

Symmetrised partial derivatives of the Riemann tensor

Here we compute the symmetrised partial derivatives $R^a_{(bcd;\underline{e})}$ in terms of the symmetrised covariant derivatives $R^a_{(bcd;\underline{e})}$. Note that the dot over an index indicates that that index does not take part in the symmetrisation.

We will use the algorithm described in section (10.3) of my lcb09-03 paper. Here we will make one small change of notation – the symbol D^a will be replaced with A^a .

We have lots of space (and no annoying editors to appease with brevity) so I will take the liberty to expand slightly on what I wrote in the lcb0-03 paper.

Our starting point is the simple identity

$$(R^a_{cdb}B^b_{a}A^cA^d)_{;e}A^e = (R^a_{cdb}B^b_{a}A^cA^d)_{;e}A^e \quad (1)$$

This is true in all frames since the quantity inside the brackets is a scalar. We are free to make any choice we like for A^a and B^a_b so let's choose A^a to be the tangent vector to any geodesic through the origin and choose the B^a_b to be constants (i.e, all partial derivatives are zero). We will also use local Riemann normal coordinates and as a consequence, the A^a will also be constant along the integral curves of A (the geodesics in an RNC are always of the form $x^a(s) = sA^a$ for some affine parameter s on the geodesic). Let df/ds be the directional derivative of the function f along the geodesics defined by A^a and assume that s is the proper length along the geodesic (although any affine parameter would be sufficient).

Thus at the origin we have, by choice,

$$\begin{aligned} 0 &= B^a_{b,c} = B^a_{b,cd} = B^a_{b,cde} = \dots \\ 0 &= dA^a/ds = d^2A^a/ds^2 = d^3A^a/ds^3 = \dots \\ 0 &= A^a_{,b}A^b = (A^a_{,b}A^b)_{;c}A^c = \left((A^a_{,b}A^b)_{;c}A^c \right)_{;d}A^d \\ 0 &= A^a_{;b}A^b = (A^a_{;b}A^b)_{;c}A^c = \left((A^a_{;b}A^b)_{;c}A^c \right)_{;d}A^d \\ df/ds &= f_{;a}A^a = f_{;a}A^a \\ d^2f/ds^2 &= (f_{;a}A^a)_{;b}A^b = (f_{;a}A^a)_{;b}A^b \\ d^3f/ds^3 &= \left((f_{;a}A^a)_{;b}A^b \right)_{;c}A^c = \left((f_{;a}A^a)_{;b}A^b \right)_{;c}A^c \end{aligned}$$

I admit I've gone overboard here in writing out more than I need to but it's handy to have all of these equations laid bare in one convenient place.

Now put $f = R^p{}_{abq} B^q{}_p A^a A^b$. Then upon taking successive derivatives, while taking full advantage of the assumptions just noted, we can easily see that

$$(R^a{}_{cdb} B^b{}_a)_{;e} A^c A^d A^e = (R^a{}_{cdb})_{;e} B^b{}_a A^c A^d A^e \quad (2)$$

This is the equation that will be computed by the following Cadabra code. All of the computations will be carried out on the left hand side (in the first version of the paper I swapped the left and right hand sides).

We will need the successive covariant derivatives of B . The first covariant derivative is just

$$B^a{}_{b;c} A^c = \Gamma^a{}_{dc} B^d{}_b A^c - \Gamma^d{}_{bc} B^a{}_d A^c$$

The quantities on the left hand side are the components of a tensor so further covariant derivatives of the right hand side can be computed (despite the presence of the Γ 's) by application of the usual rule for a covariant derivative of a mixed tensor.

Stage 1: Symmetrised partial derivatives of R

The first stage involves the expansion of the left side of (2). This leads to expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and $B^a{}_b$.

$$\begin{aligned} (R^a{}_{cdb})_{;e} B^b{}_a A^c A^d A^e &= -A^a A^b A^c B^d{}_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d{}_e g^{ef} \\ (R^a{}_{cdb})_{;ef} B^b{}_a A^c A^d A^e A^f &= -2A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c (\nabla_d B^e{}_f) g^{gf} \\ (R^a{}_{cdb})_{;efg} B^b{}_a A^c A^d A^e A^f A^g &= -3A^a A^b A^c A^d A^e \nabla_a R_{bfch} \nabla_d (\nabla_e B^f{}_g) g^{hg} - 3A^a A^b A^c A^d A^e \nabla_a B^f{}_g \nabla_b (\nabla_c R_{dfeh}) g^{gh} \\ &\quad - A^a A^b A^c A^d A^e B^f{}_g \nabla_a (\nabla_b (\nabla_c R_{dfeh})) g^{gh} - A^a A^b A^c A^d A^e R_{afbh} \nabla_c (\nabla_d (\nabla_e B^f{}_g)) g^{hg} \end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

In this stage the symmetrized covariant derivatives of $B^a{}_b$ are computed in terms of its partial derivatives (which by choice are all zero) and the connection and its partial derivatives (which in general are not zero).

$$\begin{aligned}
A^c \nabla_c (B^a_b) &= \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q \\
A^d A^c \nabla_d (\nabla_c (B^a_b)) &= A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{cd} \Gamma^c_{pq} B^p_b A^d A^q - 2 \Gamma^a_{cd} \Gamma^p_{bq} B^c_p A^d A^q + \Gamma^c_{bd} \Gamma^p_{cq} B^a_p A^d A^q \\
A^e A^d A^c \nabla_e (\nabla_d (\nabla_c (B^a_b))) &= A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q + A^c \partial_c \Gamma^a_{de} \Gamma^d_{pq} B^p_b A^e A^q + A^c \Gamma^a_{cd} \partial_e \Gamma^d_{pq} B^p_b A^e A^q \\
&\quad - 2 A^c \partial_c \Gamma^a_{de} \Gamma^p_{bq} B^d_p A^e A^q - 2 A^c \Gamma^a_{cd} \partial_e \Gamma^p_{bq} B^d_p A^e A^q + A^c \partial_c \Gamma^d_{be} \Gamma^p_{dq} B^a_p A^e A^q + A^c \Gamma^d_{bc} \partial_e \Gamma^p_{dq} B^a_p A^e A^q \\
&\quad + \Gamma^a_{ce} A^c \partial_f \Gamma^e_{pq} B^p_b A^f A^q - \Gamma^a_{ce} A^c \partial_f \Gamma^p_{bq} B^e_p A^f A^q + \Gamma^a_{cd} \Gamma^e_{cf} \Gamma^e_{pq} B^p_b A^d A^f A^q - 3 \Gamma^a_{cd} \Gamma^e_{bf} \Gamma^c_{pq} B^p_e A^d A^f A^q \\
&\quad + 3 \Gamma^a_{cd} \Gamma^e_{bf} \Gamma^p_{eq} B^c_p A^d A^f A^q - \Gamma^c_{be} A^e \partial_f \Gamma^a_{pq} B^p_c A^f A^q + \Gamma^c_{be} A^e \partial_f \Gamma^p_{cq} B^a_p A^f A^q - \Gamma^c_{bd} \Gamma^e_{cf} \Gamma^p_{eq} B^a_p A^d A^f A^q
\end{aligned}$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

Here we impose the RNC condition (that $\Gamma = 0$ while $\partial\Gamma \neq 0$).

$$\begin{aligned}
A^c \nabla_c (B^a_b) &= 0 \\
A^d A^c \nabla_d (\nabla_c (B^a_b)) &= A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q \\
A^e A^d A^c \nabla_e (\nabla_d (\nabla_c (B^a_b))) &= A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q
\end{aligned}$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

This stage uses the results from the second stage to eliminate the ∇B terms from the results of the first stage. This produces expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and the partial derivatives of the connection. In this stage we also set the B^a_b to equal 1.

$$\begin{aligned}
(R^a_{cdb})_{,e} A^c A^d A^e &= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
(R^a_{cdb})_{,ef} A^c A^d A^e A^f &= A^c A^d A^e A^f (-\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma^g_{bf} g^{ha} + R_{cbdg} \partial_e \Gamma^a_{hf} g^{gh}) \\
(R^a_{cdb})_{,efg} A^c A^d A^e A^f A^g &= A^c A^d A^e A^f A^g (-3 \nabla_c R_{dhef} \partial_f \Gamma^h_{bg} g^{ia} + 3 \nabla_c R_{dbeh} \partial_f \Gamma^a_{ig} g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_{ef} \Gamma^h_{bg} g^{ia} + R_{cbdh} \partial_{ef} \Gamma^a_{ig} g^{hi})
\end{aligned}$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

The fifth stage draws in results from `dGamma.tex` to replace the partial derivatives of Γ with partial derivatives of R_{abcd} .

$$\begin{aligned}
(R^a{}_{cdb})_{,e} A^c A^d A^e &= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
(R^a{}_{cdb})_{,ef} A^c A^d A^e A^f &= -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R^g{}_{feb} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R^a{}_{feh} g^{gh} \\
(R^a{}_{cdb})_{,efg} A^c A^d A^e A^f A^g &= -A^c A^d A^e A^f A^g R^h{}_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a{}_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h{}_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a{}_{gei} g^{hi}
\end{aligned}$$

Stage 6: Replace partial derivs of R with covariant derivs of R

The final stage is to eliminate the ∂R by using earlier results. For example, in the equation for $\partial^3 R$ we see terms involving ∂R . These first order partial derivatives can be replaced with the expression previously computed for ∂R in terms of ∇R .

$$\begin{aligned}
(R^a{}_{cdb})_{,e} A^c A^d A^e &= A^c A^d A^e \nabla_c R_{bdef} g^{af} \\
(R^a{}_{cdb})_{,ef} A^c A^d A^e A^f &= A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag} \\
(R^a{}_{cdb})_{,efg} A^c A^d A^e A^f A^g &= -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}
\end{aligned}$$

The end result are expressions for the symmetrized partial derivatives of R_{abcd} solely in terms of the symmetrized covariant derivatives of R_{abcd} .

Shared properties

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

B^{a}_{b}::Depends(\nabla{#}).
R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of R

```
def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d} -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{d} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d} -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    sort_product (obj)
    rename_dummies (obj)
    return obj

# compute the symmetric covariant derivatives of  $R^{a}_{bcd} B^{d}_{a} A^{b} A^{c}$ 

beg_stage_1 = time.time()

dRabcd00:= $R^{a}_{b c d} B^{d}_{a} A^{b} A^{c}$ .          # cdb(dRabcd00.101,dRabcd00)

dRabcd01:= $A^{a} \nabla_{a} \{ @ (dRabcd00) \}$ .      # cdb(dRabcd01.101,dRabcd01)
distribute      (dRabcd01)                    # cdb(dRabcd01.102,dRabcd01)
product_rule    (dRabcd01)                    # cdb(dRabcd01.103,dRabcd01)
distribute      (dRabcd01)                    # cdb(dRabcd01.104,dRabcd01)
substitute      (dRabcd01,$\nabla_{a} \{ A^{b} \} -> 0$) # cdb(dRabcd01.105,dRabcd01)
substitute      (dRabcd01,$\nabla_{a} \{ g^{b c} \} -> 0$) # cdb(dRabcd01.106,dRabcd01)

sort_product    (dRabcd01)
rename_dummies  (dRabcd01)
canonicalise    (dRabcd01)                    # cdb(dRabcd01.107,dRabcd01)
dRabcd01 = flatten_Rabcd (dRabcd01)           # cdb(dRabcd01.108,dRabcd01)

dRabcd02:= $A^{a} \nabla_{a} \{ @ (dRabcd01) \}$ .      # cdb(dRabcd02.101,dRabcd02)
distribute      (dRabcd02)                    # cdb(dRabcd02.102,dRabcd02)
product_rule    (dRabcd02)                    # cdb(dRabcd02.103,dRabcd02)
distribute      (dRabcd02)                    # cdb(dRabcd02.104,dRabcd02)
substitute      (dRabcd02,$\nabla_{a} \{ A^{b} \} -> 0$) # cdb(dRabcd02.105,dRabcd02)
substitute      (dRabcd02,$\nabla_{a} \{ g^{b c} \} -> 0$) # cdb(dRabcd02.106,dRabcd02)

sort_product    (dRabcd02)
```

```

rename_dummies (dRabcd02)
canonicalise    (dRabcd02)                # cdb(dRabcd02.107,dRabcd02)
dRabcd02 = flatten_Rabcd (dRabcd02)      # cdb(dRabcd02.108,dRabcd02)

dRabcd03:=A^{a}\nabla_{a}{ @ (dRabcd02) }.      # cdb(dRabcd03.101,dRabcd03)
distribute      (dRabcd03)                # cdb(dRabcd03.102,dRabcd03)
product_rule     (dRabcd03)                # cdb(dRabcd03.103,dRabcd03)
distribute      (dRabcd03)                # cdb(dRabcd03.104,dRabcd03)
substitute       (dRabcd03,$\nabla_{a}{A^{b}} -> 0$) # cdb(dRabcd03.105,dRabcd03)
substitute       (dRabcd03,$\nabla_{a}{g^{b c}} -> 0$) # cdb(dRabcd03.106,dRabcd03)

sort_product     (dRabcd03)
rename_dummies   (dRabcd03)
canonicalise     (dRabcd03)                # cdb(dRabcd03.107,dRabcd03)
dRabcd03 = flatten_Rabcd (dRabcd03)      # cdb(dRabcd03.108,dRabcd03)

dRabcd04:=A^{a}\nabla_{a}{ @ (dRabcd03) }.
distribute      (dRabcd04)
product_rule     (dRabcd04)
distribute      (dRabcd04)
substitute       (dRabcd04,$\nabla_{a}{A^{b}} -> 0$)
substitute       (dRabcd04,$\nabla_{a}{g^{b c}} -> 0$)

sort_product     (dRabcd04)
rename_dummies   (dRabcd04)
canonicalise     (dRabcd04)
dRabcd04 = flatten_Rabcd (dRabcd04)

dRabcd05:=A^{a}\nabla_{a}{ @ (dRabcd04) }.
distribute      (dRabcd05)
product_rule     (dRabcd05)
distribute      (dRabcd05)
substitute       (dRabcd05,$\nabla_{a}{A^{b}} -> 0$)
substitute       (dRabcd05,$\nabla_{a}{g^{b c}} -> 0$)

sort_product     (dRabcd05)
rename_dummies   (dRabcd05)
canonicalise     (dRabcd05)

```

```

dRabcd05 = flatten_Rabcd (dRabcd05)

def combine_nabla (obj):
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{\nabla_{s}\{\nabla_{t}\{A??}\}\}\}\}\}\rightarrow\nabla_{p\ q\ r\ s\ t}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{\nabla_{s}\{A??\}\}\}\}\}\rightarrow\nabla_{p\ q\ r\ s}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{A??\}\}\}\}\rightarrow\nabla_{p\ q\ r}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{A??\}\}\}\rightarrow\nabla_{p\ q}\{A??\}\$,repeat=True)
    return obj

dRabcd01 = combine_nabla (dRabcd01)
dRabcd02 = combine_nabla (dRabcd02)
dRabcd03 = combine_nabla (dRabcd03)
dRabcd04 = combine_nabla (dRabcd04)
dRabcd05 = combine_nabla (dRabcd05)

end_stage_1 = time.time()

```


$$\text{dRabcd00.101} := R^a{}_{bcd} B^d{}_e A^b A^c$$

$$\text{dRabcd01.101} := A^a \nabla_a (R^e{}_{bcd} B^d{}_e A^b A^c)$$

$$\text{dRabcd01.102} := A^a \nabla_a (R^e{}_{bcd} B^d{}_e A^b A^c)$$

$$\text{dRabcd01.103} := A^a (\nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c)$$

$$\text{dRabcd01.104} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + A^a R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + A^a R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c$$

$$\text{dRabcd01.105} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\text{dRabcd01.106} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\text{dRabcd01.107} := -A^a A^b A^c B^d{}_e \nabla_a R^e{}_{bcd} - A^a A^b A^c R^d{}_{be} \nabla_c B^e{}_d$$

$$\text{dRabcd01.108} := -A^a A^b A^c B^d{}_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d{}_e g^{ef}$$

$$\text{dRabcd02.101} := A^a \nabla_a (-A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef})$$

$$\text{dRabcd02.102} := -A^a \nabla_a (A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef}) - A^a \nabla_a (A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef})$$

$$\begin{aligned} \text{dRabcd02.103} := & -A^a (\nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ & + A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} + A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef}) - A^a (\nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ & + A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} + A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef}) \end{aligned}$$

$$\begin{aligned} \text{dRabcd02.104} := & -A^a \nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ & - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} - A^a \nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ & - A^a A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef} \end{aligned}$$

$$\begin{aligned} \text{dRabcd02.105} := & -A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} \\ & - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef} \end{aligned}$$

$$\text{dRabcd02.106} := -A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef}$$

$$\text{dRabcd02.107} := -2A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebf} \nabla_c (\nabla_d B^e{}_g) g^{fg}$$

$$\text{dRabcd02.108} := -2A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c (\nabla_d B^e{}_f) g^{fg}$$

$$\begin{aligned}
\text{dRabcd03.101} &:= A^a \nabla_a \left(-2A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right) \\
\text{dRabcd03.102} &:= -2A^a \nabla_a \left(A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \right) - A^a \nabla_a \left(A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \right) - A^a \nabla_a \left(A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right) \\
\text{dRabcd03.103} &:= -2A^a \left(\nabla_a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h \nabla_a A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h A^b \nabla_a A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \right. \\
&\quad + A^h A^b A^c \nabla_a A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} + A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} \\
&\quad + A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} \left. \right) - A^a \left(\nabla_a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h \nabla_a A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \right. \\
&\quad + A^h A^b \nabla_a A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h A^b A^c \nabla_a A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad + A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} + A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} \left. \right) - A^a \left(\nabla_a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right. \\
&\quad + A^h \nabla_a A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b \nabla_a A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b A^c \nabla_a A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad + A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} + A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \left. \right) \\
\text{dRabcd03.104} &:= -2A^a \nabla_a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h \nabla_a A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b \nabla_a A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \\
&\quad - 2A^a A^h A^b A^c \nabla_a A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} \\
&\quad - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} - A^a \nabla_a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h \nabla_a A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b \nabla_a A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c \nabla_a A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} - A^a \nabla_a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad - A^a A^h \nabla_a A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b \nabla_a A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c \nabla_a A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \\
\text{dRabcd03.105} &:= -2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \\
\text{dRabcd03.106} &:= -2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} \\
\text{dRabcd03.107} &:= -3A^a A^b A^c A^d A^e \nabla_a R_{bfeg} \nabla_d (\nabla_e B^f_h) g^{gh} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeg}) g^{gh} \\
&\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeg})) g^{gh} - A^a A^b A^c A^d A^e R_{afbg} \nabla_c (\nabla_d (\nabla_e B^f_h)) g^{gh} \\
\text{dRabcd03.108} &:= -3A^a A^b A^c A^d A^e \nabla_a R_{bfch} \nabla_d (\nabla_e B^f_g) g^{hg} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeg}) g^{gh} \\
&\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeg})) g^{gh} - A^a A^b A^c A^d A^e R_{afbh} \nabla_c (\nabla_d (\nabla_e B^f_g)) g^{hg}
\end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

```

# compute the covariant derivatives of  $B^{\{a\}_{\{b\}}$ , note  $B^{\{a\}_{\{b,c\}}$  is zero, by choice
# this method of computing covariant derivatives does not use auxillary fields

beg_stage_2 = time.time()

dBab00:= $B^{\{a\}_{\{b\}}$ .      # cdb(dBab00.201,dBab00)

dBab01:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab00}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab01.201,dBab01)
distribute      (dBab01)                # cdb(dBab01.202,dBab01)
product_rule    (dBab01)                # cdb(dBab01.203,dBab01)
distribute      (dBab01)                # cdb(dBab01.204,dBab01)
substitute      (dBab01, $\backslash\text{partial}_{\{a\}}\{A^{\{b\}}\} \rightarrow 0\$$ )  # cdb(dBab01.205,dBab01)
substitute      (dBab01, $\backslash\text{partial}_{\{a\}}\{B^{\{b\}}_{\{c\}}\} \rightarrow 0\$$ )  # cdb(dBab01.206,dBab01)
substitute      (dBab01, $\backslash W^{\{a\}_{\{b\}}} \rightarrow @(\text{dBab00})\$$ )  # cdb(dBab01.207,dBab01)
distribute      (dBab01)                # cdb(dBab01.208,dBab01)
canonicalise    (dBab01)                # cdb(dBab01.209,dBab01)

dBab02:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab01}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab02.201,dBab02)
distribute      (dBab02)                # cdb(dBab02.202,dBab02)
product_rule    (dBab02)                # cdb(dBab02.203,dBab02)
distribute      (dBab02)                # cdb(dBab02.204,dBab02)
substitute      (dBab02, $\backslash\text{partial}_{\{a\}}\{A^{\{b\}}\} \rightarrow 0\$$ )  # cdb(dBab02.205,dBab02)
substitute      (dBab02, $\backslash\text{partial}_{\{a\}}\{B^{\{b\}}_{\{c\}}\} \rightarrow 0\$$ )  # cdb(dBab02.206,dBab02)
substitute      (dBab02, $\backslash W^{\{a\}_{\{b\}}} \rightarrow @(\text{dBab01})\$$ )  # cdb(dBab02.207,dBab02)
distribute      (dBab02)                # cdb(dBab02.208,dBab02)
canonicalise    (dBab02)                # cdb(dBab02.209,dBab02)

dBab03:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab02}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab03.201,dBab03)
distribute      (dBab03)                # cdb(dBab03.202,dBab03)
product_rule    (dBab03)                # cdb(dBab03.203,dBab03)

```

```

distribute      (dBab03)                                # cdb(dBab03.204,dBab03)
substitute      (dBab03,$\partial_{a}\{A^{b}\} \rightarrow 0$) # cdb(dBab03.205,dBab03)
substitute      (dBab03,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$) # cdb(dBab03.206,dBab03)
substitute      (dBab03,$W^{a}_{b} \rightarrow @(dBab02)$) # cdb(dBab03.207,dBab03)
distribute      (dBab03)                                # cdb(dBab03.208,dBab03)
canonicalise    (dBab03)                                # cdb(dBab03.209,dBab03)

```

```

dBab04:=A^{c}\partial_{c}\{ @(dBab03) \} + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
      - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.

```

```

distribute      (dBab04)
product_rule    (dBab04)
distribute      (dBab04)
substitute      (dBab04,$\partial_{a}\{A^{b}\} \rightarrow 0$)
substitute      (dBab04,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$)
substitute      (dBab04,$W^{a}_{b} \rightarrow @(dBab03)$)
distribute      (dBab04)
canonicalise    (dBab04)

```

```

dBab05:=A^{c}\partial_{c}\{ @(dBab04) \} + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
      - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.

```

```

distribute      (dBab05)
product_rule    (dBab05)
distribute      (dBab05)
substitute      (dBab05,$\partial_{a}\{A^{b}\} \rightarrow 0$)
substitute      (dBab05,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$)
substitute      (dBab05,$W^{a}_{b} \rightarrow @(dBab04)$)
distribute      (dBab05)
canonicalise    (dBab05)

```

```

end_stage_2 = time.time()

```

$$\text{dBab00.201} := B^a_b$$

$$\text{dBab01.201} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.202} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.203} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.204} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.205} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.206} := \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.207} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab01.208} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab01.209} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab02.201} := A^c \partial_c (\Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.202} := A^c \partial_c (\Gamma^a_{pq} B^p_b A^q) - A^c \partial_c (\Gamma^p_{bq} B^a_p A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.203} := A^c (\partial_c \Gamma^a_{pq} B^p_b A^q + \Gamma^a_{pq} \partial_c B^p_b A^q + \Gamma^a_{pq} B^p_b \partial_c A^q) - A^c (\partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^p_{bq} \partial_c B^a_p A^q + \Gamma^p_{bq} B^a_p \partial_c A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.204} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q + A^c \Gamma^a_{pq} \partial_c B^p_b A^q + A^c \Gamma^a_{pq} B^p_b \partial_c A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q - A^c \Gamma^p_{bq} \partial_c B^a_p A^q - A^c \Gamma^p_{bq} B^a_p \partial_c A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.205} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q + A^c \Gamma^a_{pq} \partial_c B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q - A^c \Gamma^p_{bq} \partial_c B^a_p A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.206} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.207} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} (\Gamma^p_{dc} B^d_b A^c - \Gamma^d_{bc} B^p_d A^c) A^q - \Gamma^p_{bq} (\Gamma^a_{dc} B^d_p A^c - \Gamma^d_{pc} B^a_d A^c) A^q$$

$$\text{dBab02.208} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} \Gamma^p_{dc} B^d_b A^c A^q - \Gamma^a_{pq} \Gamma^d_{bc} B^p_d A^c A^q - \Gamma^p_{bq} \Gamma^a_{dc} B^d_p A^c A^q + \Gamma^p_{bq} \Gamma^d_{pc} B^a_d A^c A^q$$

$$\text{dBab02.209} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{cd} \Gamma^c_{pq} B^p_b A^d A^q - 2 \Gamma^a_{cd} \Gamma^p_{bq} B^c_p A^d A^q + \Gamma^c_{bd} \Gamma^p_{cq} B^a_p A^d A^q$$

$$\text{dBab03.201} := A^c \partial_c \left(A^e \partial_e \Gamma_{pq}^a B^p{}_b A^q - A^e \partial_e \Gamma_{bq}^p B^a{}_p A^q + \Gamma_{ed}^a \Gamma_{pq}^e B^p{}_b A^d A^q - 2 \Gamma_{ed}^a \Gamma_{bq}^p B^e{}_p A^d A^q + \Gamma_{bd}^e \Gamma_{eq}^p B^a{}_p A^d A^q \right) + \Gamma_{pq}^a W^p{}_b A^q - \Gamma_{bq}^p W^a{}_p A^q$$

$$\begin{aligned} \text{dBab03.202} := & A^c \partial_c (A^e \partial_e \Gamma_{pq}^a B^p{}_b A^q) - A^c \partial_c (A^e \partial_e \Gamma_{bq}^p B^a{}_p A^q) + A^c \partial_c (\Gamma_{ed}^a \Gamma_{pq}^e B^p{}_b A^d A^q) \\ & - 2A^c \partial_c (\Gamma_{ed}^a \Gamma_{bq}^e B^e{}_p A^d A^q) + A^c \partial_c (\Gamma_{bd}^p \Gamma_{eq}^e B^a{}_p A^d A^q) + \Gamma_{pq}^a W^p{}_b A^q - \Gamma_{bq}^p W^a{}_p A^q \end{aligned}$$

$$\begin{aligned} \text{dBab03.203} := & A^c (\partial_c A^e \partial_e \Gamma^a_{pq} B^p{}_b A^q + A^e \partial_{ce} \Gamma^a_{pq} B^p{}_b A^q + A^e \partial_e \Gamma^a_{pq} \partial_c B^p{}_b A^q + A^e \partial_e \Gamma^a_{pq} B^p{}_b \partial_c A^q) \\ & - A^c (\partial_c A^e \partial_e \Gamma^p_{bq} B^a{}_p A^q + A^e \partial_{ce} \Gamma^p_{bq} B^a{}_p A^q + A^e \partial_e \Gamma^p_{bq} \partial_c B^a{}_p A^q + A^e \partial_e \Gamma^p_{bq} B^a{}_p \partial_c A^q) \\ & + A^c (\partial_c \Gamma^a_{ed} \Gamma^e_{pq} B^p{}_b A^d A^q + \Gamma^a_{ed} \partial_c \Gamma^e_{pq} B^p{}_b A^d A^q + \Gamma^a_{ed} \Gamma^e_{pq} \partial_c B^p{}_b A^d A^q + \Gamma^a_{ed} \Gamma^e_{pq} B^p{}_b \partial_c A^d A^q + \Gamma^a_{ed} \Gamma^e_{pq} B^p{}_b A^d \partial_c A^q) \\ & - 2A^c (\partial_c \Gamma^a_{ed} \Gamma^p_{bq} B^e{}_p A^d A^q + \Gamma^a_{ed} \partial_c \Gamma^p_{bq} B^e{}_p A^d A^q + \Gamma^a_{ed} \Gamma^p_{bq} \partial_c B^e{}_p A^d A^q + \Gamma^a_{ed} \Gamma^p_{bq} B^e{}_p \partial_c A^d A^q + \Gamma^a_{ed} \Gamma^p_{bq} B^e{}_p A^d \partial_c A^q) \\ & + A^c (\partial_c \Gamma^e_{bd} \Gamma^p_{eq} B^a{}_p A^d A^q + \Gamma^e_{bd} \partial_c \Gamma^p_{eq} B^a{}_p A^d A^q + \Gamma^e_{bd} \Gamma^p_{eq} \partial_c B^a{}_p A^d A^q + \Gamma^e_{bd} \Gamma^p_{eq} B^a{}_p \partial_c A^d A^q + \Gamma^e_{bd} \Gamma^p_{eq} B^a{}_p A^d \partial_c A^q) \\ & + \Gamma^a_{pq} W^p{}_b A^q - \Gamma^p_{bq} W^a{}_p A^q \end{aligned}$$

$$\begin{aligned} \text{dBab03.204} := & A^c \partial_c A^e \partial_e \Gamma^a_{pq} B^p{}_b A^q + A^c A^e \partial_{ce} \Gamma^a_{pq} B^p{}_b A^q + A^c A^e \partial_e \Gamma^a_{pq} \partial_c B^p{}_b A^q + A^c A^e \partial_e \Gamma^a_{pq} B^p{}_b \partial_c A^q - A^c \partial_c A^e \partial_e \Gamma^p{}_{bq} B^a{}_p A^q \\ & - A^c A^e \partial_{ce} \Gamma^p{}_{bq} B^a{}_p A^q - A^c A^e \partial_e \Gamma^p{}_{bq} \partial_c B^a{}_p A^q - A^c A^e \partial_e \Gamma^p{}_{bq} B^a{}_p \partial_c A^q + A^c \partial_c \Gamma^e{}_{ed} \Gamma^e{}_{pq} B^p{}_b A^d A^q + A^c \Gamma^e{}_{ed} \partial_c \Gamma^e{}_{pq} B^p{}_b A^d A^q \\ & + A^c \Gamma^e{}_{ed} \Gamma^e{}_{pq} \partial_c B^p{}_b A^d A^q + A^c \Gamma^e{}_{ed} \Gamma^e{}_{pq} B^p{}_b \partial_c A^d A^q + A^c \Gamma^e{}_{ed} \Gamma^e{}_{pq} B^p{}_b A^d \partial_c A^q - 2 A^c \partial_c \Gamma^e{}_{ed} \Gamma^p{}_{bq} B^e{}_p A^d A^q - 2 A^c \Gamma^e{}_{ed} \partial_c \Gamma^p{}_{bq} B^e{}_p A^d A^q \\ & - 2 A^c \Gamma^e{}_{ed} \Gamma^p{}_{bq} \partial_c B^e{}_p A^d A^q - 2 A^c \Gamma^e{}_{ed} \Gamma^p{}_{bq} B^e{}_p \partial_c A^d A^q - 2 A^c \Gamma^e{}_{ed} \Gamma^p{}_{bq} B^e{}_p A^d \partial_c A^q + A^c \partial_c \Gamma^e{}_{bd} \Gamma^p{}_{eq} B^a{}_p A^d A^q \\ & + A^c \Gamma^e{}_{bd} \partial_c \Gamma^p{}_{eq} B^a{}_p A^d A^q + A^c \Gamma^e{}_{bd} \Gamma^p{}_{eq} \partial_c B^a{}_p A^d A^q + A^c \Gamma^e{}_{bd} \Gamma^p{}_{eq} B^a{}_p \partial_c A^d A^q + A^c \Gamma^e{}_{bd} \Gamma^p{}_{eq} B^a{}_p A^d \partial_c A^q + \Gamma^a{}_{pq} W^p{}_b A^q - \Gamma^p{}_{bq} W^a{}_p A^q \end{aligned}$$

$$\begin{aligned} \text{dBab03.205} := & A^c A^e \partial_{ce} \Gamma^a_{pq} B^p{}_b A^q + A^c A^e \partial_e \Gamma^a_{pq} \partial_c B^p{}_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a{}_p A^q - A^c A^e \partial_e \Gamma^p_{bq} \partial_c B^a{}_p A^q + A^c \partial_c \Gamma^a_{ed} \Gamma^e_{pq} B^p{}_b A^d A^q \\ & + A^c \Gamma^a_{ed} \partial_c \Gamma^e_{pq} B^p{}_b A^d A^q + A^c \Gamma^a_{ed} \Gamma^e_{pq} \partial_c B^p{}_b A^d A^q - 2 A^c \partial_c \Gamma^a_{ed} \Gamma^p_{bq} B^e{}_p A^d A^q - 2 A^c \Gamma^a_{ed} \partial_c \Gamma^p_{bq} B^e{}_p A^d A^q - 2 A^c \Gamma^a_{ed} \Gamma^p_{bq} \partial_c B^e{}_p A^d A^q \\ & + A^c \partial_c \Gamma^e_{bd} \Gamma^p_{eq} B^a{}_p A^d A^q + A^c \Gamma^e_{bd} \partial_c \Gamma^p_{eq} B^a{}_p A^d A^q + A^c \Gamma^e_{bd} \Gamma^p_{eq} \partial_c B^a{}_p A^d A^q + \Gamma^a_{pq} W^p{}_b A^q - \Gamma^p_{bq} W^a{}_p A^q \end{aligned}$$

$$\begin{aligned} \text{dBab03.206} := & A^c A^e \partial_{ce} \Gamma_{pq}^a B^p{}_b A^q - A^c A^e \partial_{ce} \Gamma_{bq}^p B^a{}_p A^q + A^c \partial_c \Gamma_{ed}^a \Gamma_{pq}^e B^p{}_b A^d A^q + A^c \Gamma_{ed}^a \partial_c \Gamma_{pq}^e B^p{}_b A^d A^q - 2 A^c \partial_c \Gamma_{ed}^a \Gamma_{bq}^p B^e{}_p A^d A^q \\ & - 2 A^c \Gamma_{ed}^a \partial_c \Gamma_{bq}^p B^e{}_p A^d A^q + A^c \partial_c \Gamma_{bd}^p \Gamma_{eq}^p B^a{}_p A^d A^q + A^c \Gamma_{bd}^p \partial_c \Gamma_{eq}^p B^a{}_p A^d A^q + \Gamma_{pq}^a W^p{}_b A^q - \Gamma_{bq}^p W^a{}_p A^q \end{aligned}$$

$$\begin{aligned} \text{dBab03.207} := & A^c A^e \partial_{ce} \Gamma_{pq}^a B^p{}_b A^q - A^c A^e \partial_{ce} \Gamma_{bq}^p B^a{}_p A^q + A^c \partial_c \Gamma_{ed}^a \Gamma_{pq}^e B^p{}_b A^d A^q + A^c \Gamma_{ed}^a \partial_c \Gamma_{pq}^e B^p{}_b A^d A^q \\ & - 2A^c \partial_c \Gamma_{ed}^p \Gamma_{bq}^e B^e{}_p A^d A^q - 2A^c \Gamma_{ed}^p \partial_c \Gamma_{bq}^e B^e{}_p A^d A^q + A^c \partial_c \Gamma_{bd}^e \Gamma_{eq}^p B^a{}_p A^d A^q + A^c \Gamma_{bd}^e \partial_c \Gamma_{eq}^p B^a{}_p A^d A^q \\ & + \Gamma_{pq}^a (A^c \partial_c \Gamma_{fe}^p B^f{}_b A^e - A^c \partial_c \Gamma_{be}^f B^e{}_f A^e + \Gamma_{cd}^p \Gamma_{fe}^c B^f{}_b A^d A^e - 2\Gamma_{cd}^p \Gamma_{be}^f B^c{}_f A^d A^e + \Gamma_{bd}^p \Gamma_{ce}^f B^p{}_f A^d A^e) A^q \\ & - \Gamma_{bq}^p (A^c \partial_c \Gamma_{fe}^p B^f{}_p A^e - A^c \partial_c \Gamma_{pe}^f B^a{}_f A^e + \Gamma_{cd}^p \Gamma_{fe}^c B^f{}_p A^d A^e - 2\Gamma_{cd}^p \Gamma_{pe}^f B^c{}_f A^d A^e + \Gamma_{pd}^p \Gamma_{ce}^f B^a{}_f A^d A^e) A^q \end{aligned}$$

$$\begin{aligned} \text{dBab03.208} := & A^c A^e \partial_{ce} \Gamma_{pq}^a B^p{}_b A^q - A^c A^e \partial_{ce} \Gamma_{pq}^p B^p{}_b A^q + A^c \partial_c \Gamma^a{}_{ed} \Gamma_{pq}^e B^p{}_b A^d A^q + A^e \Gamma^a{}_{ed} \partial_c \Gamma_{pq}^e B^p{}_b A^d A^q - 2 A^c \partial_c \Gamma^a{}_{ed} \Gamma_{pq}^p B^p{}_b A^d A^q \\ & - 2 A^c \Gamma^a{}_{ed} \partial_c \Gamma_{pq}^p B^p{}_b A^d A^q + A^c \partial_c \Gamma^a{}_{bd} \Gamma_{pq}^p B^p{}_b A^d A^q + A^c \Gamma^a{}_{bd} \partial_c \Gamma_{pq}^p B^p{}_b A^d A^q + \Gamma^a{}_{pq} A^c \partial_c \Gamma_{fe}^p B^f{}_b A^e A^q - \Gamma^a{}_{pq} A^c \partial_c \Gamma_{be}^f B^p{}_f A^e A^q \\ & + \Gamma^a{}_{pq} \Gamma_{cd}^p \Gamma_{fe}^c B^f{}_b A^d A^e A^q - 2 \Gamma^a{}_{pq} \Gamma_{cd}^p \Gamma_{be}^f B^c{}_f A^d A^e A^q + \Gamma^a{}_{pq} \Gamma_{cd}^p \Gamma_{ce}^f B^p{}_f A^d A^e A^q - \Gamma_{pq}^p A^c \partial_c \Gamma^a{}_{fe} B^f{}_p A^e A^q \\ & + \Gamma_{pq}^p A^c \partial_c \Gamma_{pe}^f B^a{}_f A^e A^q - \Gamma_{pq}^p \Gamma_{cd}^p \Gamma_{fe}^c B^f{}_p A^d A^e A^q + 2 \Gamma_{pq}^p \Gamma_{cd}^p \Gamma_{pe}^f B^c{}_f A^d A^e A^q - \Gamma_{pq}^p \Gamma_{pd}^p \Gamma_{ce}^f B^a{}_f A^d A^e A^q \end{aligned}$$

$$\begin{aligned}
\text{dBab03.209} := & A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q + A^c \partial_c \Gamma^a_{de} \Gamma^d_{pq} B^p_b A^e A^q + A^c \Gamma^a_{cd} \partial_e \Gamma^d_{pq} B^p_b A^e A^q \\
& - 2A^c \partial_c \Gamma^a_{de} \Gamma^p_{bq} B^d_p A^e A^q - 2A^c \Gamma^a_{cd} \partial_e \Gamma^p_{bq} B^d_p A^e A^q + A^c \partial_c \Gamma^d_{be} \Gamma^p_{dq} B^a_p A^e A^q + A^c \Gamma^d_{bc} \partial_e \Gamma^p_{dq} B^a_p A^e A^q \\
& + \Gamma^a_{ce} A^c \partial_f \Gamma^e_{pq} B^p_b A^f A^q - \Gamma^a_{ce} A^c \partial_f \Gamma^p_{bq} B^e_p A^f A^q + \Gamma^a_{cd} \Gamma^c_{ef} \Gamma^e_{pq} B^p_b A^d A^f A^q - 3\Gamma^a_{cd} \Gamma^e_{bf} \Gamma^c_{pq} B^p_e A^d A^f A^q \\
& + 3\Gamma^a_{cd} \Gamma^e_{bf} \Gamma^p_{eq} B^c_p A^d A^f A^q - \Gamma^c_{be} A^e \partial_f \Gamma^a_{pq} B^p_c A^f A^q + \Gamma^c_{be} A^e \partial_f \Gamma^p_{cq} B^a_p A^f A^q - \Gamma^c_{bd} \Gamma^e_{cf} \Gamma^p_{eq} B^a_p A^d A^f A^q
\end{aligned}$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{\{d\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{\{b\}c} \rightarrow 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{\{d\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}g}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    return obj

# switch to RNC

beg_stage_3 = time.time()

dBab01 = impose_rnc (dBab01)    # cdb (dBab01.301,dBab01)
dBab02 = impose_rnc (dBab02)    # cdb (dBab02.301,dBab02)
dBab03 = impose_rnc (dBab03)    # cdb (dBab03.301,dBab03)
dBab04 = impose_rnc (dBab04)    # cdb (dBab04.301,dBab04)
dBab05 = impose_rnc (dBab05)    # cdb (dBab05.301,dBab05)

end_stage_3 = time.time()
```


$$\text{dBab01.301} := 0$$

$$\text{dBab02.301} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab03.301} := A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q$$

$$\begin{aligned} \text{dBab04.301} := & A^c A^e A^g \partial_{ceg} \Gamma^a_{pq} B^p_b A^q - A^c A^e A^g \partial_{ceg} \Gamma^p_{bq} B^a_p A^q + 2A^c A^d \partial_c \Gamma^a_{de} \partial_g \Gamma^e_{pq} B^p_b A^g A^q - 4A^c A^d \partial_c \Gamma^a_{de} \partial_g \Gamma^p_{bq} B^e_p A^g A^q \\ & + 2A^c A^d \partial_c \Gamma^e_{bd} \partial_g \Gamma^p_{eq} B^a_p A^g A^q + A^c \partial_c \Gamma^a_{ef} A^e \partial_g \Gamma^f_{pq} B^p_b A^g A^q - 2A^c \partial_c \Gamma^a_{ef} A^e \partial_g \Gamma^p_{bq} B^f_p A^g A^q + A^c \partial_c \Gamma^e_{bf} A^f \partial_g \Gamma^p_{eq} B^a_p A^g A^q \end{aligned}$$

$$\begin{aligned} \text{dBab05.301} := & A^c A^e A^g A^i \partial_{cegi} \Gamma^a_{pq} B^p_b A^q - A^c A^e A^g A^i \partial_{cegi} \Gamma^p_{bq} B^a_p A^q + 3A^c A^d A^e \partial_{cd} \Gamma^a_{eg} \partial_i \Gamma^g_{pq} B^p_b A^i A^q + 3A^c A^d A^e \partial_c \Gamma^a_{dg} \partial_{ei} \Gamma^g_{pq} B^p_b A^i A^q \\ & - 6A^c A^d A^e \partial_{cd} \Gamma^a_{eg} \partial_i \Gamma^p_{bq} B^g_p A^i A^q - 6A^c A^d A^e \partial_c \Gamma^a_{dg} \partial_{ei} \Gamma^p_{bq} B^g_p A^i A^q + 3A^c A^d A^e \partial_{cd} \Gamma^g_{be} \partial_i \Gamma^p_{gq} B^a_p A^i A^q \\ & + 3A^c A^d A^e \partial_c \Gamma^g_{bd} \partial_{ei} \Gamma^p_{gq} B^a_p A^i A^q + A^c A^e \partial_{ce} \Gamma^a_{fg} A^f \partial_i \Gamma^g_{pq} B^p_b A^i A^q + 2A^c A^e \partial_c \Gamma^a_{ef} A^g \partial_{gi} \Gamma^f_{pq} B^p_b A^i A^q \\ & - 2A^c A^e \partial_{ce} \Gamma^a_{fg} A^f \partial_i \Gamma^p_{bq} B^g_p A^i A^q - 3A^c A^e \partial_c \Gamma^a_{ef} A^g \partial_{gi} \Gamma^p_{bq} B^f_p A^i A^q - A^c A^e \partial_c \Gamma^f_{be} A^g \partial_{gi} \Gamma^a_{pq} B^p_f A^i A^q \\ & + A^c A^e \partial_{ce} \Gamma^f_{bg} A^g \partial_i \Gamma^p_{fq} B^a_p A^i A^q + 2A^c A^e \partial_c \Gamma^f_{be} A^g \partial_{gi} \Gamma^p_{fq} B^a_p A^i A^q + A^c \partial_c \Gamma^a_{eg} A^e A^h \partial_{hi} \Gamma^g_{pq} B^p_b A^i A^q \\ & - A^c \partial_c \Gamma^a_{eg} A^e A^h \partial_{hi} \Gamma^p_{bq} B^g_p A^i A^q - A^c \partial_c \Gamma^e_{bg} A^g A^h \partial_{hi} \Gamma^a_{pq} B^p_e A^i A^q + A^c \partial_c \Gamma^e_{bg} A^g A^h \partial_{hi} \Gamma^p_{eq} B^a_p A^i A^q \end{aligned}$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

```
# substitute covariant derivs of  $B^{\{a\}_{\{b\}}$  into covariant derivs of  $R^{\{a\}_{\{bcd\}}B^{\{d\}_{\{a\}}$ 
# this produces expressions for the partial derivs of Rabcd its covariant derivs and partial derivs of Gamma
# the partial derivs of Gamma will be eliminated later by using results imported from dGamma.json

beg_stage_4 = time.time()

substitute (dRabcd01,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab01)$,repeat=True); distribute (dRabcd01)
substitute (dRabcd02,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab01)$,repeat=True); distribute (dRabcd02)
substitute (dRabcd03,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab01)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab01)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab01)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd02,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\} \{d\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab02)$,repeat=True); distribute (dRabcd02)
substitute (dRabcd03,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\} \{d\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab02)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\} \{d\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab02)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\} \{d\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab02)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd03,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\} \{d\} \{e\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab03)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\} \{d\} \{e\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab03)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\} \{d\} \{e\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab03)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}\nabla_{\{c\} \{d\} \{e\} \{f\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab04)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}\nabla_{\{c\} \{d\} \{e\} \{f\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab04)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}A^{\{g\}}\nabla_{\{c\} \{d\} \{e\} \{f\} \{g\}}\{B^{\{a\}_{\{b\}}\} \rightarrow @(dBab05)$,repeat=True); distribute (dRabcd05)

# no longer need B, so let's get rid of it

# two subtle tricks are used here
# 1) rename A and B as A002 and A001 before sort_product,
#    this ensures B will be to left of A after the sort
# 2) indices on B changed from  $B^{\{a\}_{\{b\}}$  to  $B_{\{b\}}^{\{a\}}$ ,
#    this ensures that after factor_out B will have dummy indices  $B_{\{a\}}^{\{b\}}$ 

def remove_Bab (obj):
    foo := @(obj).
```

```

substitute      (foo,$A^{a}->A002^{a},B^{a}_{b}->A001_{b}^{a}$)  # need this to sort B to the left of A
sort_product    (foo)
rename_dummies  (foo)
factor_out      (foo,$A001^{a?}_{b?},A002^{c?}$)
substitute      (foo,$A001_{a}^{b}->1,A002^{a}->A^{a}$)  # recover A and set B = 1, free indices now ^{a}_{b}
return foo

dRabcd01 = remove_Bab (dRabcd01)    # cdb(dRabcd01.401,dRabcd01)
dRabcd02 = remove_Bab (dRabcd02)    # cdb(dRabcd02.401,dRabcd02)
dRabcd03 = remove_Bab (dRabcd03)    # cdb(dRabcd03.401,dRabcd03)
dRabcd04 = remove_Bab (dRabcd04)    # cdb(dRabcd04.401,dRabcd04)
dRabcd05 = remove_Bab (dRabcd05)    # cdb(dRabcd05.401,dRabcd05)

end_stage_4 = time.time()

```

$$\begin{aligned}
\text{dRabcd01.401} &:= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
\text{dRabcd02.401} &:= A^c A^d A^e A^f \left(-\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma_{bf}^g g^{ha} + R_{cbd g} \partial_e \Gamma_{hf}^a g^{gh} \right) \\
\text{dRabcd03.401} &:= A^c A^d A^e A^f A^g \left(-3\nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3\nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_e \Gamma_{bg}^h g^{ia} + R_{cbdh} \partial_e \Gamma_{ig}^a g^{hi} \right) \\
\text{dRabcd04.401} &:= A^c A^d A^e A^f A^g A^h \left(-6\nabla_{de} R_{figj} \partial_c \Gamma_{bh}^i g^{aj} + 6\nabla_{de} R_{fbgi} \partial_c \Gamma_{jh}^a g^{ji} - 4\nabla_c R_{diej} \partial_{fg} \Gamma_{bh}^i g^{ja} + 4\nabla_c R_{dbei} \partial_{fg} \Gamma_{jh}^a g^{ij} - \nabla_{cdef} R_{gbhi} g^{ai} \right. \\
&\quad \left. - R_{cidj} \partial_{efg} \Gamma_{bh}^i g^{ja} + R_{cbdi} \partial_{efg} \Gamma_{jh}^a g^{ij} - 3R_{cidj} \partial_e \Gamma_{fk}^i \partial_g \Gamma_{bh}^k g^{ja} + 6R_{cidj} \partial_e \Gamma_{fb}^i \partial_g \Gamma_{kh}^a g^{jk} - 3R_{cbdi} \partial_e \Gamma_{kf}^j \partial_g \Gamma_{jh}^a g^{ik} \right) \\
\text{dRabcd05.401} &:= A^c A^d A^e A^f A^g A^h A^i \left(-10\nabla_{cd} R_{ejfk} \partial_{gh} \Gamma_{bi}^j g^{ka} + 10\nabla_{cd} R_{ebfj} \partial_{gh} \Gamma_{ki}^a g^{jk} - 10\nabla_{def} R_{gjhk} \partial_c \Gamma_{bi}^j g^{ak} + 10\nabla_{def} R_{gbhj} \partial_c \Gamma_{ki}^a g^{kj} \right. \\
&\quad - 5\nabla_c R_{djek} \partial_{fgh} \Gamma_{bi}^j g^{ka} + 5\nabla_c R_{dbej} \partial_{fgh} \Gamma_{ki}^a g^{jk} - 15\nabla_c R_{djek} \partial_f \Gamma_{gl}^j \partial_h \Gamma_{bi}^l g^{ka} + 30\nabla_c R_{djek} \partial_f \Gamma_{gb}^j \partial_h \Gamma_{li}^a g^{kl} - 15\nabla_c R_{dbej} \partial_f \Gamma_{lg}^k \partial_h \Gamma_{ki}^a g^{jl} \\
&\quad - \nabla_{cdefg} R_{hbij} g^{aj} - R_{cjdk} \partial_{efgh} \Gamma_{bi}^j g^{ka} + R_{cbdj} \partial_{efgh} \Gamma_{ki}^a g^{jk} - 4R_{cjdk} \partial_h \Gamma_{bi}^l \partial_{ef} \Gamma_{gl}^j g^{ka} - 6R_{cjdk} \partial_e \Gamma_{fl}^j \partial_{gh} \Gamma_{bi}^l g^{ka} \\
&\quad \left. + 8R_{cjdk} \partial_h \Gamma_{li}^a \partial_{ef} \Gamma_{gb}^j g^{kl} + 10R_{cjdk} \partial_e \Gamma_{fb}^j \partial_{gh} \Gamma_{li}^a g^{kl} - 4R_{cbdj} \partial_h \Gamma_{ki}^a \partial_{ef} \Gamma_{lg}^k g^{jl} - 6R_{cbdj} \partial_e \Gamma_{lf}^k \partial_{gh} \Gamma_{ki}^a g^{jl} + 2R_{cjdk} \partial_e \Gamma_{lf}^a \partial_{gh} \Gamma_{bi}^j g^{kl} \right)
\end{aligned}$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

```
import cdblib

beg_stage_5 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json') # cdb(dGamma01.500,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json') # cdb(dGamma02.500,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json') # cdb(dGamma03.500,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json') # cdb(dGamma04.500,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json') # cdb(dGamma05.500,dGamma05)

distribute (dRabcd01) # cdb(dRabcd01.500,dRabcd01)
distribute (dRabcd02) # cdb(dRabcd02.500,dRabcd02)
distribute (dRabcd03) # cdb(dRabcd03.500,dRabcd03)
distribute (dRabcd04) # cdb(dRabcd04.500,dRabcd04)
distribute (dRabcd05) # cdb(dRabcd05.500,dRabcd05)

# use dGamma to eliminate the partial derivs of Gamma
# this will introduces some lower order partial dervis of Rabcd on the rhs
# these extra partial derivs of Rabcd will be eliminated (later) by substiting lower order dRabcd into the higher order dRabcd

substitute (dRabcd02,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd02.501,dRabcd02)
substitute (dRabcd02,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd02.502,dRabcd02)
distribute (dRabcd02) # cdb(dRabcd02.503,dRabcd02)
sort_product (dRabcd02) # cdb(dRabcd02.504,dRabcd02)
rename_dummies (dRabcd02) # cdb(dRabcd02.505,dRabcd02)

substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(dGamma02)$,repeat=True) # cdb(dRabcd03.501,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(dGamma02)$,repeat=True) # cdb(dRabcd03.502,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd03.503,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd03.504,dRabcd03)
distribute (dRabcd03) # cdb(dRabcd03.505,dRabcd03)
sort_product (dRabcd03) # cdb(dRabcd03.506,dRabcd03)
rename_dummies (dRabcd03) # cdb(dRabcd03.507,dRabcd03)

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(dGamma03)$,repeat=True) # cdb(dRabcd04.501,dRabcd04)
```

```

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True) # cdb(dRabcd04.502,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True) # cdb(dRabcd04.503,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True) # cdb(dRabcd04.504,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True) # cdb(dRabcd04.505,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True) # cdb(dRabcd04.506,dRabcd04)
distribute (dRabcd04) # cdb(dRabcd04.507,dRabcd04)
sort_product (dRabcd04) # cdb(dRabcd04.508,dRabcd04)
rename_dummies (dRabcd04) # cdb(dRabcd04.509,dRabcd04)

substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)
distribute (dRabcd05)
sort_product (dRabcd05)
rename_dummies (dRabcd05)

end_stage_5 = time.time()

```

$$\text{dRabcd01.500} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\text{dRabcd02.500} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - A^c A^d A^e A^f R_{cgdh} \partial_e \Gamma_{bf}^g g^{ha} + A^c A^d A^e A^f R_{cbdg} \partial_e \Gamma_{hf}^a g^{gh}$$

$$\text{dRabcd02.501} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.502} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.503} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.504} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R_{feb}^g g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R_{feh}^a g^{gh}$$

$$\text{dRabcd02.505} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R_{feb}^g g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R_{feh}^a g^{gh}$$

$$\begin{aligned} \text{dRabcd03.500} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - A^c A^d A^e A^f A^g R_{chdi} \partial_{ef} \Gamma_{bg}^{ia} + A^c A^d A^e A^f A^g R_{cbdh} \partial_{ef} \Gamma_{ig}^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.501} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.502} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.503} := & -A^c A^d A^e A^g A^f R_{gfb}^h \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R_{gfi}^a \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.504} := & -A^c A^d A^e A^g A^f R_{gfb}^h \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R_{gfi}^a \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned}
\text{dRabcd03.505} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^h_{geb} R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^a_{gei} R_{cbdh} g^{hi} \\
\text{dRabcd03.506} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{gei} g^{hi} \\
\text{dRabcd03.507} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{gei} g^{hi}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.500} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} \\
&\quad + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{efg} \Gamma^i_{bh} g^{ja} \\
&\quad + A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{efg} \Gamma^a_{jh} g^{ij} - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} \\
&\quad + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.501} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
&\quad - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
&\quad - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
&\quad + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{h fj} \right) R_{cbdi} g^{ij} \\
&\quad - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.502} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
&\quad - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
&\quad - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
&\quad + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{h fj} \right) R_{cbdi} g^{ij} \\
&\quad - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.503} := & -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
& - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbe i} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik} \\
\text{dRabcd04.504} := & -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
& - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbe i} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik} \\
\text{dRabcd04.505} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcj} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbe i} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - A^c A^d A^e A^f A^h A^g R^k_{hgb} R_{cidj} \partial_e \Gamma^i_{fk} g^{ja} + 2A^c A^d A^e A^f A^h A^g R^a_{h gk} R_{cidj} \partial_e \Gamma^i_{fb} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik} \\
\text{dRabcd04.506} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcj} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbe i} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left(\frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^f A^e R^i_{fek} A^h A^g R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^f A^e R^i_{feb} A^h A^g R^a_{h gk} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.507} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcb} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{hfb} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^h A^e A^f A^g \partial_{gf} R^i_{heb} R_{cidj} g^{ja} \\
& + \frac{1}{15} A^c A^d A^h A^e A^f A^g R^i_{efk} R^k_{hgb} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^h A^e A^f A^g R^i_{egk} R^k_{hfb} R_{cidj} g^{ja} + \frac{3}{5} A^c A^d A^h A^e A^f A^g \partial_{gf} R^a_{hej} R_{cbdi} g^{ij} \\
& - \frac{1}{15} A^c A^d A^h A^e A^f A^g R^a_{efk} R^k_{hgj} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^h A^e A^f A^g R^a_{egk} R^k_{hfb} R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^f A^e R^i_{fek} A^h A^g R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^f A^e R^i_{feb} A^h A^g R^a_{hgb} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.508} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hcb} \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} \\
& + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{hfb} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{gf} R^i_{heb} g^{ja} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{efk} R^k_{hgb} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{egk} R^k_{hfb} g^{ja} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{gf} R^a_{hej} g^{ij} \\
& - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{efk} R^k_{hgj} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{egk} R^k_{hfb} g^{ij} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{fek} R^k_{hgb} g^{ja} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^a_{hgb} R^i_{feb} g^{jk} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{hgj} R^j_{fek} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.509} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hcb} \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} \\
& + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{hfb} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{gf} R^i_{heb} g^{ja} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{efk} R^k_{hgb} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{egk} R^k_{hfb} g^{ja} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{gf} R^a_{hej} g^{ij} \\
& - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{efj} R^j_{hgb} g^{ik} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{egj} R^j_{hfb} g^{ik} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{fek} R^k_{hgb} g^{ja} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^a_{hgb} R^i_{feb} g^{jk} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{hgj} R^j_{fek} g^{ik}
\end{aligned}$$

Stage 6: Replace partial derivs of R with covariant derivs of R

```
# now eliminate remaining partial derivs of Rabcd by substitution from the lower order dRabcd

# note that
#   dRabcd01 = R^a_{cdb,e} A^c A^d A^e
#   dRabcd02 = R^a_{cdb,ef} A^c A^d A^e A^f
#   dRabcd03 = R^a_{cdb,efg} A^c A^d A^e A^f A^g

# thus we can use
#   dRabcd01 to eliminate 1st partial derivs of R in dRabcd03, dRabcd04, etc.
#   dRabcd02 to eliminate 2nd partial derivs of R in dRabcd04, dRabcd05, etc.
#   dRabcd03 to eliminate 3rd partial derivs of R in dRabcd05, dRabcd06, etc.

beg_stage_6 = time.time()

substitute (dRabcd03,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(dRabcd01)$,repeat=True)      # cdb(dRabcd03.601,dRabcd03)
distribute (dRabcd03)                                                                    # cdb(dRabcd03.602,dRabcd03)

# note: dRabcd04 and dRabcd05 unused in this code (or any other code)

substitute (dRabcd04,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(dRabcd02)$,repeat=True) # cdb(dRabcd04.601,dRabcd04)
substitute (dRabcd04,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(dRabcd01)$,repeat=True)      # cdb(dRabcd04.602,dRabcd04)
distribute (dRabcd04)                                                                    # cdb(dRabcd04.603,dRabcd04)

substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c d b}\} -> @(dRabcd03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(dRabcd02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(dRabcd01)$,repeat=True)
distribute (dRabcd05)

end_stage_6 = time.time()
```

$$\begin{aligned} \text{dRabcd03.601} := & -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & + \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfb} g^{hj} R_{chdi} g^{ia} - \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfi} g^{aj} R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.602} := & -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & + \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfb} g^{hj} R_{chdi} g^{ia} - \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfi} g^{aj} R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd04.601} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hci} \nabla_{de} R_{fbgj} g^{ij} \\ & - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{h fj} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\ & - \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} - \frac{1}{3} A^h A^e A^g A^f R_{hle k} R^l_{f gb} g^{ki} + \frac{1}{3} A^h A^e A^g A^f R_{hbel} R^i_{f gk} g^{lk} \right) R_{cidj} g^{ja} \\ & + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{efk} R^k_{hgb} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{egk} R^k_{hfb} g^{ja} \\ & + \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gjfl} g^{al} - \frac{1}{3} A^h A^e A^g A^f R_{hle k} R^l_{f g j} g^{ka} + \frac{1}{3} A^h A^e A^g A^f R_{hjel} R^a_{f gk} g^{lk} \right) R_{cbdi} g^{ij} \\ & - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{efj} R^j_{h gk} g^{ik} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{egj} R^j_{h f k} g^{ik} \\ & - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{fek} R^k_{hgb} g^{ja} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^a_{h gk} R^i_{f e b} g^{jk} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{h g j} R^j_{f e k} g^{ik} \end{aligned}$$

$$\begin{aligned} \text{dRabcd04.602} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hci} \nabla_{de} R_{fbgj} g^{ij} + 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fk g b} g^{ik} \nabla_c R_{diej} g^{ja} \\ & - 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fk g j} g^{ak} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\ & - \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} - \frac{1}{3} A^h A^e A^g A^f R_{hle k} R^l_{f gb} g^{ki} + \frac{1}{3} A^h A^e A^g A^f R_{hbel} R^i_{f gk} g^{lk} \right) R_{cidj} g^{ja} \\ & + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{efk} R^k_{hgb} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{egk} R^k_{hfb} g^{ja} \\ & + \frac{3}{5} A^c A^d \left(-A^h A^e A^g A^f \nabla_{he} R_{gjfl} g^{al} - \frac{1}{3} A^h A^e A^g A^f R_{hle k} R^l_{f g j} g^{ka} + \frac{1}{3} A^h A^e A^g A^f R_{hjel} R^a_{f gk} g^{lk} \right) R_{cbdi} g^{ij} \\ & - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{efj} R^j_{h gk} g^{ik} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{egj} R^j_{h f k} g^{ik} \\ & - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{fek} R^k_{hgb} g^{ja} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^a_{h gk} R^i_{f e b} g^{jk} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{h g j} R^j_{f e k} g^{ik} \end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.603} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hci} \nabla_{de} R_{fbgj} g^{ij} + 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fkgb} g^{ik} \nabla_c R_{diej} g^{ja} \\
& - 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fk gj} g^{ak} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} R_{cidj} g^{ja} \\
& + \frac{1}{5} A^c A^d A^h A^e A^g A^f R_{hle k} R^l_{f gb} g^{ki} R_{cidj} g^{ja} - \frac{1}{5} A^c A^d A^h A^e A^g A^f R_{hbel} R^i_{f gk} g^{lk} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{efk} R^k_{hgb} g^{ja} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{egk} R^k_{hfb} g^{ja} - \frac{3}{5} A^c A^d A^h A^e A^g A^f \nabla_{he} R_{gjfl} g^{al} R_{cbdi} g^{ij} - \frac{1}{5} A^c A^d A^h A^e A^g A^f R_{hle k} R^l_{f gj} g^{ka} R_{cbdi} g^{ij} \\
& + \frac{1}{5} A^c A^d A^h A^e A^g A^f R_{hjel} R^a_{f gk} g^{lk} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{efj} R^j_{h gk} g^{ik} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{egj} R^j_{hfk} g^{ik} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^i_{fek} R^k_{hgb} g^{ja} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R^a_{h gk} R^i_{feb} g^{jk} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R^a_{h g j} R^j_{fek} g^{ik}
\end{aligned}$$

Stage 7: Reformatting

```
beg_stage_7 = time.time()

dRabcd01 = flatten_Rabcd (dRabcd01)  # cdb(dRabcd01.701,dRabcd01)
dRabcd02 = flatten_Rabcd (dRabcd02)  # cdb(dRabcd02.701,dRabcd02)
dRabcd03 = flatten_Rabcd (dRabcd03)  # cdb(dRabcd03.701,dRabcd03)
dRabcd04 = flatten_Rabcd (dRabcd04)  # cdb(dRabcd04.701,dRabcd04)
dRabcd05 = flatten_Rabcd (dRabcd05)  # cdb(dRabcd05.701,dRabcd05)

canonicalise (dRabcd01)  # cdb(dRabcd01.702,dRabcd01)
canonicalise (dRabcd02)  # cdb(dRabcd02.702,dRabcd02)
canonicalise (dRabcd03)  # cdb(dRabcd03.702,dRabcd03)
canonicalise (dRabcd04)  # cdb(dRabcd04.702,dRabcd04)
canonicalise (dRabcd05)  # cdb(dRabcd05.702,dRabcd05)

end_stage_7 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.1f} secs\\hfill\\break".format(end_stage_4-beg_stage_4))
print ("Stage 5: {:.1f} secs\\hfill\\break".format(end_stage_5-beg_stage_5))
print ("Stage 6: {:.1f} secs\\hfill\\break".format(end_stage_6-beg_stage_6))
print ("Stage 7: {:.1f} secs".format(end_stage_7-beg_stage_7))
# cdbEnd (timing)
```

$$\text{dRabcd01.701} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\text{dRabcd02.701} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R_{ifeb} g^{gi} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R_{hfei} g^{ah} g^{gi}$$

$$\begin{aligned} \text{dRabcd03.701} := & -A^c A^d A^e A^f A^g R_{hgfb} \nabla_c R_{diej} g^{ih} g^{ja} + A^c A^d A^e A^f A^g R_{hgfi} \nabla_c R_{dbej} g^{ah} g^{ji} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_g R_{ejfb} g^{hj} g^{ia} - \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \nabla_g R_{eifj} g^{ai} g^{hj} \end{aligned}$$

$$\begin{aligned} \text{dRabcd04.701} := & -2A^c A^d A^e A^f A^g A^h R_{ihcb} \nabla_{de} R_{fjgk} g^{ak} g^{ji} + 2A^c A^d A^e A^f A^g A^h R_{ihcj} \nabla_{de} R_{fbgk} g^{ai} g^{jk} + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \nabla_h R_{fkgb} g^{ik} g^{ja} \\ & - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \nabla_h R_{fjgk} g^{aj} g^{ik} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{he} R_{gbfk} g^{ik} g^{ja} \\ & + \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hkel} R_{mfgb} g^{ja} g^{li} g^{km} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hbek} R_{lfgm} g^{il} g^{ja} g^{km} \\ & + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kefl} R_{mhgb} g^{ik} g^{ja} g^{lm} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kegl} R_{mhfb} g^{ik} g^{ja} g^{lm} \\ & - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \nabla_{he} R_{gjfk} g^{ak} g^{ij} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{im} g^{ka} g^{jl} \\ & + \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{al} g^{ij} g^{km} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jefk} R_{lhgm} g^{aj} g^{im} g^{kl} \\ & - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jegk} R_{lhfm} g^{aj} g^{im} g^{kl} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kfel} R_{mhgb} g^{ik} g^{ja} g^{lm} \\ & + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{khgl} R_{mfeb} g^{ak} g^{im} g^{jl} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jhkg} R_{lfem} g^{aj} g^{im} g^{kl} \end{aligned}$$

$$\text{dRabcd01.702} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\text{dRabcd02.702} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\text{dRabcd03.702} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfg h} g^{ah}$$

$$\text{dRabcd04.702} := -\frac{7}{5} A^c A^d A^e A^f A^g A^h R_{bcdi} \nabla_{ef} R_{gjhk} g^{aj} g^{ik} + \frac{7}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{ef} R_{bghk} g^{ai} g^{jk} + A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{bghi} g^{ai}$$

$$\begin{aligned} \text{dRabcd05.702} := & -2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{bdej} \nabla_{fg} R_{hkil} g^{ak} g^{jl} + 2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{djek} \nabla_{fg} R_{bhil} g^{aj} g^{kl} \\ & - \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} \nabla_{efg} R_{hkil} g^{ak} g^{jl} + \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} \nabla_{efg} R_{bhil} g^{aj} g^{kl} \\ & + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{am} g^{jk} g^{ln} + A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{elfm} \nabla_g R_{bhin} g^{aj} g^{kl} g^{mn} \\ & - \frac{4}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{ak} g^{jm} g^{ln} + A^c A^d A^e A^f A^g A^h A^i \nabla_{cdefg} R_{bhij} g^{aj} \end{aligned}$$


```
cdblib.create ('dRabcd.json')
```

```
cdblib.put ('dRabcd01',dRabcd01,'dRabcd.json')
```

```
cdblib.put ('dRabcd02',dRabcd02,'dRabcd.json')
```

```
cdblib.put ('dRabcd03',dRabcd03,'dRabcd.json')
```

```
cdblib.put ('dRabcd04',dRabcd04,'dRabcd.json')
```

```
cdblib.put ('dRabcd05',dRabcd05,'dRabcd.json')
```

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}          -> A001^{a}          $)
    substitute (obj,$ x^{a}          -> A002^{a}          $)
    substitute (obj,$ g^{a b}        -> A003^{a b}        $)
    substitute (obj,$ \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} -> A008_{\{a b c d e f g h\}} $)
    substitute (obj,$ \nabla_{\{e f g\}}\{R_{\{a b c d\}}\}      -> A007_{\{a b c d e f g\}}  $)
    substitute (obj,$ \nabla_{\{e f\}}\{R_{\{a b c d\}}\}        -> A006_{\{a b c d e f\}}   $)
    substitute (obj,$ \nabla_{\{e\}}\{R_{\{a b c d\}}\}          -> A005_{\{a b c d e\}}    $)
    substitute (obj,$ R_{\{a b c d\}}      -> A004_{\{a b c d\}}    $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}          -> A^{a}          $)
    substitute (obj,$ A002^{a}          -> x^{a}          $)
    substitute (obj,$ A003^{a b}        -> g^{a b}        $)
    substitute (obj,$ A004_{\{a b c d\}} -> R_{\{a b c d\}}   $)
    substitute (obj,$ A005_{\{a b c d e\}} -> \nabla_{\{e\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A006_{\{a b c d e f\}} -> \nabla_{\{e f\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A007_{\{a b c d e f g\}} -> \nabla_{\{e f g\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A008_{\{a b c d e f g h\}} -> \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} $)

    return obj

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    factor_out (bah,$A^{a?}$)
    ans := @(bah).
    return ans

scaled1 = reformat (dRabcd01, 1) # cdb(scaled1.601,scaled1)
scaled2 = reformat (dRabcd02, 1) # cdb(scaled2.601,scaled2)
scaled3 = reformat (dRabcd03,-2) # cdb(scaled3.601,scaled3)
scaled4 = reformat (dRabcd04,-5) # cdb(scaled4.601,scaled4)

```

```
scaled5 = reformat (dRabcd05,-3)    # cdb(scaled5.601,scaled5)
```

Symmetrised partial derivatives of $R^a{}_{bcd}$

$$\begin{aligned}
A^c A^d A^e R^a{}_{cdb,e} &= A^c A^d A^e g^{af} \nabla_c R_{bdef} \\
A^c A^d A^e A^f R^a{}_{cdb,ef} &= A^c A^d A^e A^f g^{ag} \nabla_{cd} R_{befg} \\
-2A^c A^d A^e A^f A^g R^a{}_{cdb,efg} &= A^c A^d A^e A^f A^g (g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - 2g^{ah} \nabla_{cde} R_{bfgh}) \\
-5A^c A^d A^e A^f A^g A^h R^a{}_{cdb,efgh} &= A^c A^d A^e A^f A^g A^h (7g^{ai} g^{jk} R_{bcdj} \nabla_{ef} R_{gihk} - 7g^{ai} g^{jk} R_{cidj} \nabla_{ef} R_{bghk} - 5g^{ai} \nabla_{cdef} R_{bghi}) \\
-3A^c A^d A^e A^f A^g A^h A^i R^a{}_{cdb,efghi} &= A^c A^d A^e A^f A^g A^h A^i (6g^{aj} g^{kl} \nabla_c R_{bdek} \nabla_{fg} R_{hjil} - 6g^{aj} g^{kl} \nabla_c R_{djek} \nabla_{fg} R_{bhil} + 8g^{aj} g^{kl} R_{bcdk} \nabla_{efg} R_{hjil} \\
&\quad - 8g^{aj} g^{kl} R_{cjdk} \nabla_{efg} R_{bhil} - g^{aj} g^{kl} g^{mn} R_{bcdk} R_{elfm} \nabla_g R_{hjin} - 3g^{aj} g^{kl} g^{mn} R_{cjdk} R_{elfm} \nabla_g R_{bhin} \\
&\quad + 4g^{aj} g^{kl} g^{mn} R_{bcdk} R_{ejfm} \nabla_g R_{hlin} - 3g^{aj} \nabla_{cdefg} R_{bhij})
\end{aligned}$$

```

substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dRabcd.export')

# 6th order dRabcd, scaled
cdblib.put ('dRabcd61scaled',scaled1,'dRabcd.export')
cdblib.put ('dRabcd62scaled',scaled2,'dRabcd.export')
cdblib.put ('dRabcd63scaled',scaled3,'dRabcd.export')
cdblib.put ('dRabcd64scaled',scaled4,'dRabcd.export')
cdblib.put ('dRabcd65scaled',scaled5,'dRabcd.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

```

Timing

Stage 1: 1.2 secs

Stage 2: 3.9 secs

Stage 3: 0.5 secs

Stage 4: 112.3 secs

Stage 5: 144.7 secs

Stage 6: 181.2 secs

Stage 7: 3.6 secs