# Notes

The convention for the curvature used in these notes conforms to that of Misner-Thorne-Wheeler (MTW, eq. 11.12) , namely

$$V^a{}_{;bc} - V^a{}_{;cb} = -R^a{}_{dbc}V^d$$

Also, note the following shorthand for mixed covariant derivatives

$$\begin{aligned}
\nabla_a \left(\nabla_b\right) &= \nabla_{ab} \\
\nabla_a \left(\nabla_b \left(\nabla_c\right)\right) &= \nabla_{abc} \\
\nabla_a \left(\nabla_b \left(\nabla_c \left(\nabla_d\right)\right)\right) &= \nabla_{abcd}
\end{aligned}$$

and so on.

In terms of $\nabla$ the above MTW definition of $R^a{}_{bcd}$ can written as

$$\left(\nabla_{cb} - \nabla_{bc}\right) V^a = -R^a{}_{dbc}V^d$$

## Symmetrisation

In the following pages there will be frequent constructions of the form

$$\begin{aligned}
3A^bA^c\Gamma^a{}_{d(b,c)} &= A^bA^cR^a{}_{bcd} \\
6A^bA^cA^e\Gamma^a{}_{d(b,ce)} &= 3A^bA^cA^e\partial_e R^a{}_{bcd} \\
15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} &= A^bA^cA^eA^f\left(9\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right)
\end{aligned}$$

The vector $A^a$ has no special meaning. Its purpose is to indicate that the associciated tensor is symmetric over a selection of its indices. If the $A^a$ were not included then the right hand side would either need to be spelt out in full or some other device would be needed to denote the symmetries. The symmetrisation brackets are included on the left hand side though they are redundant (in the presence of the $A^a$).

# The metric in RNC

$$\begin{aligned} g_{ab}(x) = g_{ab} &- \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f \left(8g^{gh} R_{acdg} R_{befh} - 9\nabla_{cd} R_{aebf}\right) \\ &+ \frac{1}{90}x^c x^d x^e x^f x^g \left(2g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right) + \mathcal{O}\left(\epsilon^6\right) \end{aligned}$$

# Curvature expansion of the metric

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned} \overset{0}{g}_{ab} &= g_{ab} \\ 3\overset{2}{g}_{ab} &= -x^c x^d R_{acbd} \\ 6\overset{3}{g}_{ab} &= -x^c x^d x^e \nabla_c R_{adbe} \\ 180\overset{4}{g}_{ab} &= x^c x^d x^e x^f \left(8g^{gh} R_{acdg} R_{befh} - 9\nabla_{cd} R_{aebf}\right) \\ 90\overset{5}{g}_{ab} &= x^c x^d x^e x^f x^g \left(2g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right) \end{aligned}$$

## The inverse metric in RNC

$$g^{ab}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \frac{1}{60}x^c x^d x^e x^f \left(4g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right) + \frac{1}{90}x^c x^d x^e x^f x^g \left(3g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right) + \mathcal{O}\left(\epsilon^6\right)$$

## Curvature expansion of the inverse metric

$$g^{ab}(x) = \overset{0}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} + \overset{5}{g}{}^{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{g}{}^{ab} = g^{ab}$$

$$3\overset{2}{g}{}^{ab} = x^c x^d g^{ae} g^{bf} R_{cedf}$$

$$6\overset{3}{g}{}^{ab} = x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg}$$

$$60\overset{4}{g}{}^{ab} = x^c x^d x^e x^f \left(4g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right)$$

$$90\overset{5}{g}{}^{ab} = x^c x^d x^e x^f x^g \left(3g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right)$$

## The metric determinant in RNC

$$-\det g(x) = 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 10 R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)$$

## The metric Jacobian in RNC

$$\sqrt{-\det g(x)} = 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{360}x^a x^b x^c x^d x^e \left(-2\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - 2g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)$$

## The log of detg in RNC

$$\log\left(-\det g(x)\right) = -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)$$

## The connection in RNC

$$
\begin{aligned}
A^aA^b\Gamma^d_{ab} = \frac{2}{3}A^aA^bx^cg^{de}R_{acbe} + \frac{1}{12}A^aA^bx^cx^e\left(2g^{df}\nabla_aR_{bcef} + 4g^{df}\nabla_cR_{aebf} + g^{df}\nabla_fR_{acbe}\right) + \frac{1}{360}A^aA^bx^cx^ex^f\left(64g^{dg}g^{hi}R_{acbh}R_{egfi} - 32g^{dg}g^{hi}R_{aceh}R_{bgfi}\right. \\
- 16g^{dg}g^{hi}R_{aceh}R_{bifg} + 18g^{dg}\nabla_{ac}R_{befg} + 18g^{dg}\nabla_{ca}R_{befg} + 36g^{dg}\nabla_{ce}R_{afbg} - 16g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9g^{dg}\nabla_{gc}R_{aebf} + \left.9g^{dg}\nabla_{cg}R_{aebf}\right) \\
+ \frac{1}{180}A^aA^bx^cx^ex^fx^g\left(16g^{dh}g^{ij}R_{acbi}\nabla_eR_{fhgj} + 6g^{dh}g^{ij}R_{chei}\nabla_aR_{bfgj} + 16g^{dh}g^{ij}R_{chei}\nabla_fR_{agbj} + 5g^{dh}g^{ij}R_{chei}\nabla_jR_{afbg}\right. \\
- 8g^{dh}g^{ij}R_{ahci}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{aich}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_bR_{fhgj} - 8g^{dh}g^{ij}R_{acei}\nabla_fR_{bhgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bjgh} + 2g^{dh}\nabla_{ace}R_{bfgh} \\
+ 2g^{dh}\nabla_{cae}R_{bfgh} + 2g^{dh}\nabla_{cea}R_{bfgh} + 4g^{dh}\nabla_{cef}R_{agbh} - 4g^{dh}g^{ij}R_{achi}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_hR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bghj} \\
+ g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + \left.g^{dh}\nabla_{ceh}R_{afbg}\right)
\end{aligned}
$$

$$
\begin{aligned}
360A^aA^b\Gamma^d_{ab} = 240A^aA^bx^cg^{de}R_{acbe} + 30A^aA^bx^cx^e\left(2g^{df}\nabla_aR_{bcef} + 4g^{df}\nabla_cR_{aebf} + g^{df}\nabla_fR_{acbe}\right) + A^aA^bx^cx^ex^f\left(64g^{dg}g^{hi}R_{acbh}R_{egfi} - 32g^{dg}g^{hi}R_{aceh}R_{bgfi}\right. \\
- 16g^{dg}g^{hi}R_{aceh}R_{bifg} + 18g^{dg}\nabla_{ac}R_{befg} + 18g^{dg}\nabla_{ca}R_{befg} + 36g^{dg}\nabla_{ce}R_{afbg} - 16g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9g^{dg}\nabla_{gc}R_{aebf} + \left.9g^{dg}\nabla_{cg}R_{aebf}\right) \\
+ 2A^aA^bx^cx^ex^fx^g\left(16g^{dh}g^{ij}R_{acbi}\nabla_eR_{fhgj} + 6g^{dh}g^{ij}R_{chei}\nabla_aR_{bfgj} + 16g^{dh}g^{ij}R_{chei}\nabla_fR_{agbj} + 5g^{dh}g^{ij}R_{chei}\nabla_jR_{afbg}\right. \\
- 8g^{dh}g^{ij}R_{ahci}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{aich}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_bR_{fhgj} - 8g^{dh}g^{ij}R_{acei}\nabla_fR_{bhgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bjgh} \\
+ 2g^{dh}\nabla_{ace}R_{bfgh} + 2g^{dh}\nabla_{cae}R_{bfgh} + 2g^{dh}\nabla_{cea}R_{bfgh} + 4g^{dh}\nabla_{cef}R_{agbh} - 4g^{dh}g^{ij}R_{achi}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_hR_{bfgj} \\
- 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bghj} + g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + \left.g^{dh}\nabla_{ceh}R_{afbg}\right)
\end{aligned}
$$

# Curvature expansion of the connection

$$A^a A^b \Gamma^d_{ab} = A^a A^b \overset{2}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{3}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{4}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{5}{\Gamma}{}^d{}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$3A^a A^b \overset{2}{\Gamma}{}^d_{ab} = 2A^a A^b x^c g^{de} R_{acbe}$$

$$12A^a A^b \overset{3}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e \left(2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}\right)$$

$$\begin{aligned}360A^a A^b \overset{4}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f \big(&64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} \\ &- 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}\big)\end{aligned}$$

$$\begin{aligned}180A^a A^b \overset{5}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f x^g \big(&16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} \\ &- 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bfgh} + 2g^{dh} \nabla_{cae} R_{bfgh} \\ &+ 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} + g^{dh} \nabla_{hce} R_{afbg} \\ &+ g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg}\big)\end{aligned}$$

## Symmetrised partial derivatives of the connection

$$
\begin{aligned}
3A^bA^c\Gamma^a{}_{d(b,c)} &= A^bA^cR^a{}_{bcd} \\
6A^bA^cA^e\Gamma^a{}_{d(b,ce)} &= 3A^bA^cA^e\partial_eR^a{}_{bcd} \\
15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} &= A^bA^cA^eA^f\left(9\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right) \\
9A^bA^cA^eA^fA^g\Gamma^a{}_{d(b,cefg)} &= A^bA^cA^eA^fA^g\big(6\partial_{gfe}R^a{}_{bcd} - R^h{}_{bgd}\partial_eR^a{}_{cfh} - R^h{}_{bfd}\partial_eR^a{}_{cgh} - R^a{}_{ceh}\partial_fR^h{}_{bgd} - R^h{}_{bed}\partial_fR^a{}_{cgh} - R^a{}_{cfh}\partial_eR^h{}_{bgd} - R^a{}_{cgh}\partial_eR^h{}_{bfd}\big) \\
252A^bA^cA^eA^fA^gA^h\Gamma^a{}_{d(b,cefgh)} &= A^bA^cA^eA^fA^gA^h\big(180\partial_{hgfe}R^a{}_{bcd} - 36R^i{}_{bgd}\partial_{he}R^a{}_{cfi} + 4R^a{}_{fei}R^i{}_{chj}R^j{}_{bgd} + 4R^a{}_{fhi}R^i{}_{cej}R^j{}_{bgd} - 72R^i{}_{bfd}\partial_{he}R^a{}_{cgi} \\
&\quad + 8R^a{}_{gei}R^i{}_{chj}R^j{}_{bfd} + 8R^a{}_{ghi}R^i{}_{cej}R^j{}_{bfd} - 45\partial_eR^a{}_{cfi}\partial_gR^i{}_{bhd} - 45\partial_eR^a{}_{cgi}\partial_fR^i{}_{bhd} - 45\partial_eR^a{}_{chi}\partial_fR^i{}_{bgd} \\
&\quad - 36R^a{}_{cei}\partial_{hf}R^i{}_{bgd} + 4R^a{}_{cei}R^i{}_{gfj}R^j{}_{bhd} + 4R^a{}_{cei}R^i{}_{ghj}R^j{}_{bfd} - 36R^i{}_{bed}\partial_{hf}R^a{}_{cgi} + 4R^a{}_{gfi}R^i{}_{chj}R^j{}_{bed} \\
&\quad + 4R^a{}_{ghi}R^i{}_{cfj}R^j{}_{bed} - 45\partial_fR^a{}_{cgi}\partial_eR^i{}_{bhd} - 45\partial_fR^a{}_{chi}\partial_eR^i{}_{bgd} - 72R^a{}_{cfi}\partial_{he}R^i{}_{bgd} + 8R^a{}_{cfi}R^i{}_{gej}R^j{}_{bhd} \\
&\quad + 8R^a{}_{cfi}R^i{}_{ghj}R^j{}_{bed} - 45\partial_gR^a{}_{chi}\partial_eR^i{}_{bfd} - 36R^a{}_{cgi}\partial_{he}R^i{}_{bfd} + 4R^a{}_{cgi}R^i{}_{fej}R^j{}_{bhd} + 4R^a{}_{cgi}R^i{}_{fhj}R^j{}_{bed}\big)
\end{aligned}
$$

## Symmetrised partial derivatives of $R^a{}_{bcd}$

$$
\begin{aligned}
A^c A^d A^e R^a{}_{cdb,e} &= A^c A^d A^e g^{af} \nabla_c R_{bdef} \\
A^c A^d A^e A^f R^a{}_{cdb,ef} &= A^c A^d A^e A^f g^{ag} \nabla_{cd} R_{befg} \\
-2A^c A^d A^e A^f A^g R^a{}_{cdb,efg} &= A^c A^d A^e A^f A^g \left( g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - 2g^{ah} \nabla_{cde} R_{bfgh} \right) \\
-5A^c A^d A^e A^f A^g A^h R^a{}_{cdb,efgh} &= A^c A^d A^e A^f A^g A^h \left( 7g^{ai} g^{jk} R_{bcdj} \nabla_{ef} R_{gihk} - 7g^{ai} g^{jk} R_{cidj} \nabla_{ef} R_{bghk} - 5g^{ai} \nabla_{cdef} R_{bghi} \right) \\
-3A^c A^d A^e A^f A^g A^h A^i R^a{}_{cdb,efghi} &= A^c A^d A^e A^f A^g A^h A^i \big( 6g^{aj} g^{kl} \nabla_c R_{bdek} \nabla_{fg} R_{hjil} - 6g^{aj} g^{kl} \nabla_c R_{djek} \nabla_{fg} R_{bhil} + 8g^{aj} g^{kl} R_{bcdk} \nabla_{efg} R_{hjil} \\
&\quad - 8g^{aj} g^{kl} R_{cjdk} \nabla_{efg} R_{bhil} - g^{aj} g^{kl} g^{mn} R_{bcdk} R_{elfm} \nabla_g R_{hjin} - 3g^{aj} g^{kl} g^{mn} R_{cjdk} R_{elfm} \nabla_g R_{bhin} \\
&\quad + 4g^{aj} g^{kl} g^{mn} R_{bcdk} R_{ejfm} \nabla_g R_{hlin} - 3g^{aj} \nabla_{cdefg} R_{bhij} \big)
\end{aligned}
$$

## The generalised connection in RNC

$$A^bA^c\Gamma^a_{bc} = \frac{2}{3}A^bA^cx^dg^{ae}R_{bdce} + \frac{1}{12}A^bA^cx^dx^e\left(2g^{af}\nabla_bR_{cdef} + 4g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right) + \frac{1}{360}A^bA^cx^dx^ex^f\left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi}\right.$$
$$- 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + \left.9g^{ag}\nabla_{dg}R_{becf}\right)$$
$$+ \frac{1}{180}A^bA^cx^dx^ex^fx^g\left(16g^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg}\right.$$
$$- 8g^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh} + 2g^{ah}\nabla_{bde}R_{cfgh}$$
$$+ 2g^{ah}\nabla_{dbe}R_{cfgh} + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj}$$
$$\left.+ g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\right)$$

$$A^bA^cA^d\Gamma^a_{bcd} = \frac{1}{2}A^bA^cA^dx^eg^{af}\nabla_bR_{cedf} + \frac{1}{120}A^bA^cA^dx^ex^f\left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg}\right.$$
$$\left.+ 18g^{ag}\nabla_{be}R_{cfdg} + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right)$$
$$+ \frac{1}{180}A^bA^cA^dx^ex^fx^g\left(32g^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj}\right.$$
$$+ 2g^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} + 22g^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg}$$
$$+ 5g^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} - 12g^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_gR_{chdj}$$
$$+ 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} + 16g^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj}$$
$$+ 36g^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + 16g^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_fR_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg}$$
$$\left.+ g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20g^{ah}g^{ij}R_{beci}\nabla_jR_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}\right)$$

$$A^bA^cA^dA^e\Gamma^a_{bcde} = \frac{1}{15}A^bA^cA^dA^ex^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right)$$
$$+ \frac{1}{90}A^bA^cA^dA^ex^fx^g\left(64g^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} + 4g^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj}\right.$$
$$+ 44g^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg} - 16g^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej}$$
$$+ 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej} + 6g^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej}$$
$$\left.+ 46g^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\right)$$

$$A^bA^cA^dA^eA^f\Gamma^a_{bcdef} = \frac{1}{3}A^bA^cA^dA^eA^fx^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

## The generalised connection in RNC

This is the same as the previous page but with a small change in the format to avoid fractions.

$$
\begin{aligned}
360A^bA^c\Gamma^a_{bc} = {} & 240A^bA^cx^dg^{ae}R_{bdce} + 30A^bA^cx^dx^e\left(2g^{af}\nabla_bR_{cdef} + 4g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right)\\
& + A^bA^cx^dx^ex^f\big(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg}\\
& \qquad + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\big)\\
& + 2A^bA^cx^dx^ex^fx^g\big(16g^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg}\\
& \qquad - 8g^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh}\\
& \qquad + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh} + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj}\\
& \qquad - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\big)\\[1em]
360A^bA^cA^d\Gamma^a_{bcd} = {} & 180A^bA^cA^dx^eg^{af}\nabla_bR_{cedf} + 3A^bA^cA^dx^ex^f\big(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg}\\
& \qquad + 18g^{ag}\nabla_{be}R_{cfdg} + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\big)\\
& + 2A^bA^cA^dx^ex^fx^g\big(32g^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj}\\
& \qquad + 2g^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} + 22g^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg}\\
& \qquad + 5g^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} - 12g^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_gR_{chdj}\\
& \qquad + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh}\\
& \qquad + 16g^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + 16g^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_fR_{dghj}\\
& \qquad - 4g^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg}\\
& \qquad - 20g^{ah}g^{ij}R_{beci}\nabla_jR_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}\big)\\[1em]
90A^bA^cA^dA^e\Gamma^a_{bcde} = {} & 6A^bA^cA^dA^ex^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right)\\
& + A^bA^cA^dA^ex^fx^g\big(64g^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} + 4g^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj}\\
& \qquad + 44g^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg} - 16g^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej}\\
& \qquad + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej} + 6g^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej}\\
& \qquad + 46g^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\big)\\[1em]
3A^bA^cA^dA^eA^f\Gamma^a_{bcdef} = {} & A^bA^cA^dA^eA^fx^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)
\end{aligned}
$$

## Convert from generic (x) to local RNC coords (y)

$$y^a = \overset{0}{y}{}^a + \overset{1}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a$$

$$\begin{aligned}
\overset{0}{y}{}^a &= x^a \\
2\overset{1}{y}{}^a &= x^b x^c \Gamma^a{}_{bc} \\
6\overset{2}{y}{}^a &= x^b x^c x^d \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} + \partial_b \Gamma^a{}_{cd}\right) \\
24\overset{3}{y}{}^a &= x^b x^c x^d x^e \Big(2\Gamma^a{}_{bf}\partial_c\Gamma^f{}_{de} + \Gamma^a{}_{fg}\Gamma^f{}_{bc}\Gamma^g{}_{de} + \Gamma^f{}_{bc}\partial_f\Gamma^a{}_{de} + \partial_{bc}\Gamma^a{}_{de}\Big) \\
360\overset{4}{y}{}^a &= x^b x^c x^d x^e x^f \Big(-4\Gamma^a{}_{bg}\Gamma^g{}_{ch}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 2\Gamma^a{}_{bg}\Gamma^g{}_{ch}\partial_d\Gamma^h{}_{ef} + 3\Gamma^a{}_{bg}\Gamma^g{}_{hi}\Gamma^h{}_{cd}\Gamma^i{}_{ef} - 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_e\Gamma^g{}_{fh} + 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_h\Gamma^g{}_{ef} + 9\Gamma^a{}_{bg}\partial_{cd}\Gamma^g{}_{ef} \\
&\quad + 4\Gamma^a{}_{gh}\Gamma^g{}_{bc}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 13\Gamma^a{}_{gh}\Gamma^g{}_{bc}\partial_d\Gamma^h{}_{ef} - 4\Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_e\Gamma^a{}_{fh} + \Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_h\Gamma^a{}_{ef} + 2\partial_b\Gamma^a{}_{cg}\partial_d\Gamma^g{}_{ef} + 7\partial_g\Gamma^a{}_{bc}\partial_d\Gamma^g{}_{ef} + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_f\Gamma^a{}_{gh} \\
&\quad + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_g\Gamma^a{}_{fh} - 3\Gamma^g{}_{bc}\partial_{de}\Gamma^a{}_{fg} + 6\Gamma^g{}_{bc}\partial_{dg}\Gamma^a{}_{ef} + 3\partial_{bcd}\Gamma^a{}_{ef}\Big)
\end{aligned}$$

## The geodesic ivp

$$x^a(s) = x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A^a_{bc} + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A^a_{bcd} + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A^a_{bcde} + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A^a_{bcdef} + \cdots$$

$$\begin{aligned}
360A^a_{bc} = {} & 240x^d g^{ae} R_{bdce} + 30x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) + x^d x^e x^f \left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi}\right. \\
& - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + \left.9g^{ag}\nabla_{dg}R_{becf}\right) \\
& + 2x^d x^e x^f x^g \left(16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 8g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj}\right. \\
& - 4g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh} \\
& + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} \\
& \left. + g^{ah}\nabla_{deh}R_{bfcg}\right)
\end{aligned}$$

$$\begin{aligned}
360A^a_{bcd} = {} & 180x^e g^{af}\nabla_b R_{cedf} + 3x^e x^f \left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg}\right. \\
& \left. + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right) \\
& + 2x^e x^f x^g \left(32g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} + 2g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj}\right. \\
& + 22g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} \\
& - 12g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh} \\
& + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} + 16g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} \\
& + 16g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg} \\
& \left. + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20g^{ah}g^{ij}R_{beci}\nabla_j R_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg}\right)
\end{aligned}$$

$$\begin{aligned}
90A^a_{bcde} = {} & 6x^f \left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) + x^f x^g \left(64g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej}\right. \\
& + 4g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} + 44g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} \\
& - 16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} \\
& \left. + 6g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_j R_{dgeh}\right)
\end{aligned}$$

$$3A^a_{bcdef} = x^g \left(3g^{ah}g^{ij}R_{bgci}\nabla_d R_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

## Geodesic boundary value problem to terms linear in $R$

$$x^a(s) = x^a + sDx^a - \frac{1}{3}\left(s - s^2\right) x^b Dx^c Dx^d g^{ae} R_{bcde} + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2) x^a_2 + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \mathcal{O}\left(\epsilon^3\right)$$

$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

## Geodesic boundary value problem to terms linear in $\nabla R$

$$x^a(s) = x^a + sDx^a + \left(s - s^2\right)\left(-\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\right)$$
$$- \frac{1}{12}\left(s - s^3\right) x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} + \mathcal{O}\left(s^4, \epsilon^4\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2)x^a_2 + (s - s^3)x^a_3 + \mathcal{O}\left(s^4, \epsilon^4\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \mathcal{O}\left(\epsilon^4\right)$$
$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$
$$-24\overset{3}{x}{}^a_2 = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)$$

$$x^a_3 = \overset{3}{x}{}^a_3 + \mathcal{O}\left(\epsilon^4\right)$$
$$-12\overset{3}{x}{}^a_3 = x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef}$$

## Geodesic boundary value problem to terms linear in $\nabla^2 R$

$$
\begin{aligned}
x^a(s) = x^a + sDx^a + \left(s - s^2\right)\Bigg(&-\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right) \\
&- \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) - \frac{1}{720}x^b x^c x^d Dx^e Dx^f \big(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} \\
&- 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\big)\Bigg) \\
+ \left(s - s^3\right)\Bigg(&-\frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} - \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f \big(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} \\
&+ 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} - 48g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\big)\Bigg) \\
- \frac{1}{180}&\left(s - s^4\right)x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right) + \mathcal{O}\left(s^5, \epsilon^5\right)
\end{aligned}
$$

$$
x^a(s) = x^a + sDx^a + (s - s^2)x^a_2 + (s - s^3)x^a_3 + (s - s^4)x^a_4 + \mathcal{O}\left(s^5, \epsilon^5\right)
$$

$$
\begin{aligned}
x^a_2 &= \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \overset{4}{x}{}^a_2 + \mathcal{O}\left(\epsilon^5\right) \\
-3\overset{2}{x}{}^a_2 &= x^b Dx^c Dx^d g^{ae} R_{bcde} \\
-24\overset{3}{x}{}^a_2 &= x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right) \\
-720\overset{4}{x}{}^a_2 &= x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \big(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} \\
&\quad - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\big)
\end{aligned}
$$

$$x_3^a = \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-12\overset{3}{x}{}_3^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$-720\overset{4}{x}{}_3^a = x^b x^c Dx^d Dx^e Dx^f \left(64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg} - 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\right)$$

$$x_4^a = \overset{4}{x}{}_4^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4 g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3 g^{ag} \nabla_{cd} R_{befg}\right)$$

## Geodesic boundary value problem to terms linear in $\nabla^3 R$

The geodesic that connects the points with RNC coordinates $x^a$ and $x^a + Dx^a$ is described, for $0 \le s \le 1$, by

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + (s - s^4)x_4^a + (s - s^5)x_5^a + \mathcal{O}\left(s^6, \epsilon^6\right)$$

$$\begin{aligned}
x_2^a &= \overset{2}{x}{}_2^a + \overset{3}{x}{}_2^a + \overset{4}{x}{}_2^a + \overset{5}{x}{}_2^a + \mathcal{O}\left(\epsilon^6\right) \\
-3\overset{2}{x}{}_2^a &= x^b Dx^c Dx^d g^{ae} R_{bcde} \\
-24\overset{3}{x}{}_2^a &= x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right) \\
-720\overset{4}{x}{}_2^a &= x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right. \\
&\quad \left. - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right) \\
-360\overset{5}{x}{}_2^a &= x^b x^c x^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 20g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj}\right. \\
&\quad - 20g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 20g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} \\
&\quad \left. - 20g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 10g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj}\right) + x^b x^c Dx^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 10g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj}\right) \\
&\quad + x^b x^c x^d x^e Dx^f Dx^g \left(16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg}\right. \\
&\quad - 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 2g^{ah}\nabla_{fbc}R_{dgeh} \\
&\quad + 2g^{ah}\nabla_{bfc}R_{dgeh} + 2g^{ah}\nabla_{bcf}R_{dgeh} + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} \\
&\quad \left. - g^{ah}\nabla_{hbc}R_{dfeg} - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\right)
\end{aligned}$$

$$x_3^a = \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \overset{5}{x}{}_3^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-12\overset{3}{x}{}_3^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned}-720\overset{4}{x}{}_3^a = x^b x^c Dx^d Dx^e Dx^f \big(&64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg} \\ &- 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\big)\end{aligned}$$

$$\begin{aligned}-1080\overset{5}{x}{}_3^a = {}& x^b x^c Dx^d Dx^e Dx^f Dx^g \left(30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - 30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} - 30 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\right) \\ &+ x^b x^c x^d Dx^e Dx^f Dx^g \big(32 g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 48 g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 12 g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 18 g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\ &+ 2 g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22 g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 48 g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 12 g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} - 15 g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\ &- 5 g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} - 12 g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12 g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8 g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12 g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\ &+ 4 g^{ah} \nabla_{efb} R_{cgdh} + 4 g^{ah} \nabla_{ebf} R_{cgdh} + 6 g^{ah} \nabla_{ebc} R_{dfgh} + 4 g^{ah} \nabla_{bef} R_{cgdh} + 6 g^{ah} \nabla_{bec} R_{dfgh} + 6 g^{ah} \nabla_{bce} R_{dfgh} - 16 g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\ &- 36 g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 16 g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 4 g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 36 g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + 4 g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg} \\ &- g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg} + 20 g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 10 g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}\big)\end{aligned}$$

$$x_4^a = \overset{4}{x}{}_4^a + \overset{5}{x}{}_4^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4 g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3 g^{ag} \nabla_{cd} R_{befg}\right)$$

$$\begin{aligned}-2160\overset{5}{x}{}_4^a = x^b x^c Dx^d Dx^e Dx^f Dx^g \big(&64 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 18 g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24 g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4 g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} \\ &+ 44 g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 18 g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} + 24 g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10 g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 16 g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} \\ &+ 6 g^{ah} \nabla_{def} R_{bgch} + 8 g^{ah} \nabla_{deb} R_{cfgh} + 8 g^{ah} \nabla_{dbe} R_{cfgh} + 8 g^{ah} \nabla_{bde} R_{cfgh} - 26 g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 6 g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} \\ &- 46 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - g^{ah} \nabla_{hde} R_{bfcg} - g^{ah} \nabla_{dhe} R_{bfcg} - g^{ah} \nabla_{deh} R_{bfcg} + 40 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\big)\end{aligned}$$

$$x_5^a = \overset{5}{x}{}_5^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-360\overset{5}{x}{}_5^a = x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(3 g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3 g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh}\right)$$

## Geodesic arc-length

$$
\begin{aligned}
(\Delta s)^2 = {} & g_{ab} Dx^a Dx^b - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \\
& + \frac{1}{360} x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8 g^{gh} R_{acdg} R_{befh} - 6 \nabla_{cd} R_{aebf}\right) + \frac{1}{360} x^a x^b x^c Dx^d Dx^e Dx^f \left(16 g^{gh} R_{adbg} R_{cefh} - 9 \nabla_{da} R_{becf} - 9 \nabla_{ad} R_{becf}\right) \\
& + \frac{1}{360} x^a x^b x^c x^d Dx^e Dx^f \left(16 g^{gh} R_{aebg} R_{cfdh} - 18 \nabla_{ab} R_{cedf}\right) + \frac{1}{1080} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4 g^{hi} R_{adeh} \nabla_f R_{bgci} - 24 g^{hi} R_{adeh} \nabla_b R_{cfgi}\right. \\
& \qquad \left. + 10 g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16 g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4 \nabla_{dea} R_{bfcg} - 4 \nabla_{dae} R_{bfcg} - 4 \nabla_{ade} R_{bfcg}\right) \\
& + \frac{1}{1080} x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18 g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3 \nabla_{cde} R_{afbg}\right) \\
& + \frac{1}{1080} x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24 g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6 \nabla_{eab} R_{cfdg} - 6 \nabla_{aeb} R_{cfdg} - 6 \nabla_{abe} R_{cfdg}\right) \\
& + \frac{1}{1080} x^a x^b x^c x^d x^e Dx^f Dx^g \left(48 g^{hi} R_{afbh} \nabla_c R_{dgei} - 12 \nabla_{abc} R_{dfeg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

## Geodesic arc-length curvature expansion

$$(\Delta s)^2 = \overset{0}{\Delta} + \overset{2}{\Delta} + \overset{3}{\Delta} + \overset{4}{\Delta} + \overset{5}{\Delta} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{\Delta} = g_{ab} Dx^a Dx^b$$

$$3\overset{2}{\Delta} = -x^a x^b Dx^c Dx^d R_{acbd}$$

$$12\overset{3}{\Delta} = -x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - 2x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce}$$

$$\begin{aligned}360\overset{4}{\Delta} = {} & x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8g^{gh} R_{acdg} R_{befh} - 6\nabla_{cd} R_{aebf}\right) + x^a x^b x^c Dx^d Dx^e Dx^f \left(16g^{gh} R_{adbg} R_{cefh} - 9\nabla_{da} R_{becf} - 9\nabla_{ad} R_{becf}\right) \\ & + x^a x^b x^c x^d Dx^e Dx^f \left(16g^{gh} R_{aebg} R_{cfdh} - 18\nabla_{ab} R_{cedf}\right)\end{aligned}$$

$$\begin{aligned}1080\overset{5}{\Delta} = {} & x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4g^{hi} R_{adeh} \nabla_f R_{bgci} - 24g^{hi} R_{adeh} \nabla_b R_{cfgi} + 10g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4\nabla_{dea} R_{bfcg} - 4\nabla_{dae} R_{bfcg} \right. \\ & \left. - 4\nabla_{ade} R_{bfcg}\right) + x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3\nabla_{cde} R_{afbg}\right) \\ & + x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6\nabla_{eab} R_{cfdg} - 6\nabla_{aeb} R_{cfdg} - 6\nabla_{abe} R_{cfdg}\right) \\ & + x^a x^b x^c x^d x^e Dx^f Dx^g \left(48g^{hi} R_{afbh} \nabla_c R_{dgei} - 12\nabla_{abc} R_{dfeg}\right)\end{aligned}$$

# Tranformation between two RNC frames

$$y^a = \overset{0}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a + \overset{5}{y}{}^a + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{y}{}^a = Dx^a$$

$$\begin{aligned}
\overset{2}{y}{}^a &= \overset{2}{y}{}^a_1 \\
3\overset{2}{y}{}^a_1 &= -x^b Dx^c Dx^d g^{ae} R_{bcde}
\end{aligned}$$

$$\begin{aligned}
\overset{3}{y}{}^a &= \overset{3}{y}{}^a_1 + \overset{3}{y}{}^a_2 \\
-12\overset{3}{y}{}^a_1 &= x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} \\
-24\overset{3}{y}{}^a_2 &= x^b x^c Dx^d Dx^e \left(2g^{af} \nabla_d R_{becf} + 4g^{af} \nabla_b R_{cdef} - g^{af} \nabla_f R_{bdce}\right)
\end{aligned}$$

$$\begin{aligned}
\overset{4}{y}{}^a &= \overset{4}{y}{}^a_1 + \overset{4}{y}{}^a_2 + \overset{4}{y}{}^a_3 \\
-180\overset{4}{y}{}^a_1 &= x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3g^{ag} \nabla_{cd} R_{befg}\right) \\
-720\overset{4}{y}{}^a_2 &= x^b x^c Dx^d Dx^e Dx^f \left(32g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 16g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdch} R_{egfi} + 12g^{ag} \nabla_{de} R_{bfcg}\right. \\
&\qquad \left. + 18g^{ag} \nabla_{db} R_{cefg} + 18g^{ag} \nabla_{bd} R_{cefg} - 3g^{ag} \nabla_{gd} R_{becf} - 3g^{ag} \nabla_{dg} R_{becf}\right) \\
-720\overset{4}{y}{}^a_3 &= x^b x^c x^d Dx^e Dx^f \left(64g^{ag} g^{hi} R_{befh} R_{cgdi} - 32g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{bech} R_{dgfi} + 18g^{ag} \nabla_{eb} R_{cfdg} + 18g^{ag} \nabla_{be} R_{cfdg} + 36g^{ag} \nabla_{bc} R_{defg}\right. \\
&\qquad \left. + 16g^{ag} g^{hi} R_{bech} R_{dfgi} - 9g^{ag} \nabla_{gb} R_{cedf} - 9g^{ag} \nabla_{bg} R_{cedf}\right)
\end{aligned}$$

$$
\begin{aligned}
\overset{5}{y}{}^{a} &= \overset{5}{y}{}^{a}_{1} + \overset{5}{y}{}^{a}_{2} + \overset{5}{y}{}^{a}_{3} + \overset{5}{y}{}^{a}_{4} \\
-360\overset{5}{y}{}^{a}_{1} &= x^b Dx^c Dx^d Dx^e Dx^f Dx^g \big(3g^{ah}g^{ij}R_{bcdi}\nabla_e R_{fhgj} + 3g^{ah}g^{ij}R_{chdi}\nabla_e R_{bfgj} + g^{ah}\nabla_{cde}R_{bfgh}\big) \\
-2160\overset{5}{y}{}^{a}_{2} &= x^b x^c Dx^d Dx^e Dx^f Dx^g \big(34g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 16g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} + 14g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} \\
&\quad - 20g^{ah}g^{ij}R_{bdei}\nabla_j R_{cfgh} + 18g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} + 24g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} + 4g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 18g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj} \\
&\quad + 24g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} - 10g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}\nabla_{def}R_{bgch} + 8g^{ah}\nabla_{deb}R_{cfgh} + 8g^{ah}\nabla_{dbe}R_{cfgh} \\
&\quad + 8g^{ah}\nabla_{bde}R_{cfgh} - 6g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - g^{ah}\nabla_{hde}R_{bfcg} - g^{ah}\nabla_{dhe}R_{bfcg} - g^{ah}\nabla_{deh}R_{bfcg}\big) \\
-1080\overset{5}{y}{}^{a}_{3} &= x^b x^c x^d Dx^e Dx^f Dx^g \big(14g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 24g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} - 12g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} \\
&\quad + 2g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 24g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 14g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} \\
&\quad + 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} + 2g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 22g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} + 12g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} \\
&\quad - 12g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} - 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} - 8g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} - 12g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 4g^{ah}\nabla_{efb}R_{cgdh} + 4g^{ah}\nabla_{ebf}R_{cgdh} \\
&\quad + 6g^{ah}\nabla_{ebc}R_{dfgh} + 4g^{ah}\nabla_{bef}R_{cgdh} + 6g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} + 4g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - g^{ah}\nabla_{heb}R_{cfdg} \\
&\quad - g^{ah}\nabla_{hbe}R_{cfdg} - g^{ah}\nabla_{ehb}R_{cfdg} - g^{ah}\nabla_{bhe}R_{cfdg} - g^{ah}\nabla_{ebh}R_{cfdg} - g^{ah}\nabla_{beh}R_{cfdg}\big) \\
-360\overset{5}{y}{}^{a}_{4} &= x^b x^c x^d x^e Dx^f Dx^g \big(16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} - 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} \\
&\quad - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 2g^{ah}\nabla_{fbc}R_{dgeh} + 2g^{ah}\nabla_{bfc}R_{dgeh} \\
&\quad + 2g^{ah}\nabla_{bcf}R_{dgeh} + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} - g^{ah}\nabla_{hbc}R_{dfeg} \\
&\quad - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\big)
\end{aligned}
$$