

Checking the 2nd and 3rd order terms of Calzetta etal.

The following calculations show that my results for the RNC connection agree with those of Calzetta etal. to third order terms.

Note that I take ∇_{ab} to be $\nabla_a(\nabla_b)$.

Note also that $(LCB) R_{abcd} = -(Calzetta) R_{abcd}$. Consequently, I replace R_{abcd} with $-R_{abcd}$ in the Calzetta expressions (done as a Cadabra substitution rule).

This is relatively straightforward. We just apply a few carefully chosen applications of the first and second Bianchi identities.

Note that in this example, 2nd and 3rd order refer to the powers of x in the expression. This differs from the usage elsewhere in these examples (3rd and 4th order with respect to R and its derivatives).

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,u,v,w#}::Indices("latin",position=independent).
{\mu,\nu,\rho,\sigma,\tau,\lambda,\xi#}::Indices("greek",position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g^{a b}::Weight(label=gnum,value=1).

\delta{#}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R_{a b c d}::Depends(\nabla{#}).

x^{a}::Weight(label=xnum,value=1).

def add_tags (obj,tag):

    n = 0
    ans = Ex('0')

    for i in obj.top().terms():
        foo = obj[i]
        bah = Ex(tag+'_{'+str(n)+'}')
        ans := @(ans) + @(bah) @(foo).
        n = n + 1

    return ans

def clear_tags (obj,tag):

    ans := @(obj).
    foo = Ex(tag+'_{a?} -> 1')
    substitute (ans,foo)

    return ans

def get_xterm (obj,n):

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foo := @(obj).
bah = Ex("xnum = " + str(n))
distribute (foo)
keep_weight (foo, bah)

return foo

def get_gterm (obj,n):

foo := @(obj).
bah = Ex("gnum = " + str(n))
distribute (foo)
keep_weight (foo, bah)

return foo

def product_sort (obj):
substitute (obj,$ g^{a b} -> A001^{a b} $)
substitute (obj,$ x^{a} -> A002^{a} $)
substitute (obj,$ z^{a} -> A003^{a} $)
substitute (obj,$ R_{a b c d} -> A004_{a b c d} $)
substitute (obj,$ \nabla_{e}\{R_{a b c d}\} -> A005_{a b c d e} $)
substitute (obj,$ \nabla_{e f}\{R_{a b c d}\} -> A006_{a b c d e f} $)
sort_sum (obj)
sort_product (obj)
rename_dummies (obj)
substitute (obj,$ A001^{a b} -> g^{a b} $)
substitute (obj,$ A002^{a} -> x^{a} $)
substitute (obj,$ A003^{a} -> z^{a} $)
substitute (obj,$ A004_{a b c d} -> R_{a b c d} $)
substitute (obj,$ A005_{a b c d e} -> \nabla_{e}\{R_{a b c d}\} $)
substitute (obj,$ A006_{a b c d e f} -> \nabla_{e f}\{R_{a b c d}\} $)

return obj

def reformat (obj,scaleA,scaleB):

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foo = Ex(str(scaleA))
moo = Ex(str(scaleB))
bah := @(foo) @(obj) / @(moo).

distribute      (bah)
bah = product_sort (bah)
rename_dummies  (bah)
canonicalise    (bah)
factor_out      (bah,$g^{c? d?}$)
factor_out      (bah,$x^{a?},z^{b?}$)
ans := @(moo) @(bah) / @(foo).

return ans

# =====
# LCB

import cdblib
Gamma = cdblib.get ('Gamma','../connection.json')           # cdb(ex-12.100,Gamma)

# note that the next two lines require careful inspection of the free indices on Gamma
# expecting Gamma = \Gamma^{d}_{ab}
Gamma := z^{a} z^{b} @(Gamma).

# lower index ^{d} to _{v}

Gamma := g_{v d} @(Gamma).

distribute (Gamma)
substitute (Gamma, $g_{a d} g^{d b} -> \delta_{a}^{b}$)
eliminate_kronecker (Gamma)                                # cdb(ex-12.101,Gamma)

# change free index _{v} to _{a}

foo := tmp_{v} -> @(Gamma).                                # cdb(ex-12.191,foo)
bah := tmp_{a}.                                             # cdb(ex-12.192,bah)
substitute (bah, foo)                                       # cdb(ex-12.193,bah)

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Gamma := @(bah).                                # cdb(ex-12.102,Gamma)

Gamma = product_sort (Gamma)                    # cdb(ex-12.103,Gamma)

gam1 = get_xterm (Gamma,1)                      # cdb(ex-12.200,gam1)
gam2 = get_xterm (Gamma,2)                      # cdb(ex-12.201,gam2)
gam3 = get_xterm (Gamma,3)                      # cdb(ex-12.202,gam3)

gam30 = get_gterm (gam3,0)                      # cdb(ex-12.203,gam30)
gam31 = get_gterm (gam3,1)                      # cdb(ex-12.204,gam31)

gam1 = reformat (gam1, 3,1)                     # cdb(ex-12.300,gam1)
gam2 = reformat (gam2,12,1)                     # cdb(ex-12.301,gam2)

gam30 = reformat (gam30,40,1)                   # cdb(ex-12.302,gam30)
gam31 = reformat (gam31,45,2)                   # cdb(ex-12.303,gam31)

gam3 := @(gam30) + @(gam31).                    # cdb(ex-12.304,gam3)

Gamma := @(gam1) + @(gam2) + @(gam3).           # cdb(ex-12.305,Gamma)

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$$\begin{aligned}
\text{ex-12.100} := & x^c \left(\frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de} \right) + x^c x^e \left(\frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df} \right) \\
& + x^c x^e x^f \left(\frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \right. \\
& \quad + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \\
& \quad \left. - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg} \right) \\
& + x^c x^e x^f x^g \left(\frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} \right. \\
& \quad + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} \\
& \quad - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgj} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgj} g^{dh} \\
& \quad + \frac{1}{180} \nabla_{cea} R_{bfgj} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} \\
& \quad - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} \\
& \quad - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} \\
& \quad \left. - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh} \right)
\end{aligned}$$

ex-12.191 := tmp_v

$$\begin{aligned}
& \rightarrow \frac{1}{3}z^a z^b x^c R_{avbc} + \frac{1}{3}z^a z^b x^c R_{acbv} + \frac{1}{12}z^a z^b x^c x^e \nabla_a R_{bcev} + \frac{1}{6}z^a z^b x^c x^e \nabla_c R_{avbe} + \frac{1}{12}z^a z^b x^c x^e \nabla_b R_{acev} + \frac{1}{6}z^a z^b x^c x^e \nabla_e R_{aebv} \\
& + \frac{1}{12}z^a z^b x^c x^e \nabla_v R_{acbe} + \frac{4}{45}z^a z^b x^c x^e x^f R_{agbc} R_{evfi} g^{gi} + \frac{4}{45}z^a z^b x^c x^e x^f R_{acbg} R_{evfi} g^{gi} - \frac{2}{45}z^a z^b x^c x^e x^f R_{avch} R_{befi} g^{hi} \\
& - \frac{1}{45}z^a z^b x^c x^e x^f R_{agcv} R_{befi} g^{gi} + \frac{1}{40}z^a z^b x^c x^e x^f \nabla_{ac} R_{befv} + \frac{1}{40}z^a z^b x^c x^e x^f \nabla_{ca} R_{befv} + \frac{1}{20}z^a z^b x^c x^e x^f \nabla_{ce} R_{avbf} \\
& - \frac{2}{45}z^a z^b x^c x^e x^f R_{aceg} R_{bvfi} g^{gi} - \frac{1}{45}z^a z^b x^c x^e x^f R_{aceg} R_{bhfv} g^{gh} + \frac{1}{40}z^a z^b x^c x^e x^f \nabla_{bc} R_{aefv} + \frac{1}{40}z^a z^b x^c x^e x^f \nabla_{cb} R_{aefv} \\
& + \frac{1}{20}z^a z^b x^c x^e x^f \nabla_{ce} R_{afbv} - \frac{1}{45}z^a z^b x^c x^e x^f R_{acvh} R_{befi} g^{hi} - \frac{1}{45}z^a z^b x^c x^e x^f R_{aceg} R_{bfvi} g^{gi} + \frac{1}{40}z^a z^b x^c x^e x^f \nabla_{vc} R_{aebf} \\
& + \frac{1}{40}z^a z^b x^c x^e x^f \nabla_{cv} R_{aebf} + \frac{2}{45}z^a z^b x^c x^e x^f x^g R_{ahbc} \nabla_e R_{fv gj} g^{hj} + \frac{2}{45}z^a z^b x^c x^e x^f x^g R_{acbh} \nabla_e R_{fv gj} g^{hj} + \frac{1}{60}z^a z^b x^c x^e x^f x^g R_{cvei} \nabla_a R_{bf gj} g^{ij} \\
& + \frac{2}{45}z^a z^b x^c x^e x^f x^g R_{cvei} \nabla_f R_{aj bg} g^{ij} + \frac{1}{60}z^a z^b x^c x^e x^f x^g R_{cvei} \nabla_b R_{af gj} g^{ij} + \frac{2}{45}z^a z^b x^c x^e x^f x^g R_{cvei} \nabla_f R_{ag bj} g^{ij} \\
& + \frac{1}{36}z^a z^b x^c x^e x^f x^g R_{cvei} \nabla_j R_{af bg} g^{ij} - \frac{1}{45}z^a z^b x^c x^e x^f x^g R_{avci} \nabla_e R_{bf gj} g^{ij} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{ahcv} \nabla_e R_{bf gj} g^{hj} \\
& - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{bceh} \nabla_a R_{fv gj} g^{hj} - \frac{1}{45}z^a z^b x^c x^e x^f x^g R_{bceh} \nabla_f R_{av gj} g^{hj} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{bceh} \nabla_f R_{aigv} g^{hi} \\
& + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{ace} R_{bf gv} + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{cae} R_{bf gv} + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{cea} R_{bf gv} + \frac{1}{90}z^a z^b x^c x^e x^f x^g \nabla_{cef} R_{avbg} \\
& - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{aceh} \nabla_b R_{fv gj} g^{hj} - \frac{1}{45}z^a z^b x^c x^e x^f x^g R_{aceh} \nabla_f R_{bv gj} g^{hj} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{aceh} \nabla_f R_{bigv} g^{hi} \\
& - \frac{1}{45}z^a z^b x^c x^e x^f x^g R_{bvci} \nabla_e R_{af gj} g^{ij} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{bhcv} \nabla_e R_{af gj} g^{hj} + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{bce} R_{af gv} + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{cbe} R_{af gv} \\
& + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{ceb} R_{af gv} + \frac{1}{90}z^a z^b x^c x^e x^f x^g \nabla_{cef} R_{agbv} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{acvi} \nabla_e R_{bf gj} g^{ij} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{aceh} \nabla_v R_{bf gj} g^{hj} \\
& - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{aceh} \nabla_f R_{bgvj} g^{hj} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{bcvi} \nabla_e R_{af gj} g^{ij} - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{bceh} \nabla_v R_{af gj} g^{hj} \\
& - \frac{1}{90}z^a z^b x^c x^e x^f x^g R_{bceh} \nabla_f R_{agvj} g^{hj} + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{vce} R_{af bg} + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{cve} R_{af bg} + \frac{1}{180}z^a z^b x^c x^e x^f x^g \nabla_{cev} R_{af bg}
\end{aligned}$$

ex-12.192 := tmp_a

$$\begin{aligned}
\text{ex-12.193} := & \frac{1}{3}z^dz^bx^cR_{dabc} + \frac{1}{3}z^dz^bx^cR_{dcba} + \frac{1}{12}z^dz^bx^cx^e\nabla_dR_{bcea} + \frac{1}{6}z^dz^bx^cx^e\nabla_cR_{dabe} + \frac{1}{12}z^dz^bx^cx^e\nabla_bR_{dcea} + \frac{1}{6}z^dz^bx^cx^e\nabla_cR_{deba} \\
& + \frac{1}{12}z^dz^bx^cx^e\nabla_aR_{dcbe} + \frac{4}{45}z^dz^bx^cx^ex^fR_{dgbce}R_{eafgi}g^{gi} + \frac{4}{45}z^dz^bx^cx^ex^fR_{dcbg}R_{eafgi}g^{gi} - \frac{2}{45}z^dz^bx^cx^ex^fR_{dach}R_{befgi}g^{hi} \\
& - \frac{1}{45}z^dz^bx^cx^ex^fR_{dgca}R_{befgi}g^{gi} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{dc}R_{befa} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{cd}R_{befa} + \frac{1}{20}z^dz^bx^cx^ex^f\nabla_{ce}R_{dabf} \\
& - \frac{2}{45}z^dz^bx^cx^ex^fR_{dceg}R_{bafgi}g^{gi} - \frac{1}{45}z^dz^bx^cx^ex^fR_{dceg}R_{bhfa}g^{gh} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{bc}R_{defa} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{cb}R_{defa} \\
& + \frac{1}{20}z^dz^bx^cx^ex^f\nabla_{ce}R_{dfba} - \frac{1}{45}z^dz^bx^cx^ex^fR_{dcah}R_{befgi}g^{hi} - \frac{1}{45}z^dz^bx^cx^ex^fR_{dceg}R_{bfai}g^{gi} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{ac}R_{debf} \\
& + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{ca}R_{debf} + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{dhbc}\nabla_eR_{fagj}g^{hj} + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{dcbh}\nabla_eR_{fagj}g^{hj} + \frac{1}{60}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_dR_{bfgj}g^{ij} \\
& + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_fR_{djb g}g^{ij} + \frac{1}{60}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_bR_{dfgj}g^{ij} + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_fR_{dgbj}g^{ij} \\
& + \frac{1}{36}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_jR_{dfbg}g^{ij} - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{daci}\nabla_eR_{bfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dhca}\nabla_eR_{bfgj}g^{hj} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bce h}\nabla_dR_{fagj}g^{hj} - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{bce h}\nabla_fR_{dagj}g^{hj} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bce h}\nabla_fR_{diga}g^{hi} \\
& + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{dce}R_{bfga} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cde}R_{bfga} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{ced}R_{bfga} + \frac{1}{90}z^dz^bx^cx^ex^fx^g\nabla_{cef}R_{dabg} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dce h}\nabla_bR_{fagj}g^{hj} - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{dce h}\nabla_fR_{bagj}g^{hj} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dce h}\nabla_fR_{biga}g^{hi} \\
& - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{baci}\nabla_eR_{dfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bhca}\nabla_eR_{dfgj}g^{hj} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{bce}R_{dfga} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cbe}R_{dfga} \\
& + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{ceb}R_{dfga} + \frac{1}{90}z^dz^bx^cx^ex^fx^g\nabla_{cef}R_{dgba} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dcai}\nabla_eR_{bfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dce h}\nabla_aR_{bfgj}g^{hj} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dce h}\nabla_fR_{bgaj}g^{hj} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bcai}\nabla_eR_{dfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bce h}\nabla_aR_{dfgj}g^{hj} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bce h}\nabla_fR_{dga j}g^{hj} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{ace}R_{dfbg} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cae}R_{dfbg} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cea}R_{dfbg}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.101} := & \frac{1}{3}z^az^bx^cR_{avbc} + \frac{1}{3}z^az^bx^cR_{acbv} + \frac{1}{12}z^az^bx^cx^e\nabla_aR_{bcev} + \frac{1}{6}z^az^bx^cx^e\nabla_cR_{avbe} + \frac{1}{12}z^az^bx^cx^e\nabla_bR_{acev} + \frac{1}{6}z^az^bx^cx^e\nabla_eR_{aebv} \\
& + \frac{1}{12}z^az^bx^cx^e\nabla_vR_{acbe} + \frac{4}{45}z^az^bx^cx^ex^fR_{agbc}R_{evfi}g^{gi} + \frac{4}{45}z^az^bx^cx^ex^fR_{acbg}R_{evfi}g^{gi} - \frac{2}{45}z^az^bx^cx^ex^fR_{avch}R_{befi}g^{hi} \\
& - \frac{1}{45}z^az^bx^cx^ex^fR_{agcv}R_{befi}g^{gi} + \frac{1}{40}z^az^bx^cx^ex^f\nabla_{ac}R_{befv} + \frac{1}{40}z^az^bx^cx^ex^f\nabla_{ca}R_{befv} + \frac{1}{20}z^az^bx^cx^ex^f\nabla_{ce}R_{avbf} \\
& - \frac{2}{45}z^az^bx^cx^ex^fR_{aceg}R_{bvfi}g^{gi} - \frac{1}{45}z^az^bx^cx^ex^fR_{aceg}R_{bhfv}g^{gh} + \frac{1}{40}z^az^bx^cx^ex^f\nabla_{bc}R_{aefv} + \frac{1}{40}z^az^bx^cx^ex^f\nabla_{cb}R_{aefv} \\
& + \frac{1}{20}z^az^bx^cx^ex^f\nabla_{ce}R_{afbv} - \frac{1}{45}z^az^bx^cx^ex^fR_{acvh}R_{befi}g^{hi} - \frac{1}{45}z^az^bx^cx^ex^fR_{aceg}R_{bfvi}g^{gi} + \frac{1}{40}z^az^bx^cx^ex^f\nabla_{vc}R_{aebf} \\
& + \frac{1}{40}z^az^bx^cx^ex^f\nabla_{cv}R_{aebf} + \frac{2}{45}z^az^bx^cx^ex^fx^gR_{ahbc}\nabla_eR_{fv}g^{hj} + \frac{2}{45}z^az^bx^cx^ex^fx^gR_{acbh}\nabla_eR_{fv}g^{hj} + \frac{1}{60}z^az^bx^cx^ex^fx^gR_{cvei}\nabla_aR_{bf}g^{ij} \\
& + \frac{2}{45}z^az^bx^cx^ex^fx^gR_{cvei}\nabla_fR_{aj}g^{ij} + \frac{1}{60}z^az^bx^cx^ex^fx^gR_{cvei}\nabla_bR_{af}g^{ij} + \frac{2}{45}z^az^bx^cx^ex^fx^gR_{cvei}\nabla_fR_{ag}g^{ij} \\
& + \frac{1}{36}z^az^bx^cx^ex^fx^gR_{cvei}\nabla_jR_{af}g^{ij} - \frac{1}{45}z^az^bx^cx^ex^fx^gR_{avci}\nabla_eR_{bf}g^{ij} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{ahcv}\nabla_eR_{bf}g^{ij} \\
& - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{bceh}\nabla_aR_{fv}g^{hj} - \frac{1}{45}z^az^bx^cx^ex^fx^gR_{bceh}\nabla_fR_{av}g^{hj} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{bceh}\nabla_fR_{aigv}g^{hi} \\
& + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{ace}R_{bf}g^{ij} + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{cae}R_{bf}g^{ij} + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{cea}R_{bf}g^{ij} + \frac{1}{90}z^az^bx^cx^ex^fx^g\nabla_{cef}R_{avbg} \\
& - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{aceh}\nabla_bR_{fv}g^{hj} - \frac{1}{45}z^az^bx^cx^ex^fx^gR_{aceh}\nabla_fR_{bv}g^{hj} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{aceh}\nabla_fR_{bigv}g^{hi} \\
& - \frac{1}{45}z^az^bx^cx^ex^fx^gR_{bvci}\nabla_eR_{af}g^{ij} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{bhcv}\nabla_eR_{af}g^{ij} + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{bce}R_{af}g^{ij} + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{cbe}R_{af}g^{ij} \\
& + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{ceb}R_{af}g^{ij} + \frac{1}{90}z^az^bx^cx^ex^fx^g\nabla_{cef}R_{agbv} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{acvi}\nabla_eR_{bf}g^{ij} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{aceh}\nabla_vR_{bf}g^{ij} \\
& - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{aceh}\nabla_fR_{bgv}g^{hj} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{bcvi}\nabla_eR_{af}g^{ij} - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{bceh}\nabla_vR_{af}g^{ij} \\
& - \frac{1}{90}z^az^bx^cx^ex^fx^gR_{bceh}\nabla_fR_{agv}g^{hj} + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{vce}R_{afbg} + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{cve}R_{afbg} + \frac{1}{180}z^az^bx^cx^ex^fx^g\nabla_{cev}R_{afbg}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.102} := & \frac{1}{3}z^dz^bx^cR_{dabc} + \frac{1}{3}z^dz^bx^cR_{dcba} + \frac{1}{12}z^dz^bx^cx^e\nabla_dR_{bcea} + \frac{1}{6}z^dz^bx^cx^e\nabla_cR_{dabe} + \frac{1}{12}z^dz^bx^cx^e\nabla_bR_{dcea} + \frac{1}{6}z^dz^bx^cx^e\nabla_cR_{deba} \\
& + \frac{1}{12}z^dz^bx^cx^e\nabla_aR_{dcbe} + \frac{4}{45}z^dz^bx^cx^ex^fR_{dgbce}R_{eafgi}g^{gi} + \frac{4}{45}z^dz^bx^cx^ex^fR_{dcbg}R_{eafgi}g^{gi} - \frac{2}{45}z^dz^bx^cx^ex^fR_{dach}R_{befgi}g^{hi} \\
& - \frac{1}{45}z^dz^bx^cx^ex^fR_{dgca}R_{befgi}g^{gi} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{dc}R_{befa} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{cd}R_{befa} + \frac{1}{20}z^dz^bx^cx^ex^f\nabla_{ce}R_{dabf} \\
& - \frac{2}{45}z^dz^bx^cx^ex^fR_{dceg}R_{bafgi}g^{gi} - \frac{1}{45}z^dz^bx^cx^ex^fR_{dceg}R_{bhfa}g^{gh} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{bc}R_{defa} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{cb}R_{defa} \\
& + \frac{1}{20}z^dz^bx^cx^ex^f\nabla_{ce}R_{dfba} - \frac{1}{45}z^dz^bx^cx^ex^fR_{dcah}R_{befgi}g^{hi} - \frac{1}{45}z^dz^bx^cx^ex^fR_{dceg}R_{bfai}g^{gi} + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{ac}R_{debf} \\
& + \frac{1}{40}z^dz^bx^cx^ex^f\nabla_{ca}R_{debf} + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{dhbc}\nabla_eR_{fagj}g^{hj} + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{dcbh}\nabla_eR_{fagj}g^{hj} + \frac{1}{60}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_dR_{bfgj}g^{ij} \\
& + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_fR_{djbg}g^{ij} + \frac{1}{60}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_bR_{dfgj}g^{ij} + \frac{2}{45}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_fR_{dgbj}g^{ij} \\
& + \frac{1}{36}z^dz^bx^cx^ex^fx^gR_{cae i}\nabla_jR_{dfbg}g^{ij} - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{daci}\nabla_eR_{bfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dhca}\nabla_eR_{bfgj}g^{hj} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bceh}\nabla_dR_{fagj}g^{hj} - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{bceh}\nabla_fR_{dagj}g^{hj} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bceh}\nabla_fR_{diga}g^{hi} \\
& + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{dce}R_{bfga} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cde}R_{bfga} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{ced}R_{bfga} + \frac{1}{90}z^dz^bx^cx^ex^fx^g\nabla_{cef}R_{dabg} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dceh}\nabla_bR_{fagj}g^{hj} - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{dceh}\nabla_fR_{bagj}g^{hj} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dceh}\nabla_fR_{biga}g^{hi} \\
& - \frac{1}{45}z^dz^bx^cx^ex^fx^gR_{baci}\nabla_eR_{dfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bhca}\nabla_eR_{dfgj}g^{hj} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{bce}R_{dfga} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cbe}R_{dfga} \\
& + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{ceb}R_{dfga} + \frac{1}{90}z^dz^bx^cx^ex^fx^g\nabla_{cef}R_{dgba} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dcai}\nabla_eR_{bfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dceh}\nabla_aR_{bfgj}g^{hj} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{dceh}\nabla_fR_{bgaj}g^{hj} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bcai}\nabla_eR_{dfgj}g^{ij} - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bceh}\nabla_aR_{dfgj}g^{hj} \\
& - \frac{1}{90}z^dz^bx^cx^ex^fx^gR_{bceh}\nabla_fR_{dga j}g^{hj} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{ace}R_{dfbg} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cae}R_{dfbg} + \frac{1}{180}z^dz^bx^cx^ex^fx^g\nabla_{cea}R_{dfbg}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.103} := & \frac{1}{3}x^bz^cz^dR_{dacb} + \frac{1}{3}x^bz^cz^dR_{dbca} + \frac{1}{12}x^bx^cz^dz^e\nabla_eR_{dbca} + \frac{1}{6}x^bx^cz^dz^e\nabla_bR_{eadc} + \frac{1}{12}x^bx^cz^dz^e\nabla_aR_{ebdc} + \frac{1}{12}x^bx^cz^dz^e\nabla_dR_{ebca} \\
& + \frac{1}{6}x^bx^cz^dz^e\nabla_bR_{ecda} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{bf}R_{ecda} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fb}R_{ecda} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{faed} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{fced} \\
& + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{fced} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{eb}R_{fcda} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{be}R_{fcda} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{fdea} + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{bgc}R_{fdea} \\
& + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{bcg}R_{fdea} + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{gbc}R_{fdea} + \frac{1}{90}x^bx^cx^dx^ez^fz^g\nabla_{bcd}R_{gafe} + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{abc}R_{gdfc} \\
& + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{bac}R_{gdfc} + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{bca}R_{gdfc} + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{fbc}R_{gdea} + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{bfc}R_{gdea} \\
& + \frac{1}{180}x^bx^cx^dx^ez^fz^g\nabla_{bcf}R_{gdea} + \frac{1}{90}x^bx^cx^dx^ez^fz^g\nabla_{bcd}R_{gefa} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gef c}R_{hadb} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gef c}R_{hdab} \\
& + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{hdgb}R_{eafc} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gafc}R_{hdeb} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gfac}R_{hdeb} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gcf a}R_{hdeb} \\
& + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{hbgd}R_{eafc} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gef c}R_{hbda} - \frac{1}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{hadb}\nabla_eR_{ifgc} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{hdab}\nabla_eR_{ifgc} \\
& - \frac{1}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{hdeb}\nabla_fR_{iagc} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{hdeb}\nabla_aR_{ifgc} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{hdeb}\nabla_fR_{igac} \\
& - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{hdeb}\nabla_fR_{icga} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{hdeb}\nabla_iR_{fagc} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{hbda}\nabla_eR_{ifgc} \\
& + \frac{1}{60}g^{bc}x^dx^ex^fx^gz^hz^iR_{daeb}\nabla_iR_{hfgc} + \frac{1}{36}g^{bc}x^dx^ex^fx^gz^hz^iR_{daeb}\nabla_cR_{ifhg} + \frac{1}{60}g^{bc}x^dx^ex^fx^gz^hz^iR_{daeb}\nabla_hR_{ifgc} \\
& + \frac{2}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{daeb}\nabla_fR_{ighc} + \frac{2}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{daeb}\nabla_fR_{ichg} - \frac{1}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{iadb}\nabla_eR_{hfgc} \\
& - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{idab}\nabla_eR_{hfgc} + \frac{2}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{idhb}\nabla_eR_{fagc} - \frac{1}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{ideb}\nabla_fR_{hagc} \\
& - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{ideb}\nabla_aR_{hfgc} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{ideb}\nabla_fR_{hgac} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{ideb}\nabla_fR_{hcg a} \\
& - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{ideb}\nabla_hR_{fagc} + \frac{2}{45}g^{bc}x^dx^ex^fx^gz^hz^iR_{ibhd}\nabla_eR_{fagc} - \frac{1}{90}g^{bc}x^dx^ex^fx^gz^hz^iR_{ibda}\nabla_eR_{hfgc}
\end{aligned}$$

$$\text{ex-12.200} := \frac{1}{3}x^bz^cz^dR_{dacb} + \frac{1}{3}x^bz^cz^dR_{dbca}$$

$$\text{ex-12.201} := \frac{1}{12}x^bx^cz^dz^e\nabla_eR_{dbca} + \frac{1}{6}x^bx^cz^dz^e\nabla_bR_{eadc} + \frac{1}{12}x^bx^cz^dz^e\nabla_aR_{ebdc} + \frac{1}{12}x^bx^cz^dz^e\nabla_dR_{ebca} + \frac{1}{6}x^bx^cz^dz^e\nabla_bR_{ecda}$$

$$\begin{aligned}
\text{ex-12.202} &:= \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{bf}R_{ecda} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fb}R_{ecda} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{faed} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{fced} \\
&+ \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{fced} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{eb}R_{fcda} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{be}R_{fcda} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{fdea} \\
&- \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gefc}R_{hadb} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gefc}R_{hdab} + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{hdgb}R_{eafc} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gafc}R_{hdeb} \\
&- \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gfac}R_{hdeb} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gcfa}R_{hdeb} + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{hbgd}R_{eafc} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gefc}R_{hbda} \\
\text{ex-12.203} &:= \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{bf}R_{ecda} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fb}R_{ecda} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{faed} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ab}R_{fced} \\
&+ \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{ba}R_{fced} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{eb}R_{fcda} + \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{be}R_{fcda} + \frac{1}{20}x^bx^cx^dz^ez^f\nabla_{bc}R_{fdea} \\
\text{ex-12.204} &:= -\frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gefc}R_{hadb} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gefc}R_{hdab} + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{hdgb}R_{eafc} - \frac{2}{45}g^{bc}x^dx^ex^fz^gz^hR_{gafc}R_{hdeb} \\
&- \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gfac}R_{hdeb} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gcfa}R_{hdeb} + \frac{4}{45}g^{bc}x^dx^ex^fz^gz^hR_{hbgd}R_{eafc} - \frac{1}{45}g^{bc}x^dx^ex^fz^gz^hR_{gefc}R_{hbda}
\end{aligned}$$

$$\text{ex-12.300} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\text{ex-12.301} := \frac{1}{12}x^bx^cz^dz^e(2\nabla_dR_{abce} + 4\nabla_bR_{adce} + \nabla_aR_{bdce})$$

$$\text{ex-12.302} := \frac{1}{40}x^bx^cx^dz^ez^f(2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + 4\nabla_{bc}R_{aedf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf})$$

$$\text{ex-12.303} := \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h(-2R_{agbd}R_{cefh} - R_{adbg}R_{cefh} + R_{abdg}R_{cefh} + 4R_{adbe}R_{cgfh})$$

$$\begin{aligned}
\text{ex-12.304} &:= \frac{1}{40}x^bx^cx^dz^ez^f(2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + 4\nabla_{bc}R_{aedf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf}) \\
&+ \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h(-2R_{agbd}R_{cefh} - R_{adbg}R_{cefh} + R_{abdg}R_{cefh} + 4R_{adbe}R_{cgfh})
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.305} &:= \frac{2}{3}x^bz^cz^dR_{acbd} + \frac{1}{12}x^bx^cz^dz^e(2\nabla_dR_{abce} + 4\nabla_bR_{adce} + \nabla_aR_{bdce}) \\
&+ \frac{1}{40}x^bx^cx^dz^ez^f(2\nabla_{be}R_{acdf} + 2\nabla_{eb}R_{acdf} + 4\nabla_{bc}R_{aedf} + \nabla_{ab}R_{cedf} + \nabla_{ba}R_{cedf}) \\
&+ \frac{2}{45}g^{bc}x^dx^ex^fz^gz^h(-2R_{agbd}R_{cefh} - R_{adbg}R_{cefh} + R_{abdg}R_{cefh} + 4R_{adbe}R_{cgfh})
\end{aligned}$$

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# =====
# Calzetta
# note: \nabla_{a b} defined as \nabla_{a}\nabla_{b}

GammaBar := z^{\nu} z^{\rho} (
  (2/3) R^{\mu}_{\nu\rho\sigma} x^{\sigma}
  + (1/12) (5 \nabla_{\lambda}\{R^{\mu}_{\nu\rho\sigma}\}
    + \nabla_{\rho}\{R^{\mu}_{\sigma\nu\lambda}\}) x^{\sigma} x^{\lambda}
  + (1/6) ( (9/10) \nabla_{\tau\lambda}\{R^{\mu}_{\rho\nu\sigma}\}
    + (3/20) ( \nabla_{\tau\rho}\{R^{\mu}_{\sigma\nu\lambda}\}
      + \nabla_{\rho\tau}\{R^{\mu}_{\sigma\nu\lambda}\} )
    + (1/60) ( 21 R^{\mu}_{\lambda\xi\rho} R^{\xi}_{\sigma\nu\tau}
      + 48 R^{\mu}_{\xi\rho\lambda} R^{\xi}_{\sigma\nu\tau}
      - 37 R^{\mu}_{\sigma\xi\lambda} R^{\xi}_{\nu\rho\tau} ) ) x^{\sigma} x^{\lambda} x^{\tau} ).
# cdb(ex-12.400,GammaBar)

# convert from Greek to Latin indices

distribute (GammaBar)
rename_dummies (GammaBar,"greek","latin") # cdb(ex-12.401,GammaBar)

# lower the \mu index

GammaBar := \delta_{a \mu} @(GammaBar). # cdb(ex-12.402,GammaBar)
distribute (GammaBar) # cdb(ex-12.403,GammaBar)
eliminate_kronecker (GammaBar) # cdb(ex-12.404,GammaBar)

# sort products

GammaBar = product_sort (GammaBar) # cdb(ex-12.405,GammaBar)

# Replace R with - R (Calzetta uses the non-MTW convention for Riemann)

substitute (GammaBar, $R_{a b c d} -> - R_{a b c d}$) # cdb(ex-12.406,GammaBar)
substitute (GammaBar, $R^{\{a\}_{b c d} -> - R^{\{a\}_{b c d}$) # cdb(ex-12.407,GammaBar)

substitute (GammaBar, $R^{\{a\}_{b c d} -> g^{\{a e} R_{e b c d}$) # cdb(ex-12.408,GammaBar)

```

```

cal1 = get_xterm (GammaBar,1)           # cdb(ex-12.500,cal1)
cal2 = get_xterm (GammaBar,2)           # cdb(ex-12.501,cal2)
cal3 = get_xterm (GammaBar,3)           # cdb(ex-12.502,cal3)

cal1 = reformat (cal1,3,1)               # cdb(ex-12.600,cal1)
cal2 = reformat (cal2,12,1)              # cdb(ex-12.601,cal2)
# cal3 = reformat (cal3,360,1)           # cdb(ex-12.602,cal3)

cal30 = get_gterm (cal3,0)               # cdb(ex-12.602,cal30)
cal31 = get_gterm (cal3,1)               # cdb(ex-12.603,cal31)

cal1 = reformat (cal1, 3,1)              # cdb(ex-12.604,cal1)
cal2 = reformat (cal2,12,1)              # cdb(ex-12.605,cal2)

cal30 = reformat (cal30,40,1)            # cdb(ex-12.606,cal30)
cal31 = reformat (cal31,360,1)           # cdb(ex-12.607,cal31)

cal3 := @(cal30) + @(cal31).             # cdb(ex-12.608,cal3)

GammaBar := @(cal1) + @(cal2) + @(cal3). # cdb(ex-12.409,GammaBar)

```

$$\begin{aligned}
\text{ex-12.400} &:= z^\nu z^\rho \left(\frac{2}{3} R^\mu{}_{\nu\rho\sigma} x^\sigma + \frac{1}{12} (5 \nabla_\lambda R^\mu{}_{\nu\rho\sigma} + \nabla_\rho R^\mu{}_{\sigma\nu\lambda}) x^\sigma x^\lambda \right. \\
&\quad \left. + \frac{1}{6} \left(\frac{9}{10} \nabla_{\tau\lambda} R^\mu{}_{\rho\nu\sigma} + \frac{3}{20} \nabla_{\tau\rho} R^\mu{}_{\sigma\nu\lambda} + \frac{3}{20} \nabla_{\rho\tau} R^\mu{}_{\sigma\nu\lambda} + \frac{7}{20} R^\mu{}_{\lambda\xi\rho} R^\xi{}_{\sigma\nu\tau} + \frac{4}{5} R^\mu{}_{\xi\rho\lambda} R^\xi{}_{\sigma\nu\tau} - \frac{37}{60} R^\mu{}_{\sigma\xi\lambda} R^\xi{}_{\nu\rho\tau} \right) x^\sigma x^\lambda x^\tau \right) \\
\text{ex-12.401} &:= \frac{2}{3} z^a z^b R^\mu{}_{abc} x^c + \frac{5}{12} z^a z^b \nabla_d R^\mu{}_{abc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu{}_{abc} x^a x^c + \frac{3}{20} z^b z^a \nabla_{de} R^\mu{}_{abc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu{}_{abc} x^a x^c x^d \\
&\quad + \frac{1}{40} z^b z^d \nabla_{de} R^\mu{}_{abc} x^a x^c x^e + \frac{7}{120} z^e z^c R^\mu{}_{abc} R^b{}_{def} x^d x^a x^f + \frac{2}{15} z^e z^b R^\mu{}_{abc} R^a{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu{}_{abc} R^b{}_{def} x^a x^c x^f \\
\text{ex-12.402} &:= \delta_{a\mu} \left(\frac{2}{3} z^g z^b R^\mu{}_{gbc} x^c + \frac{5}{12} z^g z^b \nabla_d R^\mu{}_{gbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R^\mu{}_{gbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R^\mu{}_{gbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R^\mu{}_{gbc} x^g x^c x^d \right. \\
&\quad \left. + \frac{1}{40} z^b z^d \nabla_{de} R^\mu{}_{gbc} x^g x^c x^e + \frac{7}{120} z^e z^c R^\mu{}_{gbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R^\mu{}_{gbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R^\mu{}_{gbc} R^b{}_{def} x^g x^c x^f \right) \\
\text{ex-12.403} &:= \frac{2}{3} \delta_{a\mu} z^g z^b R^\mu{}_{gbc} x^c + \frac{5}{12} \delta_{a\mu} z^g z^b \nabla_d R^\mu{}_{gbc} x^c x^d + \frac{1}{12} \delta_{a\mu} z^b z^d \nabla_d R^\mu{}_{gbc} x^g x^c + \frac{3}{20} \delta_{a\mu} z^b z^g \nabla_{de} R^\mu{}_{gbc} x^c x^e x^d + \frac{1}{40} \delta_{a\mu} z^b z^e \nabla_{de} R^\mu{}_{gbc} x^g x^c x^d \\
&\quad + \frac{1}{40} \delta_{a\mu} z^b z^d \nabla_{de} R^\mu{}_{gbc} x^g x^c x^e + \frac{7}{120} \delta_{a\mu} z^e z^c R^\mu{}_{gbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} \delta_{a\mu} z^e z^b R^\mu{}_{gbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} \delta_{a\mu} z^d z^e R^\mu{}_{gbc} R^b{}_{def} x^g x^c x^f \\
\text{ex-12.404} &:= \frac{2}{3} z^g z^b R_{agbc} x^c + \frac{5}{12} z^g z^b \nabla_d R_{agbc} x^c x^d + \frac{1}{12} z^b z^d \nabla_d R_{agbc} x^g x^c + \frac{3}{20} z^b z^g \nabla_{de} R_{agbc} x^c x^e x^d + \frac{1}{40} z^b z^e \nabla_{de} R_{agbc} x^g x^c x^d \\
&\quad + \frac{1}{40} z^b z^d \nabla_{de} R_{agbc} x^g x^c x^e + \frac{7}{120} z^e z^c R_{agbc} R^b{}_{def} x^d x^g x^f + \frac{2}{15} z^e z^b R_{agbc} R^g{}_{def} x^d x^c x^f - \frac{37}{360} z^d z^e R_{agbc} R^b{}_{def} x^g x^c x^f \\
\text{ex-12.405} &:= \frac{2}{3} x^b z^c z^d R_{adcb} + \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} + \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
&\quad + \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc} \\
\text{ex-12.406} &:= -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} - \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
&\quad - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} + \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} - \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} - \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc} \\
\text{ex-12.407} &:= -\frac{2}{3} x^b z^c z^d R_{adcb} - \frac{1}{12} x^b x^c z^d z^e \nabla_e R_{adcb} - \frac{5}{12} x^b x^c z^d z^e \nabla_c R_{aedb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{fc} R_{adeb} - \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{cf} R_{adeb} \\
&\quad - \frac{3}{20} x^b x^c x^d z^e z^f \nabla_{cd} R_{afeb} - \frac{37}{360} x^b x^c x^d z^e z^f R_{adgb} R^g{}_{efc} + \frac{2}{15} x^b x^c x^d z^e z^f R_{ageb} R^g{}_{cfd} + \frac{7}{120} x^b x^c x^d z^e z^f R_{adge} R^g{}_{bfc}
\end{aligned}$$

$$\begin{aligned}
\text{ex-12.408} := & -\frac{2}{3}x^bz^cz^dR_{adcb} - \frac{1}{12}x^bx^cz^dz^e\nabla_eR_{acdb} - \frac{5}{12}x^bx^cz^dz^e\nabla_cR_{aedb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fc}R_{adeb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{cf}R_{adeb} \\
& - \frac{3}{20}x^bx^cx^dz^ez^f\nabla_{cd}R_{afeb} - \frac{37}{360}x^bx^cx^dz^ez^fR_{adgb}g^{gh}R_{hefc} + \frac{2}{15}x^bx^cx^dz^ez^fR_{ageb}g^{gh}R_{hcf d} + \frac{7}{120}x^bx^cx^dz^ez^fR_{adge}g^{gh}R_{hbfc}
\end{aligned}$$

$$\text{ex-12.500} := -\frac{2}{3}x^bz^cz^dR_{adcb}$$

$$\text{ex-12.501} := -\frac{1}{12}x^bx^cz^dz^e\nabla_eR_{acdb} - \frac{5}{12}x^bx^cz^dz^e\nabla_cR_{aedb}$$

$$\begin{aligned}\text{ex-12.502} := & -\frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fc}R_{adeb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{cf}R_{adeb} - \frac{3}{20}x^bx^cx^dz^ez^f\nabla_{cd}R_{afeb} \\ & - \frac{37}{360}x^bx^cx^dz^ez^fR_{adgb}g^{gh}R_{hefc} + \frac{2}{15}x^bx^cx^dz^ez^fR_{ageb}g^{gh}R_{hcf d} + \frac{7}{120}x^bx^cx^dz^ez^fR_{adge}g^{gh}R_{hbfc}\end{aligned}$$

$$\text{ex-12.600} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\text{ex-12.601} := \frac{1}{12}x^bx^cx^dz^e(\nabla_dR_{abce} + 5\nabla_bR_{adce})$$

$$\text{ex-12.602} := -\frac{1}{40}x^bx^cx^dz^ez^f\nabla_{fc}R_{adeb} - \frac{1}{40}x^bx^cx^dz^ez^f\nabla_{cf}R_{adeb} - \frac{3}{20}x^bx^cx^dz^ez^f\nabla_{cd}R_{afeb}$$

$$\text{ex-12.603} := -\frac{37}{360}x^bx^cx^dz^ez^fR_{adgb}g^{gh}R_{hefc} + \frac{2}{15}x^bx^cx^dz^ez^fR_{ageb}g^{gh}R_{hcf d} + \frac{7}{120}x^bx^cx^dz^ez^fR_{adge}g^{gh}R_{hbfc}$$

$$\text{ex-12.604} := \frac{2}{3}x^bz^cz^dR_{acbd}$$

$$\text{ex-12.605} := \frac{1}{12}x^bx^cx^dz^e(\nabla_dR_{abce} + 5\nabla_bR_{adce})$$

$$\text{ex-12.606} := \frac{1}{40}x^bx^cx^dz^ez^f(\nabla_{be}R_{acdf} + \nabla_{eb}R_{acdf} + 6\nabla_{bc}R_{aedf})$$

$$\text{ex-12.607} := \frac{1}{360}g^{bc}x^dx^ex^fz^gz^h(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh})$$

$$\text{ex-12.608} := \frac{1}{40}x^bx^cx^dz^ez^f(\nabla_{be}R_{acdf} + \nabla_{eb}R_{acdf} + 6\nabla_{bc}R_{aedf}) + \frac{1}{360}g^{bc}x^dx^ex^fz^gz^h(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh})$$

$$\begin{aligned}
\text{ex-12.409} := & \frac{2}{3}x^bz^cz^dR_{acbd} + \frac{1}{12}x^bx^cz^dz^e(\nabla_dR_{abce} + 5\nabla_bR_{adce}) + \frac{1}{40}x^bx^cx^dz^ez^f(\nabla_{be}R_{acdf} + \nabla_{eb}R_{acdf} + 6\nabla_{bc}R_{aedf}) \\
& + \frac{1}{360}g^{bc}x^dx^ex^fz^gz^h(37R_{adbe}R_{cgfh} - 21R_{adbg}R_{cefh} + 48R_{abdg}R_{cefh})
\end{aligned}$$

The fun begins $\Gamma - \bar{\Gamma}$

It's now time to compute the difference $\Gamma - \bar{\Gamma}$. Here it is.

```
def reformat_diff (obj):

    distribute (obj)

    obj1 = get_xterm (obj,1)
    obj2 = get_xterm (obj,2)
    obj3 = get_xterm (obj,3)

    obj30 = get_gterm (obj3,0)
    obj31 = get_gterm (obj3,1)

    obj1 = reformat (obj1, 3,1)
    obj2 = reformat (obj2,12,1)

    obj30 = reformat (obj30,40,1)
    obj31 = reformat (obj31,360,1)

    obj3 := @(obj30) + @(obj31).

    ans := @(obj1) + @(obj2) + @(obj3).

    return ans

# We could use reformat_diff here but instead we'll do it one step at a time so that
# we can see exactly what's going on. Later on we will use reformat_diff to do the job.

diff := @(Gamma) - @(GammaBar). # cdb(ex-12.diff.100,diff)
distribute (diff)

diff1 = get_xterm (diff,1) # cdb(ex-12.diff.200,diff1)
diff2 = get_xterm (diff,2) # cdb(ex-12.diff.201,diff2)
diff3 = get_xterm (diff,3) # cdb(ex-12.diff.202,diff3)

diff30 = get_gterm (diff3,0) # cdb(ex-12.diff.203,diff30)
```

```

diff31 = get_gterm (diff3,1)                                # cdb(ex-12.diff.204,diff31)

diff1  = reformat (diff1, 3,1)                               # cdb(ex-12.diff.300,diff1)
diff2  = reformat (diff2,12,1)                              # cdb(ex-12.diff.301,diff2)

diff30 = reformat (diff30,40,1)                             # cdb(ex-12.diff.302,diff30)
diff31 = reformat (diff31,360,1)                           # cdb(ex-12.diff.303,diff31)

diff3 := @(diff30) + @(diff31).                             # cdb(ex-12.diff.304,diff3)

diff := @(diff1) + @(diff2) + @(diff3).                     # cdb(ex-12.diff.305,diff)

```

$$\begin{aligned}
\text{ex-12.diff.100} := & \frac{1}{12} x^b x^c z^d z^e (2\nabla_d R_{abce} + 4\nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (2\nabla_{be} R_{acdf} + 2\nabla_{eb} R_{acdf} + 4\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\
& + \frac{2}{45} g^{bc} x^d x^e x^f z^g z^h (-2R_{agbd} R_{cefh} - R_{adbg} R_{cefh} + R_{abdg} R_{cefh} + 4R_{adbe} R_{cgfh}) - \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} + 5\nabla_b R_{adce}) \\
& - \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} + 6\nabla_{bc} R_{aedf}) - \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (37R_{adbe} R_{cgfh} - 21R_{adbg} R_{cefh} + 48R_{abdg} R_{cefh})
\end{aligned}$$

$$\text{ex-12.diff.200} := 0$$

$$\text{ex-12.diff.201} := \frac{1}{12} x^b x^c z^d z^e \nabla_d R_{abce} - \frac{1}{12} x^b x^c z^d z^e \nabla_b R_{adce} + \frac{1}{12} x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\begin{aligned}
\text{ex-12.diff.202} := & \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
& - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh}
\end{aligned}$$

$$\text{ex-12.diff.203} := \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \frac{1}{40} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf}$$

$$\text{ex-12.diff.204} := -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh}$$

$$\text{ex-12.diff.300} := 0$$

$$\text{ex-12.diff.301} := \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce})$$

$$\text{ex-12.diff.302} := \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf})$$

$$\text{ex-12.diff.303} := \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh})$$

$$\begin{aligned} \text{ex-12.diff.304} := & \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

$$\begin{aligned} \text{ex-12.diff.305} := & \frac{1}{12} x^b x^c z^d z^e (\nabla_d R_{abce} - \nabla_b R_{adce} + \nabla_a R_{bdce}) + \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ & + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32 R_{abdg} R_{cefh} + 27 R_{adbe} R_{cgfh} + 5 R_{adbg} R_{cefh} - 32 R_{agbd} R_{cefh}) \end{aligned}$$

Second order terms

```

diff2 = get_xterm (diff,2)
diff2 := 12 @(diff2).
distribute (diff2)

diff2 = add_tags (diff2,'\\mu')

# swap indices on middle term, then apply 2nd Bianchi identity

zoom      (diff2, $\\mu_{1} Q??)$
substitute (diff2, $\\nabla_{b}\\{R_{a d c e}\\} -> - \\nabla_{b}\\{R_{d a c e}\\}$)
unzoom    (diff2)

substitute (diff2, $\\mu_{1} -> \\mu_{0}, \\mu_{2} -> \\mu_{0}$)
substitute (diff2, $\\mu_{0} -> 0$)

diff2 = clear_tags (diff2,'\\mu')

diff2 := @(diff2) / 12 .

diff := @(diff1) + @(diff2) + @(diff3).

diff = reformat_diff (diff)

```

cdb (ex-12.701,diff2)

cdb (ex-12.702,diff2)

cdb (ex-12.711,diff2)

cdb (ex-12.712,diff2)

cdb (ex-12.713,diff2)

cdb (ex-12.714,diff2)

cdb (ex-12.715,diff2)

cdb (ex-12.716,diff2)

cdb(ex-12.diff.306,diff)

$$\text{ex-12.701} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.702} := x^b x^c z^d z^e \nabla_d R_{abce} - x^b x^c z^d z^e \nabla_b R_{adce} + x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.711} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_2 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.712} := \dots - \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.713} := \dots + \mu_1 x^b x^c z^d z^e \nabla_b R_{adce} + \dots$$

$$\text{ex-12.714} := \mu_0 x^b x^c z^d z^e \nabla_d R_{abce} + \mu_0 x^b x^c z^d z^e \nabla_b R_{adce} + \mu_0 x^b x^c z^d z^e \nabla_a R_{bdce}$$

$$\text{ex-12.715} := 0$$

$$\text{ex-12.716} := 0$$

$$\begin{aligned} \text{ex-12.diff.306} &:= \frac{1}{40} x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + \nabla_{eb} R_{acdf} - 2\nabla_{bc} R_{aedf} + \nabla_{ab} R_{cedf} + \nabla_{ba} R_{cedf}) \\ &\quad + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32R_{abdg} R_{cefh} + 27R_{adbe} R_{cgfh} + 5R_{adbg} R_{cefh} - 32R_{agbd} R_{cefh}) \end{aligned}$$

Third order terms, commute $\nabla\nabla R$ terms

```

diff3 = get_xterm (diff,3)
diff3 := 360 @(diff3).                                # cdb (ex-12.801,diff3)
distribute (diff3)                                    # cdb (ex-12.802,diff3)

# commutation rule for covariant derivs on Rabcd, see exrecise 3.6
# note: \nabla_{a b} defined as \nabla_a \nabla_b
CommuteNablaRiemann := \nabla_{f e}(R_{a b c d}) -> \nabla_{e f}(R_{a b c d})
+ g^{u v} R_{u a e f} R_{v b c d}
+ g^{u v} R_{u b e f} R_{a v c d}
+ g^{u v} R_{u c e f} R_{a b v d}
+ g^{u v} R_{u d e f} R_{a b c v}.

diff3 = add_tags (diff3,'\\mu')                        # cdb (ex-12.901,diff3)

# commute derivs on Rabcd so that each double deriv is of the form \nabla_{b*}

substitute (diff3, $\mu_{\{3\}} -> \mu_{\{1\}}$)        # cdb (ex-12.902,diff3)

zoom          (diff3, $\mu_{\{1\}} Q??$)                # cdb (ex-12.903,diff3)
substitute (diff3, CommuteNablaRiemann)                # cdb (ex-12.904,diff3)
unzoom        (diff3)

diff3 = clear_tags (diff3,'\\mu')
diff3 := @(diff3) / 360 .

distribute    (diff3)
canonicalise  (diff3)                                # cdb (ex-12.905,diff3)

diff := @(diff1) + @(diff2) + @(diff3).

diff = reformat_diff (diff)                            # cdb(ex-12.diff.307,diff)

```

$$\begin{aligned} \text{ex-12.801} := & 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.802} := & 9x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.901} := & 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_3 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\begin{aligned} \text{ex-12.902} := & 9\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} - 18\mu_2 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + 9\mu_4 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 27\mu_6 g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + 5\mu_7 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_8 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.903} := \dots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{eb} R_{acdf} + \dots + 9\mu_1 x^b x^c x^d z^e z^f \nabla_{ab} R_{cedf} + \dots$$

$$\begin{aligned} \text{ex-12.904} := & \dots + 9\mu_1 x^b x^c x^d z^e z^f (\nabla_{be} R_{acdf} + g^{uv} R_{uabe} R_{vcdf} + g^{uv} R_{ucbe} R_{avdf} + g^{uv} R_{udbe} R_{acvf} + g^{uv} R_{ufbe} R_{acdv}) + \dots \\ & + 9\mu_1 x^b x^c x^d z^e z^f (\nabla_{ba} R_{cedf} + g^{uv} R_{ucba} R_{vedf} + g^{uv} R_{ueba} R_{cvdf} + g^{uv} R_{udba} R_{cevf} + g^{uv} R_{ufba} R_{cedv}) + \dots \end{aligned}$$

$$\begin{aligned} \text{ex-12.905} := & \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abeu} R_{cfdv} - \frac{3}{40} x^b x^c x^d z^e z^f g^{uv} R_{abcu} R_{defv} - \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \frac{1}{20} x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\ & - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{3}{40} g^{bc} x^d x^e x^f z^g z^h R_{adbe} R_{cgfh} + \frac{1}{72} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \end{aligned}$$

$$\text{ex-12.diff.307} := \frac{1}{40} x^b x^c x^d z^e z^f (2\nabla_{be} R_{acdf} - 2\nabla_{bc} R_{aedf} + 2\nabla_{ba} R_{cedf}) + \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32R_{abdg} R_{cefh} + 32R_{adbg} R_{cefh} - 32R_{agbd} R_{cefh})$$

Third order terms, use 2nd Bianchi identity on $\nabla\nabla R$ terms

```

diff3 = get_xterm (diff,3)
diff3 := 360 @(diff3).
distribute (diff3)

diff3 = add_tags (diff3,'\\mu')

# swap indices on middle second deriv term, then apply 2nd Bianchi identity

zoom      (diff3, $\\mu_{1} Q??)$
substitute (diff3, $\\nabla_{b c}\\{R_{a e d f}\\} -> - \\nabla_{b c}\\{R_{e a d f}\\}$)
unzoom    (diff3)

substitute (diff3, $\\mu_{1} -> \\mu_{0}, \\mu_{2} -> \\mu_{0}$)
substitute (diff3, $\\mu_{0} -> 0$)

diff3 = clear_tags (diff3,'\\mu')
diff3 := @(diff3) / 360 .

distribute (diff3)
canonicalise (diff3)

diff := @(diff1) + @(diff2) + @(diff3).

diff = reformat_diff (diff)

```

cdb (ex-12.910,diff3)
cdb (ex-12.911,diff3)
cdb (ex-12.912,diff3)
cdb (ex-12.913,diff3)
cdb (ex-12.914,diff3)
cdb (ex-12.915,diff3)
cdb (ex-12.916,diff3)
cdb (ex-12.917,diff3)
cdb(ex-12.diff.308,diff)

$$\begin{aligned}
\text{ex-12.910} &:= 18x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.911} &:= 18x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.912} &:= 18\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} - 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + 18\mu_2 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.913} &:= \dots - 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{aedf} + \dots \\
\text{ex-12.914} &:= \dots + 18\mu_1 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + \dots \\
\text{ex-12.915} &:= 18\mu_0 x^b x^c x^d z^e z^f \nabla_{be} R_{acdf} + 18\mu_0 x^b x^c x^d z^e z^f \nabla_{bc} R_{eadf} + 18\mu_0 x^b x^c x^d z^e z^f \nabla_{ba} R_{cedf} \\
&\quad - 32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.916} &:= -32\mu_3 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_4 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_5 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.917} &:= -\frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - \frac{4}{45} g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh} \\
\text{ex-12.diff.308} &:= \frac{1}{360} g^{bc} x^d x^e x^f z^g z^h (-32R_{abdg} R_{cefh} + 32R_{adbg} R_{cefh} - 32R_{agbd} R_{cefh})
\end{aligned}$$

Third order terms, use 1st Bianchi identity on RR terms

```
diff3 = get_xterm (diff,3)
diff3 := 360 @(diff3).
distribute (diff3)

diff3 = add_tags (diff3,'\\mu') # cdb (ex-12.921,diff3)

# swap indices on middle term, then apply 1st Bianchi identity

zoom (diff3, $\\mu_{1} Q??$) # cdb (ex-12.922,diff3)
substitute (diff3, $R_{a d b g} R_{c e f h} -> - R_{a d g b} R_{c e f h}$) # cdb (ex-12.923,diff3)
unzoom (diff3)

substitute (diff3, $\\mu_{1} -> \\mu_{0}, \\mu_{2} -> \\mu_{0}$) # cdb (ex-12.924,diff3)
substitute (diff3, $\\mu_{0} -> 0$) # cdb (ex-12.925,diff3)

diff3 = clear_tags (diff3,'\\mu') # cdb (ex-12.926,diff3)

diff := @(diff1) + @(diff2) + @(diff3).

diff = reformat_diff (diff) # cdb(ex-12.diff.309,diff)
```

$$\text{ex-12.921} := -32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} + 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} - 32\mu_2 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.922} := \dots + 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adbg} R_{cefh} + \dots$$

$$\text{ex-12.923} := \dots - 32\mu_1 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} + \dots$$

$$\text{ex-12.924} := -32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{abdg} R_{cefh} - 32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{adgb} R_{cefh} - 32\mu_0 g^{bc} x^d x^e x^f z^g z^h R_{agbd} R_{cefh}$$

$$\text{ex-12.925} := 0$$

$$\text{ex-12.926} := 0$$

$$\text{ex-12.diff.309} := 0$$