

# The metric tensor in Riemann normal coordinates

In this notebook we compute the recursive sequences

$$g_{ab,d\underline{e}} = (g_{cb}\Gamma^c_{a(d),\underline{e}}) + (g_{ac}\Gamma^c_{b(d),\underline{e}}) \quad (1)$$

$$(n+3)\Gamma^a_{d(b,c\underline{e})} = (n+1) \left( R^a_{(bcd,\underline{e})} - (\Gamma^a_{f(c}\Gamma^f_{bd),\underline{e}}) \right) \quad (2)$$

for  $n = 1, 2, 3, \dots$ . Note in these equations that the (extended) index  $\underline{e}$  contains  $n$  normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g_{ab}(x) &= g_{ab} + g_{ab,c}x^c + \frac{1}{2!}g_{ab,cd}x^cx^d + \frac{1}{3!}g_{ab,cde}x^cx^dx^e + \dots \\ &= g_{ab} + \sum_{n=1}^{\infty} \frac{1}{n!} g_{ab,\underline{c}} x^{\underline{c}} \end{aligned}$$

## Stage 1: Symmetrised partial derivatives of $g_{ab}$

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g_{ab,c}A^c &= 0 \\ g_{ab,cd}A^cA^d &= g_{cb}\partial_e\Gamma^c_{ad}A^dA^e + g_{ac}\partial_e\Gamma^c_{bd}A^dA^e \\ g_{ab,cde}A^cA^dA^e &= g_{cb}\partial_{fe}\Gamma^c_{ad}A^dA^eA^f + g_{ac}\partial_{fe}\Gamma^c_{bd}A^dA^eA^f \end{aligned}$$

## Stage 2: Replace derivatives of $\Gamma$ with partial derivs of $R$

Now we use the results from `dGamma` to replace derivatives of  $\Gamma$  with partial derivatives of  $R$ . These were computed in `dGamma` using equation (2) above.

$$\begin{aligned}
g_{ab,c}A^c &= 0 \\
g_{ab,cd}A^cA^d &= \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb} \\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}
\end{aligned}$$

### Stage 3: Replace partial derivs of $R$ with covariant derivs of $R$

Next we use the results from `dRabcd` to replace the partial derivatives of  $R$  with covariant derivatives.

$$\begin{aligned}
g_{ab,c}A^c &= 0 \\
g_{ab,cd}A^cA^d &= -\frac{2}{3}A^cA^dR_{acbd} \\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}
\end{aligned}$$

### Stage 4: Build the Taylor series for $g_{ab}$ , reformatting and output

Each of the above expressions constitutes one term in the Taylor series for the metric. We also make the trivial change  $A \rightarrow x$ . Then we do some trivial reformatting.

$$\begin{aligned}
g_{ab}(x) &= g_{ab} + g_{ab,c}x^c + \frac{1}{2!}g_{ab,cd}x^cx^d + \frac{1}{3!}g_{ab,cde}x^cx^dx^e + \mathcal{O}(\epsilon^4) \\
&= g_{ab} - \frac{1}{3}x^cx^dR_{acbd} - \frac{1}{6}x^cx^dx^e\nabla_cR_{adbe} + \mathcal{O}(\epsilon^4)
\end{aligned}$$

# Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d} -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{d} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d} -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}\{\Gamma^{a}_{b c}\} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e}\{\Gamma^{a}_{b c}\} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f}\{\Gamma^{a}_{b c}\} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g h}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}\{\Gamma^{a}_{b c}\}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah = Ex("numx = " + str(n))
    keep_weight (foo,bah)

    return foo
```

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                                -> A001^{a}                                $)
    substitute (obj,$ x^{a}                                -> A002^{a}                                $)
    substitute (obj,$ g_{a b}                              -> A003_{a b}                              $)
    substitute (obj,$ g^{a b}                              -> A004^{a b}                              $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\}       -> A010_{a b c d e f g h}                 $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}         -> A009_{a b c d e f g}                   $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}            -> A008_{a b c d e f}                     $)
    substitute (obj,$ \nabla_e\{R_{a b c d}\}                -> A007_{a b c d e}                       $)
    substitute (obj,$ \partial_{e f g h}\{R_{a b c d}\}      -> A014_{a b c d e f g h}                 $)
    substitute (obj,$ \partial_{e f g}\{R_{a b c d}\}         -> A013_{a b c d e f g}                   $)
    substitute (obj,$ \partial_{e f}\{R_{a b c d}\}            -> A012_{a b c d e f}                     $)
    substitute (obj,$ \partial_e\{R_{a b c d}\}                -> A011_{a b c d e}                       $)
    substitute (obj,$ \partial_{e f g h}\{R^{a}_{a}_{b c d}\}   -> A018^{a}_{a}_{b c d e f g h}           $)
    substitute (obj,$ \partial_{e f g}\{R^{a}_{a}_{b c d}\}     -> A017^{a}_{a}_{b c d e f g}             $)
    substitute (obj,$ \partial_{e f}\{R^{a}_{a}_{b c d}\}        -> A016^{a}_{a}_{b c d e f}               $)
    substitute (obj,$ \partial_e\{R^{a}_{a}_{b c d}\}           -> A015^{a}_{a}_{b c d e}                 $)
    substitute (obj,$ R_{a b c d}                          -> A005_{a b c d}                          $)
    substitute (obj,$ R^{a}_{a}_{b c d}                    -> A006^{a}_{a}_{b c d}                    $)
    sort_product      (obj)
    rename_dummies    (obj)
    substitute (obj,$ A001^{a}                              -> A^{a}                                $)
    substitute (obj,$ A002^{a}                              -> x^{a}                                $)
    substitute (obj,$ A003_{a b}                            -> g_{a b}                              $)
    substitute (obj,$ A004^{a b}                            -> g^{a b}                              $)
    substitute (obj,$ A005_{a b c d}                        -> R_{a b c d}                          $)
    substitute (obj,$ A006^{a}_{a}_{b c d}                  -> R^{a}_{a}_{b c d}                      $)
    substitute (obj,$ A007_{a b c d e}                      -> \nabla_{e}\{R_{a b c d}\}                 $)
    substitute (obj,$ A008_{a b c d e f}                    -> \nabla_{e f}\{R_{a b c d}\}                $)
    substitute (obj,$ A009_{a b c d e f g}                  -> \nabla_{e f g}\{R_{a b c d}\}              $)
    substitute (obj,$ A010_{a b c d e f g h}                -> \nabla_{e f g h}\{R_{a b c d}\}            $)
    substitute (obj,$ A011_{a b c d e}                      -> \partial_e\{R_{a b c d}\}                  $)
    substitute (obj,$ A012_{a b c d e f}                    -> \partial_{e f}\{R_{a b c d}\}                $)
    substitute (obj,$ A013_{a b c d e f g}                  -> \partial_{e f g}\{R_{a b c d}\}              $)
    substitute (obj,$ A014_{a b c d e f g h}                -> \partial_{e f g h}\{R_{a b c d}\}            $)
    substitute (obj,$ A015^{a}_{a}_{b c d e}                -> \partial_e\{R^{a}_{a}_{b c d}\}              $)

```

```

substitute (obj,$ A016^{a}_{b c d e f}      -> \partial_{e f}{R^{a}_{b c d}}      $)
substitute (obj,$ A017^{a}_{b c d e f g}    -> \partial_{e f g}{R^{a}_{b c d}}    $)
substitute (obj,$ A018^{a}_{b c d e f g h}  -> \partial_{e f g h}{R^{a}_{b c d}}  $)

return obj

def reformat_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute      (bah)
    bah = product_sort (bah)
    rename_dummies  (bah)
    canonicalise    (bah)
    factor_out      (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute      (bah)
    factor_out      (bah,$x^{a?}$)
    return bah

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

```

```
\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
```

```
g_{a b}::Depends(\partial{#}).
```

```
R_{a b c d}::Depends(\partial{#}).
```

```
R^{a}_{b c d}::Depends(\partial{#}).
```

```
\Gamma^{a}_{b c}::Depends(\partial{#}).
```

```
R_{a b c d}::Depends(\nabla{#}).
```

```
R^{a}_{b c d}::Depends(\nabla{#}).
```

## Stage 1: Symmetrised partial derivatives of $g_{ab}$

```
beg_stage_1 = time.time()

# symmetrised partial derivatives of g_{ab}

gab00:=g_{a b}. # cdb (gab00.101,gab00)

gab01:=g_{c b}\Gamma^{c}_{a d} + g_{a c}\Gamma^{c}_{b d}. # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @ (gab01) }. # cdb (gab02.101,gab02)
distribute (gab02) # cdb (gab02.102,gab02)
product_rule (gab02) # cdb (gab02.103,gab02)
substitute (gab02, $\partial_{d}{g_{a b}} \rightarrow @ (gab01)$) # cdb (gab02.104,gab02)
distribute (gab02) # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @ (gab02) }. # cdb (gab03.101,gab03)
distribute (gab03) # cdb (gab03.102,gab03)
product_rule (gab03) # cdb (gab03.103,gab03)
substitute (gab03, $\partial_{d}{g_{a b}} \rightarrow @ (gab01)$) # cdb (gab03.104,gab03)
distribute (gab03) # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @ (gab03) }. # cdb (gab04.101,gab04)
distribute (gab04) # cdb (gab04.102,gab04)
product_rule (gab04) # cdb (gab04.103,gab04)
substitute (gab04, $\partial_{d}{g_{a b}} \rightarrow @ (gab01)$) # cdb (gab04.104,gab04)
distribute (gab04) # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @ (gab04) }. # cdb (gab05.101,gab05)
distribute (gab05) # cdb (gab05.102,gab05)
product_rule (gab05) # cdb (gab05.103,gab05)
substitute (gab05, $\partial_{d}{g_{a b}} \rightarrow @ (gab01)$) # cdb (gab05.104,gab05)
distribute (gab05) # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00) # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01) # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02) # cdb (gab02.106,gab02)
gab03 = impose_rnc (gab03) # cdb (gab03.106,gab03)
```

```
gab04 = impose_rnc (gab04)    # cdb (gab04.106,gab04)  
gab05 = impose_rnc (gab05)    # cdb (gab05.106,gab05)
```



$$\begin{aligned}
\text{gab00.101} &:= g_{ab} \\
\text{gab00.102} &:= g_{ab} \\
\text{gab01.101} &:= g_{cb}\Gamma^c_{ad} + g_{ac}\Gamma^c_{bd} \\
\text{gab01.102} &:= 0
\end{aligned}$$

$$\begin{aligned}
\text{gab02.101} &:= \partial_e (g_{cb}\Gamma^c_{ad} + g_{ac}\Gamma^c_{bd}) \\
\text{gab02.102} &:= \partial_e (g_{cb}\Gamma^c_{ad}) + \partial_e (g_{ac}\Gamma^c_{bd}) \\
\text{gab02.103} &:= \partial_e g_{cb}\Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + \partial_e g_{ac}\Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd} \\
\text{gab02.104} &:= (g_{fb}\Gamma^f_{ce} + g_{cf}\Gamma^f_{be}) \Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + (g_{fc}\Gamma^f_{ae} + g_{af}\Gamma^f_{ce}) \Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd} \\
\text{gab02.105} &:= g_{fb}\Gamma^f_{ce}\Gamma^c_{ad} + g_{cf}\Gamma^f_{be}\Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + g_{fc}\Gamma^f_{ae}\Gamma^c_{bd} + g_{af}\Gamma^f_{ce}\Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd} \\
\text{gab02.106} &:= g_{cb}\partial_e \Gamma^c_{ad} + g_{ac}\partial_e \Gamma^c_{bd}
\end{aligned}$$

$$\begin{aligned}
\text{gab03.101} &:= \partial_f (g_{gb}\Gamma^g_{ce}\Gamma^c_{ad} + g_{cg}\Gamma^g_{be}\Gamma^c_{ad} + g_{cb}\partial_e \Gamma^c_{ad} + g_{gc}\Gamma^g_{ae}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ac}\partial_e \Gamma^c_{bd}) \\
\text{gab03.102} &:= \partial_f (g_{gb}\Gamma^g_{ce}\Gamma^c_{ad}) + \partial_f (g_{cg}\Gamma^g_{be}\Gamma^c_{ad}) + \partial_f (g_{cb}\partial_e \Gamma^c_{ad}) + \partial_f (g_{gc}\Gamma^g_{ae}\Gamma^c_{bd}) + \partial_f (g_{ag}\Gamma^g_{ce}\Gamma^c_{bd}) + \partial_f (g_{ac}\partial_e \Gamma^c_{bd}) \\
\text{gab03.103} &:= \partial_f g_{gb}\Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\partial_f \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\Gamma^g_{ce}\partial_f \Gamma^c_{ad} + \partial_f g_{cg}\Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\partial_f \Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\Gamma^g_{be}\partial_f \Gamma^c_{ad} + \partial_f g_{cb}\partial_e \Gamma^c_{ad} + g_{cb}\partial_{fe}\Gamma^c_{ad} \\
&\quad + \partial_f g_{gc}\Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\partial_f \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\Gamma^g_{ae}\partial_f \Gamma^c_{bd} + \partial_f g_{ag}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\partial_f \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\partial_f \Gamma^c_{bd} + \partial_f g_{ac}\partial_e \Gamma^c_{bd} + g_{ac}\partial_{fe}\Gamma^c_{bd} \\
\text{gab03.104} &:= (g_{hb}\Gamma^h_{gf} + g_{gh}\Gamma^h_{bf}) \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\partial_f \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\Gamma^g_{ce}\partial_f \Gamma^c_{ad} + (g_{hg}\Gamma^h_{cf} + g_{ch}\Gamma^h_{gf}) \Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\partial_f \Gamma^g_{be}\Gamma^c_{ad} \\
&\quad + g_{cg}\Gamma^g_{be}\partial_f \Gamma^c_{ad} + (g_{gb}\Gamma^g_{cf} + g_{cg}\Gamma^g_{bf}) \partial_e \Gamma^c_{ad} + g_{cb}\partial_{fe}\Gamma^c_{ad} + (g_{hc}\Gamma^h_{gf} + g_{gh}\Gamma^h_{cf}) \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\partial_f \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\Gamma^g_{ae}\partial_f \Gamma^c_{bd} \\
&\quad + (g_{hg}\Gamma^h_{af} + g_{ah}\Gamma^h_{gf}) \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\partial_f \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\partial_f \Gamma^c_{bd} + (g_{gc}\Gamma^g_{af} + g_{ag}\Gamma^g_{cf}) \partial_e \Gamma^c_{bd} + g_{ac}\partial_{fe}\Gamma^c_{bd} \\
\text{gab03.105} &:= g_{hb}\Gamma^h_{gf}\Gamma^g_{ce}\Gamma^c_{ad} + g_{gh}\Gamma^h_{bf}\Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\partial_f \Gamma^g_{ce}\Gamma^c_{ad} + g_{gb}\Gamma^g_{ce}\partial_f \Gamma^c_{ad} + g_{hg}\Gamma^h_{cf}\Gamma^g_{be}\Gamma^c_{ad} + g_{ch}\Gamma^h_{gf}\Gamma^g_{be}\Gamma^c_{ad} + g_{cg}\partial_f \Gamma^g_{be}\Gamma^c_{ad} \\
&\quad + g_{cg}\Gamma^g_{be}\partial_f \Gamma^c_{ad} + g_{gb}\Gamma^g_{cf}\partial_e \Gamma^c_{ad} + g_{cg}\Gamma^g_{bf}\partial_e \Gamma^c_{ad} + g_{cb}\partial_{fe}\Gamma^c_{ad} + g_{hc}\Gamma^h_{gf}\Gamma^g_{ae}\Gamma^c_{bd} + g_{gh}\Gamma^h_{cf}\Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\partial_f \Gamma^g_{ae}\Gamma^c_{bd} + g_{gc}\Gamma^g_{ae}\partial_f \Gamma^c_{bd} \\
&\quad + g_{hg}\Gamma^h_{af}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ah}\Gamma^h_{gf}\Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\partial_f \Gamma^g_{ce}\Gamma^c_{bd} + g_{ag}\Gamma^g_{ce}\partial_f \Gamma^c_{bd} + g_{gc}\Gamma^g_{af}\partial_e \Gamma^c_{bd} + g_{ag}\Gamma^g_{cf}\partial_e \Gamma^c_{bd} + g_{ac}\partial_{fe}\Gamma^c_{bd} \\
\text{gab03.106} &:= g_{cb}\partial_{fe}\Gamma^c_{ad} + g_{ac}\partial_{fe}\Gamma^c_{bd}
\end{aligned}$$

$$\begin{aligned} \text{gab04.101} := & \partial_g (g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\partial_f\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\Gamma_{ce}^i\partial_f\Gamma_{ad}^c + g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{ch}\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\partial_f\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\Gamma_{be}^i\partial_f\Gamma_{ad}^c \\ & + g_{ib}\Gamma_{cf}^i\partial_e\Gamma_{ad}^c + g_{ci}\Gamma_{bf}^i\partial_e\Gamma_{ad}^c + g_{cb}\partial_{fe}\Gamma_{ad}^c + g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ic}\partial_f\Gamma_{ae}^i\Gamma_{bd}^c + g_{ic}\Gamma_{ae}^i\partial_f\Gamma_{bd}^c + g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c \\ & + g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\partial_f\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\Gamma_{ce}^i\partial_f\Gamma_{bd}^c + g_{ic}\Gamma_{af}^i\partial_e\Gamma_{bd}^c + g_{ai}\Gamma_{cf}^i\partial_e\Gamma_{bd}^c + g_{ac}\partial_{fe}\Gamma_{bd}^c) \end{aligned}$$

$$\begin{aligned} \text{gab04.102} := & \partial_g (g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c) + \partial_g (g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c) + \partial_g (g_{ib}\partial_f\Gamma_{ce}^i\Gamma_{ad}^c) + \partial_g (g_{ib}\Gamma_{ce}^i\partial_f\Gamma_{ad}^c) + \partial_g (g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c) \\ & + \partial_g (g_{ch}\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c) + \partial_g (g_{ci}\partial_f\Gamma_{be}^i\Gamma_{ad}^c) + \partial_g (g_{ci}\Gamma_{be}^i\partial_f\Gamma_{ad}^c) + \partial_g (g_{ib}\Gamma_{cf}^i\partial_e\Gamma_{ad}^c) + \partial_g (g_{ci}\Gamma_{bf}^i\partial_e\Gamma_{ad}^c) + \partial_g (g_{cb}\partial_{fe}\Gamma_{ad}^c) \\ & + \partial_g (g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c) + \partial_g (g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c) + \partial_g (g_{ic}\partial_f\Gamma_{ae}^i\Gamma_{bd}^c) + \partial_g (g_{ic}\Gamma_{ae}^i\partial_f\Gamma_{bd}^c) + \partial_g (g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c) \\ & + \partial_g (g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c) + \partial_g (g_{ai}\partial_f\Gamma_{ce}^i\Gamma_{bd}^c) + \partial_g (g_{ai}\Gamma_{ce}^i\partial_f\Gamma_{bd}^c) + \partial_g (g_{ic}\Gamma_{af}^i\partial_e\Gamma_{bd}^c) + \partial_g (g_{ai}\Gamma_{cf}^i\partial_e\Gamma_{bd}^c) + \partial_g (g_{ac}\partial_{fe}\Gamma_{bd}^c) \end{aligned}$$

$$\begin{aligned} \text{gab04.103} := & \partial_g g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{hb}\partial_g\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{hb}\Gamma_{if}^h\partial_g\Gamma_{ce}^i\Gamma_{ad}^c + g_{hb}\Gamma_{if}^h\Gamma_{ce}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c + g_{ih}\partial_g\Gamma_{bf}^h\Gamma_{ce}^i\Gamma_{ad}^c \\ & + g_{ih}\Gamma_{bf}^h\partial_g\Gamma_{ce}^i\Gamma_{ad}^c + g_{ih}\Gamma_{bf}^h\Gamma_{ce}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ib}\partial_f\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\partial_{gf}\Gamma_{ce}^i\Gamma_{ad}^c + g_{ib}\partial_f\Gamma_{ce}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ib}\Gamma_{ce}^i\partial_f\Gamma_{ad}^c + g_{ib}\partial_g\Gamma_{ce}^i\partial_f\Gamma_{ad}^c \\ & + g_{ib}\Gamma_{ce}^i\partial_{gf}\Gamma_{ad}^c + \partial_g g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{hi}\partial_g\Gamma_{cf}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{hi}\Gamma_{cf}^h\partial_g\Gamma_{be}^i\Gamma_{ad}^c + g_{hi}\Gamma_{cf}^h\Gamma_{be}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ch}\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c \\ & + g_{ch}\partial_g\Gamma_{if}^h\Gamma_{be}^i\Gamma_{ad}^c + g_{ch}\Gamma_{if}^h\partial_g\Gamma_{be}^i\Gamma_{ad}^c + g_{ch}\Gamma_{if}^h\Gamma_{be}^i\partial_g\Gamma_{ad}^c + \partial_g g_{ci}\partial_f\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\partial_{gf}\Gamma_{be}^i\Gamma_{ad}^c + g_{ci}\partial_f\Gamma_{be}^i\partial_g\Gamma_{ad}^c \\ & + \partial_g g_{ci}\Gamma_{be}^i\partial_f\Gamma_{ad}^c + g_{ci}\partial_g\Gamma_{be}^i\partial_f\Gamma_{ad}^c + g_{ci}\Gamma_{be}^i\partial_{gf}\Gamma_{ad}^c + \partial_g g_{ib}\Gamma_{cf}^i\partial_e\Gamma_{ad}^c + g_{ib}\partial_g\Gamma_{cf}^i\partial_e\Gamma_{ad}^c + g_{ib}\Gamma_{cf}^i\partial_{ge}\Gamma_{ad}^c + \partial_g g_{ci}\Gamma_{bf}^i\partial_e\Gamma_{ad}^c \\ & + g_{ci}\partial_g\Gamma_{bf}^i\partial_e\Gamma_{ad}^c + g_{ci}\Gamma_{bf}^i\partial_{ge}\Gamma_{ad}^c + \partial_g g_{cb}\partial_{fe}\Gamma_{ad}^c + g_{cb}\partial_{ge}\Gamma_{ad}^c + \partial_g g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{hc}\partial_g\Gamma_{if}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{hc}\Gamma_{if}^h\partial_g\Gamma_{ae}^i\Gamma_{bd}^c \\ & + g_{hc}\Gamma_{if}^h\Gamma_{ae}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\partial_g\Gamma_{cf}^h\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\Gamma_{cf}^h\partial_g\Gamma_{ae}^i\Gamma_{bd}^c + g_{ih}\Gamma_{cf}^h\Gamma_{ae}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ic}\partial_f\Gamma_{ae}^i\Gamma_{bd}^c \\ & + g_{ic}\partial_{gf}\Gamma_{ae}^i\Gamma_{bd}^c + g_{ic}\partial_f\Gamma_{ae}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ic}\Gamma_{ae}^i\partial_f\Gamma_{bd}^c + g_{ic}\partial_g\Gamma_{ae}^i\partial_f\Gamma_{bd}^c + g_{ic}\Gamma_{ae}^i\partial_{gf}\Gamma_{bd}^c + \partial_g g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{hi}\partial_g\Gamma_{af}^h\Gamma_{ce}^i\Gamma_{bd}^c \\ & + g_{hi}\Gamma_{af}^h\partial_g\Gamma_{ce}^i\Gamma_{bd}^c + g_{hi}\Gamma_{af}^h\Gamma_{ce}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{ah}\partial_g\Gamma_{if}^h\Gamma_{ce}^i\Gamma_{bd}^c + g_{ah}\Gamma_{if}^h\partial_g\Gamma_{ce}^i\Gamma_{bd}^c + g_{ah}\Gamma_{if}^h\Gamma_{ce}^i\partial_g\Gamma_{bd}^c \\ & + \partial_g g_{ai}\partial_f\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\partial_{gf}\Gamma_{ce}^i\Gamma_{bd}^c + g_{ai}\partial_f\Gamma_{ce}^i\partial_g\Gamma_{bd}^c + \partial_g g_{ai}\Gamma_{ce}^i\partial_f\Gamma_{bd}^c + g_{ai}\partial_g\Gamma_{ce}^i\partial_f\Gamma_{bd}^c + g_{ai}\Gamma_{ce}^i\partial_{gf}\Gamma_{bd}^c + \partial_g g_{ic}\Gamma_{af}^i\partial_e\Gamma_{bd}^c \\ & + g_{ic}\partial_g\Gamma_{af}^i\partial_e\Gamma_{bd}^c + g_{ic}\Gamma_{af}^i\partial_{ge}\Gamma_{bd}^c + \partial_g g_{ai}\Gamma_{cf}^i\partial_e\Gamma_{bd}^c + g_{ai}\partial_g\Gamma_{cf}^i\partial_e\Gamma_{bd}^c + g_{ai}\Gamma_{cf}^i\partial_{ge}\Gamma_{bd}^c + \partial_g g_{ac}\partial_{fe}\Gamma_{bd}^c + g_{ac}\partial_{gfe}\Gamma_{bd}^c \end{aligned}$$



```

# prepare first six terms in the Taylor series expansion of g_{ab}(x)

term0:= @(gab00).
distribute (term0)                # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                # cdb(term5.200,term5)

end_stage_1 = time.time()

```

$$\text{term0.200} := g_{ab}$$

$$\text{term1.200} := 0$$

$$\text{term2.200} := g_{cb} \partial_e \Gamma^c_{ad} A^d A^e + g_{ac} \partial_e \Gamma^c_{bd} A^d A^e$$

$$\text{term3.200} := g_{cb} \partial_{fe} \Gamma^c_{ad} A^d A^e A^f + g_{ac} \partial_{fe} \Gamma^c_{bd} A^d A^e A^f$$

## Stage 2: Replace derivatives of $\Gamma$ with partial derivs of $R$

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json') # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json') # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json') # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json') # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json') # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.301,term2)
substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.302,term2)
distribute (term2) # cdb(term2.303,term2)

substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.301,term3)
substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.302,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.303,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.304,term3)
distribute (term3) # cdb(term3.305,term3)

substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.301,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.302,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.303,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.304,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.305,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.306,term4)
distribute (term4) # cdb(term4.307,term4)

substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.301,term5)
substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.302,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.303,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.304,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.305,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.306,term5)
```

```

substitute (term5,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d}\}}A^{\{c\}}A^{\{b\}} -> @(dGamma01)$,repeat=True) # cdb(term5.307,term5)
substitute (term5,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b}\}}A^{\{c\}}A^{\{b\}} -> @(dGamma01)$,repeat=True) # cdb(term5.308,term5)
distribute (term5) # cdb(term5.309,term5)

end_stage_2 = time.time()

# -----
# this block of Xterms only produces formatted output, it's not part of the main computation
# -----

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
+ (1/1) @(term1) # zero
+ (1/2) @(term2)
+ (1/6) @(term3)
+ (1/24) @(term4)
+ (1/120) @(term5). # cdb(metric.301,metric)

substitute (metric,$A^{\{a\}} -> x^{\{a\}}$) # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1).
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{\{a\}} -> x^{\{a\}}$)
substitute (Xterm1,$A^{\{a\}} -> x^{\{a\}}$)
substitute (Xterm2,$A^{\{a\}} -> x^{\{a\}}$)
substitute (Xterm3,$A^{\{a\}} -> x^{\{a\}}$)
substitute (Xterm4,$A^{\{a\}} -> x^{\{a\}}$)
substitute (Xterm5,$A^{\{a\}} -> x^{\{a\}}$)

substitute (Xterm2,$g_{\{a\}b\}\partial_{\{c\}\{R^{\{b\}}_{\{d\}e\}f\}} -> \partial_{\{c\}\{R_{\{a\}d\}e\}f\}}$) # cdb(Xterm2.301,Xterm2)

```

```

substitute (Xterm3,$g_{a b} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{a b} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{a b} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm5.301,Xterm5)

substitute (Xterm2,$g_{b a} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm2.301,Xterm2)
substitute (Xterm3,$g_{b a} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{b a} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{b a} \partial_{\{c\}}\{R^{\{b\}}_{\{d e f\}}\} \rightarrow \partial_{\{c\}}\{R_{\{a d e f\}}\}) # cdb(Xterm5.301,Xterm5)

eliminate_metric (Xterm2) # cdb(Xterm2.302,Xterm2)
eliminate_metric (Xterm3) # cdb(Xterm3.302,Xterm3)
eliminate_metric (Xterm4) # cdb(Xterm4.302,Xterm4)
eliminate_metric (Xterm5) # cdb(Xterm5.302,Xterm5)

sort_product (Xterm2) # cdb(Xterm2.303,Xterm2)
sort_product (Xterm3) # cdb(Xterm3.303,Xterm3)
sort_product (Xterm4) # cdb(Xterm4.303,Xterm4)
sort_product (Xterm5) # cdb(Xterm5.303,Xterm5)

rename_dummies (Xterm2) # cdb(Xterm2.304,Xterm2)
rename_dummies (Xterm3) # cdb(Xterm3.304,Xterm3)
rename_dummies (Xterm4) # cdb(Xterm4.304,Xterm4)
rename_dummies (Xterm5) # cdb(Xterm5.304,Xterm5)

canonicalise (Xterm2) # cdb(Xterm2.305,Xterm2)
canonicalise (Xterm3) # cdb(Xterm3.305,Xterm3)
canonicalise (Xterm4) # cdb(Xterm4.305,Xterm4)
canonicalise (Xterm5) # cdb(Xterm5.305,Xterm5)

# push upper index to the left
def tidy_Rabcd (obj):
    substitute (obj,$R_{a b c}^{\{d\}} \rightarrow - R^{\{d\}}_{\{c a b\}})
    substitute (obj,$R_{a b}^{\{c\}}_{\{d\}} \rightarrow R^{\{c\}}_{\{d a b\}})
    substitute (obj,$R_{a}^{\{b\}}_{\{c d\}} \rightarrow - R^{\{b\}}_{\{a c d\}})
    return obj

Xterm0 = tidy_Rabcd (Xterm0) # cdb(Xterm0.666,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2) # cdb(Xterm2.666,Xterm2)

```

```

Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.666,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.666,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.666,Xterm5)

Xterm0 = reformat_xterm (Xterm0, 1)    # cdb(Xterm0.301,Xterm0)
Xterm2 = reformat_xterm (Xterm2, 3)    # cdb(Xterm2.301,Xterm2)
Xterm3 = reformat_xterm (Xterm3, 6)    # cdb(Xterm3.301,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)   # cdb(Xterm4.301,Xterm4)
Xterm5 = reformat_xterm (Xterm5,180)   # cdb(Xterm5.301,Xterm5)

# canonicalise from reformat_xterm will slide upper index from left hand side
# so now we slide the upper index back to the left

Xterm0 = tidy_Rabcd (Xterm0)  # cdb(Xterm0.667,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2)  # cdb(Xterm2.667,Xterm2)
Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.667,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.667,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.667,Xterm5)

# metric to 3rd, 4th, 5th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).      # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3).  # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).  # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (Metric6.301,Metric6)

# -----
# end of format block
# -----

```



$$\text{term2.301} := g_{cb}\partial_e\Gamma^c_{ad}A^dA^e + g_{ac}\partial_e\Gamma^c_{bd}A^dA^e$$

$$\text{term2.302} := \frac{1}{3}g_{cb}A^dA^eR^c_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c_{deb}$$

$$\text{term2.303} := \frac{1}{3}g_{cb}A^dA^eR^c_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c_{deb}$$

$$\text{term3.301} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.302} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.303} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.304} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\text{term3.305} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c_{dfb}$$

$$\begin{aligned} \text{term4.301} := & g_{ib}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i_{be}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{ci}\partial_g\Gamma^i_{be}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i_{bf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfa}\right) + g_{ic}\partial_f\Gamma^i_{ae}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ic}\partial_g\Gamma^i_{ae}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i_{af}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ai}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.302} := & g_{ib}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i_{be}\partial_g\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{ci}\partial_g\Gamma^i_{be}\partial_f\Gamma^c_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i_{bf}\partial_e\Gamma^c_{ad}A^dA^eA^fA^g \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfa}\right) + g_{ic}\partial_f\Gamma^i_{ae}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ic}\partial_g\Gamma^i_{ae}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i_{ce}\partial_g\Gamma^c_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i_{ce}\partial_f\Gamma^c_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i_{af}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g \\ & + g_{ai}\partial_g\Gamma^i_{cf}\partial_e\Gamma^c_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.306} := & \frac{1}{9}g_{ib}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i{}_{efb}A^dA^gR^c{}_{dga} \\ & + \frac{1}{9}g_{ci}A^eA^gR^i{}_{egb}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ib}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i{}_{fgb}A^dA^eR^c{}_{dea} \\ & + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + \frac{1}{9}g_{ic}A^eA^fR^i{}_{efa}A^dA^gR^c{}_{dgb} \\ & + \frac{1}{9}g_{ic}A^eA^gR^i{}_{ega}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ai}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i{}_{fga}A^dA^eR^c{}_{deb} \\ & + \frac{1}{9}g_{ai}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{deb} + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right) \end{aligned}$$

$$\begin{aligned}
\text{term4.307} := & \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
& + \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^fA^e\partial_{ef}R^c_{dga} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} \\
& - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
& + \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
& + \frac{3}{5}g_{ac}A^dA^gA^fA^e\partial_{ef}R^c_{dgb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}
\end{aligned}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f (-3R_{bcdg}R^g_{fae} - 13R_{acdg}R^g_{fbe} - 9g_{bg}\partial_{cd}R^g_{fae} - 9g_{ag}\partial_{cd}R^g_{fbe})$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f (-3R_{bcdg}R^g_{fae} - 13R_{acdg}R^g_{fbe} - 9g_{bg}\partial_{cd}R^g_{fae} - 9g_{ag}\partial_{cd}R^g_{fbe}) \\ + \frac{1}{180}x^c x^d x^e x^f x^g (-3R^h_{dac}\partial_e R_{bfg h} - R_{bcdh}\partial_e R^h_{gaf} - 3R^h_{dbc}\partial_e R_{afgh} - g_{bh}\partial_{cde}R^h_{gaf} - R_{acd h}\partial_e R^h_{gbf} - g_{ah}\partial_{cde}R^h_{gbf})$$

### Stage 3: Replace partial derivs of $R$ with covariant derivs of $R$

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json') # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json') # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json') # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)    # cdb(term1.401,term1)
sort_product      (term1)    # cdb(term1.402,term1)
rename_dummies    (term1)    # cdb(term1.403,term1)
canonicalise      (term1)    # cdb(term1.404,term1)

eliminate_metric (term2)    # cdb(term2.401,term2)
sort_product      (term2)    # cdb(term2.402,term2)
rename_dummies    (term2)    # cdb(term2.403,term2)
canonicalise      (term2)    # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term3.402,term3)
distribute (term3)                                                # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c d b}\} ->  @(tmp02)$,repeat=True) # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True) # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term4.404,term4)
distribute (term4)                                                # cdb(term4.405,term4)
```

```

substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c d b}\} -> @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c b d}\} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()

```

$$\text{tmp01.403} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\text{tmp02.403} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\text{tmp03.403} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfg h} g^{ah}$$

$$\text{term1.401} := 0$$

$$\text{term1.402} := 0$$

$$\text{term1.403} := 0$$

$$\text{term1.404} := 0$$

$$\text{term2.401} := \frac{1}{3}A^d A^e R_{bdea} + \frac{1}{3}A^d A^e R_{adeb}$$

$$\text{term2.402} := \frac{1}{3}A^d A^e R_{bdea} + \frac{1}{3}A^d A^e R_{adeb}$$

$$\text{term2.403} := \frac{1}{3}A^c A^d R_{bcd a} + \frac{1}{3}A^c A^d R_{acdb}$$

$$\text{term2.404} := -\frac{2}{3}A^c A^d R_{acbd}$$

$$\text{term3.401} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\text{term3.402} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\text{term3.403} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\begin{aligned} \text{term4.401} := & \frac{1}{9}g_{ib}A^e A^f R^i_{efc}A^d A^g R^c_{dga} + \frac{1}{9}g_{ib}A^e A^g R^i_{egc}A^d A^f R^c_{dfa} + \frac{1}{9}g_{ci}A^e A^f R^i_{efb}A^d A^g R^c_{dga} + \frac{1}{9}g_{ci}A^e A^g R^i_{egb}A^d A^f R^c_{dfa} \\ & + \frac{1}{9}g_{ib}A^f A^g R^i_{fgc}A^d A^e R^c_{dea} + \frac{1}{9}g_{ci}A^f A^g R^i_{fgb}A^d A^e R^c_{dea} + \frac{3}{5}g_{cb}A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} \\ & - \frac{1}{15}g_{cb}A^d A^g A^f A^e R^c_{gfh} R^h_{dea} - \frac{1}{15}g_{cb}A^d A^g A^f A^e R^c_{geh} R^h_{dfa} + \frac{1}{9}g_{ic}A^e A^f R^i_{efa}A^d A^g R^c_{dgb} + \frac{1}{9}g_{ic}A^e A^g R^i_{ega}A^d A^f R^c_{dfb} \\ & + \frac{1}{9}g_{ai}A^e A^f R^i_{efc}A^d A^g R^c_{dgb} + \frac{1}{9}g_{ai}A^e A^g R^i_{egc}A^d A^f R^c_{dfb} + \frac{1}{9}g_{ic}A^f A^g R^i_{fga}A^d A^e R^c_{deb} + \frac{1}{9}g_{ai}A^f A^g R^i_{fgc}A^d A^e R^c_{deb} \\ & + \frac{3}{5}g_{ac}A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15}g_{ac}A^d A^g A^f A^e R^c_{gfh} R^h_{deb} - \frac{1}{15}g_{ac}A^d A^g A^f A^e R^c_{geh} R^h_{dfb} \end{aligned}$$

$$\begin{aligned}
\text{term4.402} &:= \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
&+ \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
&- \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
&+ \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
&+ \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb} \\
\text{term4.403} &:= \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
&+ \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
&- \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
&+ \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
&+ \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb} \\
\text{term4.404} &:= \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
&+ \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
&- \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
&+ \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
&+ \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}
\end{aligned}$$



$$\begin{aligned}
\text{term4.405} := & \frac{1}{9}g_{ib}A^eA^fR^i_{efc}A^dA^gR^c_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i_{egc}A^dA^fR^c_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i_{efb}A^dA^gR^c_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i_{egb}A^dA^fR^c_{dfa} \\
& + \frac{1}{9}g_{ib}A^fA^gR^i_{fgc}A^dA^eR^c_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i_{fgb}A^dA^eR^c_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\
& - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{gfh}R^h_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c_{geh}R^h_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i_{efa}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i_{ega}A^dA^fR^c_{dfb} \\
& + \frac{1}{9}g_{ai}A^eA^fR^i_{efc}A^dA^gR^c_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i_{egc}A^dA^fR^c_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i_{fga}A^dA^eR^c_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i_{fgc}A^dA^eR^c_{deb} \\
& + \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{gfh}R^h_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c_{geh}R^h_{dfb}
\end{aligned}$$

## Stage 4: Build the Taylor series for $g_{ab}$ , reformatting and output

```
beg_stage_4 = time.time()
# final housekeeping

term1 = flatten_Rabcd (term1)          # cdb(term1.501,term1)
term2 = flatten_Rabcd (term2)          # cdb(term2.501,term2)
term3 = flatten_Rabcd (term3)          # cdb(term3.501,term3)
term4 = flatten_Rabcd (term4)          # cdb(term4.501,term4)
term5 = flatten_Rabcd (term5)          # cdb(term5.501,term5)

eliminate_metric (term1)
eliminate_metric (term2)
eliminate_metric (term3)
eliminate_metric (term4)
eliminate_metric (term5)

eliminate_kronecker (term1)
eliminate_kronecker (term2)
eliminate_kronecker (term3)
eliminate_kronecker (term4)
eliminate_kronecker (term5)

sort_product (term1)
sort_product (term2)
sort_product (term3)
sort_product (term4)
sort_product (term5)

rename_dummies (term1)
rename_dummies (term2)
rename_dummies (term3)
rename_dummies (term4)
rename_dummies (term5)

canonicalise (term1)                   # cdb(term1.502,term1)
canonicalise (term2)                   # cdb(term2.502,term2)
canonicalise (term3)                   # cdb(term3.502,term3)
```

```

canonicalise (term4)          # cdb(term4.502,term4)
canonicalise (term5)          # cdb(term5.502,term5)

# this is out final answer

metric:=@(term0)
  + (1/1) @(term1)
  + (1/2) @(term2)
  + (1/6) @(term3)
  + (1/24) @(term4)
  + (1/120) @(term5).          # cdb(metric.501,metric)

substitute (metric,$A^{a} -> x^{a}$) # cdb (metric.502,metric)

cdblib.create ('metric.json')

cdblib.put ('g_ab',metric,'metric.json')

# extract the terms of the metric in powers of x

term0 = get_xterm (metric,0)    # cdb(term0.503,term0)
term1 = get_xterm (metric,1)    # cdb(term1.503,term1)
term2 = get_xterm (metric,2)    # cdb(term2.503,term2)
term3 = get_xterm (metric,3)    # cdb(term3.503,term3)
term4 = get_xterm (metric,4)    # cdb(term4.503,term4)
term5 = get_xterm (metric,5)    # cdb(term5.503,term5)

cdblib.put ('g_ab_0',term0,'metric.json')
cdblib.put ('g_ab_1',term1,'metric.json')
cdblib.put ('g_ab_2',term2,'metric.json')
cdblib.put ('g_ab_3',term3,'metric.json')
cdblib.put ('g_ab_4',term4,'metric.json')
cdblib.put ('g_ab_5',term5,'metric.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3). # cdb(metric4.501,metric4)

```

$$\text{term2.501} := -\frac{2}{3}A^c A^d R_{acbd}$$

$$\text{term2.502} := -\frac{2}{3}A^c A^d R_{acbd}$$

$$\text{term3.501} := \frac{1}{2}g_{cb}A^d A^f A^e \nabla_d R_{afeg} g^{cg} + \frac{1}{2}g_{ac}A^d A^f A^e \nabla_d R_{bfeg} g^{cg}$$

$$\text{term3.502} := -A^c A^d A^e \nabla_c R_{adbe}$$

$$\begin{aligned} \text{term4.501} := & \frac{1}{9}g_{ib}A^e A^f g^{ih} R_{hefc}A^d A^g g^{cj} R_{jdga} + \frac{1}{9}g_{ib}A^e A^g g^{ih} R_{hegc}A^d A^f g^{cj} R_{jdfa} + \frac{1}{9}g_{ci}A^e A^f g^{ih} R_{hefb}A^d A^g g^{cj} R_{jdga} \\ & + \frac{1}{9}g_{ci}A^e A^g g^{ih} R_{hegb}A^d A^f g^{cj} R_{jdfa} + \frac{1}{9}g_{ib}A^f A^g g^{ih} R_{hfgc}A^d A^e g^{cj} R_{jdea} + \frac{1}{9}g_{ci}A^f A^g g^{ih} R_{hfgb}A^d A^e g^{cj} R_{jdea} \\ & + \frac{3}{5}g_{cb}A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} - \frac{1}{15}g_{cb}A^d A^g A^f A^e g^{ci} R_{igfh} g^{hj} R_{jdea} - \frac{1}{15}g_{cb}A^d A^g A^f A^e g^{ci} R_{igeh} g^{hj} R_{jdfa} \\ & + \frac{1}{9}g_{ic}A^e A^f g^{ih} R_{hefa}A^d A^g g^{cj} R_{jdg b} + \frac{1}{9}g_{ic}A^e A^g g^{ih} R_{hega}A^d A^f g^{cj} R_{jdf b} + \frac{1}{9}g_{ai}A^e A^f g^{ih} R_{hefc}A^d A^g g^{cj} R_{jdg b} \\ & + \frac{1}{9}g_{ai}A^e A^g g^{ih} R_{hegc}A^d A^f g^{cj} R_{jdf b} + \frac{1}{9}g_{ic}A^f A^g g^{ih} R_{hfga}A^d A^e g^{cj} R_{jdeb} + \frac{1}{9}g_{ai}A^f A^g g^{ih} R_{hfgc}A^d A^e g^{cj} R_{jdeb} \\ & + \frac{3}{5}g_{ac}A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15}g_{ac}A^d A^g A^f A^e g^{ci} R_{igfh} g^{hj} R_{jdeb} - \frac{1}{15}g_{ac}A^d A^g A^f A^e g^{ci} R_{igeh} g^{hj} R_{jd f b} \end{aligned}$$

$$\text{term4.502} := \frac{16}{15}A^c A^d A^e A^f R_{acd g} R_{befh} g^{gh} - \frac{6}{5}A^c A^d A^e A^f \nabla_{cd} R_{aebf}$$

$$\begin{aligned}
\text{term5.501} := & \frac{1}{6}g_{ib}A^eA^gA^f\nabla_eR_{cgfj}g^{ij}A^dA^hg^{ck}R_{kdha} + \frac{1}{6}g_{ib}A^eA^hA^f\nabla_eR_{chfj}g^{ij}A^dA^gA^cR_{kdga} + \frac{1}{6}g_{ib}A^eA^fA^gR_{kefc}A^dA^hA^g\nabla_dR_{ahgj}g^{cj} \\
& + \frac{1}{6}g_{ib}A^eA^hA^g\nabla_eR_{chgj}g^{ij}A^dA^fA^cR_{kdfa} + \frac{1}{6}g_{ib}A^eA^gA^hR_{kegc}A^dA^hA^f\nabla_dR_{ahfj}g^{cj} + \frac{1}{6}g_{ib}A^eA^hA^gR_{kehc}A^dA^gA^f\nabla_dR_{agfj}g^{cj} \\
& + \frac{1}{6}g_{ci}A^eA^gA^f\nabla_eR_{bgfj}g^{ij}A^dA^hA^cR_{kdha} + \frac{1}{6}g_{ci}A^eA^hA^f\nabla_eR_{bhfj}g^{ij}A^dA^gA^cR_{kdga} + \frac{1}{6}g_{ci}A^eA^fA^gR_{kefb}A^dA^hA^g\nabla_dR_{ahgj}g^{cj} \\
& + \frac{1}{6}g_{ci}A^eA^hA^g\nabla_eR_{bhgj}g^{ij}A^dA^fA^cR_{kdfa} + \frac{1}{6}g_{ci}A^eA^gA^hR_{kegb}A^dA^hA^f\nabla_dR_{ahfj}g^{cj} + \frac{1}{6}g_{ci}A^eA^hA^gR_{kehb}A^dA^gA^f\nabla_dR_{agfj}g^{cj} \\
& + \frac{1}{6}g_{ib}A^fA^hA^g\nabla_fR_{chgj}g^{ij}A^dA^eA^cR_{kdea} + \frac{1}{6}g_{ib}A^fA^gA^hR_{kfgc}A^dA^hA^e\nabla_dR_{ahej}g^{cj} + \frac{1}{6}g_{ib}A^fA^hA^gR_{kfhc}A^dA^gA^e\nabla_dR_{agej}g^{cj} \\
& + \frac{1}{6}g_{ci}A^fA^hA^g\nabla_fR_{bhgj}g^{ij}A^dA^eA^cR_{kdea} + \frac{1}{6}g_{ci}A^fA^gA^hR_{kfgb}A^dA^hA^e\nabla_dR_{ahej}g^{cj} + \frac{1}{6}g_{ci}A^fA^hA^gR_{kfhb}A^dA^gA^e\nabla_dR_{agej}g^{cj} \\
& + \frac{1}{6}g_{kb}A^gA^hA^cR_{jghc}A^dA^fA^e\nabla_dR_{afei}g^{ci} + \frac{1}{6}g_{ck}A^gA^hA^cR_{jghb}A^dA^fA^e\nabla_dR_{afei}g^{ci} - \frac{1}{3}g_{cb}A^dA^hA^eA^fA^gR_{adhk}\nabla_eR_{figj}g^{ci}g^{kj} \\
& + \frac{1}{3}g_{cb}A^dA^hA^eA^fA^gR_{dkhi}\nabla_eR_{afgj}g^{ck}g^{ij} + \frac{2}{3}g_{cb}A^dA^hA^eA^fA^g\nabla_{dhe}R_{afgk}g^{ck} - \frac{1}{9}g_{cb}A^dA^hA^fA^g\nabla_hR_{ifgj}g^{cj}A^eA^gR_{kdea} \\
& - \frac{1}{9}g_{cb}A^dA^hA^eA^g\nabla_hR_{iegj}g^{cj}A^fA^gR_{kdfa} - \frac{1}{9}g_{cb}A^dA^eA^f\nabla_dR_{aefj}g^{ij}A^hA^gA^cR_{khgi} - \frac{1}{9}g_{cb}A^dA^hA^eA^f\nabla_hR_{iefj}g^{cj}A^gA^hR_{kdga} \\
& - \frac{1}{9}g_{cb}A^dA^eA^g\nabla_dR_{aegj}g^{ij}A^hA^fA^cR_{khfi} - \frac{1}{9}g_{cb}A^dA^fA^g\nabla_dR_{afgj}g^{ij}A^hA^eA^cR_{khei} + \frac{1}{6}g_{ic}A^eA^gA^f\nabla_eR_{agfj}g^{ij}A^dA^hA^cR_{kdhb} \\
& + \frac{1}{6}g_{ic}A^eA^hA^f\nabla_eR_{ahfj}g^{ij}A^dA^gA^cR_{kdgb} + \frac{1}{6}g_{ic}A^eA^fA^gR_{kefa}A^dA^hA^g\nabla_dR_{bhgj}g^{cj} + \frac{1}{6}g_{ic}A^eA^hA^g\nabla_eR_{ahgj}g^{ij}A^dA^fA^cR_{kdfb} \\
& + \frac{1}{6}g_{ic}A^eA^gA^hR_{kega}A^dA^hA^f\nabla_dR_{bhfj}g^{cj} + \frac{1}{6}g_{ic}A^eA^hA^gR_{keha}A^dA^gA^f\nabla_dR_{bgfj}g^{cj} + \frac{1}{6}g_{ai}A^eA^gA^f\nabla_eR_{cgfj}g^{ij}A^dA^hA^cR_{kdhb} \\
& + \frac{1}{6}g_{ai}A^eA^hA^f\nabla_eR_{chfj}g^{ij}A^dA^gA^cR_{kdgb} + \frac{1}{6}g_{ai}A^eA^fA^gR_{kefc}A^dA^hA^g\nabla_dR_{bhgj}g^{cj} + \frac{1}{6}g_{ai}A^eA^hA^g\nabla_eR_{chgj}g^{ij}A^dA^fA^cR_{kdfb} \\
& + \frac{1}{6}g_{ai}A^eA^gA^hR_{kegc}A^dA^hA^f\nabla_dR_{bhfj}g^{cj} + \frac{1}{6}g_{ai}A^eA^hA^gR_{kehc}A^dA^gA^f\nabla_dR_{bgfj}g^{cj} + \frac{1}{6}g_{ic}A^fA^hA^g\nabla_fR_{ahgj}g^{ij}A^dA^eA^cR_{kdeb} \\
& + \frac{1}{6}g_{ic}A^fA^gA^hR_{kfga}A^dA^hA^e\nabla_dR_{bhej}g^{cj} + \frac{1}{6}g_{ic}A^fA^hA^gR_{kfha}A^dA^gA^e\nabla_dR_{bgej}g^{cj} + \frac{1}{6}g_{ai}A^fA^hA^g\nabla_fR_{chgj}g^{ij}A^dA^eA^cR_{kdeb} \\
& + \frac{1}{6}g_{ai}A^fA^gA^hR_{kfgc}A^dA^hA^e\nabla_dR_{bhej}g^{cj} + \frac{1}{6}g_{ai}A^fA^hA^gR_{kfhc}A^dA^gA^e\nabla_dR_{bgej}g^{cj} + \frac{1}{6}g_{kc}A^gA^hA^cR_{jgha}A^dA^fA^e\nabla_dR_{bf ei}g^{ci} \\
& + \frac{1}{6}g_{ak}A^gA^hA^cR_{jghc}A^dA^fA^e\nabla_dR_{bf ei}g^{ci} - \frac{1}{3}g_{ac}A^dA^hA^eA^fA^gR_{bdhk}\nabla_eR_{figj}g^{ci}g^{kj} + \frac{1}{3}g_{ac}A^dA^hA^eA^fA^gR_{dkhi}\nabla_eR_{bf gj}g^{ck}g^{ij} \\
& + \frac{2}{3}g_{ac}A^dA^hA^eA^fA^g\nabla_{dhe}R_{bf gk}g^{ck} - \frac{1}{9}g_{ac}A^dA^hA^fA^g\nabla_hR_{ifgj}g^{cj}A^eA^gR_{kdeb} - \frac{1}{9}g_{ac}A^dA^hA^eA^g\nabla_hR_{iegj}g^{cj}A^fA^gR_{kdfb} \\
& - \frac{1}{9}g_{ac}A^dA^eA^f\nabla_dR_{befj}g^{ij}A^hA^gA^cR_{khgi} - \frac{1}{9}g_{ac}A^dA^hA^eA^f\nabla_hR_{iefj}g^{cj}A^gA^hR_{kdgb} \\
& - \frac{1}{9}g_{ac}A^dA^eA^g\nabla_dR_{begj}g^{ij}A^hA^fA^cR_{khfi} - \frac{1}{9}g_{ac}A^dA^fA^g\nabla_dR_{bf gj}g^{ij}A^hA^eA^cR_{khei}
\end{aligned}$$

$$\text{term5.502} := \frac{8}{3} A^c A^d A^e A^f A^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{8}{3} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{4}{3} A^c A^d A^e A^f A^g \nabla_{cde} R_{afbg}$$

$$\begin{aligned}
\text{metric.501} &:= g_{ab} - \frac{1}{3}A^cA^dR_{acbd} - \frac{1}{6}A^cA^dA^e\nabla_cR_{adbe} + \frac{2}{45}A^cA^dA^eA^fR_{acd g}R_{befh}g^{gh} - \frac{1}{20}A^cA^dA^eA^f\nabla_{cd}R_{aebf} \\
&\quad + \frac{1}{45}A^cA^dA^eA^fA^gR_{acd h}\nabla_eR_{bfgi}g^{hi} + \frac{1}{45}A^cA^dA^eA^fA^gR_{bcd h}\nabla_eR_{afgi}g^{hi} - \frac{1}{90}A^cA^dA^eA^fA^g\nabla_{cde}R_{afbg} \\
\text{metric.502} &:= g_{ab} - \frac{1}{3}x^cx^dR_{acbd} - \frac{1}{6}x^cx^dx^e\nabla_cR_{adbe} + \frac{2}{45}x^cx^dx^ex^fR_{acd g}R_{befh}g^{gh} - \frac{1}{20}x^cx^dx^ex^f\nabla_{cd}R_{aebf} \\
&\quad + \frac{1}{45}x^cx^dx^ex^fx^gR_{acd h}\nabla_eR_{bfgi}g^{hi} + \frac{1}{45}x^cx^dx^ex^fx^gR_{bcd h}\nabla_eR_{afgi}g^{hi} - \frac{1}{90}x^cx^dx^ex^fx^g\nabla_{cde}R_{afbg}
\end{aligned}$$

$$\text{term0.503} := g_{ab}$$

$$\text{term1.503} := 0$$

$$\text{term2.503} := -\frac{1}{3}x^c x^d R_{acbd}$$

$$\text{term3.503} := -\frac{1}{6}x^c x^d x^e \nabla_c R_{adb e}$$

$$\text{term4.503} := \frac{2}{45}x^c x^d x^e x^f R_{acd g} R_{b e f h} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{a e b f}$$

$$\text{term5.503} := \frac{1}{45}x^c x^d x^e x^f x^g R_{acd h} \nabla_e R_{b f g i} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcd h} \nabla_e R_{a f g i} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{a f b g}$$



```

Xterm0 := @(term0).
Xterm1 := @(term1). # zero
Xterm2 := @(term2).
Xterm3 := @(term3).
Xterm4 := @(term4).
Xterm5 := @(term5).

Xterm0 = reformat_xterm (Xterm0, 1) # cdb(Xterm0.601,Xterm0)
Xterm2 = reformat_xterm (Xterm2, 3) # cdb(Xterm2.601,Xterm2)
Xterm3 = reformat_xterm (Xterm3, 6) # cdb(Xterm3.601,Xterm3)
Xterm4 = reformat_xterm (Xterm4,180) # cdb(Xterm4.601,Xterm4)
Xterm5 = reformat_xterm (Xterm5, 90) # cdb(Xterm5.601,Xterm5)

gab3 := @(Xterm0) + @(Xterm2). # cdb (gab3.601,gab3)
gab4 := @(Xterm0) + @(Xterm2) + @(Xterm3). # cdb (gab4.601,gab4)
gab5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4). # cdb (gab5.601,gab5)
gab6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5). # cdb (gab6.601,gab6)

Metric := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5). # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Xterm0, 1) # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Xterm2, 3) # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Xterm3, 6) # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Xterm4,180) # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Xterm5, 90) # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()

```

## The metric in Riemann normal coordinates

$$\begin{aligned} g_{ab}(x) = & g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f (8g^{gh} R_{acd g} R_{b e f h} - 9\nabla_{cd} R_{a e b f}) \\ & + \frac{1}{90}x^c x^d x^e x^f x^g (2g^{hi} R_{acd h} \nabla_e R_{b f g i} + 2g^{hi} R_{bcd h} \nabla_e R_{a f g i} - \nabla_{cde} R_{a f b g}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

## Curvature expansion of the metric

$$g_{ab}(x) = g_{ab}^0 + g_{ab}^2 + g_{ab}^3 + g_{ab}^4 + g_{ab}^5 + \mathcal{O}(\epsilon^6)$$

$$g_{ab}^0 = g_{ab}$$

$$3g_{ab}^2 = -x^c x^d R_{acbd}$$

$$6g_{ab}^3 = -x^c x^d x^e \nabla_c R_{adbe}$$

$$180g_{ab}^4 = x^c x^d x^e x^f (8g^{gh} R_{acd} R_{befh} - 9\nabla_{cd} R_{aebf})$$

$$90g_{ab}^5 = x^c x^d x^e x^f x^g (2g^{hi} R_{acd} \nabla_e R_{bfgi} + 2g^{hi} R_{bcd} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg})$$

```

cdblib.create ('metric.export')

cdblib.put ('g_ab_3',Metric3,'metric.export')  # R and \partial R
cdblib.put ('g_ab_4',Metric4,'metric.export')
cdblib.put ('g_ab_5',Metric5,'metric.export')
cdblib.put ('g_ab_6',Metric6,'metric.export')

cdblib.put ('g_ab', Metric, 'metric.export')  # R and \nabla R

cdblib.put ('g_ab_scaled0',scaled0,'metric.export')
cdblib.put ('g_ab_scaled2',scaled2,'metric.export')
cdblib.put ('g_ab_scaled3',scaled3,'metric.export')
cdblib.put ('g_ab_scaled4',scaled4,'metric.export')
cdblib.put ('g_ab_scaled5',scaled5,'metric.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)

```

# Timing

Stage 1: 3.3 secs

Stage 2: 1.2 secs

Stage 3: 97.4 secs

Stage 4: 3.5 secs

# The inverse metric tensor in Riemann normal coordinates

Here we calculate the Riemann normal expansion of the inverse metric,  $g^{ab}$ , by developing the recursive sequences

$$g^{ab}{}_{,d\underline{e}} = -\left(g^{cb}\Gamma^a{}_{c(d),\underline{e}}\right) - \left(g^{ac}\Gamma^b{}_{c(d),\underline{e}}\right) \quad (1)$$

$$(n+3)\Gamma^a{}_{d(b,c\underline{e})} = (n+1)\left(R^a{}_{(bcd,\underline{e})} - \left(\Gamma^a{}_{f(c}\Gamma^f{}_{bd),\underline{e}}\right)\right) \quad (2)$$

for  $n = 1, 2, 3, \dots$ . Note in these equations that the (extended) index  $\underline{e}$  contains  $n$  normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g^{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^c + \frac{1}{2!}g^{ab}{}_{,cd}x^cx^d + \frac{1}{3!}g^{ab}{}_{,cde}x^cx^dx^e + \dots \\ &= g^{ab} + \sum_{n=1}^{\infty} \frac{1}{n!} g^{ab}{}_{,\underline{c}} x^{\underline{c}} \end{aligned}$$

## Stage 1: Symmetrised partial derivatives of $g^{ab}$

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g^{ab}{}_{,c}A^c &= 0 \\ g^{ab}{}_{,cd}A^cA^d &= -g^{cb}\partial_e\Gamma^a{}_{cd}A^dA^e - g^{ac}\partial_e\Gamma^b{}_{cd}A^dA^e \\ g^{ab}{}_{,cde}A^cA^dA^e &= -g^{cb}\partial_{fe}\Gamma^a{}_{cd}A^dA^eA^f - g^{ac}\partial_{fe}\Gamma^b{}_{cd}A^dA^eA^f \end{aligned}$$

## Stage 2: Replace derivatives of $\Gamma$ with partial derivs of $R$

Now we use the results from `dGamma` to replace derivatives of  $\Gamma$  with partial derivatives of  $R$ . These were computed in `dGamma` using equation (2) above.

$$\begin{aligned}
g^{ab}{}_{,c}A^c &= 0 \\
g^{ab}{}_{,cd}A^cA^d &= -\frac{1}{3}g^{cb}A^dA^eR^a{}_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b{}_{dec} \\
g^{ab}{}_{,cde}A^cA^dA^e &= -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a{}_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b{}_{dfc}
\end{aligned}$$

### Stage 3: Replace partial derivs of $R$ with covariant derivs of $R$

Next we use the results from `dRabcd` to replace the partial derivatives of  $R$  with covariant derivatives.

$$\begin{aligned}
g^{ab}{}_{,c}A^c &= 0 \\
g^{ab}{}_{,cd}A^cA^d &= -\frac{1}{3}A^cA^dR^a{}_{cd}{}^b - \frac{1}{3}A^cA^dR^b{}_{cd}{}^a \\
g^{ab}{}_{,cde}A^cA^dA^e &= -\frac{1}{2}g^{cb}A^dA^fA^e\nabla_dR_{cfe}g^{ag} - \frac{1}{2}g^{ac}A^dA^fA^e\nabla_dR_{cfe}g^{bg}
\end{aligned}$$

### Stage 4: Build the Taylor series for $g_{ab}$ , reformatting and output

Each of the above expressions constitutes one term in the Taylor series for the metric. We also make the trivial change  $A \rightarrow x$ . Then we do some trivial reformatting.

$$\begin{aligned}
g_{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^c + \frac{1}{2!}g^{ab}{}_{,cd}x^cx^d + \frac{1}{3!}g^{ab}{}_{,cde}x^cx^dx^e + \mathcal{O}(\epsilon^4) \\
&= g^{ab} + \frac{1}{3}x^cx^dR_{cedf}g^{ae}g^{bf} + \frac{1}{6}x^cx^dx^e\nabla_cR_{dfe}g^{af}g^{bg} + \mathcal{O}(\epsilon^4)
\end{aligned}$$

# Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R_{a}_{b c d} -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d} -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    sort_product (obj)
    rename_dummies (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}\{\Gamma^{a}_{b c}\} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e}\{\Gamma^{a}_{b c}\} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f}\{\Gamma^{a}_{b c}\} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{e f g h}\{\Gamma^{a}_{b c}\} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{e}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{e f}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{e f g}\{\Gamma^{a}_{b c}\}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{e f g h}\{\Gamma^{a}_{b c}\}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @ (obj).
    bah = Ex("numx = " + str(n))
    keep_weight (foo,bah)
```



```

return foo

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                                -> A001^{a}                $)
    substitute (obj,$ x^{a}                                -> A002^{a}                $)
    substitute (obj,$ g_{a b}                               -> A003_{a b}              $)
    substitute (obj,$ g^{a b}                               -> A004^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\}        -> A010_{a b c d e f g h}  $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}           -> A009_{a b c d e f g}    $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}              -> A008_{a b c d e f}      $)
    substitute (obj,$ \nabla_{e}\{R_{a b c d}\}                -> A007_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}\{R_{a b c d}\}       -> A014_{a b c d e f g h}  $)
    substitute (obj,$ \partial_{e f g}\{R_{a b c d}\}          -> A013_{a b c d e f g}    $)
    substitute (obj,$ \partial_{e f}\{R_{a b c d}\}             -> A012_{a b c d e f}      $)
    substitute (obj,$ \partial_{e}\{R_{a b c d}\}               -> A011_{a b c d e}        $)
    substitute (obj,$ \partial_{e f g h}\{R^{a}_{a}_{b c d}\}    -> A018^{a}_{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}\{R^{a}_{a}_{b c d}\}       -> A017^{a}_{a}_{b c d e f g}  $)
    substitute (obj,$ \partial_{e f}\{R^{a}_{a}_{b c d}\}         -> A016^{a}_{a}_{b c d e f}    $)
    substitute (obj,$ \partial_{e}\{R^{a}_{a}_{b c d}\}           -> A015^{a}_{a}_{b c d e}      $)
    substitute (obj,$ R_{a b c d}                           -> A005_{a b c d}          $)
    substitute (obj,$ R^{a}_{a}_{b c d}                       -> A006^{a}_{a}_{b c d}      $)
    sort_product      (obj)
    rename_dummies    (obj)
    substitute (obj,$ A001^{a}                               -> A^{a}                    $)
    substitute (obj,$ A002^{a}                               -> x^{a}                    $)
    substitute (obj,$ A003_{a b}                             -> g_{a b}                  $)
    substitute (obj,$ A004^{a b}                             -> g^{a b}                  $)
    substitute (obj,$ A005_{a b c d}                         -> R_{a b c d}              $)
    substitute (obj,$ A006^{a}_{a}_{b c d}                   -> R^{a}_{a}_{b c d}        $)
    substitute (obj,$ A007_{a b c d e}                       -> \nabla_{e}\{R_{a b c d}\}  $)
    substitute (obj,$ A008_{a b c d e f}                     -> \nabla_{e f}\{R_{a b c d}\} $)
    substitute (obj,$ A009_{a b c d e f g}                   -> \nabla_{e f g}\{R_{a b c d}\} $)
    substitute (obj,$ A010_{a b c d e f g h}                 -> \nabla_{e f g h}\{R_{a b c d}\} $)
    substitute (obj,$ A011_{a b c d e}                       -> \partial_{e}\{R_{a b c d}\}  $)
    substitute (obj,$ A012_{a b c d e f}                     -> \partial_{e f}\{R_{a b c d}\}  $)
    substitute (obj,$ A013_{a b c d e f g}                   -> \partial_{e f g}\{R_{a b c d}\} $)

```

```

substitute (obj,$ A014_{a b c d e f g h} -> \partial_{e f g h}\{R_{a b c d}\} $)
substitute (obj,$ A015^{\{a\}}_{\{b c d e\}} -> \partial_{e}\{R^{\{a\}}_{\{b c d\}} $)
substitute (obj,$ A016^{\{a\}}_{\{b c d e f\}} -> \partial_{e f}\{R^{\{a\}}_{\{b c d\}} $)
substitute (obj,$ A017^{\{a\}}_{\{b c d e f g\}} -> \partial_{e f g}\{R^{\{a\}}_{\{b c d\}} $)
substitute (obj,$ A018^{\{a\}}_{\{b c d e f g h\}} -> \partial_{e f g h}\{R^{\{a\}}_{\{b c d\}} $)

return obj

def reformat_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    factor_out (bah,$x^{\{a?\}}$)
    ans := @(bah) / @(foo).
    return ans

def rescale_xterm (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    factor_out (bah,$x^{\{a?\}}$)
    return bah

def add_tags (obj,tag):
    n = 0
    ans = Ex('0')
    for i in obj.top().terms():
        foo = obj[i]
        bah = Ex(tag+'_{'+str(n)+'}')
        ans := @(ans) + @(bah) @(foo).
        n = n + 1
    return ans

def clear_tags (obj,tag):
    ans := @(obj).

```

```

foo = Ex(tag+'_{a?} -> 1')
substitute (ans,foo)
return ans

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

```

## Stage 1: Symmetrised partial derivatives of $g^{ab}$

```

beg_stage_1 = time.time()

# symmetrised partial derivatives of  $g^{ab}$ 

gab00:=g^{a b}. # cdb (gab00.101,gab00)

gab01:= - g^{c b}\Gamma^{a}_{c d} - g^{a c}\Gamma^{b}_{c d}. # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @(gab01) }. # cdb (gab02.101,gab02)
distribute (gab02) # cdb (gab02.102,gab02)
product_rule (gab02) # cdb (gab02.103,gab02)
substitute (gab02, $\partial_{d}\{g^{a b}\} \rightarrow @(gab01)$) # cdb (gab02.104,gab02)
distribute (gab02) # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @(gab02) }. # cdb (gab03.101,gab03)
distribute (gab03) # cdb (gab03.102,gab03)
product_rule (gab03) # cdb (gab03.103,gab03)
substitute (gab03, $\partial_{d}\{g^{a b}\} \rightarrow @(gab01)$) # cdb (gab03.104,gab03)
distribute (gab03) # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @(gab03) }. # cdb (gab04.101,gab04)
distribute (gab04) # cdb (gab04.102,gab04)
product_rule (gab04) # cdb (gab04.103,gab04)
substitute (gab04, $\partial_{d}\{g^{a b}\} \rightarrow @(gab01)$) # cdb (gab04.104,gab04)
distribute (gab04) # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @(gab04) }. # cdb (gab05.101,gab05)
distribute (gab05) # cdb (gab05.102,gab05)
product_rule (gab05) # cdb (gab05.103,gab05)
substitute (gab05, $\partial_{d}\{g^{a b}\} \rightarrow @(gab01)$) # cdb (gab05.104,gab05)
distribute (gab05) # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00) # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01) # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02) # cdb (gab02.106,gab02)

```

```
gab03 = impose_rnc (gab03)  # cdb (gab03.106,gab03)
gab04 = impose_rnc (gab04)  # cdb (gab04.106,gab04)
gab05 = impose_rnc (gab05)  # cdb (gab05.106,gab05)
```

$$\text{gab00.101} := g^{ab}$$

$$\text{gab00.102} := g^{ab}$$

$$\text{gab01.101} := -g^{cb}\Gamma_{cd}^a - g^{ac}\Gamma_{cd}^b$$

$$\text{gab01.102} := 0$$

$$\text{gab02.101} := \partial_e (-g^{cb}\Gamma_{cd}^a - g^{ac}\Gamma_{cd}^b)$$

$$\text{gab02.102} := -\partial_e (g^{cb}\Gamma_{cd}^a) - \partial_e (g^{ac}\Gamma_{cd}^b)$$

$$\text{gab02.103} := -\partial_e g^{cb}\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a - \partial_e g^{ac}\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab02.104} := -(-g^{fb}\Gamma_{fe}^c - g^{cf}\Gamma_{fe}^b)\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a - (-g^{fc}\Gamma_{fe}^a - g^{af}\Gamma_{fe}^c)\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab02.105} := g^{fb}\Gamma_{fe}^c\Gamma_{cd}^a + g^{cf}\Gamma_{fe}^b\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a + g^{fc}\Gamma_{fe}^a\Gamma_{cd}^b + g^{af}\Gamma_{fe}^c\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab02.106} := -g^{cb}\partial_e \Gamma_{cd}^a - g^{ac}\partial_e \Gamma_{cd}^b$$

$$\text{gab03.101} := \partial_f (g^{gb}\Gamma_{ge}^c\Gamma_{cd}^a + g^{cg}\Gamma_{ge}^b\Gamma_{cd}^a - g^{cb}\partial_e \Gamma_{cd}^a + g^{gc}\Gamma_{ge}^a\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\Gamma_{cd}^b - g^{ac}\partial_e \Gamma_{cd}^b)$$

$$\text{gab03.102} := \partial_f (g^{gb}\Gamma_{ge}^c\Gamma_{cd}^a) + \partial_f (g^{cg}\Gamma_{ge}^b\Gamma_{cd}^a) - \partial_f (g^{cb}\partial_e \Gamma_{cd}^a) + \partial_f (g^{gc}\Gamma_{ge}^a\Gamma_{cd}^b) + \partial_f (g^{ag}\Gamma_{ge}^c\Gamma_{cd}^b) - \partial_f (g^{ac}\partial_e \Gamma_{cd}^b)$$

$$\begin{aligned} \text{gab03.103} := & \partial_f g^{gb}\Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\partial_f \Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\Gamma_{ge}^c\partial_f \Gamma_{cd}^a + \partial_f g^{cg}\Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\partial_f \Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\Gamma_{ge}^b\partial_f \Gamma_{cd}^a - \partial_f g^{cb}\partial_e \Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a \\ & + \partial_f g^{gc}\Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\partial_f \Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\Gamma_{ge}^a\partial_f \Gamma_{cd}^b + \partial_f g^{ag}\Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\partial_f \Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\partial_f \Gamma_{cd}^b - \partial_f g^{ac}\partial_e \Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \end{aligned}$$

$$\begin{aligned} \text{gab03.104} := & (-g^{hb}\Gamma_{hf}^g - g^{gh}\Gamma_{hf}^b)\Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\partial_f \Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\Gamma_{ge}^c\partial_f \Gamma_{cd}^a + (-g^{hg}\Gamma_{hf}^c - g^{ch}\Gamma_{hf}^g)\Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\partial_f \Gamma_{ge}^b\Gamma_{cd}^a \\ & + g^{cg}\Gamma_{ge}^b\partial_f \Gamma_{cd}^a - (-g^{gb}\Gamma_{gf}^c - g^{cg}\Gamma_{gf}^b)\partial_e \Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a + (-g^{hc}\Gamma_{hf}^g - g^{gh}\Gamma_{hf}^c)\Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\partial_f \Gamma_{ge}^a\Gamma_{cd}^b \\ & + g^{gc}\Gamma_{ge}^a\partial_f \Gamma_{cd}^b + (-g^{hg}\Gamma_{hf}^a - g^{ah}\Gamma_{hf}^g)\Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\partial_f \Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\partial_f \Gamma_{cd}^b - (-g^{gc}\Gamma_{gf}^a - g^{ag}\Gamma_{gf}^c)\partial_e \Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \end{aligned}$$

$$\begin{aligned} \text{gab03.105} := & -g^{hb}\Gamma_{hf}^g\Gamma_{ge}^c\Gamma_{cd}^a - g^{gh}\Gamma_{hf}^b\Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\partial_f \Gamma_{ge}^c\Gamma_{cd}^a + g^{gb}\Gamma_{ge}^c\partial_f \Gamma_{cd}^a - g^{hg}\Gamma_{hf}^c\Gamma_{ge}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^g\Gamma_{ge}^b\Gamma_{cd}^a + g^{cg}\partial_f \Gamma_{ge}^b\Gamma_{cd}^a \\ & + g^{cg}\Gamma_{ge}^b\partial_f \Gamma_{cd}^a + g^{gb}\Gamma_{gf}^c\partial_e \Gamma_{cd}^a + g^{cg}\Gamma_{gf}^b\partial_e \Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{hc}\Gamma_{hf}^g\Gamma_{ge}^a\Gamma_{cd}^b - g^{gh}\Gamma_{hf}^c\Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\partial_f \Gamma_{ge}^a\Gamma_{cd}^b + g^{gc}\Gamma_{ge}^a\partial_f \Gamma_{cd}^b \\ & - g^{hg}\Gamma_{hf}^a\Gamma_{ge}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^g\Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\partial_f \Gamma_{ge}^c\Gamma_{cd}^b + g^{ag}\Gamma_{ge}^c\partial_f \Gamma_{cd}^b + g^{gc}\Gamma_{gf}^a\partial_e \Gamma_{cd}^b + g^{ag}\Gamma_{gf}^c\partial_e \Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \end{aligned}$$

$$\text{gab03.106} := -g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{ac}\partial_{fe}\Gamma_{cd}^b$$

$$\begin{aligned} \text{gab04.101} := & \partial_g \left( -g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\partial_f\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\Gamma_{ie}^c\partial_f\Gamma_{cd}^a - g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a + g^{ci}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \right. \\ & + g^{ci}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ib}\Gamma_{if}^c\partial_e\Gamma_{cd}^a + g^{ci}\Gamma_{if}^b\partial_e\Gamma_{cd}^a - g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a + g^{ic}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a + g^{ic}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a \\ & \left. - g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\partial_f\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\Gamma_{ie}^c\partial_f\Gamma_{cd}^b + g^{ic}\Gamma_{if}^a\partial_e\Gamma_{cd}^b + g^{ai}\Gamma_{if}^c\partial_e\Gamma_{cd}^b - g^{ac}\partial_{fe}\Gamma_{cd}^b \right) \end{aligned}$$

$$\begin{aligned} \text{gab04.102} := & -\partial_g \left( g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a \right) - \partial_g \left( g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a \right) + \partial_g \left( g^{ib}\partial_f\Gamma_{ie}^c\Gamma_{cd}^a \right) + \partial_g \left( g^{ib}\Gamma_{ie}^c\partial_f\Gamma_{cd}^a \right) - \partial_g \left( g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a \right) \\ & - \partial_g \left( g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ci}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ci}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a \right) + \partial_g \left( g^{ib}\Gamma_{if}^c\partial_e\Gamma_{cd}^a \right) + \partial_g \left( g^{ci}\Gamma_{if}^b\partial_e\Gamma_{cd}^a \right) - \partial_g \left( g^{cb}\partial_{fe}\Gamma_{cd}^a \right) \\ & - \partial_g \left( g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a \right) - \partial_g \left( g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ic}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \right) + \partial_g \left( g^{ic}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a \right) - \partial_g \left( g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b \right) \\ & - \partial_g \left( g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b \right) + \partial_g \left( g^{ai}\partial_f\Gamma_{ie}^c\Gamma_{cd}^b \right) + \partial_g \left( g^{ai}\Gamma_{ie}^c\partial_f\Gamma_{cd}^b \right) + \partial_g \left( g^{ic}\Gamma_{if}^a\partial_e\Gamma_{cd}^b \right) + \partial_g \left( g^{ai}\Gamma_{if}^c\partial_e\Gamma_{cd}^b \right) - \partial_g \left( g^{ac}\partial_{fe}\Gamma_{cd}^b \right) \end{aligned}$$

$$\begin{aligned} \text{gab04.103} := & -\partial_g g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a - g^{hb}\partial_g\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^a - g^{hb}\Gamma_{hf}^i\partial_g\Gamma_{ie}^c\Gamma_{cd}^a - g^{hb}\Gamma_{hf}^i\Gamma_{ie}^c\partial_g\Gamma_{cd}^a - \partial_g g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a - g^{ih}\partial_g\Gamma_{hf}^b\Gamma_{ie}^c\Gamma_{cd}^a \\ & - g^{ih}\Gamma_{hf}^b\partial_g\Gamma_{ie}^c\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^b\Gamma_{ie}^c\partial_g\Gamma_{cd}^a + \partial_g g^{ib}\partial_f\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\partial_{gf}\Gamma_{ie}^c\Gamma_{cd}^a + g^{ib}\partial_f\Gamma_{ie}^c\partial_g\Gamma_{cd}^a + \partial_g g^{ib}\Gamma_{ie}^c\partial_f\Gamma_{cd}^a + g^{ib}\partial_g\Gamma_{ie}^c\partial_f\Gamma_{cd}^a \\ & + g^{ib}\Gamma_{ie}^c\partial_{gf}\Gamma_{cd}^a - \partial_g g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{hi}\partial_g\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{hi}\Gamma_{hf}^c\partial_g\Gamma_{ie}^b\Gamma_{cd}^a - g^{hi}\Gamma_{hf}^c\Gamma_{ie}^b\partial_g\Gamma_{cd}^a - \partial_g g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a \\ & - g^{ch}\partial_g\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^i\partial_g\Gamma_{ie}^b\Gamma_{cd}^a - g^{ch}\Gamma_{hf}^i\Gamma_{ie}^b\partial_g\Gamma_{cd}^a + \partial_g g^{ci}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a + g^{ci}\partial_{gf}\Gamma_{ie}^b\Gamma_{cd}^a + g^{ci}\partial_f\Gamma_{ie}^b\partial_g\Gamma_{cd}^a \\ & + \partial_g g^{ci}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ci}\partial_g\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ci}\Gamma_{ie}^b\partial_{gf}\Gamma_{cd}^a + \partial_g g^{ib}\Gamma_{if}^c\partial_e\Gamma_{cd}^a + g^{ib}\partial_g\Gamma_{if}^c\partial_e\Gamma_{cd}^a + g^{ib}\Gamma_{if}^c\partial_{ge}\Gamma_{cd}^a + \partial_g g^{ci}\Gamma_{if}^b\partial_e\Gamma_{cd}^a \\ & + g^{ci}\partial_g\Gamma_{if}^b\partial_e\Gamma_{cd}^a + g^{ci}\Gamma_{if}^b\partial_{ge}\Gamma_{cd}^a - \partial_g g^{cb}\partial_{fe}\Gamma_{cd}^a - g^{cb}\partial_{gfe}\Gamma_{cd}^a - \partial_g g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{hc}\partial_g\Gamma_{hf}^i\Gamma_{ie}^b\Gamma_{cd}^a - g^{hc}\Gamma_{hf}^i\partial_g\Gamma_{ie}^b\Gamma_{cd}^a \\ & - g^{hc}\Gamma_{hf}^i\Gamma_{ie}^b\partial_g\Gamma_{cd}^a - \partial_g g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\partial_g\Gamma_{hf}^c\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^c\partial_g\Gamma_{ie}^b\Gamma_{cd}^a - g^{ih}\Gamma_{hf}^c\Gamma_{ie}^b\partial_g\Gamma_{cd}^a + \partial_g g^{ic}\partial_f\Gamma_{ie}^b\Gamma_{cd}^a \\ & + g^{ic}\partial_{gf}\Gamma_{ie}^b\Gamma_{cd}^a + g^{ic}\partial_f\Gamma_{ie}^b\partial_g\Gamma_{cd}^a + \partial_g g^{ic}\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ic}\partial_g\Gamma_{ie}^b\partial_f\Gamma_{cd}^a + g^{ic}\Gamma_{ie}^b\partial_{gf}\Gamma_{cd}^a - \partial_g g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b - g^{hi}\partial_g\Gamma_{hf}^a\Gamma_{ie}^c\Gamma_{cd}^b \\ & - g^{hi}\Gamma_{hf}^a\partial_g\Gamma_{ie}^c\Gamma_{cd}^b - g^{hi}\Gamma_{hf}^a\Gamma_{ie}^c\partial_g\Gamma_{cd}^b - \partial_g g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\partial_g\Gamma_{hf}^i\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^i\partial_g\Gamma_{ie}^c\Gamma_{cd}^b - g^{ah}\Gamma_{hf}^i\Gamma_{ie}^c\partial_g\Gamma_{cd}^b \\ & + \partial_g g^{ai}\partial_f\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\partial_{gf}\Gamma_{ie}^c\Gamma_{cd}^b + g^{ai}\partial_f\Gamma_{ie}^c\partial_g\Gamma_{cd}^b + \partial_g g^{ai}\Gamma_{ie}^c\partial_f\Gamma_{cd}^b + g^{ai}\partial_g\Gamma_{ie}^c\partial_f\Gamma_{cd}^b + g^{ai}\Gamma_{ie}^c\partial_{gf}\Gamma_{cd}^b + \partial_g g^{ic}\Gamma_{if}^a\partial_e\Gamma_{cd}^b \\ & + g^{ic}\partial_g\Gamma_{if}^a\partial_e\Gamma_{cd}^b + g^{ic}\Gamma_{if}^a\partial_{ge}\Gamma_{cd}^b + \partial_g g^{ai}\Gamma_{if}^c\partial_e\Gamma_{cd}^b + g^{ai}\partial_g\Gamma_{if}^c\partial_e\Gamma_{cd}^b + g^{ai}\Gamma_{if}^c\partial_{ge}\Gamma_{cd}^b - \partial_g g^{ac}\partial_{fe}\Gamma_{cd}^b - g^{ac}\partial_{gfe}\Gamma_{cd}^b \end{aligned}$$

[illegible]

[illegible]



$$\begin{aligned} \text{gab04.106} := & g^{ib} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^a + g^{ib} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^a + g^{ci} \partial_f \Gamma_{ie}^b \partial_g \Gamma_{cd}^a + g^{ci} \partial_g \Gamma_{ie}^b \partial_f \Gamma_{cd}^a + g^{ib} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^a + g^{ci} \partial_g \Gamma_{if}^b \partial_e \Gamma_{cd}^a - g^{cb} \partial_{gfe} \Gamma_{cd}^a \\ & + g^{ic} \partial_f \Gamma_{ie}^a \partial_g \Gamma_{cd}^b + g^{ic} \partial_g \Gamma_{ie}^a \partial_f \Gamma_{cd}^b + g^{ai} \partial_f \Gamma_{ie}^c \partial_g \Gamma_{cd}^b + g^{ai} \partial_g \Gamma_{ie}^c \partial_f \Gamma_{cd}^b + g^{ic} \partial_g \Gamma_{if}^a \partial_e \Gamma_{cd}^b + g^{ai} \partial_g \Gamma_{if}^c \partial_e \Gamma_{cd}^b - g^{ac} \partial_{gfe} \Gamma_{cd}^b \end{aligned}$$

*# prepare first six terms in the Taylor series expansion of  $g^{\{ab\}}(x)$*

```
term0:= @(gab00).
distribute (term0)                # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                # cdb(term5.200,term5)

end_stage_1 = time.time()
```

$$\text{term0.200} := g^{ab}$$

$$\text{term1.200} := 0$$

$$\text{term2.200} := -g^{cb} \partial_e \Gamma_{cd}^a A^d A^e - g^{ac} \partial_e \Gamma_{cd}^b A^d A^e$$

$$\text{term3.200} := -g^{cb} \partial_{fe} \Gamma_{cd}^a A^d A^e A^f - g^{ac} \partial_{fe} \Gamma_{cd}^b A^d A^e A^f$$

## Stage 2: Replace derivatives of $\Gamma$ with partial derivs of $R$

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json') # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json') # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json') # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json') # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json') # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.301,term2)
substitute (term2,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term2.302,term2)
distribute (term2) # cdb(term2.303,term2)

substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.301,term3)
substitute (term3,$\partial_{\{c\}e\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term3.302,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.303,term3)
substitute (term3,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term3.304,term3)
distribute (term3) # cdb(term3.305,term3)

substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.301,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term4.302,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.303,term4)
substitute (term4,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term4.304,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.305,term4)
substitute (term4,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term4.306,term4)
distribute (term4) # cdb(term4.307,term4)

substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.301,term5)
substitute (term5,$\partial_{\{c\}e\{f\}g\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}}A^{\{g\}} \rightarrow @(dGamma04)$,repeat=True) # cdb(term5.302,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.303,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}}A^{\{f\}} \rightarrow @(dGamma03)$,repeat=True) # cdb(term5.304,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.305,term5)
substitute (term5,$\partial_{\{c\}e\{f\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}}A^{\{e\}} \rightarrow @(dGamma02)$,repeat=True) # cdb(term5.306,term5)
```

```

substitute (term5,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{b\}d\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term5.307,term5)
substitute (term5,$\partial_{\{c\}\{\Gamma^{\{a\}}_{\{d\}b\}}A^{\{c\}}A^{\{b\}} \rightarrow @(dGamma01)$,repeat=True) # cdb(term5.308,term5)
distribute (term5) # cdb(term5.309,term5)

# -----
# this block only produces formatted output, it is not part of the main computation
# -----

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
+ (1/1) @(term1)
+ (1/2) @(term2)
+ (1/6) @(term3)
+ (1/24) @(term4)
+ (1/120) @(term5). # cdb(metric.301,metric)

substitute (metric,$A^{\{a\}} \rightarrow x^{\{a\}}$) # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1). # zero
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm1,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm2,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm3,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm4,$A^{\{a\}} \rightarrow x^{\{a\}}$)
substitute (Xterm5,$A^{\{a\}} \rightarrow x^{\{a\}}$)

# Manipulating these expressions is hampered by the presence of the partial derivative on Rabcd.
# Thus we can't freely raise/lower indices on the dRabcd terms. But we can do so on the first
# derivatives (since these are evaluated at x=0 where the connection vanishes).

```

```

substitute      (Xterm2,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm2.301,Xterm2)
substitute      (Xterm3,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm3.301,Xterm3)
substitute      (Xterm4,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm4.301,Xterm4)
substitute      (Xterm5,$g^{a b} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm5.301,Xterm5)

substitute      (Xterm2,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm2.302,Xterm2)
substitute      (Xterm3,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm3.302,Xterm3)
substitute      (Xterm4,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm4.302,Xterm4)
substitute      (Xterm5,$g^{b a} R^{c}_{d e b} -> R^{c}_{d e}^{a})$) # cdb(Xterm5.302,Xterm5)

substitute      (Xterm2,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm2.303,Xterm2)
substitute      (Xterm3,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm3.303,Xterm3)
substitute      (Xterm4,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm4.303,Xterm4)
substitute      (Xterm5,$g^{a b} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm5.303,Xterm5)

substitute      (Xterm2,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm2.304,Xterm2)
substitute      (Xterm3,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm3.304,Xterm3)
substitute      (Xterm4,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm4.304,Xterm4)
substitute      (Xterm5,$g^{b a} \partial_{c}\{R^{d}_{e f b}\} -> \partial_{c}\{R^{d}_{e f}^{a}\})$) # cdb(Xterm5.304,Xterm5)

sort_product    (Xterm2) # cdb(Xterm2.305,Xterm2)
sort_product    (Xterm3) # cdb(Xterm3.305,Xterm3)
sort_product    (Xterm4) # cdb(Xterm4.305,Xterm4)
sort_product    (Xterm5) # cdb(Xterm5.305,Xterm5)

rename_dummies  (Xterm2) # cdb(Xterm2.306,Xterm2)
rename_dummies  (Xterm3) # cdb(Xterm3.306,Xterm3)
rename_dummies  (Xterm4) # cdb(Xterm4.306,Xterm4)
rename_dummies  (Xterm5) # cdb(Xterm5.306,Xterm5)

canonicalise    (Xterm2) # cdb(Xterm2.307,Xterm2)
canonicalise    (Xterm3) # cdb(Xterm3.307,Xterm3)
canonicalise    (Xterm4) # cdb(Xterm4.307,Xterm4)
canonicalise    (Xterm5) # cdb(Xterm5.307,Xterm5)

```

# We can simplify Xterm2 and Xterm3 by careful juggling of the indices (swapping free indices on selected terms)

```

tmp = add_tags (Xterm2, '\\mu')      # cdb (tmp.001,tmp)
zoom (tmp, $\\mu_{1} Q??)$          # cdb (tmp.002,tmp)
substitute (tmp, $R^{b}_{c d}^{a} x^{c} x^{d} \to R^{a}_{c d}^{b} x^{c} x^{d}$) # cdb (tmp.003,tmp)
unzoom (tmp)
Xterm2 = clear_tags (tmp, '\\mu')    # cdb (Xterm2.401,Xterm2)

tmp = add_tags (Xterm3, '\\mu')      # cdb (tmp.011,tmp)
zoom (tmp, $\\mu_{1} Q??)$          # cdb (tmp.012,tmp)
substitute (tmp, $\\partial_{c} \{R^{b}_{d e}^{a}\} x^{c} x^{d} x^{e} \to \\partial_{c} \{R^{a}_{d e}^{b}\} x^{c} x^{d} x^{e}$) # cdb (tmp.013,
unzoom (tmp)
Xterm3 = clear_tags (tmp, '\\mu')    # cdb (Xterm3.401,Xterm3)

Xterm0 = reformat_xterm (Xterm0, 1)   # cdb(Xterm0.308,Xterm0)
Xterm2 = reformat_xterm (Xterm2, 3)   # cdb(Xterm2.308,Xterm2)
Xterm3 = reformat_xterm (Xterm3, 6)   # cdb(Xterm3.308,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)  # cdb(Xterm4.308,Xterm4)
Xterm5 = reformat_xterm (Xterm5,360)  # cdb(Xterm5.308,Xterm5)

# metric to 4th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).      # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3). # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4). # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5). # cdb (Metric6.301,Metric6)

# -----
# end of format block
# -----

end_stage_2 = time.time()

```

$$\text{term2.301} := -g^{cb}\partial_e\Gamma^a_{cd}A^dA^e - g^{ac}\partial_e\Gamma^b_{cd}A^dA^e$$

$$\text{term2.302} := -\frac{1}{3}g^{cb}A^dA^eR^a_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b_{dec}$$

$$\text{term2.303} := -\frac{1}{3}g^{cb}A^dA^eR^a_{dec} - \frac{1}{3}g^{ac}A^dA^eR^b_{dec}$$

$$\text{term3.301} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b_{dfc}$$

$$\text{term3.302} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b_{dfc}$$

$$\text{term3.303} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b_{dfc}$$

$$\text{term3.304} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b_{dfc}$$

$$\text{term3.305} := -\frac{1}{2}g^{cb}A^eA^dA^f\partial_eR^a_{dfc} - \frac{1}{2}g^{ac}A^eA^dA^f\partial_eR^b_{dfc}$$

$$\begin{aligned} \text{term4.301} := & g^{ib}\partial_f\Gamma^c_{ie}\partial_g\Gamma^a_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c_{ie}\partial_f\Gamma^a_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b_{ie}\partial_g\Gamma^a_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b_{ie}\partial_f\Gamma^a_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c_{if}\partial_e\Gamma^a_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b_{if}\partial_e\Gamma^a_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a_{geh}R^h_{dfc}\right) + g^{ic}\partial_f\Gamma^a_{ie}\partial_g\Gamma^b_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a_{ie}\partial_f\Gamma^b_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c_{ie}\partial_g\Gamma^b_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c_{ie}\partial_f\Gamma^b_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a_{if}\partial_e\Gamma^b_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c_{if}\partial_e\Gamma^b_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.302} := & g^{ib}\partial_f\Gamma^c_{ie}\partial_g\Gamma^a_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c_{ie}\partial_f\Gamma^a_{cd}A^dA^eA^fA^g + g^{ci}\partial_f\Gamma^b_{ie}\partial_g\Gamma^a_{cd}A^dA^eA^fA^g \\ & + g^{ci}\partial_g\Gamma^b_{ie}\partial_f\Gamma^a_{cd}A^dA^eA^fA^g + g^{ib}\partial_g\Gamma^c_{if}\partial_e\Gamma^a_{cd}A^dA^eA^fA^g + g^{ci}\partial_g\Gamma^b_{if}\partial_e\Gamma^a_{cd}A^dA^eA^fA^g \\ & - g^{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^a_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} - \frac{1}{15}A^dA^gA^fA^eR^a_{geh}R^h_{dfc}\right) + g^{ic}\partial_f\Gamma^a_{ie}\partial_g\Gamma^b_{cd}A^dA^eA^fA^g \\ & + g^{ic}\partial_g\Gamma^a_{ie}\partial_f\Gamma^b_{cd}A^dA^eA^fA^g + g^{ai}\partial_f\Gamma^c_{ie}\partial_g\Gamma^b_{cd}A^dA^eA^fA^g + g^{ai}\partial_g\Gamma^c_{ie}\partial_f\Gamma^b_{cd}A^dA^eA^fA^g + g^{ic}\partial_g\Gamma^a_{if}\partial_e\Gamma^b_{cd}A^dA^eA^fA^g \\ & + g^{ai}\partial_g\Gamma^c_{if}\partial_e\Gamma^b_{cd}A^dA^eA^fA^g - g^{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^b_{dgc} - \frac{1}{15}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} - \frac{1}{15}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}\right) \end{aligned}$$

$$\begin{aligned} \text{term4.306} := & \frac{1}{9} g^{ib} A^e A^f R^c_{efi} A^d A^g R^a_{dgc} + \frac{1}{9} g^{ib} A^e A^g R^c_{egi} A^d A^f R^a_{dfc} + \frac{1}{9} g^{ci} A^e A^f R^b_{efi} A^d A^g R^a_{dgc} \\ & + \frac{1}{9} g^{ci} A^e A^g R^b_{egi} A^d A^f R^a_{dfc} + \frac{1}{9} g^{ib} A^f A^g R^c_{fgi} A^d A^e R^a_{dec} + \frac{1}{9} g^{ci} A^f A^g R^b_{fgi} A^d A^e R^a_{dec} \\ & - g^{cb} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^a_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^a_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^a_{geh} R^h_{dfc} \right) + \frac{1}{9} g^{ic} A^e A^f R^a_{efi} A^d A^g R^b_{dgc} \\ & + \frac{1}{9} g^{ic} A^e A^g R^a_{egi} A^d A^f R^b_{dfc} + \frac{1}{9} g^{ai} A^e A^f R^c_{efi} A^d A^g R^b_{dgc} + \frac{1}{9} g^{ai} A^e A^g R^c_{egi} A^d A^f R^b_{dfc} + \frac{1}{9} g^{ic} A^f A^g R^a_{fgi} A^d A^e R^b_{dec} \\ & + \frac{1}{9} g^{ai} A^f A^g R^c_{fgi} A^d A^e R^b_{dec} - g^{ac} \left( \frac{3}{5} A^d A^g A^f A^e \partial_{ef} R^b_{dgc} - \frac{1}{15} A^d A^g A^f A^e R^b_{gfh} R^h_{dec} - \frac{1}{15} A^d A^g A^f A^e R^b_{geh} R^h_{dfc} \right) \end{aligned}$$

$$\begin{aligned}
\text{term4.307} := & \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
& + \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^fA^e\partial_{ef}R^a_{dgc} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} \\
& + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
& + \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
& - \frac{3}{5}g^{ac}A^dA^gA^fA^e\partial_{ef}R^b_{dgc} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}
\end{aligned}$$



$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f (7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg})$$

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \frac{1}{360}x^c x^d x^e x^f (7R^a{}_{cdg}R^g{}_{ef}{}^b + 10R^a{}_{cdg}R^b{}_{ef}{}^g - 9g^{bg}\partial_{cd}R^a{}_{efg} + 7R^b{}_{cdg}R^g{}_{ef}{}^a - 9g^{ag}\partial_{cd}R^b{}_{efg})$$

$$+ \frac{1}{360}x^c x^d x^e x^f x^g (3R^a{}_{cdh}\partial_e R^h{}_{fg}{}^b + 4\partial_c R^a{}_{deh}R^h{}_{fg}{}^b + 5\partial_c R^b{}_{deh}R^a{}_{fg}{}^h + 5\partial_c R^a{}_{deh}R^b{}_{fg}{}^h - 2g^{bh}\partial_{cde}R^a{}_{fgh} + 3R^b{}_{cdh}\partial_e R^h{}_{fg}{}^a + 4\partial_c R^b{}_{deh}R^h{}_{fg}{}^a$$

$$- 2g^{ah}\partial_{cde}R^b{}_{fgh})$$

## Stage 3: Replace partial derivs of $R$ with covariant derivs of $R$

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json') # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json') # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json') # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)    # cdb(term1.401,term1)
sort_product      (term1)    # cdb(term1.402,term1)
rename_dummies    (term1)    # cdb(term1.403,term1)
canonicalise      (term1)    # cdb(term1.404,term1)

eliminate_metric (term2)    # cdb(term2.401,term2)
sort_product      (term2)    # cdb(term2.402,term2)
rename_dummies    (term2)    # cdb(term2.403,term2)
canonicalise      (term2)    # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term3.402,term3)
distribute (term3)                                                # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c d b}\} ->  @(tmp02)$,repeat=True) # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True) # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} ->  @(tmp01)$,repeat=True)      # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)      # cdb(term4.404,term4)
distribute (term4)                                                # cdb(term4.405,term4)
```

```

substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c d b}\} -> @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c b d}\} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} -> @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c b d}\} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} -> @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c b d}\} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()

```

$$\text{tmp01.403} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\text{tmp02.403} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\text{tmp03.403} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfg h} g^{ah}$$

$$\text{term1.401} := 0$$

$$\text{term1.402} := 0$$

$$\text{term1.403} := 0$$

$$\text{term1.404} := 0$$

$$\text{term2.401} := -\frac{1}{3}A^d A^e R^a{}_{de}{}^b - \frac{1}{3}A^d A^e R^b{}_{de}{}^a$$

$$\text{term2.402} := -\frac{1}{3}A^d A^e R^a{}_{de}{}^b - \frac{1}{3}A^d A^e R^b{}_{de}{}^a$$

$$\text{term2.403} := -\frac{1}{3}A^c A^d R^a{}_{cd}{}^b - \frac{1}{3}A^c A^d R^b{}_{cd}{}^a$$

$$\text{term2.404} := -\frac{1}{3}A^c A^d R^a{}_{cd}{}^b - \frac{1}{3}A^c A^d R^b{}_{cd}{}^a$$

$$\text{term3.401} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfe} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfe} g^{bg}$$

$$\text{term3.402} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfe} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfe} g^{bg}$$

$$\text{term3.403} := -\frac{1}{2}g^{cb}A^d A^f A^e \nabla_d R_{cfe} g^{ag} - \frac{1}{2}g^{ac}A^d A^f A^e \nabla_d R_{cfe} g^{bg}$$

$$\begin{aligned} \text{term4.401} := & \frac{1}{9}g^{ib}A^e A^f R^c{}_{efi}A^d A^g R^a{}_{dgc} + \frac{1}{9}g^{ib}A^e A^g R^c{}_{egi}A^d A^f R^a{}_{dfc} + \frac{1}{9}g^{ci}A^e A^f R^b{}_{efi}A^d A^g R^a{}_{dgc} + \frac{1}{9}g^{ci}A^e A^g R^b{}_{egi}A^d A^f R^a{}_{dfc} \\ & + \frac{1}{9}g^{ib}A^f A^g R^c{}_{fgi}A^d A^e R^a{}_{dec} + \frac{1}{9}g^{ci}A^f A^g R^b{}_{fgi}A^d A^e R^a{}_{dec} - \frac{3}{5}g^{cb}A^d A^g A^e A^f \nabla_{dg} R_{cef} g^{ah} \\ & + \frac{1}{15}g^{cb}A^d A^g A^f A^e R^a{}_{gfh} R^h{}_{dec} + \frac{1}{15}g^{cb}A^d A^g A^f A^e R^a{}_{geh} R^h{}_{dfc} + \frac{1}{9}g^{ic}A^e A^f R^a{}_{efi}A^d A^g R^b{}_{dgc} + \frac{1}{9}g^{ic}A^e A^g R^a{}_{egi}A^d A^f R^b{}_{dfc} \\ & + \frac{1}{9}g^{ai}A^e A^f R^c{}_{efi}A^d A^g R^b{}_{dgc} + \frac{1}{9}g^{ai}A^e A^g R^c{}_{egi}A^d A^f R^b{}_{dfc} + \frac{1}{9}g^{ic}A^f A^g R^a{}_{fgi}A^d A^e R^b{}_{dec} + \frac{1}{9}g^{ai}A^f A^g R^c{}_{fgi}A^d A^e R^b{}_{dec} \\ & - \frac{3}{5}g^{ac}A^d A^g A^e A^f \nabla_{dg} R_{cef} g^{bh} + \frac{1}{15}g^{ac}A^d A^g A^f A^e R^b{}_{gfh} R^h{}_{dec} + \frac{1}{15}g^{ac}A^d A^g A^f A^e R^b{}_{geh} R^h{}_{dfc} \end{aligned}$$

$$\begin{aligned}
\text{term4.402} := & \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
& + \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefgh}g^{ah} \\
& + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
& + \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
& - \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefgh}g^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{term4.403} := & \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
& + \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefgh}g^{ah} \\
& + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
& + \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
& - \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefgh}g^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{term4.404} := & \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
& + \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefgh}g^{ah} \\
& + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
& + \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
& - \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefgh}g^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}
\end{aligned}$$

$$\begin{aligned}
\text{term4.405} := & \frac{1}{9}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} \\
& + \frac{1}{9}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} - \frac{3}{5}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{ah} \\
& + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} + \frac{1}{9}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} \\
& + \frac{1}{9}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
& - \frac{3}{5}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc}
\end{aligned}$$

## Stage 4: Build the Taylor series for $g_{ab}$ , reformatting and output

```
beg_stage_4 = time.time()

# final housekeeping

# lower the  $\{ab\}$  indices to  $\{uv\}$ 

tmp0 := g_{a u} g_{b v} @(term0).
tmp1 := g_{a u} g_{b v} @(term1).
tmp2 := g_{a u} g_{b v} @(term2).
tmp3 := g_{a u} g_{b v} @(term3).
tmp4 := g_{a u} g_{b v} @(term4).
tmp5 := g_{a u} g_{b v} @(term5).

distribute      (tmp1) # cdb(tmp1.501,tmp1)
eliminate_metric (tmp1) # cdb(tmp1.502,tmp1)
eliminate_kronecker (tmp1) # cdb(tmp1.503,tmp1)
tmp1 = flatten_Rabcd (tmp1)
canonicalise     (tmp1) # cdb(tmp1.506,tmp1)

distribute      (tmp2) # cdb(tmp2.501,tmp2)
eliminate_metric (tmp2) # cdb(tmp2.502,tmp2)
eliminate_kronecker (tmp2) # cdb(tmp2.503,tmp2)
tmp2 = flatten_Rabcd (tmp2)
canonicalise     (tmp2) # cdb(tmp2.506,tmp2)

distribute      (tmp3) # cdb(tmp3.501,tmp3)
eliminate_metric (tmp3) # cdb(tmp3.502,tmp3)
eliminate_kronecker (tmp3) # cdb(tmp3.503,tmp3)
tmp3 = flatten_Rabcd (tmp3)
canonicalise     (tmp3) # cdb(tmp3.506,tmp3)

distribute      (tmp4) # cdb(tmp4.501,tmp4)
eliminate_metric (tmp4) # cdb(tmp4.502,tmp4)
eliminate_kronecker (tmp4) # cdb(tmp4.503,tmp4)
tmp4 = flatten_Rabcd (tmp4)
canonicalise     (tmp4) # cdb(tmp4.506,tmp4)
```

```

distribute      (tmp5) # cdb(tmp5.501,tmp5)
eliminate_metric (tmp5) # cdb(tmp5.502,tmp5)
eliminate_kronecker (tmp5) # cdb(tmp5.503,tmp5)
tmp5 = flatten_Rabcd (tmp5)
canonicalise    (tmp5) # cdb(tmp5.506,tmp5)

# this is out final answer

# raise the  $_{uv}$  indices to  $^{ab}$ 

metric:= g^{a u} g^{b v} ( @ (tmp0)
                        + (1/1) @ (tmp1)
                        + (1/2) @ (tmp2)
                        + (1/6) @ (tmp3)
                        + (1/24) @ (tmp4)
                        + (1/120) @ (tmp5) ). # cdb(metric.500,metric)

distribute      (metric) # cdb(metric.501,metric)
eliminate_metric (metric) # cdb(metric.502,metric)
eliminate_kronecker (metric) # cdb(metric.503,metric)
metric = flatten_Rabcd (metric) # cdb(metric.504,metric)
canonicalise    (metric) # cdb(metric.505,metric)

substitute      (metric,$g_{a b} g^{b c} -> g_{a}^{c}$)
substitute      (metric,$g_{b a} g^{b c} -> g_{a}^{c}$)
substitute      (metric,$g_{b a} g^{c b} -> g_{a}^{c}$)
substitute      (metric,$g_{a b} g^{c b} -> g_{a}^{c}$)
eliminate_kronecker (metric) # cdb(metric.506,metric)
canonicalise    (metric) # cdb(metric.507,metric)

substitute (metric,$A^{a} -> x^{a}$) # cdb (metric.508,metric)

cdblib.create ('metric-inv.json')

cdblib.put ('g^{ab}',metric,'metric-inv.json')

# extract the terms of the metric in powers of x

```



```

term0 = get_xterm (metric,0)    # cdb(term0.501,term0)
term1 = get_xterm (metric,1)    # cdb(term1.501,term1)
term2 = get_xterm (metric,2)    # cdb(term2.501,term2)
term3 = get_xterm (metric,3)    # cdb(term3.501,term3)
term4 = get_xterm (metric,4)    # cdb(term4.501,term4)
term5 = get_xterm (metric,5)    # cdb(term5.501,term5)

cdblib.put ('g^ab_0',term0,'metric-inv.json')
cdblib.put ('g^ab_1',term1,'metric-inv.json')
cdblib.put ('g^ab_2',term2,'metric-inv.json')
cdblib.put ('g^ab_3',term3,'metric-inv.json')
cdblib.put ('g^ab_4',term4,'metric-inv.json')
cdblib.put ('g^ab_5',term5,'metric-inv.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3).  # cdb(metric4.501,metric4)

```

$$\text{term0.501} := g^{ab}$$

$$\text{term1.501} := 0$$

$$\text{term2.501} := \frac{1}{3}x^c x^d R_{cedf} g^{ae} g^{bf}$$

$$\text{term3.501} := \frac{1}{6}x^c x^d x^e \nabla_c R_{dfe} g^{af} g^{bg}$$

$$\text{term4.501} := \frac{1}{15}x^c x^d x^e x^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{egfh} g^{ag} g^{bh}$$

$$\text{term5.501} := \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30}x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}$$

$$\text{tmp2.501} := -\frac{1}{3}g_{au}g_{bv}A^cA^dR^a_{cd}{}^b - \frac{1}{3}g_{au}g_{bv}A^cA^dR^b_{cd}{}^a$$

$$\text{tmp2.502} := -\frac{1}{3}g_{bv}A^cA^dR_{ucd}{}^b - \frac{1}{3}g_{bv}A^cA^dR^b_{cd}{}^u$$

$$\text{tmp2.503} := -\frac{1}{3}g_{bv}A^cA^dR_{ucd}{}^b - \frac{1}{3}g_{bv}A^cA^dR^b_{cd}{}^u$$

$$\text{tmp2.506} := -\frac{2}{3}A^aA^bR_{uabc}g_{vd}g^{cd}$$

$$\text{tmp3.501} := -\frac{1}{2}g_{au}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g^{ag} - \frac{1}{2}g_{au}g_{bv}g^{ac}A^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$

$$\text{tmp3.502} := -\frac{1}{2}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cfeg}g_u{}^g - \frac{1}{2}g_{bv}g_u{}^cA^dA^fA^e\nabla_dR_{cfeg}g^{bg}$$

$$\text{tmp3.503} := -\frac{1}{2}g_{bv}g^{cb}A^dA^fA^e\nabla_dR_{cf eu} - \frac{1}{2}g_{bv}A^dA^fA^e\nabla_dR_{ufeg}g^{bg}$$

$$\text{tmp3.506} := -A^aA^bA^c\nabla_aR_{ubcd}g_{ve}g^{de}$$

$$\begin{aligned}
\text{tmp4.501} &:= \frac{1}{9}g_{au}g_{bv}g^{ib}A^eA^fR^c_{efi}A^dA^gR^a_{dgc} + \frac{1}{9}g_{au}g_{bv}g^{ib}A^eA^gR^c_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ci}A^eA^fR^b_{efi}A^dA^gR^a_{dgc} \\
&+ \frac{1}{9}g_{au}g_{bv}g^{ci}A^eA^gR^b_{egi}A^dA^fR^a_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ib}A^fA^gR^c_{fgi}A^dA^eR^a_{dec} + \frac{1}{9}g_{au}g_{bv}g^{ci}A^fA^gR^b_{fgi}A^dA^eR^a_{dec} \\
&- \frac{3}{5}g_{au}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{ah} + \frac{1}{15}g_{au}g_{bv}g^{cb}A^dA^gA^fA^eR^a_{gfh}R^h_{dec} + \frac{1}{15}g_{au}g_{bv}g^{cb}A^dA^gA^fA^eR^a_{geh}R^h_{dfc} \\
&+ \frac{1}{9}g_{au}g_{bv}g^{ic}A^eA^fR^a_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g_{au}g_{bv}g^{ic}A^eA^gR^a_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ai}A^eA^fR^c_{efi}A^dA^gR^b_{dgc} \\
&+ \frac{1}{9}g_{au}g_{bv}g^{ai}A^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{au}g_{bv}g^{ic}A^fA^gR^a_{fgi}A^dA^eR^b_{dec} + \frac{1}{9}g_{au}g_{bv}g^{ai}A^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
&- \frac{3}{5}g_{au}g_{bv}g^{ac}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g_{au}g_{bv}g^{ac}A^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g_{au}g_{bv}g^{ac}A^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{tmp4.502} &:= \frac{1}{9}g_{bv}g^{ib}A^eA^fR^c_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ib}A^eA^gR^c_{egi}A^dA^fR_{udfc} + \frac{1}{9}g_{bv}g^{ci}A^eA^fR^b_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ci}A^eA^gR^b_{egi}A^dA^fR_{udfc} \\
&+ \frac{1}{9}g_{bv}g^{ib}A^fA^gR^c_{fgi}A^dA^eR_{udec} + \frac{1}{9}g_{bv}g^{ci}A^fA^gR^b_{fgi}A^dA^eR_{udec} - \frac{3}{5}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefh}g_u^h \\
&+ \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugfh}R^h_{dec} + \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugeh}R^h_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^eA^fR_{uefi}A^dA^gR^b_{dgc} + \frac{1}{9}g_{bv}g^{ic}A^eA^gR_{uegi}A^dA^fR^b_{dfc} \\
&+ \frac{1}{9}g_{bv}g_u^iA^eA^fR^c_{efi}A^dA^gR^b_{dgc} + \frac{1}{9}g_{bv}g_u^iA^eA^gR^c_{egi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^fA^gR_{ufgi}A^dA^eR^b_{dec} + \frac{1}{9}g_{bv}g_u^iA^fA^gR^c_{fgi}A^dA^eR^b_{dec} \\
&- \frac{3}{5}g_{bv}g_u^cA^dA^gA^eA^f\nabla_{dg}R_{cefh}g^{bh} + \frac{1}{15}g_{bv}g_u^cA^dA^gA^fA^eR^b_{gfh}R^h_{dec} + \frac{1}{15}g_{bv}g_u^cA^dA^gA^fA^eR^b_{geh}R^h_{dfc} \\
\text{tmp4.503} &:= \frac{1}{9}g_{bv}g^{ib}A^eA^fR^c_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ib}A^eA^gR^c_{egi}A^dA^fR_{udfc} + \frac{1}{9}g_{bv}g^{ci}A^eA^fR^b_{efi}A^dA^gR_{udgc} + \frac{1}{9}g_{bv}g^{ci}A^eA^gR^b_{egi}A^dA^fR_{udfc} \\
&+ \frac{1}{9}g_{bv}g^{ib}A^fA^gR^c_{fgi}A^dA^eR_{udec} + \frac{1}{9}g_{bv}g^{ci}A^fA^gR^b_{fgi}A^dA^eR_{udec} - \frac{3}{5}g_{bv}g^{cb}A^dA^gA^eA^f\nabla_{dg}R_{cefu} \\
&+ \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugfh}R^h_{dec} + \frac{1}{15}g_{bv}g^{cb}A^dA^gA^fA^eR_{ugeh}R^h_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^eA^fR_{uefi}A^dA^gR^b_{dgc} \\
&+ \frac{1}{9}g_{bv}g^{ic}A^eA^gR_{uegi}A^dA^fR^b_{dfc} + \frac{1}{9}g_{bv}A^eA^fR^c_{efu}A^dA^gR^b_{dgc} + \frac{1}{9}g_{bv}A^eA^gR^c_{egu}A^dA^fR^b_{dfc} + \frac{1}{9}g_{bv}g^{ic}A^fA^gR_{ufgi}A^dA^eR^b_{dec} \\
&+ \frac{1}{9}g_{bv}A^fA^gR^c_{fgu}A^dA^eR^b_{dec} - \frac{3}{5}g_{bv}A^dA^gA^eA^f\nabla_{dg}R_{uefh}g^{bh} + \frac{1}{15}g_{bv}A^dA^gA^fA^eR^b_{gfh}R^h_{deu} + \frac{1}{15}g_{bv}A^dA^gA^fA^eR^b_{geh}R^h_{dfu} \\
\text{tmp4.506} &:= -\frac{8}{5}A^aA^bA^cA^dR_{uabe}R_{cfdg}g_{vh}g^{ef}g^{gh} - \frac{6}{5}A^aA^bA^cA^d\nabla_{ab}R_{ucde}g_{vf}g^{ef}
\end{aligned}$$

$$\text{tmp5.506} := -4A^a A^b A^c A^d A^e R_{uabf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} - 4A^a A^b A^c A^d A^e R_{afbg} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{4}{3} A^a A^b A^c A^d A^e \nabla_{abc} R_{udef} g_{vg} g^{fg}$$

$$\begin{aligned} \text{metric.500} := & g^{au} g^{bv} \left( g_{cu} g_{dv} g^{cd} - \frac{1}{3} A^e A^f R_{uefc} g_{vd} g^{cd} - \frac{1}{6} A^f A^g A^c \nabla_f R_{ugcd} g_{ve} g^{de} - \frac{1}{15} A^i A^j A^c A^d R_{uije} R_{cfdg} g_{vh} g^{ef} g^{gh} - \frac{1}{20} A^i A^j A^c A^d \nabla_{ij} R_{ucde} g_{vf} g^{ef} \right. \\ & \left. - \frac{1}{30} A^j A^k A^c A^d A^e R_{ujkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} - \frac{1}{30} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} A^j A^k A^c A^d A^e \nabla_{jkc} R_{udef} g_{vg} g^{fg} \right) \end{aligned}$$

$$\begin{aligned} \text{metric.501} := & g^{au} g^{bv} g_{cu} g_{dv} g^{cd} - \frac{1}{3} g^{au} g^{bv} A^e A^f R_{uefc} g_{vd} g^{cd} - \frac{1}{6} g^{au} g^{bv} A^f A^g A^c \nabla_f R_{ugcd} g_{ve} g^{de} - \frac{1}{15} g^{au} g^{bv} A^i A^j A^c A^d R_{uije} R_{cfdg} g_{vh} g^{ef} g^{gh} \\ & - \frac{1}{20} g^{au} g^{bv} A^i A^j A^c A^d \nabla_{ij} R_{ucde} g_{vf} g^{ef} - \frac{1}{30} g^{au} g^{bv} A^j A^k A^c A^d A^e R_{ujkf} \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\ & - \frac{1}{30} g^{au} g^{bv} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{udeh} g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{au} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R_{udef} g_{vg} g^{fg} \end{aligned}$$

$$\begin{aligned} \text{metric.502} := & g^{bv} g_c^a g_{dv} g^{cd} - \frac{1}{3} g^{bv} A^e A^f R_{efc}^a g_{vd} g^{cd} - \frac{1}{6} g^{bv} A^f A^g A^c \nabla_f R_{gcd}^a g_{ve} g^{de} - \frac{1}{15} g^{bv} A^i A^j A^c A^d R_{ije}^a R_{cfdg} g_{vh} g^{ef} g^{gh} \\ & - \frac{1}{20} g^{bv} A^i A^j A^c A^d \nabla_{ij} R_{cde}^a g_{vf} g^{ef} - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jkf}^a \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\ & - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{deh}^a g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R_{def}^a g_{vg} g^{fg} \end{aligned}$$

$$\begin{aligned} \text{metric.503} := & g^{bv} g_{dv} g^{ad} - \frac{1}{3} g^{bv} A^e A^f R_{efc}^a g_{vd} g^{cd} - \frac{1}{6} g^{bv} A^f A^g A^c \nabla_f R_{gcd}^a g_{ve} g^{de} - \frac{1}{15} g^{bv} A^i A^j A^c A^d R_{ije}^a R_{cfdg} g_{vh} g^{ef} g^{gh} \\ & - \frac{1}{20} g^{bv} A^i A^j A^c A^d \nabla_{ij} R_{cde}^a g_{vf} g^{ef} - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jkf}^a \nabla_c R_{dgeh} g_{vi} g^{fg} g^{hi} \\ & - \frac{1}{30} g^{bv} A^j A^k A^c A^d A^e R_{jfk g} \nabla_c R_{deh}^a g_{vi} g^{fh} g^{gi} - \frac{1}{90} g^{bv} A^j A^k A^c A^d A^e \nabla_{jkc} R_{def}^a g_{vg} g^{fg} \end{aligned}$$

$$\begin{aligned} \text{metric.504} := & g^{bc} g_{dc} g^{ad} - \frac{1}{3} A^c A^d R_{ecdf} g^{ae} g^{bg} g_{gh} g^{fh} - \frac{1}{6} A^c A^d A^e \nabla_d R_{fecg} g^{af} g^{bh} g_{hi} g^{gi} - \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{iefj} g^{ai} g^{bk} g_{kl} g^{jg} g^{hl} \\ & - \frac{1}{20} A^c A^d A^e A^f \nabla_{ef} R_{gcdh} g^{ag} g^{bi} g_{ij} g^{hj} - \frac{1}{30} A^c A^d A^e A^f A^g R_{hfgi} \nabla_c R_{djek} g^{ah} g^{bl} g_{lm} g^{ij} g^{km} \\ & - \frac{1}{30} A^c A^d A^e A^f A^g R_{fhgi} \nabla_c R_{jdek} g^{aj} g^{bl} g_{lm} g^{hk} g^{im} - \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{fgc} R_{hdei} g^{ah} g^{bj} g_{jk} g^{ik} \end{aligned}$$

$$\begin{aligned} \text{metric.505} := & g^{ac} g_{cd} g^{bd} + \frac{1}{3} A^c A^d R_{cedf} g^{ae} g^{bg} g_{gh} g^{fh} + \frac{1}{6} A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{bh} g_{hi} g^{gi} + \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{iefj} g^{ag} g^{bk} g_{kl} g^{hi} g^{jl} \\ & + \frac{1}{20} A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{bi} g_{ij} g^{hj} + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bl} g_{lm} g^{ij} g^{km} \\ & + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bl} g_{lm} g^{hk} g^{im} + \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{bj} g_{jk} g^{ik} \end{aligned}$$

$$\begin{aligned}
\text{metric.506} &:= g^{ba} + \frac{1}{3} A^c A^d R_{cedf} g^{ae} g^{fb} + \frac{1}{6} A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{gb} + \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{eifj} g^{ag} g^{hi} g^{jb} + \frac{1}{20} A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{hb} \\
&\quad + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{ij} g^{kb} + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{hk} g^{ib} + \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{ib} \\
\text{metric.507} &:= g^{ab} + \frac{1}{3} A^c A^d R_{cedf} g^{ae} g^{bf} + \frac{1}{6} A^c A^d A^e \nabla_c R_{dfeg} g^{af} g^{bg} + \frac{1}{15} A^c A^d A^e A^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20} A^c A^d A^e A^f \nabla_{cd} R_{egfh} g^{ag} g^{bh} \\
&\quad + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90} A^c A^d A^e A^f A^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi} \\
\text{metric.508} &:= g^{ab} + \frac{1}{3} x^c x^d R_{cedf} g^{ae} g^{bf} + \frac{1}{6} x^c x^d x^e \nabla_c R_{dfeg} g^{af} g^{bg} + \frac{1}{15} x^c x^d x^e x^f R_{cgdh} R_{eifj} g^{ag} g^{bi} g^{hj} + \frac{1}{20} x^c x^d x^e x^f \nabla_{cd} R_{egfh} g^{ag} g^{bh} \\
&\quad + \frac{1}{30} x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{ah} g^{bj} g^{ik} + \frac{1}{30} x^c x^d x^e x^f x^g R_{chdi} \nabla_e R_{fjgk} g^{aj} g^{bh} g^{ik} + \frac{1}{90} x^c x^d x^e x^f x^g \nabla_{cde} R_{fhgi} g^{ah} g^{bi}
\end{aligned}$$

```

Rterm0 := @(term0).
Rterm1 := @(term1).  # zero
Rterm2 := @(term2).
Rterm3 := @(term3).
Rterm4 := @(term4).
Rterm5 := @(term5).

Rterm0 = reformat_xterm (Rterm0, 1)      # cdb(Rterm0.601,Rterm0)
Rterm2 = reformat_xterm (Rterm2, 3)      # cdb(Rterm2.601,Rterm2)
Rterm3 = reformat_xterm (Rterm3, 6)      # cdb(Rterm3.601,Rterm3)
Rterm4 = reformat_xterm (Rterm4, 60)     # cdb(Rterm4.601,Rterm4)
Rterm5 = reformat_xterm (Rterm5, 90)     # cdb(Rterm5.601,Rterm5)

Metric := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Rterm0, 1)      # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Rterm2, 3)      # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Rterm3, 6)      # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Rterm4, 60)     # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Rterm5, 90)     # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()

```



## The inverse metric in Riemann normal coordinates

$$\begin{aligned} g^{ab}(x) = & g^{ab} + \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \frac{1}{60}x^c x^d x^e x^f (4g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3g^{ag} g^{bh} \nabla_{cd} R_{egfh}) \\ & + \frac{1}{90}x^c x^d x^e x^f x^g (3g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

## Curvature expansion of the inverse metric

$$g^{ab}(x) = g^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} + \overset{5}{g}{}^{ab} + \mathcal{O}(\epsilon^6)$$

$$\overset{0}{g}{}^{ab} = g^{ab}$$

$$3\overset{2}{g}{}^{ab} = x^c x^d g^{ae} g^{bf} R_{cedf}$$

$$6\overset{3}{g}{}^{ab} = x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg}$$

$$60\overset{4}{g}{}^{ab} = x^c x^d x^e x^f (4g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3g^{ag} g^{bh} \nabla_{cd} R_{egfh})$$

$$90\overset{5}{g}{}^{ab} = x^c x^d x^e x^f x^g (3g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi})$$

```

cdblib.create ('metric-inv.export')

cdblib.put ('g^ab_3',Metric3,'metric-inv.export')  # R and \partial R
cdblib.put ('g^ab_4',Metric4,'metric-inv.export')
cdblib.put ('g^ab_5',Metric5,'metric-inv.export')
cdblib.put ('g^ab_6',Metric6,'metric-inv.export')

cdblib.put ('g^ab', Metric, 'metric-inv.export')  # R and \nabla R

cdblib.put ('g^ab_scaled0',scaled0,'metric-inv.export')
cdblib.put ('g^ab_scaled2',scaled2,'metric-inv.export')
cdblib.put ('g^ab_scaled3',scaled3,'metric-inv.export')
cdblib.put ('g^ab_scaled4',scaled4,'metric-inv.export')
cdblib.put ('g^ab_scaled5',scaled5,'metric-inv.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)

```

# Timing

Stage 1: 3.3 secs

Stage 2: 5.8 secs

Stage 3: 97.9 secs

Stage 4: 7.2 secs

# The connection

Here we use the output from `metric.tex` and `metric-inv.tex` to compute the metric connection  $\Gamma_{ab}^d$ . We use the standard metric compatible connection

$$\Gamma_{ab}^d = \frac{1}{2} g^{dc} (g_{cb,a} + g_{ac,b} - g_{ab,c}) \quad (1)$$

Since `metric.tex` and `metric-inv.tex` generate truncated expressions for  $g_{ab}$  and  $g^{ab}$  a similar truncation must be applied to this computation of  $\Gamma_{ab}^d$ . The naive choice is to truncate  $\Gamma_{ab}^d$  *after* it has been fully evaluated on the truncated expressions for  $g_{ab}$  and  $g^{ab}$ . This will work but it wastes time and memory (big time).

A better approach is to truncate  $\Gamma_{ab}^d$  during its construction. That is, we take careful note of how the terms in the finite series for  $g_{ab}$  and  $g^{ab}$  combine to produce the terms of a particular order in the expansion of  $\Gamma_{ab}^d$ .

Suppose  $g_{ab}$  and  $g^{ab}$  are known to say fourth order. We can write each of these as follows

$$g_{ab} = g_{ab}^0 + g_{ab}^1 + g_{ab}^2 + g_{ab}^3 + g_{ab}^4 \quad (2)$$

$$g^{ab} = g^{ab0} + g^{ab1} + g^{ab2} + g^{ab3} + g^{ab4} \quad (3)$$

where  $g^n$  denotes a term of order  $\mathcal{O}(\epsilon^n)$ . A similar expansion applies for  $\Gamma_{ab}^d$ , that is

$$\Gamma_{ab}^d = \Gamma_{ab}^{d0} + \Gamma_{ab}^{d1} + \Gamma_{ab}^{d2} + \Gamma_{ab}^{d3} + \Gamma_{ab}^{d4} \quad (4)$$

After substituting these formal expansions into the equation (1) and then matching corresponding terms we obtain

$$\Gamma_{ab}^{dn} = \frac{1}{2} \sum_{i=0}^{i=n} g^{idc} \left( g_{cb,a}^{n-i} + g_{ac,b}^{n-i} - g_{ab,c}^{n-i} \right) \quad (5)$$

We use this equation to compute the successive terms in  $\Gamma_{ab}^d$ .

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

x^{a}::Depends(D{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

import cdblib

gab = cdblib.get ('g_ab','metric.json')      # cdb(gab.000,gab)
iab = cdblib.get ('g^ab','metric-inv.json')  # cdb(iab.000,iab)

defgab := g_{a b} -> @(gab).
defiab := g^{a b} -> @(iab).

dgab := D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}.  # cdb(dgab.001,dgab)

substitute (dgab,defgab)

distribute (dgab)          # cdb(dgab.002,dgab)
unwrap (dgab)              # cdb(dgab.003,dgab)
product_rule (dgab)        # cdb(dgab.004,dgab)
distribute (dgab)          # cdb(dgab.005,dgab)
substitute (dgab,$D_{a}{x^{b}}->\delta^{b}_{a}$,repeat=True) # cdb(dgab.006,dgab)

```

```
eliminate_kronecker (dgab)      # cdb(dgab.007,dgab)
sort_product      (dgab)      # cdb(dgab.008,dgab)
rename_dummies    (dgab)      # cdb(dgab.009,dgab)
canonicalise      (dgab)      # cdb(dgab.010,dgab)
```

$$\begin{aligned} \text{gab.000} := & g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45}x^c x^d x^e x^f R_{acd g} R_{bef h} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf} \\ & + \frac{1}{45}x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg} \end{aligned}$$

$$\text{dgab.001} := D_a g_{cb} + D_b g_{ac} - D_c g_{ab}$$

$$\begin{aligned} \text{dgab.002} := & D_a g_{cb} - \frac{1}{3}D_a (x^j x^d R_{c j b d}) - \frac{1}{6}D_a (x^j x^d x^e \nabla_j R_{cdbe}) + \frac{2}{45}D_a (x^j x^d x^e x^f R_{c j d g} R_{b e f h} g^{gh}) - \frac{1}{20}D_a (x^j x^d x^e x^f \nabla_{jd} R_{cebf}) \\ & + \frac{1}{45}D_a (x^j x^d x^e x^f x^g R_{c j d h} \nabla_e R_{b f g i} g^{hi}) + \frac{1}{45}D_a (x^j x^d x^e x^f x^g R_{b j d h} \nabla_e R_{c f g i} g^{hi}) - \frac{1}{90}D_a (x^j x^d x^e x^f x^g \nabla_{jde} R_{cfbg}) \\ & + D_b g_{ac} - \frac{1}{3}D_b (x^j x^d R_{a j c d}) - \frac{1}{6}D_b (x^j x^d x^e \nabla_j R_{adce}) + \frac{2}{45}D_b (x^j x^d x^e x^f R_{a j d g} R_{c e f h} g^{gh}) - \frac{1}{20}D_b (x^j x^d x^e x^f \nabla_{jd} R_{aecf}) \\ & + \frac{1}{45}D_b (x^j x^d x^e x^f x^g R_{a j d h} \nabla_e R_{c f g i} g^{hi}) + \frac{1}{45}D_b (x^j x^d x^e x^f x^g R_{c j d h} \nabla_e R_{a f g i} g^{hi}) - \frac{1}{90}D_b (x^j x^d x^e x^f x^g \nabla_{jde} R_{afcg}) \\ & - D_c g_{ab} + \frac{1}{3}D_c (x^j x^d R_{a j b d}) + \frac{1}{6}D_c (x^j x^d x^e \nabla_j R_{adbe}) - \frac{2}{45}D_c (x^j x^d x^e x^f R_{a j d g} R_{b e f h} g^{gh}) + \frac{1}{20}D_c (x^j x^d x^e x^f \nabla_{jd} R_{aebf}) \\ & - \frac{1}{45}D_c (x^j x^d x^e x^f x^g R_{a j d h} \nabla_e R_{b f g i} g^{hi}) - \frac{1}{45}D_c (x^j x^d x^e x^f x^g R_{b j d h} \nabla_e R_{a f g i} g^{hi}) + \frac{1}{90}D_c (x^j x^d x^e x^f x^g \nabla_{jde} R_{afbg}) \end{aligned}$$



$$\begin{aligned}
\text{dgab.010} := & \frac{2}{3}R_{acbd}x^d - \frac{1}{6}\nabla_a R_{bdce}x^d x^e + \frac{1}{3}\nabla_d R_{acbe}x^d x^e - \frac{4}{45}R_{acde}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adce}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adbe}R_{cfgh}g^{dg}x^e x^f x^h \\
& - \frac{1}{20}\nabla_{ad}R_{becf}x^d x^e x^f - \frac{1}{20}\nabla_{da}R_{becf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{acbf}x^d x^e x^f - \frac{2}{45}R_{acde}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i \\
& + \frac{1}{45}R_{cdef}\nabla_a R_{bghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{bdef}\nabla_a R_{cghi}g^{eh}x^d x^f x^g x^i \\
& - \frac{2}{45}R_{bdef}\nabla_g R_{achi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{eh}x^d x^f x^g x^i - \frac{1}{90}\nabla_{ade}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dae}R_{bfcg}x^d x^e x^f x^g \\
& - \frac{1}{90}\nabla_{dea}R_{bfcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{acbg}x^d x^e x^f x^g + \frac{2}{3}R_{adbc}x^d - \frac{1}{6}\nabla_b R_{adce}x^d x^e + \frac{1}{3}\nabla_d R_{aebc}x^d x^e - \frac{2}{45}R_{adbe}R_{cfgh}g^{eg}x^d x^f x^h \\
& - \frac{4}{45}R_{adef}R_{bcgh}g^{eg}x^d x^f x^h - \frac{2}{45}R_{adef}R_{bgch}g^{eg}x^d x^f x^h - \frac{1}{20}\nabla_{bd}R_{aecf}x^d x^e x^f - \frac{1}{20}\nabla_{db}R_{aecf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{afbc}x^d x^e x^f \\
& - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_b R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{adef}\nabla_g R_{bchi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{eh}x^d x^f x^g x^i \\
& - \frac{2}{45}R_{bcde}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_b R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{ei}x^d x^f x^g x^h \\
& - \frac{1}{90}\nabla_{bde}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dbe}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{deb}R_{afcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{agbc}x^d x^e x^f x^g + \frac{1}{6}\nabla_c R_{adbe}x^d x^e \\
& + \frac{2}{45}R_{adce}R_{bfgh}g^{eg}x^d x^f x^h + \frac{2}{45}R_{adef}R_{bgch}g^{eh}x^d x^f x^g + \frac{1}{20}\nabla_{cd}R_{aebf}x^d x^e x^f + \frac{1}{20}\nabla_{dc}R_{aebf}x^d x^e x^f + \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{eh}x^d x^f x^g x^i \\
& - \frac{1}{45}R_{adef}\nabla_c R_{bghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{ei}x^d x^f x^g x^h + \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_c R_{aghi}g^{eh}x^d x^f x^g x^i \\
& + \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{ei}x^d x^f x^g x^h + \frac{1}{90}\nabla_{cde}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dce}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dec}R_{afbg}x^d x^e x^f x^g
\end{aligned}$$

```

# Note:
# Computing Gamma directly by (1/2) iab dgab and *then* truncating to lower order
# is not optimal. We only want the leading order terms (to 4th order in x). But the direct
# calculation would compute *all* terms before the truncation. This does work but it
# is slower than the following code.
#
# The better approach (as adopted in this code) is to extract all of the terms of iab
# and dgab then construct the leading order terms of Gamma (to fifth order) term by term.

def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    distribute (tmp)

    substitute (tmp, $\nabla_{e f g}\{R_{a b c d}\} \rightarrow Q_{a b c d e f g}\$)
    substitute (tmp, $\nabla_{e f}\{R_{a b c d}\} \rightarrow Q_{a b c d e f}\$)
    substitute (tmp, $\nabla_{e}\{R_{a b c d}\} \rightarrow Q_{a b c d e}\$)
    substitute (tmp, $R_{a b c d} \rightarrow Q_{a b c d}\$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} \rightarrow \nabla_{e f g}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d e f} \rightarrow \nabla_{e f}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d e} \rightarrow \nabla_{e}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d} \rightarrow R_{a b c d}\$)

```

```

return foo

# terms of the curvature expansion of dg_{ab}

dgab00 = get_Rterm (dgab,0)    # cdb(dgab00.105,dgab00)  # zero
dgab01 = get_Rterm (dgab,1)    # cdb(dgab01.105,dgab01)  # zero
dgab02 = get_Rterm (dgab,2)    # cdb(dgab02.105,dgab02)
dgab03 = get_Rterm (dgab,3)    # cdb(dgab03.105,dgab03)
dgab04 = get_Rterm (dgab,4)    # cdb(dgab04.105,dgab04)
dgab05 = get_Rterm (dgab,5)    # cdb(dgab05.105,dgab05)

# Convert free indices on iab from ^{a b} to ^{d c}
# This ensures we can later build products like @(iab) @(dgab) knowing that the indices are correctly ordered.
# Without this step we would be using free indices ^{a b} and _{a b c}. Thus the product @(iab) @(dgab) would
# have just one free index _{c}. This is clearly wrong.

tmp := @(iab) \delta_{a}^{d} \delta_{b}^{c}.

distribute      (tmp)
eliminate_kronecker (tmp)
sort_product    (tmp)
rename_dummies  (tmp)
canonicalise    (tmp)

idc := @(tmp).

# terms of the curvature expansion of g^{ab}

idc00 = get_Rterm (idc,0)    # cdb(idc00.105,idc00)
idc01 = get_Rterm (idc,1)    # cdb(idc01.105,idc01)  # zero
idc02 = get_Rterm (idc,2)    # cdb(idc02.105,idc02)
idc03 = get_Rterm (idc,3)    # cdb(idc03.105,idc03)
idc04 = get_Rterm (idc,4)    # cdb(idc04.105,idc04)
idc05 = get_Rterm (idc,5)    # cdb(idc05.105,idc05)

```

$$\text{dgab00.105} := 0$$

$$\text{dgab01.105} := 0$$

$$\text{dgab02.105} := \frac{2}{3}R_{acbd}x^d + \frac{2}{3}R_{adbc}x^d$$

$$\text{dgab03.105} := -\frac{1}{6}\nabla_a R_{bdce}x^d x^e + \frac{1}{3}\nabla_d R_{acbe}x^d x^e - \frac{1}{6}\nabla_b R_{adce}x^d x^e + \frac{1}{3}\nabla_d R_{aebc}x^d x^e + \frac{1}{6}\nabla_c R_{adbe}x^d x^e$$

$$\begin{aligned} \text{dgab04.105} := & -\frac{4}{45}R_{acde}R_{bfggh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adce}R_{bfggh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adbe}R_{cfggh}g^{dg}x^e x^f x^h - \frac{1}{20}\nabla_{ad}R_{becf}x^d x^e x^f \\ & - \frac{1}{20}\nabla_{da}R_{becf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{acbf}x^d x^e x^f - \frac{2}{45}R_{adbe}R_{cfggh}g^{eg}x^d x^f x^h - \frac{4}{45}R_{ade f}R_{bcgh}g^{eg}x^d x^f x^h \\ & - \frac{2}{45}R_{ade f}R_{bgch}g^{eg}x^d x^f x^h - \frac{1}{20}\nabla_{bd}R_{aecf}x^d x^e x^f - \frac{1}{20}\nabla_{db}R_{aecf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{afbc}x^d x^e x^f \\ & + \frac{2}{45}R_{adce}R_{bfggh}g^{eg}x^d x^f x^h + \frac{2}{45}R_{ade f}R_{bgch}g^{eh}x^d x^f x^g + \frac{1}{20}\nabla_{cd}R_{aebf}x^d x^e x^f + \frac{1}{20}\nabla_{dc}R_{aebf}x^d x^e x^f \end{aligned}$$

$$\begin{aligned} \text{dgab05.105} := & -\frac{2}{45}R_{acde}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_a R_{bghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{eh}x^d x^f x^g x^i \\ & - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{bdef}\nabla_a R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bdef}\nabla_g R_{achi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{eh}x^d x^f x^g x^i \\ & - \frac{1}{90}\nabla_{ade}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dae}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dea}R_{bfcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{acbg}x^d x^e x^f x^g - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{eh}x^d x^f x^g x^i \\ & + \frac{1}{45}R_{ade f}\nabla_b R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{ade f}\nabla_g R_{bchi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{ade f}\nabla_g R_{bhci}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bcde}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i \\ & - \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_b R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{ei}x^d x^f x^g x^h - \frac{1}{90}\nabla_{bde}R_{afcg}x^d x^e x^f x^g \\ & - \frac{1}{90}\nabla_{dbe}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{deb}R_{afcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{agbc}x^d x^e x^f x^g + \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{eh}x^d x^f x^g x^i \\ & - \frac{1}{45}R_{ade f}\nabla_c R_{bghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{ade f}\nabla_g R_{bhci}g^{ei}x^d x^f x^g x^h + \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_c R_{aghi}g^{eh}x^d x^f x^g x^i \\ & + \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{ei}x^d x^f x^g x^h + \frac{1}{90}\nabla_{cde}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dce}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dec}R_{afbg}x^d x^e x^f x^g \end{aligned}$$

$$\text{idc00.105} := g^{cd}$$

$$\text{idc01.105} := 0$$

$$\text{idc02.105} := \frac{1}{3} R_{abef} g^{ca} g^{de} x^b x^f$$

$$\text{idc03.105} := \frac{1}{6} \nabla_a R_{befg} g^{cb} g^{df} x^a x^e x^g$$

$$\text{idc04.105} := \frac{1}{15} R_{abef} R_{ghij} g^{ca} g^{dg} g^{ei} x^b x^f x^h x^j + \frac{1}{20} \nabla_{ab} R_{efgh} g^{ce} g^{dg} x^a x^b x^f x^h$$

$$\text{idc05.105} := \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ch} g^{da} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ca} g^{dh} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{90} \nabla_{abe} R_{fghi} g^{cf} g^{dh} x^a x^b x^e x^g x^i$$

```

# idc = g^{d c}
# dgab = D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}

# terms of the curvature expansion of \Gamma^{d}_{a b}

# term0 := (1/2) @ (idc00) @ (dgab00).
# term1 := (1/2) (@ (idc01) @ (dgab00) + @ (idc00) @ (dgab01)).
# term2 := (1/2) (@ (idc02) @ (dgab00) + @ (idc01) @ (dgab01) + @ (idc00) @ (dgab02)).
# term3 := (1/2) (@ (idc03) @ (dgab00) + @ (idc02) @ (dgab01) + @ (idc01) @ (dgab02) + @ (idc00) @ (dgab03)).
# term4 := (1/2) (@ (idc04) @ (dgab00) + @ (idc03) @ (dgab01) + @ (idc02) @ (dgab02) + @ (idc01) @ (dgab03) + @ (idc00) @ (dgab04)).
# term5 := (1/2) (@ (idc05) @ (dgab00) + @ (idc04) @ (dgab01) + @ (idc03) @ (dgab02) + @ (idc02) @ (dgab03) + @ (idc01) @ (dgab04) + @ (idc00) @ (dgab05)).

# simplified version of the above after noting dgab00 = dgab01 = 0

term0 := 0.
term1 := 0.
term2 := (1/2) (@ (idc00) @ (dgab02)).
term3 := (1/2) (@ (idc01) @ (dgab02) + @ (idc00) @ (dgab03)).
term4 := (1/2) (@ (idc02) @ (dgab02) + @ (idc01) @ (dgab03) + @ (idc00) @ (dgab04)).
term5 := (1/2) (@ (idc03) @ (dgab02) + @ (idc02) @ (dgab03) + @ (idc01) @ (dgab04) + @ (idc00) @ (dgab05)).

def tidy_terms (obj):
  substitute (obj,$x^{a}->AA^{a}$,repeat=True) # will force AA to the left of all terms
  distribute (obj)
  sort_product (obj)
  rename_dummies (obj)
  canonicalise (obj)
  substitute (obj,$AA^{a}->x^{a}$,repeat=True) # replace AA with x
  factor_out (obj,$x^{a?}$)

  return obj

term0 = tidy_terms (term0) # cdb(term0.201,term0) # zero
term1 = tidy_terms (term1) # cdb(term1.201,term1) # zero
term2 = tidy_terms (term2) # cdb(term2.201,term2)
term3 = tidy_terms (term3) # cdb(term3.201,term3)
term4 = tidy_terms (term4) # cdb(term4.201,term4)
term5 = tidy_terms (term5) # cdb(term5.201,term5)

```

```
Gamma := @(term0) + @(term1) + @(term2) + @(term3) + @(term4) + @(term5). # cdb(Gamma.200,Gamma)
```

$$\text{term0.201} := 0$$

$$\text{term1.201} := 0$$

$$\text{term2.201} := x^c \left( \frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de} \right)$$

$$\text{term3.201} := x^c x^e \left( \frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df} \right)$$

$$\begin{aligned} \text{term4.201} := x^c x^e x^f \left( \frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \right. \\ \left. + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \right. \\ \left. - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg} \right) \end{aligned}$$

$$\begin{aligned} \text{term5.201} := x^c x^e x^f x^g \left( \frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} \right. \\ \left. + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} \right. \\ \left. - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cea} R_{bfgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} \right. \\ \left. - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} \right. \\ \left. + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} \right. \\ \left. - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{agij} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} \right. \\ \left. + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh} \right) \end{aligned}$$



$$\begin{aligned}
\text{Gamma.200} := & x^c \left( \frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de} \right) + x^c x^e \left( \frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df} \right) \\
& + x^c x^e x^f \left( \frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \right. \\
& \quad + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \\
& \quad \left. - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg} \right) \\
& + x^c x^e x^f x^g \left( \frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} \right. \\
& \quad + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} \\
& \quad - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgj} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgj} g^{dh} \\
& \quad + \frac{1}{180} \nabla_{cea} R_{bfgj} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} \\
& \quad - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} \\
& \quad - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} \\
& \quad \left. - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh} \right)
\end{aligned}$$

```
cdblib.create ('connection.json')

cdblib.put ('Gamma',Gamma,'connection.json')

cdblib.put ('GammaRterm0',term0,'connection.json')
cdblib.put ('GammaRterm1',term1,'connection.json')
cdblib.put ('GammaRterm2',term2,'connection.json')
cdblib.put ('GammaRterm3',term3,'connection.json')
cdblib.put ('GammaRterm4',term4,'connection.json')
cdblib.put ('GammaRterm5',term5,'connection.json')

checkpoint.append (term0)
checkpoint.append (term1)
checkpoint.append (term2)
checkpoint.append (term3)
checkpoint.append (term4)
checkpoint.append (term5)
```

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ g^{a b}              -> A003^{a b}              $)
    substitute (obj,$ \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} -> A008_{\{a b c d e f g h\}} $)
    substitute (obj,$ \nabla_{\{e f g\}}\{R_{\{a b c d\}}\}      -> A007_{\{a b c d e f g\}}  $)
    substitute (obj,$ \nabla_{\{e f\}}\{R_{\{a b c d\}}\}         -> A006_{\{a b c d e f\}}   $)
    substitute (obj,$ \nabla_{\{e\}}\{R_{\{a b c d\}}\}           -> A005_{\{a b c d e\}}     $)
    substitute (obj,$ R_{\{a b c d\}}        -> A004_{\{a b c d\}}     $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> A^{a}              $)
    substitute (obj,$ A002^{a}              -> x^{a}              $)
    substitute (obj,$ A003^{a b}            -> g^{a b}            $)
    substitute (obj,$ A008_{\{a b c d e f g h\}} -> \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A007_{\{a b c d e f g\}} -> \nabla_{\{e f g\}}\{R_{\{a b c d\}}\}  $)
    substitute (obj,$ A006_{\{a b c d e f\}} -> \nabla_{\{e f\}}\{R_{\{a b c d\}}\}   $)
    substitute (obj,$ A005_{\{a b c d e\}} -> \nabla_{\{e\}}\{R_{\{a b c d\}}\}    $)
    substitute (obj,$ A004_{\{a b c d\}} -> R_{\{a b c d\}}      $)

    return obj

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    factor_out (bah,$A^{a?},x^{b?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)

```

```

factor_out (bah,$A^{a?},x^{b?}$)
return bah

Rterm2 := @(term2) A^{a} A^{b}.
Rterm3 := @(term3) A^{a} A^{b}.
Rterm4 := @(term4) A^{a} A^{b}.
Rterm5 := @(term5) A^{a} A^{b}.

Rterm2 = reformat (Rterm2, 3)      # cdb(Rterm2.301,Rterm2)
Rterm3 = reformat (Rterm3, 12)     # cdb(Rterm3.301,Rterm3)
Rterm4 = reformat (Rterm4,360)     # cdb(Rterm4.301,Rterm4)
Rterm5 = reformat (Rterm5,180)     # cdb(Rterm5.301,Rterm5)

Gamma  := @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5). # cdb (Gamma.301,Gamma)
Scaled := 360 @(Gamma).           # cdb (Scaled.301,Scaled)

scaled2 = rescale (Rterm2, 3)      # cdb (scaled2.301,scaled2)
scaled3 = rescale (Rterm3, 12)     # cdb (scaled3.301,scaled3)
scaled4 = rescale (Rterm4, 360)    # cdb (scaled4.301,scaled4)
scaled5 = rescale (Rterm5, 180)    # cdb (scaled5.301,scaled5)

```

## The connection in Riemann normal coordinates

$$\begin{aligned}
A^a A^b \Gamma_{ab}^d = & \frac{2}{3} A^a A^b x^c g^{de} R_{acbe} + \frac{1}{12} A^a A^b x^c x^e (2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}) + \frac{1}{360} A^a A^b x^c x^e x^f (64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} \\
& - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}) \\
& + \frac{1}{180} A^a A^b x^c x^e x^f x^g (16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} \\
& - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bfgh} \\
& + 2g^{dh} \nabla_{cae} R_{bfgh} + 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} \\
& + g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg})
\end{aligned}$$

$$\begin{aligned}
360 A^a A^b \Gamma_{ab}^d = & 240 A^a A^b x^c g^{de} R_{acbe} + 30 A^a A^b x^c x^e (2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}) + A^a A^b x^c x^e x^f (64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} \\
& - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}) \\
& + 2 A^a A^b x^c x^e x^f x^g (16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} \\
& - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} \\
& + 2g^{dh} \nabla_{ace} R_{bfgh} + 2g^{dh} \nabla_{cae} R_{bfgh} + 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} \\
& - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} + g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg})
\end{aligned}$$

## Curvature expansion of the connection

$$A^a A^b \Gamma_{ab}^d = A^a A^b \overset{2}{\Gamma}_{ab}^d + A^a A^b \overset{3}{\Gamma}_{ab}^d + A^a A^b \overset{4}{\Gamma}_{ab}^d + A^a A^b \overset{5}{\Gamma}_{ab}^d + \mathcal{O}(\epsilon^6)$$

$$3A^a A^b \overset{2}{\Gamma}_{ab}^d = 2A^a A^b x^c g^{de} R_{acbe}$$

$$12A^a A^b \overset{3}{\Gamma}_{ab}^d = A^a A^b x^c x^e (2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe})$$

$$360A^a A^b \overset{4}{\Gamma}_{ab}^d = A^a A^b x^c x^e x^f (64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf})$$

$$180A^a A^b \overset{5}{\Gamma}_{ab}^d = A^a A^b x^c x^e x^f x^g (16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bf gj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bf gj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bf gj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bf gh} + 2g^{dh} \nabla_{cae} R_{bf gh} + 2g^{dh} \nabla_{cea} R_{bf gh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bf gj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bf gj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bg hj} + g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg})$$

# The generalised connections

The generalised connections may be computed recursively using

$$\Gamma^a_{bcd} = \Gamma^a_{(b\underline{c},d)} - (n+1)\Gamma^a_{p(\underline{c}}\Gamma^p_{bd)} \quad (1)$$

where  $\underline{c}$  contains  $n > 0$  indices. The sequence begins with the standard metric compatible connection

$$\Gamma^d_{ab} = \frac{1}{2}g^{dc}(g_{cb,a} + g_{ac,b} - g_{ab,c}) \quad (2)$$

Here we will use the results of `metric.tex` and `metric-inv.tex` to compute the metric connection  $\Gamma^d_{ab}$ . But since the  $g_{ab}$  and  $g^{ab}$  provided by those codes are truncated at a particular order in the curvatures (and thus are only approximations to the  $g_{ab}$  and  $g^{ab}$ ) similar truncations will arise in the  $\Gamma^a_{bcd}$ .

Approximations will be denoted by the addition of an overbar to an object. In this notation the metric  $g$  can be written as

$$g = \bar{g} + \mathcal{O}(\epsilon^n) \quad (3)$$

in which  $\bar{g}$  is the truncated polynomial approximation to  $g$  and  $\mathcal{O}(\epsilon^n)$  is the error term (containing terms no smaller than  $\epsilon^n$ ). The polynomial structure of  $\bar{g}$  can be expressed as

$$\bar{g} = \bar{g}^{\frac{0}{}} + \bar{g}^{\frac{1}{}} + \bar{g}^{\frac{2}{}} + \cdots + \bar{g}^{\frac{p}{}} \quad (4)$$

in which each terms like  $\bar{g}^{\frac{m}{}}$  contains only terms of order  $m$ . This notation will be applied to other quantities in particular the generalised connections.

The notation  $\mathcal{O}(\epsilon^n)$  denotes terms in the curvatures that are of order  $\epsilon^n$ . What does this actually mean? Each term in  $R$  is of order  $\epsilon^2$  while each derivative of  $R$  carries an extra power of  $\epsilon$ . Thus  $R \cdot R = \mathcal{O}(\epsilon^4)$ ,  $R \cdot R \cdot \nabla R = \mathcal{O}(\epsilon^7)$  and  $R \cdot R \cdot \nabla^2 R = \mathcal{O}(\epsilon^8)$ .

We will also adopt the convention that an object is said to be an  $\mathcal{O}(\epsilon^m)$  approximation when the corresponding error term is  $\mathcal{O}(\epsilon^{m+1})$ .

Consider the  $\mathcal{O}(\epsilon^m)$  approximation of the generalised connection, namely,

$$\bar{\Gamma}^a_{b\underline{c}_n d} = \bar{\Gamma}^a_{b\underline{c}_n d}^{\frac{0}{}} + \bar{\Gamma}^a_{b\underline{c}_n d}^{\frac{1}{}} + \bar{\Gamma}^a_{b\underline{c}_n d}^{\frac{2}{}} + \cdots + \bar{\Gamma}^a_{b\underline{c}_n d}^{\frac{m}{}} \quad (5)$$

where  $\underline{c}_n$  denotes a set of indices such as  $c_1 c_2 c_3 \dots c_n$ .

The first thing to note is that

$$0 = \bar{\Gamma}^{1+n a}_{(b\underline{c}_n, d)} \quad (6)$$

There are two proofs of this claim. For the first proof, note (by inspection) that the order  $\mathcal{O}(\epsilon^p)$  approximation for  $\bar{\Gamma}^a_{b\bar{c}_n d}$  is a polynomial in  $x$  of degree  $p - n - 1$ . Thus  $\bar{\Gamma}^{1+n a}_{(b\bar{c}_n, d)}$  is a polynomial in  $x$  of degree zero, i.e., a constant. However, we know that all generalised connections vanish at the origin of the RNC frame. Thus this constant must be zero. The second proof makes explicit use of the first (and second?) Bianchi identity, that is  $0 = R_{a(bcd)}$ . The term  $\bar{\Gamma}^{1+n a}_{(b\bar{c}_n, d)}$  will itself consist of a sum of terms built from combinations of  $x$ ,  $R$ ,  $\nabla R$  etc. The  $x^a$  will always appear in a contraction with one of the indices on  $R_{abcd}$  or one of its derivatives. Consider any one of these terms, denoted by  $A$ , and assume for the moment that  $1 + n$  is an even number, say  $1 + n = 2p$ . The indices  $(b\bar{c}_n, d)$  must somehow be assigned to the factors that comprise  $A$ . Our aim is to show that at least one  $R$  factor in  $A$  will receive 3 of these indices and thus by the Bianchi identities will be zero. If there are too many  $R$  factors then the Bianchi identities will not come into play. So how many  $R$  factors can we expect? Since  $A$  is a term in an  $\mathcal{O}(\epsilon^{(n+1)})$  approximation there can be no more than  $(n + 1)/2 = p$  Riemann factors. There will be at least one  $x$  term contracted with one of the  $p$  Riemann factors. However, we have  $n + 2 = 2p + 1$  indices to distribute amongst the  $x$  term and  $p$  Riemann factors. One of the indices is a derivative index and will have nett effect of transferring that index from  $x$  to one of the Riemann factors. The remaining  $2p$  indices must be distributed amongst the  $p$  Riemann factors. It is not possible to avoid assigning three indices to at least one of the Riemann factors. Thus, by the Bianchi identity, this  $A$  term must vanish. Similar arguments can be applied to the other cases where the  $A$  terms consists of products of  $R$  and its derivatives and in the case where  $n + 1$  is an odd number. The analysis always comes down to the distribution of the indices  $(b\bar{c}_n, d)$  amongst the factors of a typical  $A$  term. In all cases the Bianchi identity will enter the play and force  $A$  to be zero.

A corollary of the second proof is that for all  $m < n + 2$

$$0 = \bar{\Gamma}^a_{b\bar{c}_n d} \quad (7)$$

The proof follows exactly that of the second proof given above.

We can use the above results to streamline the computation of the generalised connections. We begin with the formal expression for the  $\mathcal{O}(\epsilon^m)$  approximations

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} + \bar{\Gamma}^a_{bc} + \bar{\Gamma}^a_{bc} + \cdots + \bar{\Gamma}^a_{bc} \quad (8)$$

$$\Gamma^a_{b\bar{c}} = \bar{\Gamma}^{n+1 a}_{b\bar{c}} + \bar{\Gamma}^{n+2 a}_{b\bar{c}} + \bar{\Gamma}^{n+3 a}_{b\bar{c}} + \cdots + \bar{\Gamma}^m_{b\bar{c}} \quad (9)$$

$$\Gamma^a_{b\bar{c}d} = \bar{\Gamma}^{n+2 a}_{b\bar{c}d} + \bar{\Gamma}^{n+3 a}_{b\bar{c}d} + \bar{\Gamma}^{n+4 a}_{b\bar{c}d} + \cdots + \bar{\Gamma}^m_{b\bar{c}d} \quad (10)$$

These can be substituted into equation (1) with the result

$$\Gamma^a_{b\bar{c}d} = \bar{\Gamma}^{n+1 a}_{(b\bar{c}, d)} + \bar{\Gamma}^{n+2 a}_{(b\bar{c}, d)} + \bar{\Gamma}^{n+3 a}_{(b\bar{c}, d)} + \cdots + \bar{\Gamma}^m_{(b\bar{c}, d)} - (n + 1) \left( \bar{\Gamma}^{n+1 a}_{p\bar{c}} + \bar{\Gamma}^{n+2 a}_{p\bar{c}} + \bar{\Gamma}^{n+3 a}_{p\bar{c}} + \cdots + \bar{\Gamma}^m_{p\bar{c}} \right) \left( \bar{\Gamma}^p_{bd} + \bar{\Gamma}^p_{bd} + \bar{\Gamma}^p_{bd} + \cdots + \bar{\Gamma}^p_{bd} \right) \quad (11)$$

where it is understood that in expanding the pair of bracketed terms in the last result the terms should be symmetrised over  $b\bar{c}d$  and also truncated to terms of order  $\mathcal{O}(\epsilon^m)$ . Note that the first term on the right hand side of this equation vanishes by way of the results described above.



Comparing the order  $m$  terms in equation (10) and (11) leads to the following equation

$$\bar{\Gamma}_{b\bar{c}d}^a = \bar{\Gamma}_{(b\bar{c},d)}^a - (n+1) \left( \bar{\Gamma}_{p(\bar{c}}^{m-2} \bar{\Gamma}_{bd)}^2 + \bar{\Gamma}_{p(\bar{c}}^{m-3} \bar{\Gamma}_{bd)}^3 + \bar{\Gamma}_{p(\bar{c}}^{m-4} \bar{\Gamma}_{bd)}^4 + \cdots + \bar{\Gamma}_{p(\bar{c}}^{n+1} \bar{\Gamma}_{bd)}^{m-n-1} \right) \quad (12)$$

This one equation is all that is needed to compute all of the  $\bar{\Gamma}_{b\bar{c}d}^a$  for  $p = 3, 4, 5, \dots, m$  given just the  $\bar{\Gamma}_{bd}^a$  for  $p = 2, 3, 4, \dots, m$ . For example, suppose  $m = 5$  and suppose that we are given  $\bar{\Gamma}_{bd}^a$  for  $p = 2, 3, 4, 5$ . Then with  $n = 1$  we can use equation (12) to compute in turn,  $\bar{\Gamma}_{bc_1d}^a$  for  $p = 3, 4, 5$ . Then with  $n = 2$  we compute  $\bar{\Gamma}_{bc_1c_2d}^a$  for  $p = 4, 5$  and finally with  $n = 3$  we compute  $\bar{\Gamma}_{bc_1c_2c_3d}^a$  for  $p = 5$ . There are no terms like  $\bar{\Gamma}_{bc_1c_2c_3c_4d}^a$  for  $p \leq 5$  due to the corollary given earlier.

The explicit computations for  $m = 5$  are as follows.

For  $n = 1$ ,

$$\bar{\Gamma}^a_{bc_1d} = \bar{\Gamma}^a_{(bc_1,d)} \quad (13)$$

$$\bar{\Gamma}^a_{bc_1d} = \bar{\Gamma}^a_{(bc_1,d)} - 2\bar{\Gamma}^a_{p(c_1}\bar{\Gamma}^p_{bd)} \quad (14)$$

$$\bar{\Gamma}^a_{bc_1d} = \bar{\Gamma}^a_{(bc_1,d)} - 2\bar{\Gamma}^a_{p(c_1}\bar{\Gamma}^p_{bd)} - 2\bar{\Gamma}^a_{p(c_1}\bar{\Gamma}^p_{bd)} \quad (15)$$

For  $n = 2$ ,

$$\bar{\Gamma}^a_{bc_1c_2d} = \bar{\Gamma}^a_{(bc_1c_2,d)} \quad (16)$$

$$\bar{\Gamma}^a_{bc_1c_2d} = \bar{\Gamma}^a_{(bc_1c_2,d)} - 3\bar{\Gamma}^a_{p(c_1c_2}\bar{\Gamma}^p_{bd)} \quad (17)$$

For  $n = 3$ ,

$$\bar{\Gamma}^a_{bc_1c_2c_3d} = \bar{\Gamma}^a_{(bc_1c_2c_3,d)} \quad (18)$$

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,c1,c2,c3,c4,c5,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

x^{a}::Depends(D{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

import cdblib

term0 = cdblib.get ('GammaRterm0','connection.json')
term1 = cdblib.get ('GammaRterm2','connection.json')
term2 = cdblib.get ('GammaRterm3','connection.json')
term3 = cdblib.get ('GammaRterm4','connection.json')
term4 = cdblib.get ('GammaRterm5','connection.json')

# LCB: these terms were not computed in connection.tex so set them to zero

```

```

#      maybe in the future I will compute down to term6.

term5 := 0.
term6 := 0.

# genGmn : m = eps order of Rabcd terms
#          n = number of c indices

# -----
# rules for building the genGmn

# note: after applying each rule, must symmetrise over (b c1 c2 ... cn d)

# n = 0

genG20 := genG2^{a}_{b d}.
genG30 := genG3^{a}_{b d}.
genG40 := genG4^{a}_{b d}.
genG50 := genG5^{a}_{b d}.

defG20 := genG2^{d}_{a b} -> @(term1).
defG30 := genG3^{d}_{a b} -> @(term2).
defG40 := genG4^{d}_{a b} -> @(term3).
defG50 := genG5^{d}_{a b} -> @(term4).

# LCB: rncGamma in connection.json limited to "term4" (ie. to 4th order in x)
#      so can only compute genG3*, genG4* and genG5* (at this stage)
#      but it doesn't hurt to provide the definitions for genG6*, genG7* etc. we just won't use them (at this atage)

defG60 := genG6^{d}_{a b} -> @(term5).
defG70 := genG7^{d}_{a b} -> @(term6).

# n = 1

defG31 := genG3^{a}_{b c1 d} -> D_{d}{genG3^{a}_{b c1}}.

defG41 := genG4^{a}_{b c1 d} -> D_{d}{genG4^{a}_{b c1}}
      - 2 genG2^{a}_{p c1} genG2^{p}_{b d}.

```

```

defG51 := genG5^{a}_{b c1 d} -> D_{d}{genG5^{a}_{b c1}}
      - 2 genG3^{a}_{p c1} genG2^{p}_{b d}
      - 2 genG2^{a}_{p c1} genG3^{p}_{b d}.

defG61 := genG6^{a}_{b c1 d} -> D_{d}{genG6^{a}_{b c1}}
      - 2 genG4^{a}_{p c1} genG2^{p}_{b d}
      - 2 genG3^{a}_{p c1} genG3^{p}_{b d}
      - 2 genG3^{a}_{p c1} genG4^{p}_{b d}.

defG71 := genG7^{a}_{b c1 d} -> D_{d}{genG7^{a}_{b c1}}
      - 2 genG5^{a}_{p c1} genG2^{p}_{b d}
      - 2 genG4^{a}_{p c1} genG3^{p}_{b d}
      - 2 genG3^{a}_{p c1} genG4^{p}_{b d}
      - 2 genG2^{a}_{p c1} genG5^{p}_{b d}.

# n = 2

defG42 := genG4^{a}_{b c1 c2 d} -> D_{d}{genG4^{a}_{b c1 c2}}.

defG52 := genG5^{a}_{b c1 c2 d} -> D_{d}{genG5^{a}_{b c1 c2}}
      - 3 genG3^{a}_{p c1 c2} genG2^{p}_{b d}.

defG62 := genG6^{a}_{b c1 c2 d} -> D_{d}{genG6^{a}_{b c1 c2}}
      - 3 genG4^{a}_{p c1 c2} genG2^{p}_{b d}
      - 3 genG3^{a}_{p c1 c2} genG3^{p}_{b d}.

defG72 := genG7^{a}_{b c1 c2 d} -> D_{d}{genG7^{a}_{b c1 c2}}
      - 3 genG5^{a}_{p c1 c2} genG2^{p}_{b d}
      - 3 genG4^{a}_{p c1 c2} genG3^{p}_{b d}
      - 3 genG3^{a}_{p c1 c2} genG4^{p}_{b d}.

# n = 3

defG53 := genG5^{a}_{b c1 c2 c3 d} -> D_{d}{genG5^{a}_{b c1 c2 c3}}.

defG63 := genG6^{a}_{b c1 c2 c3 d} -> D_{d}{genG6^{a}_{b c1 c2 c3}}
      - 4 genG3^{a}_{p c1 c2 c3} genG3^{p}_{b d}.

```

```

defG73 := genG7^{a}_{b c1 c2 c3 d} -> D_{d}{genG7^{a}_{b c1 c2 c3}}
      - 4 genG4^{a}_{p c1 c2 c3} genG3^{p}_{b d}
      - 4 genG3^{a}_{p c1 c2 c3} genG4^{p}_{b d}.

# n = 4

defG64 := genG6^{a}_{b c1 c2 c3 c4 d} -> D_{d}{genG6^{a}_{b c1 c2 c3 c4}}.

defG74 := genG7^{a}_{b c1 c2 c3 c4 d} -> D_{d}{genG7^{a}_{b c1 c2 c3 c4}}
      - 5 genG5^{a}_{p c1 c2 c3 c4} genG2^{p}_{b d}.

# n = 5

defG75 := genG7^{a}_{b c1 c2 c3 c4 c5 d} -> D_{d}{genG7^{a}_{b c1 c2 c3 c4 c5}}.

# -----
# build the genGmn
# =====
# n = 1

genG31 := genG3^{a}_{b c1 d}.          # cdb (genG31.000,genG31)
genG41 := genG4^{a}_{b c1 d}.          # cdb (genG41.000,genG41)
genG51 := genG5^{a}_{b c1 d}.
# genG61 := genG6^{a}_{b c1 d}.
# genG71 := genG7^{a}_{b c1 d}.

# -----
substitute      (genG20,defG20)          # cdb (genG20.001,genG20)
substitute      (genG30,defG30)          # cdb (genG30.001,genG30)
substitute      (genG40,defG40)          # cdb (genG40.001,genG40)
substitute      (genG50,defG50)          # cdb (genG50.001,genG50)

# -----
substitute      (genG31,defG31)          # cdb (genG31.001,genG31)
substitute      (genG31,defG30)          # cdb (genG31.002,genG31)

```

```

distribute      (genG31)                # cdb (genG31.002,genG31)
unwrap          (genG31)                # cdb (genG31.003,genG31)
product_rule    (genG31)                # cdb (genG31.004,genG31)
distribute      (genG31)                # cdb (genG31.005,genG31)
substitute      (genG31,$D_{a}{x^b}->\delta_{a}^{b}$) # cdb (genG31.006,genG31)
eliminate_kronecker (genG31)          # cdb (genG31.007,genG31)
sym             (genG31,$_{b},_{c1},_{d}$)
sort_product    (genG31)                # cdb (genG31.008,genG31)
rename_dummies  (genG31)                # cdb (genG31.009,genG31)
canonicalise    (genG31)                # cdb (genG31.010,genG31)

# -----
substitute      (genG41,defG41)          # cdb (genG41.001,genG41)
substitute      (genG41,defG40)          # cdb (genG41.002,genG41)
substitute      (genG41,defG20,repeat=True) # cdb (genG41.003,genG41)

distribute      (genG41)                # cdb (genG41.004,genG41)
unwrap          (genG41)                # cdb (genG41.005,genG41)
product_rule    (genG41)                # cdb (genG41.006,genG41)
distribute      (genG41)                # cdb (genG41.007,genG41)
substitute      (genG41,$D_{a}{x^b}->\delta_{a}^{b}$) # cdb (genG41.008,genG41)
eliminate_kronecker (genG41)          # cdb (genG41.009,genG41)
sym             (genG41,$_{b},_{c1},_{d}$)
sort_product    (genG41)                # cdb (genG41.010,genG41)
rename_dummies  (genG41)                # cdb (genG41.011,genG41)
canonicalise    (genG41)                # cdb (genG41.012,genG41)

# -----
substitute      (genG51,defG51)
substitute      (genG51,defG50)
substitute      (genG51,defG30,repeat=True)
substitute      (genG51,defG20,repeat=True)

distribute      (genG51)
unwrap          (genG51)
product_rule    (genG51)
distribute      (genG51)
substitute      (genG51,$D_{a}{x^b}->\delta_{a}^{b}$)

```

```

eliminate_kronecker (genG51)
sym                (genG51,${b}, ${c1}, ${d})
sort_product       (genG51)
rename_dummies     (genG51)
canonicalise       (genG51)

# update the rules

defG31 := genG3^{a}_{b c1 d} -> @(genG31).
defG41 := genG4^{a}_{b c1 d} -> @(genG41).
defG51 := genG5^{a}_{b c1 d} -> @(genG51).

# =====
# n = 2

genG42 := genG4^{a}_{b c1 c2 d}.
genG52 := genG5^{a}_{b c1 c2 d}.
# genG62 := genG6^{a}_{b c1 c2 d}.
# genG72 := genG7^{a}_{b c1 c2 d}.

# -----
substitute      (genG42,defG42)
substitute      (genG42,defG41)

distribute      (genG42)
unwrap          (genG42)
product_rule     (genG42)
distribute      (genG42)
substitute      (genG42,$D_{a}{x^b}->\delta_{a}^{b})
eliminate_kronecker (genG42)
sym             (genG42,${b}, ${c1}, ${c2}, ${d})
sort_product     (genG42)
rename_dummies   (genG42)
canonicalise     (genG42)

# -----
substitute      (genG52,defG52)
substitute      (genG52,defG51)

```



```

substitute      (genG52,defG31,repeat=True)
substitute      (genG52,defG20,repeat=True)

distribute      (genG52)
unwrap          (genG52)
product_rule    (genG52)
distribute      (genG52)
substitute      (genG52,$D_{a}{x^b}->\delta_{a}^{b}$)
eliminate_kronecker (genG52)
sym             (genG52,$_{b},_{c1},_{c2},_{d}$)
sort_product    (genG52)
rename_dummies  (genG52)
canonicalise    (genG52)                                # cdb (genG52.001,genG52)

# update the rules

defG42 := genG4^{a}_{b c1 c2 d} -> @(genG42).
defG52 := genG5^{a}_{b c1 c2 d} -> @(genG52).

# =====
# n = 3

genG53 := genG5^{a}_{b c1 c2 c3 d}.
# genG63 := genG6^{a}_{b c1 c2 c3 d}.
# genG73 := genG7^{a}_{b c1 c2 c3 d}.

# -----
substitute      (genG53,defG53)
substitute      (genG53,defG52)

distribute      (genG53)
unwrap          (genG53)
product_rule    (genG53)
distribute      (genG53)
substitute      (genG53,$D_{a}{x^b}->\delta_{a}^{b}$)
eliminate_kronecker (genG53)
sym             (genG53,$_{b},_{c1},_{c2},_{c3},_{d}$)
sort_product    (genG53)

```

```
rename_dummies (genG53)
canonicalise    (genG53)                # cdb (genG53.001,genG53)

# update the rules

defG53 := genG5^{\{a\}_{b\ c1\ c2\ c3\ d}} -> @(genG53).
```

$$\text{genG31.000} := \text{gen}G3^a{}_{bcd}$$

$$\text{genG31.001} := D_d(\text{gen}G3^a{}_{bcd})$$

$$\text{genG31.002} := \frac{1}{12}D_d(x^c x^e \nabla_b R_{c1cef} g^{af}) + \frac{1}{6}D_d(x^c x^e \nabla_c R_{bfc1e} g^{af}) + \frac{1}{12}D_d(x^c x^e \nabla_{c1} R_{bcef} g^{af}) + \frac{1}{6}D_d(x^c x^e \nabla_c R_{bec1f} g^{af}) + \frac{1}{12}D_d(x^c x^e \nabla_f R_{bcc1e} g^{af})$$

$$\text{genG31.003} := \frac{1}{12}\nabla_b R_{c1cef} g^{af} D_d(x^c x^e) + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} D_d(x^c x^e) + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} D_d(x^c x^e) + \frac{1}{6}\nabla_c R_{bec1f} g^{af} D_d(x^c x^e) + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} D_d(x^c x^e)$$

$$\begin{aligned} \text{genG31.004} := & \frac{1}{12}\nabla_b R_{c1cef} g^{af} (D_d x^c x^e + x^c D_d x^e) + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} (D_d x^c x^e + x^c D_d x^e) + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} (D_d x^c x^e + x^c D_d x^e) \\ & + \frac{1}{6}\nabla_c R_{bec1f} g^{af} (D_d x^c x^e + x^c D_d x^e) + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} (D_d x^c x^e + x^c D_d x^e) \end{aligned}$$

$$\begin{aligned} \text{genG31.005} := & \frac{1}{12}\nabla_b R_{c1cef} g^{af} D_d x^c x^e + \frac{1}{12}\nabla_b R_{c1cef} g^{af} x^c D_d x^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} D_d x^c x^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} x^c D_d x^e + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} D_d x^c x^e \\ & + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} x^c D_d x^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} D_d x^c x^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} x^c D_d x^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} D_d x^c x^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} x^c D_d x^e \end{aligned}$$

$$\begin{aligned} \text{genG31.006} := & \frac{1}{12}\nabla_b R_{c1cef} g^{af} \delta_d^c x^e + \frac{1}{12}\nabla_b R_{c1cef} g^{af} x^c \delta_d^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} \delta_d^c x^e + \frac{1}{6}\nabla_c R_{bfc1e} g^{af} x^c \delta_d^e + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} \delta_d^c x^e \\ & + \frac{1}{12}\nabla_{c1} R_{bcef} g^{af} x^c \delta_d^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} \delta_d^c x^e + \frac{1}{6}\nabla_c R_{bec1f} g^{af} x^c \delta_d^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} \delta_d^c x^e + \frac{1}{12}\nabla_f R_{bcc1e} g^{af} x^c \delta_d^e \end{aligned}$$

$$\begin{aligned} \text{genG31.007} := & \frac{1}{12}\nabla_b R_{c1def} g^{af} x^e + \frac{1}{12}\nabla_b R_{c1cdf} g^{af} x^c + \frac{1}{6}\nabla_d R_{bfc1e} g^{af} x^e + \frac{1}{6}\nabla_c R_{bfc1d} g^{af} x^c + \frac{1}{12}\nabla_{c1} R_{bdef} g^{af} x^e \\ & + \frac{1}{12}\nabla_{c1} R_{bcd} g^{af} x^c + \frac{1}{6}\nabla_d R_{bec1f} g^{af} x^e + \frac{1}{6}\nabla_c R_{bcd1f} g^{af} x^c + \frac{1}{12}\nabla_f R_{bcd1e} g^{af} x^e + \frac{1}{12}\nabla_f R_{bcc1d} g^{af} x^c \end{aligned}$$

$$\begin{aligned}
\text{genG31.008} := & \frac{1}{36} \nabla_b R_{c1def} g^{af} x^e + \frac{1}{36} \nabla_b R_{dc1ef} g^{af} x^e + \frac{1}{36} \nabla_{c1} R_{bdef} g^{af} x^e + \frac{1}{36} \nabla_{c1} R_{dbef} g^{af} x^e + \frac{1}{36} \nabla_d R_{bc1ef} g^{af} x^e + \frac{1}{36} \nabla_d R_{c1bef} g^{af} x^e \\
& + \frac{1}{36} \nabla_b R_{c1cdf} g^{af} x^c + \frac{1}{36} \nabla_b R_{dcc1f} g^{af} x^c + \frac{1}{36} \nabla_{c1} R_{bcd f} g^{af} x^c + \frac{1}{36} \nabla_{c1} R_{dc b f} g^{af} x^c + \frac{1}{36} \nabla_d R_{bcc1f} g^{af} x^c + \frac{1}{36} \nabla_d R_{c1cbf} g^{af} x^c \\
& + \frac{1}{36} \nabla_d R_{bf c1e} g^{af} x^e + \frac{1}{36} \nabla_{c1} R_{bf de} g^{af} x^e + \frac{1}{36} \nabla_d R_{c1f b e} g^{af} x^e + \frac{1}{36} \nabla_b R_{c1f d e} g^{af} x^e + \frac{1}{36} \nabla_{c1} R_{df b e} g^{af} x^e + \frac{1}{36} \nabla_b R_{df c1e} g^{af} x^e \\
& + \frac{1}{6} \nabla_c \left( \frac{1}{6} R_{bf c1d} + \frac{1}{6} R_{bf d c1} + \frac{1}{6} R_{c1f b d} + \frac{1}{6} R_{c1f d b} + \frac{1}{6} R_{df b c1} + \frac{1}{6} R_{df c1b} \right) g^{af} x^c + \frac{1}{36} \nabla_d R_{bec1f} g^{af} x^e + \frac{1}{36} \nabla_{c1} R_{bedf} g^{af} x^e + \frac{1}{36} \nabla_d R_{c1ebf} g^{af} x^e \\
& + \frac{1}{36} \nabla_b R_{c1edf} g^{af} x^e + \frac{1}{36} \nabla_{c1} R_{debf} g^{af} x^e + \frac{1}{36} \nabla_b R_{dec1f} g^{af} x^e + \frac{1}{6} \nabla_c \left( \frac{1}{6} R_{bdc1f} + \frac{1}{6} R_{bc1df} + \frac{1}{6} R_{c1dbf} + \frac{1}{6} R_{c1bdf} + \frac{1}{6} R_{dc1bf} + \frac{1}{6} R_{dbc1f} \right) g^{af} x^c \\
& + \frac{1}{12} \nabla_f \left( \frac{1}{6} R_{bdc1e} + \frac{1}{6} R_{bc1de} + \frac{1}{6} R_{c1dbe} + \frac{1}{6} R_{c1bde} + \frac{1}{6} R_{dc1be} + \frac{1}{6} R_{dbc1e} \right) g^{af} x^e \\
& + \frac{1}{12} \nabla_f \left( \frac{1}{6} R_{bcc1d} + \frac{1}{6} R_{bcd c1} + \frac{1}{6} R_{c1cbd} + \frac{1}{6} R_{c1cdb} + \frac{1}{6} R_{dcb c1} + \frac{1}{6} R_{dcc1b} \right) g^{af} x^c \\
\\
\text{genG31.009} := & \frac{1}{36} \nabla_b R_{c1dce} g^{ae} x^c + \frac{1}{36} \nabla_b R_{dc1ce} g^{ae} x^c + \frac{1}{36} \nabla_{c1} R_{bdce} g^{ae} x^c + \frac{1}{36} \nabla_{c1} R_{dbce} g^{ae} x^c + \frac{1}{36} \nabla_d R_{bc1ce} g^{ae} x^c + \frac{1}{36} \nabla_d R_{c1bce} g^{ae} x^c \\
& + \frac{1}{18} \nabla_b R_{c1cde} g^{ae} x^c + \frac{1}{18} \nabla_b R_{dcc1e} g^{ae} x^c + \frac{1}{18} \nabla_{c1} R_{bcde} g^{ae} x^c + \frac{1}{18} \nabla_{c1} R_{dcbe} g^{ae} x^c + \frac{1}{18} \nabla_d R_{bcc1e} g^{ae} x^c \\
& + \frac{1}{18} \nabla_d R_{c1cbe} g^{ae} x^c + \frac{1}{36} \nabla_d R_{bcc1e} g^{ac} x^e + \frac{1}{36} \nabla_{c1} R_{bcde} g^{ac} x^e + \frac{1}{36} \nabla_d R_{c1cbe} g^{ac} x^e + \frac{1}{36} \nabla_b R_{c1cde} g^{ac} x^e + \frac{1}{36} \nabla_{c1} R_{dcbe} g^{ac} x^e \\
& + \frac{1}{36} \nabla_b R_{dcc1e} g^{ac} x^e + \frac{1}{6} \nabla_e \left( \frac{1}{6} R_{bcc1d} + \frac{1}{6} R_{bcd c1} + \frac{1}{6} R_{c1cbd} + \frac{1}{6} R_{c1cdb} + \frac{1}{6} R_{dcb c1} + \frac{1}{6} R_{dcc1b} \right) g^{ac} x^e \\
& + \frac{1}{6} \nabla_e \left( \frac{1}{6} R_{bdc1c} + \frac{1}{6} R_{bc1dc} + \frac{1}{6} R_{c1dbc} + \frac{1}{6} R_{c1bdc} + \frac{1}{6} R_{dc1bc} + \frac{1}{6} R_{dbc1c} \right) g^{ac} x^e \\
& + \frac{1}{12} \nabla_e \left( \frac{1}{6} R_{bdc1c} + \frac{1}{6} R_{bc1dc} + \frac{1}{6} R_{c1dbc} + \frac{1}{6} R_{c1bdc} + \frac{1}{6} R_{dc1bc} + \frac{1}{6} R_{dbc1c} \right) g^{ae} x^c \\
& + \frac{1}{12} \nabla_e \left( \frac{1}{6} R_{bcc1d} + \frac{1}{6} R_{bcd c1} + \frac{1}{6} R_{c1cbd} + \frac{1}{6} R_{c1cdb} + \frac{1}{6} R_{dcb c1} + \frac{1}{6} R_{dcc1b} \right) g^{ae} x^c \\
\\
\text{genG31.010} := & \frac{1}{12} \nabla_b R_{c1cde} g^{ae} x^c + \frac{1}{12} \nabla_b R_{c1cde} g^{ac} x^e + \frac{1}{12} \nabla_{c1} R_{bcde} g^{ae} x^c + \frac{1}{12} \nabla_{c1} R_{bcde} g^{ac} x^e + \frac{1}{12} \nabla_d R_{bcc1e} g^{ae} x^c + \frac{1}{12} \nabla_d R_{bcc1e} g^{ac} x^e
\end{aligned}$$

$$\text{genG41.000} := \text{gen}G4^a{}_{bc1d}$$

$$\text{genG41.001} := D_d \left( \text{gen}G4^a{}_{bc1} \right) - 2\text{gen}G2^a{}_{pc1}\text{gen}G2^p{}_{bd}$$

$$\begin{aligned} \text{genG41.002} := D_d \left( x^c x^e x^f \left( \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} - \frac{2}{45} R_{bgch} R_{c1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bgch} R_{c1efi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{bc} R_{c1efg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{40} \nabla_{cb} R_{c1efg} g^{ag} + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} - \frac{2}{45} R_{bceg} R_{c1hfi} g^{ah} g^{gi} - \frac{1}{45} R_{bceg} R_{c1hfi} g^{ai} g^{gh} + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} - \frac{1}{45} R_{bcgh} R_{c1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bceg} R_{c1fhi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} \right) \right) - 2\text{gen}G2^a{}_{pc1}\text{gen}G2^p{}_{bd} \end{aligned}$$

$$\begin{aligned} \text{genG41.003} := D_d \left( x^c x^e x^f \left( \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} - \frac{2}{45} R_{bgch} R_{c1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bgch} R_{c1efi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{bc} R_{c1efg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{40} \nabla_{cb} R_{c1efg} g^{ag} + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} - \frac{2}{45} R_{bceg} R_{c1hfi} g^{ah} g^{gi} - \frac{1}{45} R_{bceg} R_{c1hfi} g^{ai} g^{gh} + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} - \frac{1}{45} R_{bcgh} R_{c1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bceg} R_{c1fhi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} \right) \right) \\ - 2x^c \left( \frac{1}{3} R_{pec1c} g^{ae} + \frac{1}{3} R_{pcc1e} g^{ae} \right) x^f \left( \frac{1}{3} R_{bgdf} g^{pg} + \frac{1}{3} R_{bfdg} g^{pg} \right) \end{aligned}$$

$$\begin{aligned} \text{genG41.004} := \frac{4}{45} D_d \left( x^c x^e x^f R_{bgc1c} R_{ehfi} g^{ah} g^{gi} \right) + \frac{4}{45} D_d \left( x^c x^e x^f R_{bcc1g} R_{ehfi} g^{ah} g^{gi} \right) - \frac{2}{45} D_d \left( x^c x^e x^f R_{bgch} R_{c1efi} g^{ag} g^{hi} \right) \\ - \frac{1}{45} D_d \left( x^c x^e x^f R_{bgch} R_{c1efi} g^{ah} g^{gi} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{bc} R_{c1efg} g^{ag} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{cb} R_{c1efg} g^{ag} \right) + \frac{1}{20} D_d \left( x^c x^e x^f \nabla_{ce} R_{bgc1f} g^{ag} \right) \\ - \frac{2}{45} D_d \left( x^c x^e x^f R_{bceg} R_{c1hfi} g^{ah} g^{gi} \right) - \frac{1}{45} D_d \left( x^c x^e x^f R_{bceg} R_{c1hfi} g^{ai} g^{gh} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{c1c} R_{befg} g^{ag} \right) \\ + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{cc1} R_{befg} g^{ag} \right) + \frac{1}{20} D_d \left( x^c x^e x^f \nabla_{ce} R_{bfc1g} g^{ag} \right) - \frac{1}{45} D_d \left( x^c x^e x^f R_{bcgh} R_{c1efi} g^{ag} g^{hi} \right) \\ - \frac{1}{45} D_d \left( x^c x^e x^f R_{bceg} R_{c1fhi} g^{ah} g^{gi} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{gc} R_{bec1f} g^{ag} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{cg} R_{bec1f} g^{ag} \right) \\ - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg} \end{aligned}$$

$$\begin{aligned}
\text{genG41.005} := & \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} D_d (x^c x^e x^f) + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} D_d (x^c x^e x^f) - \frac{2}{45} R_{bgch} R_{c1efi} g^{ag} g^{hi} D_d (x^c x^e x^f) \\
& - \frac{1}{45} R_{bgch} R_{c1efi} g^{ah} g^{gi} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{bc} R_{c1efg} g^{ag} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{cb} R_{c1efg} g^{ag} D_d (x^c x^e x^f) + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} D_d (x^c x^e x^f) \\
& - \frac{2}{45} R_{bceg} R_{c1hfi} g^{ah} g^{gi} D_d (x^c x^e x^f) - \frac{1}{45} R_{bceg} R_{c1hfi} g^{ai} g^{gh} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} D_d (x^c x^e x^f) \\
& + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} D_d (x^c x^e x^f) + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} D_d (x^c x^e x^f) - \frac{1}{45} R_{bcgh} R_{c1efi} g^{ag} g^{hi} D_d (x^c x^e x^f) \\
& - \frac{1}{45} R_{bceg} R_{c1fhi} g^{ah} g^{gi} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} D_d (x^c x^e x^f) \\
& - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg} \\
\\
\text{genG41.006} := & \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{2}{45} R_{bgch} R_{c1efi} g^{ag} g^{hi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) - \frac{1}{45} R_{bgch} R_{c1efi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{40} \nabla_{bc} R_{c1efg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{40} \nabla_{cb} R_{c1efg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) - \frac{2}{45} R_{bceg} R_{c1hfi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{1}{45} R_{bceg} R_{c1hfi} g^{ai} g^{gh} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{1}{45} R_{bcgh} R_{c1efi} g^{ag} g^{hi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) - \frac{1}{45} R_{bceg} R_{c1fhi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$

$$\begin{aligned}
\text{genG41.007} := & \frac{4}{45} R_{bgclc} R_{ehfi} g^{ah} g^{gi} D_d x^c x^e x^f + \frac{4}{45} R_{bgclc} R_{ehfi} g^{ah} g^{gi} x^c D_d x^e x^f + \frac{4}{45} R_{bgclc} R_{ehfi} g^{ah} g^{gi} x^c x^e D_d x^f + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} D_d x^c x^e x^f \\
& + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} x^c D_d x^e x^f + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} x^c x^e D_d x^f - \frac{2}{45} R_{bgch} R_{clefi} g^{ag} g^{hi} D_d x^c x^e x^f - \frac{2}{45} R_{bgch} R_{clefi} g^{ag} g^{hi} x^c D_d x^e x^f \\
& - \frac{2}{45} R_{bgch} R_{clefi} g^{ag} g^{hi} x^c x^e D_d x^f - \frac{1}{45} R_{bgch} R_{clefi} g^{ah} g^{gi} D_d x^c x^e x^f - \frac{1}{45} R_{bgch} R_{clefi} g^{ah} g^{gi} x^c D_d x^e x^f - \frac{1}{45} R_{bgch} R_{clefi} g^{ah} g^{gi} x^c x^e D_d x^f \\
& + \frac{1}{40} \nabla_{bc} R_{clefg} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{bc} R_{clefg} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{bc} R_{clefg} g^{ag} x^c x^e D_d x^f + \frac{1}{40} \nabla_{cb} R_{clefg} g^{ag} D_d x^c x^e x^f \\
& + \frac{1}{40} \nabla_{cb} R_{clefg} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{cb} R_{clefg} g^{ag} x^c x^e D_d x^f + \frac{1}{20} \nabla_{ce} R_{bgclf} g^{ag} D_d x^c x^e x^f + \frac{1}{20} \nabla_{ce} R_{bgclf} g^{ag} x^c D_d x^e x^f \\
& + \frac{1}{20} \nabla_{ce} R_{bgclf} g^{ag} x^c x^e D_d x^f - \frac{2}{45} R_{bceg} R_{clhfi} g^{ah} g^{gi} D_d x^c x^e x^f - \frac{2}{45} R_{bceg} R_{clhfi} g^{ah} g^{gi} x^c D_d x^e x^f - \frac{2}{45} R_{bceg} R_{clhfi} g^{ah} g^{gi} x^c x^e D_d x^f \\
& - \frac{1}{45} R_{bceg} R_{clhfi} g^{ai} g^{gh} D_d x^c x^e x^f - \frac{1}{45} R_{bceg} R_{clhfi} g^{ai} g^{gh} x^c D_d x^e x^f - \frac{1}{45} R_{bceg} R_{clhfi} g^{ai} g^{gh} x^c x^e D_d x^f + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} D_d x^c x^e x^f \\
& + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} x^c x^e D_d x^f + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} x^c D_d x^e x^f \\
& + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} x^c x^e D_d x^f + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} D_d x^c x^e x^f + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} x^c D_d x^e x^f + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} x^c x^e D_d x^f \\
& - \frac{1}{45} R_{bcgh} R_{clefi} g^{ag} g^{hi} D_d x^c x^e x^f - \frac{1}{45} R_{bcgh} R_{clefi} g^{ag} g^{hi} x^c D_d x^e x^f - \frac{1}{45} R_{bcgh} R_{clefi} g^{ag} g^{hi} x^c x^e D_d x^f - \frac{1}{45} R_{bceg} R_{clfhi} g^{ah} g^{gi} D_d x^c x^e x^f \\
& - \frac{1}{45} R_{bceg} R_{clfhi} g^{ah} g^{gi} x^c D_d x^e x^f - \frac{1}{45} R_{bceg} R_{clfhi} g^{ah} g^{gi} x^c x^e D_d x^f + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} x^c D_d x^e x^f \\
& + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} x^c x^e D_d x^f + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} x^c x^e D_d x^f \\
& - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$

$$\begin{aligned}
\text{genG41.008} := & \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} \delta_d^c x^e x^f + \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} x^c \delta_d^e x^f + \frac{4}{45} R_{bgc1c} R_{ehfi} g^{ah} g^{gi} x^c x^e \delta_d^f + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} \delta_d^c x^e x^f \\
& + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} x^c \delta_d^e x^f + \frac{4}{45} R_{bcc1g} R_{ehfi} g^{ah} g^{gi} x^c x^e \delta_d^f - \frac{2}{45} R_{bgch} R_{clefi} g^{ag} g^{hi} \delta_d^c x^e x^f - \frac{2}{45} R_{bgch} R_{clefi} g^{ag} g^{hi} x^c \delta_d^e x^f \\
& - \frac{2}{45} R_{bgch} R_{clefi} g^{ag} g^{hi} x^c x^e \delta_d^f - \frac{1}{45} R_{bgch} R_{clefi} g^{ah} g^{gi} \delta_d^c x^e x^f - \frac{1}{45} R_{bgch} R_{clefi} g^{ah} g^{gi} x^c \delta_d^e x^f - \frac{1}{45} R_{bgch} R_{clefi} g^{ah} g^{gi} x^c x^e \delta_d^f \\
& + \frac{1}{40} \nabla_{bc} R_{clefg} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{bc} R_{clefg} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{bc} R_{clefg} g^{ag} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{cb} R_{clefg} g^{ag} \delta_d^c x^e x^f \\
& + \frac{1}{40} \nabla_{cb} R_{clefg} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{cb} R_{clefg} g^{ag} x^c x^e \delta_d^f + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} \delta_d^c x^e x^f + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} x^c \delta_d^e x^f + \frac{1}{20} \nabla_{ce} R_{bgc1f} g^{ag} x^c x^e \delta_d^f \\
& - \frac{2}{45} R_{bceg} R_{clhfi} g^{ah} g^{gi} \delta_d^c x^e x^f - \frac{2}{45} R_{bceg} R_{clhfi} g^{ah} g^{gi} x^c \delta_d^e x^f - \frac{2}{45} R_{bceg} R_{clhfi} g^{ah} g^{gi} x^c x^e \delta_d^f - \frac{1}{45} R_{bceg} R_{clhfi} g^{ai} g^{gh} \delta_d^c x^e x^f \\
& - \frac{1}{45} R_{bceg} R_{clhfi} g^{ai} g^{gh} x^c \delta_d^e x^f - \frac{1}{45} R_{bceg} R_{clhfi} g^{ai} g^{gh} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} x^c \delta_d^e x^f \\
& + \frac{1}{40} \nabla_{c1c} R_{befg} g^{ag} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{cc1} R_{befg} g^{ag} x^c x^e \delta_d^f + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} \delta_d^c x^e x^f \\
& + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} x^c \delta_d^e x^f + \frac{1}{20} \nabla_{ce} R_{bfc1g} g^{ag} x^c x^e \delta_d^f - \frac{1}{45} R_{bcgh} R_{clefi} g^{ag} g^{hi} \delta_d^c x^e x^f - \frac{1}{45} R_{bcgh} R_{clefi} g^{ag} g^{hi} x^c \delta_d^e x^f \\
& - \frac{1}{45} R_{bcgh} R_{clefi} g^{ag} g^{hi} x^c x^e \delta_d^f - \frac{1}{45} R_{bceg} R_{clfhi} g^{ah} g^{gi} \delta_d^c x^e x^f - \frac{1}{45} R_{bceg} R_{clfhi} g^{ah} g^{gi} x^c \delta_d^e x^f - \frac{1}{45} R_{bceg} R_{clfhi} g^{ah} g^{gi} x^c x^e \delta_d^f \\
& + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{gc} R_{bec1f} g^{ag} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} x^c \delta_d^e x^f \\
& + \frac{1}{40} \nabla_{cg} R_{bec1f} g^{ag} x^c x^e \delta_d^f - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$



$$\begin{aligned}
\text{genG41.009} := & \frac{4}{45} R_{bgc1d} R_{ehfi} g^{ah} g^{gi} x^e x^f + \frac{4}{45} R_{bgc1c} R_{dhfi} g^{ah} g^{gi} x^c x^f + \frac{4}{45} R_{bgc1c} R_{ehdi} g^{ah} g^{gi} x^c x^e + \frac{4}{45} R_{bdc1g} R_{ehfi} g^{ah} g^{gi} x^e x^f \\
& + \frac{4}{45} R_{bcc1g} R_{dhfi} g^{ah} g^{gi} x^c x^f + \frac{4}{45} R_{bcc1g} R_{ehdi} g^{ah} g^{gi} x^c x^e - \frac{2}{45} R_{bgdh} R_{c1efi} g^{ag} g^{hi} x^e x^f - \frac{2}{45} R_{bgch} R_{c1dfi} g^{ag} g^{hi} x^c x^f \\
& - \frac{2}{45} R_{bgch} R_{c1edi} g^{ag} g^{hi} x^c x^e - \frac{1}{45} R_{bgdh} R_{c1efi} g^{ah} g^{gi} x^e x^f - \frac{1}{45} R_{bgch} R_{c1dfi} g^{ah} g^{gi} x^c x^f - \frac{1}{45} R_{bgch} R_{c1edi} g^{ah} g^{gi} x^c x^e + \frac{1}{40} \nabla_{bd} R_{c1efg} g^{ag} x^e x^f \\
& + \frac{1}{40} \nabla_{bc} R_{c1dfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{bc} R_{c1edg} g^{ag} x^c x^e + \frac{1}{40} \nabla_{db} R_{c1efg} g^{ag} x^e x^f + \frac{1}{40} \nabla_{cb} R_{c1dfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{cb} R_{c1edg} g^{ag} x^c x^e \\
& + \frac{1}{20} \nabla_{de} R_{bgc1f} g^{ag} x^e x^f + \frac{1}{20} \nabla_{cd} R_{bgc1f} g^{ag} x^c x^f + \frac{1}{20} \nabla_{ce} R_{bgc1d} g^{ag} x^c x^e - \frac{2}{45} R_{bdeg} R_{c1hfi} g^{ah} g^{gi} x^e x^f - \frac{2}{45} R_{bcdg} R_{c1hfi} g^{ah} g^{gi} x^c x^f \\
& - \frac{2}{45} R_{bceg} R_{c1hdi} g^{ah} g^{gi} x^c x^e - \frac{1}{45} R_{bdeg} R_{c1hfi} g^{ai} g^{gh} x^e x^f - \frac{1}{45} R_{bcdg} R_{c1hfi} g^{ai} g^{gh} x^c x^f - \frac{1}{45} R_{bceg} R_{c1hdi} g^{ai} g^{gh} x^c x^e + \frac{1}{40} \nabla_{c1d} R_{befg} g^{ag} x^e x^f \\
& + \frac{1}{40} \nabla_{c1c} R_{bdfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{c1c} R_{bedg} g^{ag} x^c x^e + \frac{1}{40} \nabla_{dc1} R_{befg} g^{ag} x^e x^f + \frac{1}{40} \nabla_{cc1} R_{bdfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{cc1} R_{bedg} g^{ag} x^c x^e \\
& + \frac{1}{20} \nabla_{de} R_{bfc1g} g^{ag} x^e x^f + \frac{1}{20} \nabla_{cd} R_{bfc1g} g^{ag} x^c x^f + \frac{1}{20} \nabla_{ce} R_{bdc1g} g^{ag} x^c x^e - \frac{1}{45} R_{bdgh} R_{c1efi} g^{ag} g^{hi} x^e x^f - \frac{1}{45} R_{bcgh} R_{c1dfi} g^{ag} g^{hi} x^c x^f \\
& - \frac{1}{45} R_{bcgh} R_{c1edi} g^{ag} g^{hi} x^c x^e - \frac{1}{45} R_{bdeg} R_{c1fhi} g^{ah} g^{gi} x^e x^f - \frac{1}{45} R_{bcdg} R_{c1fhi} g^{ah} g^{gi} x^c x^f - \frac{1}{45} R_{bceg} R_{c1dhi} g^{ah} g^{gi} x^c x^e \\
& + \frac{1}{40} \nabla_{gd} R_{bec1f} g^{ag} x^e x^f + \frac{1}{40} \nabla_{gc} R_{bdc1f} g^{ag} x^c x^f + \frac{1}{40} \nabla_{gc} R_{bec1d} g^{ag} x^c x^e + \frac{1}{40} \nabla_{dg} R_{bec1f} g^{ag} x^e x^f + \frac{1}{40} \nabla_{cg} R_{bdc1f} g^{ag} x^c x^f \\
& + \frac{1}{40} \nabla_{cg} R_{bec1d} g^{ag} x^c x^e - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$

$$\begin{aligned}
\text{genG41.012} := & -\frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eg}x^fx^h \\
& - \frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eh}x^fx^g - \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{cf}x^ex^h - \frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{cf}x^ex^h \\
& - \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{cg}x^fx^h - \frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{ch}x^fx^g \\
& + \frac{1}{45}R_{bcde}R_{c1fgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcef}R_{c1gdh}g^{ag}g^{eh}x^cx^f + \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ae}g^{cg}x^fx^h \\
& + \frac{1}{45}R_{bcef}R_{c1gdh}g^{ah}g^{eg}x^cx^f + \frac{1}{45}R_{bcde}R_{c1fgh}g^{ae}g^{cg}x^fx^h - \frac{1}{60}\nabla_{bd}R_{c1cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{bc1}R_{dcef}g^{ae}x^cx^f \\
& - \frac{1}{60}\nabla_{c1d}R_{bcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c1b}R_{dcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{dc1}R_{bcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{db}R_{c1cef}g^{ae}x^cx^f + \frac{1}{40}\nabla_{bc}R_{c1edf}g^{af}x^cx^e \\
& + \frac{1}{40}\nabla_{bc}R_{c1edf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{c1c}R_{bedf}g^{af}x^cx^e + \frac{1}{40}\nabla_{c1c}R_{bedf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{dc}R_{bec1f}g^{af}x^cx^e + \frac{1}{40}\nabla_{dc}R_{bec1f}g^{ae}x^cx^f \\
& + \frac{1}{40}\nabla_{cb}R_{c1edf}g^{af}x^cx^e + \frac{1}{40}\nabla_{cb}R_{c1edf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cc1}R_{bedf}g^{af}x^cx^e + \frac{1}{40}\nabla_{cc1}R_{bedf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cd}R_{bec1f}g^{af}x^cx^e \\
& + \frac{1}{40}\nabla_{cd}R_{bec1f}g^{ae}x^cx^f + \frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fh}x^cx^g + \frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fg}x^cx^h + \frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{eh}x^cx^f \\
& + \frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{ch}x^ex^f + \frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{eh}x^cx^f + \frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{ch}x^ex^f + \frac{1}{120}\nabla_{cd}R_{bec1f}g^{ac}x^ex^f \\
& + \frac{1}{120}\nabla_{cc1}R_{bedf}g^{ac}x^ex^f + \frac{1}{120}\nabla_{cb}R_{c1edf}g^{ac}x^ex^f + \frac{1}{120}\nabla_{dc}R_{bec1f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{c1c}R_{bedf}g^{ac}x^ex^f + \frac{1}{120}\nabla_{bc}R_{c1edf}g^{ac}x^ex^f
\end{aligned}$$

$$\text{genG42.000} := \text{gen}G4^a_{bc1c2d}$$

$$\text{genG42.001} := D_d(\text{gen}G4^a_{bc1c2})$$

$$\begin{aligned} \text{genG42.002} := D_d \bigg( & -\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eg}x^fx^h \\ & - \frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eh}x^fx^g - \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{cf}x^ex^h - \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{cf}x^ex^h \\ & - \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{cg}x^fx^h - \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{ch}x^fx^g \\ & + \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ag}g^{eh}x^cx^f + \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ae}g^{cg}x^fx^h \\ & + \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ah}g^{eg}x^cx^f + \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ae}g^{cg}x^fx^h - \frac{1}{60}\nabla_{bc2}R_{c1cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{bc1}R_{c2cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c1c2}R_{bcef}g^{ae}x^cx^f \\ & - \frac{1}{60}\nabla_{c1b}R_{c2cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c2c1}R_{bcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c2b}R_{c1cef}g^{ae}x^cx^f + \frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{af}x^cx^e + \frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{ae}x^cx^f \\ & + \frac{1}{40}\nabla_{c1c}R_{bec2f}g^{af}x^cx^e + \frac{1}{40}\nabla_{c1c}R_{bec2f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{c2c}R_{bec1f}g^{af}x^cx^e + \frac{1}{40}\nabla_{c2c}R_{bec1f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{af}x^cx^e \\ & + \frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cc1}R_{bec2f}g^{af}x^cx^e + \frac{1}{40}\nabla_{cc1}R_{bec2f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cc2}R_{bec1f}g^{af}x^cx^e + \frac{1}{40}\nabla_{cc2}R_{bec1f}g^{ae}x^cx^f \\ & + \frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fh}x^cx^g + \frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fg}x^cx^h + \frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{eh}x^cx^f + \frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{ch}x^ex^f \\ & + \frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{eh}x^cx^f + \frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{ch}x^ex^f + \frac{1}{120}\nabla_{cc2}R_{bec1f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{cc1}R_{bec2f}g^{ac}x^ex^f \\ & + \frac{1}{120}\nabla_{cb}R_{c1ec2f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{c2c}R_{bec1f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{c1c}R_{bec2f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{bc}R_{c1ec2f}g^{ac}x^ex^f \bigg) \end{aligned}$$

$$\begin{aligned}
\text{genG42.003} := & -\frac{4}{45}D_d(R_{bcc1e}R_{c2fgh}g^{af}g^{cg}x^ex^h) - \frac{4}{45}D_d(R_{bcc2e}R_{c1fgh}g^{af}g^{cg}x^ex^h) - \frac{4}{45}D_d(R_{bcc1e}R_{c2fgh}g^{af}g^{eg}x^cx^h) \\
& - \frac{4}{45}D_d(R_{bcef}R_{c1gc2h}g^{ac}g^{eg}x^fx^h) - \frac{4}{45}D_d(R_{bcc2e}R_{c1fgh}g^{af}g^{eg}x^cx^h) - \frac{4}{45}D_d(R_{bcef}R_{c1gc2h}g^{ac}g^{eh}x^fx^g) \\
& - \frac{1}{45}D_d(R_{bcc1e}R_{c2fgh}g^{ag}g^{cf}x^ex^h) - \frac{1}{45}D_d(R_{bcc2e}R_{c1fgh}g^{ag}g^{cf}x^ex^h) - \frac{1}{45}D_d(R_{bcc1e}R_{c2fgh}g^{ag}g^{ef}x^cx^h) \\
& - \frac{1}{45}D_d(R_{bcef}R_{c1gc2h}g^{ae}g^{cg}x^fx^h) - \frac{1}{45}D_d(R_{bcc2e}R_{c1fgh}g^{ag}g^{ef}x^cx^h) - \frac{1}{45}D_d(R_{bcef}R_{c1gc2h}g^{ae}g^{ch}x^fx^g) \\
& + \frac{1}{45}D_d(R_{bcc2e}R_{c1fgh}g^{ac}g^{eg}x^fx^h) + \frac{1}{45}D_d(R_{bcc1e}R_{c2fgh}g^{ac}g^{eg}x^fx^h) + \frac{1}{45}D_d(R_{bcef}R_{c1gc2h}g^{ag}g^{eh}x^cx^f) \\
& + \frac{1}{45}D_d(R_{bcc1e}R_{c2fgh}g^{ae}g^{cg}x^fx^h) + \frac{1}{45}D_d(R_{bcef}R_{c1gc2h}g^{ah}g^{eg}x^cx^f) + \frac{1}{45}D_d(R_{bcc2e}R_{c1fgh}g^{ae}g^{cg}x^fx^h) \\
& - \frac{1}{60}D_d(\nabla_{bc2}R_{c1cef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{bc1}R_{c2cef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{c1c2}R_{bcef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{c1b}R_{c2cef}g^{ae}x^cx^f) \\
& - \frac{1}{60}D_d(\nabla_{c2c1}R_{bcef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{c2b}R_{c1cef}g^{ae}x^cx^f) + \frac{1}{40}D_d(\nabla_{bc}R_{c1ec2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{bc}R_{c1ec2f}g^{ae}x^cx^f) \\
& + \frac{1}{40}D_d(\nabla_{c1c}R_{bec2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{c1c}R_{bec2f}g^{ae}x^cx^f) + \frac{1}{40}D_d(\nabla_{c2c}R_{bec1f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{c2c}R_{bec1f}g^{ae}x^cx^f) \\
& + \frac{1}{40}D_d(\nabla_{cb}R_{c1ec2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{cb}R_{c1ec2f}g^{ae}x^cx^f) + \frac{1}{40}D_d(\nabla_{cc1}R_{bec2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{cc1}R_{bec2f}g^{ae}x^cx^f) \\
& + \frac{1}{40}D_d(\nabla_{cc2}R_{bec1f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{cc2}R_{bec1f}g^{ae}x^cx^f) + \frac{1}{15}D_d(R_{bcef}R_{c1gc2h}g^{ae}g^{fh}x^cx^g) + \frac{1}{15}D_d(R_{bcef}R_{c1gc2h}g^{ae}g^{fg}x^cx^h) \\
& + \frac{1}{15}D_d(R_{bcc2e}R_{c1fgh}g^{ag}g^{eh}x^cx^f) + \frac{1}{15}D_d(R_{bcc2e}R_{c1fgh}g^{ag}g^{ch}x^ex^f) + \frac{1}{15}D_d(R_{bcc1e}R_{c2fgh}g^{ag}g^{eh}x^cx^f) \\
& + \frac{1}{15}D_d(R_{bcc1e}R_{c2fgh}g^{ag}g^{ch}x^ex^f) + \frac{1}{120}D_d(\nabla_{cc2}R_{bec1f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{cc1}R_{bec2f}g^{ac}x^ex^f) \\
& + \frac{1}{120}D_d(\nabla_{cb}R_{c1ec2f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{c2c}R_{bec1f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{c1c}R_{bec2f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{bc}R_{c1ec2f}g^{ac}x^ex^f)
\end{aligned}$$

$$\begin{aligned}
\text{genG42.004} := & -\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{cg}D_d(x^ex^h) - \frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{cg}D_d(x^ex^h) - \frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{eg}D_d(x^cx^h) \\
& - \frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eg}D_d(x^fx^h) - \frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{eg}D_d(x^cx^h) - \frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eh}D_d(x^fx^g) \\
& - \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{cf}D_d(x^ex^h) - \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{cf}D_d(x^ex^h) - \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{ef}D_d(x^cx^h) \\
& - \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{cg}D_d(x^fx^h) - \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{ef}D_d(x^cx^h) - \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{ch}D_d(x^fx^g) \\
& + \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ac}g^{eg}D_d(x^fx^h) + \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ac}g^{eg}D_d(x^fx^h) + \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ag}g^{eh}D_d(x^cx^f) \\
& + \frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ae}g^{cg}D_d(x^fx^h) + \frac{1}{45}R_{bcef}R_{c1gc2h}g^{ah}g^{eg}D_d(x^cx^f) + \frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ae}g^{cg}D_d(x^fx^h) \\
& - \frac{1}{60}\nabla_{bc2}R_{c1cef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{bc1}R_{c2cef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{c1c2}R_{bcef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{c1b}R_{c2cef}g^{ae}D_d(x^cx^f) \\
& - \frac{1}{60}\nabla_{c2c1}R_{bcef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{c2b}R_{c1cef}g^{ae}D_d(x^cx^f) + \frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{ae}D_d(x^cx^f) \\
& + \frac{1}{40}\nabla_{c1c}R_{bec2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{c1c}R_{bec2f}g^{ae}D_d(x^cx^f) + \frac{1}{40}\nabla_{c2c}R_{bec1f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{c2c}R_{bec1f}g^{ae}D_d(x^cx^f) \\
& + \frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{ae}D_d(x^cx^f) + \frac{1}{40}\nabla_{cc1}R_{bec2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{cc1}R_{bec2f}g^{ae}D_d(x^cx^f) \\
& + \frac{1}{40}\nabla_{cc2}R_{bec1f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{cc2}R_{bec1f}g^{ae}D_d(x^cx^f) + \frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fh}D_d(x^cx^g) + \frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fg}D_d(x^cx^h) \\
& + \frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{eh}D_d(x^cx^f) + \frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{ch}D_d(x^ex^f) + \frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{eh}D_d(x^cx^f) \\
& + \frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{ch}D_d(x^ex^f) + \frac{1}{120}\nabla_{cc2}R_{bec1f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{cc1}R_{bec2f}g^{ac}D_d(x^ex^f) \\
& + \frac{1}{120}\nabla_{cb}R_{c1ec2f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{c2c}R_{bec1f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{c1c}R_{bec2f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{bc}R_{c1ec2f}g^{ac}D_d(x^ex^f)
\end{aligned}$$

$$\begin{aligned}
\text{genG42.005} := & -\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{cg}(D_dx^ex^h+x^eD_dx^h)-\frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{cg}(D_dx^ex^h+x^eD_dx^h)-\frac{4}{45}R_{bcc1e}R_{c2fgh}g^{af}g^{eg}(D_dx^cx^h+x^cD_dx^h) \\
& -\frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eg}(D_dx^fx^h+x^fD_dx^h)-\frac{4}{45}R_{bcc2e}R_{c1fgh}g^{af}g^{eg}(D_dx^cx^h+x^cD_dx^h)-\frac{4}{45}R_{bcef}R_{c1gc2h}g^{ac}g^{eh}(D_dx^fx^g+x^fD_dx^g) \\
& -\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{cf}(D_dx^ex^h+x^eD_dx^h)-\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{cf}(D_dx^ex^h+x^eD_dx^h)-\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ag}g^{ef}(D_dx^cx^h+x^cD_dx^h) \\
& -\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{cg}(D_dx^fx^h+x^fD_dx^h)-\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ag}g^{ef}(D_dx^cx^h+x^cD_dx^h)-\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ae}g^{ch}(D_dx^fx^g+x^fD_dx^g) \\
& +\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ac}g^{eg}(D_dx^fx^h+x^fD_dx^h)+\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ac}g^{eg}(D_dx^fx^h+x^fD_dx^h)+\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ag}g^{eh}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{45}R_{bcc1e}R_{c2fgh}g^{ae}g^{cg}(D_dx^fx^h+x^fD_dx^h)+\frac{1}{45}R_{bcef}R_{c1gc2h}g^{ah}g^{eg}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{45}R_{bcc2e}R_{c1fgh}g^{ae}g^{cg}(D_dx^fx^h+x^fD_dx^h) \\
& -\frac{1}{60}\nabla_{bc2}R_{c1cef}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{bc1}R_{c2cef}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{c1c2}R_{bcef}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& -\frac{1}{60}\nabla_{c1b}R_{c2cef}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{c2c1}R_{bcef}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{c2b}R_{c1cef}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{bc}R_{c1ec2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{c1c}R_{bec2f}g^{af}(D_dx^cx^e+x^cD_dx^e) \\
& +\frac{1}{40}\nabla_{c1c}R_{bec2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{c2c}R_{bec1f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{c2c}R_{bec1f}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{cb}R_{c1ec2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{cc1}R_{bec2f}g^{af}(D_dx^cx^e+x^cD_dx^e) \\
& +\frac{1}{40}\nabla_{cc1}R_{bec2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{cc2}R_{bec1f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{cc2}R_{bec1f}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fh}(D_dx^cx^g+x^cD_dx^g)+\frac{1}{15}R_{bcef}R_{c1gc2h}g^{ae}g^{fg}(D_dx^cx^h+x^cD_dx^h)+\frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{eh}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{15}R_{bcc2e}R_{c1fgh}g^{ag}g^{ch}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{eh}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{15}R_{bcc1e}R_{c2fgh}g^{ag}g^{ch}(D_dx^ex^f+x^eD_dx^f) \\
& +\frac{1}{120}\nabla_{cc2}R_{bec1f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{cc1}R_{bec2f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{cb}R_{c1ec2f}g^{ac}(D_dx^ex^f+x^eD_dx^f) \\
& +\frac{1}{120}\nabla_{c2c}R_{bec1f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{c1c}R_{bec2f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{bc}R_{c1ec2f}g^{ac}(D_dx^ex^f+x^eD_dx^f)
\end{aligned}$$

$$\begin{aligned}
\text{genG42.011} := & \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{af} g^{cg} x^e + \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{ag} g^{cf} x^e + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{af} g^{cg} x^e + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{ag} g^{cf} x^e + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{af} g^{cg} x^e \\
& + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{ag} g^{cf} x^e + \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{af} g^{eg} x^c + \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{ag} g^{ef} x^c + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{ac} g^{ef} x^g + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{ae} g^{cf} x^g \\
& + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{ac} g^{ef} x^g + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{ae} g^{cf} x^g + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{af} g^{eg} x^c + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{ag} g^{ef} x^c + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{ac} g^{eg} x^f \\
& + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{ae} g^{cg} x^f + \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{ac} g^{ef} x^g + \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{ae} g^{cf} x^g + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{af} g^{eg} x^c + \frac{1}{45} R_{bcde} R_{c1fc2g} g^{ag} g^{ef} x^c \\
& + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{ac} g^{eg} x^f + \frac{1}{45} R_{bcc2e} R_{c1fdg} g^{ae} g^{cg} x^f + \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{ac} g^{eg} x^f + \frac{1}{45} R_{bcc1e} R_{c2fdg} g^{ae} g^{cg} x^f + \frac{1}{60} \nabla_{bc2} R_{c1cde} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{bd} R_{c1cc2e} g^{ae} x^c + \frac{1}{60} \nabla_{bc1} R_{c2cde} g^{ae} x^c + \frac{1}{60} \nabla_{bd} R_{c1cc2e} g^{ac} x^e + \frac{1}{60} \nabla_{bc1} R_{c2cde} g^{ac} x^e + \frac{1}{60} \nabla_{bc2} R_{c1cde} g^{ac} x^e + \frac{1}{60} \nabla_{c1c2} R_{bcde} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{c1d} R_{bcc2e} g^{ae} x^c + \frac{1}{60} \nabla_{c1b} R_{c2cde} g^{ae} x^c + \frac{1}{60} \nabla_{c1d} R_{bcc2e} g^{ac} x^e + \frac{1}{60} \nabla_{c1b} R_{c2cde} g^{ac} x^e + \frac{1}{60} \nabla_{c1c2} R_{bcde} g^{ac} x^e + \frac{1}{60} \nabla_{c2c1} R_{bcde} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{c2d} R_{bcc1e} g^{ae} x^c + \frac{1}{60} \nabla_{c2b} R_{c1cde} g^{ae} x^c + \frac{1}{60} \nabla_{c2d} R_{bcc1e} g^{ac} x^e + \frac{1}{60} \nabla_{c2b} R_{c1cde} g^{ac} x^e + \frac{1}{60} \nabla_{c2c1} R_{bcde} g^{ac} x^e + \frac{1}{60} \nabla_{dc1} R_{bcc2e} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{dc2} R_{bcc1e} g^{ae} x^c + \frac{1}{60} \nabla_{db} R_{c1cc2e} g^{ae} x^c + \frac{1}{60} \nabla_{dc2} R_{bcc1e} g^{ac} x^e + \frac{1}{60} \nabla_{db} R_{c1cc2e} g^{ac} x^e + \frac{1}{60} \nabla_{dc1} R_{bcc2e} g^{ac} x^e
\end{aligned}$$

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ g^{a b}              -> A003^{a b}              $)
    substitute (obj,$ \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} -> A008_{\{a b c d e f g h\}} $)
    substitute (obj,$ \nabla_{\{e f g\}}\{R_{\{a b c d\}}\}      -> A007_{\{a b c d e f g\}}  $)
    substitute (obj,$ \nabla_{\{e f\}}\{R_{\{a b c d\}}\}         -> A006_{\{a b c d e f\}}   $)
    substitute (obj,$ \nabla_{\{e\}}\{R_{\{a b c d\}}\}           -> A005_{\{a b c d e\}}     $)
    substitute (obj,$ R_{\{a b c d\}}        -> A004_{\{a b c d\}}     $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> A^{a}              $)
    substitute (obj,$ A002^{a}              -> x^{a}              $)
    substitute (obj,$ A003^{a b}            -> g^{a b}            $)
    substitute (obj,$ A004_{\{a b c d\}}     -> R_{\{a b c d\}}     $)
    substitute (obj,$ A005_{\{a b c d e\}}   -> \nabla_{\{e\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A006_{\{a b c d e f\}} -> \nabla_{\{e f\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A007_{\{a b c d e f g\}} -> \nabla_{\{e f g\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A008_{\{a b c d e f g h\}} -> \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} $)

    return obj

# -----
symG20 := @(genG20) A^{b} A^{d}.                # cdb (symG20.100,symG20)

distribute (symG20)                             # cdb (symG20.101,symG20)
symG20 = product_sort (symG20)                  # cdb (symG20.102,symG20)
rename_dummies (symG20)                         # cdb (symG20.103,symG20)
canonicalise (symG20)                           # cdb (symG20.104,symG20)

# -----
symG30 := @(genG30) A^{b} A^{d}.                # cdb (symG30.100,symG30)

distribute (symG30)                             # cdb (symG30.101,symG30)
symG30 = product_sort (symG30)                  # cdb (symG30.102,symG30)
rename_dummies (symG30)                         # cdb (symG30.103,symG30)
canonicalise (symG30)                           # cdb (symG30.104,symG30)

```



```

# -----
symG40 := @(genG40) A{b} A{d}. # cdb (symG40.100,symG40)

distribute (symG40) # cdb (symG40.101,symG40)
symG40 = product_sort (symG40) # cdb (symG40.102,symG40)
rename_dummies (symG40) # cdb (symG40.103,symG40)
canonicalise (symG40) # cdb (symG40.104,symG40)

# -----
symG50 := @(genG50) A{b} A{d}. # cdb (symG50.100,symG50)

distribute (symG50) # cdb (symG50.101,symG50)
symG50 = product_sort (symG50) # cdb (symG50.102,symG50)
rename_dummies (symG50) # cdb (symG50.103,symG50)
canonicalise (symG50) # cdb (symG50.104,symG50)

# -----
symG31 := @(genG31) A{b} A{c1} A{d}. # cdb (symG31.100,symG31)

distribute (symG31) # cdb (symG31.101,symG31)
symG31 = product_sort (symG31) # cdb (symG31.102,symG31)
rename_dummies (symG31) # cdb (symG31.103,symG31)
canonicalise (symG31) # cdb (symG31.104,symG31)

# -----
symG41 := @(genG41) A{b} A{c1} A{d}. # cdb (symG41.100,symG41)

distribute (symG41) # cdb (symG41.101,symG41)
symG41 = product_sort (symG41) # cdb (symG41.102,symG41)
rename_dummies (symG41) # cdb (symG41.103,symG41)
canonicalise (symG41) # cdb (symG41.104,symG41)

# -----
symG51 := @(genG51) A{b} A{c1} A{d}. # cdb (symG51.100,symG51)

distribute (symG51) # cdb (symG51.101,symG51)
symG51 = product_sort (symG51) # cdb (symG51.102,symG51)

```

```

rename_dummies      (symG51)                # cdb (symG51.103,symG51)
canonicalise        (symG51)                # cdb (symG51.104,symG51)

# -----
symG42 := @(genG42) A^{b} A^{c1} A^{c2} A^{d}.      # cdb (symG42.100,symG42)

distribute          (symG42)                # cdb (symG42.101,symG42)
symG42 = product_sort (symG42)              # cdb (symG42.102,symG42)
rename_dummies      (symG42)                # cdb (symG42.103,symG42)
canonicalise        (symG42)                # cdb (symG42.104,symG42)

# -----
symG52 := @(genG52) A^{b} A^{c1} A^{c2} A^{d}.      # cdb (symG52.100,symG52)

distribute          (symG52)                # cdb (symG52.101,symG52)
symG52 = product_sort (symG52)              # cdb (symG52.102,symG52)
rename_dummies      (symG52)                # cdb (symG52.103,symG52)
canonicalise        (symG52)                # cdb (symG52.104,symG52)

# -----
symG53 := @(genG53) A^{b} A^{c1} A^{c2} A^{c3} A^{d}.  # cdb (symG53.100,symG53)

distribute          (symG53)                # cdb (symG53.101,symG53)
symG53 = product_sort (symG53)              # cdb (symG53.102,symG53)
rename_dummies      (symG53)                # cdb (symG53.103,symG53)
canonicalise        (symG53)                # cdb (symG53.104,symG53)

```

$$\begin{aligned}
\text{symG31.100} &:= \left( \frac{1}{12} \nabla_b R_{c1cde} g^{ae} x^c + \frac{1}{12} \nabla_b R_{c1cde} g^{ac} x^e + \frac{1}{12} \nabla_{c1} R_{bcde} g^{ae} x^c + \frac{1}{12} \nabla_{c1} R_{bcde} g^{ac} x^e + \frac{1}{12} \nabla_d R_{bcc1e} g^{ae} x^c + \frac{1}{12} \nabla_d R_{bcc1e} g^{ac} x^e \right) A^b A^{c1} A^d \\
\text{symG31.101} &:= \frac{1}{12} \nabla_b R_{c1cde} g^{ae} x^c A^b A^{c1} A^d + \frac{1}{12} \nabla_b R_{c1cde} g^{ac} x^e A^b A^{c1} A^d + \frac{1}{12} \nabla_{c1} R_{bcde} g^{ae} x^c A^b A^{c1} A^d \\
&\quad + \frac{1}{12} \nabla_{c1} R_{bcde} g^{ac} x^e A^b A^{c1} A^d + \frac{1}{12} \nabla_d R_{bcc1e} g^{ae} x^c A^b A^{c1} A^d + \frac{1}{12} \nabla_d R_{bcc1e} g^{ac} x^e A^b A^{c1} A^d \\
\text{symG31.102} &:= \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_b R_{cfde} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_c R_{bedf} \\
&\quad + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_c R_{bfde} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_d R_{becf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_d R_{bfce} \\
\text{symG31.103} &:= \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} + \frac{1}{12} A^b A^c A^d x^f g^{ae} \nabla_b R_{cedf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_c R_{bedf} \\
&\quad + \frac{1}{12} A^b A^c A^d x^f g^{ae} \nabla_c R_{bedf} + \frac{1}{12} A^b A^c A^d x^e g^{af} \nabla_d R_{becf} + \frac{1}{12} A^b A^c A^d x^f g^{ae} \nabla_d R_{becf} \\
\text{symG31.104} &:= \frac{1}{2} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf}
\end{aligned}$$

$$\begin{aligned}
\text{symG41.100} := & \left( -\frac{4}{45} R_{bcc1e} R_{dfgh} g^{af} g^{cg} x^e x^h - \frac{4}{45} R_{bcde} R_{c1fgh} g^{af} g^{cg} x^e x^h - \frac{4}{45} R_{bcc1e} R_{dfgh} g^{af} g^{eg} x^c x^h - \frac{4}{45} R_{bcef} R_{c1gdh} g^{ac} g^{eg} x^f x^h \right. \\
& - \frac{4}{45} R_{bcde} R_{c1fgh} g^{af} g^{eg} x^c x^h - \frac{4}{45} R_{bcef} R_{c1gdh} g^{ac} g^{eh} x^f x^g - \frac{1}{45} R_{bcc1e} R_{dfgh} g^{ag} g^{cf} x^e x^h - \frac{1}{45} R_{bcde} R_{c1fgh} g^{ag} g^{cf} x^e x^h \\
& - \frac{1}{45} R_{bcc1e} R_{dfgh} g^{ag} g^{ef} x^c x^h - \frac{1}{45} R_{bcef} R_{c1gdh} g^{ae} g^{cg} x^f x^h - \frac{1}{45} R_{bcde} R_{c1fgh} g^{ag} g^{ef} x^c x^h - \frac{1}{45} R_{bcef} R_{c1gdh} g^{ae} g^{ch} x^f x^g \\
& + \frac{1}{45} R_{bcde} R_{c1fgh} g^{ac} g^{eg} x^f x^h + \frac{1}{45} R_{bcc1e} R_{dfgh} g^{ac} g^{eg} x^f x^h + \frac{1}{45} R_{bcef} R_{c1gdh} g^{ag} g^{eh} x^c x^f + \frac{1}{45} R_{bcc1e} R_{dfgh} g^{ae} g^{cg} x^f x^h \\
& + \frac{1}{45} R_{bcef} R_{c1gdh} g^{ah} g^{eg} x^c x^f + \frac{1}{45} R_{bcde} R_{c1fgh} g^{ae} g^{cg} x^f x^h - \frac{1}{60} \nabla_{bd} R_{c1cef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{bc1} R_{dcef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{c1d} R_{bcef} g^{ae} x^c x^f \\
& - \frac{1}{60} \nabla_{c1b} R_{dcef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{dc1} R_{bcef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{db} R_{c1cef} g^{ae} x^c x^f + \frac{1}{40} \nabla_{bc} R_{c1edf} g^{af} x^c x^e + \frac{1}{40} \nabla_{bc} R_{c1edf} g^{ae} x^c x^f \\
& + \frac{1}{40} \nabla_{c1c} R_{bedf} g^{af} x^c x^e + \frac{1}{40} \nabla_{c1c} R_{bedf} g^{ae} x^c x^f + \frac{1}{40} \nabla_{dc} R_{bec1f} g^{af} x^c x^e + \frac{1}{40} \nabla_{dc} R_{bec1f} g^{ae} x^c x^f + \frac{1}{40} \nabla_{cb} R_{c1edf} g^{af} x^c x^e \\
& + \frac{1}{40} \nabla_{cb} R_{c1edf} g^{ae} x^c x^f + \frac{1}{40} \nabla_{cc1} R_{bedf} g^{af} x^c x^e + \frac{1}{40} \nabla_{cc1} R_{bedf} g^{ae} x^c x^f + \frac{1}{40} \nabla_{cd} R_{bec1f} g^{af} x^c x^e + \frac{1}{40} \nabla_{cd} R_{bec1f} g^{ae} x^c x^f \\
& + \frac{1}{15} R_{bcef} R_{c1gdh} g^{ae} g^{fh} x^c x^g + \frac{1}{15} R_{bcef} R_{c1gdh} g^{ae} g^{fg} x^c x^h + \frac{1}{15} R_{bcde} R_{c1fgh} g^{ag} g^{eh} x^c x^f + \frac{1}{15} R_{bcde} R_{c1fgh} g^{ag} g^{ch} x^e x^f \\
& + \frac{1}{15} R_{bcc1e} R_{dfgh} g^{ag} g^{eh} x^c x^f + \frac{1}{15} R_{bcc1e} R_{dfgh} g^{ag} g^{ch} x^e x^f + \frac{1}{120} \nabla_{cd} R_{bec1f} g^{ac} x^e x^f + \frac{1}{120} \nabla_{cc1} R_{bedf} g^{ac} x^e x^f + \frac{1}{120} \nabla_{cb} R_{c1edf} g^{ac} x^e x^f \\
& \left. + \frac{1}{120} \nabla_{dc} R_{bec1f} g^{ac} x^e x^f + \frac{1}{120} \nabla_{c1c} R_{bedf} g^{ac} x^e x^f + \frac{1}{120} \nabla_{bc} R_{c1edf} g^{ac} x^e x^f \right) A^b A^{c1} A^d
\end{aligned}$$

$$\begin{aligned}
\text{symG41.101} := & -\frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{cg}x^ex^hA^bA^{c1}A^d - \frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{cg}x^ex^hA^bA^{c1}A^d - \frac{4}{45}R_{bcc1e}R_{dfgh}g^{af}g^{eg}x^cx^hA^bA^{c1}A^d \\
& - \frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eg}x^fx^hA^bA^{c1}A^d - \frac{4}{45}R_{bcde}R_{c1fgh}g^{af}g^{eg}x^cx^hA^bA^{c1}A^d - \frac{4}{45}R_{bcef}R_{c1gdh}g^{ac}g^{eh}x^fx^gA^bA^{c1}A^d \\
& - \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{cf}x^ex^hA^bA^{c1}A^d - \frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{cf}x^ex^hA^bA^{c1}A^d - \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ag}g^{ef}x^cx^hA^bA^{c1}A^d \\
& - \frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{cg}x^fx^hA^bA^{c1}A^d - \frac{1}{45}R_{bcde}R_{c1fgh}g^{ag}g^{ef}x^cx^hA^bA^{c1}A^d - \frac{1}{45}R_{bcef}R_{c1gdh}g^{ae}g^{ch}x^fx^gA^bA^{c1}A^d \\
& + \frac{1}{45}R_{bcde}R_{c1fgh}g^{ac}g^{eg}x^fx^hA^bA^{c1}A^d + \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ac}g^{eg}x^fx^hA^bA^{c1}A^d + \frac{1}{45}R_{bcef}R_{c1gdh}g^{ag}g^{eh}x^cx^fA^bA^{c1}A^d \\
& + \frac{1}{45}R_{bcc1e}R_{dfgh}g^{ae}g^{cg}x^fx^hA^bA^{c1}A^d + \frac{1}{45}R_{bcef}R_{c1gdh}g^{ah}g^{eg}x^cx^fA^bA^{c1}A^d + \frac{1}{45}R_{bcde}R_{c1fgh}g^{ae}g^{cg}x^fx^hA^bA^{c1}A^d \\
& - \frac{1}{60}\nabla_{bd}R_{c1cef}g^{ae}x^cx^fA^bA^{c1}A^d - \frac{1}{60}\nabla_{bc1}R_{dcef}g^{ae}x^cx^fA^bA^{c1}A^d - \frac{1}{60}\nabla_{c1d}R_{bcef}g^{ae}x^cx^fA^bA^{c1}A^d \\
& - \frac{1}{60}\nabla_{c1b}R_{dcef}g^{ae}x^cx^fA^bA^{c1}A^d - \frac{1}{60}\nabla_{dc1}R_{bcef}g^{ae}x^cx^fA^bA^{c1}A^d - \frac{1}{60}\nabla_{db}R_{c1cef}g^{ae}x^cx^fA^bA^{c1}A^d \\
& + \frac{1}{40}\nabla_{bc}R_{c1edf}g^{af}x^cx^eA^bA^{c1}A^d + \frac{1}{40}\nabla_{bc}R_{c1edf}g^{ae}x^cx^fA^bA^{c1}A^d + \frac{1}{40}\nabla_{c1c}R_{bedf}g^{af}x^cx^eA^bA^{c1}A^d + \frac{1}{40}\nabla_{c1c}R_{bedf}g^{ae}x^cx^fA^bA^{c1}A^d \\
& + \frac{1}{40}\nabla_{dc}R_{bec1f}g^{af}x^cx^eA^bA^{c1}A^d + \frac{1}{40}\nabla_{dc}R_{bec1f}g^{ae}x^cx^fA^bA^{c1}A^d + \frac{1}{40}\nabla_{cb}R_{c1edf}g^{af}x^cx^eA^bA^{c1}A^d + \frac{1}{40}\nabla_{cb}R_{c1edf}g^{ae}x^cx^fA^bA^{c1}A^d \\
& + \frac{1}{40}\nabla_{cc1}R_{bedf}g^{af}x^cx^eA^bA^{c1}A^d + \frac{1}{40}\nabla_{cc1}R_{bedf}g^{ae}x^cx^fA^bA^{c1}A^d + \frac{1}{40}\nabla_{cd}R_{bec1f}g^{af}x^cx^eA^bA^{c1}A^d + \frac{1}{40}\nabla_{cd}R_{bec1f}g^{ae}x^cx^fA^bA^{c1}A^d \\
& + \frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fh}x^cx^gA^bA^{c1}A^d + \frac{1}{15}R_{bcef}R_{c1gdh}g^{ae}g^{fg}x^cx^hA^bA^{c1}A^d + \frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{eh}x^cx^fA^bA^{c1}A^d \\
& + \frac{1}{15}R_{bcde}R_{c1fgh}g^{ag}g^{ch}x^ex^fA^bA^{c1}A^d + \frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{eh}x^cx^fA^bA^{c1}A^d + \frac{1}{15}R_{bcc1e}R_{dfgh}g^{ag}g^{ch}x^ex^fA^bA^{c1}A^d \\
& + \frac{1}{120}\nabla_{cd}R_{bec1f}g^{ac}x^ex^fA^bA^{c1}A^d + \frac{1}{120}\nabla_{cc1}R_{bedf}g^{ac}x^ex^fA^bA^{c1}A^d + \frac{1}{120}\nabla_{cb}R_{c1edf}g^{ac}x^ex^fA^bA^{c1}A^d \\
& + \frac{1}{120}\nabla_{dc}R_{bec1f}g^{ac}x^ex^fA^bA^{c1}A^d + \frac{1}{120}\nabla_{c1c}R_{bedf}g^{ac}x^ex^fA^bA^{c1}A^d + \frac{1}{120}\nabla_{bc}R_{c1edf}g^{ac}x^ex^fA^bA^{c1}A^d
\end{aligned}$$

$$\begin{aligned}
\text{symG41.102} := & -\frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhce}R_{dgif} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhde}R_{cgif} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bech}R_{dgif} \\
& - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bghe}R_{cidf} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bedh}R_{cgif} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bghe}R_{cfdi} \\
& - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhce}R_{digf} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhde}R_{cgif} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bech}R_{digf} \\
& - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhge}R_{cidf} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bedh}R_{cgif} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhge}R_{cfdi} \\
& + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bgdh}R_{ceif} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bgch}R_{deif} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{behf}R_{cgdi} \\
& + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhcg}R_{deif} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{behf}R_{cidg} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhdg}R_{ceif} \\
& - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{bd}R_{ceg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{bc}R_{deg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{cd}R_{beg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{cb}R_{deg f} \\
& - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{dc}R_{beg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{db}R_{ceg f} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{be}R_{cf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{be}R_{cgdf} \\
& + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ce}R_{bf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ce}R_{bgdf} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{de}R_{bf cg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{de}R_{bgcf} \\
& + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{eb}R_{cf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{eb}R_{cgdf} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ec}R_{bf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ec}R_{bgdf} \\
& + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ed}R_{bf cg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ed}R_{bgcf} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{begh}R_{cf di} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{begh}R_{cidf} \\
& + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bedh}R_{cf gi} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bhde}R_{cf gi} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bech}R_{df gi} \\
& + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bhce}R_{df gi} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{gd}R_{becf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{gc}R_{bedf} \\
& + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{gb}R_{cedf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{dg}R_{becf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{cg}R_{bedf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{bg}R_{cedf}
\end{aligned}$$

$$\begin{aligned}
\text{symG41.103} := & -\frac{4}{45}A^bA^cA^dA^fx^ig^{ag}g^{eh}R_{becf}R_{dghi} - \frac{4}{45}A^bA^cA^dA^fx^ig^{ag}g^{eh}R_{bedf}R_{cg hi} - \frac{4}{45}A^bA^cA^dA^ex^ig^{ag}g^{fh}R_{becf}R_{dghi} \\
& - \frac{4}{45}A^bA^cA^dA^gx^ig^{ae}g^{fh}R_{befg}R_{chdi} - \frac{4}{45}A^bA^cA^dA^ex^ig^{ag}g^{fh}R_{bedf}R_{cg hi} - \frac{4}{45}A^bA^cA^dA^gx^hg^{ae}g^{fi}R_{befg}R_{chdi} \\
& - \frac{1}{45}A^bA^cA^dA^fx^ig^{ah}g^{eg}R_{becf}R_{dghi} - \frac{1}{45}A^bA^cA^dA^fx^ig^{ah}g^{eg}R_{bedf}R_{cg hi} - \frac{1}{45}A^bA^cA^dA^ex^ig^{ah}g^{fg}R_{becf}R_{dghi} \\
& - \frac{1}{45}A^bA^cA^dA^gx^ig^{af}g^{eh}R_{befg}R_{chdi} - \frac{1}{45}A^bA^cA^dA^ex^ig^{ah}g^{fg}R_{bedf}R_{cg hi} - \frac{1}{45}A^bA^cA^dA^gx^hg^{af}g^{ei}R_{befg}R_{chdi} \\
& + \frac{1}{45}A^bA^cA^dA^gx^ig^{ae}g^{fh}R_{bedf}R_{cg hi} + \frac{1}{45}A^bA^cA^dA^gx^ig^{ae}g^{fh}R_{becf}R_{dghi} + \frac{1}{45}A^bA^cA^dA^ex^ig^{ah}g^{fi}R_{befg}R_{chdi} \\
& + \frac{1}{45}A^bA^cA^dA^gx^ig^{af}g^{eh}R_{becf}R_{dghi} + \frac{1}{45}A^bA^cA^dA^ex^ig^{ai}g^{fh}R_{befg}R_{chdi} + \frac{1}{45}A^bA^cA^dA^gx^ig^{af}g^{eh}R_{bedf}R_{cg hi} \\
& - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{bd}R_{cefg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{bc}R_{defg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{cd}R_{befg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{cb}R_{defg} \\
& - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{dc}R_{befg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{db}R_{cefg} + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{bg}R_{cedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{bg}R_{cedf} \\
& + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{cg}R_{bedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{cg}R_{bedf} + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{dg}R_{becf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{dg}R_{becf} \\
& + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{gc}R_{bedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{gc}R_{bedf} \\
& + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{gd}R_{becf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{gd}R_{becf} + \frac{1}{15}A^bA^cA^dA^ex^hg^{af}g^{gi}R_{befg}R_{chdi} + \frac{1}{15}A^bA^cA^dA^ex^ig^{af}g^{gh}R_{befg}R_{chdi} \\
& + \frac{1}{15}A^bA^cA^dA^ex^ig^{ah}g^{fi}R_{bedf}R_{cg hi} + \frac{1}{15}A^bA^cA^dA^fx^hg^{ah}g^{ei}R_{bedf}R_{cg hi} + \frac{1}{15}A^bA^cA^dA^ex^ig^{ah}g^{fi}R_{becf}R_{dghi} \\
& + \frac{1}{15}A^bA^cA^dA^fx^hg^{ah}g^{ei}R_{becf}R_{dghi} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{gd}R_{becf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{gc}R_{bedf} \\
& + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{gb}R_{cedf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{dg}R_{becf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{cg}R_{bedf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{bg}R_{cedf} \\
\text{symG41.104} := & \frac{8}{15}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{bech}R_{dghi} + \frac{2}{15}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{bech}R_{difg} - \frac{2}{15}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{befh}R_{cg di} \\
& + \frac{1}{10}A^bA^cA^dA^ex^fg^{ag}\nabla_{bc}R_{defg} + \frac{3}{20}A^bA^cA^dA^ex^fg^{ag}\nabla_{be}R_{cfdg} + \frac{3}{20}A^bA^cA^dA^ex^fg^{ag}\nabla_{eb}R_{cfdg} \\
& + \frac{2}{5}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{bech}R_{dfgi} + \frac{1}{40}A^bA^cA^dA^ex^fg^{ag}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dA^ex^fg^{ag}\nabla_{bg}R_{cedf}
\end{aligned}$$

$$\begin{aligned}
\text{symG51.104} := & \frac{8}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + \frac{4}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} \\
& + \frac{1}{10} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} + \frac{1}{90} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + \frac{11}{90} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} \\
& + \frac{4}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\
& + \frac{1}{12} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} + \frac{1}{36} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\
& - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - \frac{2}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} \\
& - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bce} R_{dfgh} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bec} R_{dfgh} \\
& + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bef} R_{cgdh} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebc} R_{dfgh} + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebf} R_{cgdh} \\
& + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{efb} R_{cgdh} + \frac{4}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + \frac{1}{5} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\
& + \frac{4}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + \frac{1}{5} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} \\
& - \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{hbe} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{heb} R_{cfdg} \\
& + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bhe} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ehb} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{beh} R_{cfdg} \\
& + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebh} R_{cfdg} - \frac{1}{9} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} + \frac{1}{18} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}
\end{aligned}$$



$$\text{symG42.104} := \frac{8}{15} A^b A^c A^d A^e x^f g^{ag} g^{hi} R_{bfch} R_{dgei} + \frac{2}{5} A^b A^c A^d A^e x^f g^{ag} \nabla_{bc} R_{dfeg}$$

$$\begin{aligned} \text{symG52.104} := & \frac{32}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + \frac{1}{5} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{4}{15} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} \\ & + \frac{2}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + \frac{22}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} \\ & + \frac{1}{5} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{4}{15} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + \frac{1}{9} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\ & - \frac{8}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + \frac{1}{15} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bcd} R_{efgh} + \frac{4}{45} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bcf} R_{dgeh} \\ & + \frac{4}{45} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bfc} R_{dgeh} + \frac{4}{45} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{fbc} R_{dgeh} + \frac{13}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} \\ & + \frac{1}{15} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + \frac{23}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + \frac{1}{90} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{hbc} R_{dfeg} \\ & + \frac{1}{90} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bhc} R_{dfeg} + \frac{1}{90} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bch} R_{dfeg} - \frac{4}{9} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh} \end{aligned}$$

$$\text{symG53.104} := A^b A^c A^d A^e A^f x^g g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + A^b A^c A^d A^e A^f x^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + \frac{1}{3} A^b A^c A^d A^e A^f x^g g^{ah} \nabla_{bcd} R_{egfh}$$

```

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    factor_out (bah,$A^{a?},x^{b?}$)
    ans := @(bah) / @(foo).
    return ans

fooG20 = reformat (symG20,3)
fooG30 = reformat (symG30,12)
fooG40 = reformat (symG40,360)
fooG50 = reformat (symG50,180)

fooG31 = reformat (symG31,2)
fooG41 = reformat (symG41,120)
fooG51 = reformat (symG51,180)

fooG42 = reformat (symG42,15)
fooG52 = reformat (symG52,90)

fooG53 = reformat (symG53,3)

genGamma0 := @(fooG20) + @(fooG30) + @(fooG40) + @(fooG50). # cdb (genGamma0.000,genGamma0)
genGamma1 := @(fooG31) + @(fooG41) + @(fooG51). # cdb (genGamma1.000,genGamma1)
genGamma2 := @(fooG42) + @(fooG52). # cdb (genGamma2.000,genGamma2)
genGamma3 := @(fooG53). # cdb (genGamma3.000,genGamma3)

cdblib.create ('genGamma.json')

cdblib.put ('genGamma0',genGamma0,'genGamma.json')
cdblib.put ('genGamma1',genGamma1,'genGamma.json')
cdblib.put ('genGamma2',genGamma2,'genGamma.json')
cdblib.put ('genGamma3',genGamma3,'genGamma.json')

cdblib.put ('genGamma01',fooG20,'genGamma.json')
cdblib.put ('genGamma02',fooG30,'genGamma.json')
cdblib.put ('genGamma03',fooG40,'genGamma.json')
cdblib.put ('genGamma04',fooG50,'genGamma.json')

```

```
cdblib.put ('genGamma11',fooG31,'genGamma.json')
cdblib.put ('genGamma12',fooG41,'genGamma.json')
cdblib.put ('genGamma13',fooG51,'genGamma.json')

cdblib.put ('genGamma21',fooG42,'genGamma.json')
cdblib.put ('genGamma22',fooG52,'genGamma.json')

cdblib.put ('genGamma31',fooG53,'genGamma.json')
```

# The generalised connection in Riemann normal coordinates

$$\begin{aligned}
A^b A^c \Gamma_{bc}^a(x) &= \frac{2}{3} A^b A^c x^d g^{ae} R_{bdce} + \frac{1}{12} A^b A^c x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce}) \\
&\quad + \frac{1}{360} A^b A^c x^d x^e x^f (64g^{ag} g^{hi} R_{bdch} R_{egfi} - 32g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18g^{ag} \nabla_{bd} R_{cefg} + 18g^{ag} \nabla_{db} R_{cefg} + 36g^{ag} \nabla_{de} R_{bfcg} \\
&\quad - 16g^{ag} g^{hi} R_{bdeh} R_{cfig} + 9g^{ag} \nabla_{gd} R_{becf} + 9g^{ag} \nabla_{dg} R_{becf}) + \frac{1}{180} A^b A^c x^d x^e x^f x^g (16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} \\
&\quad + 16g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 8g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} \\
&\quad - 8g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 2g^{ah} \nabla_{bde} R_{cfgh} + 2g^{ah} \nabla_{dbe} R_{cfgh} + 2g^{ah} \nabla_{deb} R_{cfgh} + 4g^{ah} \nabla_{def} R_{bgch} \\
&\quad - 4g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + g^{ah} \nabla_{hde} R_{bfcg} + g^{ah} \nabla_{dhe} R_{bfcg} + g^{ah} \nabla_{deh} R_{bfcg}) \\
A^b A^c A^d \Gamma_{bcd}^a(x) &= \frac{1}{2} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} + \frac{1}{120} A^b A^c A^d x^e x^f (64g^{ag} g^{hi} R_{bech} R_{dgfi} + 16g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{befh} R_{cgdi} + 12g^{ag} \nabla_{bc} R_{defg} \\
&\quad + 18g^{ag} \nabla_{be} R_{cfdg} + 18g^{ag} \nabla_{eb} R_{cfdg} + 48g^{ag} g^{hi} R_{bech} R_{dfgi} + 3g^{ag} \nabla_{gb} R_{cedf} + 3g^{ag} \nabla_{bg} R_{cedf}) \\
&\quad + \frac{1}{180} A^b A^c A^d x^e x^f x^g (32g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 48g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + 18g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
&\quad + 2g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 22g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} \\
&\quad + 5g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} - 12g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - 8g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - 12g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} \\
&\quad + 4g^{ah} \nabla_{bce} R_{dfgh} + 4g^{ah} \nabla_{bec} R_{dfgh} + 6g^{ah} \nabla_{bef} R_{cgdh} + 4g^{ah} \nabla_{ebc} R_{dfgh} + 6g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{efb} R_{cgdh} + 16g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} \\
&\quad + 36g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + 16g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 4g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 36g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - 4g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} \\
&\quad + g^{ah} \nabla_{hbe} R_{cfdg} + g^{ah} \nabla_{heb} R_{cfdg} + g^{ah} \nabla_{bhe} R_{cfdg} + g^{ah} \nabla_{ehb} R_{cfdg} + g^{ah} \nabla_{beh} R_{cfdg} + g^{ah} \nabla_{ebh} R_{cfdg} - 20g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} \\
&\quad + 10g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}) \\
A^b A^c A^d A^e \Gamma_{bcde}^a(x) &= \frac{1}{15} A^b A^c A^d A^e x^f (8g^{ag} g^{hi} R_{bfch} R_{dgei} + 6g^{ag} \nabla_{bc} R_{dfeg}) \\
&\quad + \frac{1}{90} A^b A^c A^d A^e x^f x^g (64g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + 4g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} \\
&\quad + 44g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 10g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} - 16g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} \\
&\quad + 6g^{ah} \nabla_{bcd} R_{efgh} + 8g^{ah} \nabla_{bcf} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{fbc} R_{dgeh} + 26g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} + 6g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgeh} \\
&\quad + 46g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah} \nabla_{hbc} R_{dfeg} + g^{ah} \nabla_{bhc} R_{dfeg} + g^{ah} \nabla_{bch} R_{dfeg} - 40g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh}) \\
A^b A^c A^d A^e A^f \Gamma_{bcdef}^a(x) &= \frac{1}{3} A^b A^c A^d A^e A^f x^g (3g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah} \nabla_{bcd} R_{egfh})
\end{aligned}$$

```
scaledGamma0 := 360 @(genGamma0). # cdb (scaledGamma0.001,scaledGamma0)
scaledGamma1 := 360 @(genGamma1). # cdb (scaledGamma1.001,scaledGamma1)
scaledGamma2 := 90  @(genGamma2). # cdb (scaledGamma2.001,scaledGamma2)
scaledGamma3 := 3   @(genGamma3). # cdb (scaledGamma3.001,scaledGamma3)
```

# The generalised connection in Riemann normal coordinates

This is the same as the previous page but with a small change in the format to avoid fractions.

$$\begin{aligned}
360A^bA^c\Gamma_{bc}^a(x) = & 240A^bA^c x^d g^{ae} R_{bdce} + 30A^bA^c x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce}) \\
& + A^bA^c x^d x^e x^f (64g^{ag} g^{hi} R_{bdch} R_{egfi} - 32g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdeh} R_{cfig} + 18g^{ag} \nabla_{bd} R_{cefg} \\
& \quad + 18g^{ag} \nabla_{db} R_{cefg} + 36g^{ag} \nabla_{de} R_{bfeg} - 16g^{ag} g^{hi} R_{bdeh} R_{cfig} + 9g^{ag} \nabla_{gd} R_{becf} + 9g^{ag} \nabla_{dg} R_{becf}) \\
& + 2A^bA^c x^d x^e x^f x^g (16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} + 16g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfeg} \\
& \quad - 8g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} - 8g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} \\
& \quad + 2g^{ah} \nabla_{bde} R_{cfgh} + 2g^{ah} \nabla_{dbe} R_{cfgh} + 2g^{ah} \nabla_{deb} R_{cfgh} + 4g^{ah} \nabla_{def} R_{bgch} - 4g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} \\
& \quad - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cgjh} + g^{ah} \nabla_{hde} R_{bfeg} + g^{ah} \nabla_{dhe} R_{bfeg} + g^{ah} \nabla_{deh} R_{bfeg})
\end{aligned}$$

$$\begin{aligned}
360A^bA^cA^d\Gamma_{bcd}^a(x) = & 180A^bA^cA^d x^e g^{af} \nabla_b R_{cedf} + 3A^bA^cA^d x^e x^f (64g^{ag} g^{hi} R_{bech} R_{dghi} + 16g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{befh} R_{cgdi} \\
& \quad + 12g^{ag} \nabla_{bc} R_{defg} + 18g^{ag} \nabla_{be} R_{cfdg} + 18g^{ag} \nabla_{eb} R_{cfdg} + 48g^{ag} g^{hi} R_{bech} R_{dfgi} + 3g^{ag} \nabla_{gb} R_{cedf} + 3g^{ag} \nabla_{bg} R_{cedf}) \\
& + 2A^bA^cA^d x^e x^f x^g (32g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 48g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + 18g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
& \quad + 2g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 22g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} \\
& \quad + 5g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} - 12g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - 8g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - 12g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} \\
& \quad + 4g^{ah} \nabla_{bce} R_{dfgh} + 4g^{ah} \nabla_{bec} R_{dfgh} + 6g^{ah} \nabla_{bef} R_{cgdh} + 4g^{ah} \nabla_{ebc} R_{dfgh} + 6g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{efb} R_{cgdh} \\
& \quad + 16g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + 36g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + 16g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 4g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 36g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} \\
& \quad - 4g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + g^{ah} \nabla_{hbe} R_{cfdg} + g^{ah} \nabla_{heb} R_{cfdg} + g^{ah} \nabla_{bhe} R_{cfdg} + g^{ah} \nabla_{ehb} R_{cfdg} + g^{ah} \nabla_{beh} R_{cfdg} + g^{ah} \nabla_{ebh} R_{cfdg} \\
& \quad - 20g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} + 10g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg})
\end{aligned}$$

$$\begin{aligned}
90A^bA^cA^dA^e\Gamma_{bcde}^a(x) = & 6A^bA^cA^dA^e x^f (8g^{ag} g^{hi} R_{bfch} R_{dgei} + 6g^{ag} \nabla_{bc} R_{dfeg}) \\
& + A^bA^cA^dA^e x^f x^g (64g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + 4g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} \\
& \quad + 44g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 10g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\
& \quad - 16g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + 6g^{ah} \nabla_{bcd} R_{efgh} + 8g^{ah} \nabla_{bcf} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{fbc} R_{dgeh} + 26g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} \\
& \quad + 6g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + 46g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah} \nabla_{hbc} R_{dfeg} + g^{ah} \nabla_{bhc} R_{dfeg} + g^{ah} \nabla_{bch} R_{dfeg} - 40g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh})
\end{aligned}$$

$$3A^bA^cA^dA^eA^f\Gamma_{bcdef}^a(x) = A^bA^cA^dA^eA^f x^g (3g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah} \nabla_{bcd} R_{egfh})$$

```

tmp0 := @(fooG20) + @(fooG30).
tmp1 := @(fooG31).

alt0 := @(genGamma0).
alt1 := @(genGamma1).
alt2 := @(genGamma2).
alt3 := @(genGamma3).

alt0scaled := @(scaledGamma0).
alt1scaled := @(scaledGamma1).
alt2scaled := @(scaledGamma2).
alt3scaled := @(scaledGamma3).

substitute (tmp0, $A^{a}->1$)
substitute (tmp1, $A^{a}->1$)

substitute (alt0, $A^{a}->1$)
substitute (alt1, $A^{a}->1$)
substitute (alt2, $A^{a}->1$)
substitute (alt3, $A^{a}->1$)

substitute (alt0scaled, $A^{a}->1$)
substitute (alt1scaled, $A^{a}->1$)
substitute (alt2scaled, $A^{a}->1$)
substitute (alt3scaled, $A^{a}->1$)

cdblib.create ('genGamma.export')

# 4th order gen gamma
cdblib.put ('gen_gamma_0_4th',tmp0,'genGamma.export')
cdblib.put ('gen_gamma_1_4th',tmp1,'genGamma.export')

# 6th order gen gamma
cdblib.put ('gen_gamma_0',alt0,'genGamma.export')
cdblib.put ('gen_gamma_1',alt1,'genGamma.export')
cdblib.put ('gen_gamma_2',alt2,'genGamma.export')
cdblib.put ('gen_gamma_3',alt3,'genGamma.export')

```



```
# 6th order gen gamma scaled
cdblib.put ('gen_gamma_0_scaled',alt0scaled,'genGamma.export')
cdblib.put ('gen_gamma_1_scaled',alt1scaled,'genGamma.export')
cdblib.put ('gen_gamma_2_scaled',alt2scaled,'genGamma.export')
cdblib.put ('gen_gamma_3_scaled',alt3scaled,'genGamma.export')

checkpoint.append (tmp0)
checkpoint.append (tmp1)

checkpoint.append (alt0)
checkpoint.append (alt1)
checkpoint.append (alt2)
checkpoint.append (alt3)

checkpoint.append (alt0scaled)
checkpoint.append (alt1scaled)
checkpoint.append (alt2scaled)
checkpoint.append (alt3scaled)
```

# Symmetrized partial derivatives of the connection

Here we calculate the recursive sequences

$$(n+3)\Gamma^a_{d(b,c\bar{e}_n)} = (n+1) \left( R^a_{(bcd,\bar{e}_n)} - (\Gamma^a_{fc}\Gamma^f_{bd})_{,\bar{e}_n} \right)$$

for  $n = 1, 2, 3, \dots$ . Note that the (extended) index  $\bar{e}_n$  contains  $n$  normal indices.

The result will be expressions for the  $\Gamma^a_{d(b,c\bar{e}_n)}$  in terms of the Riemann tensor and its partial derivatives.

## Stage 1: Compute symmetrised derivatives

In the first stage we simply apply the above recursive equation using a simple trick to impose the symmetries. Start with the original equation and dot out the symmetric indices with  $A^a$  then factor out the partial derivatives. This leads to

$$(n+3)\Gamma^a_{db,c\bar{e}_n} A^b A^c A^{\bar{e}_n} = (n+1) \left( R^a_{bcd} - \Gamma^a_{fc}\Gamma^f_{bd} \right)_{,\bar{e}_n} A^b A^c A^{\bar{e}_n} \quad (1)$$

Thus we also have (for the next iteration)

$$(n+4)\Gamma^a_{db,c\bar{e}_{n+1}} A^b A^c A^{\bar{e}_{n+1}} = (n+2) \left( R^a_{bcd} - \Gamma^a_{fc}\Gamma^f_{bd} \right)_{,\bar{e}_{n+1}} A^b A^c A^{\bar{e}_{n+1}} \quad (2)$$

The  $A^a$  can be freely chosen so choose  $A^a$  to be a constant (i.e., zero derivative). Now define  $P_n$  by

$$P_n = \Gamma^a_{db,c\bar{e}_n} A^b A^c A^{\bar{e}_n} \quad (3)$$

then the above pair of equations can be combined to give

$$P_{n+1} = \frac{(n+2)(n+3)}{(n+4)(n+1)} A^f \partial_f (P_n) \quad (4)$$

This is a very easy equation to compute as it just requires successive rounds of differentiation.

The first term in the sequence is  $P_0$  given by

$$P_0 = \frac{1}{3} A^b A^c (R^a_{bcd} - \Gamma^a_{ce}\Gamma^e_{bd}) \quad (5)$$

The first few results are

$$\begin{aligned}
P_0 &= A^b A^c \Gamma^a_{d(b,c)} = \frac{1}{3} A^b A^c (R^a_{bcd} - \Gamma^a_{ce} \Gamma^e_{bd}) \\
P_1 &= A^b A^c A^e \Gamma^a_{d(b,ce)} = \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} \\
P_2 &= A^b A^c A^e A^f \Gamma^a_{d(b,cef)} = \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} \\
&\quad - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd}
\end{aligned}$$

## Stage 2: Impose Riemann normal coordinates

Here we impose the RNC condition by setting the  $\Gamma^a_{bc}$  to zero (but not their derivatives).

$$\begin{aligned}
A^b A^c \Gamma^a_{d(b,c)} &= \frac{1}{3} A^b A^c R^a_{bcd} \\
A^b A^c A^e \Gamma^a_{d(b,ce)} &= \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} \\
A^b A^c A^e A^f \Gamma^a_{d(b,cef)} &= \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd}
\end{aligned}$$

## Stage 3: Replace partial derivatives of $\Gamma$ with partial derivatives of $R$

The key point to note is that the partial derivatives of  $\Gamma$  on the right hand side are all symmetrized in exactly the same manner as the partial derivatives on the left hand side. Thus results from the lower order equations can be fed into the later equations to completely eliminate the partial derivatives of  $\Gamma$ .

$$\begin{aligned}
A^b A^c \Gamma^a_{d(b,c)} &= \frac{1}{3} A^b A^c R^a_{bcd} \\
A^b A^c A^e \Gamma^a_{d(b,ce)} &= \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} \\
A^b A^c A^e A^f \Gamma^a_{d(b,cef)} &= \frac{3}{5} A^b A^c A^e A^f \partial_{fe} R^a_{bcd} - \frac{1}{15} A^b A^c A^e A^f R^a_{ceg} R^g_{bfd} - \frac{1}{15} A^b A^c A^e A^f R^a_{cfg} R^g_{bed}
\end{aligned}$$

## Stage 4: Reformatting

This is just simple reformatting.

$$3A^b A^c \Gamma^a_{d(b,c)} = A^b A^c R^a_{bcd}$$

$$6A^b A^c A^e \Gamma^a_{d(b,ce)} = 3A^b A^c A^e \partial_e R^a_{bcd}$$

$$15A^b A^c A^e A^f \Gamma^a_{d(b,cef)} = A^b A^c A^e A^f (9\partial_{fe} R^a_{bcd} - R^a_{ceg} R^g_{bfd} - R^a_{cfg} R^g_{bed})$$

## Stage 1: Compute symmetrised derivatives

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

# symmetrized partial derivatives of \Gamma

dGamma01:= (1/3) A^{b} A^{c} ( R^{a}_{b c d} - \Gamma^{a}_{b c} \Gamma^{e}_{d} ).
# cdb (dGamma01.101,dGamma01)

dGamma02:= (6/4) A^{a} \partial_{a} { @ (dGamma01) }. # cdb (dGamma02.101,dGamma02)
distribute (dGamma02) # cdb (dGamma02.102,dGamma02)
product_rule (dGamma02) # cdb (dGamma02.103,dGamma02)
unwrap (dGamma02) # cdb (dGamma02.104,dGamma02)
distribute (dGamma02) # cdb (dGamma02.105,dGamma02)

dGamma03:= (12/10) A^{a} \partial_{a} { @ (dGamma02) }. # cdb (dGamma03.101,dGamma03)
distribute (dGamma03) # cdb (dGamma03.102,dGamma03)
product_rule (dGamma03) # cdb (dGamma03.103,dGamma03)
unwrap (dGamma03) # cdb (dGamma03.104,dGamma03)
```

```

distribute      (dGamma03)                      # cdb (dGamma03.105,dGamma03)

dGamma04:= (20/18) A^{a}\partial_{a}{ @(dGamma03) }. # cdb (dGamma04.101,dGamma04)
distribute      (dGamma04)                      # cdb (dGamma04.102,dGamma04)
product_rule    (dGamma04)                      # cdb (dGamma04.103,dGamma04)
unwrap          (dGamma04)                      # cdb (dGamma04.104,dGamma04)
distribute      (dGamma04)                      # cdb (dGamma04.105,dGamma04)

dGamma05:= (30/28) A^{a}\partial_{a}{ @(dGamma04) }. # cdb (dGamma05.101,dGamma05)
distribute      (dGamma05)                      # cdb (dGamma05.102,dGamma05)
product_rule    (dGamma05)                      # cdb (dGamma05.103,dGamma05)
unwrap          (dGamma05)                      # cdb (dGamma05.104,dGamma05)
distribute      (dGamma05)                      # cdb (dGamma05.105,dGamma05)

```

$$\text{dGamma01.101} := \frac{1}{3} A^b A^c (R^a_{bcd} - \Gamma^a_{ce} \Gamma^e_{bd})$$

$$\text{dGamma02.101} := \frac{1}{2} A^f \partial_f (A^b A^c (R^a_{bcd} - \Gamma^a_{ce} \Gamma^e_{bd}))$$

$$\text{dGamma02.102} := \frac{1}{2} A^f \partial_f (A^b A^c R^a_{bcd}) - \frac{1}{2} A^f \partial_f (A^b A^c \Gamma^a_{ce} \Gamma^e_{bd})$$

$$\text{dGamma02.103} := \frac{1}{2} A^f (\partial_f A^b A^c R^a_{bcd} + A^b \partial_f A^c R^a_{bcd} + A^b A^c \partial_f R^a_{bcd}) - \frac{1}{2} A^f (\partial_f A^b A^c \Gamma^a_{ce} \Gamma^e_{bd} + A^b \partial_f A^c \Gamma^a_{ce} \Gamma^e_{bd} + A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd})$$

$$\text{dGamma02.104} := \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f (A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd})$$

$$\text{dGamma02.105} := \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd}$$

$$\text{dGamma03.101} := \frac{6}{5} A^g \partial_g \left( \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd} - \frac{1}{2} A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} - \frac{1}{2} A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} \right)$$

$$\text{dGamma03.102} := \frac{3}{5} A^g \partial_g (A^f A^b A^c \partial_f R^a_{bcd}) - \frac{3}{5} A^g \partial_g (A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd}) - \frac{3}{5} A^g \partial_g (A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd})$$

$$\begin{aligned} \text{dGamma03.103} := & \frac{3}{5} A^g (\partial_g A^f A^b A^c \partial_f R^a_{bcd} + A^f \partial_g A^b A^c \partial_f R^a_{bcd} + A^f A^b \partial_g A^c \partial_f R^a_{bcd} + A^f A^b A^c \partial_{gf} R^a_{bcd}) \\ & - \frac{3}{5} A^g (\partial_g A^f A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^f \partial_g A^b A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b \partial_g A^c \partial_f \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd}) \\ & - \frac{3}{5} A^g (\partial_g A^f A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f \partial_g A^b A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b \partial_g A^c \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd}) \end{aligned}$$

$$\text{dGamma03.104} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g (A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd}) - \frac{3}{5} A^g (A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd})$$

$$\text{dGamma03.105} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd}$$

$$\text{dGamma04.101} := \frac{10}{9} A^h \partial_h \left( \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} \right. \\ \left. - \frac{3}{5} A^g A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd} \right)$$

$$\text{dGamma04.102} := \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_{gf} R^a_{bcd}) - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd}) - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd}) \\ - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd}) - \frac{2}{3} A^h \partial_h (A^g A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd})$$

$$\text{dGamma04.103} := \frac{2}{3} A^h (\partial_h A^g A^f A^b A^c \partial_{gf} R^a_{bcd} + A^g \partial_h A^f A^b A^c \partial_{gf} R^a_{bcd} + A^g A^f \partial_h A^b A^c \partial_{gf} R^a_{bcd} + A^g A^f A^b \partial_h A^c \partial_{gf} R^a_{bcd} + A^g A^f A^b A^c \partial_{hgf} R^a_{bcd}) \\ - \frac{2}{3} A^h (\partial_h A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g \partial_h A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g A^f \partial_h A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g A^f A^b \partial_h A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} \\ + A^g A^f A^b A^c \partial_{hgf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \partial_h \Gamma^e_{bd} + A^g A^f \partial_h A^b A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g A^f A^b \partial_h A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} \\ + A^g A^f A^b A^c \partial_{hgf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \partial_h \Gamma^e_{bd} + A^g A^f A^b \partial_h A^c \partial_{gf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g A^f A^b A^c \partial_{hgf} \Gamma^a_{ce} \Gamma^e_{bd} \\ + A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} + A^g A^f A^b \partial_h A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} + A^g A^f A^b A^c \partial_{hgf} \Gamma^a_{ce} \partial_g \Gamma^e_{bd} + A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_{hg} \Gamma^e_{bd}) \\ - \frac{2}{3} A^h (\partial_h A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^g \partial_h A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^g A^f \partial_h A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^g A^f A^b \partial_h A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd} \\ + A^g A^f A^b A^c \partial_{hg} \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_{hf} \Gamma^e_{bd}) - \frac{2}{3} A^h (\partial_h A^g A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd} + A^g \partial_h A^f A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd} \\ + A^g A^f \partial_h A^b A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd} + A^g A^f A^b \partial_h A^c \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd} + A^g A^f A^b A^c \Gamma^a_{ce} \partial_{hgf} \Gamma^e_{bd})$$

$$\text{dGamma04.104} := \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a_{bcd} - \frac{2}{3} A^h (A^g A^f A^b A^c \partial_{hgf} \Gamma^a_{ce} \Gamma^e_{bd} + A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \partial_h \Gamma^e_{bd}) \\ - \frac{2}{3} A^h (A^g A^f A^b A^c \partial_{hf} \Gamma^a_{ce} \partial_g \Gamma^e_{bd} + A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_{hg} \Gamma^e_{bd}) \\ - \frac{2}{3} A^h (A^g A^f A^b A^c \partial_{hg} \Gamma^a_{ce} \partial_f \Gamma^e_{bd} + A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_{hf} \Gamma^e_{bd}) - \frac{2}{3} A^h (A^g A^f A^b A^c \partial_h \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd} + A^g A^f A^b A^c \Gamma^a_{ce} \partial_{hgf} \Gamma^e_{bd})$$

$$\text{dGamma04.105} := \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a_{ce} \Gamma^e_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a_{ce} \partial_h \Gamma^e_{bd} \\ - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_{hg} \Gamma^e_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a_{ce} \partial_f \Gamma^e_{bd} \\ - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_{hf} \Gamma^e_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a_{ce} \partial_{gf} \Gamma^e_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \Gamma^a_{ce} \partial_{hgf} \Gamma^e_{bd}$$



## Stage 2: Impose Riemann normal coordinates

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{\{d\}}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g}^{\{a\}}_{\{b\}c}\$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{\{a\}}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}}_{\{b\}c}\$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{\{a\}}_{\{b\}c} \rightarrow 0\$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{\{d\}}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e\{f\}}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e\{f\}g}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}}_{\{b\}c} \rightarrow \partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{\{a\}}_{\{b\}c}\}\$,repeat=True)
    return obj

# switch to RNC

dGamma01 = impose_rnc (dGamma01)    # cdb (dGamma01.201,dGamma01)
dGamma02 = impose_rnc (dGamma02)    # cdb (dGamma02.202,dGamma02)
dGamma03 = impose_rnc (dGamma03)    # cdb (dGamma03.203,dGamma03)
dGamma04 = impose_rnc (dGamma04)    # cdb (dGamma04.204,dGamma04)
dGamma05 = impose_rnc (dGamma05)    # cdb (dGamma05.205,dGamma05)
```

$$d\Gamma_{01.201} := \frac{1}{3} A^b A^c R^a_{bcd}$$

$$d\Gamma_{02.202} := \frac{1}{2} A^f A^b A^c \partial_f R^a_{bcd}$$

$$d\Gamma_{03.203} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a_{bcd} - \frac{3}{5} A^g A^f A^b A^c \partial_f \Gamma^a_{ce} \partial_g \Gamma^e_{bd} - \frac{3}{5} A^g A^f A^b A^c \partial_g \Gamma^a_{ce} \partial_f \Gamma^e_{bd}$$

$$\begin{aligned}
\text{dGamma04.204} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} \\
& - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.205} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ifh} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{fhg} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgh} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{ihf} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{igf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_i \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd}
\end{aligned}$$

## Stage 3: Replace partial derivatives of $\Gamma$ with partial derivatives of $R$

```
# use lower equations to eliminate partial derivs of Gamma from rhs

# this produces expressions for the partial derivs of the Gamma's in terms of the Rabcd and its partial derivs

substitute (dGamma03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma03.301,dGamma03.302)
substitute (dGamma03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma03.302,dGamma03.303)
distribute (dGamma03)                                                                # cdb(dGamma03.303,dGamma03.304)

substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True)  # cdb(dGamma04.301,dGamma04.302)
substitute (dGamma04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True)  # cdb(dGamma04.302,dGamma04.303)
substitute (dGamma04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma04.303,dGamma04.304)
substitute (dGamma04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)      # cdb(dGamma04.304,dGamma04.305)
distribute (dGamma04)                                                                # cdb(dGamma04.305,dGamma04.306)

substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma03)$,repeat=True)  # cdb(dGamma05.301,dGamma05.302)
substitute (dGamma05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True)  # cdb(dGamma05.302,dGamma05.303)
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True)      # cdb(dGamma05.303,dGamma05.304)
substitute (dGamma05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True)      # cdb(dGamma05.304,dGamma05.305)
substitute (dGamma05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)          # cdb(dGamma05.305,dGamma05.306)
substitute (dGamma05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)          # cdb(dGamma05.306,dGamma05.307)
distribute (dGamma05)                                                                # cdb(dGamma05.307,dGamma05.308)
```

$$\text{dGamma03.301} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\text{dGamma03.302} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\text{dGamma03.303} := \frac{3}{5} A^g A^f A^b A^c \partial_{gf} R^a{}_{bcd} - \frac{1}{15} A^b A^g R^e{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{15} A^c A^g R^a{}_{cge} A^b A^f R^e{}_{bfd}$$

$$\begin{aligned} \text{dGamma04.301} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{gf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_f \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} \\ & - \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{2}{3} A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{gf} \Gamma^e{}_{bd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.302} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{3} A^h A^f A^c A^g \partial_f R^a{}_{cge} A^b \partial_h \Gamma^e{}_{bd} - \frac{1}{3} A^f A^c A^h \partial_f R^a{}_{che} A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{1}{3} A^g A^b A^h \partial_g R^e{}_{bhd} A^f A^c \partial_f \Gamma^a{}_{ce} \\ & - \frac{1}{3} A^g A^c A^h \partial_g R^a{}_{che} A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{1}{3} A^f A^b A^h \partial_f R^e{}_{bhd} A^g A^c \partial_g \Gamma^a{}_{ce} - \frac{1}{3} A^h A^f A^b A^g \partial_f R^e{}_{bgd} A^c \partial_h \Gamma^a{}_{ce} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.303} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.304} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.305} := & \frac{2}{3} A^h A^g A^f A^b A^c \partial_{hgf} R^a{}_{bcd} - \frac{1}{9} A^b A^h R^e{}_{bhd} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{9} A^f A^c A^h \partial_f R^a{}_{che} A^b A^g R^e{}_{bgd} - \frac{1}{9} A^g A^b A^h \partial_g R^e{}_{bhd} A^c A^f R^a{}_{cfe} \\ & - \frac{1}{9} A^g A^c A^h \partial_g R^a{}_{che} A^b A^f R^e{}_{bfd} - \frac{1}{9} A^f A^b A^h \partial_f R^e{}_{bhd} A^c A^g R^a{}_{cge} - \frac{1}{9} A^c A^h R^a{}_{che} A^f A^b A^g \partial_f R^e{}_{bgd} \end{aligned}$$

$$\begin{aligned}
\text{dGamma05.301} &:= \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_i \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{igf} \Gamma^a{}_{ce} \partial_h \Gamma^e{}_{bd} \\
&- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihf} \Gamma^a{}_{ce} \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd} \\
&- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ifh} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{fh} \Gamma^a{}_{ce} \partial_{ihg} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihg} \Gamma^a{}_{ce} \partial_f \Gamma^e{}_{bd} \\
&- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_g \Gamma^a{}_{ce} \partial_{ihf} \Gamma^e{}_{bd} \\
&- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gfh} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_h \Gamma^a{}_{ce} \partial_{igf} \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_i \Gamma^a{}_{ce} \partial_{hgf} \Gamma^e{}_{bd} \\
\text{dGamma05.302} &:= \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jfh} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cjh} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hjh} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd} \\
&- \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jfh} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cjh} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gfh} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} \\
&- \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jfh} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cjh} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hfh} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd} \\
&- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ifh} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jgh} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce} \\
&- \frac{5}{7} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jgh} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cjh} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hg} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} \\
&- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hf} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jfh} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce} \\
&- \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gfh} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jfh} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce} \\
&- \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jfh} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hjh} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.303} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hfg} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{gff} \Gamma^a{}_{ce} \partial_{ih} \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hff} \Gamma^a{}_{ce} \partial_{ig} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{iff} \Gamma^a{}_{ce} \partial_{hg} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{hgf} \Gamma^a{}_{ce} \partial_{if} \Gamma^e{}_{bd} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ig} \Gamma^a{}_{ce} \partial_{hff} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ih} \Gamma^a{}_{ce} \partial_{gff} \Gamma^e{}_{bd} - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hfg} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.304} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a{}_{che} - \frac{1}{15} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} - \frac{1}{15} A^h A^j R^a{}_{hfg} A^c A^f R^g{}_{cfe} \right) A^b \partial_i \Gamma^e{}_{bd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} - \frac{1}{15} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} \right) A^h A^b \partial_h \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} \right) A^g A^b \partial_g \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} \right) A^f A^c \partial_f \Gamma^a{}_{ce} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} - \frac{1}{15} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} - \frac{1}{15} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} \right) A^f A^b \partial_f \Gamma^e{}_{bd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} - \frac{1}{15} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} \right) A^g A^c \partial_g \Gamma^a{}_{ce} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} \\
& - \frac{5}{7} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} - \frac{1}{15} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} \right) A^h A^c \partial_h \Gamma^a{}_{ce} \\
& - \frac{5}{7} A^i \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e{}_{bhd} - \frac{1}{15} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} - \frac{1}{15} A^h A^j R^e{}_{hfg} A^b A^f R^g{}_{bfd} \right) A^c \partial_i \Gamma^a{}_{ce}
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.305} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a_{bcd} - \frac{5}{21} A^b A^i R^e_{bid} \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a_{che} - \frac{1}{15} A^c A^j R^g_{cje} A^h A^f R^a_{hfg} - \frac{1}{15} A^h A^j R^a_{hfg} A^c A^f R^g_{cfe} \right) \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a_{cie} - \frac{1}{15} A^c A^j R^g_{cje} A^i A^f R^a_{ifg} - \frac{1}{15} A^i A^j R^a_{ijg} A^c A^f R^g_{cfe} \right) A^b A^h R^e_{bhd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e_{bid} A^f A^c A^g \partial_f R^a_{cge} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a_{cie} - \frac{1}{15} A^c A^j R^h_{cje} A^i A^f R^a_{ifh} - \frac{1}{15} A^i A^j R^a_{ijh} A^c A^f R^h_{cfe} \right) A^b A^g R^e_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e_{bid} A^f A^c A^h \partial_f R^a_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a_{cie} A^g A^b A^h \partial_g R^e_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e_{bid} - \frac{1}{15} A^b A^j R^h_{bjd} A^i A^g R^e_{igh} - \frac{1}{15} A^i A^j R^e_{ijh} A^b A^g R^h_{bgd} \right) A^c A^f R^a_{cfe} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a_{cie} - \frac{1}{15} A^c A^j R^h_{cje} A^i A^g R^a_{igh} - \frac{1}{15} A^i A^j R^a_{ijh} A^c A^g R^h_{cge} \right) A^b A^f R^e_{bfd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e_{bid} A^g A^c A^h \partial_g R^a_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a_{cie} A^f A^b A^h \partial_f R^e_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e_{bid} - \frac{1}{15} A^b A^j R^h_{bjd} A^i A^f R^e_{ifh} - \frac{1}{15} A^i A^j R^e_{ijh} A^b A^f R^h_{bfd} \right) A^c A^g R^a_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a_{cie} A^f A^b A^g \partial_f R^e_{bgd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e_{bid} - \frac{1}{15} A^b A^j R^g_{bjd} A^i A^f R^e_{ifg} - \frac{1}{15} A^i A^j R^e_{ijg} A^b A^f R^g_{bfd} \right) A^c A^h R^a_{che} \\
& - \frac{5}{21} A^c A^i R^a_{cie} \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e_{bhd} - \frac{1}{15} A^b A^j R^g_{bjd} A^h A^f R^e_{hfg} - \frac{1}{15} A^h A^j R^e_{hfg} A^b A^f R^g_{bfd} \right)
\end{aligned}$$



$$\begin{aligned}
\text{dGamma05.306} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a_{bcd} - \frac{5}{21} A^b A^i R^e_{bid} \left( \frac{3}{5} A^j A^f A^c A^h \partial_{jff} R^a_{che} - \frac{1}{15} A^c A^j R^g_{cje} A^h A^f R^a_{hfg} - \frac{1}{15} A^h A^j R^a_{hfg} A^c A^f R^g_{cfe} \right) \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a_{cie} - \frac{1}{15} A^c A^j R^g_{cje} A^i A^f R^a_{ifg} - \frac{1}{15} A^i A^j R^a_{ijg} A^c A^f R^g_{cfe} \right) A^b A^h R^e_{bhd} \\
& - \frac{5}{28} A^h A^b A^i \partial_h R^e_{bid} A^f A^c A^g \partial_f R^a_{cge} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^c A^i \partial_{jff} R^a_{cie} - \frac{1}{15} A^c A^j R^h_{cje} A^i A^f R^a_{ifh} - \frac{1}{15} A^i A^j R^a_{ijh} A^c A^f R^h_{cfe} \right) A^b A^g R^e_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e_{bid} A^f A^c A^h \partial_f R^a_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a_{cie} A^g A^b A^h \partial_g R^e_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^b A^i \partial_{jg} R^e_{bid} - \frac{1}{15} A^b A^j R^h_{bjd} A^i A^g R^e_{igh} - \frac{1}{15} A^i A^j R^e_{ijh} A^b A^g R^h_{bgd} \right) A^c A^f R^a_{cfe} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^g A^c A^i \partial_{jg} R^a_{cie} - \frac{1}{15} A^c A^j R^h_{cje} A^i A^g R^a_{igh} - \frac{1}{15} A^i A^j R^a_{ijh} A^c A^g R^h_{cge} \right) A^b A^f R^e_{bfd} \\
& - \frac{5}{28} A^f A^b A^i \partial_f R^e_{bid} A^g A^c A^h \partial_g R^a_{che} - \frac{5}{28} A^g A^c A^i \partial_g R^a_{cie} A^f A^b A^h \partial_f R^e_{bhd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e_{bid} - \frac{1}{15} A^b A^j R^h_{bjd} A^i A^f R^e_{ifh} - \frac{1}{15} A^i A^j R^e_{ijh} A^b A^f R^h_{bfd} \right) A^c A^g R^a_{cge} \\
& - \frac{5}{28} A^h A^c A^i \partial_h R^a_{cie} A^f A^b A^g \partial_f R^e_{bgd} \\
& - \frac{5}{21} \left( \frac{3}{5} A^j A^f A^b A^i \partial_{jff} R^e_{bid} - \frac{1}{15} A^b A^j R^g_{bjd} A^i A^f R^e_{ifg} - \frac{1}{15} A^i A^j R^e_{ijg} A^b A^f R^g_{bfd} \right) A^c A^h R^a_{che} \\
& - \frac{5}{21} A^c A^i R^a_{cie} \left( \frac{3}{5} A^j A^f A^b A^h \partial_{jff} R^e_{bhd} - \frac{1}{15} A^b A^j R^g_{bjd} A^h A^f R^e_{hfg} - \frac{1}{15} A^h A^j R^e_{hfg} A^b A^f R^g_{bfd} \right)
\end{aligned}$$

$$\begin{aligned}
\text{dGamma05.307} := & \frac{5}{7} A^i A^h A^g A^f A^b A^c \partial_{ihgf} R^a{}_{bcd} - \frac{1}{7} A^b A^i R^e{}_{bid} A^j A^f A^c A^h \partial_{jff} R^a{}_{che} + \frac{1}{63} A^b A^i R^e{}_{bid} A^c A^j R^g{}_{cje} A^h A^f R^a{}_{hfg} \\
& + \frac{1}{63} A^b A^i R^e{}_{bid} A^h A^j R^a{}_{hfg} A^c A^f R^g{}_{cfe} - \frac{1}{7} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} A^b A^h R^e{}_{bhd} + \frac{1}{63} A^c A^j R^g{}_{cje} A^i A^f R^a{}_{ifg} A^b A^h R^e{}_{bhd} \\
& + \frac{1}{63} A^i A^j R^a{}_{ijg} A^c A^f R^g{}_{cfe} A^b A^h R^e{}_{bhd} - \frac{5}{28} A^h A^b A^i \partial_h R^e{}_{bid} A^f A^c A^g \partial_f R^a{}_{cge} - \frac{1}{7} A^j A^f A^c A^i \partial_{jff} R^a{}_{cie} A^b A^g R^e{}_{bgd} \\
& + \frac{1}{63} A^c A^j R^h{}_{cje} A^i A^f R^a{}_{ifh} A^b A^g R^e{}_{bgd} + \frac{1}{63} A^i A^j R^a{}_{ijh} A^c A^f R^h{}_{cfe} A^b A^g R^e{}_{bgd} \\
& - \frac{5}{28} A^g A^b A^i \partial_g R^e{}_{bid} A^f A^c A^h \partial_f R^a{}_{che} - \frac{5}{28} A^f A^c A^i \partial_f R^a{}_{cie} A^g A^b A^h \partial_g R^e{}_{bhd} - \frac{1}{7} A^j A^g A^b A^i \partial_{jg} R^e{}_{bid} A^c A^f R^a{}_{cfe} \\
& + \frac{1}{63} A^b A^j R^h{}_{bjd} A^i A^g R^e{}_{igh} A^c A^f R^a{}_{cfe} + \frac{1}{63} A^i A^j R^e{}_{ijh} A^b A^g R^h{}_{bgd} A^c A^f R^a{}_{cfe} - \frac{1}{7} A^j A^g A^c A^i \partial_{jg} R^a{}_{cie} A^b A^f R^e{}_{bfd} \\
& + \frac{1}{63} A^c A^j R^h{}_{cje} A^i A^g R^a{}_{igh} A^b A^f R^e{}_{bfd} + \frac{1}{63} A^i A^j R^a{}_{ijh} A^c A^g R^h{}_{cge} A^b A^f R^e{}_{bfd} - \frac{5}{28} A^f A^b A^i \partial_f R^e{}_{bid} A^g A^c A^h \partial_g R^a{}_{che} \\
& - \frac{5}{28} A^g A^c A^i \partial_g R^a{}_{cie} A^f A^b A^h \partial_f R^e{}_{bhd} - \frac{1}{7} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} A^c A^g R^a{}_{cge} + \frac{1}{63} A^b A^j R^h{}_{bjd} A^i A^f R^e{}_{ifh} A^c A^g R^a{}_{cge} \\
& + \frac{1}{63} A^i A^j R^e{}_{ijh} A^b A^f R^h{}_{bfd} A^c A^g R^a{}_{cge} - \frac{5}{28} A^h A^c A^i \partial_h R^a{}_{cie} A^f A^b A^g \partial_f R^e{}_{bgd} - \frac{1}{7} A^j A^f A^b A^i \partial_{jff} R^e{}_{bid} A^c A^h R^a{}_{che} \\
& + \frac{1}{63} A^b A^j R^g{}_{bjd} A^i A^f R^e{}_{ifg} A^c A^h R^a{}_{che} + \frac{1}{63} A^i A^j R^e{}_{ijg} A^b A^f R^g{}_{bfd} A^c A^h R^a{}_{che} - \frac{1}{7} A^c A^i R^a{}_{cie} A^j A^f A^b A^h \partial_{jff} R^e{}_{bhd} \\
& + \frac{1}{63} A^c A^i R^a{}_{cie} A^b A^j R^g{}_{bjd} A^h A^f R^e{}_{hfg} + \frac{1}{63} A^c A^i R^a{}_{cie} A^h A^j R^e{}_{hfg} A^b A^f R^g{}_{bfd}
\end{aligned}$$

```

# note:
# canonicalise must not be used here because it may make changes like
#    $R^{\{a\}_{\{b\}c}d} \rightarrow -R_{\{b\}}^{\{a\}_{\{c\}d}}$ 
# these changes can not be applied inside a \partial, must defer use
# of canonicalise until we have \nabla acting on curvatures

sort_product    (dGamma03) # cdb(dGamma03.401,dGamma03)
rename_dummies  (dGamma03) # cdb(dGamma03.402,dGamma03)
# canonicalise   (dGamma03) # cdb(dGamma03.403,dGamma03)

sort_product    (dGamma04) # cdb(dGamma04.401,dGamma04)
rename_dummies  (dGamma04) # cdb(dGamma04.402,dGamma04)
# canonicalise   (dGamma04) # cdb(dGamma04.403,dGamma04)

sort_product    (dGamma05) # cdb(dGamma05.401,dGamma05)
rename_dummies  (dGamma05) # cdb(dGamma05.402,dGamma05)
# canonicalise   (dGamma05) # cdb(dGamma05.403,dGamma05)

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$$\text{dGamma03.401} := \frac{3}{5} A^b A^c A^f A^g \partial_{gf} R^a_{bcd} - \frac{1}{15} A^b A^c A^f A^g R^a_{cfe} R^e_{bgd} - \frac{1}{15} A^b A^c A^f A^g R^a_{cge} R^e_{bfd}$$

$$\text{dGamma03.402} := \frac{3}{5} A^b A^c A^e A^f \partial_{fe} R^a_{bcd} - \frac{1}{15} A^b A^c A^e A^f R^a_{ceg} R^g_{bfd} - \frac{1}{15} A^b A^c A^e A^f R^a_{cfg} R^g_{bed}$$

$$\begin{aligned} \text{dGamma04.401} := & \frac{2}{3} A^b A^c A^f A^g A^h \partial_{hgf} R^a_{bcd} - \frac{1}{9} A^b A^c A^f A^g A^h R^e_{bhd} \partial_f R^a_{cge} - \frac{1}{9} A^b A^c A^f A^g A^h R^e_{bgd} \partial_f R^a_{che} - \frac{1}{9} A^b A^c A^f A^g A^h R^a_{cfe} \partial_g R^e_{bhd} \\ & - \frac{1}{9} A^b A^c A^f A^g A^h R^e_{bfd} \partial_g R^a_{che} - \frac{1}{9} A^b A^c A^f A^g A^h R^a_{cge} \partial_f R^e_{bhd} - \frac{1}{9} A^b A^c A^f A^g A^h R^a_{che} \partial_f R^e_{bgd} \end{aligned}$$

$$\begin{aligned} \text{dGamma04.402} := & \frac{2}{3} A^b A^c A^e A^f A^g \partial_{gfe} R^a_{bcd} - \frac{1}{9} A^b A^c A^e A^f A^g R^h_{bgd} \partial_e R^a_{cfh} - \frac{1}{9} A^b A^c A^e A^f A^g R^h_{bfd} \partial_e R^a_{cgh} - \frac{1}{9} A^b A^c A^e A^f A^g R^a_{ceh} \partial_f R^h_{bgd} \\ & - \frac{1}{9} A^b A^c A^e A^f A^g R^h_{bed} \partial_f R^a_{cgh} - \frac{1}{9} A^b A^c A^e A^f A^g R^a_{cfh} \partial_e R^h_{bgd} - \frac{1}{9} A^b A^c A^e A^f A^g R^a_{cgh} \partial_e R^h_{bfd} \end{aligned}$$

$$\begin{aligned} \text{dGamma05.401} := & \frac{5}{7} A^b A^c A^f A^g A^h A^i \partial_{ihgf} R^a_{bcd} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^e_{bid} \partial_{jf} R^a_{che} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{hfg} R^e_{bid} R^g_{cje} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{hfg} R^e_{bid} R^g_{cfe} - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^e_{bhd} \partial_{jf} R^a_{cie} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{ifg} R^e_{bhd} R^g_{cje} \\ & + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{ijg} R^e_{bhd} R^g_{cfe} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a_{cge} \partial_h R^e_{bid} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^e_{bgd} \partial_{jf} R^a_{cie} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{ifh} R^e_{bgd} R^h_{cje} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{ijh} R^e_{bgd} R^h_{cfe} \\ & - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a_{che} \partial_g R^e_{bid} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^a_{cie} \partial_g R^e_{bhd} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^a_{cfe} \partial_{jg} R^e_{bid} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cfe} R^e_{igh} R^h_{bjd} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cfe} R^e_{ijh} R^h_{bgd} \\ & - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^e_{bfd} \partial_{jg} R^a_{cie} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{igh} R^e_{bfd} R^h_{cje} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{ijh} R^e_{bfd} R^h_{cge} \\ & - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e_{bid} \partial_g R^a_{che} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e_{bhd} \partial_g R^a_{cie} - \frac{1}{7} A^b A^c A^f A^g A^i A^j R^a_{cge} \partial_{jf} R^e_{bid} \\ & + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cge} R^e_{ifh} R^h_{bjd} + \frac{1}{63} A^b A^c A^f A^g A^i A^j R^a_{cge} R^e_{ijh} R^h_{bfd} - \frac{5}{28} A^b A^c A^f A^g A^h A^i \partial_f R^e_{bgd} \partial_h R^a_{cie} \\ & - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^a_{che} \partial_{jf} R^e_{bid} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{che} R^e_{ifg} R^g_{bjd} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{che} R^e_{ijg} R^g_{bfd} \\ & - \frac{1}{7} A^b A^c A^f A^h A^i A^j R^a_{cie} \partial_{jf} R^e_{bhd} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{cie} R^e_{hfg} R^g_{bjd} + \frac{1}{63} A^b A^c A^f A^h A^i A^j R^a_{cie} R^e_{hfg} R^g_{bfd} \end{aligned}$$

$$\begin{aligned}
\text{dGamma05.402} := & \frac{5}{7} A^b A^c A^e A^f A^g A^h \partial_{hgfe} R^a_{bcd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i_{bgd} \partial_{he} R^a_{cfi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{fei} R^j_{bgd} R^i_{chj} \\
& + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{fhi} R^j_{bgd} R^i_{cej} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^i_{bfd} \partial_{he} R^a_{cgi} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{gei} R^j_{bfd} R^i_{chj} \\
& + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{ghi} R^j_{bfd} R^i_{cej} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a_{cfi} \partial_g R^i_{bhd} \\
& - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a_{cgi} \partial_f R^i_{bhd} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^a_{chi} \partial_f R^i_{bgd} - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a_{cei} \partial_{hf} R^i_{bgd} \\
& + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cei} R^i_{gfj} R^j_{bhd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cei} R^i_{ghj} R^j_{bfd} \\
& - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^i_{bed} \partial_{hf} R^a_{cgi} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{gfi} R^j_{bed} R^i_{chj} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{ghi} R^j_{bed} R^i_{cfj} \\
& - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i_{bhd} \partial_f R^a_{cgi} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i_{bgd} \partial_f R^a_{chi} - \frac{2}{7} A^b A^c A^e A^f A^g A^h R^a_{cfi} \partial_{he} R^i_{bgd} \\
& + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{cfi} R^i_{gej} R^j_{bhd} + \frac{2}{63} A^b A^c A^e A^f A^g A^h R^a_{cfi} R^i_{ghj} R^j_{bed} - \frac{5}{28} A^b A^c A^e A^f A^g A^h \partial_e R^i_{bfd} \partial_g R^a_{chi} \\
& - \frac{1}{7} A^b A^c A^e A^f A^g A^h R^a_{cgi} \partial_{he} R^i_{bfd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cgi} R^i_{fej} R^j_{bhd} + \frac{1}{63} A^b A^c A^e A^f A^g A^h R^a_{cgi} R^i_{fhj} R^j_{bed}
\end{aligned}$$

```
import cdblib

cdblib.create ('dGamma.json')

cdblib.put ('dGamma01',dGamma01,'dGamma.json')
cdblib.put ('dGamma02',dGamma02,'dGamma.json')
cdblib.put ('dGamma03',dGamma03,'dGamma.json')
cdblib.put ('dGamma04',dGamma04,'dGamma.json')
cdblib.put ('dGamma05',dGamma05,'dGamma.json')
```

## Stage 4: Reformatting

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                                -> A001^{a}                                $)
    substitute (obj,$ x^{a}                                -> A002^{a}                                $)
    substitute (obj,$ g^{a b}                              -> A003^{a b}                              $)
    substitute (obj,$ \partial_{e f g h}\{R^{a}_{b c d}\}    -> A008^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}\{R^{a}_{b c d}\}       -> A007^{a}_{b c d e f g}  $)
    substitute (obj,$ \partial_{e f}\{R^{a}_{b c d}\}         -> A006^{a}_{b c d e f}   $)
    substitute (obj,$ \partial_{e}\{R^{a}_{b c d}\}          -> A005^{a}_{b c d e}    $)
    substitute (obj,$ R^{a}_{b c d}                        -> A004^{a}_{b c d}      $)
    sort_product      (obj)
    rename_dummies   (obj)
    substitute (obj,$ A001^{a}                            -> A^{a}                            $)
    substitute (obj,$ A002^{a}                            -> x^{a}                            $)
    substitute (obj,$ A003^{a b}                          -> g^{a b}                          $)
    substitute (obj,$ A004^{a}_{b c d}                    -> R^{a}_{b c d}                    $)
    substitute (obj,$ A005^{a}_{b c d e}                  -> \partial_{e}\{R^{a}_{b c d}\}      $)
    substitute (obj,$ A006^{a}_{b c d e f}                -> \partial_{e f}\{R^{a}_{b c d}\}     $)
    substitute (obj,$ A007^{a}_{b c d e f g}              -> \partial_{e f g}\{R^{a}_{b c d}\}    $)
    substitute (obj,$ A008^{a}_{b c d e f g h}            -> \partial_{e f g h}\{R^{a}_{b c d}\}  $)

    return obj

def reformat (obj,scale):
    bah = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute      (tmp)
    tmp = product_sort (tmp)
    rename_dummies  (tmp)
    factor_out      (tmp,$A^{a?}$)
    return tmp

def get_term (obj,n):

    A^{a}::Weight(label=numA).
```

```

foo := @(obj).
bah = Ex("numA = " + str(n))
distribute (foo)
keep_weight (foo, bah)

return foo

Gterm01 := @(dGamma01).
Gterm02 := @(dGamma02).
Gterm03 := @(dGamma03).
Gterm04 := @(dGamma04).
Gterm05 := @(dGamma05).

scaled1 = reformat (Gterm01, 3) # cdb (scaled1.002,scaled1)
scaled2 = reformat (Gterm02, 6) # cdb (scaled2.002,scaled2)
scaled3 = reformat (Gterm03, 15) # cdb (scaled3.002,scaled3)
scaled4 = reformat (Gterm04, 9) # cdb (scaled4.002,scaled4)
scaled5 = reformat (Gterm05, 252) # cdb (scaled5.002,scaled5)

```



## Symmetrised partial derivatives of the connection

$$3A^b A^c \Gamma^a_{d(b,c)} = A^b A^c R^a_{bcd}$$

$$6A^b A^c A^e \Gamma^a_{d(b,ce)} = 3A^b A^c A^e \partial_e R^a_{bcd}$$

$$15A^b A^c A^e A^f \Gamma^a_{d(b,cef)} = A^b A^c A^e A^f (9\partial_{fe} R^a_{bcd} - R^a_{ceg} R^g_{bfd} - R^a_{cfg} R^g_{bed})$$

$$9A^b A^c A^e A^f A^g \Gamma^a_{d(b,cefg)} = A^b A^c A^e A^f A^g (6\partial_{gfe} R^a_{bcd} - R^h_{bgd} \partial_e R^a_{cfh} - R^h_{bfd} \partial_e R^a_{cgh} - R^a_{ceh} \partial_f R^h_{bgd} - R^h_{bed} \partial_f R^a_{cgh} - R^a_{cfh} \partial_e R^h_{bgd} - R^a_{cgh} \partial_e R^h_{bfd})$$

$$\begin{aligned} 252A^b A^c A^e A^f A^g A^h \Gamma^a_{d(b,cefg h)} = & A^b A^c A^e A^f A^g A^h (180\partial_{hgfe} R^a_{bcd} - 36R^i_{bgd} \partial_{he} R^a_{cfi} + 4R^a_{fei} R^i_{chj} R^j_{bgd} + 4R^a_{fhi} R^i_{cej} R^j_{bgd} - 72R^i_{bfd} \partial_{he} R^a_{cgi} \\ & + 8R^a_{gei} R^i_{chj} R^j_{bfd} + 8R^a_{ghi} R^i_{cej} R^j_{bfd} - 45\partial_e R^a_{cfi} \partial_g R^i_{bhd} - 45\partial_e R^a_{cgi} \partial_f R^i_{bhd} - 45\partial_e R^a_{chi} \partial_f R^i_{bgd} \\ & - 36R^a_{cei} \partial_{hf} R^i_{bgd} + 4R^a_{cei} R^i_{g fj} R^j_{bhd} + 4R^a_{cei} R^i_{ghj} R^j_{bfd} - 36R^i_{bed} \partial_{hf} R^a_{cgi} + 4R^a_{gfi} R^i_{chj} R^j_{bed} \\ & + 4R^a_{ghi} R^i_{cfj} R^j_{bed} - 45\partial_f R^a_{cgi} \partial_e R^i_{bhd} - 45\partial_f R^a_{chi} \partial_e R^i_{bgd} - 72R^a_{cfi} \partial_{he} R^i_{bgd} + 8R^a_{cfi} R^i_{gej} R^j_{bhd} \\ & + 8R^a_{cfi} R^i_{ghj} R^j_{bed} - 45\partial_g R^a_{chi} \partial_e R^i_{bfd} - 36R^a_{cgi} \partial_{he} R^i_{bfd} + 4R^a_{cgi} R^i_{fej} R^j_{bhd} + 4R^a_{cgi} R^i_{fhj} R^j_{bed}) \end{aligned}$$

```

substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dGamma.export')

# 6th order dGamma, scaled
cdblib.put ('dGamma61scaled',scaled1,'dGamma.export')
cdblib.put ('dGamma62scaled',scaled2,'dGamma.export')
cdblib.put ('dGamma63scaled',scaled3,'dGamma.export')
cdblib.put ('dGamma64scaled',scaled4,'dGamma.export')
cdblib.put ('dGamma65scaled',scaled5,'dGamma.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

```

# Symmetrised partial derivatives of the Riemann tensor

Here we compute the symmetrised partial derivatives  $R^a_{(bcd;\underline{e})}$  in terms of the symmetrised covariant derivatives  $R^a_{(bcd;\underline{e})}$ . Note that the dot over an index indicates that that index does not take part in the symmetrisation.

We will use the algorithm described in section (10.3) of my lcb09-03 paper. Here we will make one small change of notation – the symbol  $D^a$  will be replaced with  $A^a$ .

We have lots of space (and no annoying editors to appease with brevity) so I will take the liberty to expand slightly on what I wrote in the lcb0-03 paper.

Our starting point is the simple identity

$$(R^a_{\phantom{a}cdb}B^b_{\phantom{b}a}A^cA^d)_{;e}A^e = (R^a_{\phantom{a}cdb}B^b_{\phantom{b}a}A^cA^d)_{;e}A^e \quad (1)$$

This is true in all frames since the quantity inside the brackets is a scalar. We are free to make any choice we like for  $A^a$  and  $B^a_b$  so let's choose  $A^a$  to be the tangent vector to any geodesic through the origin and choose the  $B^a_b$  to be constants (i.e, all partial derivatives are zero). We will also use local Riemann normal coordinates and as a consequence, the  $A^a$  will also be constant along the integral curves of  $A$  (the geodesics in an RNC are always of the form  $x^a(s) = sA^a$  for some affine parameter  $s$  on the geodesic). Let  $df/ds$  be the directional derivative of the function  $f$  along the geodesics defined by  $A^a$  and assume that  $s$  is the proper length along the geodesic (although any affine parameter would be sufficient).

Thus at the origin we have, by choice,

$$\begin{aligned} 0 &= B^a_{b,c} = B^a_{b,cd} = B^a_{b,cde} = \dots \\ 0 &= dA^a/ds = d^2A^a/ds^2 = d^3A^a/ds^3 = \dots \\ 0 &= A^a_{;b}A^b = (A^a_{;b}A^b)_{;c}A^c = \left((A^a_{;b}A^b)_{;c}A^c\right)_{;d}A^d \\ 0 &= A^a_{;b}A^b = (A^a_{;b}A^b)_{;c}A^c = \left((A^a_{;b}A^b)_{;c}A^c\right)_{;d}A^d \\ df/ds &= f_{;a}A^a = f_{;a}A^a \\ d^2f/ds^2 &= (f_{;a}A^a)_{;b}A^b = (f_{;a}A^a)_{;b}A^b \\ d^3f/ds^3 &= \left((f_{;a}A^a)_{;b}A^b\right)_{;c}A^c = \left((f_{;a}A^a)_{;b}A^b\right)_{;c}A^c \end{aligned}$$

I admit I've gone overboard here in writing out more than I need to but it's handy to have all of these equations laid bare in one convenient place.

Now put  $f = R^p_{abq} B^q_p A^a A^b$ . Then upon taking successive derivatives, while taking full advantage of the assumptions just noted, we can easily see that

$$(R^a_{cdb} B^b_a)_{;e} A^c A^d A^e = (R^a_{cdb})_{;e} B^b_a A^c A^d A^e \quad (2)$$

This is the equation that will be computed by the following Cadabra code. All of the computations will be carried out on the left hand side (in the first version of the paper I swapped the left and right hand sides).

We will need the successive covariant derivatives of  $B$ . The first covariant derivative is just

$$B^a_{b;c} A^c = \Gamma^a_{dc} B^d_b A^c - \Gamma^d_{bc} B^a_d A^c$$

The quantities on the left hand side are the components of a tensor so further covariant derivatives of the right hand side can be computed (despite the presence of the  $\Gamma$ 's) by application of the usual rule for a covariant derivative of a mixed tensor.

## Stage 1: Symmetrised partial derivatives of $R$

The first stage involves the expansion of the left side of (2). This leads to expressions for the symmetrized partial derivatives of  $R_{abcd}$  in terms of the symmetrized covariant derivatives of  $R_{abcd}$  and  $B^a_b$ .

$$\begin{aligned} (R^a_{cdb})_{;e} B^b_a A^c A^d A^e &= -A^a A^b A^c B^d_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d_e g^{ef} \\ (R^a_{cdb})_{;ef} B^b_a A^c A^d A^e A^f &= -2A^a A^b A^c A^d \nabla_a B^e_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\ (R^a_{cdb})_{;efg} B^b_a A^c A^d A^e A^f A^g &= -3A^a A^b A^c A^d A^e \nabla_a R_{bfch} \nabla_d (\nabla_e B^f_g) g^{hg} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeh}) g^{gh} \\ &\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeh})) g^{gh} - A^a A^b A^c A^d A^e R_{afbh} \nabla_c (\nabla_d (\nabla_e B^f_g)) g^{hg} \end{aligned}$$

## Stage 2: Symmetrised covariant derivatives of $B$

In this stage the symmetrized covariant derivatives of  $B^a_b$  are computed in terms of its partial derivatives (which by choice are all zero) and the connection and its partial derivatives (which in general are not zero).

$$\begin{aligned}
A^c \nabla_c (B^a_b) &= \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q \\
A^d A^c \nabla_d (\nabla_c (B^a_b)) &= A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{cd} \Gamma^c_{pq} B^p_b A^d A^q - 2 \Gamma^a_{cd} \Gamma^p_{bq} B^c_p A^d A^q + \Gamma^c_{bd} \Gamma^p_{cq} B^a_p A^d A^q \\
A^e A^d A^c \nabla_e (\nabla_d (\nabla_c (B^a_b))) &= A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q + A^c \partial_c \Gamma^a_{de} \Gamma^d_{pq} B^p_b A^e A^q + A^c \Gamma^a_{cd} \partial_e \Gamma^d_{pq} B^p_b A^e A^q \\
&\quad - 2 A^c \partial_c \Gamma^a_{de} \Gamma^p_{bq} B^d_p A^e A^q - 2 A^c \Gamma^a_{cd} \partial_e \Gamma^p_{bq} B^d_p A^e A^q + A^c \partial_c \Gamma^d_{be} \Gamma^p_{dq} B^a_p A^e A^q + A^c \Gamma^d_{bc} \partial_e \Gamma^p_{dq} B^a_p A^e A^q \\
&\quad + \Gamma^a_{ce} A^c \partial_f \Gamma^e_{pq} B^p_b A^f A^q - \Gamma^a_{ce} A^c \partial_f \Gamma^p_{bq} B^e_p A^f A^q + \Gamma^a_{cd} \Gamma^e_{cf} \Gamma^e_{pq} B^p_b A^d A^f A^q - 3 \Gamma^a_{cd} \Gamma^e_{bf} \Gamma^c_{pq} B^p_e A^d A^f A^q \\
&\quad + 3 \Gamma^a_{cd} \Gamma^e_{bf} \Gamma^p_{eq} B^c_p A^d A^f A^q - \Gamma^c_{be} A^e \partial_f \Gamma^a_{pq} B^p_c A^f A^q + \Gamma^c_{be} A^e \partial_f \Gamma^p_{cq} B^a_p A^f A^q - \Gamma^c_{bd} \Gamma^e_{cf} \Gamma^p_{eq} B^a_p A^d A^f A^q
\end{aligned}$$

### Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of $B$

Here we impose the RNC condition (that  $\Gamma = 0$  while  $\partial\Gamma \neq 0$ ).

$$\begin{aligned}
A^c \nabla_c (B^a_b) &= 0 \\
A^d A^c \nabla_d (\nabla_c (B^a_b)) &= A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q \\
A^e A^d A^c \nabla_e (\nabla_d (\nabla_c (B^a_b))) &= A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q
\end{aligned}$$

### Stage 4: Replace covariant derivs of $B$ with partial derivs of $\Gamma$

This stage uses the results from the second stage to eliminate the  $\nabla B$  terms from the results of the first stage. This produces expressions for the symmetrized partial derivatives of  $R_{abcd}$  in terms of the symmetrized covariant derivatives of  $R_{abcd}$  and the partial derivatives of the connection. In this stage we also set the  $B^a_b$  to equal 1.

$$\begin{aligned}
(R^a_{cdb})_{,e} A^c A^d A^e &= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
(R^a_{cdb})_{,ef} A^c A^d A^e A^f &= A^c A^d A^e A^f (-\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma^g_{bf} g^{ha} + R_{cbdg} \partial_e \Gamma^a_{hf} g^{gh}) \\
(R^a_{cdb})_{,efg} A^c A^d A^e A^f A^g &= A^c A^d A^e A^f A^g (-3 \nabla_c R_{dhef} \partial_f \Gamma^h_{bg} g^{ia} + 3 \nabla_c R_{dbeh} \partial_f \Gamma^a_{ig} g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_{ef} \Gamma^h_{bg} g^{ia} + R_{cbdh} \partial_{ef} \Gamma^a_{ig} g^{hi})
\end{aligned}$$

## Stage 5: Replace partial derivs of $\Gamma$ with partial derivs of $R$

The fifth stage draws in results from `dGamma.tex` to replace the partial derivatives of  $\Gamma$  with partial derivatives of  $R_{abcd}$ .

$$\begin{aligned}
(R^a_{\text{cdb}})_{,e} A^c A^d A^e &= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
(R^a_{\text{cdb}})_{,ef} A^c A^d A^e A^f &= -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R^g_{feb} R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R^a_{feg} R_{cbdh} g^{hg} \\
(R^a_{\text{cdb}})_{,efg} A^c A^d A^e A^f A^g &= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{gei} g^{hi}
\end{aligned}$$

## Stage 6: Replace partial derivs of $R$ with covariant derivs of $R$

The final stage is to eliminate the  $\partial R$  by using earlier results. For example, in the equation for  $\partial^3 R$  we see terms involving  $\partial R$ . These first order partial derivatives can be replaced with the expression previously computed for  $\partial R$  in terms of  $\nabla R$ .

$$\begin{aligned}
(R^a_{\text{cdb}})_{,e} A^c A^d A^e &= A^c A^d A^e \nabla_c R_{bdef} g^{af} \\
(R^a_{\text{cdb}})_{,ef} A^c A^d A^e A^f &= A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag} \\
(R^a_{\text{cdb}})_{,efg} A^c A^d A^e A^f A^g &= -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}
\end{aligned}$$

The end result are expressions for the symmetrized partial derivatives of  $R_{abcd}$  solely in terms of the symmetrized covariant derivatives of  $R_{abcd}$ .

# Shared properties

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

B^{a}_{b}::Depends(\nabla{#}).
R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

## Stage 1: Symmetrised partial derivatives of $R$

```
def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d} -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{d} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d} -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    sort_product (obj)
    rename_dummies (obj)
    return obj

# compute the symmetric covariant derivatives of  $R^{a}_{bcd} B^{d}_{a} A^{b} A^{c}$ 

beg_stage_1 = time.time()

dRabcd00:= $R^{a}_{b c d} B^{d}_{a} A^{b} A^{c}$ .          # cdb(dRabcd00.101,dRabcd00)

dRabcd01:= $A^{a} \nabla_{a} \{ @ (dRabcd00) \}$ .      # cdb(dRabcd01.101,dRabcd01)
distribute      (dRabcd01)                    # cdb(dRabcd01.102,dRabcd01)
product_rule     (dRabcd01)                    # cdb(dRabcd01.103,dRabcd01)
distribute      (dRabcd01)                    # cdb(dRabcd01.104,dRabcd01)
substitute      (dRabcd01,$\nabla_{a} \{ A^{b} \} -> 0$) # cdb(dRabcd01.105,dRabcd01)
substitute      (dRabcd01,$\nabla_{a} \{ g^{b c} \} -> 0$) # cdb(dRabcd01.106,dRabcd01)

sort_product     (dRabcd01)
rename_dummies   (dRabcd01)
canonicalise     (dRabcd01)                    # cdb(dRabcd01.107,dRabcd01)
dRabcd01 = flatten_Rabcd (dRabcd01)            # cdb(dRabcd01.108,dRabcd01)

dRabcd02:= $A^{a} \nabla_{a} \{ @ (dRabcd01) \}$ .      # cdb(dRabcd02.101,dRabcd02)
distribute      (dRabcd02)                    # cdb(dRabcd02.102,dRabcd02)
product_rule     (dRabcd02)                    # cdb(dRabcd02.103,dRabcd02)
distribute      (dRabcd02)                    # cdb(dRabcd02.104,dRabcd02)
substitute      (dRabcd02,$\nabla_{a} \{ A^{b} \} -> 0$) # cdb(dRabcd02.105,dRabcd02)
substitute      (dRabcd02,$\nabla_{a} \{ g^{b c} \} -> 0$) # cdb(dRabcd02.106,dRabcd02)

sort_product     (dRabcd02)
```



```

rename_dummies (dRabcd02)
canonicalise    (dRabcd02)                # cdb(dRabcd02.107,dRabcd02)
dRabcd02 = flatten_Rabcd (dRabcd02)      # cdb(dRabcd02.108,dRabcd02)

dRabcd03:=A^{a}\nabla_{a}{ @ (dRabcd02) }. # cdb(dRabcd03.101,dRabcd03)
distribute      (dRabcd03)                # cdb(dRabcd03.102,dRabcd03)
product_rule    (dRabcd03)                # cdb(dRabcd03.103,dRabcd03)
distribute      (dRabcd03)                # cdb(dRabcd03.104,dRabcd03)
substitute      (dRabcd03,$\nabla_{a}{A^{b}} -> 0$) # cdb(dRabcd03.105,dRabcd03)
substitute      (dRabcd03,$\nabla_{a}{g^{b c}} -> 0$) # cdb(dRabcd03.106,dRabcd03)

sort_product    (dRabcd03)
rename_dummies  (dRabcd03)
canonicalise    (dRabcd03)                # cdb(dRabcd03.107,dRabcd03)
dRabcd03 = flatten_Rabcd (dRabcd03)      # cdb(dRabcd03.108,dRabcd03)

dRabcd04:=A^{a}\nabla_{a}{ @ (dRabcd03) }.
distribute      (dRabcd04)
product_rule    (dRabcd04)
distribute      (dRabcd04)
substitute      (dRabcd04,$\nabla_{a}{A^{b}} -> 0$)
substitute      (dRabcd04,$\nabla_{a}{g^{b c}} -> 0$)

sort_product    (dRabcd04)
rename_dummies  (dRabcd04)
canonicalise    (dRabcd04)
dRabcd04 = flatten_Rabcd (dRabcd04)

dRabcd05:=A^{a}\nabla_{a}{ @ (dRabcd04) }.
distribute      (dRabcd05)
product_rule    (dRabcd05)
distribute      (dRabcd05)
substitute      (dRabcd05,$\nabla_{a}{A^{b}} -> 0$)
substitute      (dRabcd05,$\nabla_{a}{g^{b c}} -> 0$)

sort_product    (dRabcd05)
rename_dummies  (dRabcd05)
canonicalise    (dRabcd05)

```

```

dRabcd05 = flatten_Rabcd (dRabcd05)

def combine_nabla (obj):
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{\nabla_{s}\{\nabla_{t}\{A??}\}}\}}\}\rightarrow\nabla_{p\ q\ r\ s\ t}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{\nabla_{s}\{A??}\}}\}\}\rightarrow\nabla_{p\ q\ r\ s}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{\nabla_{r}\{A??\}\}}\}\rightarrow\nabla_{p\ q\ r}\{A??\}\$,repeat=True)
    substitute (obj,$\nabla_{p}\{\nabla_{q}\{A??\}\}\}\rightarrow\nabla_{p\ q}\{A??\}\$,repeat=True)
    return obj

dRabcd01 = combine_nabla (dRabcd01)
dRabcd02 = combine_nabla (dRabcd02)
dRabcd03 = combine_nabla (dRabcd03)
dRabcd04 = combine_nabla (dRabcd04)
dRabcd05 = combine_nabla (dRabcd05)

end_stage_1 = time.time()

```

$$\text{dRabcd00.101} := R^a{}_{bcd} B^d{}_e A^b A^c$$

$$\text{dRabcd01.101} := A^a \nabla_a (R^e{}_{bcd} B^d{}_e A^b A^c)$$

$$\text{dRabcd01.102} := A^a \nabla_a (R^e{}_{bcd} B^d{}_e A^b A^c)$$

$$\text{dRabcd01.103} := A^a (\nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c)$$

$$\text{dRabcd01.104} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c + A^a R^e{}_{bcd} B^d{}_e \nabla_a A^b A^c + A^a R^e{}_{bcd} B^d{}_e A^b \nabla_a A^c$$

$$\text{dRabcd01.105} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\text{dRabcd01.106} := A^a \nabla_a R^e{}_{bcd} B^d{}_e A^b A^c + A^a R^e{}_{bcd} \nabla_a B^d{}_e A^b A^c$$

$$\text{dRabcd01.107} := -A^a A^b A^c B^d{}_e \nabla_a R^e{}_{bcd} - A^a A^b A^c R^d{}_{be} \nabla_c B^e{}_d$$

$$\text{dRabcd01.108} := -A^a A^b A^c B^d{}_e \nabla_a R_{bfcd} g^{ef} - A^a A^b A^c R_{afbd} \nabla_c B^d{}_e g^{ef}$$

$$\text{dRabcd02.101} := A^a \nabla_a (-A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef})$$

$$\text{dRabcd02.102} := -A^a \nabla_a (A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef}) - A^a \nabla_a (A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef})$$

$$\begin{aligned} \text{dRabcd02.103} := & -A^a (\nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} + A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ & + A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} + A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef}) - A^a (\nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ & + A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} + A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} + A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef}) \end{aligned}$$

$$\begin{aligned} \text{dRabcd02.104} := & -A^a \nabla_a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g \nabla_a A^b A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b \nabla_a A^c B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} \\ & - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} - A^a \nabla_a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g \nabla_a A^b A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} \\ & - A^a A^g A^b \nabla_a A^c R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef} \end{aligned}$$

$$\begin{aligned} \text{dRabcd02.105} := & -A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_g R_{bfcd} \nabla_a g^{ef} \\ & - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_c B^d{}_e \nabla_a g^{ef} \end{aligned}$$

$$\text{dRabcd02.106} := -A^a A^g A^b A^c \nabla_a B^d{}_e \nabla_g R_{bfcd} g^{ef} - A^a A^g A^b A^c B^d{}_e \nabla_a (\nabla_g R_{bfcd}) g^{ef} - A^a A^g A^b A^c \nabla_a R_{gfbd} \nabla_c B^d{}_e g^{ef} - A^a A^g A^b A^c R_{gfbd} \nabla_a (\nabla_c B^d{}_e) g^{ef}$$

$$\text{dRabcd02.107} := -2A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebf} \nabla_c (\nabla_d B^e{}_g) g^{fg}$$

$$\text{dRabcd02.108} := -2A^a A^b A^c A^d \nabla_a B^e{}_f \nabla_b R_{cedg} g^{fg} - A^a A^b A^c A^d B^e{}_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^b A^c A^d R_{aebg} \nabla_c (\nabla_d B^e{}_f) g^{fg}$$

$$\begin{aligned}
\text{dRabcd03.101} &:= A^a \nabla_a \left( -2A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right) \\
\text{dRabcd03.102} &:= -2A^a \nabla_a \left( A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \right) - A^a \nabla_a \left( A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \right) - A^a \nabla_a \left( A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right) \\
\text{dRabcd03.103} &:= -2A^a \left( \nabla_a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h \nabla_a A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h A^b \nabla_a A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \right. \\
&\quad + A^h A^b A^c \nabla_a A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} + A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} + A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} \\
&\quad + A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} \left. \right) - A^a \left( \nabla_a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h \nabla_a A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \right. \\
&\quad + A^h A^b \nabla_a A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h A^b A^c \nabla_a A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} + A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad + A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} + A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} \left. \right) - A^a \left( \nabla_a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \right. \\
&\quad + A^h \nabla_a A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b \nabla_a A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b A^c \nabla_a A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad + A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} + A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} + A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \left. \right) \\
\text{dRabcd03.104} &:= -2A^a \nabla_a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h \nabla_a A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b \nabla_a A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} \\
&\quad - 2A^a A^h A^b A^c \nabla_a A^d \nabla_h B^e_f \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} \\
&\quad - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} - A^a \nabla_a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h \nabla_a A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b \nabla_a A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c \nabla_a A^d B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} - A^a \nabla_a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad - A^a A^h \nabla_a A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b \nabla_a A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c \nabla_a A^d R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \\
\text{dRabcd03.105} &:= -2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_b R_{cedg} \nabla_a g^{fg} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d B^e_f \nabla_h (\nabla_b R_{cedg}) \nabla_a g^{fg} \\
&\quad - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_c (\nabla_d B^e_f) \nabla_a g^{gf} \\
\text{dRabcd03.106} &:= -2A^a A^h A^b A^c A^d \nabla_a (\nabla_h B^e_f) \nabla_b R_{cedg} g^{fg} - 2A^a A^h A^b A^c A^d \nabla_h B^e_f \nabla_a (\nabla_b R_{cedg}) g^{fg} - A^a A^h A^b A^c A^d \nabla_a B^e_f \nabla_h (\nabla_b R_{cedg}) g^{fg} \\
&\quad - A^a A^h A^b A^c A^d B^e_f \nabla_a (\nabla_h (\nabla_b R_{cedg})) g^{fg} - A^a A^h A^b A^c A^d \nabla_a R_{hebg} \nabla_c (\nabla_d B^e_f) g^{gf} - A^a A^h A^b A^c A^d R_{hebg} \nabla_a (\nabla_c (\nabla_d B^e_f)) g^{gf} \\
\text{dRabcd03.107} &:= -3A^a A^b A^c A^d A^e \nabla_a R_{bfcg} \nabla_d (\nabla_e B^f_h) g^{gh} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeh}) g^{gh} \\
&\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeh})) g^{gh} - A^a A^b A^c A^d A^e R_{afbg} \nabla_c (\nabla_d (\nabla_e B^f_h)) g^{gh} \\
\text{dRabcd03.108} &:= -3A^a A^b A^c A^d A^e \nabla_a R_{bfcg} \nabla_d (\nabla_e B^f_g) g^{hg} - 3A^a A^b A^c A^d A^e \nabla_a B^f_g \nabla_b (\nabla_c R_{dfeh}) g^{hg} \\
&\quad - A^a A^b A^c A^d A^e B^f_g \nabla_a (\nabla_b (\nabla_c R_{dfeh})) g^{hg} - A^a A^b A^c A^d A^e R_{afbh} \nabla_c (\nabla_d (\nabla_e B^f_g)) g^{hg}
\end{aligned}$$

## Stage 2: Symmetrised covariant derivatives of $B$

```

# compute the covariant derivatives of  $B^{\{a\}_{\{b\}}$ , note  $B^{\{a\}_{\{b,c\}}$  is zero, by choice
# this method of computing covariant derivatives does not use auxillary fields

beg_stage_2 = time.time()

dBab00:= $B^{\{a\}_{\{b\}}$ .      # cdb(dBab00.201,dBab00)

dBab01:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab00}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab01.201,dBab01)
distribute      (dBab01)                # cdb(dBab01.202,dBab01)
product_rule    (dBab01)                # cdb(dBab01.203,dBab01)
distribute      (dBab01)                # cdb(dBab01.204,dBab01)
substitute      (dBab01,$\text{partial}_{\{a\}}\{A^{\{b\}}\} \rightarrow 0\$) # cdb(dBab01.205,dBab01)
substitute      (dBab01,$\text{partial}_{\{a\}}\{B^{\{b\}}_{\{c\}}\} \rightarrow 0\$) # cdb(dBab01.206,dBab01)
substitute      (dBab01,$W^{\{a\}_{\{b\}}} \rightarrow @(\text{dBab00})\$) # cdb(dBab01.207,dBab01)
distribute      (dBab01)                # cdb(dBab01.208,dBab01)
canonicalise    (dBab01)                # cdb(dBab01.209,dBab01)

dBab02:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab01}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab02.201,dBab02)
distribute      (dBab02)                # cdb(dBab02.202,dBab02)
product_rule    (dBab02)                # cdb(dBab02.203,dBab02)
distribute      (dBab02)                # cdb(dBab02.204,dBab02)
substitute      (dBab02,$\text{partial}_{\{a\}}\{A^{\{b\}}\} \rightarrow 0\$) # cdb(dBab02.205,dBab02)
substitute      (dBab02,$\text{partial}_{\{a\}}\{B^{\{b\}}_{\{c\}}\} \rightarrow 0\$) # cdb(dBab02.206,dBab02)
substitute      (dBab02,$W^{\{a\}_{\{b\}}} \rightarrow @(\text{dBab01})\$) # cdb(dBab02.207,dBab02)
distribute      (dBab02)                # cdb(dBab02.208,dBab02)
canonicalise    (dBab02)                # cdb(dBab02.209,dBab02)

dBab03:= $A^{\{c\}}\backslash\text{partial}_{\{c\}}\{ @(\text{dBab02}) \} + \backslash\Gamma^{\{a\}_{\{p\}}}_{\{q\}} W^{\{p\}_{\{b\}}} A^{\{q\}}$ 
      -  $\backslash\Gamma^{\{p\}_{\{b\}}}_{\{q\}} W^{\{a\}_{\{p\}}} A^{\{q\}}$ .
                                     # cdb(dBab03.201,dBab03)
distribute      (dBab03)                # cdb(dBab03.202,dBab03)
product_rule    (dBab03)                # cdb(dBab03.203,dBab03)

```

```

distribute      (dBab03)                                # cdb(dBab03.204,dBab03)
substitute      (dBab03,$\partial_{a}\{A^{b}\} \rightarrow 0$) # cdb(dBab03.205,dBab03)
substitute      (dBab03,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$) # cdb(dBab03.206,dBab03)
substitute      (dBab03,$W^{a}_{b} \rightarrow @(dBab02)$) # cdb(dBab03.207,dBab03)
distribute      (dBab03)                                # cdb(dBab03.208,dBab03)
canonicalise    (dBab03)                                # cdb(dBab03.209,dBab03)

dBab04:=A^{c}\partial_{c}\{ @(dBab03) \} + \Gamma^{a}_{p\ q} W^{p}_{b} A^{q}
              - \Gamma^{p}_{b\ q} W^{a}_{p} A^{q}.

distribute      (dBab04)
product_rule    (dBab04)
distribute      (dBab04)
substitute      (dBab04,$\partial_{a}\{A^{b}\} \rightarrow 0$)
substitute      (dBab04,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$)
substitute      (dBab04,$W^{a}_{b} \rightarrow @(dBab03)$)
distribute      (dBab04)
canonicalise    (dBab04)

dBab05:=A^{c}\partial_{c}\{ @(dBab04) \} + \Gamma^{a}_{p\ q} W^{p}_{b} A^{q}
              - \Gamma^{p}_{b\ q} W^{a}_{p} A^{q}.

distribute      (dBab05)
product_rule    (dBab05)
distribute      (dBab05)
substitute      (dBab05,$\partial_{a}\{A^{b}\} \rightarrow 0$)
substitute      (dBab05,$\partial_{a}\{B^{b}\}_{c}\} \rightarrow 0$)
substitute      (dBab05,$W^{a}_{b} \rightarrow @(dBab04)$)
distribute      (dBab05)
canonicalise    (dBab05)

end_stage_2 = time.time()

```

$$\text{dBab00.201} := B^a_b$$

$$\text{dBab01.201} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.202} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.203} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.204} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.205} := A^c \partial_c B^a_b + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.206} := \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab01.207} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab01.208} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab01.209} := \Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab02.201} := A^c \partial_c (\Gamma^a_{pq} B^p_b A^q - \Gamma^p_{bq} B^a_p A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.202} := A^c \partial_c (\Gamma^a_{pq} B^p_b A^q) - A^c \partial_c (\Gamma^p_{bq} B^a_p A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.203} := A^c (\partial_c \Gamma^a_{pq} B^p_b A^q + \Gamma^a_{pq} \partial_c B^p_b A^q + \Gamma^a_{pq} B^p_b \partial_c A^q) - A^c (\partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^p_{bq} \partial_c B^a_p A^q + \Gamma^p_{bq} B^a_p \partial_c A^q) + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.204} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q + A^c \Gamma^a_{pq} \partial_c B^p_b A^q + A^c \Gamma^a_{pq} B^p_b \partial_c A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q - A^c \Gamma^p_{bq} \partial_c B^a_p A^q - A^c \Gamma^p_{bq} B^a_p \partial_c A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.205} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q + A^c \Gamma^a_{pq} \partial_c B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q - A^c \Gamma^p_{bq} \partial_c B^a_p A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.206} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} W^p_b A^q - \Gamma^p_{bq} W^a_p A^q$$

$$\text{dBab02.207} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} (\Gamma^p_{dc} B^d_b A^c - \Gamma^d_{bc} B^p_d A^c) A^q - \Gamma^p_{bq} (\Gamma^a_{dc} B^d_p A^c - \Gamma^d_{pc} B^a_d A^c) A^q$$

$$\text{dBab02.208} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{pq} \Gamma^p_{dc} B^d_b A^c A^q - \Gamma^a_{pq} \Gamma^d_{bc} B^p_d A^c A^q - \Gamma^p_{bq} \Gamma^a_{dc} B^d_p A^c A^q + \Gamma^p_{bq} \Gamma^d_{pc} B^a_d A^c A^q$$

$$\text{dBab02.209} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q + \Gamma^a_{cd} \Gamma^c_{pq} B^p_b A^d A^q - 2 \Gamma^a_{cd} \Gamma^p_{bq} B^c_p A^d A^q + \Gamma^c_{bd} \Gamma^p_{cq} B^a_p A^d A^q$$





$$\begin{aligned}
\text{dBab03.209} := & A^c A^e \partial_{ce} \Gamma^a_{pq} B^p{}_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a{}_p A^q + A^c \partial_c \Gamma^a_{de} \Gamma^d_{pq} B^p{}_b A^e A^q + A^c \Gamma^a_{cd} \partial_e \Gamma^d_{pq} B^p{}_b A^e A^q \\
& - 2A^c \partial_c \Gamma^a_{de} \Gamma^p_{bq} B^d{}_p A^e A^q - 2A^c \Gamma^a_{cd} \partial_e \Gamma^p_{bq} B^d{}_p A^e A^q + A^c \partial_c \Gamma^d_{be} \Gamma^p_{dq} B^a{}_p A^e A^q + A^c \Gamma^d_{bc} \partial_e \Gamma^p_{dq} B^a{}_p A^e A^q \\
& + \Gamma^a_{ce} A^c \partial_f \Gamma^e_{pq} B^p{}_b A^f A^q - \Gamma^a_{ce} A^c \partial_f \Gamma^p_{bq} B^e{}_p A^f A^q + \Gamma^a_{cd} \Gamma^c_{ef} \Gamma^e_{pq} B^p{}_b A^d A^f A^q - 3\Gamma^a_{cd} \Gamma^e_{bf} \Gamma^c_{pq} B^p{}_e A^d A^f A^q \\
& + 3\Gamma^a_{cd} \Gamma^e_{bf} \Gamma^p_{eq} B^c{}_p A^d A^f A^q - \Gamma^c_{be} A^e \partial_f \Gamma^a_{pq} B^p{}_c A^f A^q + \Gamma^c_{be} A^e \partial_f \Gamma^p_{cq} B^a{}_p A^f A^q - \Gamma^c_{bd} \Gamma^e_{cf} \Gamma^p_{eq} B^a{}_p A^d A^f A^q
\end{aligned}$$

### Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of $B$

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{\{d\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g}^{\{a\}_{\{b\}c}}$,repeat=True)
    substitute (obj,$\partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{a}_{\{b\}c}\} \rightarrow zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}_{\{b\}c}}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{\{b\}c} \rightarrow 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{\{d\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}g}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    substitute (obj,$zzz_{\{d\}e\{f\}g\{h\}}^{\{a\}_{\{b\}c}} \rightarrow \partial_{\{d\}e\{f\}g\{h\}}\{\Gamma^{a}_{\{b\}c}\}$,repeat=True)
    return obj

# switch to RNC

beg_stage_3 = time.time()

dBab01 = impose_rnc (dBab01)    # cdb (dBab01.301,dBab01)
dBab02 = impose_rnc (dBab02)    # cdb (dBab02.301,dBab02)
dBab03 = impose_rnc (dBab03)    # cdb (dBab03.301,dBab03)
dBab04 = impose_rnc (dBab04)    # cdb (dBab04.301,dBab04)
dBab05 = impose_rnc (dBab05)    # cdb (dBab05.301,dBab05)

end_stage_3 = time.time()
```

$$\text{dBab01.301} := 0$$

$$\text{dBab02.301} := A^c \partial_c \Gamma^a_{pq} B^p_b A^q - A^c \partial_c \Gamma^p_{bq} B^a_p A^q$$

$$\text{dBab03.301} := A^c A^e \partial_{ce} \Gamma^a_{pq} B^p_b A^q - A^c A^e \partial_{ce} \Gamma^p_{bq} B^a_p A^q$$

$$\begin{aligned} \text{dBab04.301} := & A^c A^e A^g \partial_{ceg} \Gamma^a_{pq} B^p_b A^q - A^c A^e A^g \partial_{ceg} \Gamma^p_{bq} B^a_p A^q + 2A^c A^d \partial_c \Gamma^a_{de} \partial_g \Gamma^e_{pq} B^p_b A^g A^q - 4A^c A^d \partial_c \Gamma^a_{de} \partial_g \Gamma^p_{bq} B^e_p A^g A^q \\ & + 2A^c A^d \partial_c \Gamma^e_{bd} \partial_g \Gamma^p_{eq} B^a_p A^g A^q + A^c \partial_c \Gamma^a_{ef} A^e \partial_g \Gamma^f_{pq} B^p_b A^g A^q - 2A^c \partial_c \Gamma^a_{ef} A^e \partial_g \Gamma^p_{bq} B^f_p A^g A^q + A^c \partial_c \Gamma^e_{bf} A^f \partial_g \Gamma^p_{eq} B^a_p A^g A^q \end{aligned}$$

$$\begin{aligned} \text{dBab05.301} := & A^c A^e A^g A^i \partial_{cegi} \Gamma^a_{pq} B^p_b A^q - A^c A^e A^g A^i \partial_{cegi} \Gamma^p_{bq} B^a_p A^q + 3A^c A^d A^e \partial_{cd} \Gamma^a_{eg} \partial_i \Gamma^g_{pq} B^p_b A^i A^q + 3A^c A^d A^e \partial_c \Gamma^a_{dg} \partial_{ei} \Gamma^g_{pq} B^p_b A^i A^q \\ & - 6A^c A^d A^e \partial_{cd} \Gamma^a_{eg} \partial_i \Gamma^p_{bq} B^g_p A^i A^q - 6A^c A^d A^e \partial_c \Gamma^a_{dg} \partial_{ei} \Gamma^p_{bq} B^g_p A^i A^q + 3A^c A^d A^e \partial_{cd} \Gamma^g_{be} \partial_i \Gamma^p_{gq} B^a_p A^i A^q \\ & + 3A^c A^d A^e \partial_c \Gamma^g_{bd} \partial_{ei} \Gamma^p_{gq} B^a_p A^i A^q + A^c A^e \partial_{ce} \Gamma^a_{fg} A^f \partial_i \Gamma^g_{pq} B^p_b A^i A^q + 2A^c A^e \partial_c \Gamma^a_{ef} A^g \partial_{gi} \Gamma^f_{pq} B^p_b A^i A^q \\ & - 2A^c A^e \partial_{ce} \Gamma^a_{fg} A^f \partial_i \Gamma^p_{bq} B^g_p A^i A^q - 3A^c A^e \partial_c \Gamma^a_{ef} A^g \partial_{gi} \Gamma^p_{bq} B^f_p A^i A^q - A^c A^e \partial_c \Gamma^f_{be} A^g \partial_{gi} \Gamma^a_{pq} B^p_f A^i A^q \\ & + A^c A^e \partial_{ce} \Gamma^f_{bg} A^g \partial_i \Gamma^p_{fq} B^a_p A^i A^q + 2A^c A^e \partial_c \Gamma^f_{be} A^g \partial_{gi} \Gamma^p_{fq} B^a_p A^i A^q + A^c \partial_c \Gamma^a_{eg} A^e A^h \partial_{hi} \Gamma^g_{pq} B^p_b A^i A^q \\ & - A^c \partial_c \Gamma^a_{eg} A^e A^h \partial_{hi} \Gamma^p_{bq} B^g_p A^i A^q - A^c \partial_c \Gamma^e_{bg} A^g A^h \partial_{hi} \Gamma^a_{pq} B^p_e A^i A^q + A^c \partial_c \Gamma^e_{bg} A^g A^h \partial_{hi} \Gamma^p_{eq} B^a_p A^i A^q \end{aligned}$$

## Stage 4: Replace covariant derivs of $B$ with partial derivs of $\Gamma$

```
# substitute covariant derivs of  $B^{\{a\}_{\{b\}}$  into covariant derivs of  $R^{\{a\}_{\{bcd\}}B^{\{d\}_{\{a\}}$ 
# this produces expressions for the partial derivs of Rabcd its covariant derivs and partial derivs of Gamma
# the partial derivs of Gamma will be eliminated later by using results imported from dGamma.json

beg_stage_4 = time.time()

substitute (dRabcd01,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd01)
substitute (dRabcd02,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd02)
substitute (dRabcd03,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}\nabla_{\{c\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab01)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd02,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}d}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd02)
substitute (dRabcd03,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}d}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}d}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}\nabla_{\{c\}d}\{B^{\{a\}_{\{b\}}}\} -> @(dBab02)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd03,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\}d\{e\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab03)$,repeat=True); distribute (dRabcd03)
substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\}d\{e\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab03)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}\nabla_{\{c\}d\{e\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab03)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd04,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}\nabla_{\{c\}d\{e\}f}\{B^{\{a\}_{\{b\}}}\} -> @(dBab04)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}\nabla_{\{c\}d\{e\}f}\{B^{\{a\}_{\{b\}}}\} -> @(dBab04)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd05,$A^{\{c\}}A^{\{d\}}A^{\{e\}}A^{\{f\}}A^{\{g\}}\nabla_{\{c\}d\{e\}f\{g\}}\{B^{\{a\}_{\{b\}}}\} -> @(dBab05)$,repeat=True); distribute (dRabcd05)

# no longer need B, so let's get rid of it

# two subtle tricks are used here
# 1) rename A and B as A002 and A001 before sort_product,
#    this ensures B will be to left of A after the sort
# 2) indices on B changed from  $B^{\{a\}_{\{b\}}$  to  $B_{\{b\}}^{\{a\}}$ ,
#    this ensures that after factor_out B will have dummy indices  $B_{\{a\}}^{\{b\}}$ 

def remove_Bab (obj):
    foo := @(obj).
```

```

substitute      (foo,$A^{a}->A002^{a},B^{a}_{b}->A001_{b}^{a})  # need this to sort B to the left of A
sort_product    (foo)
rename_dummies  (foo)
factor_out      (foo,$A001^{a?}_{b?},A002^{c?})
substitute      (foo,$A001_{a}^{b}->1,A002^{a}->A^{a})  # recover A and set B = 1, free indices now ^{a}_{b}
return foo

dRabcd01 = remove_Bab (dRabcd01)    # cdb(dRabcd01.401,dRabcd01)
dRabcd02 = remove_Bab (dRabcd02)    # cdb(dRabcd02.401,dRabcd02)
dRabcd03 = remove_Bab (dRabcd03)    # cdb(dRabcd03.401,dRabcd03)
dRabcd04 = remove_Bab (dRabcd04)    # cdb(dRabcd04.401,dRabcd04)
dRabcd05 = remove_Bab (dRabcd05)    # cdb(dRabcd05.401,dRabcd05)

end_stage_4 = time.time()

```

$$\text{dRabcd01.401} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\text{dRabcd02.401} := A^c A^d A^e A^f \left( -\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma_{bf}^g g^{ha} + R_{cbdg} \partial_e \Gamma_{hf}^a g^{gh} \right)$$

$$\text{dRabcd03.401} := A^c A^d A^e A^f A^g \left( -3\nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3\nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_e \Gamma_{bg}^h g^{ia} + R_{cbdh} \partial_e \Gamma_{ig}^a g^{hi} \right)$$

$$\begin{aligned} \text{dRabcd04.401} := & A^c A^d A^e A^f A^g A^h \left( -6\nabla_{de} R_{figj} \partial_c \Gamma_{bh}^i g^{aj} + 6\nabla_{de} R_{fbgi} \partial_c \Gamma_{jh}^a g^{ji} - 4\nabla_c R_{diej} \partial_{fg} \Gamma_{bh}^i g^{ja} + 4\nabla_c R_{dbei} \partial_{fg} \Gamma_{jh}^a g^{ij} - \nabla_{cdef} R_{gbhi} g^{ai} \right. \\ & \left. - R_{cidj} \partial_{efg} \Gamma_{bh}^i g^{ja} + R_{cbdi} \partial_{efg} \Gamma_{jh}^a g^{ij} - 3R_{cidj} \partial_e \Gamma_{fk}^i \partial_g \Gamma_{bh}^k g^{ja} + 6R_{cidj} \partial_e \Gamma_{fb}^i \partial_g \Gamma_{kh}^a g^{jk} - 3R_{cbdi} \partial_e \Gamma_{kf}^j \partial_g \Gamma_{jh}^a g^{ik} \right) \end{aligned}$$

$$\begin{aligned} \text{dRabcd05.401} := & A^c A^d A^e A^f A^g A^h A^i \left( -10\nabla_{cd} R_{ejfk} \partial_{gh} \Gamma_{bi}^j g^{ka} + 10\nabla_{cd} R_{ebfj} \partial_{gh} \Gamma_{ki}^a g^{jk} - 10\nabla_{def} R_{gjhk} \partial_c \Gamma_{bi}^j g^{ak} + 10\nabla_{def} R_{gbhj} \partial_c \Gamma_{ki}^a g^{kj} \right. \\ & - 5\nabla_c R_{djek} \partial_{fgh} \Gamma_{bi}^j g^{ka} + 5\nabla_c R_{dbej} \partial_{fgh} \Gamma_{ki}^a g^{jk} - 15\nabla_c R_{djek} \partial_f \Gamma_{gl}^j \partial_h \Gamma_{bi}^l g^{ka} + 30\nabla_c R_{djek} \partial_f \Gamma_{gb}^j \partial_h \Gamma_{li}^a g^{kl} - 15\nabla_c R_{dbej} \partial_f \Gamma_{lg}^k \partial_h \Gamma_{ki}^a g^{jl} \\ & - \nabla_{cdefg} R_{hbij} g^{aj} - R_{cjdk} \partial_{efgh} \Gamma_{bi}^j g^{ka} + R_{cbdj} \partial_{efgh} \Gamma_{ki}^a g^{jk} - 4R_{cjdk} \partial_h \Gamma_{bi}^l \partial_{ef} \Gamma_{gl}^j g^{ka} - 6R_{cjdk} \partial_e \Gamma_{fl}^j \partial_{gh} \Gamma_{bi}^l g^{ka} \\ & \left. + 8R_{cjdk} \partial_h \Gamma_{li}^a \partial_{ef} \Gamma_{gb}^j g^{kl} + 10R_{cjdk} \partial_e \Gamma_{fb}^j \partial_{gh} \Gamma_{li}^a g^{kl} - 4R_{cbdj} \partial_h \Gamma_{ki}^a \partial_{ef} \Gamma_{lg}^k g^{jl} - 6R_{cbdj} \partial_e \Gamma_{lf}^k \partial_{gh} \Gamma_{ki}^a g^{jl} + 2R_{cjdk} \partial_e \Gamma_{lf}^a \partial_{gh} \Gamma_{bi}^j g^{kl} \right) \end{aligned}$$

## Stage 5: Replace partial derivs of $\Gamma$ with partial derivs of $R$

```
import cdblib

beg_stage_5 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json') # cdb(dGamma01.500,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json') # cdb(dGamma02.500,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json') # cdb(dGamma03.500,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json') # cdb(dGamma04.500,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json') # cdb(dGamma05.500,dGamma05)

distribute (dRabcd01) # cdb(dRabcd01.500,dRabcd01)
distribute (dRabcd02) # cdb(dRabcd02.500,dRabcd02)
distribute (dRabcd03) # cdb(dRabcd03.500,dRabcd03)
distribute (dRabcd04) # cdb(dRabcd04.500,dRabcd04)
distribute (dRabcd05) # cdb(dRabcd05.500,dRabcd05)

# use dGamma to eliminate the partial derivs of Gamma
# this will introduces some lower order partial dervis of Rabcd on the rhs
# these extra partial derivs of Rabcd will be eliminated (later) by substiting lower order dRabcd into the higher order dRabcd

substitute (dRabcd02,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd02.501,dRabcd02)
substitute (dRabcd02,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd02.502,dRabcd02)
distribute (dRabcd02) # cdb(dRabcd02.503,dRabcd02)
sort_product (dRabcd02) # cdb(dRabcd02.504,dRabcd02)
rename_dummies (dRabcd02) # cdb(dRabcd02.505,dRabcd02)

substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(dGamma02)$,repeat=True) # cdb(dRabcd03.501,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(dGamma02)$,repeat=True) # cdb(dRabcd03.502,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd03.503,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(dGamma01)$,repeat=True) # cdb(dRabcd03.504,dRabcd03)
distribute (dRabcd03) # cdb(dRabcd03.505,dRabcd03)
sort_product (dRabcd03) # cdb(dRabcd03.506,dRabcd03)
rename_dummies (dRabcd03) # cdb(dRabcd03.507,dRabcd03)

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(dGamma03)$,repeat=True) # cdb(dRabcd04.501,dRabcd04)
```

```

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True) # cdb(dRabcd04.502,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True) # cdb(dRabcd04.503,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True) # cdb(dRabcd04.504,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True) # cdb(dRabcd04.505,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True) # cdb(dRabcd04.506,dRabcd04)
distribute (dRabcd04) # cdb(dRabcd04.507,dRabcd04)
sort_product (dRabcd04) # cdb(dRabcd04.508,dRabcd04)
rename_dummies (dRabcd04) # cdb(dRabcd04.509,dRabcd04)

substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} \rightarrow @(dGamma01)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} \rightarrow @(dGamma01)$,repeat=True)
distribute (dRabcd05)
sort_product (dRabcd05)
rename_dummies (dRabcd05)

end_stage_5 = time.time()

```



$$\text{dRabcd01.500} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\text{dRabcd02.500} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - A^c A^d A^e A^f R_{cgdh} \partial_e \Gamma_{bf}^g g^{ha} + A^c A^d A^e A^f R_{cbdg} \partial_e \Gamma_{hf}^a g^{gh}$$

$$\text{dRabcd02.501} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.502} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.503} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^f A^e R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^f A^e R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.504} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{feh}^a R_{cbdg} g^{gh}$$

$$\text{dRabcd02.505} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{feb}^g R_{cgdh} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{feh}^a R_{cbdh} g^{hg}$$

$$\begin{aligned} \text{dRabcd03.500} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - A^c A^d A^e A^f A^g R_{chdi} \partial_{ef} \Gamma_{bg}^h g^{ia} + A^c A^d A^e A^f A^g R_{cbdh} \partial_{ef} \Gamma_{ig}^a g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.501} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.502} := & -3A^c A^d A^e A^f A^g \nabla_c R_{dhei} \partial_f \Gamma_{bg}^h g^{ia} + 3A^c A^d A^e A^f A^g \nabla_c R_{dbeh} \partial_f \Gamma_{ig}^a g^{hi} \\ & - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.503} := & -A^c A^d A^e A^g A^f R_{gfb}^h \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R_{gfi}^a \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.504} := & -A^c A^d A^e A^g A^f R_{gfb}^h \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^g A^f R_{gfi}^a \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{geb}^h R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R_{gei}^a R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned}
\text{dRabcd03.505} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^h_{geb} R_{chdi} g^{ia} + \frac{1}{2} A^c A^d A^f A^g A^e \partial_f R^a_{gei} R_{cbdh} g^{hi} \\
\text{dRabcd03.506} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfi} \nabla_c R_{dbeh} g^{hi} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{gei} g^{hi} \\
\text{dRabcd03.507} &:= -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^h_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^a_{gei} g^{hi}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.500} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} \\
&\quad + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cde} R_{gbhi} g^{ai} - A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{efg} \Gamma^i_{bh} g^{ja} \\
&\quad + A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{efg} \Gamma^a_{jh} g^{ij} - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} \\
&\quad + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.501} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
&\quad - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cde} R_{gbhi} g^{ai} \\
&\quad - A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
&\quad + A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{h fj} \right) R_{cbdi} g^{ij} \\
&\quad - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.502} &:= -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
&\quad - 4A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_{fg} \Gamma^i_{bh} g^{ja} + 4A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_{fg} \Gamma^a_{jh} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cde} R_{gbhi} g^{ai} \\
&\quad - A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
&\quad + A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{h fj} \right) R_{cbdi} g^{ij} \\
&\quad - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.503} := & -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
& - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik} \\
\text{dRabcd04.504} := & -6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{figj} \partial_c \Gamma^i_{bh} g^{aj} + 6A^c A^d A^e A^f A^g A^h \nabla_{de} R_{fbgi} \partial_c \Gamma^a_{jh} g^{ji} \\
& - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - 3A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fk} \partial_g \Gamma^k_{bh} g^{ja} + 6A^c A^d A^e A^f A^g A^h R_{cidj} \partial_e \Gamma^i_{fb} \partial_g \Gamma^a_{kh} g^{jk} - 3A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_e \Gamma^j_{kf} \partial_g \Gamma^a_{jh} g^{ik} \\
\text{dRabcd04.505} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcb} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - A^c A^d A^e A^f A^h A^g R^k_{hgb} R_{cidj} \partial_e \Gamma^i_{fk} g^{ja} + 2A^c A^d A^e A^f A^h A^g R^a_{hgb} R_{cidj} \partial_e \Gamma^i_{fb} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik} \\
\text{dRabcd04.506} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcb} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{h fj} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} \\
& - A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^i_{heb} - \frac{1}{15} A^h A^e A^f A^g R^i_{efk} R^k_{hgb} - \frac{1}{15} A^h A^e A^f A^g R^i_{egk} R^k_{hfb} \right) R_{cidj} g^{ja} \\
& + A^c A^d \left( \frac{3}{5} A^h A^e A^f A^g \partial_{gf} R^a_{hej} - \frac{1}{15} A^h A^e A^f A^g R^a_{efk} R^k_{hgj} - \frac{1}{15} A^h A^e A^f A^g R^a_{egk} R^k_{hfj} \right) R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^f A^e R^i_{fek} A^h A^g R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^f A^e R^i_{feb} A^h A^g R^a_{hgb} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgj} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.507} := & -2A^h A^c R^i_{hcb} A^d A^e A^f A^g \nabla_{de} R_{figj} g^{aj} + 2A^h A^c R^a_{hcb} A^d A^e A^f A^g \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^g A^h A^f \partial_g R^i_{hfb} \nabla_c R_{diej} g^{ja} \\
& + 2A^c A^d A^e A^g A^h A^f \partial_g R^a_{hfb} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^h A^e A^f A^g \partial_{gf} R^i_{heb} R_{cidj} g^{ja} \\
& + \frac{1}{15} A^c A^d A^h A^e A^f A^g R^i_{efk} R^k_{hgb} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^h A^e A^f A^g R^i_{egk} R^k_{hfb} R_{cidj} g^{ja} + \frac{3}{5} A^c A^d A^h A^e A^f A^g \partial_{gf} R^a_{hej} R_{cbdi} g^{ij} \\
& - \frac{1}{15} A^c A^d A^h A^e A^f A^g R^a_{efk} R^k_{hgb} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^h A^e A^f A^g R^a_{egk} R^k_{hfb} R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^f A^e R^i_{fek} A^h A^g R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^f A^e R^i_{feb} A^h A^g R^a_{hgb} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^f A^e R^j_{fek} A^h A^g R^a_{hgb} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.508} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hcb} \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} \\
& + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{hfb} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{gf} R^i_{heb} g^{ja} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efk} R^k_{hgb} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egk} R^k_{hfb} R_{cidj} g^{ja} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{gf} R^a_{hej} g^{ij} \\
& - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efk} R^k_{hgb} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egk} R^k_{hfb} R_{cbdi} g^{ij} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fek} R^k_{hgb} R_{cidj} g^{ja} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgb} R^i_{feb} R_{cidj} g^{jk} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgb} R^j_{fek} R_{cbdi} g^{ik}
\end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.509} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hcb} \nabla_{de} R_{fbgi} g^{ji} - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} \\
& + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{hfb} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \partial_{gf} R^i_{heb} g^{ja} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efj} R^j_{hgb} R_{cidk} g^{ka} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egj} R^j_{hfb} R_{cidk} g^{ka} \\
& + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \partial_{gf} R^a_{hej} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efi} R^i_{hgb} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egi} R^i_{hfb} R_{cbdk} g^{kj} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fej} R^j_{hgb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^j_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^i_{fej} R_{cbdk} g^{kj}
\end{aligned}$$

## Stage 6: Replace partial derivs of $R$ with covariant derivs of $R$

```
# now eliminate remaining partial derivs of Rabcd by substitution from the lower order dRabcd

# note that
#   dRabcd01 =  $R^{a_{cdb,e}} A^c A^d A^e$ 
#   dRabcd02 =  $R^{a_{cdb,ef}} A^c A^d A^e A^f$ 
#   dRabcd03 =  $R^{a_{cdb,efg}} A^c A^d A^e A^f A^g$ 

# thus we can use
#   dRabcd01 to eliminate 1st partial derivs of R in dRabcd03, dRabcd04, etc.
#   dRabcd02 to eliminate 2nd partial derivs of R in dRabcd04, dRabcd05, etc.
#   dRabcd03 to eliminate 3rd partial derivs of R in dRabcd05, dRabcd06, etc.

beg_stage_6 = time.time()

substitute (dRabcd03,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} \rightarrow @(dRabcd01)$,repeat=True)      # cdb(dRabcd03.601,dRabcd03)
distribute (dRabcd03)                                                                    # cdb(dRabcd03.602,dRabcd03)

# note: dRabcd04 and dRabcd05 unused in this code (or any other code)

substitute (dRabcd04,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} \rightarrow @(dRabcd02)$,repeat=True) # cdb(dRabcd04.601,dRabcd04)
substitute (dRabcd04,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} \rightarrow @(dRabcd01)$,repeat=True)      # cdb(dRabcd04.602,dRabcd04)
distribute (dRabcd04)                                                                    # cdb(dRabcd04.603,dRabcd04)

substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{efg}\{R^{a}_{c d b}\} \rightarrow @(dRabcd03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}\partial_{ef}\{R^{a}_{c d b}\} \rightarrow @(dRabcd02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}\partial_{e}\{R^{a}_{c d b}\} \rightarrow @(dRabcd01)$,repeat=True)
distribute (dRabcd05)

end_stage_6 = time.time()
```

$$\begin{aligned} \text{dRabcd03.601} := & -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & + \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfb} g^{hj} R_{chdi} g^{ia} - \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfi} g^{aj} R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd03.602} := & -A^c A^d A^e A^f A^g R^h_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^a_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & + \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfb} g^{hj} R_{chdi} g^{ia} - \frac{1}{2} A^c A^d A^g A^e A^f \nabla_g R_{ejfi} g^{aj} R_{cbdh} g^{hi} \end{aligned}$$

$$\begin{aligned} \text{dRabcd04.601} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hci} \nabla_{de} R_{fbgj} g^{ij} \\ & - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \partial_g R^i_{hfb} g^{ja} + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \partial_g R^a_{h fj} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cde} R_{gbhi} g^{ai} \\ & - \frac{3}{5} A^c A^d \left( -A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} - \frac{1}{3} A^h A^e A^g A^f R^l_{fgb} R_{hle k} g^{ki} + \frac{1}{3} A^h A^e A^g A^f R^i_{fgl} R_{hbek} g^{kl} \right) R_{cidj} g^{ja} \\ & + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efj} R^j_{hgb} R_{cidk} g^{ka} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egj} R^j_{hfb} R_{cidk} g^{ka} \\ & + \frac{3}{5} A^c A^d \left( -A^h A^e A^g A^f \nabla_{he} R_{gjfl} g^{al} - \frac{1}{3} A^h A^e A^g A^f R^l_{fgj} R_{hle k} g^{ka} + \frac{1}{3} A^h A^e A^g A^f R^a_{fgl} R_{hjek} g^{kl} \right) R_{cbdi} g^{ij} \\ & - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efi} R^i_{hgj} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egi} R^i_{hfj} R_{cbdk} g^{kj} \\ & - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fej} R^j_{hgb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^j_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^i_{fej} R_{cbdk} g^{kj} \end{aligned}$$

$$\begin{aligned} \text{dRabcd04.602} := & -2A^c A^d A^e A^f A^g A^h R^i_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a_{hci} \nabla_{de} R_{fbgj} g^{ij} + 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fkgb} g^{ik} \nabla_c R_{diej} g^{ja} \\ & - 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fk gj} g^{ak} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cde} R_{gbhi} g^{ai} \\ & - \frac{3}{5} A^c A^d \left( -A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} - \frac{1}{3} A^h A^e A^g A^f R^l_{fgb} R_{hle k} g^{ki} + \frac{1}{3} A^h A^e A^g A^f R^i_{fgl} R_{hbek} g^{kl} \right) R_{cidj} g^{ja} \\ & + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{efj} R^j_{hgb} R_{cidk} g^{ka} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i_{egj} R^j_{hfb} R_{cidk} g^{ka} \\ & + \frac{3}{5} A^c A^d \left( -A^h A^e A^g A^f \nabla_{he} R_{gjfl} g^{al} - \frac{1}{3} A^h A^e A^g A^f R^l_{fgj} R_{hle k} g^{ka} + \frac{1}{3} A^h A^e A^g A^f R^a_{fgl} R_{hjek} g^{kl} \right) R_{cbdi} g^{ij} \\ & - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{efi} R^i_{hgj} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a_{egi} R^i_{hfj} R_{cbdk} g^{kj} \\ & - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i_{fej} R^j_{hgb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^j_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a_{hgi} R^i_{fej} R_{cbdk} g^{kj} \end{aligned}$$

$$\begin{aligned}
\text{dRabcd04.603} := & -2A^c A^d A^e A^f A^g A^h R^i{}_{hcb} \nabla_{de} R_{figj} g^{aj} + 2A^c A^d A^e A^f A^g A^h R^a{}_{hci} \nabla_{de} R_{fbgj} g^{ij} + 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fkgb} g^{ik} \nabla_c R_{diej} g^{ja} \\
& - 2A^c A^d A^e A^h A^f A^g \nabla_h R_{fkgj} g^{ak} \nabla_c R_{dbei} g^{ij} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^h A^e A^g A^f \nabla_{he} R_{gbfl} g^{il} R_{cidj} g^{ja} \\
& + \frac{1}{5} A^c A^d A^h A^e A^g A^f R^l{}_{f gb} R_{hle k} g^{ki} R_{cidj} g^{ja} - \frac{1}{5} A^c A^d A^h A^e A^g A^f R^i{}_{f gl} R_{hbek} g^{kl} R_{cidj} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i{}_{efj} R^j{}_{hgb} R_{cidk} g^{ka} \\
& + \frac{1}{15} A^c A^d A^e A^f A^g A^h R^i{}_{egj} R^j{}_{hfb} R_{cidk} g^{ka} - \frac{3}{5} A^c A^d A^h A^e A^g A^f \nabla_{he} R_{g jfl} g^{al} R_{cbdi} g^{ij} - \frac{1}{5} A^c A^d A^h A^e A^g A^f R^l{}_{fgj} R_{hle k} g^{ka} R_{cbdi} g^{ij} \\
& + \frac{1}{5} A^c A^d A^h A^e A^g A^f R^a{}_{f gl} R_{hjek} g^{kl} R_{cbdi} g^{ij} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a{}_{efi} R^i{}_{hgj} R_{cbdk} g^{kj} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R^a{}_{egi} R^i{}_{hfj} R_{cbdk} g^{kj} \\
& - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^i{}_{fej} R^j{}_{hgb} R_{cidk} g^{ka} + \frac{2}{3} A^c A^d A^e A^f A^g A^h R^a{}_{hgi} R^j{}_{feb} R_{cjdk} g^{ki} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R^a{}_{hgi} R^i{}_{fej} R_{cbdk} g^{kj}
\end{aligned}$$

## Stage 7: Reformatting

```
beg_stage_7 = time.time()

dRabcd01 = flatten_Rabcd (dRabcd01) # cdb(dRabcd01.701,dRabcd01)
dRabcd02 = flatten_Rabcd (dRabcd02) # cdb(dRabcd02.701,dRabcd02)
dRabcd03 = flatten_Rabcd (dRabcd03) # cdb(dRabcd03.701,dRabcd03)
dRabcd04 = flatten_Rabcd (dRabcd04) # cdb(dRabcd04.701,dRabcd04)
dRabcd05 = flatten_Rabcd (dRabcd05) # cdb(dRabcd05.701,dRabcd05)

canonicalise (dRabcd01) # cdb(dRabcd01.702,dRabcd01)
canonicalise (dRabcd02) # cdb(dRabcd02.702,dRabcd02)
canonicalise (dRabcd03) # cdb(dRabcd03.702,dRabcd03)
canonicalise (dRabcd04) # cdb(dRabcd04.702,dRabcd04)
canonicalise (dRabcd05) # cdb(dRabcd05.702,dRabcd05)

end_stage_7 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.1f} secs\\hfill\\break".format(end_stage_4-beg_stage_4))
print ("Stage 5: {:.1f} secs\\hfill\\break".format(end_stage_5-beg_stage_5))
print ("Stage 6: {:.1f} secs\\hfill\\break".format(end_stage_6-beg_stage_6))
print ("Stage 7: {:.1f} secs".format(end_stage_7-beg_stage_7))
# cdbEnd (timing)
```



$$\text{dRabcd01.701} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\text{dRabcd02.701} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R_{ifeb} g^{gi} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R_{hfei} g^{ah} g^{gi}$$

$$\begin{aligned} \text{dRabcd03.701} := & -A^c A^d A^e A^f A^g R_{hgfb} \nabla_c R_{diej} g^{ih} g^{ja} + A^c A^d A^e A^f A^g R_{hgfi} \nabla_c R_{dbej} g^{ah} g^{ji} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ & + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_g R_{ejfb} g^{hj} g^{ia} - \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \nabla_g R_{eifj} g^{ai} g^{hj} \end{aligned}$$

$$\begin{aligned} \text{dRabcd04.701} := & -2A^c A^d A^e A^f A^g A^h R_{ihcb} \nabla_{de} R_{fjgk} g^{ak} g^{ji} + 2A^c A^d A^e A^f A^g A^h R_{ihcj} \nabla_{de} R_{fbgk} g^{ai} g^{jk} + 2A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \nabla_h R_{fkgb} g^{ik} g^{ja} \\ & - 2A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \nabla_h R_{fjgk} g^{aj} g^{ik} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{he} R_{gbfk} g^{ik} g^{ja} \\ & + \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hkel} R_{mfgb} g^{ja} g^{li} g^{km} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hbek} R_{lfgm} g^{il} g^{ja} g^{km} \\ & + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kefl} R_{mhgb} g^{ik} g^{lm} g^{ja} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kegl} R_{mhfb} g^{ik} g^{lm} g^{ja} \\ & - \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \nabla_{he} R_{gjfk} g^{ak} g^{ij} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{im} g^{ka} g^{jl} \\ & + \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{al} g^{ij} g^{km} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jefk} R_{lhgm} g^{aj} g^{kl} g^{im} \\ & - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jegk} R_{lhfm} g^{aj} g^{kl} g^{im} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kfel} R_{mhgb} g^{ik} g^{lm} g^{ja} \\ & + \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{khgl} R_{mfeb} g^{ak} g^{im} g^{jl} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jhkg} R_{lfem} g^{aj} g^{kl} g^{im} \end{aligned}$$

$$\text{dRabcd01.702} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\text{dRabcd02.702} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\text{dRabcd03.702} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfg h} g^{ah}$$

$$\text{dRabcd04.702} := -\frac{7}{5} A^c A^d A^e A^f A^g A^h R_{bcdi} \nabla_{ef} R_{gjhk} g^{aj} g^{ik} + \frac{7}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{ef} R_{bghk} g^{ai} g^{jk} + A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{bghi} g^{ai}$$

$$\begin{aligned} \text{dRabcd05.702} := & -2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{bdej} \nabla_{fg} R_{hkil} g^{ak} g^{jl} + 2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{djek} \nabla_{fg} R_{bhil} g^{aj} g^{kl} \\ & - \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} \nabla_{efg} R_{hkil} g^{ak} g^{jl} + \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} \nabla_{efg} R_{bhil} g^{aj} g^{kl} \\ & + \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{am} g^{jk} g^{ln} + A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{elfm} \nabla_g R_{bhin} g^{aj} g^{kl} g^{mn} \\ & - \frac{4}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{ak} g^{jm} g^{ln} + A^c A^d A^e A^f A^g A^h A^i \nabla_{cdefg} R_{bhij} g^{aj} \end{aligned}$$

```
cdblib.create ('dRabcd.json')
```

```
cdblib.put ('dRabcd01',dRabcd01,'dRabcd.json')
```

```
cdblib.put ('dRabcd02',dRabcd02,'dRabcd.json')
```

```
cdblib.put ('dRabcd03',dRabcd03,'dRabcd.json')
```

```
cdblib.put ('dRabcd04',dRabcd04,'dRabcd.json')
```

```
cdblib.put ('dRabcd05',dRabcd05,'dRabcd.json')
```

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ g^{a b}              -> A003^{a b}              $)
    substitute (obj,$ \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} -> A008_{\{a b c d e f g h\}} $)
    substitute (obj,$ \nabla_{\{e f g\}}\{R_{\{a b c d\}}\}      -> A007_{\{a b c d e f g\}}  $)
    substitute (obj,$ \nabla_{\{e f\}}\{R_{\{a b c d\}}\}         -> A006_{\{a b c d e f\}}   $)
    substitute (obj,$ \nabla_{\{e\}}\{R_{\{a b c d\}}\}          -> A005_{\{a b c d e\}}     $)
    substitute (obj,$ R_{\{a b c d\}}        -> A004_{\{a b c d\}}      $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> A^{a}              $)
    substitute (obj,$ A002^{a}              -> x^{a}              $)
    substitute (obj,$ A003^{a b}            -> g^{a b}            $)
    substitute (obj,$ A004_{\{a b c d\}}     -> R_{\{a b c d\}}     $)
    substitute (obj,$ A005_{\{a b c d e\}}   -> \nabla_{\{e\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A006_{\{a b c d e f\}} -> \nabla_{\{e f\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A007_{\{a b c d e f g\}} -> \nabla_{\{e f g\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A008_{\{a b c d e f g h\}} -> \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} $)

    return obj

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    factor_out (bah,$A^{a?}$)
    ans := @(bah).
    return ans

scaled1 = reformat (dRabcd01, 1) # cdb(scaled1.601,scaled1)
scaled2 = reformat (dRabcd02, 1) # cdb(scaled2.601,scaled2)
scaled3 = reformat (dRabcd03,-2) # cdb(scaled3.601,scaled3)
scaled4 = reformat (dRabcd04,-5) # cdb(scaled4.601,scaled4)

```

```
scaled5 = reformat (dRabcd05,-3)    # cdb(scaled5.601,scaled5)
```

## Symmetrised partial derivatives of $R^a{}_{bcd}$

$$\begin{aligned}
A^c A^d A^e R^a{}_{cdb,e} &= A^c A^d A^e g^{af} \nabla_c R_{bdef} \\
A^c A^d A^e A^f R^a{}_{cdb,ef} &= A^c A^d A^e A^f g^{ag} \nabla_{cd} R_{befg} \\
-2A^c A^d A^e A^f A^g R^a{}_{cdb,efg} &= A^c A^d A^e A^f A^g (g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - 2g^{ah} \nabla_{cde} R_{bfgh}) \\
-5A^c A^d A^e A^f A^g A^h R^a{}_{cdb,efgh} &= A^c A^d A^e A^f A^g A^h (7g^{ai} g^{jk} R_{bcdj} \nabla_{ef} R_{gihk} - 7g^{ai} g^{jk} R_{cidj} \nabla_{ef} R_{bghk} - 5g^{ai} \nabla_{cdef} R_{bghi}) \\
-3A^c A^d A^e A^f A^g A^h A^i R^a{}_{cdb,efghi} &= A^c A^d A^e A^f A^g A^h A^i (6g^{aj} g^{kl} \nabla_c R_{bdek} \nabla_{fg} R_{hji l} - 6g^{aj} g^{kl} \nabla_c R_{djek} \nabla_{fg} R_{bhil} + 8g^{aj} g^{kl} R_{bcdk} \nabla_{efg} R_{hji l} \\
&\quad - 8g^{aj} g^{kl} R_{cjdk} \nabla_{efg} R_{bhil} - g^{aj} g^{kl} g^{mn} R_{bcdk} R_{elfm} \nabla_g R_{hjin} - 3g^{aj} g^{kl} g^{mn} R_{cjdk} R_{elfm} \nabla_g R_{bhin} \\
&\quad + 4g^{aj} g^{kl} g^{mn} R_{bcdk} R_{ejfm} \nabla_g R_{hlin} - 3g^{aj} \nabla_{cdefg} R_{bhij})
\end{aligned}$$

```

substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dRabcd.export')

# 6th order dRabcd, scaled
cdblib.put ('dRabcd61scaled',scaled1,'dRabcd.export')
cdblib.put ('dRabcd62scaled',scaled2,'dRabcd.export')
cdblib.put ('dRabcd63scaled',scaled3,'dRabcd.export')
cdblib.put ('dRabcd64scaled',scaled4,'dRabcd.export')
cdblib.put ('dRabcd65scaled',scaled5,'dRabcd.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

```

# Timing

Stage 1: 2.4 secs

Stage 2: 6.3 secs

Stage 3: 0.5 secs

Stage 4: 177.4 secs

Stage 5: 195.4 secs

Stage 6: 241.7 secs

Stage 7: 8.4 secs



# Geodesic BVP

Consider a geodesic that connects two points  $P_i$  and  $P_j$  with RNC coordinates  $x_i^a$  and  $x_j^a$ . Our aim is to construct a solution  $x^a(s)$  of the geodesic equation such that  $x^a(0) = x_i^a$  and  $x^a(1) = x_j^a$ .

We will do this in two stages. First we will solve

$$x_j^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{\underline{b}_k}^a y^{\underline{b}_k} \quad (1)$$

for  $y^a$  as an explicit polynomial in  $x_i^a$  and  $x_j^a$ . The functions  $\Gamma_{\underline{b}_k}^a$  are the generalised connections for the RNC frame evaluated at  $x^a = x_i^a$ .

In the second stage, we will substitute our expression for  $y^a$  into

$$x^a(s) = x_i^a + sy^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{\underline{b}_k}^a y^{\underline{b}_k} s^k \quad (2)$$

to obtain the desired solution to the two point boundary value problem.

## Stage 1: The fixed point iteration scheme

First we rewrite the main equation (1) in the suggestive form

$$y^a = \Delta x^a + \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{\underline{b}_k}^a y^{\underline{b}_k}$$

where  $\Delta x^a = x_j^a - x_i^a$ . Our approximate solution for  $y^a$  will be taken to be the partial sums for the infinite series. Thus we will solve

$$y^a = \Delta x^a + \sum_{k=2}^n \frac{1}{k!} \Gamma_{\underline{b}_k}^a y^{\underline{b}_k}$$

for  $y^a$ . Note that in the last term of the sum, the  $\Gamma_{\underline{b}_n}^a$  will contain curvature terms of order  $\mathcal{O}(\epsilon^n)$ . Thus in truncating the series at this point we will loose contributions to the curvature terms of order  $\mathcal{O}(\epsilon^{n+1})$  and higher. So to be consistent we must truncate all terms of the partial sum to order  $\mathcal{O}(\epsilon^n)$  (i.e., exclude any contributions from terms  $\mathcal{O}(\epsilon^{n+1})$  and higher, these are the terms that would couple with the terms that we

excluded when truncating the original infinite series). Let  $\overset{k}{T}$  the operator that truncates its argument to contain terms no higher than  $\mathcal{O}(\epsilon^n)$ . Then we have the following modified version of the equation for  $\overset{n}{y}^a$

$$\overset{n}{y}^a = \Delta x^a + \sum_{k=2}^n \frac{1}{k!} \overset{k}{T} \left( \Gamma_{\underline{b}_k}^a \overset{n}{y}^{\underline{b}_k} \right)$$

Finally we note that since  $\Gamma_{\underline{b}_k}^a = \mathcal{O}(\epsilon^k)$ , we can use lower order estimates for the  $\overset{k}{y}^a$  in the right hand side of the sum. This allows us to compute  $\overset{n}{y}^a$  by successive approximations such as

$$\begin{aligned} \overset{0}{y}^a &= \Delta x^a \\ \overset{2}{y}^a &= \overset{0}{y}^a + \frac{1}{2!} \overset{2}{T} \left( \Gamma_{bc}^a \overset{0}{y}^b \overset{0}{y}^c \right) \\ \overset{3}{y}^a &= \overset{0}{y}^a + \frac{1}{2!} \overset{3}{T} \left( \Gamma_{bc}^a \overset{2}{y}^b \overset{2}{y}^c \right) + \frac{1}{3!} \overset{3}{T} \left( \Gamma_{bcd}^a \overset{0}{y}^b \overset{0}{y}^c \overset{0}{y}^d \right) \\ \overset{4}{y}^a &= \overset{0}{y}^a + \frac{1}{2!} \overset{4}{T} \left( \Gamma_{bc}^a \overset{3}{y}^b \overset{3}{y}^c \right) + \frac{1}{3!} \overset{4}{T} \left( \Gamma_{bcd}^a \overset{2}{y}^b \overset{2}{y}^c \overset{2}{y}^d \right) + \frac{1}{4!} \overset{4}{T} \left( \Gamma_{bcde}^a \overset{0}{y}^b \overset{0}{y}^c \overset{0}{y}^d \overset{0}{y}^e \right) \\ \overset{5}{y}^a &= \overset{0}{y}^a + \frac{1}{2!} \overset{5}{T} \left( \Gamma_{bc}^a \overset{4}{y}^b \overset{4}{y}^c \right) + \frac{1}{3!} \overset{5}{T} \left( \Gamma_{bcd}^a \overset{3}{y}^b \overset{3}{y}^c \overset{3}{y}^d \right) + \frac{1}{4!} \overset{5}{T} \left( \Gamma_{bcde}^a \overset{2}{y}^b \overset{2}{y}^c \overset{2}{y}^d \overset{2}{y}^e \right) + \frac{1}{5!} \overset{5}{T} \left( \Gamma_{bcdef}^a \overset{0}{y}^b \overset{0}{y}^c \overset{0}{y}^d \overset{0}{y}^e \overset{0}{y}^f \right) \end{aligned}$$

and so on. Note that there are no  $\overset{1}{y}^a$  terms.

## Stage 2: Introduce the generalised connections

This is the final stage – it introduces the generalised connection after the completion of the fixed point scheme.

All that needs be done is to substitute our expression for  $y^a$  into

$$x^a(s) = x_i^a + sy^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{\underline{b}_k}^a y^{\underline{b}_k} s^k \quad (3)$$

to obtain the desired solution to the two point boundary value problem.

The generalised connections  $\Gamma_{\underline{b}_k}^a$  are taken from the results of the `genGamma` code.

## Stage 1: The fixed point iteration scheme

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.
R_{a b c d}::Depends(\nabla{#}).

{Gam22^{a}_{b c},Gam23^{a}_{b c},Gam24^{a}_{b c},Gam25^{a}_{b c}}::TableauSymmetry(shape={2}, indices={1,2}).
{Gam33^{a}_{b c d},Gam34^{a}_{b c d},Gam35^{a}_{b c d}}::TableauSymmetry(shape={3}, indices={1,2,3}).
{Gam44^{a}_{b c d e},Gam45^{a}_{b c d e}}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
{Gam55^{a}_{b c d e f}}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).

{Gam22^{a}_{b c}}::Weight(label=eps,value=2).
{Gam23^{a}_{b c},Gam33^{a}_{b c d}}::Weight(label=eps,value=3).
{Gam24^{a}_{b c},Gam34^{a}_{b c d},Gam44^{a}_{b c d e}}::Weight(label=eps,value=4).
{Gam25^{a}_{b c},Gam35^{a}_{b c d},Gam45^{a}_{b c d e},Gam55^{a}_{b c d e f}}::Weight(label=eps,value=5).

{Dx^{a}}::Weight(label=eps,value=0).

{y00^{a},y20^{a},y30^{a},y40^{a},y50^{a}}::Weight(label=eps,value=0).
{y22^{a},y32^{a},y42^{a},y52^{a}}::Weight(label=eps,value=2).
{y33^{a},y43^{a},y53^{a}}::Weight(label=eps,value=3).
{y44^{a},y54^{a}}::Weight(label=eps,value=4).
{y55^{a}}::Weight(label=eps,value=5).

# Dx{#}::LaTeXForm{"{\Dx}"}. # LCB: currently causes a bug, it kills ::KeepWeight for Dx

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}          -> A001^{a}          $)
    substitute (obj,$ Dx^{a}         -> A002^{a}          $)
```

```

substitute (obj,$ g^{a b}                -> A003^{a b}                $)
substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\} -> A008_{a b c d e f g h} $)
substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}    -> A007_{a b c d e f g}    $)
substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}      -> A006_{a b c d e f}      $)
substitute (obj,$ \nabla_{e}\{R_{a b c d}\}        -> A005_{a b c d e}        $)
substitute (obj,$ R_{a b c d}                -> A004_{a b c d}                $)
sort_product (obj)
rename_dummies (obj)
substitute (obj,$ A001^{a}                  -> x^{a}                  $)
substitute (obj,$ A002^{a}                  -> Dx^{a}                  $)
substitute (obj,$ A003^{a b}                -> g^{a b}                $)
substitute (obj,$ A004_{a b c d}            -> R_{a b c d}            $)
substitute (obj,$ A005_{a b c d e}          -> \nabla_{e}\{R_{a b c d}\} $)
substitute (obj,$ A006_{a b c d e f}        -> \nabla_{e f}\{R_{a b c d}\} $)
substitute (obj,$ A007_{a b c d e f g}      -> \nabla_{e f g}\{R_{a b c d}\} $)
substitute (obj,$ A008_{a b c d e f g h}    -> \nabla_{e f g h}\{R_{a b c d}\} $)

```

```

return obj

```

```

def get_term (obj,n):

```

```

    tmp := @(obj).
    foo = Ex("eps = " + str(n))
    distribute (tmp)
    keep_weight (tmp, foo)

```

```

    return tmp

```

```

def truncate (obj,n):

```

```

    ans = Ex(0)

```

```

    for i in range (0,n+1):
        foo := @(obj).
        bah = Ex("eps = " + str(i))
        distribute (foo)
        keep_weight (foo, bah)
        ans = ans + foo

```

```

return ans

def substitute_eps (obj):
    substitute (obj, epsy0)
    substitute (obj, epsy2)
    substitute (obj, epsy3)
    substitute (obj, epsy4)
    substitute (obj, epsy5)
    substitute (obj, epsGam2)
    substitute (obj, epsGam3)
    substitute (obj, epsGam4)
    substitute (obj, epsGam5)
    distribute (obj)
    obj = truncate (obj, 5)
    obj = product_sort (obj)
    rename_dummies (obj)
    canonicalise (obj)

    return obj

beg_stage_1 = time.time()

# yn = y expanded to terms upto and including O(eps^n)

y0 := Dx^{a}.
y2 := Dx^{a} + (1/2) Gam^{a}_{b c} y0^{b} y0^{c}.
y3 := Dx^{a} + (1/2) Gam^{a}_{b c} y2^{b} y2^{c}
      + (1/6) Gam^{a}_{b c d} y0^{b} y0^{c} y0^{d}.
y4 := Dx^{a} + (1/2) Gam^{a}_{b c} y3^{b} y3^{c}
      + (1/6) Gam^{a}_{b c d} y2^{b} y2^{c} y2^{d}
      + (1/24) Gam^{a}_{b c d e} y0^{b} y0^{c} y0^{d} y0^{e}.
y5 := Dx^{a} + (1/2) Gam^{a}_{b c} y4^{b} y4^{c}
      + (1/6) Gam^{a}_{b c d} y3^{b} y3^{c} y3^{d}
      + (1/24) Gam^{a}_{b c d e} y2^{b} y2^{c} y2^{d} y2^{e}
      + (1/120) Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

# epsyN = y expanded to terms upto and including O(eps^N)

```

```

# yPQ = O(eps^Q) term of epsyP

# expand to O(eps^5)

epsy0 := y0^{a} -> y00^{a}.
epsy2 := y2^{a} -> y20^{a}+y22^{a}.
epsy3 := y3^{a} -> y30^{a}+y32^{a}+y33^{a}.
epsy4 := y4^{a} -> y40^{a}+y42^{a}+y43^{a}+y44^{a}.
epsy5 := y5^{a} -> y50^{a}+y52^{a}+y53^{a}+y54^{a}+y55^{a}.

# epsGamN = gen. gamma with N lower indices (epsGam2 = the connection)
# epsGamPQ = O(eps^Q) term of epsGamP

epsGam2 := Gam^{a}_{b c} -> Gam22^{a}_{b c}+Gam23^{a}_{b c}+Gam24^{a}_{b c}+Gam25^{a}_{b c}.
epsGam3 := Gam^{a}_{b c d} -> Gam33^{a}_{b c d}+Gam34^{a}_{b c d}+Gam35^{a}_{b c d}.
epsGam4 := Gam^{a}_{b c d e} -> Gam44^{a}_{b c d e}+Gam45^{a}_{b c d e}.
epsGam5 := Gam^{a}_{b c d e f} -> Gam55^{a}_{b c d e f}.

y0 = substitute_eps (y0)    # cdb (y0.001,y0)
y2 = substitute_eps (y2)    # cdb (y2.001,y2)
y3 = substitute_eps (y3)    # cdb (y3.001,y3)
y4 = substitute_eps (y4)    # cdb (y4.001,y4)
y5 = substitute_eps (y5)    # cdb (y5.001,y5)

y0 = truncate (y0,1)        # cdb (y0.002,y0)
y2 = truncate (y2,2)        # cdb (y2.002,y2)
y3 = truncate (y3,3)        # cdb (y3.002,y3)
y4 = truncate (y4,4)        # cdb (y4.002,y4)
y5 = truncate (y5,5)        # cdb (y5.002,y5)

defy0 := y0^{a} -> @(y0).
defy2 := y2^{a} -> @(y2).
defy3 := y3^{a} -> @(y3).
defy4 := y4^{a} -> @(y4).
defy5 := y5^{a} -> @(y5).

# -----
def tidy (obj):

```

```

obj = product_sort (obj)
rename_dummies      (obj)
canonicalise        (obj)
return obj

# -----
# y0

y00 := @(y0).          # cdb (y00.101,y00)

defy00 := y00^{a} -> @(y00).

# -----
# y2

substitute (y2,defy00)

distribute (y2)

y20 = get_term (y2,0)   # cdb (y20.101,y20)
y22 = get_term (y2,2)   # cdb (y22.101,y22)

y20 = tidy (y20)        # cdb (y20.201,y20)
y22 = tidy (y22)        # cdb (y22.201,y22)

defy20 := y20^{a} -> @(y20).
defy22 := y22^{a} -> @(y22).

# -----
# y3

substitute (y3,defy00)

substitute (y3,defy20)
substitute (y3,defy22)

distribute (y3)

```

```

y30 = get_term (y3,0)    # cdb (y30.101,y30)
y32 = get_term (y3,2)    # cdb (y32.101,y32)
y33 = get_term (y3,3)    # cdb (y33.101,y33)

y30 = tidy (y30)         # cdb (y30.201,y30)
y32 = tidy (y32)         # cdb (y32.201,y32)
y33 = tidy (y33)         # cdb (y33.201,y33)

defy30 := y30^{a} -> @(y30).
defy32 := y32^{a} -> @(y32).
defy33 := y33^{a} -> @(y33).

# -----
# y4

substitute (y4,defy00)

substitute (y4,defy20)
substitute (y4,defy22)

substitute (y4,defy30)
substitute (y4,defy32)
substitute (y4,defy33)

distribute (y4)

y40 = get_term (y4,0)    # cdb (y40.101,y40)
y42 = get_term (y4,2)    # cdb (y42.101,y42)
y43 = get_term (y4,3)    # cdb (y43.101,y43)
y44 = get_term (y4,4)    # cdb (y44.101,y44)

y40 = tidy (y40)         # cdb (y40.201,y40)
y42 = tidy (y42)         # cdb (y42.201,y42)
y43 = tidy (y43)         # cdb (y43.201,y43)
y44 = tidy (y44)         # cdb (y44.201,y44)

defy40 := y40^{a} -> @(y40).
defy42 := y42^{a} -> @(y42).

```



```

defy43 := y43^{a} -> @(y43).
defy44 := y44^{a} -> @(y44).

# -----
# y5

substitute (y5,defy00)

substitute (y5,defy20)
substitute (y5,defy22)

substitute (y5,defy30)
substitute (y5,defy32)
substitute (y5,defy33)

substitute (y5,defy40)
substitute (y5,defy42)
substitute (y5,defy43)
substitute (y5,defy44)

distribute (y5)

y50 = get_term (y5,0)   # cdb (y50.101,y50)
y52 = get_term (y5,2)   # cdb (y52.101,y52)
y53 = get_term (y5,3)   # cdb (y53.101,y53)
y54 = get_term (y5,4)   # cdb (y54.101,y54)
y55 = get_term (y5,5)   # cdb (y55.101,y55)

y50 = tidy (y50)        # cdb (y50.201,y50)
y52 = tidy (y52)        # cdb (y52.201,y52)
y53 = tidy (y53)        # cdb (y53.201,y53)
y54 = tidy (y54)        # cdb (y54.201,y54)
y55 = tidy (y55)        # cdb (y55.201,y55)

defy50 := y50^{a} -> @(y50).
defy52 := y52^{a} -> @(y52).
defy53 := y53^{a} -> @(y53).
defy54 := y54^{a} -> @(y54).

```

```
defy55 := y55^{a} -> @(y55).
```

```
end_stage_1 = time.time()
```

$$y0.001 := Dx^a$$

$$y2.001 := Dx^a + \frac{1}{2}Gam22^a_{bc}y00^by00^c + \frac{1}{2}Gam23^a_{bc}y00^by00^c + \frac{1}{2}Gam24^a_{bc}y00^by00^c + \frac{1}{2}Gam25^a_{bc}y00^by00^c$$

$$y3.001 := Dx^a + \frac{1}{2}Gam22^a_{bc}y20^by20^c + \frac{1}{2}Gam23^a_{bc}y20^by20^c + \frac{1}{6}Gam33^a_{bcd}y00^by00^cy00^d + Gam22^a_{bc}y20^by22^c + \frac{1}{2}Gam24^a_{bc}y20^by20^c \\ + \frac{1}{6}Gam34^a_{bcd}y00^by00^cy00^d + Gam23^a_{bc}y20^by22^c + \frac{1}{2}Gam25^a_{bc}y20^by20^c + \frac{1}{6}Gam35^a_{bcd}y00^by00^cy00^d$$

$$y4.001 := Dx^a + \frac{1}{2}Gam22^a_{bc}y30^by30^c + \frac{1}{2}Gam23^a_{bc}y30^by30^c + \frac{1}{6}Gam33^a_{bcd}y20^by20^cy20^d + Gam22^a_{bc}y30^by32^c + \frac{1}{2}Gam24^a_{bc}y30^by30^c \\ + \frac{1}{6}Gam34^a_{bcd}y20^by20^cy20^d + \frac{1}{24}Gam44^a_{bcde}y00^by00^cy00^dy00^e + Gam22^a_{bc}y30^by33^c + Gam23^a_{bc}y30^by32^c \\ + \frac{1}{2}Gam25^a_{bc}y30^by30^c + \frac{1}{2}Gam33^a_{bcd}y20^by20^cy22^d + \frac{1}{6}Gam35^a_{bcd}y20^by20^cy20^d + \frac{1}{24}Gam45^a_{bcde}y00^by00^cy00^dy00^e$$

$$y5.001 := Dx^a + \frac{1}{2}Gam22^a_{bc}y40^by40^c + \frac{1}{2}Gam23^a_{bc}y40^by40^c + \frac{1}{6}Gam33^a_{bcd}y30^by30^cy30^d + Gam22^a_{bc}y40^by42^c + \frac{1}{2}Gam24^a_{bc}y40^by40^c \\ + \frac{1}{6}Gam34^a_{bcd}y30^by30^cy30^d + \frac{1}{24}Gam44^a_{bcde}y20^by20^cy20^dy20^e + Gam22^a_{bc}y40^by43^c + Gam23^a_{bc}y40^by42^c + \frac{1}{2}Gam25^a_{bc}y40^by40^c \\ + \frac{1}{2}Gam33^a_{bcd}y30^by30^cy32^d + \frac{1}{6}Gam35^a_{bcd}y30^by30^cy30^d + \frac{1}{24}Gam45^a_{bcde}y20^by20^cy20^dy20^e + \frac{1}{120}Gam55^a_{bcdef}y00^by00^cy00^dy00^ey00^f$$

$$y0.002 := Dx^a$$

$$y2.002 := Dx^a + \frac{1}{2}Gam22^a_{bc}y00^by00^c$$

$$y3.002 := Dx^a + \frac{1}{2}Gam22^a_{bc}y20^by20^c + \frac{1}{2}Gam23^a_{bc}y20^by20^c + \frac{1}{6}Gam33^a_{bcd}y00^by00^cy00^d$$

$$y4.002 := Dx^a + \frac{1}{2}Gam22^a_{bc}y30^by30^c + \frac{1}{2}Gam23^a_{bc}y30^by30^c + \frac{1}{6}Gam33^a_{bcd}y20^by20^cy20^d + Gam22^a_{bc}y30^by32^c \\ + \frac{1}{2}Gam24^a_{bc}y30^by30^c + \frac{1}{6}Gam34^a_{bcd}y20^by20^cy20^d + \frac{1}{24}Gam44^a_{bcde}y00^by00^cy00^dy00^e$$

$$y5.002 := Dx^a + \frac{1}{2}Gam22^a_{bc}y40^by40^c + \frac{1}{2}Gam23^a_{bc}y40^by40^c + \frac{1}{6}Gam33^a_{bcd}y30^by30^cy30^d + Gam22^a_{bc}y40^by42^c + \frac{1}{2}Gam24^a_{bc}y40^by40^c \\ + \frac{1}{6}Gam34^a_{bcd}y30^by30^cy30^d + \frac{1}{24}Gam44^a_{bcde}y20^by20^cy20^dy20^e + Gam22^a_{bc}y40^by43^c + Gam23^a_{bc}y40^by42^c + \frac{1}{2}Gam25^a_{bc}y40^by40^c \\ + \frac{1}{2}Gam33^a_{bcd}y30^by30^cy32^d + \frac{1}{6}Gam35^a_{bcd}y30^by30^cy30^d + \frac{1}{24}Gam45^a_{bcde}y20^by20^cy20^dy20^e + \frac{1}{120}Gam55^a_{bcdef}y00^by00^cy00^dy00^ey00^f$$

$$y00.101 := Dx^a$$

$$y20.201 := Dx^a$$

$$y22.201 := \frac{1}{2} Dx^b Dx^c Gam22^a_{bc}$$

$$y30.201 := Dx^a$$

$$y32.201 := \frac{1}{2} Dx^b Dx^c Gam22^a_{bc}$$

$$y33.201 := \frac{1}{2} Dx^b Dx^c Gam23^a_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a_{bcd}$$

$$y40.201 := Dx^a$$

$$y42.201 := \frac{1}{2} Dx^b Dx^c Gam22^a_{bc}$$

$$y43.201 := \frac{1}{2} Dx^b Dx^c Gam23^a_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a_{bcd}$$

$$y44.201 := \frac{1}{2} Dx^b Dx^c Dx^d Gam22^a_{be} Gam22^e_{cd} + \frac{1}{2} Dx^b Dx^c Gam24^a_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam34^a_{bcd} + \frac{1}{24} Dx^b Dx^c Dx^d Dx^e Gam44^a_{bcde}$$

$$y50.201 := Dx^a$$

$$y52.201 := \frac{1}{2} Dx^b Dx^c Gam22^a_{bc}$$

$$y53.201 := \frac{1}{2} Dx^b Dx^c Gam23^a_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam33^a_{bcd}$$

$$y54.201 := \frac{1}{2} Dx^b Dx^c Dx^d Gam22^a_{be} Gam22^e_{cd} + \frac{1}{2} Dx^b Dx^c Gam24^a_{bc} + \frac{1}{6} Dx^b Dx^c Dx^d Gam34^a_{bcd} + \frac{1}{24} Dx^b Dx^c Dx^d Dx^e Gam44^a_{bcde}$$

$$\begin{aligned} y55.201 := & \frac{1}{2} Dx^b Dx^c Dx^d Gam22^a_{be} Gam23^e_{cd} + \frac{1}{6} Dx^b Dx^c Dx^d Dx^e Gam22^a_{bf} Gam33^f_{cde} + \frac{1}{2} Dx^b Dx^c Dx^d Gam22^e_{bc} Gam23^a_{de} \\ & + \frac{1}{2} Dx^b Dx^c Gam25^a_{bc} + \frac{1}{4} Dx^b Dx^c Dx^d Dx^e Gam22^f_{bc} Gam33^a_{def} + \frac{1}{6} Dx^b Dx^c Dx^d Gam35^a_{bcd} \\ & + \frac{1}{24} Dx^b Dx^c Dx^d Dx^e Gam45^a_{bcde} + \frac{1}{120} Dx^b Dx^c Dx^d Dx^e Dx^f Gam55^a_{bcdef} \end{aligned}$$

## Stage 2a: Introduce the generalised connections, build terms of $y^a$

```
def substitute_gam (obj):

    substitute      (obj,defGam22)
    substitute      (obj,defGam23)
    substitute      (obj,defGam24)
    substitute      (obj,defGam25)

    substitute      (obj,defGam33)
    substitute      (obj,defGam34)
    substitute      (obj,defGam35)

    substitute      (obj,defGam44)
    substitute      (obj,defGam45)

    substitute      (obj,defGam55)

    distribute      (obj)
    return obj

import cdblib

beg_stage_2a = time.time()

Gam22 = cdblib.get ('genGamma01','genGamma.json')
Gam23 = cdblib.get ('genGamma02','genGamma.json')
Gam24 = cdblib.get ('genGamma03','genGamma.json')
Gam25 = cdblib.get ('genGamma04','genGamma.json')

Gam33 = cdblib.get ('genGamma11','genGamma.json')
Gam34 = cdblib.get ('genGamma12','genGamma.json')
Gam35 = cdblib.get ('genGamma13','genGamma.json')

Gam44 = cdblib.get ('genGamma21','genGamma.json')
Gam45 = cdblib.get ('genGamma22','genGamma.json')

Gam55 = cdblib.get ('genGamma31','genGamma.json')
```

```
# peel off the A{a}, must then symmetrise over revealed indices
```

```
substitute (Gam22,$A{a}->1$)  
substitute (Gam23,$A{a}->1$)  
substitute (Gam24,$A{a}->1$)  
substitute (Gam25,$A{a}->1$)
```

```
substitute (Gam33,$A{a}->1$)  
substitute (Gam34,$A{a}->1$)  
substitute (Gam35,$A{a}->1$)
```

```
substitute (Gam44,$A{a}->1$)  
substitute (Gam45,$A{a}->1$)
```

```
substitute (Gam55,$A{a}->1$)
```

```
# now symmetrise
```

```
sym (Gam22,$_{b},_{c}$)  
sym (Gam23,$_{b},_{c}$)  
sym (Gam24,$_{b},_{c}$)  
sym (Gam25,$_{b},_{c}$)
```

```
sym (Gam33,$_{b},_{c},_{d}$)  
sym (Gam34,$_{b},_{c},_{d}$)  
sym (Gam35,$_{b},_{c},_{d}$)
```

```
sym (Gam44,$_{b},_{c},_{d},_{e}$)  
sym (Gam45,$_{b},_{c},_{d},_{e}$)
```

```
sym (Gam55,$_{b},_{c},_{d},_{e},_{f}$)
```

```
defGam22 := Gam22{a}_{b c} -> @(Gam22).  
defGam23 := Gam23{a}_{b c} -> @(Gam23).  
defGam24 := Gam24{a}_{b c} -> @(Gam24).  
defGam25 := Gam25{a}_{b c} -> @(Gam25).
```



```

defGam33 := Gam33^{a}_{b c d} -> @(Gam33).
defGam34 := Gam34^{a}_{b c d} -> @(Gam34).
defGam35 := Gam35^{a}_{b c d} -> @(Gam35).

defGam44 := Gam44^{a}_{b c d e} -> @(Gam44).
defGam45 := Gam45^{a}_{b c d e} -> @(Gam45).

defGam55 := Gam55^{a}_{b c d e f} -> @(Gam55).

# -----
# y2

y22 = substitute_gam (y22)

y22 = tidy (y22) # cdb (y22.301,y22)

y2 := @(y20) + @(y22). # cdb (y2.301,y2)

# -----
# y3

y32 = substitute_gam (y32)
y33 = substitute_gam (y33)

y32 = tidy (y32) # cdb (y32.301,y32)
y33 = tidy (y33) # cdb (y33.301,y33)

y3 := @(y30) + @(y32) + @(y33). # cdb (y3.301,y3)

# -----
# y4

y42 = substitute_gam (y42)
y43 = substitute_gam (y43)
y44 = substitute_gam (y44)

y42 = tidy (y42) # cdb (y42.301,y42)
y43 = tidy (y43) # cdb (y43.301,y43)

```

```

y44 = tidy (y44)                                # cdb (y44.301,y44)

y4 := @(y40) + @(y42) + @(y43) + @(y44).        # cdb (y4.301,y4)

# -----
# y5

y52 = substitute_gam (y52)
y53 = substitute_gam (y53)
y54 = substitute_gam (y54)
y55 = substitute_gam (y55)

y52 = tidy (y52)                                # cdb (y52.301,y52)
y53 = tidy (y53)                                # cdb (y53.301,y53)
y54 = tidy (y54)                                # cdb (y54.301,y54)
y55 = tidy (y55)                                # cdb (y55.301,y55)

y5 := @(y50) + @(y52) + @(y53) + @(y54) + @(y55). # cdb (y5.301,y5)

# -----
cdblib.create ('geodesic-bvp.json')

cdblib.put ('y2',y2,'geodesic-bvp.json')
cdblib.put ('y3',y3,'geodesic-bvp.json')
cdblib.put ('y4',y4,'geodesic-bvp.json')
cdblib.put ('y5',y5,'geodesic-bvp.json')

cdblib.put ('y20',y20,'geodesic-bvp.json')
cdblib.put ('y22',y22,'geodesic-bvp.json')

cdblib.put ('y30',y30,'geodesic-bvp.json')
cdblib.put ('y32',y32,'geodesic-bvp.json')
cdblib.put ('y33',y33,'geodesic-bvp.json')

cdblib.put ('y40',y40,'geodesic-bvp.json')
cdblib.put ('y42',y42,'geodesic-bvp.json')
cdblib.put ('y43',y43,'geodesic-bvp.json')
cdblib.put ('y44',y44,'geodesic-bvp.json')

```

```
cdblib.put ('y50',y50,'geodesic-bvp.json')
cdblib.put ('y52',y52,'geodesic-bvp.json')
cdblib.put ('y53',y53,'geodesic-bvp.json')
cdblib.put ('y54',y54,'geodesic-bvp.json')
cdblib.put ('y55',y55,'geodesic-bvp.json')

end_stage_2a = time.time()
```

$$\text{y50.301} := Dx^a$$

$$\text{y52.301} := -\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\text{y53.301} := -\frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} + \frac{1}{24}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned} \text{y54.301} := & -\frac{2}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cfdg} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{4}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{befh} R_{cgdi} \\ & + \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dgfi} - \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{eb} R_{cfdg} \\ & - \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bc} R_{defg} - \frac{1}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dfgi} \\ & + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bg} R_{cedf} - \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cgfi} \\ & + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{60}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{de} R_{bfcg} - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{db} R_{cefg} \\ & - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{gd} R_{becf} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag} \nabla_{dg} R_{becf} \\ & - \frac{1}{45}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{1}{60}x^b Dx^c Dx^d Dx^e Dx^f g^{ag} \nabla_{cd} R_{befg} \end{aligned}$$

$$\begin{aligned}
\text{y55.301} := & -\frac{7}{540}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} - \frac{1}{45}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} \\
& + \frac{1}{216}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} \\
& - \frac{17}{1080}x^bx^cDx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} + \frac{1}{135}x^bx^cDx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} \\
& - \frac{1}{540}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} + \frac{1}{108}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} \\
& - \frac{1}{45}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} - \frac{7}{540}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} \\
& - \frac{2}{45}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} - \frac{1}{60}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} - \frac{2}{45}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} \\
& + \frac{1}{72}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} + \frac{1}{45}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} \\
& + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} + \frac{1}{45}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} \\
& - \frac{1}{180}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{fbc}R_{dgeh} - \frac{1}{180}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bfc}R_{dgeh} - \frac{1}{180}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bcf}R_{dgeh} \\
& - \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bcd}R_{efgh} - \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} - \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} \\
& - \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} + \frac{1}{360}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{hbc}R_{dfeg} + \frac{1}{360}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bhc}R_{dfeg} \\
& + \frac{1}{360}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bch}R_{dfeg} - \frac{7}{1080}x^bx^cDx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} - \frac{1}{540}x^bx^cDx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} \\
& + \frac{1}{108}x^bx^cDx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bdei}\nabla_j R_{cfgh} - \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} \\
& - \frac{1}{540}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} - \frac{11}{540}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} - \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} \\
& + \frac{1}{216}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} \\
& + \frac{1}{135}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} + \frac{1}{90}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} - \frac{1}{270}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{efb}R_{cgdh} \\
& - \frac{1}{270}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{ebf}R_{cgdh} - \frac{1}{180}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{ebc}R_{dfgh} - \frac{1}{270}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bef}R_{cgdh} \\
& - \frac{1}{180}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bec}R_{dfgh} - \frac{1}{180}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bce}R_{dfgh} - \frac{1}{270}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} \\
& - \frac{1}{270}x^bx^cx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} + \frac{1}{1080}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{heb}R_{cfdg} + \frac{1}{1080}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{hbe}R_{cfdg} \\
& + \frac{1}{1080}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{ehb}R_{cfdg} + \frac{1}{1080}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{bhe}R_{cfdg} + \frac{1}{1080}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{ebh}R_{cfdg} \\
& + \frac{1}{1080}x^bx^cx^dDx^eDx^fDx^g g^{ah}\nabla_{beh}R_{cfdg} - \frac{1}{120}x^bx^cDx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} - \frac{1}{90}x^bx^cDx^dDx^eDx^fDx^g g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj}
\end{aligned}$$

## Stage 2b: Building the terms of $x^a(s)$

```
def substitute_y (obj):
    substitute (obj,defy00)
    substitute (obj,defy20)
    substitute (obj,defy30)
    substitute (obj,defy32)
    substitute (obj,defy40)
    substitute (obj,defy42)
    substitute (obj,defy43)
    distribute (obj)
    return obj

beg_stage_2b = time.time()

term2 := Gam^{a}_{b c} y4^{b} y4^{c}.
term3 := Gam^{a}_{b c d} y3^{b} y3^{c} y3^{d}.
term4 := Gam^{a}_{b c d e} y2^{b} y2^{c} y2^{d} y2^{e}.
term5 := Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

term2 = substitute_eps (term2)    # cdb (term2.401,term2)
term3 = substitute_eps (term3)    # cdb (term3.401,term3)
term4 = substitute_eps (term4)    # cdb (term4.401,term4)
term5 = substitute_eps (term5)    # cdb (term5.401,term5)

term2 = substitute_y (term2)
term3 = substitute_y (term3)
term4 = substitute_y (term4)
term5 = substitute_y (term5)

term2 = substitute_gam (term2)
term3 = substitute_gam (term3)
term4 = substitute_gam (term4)
term5 = substitute_gam (term5)

term2 = tidy (term2)    # cdb (term2.501,term2)
term3 = tidy (term3)    # cdb (term3.501,term3)
term4 = tidy (term4)    # cdb (term4.501,term4)
```

```
term5 = tidy (term5)  # cdb (term5.501,term5)
```

$$\begin{aligned} \text{term2.401} := & Gam22^a_{bc}y40^by40^c + Gam23^a_{bc}y40^by40^c + 2Gam22^a_{bc}y40^by42^c + Gam24^a_{bc}y40^by40^c \\ & + 2Gam22^a_{bc}y40^by43^c + 2Gam23^a_{bc}y40^by42^c + Gam25^a_{bc}y40^by40^c \end{aligned}$$

$$\text{term3.401} := Gam33^a_{bcd}y30^by30^cy30^d + Gam34^a_{bcd}y30^by30^cy30^d + 3Gam33^a_{bcd}y30^by30^cy32^d + Gam35^a_{bcd}y30^by30^cy30^d$$

$$\text{term4.401} := Gam44^a_{bcde}y20^by20^cy20^dy20^e + Gam45^a_{bcde}y20^by20^cy20^dy20^e$$

$$\text{term5.401} := Gam55^a_{bcdef}y00^by00^cy00^dy00^ey00^f$$



$$\begin{aligned}
\text{term2.501} := & -\frac{2}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{3}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} + \frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} \\
& - \frac{2}{9}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{2}{9}x^b x^c Dx^d Dx^e Dx^f g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{8}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{befh} R_{cgdi} \\
& + \frac{4}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{difg} + \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dvgi} - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{eb} R_{cf dg} \\
& - \frac{1}{20}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{be} R_{cf dg} - \frac{1}{10}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bc} R_{defg} - \frac{2}{45}x^b x^c x^d Dx^e Dx^f g^{ag} g^{hi} R_{bech} R_{dfgi} \\
& + \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{40}x^b x^c x^d Dx^e Dx^f g^{ag} \nabla_{bg} R_{cedf} - \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\
& - \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + \frac{1}{36}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cf dg} + \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\
& + \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} - \frac{1}{36}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bieh} \nabla_j R_{cf dg} \\
& - \frac{1}{18}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cf gj} + \frac{1}{18}x^b x^c Dx^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{bidh} \nabla_e R_{cf gj} \\
& + \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} - \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} \\
& + \frac{1}{9}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - \frac{1}{18}x^b x^c x^d Dx^e Dx^f Dx^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} - \frac{4}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bf gi} \nabla_c R_{dhej} \\
& - \frac{1}{30}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} - \frac{4}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + \frac{1}{36}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\
& + \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} \\
& + \frac{2}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{fbc} R_{dgeh} \\
& - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bfc} R_{dgeh} - \frac{1}{90}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcf} R_{dgeh} - \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bcd} R_{efgh} \\
& - \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} - \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} - \frac{1}{45}x^b x^c x^d x^e Dx^f Dx^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} \\
& + \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{hbc} R_{dfeg} + \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bhc} R_{dfeg} + \frac{1}{180}x^b x^c x^d x^e Dx^f Dx^g g^{ah} \nabla_{bch} R_{dfeg}
\end{aligned}$$

$$\begin{aligned}
\text{term3.501} := & -\frac{1}{2}x^b D x^c D x^d D x^e g^{af} \nabla_c R_{bdef} - \frac{8}{15}x^b x^c D x^d D x^e D x^f g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{2}{15}x^b x^c D x^d D x^e D x^f g^{ag} g^{hi} R_{bdeh} R_{cgfi} \\
& + \frac{2}{15}x^b x^c D x^d D x^e D x^f g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{10}x^b x^c D x^d D x^e D x^f g^{ag} \nabla_{de} R_{bfcg} - \frac{3}{20}x^b x^c D x^d D x^e D x^f g^{ag} \nabla_{db} R_{cefg} \\
& - \frac{3}{20}x^b x^c D x^d D x^e D x^f g^{ag} \nabla_{bd} R_{cefg} + \frac{2}{5}x^b x^c D x^d D x^e D x^f g^{ag} g^{hi} R_{bdeh} R_{cfig} + \frac{1}{40}x^b x^c D x^d D x^e D x^f g^{ag} \nabla_{gd} R_{becf} \\
& + \frac{1}{40}x^b x^c D x^d D x^e D x^f g^{ag} \nabla_{dg} R_{becf} - \frac{1}{6}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + \frac{1}{6}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} \\
& + \frac{1}{6}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh} - \frac{8}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} \\
& - \frac{4}{15}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - \frac{1}{15}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - \frac{1}{10}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\
& - \frac{1}{90}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} - \frac{11}{90}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{4}{15}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} \\
& - \frac{1}{15}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} + \frac{1}{12}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} + \frac{1}{36}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} \\
& + \frac{1}{15}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + \frac{1}{15}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + \frac{2}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} \\
& + \frac{1}{15}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} - \frac{1}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{efb} R_{cgdh} - \frac{1}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{ebf} R_{cgdh} \\
& - \frac{1}{30}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{ebc} R_{dfgh} - \frac{1}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{bef} R_{cgdh} - \frac{1}{30}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{bec} R_{dfgh} \\
& - \frac{1}{30}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{bce} R_{dfgh} + \frac{4}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + \frac{1}{5}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} \\
& + \frac{4}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} - \frac{1}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} + \frac{1}{5}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} \\
& - \frac{1}{45}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} + \frac{1}{180}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{heb} R_{cfdg} + \frac{1}{180}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{hbe} R_{cfdg} \\
& + \frac{1}{180}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{ehb} R_{cfdg} + \frac{1}{180}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{bhe} R_{cfdg} + \frac{1}{180}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{ebh} R_{cfdg} \\
& + \frac{1}{180}x^b x^c x^d D x^e D x^f D x^g g^{ah} \nabla_{beh} R_{cfdg} - \frac{1}{9}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} - \frac{1}{18}x^b x^c x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}
\end{aligned}$$

$$\begin{aligned}
\text{term4.501} := & -\frac{8}{15}x^b D x^c D x^d D x^e D x^f g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{2}{5}x^b D x^c D x^d D x^e D x^f g^{ag} \nabla_{cd} R_{befg} - \frac{32}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} \\
& - \frac{1}{5}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - \frac{4}{15}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} \\
& - \frac{2}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} - \frac{22}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} \\
& - \frac{1}{5}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} - \frac{4}{15}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} \\
& + \frac{1}{9}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} + \frac{8}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} - \frac{1}{15}x^b x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{def} R_{bgch} \\
& - \frac{4}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{deb} R_{cfgh} - \frac{4}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{dbe} R_{cfgh} - \frac{4}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{bde} R_{cfgh} \\
& + \frac{13}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} + \frac{1}{15}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} \\
& + \frac{23}{45}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_f R_{cgjh} + \frac{1}{90}x^b x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{hde} R_{bfcg} \\
& + \frac{1}{90}x^b x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{dhe} R_{bfcg} + \frac{1}{90}x^b x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{deh} R_{bfcg} - \frac{4}{9}x^b x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh} \\
\text{term5.501} := & -x^b D x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - x^b D x^c D x^d D x^e D x^f D x^g g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - \frac{1}{3}x^b D x^c D x^d D x^e D x^f D x^g g^{ah} \nabla_{cde} R_{bfgh}
\end{aligned}$$

```

# Check:
#   x^{a} at s=1 should equal x^{a} + Dx^{a}
#   but x^{a}(s) = x^{a} + s y^{a} - \sum (1/n!) @ (termn) s^n
#   thus foo should equal Dx^{a} and it does (yeah)

foo := @(y5)
- (1/2) @(term2)
- (1/6) @(term3)
- (1/24) @(term4)
- (1/120) @(term5).

distribute      (foo)
obj = product_sort (foo)
rename_dummies  (foo)
canonicalise    (foo)      # cdb (foo.001,foo)

term2 := (1/2) @(term2).  # cdb(term2.502,term2)
term3 := (1/6) @(term3).  # cdb(term3.502,term3)
term4 := (1/24) @(term4). # cdb(term4.502,term4)
term5 := (1/120) @(term5). # cdb(term5.502,term5)

end_stage_2b = time.time()

```

$$\text{foo.001} := Dx^a$$

$$\text{y2.301} := Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\begin{aligned} \text{y3.301} := & Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{12}x^b x^c Dx^d Dx^e g^{af} \nabla_d R_{becf} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af} \nabla_b R_{cdef} \\ & + \frac{1}{24}x^b x^c Dx^d Dx^e g^{af} \nabla_f R_{bdce} - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} \end{aligned}$$

$$\begin{aligned}
\text{y4.301} := & Dx^a - \frac{1}{3}x^bDx^cDx^d g^{ae}R_{bcde} - \frac{1}{12}x^b x^c Dx^d Dx^e g^{af}\nabla_d R_{becf} - \frac{1}{6}x^b x^c Dx^d Dx^e g^{af}\nabla_b R_{cdef} + \frac{1}{24}x^b x^c Dx^d Dx^e g^{af}\nabla_f R_{bdce} \\
& - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} - \frac{2}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bdeh}R_{cfgi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bdeh}R_{cifg} \\
& - \frac{4}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{befh}R_{cgdi} + \frac{2}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bec h}R_{difg} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bec h}R_{dgfi} \\
& - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{eb}R_{cfdg} - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{be}R_{cfdg} - \frac{1}{20}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{bc}R_{defg} \\
& - \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bec h}R_{dfgi} + \frac{1}{80}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{gb}R_{cedf} + \frac{1}{80}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{bg}R_{cedf} \\
& - \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bdeh}R_{cgfi} + \frac{1}{45}x^b x^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bdch}R_{egfi} - \frac{1}{60}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{de}R_{bfeg} \\
& - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{db}R_{cefg} - \frac{1}{40}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{bd}R_{cefg} + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{gd}R_{becf} \\
& + \frac{1}{240}x^b x^c Dx^d Dx^e Dx^f g^{ag}\nabla_{dg}R_{becf} - \frac{1}{45}x^b Dx^c Dx^d Dx^e Dx^f g^{ag}g^{hi}R_{bcdh}R_{egfi} - \frac{1}{60}x^b Dx^c Dx^d Dx^e Dx^f g^{ag}\nabla_{cd}R_{befg}
\end{aligned}$$

## Stage 3: Reformatting and output

```
def get_Rterm (obj,n):

    # I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
    # what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
    # It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    distribute (tmp)

    substitute (tmp, $\nabla_{e f g}\{R_{a b c d}\} \rightarrow Q_{a b c d e f g}\$)
    substitute (tmp, $\nabla_{e f}\{R_{a b c d}\} \rightarrow Q_{a b c d e f}\$)
    substitute (tmp, $\nabla_{e}\{R_{a b c d}\} \rightarrow Q_{a b c d e}\$)
    substitute (tmp, $R_{a b c d} \rightarrow Q_{a b c d}\$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} \rightarrow \nabla_{e f g}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d e f} \rightarrow \nabla_{e f}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d e} \rightarrow \nabla_{e}\{R_{a b c d}\}\$)
    substitute (foo, $Q_{a b c d} \rightarrow R_{a b c d}\$)

    return foo

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
```

```

rename_dummies (bah)
canonicalise    (bah)
substitute      (bah,$Dx^{b}->zzz^{b}$)
factor_out      (bah,$x^{a?},zzz^{b?}$)
substitute      (bah,$zzz^{b}->Dx^{b}$)
ans := @(bah) / @(foo).
return ans

def rescale (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute    (bah)
    substitute     (bah,$Dx^{b}->zzz^{b}$)
    factor_out     (bah,$x^{a?},zzz^{b?}$)
    substitute     (bah,$zzz^{b}->Dx^{b}$)
    return bah

beg_stage_3 = time.time()

Rterm22 = get_Rterm (term2,2)      # cdb(Rterm22.101,Rterm22)
Rterm23 = get_Rterm (term2,3)      # cdb(Rterm23.101,Rterm23)
Rterm24 = get_Rterm (term2,4)      # cdb(Rterm24.101,Rterm24)
Rterm25 = get_Rterm (term2,5)      # cdb(Rterm25.101,Rterm25)

Rterm32 = get_Rterm (term3,2)      # cdb(Rterm32.101,Rterm32) # zero
Rterm33 = get_Rterm (term3,3)      # cdb(Rterm33.101,Rterm33)
Rterm34 = get_Rterm (term3,4)      # cdb(Rterm34.101,Rterm34)
Rterm35 = get_Rterm (term3,5)      # cdb(Rterm35.101,Rterm35)

Rterm42 = get_Rterm (term4,2)      # cdb(Rterm42.101,Rterm42) # zero
Rterm43 = get_Rterm (term4,3)      # cdb(Rterm43.101,Rterm43) # zero
Rterm44 = get_Rterm (term4,4)      # cdb(Rterm44.101,Rterm44)
Rterm45 = get_Rterm (term4,5)      # cdb(Rterm45.101,Rterm45)

Rterm52 = get_Rterm (term5,2)      # cdb(Rterm52.101,Rterm52) # zero
Rterm53 = get_Rterm (term5,3)      # cdb(Rterm53.101,Rterm53) # zero
Rterm54 = get_Rterm (term5,4)      # cdb(Rterm54.101,Rterm54) # zero
Rterm55 = get_Rterm (term5,5)      # cdb(Rterm55.101,Rterm55)

```

```

Rterm22 = rescale ( reformat (Rterm22,  -3),  -3 )  # cdb(Rterm22.102,Rterm22)
Rterm23 = rescale ( reformat (Rterm23, -24), -24 )  # cdb(Rterm23.102,Rterm23)
Rterm24 = rescale ( reformat (Rterm24, -720), -720 ) # cdb(Rterm24.102,Rterm24)
Rterm25 = rescale ( reformat (Rterm25, -360), -360 ) # cdb(Rterm25.102,Rterm25)

Rterm33 = rescale ( reformat (Rterm33, -12),  -12 ) # cdb(Rterm33.102,Rterm33)
Rterm34 = rescale ( reformat (Rterm34, -720), -720 ) # cdb(Rterm34.102,Rterm34)
Rterm35 = rescale ( reformat (Rterm35,-1080), -1080 ) # cdb(Rterm35.102,Rterm35)

Rterm44 = rescale ( reformat (Rterm44, -180), -180 ) # cdb(Rterm44.102,Rterm44)
Rterm45 = rescale ( reformat (Rterm45,-2160), -2160 ) # cdb(Rterm45.102,Rterm45)

Rterm55 = rescale ( reformat (Rterm55, -360), -360 ) # cdb(Rterm55.102,Rterm55)

```



```

# -----
# bvp to terms linear in R

tmp2 := -(1/3) @(Rterm22).

bvp2 := x^{a}
      + s Dx^{a}
      + (s-s**2) @(tmp2).                                # cdb(bvp.601,bvp2)

cdblib.put ('bvp2',bvp2,'geodesic-bvp.json')
cdblib.put ('bvp22',tmp2,'geodesic-bvp.json')

y2 := Dx^{a} + @(tmp2).                                    # cdb(y2.600,y2)

# -----
# bvp to terms linear in dR

tmp2 := -(1/3) @(Rterm22) - (1/24) @(Rterm23).
tmp3 := -(1/12) @(Rterm33).

bvp3 := x^{a}
      + s Dx^{a}
      + (s-s**2) @(tmp2)
      + (s-s**3) @(tmp3).                                # cdb(bvp.602,bvp3)

cdblib.put ('bvp3',bvp3,'geodesic-bvp.json')
cdblib.put ('bvp32',tmp2,'geodesic-bvp.json')
cdblib.put ('bvp33',tmp3,'geodesic-bvp.json')

y3 := Dx^{a} + @(tmp2) + @(tmp3).                        # cdb(y3.600,y3)

# -----
# bvp to terms linear in d^2 R

tmp2 := -(1/3) @(Rterm22) - (1/24) @(Rterm23) - (1/720) @(Rterm24).
tmp3 := -(1/12) @(Rterm33) - (1/720) @(Rterm34).
tmp4 := -(1/180) @(Rterm44).

```

```

bvp4 := x^{a}
      + s Dx^{a}
      + (s-s**2) @(tmp2)
      + (s-s**3) @(tmp3)
      + (s-s**4) @(tmp4).                                # cdb(bvp.603,bvp4)

cdblib.put ('bvp4',bvp4,'geodesic-bvp.json')
cdblib.put ('bvp42',tmp2,'geodesic-bvp.json')
cdblib.put ('bvp43',tmp3,'geodesic-bvp.json')
cdblib.put ('bvp44',tmp4,'geodesic-bvp.json')

y4 := Dx^{a} + @(tmp2) + @(tmp3) + @(tmp4).              # cdb(y4.600,y4)

# -----
# bvp to terms linear in d^3 R

tmp2 := @(term2).
tmp3 := @(term3).
tmp4 := @(term4).
tmp5 := @(term5).

bvp5 := x^{a}
      + s Dx^{a}
      + (s-s**2) @(tmp2)
      + (s-s**3) @(tmp3)
      + (s-s**4) @(tmp4)
      + (s-s**5) @(tmp5).                                # cdb(bvp.604,bvp5)

cdblib.put ('bvp5',bvp5,'geodesic-bvp.json')
cdblib.put ('bvp52',term2,'geodesic-bvp.json')
cdblib.put ('bvp53',term3,'geodesic-bvp.json')
cdblib.put ('bvp54',term4,'geodesic-bvp.json')
cdblib.put ('bvp55',term5,'geodesic-bvp.json')

y5 := Dx^{a} + @(tmp2) + @(tmp3) + @(tmp4) + @(tmp5).    # cdb(y5.600,y5)

end_stage_3 = time.time()

```

```
# cdbBeg (timing)
print ("Stage 1:  {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2a: {:7.1f} secs\\hfill\\break".format(end_stage_2a-beg_stage_2a))
print ("Stage 2b: {:7.1f} secs\\hfill\\break".format(end_stage_2b-beg_stage_2b))
print ("Stage 3:  {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
# cdbEnd (timing)
```

## Non-unit tangent vectors at $P$

These are not unit vectors, their length is the geodesic distance from  $P$  to  $Q$

$$y2.600 := Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$y3.600 := Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e (2g^{af} \nabla_d R_{becf} + 4g^{af} \nabla_b R_{cdef} - g^{af} \nabla_f R_{bdce}) - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned} y4.600 := & Dx^a - \frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e (2g^{af} \nabla_d R_{becf} + 4g^{af} \nabla_b R_{cdef} - g^{af} \nabla_f R_{bdce}) \\ & - \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f (80g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 80g^{ag} g^{hi} R_{bdeh} R_{cifg}) \\ & - \frac{1}{720}x^b x^c x^d Dx^e Dx^f (64g^{ag} g^{hi} R_{befh} R_{cgdi} - 32g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{bech} R_{dgfi} + 18g^{ag} \nabla_{eb} R_{cfdg} + 18g^{ag} \nabla_{be} R_{cfdg} \\ & \quad + 36g^{ag} \nabla_{bc} R_{defg} + 16g^{ag} g^{hi} R_{bech} R_{dfgi} - 9g^{ag} \nabla_{gb} R_{cedf} - 9g^{ag} \nabla_{bg} R_{cedf}) - \frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} \\ & - \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f (64g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdch} R_{egfi} + 12g^{ag} \nabla_{de} R_{bfcg} + 18g^{ag} \nabla_{db} R_{cefg} \\ & \quad + 18g^{ag} \nabla_{bd} R_{cefg} - 48g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3g^{ag} \nabla_{gd} R_{becf} - 3g^{ag} \nabla_{dg} R_{becf}) - \frac{1}{180}x^b Dx^c Dx^d Dx^e Dx^f (4g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3g^{ag} \nabla_{cd} R_{befg}) \end{aligned}$$

## Geodesic boundary value problem to terms linear in $R$

$$x^a(s) = x^a + sDx^a - \frac{1}{3}(s - s^2)x^b Dx^c Dx^d g^{ae} R_{bcde} + \mathcal{O}(s^3, \epsilon^3)$$

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + \mathcal{O}(s^3, \epsilon^3)$$

$$x_2^a = \overset{2}{x}_2^a + \mathcal{O}(\epsilon^3)$$

$$-3\overset{2}{x}_2^a = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

## Geodesic boundary value problem to terms linear in $\nabla R$

$$x^a(s) = x^a + sDx^a + (s - s^2) \left( -\frac{1}{3}x^bDx^cDx^dg^{ae}R_{bcde} - \frac{1}{24}x^bx^cDx^dDx^e(2g^{af}\nabla_dR_{becf} + 4g^{af}\nabla_bR_{cdef} - g^{af}\nabla_fR_{bdce}) \right) \\ - \frac{1}{12}(s - s^3)x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef} + \mathcal{O}(s^4, \epsilon^4)$$

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + \mathcal{O}(s^4, \epsilon^4)$$

$$x_2^a = \overset{2}{x}_2^a + \overset{3}{x}_2^a + \mathcal{O}(\epsilon^4)$$

$$-3\overset{2}{x}_2^a = x^bDx^cDx^dg^{ae}R_{bcde}$$

$$-24\overset{3}{x}_2^a = x^bx^cDx^dDx^e(2g^{af}\nabla_dR_{becf} + 4g^{af}\nabla_bR_{cdef} - g^{af}\nabla_fR_{bdce})$$

$$x_3^a = \overset{3}{x}_3^a + \mathcal{O}(\epsilon^4)$$

$$-12\overset{3}{x}_3^a = x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef}$$

## Geodesic boundary value problem to terms linear in $\nabla^2 R$

$$\begin{aligned}
x^a(s) = & x^a + sDx^a + (s - s^2) \left( -\frac{1}{3}x^bDx^cDx^dg^{ae}R_{bcde} - \frac{1}{24}x^bx^cDx^dDx^e(2g^{af}\nabla_dR_{becf} + 4g^{af}\nabla_bR_{cdef} - g^{af}\nabla_fR_{bdce}) \right. \\
& - \frac{1}{720}x^bx^cDx^dDx^eDx^f(80g^{ag}g^{hi}R_{bdeh}R_{cfdgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}) - \frac{1}{720}x^bx^cDx^dDx^eDx^f(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} \\
& \left. - 16g^{ag}g^{hi}R_{bech}R_{dghi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}) \right) \\
& + (s - s^3) \left( -\frac{1}{12}x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef} - \frac{1}{720}x^bx^cDx^dDx^eDx^f(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} \right. \\
& \left. + 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} - 48g^{ag}g^{hi}R_{bdeh}R_{cfdgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}) \right) \\
& - \frac{1}{180}(s - s^4)x^bDx^cDx^dDx^eDx^f(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}) + \mathcal{O}(s^5, \epsilon^5)
\end{aligned}$$

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + (s - s^4)x_4^a + \mathcal{O}(s^5, \epsilon^5)$$

$$x_2^a = \dot{x}_2^a + \ddot{x}_2^a + \ddot{x}_2^a + \mathcal{O}(\epsilon^5)$$

$$-3\ddot{x}_2^a = x^bDx^cDx^dg^{ae}R_{bcde}$$

$$-24\ddot{x}_2^a = x^bx^cDx^dDx^e(2g^{af}\nabla_dR_{becf} + 4g^{af}\nabla_bR_{cdef} - g^{af}\nabla_fR_{bdce})$$

$$\begin{aligned}
-720\ddot{x}_2^a = & x^bx^cDx^dDx^eDx^f(80g^{ag}g^{hi}R_{bdeh}R_{cfdgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}) + x^bx^cDx^dDx^eDx^f(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} \\
& - 16g^{ag}g^{hi}R_{bech}R_{dghi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf})
\end{aligned}$$

$$x_3^a = \dot{x}_3^a + \ddot{x}_3^a + \mathcal{O}(\epsilon^5)$$

$$-12\ddot{x}_3^a = x^bDx^cDx^dDx^eg^{af}\nabla_cR_{bdef}$$

$$\begin{aligned}
-720\ddot{x}_3^a = & x^bx^cDx^dDx^eDx^f(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} + 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} \\
& - 48g^{ag}g^{hi}R_{bdeh}R_{cfdgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf})
\end{aligned}$$

$$x_4^a = x_4^a + \mathcal{O}(\epsilon^5)$$

$$-180x_4^a = x^b D x^c D x^d D x^e D x^f (4g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3g^{ag} \nabla_{cd} R_{befg})$$



# Geodesic boundary value problem to terms linear in $\nabla^3 R$

The geodesic that connects the points with RNC coordinates  $x^a$  and  $x^a + Dx^a$  is described, for  $0 \leq s \leq 1$ , by

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + (s - s^4)x_4^a + (s - s^5)x_5^a + \mathcal{O}(s^6, \epsilon^6)$$

$$x_2^a = \overset{2}{x}_2^a + \overset{3}{x}_2^a + \overset{4}{x}_2^a + \overset{5}{x}_2^a + \mathcal{O}(\epsilon^6)$$

$$-3\overset{2}{x}_2^a = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$-24\overset{3}{x}_2^a = x^b x^c Dx^d Dx^e (2g^{af} \nabla_d R_{becf} + 4g^{af} \nabla_b R_{cdef} - g^{af} \nabla_f R_{bdce})$$

$$\begin{aligned} -720\overset{4}{x}_2^a = & x^b x^c Dx^d Dx^e Dx^f (80g^{ag} g^{hi} R_{bdeh} R_{cfdgi} - 80g^{ag} g^{hi} R_{bdeh} R_{cifg}) + x^b x^c x^d Dx^e Dx^f (64g^{ag} g^{hi} R_{befh} R_{cgdi} - 32g^{ag} g^{hi} R_{bech} R_{difg} \\ & - 16g^{ag} g^{hi} R_{bech} R_{dvgfi} + 18g^{ag} \nabla_{eb} R_{cfdg} + 18g^{ag} \nabla_{be} R_{cfdg} + 36g^{ag} \nabla_{bc} R_{defg} + 16g^{ag} g^{hi} R_{bech} R_{dfgi} - 9g^{ag} \nabla_{gb} R_{cedf} - 9g^{ag} \nabla_{bg} R_{cedf}) \end{aligned}$$

$$\begin{aligned} -360\overset{5}{x}_2^a = & x^b x^c x^d Dx^e Dx^f Dx^g (10g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + 20g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 5g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} - 10g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\ & - 20g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 5g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} - 10g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} - 10g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 20g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} \\ & - 20g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 10g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj}) + x^b x^c Dx^d Dx^e Dx^f Dx^g (10g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 10g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj}) \\ & + x^b x^c x^d x^e Dx^f Dx^g (16g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + 6g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 16g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} - 5g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\ & - 8g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} - 4g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} - 4g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} - 8g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} - 4g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 2g^{ah} \nabla_{fbc} R_{dgeh} \\ & + 2g^{ah} \nabla_{bfc} R_{dgeh} + 2g^{ah} \nabla_{bcf} R_{dgeh} + 4g^{ah} \nabla_{bcd} R_{efgh} + 4g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} + 4g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + 4g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} \\ & - g^{ah} \nabla_{hbc} R_{dfeg} - g^{ah} \nabla_{bhc} R_{dfeg} - g^{ah} \nabla_{bch} R_{dfeg}) \end{aligned}$$

$$x_3^a = \overset{3}{x}_3^a + \overset{4}{x}_3^a + \overset{5}{x}_3^a + \mathcal{O}(\epsilon^6)$$

$$-12\overset{3}{x}_3^a = x^b D x^c D x^d D x^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned} -720\overset{4}{x}_3^a = & x^b x^c D x^d D x^e D x^f (64g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdch} R_{egfi} + 12g^{ag} \nabla_{de} R_{bfcg} + 18g^{ag} \nabla_{db} R_{cefg} + 18g^{ag} \nabla_{bd} R_{cefg} \\ & - 48g^{ag} g^{hi} R_{bdeh} R_{cfigi} - 3g^{ag} \nabla_{gd} R_{becf} - 3g^{ag} \nabla_{dg} R_{becf}) \end{aligned}$$

$$\begin{aligned} -1080\overset{5}{x}_3^a = & x^b x^c D x^d D x^e D x^f D x^g (30g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - 30g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} - 30g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}) \\ & + x^b x^c x^d D x^e D x^f D x^g (32g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 48g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 18g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\ & + 2g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 48g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 12g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} - 15g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\ & - 5g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} - 12g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\ & + 4g^{ah} \nabla_{efb} R_{cgdh} + 4g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{ebc} R_{dfgh} + 4g^{ah} \nabla_{bef} R_{cgdh} + 6g^{ah} \nabla_{bec} R_{dfgh} + 6g^{ah} \nabla_{bce} R_{dfgh} - 16g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\ & - 36g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 16g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 4g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 36g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + 4g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg} \\ & - g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg} + 20g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 10g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}) \end{aligned}$$

$$x_4^a = \overset{4}{x}_4^a + \overset{5}{x}_4^a + \mathcal{O}(\epsilon^6)$$

$$-180\overset{4}{x}_4^a = x^b D x^c D x^d D x^e D x^f (4g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3g^{ag} \nabla_{cd} R_{befg})$$

$$\begin{aligned} -2160\overset{5}{x}_4^a = & x^b x^c D x^d D x^e D x^f D x^g (64g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 18g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} \\ & + 44g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 18g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} + 24g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} \\ & + 6g^{ah} \nabla_{def} R_{bgch} + 8g^{ah} \nabla_{deb} R_{cfgh} + 8g^{ah} \nabla_{dbe} R_{cfgh} + 8g^{ah} \nabla_{bde} R_{cfgh} - 26g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 6g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} \\ & - 46g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - g^{ah} \nabla_{hde} R_{bfcg} - g^{ah} \nabla_{dhe} R_{bfcg} - g^{ah} \nabla_{deh} R_{bfcg} + 40g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}) \end{aligned}$$

$$x_5^a = \overset{5}{x}_5^a + \mathcal{O}(\epsilon^6)$$

$$-360\overset{5}{x}_5^a = x^b D x^c D x^d D x^e D x^f D x^g (3g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh})$$

```

tmp2 := 8 @(Rterm22) + @(Rterm23).
tmp3 := @(Rterm33).

factor_out      (tmp2,$Dx^{a?}$) # cdb(tmp2.001,tmp2)
rename_dummies (tmp2)
factor_out      (tmp2,$Dx^{a?}$) # cdb(tmp2.002,tmp2)

bvp4 := x^{a}
      + \lam Dx^{a}
      - (1/24) (\lam-\lam**2) @(tmp2)
      - (1/12) (\lam-\lam**3) @(tmp3).      # cdb(bvp4,bvp4)

cdblib.create ('geodesic-bvp.export')

# 4th order bvp
cdblib.put ('bvp4',bvp4,'geodesic-bvp.export')

# 6th order bvp terms, scaled
cdblib.put ('bvp622',Rterm22,'geodesic-bvp.export')
cdblib.put ('bvp623',Rterm23,'geodesic-bvp.export')
cdblib.put ('bvp624',Rterm24,'geodesic-bvp.export')
cdblib.put ('bvp625',Rterm25,'geodesic-bvp.export')

cdblib.put ('bvp633',Rterm33,'geodesic-bvp.export')
cdblib.put ('bvp634',Rterm34,'geodesic-bvp.export')
cdblib.put ('bvp635',Rterm35,'geodesic-bvp.export')

cdblib.put ('bvp644',Rterm44,'geodesic-bvp.export')
cdblib.put ('bvp645',Rterm45,'geodesic-bvp.export')

cdblib.put ('bvp655',Rterm55,'geodesic-bvp.export')

checkpoint.append (bvp4)

checkpoint.append (Rterm22)
checkpoint.append (Rterm23)
checkpoint.append (Rterm24)
checkpoint.append (Rterm25)

```

```
checkpoint.append (Rterm33)
checkpoint.append (Rterm34)
checkpoint.append (Rterm35)

checkpoint.append (Rterm44)
checkpoint.append (Rterm45)

checkpoint.append (Rterm55)
```

# Timing

Stage 1: 5.8 secs

Stage 2a: 69.4 secs

Stage 2b: 67.5 secs

Stage 3: 13.7 secs

# Geodesic IVP

Our game here is to find the solution of

$$0 = \frac{d^2 x^a}{ds^2} + \Gamma_{bc}^a(x) \frac{dx^b}{ds} \frac{dx^c}{ds}$$

subject to the initial conditions  $x^a(s) = x^a$  and  $dx^a(s)/ds = \dot{x}^a$  at  $s = 0$ .

## Algorithm

By successive differentiation of the above equation we can compute

$$\frac{d^n x^a}{ds^n} = -\Gamma_{\underline{d}_n}^a \frac{dx^{\underline{d}_n}}{ds}$$

at  $s = 0$  for  $n = 2, 3, 4, \dots$ . The  $\Gamma_{\underline{d}_n}^a$  are the *generalised connections*.

We can then construct the Taylor series solution for  $x^a(s)$

$$x^a(s) = x^a + s\dot{x}^a - \sum_{k=2}^{\infty} \frac{s^k}{k!} \Gamma_{\underline{d}_k}^a \dot{x}^{\underline{d}_k}$$

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

import cdblib

sterm21 = cdblib.get ('genGamma01','genGamma.json')
sterm22 = cdblib.get ('genGamma02','genGamma.json')
sterm23 = cdblib.get ('genGamma03','genGamma.json')
sterm24 = cdblib.get ('genGamma04','genGamma.json')

sterm31 = cdblib.get ('genGamma11','genGamma.json')
sterm32 = cdblib.get ('genGamma12','genGamma.json')
sterm33 = cdblib.get ('genGamma13','genGamma.json')

sterm41 = cdblib.get ('genGamma21','genGamma.json')
sterm42 = cdblib.get ('genGamma22','genGamma.json')

sterm51 = cdblib.get ('genGamma31','genGamma.json')

sterm2 := @(sterm21) + @(sterm22) + @(sterm23) + @(sterm24). # cdb (sterm2.000,sterm2)
sterm3 := @(sterm31) + @(sterm32) + @(sterm33). # cdb (sterm3.000,sterm3)
sterm4 := @(sterm41) + @(sterm42). # cdb (sterm4.000,sterm4)
sterm5 := @(sterm51). # cdb (sterm5.000,sterm5)

factor_out (sterm2,$A^{a?}$) # cdb (sterm2.001,sterm2)
factor_out (sterm3,$A^{a?}$) # cdb (sterm3.001,sterm3)
factor_out (sterm4,$A^{a?}$) # cdb (sterm4.001,sterm4)
factor_out (sterm5,$A^{a?}$) # cdb (sterm5.001,sterm5)

sterm2 := 360 @(sterm2).
sterm3 := 360 @(sterm3).
sterm4 := 90 @(sterm4).
sterm5 := 3 @(sterm5).

substitute (sterm2,$A^{a}->1$) # cdb (sterm2.002,sterm2)
substitute (sterm3,$A^{a}->1$) # cdb (sterm3.002,sterm3)
substitute (sterm4,$A^{a}->1$) # cdb (sterm4.002,sterm4)

```

```
substitute (sterm5,$A^{a}->1$)
```

```
# cdb (sterm5.002,sterm5)
```



# The geodesic ivp

$$x^d(s) = x^d + s\dot{x}^d + \frac{s^2}{2!}\dot{x}^a\dot{x}^b A_{ab}^d + \frac{s^3}{3!}\dot{x}^a\dot{x}^b\dot{x}^c A_{abc}^d + \frac{s^4}{4!}\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^d A_{abce}^d + \frac{s^5}{5!}\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A_{abcef}^d + \dots$$

$$\begin{aligned} 360A_{ab}^d = & 240x^d g^{ae} R_{bdce} + 30x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce}) + x^d x^e x^f (64g^{ag} g^{hi} R_{bdch} R_{egfi} - 32g^{ag} g^{hi} R_{bdeh} R_{cgfi} \\ & - 16g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18g^{ag} \nabla_{bd} R_{cef g} + 18g^{ag} \nabla_{db} R_{cef g} + 36g^{ag} \nabla_{de} R_{bfcg} - 16g^{ag} g^{hi} R_{bdeh} R_{cf gi} + 9g^{ag} \nabla_{gd} R_{becf} + 9g^{ag} \nabla_{dg} R_{becf}) \\ & + 2x^d x^e x^f x^g (16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah} g^{ij} R_{dhei} \nabla_b R_{cf gj} + 16g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 8g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cf gj} \\ & - 4g^{ah} g^{ij} R_{bidh} \nabla_e R_{cf gj} - 4g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} - 8g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 2g^{ah} \nabla_{bde} R_{cf gh} + 2g^{ah} \nabla_{dbe} R_{cf gh} \\ & + 2g^{ah} \nabla_{deb} R_{cf gh} + 4g^{ah} \nabla_{def} R_{bgch} - 4g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cf gj} - 4g^{ah} g^{ij} R_{bdei} \nabla_h R_{cf gj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + g^{ah} \nabla_{hde} R_{bfcg} + g^{ah} \nabla_{dhe} R_{bfcg} \\ & + g^{ah} \nabla_{deh} R_{bfcg}) \end{aligned}$$

$$\begin{aligned} 360A_{abc}^d = & 180x^e g^{af} \nabla_b R_{cedf} + 3x^e x^f (64g^{ag} g^{hi} R_{bech} R_{d gfi} + 16g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{befh} R_{cgdi} + 12g^{ag} \nabla_{bc} R_{defg} + 18g^{ag} \nabla_{be} R_{cf dg} \\ & + 18g^{ag} \nabla_{eb} R_{cf dg} + 48g^{ag} g^{hi} R_{bech} R_{df gi} + 3g^{ag} \nabla_{gb} R_{cedf} + 3g^{ag} \nabla_{bg} R_{cedf}) \\ & + 2x^e x^f x^g (32g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 48g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + 18g^{ah} g^{ij} R_{bhei} \nabla_c R_{df gj} + 2g^{ah} g^{ij} R_{bieh} \nabla_c R_{df gj} \\ & + 22g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15g^{ah} g^{ij} R_{bhei} \nabla_j R_{cf dg} + 5g^{ah} g^{ij} R_{bieh} \nabla_j R_{cf dg} \\ & - 12g^{ah} g^{ij} R_{bhci} \nabla_e R_{df gj} - 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - 8g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - 12g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 4g^{ah} \nabla_{bce} R_{df gh} + 4g^{ah} \nabla_{bec} R_{df gh} \\ & + 6g^{ah} \nabla_{bef} R_{cgdh} + 4g^{ah} \nabla_{ebc} R_{df gh} + 6g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{efb} R_{cgdh} + 16g^{ah} g^{ij} R_{behi} \nabla_c R_{df gj} + 36g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\ & + 16g^{ah} g^{ij} R_{beci} \nabla_h R_{df gj} - 4g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 36g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - 4g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + g^{ah} \nabla_{hbe} R_{cf dg} + g^{ah} \nabla_{heb} R_{cf dg} \\ & + g^{ah} \nabla_{bhe} R_{cf dg} + g^{ah} \nabla_{ehb} R_{cf dg} + g^{ah} \nabla_{beh} R_{cf dg} + g^{ah} \nabla_{ebh} R_{cf dg} - 20g^{ah} g^{ij} R_{beci} \nabla_j R_{df gh} + 10g^{ah} g^{ij} R_{behi} \nabla_j R_{cf dg}) \end{aligned}$$

$$\begin{aligned} 90A_{abce}^d = & 6x^f (8g^{ag} g^{hi} R_{bfch} R_{dgei} + 6g^{ag} \nabla_{bc} R_{df eg}) + x^f x^g (64g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} \\ & + 4g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + 44g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 10g^{ah} g^{ij} R_{bhci} \nabla_j R_{df eg} \\ & - 16g^{ah} g^{ij} R_{bf gi} \nabla_c R_{dhej} + 6g^{ah} \nabla_{bcd} R_{ef gh} + 8g^{ah} \nabla_{bcf} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{fbc} R_{dgeh} + 26g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} \\ & + 6g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + 46g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah} \nabla_{hbc} R_{df eg} + g^{ah} \nabla_{bhc} R_{df eg} + g^{ah} \nabla_{bch} R_{df eg} - 40g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh}) \end{aligned}$$

$$3A_{abcef}^d = x^g (3g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah} \nabla_{bcd} R_{egfh})$$

```

sterm2short := @(sterm21) + @(sterm22).          # cdb (sterm2.short.001,sterm2short)
sterm3short := @(sterm31).                        # cdb (sterm3.short.001,sterm3short)
sterm2shortscaled := 12 @(sterm2short).           # cdb (sterm2.short.scaled.002,sterm2shortscaled)
sterm3shortscaled := 2 @(sterm3short).            # cdb (sterm3.short.scaled.002,sterm3shortscaled)

substitute (sterm2shortscaled,$A^{a}->1$)        # cdb (sterm2.short.scaled.003,sterm2shortscaled)
substitute (sterm3shortscaled,$A^{a}->1$)        # cdb (sterm3.short.scaled.003,sterm3shortscaled)

cdblib.create ('geodesic-ivp.export')

# 4th order ivp terms scaled
cdblib.put ('ivp42',sterm2shortscaled,'geodesic-ivp.export')
cdblib.put ('ivp43',sterm3shortscaled,'geodesic-ivp.export')

# 6th order ivp terms scaled
cdblib.put ('ivp62',sterm2,'geodesic-ivp.export')
cdblib.put ('ivp63',sterm3,'geodesic-ivp.export')
cdblib.put ('ivp64',sterm4,'geodesic-ivp.export')
cdblib.put ('ivp65',sterm5,'geodesic-ivp.export')

checkpoint.append (sterm2shortscaled)
checkpoint.append (sterm3shortscaled)

checkpoint.append (sterm2)
checkpoint.append (sterm3)
checkpoint.append (sterm4)
checkpoint.append (sterm5)

```

$$\text{sterm2.short.001} := \frac{2}{3} A^b A^c x^d g^{ae} R_{bdce} + \frac{1}{12} A^b A^c x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce})$$

$$\text{sterm3.short.001} := \frac{1}{2} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf}$$

$$\text{sterm2.short.scaled.002} := 8A^b A^c x^d g^{ae} R_{bdce} + A^b A^c x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce})$$

$$\text{sterm3.short.scaled.002} := A^b A^c A^d x^e g^{af} \nabla_b R_{cedf}$$

$$\text{sterm2.short.scaled.003} := 8x^d g^{ae} R_{bdce} + x^d x^e \left( 2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce} \right)$$

$$\text{sterm3.short.scaled.003} := x^e g^{af} \nabla_b R_{cedf}$$

# Geodesic arc-length

Given a pair of points  $P$  and  $Q$  the geodesic arc-length can be computed using

$$L_{PQ} = \int_P^Q \left( g_{ab}(x) \frac{dx^a}{ds} \frac{dx^b}{ds} \right)^{1/2} ds \quad (1)$$

Since the path is a geodesic the integrand is constant and thus

$$L_{PQ}^2 = g_{ab}(x) \frac{dx^a}{ds} \frac{dx^b}{ds} \Big|_P \quad (2)$$

where  $s$  is a re-scaled parameter (0 at  $P$  and 1 at  $Q$ ). The point  $P$  has RNC coordinates  $x^a$  while the point  $Q$  has coordinates  $x^a + Dx^a$ .

The vector  $dx^a/ds$  at  $P$  is given by the solution of the geodesic boundary value problem. This was found in the previous code (`geodesic-bvp`). That is

$$\frac{dx^b}{ds} \Big|_P = y^a \quad (3)$$

and thus

$$L_{PQ}^2 = g_{ab}(x) y^a y^b \quad (4)$$

It is possible to directly evaluate the right hand side of (4) using the results from the `geodesic-bvp` and `metric` codes. The result would need to be truncated (to an order consistent with the results from those codes). But doing so would be computationally expensive as at least half of the terms will be thrown away. A better approach is compute just the terms that will survive the truncation. This is done by expanding  $g_{ab}(x)$  and  $y^a$  as a truncated series in the curvatures and its derivatives.

The  $g_{ab}(x)$  and  $y^a$  are written in a (truncated) formal power series in the curvature and its derivatives

$$y^a = \overset{0}{y}^a + \overset{2}{y}^a + \overset{3}{y}^a + \overset{4}{y}^a + \overset{5}{y}^a + \mathcal{O}(\epsilon^6) \quad (5)$$

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}(\epsilon^6) \quad (6)$$

Note that this use of  $\overset{i}{y}$  differs from that used in `geodesic-bvp`. Here the index above  $y^a$  denotes a particular term in the curvature expansion while in `geodesic-bvp` the index denoted the iteration number (in the fixed point scheme used to solve the BVP for  $y^a$ ).

# Stage 1

The formal curvature expansions are substituted into equation (4), expanded and truncated to retain terms of order  $\mathcal{O}(\epsilon^5)$  or less. The expansion to 4th order terms is as follows.

$$L_{PQ}^2 = {}^0g_{ab}{}^0y^a{}^0y^b + 2{}^0g_{ab}{}^0y^a{}^2y^b + {}^2g_{ab}{}^0y^a{}^0y^b + 2{}^0g_{ab}{}^0y^a{}^3y^b + {}^3g_{ab}{}^0y^a{}^0y^b + 2{}^0g_{ab}{}^0y^a{}^4y^b + {}^0g_{ab}{}^2y^a{}^2y^b + 2{}^2g_{ab}{}^0y^a{}^2y^b + {}^4g_{ab}{}^0y^a{}^0y^b$$

From `geodesic-bvp` (actually from `rnc2rnc` which reformatted the results nicely) we have

$${}^0y^a = Dx^a$$

$${}^2y^a = -\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$${}^3y^a = x^b x^c Dx^d Dx^e \left( -\frac{1}{12} g^{af} \nabla_d R_{becf} - \frac{1}{6} g^{af} \nabla_b R_{cdef} + \frac{1}{24} g^{af} \nabla_f R_{bdce} \right) - \frac{1}{12} x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned} {}^4y^a = & x^b x^c Dx^d Dx^e Dx^f \left( -\frac{2}{45} g^{ag} g^{hi} R_{bdeh} R_{cfdgi} + \frac{1}{45} g^{ag} g^{hi} R_{bdeh} R_{cifgi} - \frac{1}{45} g^{ag} g^{hi} R_{bdeh} R_{cgfhi} + \frac{1}{45} g^{ag} g^{hi} R_{bdch} R_{egfhi} - \frac{1}{60} g^{ag} \nabla_{de} R_{bfch} \right. \\ & \left. - \frac{1}{40} g^{ag} \nabla_{db} R_{cefg} - \frac{1}{40} g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240} g^{ag} \nabla_{gd} R_{becf} + \frac{1}{240} g^{ag} \nabla_{dg} R_{becf} \right) \\ & + x^b x^c x^d Dx^e Dx^f \left( -\frac{4}{45} g^{ag} g^{hi} R_{befh} R_{cgdi} + \frac{2}{45} g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45} g^{ag} g^{hi} R_{bech} R_{dghfi} - \frac{1}{40} g^{ag} \nabla_{eb} R_{cfdg} - \frac{1}{40} g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20} g^{ag} \nabla_{bc} R_{defg} \right. \\ & \left. - \frac{1}{45} g^{ag} g^{hi} R_{bech} R_{dfgi} + \frac{1}{80} g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80} g^{ag} \nabla_{bg} R_{cedf} \right) + x^b Dx^c Dx^d Dx^e Dx^f \left( -\frac{1}{45} g^{ag} g^{hi} R_{bcdh} R_{egfhi} - \frac{1}{60} g^{ag} \nabla_{cd} R_{befg} \right) \end{aligned}$$

and from `metric` we have

$${}^0g_{ab} = g_{ab}$$

$$3{}^2g_{ab} = -x^c x^d R_{acbd}$$

$$6{}^3g_{ab} = -x^c x^d x^e \nabla_c R_{adbe}$$

$$180{}^4g_{ab} = x^c x^d x^e x^f (8g^{gh} R_{acdgh} R_{befh} - 9\nabla_{cd} R_{aebf})$$

## Stage 2

The results from the `geodesic-bvp` and `metric` codes are read to provide values for the  $\bar{y}^a$  and  $\bar{g}_{ab}$ . These are substituted into the result from Stage 1, et volia, the final answer. To 4th-order terms the result is given by

$$\begin{aligned}
L_{PQ}^2 = & g_{ab} D x^a D x^b - \frac{1}{3} x^a x^b D x^c D x^d R_{acbd} - \frac{1}{12} x^a x^b D x^c D x^d D x^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c D x^d D x^e \nabla_a R_{bdce} \\
& + \frac{1}{360} x^a x^b D x^c D x^d D x^e D x^f (-8g^{gh} R_{acdg} R_{befh} - 6\nabla_{cd} R_{aebf}) + \frac{1}{360} x^a x^b x^c D x^d D x^e D x^f (16g^{gh} R_{adbg} R_{cefh} - 9\nabla_{da} R_{becf} - 9\nabla_{ad} R_{becf}) \\
& + \frac{1}{360} x^a x^b x^c x^d D x^e D x^f (16g^{gh} R_{aebg} R_{cf dh} - 18\nabla_{ab} R_{cedf}) + \frac{1}{1080} x^a x^b x^c D x^d D x^e D x^f D x^g (-4g^{hi} R_{adeh} \nabla_f R_{bgci} - 24g^{hi} R_{adeh} \nabla_b R_{cf gi} \\
& \quad + 10g^{hi} R_{adeh} \nabla_i R_{bf cg} + 16g^{hi} R_{adbh} \nabla_e R_{cf gi} - 4\nabla_{dea} R_{bf cg} - 4\nabla_{dae} R_{bf cg} - 4\nabla_{ade} R_{bf cg}) \\
& + \frac{1}{1080} x^a x^b D x^c D x^d D x^e D x^f D x^g (-18g^{hi} R_{acdh} \nabla_e R_{bf gi} - 3\nabla_{cde} R_{af bg}) \\
& + \frac{1}{1080} x^a x^b x^c x^d D x^e D x^f D x^g (24g^{hi} R_{aefh} \nabla_b R_{cg di} + 24g^{hi} R_{aebh} \nabla_f R_{cg di} + 24g^{hi} R_{aebh} \nabla_c R_{df gi} - 6\nabla_{eab} R_{cf dg} - 6\nabla_{aeb} R_{cf dg} - 6\nabla_{abe} R_{cf dg}) \\
& + \frac{1}{1080} x^a x^b x^c x^d x^e D x^f D x^g (48g^{hi} R_{afbh} \nabla_c R_{dgei} - 12\nabla_{abc} R_{df eg}) + \mathcal{O}(\epsilon^5)
\end{aligned}$$

## Shared properties

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
\Gamma^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
\Gamma^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).

x^{a}::Depends(D{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

g0{#}::LaTeXForm ("\\ngab{0}").
g2{#}::LaTeXForm ("\\ngab{2}").
g3{#}::LaTeXForm ("\\ngab{3}").
g4{#}::LaTeXForm ("\\ngab{4}").
```

```
g5{#}::LaTeXForm ("\\ngab{5}").
```

```
y0{#}::LaTeXForm ("\\ny{0}").
```

```
y2{#}::LaTeXForm ("\\ny{2}").
```

```
y3{#}::LaTeXForm ("\\ny{3}").
```

```
y4{#}::LaTeXForm ("\\ny{4}").
```

```
y5{#}::LaTeXForm ("\\ny{5}").
```



## Stage 1: The formal expansion

```
g0_{a b}::Symmetric.
g2_{a b}::Symmetric.
g3_{a b}::Symmetric.
g4_{a b}::Symmetric.
g5_{a b}::Symmetric.

g0_{a b}::Weight(label=num,value=0).
g2_{a b}::Weight(label=num,value=2).
g3_{a b}::Weight(label=num,value=3).
g4_{a b}::Weight(label=num,value=4).
g5_{a b}::Weight(label=num,value=5).

y0^{a}::Weight(label=num,value=0).
y2^{a}::Weight(label=num,value=2).
y3^{a}::Weight(label=num,value=3).
y4^{a}::Weight(label=num,value=4).
y5^{a}::Weight(label=num,value=5).

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ Dx^{a}               -> A003^{a}                $)
    substitute (obj,$ g_{a b}              -> A004_{a b}              $)
    substitute (obj,$ g^{a b}              -> A005^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}} -> A010_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}   -> A009_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}     -> A008_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}       -> A007_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}           -> A006_{a b c d}       $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}             -> A^{a}             $)
    substitute (obj,$ A002^{a}             -> x^{a}             $)
    substitute (obj,$ A003^{a}             -> Dx^{a}            $)
    substitute (obj,$ A004_{a b}           -> g_{a b}           $)
```

```

substitute (obj,$ A005^{a b}          -> g^{a b}          $)
substitute (obj,$ A006_{a b c d}      -> R_{a b c d}      $)
substitute (obj,$ A007_{a b c d e}    -> \nabla_{e}\{R_{a b c d}\} $)
substitute (obj,$ A008_{a b c d e f}  -> \nabla_{e f}\{R_{a b c d}\} $)
substitute (obj,$ A009_{a b c d e f g}-> \nabla_{e f g}\{R_{a b c d}\} $)
substitute (obj,$ A010_{a b c d e f g h}-> \nabla_{e f g h}\{R_{a b c d}\} $)

return obj

def truncate (obj,n):
    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah = Ex("num = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    return ans

# expansions wrt the curvature

defgab := g_{a b} -> g0_{a b} + g2_{a b} + g3_{a b} + g4_{a b} + g5_{a b}.
defy   := y^{a}   -> y0^{a} + y2^{a} + y3^{a} + y4^{a} + y5^{a}.

lsq    := g_{a b} y^{a} y^{b}.

substitute (lsq,defgab)
substitute (lsq,defy)
distribute (lsq)

def tidy (obj):
    foo := @(obj).
    sort_product      (foo)
    rename_dummies    (foo)
    canonicalise      (foo)
    return foo

```

```

lsq0 = tidy ( truncate (lsq,0) ) # cdb (lsq0.002,lsq0)
lsq2 = tidy ( truncate (lsq,2) ) # cdb (lsq2.002,lsq2)
lsq3 = tidy ( truncate (lsq,3) ) # cdb (lsq3.002,lsq3)
lsq4 = tidy ( truncate (lsq,4) ) # cdb (lsq4.002,lsq4)
lsq5 = tidy ( truncate (lsq,5) ) # cdb (lsq5.002,lsq5)

d20 := @(lsq2) - @(lsq0).      # cdb (d20.001,d20)  # check, should contain only O(2) terms
d32 := @(lsq3) - @(lsq2).      # cdb (d32.001,d32)  # check, should contain only O(3) terms
d43 := @(lsq4) - @(lsq3).      # cdb (d43.001,d43)  # check, should contain only O(4) terms
d54 := @(lsq5) - @(lsq4).      # cdb (d54.001,d54)  # check, should contain only O(5) terms

d5 := @(lsq5) - @(lsq).        # cdb (d5.001,d5)
d5 = tidy (d5)                 # cdb (d5.002,d5)  # all higher order terms, should see no O(5) terms

```

$$\text{lsq0.002} := g_{ab}^0 y^a y^b$$

$$\text{lsq2.002} := g_{ab}^0 y^a y^b + 2g_{ab}^0 y^a y^b + g_{ab}^2 y^a y^b$$

$$\text{lsq3.002} := g_{ab}^0 y^a y^b + 2 g_{ab}^0 y^a y^2 b + g_{ab}^2 y^a y^b + 2 g_{ab}^0 y^a y^3 b + g_{ab}^3 y^a y^b$$

$$\text{lsq4.002} := g_{ab}^0 y^a y^b + 2g_{ab}^0 y^a y^2 b + g_{ab}^2 y^a y^b + 2g_{ab}^0 y^a y^3 b + g_{ab}^3 y^a y^b + 2g_{ab}^0 y^a y^4 b + g_{ab}^0 y^2 y^2 b + 2g_{ab}^2 y^a y^2 b + g_{ab}^4 y^a y^b$$

$$\begin{aligned} \text{lsq5.002} := & \overset{0}{g}_{ab}\overset{0}{y}^a\overset{0}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{2}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{0}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{3}{y}^b + \overset{3}{g}_{ab}\overset{0}{y}^a\overset{0}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{4}{y}^b + \overset{0}{g}_{ab}\overset{2}{y}^a\overset{2}{y}^b \\ & + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{2}{y}^b + \overset{4}{g}_{ab}\overset{0}{y}^a\overset{0}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{5}{y}^b + \overset{2}{g}_{ab}\overset{2}{y}^a\overset{3}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{3}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{2}{y}^b + \overset{5}{g}_{ab}\overset{0}{y}^a\overset{0}{y}^b \end{aligned}$$

$$\text{d20.001} := 2\overset{0}{g}_{ab}\overset{0}{y}^a\overset{2}{y}^b + \overset{2}{g}_{ab}\overset{0}{y}^a\overset{0}{y}^b$$

$$\text{d32.001} := 2g_{ab}^0 y^a y^b + g_{ab}^3 y^a y^b$$

$$\mathbf{d43.001} := 2g_{ab}^0 y^a y^b + g_{ab}^0 y^a y^b + 2g_{ab}^2 y^a y^b + g_{ab}^4 y^a y^b$$

$$\text{d54.001} := 2g_{ab}^0 y^a y^b + 2g_{ab}^1 y^a y^b + 2g_{ab}^2 y^a y^b + 2g_{ab}^3 y^a y^b + g_{ab}^5 y^a y^b$$

$$\begin{aligned} \text{d5.002} := & -2g_{ab}^0 y^2 a^4 b - 2g_{ab}^0 y^2 a^5 b - g_{ab}^0 y^3 a^3 b - 2g_{ab}^0 y^3 a^4 b - 2g_{ab}^0 y^3 a^5 b - g_{ab}^0 y^4 a^4 b - 2g_{ab}^0 y^4 a^5 b - g_{ab}^0 y^5 a^5 b - 2g_{ab}^2 y^0 a^4 b - 2g_{ab}^2 y^0 a^5 b - g_{ab}^2 y^2 a^2 b - 2g_{ab}^2 y^2 a^3 b - 2g_{ab}^2 y^2 a^4 b \\ & - 2g_{ab}^2 y^2 a^5 b - g_{ab}^2 y^3 a^3 b - 2g_{ab}^2 y^3 a^4 b - 2g_{ab}^2 y^3 a^5 b - g_{ab}^2 y^4 a^4 b - 2g_{ab}^2 y^4 a^5 b - g_{ab}^2 y^5 a^5 b - 2g_{ab}^3 y^0 a^3 b - 2g_{ab}^3 y^0 a^4 b - 2g_{ab}^3 y^0 a^5 b - g_{ab}^3 y^2 a^2 b - 2g_{ab}^3 y^2 a^3 b \\ & - 2g_{ab}^3 y^2 a^4 b - 2g_{ab}^3 y^2 a^5 b - g_{ab}^3 y^3 a^3 b - 2g_{ab}^3 y^3 a^4 b - 2g_{ab}^3 y^3 a^5 b - g_{ab}^3 y^4 a^4 b - 2g_{ab}^3 y^4 a^5 b - g_{ab}^3 y^5 a^5 b - 2g_{ab}^4 y^0 a^2 b - 2g_{ab}^4 y^0 a^3 b - 2g_{ab}^4 y^0 a^4 b - 2g_{ab}^4 y^0 a^5 b \\ & - g_{ab}^4 y^2 a^2 b - 2g_{ab}^4 y^2 a^3 b - 2g_{ab}^4 y^2 a^4 b - 2g_{ab}^4 y^2 a^5 b - g_{ab}^4 y^3 a^3 b - 2g_{ab}^4 y^3 a^4 b - 2g_{ab}^4 y^3 a^5 b - g_{ab}^4 y^4 a^4 b - 2g_{ab}^4 y^4 a^5 b - g_{ab}^4 y^5 a^5 b - 2g_{ab}^5 y^0 a^2 b - 2g_{ab}^5 y^0 a^3 b \\ & - 2g_{ab}^5 y^0 a^4 b - 2g_{ab}^5 y^0 a^5 b - g_{ab}^5 y^2 a^2 b - 2g_{ab}^5 y^2 a^3 b - 2g_{ab}^5 y^2 a^4 b - 2g_{ab}^5 y^2 a^5 b - g_{ab}^5 y^3 a^3 b - 2g_{ab}^5 y^3 a^4 b - 2g_{ab}^5 y^3 a^5 b - g_{ab}^5 y^4 a^4 b - 2g_{ab}^5 y^4 a^5 b - g_{ab}^5 y^5 a^5 b \end{aligned}$$

## Stage 2: Substution of $y^a$ and $g_{ab}$

```
import cdblib

g0ab = cdblib.get('g_ab_0', 'metric.json')
g2ab = cdblib.get('g_ab_2', 'metric.json')
g3ab = cdblib.get('g_ab_3', 'metric.json')
g4ab = cdblib.get('g_ab_4', 'metric.json')
g5ab = cdblib.get('g_ab_5', 'metric.json')

defg0ab := g0_{a b} -> @(g0ab).
defg2ab := g2_{a b} -> @(g2ab).
defg3ab := g3_{a b} -> @(g3ab).
defg4ab := g4_{a b} -> @(g4ab).
defg5ab := g5_{a b} -> @(g5ab).

y0a = cdblib.get('y50', 'geodesic-bvp.json')
y2a = cdblib.get('y52', 'geodesic-bvp.json')
y3a = cdblib.get('y53', 'geodesic-bvp.json')
y4a = cdblib.get('y54', 'geodesic-bvp.json')
y5a = cdblib.get('y55', 'geodesic-bvp.json')

defy0a := y0^{a} -> @(y0a).
defy2a := y2^{a} -> @(y2a).
defy3a := y3^{a} -> @(y3a).
defy4a := y4^{a} -> @(y4a).
defy5a := y5^{a} -> @(y5a).

def substitute_gab_ya (obj):

    foo := @(obj).

    substitute (foo, defg0ab)
    substitute (foo, defg2ab)
    substitute (foo, defg3ab)
    substitute (foo, defg4ab)
    substitute (foo, defg5ab)
```

```

substitute (foo,defy0a)
substitute (foo,defy2a)
substitute (foo,defy3a)
substitute (foo,defy4a)
substitute (foo,defy5a)

distribute      (foo)
sort_product    (foo)
rename_dummies  (foo)
canonicalise    (foo)

substitute      (foo,$g_{a b} g^{c b} -> \delta^{c}_{a}$)
eliminate_kronecker (foo)
foo = product_sort (foo)
rename_dummies   (foo)
canonicalise     (foo)

return foo

def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

Q_{a b c d}::Weight(label=numR,value=2).
Q_{a b c d e}::Weight(label=numR,value=3).
Q_{a b c d e f}::Weight(label=numR,value=4).
Q_{a b c d e f g}::Weight(label=numR,value=5).

tmp := @(obj).

distribute (tmp)

substitute (tmp, $\nabla_{e f g}\{R_{a b c d}\} -> Q_{a b c d e f g}$)
substitute (tmp, $\nabla_{e f}\{R_{a b c d}\} -> Q_{a b c d e f}$)
substitute (tmp, $\nabla_{e}\{R_{a b c d}\} -> Q_{a b c d e}$)

```

```

substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

foo := @(tmp).
bah = Ex("numR = " + str(n))
keep_weight (foo, bah)

substitute (foo, $Q_{a b c d e f g} -> \nabla_{e f g}\{R_{a b c d}\}$)
substitute (foo, $Q_{a b c d e f} -> \nabla_{e f}\{R_{a b c d}\}$)
substitute (foo, $Q_{a b c d e} -> \nabla_e\{R_{a b c d}\}$)
substitute (foo, $Q_{a b c d} -> R_{a b c d}$)

return foo

lsq2 = substitute_gab_ya (lsq2) # cdb (lsq2.101,lsq2)
lsq3 = substitute_gab_ya (lsq3) # cdb (lsq3.101,lsq3)
lsq4 = substitute_gab_ya (lsq4) # cdb (lsq4.101,lsq4)
lsq5 = substitute_gab_ya (lsq5) # cdb (lsq5.101,lsq5)

lsq50 = get_Rterm (lsq5,0)
lsq52 = get_Rterm (lsq5,2)
lsq53 = get_Rterm (lsq5,3)
lsq54 = get_Rterm (lsq5,4)
lsq55 = get_Rterm (lsq5,5)

cdblib.create ('geodesic-lsq.json')

cdblib.put ('lsq2',lsq2,'geodesic-lsq.json')
cdblib.put ('lsq3',lsq3,'geodesic-lsq.json')
cdblib.put ('lsq4',lsq4,'geodesic-lsq.json')
cdblib.put ('lsq5',lsq5,'geodesic-lsq.json')

cdblib.put ('lsq50',lsq50,'geodesic-lsq.json')
cdblib.put ('lsq52',lsq52,'geodesic-lsq.json')
cdblib.put ('lsq53',lsq53,'geodesic-lsq.json')
cdblib.put ('lsq54',lsq54,'geodesic-lsq.json')
cdblib.put ('lsq55',lsq55,'geodesic-lsq.json')

```

$$\text{lsq2.101} := Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd}$$

$$\text{lsq3.101} := Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce}$$

$$\begin{aligned} \text{lsq4.101} := & Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} - \frac{1}{45} x^a x^b Dx^c Dx^d Dx^e Dx^f g^{gh} R_{acd g} R_{bef h} \\ & + \frac{2}{45} x^a x^b x^c Dx^d Dx^e Dx^f g^{gh} R_{adbg} R_{cef h} - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{da} R_{becf} - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{ad} R_{becf} \\ & - \frac{1}{60} x^a x^b Dx^c Dx^d Dx^e Dx^f \nabla_{cd} R_{aebf} + \frac{2}{45} x^a x^b x^c Dx^d Dx^e Dx^f g^{gh} R_{aebg} R_{cf dh} - \frac{1}{20} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{ab} R_{cedf} \end{aligned}$$

$$\begin{aligned} \text{lsq5.101} := & Dx^a Dx^b g_{ab} - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \\ & - \frac{1}{45} x^a x^b Dx^c Dx^d Dx^e Dx^f g^{gh} R_{acd g} R_{bef h} + \frac{2}{45} x^a x^b x^c Dx^d Dx^e Dx^f g^{gh} R_{adbg} R_{cef h} \\ & - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{da} R_{becf} - \frac{1}{40} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{ad} R_{becf} - \frac{1}{60} x^a x^b Dx^c Dx^d Dx^e Dx^f \nabla_{cd} R_{aebf} \\ & + \frac{2}{45} x^a x^b x^c Dx^d Dx^e Dx^f g^{gh} R_{aebg} R_{cf dh} - \frac{1}{20} x^a x^b x^c Dx^d Dx^e Dx^f \nabla_{ab} R_{cedf} - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adeh} \nabla_f R_{bgci} \\ & - \frac{1}{45} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adeh} \nabla_b R_{cf gi} + \frac{1}{108} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adeh} \nabla_i R_{bf cg} \\ & - \frac{1}{60} x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{acd h} \nabla_e R_{bf gi} + \frac{1}{45} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{aef h} \nabla_b R_{cg di} \\ & + \frac{1}{45} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{aebh} \nabla_f R_{cg di} + \frac{1}{45} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{aebh} \nabla_c R_{df gi} - \frac{1}{180} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{eab} R_{cf dg} \\ & - \frac{1}{180} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{aeb} R_{cf dg} - \frac{1}{180} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{abe} R_{cf dg} + \frac{2}{135} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{adbh} \nabla_e R_{cf gi} \\ & - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{dea} R_{bf cg} - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{dae} R_{bf cg} - \frac{1}{270} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{ade} R_{bf cg} \\ & - \frac{1}{360} x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \nabla_{cde} R_{af bg} + \frac{2}{45} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g g^{hi} R_{af bh} \nabla_c R_{dgei} - \frac{1}{90} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \nabla_{abc} R_{df eg} \end{aligned}$$



## Stage 3: Reformatting

```
def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    substitute (bah,$Dx^{b}->zzz^{b}$)
    factor_out (bah,$x^{a?},zzz^{b?}$)
    substitute (bah,$zzz^{b}->Dx^{b}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    substitute (bah,$Dx^{b}->zzz^{b}$)
    factor_out (bah,$x^{a?},zzz^{b?}$)
    substitute (bah,$zzz^{b}->Dx^{b}$)
    return bah

Rterm0 := @(lsq50).
Rterm2 := @(lsq52).
Rterm3 := @(lsq53).
Rterm4 := @(lsq54).
Rterm5 := @(lsq55).

Rterm0 = reformat (Rterm0, 1)      # cdb(Rterm0.301,Rterm0) # LCB: returns Dx before g, not what I want
Rterm2 = reformat (Rterm2, 3)      # cdb(Rterm2.301,Rterm2)
Rterm3 = reformat (Rterm3, 12)     # cdb(Rterm3.301,Rterm3)
Rterm4 = reformat (Rterm4, 360)    # cdb(Rterm4.301,Rterm4)
Rterm5 = reformat (Rterm5,1080)    # cdb(Rterm5.301,Rterm5)

Rterm0 := g_{a b} Dx^{a} Dx^{b}.  # LCB: fixes the order of terms, g before Dx,
```

```

lsq3 := @(Rterm0) + @(Rterm2). # cdb (lsq4.301,lsq3)
lsq4 := @(Rterm0) + @(Rterm2) + @(Rterm3). # cdb (lsq4.301,lsq4)
lsq5 := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4). # cdb (lsq5.301,lsq5)
lsq6 := @(Rterm0) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5). # cdb (lsq5.301,lsq6)

lsq := @(lsq6). # cdb (lsq.301,lsq)

scaled0 = rescale (Rterm0, 1) # cdb (scaled0.301,scaled0) # LCB: returns Dx before g, not what I want
scaled2 = rescale (Rterm2, 3) # cdb (scaled2.301,scaled2)
scaled3 = rescale (Rterm3, 12) # cdb (scaled3.301,scaled3)
scaled4 = rescale (Rterm4, 360) # cdb (scaled4.301,scaled4)
scaled5 = rescale (Rterm5, 1080) # cdb (scaled5.301,scaled5)

scaled0 := g_{a b} Dx^{a} Dx^{b}. # cdb (scaled0.301,scaled0) # LCB: fixes the order of terms, g before Dx, good

```

## Geodesic arc-length

$$\begin{aligned}
(\Delta s)^2 = & g_{ab} D x^a D x^b - \frac{1}{3} x^a x^b D x^c D x^d R_{acbd} - \frac{1}{12} x^a x^b D x^c D x^d D x^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c D x^d D x^e \nabla_a R_{bdce} \\
& + \frac{1}{360} x^a x^b D x^c D x^d D x^e D x^f (-8g^{gh} R_{acd g} R_{b e f h} - 6\nabla_{cd} R_{a e b f}) + \frac{1}{360} x^a x^b x^c D x^d D x^e D x^f (16g^{gh} R_{ad b g} R_{c e f h} - 9\nabla_{da} R_{b e c f} - 9\nabla_{ad} R_{b e c f}) \\
& + \frac{1}{360} x^a x^b x^c x^d D x^e D x^f (16g^{gh} R_{a e b g} R_{c f d h} - 18\nabla_{ab} R_{c e d f}) + \frac{1}{1080} x^a x^b x^c D x^d D x^e D x^f D x^g (-4g^{hi} R_{a d e h} \nabla_f R_{b g c i} - 24g^{hi} R_{a d e h} \nabla_b R_{c f g i} \\
& \quad + 10g^{hi} R_{a d e h} \nabla_i R_{b f c g} + 16g^{hi} R_{a d b h} \nabla_e R_{c f g i} - 4\nabla_{d e a} R_{b f c g} - 4\nabla_{d a e} R_{b f c g} - 4\nabla_{a d e} R_{b f c g}) \\
& + \frac{1}{1080} x^a x^b D x^c D x^d D x^e D x^f D x^g (-18g^{hi} R_{a c d h} \nabla_e R_{b f g i} - 3\nabla_{c d e} R_{a f b g}) \\
& + \frac{1}{1080} x^a x^b x^c x^d D x^e D x^f D x^g (24g^{hi} R_{a e f h} \nabla_b R_{c g d i} + 24g^{hi} R_{a e b h} \nabla_f R_{c g d i} + 24g^{hi} R_{a e b h} \nabla_c R_{d f g i} - 6\nabla_{e a b} R_{c f d g} - 6\nabla_{a e b} R_{c f d g} - 6\nabla_{a b e} R_{c f d g}) \\
& + \frac{1}{1080} x^a x^b x^c x^d x^e D x^f D x^g (48g^{hi} R_{a f b h} \nabla_c R_{d g e i} - 12\nabla_{a b c} R_{d f e g}) + \mathcal{O}(\epsilon^6)
\end{aligned}$$

# Geodesic arc-length curvature expansion

$$(\Delta s)^2 = \overset{0}{\Delta} + \overset{2}{\Delta} + \overset{3}{\Delta} + \overset{4}{\Delta} + \overset{5}{\Delta} + \mathcal{O}(\epsilon^6)$$

$$\overset{0}{\Delta} = g_{ab} D x^a D x^b$$

$$3\overset{2}{\Delta} = -x^a x^b D x^c D x^d R_{acbd}$$

$$12\overset{3}{\Delta} = -x^a x^b D x^c D x^d D x^e \nabla_c R_{adbe} - 2x^a x^b x^c D x^d D x^e \nabla_a R_{bdce}$$

$$360\overset{4}{\Delta} = x^a x^b D x^c D x^d D x^e D x^f (-8g^{gh} R_{acd g} R_{be f h} - 6\nabla_{cd} R_{aeb f}) + x^a x^b x^c D x^d D x^e D x^f (16g^{gh} R_{adb g} R_{ce f h} - 9\nabla_{da} R_{bec f} - 9\nabla_{ad} R_{bec f}) \\ + x^a x^b x^c x^d D x^e D x^f (16g^{gh} R_{aeb g} R_{cf d h} - 18\nabla_{ab} R_{ced f})$$

$$1080\overset{5}{\Delta} = x^a x^b x^c D x^d D x^e D x^f D x^g (-4g^{hi} R_{ade h} \nabla_f R_{bgci} - 24g^{hi} R_{ade h} \nabla_b R_{cf gi} + 10g^{hi} R_{ade h} \nabla_i R_{bf cg} + 16g^{hi} R_{adb h} \nabla_e R_{cf gi} - 4\nabla_{dea} R_{bf cg} - 4\nabla_{dae} R_{bf cg} \\ - 4\nabla_{ade} R_{bf cg}) + x^a x^b D x^c D x^d D x^e D x^f D x^g (-18g^{hi} R_{acd h} \nabla_e R_{bf gi} - 3\nabla_{cde} R_{af bg}) \\ + x^a x^b x^c x^d D x^e D x^f D x^g (24g^{hi} R_{aef h} \nabla_b R_{cg di} + 24g^{hi} R_{aeb h} \nabla_f R_{cg di} + 24g^{hi} R_{aeb h} \nabla_c R_{df gi} - 6\nabla_{eab} R_{cf dg} - 6\nabla_{aeb} R_{cf dg} - 6\nabla_{abe} R_{cf dg}) \\ + x^a x^b x^c x^d x^e D x^f D x^g (48g^{hi} R_{af bh} \nabla_c R_{dgei} - 12\nabla_{abc} R_{df eg})$$

```
cdblib.create ('geodesic-lsq.export')

# 3rd to 6th order lsq
cdblib.put ('lsq3',lsq3,'geodesic-lsq.export')
cdblib.put ('lsq4',lsq4,'geodesic-lsq.export')
cdblib.put ('lsq5',lsq5,'geodesic-lsq.export')
cdblib.put ('lsq6',lsq6,'geodesic-lsq.export')

# 6th order lsq terms, scaled
cdblib.put ('lsq60',scaled0,'geodesic-lsq.export')
cdblib.put ('lsq62',scaled2,'geodesic-lsq.export')
cdblib.put ('lsq63',scaled3,'geodesic-lsq.export')
cdblib.put ('lsq64',scaled4,'geodesic-lsq.export')
cdblib.put ('lsq65',scaled5,'geodesic-lsq.export')

checkpoint.append (lsq4)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```

## Geodesic mid-point for arc-length

This code uses the results of `geodesic-lsq` and `metric` to show that the 2nd and 3rd order estimates for  $L_{PQ}^2$  can be recovered using a mid-point estimate. For the 3rd order estimate we have

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adb e} + \mathcal{O}(\epsilon^4)$$
$$L_{PQ}^2 = g_{ab} D x^a D x^b - \frac{1}{3}x^a x^b D x^c D x^d R_{acbd} - \frac{1}{12}x^a x^b D x^c D x^d D x^e \nabla_c R_{adb e} - \frac{1}{6}x^a x^b x^c D x^d D x^e \nabla_a R_{bdce} + \mathcal{O}(\epsilon^4)$$

The code below verifies that

$$L_{PQ}^2 = g_{ab}(\bar{x}) D x^a D x^b + \mathcal{O}(\epsilon^4)$$

where  $\bar{x}$  is the *coordinate* midpoint of the geodesic

$$\bar{x}^a = \frac{1}{2} (x_P^a + x_Q^a)$$

This result holds true only for the 2nd and 3rd order estimates. Note that the *coordinate* midpoint is not the *geometric* midpoint of the geodesic.

It might be interesting to see if the higher order estimates could be recovered by sampling the metric at points other than the mid point.

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.

import cdblib

gab = cdblib.get('g_ab','metric.json')

lsq2 = cdblib.get('lsq2','geodesic-lsq.json')
lsq3 = cdblib.get('lsq3','geodesic-lsq.json')
lsq4 = cdblib.get('lsq4','geodesic-lsq.json')
lsq5 = cdblib.get('lsq5','geodesic-lsq.json')

substitute (gab,$x^{a}->(p^{a}+q^{a})/2$)  # evaluate rnc gab at mid-point
distribute (gab)

defgab := g_{a b} -> @(gab).

mid := g_{a b} (q^{a}-p^{a}) (q^{b}-p^{b}).

substitute      (mid,defgab)
distribute      (mid)
sort_product    (mid)
rename_dummies  (mid)
canonicalise    (mid)

tst2 := @(lsq2) - @(mid).           # cdb (tst2.201,tst2)
tst3 := @(lsq3) - @(mid).           # cdb (tst3.201,tst3)
tst4 := @(lsq4) - @(mid).           # cdb (tst4.201,tst4)
tst5 := @(lsq5) - @(mid).           # cdb (tst5.201,tst5)

substitute      (tst2,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst2,$x^{a} -> p^{a}$)

```

```

distribute      (tst2)
sort_product    (tst2)
rename_dummies  (tst2)
canonicalise    (tst2)                                # cdb (tst2.202,tst2)

substitute      (tst3,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst3,$x^{a} -> p^{a}$)
distribute      (tst3)
sort_product    (tst3)
rename_dummies  (tst3)
canonicalise    (tst3)                                # cdb (tst3.202,tst3)

substitute      (tst4,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst4,$x^{a} -> p^{a}$)
distribute      (tst4)
sort_product    (tst4)
rename_dummies  (tst4)
canonicalise    (tst4)                                # cdb (tst4.202,tst4)

substitute      (tst5,$Dx^{a} -> q^{a}-p^{a}$)
substitute      (tst5,$x^{a} -> p^{a}$)
distribute      (tst5)
sort_product    (tst5)
rename_dummies  (tst5)
canonicalise    (tst5)                                # cdb (tst5.202,tst5)

```



# Reformatting

```
def truncateR (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    substitute (tmp, $\nabla_{e f g}\{R_{a b c d}\} \rightarrow Q_{a b c d e f g}\$)
    substitute (tmp, $\nabla_{e f}\{R_{a b c d}\} \rightarrow Q_{a b c d e f}\$)
    substitute (tmp, $\nabla_e\{R_{a b c d}\} \rightarrow Q_{a b c d e}\$)
    substitute (tmp, $R_{a b c d} \rightarrow Q_{a b c d}\$)

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(tmp).
        bah = Ex("numR = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    substitute (ans, $Q_{a b c d e f g} \rightarrow \nabla_{e f g}\{R_{a b c d}\}\$)
    substitute (ans, $Q_{a b c d e f} \rightarrow \nabla_{e f}\{R_{a b c d}\}\$)
    substitute (ans, $Q_{a b c d e} \rightarrow \nabla_e\{R_{a b c d}\}\$)
    substitute (ans, $Q_{a b c d} \rightarrow R_{a b c d}\$)

    return ans

tst2 = truncateR (tst2,2) # cdb (tst2.301,tst2)
tst3 = truncateR (tst3,3) # cdb (tst3.301,tst3)
tst4 = truncateR (tst4,4) # cdb (tst4.301,tst4)
```

```
tst5 = truncateR (tst5,5)  # cdb (tst5.301,tst5)
```

## Errors is mid-point estimates for $L_{PQ}^2$

$$(L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_2 = 0$$

$$(L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_3 = 0$$

$$\begin{aligned} (L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_4 = & -\frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^cp^gq^bq^dq^fq^h + \frac{1}{15}R_{abcd}R_{efgh}g^{ae}p^bp^cp^gq^dq^fq^h - \frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^bp^cp^fp^gq^dq^h \\ & + \frac{1}{240}\nabla_{ab}R_{cdef}p^bp^cp^eq^aq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^bp^cp^eq^dq^f + \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^cp^eq^bq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^cp^eq^aq^bq^dq^f \end{aligned}$$

$$\begin{aligned} (L_{PQ}^2 - g_{ab}(\bar{x})Dx^aDx^b)_5 = & -\frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^cp^gq^bq^dq^fq^h + \frac{1}{15}R_{abcd}R_{efgh}g^{ae}p^bp^cp^gq^dq^fq^h - \frac{1}{30}R_{abcd}R_{efgh}g^{ae}p^bp^cp^fp^gq^dq^h \\ & + \frac{1}{240}\nabla_{ab}R_{cdef}p^bp^cp^eq^aq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^bp^cp^eq^dq^f + \frac{1}{240}\nabla_{ab}R_{cdef}p^ap^cp^eq^bq^dq^f - \frac{1}{240}\nabla_{ab}R_{cdef}p^cp^eq^aq^bq^dq^f \\ & + \frac{1}{135}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^gp^hq^bq^dq^eq^i + \frac{7}{270}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^ep^gp^hq^bq^dq^i - \frac{1}{90}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^gp^hq^bq^dq^eq^i \\ & - \frac{1}{45}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^ep^gp^hq^bq^dq^i - \frac{1}{90}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^ep^hq^bq^dq^gq^i + \frac{1}{135}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^ep^hq^bq^dq^gq^i \\ & - \frac{1}{108}R_{abcd}\nabla_eR_{fghi}g^{ae}p^cp^fp^hq^bq^dq^gq^i + \frac{1}{108}R_{abcd}\nabla_eR_{fghi}g^{ae}p^bp^cp^fp^hq^bq^dq^gq^i - \frac{1}{45}R_{abcd}\nabla_eR_{fghi}g^{af}p^cp^hq^bq^dq^eq^gq^i \\ & + \frac{7}{270}R_{abcd}\nabla_eR_{fghi}g^{af}p^bp^cp^hq^bq^dq^eq^gq^i + \frac{1}{2160}\nabla_{abc}R_{defg}p^bp^cp^dp^fp^aq^eq^g - \frac{1}{720}\nabla_{abc}R_{defg}p^ap^bp^cp^dp^fp^aq^eq^g \\ & + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^cp^dp^fp^bq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^bp^dp^fp^cq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^cp^dp^fp^aq^bq^eq^g \\ & + \frac{1}{2160}\nabla_{abc}R_{defg}p^bp^dp^fp^aq^cq^eq^g + \frac{1}{2160}\nabla_{abc}R_{defg}p^ap^dp^fp^bq^cq^eq^g - \frac{1}{720}\nabla_{abc}R_{defg}p^dp^fp^aq^bq^cq^eq^g \end{aligned}$$

# Converting from generic to rnc coordinates

The following is based on the approach used in the `geodesic-bvp.tex` code. The main difference here is that this time we will *not* be assuming that the coordinates are in Riemann normal form. This will be apparent in the expression for the generalised connections – they will be expressed in terms of the partial derivatives of the connection rather the covariant derivatives of the Riemann tensor. There will also be a change in the way the Taylor series are developed. In this case the expansion parameter  $\epsilon$  will be associated with the connection and its derivatives rather than the Riemann tensor. We will use

$$\Gamma^a_{bc} = \mathcal{O}(\epsilon) , \quad \Gamma^a_{bc,d} = \mathcal{O}(\epsilon^2) , \quad \Gamma^a_{bc,de} = \mathcal{O}(\epsilon^3) , \quad \text{etc.}$$

The generalised connections are defined recursively by

$$\Gamma^a_{bcd} = \Gamma^a_{(bc,d)} - (n+1)\Gamma^a_{p(\underline{c}}\Gamma^p_{bd)} \quad (1)$$

where  $\underline{c}$  contains  $n > 0$  indices. It is easy to see from this equation that the generalised connections will behave much the same as the connection, that is

$$\Gamma^a_{bc} = \mathcal{O}(\epsilon) , \quad \Gamma^a_{bcd} = \mathcal{O}(\epsilon^2) , \quad \Gamma^a_{bcde} = \mathcal{O}(\epsilon^3) , \quad \text{etc.}$$

This allows us to represent each generalised connection by a single expression (typically `GamNN`).

The situation is slightly different in `geodesic-bvp.tex`. In that code the connection and the generalised connection are expanded as a series in the Riemann tensor and its derivatives. Thus each connection is written in the form

$$\bar{\Gamma}^a_{\underline{c}_n} = \bar{\Gamma}^a_{\underline{c}_n}^{(0)} + \bar{\Gamma}^a_{\underline{c}_n}^{(1)} + \bar{\Gamma}^a_{\underline{c}_n}^{(2)} + \cdots + \bar{\Gamma}^a_{\underline{c}_n}^{(m)} \quad (2)$$

where  $\underline{c}_n$  denotes a set of indices such as  $c_1c_2c_3 \dots c_n$ . The terms of the RHS are each of a different weight in  $\epsilon$ .

## Stage 1: The generalised connections

The generalised connections  $\Gamma^a_{\underline{c}_n}$  could be computed directly by successive application of equation (1). But a more effiecent method exists and its basis lies in the original definition of the generalised connections. Recall that the generalised connections arose when buidling a formal power series solution of the geodesic equation

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} \quad (3)$$

The key idea was that the coefficients  $c_n$  in the formal power series

$$x^a = c_0^a + s c_1^a + s^2 c_2^a + \dots \quad (4)$$

could be computed using

$$c_n^a = \frac{1}{n!} \left. \frac{d^n x^a}{ds^n} \right|_{s=0} \quad (5)$$

with the second, third and higher derivatives of  $x^a$  found by successive differentiation of the geodesic equation. The generalised connections were introduced as part of this algorithm, leading to

$$c_n^a = - \Gamma_{\underline{c}_n}^a A^{\underline{c}_n} \Big|_{s=0} \quad n = 2, 3, 4 \dots \quad (6)$$

and

$$\Gamma_{\underline{c}_{n+1}}^a A^{\underline{c}_{n+1}} = \frac{d}{ds} \left( \Gamma_{\underline{c}_n}^a A^{\underline{c}_n} \right) \quad (7)$$

with  $d/ds = A^a \partial_a$ ,  $A^a = dx^a/ds$  and  $dA^a/ds = -\Gamma_{bc}^a A^b A^c$ .

The upshot is that computing the  $\Gamma_{\underline{c}_n}^a A^{\underline{c}_n}$  requires little more than successive rounds of differentiation (and a few substitutions for the derivatives of  $A^a$ ).

Note that the coefficients  $c_0$  and  $c_1$  must be determined from the initial conditions. Suppose that  $x^a = x_i^a$  at  $s = 0$  then  $c_0 = x_i^a$  while  $c_1 = A^a$ .

The Riemann normal coordinates of the point  $j$  (where  $s = 1$ ) are introduced by setting

$$y^a = A^a \quad (8)$$

This leads to

$$x_j^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{\underline{b}_k}^a y^{\underline{b}_k} \quad (9)$$

Note that given two points  $i$  and  $j$ , the  $y^a$  would be found as a root of this non-linear equation for  $y^a$ .

## Stage 2: The fixed point scheme for $y^a$

This second stage is almost exactly the same as the corresponding stage in `geodesic-bvp`. The difference here is that the generalised connections involve partial derivatives of the connection. In `contrat`, the `geodesic-bvp` code is specific to RNC and thus uses the generalised connections based on covariant derivatives of the Riemann tensor.

We begin this second stage by rewriting the equation (9) in the suggestive form

$$y^a = x_j^a - x_i^a + \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\underline{b}_k}$$

and then use this as the basis of a fixed point iteration scheme.

Start with the first approximation  $y_1^a = x_j^a - x_i^a = \Delta x^a$ , then compute the successive approximations

$$y_1^a = \Delta x^a$$

$$y_2^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_1^b y_1^c$$

$$y_3^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_2^b y_2^c + \frac{1}{3!} \Gamma^a_{bcd} y_1^b y_1^c y_1^d$$

$$y_4^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_3^b y_3^c + \frac{1}{3!} \Gamma^a_{bcd} y_2^b y_2^c y_2^d + \frac{1}{4!} \Gamma^a_{bcde} y_1^b y_1^c y_1^d y_1^e$$

$$y_5^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_4^b y_4^c + \frac{1}{3!} \Gamma^a_{bcd} y_3^b y_3^c y_3^d + \frac{1}{4!} \Gamma^a_{bcde} y_2^b y_2^c y_2^d y_2^e + \frac{1}{5!} \Gamma^a_{bcdef} y_1^b y_1^c y_1^d y_1^e y_1^f$$

and so on. Not that the  $\Gamma^a_{bc}$ ,  $\Gamma^a_{bcd}$ ,  $\Gamma^a_{bcde}$  etc. will all depend on the original coordinates  $x^a$  at the initial point (i.e.,  $P = x_i^a$ ).

## Stage3: Introduce the generalised connections from Stage 1

This is the final stage – it introduces the generalised connection after the completion of the fixed point scheme.

The result will be an equation for the  $y^a$  in terms of the original coordinates  $x^a$  and the connections (and its derivatives) at a chosen point  $s = 0$  (aka  $i$ ).

The  $y^a$  define an RNC frame in the neighbourhood of the chosen point  $i$ .

## Stage 1: The generalised connections

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

A^{a}::Depends(\partial{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).

\Gamma^{a}_{b c}::Depends(\partial{#}).

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
\Gamma^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
\Gamma^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
\Gamma^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

\Gamma^{p}_{a b}::Weight(label=numG,value=1).
\Gamma^{p}_{a b c}::Weight(label=numG,value=2).
\Gamma^{p}_{a b c d}::Weight(label=numG,value=3).
```

```

\Gamma^{p}_{\{a b c d e\}}::Weight(label=numG,value=4).
\Gamma^{p}_{\{a b c d e f\}}::Weight(label=numG,value=5).

def product_sort (obj):
    substitute (obj,$ A^{\{a\}}                -> A001^{\{a\}}                $)
    substitute (obj,$ x^{\{a\}}                  -> A002^{\{a\}}                  $)
    substitute (obj,$ g^{\{a b\}}                 -> A003^{\{a b\}}                 $)
    substitute (obj,$ \Gamma^{p}_{\{a b\}}          -> A004^{\{p\}}_{\{a b\}}          $)
    substitute (obj,$ \Gamma^{p}_{\{a b c\}}         -> A005^{\{p\}}_{\{a b c\}}         $)
    substitute (obj,$ \Gamma^{p}_{\{a b c d\}}        -> A006^{\{p\}}_{\{a b c d\}}        $)
    substitute (obj,$ \Gamma^{p}_{\{a b c d e\}}       -> A007^{\{p\}}_{\{a b c d e\}}       $)
    substitute (obj,$ \Gamma^{p}_{\{a b c d e f\}}      -> A008^{\{p\}}_{\{a b c d e f\}}      $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{\{a\}}                -> A^{\{a\}}                $)
    substitute (obj,$ A002^{\{a\}}                -> x^{\{a\}}                $)
    substitute (obj,$ A003^{\{a b\}}               -> g^{\{a b\}}               $)
    substitute (obj,$ A004^{\{p\}}_{\{a b\}}          -> \Gamma^{p}_{\{a b\}}          $)
    substitute (obj,$ A005^{\{p\}}_{\{a b c\}}         -> \Gamma^{p}_{\{a b c\}}         $)
    substitute (obj,$ A006^{\{p\}}_{\{a b c d\}}        -> \Gamma^{p}_{\{a b c d\}}        $)
    substitute (obj,$ A007^{\{p\}}_{\{a b c d e\}}       -> \Gamma^{p}_{\{a b c d e\}}       $)
    substitute (obj,$ A008^{\{p\}}_{\{a b c d e f\}}      -> \Gamma^{p}_{\{a b c d e f\}}      $)

    return obj

def truncateGam (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah = Ex("numG = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    return ans

beg_stage_1 = time.time()

```



```

# note that we use A^{a} in place of dx^a/ds

Gamma := \Gamma^{d}_{a b} A^{a} A^{b}.

# the geodesic equation

dAds := A^{c} \partial_{c} A^{d} - \Gamma^{d}_{ab} A^{a} A^{b}.

# eq0, eq1, eq2 ... are the successive derivatives of Gamma
# thus they are the generalised gamma's dotted into (multiple copies of) A^{a} = dx^a/ds

# =====
eq0 := \Gamma^{d}_{ab} A^{a} A^{b}. # cdb (eq0.000,eq0)

# =====
eq1 := A^{c} \partial_{c} \Gamma^{d}_{ab} A^{a} A^{b}. # cdb (eq1.000,eq1)

distribute      (eq1) # cdb (eq1.001,eq1)
unwrap          (eq1) # cdb (eq1.002,eq1)
product_rule    (eq1) # cdb (eq1.003,eq1)
distribute      (eq1) # cdb (eq1.004,eq1)
substitute      (eq1,dAds) # cdb (eq1.005,eq1)
distribute      (eq1) # cdb (eq1.006,eq1)
eq1 = truncateGam (eq1,5) # cdb (eq1.007,eq1)
sort_product    (eq1) # cdb (eq1.008,eq1)
rename_dummies  (eq1) # cdb (eq1.009,eq1)
canonicalise    (eq1) # cdb (eq1.010,eq1)

# =====
eq2 := A^{c} \partial_{c} A^{d} \Gamma^{e}_{ab} A^{a} A^{b}. # cdb (eq2.000,eq2)

distribute      (eq2) # cdb (eq2.001,eq2)
unwrap          (eq2) # cdb (eq2.002,eq2)
product_rule    (eq2) # cdb (eq2.003,eq2)
distribute      (eq2) # cdb (eq2.004,eq2)
substitute      (eq2,dAds) # cdb (eq2.005,eq2)
distribute      (eq2) # cdb (eq2.006,eq2)

```

```

eq2 = truncateGam (eq2,5)           # cdb (eq2.007,eq2)
sort_product      (eq2)             # cdb (eq2.008,eq2)
rename_dummies    (eq2)             # cdb (eq2.009,eq2)
canonicalise      (eq2)             # cdb (eq2.010,eq2)

# =====
eq3 := A^{c} \partial_{c}{@(eq2)}.    # cdb (eq3.000,eq3)

distribute        (eq3)             # cdb (eq3.001,eq3)
unwrap            (eq3)             # cdb (eq3.002,eq3)
product_rule      (eq3)             # cdb (eq3.003,eq3)
distribute        (eq3)             # cdb (eq3.004,eq3)
substitute        (eq3,dAds)        # cdb (eq3.005,eq3)
distribute        (eq3)             # cdb (eq3.006,eq3)
eq3 = truncateGam (eq3,5)           # cdb (eq3.007,eq3)
sort_product      (eq3)             # cdb (eq3.008,eq3)
rename_dummies    (eq3)             # cdb (eq3.009,eq3)
canonicalise      (eq3)             # cdb (eq3.010,eq3)

# =====
eq4 := A^{c} \partial_{c}{@(eq3)}.    # cdb (eq4.000,eq4)

distribute        (eq4)             # cdb (eq4.001,eq4)
unwrap            (eq4)             # cdb (eq4.002,eq4)
product_rule      (eq4)             # cdb (eq4.003,eq4)
distribute        (eq4)             # cdb (eq4.004,eq4)
substitute        (eq4,dAds)        # cdb (eq4.005,eq4)
distribute        (eq4)             # cdb (eq4.006,eq4)
eq4 = truncateGam (eq4,5)           # cdb (eq4.007,eq4)
sort_product      (eq4)             # cdb (eq4.008,eq4)
rename_dummies    (eq4)             # cdb (eq4.009,eq4)
canonicalise      (eq4)             # cdb (eq4.010,eq4)

end_stage_1 = time.time()

```

$$\text{eq0.000} := \Gamma^d_{ab} A^a A^b$$

$$\text{eq1.000} := A^c \partial_c (\Gamma_{ab}^d A^a A^b)$$

$$\text{eq1.001} := A^c \partial_c (\Gamma_{ab}^d A^a A^b)$$

$$\text{eq1.002} := A^c \partial_c (\Gamma_{ab}^d A^a A^b)$$

$$\text{eq1.003} := A^c (\partial_c \Gamma_{ab}^d A^a A^b + \Gamma_{ab}^d \partial_c A^a A^b + \Gamma_{ab}^d A^a \partial_c A^b)$$

$$\text{eq1.004} := A^c \partial_c \Gamma_{ab}^d A^a A^b + A^c \Gamma_{ab}^d \partial_c A^a A^b + A^c \Gamma_{ab}^d A^a \partial_c A^b$$

$$\text{eq1.005} := A^c \partial_c \Gamma_{ab}^d A^a A^b - \Gamma_{ce}^a A^c A^e \Gamma_{ab}^d A^b - \Gamma_{ec}^b A^e A^c \Gamma_{ab}^d A^a$$

$$\text{eq1.006} := A^c \partial_c \Gamma_{ab}^d A^a A^b - \Gamma_{ce}^a A^c A^e \Gamma_{ab}^d A^b - \Gamma_{ec}^b A^e A^c \Gamma_{ab}^d A^a$$

$$\text{eq1.007} := A^c \partial_c \Gamma_{ab}^d A^a A^b - \Gamma_{ce}^a A^c A^e \Gamma_{ab}^d A^b - \Gamma_{ec}^b A^e A^c \Gamma_{ab}^d A^a$$

$$\text{eq1.008} := A^a A^b A^c \partial_c \Gamma_{ab}^d - A^b A^c A^e \Gamma_{ce}^a \Gamma_{ab}^d - A^a A^c A^e \Gamma_{ec}^b \Gamma_{ab}^d$$

$$\text{eq1.009} := A^a A^b A^c \partial_c \Gamma_{ab}^d - A^a A^b A^c \Gamma_{bc}^e \Gamma_{ea}^d - A^a A^b A^c \Gamma_{cb}^e \Gamma_{ae}^d$$

$$\text{eq1.010} := A^a A^b A^c \partial_a \Gamma_{bc}^d - 2A^a A^b A^c \Gamma_{ae}^d \Gamma_{bc}^e$$

$$\text{eq2.000} := A^c \partial_c (A^a A^b A^f \partial_a \Gamma_{bf}^d - 2A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e)$$

$$\text{eq2.001} := A^c \partial_c (A^a A^b A^f \partial_a \Gamma_{bf}^d) - 2A^c \partial_c (A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e)$$

$$\text{eq2.002} := A^c \partial_c (A^a A^b A^f \partial_a \Gamma_{bf}^d) - 2A^c \partial_c (A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e)$$

$$\begin{aligned} \text{eq2.003} := & A^c (\partial_c A^a A^b A^f \partial_a \Gamma_{bf}^d + A^a \partial_c A^b A^f \partial_a \Gamma_{bf}^d + A^a A^b \partial_c A^f \partial_a \Gamma_{bf}^d + A^a A^b A^f \partial_{ca} \Gamma_{bf}^d) \\ & - 2A^c (\partial_c A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e + A^a \partial_c A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e + A^a A^b \partial_c A^f \Gamma_{ae}^d \Gamma_{bf}^e + A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e + A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e) \end{aligned}$$

$$\begin{aligned} \text{eq2.004} := & A^c \partial_c A^a A^b A^f \partial_a \Gamma_{bf}^d + A^c A^a \partial_c A^b A^f \partial_a \Gamma_{bf}^d + A^c A^a A^b \partial_c A^f \partial_a \Gamma_{bf}^d + A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d - 2A^c \partial_c A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e \\ & - 2A^c A^a \partial_c A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b \partial_c A^f \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.005} := & -\Gamma_{ce}^a A^c A^e A^b A^f \partial_a \Gamma_{bf}^d - \Gamma_{ec}^b A^e A^c A^a A^f \partial_a \Gamma_{bf}^d - \Gamma_{ce}^f A^c A^e A^a A^b \partial_a \Gamma_{bf}^d + A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d + 2\Gamma_{cg}^a A^c A^g A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e \\ & + 2\Gamma_{gc}^b A^g A^c A^a A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{cg}^f A^c A^g A^a A^b \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.006} := & -\Gamma_{ce}^a A^c A^e A^b A^f \partial_a \Gamma_{bf}^d - \Gamma_{ec}^b A^e A^c A^a A^f \partial_a \Gamma_{bf}^d - \Gamma_{ce}^f A^c A^e A^a A^b \partial_a \Gamma_{bf}^d + A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d + 2\Gamma_{cg}^a A^c A^g A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e \\ & + 2\Gamma_{gc}^b A^g A^c A^a A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{cg}^f A^c A^g A^a A^b \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.007} := & A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d - \Gamma_{ce}^a A^c A^e A^b A^f \partial_a \Gamma_{bf}^d - \Gamma_{ec}^b A^e A^c A^a A^f \partial_a \Gamma_{bf}^d - \Gamma_{ce}^f A^c A^e A^a A^b \partial_a \Gamma_{bf}^d - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e \\ & - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e + 2\Gamma_{cg}^a A^c A^g A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{gc}^b A^g A^c A^a A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{cg}^f A^c A^g A^a A^b \Gamma_{ae}^d \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.008} := & A^a A^b A^c A^f \partial_{ca} \Gamma_{bf}^d - A^b A^c A^e A^f \Gamma_{ce}^a \partial_a \Gamma_{bf}^d - A^a A^c A^e A^f \Gamma_{ec}^b \partial_a \Gamma_{bf}^d - A^a A^b A^c A^e \Gamma_{ce}^f \partial_a \Gamma_{bf}^d - 2A^a A^b A^c A^f \Gamma_{bf}^e \partial_c \Gamma_{ae}^d \\ & - 2A^a A^b A^c A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e + 2A^b A^c A^f A^g \Gamma_{cg}^a \Gamma_{ae}^d \Gamma_{bf}^e + 2A^a A^c A^f A^g \Gamma_{gc}^b \Gamma_{ae}^d \Gamma_{bf}^e + 2A^a A^b A^c A^g \Gamma_{ae}^d \Gamma_{bf}^e \Gamma_{cg}^f \end{aligned}$$

$$\begin{aligned} \text{eq2.009} := & A^a A^b A^c A^e \partial_{ca} \Gamma_{be}^d - A^a A^b A^c A^e \Gamma_{bc}^f \partial_f \Gamma_{ae}^d - A^a A^b A^c A^e \Gamma_{cb}^f \partial_a \Gamma_{fe}^d - A^a A^b A^c A^e \Gamma_{ce}^f \partial_a \Gamma_{bf}^d - 2A^a A^b A^c A^e \Gamma_{be}^f \partial_c \Gamma_{af}^d \\ & - 2A^a A^b A^c A^e \Gamma_{af}^d \partial_c \Gamma_{be}^f + 2A^a A^b A^c A^e \Gamma_{be}^f \Gamma_{fg}^d \Gamma_{ac}^g + 2A^a A^b A^c A^e \Gamma_{eb}^f \Gamma_{ag}^d \Gamma_{fc}^g + 2A^a A^b A^c A^e \Gamma_{af}^d \Gamma_{bg}^f \Gamma_{ce}^g \end{aligned}$$

$$\begin{aligned} \text{eq2.010} := & A^a A^b A^c A^e \partial_{ab} \Gamma_{ce}^d - A^a A^b A^c A^e \Gamma_{ab}^f \partial_f \Gamma_{ce}^d - 4A^a A^b A^c A^e \Gamma_{ab}^f \partial_c \Gamma_{ef}^d \\ & - 2A^a A^b A^c A^e \Gamma_{af}^d \partial_b \Gamma_{ce}^f + 2A^a A^b A^c A^e \Gamma_{fg}^d \Gamma_{ab}^f \Gamma_{ce}^g + 4A^a A^b A^c A^e \Gamma_{af}^d \Gamma_{bg}^f \Gamma_{ce}^g \end{aligned}$$

$$\begin{aligned}
\text{eq3.010} := & A^a A^b A^c A^e A^f \partial_{abc} \Gamma_{ef}^d - A^a A^b A^c A^e A^f \partial_g \Gamma_{ab}^d \partial_c \Gamma_{ef}^g - 6 A^a A^b A^c A^e A^f \partial_a \Gamma_{bg}^d \partial_c \Gamma_{ef}^g - 3 A^a A^b A^c A^e A^f \Gamma_{ab}^g \partial_{cg} \Gamma_{ef}^d \\
& - 6 A^a A^b A^c A^e A^f \Gamma_{ab}^g \partial_{ce} \Gamma_{fg}^d - 2 A^a A^b A^c A^e A^f \Gamma_{ag}^d \partial_{bc} \Gamma_{ef}^g + 2 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{cg}^h \partial_h \Gamma_{ef}^d + 6 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{ce}^h \partial_g \Gamma_{fh}^d \\
& + 12 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{cg}^h \partial_e \Gamma_{fh}^d + 6 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{ce}^h \partial_f \Gamma_{gh}^d + 6 A^a A^b A^c A^e A^f \Gamma_{gh}^d \Gamma_{ab}^g \partial_c \Gamma_{ef}^h \\
& + 2 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bc}^h \partial_h \Gamma_{ef}^g + 8 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bc}^h \partial_e \Gamma_{fh}^g + 4 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bh}^g \partial_c \Gamma_{ef}^h \\
& - 12 A^a A^b A^c A^e A^f \Gamma_{gh}^d \Gamma_{ab}^g \Gamma_{ci}^h \Gamma_{ef}^i - 4 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{hi}^g \Gamma_{bc}^h \Gamma_{ef}^i - 8 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bh}^g \Gamma_{ci}^h \Gamma_{ef}^i
\end{aligned}$$

$$\begin{aligned}
\text{eq4.010} := & A^a A^b A^c A^e A^f A^g \partial_{abce} \Gamma^d_{fg} - 4A^a A^b A^c A^e A^f A^g \partial_a \Gamma^h_{bc} \partial_{eh} \Gamma^d_{fg} - A^a A^b A^c A^e A^f A^g \partial_h \Gamma^d_{ab} \partial_{ce} \Gamma^h_{fg} - 12A^a A^b A^c A^e A^f A^g \partial_a \Gamma^h_{bc} \partial_{ef} \Gamma^d_{gh} \\
& - 8A^a A^b A^c A^e A^f A^g \partial_a \Gamma^d_{bh} \partial_{ce} \Gamma^h_{fg} - 6A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_{ceh} \Gamma^d_{fg} - 8A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_{cef} \Gamma^d_{gh} \\
& + 8A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_i \Gamma^d_{ch} \partial_e \Gamma^i_{fg} + A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_i \Gamma^d_{ce} \partial_h \Gamma^i_{fg} + 4A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_i \Gamma^d_{ce} \partial_f \Gamma^i_{gh} \\
& + 12A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_h \Gamma^d_{ci} \partial_e \Gamma^i_{fg} + 24A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_c \Gamma^d_{hi} \partial_e \Gamma^i_{fg} + 8A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_c \Gamma^d_{ei} \partial_h \Gamma^i_{fg} \\
& + 32A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \partial_c \Gamma^d_{ei} \partial_f \Gamma^i_{gh} - 2A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \partial_{bce} \Gamma^h_{fg} + 2A^a A^b A^c A^e A^f A^g \Gamma^h_{ai} \partial_h \Gamma^d_{bc} \partial_e \Gamma^i_{fg} \\
& + 16A^a A^b A^c A^e A^f A^g \Gamma^h_{ai} \partial_b \Gamma^d_{ch} \partial_e \Gamma^i_{fg} + 6A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \partial_a \Gamma^h_{bc} \partial_e \Gamma^i_{fg} + 2A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \partial_b \Gamma^i_{ce} \partial_i \Gamma^h_{fg} \\
& + 12A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \partial_b \Gamma^h_{ci} \partial_e \Gamma^i_{fg} + 8A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ch} \partial_{ei} \Gamma^d_{fg} + 3A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \partial_{hi} \Gamma^d_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \partial_{fh} \Gamma^d_{gi} + 24A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ch} \partial_{ef} \Gamma^d_{gi} + 12A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \partial_{fg} \Gamma^d_{hi} \\
& + 8A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{ab} \partial_{ce} \Gamma^i_{fg} + 6A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^i_{bc} \partial_{ei} \Gamma^h_{fg} + 12A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^i_{bc} \partial_{ef} \Gamma^h_{gi} \\
& + 4A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{bi} \partial_{ce} \Gamma^i_{fg} - 4A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ch} \Gamma^j_{ei} \partial_j \Gamma^d_{fg} - 2A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \Gamma^j_{hi} \partial_j \Gamma^d_{fg} \\
& - 16A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \Gamma^j_{fh} \partial_j \Gamma^d_{gi} - 24A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \Gamma^j_{fh} \partial_i \Gamma^d_{gj} - 12A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \Gamma^j_{fg} \partial_h \Gamma^d_{ij} \\
& - 32A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ch} \Gamma^j_{ei} \partial_f \Gamma^d_{gj} - 16A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \Gamma^j_{hi} \partial_f \Gamma^d_{gj} - 48A^a A^b A^c A^e A^f A^g \Gamma^h_{ab} \Gamma^i_{ce} \Gamma^j_{fh} \partial_g \Gamma^d_{ij} \\
& - 24A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{aj} \Gamma^j_{bc} \partial_e \Gamma^i_{fg} - 8A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{ab} \Gamma^j_{ce} \partial_j \Gamma^i_{fg} - 32A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{ab} \Gamma^j_{ce} \partial_f \Gamma^i_{gj} \\
& - 4A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^i_{bc} \Gamma^j_{ei} \partial_j \Gamma^h_{fg} - 12A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^i_{bc} \Gamma^j_{ef} \partial_i \Gamma^h_{gj} - 24A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^i_{bc} \Gamma^j_{ei} \partial_f \Gamma^h_{gj} \\
& - 12A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^i_{bc} \Gamma^j_{ef} \partial_g \Gamma^h_{ij} - 16A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{ab} \Gamma^i_{cj} \partial_e \Gamma^j_{fg} - 12A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{ij} \Gamma^i_{bc} \partial_e \Gamma^j_{fg} \\
& - 4A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{bi} \Gamma^j_{ce} \partial_j \Gamma^i_{fg} - 16A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{bi} \Gamma^j_{ce} \partial_f \Gamma^i_{gj} - 8A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{bi} \Gamma^j_{cj} \partial_e \Gamma^j_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{aj} \Gamma^i_{bk} \Gamma^j_{ce} \Gamma^k_{fg} + 16A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{ab} \Gamma^i_{jk} \Gamma^j_{ce} \Gamma^k_{fg} + 32A^a A^b A^c A^e A^f A^g \Gamma^d_{hi} \Gamma^h_{ab} \Gamma^i_{cj} \Gamma^j_{ek} \Gamma^k_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{ij} \Gamma^i_{bc} \Gamma^j_{ek} \Gamma^k_{fg} + 8A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{bi} \Gamma^i_{jk} \Gamma^j_{ce} \Gamma^k_{fg} + 16A^a A^b A^c A^e A^f A^g \Gamma^d_{ah} \Gamma^h_{bi} \Gamma^i_{cj} \Gamma^j_{ek} \Gamma^k_{fg}
\end{aligned}$$

## Stage 2: The fixed point scheme for $y^a$

```
{x^{a}}::Weight(label=eps,value=0).

{y00^{a},y10^{a},y20^{a},y30^{a},y40^{a}}::Weight(label=eps,value=0).
{y11^{a},y21^{a},y31^{a},y41^{a}}::Weight(label=eps,value=1).
{y22^{a},y32^{a},y42^{a}}::Weight(label=eps,value=2).
{y33^{a},y43^{a}}::Weight(label=eps,value=3).
{y44^{a}}::Weight(label=eps,value=4).

{Gam11^{a}_{b c}}::Weight(label=eps,value=1).
{Gam22^{a}_{b c d}}::Weight(label=eps,value=2).
{Gam33^{a}_{b c d e}}::Weight(label=eps,value=3).
{Gam44^{a}_{b c d e f}}::Weight(label=eps,value=4).
{Gam55^{a}_{b c d e f g}}::Weight(label=eps,value=5).

Gam11^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
Gam22^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
Gam33^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
Gam44^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
Gam55^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

y00{#}::LaTeXForm ("\\ny{00}").
y10{#}::LaTeXForm ("\\ny{10}").
y20{#}::LaTeXForm ("\\ny{20}").
y30{#}::LaTeXForm ("\\ny{30}").
y40{#}::LaTeXForm ("\\ny{40}").
y11{#}::LaTeXForm ("\\ny{11}").
y21{#}::LaTeXForm ("\\ny{21}").
y31{#}::LaTeXForm ("\\ny{31}").
y41{#}::LaTeXForm ("\\ny{41}").
y22{#}::LaTeXForm ("\\ny{22}").
y32{#}::LaTeXForm ("\\ny{32}").
y42{#}::LaTeXForm ("\\ny{42}").
y33{#}::LaTeXForm ("\\ny{33}").
y43{#}::LaTeXForm ("\\ny{43}").
y44{#}::LaTeXForm ("\\ny{44}").
```



```

Gam11{#}::LaTeXForm ("\\nGamma{11}").
Gam22{#}::LaTeXForm ("\\nGamma{22}").
Gam33{#}::LaTeXForm ("\\nGamma{33}").
Gam44{#}::LaTeXForm ("\\nGamma{44}").
Gam55{#}::LaTeXForm ("\\nGamma{55}").

```

```

def get_term (obj,n):

```

```

    foo := @(obj).
    bah = Ex("eps = " + str(n))
    distribute (foo)
    keep_weight (foo, bah)

```

```

    return foo

```

```

def truncateEps (obj,n):

```

```

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah = Ex("eps = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

```

```

    return ans

```

```

def substitute_eps (obj):

```

```

    substitute (obj,epsy0)
    substitute (obj,epsy1)
    substitute (obj,epsy2)
    substitute (obj,epsy3)
    substitute (obj,epsy4)
    substitute (obj,epsGam1)
    substitute (obj,epsGam2)
    substitute (obj,epsGam3)
    substitute (obj,epsGam4)
    substitute (obj,epsGam5)

```

```

    distribute      (obj)
    obj = truncateEps (obj,4)
    obj = product_sort (obj)
    rename_dummies  (obj)
    canonicalise    (obj)

    return obj

def tidy (obj):
    obj = product_sort (obj)
    rename_dummies (obj)
    canonicalise    (obj)
    return obj

beg_stage_2 = time.time()

y0 := x^{a}.
y1 := x^{a} + (1/2) Gam^{a}_{b c} y0^{b} y0^{c}.
y2 := x^{a} + (1/2) Gam^{a}_{b c} y1^{b} y1^{c}
      + (1/6) Gam^{a}_{b c d} y0^{b} y0^{c} y0^{d}.
y3 := x^{a} + (1/2) Gam^{a}_{b c} y2^{b} y2^{c}
      + (1/6) Gam^{a}_{b c d} y1^{b} y1^{c} y1^{d}
      + (1/24) Gam^{a}_{b c d e} y0^{b} y0^{c} y0^{d} y0^{e}.
y4 := x^{a} + (1/2) Gam^{a}_{b c} y3^{b} y3^{c}
      + (1/6) Gam^{a}_{b c d} y2^{b} y2^{c} y2^{d}
      + (1/24) Gam^{a}_{b c d e} y1^{b} y1^{c} y1^{d} y1^{e}
      + (1/120) Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

# note that:
# y00 = y10 = y20 = y30 = y40
# y11 = y21 = y31 = y41
# y22 = y32 = y42
# y33 = y43
# y44

# expand each y in powers of eps

epsy0 := y0^{a} -> y00^{a}.

```

```

epsy1 := y1^{a} -> y10^{a}+y11^{a}.
epsy2 := y2^{a} -> y20^{a}+y21^{a}+y22^{a}.
epsy3 := y3^{a} -> y30^{a}+y31^{a}+y32^{a}+y33^{a}.
epsy4 := y4^{a} -> y40^{a}+y41^{a}+y42^{a}+y43^{a}+y44^{a}.

epsGam1 := Gam^{a}_{b c} -> Gam11^{a}_{b c}.
epsGam2 := Gam^{a}_{b c d} -> Gam22^{a}_{b c d}.
epsGam3 := Gam^{a}_{b c d e} -> Gam33^{a}_{b c d e}.
epsGam4 := Gam^{a}_{b c d e f} -> Gam44^{a}_{b c d e f}.
epsGam5 := Gam^{a}_{b c d e f g} -> Gam55^{a}_{b c d e f g}.

y0 = substitute_eps (y0)      # cdb (y0.001,y0)
y1 = substitute_eps (y1)      # cdb (y1.001,y1)
y2 = substitute_eps (y2)      # cdb (y2.001,y2)
y3 = substitute_eps (y3)      # cdb (y3.001,y3)
y4 = substitute_eps (y4)      # cdb (y4.001,y4)

defy0 := y0^{a} -> @(y0).
defy1 := y1^{a} -> @(y1).
defy2 := y2^{a} -> @(y2).
defy3 := y3^{a} -> @(y3).
defy4 := y4^{a} -> @(y4).

# -----
# y0

y00 := @(y0).                # cdb (y00.101,y00)

defy00 := y00^{a} -> @(y00).

# -----
# y1

substitute (y1,defy00)

distribute (y1)

y10 = get_term (y1,0)        # cdb (y10.101,y10)

```

```

y11 = get_term (y1,1)    # cdb (y11.101,y11)

defy10 := y10^{a} -> @(y10).
defy11 := y11^{a} -> @(y11).

# -----
# y2

substitute (y2,defy00)

substitute (y2,defy10)
substitute (y2,defy11)

distribute (y2)

y20 = get_term (y2,0)    # cdb (y20.101,y20)
y21 = get_term (y2,1)    # cdb (y21.101,y21)
y22 = get_term (y2,2)    # cdb (y22.101,y22)

y20 = tidy (y20)        # cdb (y20.201,y20)
y21 = tidy (y21)        # cdb (y21.201,y21)
y22 = tidy (y22)        # cdb (y22.201,y22)

defy20 := y20^{a} -> @(y20).
defy21 := y21^{a} -> @(y21).
defy22 := y22^{a} -> @(y22).

# -----
# y3

substitute (y3,defy00)

substitute (y3,defy10)
substitute (y3,defy11)

substitute (y3,defy20)
substitute (y3,defy21)
substitute (y3,defy22)

```

```

distribute (y3)

y30 = get_term (y3,0)    # cdb (y30.101,y30)
y31 = get_term (y3,1)    # cdb (y31.101,y31)
y32 = get_term (y3,2)    # cdb (y32.101,y32)
y33 = get_term (y3,3)    # cdb (y33.101,y33)

y30 = tidy (y30)        # cdb (y30.201,y30)
y31 = tidy (y31)        # cdb (y31.201,y31)
y32 = tidy (y32)        # cdb (y32.201,y32)
y33 = tidy (y33)        # cdb (y33.201,y33)

defy30 := y30^{a} -> @(y30).
defy31 := y31^{a} -> @(y31).
defy32 := y32^{a} -> @(y32).
defy33 := y33^{a} -> @(y33).

# -----
# y4

substitute (y4,defy00)

substitute (y4,defy10)
substitute (y4,defy11)

substitute (y4,defy20)
substitute (y4,defy21)
substitute (y4,defy22)

substitute (y4,defy30)
substitute (y4,defy31)
substitute (y4,defy32)
substitute (y4,defy33)

distribute (y4)

y40 = get_term (y4,0)    # cdb (y40.101,y40)

```

```

y41 = get_term (y4,1)    # cdb (y41.101,y41)
y42 = get_term (y4,2)    # cdb (y42.101,y42)
y43 = get_term (y4,3)    # cdb (y43.101,y43)
y44 = get_term (y4,4)    # cdb (y44.101,y44)

y40 = tidy (y40)        # cdb (y40.201,y40)
y41 = tidy (y41)        # cdb (y41.201,y41)
y42 = tidy (y42)        # cdb (y42.201,y42)
y43 = tidy (y43)        # cdb (y43.201,y43)
y44 = tidy (y44)        # cdb (y44.201,y44)

defy40 := y40^{a} -> @(y40).
defy41 := y41^{a} -> @(y41).
defy42 := y42^{a} -> @(y42).
defy43 := y43^{a} -> @(y43).
defy44 := y44^{a} -> @(y44).

end_stage_2 = time.time()

```

$$\text{y1.001} := x^a + \frac{1}{2} \Gamma^a{}_{bc}{}^{00} b^{00} y^c$$

$$\text{y2.001} := x^a + \frac{1}{2} \Gamma^a{}_{bc}{}^{10} b^{10} y^c + \Gamma^a{}_{bc}{}^{10} b^{11} y^c + \frac{1}{6} \Gamma^a{}_{bcd}{}^{00} b^{00} y^c y^d + \frac{1}{2} \Gamma^a{}_{bc}{}^{11} b^{11} y^c$$

$$\text{y3.001} := x^a + \frac{1}{2} \Gamma^a{}_{bc}{}^{20} b^{20} y^c + \Gamma^a{}_{bc}{}^{20} b^{21} y^c + \frac{1}{6} \Gamma^a{}_{bcd}{}^{10} b^{10} y^c y^d + \Gamma^a{}_{bc}{}^{20} b^{22} y^c + \frac{1}{2} \Gamma^a{}_{bc}{}^{21} b^{21} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{10} b^{10} y^c y^d + \frac{1}{24} \Gamma^a{}_{bcde}{}^{00} b^{00} y^c y^d y^e + \Gamma^a{}_{bc}{}^{21} b^{22} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{10} b^{11} y^c y^d$$

$$\begin{aligned} \text{y4.001} := x^a &+ \frac{1}{2} \Gamma^a{}_{bc}{}^{30} b^{30} y^c + \Gamma^a{}_{bc}{}^{30} b^{31} y^c + \frac{1}{6} \Gamma^a{}_{bcd}{}^{20} b^{20} y^c y^d + \Gamma^a{}_{bc}{}^{30} b^{32} y^c + \frac{1}{2} \Gamma^a{}_{bc}{}^{31} b^{31} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{20} b^{20} y^c y^d + \frac{1}{24} \Gamma^a{}_{bcde}{}^{10} b^{10} y^c y^d y^e \\ &+ \Gamma^a{}_{bc}{}^{30} b^{33} y^c + \Gamma^a{}_{bc}{}^{31} b^{32} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{20} b^{20} y^c y^d + \frac{1}{2} \Gamma^a{}_{bcd}{}^{21} b^{21} y^c y^d + \frac{1}{6} \Gamma^a{}_{bcde}{}^{10} b^{10} y^c y^d y^e + \frac{1}{120} \Gamma^a{}_{bcdef}{}^{00} b^{00} y^c y^d y^e y^f \end{aligned}$$

$$y_{10.101} := x^a$$

$$y_{11.101} := \frac{1}{2} \Gamma_{bc}^{11a} x^b x^c$$

$$y_{20.201} := x^a$$

$$y_{21.201} := \frac{1}{2} x^b x^c \Gamma_{bc}^{11a}$$

$$y_{22.201} := \frac{1}{2} x^b x^c x^d \Gamma_{be}^{11a} \Gamma_{cd}^{11e} + \frac{1}{6} x^b x^c x^d \Gamma_{bcd}^{22a}$$

$$y_{30.201} := x^a$$

$$y_{31.201} := \frac{1}{2} x^b x^c \Gamma_{bc}^{11a}$$

$$y_{32.201} := \frac{1}{2} x^b x^c x^d \Gamma_{be}^{11a} \Gamma_{cd}^{11e} + \frac{1}{6} x^b x^c x^d \Gamma_{bcd}^{22a}$$

$$y_{33.201} := \frac{1}{2} x^b x^c x^d x^e \Gamma_{bf}^{11a} \Gamma_{cg}^{11f} \Gamma_{de}^{11g} + \frac{1}{6} x^b x^c x^d x^e \Gamma_{bf}^{11a} \Gamma_{cde}^{22f} + \frac{1}{8} x^b x^c x^d x^e \Gamma_{fg}^{11a} \Gamma_{bc}^{11f} \Gamma_{de}^{11g} + \frac{1}{4} x^b x^c x^d x^e \Gamma_{bc}^{11f} \Gamma_{def}^{22a} + \frac{1}{24} x^b x^c x^d x^e \Gamma_{bcde}^{33a}$$



$$\text{y40.201} := x^a$$

$$\text{y41.201} := \frac{1}{2}x^bx^c\Gamma_{bc}^{11a}$$

$$\text{y42.201} := \frac{1}{2}x^bx^cx^d\Gamma_{be}^{11a}\Gamma_{cd}^{11e} + \frac{1}{6}x^bx^cx^d\Gamma_{bcd}^{22a}$$

$$\text{y43.201} := \frac{1}{2}x^bx^cx^dx^e\Gamma_{bf}^{11a}\Gamma_{cg}^{11f}\Gamma_{de}^{11g} + \frac{1}{6}x^bx^cx^dx^e\Gamma_{bf}^{11a}\Gamma_{cde}^{22f} + \frac{1}{8}x^bx^cx^dx^e\Gamma_{fg}^{11a}\Gamma_{bc}^{11f}\Gamma_{de}^{11g} + \frac{1}{4}x^bx^cx^dx^e\Gamma_{bc}^{11f}\Gamma_{def}^{22a} + \frac{1}{24}x^bx^cx^dx^e\Gamma_{bcde}^{33a}$$

$$\begin{aligned} \text{y44.201} := & \frac{1}{2}x^bx^cx^dx^ex^f\Gamma_{bg}^{11a}\Gamma_{ch}^{11g}\Gamma_{di}^{11h}\Gamma_{ef}^{11i} + \frac{1}{6}x^bx^cx^dx^ex^f\Gamma_{bg}^{11a}\Gamma_{ch}^{11g}\Gamma_{def}^{22h} + \frac{1}{8}x^bx^cx^dx^ex^f\Gamma_{bg}^{11a}\Gamma_{hi}^{11g}\Gamma_{cd}^{11h}\Gamma_{ef}^{11i} + \frac{1}{4}x^bx^cx^dx^ex^f\Gamma_{bg}^{11a}\Gamma_{cd}^{11h}\Gamma_{efh}^{22g} \\ & + \frac{1}{24}x^bx^cx^dx^ex^f\Gamma_{bg}^{11a}\Gamma_{cdef}^{33g} + \frac{1}{4}x^bx^cx^dx^ex^f\Gamma_{gh}^{11a}\Gamma_{bc}^{11g}\Gamma_{di}^{11h}\Gamma_{ef}^{11i} + \frac{1}{12}x^bx^cx^dx^ex^f\Gamma_{gh}^{11a}\Gamma_{bc}^{11g}\Gamma_{def}^{22h} + \frac{1}{4}x^bx^cx^dx^ex^f\Gamma_{bc}^{11g}\Gamma_{dgh}^{11h}\Gamma_{efh}^{22a} \\ & + \frac{1}{12}x^bx^cx^dx^ex^f\Gamma_{bcg}^{22a}\Gamma_{def}^{22g} + \frac{1}{8}x^bx^cx^dx^ex^f\Gamma_{bc}^{11g}\Gamma_{de}^{11h}\Gamma_{fgh}^{22a} + \frac{1}{12}x^bx^cx^dx^ex^f\Gamma_{bc}^{11g}\Gamma_{defg}^{33a} + \frac{1}{120}x^bx^cx^dx^ex^f\Gamma_{bcdef}^{44a} \end{aligned}$$

## Stage3: Introduce the generalised connections from Stage 1

```
def substitute_gam (obj):
    substitute (obj,defGam11)
    substitute (obj,defGam22)
    substitute (obj,defGam33)
    substitute (obj,defGam44)
    substitute (obj,defGam55)
    distribute (obj)

    return obj

beg_stage_3 = time.time()

Gam11 := @(eq0).
Gam22 := @(eq1).
Gam33 := @(eq2).
Gam44 := @(eq3).
Gam55 := @(eq4).

# peel off the  $A^{\{a\}}$ , must then symmetrise over revealed indices

substitute (Gam11,$A^{\{a\}}\rightarrow 1$)
substitute (Gam22,$A^{\{a\}}\rightarrow 1$)
substitute (Gam33,$A^{\{a\}}\rightarrow 1$)
substitute (Gam44,$A^{\{a\}}\rightarrow 1$)
substitute (Gam55,$A^{\{a\}}\rightarrow 1$)

# now symmetrise

sym (Gam11,$_{\{a\}},_{\{b\}}$)
sym (Gam22,$_{\{a\}},_{\{b\}},_{\{c\}}$)
sym (Gam33,$_{\{a\}},_{\{b\}},_{\{c\}},_{\{e\}}$)
sym (Gam44,$_{\{a\}},_{\{b\}},_{\{c\}},_{\{e\}},_{\{f\}}$)
sym (Gam55,$_{\{a\}},_{\{b\}},_{\{c\}},_{\{e\}},_{\{f\}},_{\{g\}}$)

defGam11 := Gam11^{\{d\}}_{\{a\} b} -> @(Gam11).
defGam22 := Gam22^{\{d\}}_{\{a\} b c} -> @(Gam22).
```

```

defGam33 := Gam33^{d}_{a b c e} -> @(Gam33).
defGam44 := Gam44^{d}_{a b c e f} -> @(Gam44).
defGam55 := Gam55^{d}_{a b c e f g} -> @(Gam55).

y31 = substitute_gam (y31)
y32 = substitute_gam (y32)
y33 = substitute_gam (y33)

y31 = tidy (y31)      # cdb (y31.301,y31)
y32 = tidy (y32)      # cdb (y32.301,y32)
y33 = tidy (y33)      # cdb (y33.301,y33)

y3 := @(y30) + @(y31) + @(y32) + @(y33).

y41 = substitute_gam (y41)
y42 = substitute_gam (y42)
y43 = substitute_gam (y43)
y44 = substitute_gam (y44)

y41 = tidy (y41)      # cdb (y41.301,y41)
y42 = tidy (y42)      # cdb (y42.301,y42)
y43 = tidy (y43)      # cdb (y43.301,y43)
y44 = tidy (y44)      # cdb (y44.301,y44)

y4 := @(y40) + @(y41) + @(y42) + @(y43) + @(y44).

end_stage_3 = time.time()

```

$$\text{y30.201} := x^a$$

$$\text{y31.301} := \frac{1}{2}x^b x^c \Gamma^a_{bc}$$

$$\text{y32.301} := \frac{1}{6}x^b x^c x^d \Gamma^a_{be} \Gamma^e_{cd} + \frac{1}{6}x^b x^c x^d \partial_b \Gamma^a_{cd}$$

$$\text{y33.301} := \frac{1}{12}x^b x^c x^d x^e \Gamma^a_{bf} \partial_c \Gamma^f_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^a_{fg} \Gamma^f_{bc} \Gamma^g_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^f_{bc} \partial_f \Gamma^a_{de} + \frac{1}{24}x^b x^c x^d x^e \partial_{bc} \Gamma^a_{de}$$

$$\text{y40.201} := x^a$$

$$\text{y41.301} := \frac{1}{2}x^b x^c \Gamma^a_{bc}$$

$$\text{y42.301} := \frac{1}{6}x^b x^c x^d \Gamma^a_{be} \Gamma^e_{cd} + \frac{1}{6}x^b x^c x^d \partial_b \Gamma^a_{cd}$$

$$\text{y43.301} := \frac{1}{12}x^b x^c x^d x^e \Gamma^a_{bf} \partial_c \Gamma^f_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^a_{fg} \Gamma^f_{bc} \Gamma^g_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^f_{bc} \partial_f \Gamma^a_{de} + \frac{1}{24}x^b x^c x^d x^e \partial_{bc} \Gamma^a_{de}$$

$$\begin{aligned} \text{y44.301} := & -\frac{1}{90}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^g_{ch} \Gamma^h_{di} \Gamma^i_{ef} + \frac{1}{180}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^g_{ch} \partial_d \Gamma^h_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^g_{hi} \Gamma^h_{cd} \Gamma^i_{ef} \\ & - \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^h_{cd} \partial_e \Gamma^g_{fh} + \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^h_{cd} \partial_h \Gamma^g_{ef} + \frac{1}{40}x^b x^c x^d x^e x^f \Gamma^a_{bg} \partial_{cd} \Gamma^g_{ef} + \frac{1}{90}x^b x^c x^d x^e x^f \Gamma^a_{gh} \Gamma^g_{bc} \Gamma^h_{di} \Gamma^i_{ef} \\ & + \frac{13}{360}x^b x^c x^d x^e x^f \Gamma^a_{gh} \Gamma^g_{bc} \partial_d \Gamma^h_{ef} - \frac{1}{90}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{dg} \partial_e \Gamma^a_{fh} + \frac{1}{360}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{dg} \partial_h \Gamma^a_{ef} \\ & + \frac{1}{180}x^b x^c x^d x^e x^f \partial_b \Gamma^a_{cg} \partial_d \Gamma^g_{ef} + \frac{7}{360}x^b x^c x^d x^e x^f \partial_g \Gamma^a_{bc} \partial_d \Gamma^g_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{de} \partial_f \Gamma^a_{gh} \\ & + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{de} \partial_g \Gamma^a_{fh} - \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g_{bc} \partial_{de} \Gamma^a_{fg} + \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^g_{bc} \partial_{dg} \Gamma^a_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \partial_{bcd} \Gamma^a_{ef} \end{aligned}$$

## Stage4: Reformatting and output

```
{x^{a}}>::Weight(label=numx).
\Gamma^{a}_{b c}>::TableauSymmetry(shape={2}, indices={1,2}).

def reformat (obj,scale):

    bah = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute      (tmp)
    tmp = product_sort (tmp)
    rename_dummies  (tmp)
    canonicalise    (tmp)
    factor_out      (tmp,$x^{a?}$)

    return tmp

def get_term (obj,n):

    tmp := @(obj).
    foo = Ex("numx = " + str(n))
    distribute      (tmp)
    keep_weight     (tmp, foo)

    return tmp

beg_stage_4 = time.time()

rnc := x^{a}
      + @(y41)
      + @(y42)
      + @(y43)
      + @(y44).

# substitute (rnc,$A^{a}->x^{a}$)

rnc1 = get_term (rnc,1)          # cdb (rnc1.001,rnc1)
rnc2 = get_term (rnc,2)          # cdb (rnc2.001,rnc2)
```

```

rnc3 = get_term (rnc,3)          # cdb (rnc3.001,rnc3)
rnc4 = get_term (rnc,4)          # cdb (rnc4.001,rnc4)
rnc5 = get_term (rnc,5)          # cdb (rnc5.001,rnc5)

scaled1 = reformat (rnc1,  1)    # cdb (scaled1.002,scaled1)
scaled2 = reformat (rnc2,  2)    # cdb (scaled2.002,scaled2)
scaled3 = reformat (rnc3,  6)    # cdb (scaled3.002,scaled3)
scaled4 = reformat (rnc4, 24)    # cdb (scaled4.002,scaled4)
scaled5 = reformat (rnc5, 360)   # cdb (scaled5.002,scaled5)

import cdblib

cdblib.create ('gen2rnc.json')

cdblib.put ('rnc',rnc,'gen2rnc.json')

cdblib.put ('rnc1',rnc1,'gen2rnc.json')
cdblib.put ('rnc2',rnc2,'gen2rnc.json')
cdblib.put ('rnc3',rnc3,'gen2rnc.json')
cdblib.put ('rnc4',rnc4,'gen2rnc.json')
cdblib.put ('rnc5',rnc5,'gen2rnc.json')

end_stage_4 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:.7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)

```

# Timing

Stage 1: 14.4 secs  
Stage 2: 3.1 secs  
Stage 3: 448.7 secs  
Stage 4: 1.7 secs



## Convert from generic (x) to local RNC coords (y)

$$y^a = {}^0y^a + {}^1y^a + {}^2y^a + {}^3y^a + {}^4y^a$$

$${}^0y^a = x^a$$

$$2{}^1y^a = x^b x^c \Gamma^a_{bc}$$

$$6{}^2y^a = x^b x^c x^d (\Gamma^a_{be} \Gamma^e_{cd} + \partial_b \Gamma^a_{cd})$$

$$24{}^3y^a = x^b x^c x^d x^e (2\Gamma^a_{bf} \partial_c \Gamma^f_{de} + \Gamma^a_{fg} \Gamma^f_{bc} \Gamma^g_{de} + \Gamma^f_{bc} \partial_f \Gamma^a_{de} + \partial_{bc} \Gamma^a_{de})$$

$$\begin{aligned} 360{}^4y^a = & x^b x^c x^d x^e x^f (-4\Gamma^a_{bg} \Gamma^g_{ch} \Gamma^h_{di} \Gamma^i_{ef} + 2\Gamma^a_{bg} \Gamma^g_{ch} \partial_d \Gamma^h_{ef} + 3\Gamma^a_{bg} \Gamma^g_{hi} \Gamma^h_{cd} \Gamma^i_{ef} - 6\Gamma^a_{bg} \Gamma^h_{cd} \partial_e \Gamma^g_{fh} + 6\Gamma^a_{bg} \Gamma^h_{cd} \partial_h \Gamma^g_{ef} + 9\Gamma^a_{bg} \partial_{cd} \Gamma^g_{ef} \\ & + 4\Gamma^a_{gh} \Gamma^g_{bc} \Gamma^h_{di} \Gamma^i_{ef} + 13\Gamma^a_{gh} \Gamma^g_{bc} \partial_d \Gamma^h_{ef} - 4\Gamma^g_{bc} \Gamma^h_{dg} \partial_e \Gamma^a_{fh} + \Gamma^g_{bc} \Gamma^h_{dg} \partial_h \Gamma^a_{ef} + 2\partial_b \Gamma^a_{cg} \partial_d \Gamma^g_{ef} + 7\partial_g \Gamma^a_{bc} \partial_d \Gamma^g_{ef} + 3\Gamma^g_{bc} \Gamma^h_{de} \partial_f \Gamma^a_{gh} \\ & + 3\Gamma^g_{bc} \Gamma^h_{de} \partial_g \Gamma^a_{fh} - 3\Gamma^g_{bc} \partial_{de} \Gamma^a_{fg} + 6\Gamma^g_{bc} \partial_{dg} \Gamma^a_{ef} + 3\partial_{bcd} \Gamma^a_{ef}) \end{aligned}$$

```
cdblib.create ('gen2rnc.export')

# 6th order terms, scaled
cdblib.put ('rnc61scaled',scaled1,'gen2rnc.export')
cdblib.put ('rnc62scaled',scaled2,'gen2rnc.export')
cdblib.put ('rnc63scaled',scaled3,'gen2rnc.export')
cdblib.put ('rnc64scaled',scaled4,'gen2rnc.export')
cdblib.put ('rnc65scaled',scaled5,'gen2rnc.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```

# Convert from rnc to generic coordinates

The following code is based on the `gen2rnc.tex` code.

It is common to do some computations in a local RNC. Doing so makes various parts of the computations much easier to manage than in the original non-RNC coordinates. One simple example is the proof of the second Bianchi identities.

This code develops the inverse transformation, that is from the local RNC coordinates back to generic coordinates. The key equation (drawn from `gen2rnc.tex`) is

$$x_j^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{b_k}^a y^{b_k} \quad (1)$$

In `gen2rnc.tex` this equation was solved for the RNC coordinates  $y$  given the generic coordinates  $x_j$  and  $x_i$ . Here we will instead take  $x_i$  and  $y$  as given and use this equation to compute  $x_j$ . The first change we will make is to replace  $x_j$  with  $x$  (as the subscript  $j$  serves no useful purpose).

Thus our job will be to compute

$$x^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{b_k}^a y^{b_k} \quad (2)$$

given  $x_i$  and  $y$ . The generalised connections will be computed recursively by

$$\Gamma_{bcd}^a = \Gamma_{(b\bar{c},d)}^a - (n+1)\Gamma_{p(\bar{c}}^a \Gamma_{bd)}^p \quad (3)$$

As noted in `gen2rnc.tex`, the generalised connections will scale with the expansions parameter  $\epsilon$  according to

$$\Gamma_{bc}^a = \mathcal{O}(\epsilon) \ , \quad \Gamma_{bcd}^a = \mathcal{O}(\epsilon^2) \ , \quad \Gamma_{bcde}^a = \mathcal{O}(\epsilon^3) \ , \quad \text{etc.}$$

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

A^{a}::Depends(\partial{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).

Q^{a}_{b c}::Depends(\partial{#}).

Q^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
Q^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
Q^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
Q^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
Q^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

Q^{p}_{a b}::Weight(label=numQ,value=1).
Q^{p}_{a b c}::Weight(label=numQ,value=2).
Q^{p}_{a b c d}::Weight(label=numQ,value=3).
Q^{p}_{a b c d e}::Weight(label=numQ,value=4).
Q^{p}_{a b c d e f}::Weight(label=numQ,value=5).

def product_sort (obj):

```

```

substitute (obj,$ A^{a}          -> A001^{a}          $)
substitute (obj,$ x^{a}          -> A002^{a}          $)
substitute (obj,$ g^{a b}        -> A003^{a b}        $)
substitute (obj,$ Q^{p}_{a b}     -> A004^{p}_{a b}     $)
substitute (obj,$ Q^{p}_{a b c}    -> A005^{p}_{a b c}    $)
substitute (obj,$ Q^{p}_{a b c d}  -> A006^{p}_{a b c d}  $)
substitute (obj,$ Q^{p}_{a b c d e}-> A007^{p}_{a b c d e} $)
substitute (obj,$ Q^{p}_{a b c d e f}-> A008^{p}_{a b c d e f} $)
sort_product (obj)
rename_dummies (obj)
substitute (obj,$ A001^{a}        -> A^{a}          $)
substitute (obj,$ A002^{a}        -> x^{a}          $)
substitute (obj,$ A003^{a b}      -> g^{a b}        $)
substitute (obj,$ A004^{p}_{a b}   -> Q^{p}_{a b}     $)
substitute (obj,$ A005^{p}_{a b c} -> Q^{p}_{a b c}   $)
substitute (obj,$ A006^{p}_{a b c d}-> Q^{p}_{a b c d}  $)
substitute (obj,$ A007^{p}_{a b c d e}-> Q^{p}_{a b c d e} $)
substitute (obj,$ A008^{p}_{a b c d e f}-> Q^{p}_{a b c d e f} $)

return obj

def truncateQ (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah  = Ex("numQ = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    return ans

# A^{a} = dx^a/ds

Gamma := Q^{d}_{a b} A^{a} A^{b}.

dAds := A^{c} \partial_{c}{A^{d}}-> - @(Gamma).

```

```

# =====
eq0 := @(Gamma).                # cdb (eq0.000,eq0)

# =====
eq1 := A^{c} \partial_{c}{@(eq0)}.    # cdb (eq1.000,eq1)

distribute      (eq1)                # cdb (eq1.001,eq1)
unwrap          (eq1)                # cdb (eq1.002,eq1)
product_rule    (eq1)                # cdb (eq1.003,eq1)
distribute      (eq1)                # cdb (eq1.004,eq1)
substitute      (eq1,dAds)           # cdb (eq1.005,eq1)
distribute      (eq1)                # cdb (eq1.006,eq1)
eq1 = truncateQ (eq1,5)              # cdb (eq1.007,eq1)
sort_product    (eq1)                # cdb (eq1.008,eq1)
rename_dummies  (eq1)                # cdb (eq1.009,eq1)
canonicalise    (eq1)                # cdb (eq1.010,eq1)

# =====
eq2 := A^{c} \partial_{c}{@(eq1)}.    # cdb (eq2.000,eq2)

distribute      (eq2)                # cdb (eq2.001,eq2)
unwrap          (eq2)                # cdb (eq2.002,eq2)
product_rule    (eq2)                # cdb (eq2.003,eq2)
distribute      (eq2)                # cdb (eq2.004,eq2)
substitute      (eq2,dAds)           # cdb (eq2.005,eq2)
distribute      (eq2)                # cdb (eq2.006,eq2)
eq2 = truncateQ (eq2,5)              # cdb (eq2.007,eq2)
sort_product    (eq2)                # cdb (eq2.008,eq2)
rename_dummies  (eq2)                # cdb (eq2.009,eq2)
canonicalise    (eq2)                # cdb (eq2.010,eq2)

# =====
eq3 := A^{c} \partial_{c}{@(eq2)}.    # cdb (eq3.000,eq3)

distribute      (eq3)                # cdb (eq3.001,eq3)
unwrap          (eq3)                # cdb (eq3.002,eq3)
product_rule    (eq3)                # cdb (eq3.003,eq3)

```

```

distribute      (eq3)                # cdb (eq3.004,eq3)
substitute      (eq3,dAds)           # cdb (eq3.005,eq3)
distribute      (eq3)                # cdb (eq3.006,eq3)
eq3 = truncateQ (eq3,5)              # cdb (eq3.007,eq3)
sort_product    (eq3)                # cdb (eq3.008,eq3)
rename_dummies  (eq3)                # cdb (eq3.009,eq3)
canonicalise    (eq3)                # cdb (eq3.010,eq3)

# =====
eq4 := A^{c} \partial_{c}{@(eq3)}.    # cdb (eq4.000,eq4)

distribute      (eq4)                # cdb (eq4.001,eq4)
unwrap          (eq4)                # cdb (eq4.002,eq4)
product_rule    (eq4)                # cdb (eq4.003,eq4)
distribute      (eq4)                # cdb (eq4.004,eq4)
substitute      (eq4,dAds)           # cdb (eq4.005,eq4)
distribute      (eq4)                # cdb (eq4.006,eq4)
eq4 = truncateQ (eq4,5)              # cdb (eq4.007,eq4)
sort_product    (eq4)                # cdb (eq4.008,eq4)
rename_dummies  (eq4)                # cdb (eq4.009,eq4)
canonicalise    (eq4)                # cdb (eq4.010,eq4)

```

$$\text{eq0.000} := Q^d_{ab} A^a A^b$$



$$\text{eq1.000} := A^c \partial_c (Q^d_{ab} A^a A^b)$$

$$\text{eq1.001} := A^c \partial_c (Q^d_{ab} A^a A^b)$$

$$\text{eq1.002} := A^c \partial_c (Q^d_{ab} A^a A^b)$$

$$\text{eq1.003} := A^c (\partial_c Q^d_{ab} A^a A^b + Q^d_{ab} \partial_c A^a A^b + Q^d_{ab} A^a \partial_c A^b)$$

$$\text{eq1.004} := A^c \partial_c Q^d_{ab} A^a A^b + A^c Q^d_{ab} \partial_c A^a A^b + A^c Q^d_{ab} A^a \partial_c A^b$$

$$\text{eq1.005} := A^c \partial_c Q^d_{ab} A^a A^b - Q^a_{ce} A^c A^e Q^d_{ab} A^b - Q^b_{ec} A^e A^c Q^d_{ab} A^a$$

$$\text{eq1.006} := A^c \partial_c Q^d_{ab} A^a A^b - Q^a_{ce} A^c A^e Q^d_{ab} A^b - Q^b_{ec} A^e A^c Q^d_{ab} A^a$$

$$\text{eq1.007} := A^c \partial_c Q^d_{ab} A^a A^b - Q^a_{ce} A^c A^e Q^d_{ab} A^b - Q^b_{ec} A^e A^c Q^d_{ab} A^a$$

$$\text{eq1.008} := A^a A^b A^c \partial_c Q^d_{ab} - A^b A^c A^e Q^a_{ce} Q^d_{ab} - A^a A^c A^e Q^b_{ec} Q^d_{ab}$$

$$\text{eq1.009} := A^a A^b A^c \partial_c Q^d_{ab} - A^a A^b A^c Q^e_{bc} Q^d_{ea} - A^a A^b A^c Q^e_{cb} Q^d_{ae}$$

$$\text{eq1.010} := A^a A^b A^c \partial_a Q^d_{bc} - 2A^a A^b A^c Q^d_{ae} Q^e_{bc}$$

$$\text{eq2.000} := A^c \partial_c (A^a A^b A^f \partial_a Q^d_{bf} - 2A^a A^b A^f Q^d_{ae} Q^e_{bf})$$

$$\text{eq2.001} := A^c \partial_c (A^a A^b A^f \partial_a Q^d_{bf}) - 2A^c \partial_c (A^a A^b A^f Q^d_{ae} Q^e_{bf})$$

$$\text{eq2.002} := A^c \partial_c (A^a A^b A^f \partial_a Q^d_{bf}) - 2A^c \partial_c (A^a A^b A^f Q^d_{ae} Q^e_{bf})$$

$$\begin{aligned} \text{eq2.003} := & A^c (\partial_c A^a A^b A^f \partial_a Q^d_{bf} + A^a \partial_c A^b A^f \partial_a Q^d_{bf} + A^a A^b \partial_c A^f \partial_a Q^d_{bf} + A^a A^b A^f \partial_{ca} Q^d_{bf}) \\ & - 2A^c (\partial_c A^a A^b A^f Q^d_{ae} Q^e_{bf} + A^a \partial_c A^b A^f Q^d_{ae} Q^e_{bf} + A^a A^b \partial_c A^f Q^d_{ae} Q^e_{bf} + A^a A^b A^f \partial_c Q^d_{ae} Q^e_{bf} + A^a A^b A^f Q^d_{ae} \partial_c Q^e_{bf}) \end{aligned}$$

$$\begin{aligned} \text{eq2.004} := & A^c \partial_c A^a A^b A^f \partial_a Q^d_{bf} + A^c A^a \partial_c A^b A^f \partial_a Q^d_{bf} + A^c A^a A^b \partial_c A^f \partial_a Q^d_{bf} + A^c A^a A^b A^f \partial_{ca} Q^d_{bf} - 2A^c \partial_c A^a A^b A^f Q^d_{ae} Q^e_{bf} \\ & - 2A^c A^a \partial_c A^b A^f Q^d_{ae} Q^e_{bf} - 2A^c A^a A^b \partial_c A^f Q^d_{ae} Q^e_{bf} - 2A^c A^a A^b A^f \partial_c Q^d_{ae} Q^e_{bf} - 2A^c A^a A^b A^f Q^d_{ae} \partial_c Q^e_{bf} \end{aligned}$$

$$\begin{aligned} \text{eq2.005} := & -Q^a_{ce} A^c A^e A^b A^f \partial_a Q^d_{bf} - Q^b_{ec} A^e A^c A^a A^f \partial_a Q^d_{bf} - Q^f_{ce} A^c A^e A^a A^b \partial_a Q^d_{bf} + A^c A^a A^b A^f \partial_{ca} Q^d_{bf} + 2Q^a_{cg} A^c A^g A^b A^f Q^d_{ae} Q^e_{bf} \\ & + 2Q^b_{gc} A^g A^c A^a A^f Q^d_{ae} Q^e_{bf} + 2Q^f_{cg} A^c A^g A^a A^b Q^d_{ae} Q^e_{bf} - 2A^c A^a A^b A^f \partial_c Q^d_{ae} Q^e_{bf} - 2A^c A^a A^b A^f Q^d_{ae} \partial_c Q^e_{bf} \end{aligned}$$

$$\begin{aligned} \text{eq2.006} := & -Q^a_{ce} A^c A^e A^b A^f \partial_a Q^d_{bf} - Q^b_{ec} A^e A^c A^a A^f \partial_a Q^d_{bf} - Q^f_{ce} A^c A^e A^a A^b \partial_a Q^d_{bf} + A^c A^a A^b A^f \partial_{ca} Q^d_{bf} + 2Q^a_{cg} A^c A^g A^b A^f Q^d_{ae} Q^e_{bf} \\ & + 2Q^b_{gc} A^g A^c A^a A^f Q^d_{ae} Q^e_{bf} + 2Q^f_{cg} A^c A^g A^a A^b Q^d_{ae} Q^e_{bf} - 2A^c A^a A^b A^f \partial_c Q^d_{ae} Q^e_{bf} - 2A^c A^a A^b A^f Q^d_{ae} \partial_c Q^e_{bf} \end{aligned}$$

$$\begin{aligned} \text{eq2.007} := & A^c A^a A^b A^f \partial_{ca} Q^d_{bf} - Q^a_{ce} A^c A^e A^b A^f \partial_a Q^d_{bf} - Q^b_{ec} A^e A^c A^a A^f \partial_a Q^d_{bf} - Q^f_{ce} A^c A^e A^a A^b \partial_a Q^d_{bf} - 2A^c A^a A^b A^f \partial_c Q^d_{ae} Q^e_{bf} \\ & - 2A^c A^a A^b A^f Q^d_{ae} \partial_c Q^e_{bf} + 2Q^a_{cg} A^c A^g A^b A^f Q^d_{ae} Q^e_{bf} + 2Q^b_{gc} A^g A^c A^a A^f Q^d_{ae} Q^e_{bf} + 2Q^f_{cg} A^c A^g A^a A^b Q^d_{ae} Q^e_{bf} \end{aligned}$$

$$\begin{aligned} \text{eq2.008} := & A^a A^b A^c A^f \partial_{ca} Q^d_{bf} - A^b A^c A^e A^f Q^a_{ce} \partial_a Q^d_{bf} - A^a A^c A^e A^f Q^b_{ec} \partial_a Q^d_{bf} - A^a A^b A^c A^e Q^f_{ce} \partial_a Q^d_{bf} - 2A^a A^b A^c A^f Q^e_{bf} \partial_c Q^d_{ae} \\ & - 2A^a A^b A^c A^f Q^d_{ae} \partial_c Q^e_{bf} + 2A^b A^c A^f A^g Q^a_{cg} Q^d_{ae} Q^e_{bf} + 2A^a A^c A^f A^g Q^b_{gc} Q^d_{ae} Q^e_{bf} + 2A^a A^b A^c A^g Q^d_{ae} Q^e_{bf} Q^f_{cg} \end{aligned}$$

$$\begin{aligned} \text{eq2.009} := & A^a A^b A^c A^e \partial_{ca} Q^d_{be} - A^a A^b A^c A^e Q^f_{bc} \partial_f Q^d_{ae} - A^a A^b A^c A^e Q^f_{cb} \partial_a Q^d_{fe} - A^a A^b A^c A^e Q^f_{ce} \partial_a Q^d_{bf} - 2A^a A^b A^c A^e Q^f_{be} \partial_c Q^d_{af} \\ & - 2A^a A^b A^c A^e Q^d_{af} \partial_c Q^f_{be} + 2A^a A^b A^c A^e Q^f_{be} Q^d_{fg} Q^g_{ac} + 2A^a A^b A^c A^e Q^f_{eb} Q^d_{ag} Q^g_{fc} + 2A^a A^b A^c A^e Q^d_{af} Q^f_{bg} Q^g_{ce} \end{aligned}$$

$$\begin{aligned} \text{eq2.010} := & A^a A^b A^c A^e \partial_{ab} Q^d_{ce} - A^a A^b A^c A^e Q^f_{ab} \partial_f Q^d_{ce} - 4A^a A^b A^c A^e Q^f_{ab} \partial_c Q^d_{ef} \\ & - 2A^a A^b A^c A^e Q^d_{af} \partial_b Q^f_{ce} + 2A^a A^b A^c A^e Q^d_{fg} Q^f_{ab} Q^g_{ce} + 4A^a A^b A^c A^e Q^d_{af} Q^f_{bg} Q^g_{ce} \end{aligned}$$

$$\begin{aligned}
\text{eq3.010} := & A^a A^b A^c A^e A^f \partial_{abc} Q^d_{ef} - A^a A^b A^c A^e A^f \partial_g Q^d_{ab} \partial_c Q^g_{ef} - 6 A^a A^b A^c A^e A^f \partial_a Q^d_{bg} \partial_c Q^g_{ef} - 3 A^a A^b A^c A^e A^f Q^g_{ab} \partial_{cg} Q^d_{ef} \\
& - 6 A^a A^b A^c A^e A^f Q^g_{ab} \partial_{ce} Q^d_{fg} - 2 A^a A^b A^c A^e A^f Q^d_{ag} \partial_{bc} Q^g_{ef} + 2 A^a A^b A^c A^e A^f Q^g_{ab} Q^h_{cg} \partial_h Q^d_{ef} + 6 A^a A^b A^c A^e A^f Q^g_{ab} Q^h_{ce} \partial_g Q^d_{fh} \\
& + 12 A^a A^b A^c A^e A^f Q^g_{ab} Q^h_{cg} \partial_e Q^d_{fh} + 6 A^a A^b A^c A^e A^f Q^g_{ab} Q^h_{ce} \partial_f Q^d_{gh} + 6 A^a A^b A^c A^e A^f Q^d_{gh} Q^g_{ab} \partial_c Q^h_{ef} \\
& + 2 A^a A^b A^c A^e A^f Q^d_{ag} Q^h_{bc} \partial_h Q^g_{ef} + 8 A^a A^b A^c A^e A^f Q^d_{ag} Q^h_{bc} \partial_e Q^g_{fh} + 4 A^a A^b A^c A^e A^f Q^d_{ag} Q^g_{bh} \partial_c Q^h_{ef} \\
& - 12 A^a A^b A^c A^e A^f Q^d_{gh} Q^g_{ab} Q^h_{ci} Q^i_{ef} - 4 A^a A^b A^c A^e A^f Q^d_{ag} Q^g_{hi} Q^h_{bc} Q^i_{ef} - 8 A^a A^b A^c A^e A^f Q^d_{ag} Q^g_{bh} Q^h_{ci} Q^i_{ef}
\end{aligned}$$

$$\begin{aligned}
\text{eq4.010} := & A^a A^b A^c A^e A^f A^g \partial_{abce} Q^d_{fg} - 4A^a A^b A^c A^e A^f A^g \partial_a Q^h_{bc} \partial_{eh} Q^d_{fg} - A^a A^b A^c A^e A^f A^g \partial_h Q^d_{ab} \partial_{ce} Q^h_{fg} - 12A^a A^b A^c A^e A^f A^g \partial_a Q^h_{bc} \partial_{ef} Q^d_{gh} \\
& - 8A^a A^b A^c A^e A^f A^g \partial_a Q^d_{bh} \partial_{ce} Q^h_{fg} - 6A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_{ceh} Q^d_{fg} - 8A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_{cef} Q^d_{gh} \\
& + 8A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_i Q^d_{ch} \partial_e Q^i_{fg} + A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_i Q^d_{ce} \partial_h Q^i_{fg} + 4A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_i Q^d_{ce} \partial_f Q^i_{gh} \\
& + 12A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_h Q^d_{ci} \partial_e Q^i_{fg} + 24A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_c Q^d_{hi} \partial_e Q^i_{fg} + 8A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_c Q^d_{ei} \partial_h Q^i_{fg} \\
& + 32A^a A^b A^c A^e A^f A^g Q^h_{ab} \partial_c Q^d_{ei} \partial_f Q^i_{gh} - 2A^a A^b A^c A^e A^f A^g Q^d_{ah} \partial_{bce} Q^h_{fg} + 2A^a A^b A^c A^e A^f A^g Q^h_{ai} \partial_h Q^d_{bc} \partial_e Q^i_{fg} \\
& + 16A^a A^b A^c A^e A^f A^g Q^h_{ai} \partial_b Q^d_{ch} \partial_e Q^i_{fg} + 6A^a A^b A^c A^e A^f A^g Q^d_{hi} \partial_a Q^h_{bc} \partial_e Q^i_{fg} + 2A^a A^b A^c A^e A^f A^g Q^d_{ah} \partial_b Q^i_{ce} \partial_i Q^h_{fg} \\
& + 12A^a A^b A^c A^e A^f A^g Q^d_{ah} \partial_b Q^h_{ci} \partial_e Q^i_{fg} + 8A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ch} \partial_{ei} Q^d_{fg} + 3A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} \partial_{hi} Q^d_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} \partial_{fh} Q^d_{gi} + 24A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ch} \partial_{ef} Q^d_{gi} + 12A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} \partial_{fg} Q^d_{hi} \\
& + 8A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{ab} \partial_{ce} Q^i_{fg} + 6A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^i_{bc} \partial_{ei} Q^h_{fg} + 12A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^i_{bc} \partial_{ef} Q^h_{gi} \\
& + 4A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{bi} \partial_{ce} Q^i_{fg} - 4A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ch} Q^j_{ei} \partial_j Q^d_{fg} - 2A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} Q^j_{hi} \partial_j Q^d_{fg} \\
& - 16A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} Q^j_{fh} \partial_j Q^d_{gi} - 24A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} Q^j_{fh} \partial_i Q^d_{gj} - 12A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} Q^j_{fg} \partial_h Q^d_{ij} \\
& - 32A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ch} Q^j_{ei} \partial_f Q^d_{gj} - 16A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} Q^j_{hi} \partial_f Q^d_{gj} - 48A^a A^b A^c A^e A^f A^g Q^h_{ab} Q^i_{ce} Q^j_{fh} \partial_g Q^d_{ij} \\
& - 24A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{aj} Q^j_{bc} \partial_e Q^i_{fg} - 8A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{ab} Q^j_{ce} \partial_j Q^i_{fg} - 32A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{ab} Q^j_{ce} \partial_f Q^i_{gj} \\
& - 4A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^i_{bc} Q^j_{ei} \partial_j Q^h_{fg} - 12A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^i_{bc} Q^j_{ef} \partial_i Q^h_{gj} - 24A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^i_{bc} Q^j_{ei} \partial_f Q^h_{gj} \\
& - 12A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^i_{bc} Q^j_{ef} \partial_g Q^h_{ij} - 16A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{ab} Q^i_{cj} \partial_e Q^j_{fg} - 12A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{ij} Q^i_{bc} \partial_e Q^j_{fg} \\
& - 4A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{bi} Q^j_{ce} \partial_j Q^i_{fg} - 16A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{bi} Q^j_{ce} \partial_f Q^i_{gj} - 8A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{bi} Q^i_{cj} \partial_e Q^j_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{aj} Q^i_{bk} Q^j_{ce} Q^k_{fg} + 16A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{ab} Q^i_{jk} Q^j_{ce} Q^k_{fg} + 32A^a A^b A^c A^e A^f A^g Q^d_{hi} Q^h_{ab} Q^i_{cj} Q^j_{ek} Q^k_{fg} \\
& + 24A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{ij} Q^i_{bc} Q^j_{ek} Q^k_{fg} + 8A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{bi} Q^i_{jk} Q^j_{ce} Q^k_{fg} + 16A^a A^b A^c A^e A^f A^g Q^d_{ah} Q^h_{bi} Q^i_{cj} Q^j_{ek} Q^k_{fg}
\end{aligned}$$

```

def reformat (obj):
    bah := @(obj).
    distribute      (bah)
    bah = product_sort (bah)
    rename_dummies  (bah)
    canonicalise     (bah)
    factor_out       (bah,$A^{a?}$)
    substitute       (bah,$A^{a}->y^{a}$)
    substitute       (bah,$Q^{a}_{b\ c}->\Gamma^{a}_{b\ c}$)
    ans := @(bah).
    return ans

eq0 = reformat(eq0)  # cdb (eq0.100,eq0)
eq1 = reformat(eq1)  # cdb (eq1.100,eq1)
eq2 = reformat(eq2)  # cdb (eq2.100,eq2)
eq3 = reformat(eq3)  # cdb (eq3.100,eq3)
eq4 = reformat(eq4)  # cdb (eq4.100,eq4)

checkpoint.append (eq0)
checkpoint.append (eq1)
checkpoint.append (eq2)
checkpoint.append (eq3)
checkpoint.append (eq4)

```

## Convert from local RNC coords (y) to generic (x)

$$x^a = x_i^a + \overset{0}{x}^a - \overset{1}{x}^a - \overset{2}{x}^a - \overset{3}{x}^a - \overset{4}{x}^a - \overset{5}{x}^a$$

$$\overset{0}{x}^a = y^a$$

$$2! \overset{1}{x}^a = y^a y^b \Gamma_{ab}^d$$

$$3! \overset{2}{x}^a = y^a y^b y^c (\partial_a \Gamma_{bc}^d - 2\Gamma_{ae}^d \Gamma_{bc}^e)$$

$$4! \overset{3}{x}^a = y^a y^b y^c y^e (\partial_{ab} \Gamma_{ce}^d - \Gamma_{ab}^f \partial_f \Gamma_{ce}^d - 4\Gamma_{ab}^f \partial_c \Gamma_{ef}^d - 2\Gamma_{af}^d \partial_b \Gamma_{ce}^f + 2\Gamma_{fg}^d \Gamma_{ab}^f \Gamma_{ce}^g + 4\Gamma_{af}^d \Gamma_{bg}^f \Gamma_{ce}^g)$$

$$5! \overset{4}{x}^a = y^a y^b y^c y^e y^f (\partial_{abc} \Gamma_{ef}^d - \partial_g \Gamma_{ab}^d \partial_c \Gamma_{ef}^g - 6\partial_a \Gamma_{bg}^d \partial_c \Gamma_{ef}^g - 3\Gamma_{ab}^g \partial_{cg} \Gamma_{ef}^d - 6\Gamma_{ab}^g \partial_{ce} \Gamma_{fg}^d - 2\Gamma_{ag}^d \partial_{bc} \Gamma_{ef}^g + 2\Gamma_{ab}^g \Gamma_{cg}^h \partial_h \Gamma_{ef}^d + 6\Gamma_{ab}^g \Gamma_{ce}^h \partial_g \Gamma_{fh}^d \\ + 12\Gamma_{ab}^g \Gamma_{cg}^h \partial_e \Gamma_{fh}^d + 6\Gamma_{ab}^g \Gamma_{ce}^h \partial_f \Gamma_{gh}^d + 6\Gamma_{gh}^d \Gamma_{ab}^g \partial_c \Gamma_{ef}^h + 2\Gamma_{ag}^d \Gamma_{bc}^h \partial_h \Gamma_{ef}^g + 8\Gamma_{ag}^d \Gamma_{bc}^h \partial_e \Gamma_{fh}^g + 4\Gamma_{ag}^d \Gamma_{bh}^g \partial_c \Gamma_{ef}^h - 12\Gamma_{gh}^d \Gamma_{ab}^g \Gamma_{ci}^h \Gamma_{ef}^i \\ - 4\Gamma_{ag}^d \Gamma_{hi}^g \Gamma_{bc}^h \Gamma_{ef}^i - 8\Gamma_{ag}^d \Gamma_{bh}^g \Gamma_{ci}^h \Gamma_{ef}^i)$$

$$6! \overset{5}{x}^a = y^a y^b y^c y^e y^f y^g (\partial_{abce} \Gamma_{fg}^d - 4\partial_a \Gamma_{bc}^h \partial_{eh} \Gamma_{fg}^d - \partial_h \Gamma_{ab}^d \partial_{ce} \Gamma_{fg}^h - 12\partial_a \Gamma_{bc}^h \partial_{ef} \Gamma_{gh}^d - 8\partial_a \Gamma_{bh}^d \partial_{ce} \Gamma_{fg}^h - 6\Gamma_{ab}^h \partial_{ceh} \Gamma_{fg}^d - 8\Gamma_{ab}^h \partial_{cef} \Gamma_{gh}^d \\ + 8\Gamma_{ab}^h \partial_i \Gamma_{ch}^d \partial_e \Gamma_{fg}^i + \Gamma_{ab}^h \partial_i \Gamma_{ce}^d \partial_h \Gamma_{fg}^i + 4\Gamma_{ab}^h \partial_i \Gamma_{ce}^d \partial_f \Gamma_{gh}^i + 12\Gamma_{ab}^h \partial_h \Gamma_{ci}^d \partial_e \Gamma_{fg}^i + 24\Gamma_{ab}^h \partial_c \Gamma_{hi}^d \partial_e \Gamma_{fg}^i + 8\Gamma_{ab}^h \partial_c \Gamma_{ei}^d \partial_h \Gamma_{fg}^i \\ + 32\Gamma_{ab}^h \partial_c \Gamma_{ei}^d \partial_f \Gamma_{gh}^i - 2\Gamma_{ah}^d \partial_{bce} \Gamma_{fg}^h + 2\Gamma_{ai}^h \partial_h \Gamma_{bc}^d \partial_e \Gamma_{fg}^i + 16\Gamma_{ai}^h \partial_b \Gamma_{ch}^d \partial_e \Gamma_{fg}^i + 6\Gamma_{hi}^d \partial_a \Gamma_{bc}^h \partial_e \Gamma_{fg}^i + 2\Gamma_{ah}^d \partial_b \Gamma_{ce}^h \partial_i \Gamma_{fg}^h \\ + 12\Gamma_{ah}^d \partial_b \Gamma_{ci}^h \partial_e \Gamma_{fg}^i + 8\Gamma_{ab}^h \Gamma_{ch}^i \partial_{ei} \Gamma_{fg}^d + 3\Gamma_{ab}^h \Gamma_{ce}^i \partial_{hi} \Gamma_{fg}^d + 24\Gamma_{ab}^h \Gamma_{ce}^i \partial_{fh} \Gamma_{gi}^d + 24\Gamma_{ab}^h \Gamma_{ch}^i \partial_{ef} \Gamma_{gi}^d + 12\Gamma_{ab}^h \Gamma_{ce}^i \partial_{fg} \Gamma_{hi}^d \\ + 8\Gamma_{hi}^d \Gamma_{ab}^h \partial_{ce} \Gamma_{fg}^i + 6\Gamma_{ah}^d \Gamma_{bc}^i \partial_{ei} \Gamma_{fg}^h + 12\Gamma_{ah}^d \Gamma_{bc}^i \partial_{ef} \Gamma_{gi}^h + 4\Gamma_{ah}^d \Gamma_{bi}^h \partial_{ce} \Gamma_{fg}^i - 4\Gamma_{ab}^h \Gamma_{ch}^i \Gamma_{ei}^j \partial_j \Gamma_{fg}^d - 2\Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{hi}^j \partial_j \Gamma_{fg}^d \\ - 16\Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fh}^j \partial_j \Gamma_{gi}^d - 24\Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fh}^j \partial_i \Gamma_{gj}^d - 12\Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fg}^j \partial_h \Gamma_{ij}^d - 32\Gamma_{ab}^h \Gamma_{ch}^i \Gamma_{ei}^j \partial_f \Gamma_{gj}^d - 16\Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{hi}^j \partial_f \Gamma_{gj}^d \\ - 48\Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fh}^j \partial_g \Gamma_{ij}^d - 24\Gamma_{hi}^d \Gamma_{aj}^h \Gamma_{bc}^j \partial_e \Gamma_{fg}^i - 8\Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{ce}^j \partial_j \Gamma_{fg}^i - 32\Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{ce}^j \partial_f \Gamma_{gj}^i - 4\Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ei}^j \partial_j \Gamma_{fg}^h \\ - 12\Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ef}^j \partial_i \Gamma_{gj}^h - 24\Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ei}^j \partial_f \Gamma_{gj}^h - 12\Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ef}^j \partial_g \Gamma_{ij}^h - 16\Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{cj}^i \partial_e \Gamma_{fg}^j - 12\Gamma_{ah}^d \Gamma_{ij}^h \Gamma_{bc}^i \partial_e \Gamma_{fg}^j \\ - 4\Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{ce}^j \partial_j \Gamma_{fg}^i - 16\Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{ce}^j \partial_f \Gamma_{gj}^i - 8\Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{cj}^i \partial_e \Gamma_{fg}^j + 24\Gamma_{hi}^d \Gamma_{aj}^h \Gamma_{bk}^i \Gamma_{ce}^j \Gamma_{fg}^k + 16\Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{jk}^i \Gamma_{ce}^j \Gamma_{fg}^k \\ + 32\Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{cj}^i \Gamma_{ek}^j \Gamma_{fg}^k + 24\Gamma_{ah}^d \Gamma_{ij}^h \Gamma_{bc}^i \Gamma_{ek}^j \Gamma_{fg}^k + 8\Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{jk}^i \Gamma_{ce}^j \Gamma_{fg}^k + 16\Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{cj}^i \Gamma_{ek}^j \Gamma_{fg}^k)$$

# From one RNC to another

Consider an RNC frame with RNC coordinates  $x^a$ .

In the `geodesic-bvp` code the two point boundary value problem (for the geodesic connecting two points) was solved. There is a bonus in that calculation – it can be trivially adapted to the case of transforming from one RNC into another.

The starting point is the basic equation for the geodesic connecting  $P$  (with coordinates  $x^a$ ) to  $Q$  (with coordinates  $x^a + Dx^a$ )

$$x^a(s) = x_i^a + sy^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\underline{b}_k} s^k$$

The affine parameter  $s$  varies from 0 (at  $P$ ) to 1 (at  $Q$ ).

A new RNC frame, with origin at  $P$ , can be defined via the  $y^a$  with the coordinates of  $Q$  in the new RNC frame defined by  $y^a$  (since  $s = 1$  at  $Q$ ). Recall that in an RNC all geodesics through the origin are described by  $y^a(s) = sy^a$ . Thus the transformation from  $x^a$  to  $y^a$  satisfies

$$x^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\underline{b}_k}$$

where the  $\Gamma^a_{\underline{b}_k}$  are the generalised connections of the  $x^a$  frame evaluated at  $x^a = 0$ . This equation can be inverted to express  $y^a$  in terms of  $x^a$ . This computation is done in the `geodesic-bvp` code – we only quote the results here (at the end).

The new  $y^a$  frame has origin at  $P$ . Its coordinate axes are aligned with those (at  $P$ ) of the original RNC frame. To see this just note that  $\partial x^a / \partial y^b = \delta_b^a$  at  $P$ . Thus the metric at  $P$  in the new frame has values  $g_{ab}(x)$  (i.e., exactly those of the original RNC frame). Note that this means that the coordinate axes of the new frame are not necessarily orthogonal.

The calculations in this code are trivial. It uses the  $y^a$  found in `geodesic-bvp` as the basis of the transformation from  $x^a$  to  $y^a$ . Most of the code involves reformatting the  $y^a$ .

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

# Dx{#}::LaTeXForm{"{\Dx}"}. # LCB: currently causes a bug, it kills ::KeepWeight for Dx

import cdblib

Y5 = cdblib.get ('y5','geodesic-bvp.json')

Y50 = cdblib.get ('y50','geodesic-bvp.json')
Y52 = cdblib.get ('y52','geodesic-bvp.json')
Y53 = cdblib.get ('y53','geodesic-bvp.json')
Y54 = cdblib.get ('y54','geodesic-bvp.json')
Y55 = cdblib.get ('y55','geodesic-bvp.json')

# this copies y5* from geodesic-bvp.json to rnc2rnc.json

cdblib.create ('rnc2rnc.json')

cdblib.put ('rnc2rnc',Y5,'rnc2rnc.json')

cdblib.put ('rnc2rnc0',Y50,'rnc2rnc.json')
cdblib.put ('rnc2rnc2',Y52,'rnc2rnc.json')
cdblib.put ('rnc2rnc3',Y53,'rnc2rnc.json')
cdblib.put ('rnc2rnc4',Y54,'rnc2rnc.json')
cdblib.put ('rnc2rnc5',Y55,'rnc2rnc.json')

```



```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}                                -> A001^{a}                $)
    substitute (obj,$ Dx^{a}                                -> A002^{a}                $)
    substitute (obj,$ g^{a b}                               -> A003^{a b}              $)
    substitute (obj,$ \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} -> A008_{\{a b c d e f g h\}} $)
    substitute (obj,$ \nabla_{\{e f g\}}\{R_{\{a b c d\}}\}      -> A007_{\{a b c d e f g\}} $)
    substitute (obj,$ \nabla_{\{e f\}}\{R_{\{a b c d\}}\}         -> A006_{\{a b c d e f\}}  $)
    substitute (obj,$ \nabla_{\{e\}}\{R_{\{a b c d\}}\}           -> A005_{\{a b c d e\}}    $)
    substitute (obj,$ R_{\{a b c d\}}                        -> A004_{\{a b c d\}}      $)
    sort_product      (obj)
    rename_dummies    (obj)
    substitute (obj,$ A001^{a}                                -> x^{a}                $)
    substitute (obj,$ A002^{a}                                -> Dx^{a}               $)
    substitute (obj,$ A003^{a b}                               -> g^{a b}              $)
    substitute (obj,$ A004_{\{a b c d\}}                       -> R_{\{a b c d\}}        $)
    substitute (obj,$ A005_{\{a b c d e\}}                     -> \nabla_{\{e\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A006_{\{a b c d e f\}}                   -> \nabla_{\{e f\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A007_{\{a b c d e f g\}}                 -> \nabla_{\{e f g\}}\{R_{\{a b c d\}}\} $)
    substitute (obj,$ A008_{\{a b c d e f g h\}}               -> \nabla_{\{e f g h\}}\{R_{\{a b c d\}}\} $)

    return obj

def get_xDxterm (obj,n,m):

    x^{a}::Weight(label=numx,value=1).
    Dx^{a}::Weight(label=numDx,value=1).

    tmp := @(obj).
    distribute (tmp)

    foo = Ex("numx = " + str(n))
    bah = Ex("numDx = " + str(m))
    keep_weight (tmp, foo)
    keep_weight (tmp, bah)

    return tmp

```

```

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    substitute (bah,$Dx^{b}->zzz^{b}$)
    factor_out (bah,$x^{a?},zzz^{b?}$)
    substitute (bah,$zzz^{b}->Dx^{b}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    substitute (bah,$Dx^{b}->zzz^{b}$)
    factor_out (bah,$x^{a?},zzz^{b?}$)
    substitute (bah,$zzz^{b}->Dx^{b}$)
    return bah

term0 := @(Y50). # cdb (term0.101,term0)
term2 := @(Y52). # cdb (term2.101,term2)
term3 := @(Y53). # cdb (term3.101,term3)
term4 := @(Y54). # cdb (term4.101,term4)
term5 := @(Y55). # cdb (term5.101,term5)

term0 = reformat (term0,1) # cdb (term0.102,term0)
term2 = reformat (term2,1) # cdb (term2.102,term2)
term3 = reformat (term3,1) # cdb (term3.102,term3)
term4 = reformat (term4,1) # cdb (term4.102,term4)
term5 = reformat (term5,1) # cdb (term5.102,term5)

xDxterm12 = get_xDxterm (term2,1,2) # cdb(xDxterm12.101,xDxterm12)

xDxterm13 = get_xDxterm (term3,1,3) # cdb(xDxterm13.101,xDxterm13)
xDxterm22 = get_xDxterm (term3,2,2) # cdb(xDxterm22.101,xDxterm22)

```

```

xDxterm14 = get_xDxterm (term4,1,4)    # cdb(xDxterm14.101,xDxterm14)
xDxterm23 = get_xDxterm (term4,2,3)    # cdb(xDxterm23.101,xDxterm23)
xDxterm32 = get_xDxterm (term4,3,2)    # cdb(xDxterm32.101,xDxterm32)

xDxterm15 = get_xDxterm (term5,1,5)    # cdb(xDxterm15.101,xDxterm15)
xDxterm24 = get_xDxterm (term5,2,4)    # cdb(xDxterm24.101,xDxterm24)
xDxterm33 = get_xDxterm (term5,3,3)    # cdb(xDxterm33.101,xDxterm33)
xDxterm42 = get_xDxterm (term5,4,2)    # cdb(xDxterm42.101,xDxterm42)

xDxterm12 = rescale ( reformat (xDxterm12,    3),    3 )    # cdb(xDxterm12.102,xDxterm12)

xDxterm13 = rescale ( reformat (xDxterm13,    12),    -12 )    # cdb(xDxterm13.102,xDxterm13)
xDxterm22 = rescale ( reformat (xDxterm22,    24),    -24 )    # cdb(xDxterm22.102,xDxterm22)

xDxterm14 = rescale ( reformat (xDxterm14,    180),    -180 )    # cdb(xDxterm14.102,xDxterm14)
xDxterm23 = rescale ( reformat (xDxterm23,    720),    -720 )    # cdb(xDxterm23.102,xDxterm23)
xDxterm32 = rescale ( reformat (xDxterm32,    720),    -720 )    # cdb(xDxterm32.102,xDxterm32)

xDxterm15 = rescale ( reformat (xDxterm15,    360),    -360 )    # cdb(xDxterm15.102,xDxterm15)
xDxterm24 = rescale ( reformat (xDxterm24,    2160),    -2160 )    # cdb(xDxterm24.102,xDxterm24)
xDxterm33 = rescale ( reformat (xDxterm33,    1080),    -1080 )    # cdb(xDxterm33.102,xDxterm33)
xDxterm42 = rescale ( reformat (xDxterm42,    360),    -360 )    # cdb(xDxterm42.102,xDxterm42)

checkpoint.append (term0)
checkpoint.append (term2)
checkpoint.append (term3)
checkpoint.append (term4)
checkpoint.append (term5)

```

## Transformation between two RNC frames

$$y^a = \overset{0}{y}^a + \overset{2}{y}^a + \overset{3}{y}^a + \overset{4}{y}^a + \overset{5}{y}^a + \mathcal{O}(\epsilon^6)$$

$$\overset{0}{y}^a = Dx^a$$

$$\overset{2}{y}^a = -\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$\overset{3}{y}^a = x^b x^c Dx^d Dx^e \left( -\frac{1}{12} g^{af} \nabla_d R_{becf} - \frac{1}{6} g^{af} \nabla_b R_{cdef} + \frac{1}{24} g^{af} \nabla_f R_{bdce} \right) - \frac{1}{12} x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned} \overset{4}{y}^a = & x^b x^c Dx^d Dx^e Dx^f \left( -\frac{2}{45} g^{ag} g^{hi} R_{bdeh} R_{cfgi} + \frac{1}{45} g^{ag} g^{hi} R_{bdeh} R_{cifg} - \frac{1}{45} g^{ag} g^{hi} R_{bdeh} R_{cgfi} + \frac{1}{45} g^{ag} g^{hi} R_{bdch} R_{egfi} - \frac{1}{60} g^{ag} \nabla_{de} R_{bfcg} - \frac{1}{40} g^{ag} \nabla_{db} R_{cefg} \right. \\ & \left. - \frac{1}{40} g^{ag} \nabla_{bd} R_{cefg} + \frac{1}{240} g^{ag} \nabla_{gd} R_{becf} + \frac{1}{240} g^{ag} \nabla_{dg} R_{becf} \right) \\ & + x^b x^c x^d Dx^e Dx^f \left( -\frac{4}{45} g^{ag} g^{hi} R_{befh} R_{cgdi} + \frac{2}{45} g^{ag} g^{hi} R_{bech} R_{difg} + \frac{1}{45} g^{ag} g^{hi} R_{bech} R_{dgfi} - \frac{1}{40} g^{ag} \nabla_{eb} R_{cfdg} - \frac{1}{40} g^{ag} \nabla_{be} R_{cfdg} - \frac{1}{20} g^{ag} \nabla_{bc} R_{defg} \right. \\ & \left. - \frac{1}{45} g^{ag} g^{hi} R_{bech} R_{dfgi} + \frac{1}{80} g^{ag} \nabla_{gb} R_{cedf} + \frac{1}{80} g^{ag} \nabla_{bg} R_{cedf} \right) + x^b Dx^c Dx^d Dx^e Dx^f \left( -\frac{1}{45} g^{ag} g^{hi} R_{bcdh} R_{egfi} - \frac{1}{60} g^{ag} \nabla_{cd} R_{befg} \right) \end{aligned}$$

$$\begin{aligned}
\tilde{y}^a = & x^b x^c x^d D x^e D x^f D x^g \left( -\frac{7}{540} g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} - \frac{1}{45} g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + \frac{1}{216} g^{ah} g^{ij} R_{behi} \nabla_j R_{cf dg} + \frac{1}{90} g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \right. \\
& + \frac{1}{90} g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} - \frac{1}{540} g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{108} g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} - \frac{1}{45} g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + \frac{1}{90} g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} \\
& - \frac{7}{540} g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} - \frac{1}{90} g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - \frac{1}{540} g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} - \frac{11}{540} g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{1}{90} g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
& + \frac{1}{216} g^{ah} g^{ij} R_{bhei} \nabla_j R_{cf dg} + \frac{1}{90} g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + \frac{1}{90} g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + \frac{1}{135} g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + \frac{1}{90} g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\
& - \frac{1}{270} g^{ah} \nabla_{efb} R_{cgdh} - \frac{1}{270} g^{ah} \nabla_{ebf} R_{cgdh} - \frac{1}{180} g^{ah} \nabla_{ebc} R_{dfgh} - \frac{1}{270} g^{ah} \nabla_{bef} R_{cgdh} - \frac{1}{180} g^{ah} \nabla_{bec} R_{dfgh} - \frac{1}{180} g^{ah} \nabla_{bce} R_{dfgh} - \frac{1}{270} g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} \\
& - \frac{1}{270} g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} + \frac{1}{1080} g^{ah} \nabla_{heb} R_{cf dg} + \frac{1}{1080} g^{ah} \nabla_{hbe} R_{cf dg} + \frac{1}{1080} g^{ah} \nabla_{ehb} R_{cf dg} + \frac{1}{1080} g^{ah} \nabla_{bhe} R_{cf dg} + \frac{1}{1080} g^{ah} \nabla_{ebh} R_{cf dg} \\
& + \frac{1}{1080} g^{ah} \nabla_{beh} R_{cf dg} \Big) + x^b x^c D x^d D x^e D x^f D x^g \left( -\frac{17}{1080} g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cf gj} + \frac{1}{135} g^{ah} g^{ij} R_{bidh} \nabla_e R_{cf gj} - \frac{7}{1080} g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} \right. \\
& - \frac{1}{540} g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + \frac{1}{108} g^{ah} g^{ij} R_{bdei} \nabla_j R_{cf gh} - \frac{1}{120} g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - \frac{1}{90} g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} - \frac{1}{540} g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} \\
& - \frac{1}{120} g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cf gj} - \frac{1}{90} g^{ah} g^{ij} R_{dhei} \nabla_b R_{cf gj} + \frac{1}{216} g^{ah} g^{ij} R_{dhei} \nabla_j R_{bf cg} + \frac{1}{135} g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} - \frac{1}{360} g^{ah} \nabla_{def} R_{bgch} - \frac{1}{270} g^{ah} \nabla_{deb} R_{cf gh} \\
& - \frac{1}{270} g^{ah} \nabla_{dbe} R_{cf gh} - \frac{1}{270} g^{ah} \nabla_{bde} R_{cf gh} + \frac{1}{360} g^{ah} g^{ij} R_{bdei} \nabla_h R_{cf gj} + \frac{1}{2160} g^{ah} \nabla_{hde} R_{bf cg} + \frac{1}{2160} g^{ah} \nabla_{dhe} R_{bf cg} + \frac{1}{2160} g^{ah} \nabla_{deh} R_{bf cg} \Big) \\
& + x^b x^c x^d x^e D x^f D x^g \left( -\frac{2}{45} g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} - \frac{1}{60} g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} - \frac{2}{45} g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + \frac{1}{72} g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \right. \\
& + \frac{1}{45} g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{1}{90} g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + \frac{1}{45} g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} \\
& - \frac{1}{180} g^{ah} \nabla_{fbc} R_{dgeh} - \frac{1}{180} g^{ah} \nabla_{bfc} R_{dgeh} - \frac{1}{180} g^{ah} \nabla_{bcf} R_{dgeh} - \frac{1}{90} g^{ah} \nabla_{bcd} R_{efgh} - \frac{1}{90} g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} - \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} \\
& - \frac{1}{90} g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + \frac{1}{360} g^{ah} \nabla_{hbc} R_{dfeg} + \frac{1}{360} g^{ah} \nabla_{bhc} R_{dfeg} + \frac{1}{360} g^{ah} \nabla_{bch} R_{dfeg} \Big) \\
& + x^b D x^c D x^d D x^e D x^f D x^g \left( -\frac{1}{120} g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - \frac{1}{120} g^{ah} g^{ij} R_{chdi} \nabla_e R_{bf gj} - \frac{1}{360} g^{ah} \nabla_{cde} R_{bf gh} \right)
\end{aligned}$$

# Tranformation between two RNC frames

Same as before but with an improved format (maybe) for the expressions.

$$y^a = {}^0y^a + {}^2y^a + {}^3y^a + {}^4y^a + {}^5y^a + \mathcal{O}(\epsilon^6) \quad (1)$$

$${}^0y^a = Dx^a \quad (2a)$$

$${}^2y^a = {}^2y_1^a \quad (3a)$$

$$3{}^2y_1^a = -x^b Dx^c Dx^d g^{ae} R_{bcde} \quad (3b)$$

$${}^3y^a = {}^3y_1^a + {}^3y_2^a \quad (4a)$$

$$-12{}^3y_1^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} \quad (4b)$$

$$-24{}^3y_2^a = x^b x^c Dx^d Dx^e (2g^{af} \nabla_d R_{becf} + 4g^{af} \nabla_b R_{cdef} - g^{af} \nabla_f R_{bdce}) \quad (4c)$$

$${}^4y^a = {}^4y_1^a + {}^4y_2^a + {}^4y_3^a \quad (5a)$$

$$-180{}^4y_1^a = x^b Dx^c Dx^d Dx^e Dx^f (4g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3g^{ag} \nabla_{cd} R_{befg}) \quad (5b)$$

$$\begin{aligned} -720{}^4y_2^a = x^b x^c Dx^d Dx^e Dx^f (32g^{ag} g^{hi} R_{bdeh} R_{cfdi} - 16g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdch} R_{egfi} + 12g^{ag} \nabla_{de} R_{bfcg} \\ + 18g^{ag} \nabla_{db} R_{cefg} + 18g^{ag} \nabla_{bd} R_{cefg} - 3g^{ag} \nabla_{gd} R_{becf} - 3g^{ag} \nabla_{dg} R_{becf}) \end{aligned} \quad (5c)$$

$$\begin{aligned} -720{}^4y_3^a = x^b x^c x^d Dx^e Dx^f (64g^{ag} g^{hi} R_{befh} R_{cgdi} - 32g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{bech} R_{dgfi} + 18g^{ag} \nabla_{eb} R_{cfdg} + 18g^{ag} \nabla_{be} R_{cfdg} \\ + 36g^{ag} \nabla_{bc} R_{defg} + 16g^{ag} g^{hi} R_{bech} R_{dfgi} - 9g^{ag} \nabla_{gb} R_{cedf} - 9g^{ag} \nabla_{bg} R_{cedf}) \end{aligned} \quad (5d)$$

$$\overset{5}{y}^a = \overset{5}{y}_1^a + \overset{5}{y}_2^a + \overset{5}{y}_3^a + \overset{5}{y}_4^a \quad (6a)$$

$$-360\overset{5}{y}_1^a = x^b D x^c D x^d D x^e D x^f D x^g (3g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh}) \quad (6b)$$

$$\begin{aligned} -2160\overset{5}{y}_2^a = & x^b x^c D x^d D x^e D x^f D x^g (34g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 16g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 14g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjh} \\ & - 20g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh} + 18g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 18g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} \\ & + 24g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfeg} - 16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah} \nabla_{def} R_{bgch} + 8g^{ah} \nabla_{deb} R_{cfgh} + 8g^{ah} \nabla_{dbe} R_{cfgh} \\ & + 8g^{ah} \nabla_{bde} R_{cfgh} - 6g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} - g^{ah} \nabla_{hde} R_{bfeg} - g^{ah} \nabla_{dhe} R_{bfeg} - g^{ah} \nabla_{deh} R_{bfeg}) \end{aligned} \quad (6c)$$

$$\begin{aligned} -1080\overset{5}{y}_3^a = & x^b x^c x^d D x^e D x^f D x^g (14g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + 24g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 5g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg} - 12g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\ & - 12g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 2g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} - 10g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 24g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} - 12g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} \\ & + 14g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 2g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 12g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\ & - 5g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} - 12g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\ & + 4g^{ah} \nabla_{efb} R_{cgdh} + 4g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{ebc} R_{dfgh} + 4g^{ah} \nabla_{bef} R_{cgdh} + 6g^{ah} \nabla_{bec} R_{dfgh} + 6g^{ah} \nabla_{bce} R_{dfgh} + 4g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} \\ & + 4g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg} - g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg}) \end{aligned} \quad (6d)$$

$$\begin{aligned} -360\overset{5}{y}_4^a = & x^b x^c x^d x^e D x^f D x^g (16g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + 6g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 16g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} - 5g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\ & - 8g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} - 4g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} - 4g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} - 8g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} - 4g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} \\ & + 2g^{ah} \nabla_{fbc} R_{dgeh} + 2g^{ah} \nabla_{bfc} R_{dgeh} + 2g^{ah} \nabla_{bcf} R_{dgeh} + 4g^{ah} \nabla_{bcd} R_{efgh} + 4g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} + 4g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} \\ & + 4g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} - g^{ah} \nabla_{hbc} R_{dfeg} - g^{ah} \nabla_{bhc} R_{dfeg} - g^{ah} \nabla_{bch} R_{dfeg}) \end{aligned} \quad (6e)$$

# The determinant of the metric

Our game here is to compute (the leading terms) in  $\det g$  of the metric in RNC form

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45}x^c x^d x^e x^f R_{acd g} R_{bef h} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf} \\ + \frac{1}{45}x^c x^d x^e x^f x^g R_{acd h} \nabla_e R_{bf g i} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcd h} \nabla_e R_{af g i} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg} + \mathcal{O}(\epsilon^5)$$

For the sake of simplicity let's assume that we are working in 3-dimensions. The following analysis is easily generalised to other dimensions (and the final answers for  $\det g$  and friends are unchanged).

Define  $\epsilon_{ijk}^{abc}$  by

$$\epsilon_{ijk}^{abc} = \delta_i^a \delta_j^b \delta_k^c - \delta_i^b \delta_j^a \delta_k^c + \delta_i^c \delta_j^a \delta_k^b - \delta_i^c \delta_j^b \delta_k^a + \delta_i^b \delta_j^c \delta_k^a - \delta_i^a \delta_j^c \delta_k^b \quad (1)$$

It is easy to see that  $\epsilon_{ijk}^{abc}$  is anti-symmetric in both its upper and lower indices. A trivial computation shows that for any  $3 \times 3$  square matrix  $M_{ab}$ ,

$$\epsilon_{123}^{abc} M_{1a} M_{2b} M_{3c} = (\delta_1^a \delta_2^b \delta_3^c - \delta_1^b \delta_2^a \delta_3^c + \delta_1^c \delta_2^a \delta_3^b - \delta_1^c \delta_2^b \delta_3^a + \delta_1^b \delta_2^c \delta_3^a - \delta_1^a \delta_2^c \delta_3^b) M_{1a} M_{2b} M_{3c} = \det M \quad (2)$$

This can be easily generalised to

$$\epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} = \begin{cases} \pm \det M & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The  $\pm$  sign in the above depends on the particular permutations of  $(ijk)$  and  $(pqr)$ . If both permutations are even or both odd then the sign is  $+1$  otherwise the sign is  $-1$ . The same arguments can also be applied to a matrix inverse  $N^{-1}$  leading to

$$\epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} = \begin{cases} \pm \det N^{-1} & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Note that the  $\pm$  in this case will match exactly that for the case of  $\det M$ . Thus, multiplying both expressions and summing over all choices for  $(ijk)$  and  $(pqr)$  leads to

$$\sum_{\substack{(ijk) \\ (pqr)}} (\det N^{-1}) \det M = \epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} \epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} \quad (5)$$



where the sum on the left hand side includes just those  $(ijk)$  and  $(prq)$  that are permutations of  $(123)$ . There are  $3!$  choices for  $(ijk)$  and  $3!$  choices for  $(prq)$  and thus the left hand side is easily reduced to  $(3!)^2 \det M / \det N$  where  $\det N = 1 / \det N^{-1}$ . For the right hand side notice that

$$\epsilon_{uvw}^{ijk} \epsilon_{ijk}^{abc} = 3! \epsilon_{uvw}^{abc} \quad (6)$$

which leads to

$$\det M = \frac{1}{3!} \det N \epsilon_{uvw}^{abc} M_{pa} M_{qb} M_{rc} N^{pu} N^{qv} N^{rw} \quad (7)$$

For our RNC metric we will set  $N^{ab} = g^{ab}$  and  $M_{ij} = g_{ij}(x)$ . Since  $g^{ab}$  is of the form  $\text{diag}(-1, 1, 1, 1)$  we have  $\det g = -1$  and thus

$$\det g(x) = -\frac{1}{3!} \epsilon_{ijk}^{abc} g_{pa}(x) g_{qb}(x) g_{rc}(x) g^{ip} g^{jq} g^{kr} \quad (8)$$

The  $\epsilon_{ijk}^{abc}$  can be constructed in Cadabra by applying the `asym` algorithm to the upper indices of  $\delta_i^a \delta_j^b \delta_k^c$ . Note that `asym` will include the  $1/3!$  coefficient as part of its output.

The following code computes  $-\det g$  rather than  $\det g$ .

**Note** that Calzetta et al. use an opposite sign for  $R_{abcd}$  so when comparing the following results against Calzetta do take note of this flipped sign in  $R_{abcd}$ .

```

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Integer(1..2).

\nabla{#}::Derivative.

d{#}::KroneckerDelta.

g^{a b}::Symmetric.
g_{a b}::Symmetric.

R_{a b c d}::RiemannTensor.

x^{a}::Weight(label=numx,value=1).

def truncate (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah = Ex("numx = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    return ans

import cdblib

g0ab = cdblib.get('g_ab_0','metric.json')
g1ab = cdblib.get('g_ab_1','metric.json')  # zero in RNC
g2ab = cdblib.get('g_ab_2','metric.json')
g3ab = cdblib.get('g_ab_3','metric.json')
g4ab = cdblib.get('g_ab_4','metric.json')
g5ab = cdblib.get('g_ab_5','metric.json')

gab := @(g0ab) + @(g1ab) + @(g2ab) + @(g3ab) + @(g4ab) + @(g5ab).  # cdb (gab.001,gab)
gxab := gx_{a b} -> @(gab).

```

```

eps := d^{a}_{i} d^{b}_{j}.      # cdb(eps.001,eps)
asym (eps,$^{a},^{b}$)          # cdb(eps.002,eps) # includes a factor of 1/2!

# compute negative Ndetg rather than det g
Ndetg := @(eps) gx_{p a} gx_{q b} g^{i p} g^{j q}. # note 1/2! included in eps

substitute      (Ndetg,gxab)
distribute      (Ndetg)
Ndetg = truncate (Ndetg,5)                                     # cdb (Ndetg.001,Ndetg)
substitute      (Ndetg,$g^{a b} g_{b c} -> d^{a}_{c}$,repeat=True) # cdb (Ndetg.002,Ndetg)
eliminate_kronecker (Ndetg)                                    # cdb (Ndetg.003,Ndetg)
sort_product     (Ndetg)                                       # cdb (Ndetg.004,Ndetg)
rename_dummies   (Ndetg)                                       # cdb (Ndetg.005,Ndetg)
canonicalise     (Ndetg)                                       # cdb (Ndetg.006,Ndetg)

# introduce the Ricci tensor

substitute      (Ndetg,$R_{a b c d} g^{a c} -> R_{b d}$,repeat=True) # cdb (Ndetg.101,Ndetg)
substitute      (Ndetg,$\nabla_{a}\{R_{b c d e}\} g^{b d} -> \nabla_{a}\{R_{c e}\}$,repeat=True) # cdb (Ndetg.102,Ndetg)
substitute      (Ndetg,$\nabla_{a b}\{R_{c d e f}\} g^{c e} -> \nabla_{a b}\{R_{d f}\}$,repeat=True) # cdb (Ndetg.103,Ndetg)
substitute      (Ndetg,$\nabla_{a b c}\{R_{d e f g}\} g^{d f} -> \nabla_{a b c}\{R_{e g}\}$,repeat=True) # cdb (Ndetg.104,Ndetg)

# the following are based on sqrt-Ndetg.tex

sqrtNdetg := 1/2 + (1/2) @(Ndetg)
            - (1/8) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
            - (1/4) (1/18) R_{a b} \nabla_{c}\{R_{d e}\} x^{a} x^{b} x^{c} x^{d} x^{e}.
            # cdb (sqrtNdetg.001,sqrtNdetg)

sort_product     (sqrtNdetg)                                     # cdb (sqrtNdetg.002,sqrtNdetg)
rename_dummies   (sqrtNdetg)                                     # cdb (sqrtNdetg.003,sqrtNdetg)
canonicalise     (sqrtNdetg)                                     # cdb (sqrtNdetg.004,sqrtNdetg)

logNdetg := -1 + @(Ndetg)
            - (1/2) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
            - (1/18) R_{a b} \nabla_{c}\{R_{d e}\} x^{a} x^{b} x^{c} x^{d} x^{e}.
            # cdb (logNdetg.001,logNdetg)

```

```
sort_product      (logNdetg)                # cdb (logNdetg.002,logNdetg)
rename_dummies    (logNdetg)                # cdb (logNdetg.003,logNdetg)
canonicalise      (logNdetg)                # cdb (logNdetg.004,logNdetg)
```

$$\text{eps.001} := d^a_i d^b_j$$

$$\text{eps.002} := \frac{1}{2} d^a_i d^b_j - \frac{1}{2} d^b_i d^a_j$$

$$\begin{aligned} \text{Ndetg.001} := & \frac{1}{2} d^a_i d^b_j g_{pa} g_{qb} g^{ip} g^{jq} - \frac{1}{2} d^b_i d^a_j g_{pa} g_{qb} g^{ip} g^{jq} - \frac{1}{6} d^a_i d^b_j g_{pa} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{6} d^a_i d^b_j x^c x^d R_{pcad} g_{qb} g^{ip} g^{jq} + \frac{1}{6} d^b_i d^a_j g_{pa} x^l x^m R_{qlbm} g^{ip} g^{jq} \\ & + \frac{1}{6} d^b_i d^a_j x^c x^d R_{pcad} g_{qb} g^{ip} g^{jq} - \frac{1}{12} d^a_i d^b_j g_{pa} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} - \frac{1}{12} d^a_i d^b_j x^c x^d x^e \nabla_c R_{pdae} g_{qb} g^{ip} g^{jq} \\ & + \frac{1}{12} d^b_i d^a_j g_{pa} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} + \frac{1}{12} d^b_i d^a_j x^c x^d x^e \nabla_c R_{pdae} g_{qb} g^{ip} g^{jq} + \frac{1}{45} d^a_i d^b_j g_{pa} x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} g^{ip} g^{jq} \\ & - \frac{1}{40} d^a_i d^b_j g_{pa} x^l x^m x^n x^o \nabla_{lm} R_{qnbo} g^{ip} g^{jq} + \frac{1}{18} d^a_i d^b_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} + \frac{1}{45} d^a_i d^b_j x^c x^d x^e x^f R_{pcdg} R_{ae fh} g^{gh} g_{qb} g^{ip} g^{jq} \\ & - \frac{1}{40} d^a_i d^b_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g_{qb} g^{ip} g^{jq} - \frac{1}{45} d^b_i d^a_j g_{pa} x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} g^{ip} g^{jq} + \frac{1}{40} d^b_i d^a_j g_{pa} x^l x^m x^n x^o \nabla_{lm} R_{qnbo} g^{ip} g^{jq} \\ & - \frac{1}{18} d^b_i d^a_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{45} d^b_i d^a_j x^c x^d x^e x^f R_{pcdg} R_{ae fh} g^{gh} g_{qb} g^{ip} g^{jq} + \frac{1}{40} d^b_i d^a_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g_{qb} g^{ip} g^{jq} \\ & + \frac{1}{90} d^a_i d^b_j g_{pa} x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} g^{ip} g^{jq} + \frac{1}{90} d^a_i d^b_j g_{pa} x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} g^{ip} g^{jq} \\ & - \frac{1}{180} d^a_i d^b_j g_{pa} x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} g^{ip} g^{jq} + \frac{1}{36} d^a_i d^b_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} \\ & + \frac{1}{36} d^a_i d^b_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} + \frac{1}{90} d^a_i d^b_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g_{qb} g^{ip} g^{jq} \\ & + \frac{1}{90} d^a_i d^b_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g_{qb} g^{ip} g^{jq} - \frac{1}{180} d^a_i d^b_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g_{qb} g^{ip} g^{jq} \\ & - \frac{1}{90} d^b_i d^a_j g_{pa} x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} g^{ip} g^{jq} - \frac{1}{90} d^b_i d^a_j g_{pa} x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} g^{ip} g^{jq} \\ & + \frac{1}{180} d^b_i d^a_j g_{pa} x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} g^{ip} g^{jq} - \frac{1}{36} d^b_i d^a_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} \\ & - \frac{1}{36} d^b_i d^a_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{90} d^b_i d^a_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g_{qb} g^{ip} g^{jq} \\ & - \frac{1}{90} d^b_i d^a_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g_{qb} g^{ip} g^{jq} + \frac{1}{180} d^b_i d^a_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g_{qb} g^{ip} g^{jq} \end{aligned}$$

$$\begin{aligned}
\text{Ndetg.002} := & \frac{1}{2}d^a{}_i d^b{}_j d^i{}_a d^j{}_b - \frac{1}{2}d^b{}_i d^a{}_j d^i{}_a d^j{}_b - \frac{1}{6}d^a{}_i d^b{}_j x^l x^m R_{qlbm} d^i{}_a g^{jq} - \frac{1}{6}d^a{}_i d^b{}_j x^c x^d R_{pcad} g^{ip} d^j{}_b + \frac{1}{6}d^b{}_i d^a{}_j x^l x^m R_{qlbm} d^i{}_a g^{jq} \\
& + \frac{1}{6}d^b{}_i d^a{}_j x^c x^d R_{pcad} g^{ip} d^j{}_b - \frac{1}{12}d^a{}_i d^b{}_j x^l x^m x^n \nabla_l R_{qmbn} d^i{}_a g^{jq} - \frac{1}{12}d^a{}_i d^b{}_j x^c x^d x^e \nabla_c R_{pdae} g^{ip} d^j{}_b \\
& + \frac{1}{12}d^b{}_i d^a{}_j x^l x^m x^n \nabla_l R_{qmbn} d^i{}_a g^{jq} + \frac{1}{12}d^b{}_i d^a{}_j x^c x^d x^e \nabla_c R_{pdae} g^{ip} d^j{}_b + \frac{1}{45}d^a{}_i d^b{}_j x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} d^i{}_a g^{jq} \\
& - \frac{1}{40}d^a{}_i d^b{}_j x^l x^m x^n x^o \nabla_{lm} R_{qnbo} d^i{}_a g^{jq} + \frac{1}{18}d^a{}_i d^b{}_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} + \frac{1}{45}d^a{}_i d^b{}_j x^c x^d x^e x^f R_{pcdg} R_{aefh} g^{gh} g^{ip} d^j{}_b \\
& - \frac{1}{40}d^a{}_i d^b{}_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g^{ip} d^j{}_b - \frac{1}{45}d^b{}_i d^a{}_j x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} d^i{}_a g^{jq} + \frac{1}{40}d^b{}_i d^a{}_j x^l x^m x^n x^o \nabla_{lm} R_{qnbo} d^i{}_a g^{jq} \\
& - \frac{1}{18}d^b{}_i d^a{}_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{45}d^b{}_i d^a{}_j x^c x^d x^e x^f R_{pcdg} R_{aefh} g^{gh} g^{ip} d^j{}_b + \frac{1}{40}d^b{}_i d^a{}_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g^{ip} d^j{}_b \\
& + \frac{1}{90}d^a{}_i d^b{}_j x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} d^i{}_a g^{jq} + \frac{1}{90}d^a{}_i d^b{}_j x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} d^i{}_a g^{jq} \\
& - \frac{1}{180}d^a{}_i d^b{}_j x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} d^i{}_a g^{jq} + \frac{1}{36}d^a{}_i d^b{}_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} + \frac{1}{36}d^a{}_i d^b{}_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} \\
& + \frac{1}{90}d^a{}_i d^b{}_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g^{ip} d^j{}_b + \frac{1}{90}d^a{}_i d^b{}_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g^{ip} d^j{}_b \\
& - \frac{1}{180}d^a{}_i d^b{}_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g^{ip} d^j{}_b - \frac{1}{90}d^b{}_i d^a{}_j x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} d^i{}_a g^{jq} \\
& - \frac{1}{90}d^b{}_i d^a{}_j x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} d^i{}_a g^{jq} + \frac{1}{180}d^b{}_i d^a{}_j x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} d^i{}_a g^{jq} - \frac{1}{36}d^b{}_i d^a{}_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} \\
& - \frac{1}{36}d^b{}_i d^a{}_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{90}d^b{}_i d^a{}_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g^{ip} d^j{}_b \\
& - \frac{1}{90}d^b{}_i d^a{}_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g^{ip} d^j{}_b + \frac{1}{180}d^b{}_i d^a{}_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g^{ip} d^j{}_b
\end{aligned}$$

$$\begin{aligned}
\text{Ndetg.003} := & 1 - \frac{1}{6}x^lx^mR_{qljm}g^{jq} - \frac{1}{3}x^cx^dR_{pcid}g^{ip} + \frac{1}{6}x^cx^dR_{pcbd}g^{bp} - \frac{1}{12}x^lx^mx^n\nabla_lR_{qmjn}g^{jq} - \frac{1}{6}x^cx^dx^e\nabla_cR_{pdie}g^{ip} \\
& + \frac{1}{12}x^cx^dx^e\nabla_cR_{pdbe}g^{bp} + \frac{1}{45}x^lx^mx^nx^oR_{qlmr}R_{jnos}g^{rs}g^{jq} - \frac{1}{40}x^lx^mx^nx^o\nabla_{lm}R_{qnjo}g^{jq} + \frac{1}{18}x^cx^dR_{pcid}x^lx^mR_{qljm}g^{ip}g^{jq} \\
& + \frac{2}{45}x^cx^dx^ex^fR_{pcdg}R_{iefh}g^{gh}g^{ip} - \frac{1}{20}x^cx^dx^ex^f\nabla_{cd}R_{peif}g^{ip} - \frac{1}{18}x^cx^dR_{pcjd}x^lx^mR_{qlim}g^{ip}g^{jq} \\
& - \frac{1}{45}x^cx^dx^ex^fR_{pcdg}R_{befh}g^{gh}g^{bp} + \frac{1}{40}x^cx^dx^ex^f\nabla_{cd}R_{pebf}g^{bp} + \frac{1}{90}x^lx^mx^nx^ox^rR_{qlms}\nabla_nR_{jort}g^{st}g^{jq} \\
& + \frac{1}{90}x^lx^mx^nx^ox^rR_{jlm}\nabla_nR_{qort}g^{st}g^{jq} - \frac{1}{180}x^lx^mx^nx^ox^r\nabla_{lmn}R_{qojr}g^{jq} + \frac{1}{36}x^cx^dR_{pcid}x^lx^m\nabla_lR_{qmjn}g^{ip}g^{jq} \\
& + \frac{1}{36}x^cx^dx^e\nabla_cR_{pdie}x^lx^mR_{qljm}g^{ip}g^{jq} + \frac{1}{45}x^cx^dx^ex^fgR_{pcdh}\nabla_eR_{ifgk}g^{hk}g^{ip} + \frac{1}{45}x^cx^dx^ex^fgR_{icdh}\nabla_eR_{pfgk}g^{hk}g^{ip} \\
& - \frac{1}{90}x^cx^dx^ex^fg\nabla_{cde}R_{pfig}g^{ip} - \frac{1}{36}x^cx^dR_{pcjd}x^lx^m\nabla_lR_{qmin}g^{ip}g^{jq} - \frac{1}{36}x^cx^dx^e\nabla_cR_{pdje}x^lx^mR_{qlim}g^{ip}g^{jq} \\
& - \frac{1}{90}x^cx^dx^ex^fgR_{pcdh}\nabla_eR_{bfgk}g^{hk}g^{bp} - \frac{1}{90}x^cx^dx^ex^fgR_{bcdh}\nabla_eR_{pfgk}g^{hk}g^{bp} + \frac{1}{180}x^cx^dx^ex^fg\nabla_{cde}R_{pfbg}g^{bp}
\end{aligned}$$

$$\begin{aligned}
\text{Ndetg.004} := & 1 - \frac{1}{6}R_{qljm}g^{jq}x^lx^m - \frac{1}{3}R_{pcid}g^{ip}x^cx^d + \frac{1}{6}R_{pcbd}g^{bp}x^cx^d - \frac{1}{12}\nabla_lR_{qmjn}g^{jq}x^lx^mx^n - \frac{1}{6}\nabla_cR_{pdie}g^{ip}x^cx^dx^e \\
& + \frac{1}{12}\nabla_cR_{pdbe}g^{bp}x^cx^dx^e + \frac{1}{45}R_{jnos}R_{qlmr}g^{jq}g^{rs}x^lx^mx^nx^o - \frac{1}{40}\nabla_{lm}R_{qnjo}g^{jq}x^lx^mx^nx^o + \frac{1}{18}R_{pcid}R_{qljm}g^{ip}g^{jq}x^cx^dx^lx^m \\
& + \frac{2}{45}R_{iefh}R_{pcdg}g^{gh}g^{ip}x^cx^dx^ex^f - \frac{1}{20}\nabla_{cd}R_{peif}g^{ip}x^cx^dx^ex^f - \frac{1}{18}R_{pcjd}R_{qlim}g^{ip}g^{jq}x^cx^dx^lx^m \\
& - \frac{1}{45}R_{befh}R_{pcdg}g^{bp}g^{gh}x^cx^dx^ex^f + \frac{1}{40}\nabla_{cd}R_{pebf}g^{bp}x^cx^dx^ex^f + \frac{1}{90}R_{qlms}\nabla_nR_{jort}g^{jq}g^{st}x^lx^mx^nx^ox^r \\
& + \frac{1}{90}R_{jlm}\nabla_nR_{qort}g^{jq}g^{st}x^lx^mx^nx^ox^r - \frac{1}{180}\nabla_{lmn}R_{qojr}g^{jq}x^lx^mx^nx^ox^r + \frac{1}{36}R_{pcid}\nabla_lR_{qmjn}g^{ip}g^{jq}x^cx^dx^lx^mx^n \\
& + \frac{1}{36}R_{qljm}\nabla_cR_{pdie}g^{ip}g^{jq}x^cx^dx^ex^lx^m + \frac{1}{45}R_{pcdh}\nabla_eR_{ifgk}g^{hk}g^{ip}x^cx^dx^ex^fx^g + \frac{1}{45}R_{icdh}\nabla_eR_{pfgk}g^{hk}g^{ip}x^cx^dx^ex^fx^g \\
& - \frac{1}{90}\nabla_{cde}R_{pfig}g^{ip}x^cx^dx^ex^fx^g - \frac{1}{36}R_{pcjd}\nabla_lR_{qmin}g^{ip}g^{jq}x^cx^dx^lx^mx^n - \frac{1}{36}R_{qlim}\nabla_cR_{pdje}g^{ip}g^{jq}x^cx^dx^ex^lx^m \\
& - \frac{1}{90}R_{pcdh}\nabla_eR_{bfgk}g^{bp}g^{hk}x^cx^dx^ex^fx^g - \frac{1}{90}R_{bcdh}\nabla_eR_{pfgk}g^{bp}g^{hk}x^cx^dx^ex^fx^g + \frac{1}{180}\nabla_{cde}R_{pfbg}g^{bp}x^cx^dx^ex^fx^g
\end{aligned}$$

$$\begin{aligned}
\text{Ndetg.005} &:= 1 - \frac{1}{3}R_{abcd}g^{ca}x^bx^d - \frac{1}{6}\nabla_e R_{abcd}g^{ca}x^ex^bx^d - \frac{1}{20}\nabla_{ef}R_{abcd}g^{ca}x^ex^fx^bx^d + \frac{1}{18}R_{abcd}R_{efgh}g^{ca}g^{ge}x^bx^dx^fx^h \\
&\quad + \frac{2}{45}R_{abcd}R_{efgh}g^{hd}g^{ae}x^fx^gx^bx^c - \frac{1}{18}R_{abcd}R_{efgh}g^{ga}g^{ce}x^bx^dx^fx^h - \frac{1}{90}\nabla_{efg}R_{abcd}g^{ca}x^ex^fx^gx^bx^d \\
&\quad + \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ca}g^{ge}x^bx^dx^ix^fx^h + \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ge}g^{ca}x^ix^fx^hx^bx^d + \frac{1}{45}R_{abcd}\nabla_i R_{efgh}g^{dh}g^{ea}x^bx^cx^ix^fx^g \\
&\quad + \frac{1}{45}R_{abcd}\nabla_i R_{efgh}g^{dh}g^{ae}x^bx^cx^ix^fx^g - \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ga}g^{ce}x^bx^dx^ix^fx^h - \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ce}g^{ga}x^ix^fx^hx^bx^d \\
\text{Ndetg.006} &:= 1 - \frac{1}{3}R_{abcd}g^{ac}x^bx^d - \frac{1}{6}\nabla_a R_{bcde}g^{bd}x^ax^cx^e - \frac{1}{20}\nabla_{ab}R_{cdef}g^{ce}x^ax^bx^dx^f + \frac{1}{18}R_{abcd}R_{efgh}g^{ac}g^{eg}x^bx^dx^fx^h - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{90}\nabla_{abc}R_{defg}g^{df}x^ax^bx^cx^ex^g + \frac{1}{18}R_{abcd}\nabla_e R_{fghi}g^{ac}g^{fh}x^bx^dx^ex^gx^i - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i \\
\text{Ndetg.104} &:= 1 - \frac{1}{3}R_{bd}x^bx^d - \frac{1}{6}\nabla_a R_{ce}x^ax^cx^e - \frac{1}{20}\nabla_{ab}R_{df}x^ax^bx^dx^f + \frac{1}{18}R_{bd}R_{fh}x^bx^dx^fx^h - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{90}\nabla_{abc}R_{eg}x^ax^bx^cx^ex^g + \frac{1}{18}R_{bd}\nabla_e R_{gi}x^bx^dx^ex^gx^i - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i \\
\text{sqrtNdetg.004} &:= 1 - \frac{1}{6}R_{ab}x^ax^b - \frac{1}{12}\nabla_a R_{bc}x^ax^bx^c - \frac{1}{40}\nabla_{ab}R_{cd}x^ax^bx^cx^d + \frac{1}{72}R_{ab}R_{cd}x^ax^bx^cx^d - \frac{1}{180}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{180}\nabla_{abc}R_{de}x^ax^bx^cx^dx^e + \frac{1}{72}R_{ab}\nabla_c R_{de}x^ax^bx^cx^dx^e - \frac{1}{180}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i \\
\text{logNdetg.004} &:= -\frac{1}{3}R_{ab}x^ax^b - \frac{1}{6}\nabla_a R_{bc}x^ax^bx^c - \frac{1}{20}\nabla_{ab}R_{cd}x^ax^bx^cx^d - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{90}\nabla_{abc}R_{de}x^ax^bx^cx^dx^e - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i
\end{aligned}$$



```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}                                -> A000^{a}                $)
    substitute (obj,$ g^{a b}                              -> A001^{a b}                $)
    substitute (obj,$ \nabla_{c d e f}\{R_{a b}\}           -> A007_{a b c d e f}      $)
    substitute (obj,$ \nabla_{c d e}\{R_{a b}\}              -> A006_{a b c d e}        $)
    substitute (obj,$ \nabla_{c d}\{R_{a b}\}                -> A005_{a b c d}          $)
    substitute (obj,$ \nabla_{c}\{R_{a b}\}                  -> A004_{a b c}            $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\}         -> A011_{a b c d e f g h}  $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}          -> A010_{a b c d e f g}    $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}            -> A009_{a b c d e f}      $)
    substitute (obj,$ \nabla_{e}\{R_{a b c d}\}              -> A008_{a b c d e}        $)
    substitute (obj,$ R_{a b}                               -> A002_{a b}              $)
    substitute (obj,$ R_{a b c d}                           -> A003_{a b c d}          $)
    sort_product      (obj)
    rename_dummies    (obj)
    substitute (obj,$ A000^{a}                                -> x^{a}                $)
    substitute (obj,$ A001^{a b}                              -> g^{a b}              $)
    substitute (obj,$ A002_{a b}                              -> R_{a b}              $)
    substitute (obj,$ A003_{a b c d}                          -> R_{a b c d}          $)
    substitute (obj,$ A004_{a b c}                            -> \nabla_{c}\{R_{a b}\}    $)
    substitute (obj,$ A005_{a b c d}                          -> \nabla_{c d}\{R_{a b}\}   $)
    substitute (obj,$ A006_{a b c d e}                        -> \nabla_{c d e}\{R_{a b}\} $)
    substitute (obj,$ A007_{a b c d e f}                      -> \nabla_{c d e f}\{R_{a b}\} $)
    substitute (obj,$ A008_{a b c d e}                        -> \nabla_{e}\{R_{a b c d}\} $)
    substitute (obj,$ A009_{a b c d e f}                      -> \nabla_{e f}\{R_{a b c d}\} $)
    substitute (obj,$ A010_{a b c d e f g}                    -> \nabla_{e f g}\{R_{a b c d}\} $)
    substitute (obj,$ A011_{a b c d e f g h}                  -> \nabla_{e f g h}\{R_{a b c d}\} $)

    return obj

def get_term (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah  = Ex("numx = " + str(n))
    keep_weight (foo,bah)

```

```

    return foo

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    sort_sum (bah)
    factor_out (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    factor_out (bah,$x^{a?}$)
    return bah

# -----
# reformat Ndetg

Rterm0 = get_term (Ndetg,0)      # cdb(Rterm0.701,Rterm0)
Rterm1 = get_term (Ndetg,1)      # cdb(Rterm1.701,Rterm1)
Rterm2 = get_term (Ndetg,2)      # cdb(Rterm2.701,Rterm2)
Rterm3 = get_term (Ndetg,3)      # cdb(Rterm3.701,Rterm3)
Rterm4 = get_term (Ndetg,4)      # cdb(Rterm4.701,Rterm4)
Rterm5 = get_term (Ndetg,5)      # cdb(Rterm5.701,Rterm5)

Rterm0 = reformat (Rterm0, 1)     # cdb(Rterm0.702,Rterm0)
Rterm1 = reformat (Rterm1, 1)     # cdb(Rterm1.702,Rterm1)
Rterm2 = reformat (Rterm2, 3)     # cdb(Rterm2.702,Rterm2)
Rterm3 = reformat (Rterm3, 6)     # cdb(Rterm3.702,Rterm3)
Rterm4 = reformat (Rterm4,180)    # cdb(Rterm4.702,Rterm4)
Rterm5 = reformat (Rterm5, 90)    # cdb(Rterm5.702,Rterm5)

```

```

Ndetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Ndetg.701,Ndetg)

# -----
# reformat sqrtNdetg

Rterm0 = get_term (sqrtNdetg,0)  # cdb(Rterm0.801,Rterm0)
Rterm1 = get_term (sqrtNdetg,1)  # cdb(Rterm1.801,Rterm1)
Rterm2 = get_term (sqrtNdetg,2)  # cdb(Rterm2.801,Rterm2)
Rterm3 = get_term (sqrtNdetg,3)  # cdb(Rterm3.801,Rterm3)
Rterm4 = get_term (sqrtNdetg,4)  # cdb(Rterm4.801,Rterm4)
Rterm5 = get_term (sqrtNdetg,5)  # cdb(Rterm5.801,Rterm5)

Rterm0 = reformat (Rterm0, 1)  # cdb(Rterm0.802,Rterm0)
Rterm1 = reformat (Rterm1, 1)  # cdb(Rterm1.802,Rterm1)
Rterm2 = reformat (Rterm2, 6)  # cdb(Rterm2.802,Rterm2)
Rterm3 = reformat (Rterm3, 12) # cdb(Rterm3.802,Rterm3)
Rterm4 = reformat (Rterm4,360) # cdb(Rterm4.802,Rterm4)
Rterm5 = reformat (Rterm5,360) # cdb(Rterm5.802,Rterm5)

sqrtNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (sqrtNdetg.801,sqrtNdetg)

# -----
# reformat logNdetg

Rterm0 = get_term (logNdetg,0)  # cdb(Rterm0.901,Rterm0)
Rterm1 = get_term (logNdetg,1)  # cdb(Rterm1.901,Rterm1)
Rterm2 = get_term (logNdetg,2)  # cdb(Rterm2.901,Rterm2)
Rterm3 = get_term (logNdetg,3)  # cdb(Rterm3.901,Rterm3)
Rterm4 = get_term (logNdetg,4)  # cdb(Rterm4.901,Rterm4)
Rterm5 = get_term (logNdetg,5)  # cdb(Rterm5.901,Rterm5)

Rterm0 = reformat (Rterm0, 1)  # cdb(Rterm0.902,Rterm0)
Rterm1 = reformat (Rterm1, 1)  # cdb(Rterm1.902,Rterm1)
Rterm2 = reformat (Rterm2, 3)  # cdb(Rterm2.902,Rterm2)
Rterm3 = reformat (Rterm3, 6)  # cdb(Rterm3.902,Rterm3)
Rterm4 = reformat (Rterm4,180) # cdb(Rterm4.902,Rterm4)
Rterm5 = reformat (Rterm5, 90) # cdb(Rterm5.902,Rterm5)

```

```
logNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5). # cdb (logNdetg.901,logNdetg)
```

## The metric determinant in Riemann normal coordinates

$$\begin{aligned} -\det g(x) = & 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) \\ & + \frac{1}{90}x^a x^b x^c x^d x^e (-\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

## The volume element in Riemann normal coordinates

If  $-\det g(x)$  is non-negative then we also have

$$\begin{aligned} \sqrt{-\det g(x)} = & 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) \\ & + \frac{1}{360}x^a x^b x^c x^d x^e (-2\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - 2g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

## The log of -detg in Riemann normal coordinates

Apart from the signs, this matches exactly the expression given by Calzetta et al. (eq. A14)

$$\begin{aligned} \log(-\det g(x)) = & -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) \\ & + \frac{1}{90}x^a x^b x^c x^d x^e (-\nabla_{abc}R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

```
cdblib.create ('detg2.export')

cdblib.put ('Ndetg',    Ndetg,    'detg2.export')
cdblib.put ('sqrtNdetg',sqrtNdetg,'detg2.export')
cdblib.put ('logNdetg', logNdetg, 'detg2.export')

checkpoint.append (Ndetg)
checkpoint.append (sqrtNdetg)
checkpoint.append (logNdetg)
```