.102,term3)
term4 = reformat (term4,1)  # cdb (term4.102,term4)
term5 = reformat (term5,1)  # cdb (term5.102,term5)

xDxterm12 = get_xDxterm (term2,1,2)   # cdb(xDxterm12.101,xDxterm12)

xDxterm13 = get_xDxterm (term3,1,3)   # cdb(xDxterm13.101,xDxterm13)
xDxterm22 = get_xDxterm (term3,2,2)   # cdb(xDxterm22.101,xDxterm22)
```

```
xDxterm14 = get_xDxterm (term4,1,4)   # cdb(xDxterm14.101,xDxterm14)
xDxterm23 = get_xDxterm (term4,2,3)   # cdb(xDxterm23.101,xDxterm23)
xDxterm32 = get_xDxterm (term4,3,2)   # cdb(xDxterm32.101,xDxterm32)

xDxterm15 = get_xDxterm (term5,1,5)   # cdb(xDxterm15.101,xDxterm15)
xDxterm24 = get_xDxterm (term5,2,4)   # cdb(xDxterm24.101,xDxterm24)
xDxterm33 = get_xDxterm (term5,3,3)   # cdb(xDxterm33.101,xDxterm33)
xDxterm42 = get_xDxterm (term5,4,2)   # cdb(xDxterm42.101,xDxterm42)


xDxterm12 = rescale ( reformat (xDxterm12,    3),     3 )   # cdb(xDxterm12.102,xDxterm12)

xDxterm13 = rescale ( reformat (xDxterm13,   12),   -12 )   # cdb(xDxterm13.102,xDxterm13)
xDxterm22 = rescale ( reformat (xDxterm22,   24),   -24 )   # cdb(xDxterm22.102,xDxterm22)

xDxterm14 = rescale ( reformat (xDxterm14,  180),  -180 )   # cdb(xDxterm14.102,xDxterm14)
xDxterm23 = rescale ( reformat (xDxterm23,  720),  -720 )   # cdb(xDxterm23.102,xDxterm23)
xDxterm32 = rescale ( reformat (xDxterm32,  720),  -720 )   # cdb(xDxterm32.102,xDxterm32)

xDxterm15 = rescale ( reformat (xDxterm15,  360),  -360 )   # cdb(xDxterm15.102,xDxterm15)
xDxterm24 = rescale ( reformat (xDxterm24, 2160), -2160 )   # cdb(xDxterm24.102,xDxterm24)
xDxterm33 = rescale ( reformat (xDxterm33, 1080), -1080 )   # cdb(xDxterm33.102,xDxterm33)
xDxterm42 = rescale ( reformat (xDxterm42,  360),  -360 )   # cdb(xDxterm42.102,xDxterm42)

checkpoint.append (term0)
checkpoint.append (term2)
checkpoint.append (term3)
checkpoint.append (term4)
checkpoint.append (term5)
```

Tranformation between two RNC frames

$$y^a = \overset{0}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a + \overset{5}{y}{}^a + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{y}{}^a = Dx^a$$

$$\overset{2}{y}{}^a = -\frac{1}{3} x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\overset{3}{y}{}^a = x^b x^c Dx^d Dx^e \left(-\frac{1}{12} g^{af} \nabla_d R_{becf} - \frac{1}{6} g^{af} \nabla_b R_{cdef} + \frac{1}{24} g^{af} \nabla_f R_{bdce} \right) - \frac{1}{12} x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned}
\overset{4}{y}{}^a = x^b x^c Dx^d Dx^e Dx^f & \left(-\frac{2}{45} g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{1}{45} g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{1}{45} g^{ag} g^{hi} R_{bdeh} R_{cgfi} + \frac{1}{45} g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{60} g^{ag} \nabla_{de} R_{bfcg} - \frac{1}{40} g^{ag} \nabla_{db} R_{cefg} \right. \\
& \left. - \frac{1}{40} g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240} g^{ag} \nabla_{gd} R_{becf} + \frac{1}{240} g^{ag} \nabla_{dg} R_{becf} \right) \\
+ x^b x^c x^d Dx^e Dx^f & \left(-\frac{4}{45} g^{ag} g^{hi} R_{befh} R_{cgdi} + \frac{2}{45} g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45} g^{ag} g^{hi} R_{bech} R_{dgfi} - \frac{1}{40} g^{ag} \nabla_{eb} R_{cfdg} - \frac{1}{40} g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20} g^{ag} \nabla_{bc} R_{defg} \right. \\
& \left. - \frac{1}{45} g^{ag} g^{hi} R_{bech} R_{dfgi} + \frac{1}{80} g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80} g^{ag} \nabla_{bg} R_{cedf} \right) + x^b Dx^c Dx^d Dx^e Dx^f \left(-\frac{1}{45} g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{1}{60} g^{ag} \nabla_{cd} R_{befg} \right)
\end{aligned}$$

$$
\begin{aligned}
\overset{5}{y}{}^a = x^b x^c x^d Dx^e Dx^f Dx^g \Big(& -\frac{7}{540} g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} - \frac{1}{45} g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + \frac{1}{216} g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} + \frac{1}{90} g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\
& + \frac{1}{90} g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} - \frac{1}{540} g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{108} g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} - \frac{1}{45} g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + \frac{1}{90} g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} \\
& - \frac{7}{540} g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} - \frac{1}{90} g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - \frac{1}{540} g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} - \frac{11}{540} g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{1}{90} g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
& + \frac{1}{216} g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} + \frac{1}{90} g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + \frac{1}{90} g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + \frac{1}{135} g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + \frac{1}{90} g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\
& - \frac{1}{270} g^{ah} \nabla_{efb} R_{cgdh} - \frac{1}{270} g^{ah} \nabla_{ebf} R_{cgdh} - \frac{1}{180} g^{ah} \nabla_{ebc} R_{dfgh} - \frac{1}{270} g^{ah} \nabla_{bef} R_{cgdh} - \frac{1}{180} g^{ah} \nabla_{bec} R_{dfgh} - \frac{1}{180} g^{ah} \nabla_{bce} R_{dfgh} - \frac{1}{270} g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} \\
& - \frac{1}{270} g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} + \frac{1}{1080} g^{ah} \nabla_{heb} R_{cfdg} + \frac{1}{1080} g^{ah} \nabla_{hbe} R_{cfdg} + \frac{1}{1080} g^{ah} \nabla_{ehb} R_{cfdg} + \frac{1}{1080} g^{ah} \nabla_{bhe} R_{cfdg} + \frac{1}{1080} g^{ah} \nabla_{ebh} R_{cfdg} \\
& + \frac{1}{1080} g^{ah} \nabla_{beh} R_{cfdg} \Big) + x^b x^c Dx^d Dx^e Dx^f Dx^g \Big(-\frac{17}{1080} g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} + \frac{1}{135} g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} - \frac{7}{1080} g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} \\
& - \frac{1}{540} g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + \frac{1}{108} g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh} - \frac{1}{120} g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - \frac{1}{90} g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} - \frac{1}{540} g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} \\
& - \frac{1}{120} g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} - \frac{1}{90} g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} + \frac{1}{216} g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} + \frac{1}{135} g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} - \frac{1}{360} g^{ah} \nabla_{def} R_{bgch} - \frac{1}{270} g^{ah} \nabla_{deb} R_{cfgh} \\
& - \frac{1}{270} g^{ah} \nabla_{dbe} R_{cfgh} - \frac{1}{270} g^{ah} \nabla_{bde} R_{cfgh} + \frac{1}{360} g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} + \frac{1}{2160} g^{ah} \nabla_{hde} R_{bfcg} + \frac{1}{2160} g^{ah} \nabla_{dhe} R_{bfcg} + \frac{1}{2160} g^{ah} \nabla_{deh} R_{bfcg} \Big) \\
& + x^b x^c x^d x^e Dx^f Dx^g \Big(-\frac{2}{45} g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} - \frac{1}{60} g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} - \frac{2}{45} g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + \frac{1}{72} g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\
& + \frac{1}{45} g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{1}{90} g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + \frac{1}{45} g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} \\
& - \frac{1}{180} g^{ah} \nabla_{fbc} R_{dgeh} - \frac{1}{180} g^{ah} \nabla_{bfc} R_{dgeh} - \frac{1}{180} g^{ah} \nabla_{bcf} R_{dgeh} - \frac{1}{90} g^{ah} \nabla_{bcd} R_{efgh} - \frac{1}{90} g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} - \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} \\
& - \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + \frac{1}{360} g^{ah} \nabla_{hbc} R_{dfeg} + \frac{1}{360} g^{ah} \nabla_{bhc} R_{dfeg} + \frac{1}{360} g^{ah} \nabla_{bch} R_{dfeg} \Big) \\
& + x^b Dx^c Dx^d Dx^e Dx^f Dx^g \Big(-\frac{1}{120} g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - \frac{1}{120} g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - \frac{1}{360} g^{ah} \nabla_{cde} R_{bfgh} \Big)
\end{aligned}
$$

Tranformation between two RNC frames

Same as before but with an improved format (maybe) for the expressions.

$$y^a = \overset{0}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a + \overset{5}{y}{}^a + \mathcal{O}\left(\epsilon^6\right) \tag{1}$$

$$\overset{0}{y}{}^a = Dx^a \tag{2a}$$

$$\overset{2}{y}{}^a = \overset{2}{y}{}^a_1 \tag{3a}$$

$$3\overset{2}{y}{}^a_1 = -x^b Dx^c Dx^d g^{ae} R_{bcde} \tag{3b}$$

$$\overset{3}{y}{}^a = \overset{3}{y}{}^a_1 + \overset{3}{y}{}^a_2 \tag{4a}$$

$$-12\overset{3}{y}{}^a_1 = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} \tag{4b}$$

$$-24\overset{3}{y}{}^a_2 = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right) \tag{4c}$$

$$\overset{4}{y}{}^a = \overset{4}{y}{}^a_1 + \overset{4}{y}{}^a_2 + \overset{4}{y}{}^a_3 \tag{5a}$$

$$-180\overset{4}{y}{}^a_1 = x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right) \tag{5b}$$

$$\begin{aligned}-720\overset{4}{y}{}^a_2 = x^b x^c Dx^d Dx^e Dx^f \big(&32g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} + 12g^{ag}\nabla_{de}R_{bfcg} \\ &+ 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\big)\end{aligned} \tag{5c}$$

$$\begin{aligned}-720\overset{4}{y}{}^a_3 = x^b x^c x^d Dx^e Dx^f \big(&64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} \\ &+ 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\big)\end{aligned} \tag{5d}$$

$$\overset{5}{y}{}^a = \overset{5}{y}{}^a_1 + \overset{5}{y}{}^a_2 + \overset{5}{y}{}^a_3 + \overset{5}{y}{}^a_4 \tag{6a}$$

$$-360\overset{5}{y}{}^a_1 = x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(3g^{ah}g^{ij}R_{bcdi}\nabla_e R_{fhgj} + 3g^{ah}g^{ij}R_{chdi}\nabla_e R_{bfgj} + g^{ah}\nabla_{cde}R_{bfgh}\right) \tag{6b}$$

$$\begin{aligned}
-2160\overset{5}{y}{}^a_2 = x^b x^c Dx^d Dx^e Dx^f Dx^g \big(&34g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 16g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} + 14g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} \\
&- 20g^{ah}g^{ij}R_{bdei}\nabla_j R_{cfgh} + 18g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} + 24g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} + 4g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 18g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj} \\
&+ 24g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} - 10g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}\nabla_{def}R_{bgch} + 8g^{ah}\nabla_{deb}R_{cfgh} + 8g^{ah}\nabla_{dbe}R_{cfgh} \\
&+ 8g^{ah}\nabla_{bde}R_{cfgh} - 6g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - g^{ah}\nabla_{hde}R_{bfcg} - g^{ah}\nabla_{dhe}R_{bfcg} - g^{ah}\nabla_{deh}R_{bfcg}\big)
\end{aligned} \tag{6c}$$

$$\begin{aligned}
-1080\overset{5}{y}{}^a_3 = x^b x^c x^d Dx^e Dx^f Dx^g \big(&14g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 24g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} \\
&- 12g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} + 2g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 24g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} \\
&+ 14g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} + 2g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 22g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} + 12g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} \\
&- 5g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} - 12g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} - 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} - 8g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} - 12g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} \\
&+ 4g^{ah}\nabla_{efb}R_{cgdh} + 4g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{ebc}R_{dfgh} + 4g^{ah}\nabla_{bef}R_{cgdh} + 6g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} \\
&+ 4g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - g^{ah}\nabla_{heb}R_{cfdg} - g^{ah}\nabla_{hbe}R_{cfdg} - g^{ah}\nabla_{ehb}R_{cfdg} - g^{ah}\nabla_{bhe}R_{cfdg} - g^{ah}\nabla_{ebh}R_{cfdg} - g^{ah}\nabla_{beh}R_{cfdg}\big)
\end{aligned} \tag{6d}$$

$$\begin{aligned}
-360\overset{5}{y}{}^a_4 = x^b x^c x^d x^e Dx^f Dx^g \big(&16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} \\
&- 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} \\
&+ 2g^{ah}\nabla_{fbc}R_{dgeh} + 2g^{ah}\nabla_{bfc}R_{dgeh} + 2g^{ah}\nabla_{bcf}R_{dgeh} + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} \\
&+ 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} - g^{ah}\nabla_{hbc}R_{dfeg} - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\big)
\end{aligned} \tag{6e}$$

The determinant of the metric

Our game here is to compute (the leading terms) in $\det g$ of the metric in RNC form

$$
\begin{aligned}
g_{ab}(x) = g_{ab} &- \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45}x^c x^d x^e x^f R_{acdg}R_{befh}g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd}R_{aebf}\\
&+ \frac{1}{45}x^c x^d x^e x^f x^g R_{acdh}\nabla_e R_{bfgi}g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcdh}\nabla_e R_{afgi}g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde}R_{afbg} + \mathcal{O}\left(\epsilon^5\right)
\end{aligned}
$$

For the sake of simplicity let's assume that we are working in 3-dimensions. The following analysis is easily generalsied to other dimensions (and the final answers for $\det g$ and friends are unchanged).

Define ϵ^{abc}_{ijk} by

$$\epsilon^{abc}_{ijk} = \delta^a_i\delta^b_j\delta^c_k - \delta^b_i\delta^a_j\delta^c_k + \delta^c_i\delta^a_j\delta^b_k - \delta^c_i\delta^b_j\delta^a_k + \delta^b_i\delta^c_j\delta^a_k - \delta^a_i\delta^c_j\delta^b_k \tag{1}$$

It is easy to see that ϵ^{abc}_{ijk} is anti-symmetric in both its upper and lower indices. A trivial computation shows that for any 3×3 square matrix M_{ab},

$$\epsilon^{abc}_{123}M_{1a}M_{2b}M_{3c} = \left(\delta^a_1\delta^b_2\delta^c_3 - \delta^b_1\delta^a_2\delta^c_3 + \delta^c_1\delta^a_2\delta^b_3 - \delta^c_1\delta^b_2\delta^a_3 + \delta^b_1\delta^c_2\delta^a_3 - \delta^a_1\delta^c_2\delta^b_3\right) M_{1a}M_{2b}M_{3c} = \det M \tag{2}$$

This can be easily generalised to

$$\epsilon^{abc}_{ijk}M_{pa}M_{qb}M_{rc} = \begin{cases} \pm\det M & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123)\\ 0 & \text{otherwise}\end{cases} \tag{3}$$

The $\pm$ sign in the above depends on the particular permutations of (ijk) and (pqr). If both permutations are even or both odd then the sign is $+1$ otherwise the sign is -1. The same arguments can also be applied to a matrix inverse N^{-1} leading to

$$\epsilon^{ijk}_{uvw}N^{pu}N^{qv}M^{rw} = \begin{cases} \pm\det N^{-1} & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123)\\ 0 & \text{otherwise}\end{cases} \tag{4}$$

Note that the $\pm$ in this case will match exactly that for the case of $\det M$. Thus, multiplying both expressions and summing over all choices for (ijk) and (pqr) leads to

$$\sum_{\substack{(ijk)\\(pqr)}} \left(\det N^{-1}\right)\det M = \epsilon^{ijk}_{uvw}N^{pu}N^{qv}M^{rw}\epsilon^{abc}_{ijk}M_{pa}M_{qb}M_{rc} \tag{5}$$

where the sum on the left hand side includes just those (ijk) and (prq) that are permutations of (123). There are 3! choices for (ijk) and 3! choices for (pqr) and thus the left hand side is easily reduced to $(3!)^2 \det M / \det N$ where $\det N = 1/\det N^{-1}$. For the right hand side notice that

$$\epsilon^{ijk}_{uvw} \epsilon^{abc}_{ijk} = 3!\, \epsilon^{abc}_{uvw} \tag{6}$$

which leads to

$$\det M = \frac{1}{3!} \det N \epsilon^{abc}_{uvw} M_{pa} M_{qb} M_{rc} N^{pu} N^{qv} N^{rw} \tag{7}$$

For our RNC metric we will set $N^{ab} = g^{ab}$ and $M_{ij} = g_{ij}(x)$. Since g^{ab} is of the form $\mathrm{diag}(-1, 1, 1, 1)$ we have $\det g = -1$ and thus

$$\det g(x) = -\frac{1}{3!} \epsilon^{abc}_{ijk}\, g_{pa}(x)\, g_{qb}(x)\, g_{rc}(x)\, g^{ip} g^{jq} g^{kr} \tag{8}$$

The ϵ^{abc}_{ijk} can be constructed in Cadabra by applying the `asym` algorithm to the upper indices of $\delta^a_i \delta^b_j \delta^c_k$. Note that `asym` will include the 1/3! coeffcient as part of its output.

The following code computes $-\det g$ rather than $\det g$.

Note that Calzetta etal. use an opposite sign for R_{abcd} so when comparing the following results against Calzetta do take note of this flipped sign in R_{abcd}.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Integer(1..2).

\nabla{#}::Derivative.

d{#}::KroneckerDelta.

g^{a b}::Symmetric.
g_{a b}::Symmetric.

R_{a b c d}::RiemannTensor.

x^{a}::Weight(label=numx,value=1).

def truncate (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah = Ex("numx = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

import cdblib

g0ab = cdblib.get('g_ab_0','metric.json')
g1ab = cdblib.get('g_ab_1','metric.json')  # zero in RNC
g2ab = cdblib.get('g_ab_2','metric.json')
g3ab = cdblib.get('g_ab_3','metric.json')
g4ab = cdblib.get('g_ab_4','metric.json')
g5ab = cdblib.get('g_ab_5','metric.json')

gab := @(g0ab) + @(g1ab) + @(g2ab) + @(g3ab) + @(g4ab) + @(g5ab).  # cdb (gab.001,gab)
gxab := gx_{a b} -> @(gab).
```

```
eps := d^{a}_{i} d^{b}_{j}.   # cdb(eps.001,eps)
asym (eps,$^{a},^{b}$)        # cdb(eps.002,eps) # includes a factor of 1/2!

# compute negative Ndetg rather than det g
Ndetg := @(eps) gx_{p a} gx_{q b} g^{i p} g^{j q}.  # note 1/2! included in eps

substitute        (Ndetg,gxab)
distribute        (Ndetg)
Ndetg = truncate (Ndetg,5)                                      # cdb (Ndetg.001,Ndetg)
substitute        (Ndetg,$g^{a b} g_{b c} -> d^{a}_{c}$,repeat=True) # cdb (Ndetg.002,Ndetg)
eliminate_kronecker (Ndetg)                                     # cdb (Ndetg.003,Ndetg)
sort_product      (Ndetg)                                       # cdb (Ndetg.004,Ndetg)
rename_dummies    (Ndetg)                                       # cdb (Ndetg.005,Ndetg)
canonicalise      (Ndetg)                                       # cdb (Ndetg.006,Ndetg)

# introduce the Ricci tensor

substitute     (Ndetg,$R_{a b c d} g^{a c} -> R_{b d}$,repeat=True)                          # cdb (Ndetg.101,Ndetg)
substitute     (Ndetg,$\nabla_{a}{R_{b c d e}} g^{b d}  -> \nabla_{a}{R_{c e}}$,repeat=True)         # cdb (Ndetg.102,Ndetg)
substitute     (Ndetg,$\nabla_{a b}{R_{c d e f}} g^{c e}  -> \nabla_{a b}{R_{d f}}$,repeat=True)     # cdb (Ndetg.103,Ndetg)
substitute     (Ndetg,$\nabla_{a b c}{R_{d e f g}} g^{d f}  -> \nabla_{a b c}{R_{e g}}$,repeat=True) # cdb (Ndetg.104,Ndetg)

# the following are based on sqrt-Ndetg.tex

sqrtNdetg := 1/2 + (1/2) @(Ndetg)
            - (1/8) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
            - (1/4) (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
            # cdb (sqrtNdetg.001,sqrtNdetg)

sort_product   (sqrtNdetg)                                      # cdb (sqrtNdetg.002,sqrtNdetg)
rename_dummies (sqrtNdetg)                                      # cdb (sqrtNdetg.003,sqrtNdetg)
canonicalise   (sqrtNdetg)                                      # cdb (sqrtNdetg.004,sqrtNdetg)

logNdetg := -1 + @(Ndetg)
            - (1/2) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
            - (1/18) R_{a b} \nabla_{c}{R_{d e}} x^{a} x^{b} x^{c} x^{d} x^{e}.
            # cdb (logNdetg.001,logNdetg)
```

```
sort_product    (logNdetg)                                  # cdb (logNdetg.002,logNdetg)
rename_dummies (logNdetg)                                  # cdb (logNdetg.003,logNdetg)
canonicalise    (logNdetg)                                  # cdb (logNdetg.004,logNdetg)
```

$$\texttt{eps.001} := {d^{a}}_{i}{d^{b}}_{j}$$

$$\texttt{eps.002} := \frac{1}{2}{d^{a}}_{i}{d^{b}}_{j} - \frac{1}{2}{d^{b}}_{i}{d^{a}}_{j}$$

$$\begin{aligned}
\texttt{Ndetg.001} := {} & \frac{1}{2}{d^{a}}_{i}{d^{b}}_{j}g_{pa}g_{qb}g^{ip}g^{jq} - \frac{1}{2}{d^{b}}_{i}{d^{a}}_{j}g_{pa}g_{qb}g^{ip}g^{jq} - \frac{1}{6}{d^{a}}_{i}{d^{b}}_{j}g_{pa}x^{l}x^{m}R_{qlbm}g^{ip}g^{jq} - \frac{1}{6}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}R_{pcad}g_{qb}g^{ip}g^{jq} + \frac{1}{6}{d^{b}}_{i}{d^{a}}_{j}g_{pa}x^{l}x^{m}R_{qlbm}g^{ip}g^{jq} \\
& + \frac{1}{6}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}R_{pcad}g_{qb}g^{ip}g^{jq} - \frac{1}{12}{d^{a}}_{i}{d^{b}}_{j}g_{pa}x^{l}x^{m}x^{n}\nabla_{l}R_{qmbn}g^{ip}g^{jq} - \frac{1}{12}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}x^{e}\nabla_{c}R_{pdae}g_{qb}g^{ip}g^{jq} \\
& + \frac{1}{12}{d^{b}}_{i}{d^{a}}_{j}g_{pa}x^{l}x^{m}x^{n}\nabla_{l}R_{qmbn}g^{ip}g^{jq} + \frac{1}{12}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}x^{e}\nabla_{c}R_{pdae}g_{qb}g^{ip}g^{jq} + \frac{1}{45}{d^{a}}_{i}{d^{b}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}R_{qlmr}R_{bnos}g^{rs}g^{ip}g^{jq} \\
& - \frac{1}{40}{d^{a}}_{i}{d^{b}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}\nabla_{lm}R_{qnbo}g^{ip}g^{jq} + \frac{1}{18}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}R_{pcad}x^{l}x^{m}R_{qlbm}g^{ip}g^{jq} + \frac{1}{45}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}x^{e}x^{f}R_{pcdg}R_{aefh}g^{gh}g_{qb}g^{ip}g^{jq} \\
& - \frac{1}{40}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}x^{e}x^{f}\nabla_{cd}R_{peaf}g_{qb}g^{ip}g^{jq} - \frac{1}{45}{d^{b}}_{i}{d^{a}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}R_{qlmr}R_{bnos}g^{rs}g^{ip}g^{jq} + \frac{1}{40}{d^{b}}_{i}{d^{a}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}\nabla_{lm}R_{qnbo}g^{ip}g^{jq} \\
& - \frac{1}{18}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}R_{pcad}x^{l}x^{m}R_{qlbm}g^{ip}g^{jq} - \frac{1}{45}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}x^{e}x^{f}R_{pcdg}R_{aefh}g^{gh}g_{qb}g^{ip}g^{jq} + \frac{1}{40}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}x^{e}x^{f}\nabla_{cd}R_{peaf}g_{qb}g^{ip}g^{jq} \\
& + \frac{1}{90}{d^{a}}_{i}{d^{b}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}x^{r}R_{qlms}\nabla_{n}R_{bort}g^{st}g^{ip}g^{jq} + \frac{1}{90}{d^{a}}_{i}{d^{b}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}x^{r}R_{blms}\nabla_{n}R_{qort}g^{st}g^{ip}g^{jq} \\
& - \frac{1}{180}{d^{a}}_{i}{d^{b}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}x^{r}\nabla_{lmn}R_{qobr}g^{ip}g^{jq} + \frac{1}{36}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}R_{pcad}x^{l}x^{m}x^{n}\nabla_{l}R_{qmbn}g^{ip}g^{jq} \\
& + \frac{1}{36}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}x^{e}\nabla_{c}R_{pdae}x^{l}x^{m}R_{qlbm}g^{ip}g^{jq} + \frac{1}{90}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}x^{e}x^{f}x^{g}R_{pcdh}\nabla_{e}R_{afgk}g^{hk}g_{qb}g^{ip}g^{jq} \\
& + \frac{1}{90}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}x^{e}x^{f}x^{g}R_{acdh}\nabla_{e}R_{pfgk}g^{hk}g_{qb}g^{ip}g^{jq} - \frac{1}{180}{d^{a}}_{i}{d^{b}}_{j}x^{c}x^{d}x^{e}x^{f}x^{g}\nabla_{cde}R_{pfag}g_{qb}g^{ip}g^{jq} \\
& - \frac{1}{90}{d^{b}}_{i}{d^{a}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}x^{r}R_{qlms}\nabla_{n}R_{bort}g^{st}g^{ip}g^{jq} - \frac{1}{90}{d^{b}}_{i}{d^{a}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}x^{r}R_{blms}\nabla_{n}R_{qort}g^{st}g^{ip}g^{jq} \\
& + \frac{1}{180}{d^{b}}_{i}{d^{a}}_{j}g_{pa}x^{l}x^{m}x^{n}x^{o}x^{r}\nabla_{lmn}R_{qobr}g^{ip}g^{jq} - \frac{1}{36}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}R_{pcad}x^{l}x^{m}x^{n}\nabla_{l}R_{qmbn}g^{ip}g^{jq} \\
& - \frac{1}{36}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}x^{e}\nabla_{c}R_{pdae}x^{l}x^{m}R_{qlbm}g^{ip}g^{jq} - \frac{1}{90}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}x^{e}x^{f}x^{g}R_{pcdh}\nabla_{e}R_{afgk}g^{hk}g_{qb}g^{ip}g^{jq} \\
& - \frac{1}{90}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}x^{e}x^{f}x^{g}R_{acdh}\nabla_{e}R_{pfgk}g^{hk}g_{qb}g^{ip}g^{jq} + \frac{1}{180}{d^{b}}_{i}{d^{a}}_{j}x^{c}x^{d}x^{e}x^{f}x^{g}\nabla_{cde}R_{pfag}g_{qb}g^{ip}g^{jq}
\end{aligned}$$

$$
\begin{aligned}
\texttt{Ndetg.002} := {} & \frac{1}{2} d^{a}{}_{i} d^{b}{}_{j} d^{i}{}_{a} d^{j}{}_{b} - \frac{1}{2} d^{b}{}_{i} d^{a}{}_{j} d^{i}{}_{a} d^{j}{}_{b} - \frac{1}{6} d^{a}{}_{i} d^{b}{}_{j} x^{l} x^{m} R_{qlbm} d^{i}{}_{a} g^{jq} - \frac{1}{6} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} R_{pcad} g^{ip} d^{j}{}_{b} + \frac{1}{6} d^{b}{}_{i} d^{a}{}_{j} x^{l} x^{m} R_{qlbm} d^{i}{}_{a} g^{jq} \\
& + \frac{1}{6} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} R_{pcad} g^{ip} d^{j}{}_{b} - \frac{1}{12} d^{a}{}_{i} d^{b}{}_{j} x^{l} x^{m} x^{n} \nabla_{l} R_{qmbn} d^{i}{}_{a} g^{jq} - \frac{1}{12} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} x^{e} \nabla_{c} R_{pdae} g^{ip} d^{j}{}_{b} \\
& + \frac{1}{12} d^{b}{}_{i} d^{a}{}_{j} x^{l} x^{m} x^{n} \nabla_{l} R_{qmbn} d^{i}{}_{a} g^{jq} + \frac{1}{12} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} x^{e} \nabla_{c} R_{pdae} g^{ip} d^{j}{}_{b} + \frac{1}{45} d^{a}{}_{i} d^{b}{}_{j} x^{l} x^{m} x^{n} x^{o} R_{qlmr} R_{bnos} g^{rs} d^{i}{}_{a} g^{jq} \\
& - \frac{1}{40} d^{a}{}_{i} d^{b}{}_{j} x^{l} x^{m} x^{n} x^{o} \nabla_{lm} R_{qnbo} d^{i}{}_{a} g^{jq} + \frac{1}{18} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} R_{pcad} x^{l} x^{m} R_{qlbm} g^{ip} g^{jq} + \frac{1}{45} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} x^{e} x^{f} R_{pcdg} R_{aefh} g^{gh} g^{ip} d^{j}{}_{b} \\
& - \frac{1}{40} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} x^{e} x^{f} \nabla_{cd} R_{peaf} g^{ip} d^{j}{}_{b} - \frac{1}{45} d^{b}{}_{i} d^{a}{}_{j} x^{l} x^{m} x^{n} x^{o} R_{qlmr} R_{bnos} g^{rs} d^{i}{}_{a} g^{jq} + \frac{1}{40} d^{b}{}_{i} d^{a}{}_{j} x^{l} x^{m} x^{n} x^{o} \nabla_{lm} R_{qnbo} d^{i}{}_{a} g^{jq} \\
& - \frac{1}{18} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} R_{pcad} x^{l} x^{m} R_{qlbm} g^{ip} g^{jq} - \frac{1}{45} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} x^{e} x^{f} R_{pcdg} R_{aefh} g^{gh} g^{ip} d^{j}{}_{b} + \frac{1}{40} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} x^{e} x^{f} \nabla_{cd} R_{peaf} g^{ip} d^{j}{}_{b} \\
& + \frac{1}{90} d^{a}{}_{i} d^{b}{}_{j} x^{l} x^{m} x^{n} x^{o} x^{r} R_{qlms} \nabla_{n} R_{bort} g^{st} d^{i}{}_{a} g^{jq} + \frac{1}{90} d^{a}{}_{i} d^{b}{}_{j} x^{l} x^{m} x^{n} x^{o} x^{r} R_{blms} \nabla_{n} R_{qort} g^{st} d^{i}{}_{a} g^{jq} \\
& - \frac{1}{180} d^{a}{}_{i} d^{b}{}_{j} x^{l} x^{m} x^{n} x^{o} x^{r} \nabla_{lmn} R_{qobr} d^{i}{}_{a} g^{jq} + \frac{1}{36} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} R_{pcad} x^{l} x^{m} x^{n} \nabla_{l} R_{qmbn} g^{ip} g^{jq} + \frac{1}{36} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} x^{e} \nabla_{c} R_{pdae} x^{l} x^{m} R_{qlbm} g^{ip} g^{jq} \\
& + \frac{1}{90} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} x^{e} x^{f} x^{g} R_{pcdh} \nabla_{e} R_{afgk} g^{hk} g^{ip} d^{j}{}_{b} + \frac{1}{90} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} x^{e} x^{f} x^{g} R_{acdh} \nabla_{e} R_{pfgk} g^{hk} g^{ip} d^{j}{}_{b} \\
& - \frac{1}{180} d^{a}{}_{i} d^{b}{}_{j} x^{c} x^{d} x^{e} x^{f} x^{g} \nabla_{cde} R_{pfag} g^{ip} d^{j}{}_{b} - \frac{1}{90} d^{b}{}_{i} d^{a}{}_{j} x^{l} x^{m} x^{n} x^{o} x^{r} R_{qlms} \nabla_{n} R_{bort} g^{st} d^{i}{}_{a} g^{jq} \\
& - \frac{1}{90} d^{b}{}_{i} d^{a}{}_{j} x^{l} x^{m} x^{n} x^{o} x^{r} R_{blms} \nabla_{n} R_{qort} g^{st} d^{i}{}_{a} g^{jq} + \frac{1}{180} d^{b}{}_{i} d^{a}{}_{j} x^{l} x^{m} x^{n} x^{o} x^{r} \nabla_{lmn} R_{qobr} d^{i}{}_{a} g^{jq} - \frac{1}{36} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} R_{pcad} x^{l} x^{m} x^{n} \nabla_{l} R_{qmbn} g^{ip} g^{jq} \\
& - \frac{1}{36} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} x^{e} \nabla_{c} R_{pdae} x^{l} x^{m} R_{qlbm} g^{ip} g^{jq} - \frac{1}{90} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} x^{e} x^{f} x^{g} R_{pcdh} \nabla_{e} R_{afgk} g^{hk} g^{ip} d^{j}{}_{b} \\
& - \frac{1}{90} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} x^{e} x^{f} x^{g} R_{acdh} \nabla_{e} R_{pfgk} g^{hk} g^{ip} d^{j}{}_{b} + \frac{1}{180} d^{b}{}_{i} d^{a}{}_{j} x^{c} x^{d} x^{e} x^{f} x^{g} \nabla_{cde} R_{pfag} g^{ip} d^{j}{}_{b}
\end{aligned}
$$

$$\texttt{Ndetg.003} := 1 - \frac{1}{6}x^l x^m R_{qljm} g^{jq} - \frac{1}{3}x^c x^d R_{pcid} g^{ip} + \frac{1}{6}x^c x^d R_{pcbd} g^{bp} - \frac{1}{12}x^l x^m x^n \nabla_l R_{qmjn} g^{jq} - \frac{1}{6}x^c x^d x^e \nabla_c R_{pdie} g^{ip}$$
$$+ \frac{1}{12}x^c x^d x^e \nabla_c R_{pdbe} g^{bp} + \frac{1}{45}x^l x^m x^n x^o R_{qlmr} R_{jnos} g^{rs} g^{jq} - \frac{1}{40}x^l x^m x^n x^o \nabla_{lm} R_{qnjo} g^{jq} + \frac{1}{18}x^c x^d R_{pcid} x^l x^m R_{qljm} g^{ip} g^{jq}$$
$$+ \frac{2}{45}x^c x^d x^e x^f R_{pcdg} R_{iefh} g^{gh} g^{ip} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{peif} g^{ip} - \frac{1}{18}x^c x^d R_{pcjd} x^l x^m R_{qlim} g^{ip} g^{jq}$$
$$- \frac{1}{45}x^c x^d x^e x^f R_{pcdg} R_{befh} g^{gh} g^{bp} + \frac{1}{40}x^c x^d x^e x^f \nabla_{cd} R_{pebf} g^{bp} + \frac{1}{90}x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{jort} g^{st} g^{jq}$$
$$+ \frac{1}{90}x^l x^m x^n x^o x^r R_{jlms} \nabla_n R_{qort} g^{st} g^{jq} - \frac{1}{180}x^l x^m x^n x^o x^r \nabla_{lmn} R_{qojr} g^{jq} + \frac{1}{36}x^c x^d R_{pcid} x^l x^m x^n \nabla_l R_{qmjn} g^{ip} g^{jq}$$
$$+ \frac{1}{36}x^c x^d x^e \nabla_c R_{pdie} x^l x^m R_{qljm} g^{ip} g^{jq} + \frac{1}{45}x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{ifgk} g^{hk} g^{ip} + \frac{1}{45}x^c x^d x^e x^f x^g R_{icdh} \nabla_e R_{pfgk} g^{hk} g^{ip}$$
$$- \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{pfig} g^{ip} - \frac{1}{36}x^c x^d R_{pcjd} x^l x^m x^n \nabla_l R_{qmin} g^{ip} g^{jq} - \frac{1}{36}x^c x^d x^e \nabla_c R_{pdje} x^l x^m R_{qlim} g^{ip} g^{jq}$$
$$- \frac{1}{90}x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{bfgk} g^{hk} g^{bp} - \frac{1}{90}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{pfgk} g^{hk} g^{bp} + \frac{1}{180}x^c x^d x^e x^f x^g \nabla_{cde} R_{pfbg} g^{bp}$$

$$\texttt{Ndetg.004} := 1 - \frac{1}{6}R_{qljm} g^{jq} x^l x^m - \frac{1}{3}R_{pcid} g^{ip} x^c x^d + \frac{1}{6}R_{pcbd} g^{bp} x^c x^d - \frac{1}{12}\nabla_l R_{qmjn} g^{jq} x^l x^m x^n - \frac{1}{6}\nabla_c R_{pdie} g^{ip} x^c x^d x^e$$
$$+ \frac{1}{12}\nabla_c R_{pdbe} g^{bp} x^c x^d x^e + \frac{1}{45}R_{jnos} R_{qlmr} g^{jq} g^{rs} x^l x^m x^n x^o - \frac{1}{40}\nabla_{lm} R_{qnjo} g^{jq} x^l x^m x^n x^o + \frac{1}{18}R_{pcid} R_{qljm} g^{ip} g^{jq} x^c x^d x^l x^m$$
$$+ \frac{2}{45}R_{iefh} R_{pcdg} g^{gh} g^{ip} x^c x^d x^e x^f - \frac{1}{20}\nabla_{cd} R_{peif} g^{ip} x^c x^d x^e x^f - \frac{1}{18}R_{pcjd} R_{qlim} g^{ip} g^{jq} x^c x^d x^l x^m$$
$$- \frac{1}{45}R_{befh} R_{pcdg} g^{bp} g^{gh} x^c x^d x^e x^f + \frac{1}{40}\nabla_{cd} R_{pebf} g^{bp} x^c x^d x^e x^f + \frac{1}{90}R_{qlms} \nabla_n R_{jort} g^{jq} g^{st} x^l x^m x^n x^o x^r$$
$$+ \frac{1}{90}R_{jlms} \nabla_n R_{qort} g^{jq} g^{st} x^l x^m x^n x^o x^r - \frac{1}{180}\nabla_{lmn} R_{qojr} g^{jq} x^l x^m x^n x^o x^r + \frac{1}{36}R_{pcid} \nabla_l R_{qmjn} g^{ip} g^{jq} x^c x^d x^l x^m x^n$$
$$+ \frac{1}{36}R_{qljm} \nabla_c R_{pdie} g^{ip} g^{jq} x^c x^d x^e x^l x^m + \frac{1}{45}R_{pcdh} \nabla_e R_{ifgk} g^{hk} g^{ip} x^c x^d x^e x^f x^g + \frac{1}{45}R_{icdh} \nabla_e R_{pfgk} g^{hk} g^{ip} x^c x^d x^e x^f x^g$$
$$- \frac{1}{90}\nabla_{cde} R_{pfig} g^{ip} x^c x^d x^e x^f x^g - \frac{1}{36}R_{pcjd} \nabla_l R_{qmin} g^{ip} g^{jq} x^c x^d x^l x^m x^n - \frac{1}{36}R_{qlim} \nabla_c R_{pdje} g^{ip} g^{jq} x^c x^d x^e x^l x^m$$
$$- \frac{1}{90}R_{pcdh} \nabla_e R_{bfgk} g^{bp} g^{hk} x^c x^d x^e x^f x^g - \frac{1}{90}R_{bcdh} \nabla_e R_{pfgk} g^{bp} g^{hk} x^c x^d x^e x^f x^g + \frac{1}{180}\nabla_{cde} R_{pfbg} g^{bp} x^c x^d x^e x^f x^g$$

$$\texttt{Ndetg.005} := 1 - \frac{1}{3}R_{abcd}g^{ca}x^b x^d - \frac{1}{6}\nabla_e R_{abcd}g^{ca}x^e x^b x^d - \frac{1}{20}\nabla_{ef}R_{abcd}g^{ca}x^e x^f x^b x^d + \frac{1}{18}R_{abcd}R_{efgh}g^{ca}g^{ge}x^b x^d x^f x^h$$
$$+ \frac{2}{45}R_{abcd}R_{efgh}g^{hd}g^{ae}x^f x^g x^b x^c - \frac{1}{18}R_{abcd}R_{efgh}g^{ga}g^{ce}x^b x^d x^f x^h - \frac{1}{90}\nabla_{efg}R_{abcd}g^{ca}x^e x^f x^g x^b x^d$$
$$+ \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ca}g^{ge}x^b x^d x^i x^f x^h + \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ge}g^{ca}x^i x^f x^h x^b x^d + \frac{1}{45}R_{abcd}\nabla_i R_{efgh}g^{dh}g^{ea}x^b x^c x^i x^f x^g$$
$$+ \frac{1}{45}R_{abcd}\nabla_i R_{efgh}g^{dh}g^{ae}x^b x^c x^i x^f x^g - \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ga}g^{ce}x^b x^d x^i x^f x^h - \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ce}g^{ga}x^i x^f x^h x^b x^d$$

$$\texttt{Ndetg.006} := 1 - \frac{1}{3}R_{abcd}g^{ac}x^b x^d - \frac{1}{6}\nabla_a R_{bcde}g^{bd}x^a x^c x^e - \frac{1}{20}\nabla_{ab}R_{cdef}g^{ce}x^a x^b x^d x^f + \frac{1}{18}R_{abcd}R_{efgh}g^{ac}g^{eg}x^b x^d x^f x^h - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^b x^d x^f x^h$$
$$- \frac{1}{90}\nabla_{abc}R_{defg}g^{df}x^a x^b x^c x^e x^g + \frac{1}{18}R_{abcd}\nabla_e R_{fghi}g^{ac}g^{fh}x^b x^d x^e x^g x^i - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^b x^d x^e x^g x^i$$

$$\texttt{Ndetg.104} := 1 - \frac{1}{3}R_{bd}x^b x^d - \frac{1}{6}\nabla_a R_{ce}x^a x^c x^e - \frac{1}{20}\nabla_{ab}R_{df}x^a x^b x^d x^f + \frac{1}{18}R_{bd}R_{fh}x^b x^d x^f x^h - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^b x^d x^f x^h$$
$$- \frac{1}{90}\nabla_{abc}R_{eg}x^a x^b x^c x^e x^g + \frac{1}{18}R_{bd}\nabla_e R_{gi}x^b x^d x^e x^g x^i - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^b x^d x^e x^g x^i$$

$$\texttt{sqrtNdetg.004} := 1 - \frac{1}{6}R_{ab}x^a x^b - \frac{1}{12}\nabla_a R_{bc}x^a x^b x^c - \frac{1}{40}\nabla_{ab}R_{cd}x^a x^b x^c x^d + \frac{1}{72}R_{ab}R_{cd}x^a x^b x^c x^d - \frac{1}{180}R_{abcd}R_{efgh}g^{ae}g^{cg}x^b x^d x^f x^h$$
$$- \frac{1}{180}\nabla_{abc}R_{de}x^a x^b x^c x^d x^e + \frac{1}{72}R_{ab}\nabla_c R_{de}x^a x^b x^c x^d x^e - \frac{1}{180}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^b x^d x^e x^g x^i$$

$$\texttt{logNdetg.004} := -\frac{1}{3}R_{ab}x^a x^b - \frac{1}{6}\nabla_a R_{bc}x^a x^b x^c - \frac{1}{20}\nabla_{ab}R_{cd}x^a x^b x^c x^d - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^b x^d x^f x^h$$
$$- \frac{1}{90}\nabla_{abc}R_{de}x^a x^b x^c x^d x^e - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^b x^d x^e x^g x^i$$

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}                              -> A000^{a}                 $)
    substitute (obj,$ g^{a b}                            -> A001^{a b}               $)
    substitute (obj,$ \nabla_{c d e f}{R_{a b}}          -> A007_{a b c d e f}       $)
    substitute (obj,$ \nabla_{c d e}{R_{a b}}            -> A006_{a b c d e}         $)
    substitute (obj,$ \nabla_{c d}{R_{a b}}              -> A005_{a b c d}           $)
    substitute (obj,$ \nabla_{c}{R_{a b}}                -> A004_{a b c}             $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A011_{a b c d e f g h}   $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A010_{a b c d e f g}     $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A009_{a b c d e f}       $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A008_{a b c d e}         $)
    substitute (obj,$ R_{a b}                            -> A002_{a b}               $)
    substitute (obj,$ R_{a b c d}                        -> A003_{a b c d}           $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A000^{a}                 -> x^{a}                              $)
    substitute (obj,$ A001^{a b}               -> g^{a b}                            $)
    substitute (obj,$ A002_{a b}               -> R_{a b}                            $)
    substitute (obj,$ A003_{a b c d}           -> R_{a b c d}                        $)
    substitute (obj,$ A004_{a b c}             -> \nabla_{c}{R_{a b}}                $)
    substitute (obj,$ A005_{a b c d}           -> \nabla_{c d}{R_{a b}}              $)
    substitute (obj,$ A006_{a b c d e}         -> \nabla_{c d e}{R_{a b}}            $)
    substitute (obj,$ A007_{a b c d e f}       -> \nabla_{c d e f}{R_{a b}}          $)
    substitute (obj,$ A008_{a b c d e}         -> \nabla_{e}{R_{a b c d}}            $)
    substitute (obj,$ A009_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}}          $)
    substitute (obj,$ A010_{a b c d e f g}     -> \nabla_{e f g}{R_{a b c d}}        $)
    substitute (obj,$ A011_{a b c d e f g h}   -> \nabla_{e f g h}{R_{a b c d}}      $)

    return obj

def get_term (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah  = Ex("numx = " + str(n))
    keep_weight (foo,bah)
```

```
    return foo

def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    sort_sum       (bah)
    factor_out     (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute  (bah)
    factor_out  (bah,$x^{a?}$)
    return bah

# ---------------------------------------------------------------
# reformat Ndetg

Rterm0 = get_term (Ndetg,0)       # cdb(Rterm0.701,Rterm0)
Rterm1 = get_term (Ndetg,1)       # cdb(Rterm1.701,Rterm1)
Rterm2 = get_term (Ndetg,2)       # cdb(Rterm2.701,Rterm2)
Rterm3 = get_term (Ndetg,3)       # cdb(Rterm3.701,Rterm3)
Rterm4 = get_term (Ndetg,4)       # cdb(Rterm4.701,Rterm4)
Rterm5 = get_term (Ndetg,5)       # cdb(Rterm5.701,Rterm5)

Rterm0 = reformat (Rterm0,  1)    # cdb(Rterm0.702,Rterm0)
Rterm1 = reformat (Rterm1,  1)    # cdb(Rterm1.702,Rterm1)
Rterm2 = reformat (Rterm2,  3)    # cdb(Rterm2.702,Rterm2)
Rterm3 = reformat (Rterm3,  6)    # cdb(Rterm3.702,Rterm3)
Rterm4 = reformat (Rterm4,180)    # cdb(Rterm4.702,Rterm4)
Rterm5 = reformat (Rterm5, 90)    # cdb(Rterm5.702,Rterm5)
```

```
Ndetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Ndetg.701,Ndetg)

# ----------------------------------------------------------------
# reformat sqrtNdetg

Rterm0 = get_term (sqrtNdetg,0)   # cdb(Rterm0.801,Rterm0)
Rterm1 = get_term (sqrtNdetg,1)   # cdb(Rterm1.801,Rterm1)
Rterm2 = get_term (sqrtNdetg,2)   # cdb(Rterm2.801,Rterm2)
Rterm3 = get_term (sqrtNdetg,3)   # cdb(Rterm3.801,Rterm3)
Rterm4 = get_term (sqrtNdetg,4)   # cdb(Rterm4.801,Rterm4)
Rterm5 = get_term (sqrtNdetg,5)   # cdb(Rterm5.801,Rterm5)

Rterm0 = reformat (Rterm0,  1)    # cdb(Rterm0.802,Rterm0)
Rterm1 = reformat (Rterm1,  1)    # cdb(Rterm1.802,Rterm1)
Rterm2 = reformat (Rterm2,  6)    # cdb(Rterm2.802,Rterm2)
Rterm3 = reformat (Rterm3, 12)    # cdb(Rterm3.802,Rterm3)
Rterm4 = reformat (Rterm4,360)    # cdb(Rterm4.802,Rterm4)
Rterm5 = reformat (Rterm5,360)    # cdb(Rterm5.802,Rterm5)

sqrtNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (sqrtNdetg.801,sqrtNdetg)

# ----------------------------------------------------------------
# reformat logNdetg

Rterm0 = get_term (logNdetg,0)    # cdb(Rterm0.901,Rterm0)
Rterm1 = get_term (logNdetg,1)    # cdb(Rterm1.901,Rterm1)
Rterm2 = get_term (logNdetg,2)    # cdb(Rterm2.901,Rterm2)
Rterm3 = get_term (logNdetg,3)    # cdb(Rterm3.901,Rterm3)
Rterm4 = get_term (logNdetg,4)    # cdb(Rterm4.901,Rterm4)
Rterm5 = get_term (logNdetg,5)    # cdb(Rterm5.901,Rterm5)

Rterm0 = reformat (Rterm0,  1)    # cdb(Rterm0.902,Rterm0)
Rterm1 = reformat (Rterm1,  1)    # cdb(Rterm1.902,Rterm1)
Rterm2 = reformat (Rterm2,  3)    # cdb(Rterm2.902,Rterm2)
Rterm3 = reformat (Rterm3,  6)    # cdb(Rterm3.902,Rterm3)
Rterm4 = reformat (Rterm4,180)    # cdb(Rterm4.902,Rterm4)
Rterm5 = reformat (Rterm5, 90)    # cdb(Rterm5.902,Rterm5)
```

```
logNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (logNdetg.901,logNdetg)
```

The metric determinant in Riemann normal coordinates

$$
\begin{aligned}
-\det g(x) = 1 &- \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) \\
&+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

The volume element in Riemann normal coordinates

If $-\det g(x)$ is non-negative then we also have

$$
\begin{aligned}
\sqrt{-\det g(x)} = 1 &- \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) \\
&+ \frac{1}{360}x^a x^b x^c x^d x^e \left(-2\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - 2g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

The log of -detg in Riemann normal coordinates

Apart from the signs, this matches exactly the expression given by Calzetta etal. (eq. A14)

$$
\begin{aligned}
\log\left(-\det g(x)\right) = &-\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right) \\
&+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

```
cdblib.create ('detg2.export')

cdblib.put ('Ndetg',    Ndetg,    'detg2.export')
cdblib.put ('sqrtNdetg',sqrtNdetg,'detg2.export')
cdblib.put ('logNdetg', logNdetg, 'detg2.export')

checkpoint.append (Ndetg)
checkpoint.append (sqrtNdetg)
checkpoint.append (logNdetg)
```