

The connection

Here we use the output from `metric.tex` and `metric-inv.tex` to compute the metric connection Γ_{ab}^d . We use the standard metric compatible connection

$$\Gamma_{ab}^d = \frac{1}{2} g^{dc} (g_{cb,a} + g_{ac,b} - g_{ab,c}) \quad (1)$$

Since `metric.tex` and `metric-inv.tex` generate truncated expressions for g_{ab} and g^{ab} a similar truncation must be applied to this computation of Γ_{ab}^d . The naive choice is to truncate Γ_{ab}^d *after* it has been fully evaluated on the truncated expressions for g_{ab} and g^{ab} . This will work but it wastes time and memory (big time).

A better approach is to truncate Γ_{ab}^d during its construction. That is, we take careful note of how the terms in the finite series for g_{ab} and g^{ab} combine to produce the terms of a particular order in the expansion of Γ_{ab}^d .

Suppose g_{ab} and g^{ab} are known to say fourth order. We can write each of these as follows

$$g_{ab} = g_{ab}^0 + g_{ab}^1 + g_{ab}^2 + g_{ab}^3 + g_{ab}^4 \quad (2)$$

$$g^{ab} = g^{ab0} + g^{ab1} + g^{ab2} + g^{ab3} + g^{ab4} \quad (3)$$

where g^n denotes a term of order $\mathcal{O}(\epsilon^n)$. A similar expansion applies for Γ_{ab}^d , that is

$$\Gamma_{ab}^d = \Gamma_{ab}^{d0} + \Gamma_{ab}^{d1} + \Gamma_{ab}^{d2} + \Gamma_{ab}^{d3} + \Gamma_{ab}^{d4} \quad (4)$$

After substituting these formal expansions into the equation (1) and then matching corresponding terms we obtain

$$\Gamma_{ab}^{dn} = \frac{1}{2} \sum_{i=0}^{i=n} g^{idc} \left(g_{cb,a}^{n-i} + g_{ac,b}^{n-i} - g_{ab,c}^{n-i} \right) \quad (5)$$

We use this equation to compute the successive terms in Γ_{ab}^d .

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

x^{a}::Depends(D{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

import cdblib

gab = cdblib.get ('g_ab','metric.json')      # cdb(gab.000,gab)
iab = cdblib.get ('g^ab','metric-inv.json')  # cdb(iab.000,iab)

defgab := g_{a b} -> @(gab).
defiab := g^{a b} -> @(iab).

dgab := D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}.  # cdb(dgab.001,dgab)

substitute (dgab,defgab)

distribute (dgab)          # cdb(dgab.002,dgab)
unwrap (dgab)              # cdb(dgab.003,dgab)
product_rule (dgab)        # cdb(dgab.004,dgab)
distribute (dgab)          # cdb(dgab.005,dgab)
substitute (dgab,$D_{a}{x^{b}}->\delta^{b}_{a}$,repeat=True) # cdb(dgab.006,dgab)

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eliminate_kronecker (dgab)      # cdb(dgab.007,dgab)
sort_product      (dgab)      # cdb(dgab.008,dgab)
rename_dummies   (dgab)      # cdb(dgab.009,dgab)
canonicalise     (dgab)      # cdb(dgab.010,dgab)
```

$$\begin{aligned} \text{gab.000} := & g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45}x^c x^d x^e x^f R_{acd g} R_{bef h} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf} \\ & + \frac{1}{45}x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afb g} \end{aligned}$$

$$\text{dgab.001} := D_a g_{cb} + D_b g_{ac} - D_c g_{ab}$$

$$\begin{aligned} \text{dgab.002} := & D_a g_{cb} - \frac{1}{3}D_a (x^j x^d R_{c j b d}) - \frac{1}{6}D_a (x^j x^d x^e \nabla_j R_{cdbe}) + \frac{2}{45}D_a (x^j x^d x^e x^f R_{c j d g} R_{bef h} g^{gh}) - \frac{1}{20}D_a (x^j x^d x^e x^f \nabla_{jd} R_{cebf}) \\ & + \frac{1}{45}D_a (x^j x^d x^e x^f x^g R_{c j d h} \nabla_e R_{bfgi} g^{hi}) + \frac{1}{45}D_a (x^j x^d x^e x^f x^g R_{b j d h} \nabla_e R_{cfgi} g^{hi}) - \frac{1}{90}D_a (x^j x^d x^e x^f x^g \nabla_{jde} R_{cfbg}) \\ & + D_b g_{ac} - \frac{1}{3}D_b (x^j x^d R_{a j c d}) - \frac{1}{6}D_b (x^j x^d x^e \nabla_j R_{adce}) + \frac{2}{45}D_b (x^j x^d x^e x^f R_{a j d g} R_{cef h} g^{gh}) - \frac{1}{20}D_b (x^j x^d x^e x^f \nabla_{jd} R_{aecf}) \\ & + \frac{1}{45}D_b (x^j x^d x^e x^f x^g R_{a j d h} \nabla_e R_{cfgi} g^{hi}) + \frac{1}{45}D_b (x^j x^d x^e x^f x^g R_{c j d h} \nabla_e R_{afgi} g^{hi}) - \frac{1}{90}D_b (x^j x^d x^e x^f x^g \nabla_{jde} R_{afcg}) \\ & - D_c g_{ab} + \frac{1}{3}D_c (x^j x^d R_{a j b d}) + \frac{1}{6}D_c (x^j x^d x^e \nabla_j R_{adbe}) - \frac{2}{45}D_c (x^j x^d x^e x^f R_{a j d g} R_{bef h} g^{gh}) + \frac{1}{20}D_c (x^j x^d x^e x^f \nabla_{jd} R_{aebf}) \\ & - \frac{1}{45}D_c (x^j x^d x^e x^f x^g R_{a j d h} \nabla_e R_{bfgi} g^{hi}) - \frac{1}{45}D_c (x^j x^d x^e x^f x^g R_{b j d h} \nabla_e R_{afgi} g^{hi}) + \frac{1}{90}D_c (x^j x^d x^e x^f x^g \nabla_{jde} R_{afb g}) \end{aligned}$$

$$\begin{aligned}
\text{dgab.010} := & \frac{2}{3}R_{acbd}x^d - \frac{1}{6}\nabla_a R_{bdce}x^d x^e + \frac{1}{3}\nabla_d R_{acbe}x^d x^e - \frac{4}{45}R_{acde}R_{bfggh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adce}R_{bfggh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adbe}R_{cfgh}g^{dg}x^e x^f x^h \\
& - \frac{1}{20}\nabla_{ad}R_{becf}x^d x^e x^f - \frac{1}{20}\nabla_{da}R_{becf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{acbf}x^d x^e x^f - \frac{2}{45}R_{acde}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i \\
& + \frac{1}{45}R_{cdef}\nabla_a R_{bghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{bdef}\nabla_a R_{cghi}g^{eh}x^d x^f x^g x^i \\
& - \frac{2}{45}R_{bdef}\nabla_g R_{achi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{eh}x^d x^f x^g x^i - \frac{1}{90}\nabla_{ade}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dae}R_{bfcg}x^d x^e x^f x^g \\
& - \frac{1}{90}\nabla_{dea}R_{bfcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{acbg}x^d x^e x^f x^g + \frac{2}{3}R_{adbc}x^d - \frac{1}{6}\nabla_b R_{adce}x^d x^e + \frac{1}{3}\nabla_d R_{aebc}x^d x^e - \frac{2}{45}R_{adbe}R_{cfgh}g^{eg}x^d x^f x^h \\
& - \frac{4}{45}R_{adef}R_{bcgh}g^{eg}x^d x^f x^h - \frac{2}{45}R_{adef}R_{bgch}g^{eg}x^d x^f x^h - \frac{1}{20}\nabla_{bd}R_{aecf}x^d x^e x^f - \frac{1}{20}\nabla_{db}R_{aecf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{afbc}x^d x^e x^f \\
& - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_b R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{adef}\nabla_g R_{bchi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{eh}x^d x^f x^g x^i \\
& - \frac{2}{45}R_{bcde}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_b R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{ei}x^d x^f x^g x^h \\
& - \frac{1}{90}\nabla_{bde}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dbe}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{deb}R_{afcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{agbc}x^d x^e x^f x^g + \frac{1}{6}\nabla_c R_{adbe}x^d x^e \\
& + \frac{2}{45}R_{adce}R_{bfggh}g^{eg}x^d x^f x^h + \frac{2}{45}R_{adef}R_{bgch}g^{eh}x^d x^f x^g + \frac{1}{20}\nabla_{cd}R_{aebf}x^d x^e x^f + \frac{1}{20}\nabla_{dc}R_{aebf}x^d x^e x^f + \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{eh}x^d x^f x^g x^i \\
& - \frac{1}{45}R_{adef}\nabla_c R_{bghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{ei}x^d x^f x^g x^h + \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_c R_{aghi}g^{eh}x^d x^f x^g x^i \\
& + \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{ei}x^d x^f x^g x^h + \frac{1}{90}\nabla_{cde}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dce}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dec}R_{afbg}x^d x^e x^f x^g
\end{aligned}$$

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# Note:
# Computing Gamma directly by (1/2) iab dgab and *then* truncating to lower order
# is not optimal. We only want the leading order terms (to 4th order in x). But the direct
# calculation would compute *all* terms before the truncation. This does work but it
# is slower than the following code.
#
# The better approach (as adopted in this code) is to extract all of the terms of iab
# and dgab then construct the leading order terms of Gamma (to fifth order) term by term.

def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    distribute (tmp)

    substitute (tmp, $\nabla_{e f g}\{R_{a b c d}\} \rightarrow Q_{a b c d e f g}\$)
    substitute (tmp, $\nabla_{e f}\{R_{a b c d}\} \rightarrow Q_{a b c d e f}\$)
    substitute (tmp, $\nabla_{e}\{R_{a b c d}\} \rightarrow Q_{a b c d e}\$)
    substitute (tmp, $R_{a b c d} \rightarrow Q_{a b c d}\$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} \rightarrow \nabla_{e f g}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d e f} \rightarrow \nabla_{e f}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d e} \rightarrow \nabla_{e}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d} \rightarrow R_{a b c d}\$)

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return foo

# terms of the curvature expansion of dg_{ab}

dgab00 = get_Rterm (dgab,0)    # cdb(dgab00.105,dgab00)  # zero
dgab01 = get_Rterm (dgab,1)    # cdb(dgab01.105,dgab01)  # zero
dgab02 = get_Rterm (dgab,2)    # cdb(dgab02.105,dgab02)
dgab03 = get_Rterm (dgab,3)    # cdb(dgab03.105,dgab03)
dgab04 = get_Rterm (dgab,4)    # cdb(dgab04.105,dgab04)
dgab05 = get_Rterm (dgab,5)    # cdb(dgab05.105,dgab05)

# Convert free indices on iab from ^{a b} to ^{d c}
# This ensures we can later build products like @(iab) @(dgab) knowing that the indices are correctly ordered.
# Without this step we would be using free indices ^{a b} and _{a b c}. Thus the product @(iab) @(dgab) would
# have just one free index _{c}. This is clearly wrong.

tmp := @(iab) \delta_{a}^{d} \delta_{b}^{c}.

distribute      (tmp)
eliminate_kronecker (tmp)
sort_product    (tmp)
rename_dummies  (tmp)
canonicalise    (tmp)

idc := @(tmp).

# terms of the curvature expansion of g^{ab}

idc00 = get_Rterm (idc,0)    # cdb(idc00.105,idc00)
idc01 = get_Rterm (idc,1)    # cdb(idc01.105,idc01)  # zero
idc02 = get_Rterm (idc,2)    # cdb(idc02.105,idc02)
idc03 = get_Rterm (idc,3)    # cdb(idc03.105,idc03)
idc04 = get_Rterm (idc,4)    # cdb(idc04.105,idc04)
idc05 = get_Rterm (idc,5)    # cdb(idc05.105,idc05)

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$$\text{dgab00.105} := 0$$

$$\text{dgab01.105} := 0$$

$$\text{dgab02.105} := \frac{2}{3}R_{acbd}x^d + \frac{2}{3}R_{adbc}x^d$$

$$\text{dgab03.105} := -\frac{1}{6}\nabla_a R_{bdce}x^d x^e + \frac{1}{3}\nabla_d R_{acbe}x^d x^e - \frac{1}{6}\nabla_b R_{adce}x^d x^e + \frac{1}{3}\nabla_d R_{aebc}x^d x^e + \frac{1}{6}\nabla_c R_{adbe}x^d x^e$$

$$\begin{aligned} \text{dgab04.105} := & -\frac{4}{45}R_{acde}R_{bfggh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adce}R_{bfggh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adbe}R_{cfggh}g^{dg}x^e x^f x^h - \frac{1}{20}\nabla_{ad}R_{becf}x^d x^e x^f \\ & - \frac{1}{20}\nabla_{da}R_{becf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{acbf}x^d x^e x^f - \frac{2}{45}R_{adbe}R_{cfggh}g^{eg}x^d x^f x^h - \frac{4}{45}R_{adef}R_{bcgh}g^{eg}x^d x^f x^h \\ & - \frac{2}{45}R_{adef}R_{bgch}g^{eg}x^d x^f x^h - \frac{1}{20}\nabla_{bd}R_{aecf}x^d x^e x^f - \frac{1}{20}\nabla_{db}R_{aecf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{afbc}x^d x^e x^f \\ & + \frac{2}{45}R_{adce}R_{bfggh}g^{eg}x^d x^f x^h + \frac{2}{45}R_{adef}R_{bgch}g^{eh}x^d x^f x^g + \frac{1}{20}\nabla_{cd}R_{aebf}x^d x^e x^f + \frac{1}{20}\nabla_{dc}R_{aebf}x^d x^e x^f \end{aligned}$$

$$\begin{aligned} \text{dgab05.105} := & -\frac{2}{45}R_{acde}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_a R_{bghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{eh}x^d x^f x^g x^i \\ & - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{bdef}\nabla_a R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bdef}\nabla_g R_{achi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{eh}x^d x^f x^g x^i \\ & - \frac{1}{90}\nabla_{ade}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dae}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dea}R_{bfcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{acbg}x^d x^e x^f x^g - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{eh}x^d x^f x^g x^i \\ & + \frac{1}{45}R_{adef}\nabla_b R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{adef}\nabla_g R_{bchi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bcde}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i \\ & - \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_b R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{ei}x^d x^f x^g x^h - \frac{1}{90}\nabla_{bde}R_{afcg}x^d x^e x^f x^g \\ & - \frac{1}{90}\nabla_{dbe}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{deb}R_{afcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{agbc}x^d x^e x^f x^g + \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{eh}x^d x^f x^g x^i \\ & - \frac{1}{45}R_{adef}\nabla_c R_{bghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{ei}x^d x^f x^g x^h + \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_c R_{aghi}g^{eh}x^d x^f x^g x^i \\ & + \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{ei}x^d x^f x^g x^h + \frac{1}{90}\nabla_{cde}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dce}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dec}R_{afbg}x^d x^e x^f x^g \end{aligned}$$

$$\text{idc00.105} := g^{cd}$$

$$\text{idc01.105} := 0$$

$$\text{idc02.105} := \frac{1}{3} R_{abef} g^{ca} g^{de} x^b x^f$$

$$\text{idc03.105} := \frac{1}{6} \nabla_a R_{befg} g^{cb} g^{df} x^a x^e x^g$$

$$\text{idc04.105} := \frac{1}{15} R_{abef} R_{ghij} g^{ca} g^{dg} g^{ei} x^b x^f x^h x^j + \frac{1}{20} \nabla_{ab} R_{efgh} g^{ce} g^{dg} x^a x^b x^f x^h$$

$$\text{idc05.105} := \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ch} g^{da} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ca} g^{dh} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{90} \nabla_{abe} R_{fghi} g^{cf} g^{dh} x^a x^b x^e x^g x^i$$

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# idc = g^{d c}
# dgab = D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}

# terms of the curvature expansion of \Gamma^d_{a b}

# term0 := (1/2) @ (idc00) @ (dgab00).
# term1 := (1/2) (@ (idc01) @ (dgab00) + @ (idc00) @ (dgab01)).
# term2 := (1/2) (@ (idc02) @ (dgab00) + @ (idc01) @ (dgab01) + @ (idc00) @ (dgab02)).
# term3 := (1/2) (@ (idc03) @ (dgab00) + @ (idc02) @ (dgab01) + @ (idc01) @ (dgab02) + @ (idc00) @ (dgab03)).
# term4 := (1/2) (@ (idc04) @ (dgab00) + @ (idc03) @ (dgab01) + @ (idc02) @ (dgab02) + @ (idc01) @ (dgab03) + @ (idc00) @ (dgab04)).
# term5 := (1/2) (@ (idc05) @ (dgab00) + @ (idc04) @ (dgab01) + @ (idc03) @ (dgab02) + @ (idc02) @ (dgab03) + @ (idc01) @ (dgab04) + @ (idc00) @ (dgab05)).

# simplified version of the above after noting dgab00 = dgab01 = 0

term0 := 0.
term1 := 0.
term2 := (1/2) (@ (idc00) @ (dgab02)).
term3 := (1/2) (@ (idc01) @ (dgab02) + @ (idc00) @ (dgab03)).
term4 := (1/2) (@ (idc02) @ (dgab02) + @ (idc01) @ (dgab03) + @ (idc00) @ (dgab04)).
term5 := (1/2) (@ (idc03) @ (dgab02) + @ (idc02) @ (dgab03) + @ (idc01) @ (dgab04) + @ (idc00) @ (dgab05)).

def tidy_terms (obj):
  substitute (obj,$x^{a}->AA^{a}$,repeat=True) # will force AA to the left of all terms
  distribute (obj)
  sort_product (obj)
  rename_dummies (obj)
  canonicalise (obj)
  substitute (obj,$AA^{a}->x^{a}$,repeat=True) # replace AA with x
  factor_out (obj,$x^{a?}$)

  return obj

term0 = tidy_terms (term0) # cdb(term0.201,term0) # zero
term1 = tidy_terms (term1) # cdb(term1.201,term1) # zero
term2 = tidy_terms (term2) # cdb(term2.201,term2)
term3 = tidy_terms (term3) # cdb(term3.201,term3)
term4 = tidy_terms (term4) # cdb(term4.201,term4)
term5 = tidy_terms (term5) # cdb(term5.201,term5)

```

```
Gamma := @(term0) + @(term1) + @(term2) + @(term3) + @(term4) + @(term5). # cdb(Gamma.200,Gamma)
```

$$\text{term0.201} := 0$$

$$\text{term1.201} := 0$$

$$\text{term2.201} := x^c \left(\frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de} \right)$$

$$\text{term3.201} := x^c x^e \left(\frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df} \right)$$

$$\begin{aligned} \text{term4.201} := x^c x^e x^f \left(\frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \right. \\ \left. + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \right. \\ \left. - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg} \right) \end{aligned}$$

$$\begin{aligned} \text{term5.201} := x^c x^e x^f x^g \left(\frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} \right. \\ \left. + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} \right. \\ \left. - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cea} R_{bfgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} \right. \\ \left. - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} \right. \\ \left. + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} \right. \\ \left. - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{agij} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} \right. \\ \left. + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh} \right) \end{aligned}$$

$$\begin{aligned}
\text{Gamma.200} := & x^c \left(\frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de} \right) + x^c x^e \left(\frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df} \right) \\
& + x^c x^e x^f \left(\frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \right. \\
& \quad + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \\
& \quad \left. - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg} \right) \\
& + x^c x^e x^f x^g \left(\frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} \right. \\
& \quad + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} \\
& \quad - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgj} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgj} g^{dh} \\
& \quad + \frac{1}{180} \nabla_{cea} R_{bfgj} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} \\
& \quad - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} \\
& \quad - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} \\
& \quad \left. - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh} \right)
\end{aligned}$$

```
cdblib.create ('connection.json')

cdblib.put ('Gamma',Gamma,'connection.json')

cdblib.put ('GammaRterm0',term0,'connection.json')
cdblib.put ('GammaRterm1',term1,'connection.json')
cdblib.put ('GammaRterm2',term2,'connection.json')
cdblib.put ('GammaRterm3',term3,'connection.json')
cdblib.put ('GammaRterm4',term4,'connection.json')
cdblib.put ('GammaRterm5',term5,'connection.json')

checkpoint.append (term0)
checkpoint.append (term1)
checkpoint.append (term2)
checkpoint.append (term3)
checkpoint.append (term4)
checkpoint.append (term5)
```

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ g^{a b}              -> A003^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\} -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}    -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}      -> A006_{a b c d e f}    $)
    substitute (obj,$ \nabla_e\{R_{a b c d}\}         -> A005_{a b c d e}     $)
    substitute (obj,$ R_{a b c d}           -> A004_{a b c d}      $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> A^{a}              $)
    substitute (obj,$ A002^{a}              -> x^{a}              $)
    substitute (obj,$ A003^{a b}            -> g^{a b}            $)
    substitute (obj,$ A008_{a b c d e f g h} -> \nabla_{e f g h}\{R_{a b c d}\} $)
    substitute (obj,$ A007_{a b c d e f g}   -> \nabla_{e f g}\{R_{a b c d}\}   $)
    substitute (obj,$ A006_{a b c d e f}     -> \nabla_{e f}\{R_{a b c d}\}    $)
    substitute (obj,$ A005_{a b c d e}       -> \nabla_e\{R_{a b c d}\}      $)
    substitute (obj,$ A004_{a b c d}         -> R_{a b c d}          $)

    return obj

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    factor_out (bah,$A^{a?},x^{b?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)

```

```

factor_out (bah,$A^{a?},x^{b?}$)
return bah

Rterm2 := @(term2) A^{a} A^{b}.
Rterm3 := @(term3) A^{a} A^{b}.
Rterm4 := @(term4) A^{a} A^{b}.
Rterm5 := @(term5) A^{a} A^{b}.

Rterm2 = reformat (Rterm2, 3)      # cdb(Rterm2.301,Rterm2)
Rterm3 = reformat (Rterm3, 12)     # cdb(Rterm3.301,Rterm3)
Rterm4 = reformat (Rterm4,360)     # cdb(Rterm4.301,Rterm4)
Rterm5 = reformat (Rterm5,180)     # cdb(Rterm5.301,Rterm5)

Gamma  := @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5). # cdb (Gamma.301,Gamma)
Scaled := 360 @(Gamma).                # cdb (Scaled.301,Scaled)

scaled2 = rescale (Rterm2, 3)      # cdb (scaled2.301,scaled2)
scaled3 = rescale (Rterm3, 12)     # cdb (scaled3.301,scaled3)
scaled4 = rescale (Rterm4, 360)    # cdb (scaled4.301,scaled4)
scaled5 = rescale (Rterm5, 180)    # cdb (scaled5.301,scaled5)

```


The connection in Riemann normal coordinates

$$\begin{aligned}
A^a A^b \Gamma_{ab}^d = & \frac{2}{3} A^a A^b x^c g^{de} R_{acbe} + \frac{1}{12} A^a A^b x^c x^e (2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}) + \frac{1}{360} A^a A^b x^c x^e x^f (64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} \\
& - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}) \\
& + \frac{1}{180} A^a A^b x^c x^e x^f x^g (16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} \\
& - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bfgh} \\
& + 2g^{dh} \nabla_{cae} R_{bfgh} + 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} \\
& + g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg})
\end{aligned}$$

$$\begin{aligned}
360 A^a A^b \Gamma_{ab}^d = & 240 A^a A^b x^c g^{de} R_{acbe} + 30 A^a A^b x^c x^e (2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}) + A^a A^b x^c x^e x^f (64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} \\
& - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}) \\
& + 2 A^a A^b x^c x^e x^f x^g (16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} \\
& - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} \\
& + 2g^{dh} \nabla_{ace} R_{bfgh} + 2g^{dh} \nabla_{cae} R_{bfgh} + 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} \\
& - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} + g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg})
\end{aligned}$$

Curvature expansion of the connection

$$A^a A^b \Gamma_{ab}^d = A^a A^b \overset{2}{\Gamma}_{ab}^d + A^a A^b \overset{3}{\Gamma}_{ab}^d + A^a A^b \overset{4}{\Gamma}_{ab}^d + A^a A^b \overset{5}{\Gamma}_{ab}^d + \mathcal{O}(\epsilon^6)$$

$$3A^a A^b \overset{2}{\Gamma}_{ab}^d = 2A^a A^b x^c g^{de} R_{acbe}$$

$$12A^a A^b \overset{3}{\Gamma}_{ab}^d = A^a A^b x^c x^e (2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe})$$

$$360A^a A^b \overset{4}{\Gamma}_{ab}^d = A^a A^b x^c x^e x^f (64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf})$$

$$180A^a A^b \overset{5}{\Gamma}_{ab}^d = A^a A^b x^c x^e x^f x^g (16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bf gj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bf gj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bf gj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bf gh} + 2g^{dh} \nabla_{cae} R_{bf gh} + 2g^{dh} \nabla_{cea} R_{bf gh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bf gj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bf gj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bg hj} + g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg})$$