Geodesic IVP

Our game here is to find the solution of

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a(s)/ds = \dot{x}^a$ at $s = 0$.

Algorithm

By successive differentiation of the above equation we can compute

$$\frac{d^nx^a}{ds^n} = -\Gamma^a_{\underline{d}_n}\frac{dx^{\underline{d}_n}}{ds}$$

at $s = 0$ for $n = 2, 3, 4, \ldots$. The $\Gamma^a_{\underline{d}_n}$ are the *generalised connections*.

We can then construct the Taylor series solution for $x^a(s)$

$$x^a(s) = x^a + s\dot{x}^a - \sum_{k=2}^{\infty}\frac{s^k}{k!}\Gamma^a_{\underline{d}_k}\dot{x}^{\underline{d}_k}$$

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

import cdblib

# change signs to account for - sign in front of the sum for x^a(s), see above preamble

def flip_sign (obj):
    return Ex(0) - obj

sterm21 = flip_sign (cdblib.get ('genGamma01','genGamma.json'))
sterm22 = flip_sign (cdblib.get ('genGamma02','genGamma.json'))
sterm23 = flip_sign (cdblib.get ('genGamma03','genGamma.json'))
sterm24 = flip_sign (cdblib.get ('genGamma04','genGamma.json'))

sterm31 = flip_sign (cdblib.get ('genGamma11','genGamma.json'))
sterm32 = flip_sign (cdblib.get ('genGamma12','genGamma.json'))
sterm33 = flip_sign (cdblib.get ('genGamma13','genGamma.json'))

sterm41 = flip_sign (cdblib.get ('genGamma21','genGamma.json'))
sterm42 = flip_sign (cdblib.get ('genGamma22','genGamma.json'))

sterm51 = flip_sign (cdblib.get ('genGamma31','genGamma.json'))

sterm2 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24).  # cdb (sterm2.000,sterm2)
sterm3 := @(sterm31) + @(sterm32) + @(sterm33).              # cdb (sterm3.000,sterm3)
sterm4 := @(sterm41) + @(sterm42).                           # cdb (sterm4.000,sterm4)
sterm5 := @(sterm51).                                        # cdb (sterm5.000,sterm5)

factor_out (sterm2,$A^{a?}$)                                 # cdb (sterm2.001,sterm2)
factor_out (sterm3,$A^{a?}$)                                 # cdb (sterm3.001,sterm3)
factor_out (sterm4,$A^{a?}$)                                 # cdb (sterm4.001,sterm4)
factor_out (sterm5,$A^{a?}$)                                 # cdb (sterm5.001,sterm5)

sterm2 := 360 @(sterm2).
sterm3 := 360 @(sterm3).
sterm4 :=  90 @(sterm4).
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sterm5 :=   3 @(sterm5).

substitute (sterm2,$A^{a}->1$)                          # cdb (sterm2.002,sterm2)
substitute (sterm3,$A^{a}->1$)                          # cdb (sterm3.002,sterm3)
substitute (sterm4,$A^{a}->1$)                          # cdb (sterm4.002,sterm4)
substitute (sterm5,$A^{a}->1$)                          # cdb (sterm5.002,sterm5)
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The geodesic ivp

$$x^a(s) = x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A^a_{bc} + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A^a_{bcd} + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A^a_{bcde} + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A^a_{bcdef} + \cdots$$

$$\begin{aligned}
360A^a_{bc} = &-240x^d g^{ae}R_{bdce} - 30x^d x^e\left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) - x^d x^e x^f\left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi}\right.\\
&\left. - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\right)\\
&- 2x^d x^e x^f x^g\left(16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 8g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj}\right.\\
&- 4g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh}\\
&+ 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg}\\
&\left. + g^{ah}\nabla_{deh}R_{bfcg}\right)
\end{aligned}$$

$$\begin{aligned}
360A^a_{bcd} = &-180x^e g^{af}\nabla_b R_{cedf} - 3x^e x^f\left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg}\right.\\
&\left. + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right)\\
&- 2x^e x^f x^g\left(32g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} + 2g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj}\right.\\
&+ 22g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg}\\
&- 12g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh}\\
&+ 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} + 16g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj}\\
&+ 16g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg}\\
&\left. + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20g^{ah}g^{ij}R_{beci}\nabla_j R_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg}\right)
\end{aligned}$$

$$\begin{aligned}
90A^a_{bcde} = &-6x^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) - x^f x^g\left(64g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej}\right.\\
&+ 4g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} + 44g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg}\\
&- 16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej}\\
&\left. + 6g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_j R_{dgeh}\right)
\end{aligned}$$

$$3A^a_{bcdef} = -x^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_d R_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

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sterm2short := @(sterm21) + @(sterm22).              # cdb (sterm2.short.001,sterm2short)
sterm3short := @(sterm31).                           # cdb (sterm3.short.001,sterm3short)
sterm2shortscaled := 12 @(sterm2short).              # cdb (sterm2.short.scaled.002,sterm2shortscaled)
sterm3shortscaled :=  2 @(sterm3short).              # cdb (sterm3.short.scaled.002,sterm3shortscaled)

substitute (sterm2shortscaled,$A^{a}->1$)            # cdb (sterm2.short.scaled.003,sterm2shortscaled)
substitute (sterm3shortscaled,$A^{a}->1$)            # cdb (sterm3.short.scaled.003,sterm3shortscaled)

cdblib.create ('geodesic-ivp.export')

# 4th order ivp terms scaled
cdblib.put ('ivp42',sterm2shortscaled,'geodesic-ivp.export')
cdblib.put ('ivp43',sterm3shortscaled,'geodesic-ivp.export')

# 6th order ivp terms scaled
cdblib.put ('ivp62',sterm2,'geodesic-ivp.export')
cdblib.put ('ivp63',sterm3,'geodesic-ivp.export')
cdblib.put ('ivp64',sterm4,'geodesic-ivp.export')
cdblib.put ('ivp65',sterm5,'geodesic-ivp.export')

checkpoint.append (sterm2shortscaled)
checkpoint.append (sterm3shortscaled)

checkpoint.append (sterm2)
checkpoint.append (sterm3)
checkpoint.append (sterm4)
checkpoint.append (sterm5)
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$$\texttt{sterm2.short.001} := -\frac{2}{3}A^b A^c x^d g^{ae} R_{bdce} - \frac{1}{12}A^b A^c x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$$

$$\texttt{sterm3.short.001} := -\frac{1}{2}A^b A^c A^d x^e g^{af}\nabla_b R_{cedf}$$

$$\texttt{sterm2.short.scaled.002} := -8A^b A^c x^d g^{ae} R_{bdce} - A^b A^c x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$$

$$\texttt{sterm3.short.scaled.002} := -A^b A^c A^d x^e g^{af}\nabla_b R_{cedf}$$

$\texttt{sterm2.short.scaled.003} := -8x^d g^{ae} R_{bdce} - x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)$

$\texttt{sterm3.short.scaled.003} := -x^e g^{af}\nabla_b R_{cedf}$