# Geodesic arc-length

Give a pair of points $P$ and $Q$ the geodesic arc-length can be computed using

$$L_{PQ} = \int_P^Q \left( g_{ab}(x) \frac{dx^a}{ds} \frac{dx^b}{ds} \right)^{1/2} ds \tag{1}$$

Since the path is a geodesic the integrand is constant and thus

$$L_{PQ}^2 = g_{ab}(x) \frac{dx^a}{ds} \frac{dx^b}{ds} \bigg|_P \tag{2}$$

where $s$ is a re-scaled parameter (0 at $P$ and 1 at $Q$). The point $P$ has RNC coordinates $x^a$ while the point $Q$ has coordinates $x^a + Dx^a$.

The vector $dx^a/ds$ at $P$ is given by the solution of the geodesic boundary value problem. This was found in the previous code (`geodesic-bvp`). That is

$$\frac{dx^b}{ds} \bigg|_P = y^a \tag{3}$$

and thus

$$L_{PQ}^2 = g_{ab}(x) y^a y^b \tag{4}$$

It is possible to directly evaluate the right hand side of (4) using the results from the `geodesic-bvp` and `metric` codes. The result would need to be truncated (to an order consistent with the results form those codes). But doing so would be computationaly expensive as at least half of the terms will be thrown away. A better approach is compute just the terms that will survive the truncation. This is done by expanding $g_{ab}(x)$ and $y^a$ as a truncated series in the curvatures and its derivatives.

The $g_{ab}(x)$ and $y^a$ are written in a (truncated) formal power series in the curvature and its derivatives

$$y^a = \overset{0}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a + \overset{5}{y}{}^a + \mathcal{O}\left(\epsilon^6\right) \tag{5}$$

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}\left(\epsilon^6\right) \tag{6}$$

Note that this use of $\overset{i}{y}$ differs from that used in `geodesic-bvp`. Here the index above $y^a$ denotes a particular term in the curvature expansion while in `geodesic-bvp` the index denoted the iteration number (in the fixed point scheme used to solve the BVP for $y^a$).

## Stage 1

The formal curvature expansions are substituted into equation (4), expanded and truncated to retain terms of order $\mathcal{O}\left(\epsilon^5\right)$ or less. The expansion to 4th order terms is as follows.

$$L^2_{PQ} = \overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b + 2\overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{2}{y}{}^b + \overset{2}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b + 2\overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{3}{y}{}^b + \overset{3}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b + 2\overset{0}{g}_{ab}\overset{0}{y}{}^a\overset{4}{y}{}^b + \overset{0}{g}_{ab}\overset{2}{y}{}^a\overset{2}{y}{}^b + 2\overset{2}{g}_{ab}\overset{0}{y}{}^a\overset{2}{y}{}^b + \overset{4}{g}_{ab}\overset{0}{y}{}^a\overset{0}{y}{}^b$$

From `geodesic-bvp` (actually from `rnc2rnc which reformatted the results nicely`) we have

$$\overset{0}{y}{}^a = Dx^a$$

$$\overset{2}{y}{}^a = -\frac{1}{3}x^bDx^cDx^dg^{ae}R_{bcde}$$

$$\overset{3}{y}{}^a = x^bx^cDx^dDx^e\left(-\frac{1}{12}g^{af}\nabla_dR_{becf} - \frac{1}{6}g^{af}\nabla_bR_{cdef} + \frac{1}{24}g^{af}\nabla_fR_{bdce}\right) - \frac{1}{12}x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef}$$

$$\begin{aligned}\overset{4}{y}{}^a = {}& x^bx^cDx^dDx^eDx^f\left(-\frac{2}{45}g^{ag}g^{hi}R_{bdeh}R_{cfgi} + \frac{1}{45}g^{ag}g^{hi}R_{bdeh}R_{cifg} - \frac{1}{45}g^{ag}g^{hi}R_{bdeh}R_{cgfi} + \frac{1}{45}g^{ag}g^{hi}R_{bdch}R_{egfi} - \frac{1}{60}g^{ag}\nabla_{de}R_{bfcg}\right.\\ &\left. - \frac{1}{40}g^{ag}\nabla_{db}R_{cefg} - \frac{1}{40}g^{ag}\nabla_{bd}R_{cefg} + \frac{1}{240}g^{ag}\nabla_{gd}R_{becf} + \frac{1}{240}g^{ag}\nabla_{dg}R_{becf}\right)\\ &+ x^bx^cx^dDx^eDx^f\left(-\frac{4}{45}g^{ag}g^{hi}R_{befh}R_{cgdi} + \frac{2}{45}g^{ag}g^{hi}R_{bech}R_{difg} + \frac{1}{45}g^{ag}g^{hi}R_{bech}R_{dgfi} - \frac{1}{40}g^{ag}\nabla_{eb}R_{cfdg} - \frac{1}{40}g^{ag}\nabla_{be}R_{cfdg} - \frac{1}{20}g^{ag}\nabla_{bc}R_{defg}\right.\\ &\left. - \frac{1}{45}g^{ag}g^{hi}R_{bech}R_{dfgi} + \frac{1}{80}g^{ag}\nabla_{gb}R_{cedf} + \frac{1}{80}g^{ag}\nabla_{bg}R_{cedf}\right) + x^bDx^cDx^dDx^eDx^f\left(-\frac{1}{45}g^{ag}g^{hi}R_{bcdh}R_{egfi} - \frac{1}{60}g^{ag}\nabla_{cd}R_{befg}\right)\end{aligned}$$

and from `metric` we have

$$\overset{0}{g}_{ab} = g_{ab}$$

$$3\overset{2}{g}_{ab} = -x^cx^dR_{acbd}$$

$$6\overset{3}{g}_{ab} = -x^cx^dx^e\nabla_cR_{adbe}$$

$$180\overset{4}{g}_{ab} = x^cx^dx^ex^f\left(8g^{gh}R_{acdg}R_{befh} - 9\nabla_{cd}R_{aebf}\right)$$

# Stage 2

The results from the `geodesic-bvp` and `metric` codes are read to provide values for the $\overset{n}{y}{}^a$ and $\overset{m}{g}{}_{ab}$. These are substituted into the result from Stage 1, et volia, the final answer. To 4th-order terms the result is given by

$$
\begin{aligned}
L^2_{PQ} = {} & g_{ab}Dx^aDx^b - \frac{1}{3}x^ax^bDx^cDx^dR_{acbd} - \frac{1}{12}x^ax^bDx^cDx^dDx^e\nabla_cR_{adbe} - \frac{1}{6}x^ax^bx^cDx^dDx^e\nabla_aR_{bdce} \\
& + \frac{1}{360}x^ax^bDx^cDx^dDx^eDx^f\left(-8g^{gh}R_{acdg}R_{befh} - 6\nabla_{cd}R_{aebf}\right) + \frac{1}{360}x^ax^bx^cDx^dDx^eDx^f\left(16g^{gh}R_{adbg}R_{cefh} - 9\nabla_{da}R_{becf} - 9\nabla_{ad}R_{becf}\right) \\
& + \frac{1}{360}x^ax^bx^cx^dDx^eDx^f\left(16g^{gh}R_{aebg}R_{cfdh} - 18\nabla_{ab}R_{cedf}\right) + \frac{1}{1080}x^ax^bx^cDx^dDx^eDx^fDx^g\left(-4g^{hi}R_{adeh}\nabla_fR_{bgci} - 24g^{hi}R_{adeh}\nabla_bR_{cfgi}\right. \\
& \qquad \left. + 10g^{hi}R_{adeh}\nabla_iR_{bfcg} + 16g^{hi}R_{adbh}\nabla_eR_{cfgi} - 4\nabla_{dea}R_{bfcg} - 4\nabla_{dae}R_{bfcg} - 4\nabla_{ade}R_{bfcg}\right) \\
& + \frac{1}{1080}x^ax^bDx^cDx^dDx^eDx^fDx^g\left(-18g^{hi}R_{acdh}\nabla_eR_{bfgi} - 3\nabla_{cde}R_{afbg}\right) \\
& + \frac{1}{1080}x^ax^bx^cx^dDx^eDx^fDx^g\left(24g^{hi}R_{aefh}\nabla_bR_{cgdi} + 24g^{hi}R_{aebh}\nabla_fR_{cgdi} + 24g^{hi}R_{aebh}\nabla_cR_{dfgi} - 6\nabla_{eab}R_{cfdg} - 6\nabla_{aeb}R_{cfdg} - 6\nabla_{abe}R_{cfdg}\right) \\
& + \frac{1}{1080}x^ax^bx^cx^dx^eDx^fDx^g\left(48g^{hi}R_{afbh}\nabla_cR_{dgei} - 12\nabla_{abc}R_{dfeg}\right) + \mathcal{O}\left(\epsilon^5\right)
\end{aligned}
$$

# Shared properties

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
\Gamma^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
\Gamma^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).

x^{a}::Depends(D{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

g0{#}::LaTeXForm("\ngab{0}").
g2{#}::LaTeXForm("\ngab{2}").
g3{#}::LaTeXForm("\ngab{3}").
g4{#}::LaTeXForm("\ngab{4}").
```

```
g5{#}::LaTeXForm("\ngab{5}").

y0{#}::LaTeXForm("\ny{0}").
y2{#}::LaTeXForm("\ny{2}").
y3{#}::LaTeXForm("\ny{3}").
y4{#}::LaTeXForm("\ny{4}").
y5{#}::LaTeXForm("\ny{5}").
```

## Stage 1: The formal expansion

```
g0_{a b}::Symmetric.
g2_{a b}::Symmetric.
g3_{a b}::Symmetric.
g4_{a b}::Symmetric.
g5_{a b}::Symmetric.

g0_{a b}::Weight(label=num,value=0).
g2_{a b}::Weight(label=num,value=2).
g3_{a b}::Weight(label=num,value=3).
g4_{a b}::Weight(label=num,value=4).
g5_{a b}::Weight(label=num,value=5).

y0^{a}::Weight(label=num,value=0).
y2^{a}::Weight(label=num,value=2).
y3^{a}::Weight(label=num,value=3).
y4^{a}::Weight(label=num,value=4).
y5^{a}::Weight(label=num,value=5).

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                           -> A001^{a}               $)
    substitute (obj,$ x^{a}                           -> A002^{a}               $)
    substitute (obj,$ Dx^{a}                          -> A003^{a}               $)
    substitute (obj,$ g_{a b}                         -> A004_{a b}             $)
    substitute (obj,$ g^{a b}                         -> A005^{a b}             $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}   -> A010_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}     -> A009_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}       -> A008_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}         -> A007_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                     -> A006_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                 -> A^{a}                         $)
    substitute (obj,$ A002^{a}                 -> x^{a}                         $)
    substitute (obj,$ A003^{a}                 -> Dx^{a}                        $)
    substitute (obj,$ A004_{a b}               -> g_{a b}                       $)
```

```
    substitute (obj,$ A005^{a b}                -> g^{a b}                    $)
    substitute (obj,$ A006_{a b c d}            -> R_{a b c d}                $)
    substitute (obj,$ A007_{a b c d e}          -> \nabla_{e}{R_{a b c d}}    $)
    substitute (obj,$ A008_{a b c d e f}        -> \nabla_{e f}{R_{a b c d}}  $)
    substitute (obj,$ A009_{a b c d e f g}      -> \nabla_{e f g}{R_{a b c d}}  $)
    substitute (obj,$ A010_{a b c d e f g h}    -> \nabla_{e f g h}{R_{a b c d}} $)

    return obj

def truncate (obj,n):
    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(obj).
       bah = Ex("num = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    return ans

# expansions wrt the curvature

defgab := g_{a b} -> g0_{a b} + g2_{a b} + g3_{a b} + g4_{a b} + g5_{a b}.
defy   := y^{a}   -> y0^{a} + y2^{a} + y3^{a} + y4^{a} + y5^{a}.

lsq    := g_{a b} y^{a} y^{b}.

substitute (lsq,defgab)
substitute (lsq,defy)
distribute (lsq)

def tidy (obj):
    foo := @(obj).
    sort_product    (foo)
    rename_dummies  (foo)
    canonicalise    (foo)
    return foo
```

```
lsq0 = tidy ( truncate (lsq,0) )  # cdb (lsq0.002,lsq0)
lsq2 = tidy ( truncate (lsq,2) )  # cdb (lsq2.002,lsq2)
lsq3 = tidy ( truncate (lsq,3) )  # cdb (lsq3.002,lsq3)
lsq4 = tidy ( truncate (lsq,4) )  # cdb (lsq4.002,lsq4)
lsq5 = tidy ( truncate (lsq,5) )  # cdb (lsq5.002,lsq5)

d20 := @(lsq2) - @(lsq0).         # cdb (d20.001,d20)   # check, should contain only O(2) terms
d32 := @(lsq3) - @(lsq2).         # cdb (d32.001,d32)   # check, should contain only O(3) terms
d43 := @(lsq4) - @(lsq3).         # cdb (d43.001,d43)   # check, should contain only O(4) terms
d54 := @(lsq5) - @(lsq4).         # cdb (d54.001,d54)   # check, should contain only O(5) terms

d5 := @(lsq5) - @(lsq).           # cdb (d5.001,d5)
d5  = tidy (d5)                   # cdb (d5.002,d5)  # all higher order terms, should see no O(5) terms
```

$$\texttt{lsq0.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{lsq2.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{lsq3.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{lsq4.002} := \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} + \overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\begin{aligned}\texttt{lsq5.002} := {} & \overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} + \overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} \\ & + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b} + 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} + 2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}\end{aligned}$$

$$\texttt{d20.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{d32.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + \overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{d43.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} + \overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\texttt{d54.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} + 2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} + 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} + 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} + \overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{0}{y}^{b}$$

$$\begin{aligned}\texttt{d5.002} := {} & -2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{0}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{0}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{0}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{0}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{0}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{0}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{0}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{2}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} - 2\overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} \\ & - 2\overset{2}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{2}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{2}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{2}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{2}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} - 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} \\ & - 2\overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{3}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{3}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{3}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} \\ & - \overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} - 2\overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{4}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{4}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{4}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{4}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{4}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b} - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{2}{y}^{b} - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{3}{y}^{b} \\ & - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{0}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{2}{y}^{b} - 2\overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{3}{y}^{b} - 2\overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{2}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{3}{y}^{a}\overset{3}{y}^{b} - 2\overset{5}{g}_{ab}\overset{3}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{3}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{4}{y}^{a}\overset{4}{y}^{b} - 2\overset{5}{g}_{ab}\overset{4}{y}^{a}\overset{5}{y}^{b} - \overset{5}{g}_{ab}\overset{5}{y}^{a}\overset{5}{y}^{b}\end{aligned}$$

## Stage 2: Substution of $\overset{n}{y}^a$ and $\overset{m}{g}_{ab}$

```
import cdblib

g0ab = cdblib.get('g_ab_0','metric.json')
g2ab = cdblib.get('g_ab_2','metric.json')
g3ab = cdblib.get('g_ab_3','metric.json')
g4ab = cdblib.get('g_ab_4','metric.json')
g5ab = cdblib.get('g_ab_5','metric.json')

defg0ab := g0_{a b} -> @(g0ab).
defg2ab := g2_{a b} -> @(g2ab).
defg3ab := g3_{a b} -> @(g3ab).
defg4ab := g4_{a b} -> @(g4ab).
defg5ab := g5_{a b} -> @(g5ab).

y0a = cdblib.get('y50','geodesic-bvp.json')
y2a = cdblib.get('y52','geodesic-bvp.json')
y3a = cdblib.get('y53','geodesic-bvp.json')
y4a = cdblib.get('y54','geodesic-bvp.json')
y5a = cdblib.get('y55','geodesic-bvp.json')

defy0a := y0^{a} -> @(y0a).
defy2a := y2^{a} -> @(y2a).
defy3a := y3^{a} -> @(y3a).
defy4a := y4^{a} -> @(y4a).
defy5a := y5^{a} -> @(y5a).

def substitute_gab_ya (obj):

   foo := @(obj).

   substitute (foo,defg0ab)
   substitute (foo,defg2ab)
   substitute (foo,defg3ab)
   substitute (foo,defg4ab)
   substitute (foo,defg5ab)
```

```
    substitute (foo,defy0a)
    substitute (foo,defy2a)
    substitute (foo,defy3a)
    substitute (foo,defy4a)
    substitute (foo,defy5a)

    distribute     (foo)
    sort_product   (foo)
    rename_dummies (foo)
    canonicalise   (foo)

    substitute     (foo,$g_{a b} g^{c b} -> \delta^{c}_{a}$)
    eliminate_kronecker (foo)
    foo = product_sort  (foo)
    rename_dummies      (foo)
    canonicalise        (foo)

    return foo

def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

     Q_{a b c d}::Weight(label=numR,value=2).
     Q_{a b c d e}::Weight(label=numR,value=3).
     Q_{a b c d e f}::Weight(label=numR,value=4).
     Q_{a b c d e f g}::Weight(label=numR,value=5).

     tmp := @(obj).

     distribute (tmp)

     substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
     substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
     substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
```

```
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d} -> R_{a b c d}$)

    return foo

lsq2 = substitute_gab_ya (lsq2)  # cdb (lsq2.101,lsq2)
lsq3 = substitute_gab_ya (lsq3)  # cdb (lsq3.101,lsq3)
lsq4 = substitute_gab_ya (lsq4)  # cdb (lsq4.101,lsq4)
lsq5 = substitute_gab_ya (lsq5)  # cdb (lsq5.101,lsq5)

lsq50 = get_Rterm (lsq5,0)
lsq52 = get_Rterm (lsq5,2)
lsq53 = get_Rterm (lsq5,3)
lsq54 = get_Rterm (lsq5,4)
lsq55 = get_Rterm (lsq5,5)

cdblib.create ('geodesic-lsq.json')

cdblib.put ('lsq2',lsq2,'geodesic-lsq.json')
cdblib.put ('lsq3',lsq3,'geodesic-lsq.json')
cdblib.put ('lsq4',lsq4,'geodesic-lsq.json')
cdblib.put ('lsq5',lsq5,'geodesic-lsq.json')

cdblib.put ('lsq50',lsq50,'geodesic-lsq.json')
cdblib.put ('lsq52',lsq52,'geodesic-lsq.json')
cdblib.put ('lsq53',lsq53,'geodesic-lsq.json')
cdblib.put ('lsq54',lsq54,'geodesic-lsq.json')
cdblib.put ('lsq55',lsq55,'geodesic-lsq.json')
```

$$\texttt{lsq2.101} := Dx^aDx^bg_{ab} - \frac{1}{3}x^ax^bDx^cDx^dR_{acbd}$$

$$\texttt{lsq3.101} := Dx^aDx^bg_{ab} - \frac{1}{3}x^ax^bDx^cDx^dR_{acbd} - \frac{1}{12}x^ax^bDx^cDx^dDx^e\nabla_cR_{adbe} - \frac{1}{6}x^ax^bx^cDx^dDx^e\nabla_aR_{bdce}$$

$$\begin{aligned}\texttt{lsq4.101} := {} & Dx^aDx^bg_{ab} - \frac{1}{3}x^ax^bDx^cDx^dR_{acbd} - \frac{1}{12}x^ax^bDx^cDx^dDx^e\nabla_cR_{adbe} - \frac{1}{6}x^ax^bx^cDx^dDx^e\nabla_aR_{bdce} - \frac{1}{45}x^ax^bDx^cDx^dDx^eDx^fg^{gh}R_{acdg}R_{befh}\\
& + \frac{2}{45}x^ax^bx^cDx^dDx^eDx^fg^{gh}R_{adbg}R_{cefh} - \frac{1}{40}x^ax^bx^cDx^dDx^eDx^f\nabla_{da}R_{becf} - \frac{1}{40}x^ax^bx^cDx^dDx^eDx^f\nabla_{ad}R_{becf}\\
& - \frac{1}{60}x^ax^bDx^cDx^dDx^eDx^f\nabla_{cd}R_{aebf} + \frac{2}{45}x^ax^bx^cx^dDx^eDx^fg^{gh}R_{aebg}R_{cfdh} - \frac{1}{20}x^ax^bx^cx^dDx^eDx^f\nabla_{ab}R_{cedf}\end{aligned}$$

$$\begin{aligned}\texttt{lsq5.101} := {} & Dx^aDx^bg_{ab} - \frac{1}{3}x^ax^bDx^cDx^dR_{acbd} - \frac{1}{12}x^ax^bDx^cDx^dDx^e\nabla_cR_{adbe} - \frac{1}{6}x^ax^bx^cDx^dDx^e\nabla_aR_{bdce}\\
& - \frac{1}{45}x^ax^bDx^cDx^dDx^eDx^fg^{gh}R_{acdg}R_{befh} + \frac{2}{45}x^ax^bx^cDx^dDx^eDx^fg^{gh}R_{adbg}R_{cefh}\\
& - \frac{1}{40}x^ax^bx^cDx^dDx^eDx^f\nabla_{da}R_{becf} - \frac{1}{40}x^ax^bx^cDx^dDx^eDx^f\nabla_{ad}R_{becf} - \frac{1}{60}x^ax^bDx^cDx^dDx^eDx^f\nabla_{cd}R_{aebf}\\
& + \frac{2}{45}x^ax^bx^cx^dDx^eDx^fg^{gh}R_{aebg}R_{cfdh} - \frac{1}{20}x^ax^bx^cx^dDx^eDx^f\nabla_{ab}R_{cedf} - \frac{1}{270}x^ax^bx^cDx^dDx^eDx^fDx^gg^{hi}R_{adeh}\nabla_fR_{bgci}\\
& - \frac{1}{45}x^ax^bx^cDx^dDx^eDx^fDx^gg^{hi}R_{adeh}\nabla_bR_{cfgi} + \frac{1}{108}x^ax^bx^cDx^dDx^eDx^fDx^gg^{hi}R_{adeh}\nabla_iR_{bfcg}\\
& - \frac{1}{60}x^ax^bDx^cDx^dDx^eDx^fDx^gg^{hi}R_{acdh}\nabla_eR_{bfgi} + \frac{1}{45}x^ax^bx^cx^dDx^eDx^fDx^gg^{hi}R_{aefh}\nabla_bR_{cgdi}\\
& + \frac{1}{45}x^ax^bx^cx^dDx^eDx^fDx^gg^{hi}R_{aebh}\nabla_fR_{cgdi} + \frac{1}{45}x^ax^bx^cx^dDx^eDx^fDx^gg^{hi}R_{aebh}\nabla_cR_{dfgi} - \frac{1}{180}x^ax^bx^cx^dDx^eDx^fDx^g\nabla_{eab}R_{cfdg}\\
& - \frac{1}{180}x^ax^bx^cx^dDx^eDx^fDx^g\nabla_{aeb}R_{cfdg} - \frac{1}{180}x^ax^bx^cx^dDx^eDx^fDx^g\nabla_{abe}R_{cfdg} + \frac{2}{135}x^ax^bx^cDx^dDx^eDx^fDx^gg^{hi}R_{adbh}\nabla_eR_{cfgi}\\
& - \frac{1}{270}x^ax^bx^cDx^dDx^eDx^fDx^g\nabla_{dea}R_{bfcg} - \frac{1}{270}x^ax^bx^cDx^dDx^eDx^fDx^g\nabla_{dae}R_{bfcg} - \frac{1}{270}x^ax^bx^cDx^dDx^eDx^fDx^g\nabla_{ade}R_{bfcg}\\
& - \frac{1}{360}x^ax^bDx^cDx^dDx^eDx^fDx^g\nabla_{cde}R_{afbg} + \frac{2}{45}x^ax^bx^cx^dx^eDx^fDx^gg^{hi}R_{afbh}\nabla_cR_{dgei} - \frac{1}{90}x^ax^bx^cx^dx^eDx^fDx^g\nabla_{abc}R_{dfeg}\end{aligned}$$

## Stage 3: Reformatting

```
def reformat (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute     (bah)
   bah = product_sort (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   substitute     (bah,$Dx^{b}->zzz^{b}$)
   factor_out     (bah,$x^{a?},zzz^{b?}$)
   substitute     (bah,$zzz^{b}->Dx^{b}$)
   ans := @(bah) / @(foo).
   return ans

def rescale (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute  (bah)
   substitute  (bah,$Dx^{b}->zzz^{b}$)
   factor_out  (bah,$x^{a?},zzz^{b?}$)
   substitute  (bah,$zzz^{b}->Dx^{b}$)
   return bah

Rterm0 := @(lsq50).
Rterm2 := @(lsq52).
Rterm3 := @(lsq53).
Rterm4 := @(lsq54).
Rterm5 := @(lsq55).

Rterm0 = reformat (Rterm0,   1)    # cdb(Rterm0.301,Rterm0) # LCB: returns Dx before g, not what I want
Rterm2 = reformat (Rterm2,   3)    # cdb(Rterm2.301,Rterm2)
Rterm3 = reformat (Rterm3,  12)    # cdb(Rterm3.301,Rterm3)
Rterm4 = reformat (Rterm4, 360)    # cdb(Rterm4.301,Rterm4)
Rterm5 = reformat (Rterm5,1080)    # cdb(Rterm5.301,Rterm5)

Rterm0 := g_{a b} Dx^{a} Dx^{b}.   # LCB: fixes the order of terms, g before Dx,
```

```
lsq3 := @(Rterm0) + @(Rterm2).                                  # cdb (lsq4.301,lsq3)
lsq4 := @(Rterm0) + @(Rterm2) + @(Rterm3).                      # cdb (lsq4.301,lsq4)
lsq5 := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4).          # cdb (lsq5.301,lsq5)
lsq6 := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (lsq5.301,lsq6)

lsq  := @(lsq6).                  # cdb (lsq.301,lsq)

scaled0 = rescale (Rterm0,    1)   # cdb (scaled0.301,scaled0) # LCB: returns Dx before g, not what I want
scaled2 = rescale (Rterm2,    3)   # cdb (scaled2.301,scaled2)
scaled3 = rescale (Rterm3,   12)   # cdb (scaled3.301,scaled3)
scaled4 = rescale (Rterm4,  360)   # cdb (scaled4.301,scaled4)
scaled5 = rescale (Rterm5, 1080)   # cdb (scaled5.301,scaled5)

scaled0 := g_{a b} Dx^{a} Dx^{b}.  # cdb (scaled0.301,scaled0) # LCB: fixes the order of terms, g before Dx, good
```

## Geodesic arc-length

$$
\begin{aligned}
(\Delta s)^2 = {} & g_{ab}Dx^aDx^b - \frac{1}{3}x^ax^bDx^cDx^dR_{acbd} - \frac{1}{12}x^ax^bDx^cDx^dDx^e\nabla_cR_{adbe} - \frac{1}{6}x^ax^bx^cDx^dDx^e\nabla_aR_{bdce} \\
& + \frac{1}{360}x^ax^bDx^cDx^dDx^eDx^f\left(-8g^{gh}R_{acdg}R_{befh} - 6\nabla_{cd}R_{aebf}\right) + \frac{1}{360}x^ax^bx^cDx^dDx^eDx^f\left(16g^{gh}R_{adbg}R_{cefh} - 9\nabla_{da}R_{becf} - 9\nabla_{ad}R_{becf}\right) \\
& + \frac{1}{360}x^ax^bx^cx^dDx^eDx^f\left(16g^{gh}R_{aebg}R_{cfdh} - 18\nabla_{ab}R_{cedf}\right) + \frac{1}{1080}x^ax^bx^cDx^dDx^eDx^fDx^g\left(-4g^{hi}R_{adeh}\nabla_fR_{bgci} - 24g^{hi}R_{adeh}\nabla_bR_{cfgi}\right. \\
& \qquad \left. + 10g^{hi}R_{adeh}\nabla_iR_{bfcg} + 16g^{hi}R_{adbh}\nabla_eR_{cfgi} - 4\nabla_{dea}R_{bfcg} - 4\nabla_{dae}R_{bfcg} - 4\nabla_{ade}R_{bfcg}\right) \\
& + \frac{1}{1080}x^ax^bDx^cDx^dDx^eDx^fDx^g\left(-18g^{hi}R_{acdh}\nabla_eR_{bfgi} - 3\nabla_{cde}R_{afbg}\right) \\
& + \frac{1}{1080}x^ax^bx^cx^dDx^eDx^fDx^g\left(24g^{hi}R_{aefh}\nabla_bR_{cgdi} + 24g^{hi}R_{aebh}\nabla_fR_{cgdi} + 24g^{hi}R_{aebh}\nabla_cR_{dfgi} - 6\nabla_{eab}R_{cfdg} - 6\nabla_{aeb}R_{cfdg} - 6\nabla_{abe}R_{cfdg}\right) \\
& + \frac{1}{1080}x^ax^bx^cx^dx^eDx^fDx^g\left(48g^{hi}R_{afbh}\nabla_cR_{dgei} - 12\nabla_{abc}R_{dfeg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

## Geodesic arc-length curvature expansion

$$(\Delta s)^2 = \overset{0}{\Delta} + \overset{2}{\Delta} + \overset{3}{\Delta} + \overset{4}{\Delta} + \overset{5}{\Delta} + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned}
\overset{0}{\Delta} &= g_{ab} Dx^a Dx^b \\
3\overset{2}{\Delta} &= -x^a x^b Dx^c Dx^d R_{acbd} \\
12\overset{3}{\Delta} &= -x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - 2x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \\
360\overset{4}{\Delta} &= x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8g^{gh} R_{acdg} R_{befh} - 6\nabla_{cd} R_{aebf}\right) + x^a x^b x^c Dx^d Dx^e Dx^f \left(16g^{gh} R_{adbg} R_{cefh} - 9\nabla_{da} R_{becf} - 9\nabla_{ad} R_{becf}\right) \\
&\quad + x^a x^b x^c x^d Dx^e Dx^f \left(16g^{gh} R_{aebg} R_{cfdh} - 18\nabla_{ab} R_{cedf}\right) \\
1080\overset{5}{\Delta} &= x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4g^{hi} R_{adeh} \nabla_f R_{bgci} - 24g^{hi} R_{adeh} \nabla_b R_{cfgi} + 10g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4\nabla_{dea} R_{bfcg} - 4\nabla_{dae} R_{bfcg} \right. \\
&\qquad \left. - 4\nabla_{ade} R_{bfcg}\right) + x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3\nabla_{cde} R_{afbg}\right) \\
&\quad + x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6\nabla_{eab} R_{cfdg} - 6\nabla_{aeb} R_{cfdg} - 6\nabla_{abe} R_{cfdg}\right) \\
&\quad + x^a x^b x^c x^d x^e Dx^f Dx^g \left(48g^{hi} R_{afbh} \nabla_c R_{dgei} - 12\nabla_{abc} R_{dfeg}\right)
\end{aligned}$$

```
cdblib.create ('geodesic-lsq.export')

# 3rd to 6th order lsq
cdblib.put ('lsq3',lsq3,'geodesic-lsq.export')
cdblib.put ('lsq4',lsq4,'geodesic-lsq.export')
cdblib.put ('lsq5',lsq5,'geodesic-lsq.export')
cdblib.put ('lsq6',lsq6,'geodesic-lsq.export')

# 6th order lsq terms, scaled
cdblib.put ('lsq60',scaled0,'geodesic-lsq.export')
cdblib.put ('lsq62',scaled2,'geodesic-lsq.export')
cdblib.put ('lsq63',scaled3,'geodesic-lsq.export')
cdblib.put ('lsq64',scaled4,'geodesic-lsq.export')
cdblib.put ('lsq65',scaled5,'geodesic-lsq.export')

checkpoint.append (lsq4)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```