

Converting from generic to rnc coordinates

The following is based on the approach used in the `geodesic-bvp.tex` code. The main difference here is that this time we will *not* be assuming that the coordinates are in Riemann normal form. This will be apparent in the expression for the generalised connections – they will be expressed in terms of the partial derivatives of the connection rather the covariant derivatives of the Riemann tensor. There will also be a change in the way the Taylor series are developed. In this case the expansion parameter ϵ will be associated with the connection and its derivatives rather than the Riemann tensor. We will use

$$\Gamma^a_{bc} = \mathcal{O}(\epsilon) , \quad \Gamma^a_{bc,d} = \mathcal{O}(\epsilon^2) , \quad \Gamma^a_{bc,de} = \mathcal{O}(\epsilon^3) , \quad \text{etc.}$$

The generalised connections are defined recursively by

$$\Gamma^a_{bcd} = \Gamma^a_{(b\bar{c},d)} - (n+1)\Gamma^a_{p(\bar{c}}\Gamma^p_{bd)} \quad (1)$$

where \bar{c} contains $n > 0$ indices. It is easy to see from this equation that the generalised connections will behave much the same as the connection, that is

$$\Gamma^a_{bc} = \mathcal{O}(\epsilon) , \quad \Gamma^a_{bcd} = \mathcal{O}(\epsilon^2) , \quad \Gamma^a_{bcde} = \mathcal{O}(\epsilon^3) , \quad \text{etc.}$$

This allows us to represent each generalised connection by a single expression (typically `GamNN`).

The situation is slightly different in `geodesic-bvp.tex`. In that code the connection and the generalised connection are expanded as a series in the Riemann tensor and its derivatives. Thus each connection is written in the form

$$\bar{\Gamma}^a_{\bar{c}_n} = \bar{\Gamma}^a_{\bar{c}_n}^{(0)} + \bar{\Gamma}^a_{\bar{c}_n}^{(1)} + \bar{\Gamma}^a_{\bar{c}_n}^{(2)} + \cdots + \bar{\Gamma}^a_{\bar{c}_n}^{(m)} \quad (2)$$

where \bar{c}_n denotes a set of indices such as $c_1c_2c_3 \dots c_n$. The terms of the RHS are each of a different weight in ϵ .

Stage 1: The generalised connections

The generalised connections $\Gamma^a_{\bar{c}_n}$ could be computed directly by successive application of equation (1). But a more effiecent method exists and its basis lies in the original definition of the generalised connections. Recall that the generalised connections arose when buidling a formal power series solution of the geodesic equation

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} \quad (3)$$

The key idea was that the coefficients c_n in the formal power series

$$x^a = c_0^a + s c_1^a + s^2 c_2^a + \dots \quad (4)$$

could be computed using

$$c_n^a = \frac{1}{n!} \left. \frac{d^n x^a}{ds^n} \right|_{s=0} \quad (5)$$

with the second, third and higher derivatives of x^a found by successive differentiation of the geodesic equation. The generalised connections were introduced as part of this algorithm, leading to

$$c_n^a = - \Gamma_{\underline{c}_n}^a A^{\underline{c}_n} \Big|_{s=0} \quad n = 2, 3, 4 \dots \quad (6)$$

and

$$\Gamma_{\underline{c}_{n+1}}^a A^{\underline{c}_{n+1}} = \frac{d}{ds} \left(\Gamma_{\underline{c}_n}^a A^{\underline{c}_n} \right) \quad (7)$$

with $d/ds = A^a \partial_a$, $A^a = dx^a/ds$ and $dA^a/ds = -\Gamma_{bc}^a A^b A^c$.

The upshot is that computing the $\Gamma_{\underline{c}_n}^a A^{\underline{c}_n}$ requires little more than successive rounds of differentiation (and a few substitutions for the derivatives of A^a).

Note that the coefficients c_0 and c_1 must be determined from the initial conditions. Suppose that $x^a = x_i^a$ at $s = 0$ then $c_0 = x_i^a$ while $c_1 = A^a$.

The Riemann normal coordinates of the point j (where $s = 1$) are introduced by setting

$$y^a = A^a \quad (8)$$

This leads to

$$x_j^a = x_i^a + y^a - \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_{\underline{b}_k}^a y^{\underline{b}_k} \quad (9)$$

Note that given two points i and j , the y^a would be found as a root of this non-linear equation for y^a .

Stage 2: The fixed point scheme for y^a

This second stage is almost exactly the same as the corresponding stage in `geodesic-bvp`. The difference here is that the generalised connections involve partial derivatives of the connection. In `contrat`, the `geodesic-bvp` code is specific to RNC and thus uses the generalised connections based on covariant derivatives of the Riemann tensor.

We begin this second stage by rewriting the equation (9) in the suggestive form

$$y^a = x_j^a - x_i^a + \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma^a_{\underline{b}_k} y^{\underline{b}_k}$$

and then use this as the basis of a fixed point iteration scheme.

Start with the first approximation $y_1^a = x_j^a - x_i^a = \Delta x^a$, then compute the successive approximations

$$y_1^a = \Delta x^a$$

$$y_2^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_1^b y_1^c$$

$$y_3^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_2^b y_2^c + \frac{1}{3!} \Gamma^a_{bcd} y_1^b y_1^c y_1^d$$

$$y_4^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_3^b y_3^c + \frac{1}{3!} \Gamma^a_{bcd} y_2^b y_2^c y_2^d + \frac{1}{4!} \Gamma^a_{bcde} y_1^b y_1^c y_1^d y_1^e$$

$$y_5^a = y_1^a + \frac{1}{2!} \Gamma^a_{bc} y_4^b y_4^c + \frac{1}{3!} \Gamma^a_{bcd} y_3^b y_3^c y_3^d + \frac{1}{4!} \Gamma^a_{bcde} y_2^b y_2^c y_2^d y_2^e + \frac{1}{5!} \Gamma^a_{bcdef} y_1^b y_1^c y_1^d y_1^e y_1^f$$

and so on. Not that the Γ^a_{bc} , Γ^a_{bcd} , Γ^a_{bcde} etc. will all depend on the original coordinates x^a at the initial point (i.e., $P = x_i^a$).

Stage3: Introduce the generalised connections from Stage 1

This is the final stage – it introduces the generalised connection after the completion of the fixed point scheme.

The result will be an equation for the y^a in terms of the original coordinates x^a and the connections (and its derivatives) at a chosen point $s = 0$ (aka i).

The y^a define an RNC frame in the neighbourhood of the chosen point i .

Stage 1: The generalised connections

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

A^{a}::Depends(\partial{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).

\Gamma^{a}_{b c}::Depends(\partial{#}).

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
\Gamma^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
\Gamma^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
\Gamma^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
\Gamma^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

\Gamma^{p}_{a b}::Weight(label=numG,value=1).
\Gamma^{p}_{a b c}::Weight(label=numG,value=2).
\Gamma^{p}_{a b c d}::Weight(label=numG,value=3).
```

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\Gamma^{p}_{\{a b c d e\}}::Weight(label=numG,value=4).
\Gamma^{p}_{\{a b c d e f\}}::Weight(label=numG,value=5).

def product_sort (obj):
    substitute (obj,$ A^{\{a\}}                -> A001^{\{a\}}                $)
    substitute (obj,$ x^{\{a\}}                  -> A002^{\{a\}}                  $)
    substitute (obj,$ g^{\{a b\}}                 -> A003^{\{a b\}}                 $)
    substitute (obj,$ \Gamma^{p}_{\{a b\}}          -> A004^{p}_{\{a b\}}          $)
    substitute (obj,$ \Gamma^{p}_{\{a b c\}}         -> A005^{p}_{\{a b c\}}         $)
    substitute (obj,$ \Gamma^{p}_{\{a b c d\}}        -> A006^{p}_{\{a b c d\}}        $)
    substitute (obj,$ \Gamma^{p}_{\{a b c d e\}}       -> A007^{p}_{\{a b c d e\}}       $)
    substitute (obj,$ \Gamma^{p}_{\{a b c d e f\}}      -> A008^{p}_{\{a b c d e f\}}      $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{\{a\}}                -> A^{\{a\}}                $)
    substitute (obj,$ A002^{\{a\}}                  -> x^{\{a\}}                  $)
    substitute (obj,$ A003^{\{a b\}}                 -> g^{\{a b\}}                 $)
    substitute (obj,$ A004^{p}_{\{a b\}}          -> \Gamma^{p}_{\{a b\}}          $)
    substitute (obj,$ A005^{p}_{\{a b c\}}         -> \Gamma^{p}_{\{a b c\}}         $)
    substitute (obj,$ A006^{p}_{\{a b c d\}}        -> \Gamma^{p}_{\{a b c d\}}        $)
    substitute (obj,$ A007^{p}_{\{a b c d e\}}       -> \Gamma^{p}_{\{a b c d e\}}       $)
    substitute (obj,$ A008^{p}_{\{a b c d e f\}}      -> \Gamma^{p}_{\{a b c d e f\}}      $)

    return obj

def truncateGam (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah  = Ex("numG = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    return ans

beg_stage_1 = time.time()

```

```

# note that we use A^{a} in place of dx^a/ds

Gamma := \Gamma^{d}_{a b} A^{a} A^{b}.

# the geodesic equation

dAds := A^{c} \partial_{c} A^{d} - \Gamma^{d}_{ab} A^{a} A^{b}.

# eq0, eq1, eq2 ... are the successive derivatives of Gamma
# thus they are the generalised gamma's dotted into (multiple copies of) A^{a} = dx^a/ds

# =====
eq0 := \Gamma^{d}_{ab} A^{a} A^{b}. # cdb (eq0.000,eq0)

# =====
eq1 := A^{c} \partial_{c} \Gamma^{d}_{ab} A^{a} A^{b}. # cdb (eq1.000,eq1)

distribute      (eq1) # cdb (eq1.001,eq1)
unwrap          (eq1) # cdb (eq1.002,eq1)
product_rule    (eq1) # cdb (eq1.003,eq1)
distribute      (eq1) # cdb (eq1.004,eq1)
substitute      (eq1,dAds) # cdb (eq1.005,eq1)
distribute      (eq1) # cdb (eq1.006,eq1)
eq1 = truncateGam (eq1,5) # cdb (eq1.007,eq1)
sort_product    (eq1) # cdb (eq1.008,eq1)
rename_dummies  (eq1) # cdb (eq1.009,eq1)
canonicalise    (eq1) # cdb (eq1.010,eq1)

# =====
eq2 := A^{c} \partial_{c} A^{d} \Gamma^{e}_{ab} A^{a} A^{b}. # cdb (eq2.000,eq2)

distribute      (eq2) # cdb (eq2.001,eq2)
unwrap          (eq2) # cdb (eq2.002,eq2)
product_rule    (eq2) # cdb (eq2.003,eq2)
distribute      (eq2) # cdb (eq2.004,eq2)
substitute      (eq2,dAds) # cdb (eq2.005,eq2)
distribute      (eq2) # cdb (eq2.006,eq2)

```

```

eq2 = truncateGam (eq2,5)           # cdb (eq2.007,eq2)
sort_product      (eq2)             # cdb (eq2.008,eq2)
rename_dummies    (eq2)             # cdb (eq2.009,eq2)
canonicalise      (eq2)             # cdb (eq2.010,eq2)

# =====
eq3 := A^{c} \partial_{c}{@(eq2)}.    # cdb (eq3.000,eq3)

distribute        (eq3)             # cdb (eq3.001,eq3)
unwrap            (eq3)             # cdb (eq3.002,eq3)
product_rule      (eq3)             # cdb (eq3.003,eq3)
distribute        (eq3)             # cdb (eq3.004,eq3)
substitute        (eq3,dAds)        # cdb (eq3.005,eq3)
distribute        (eq3)             # cdb (eq3.006,eq3)
eq3 = truncateGam (eq3,5)           # cdb (eq3.007,eq3)
sort_product      (eq3)             # cdb (eq3.008,eq3)
rename_dummies    (eq3)             # cdb (eq3.009,eq3)
canonicalise      (eq3)             # cdb (eq3.010,eq3)

# =====
eq4 := A^{c} \partial_{c}{@(eq3)}.    # cdb (eq4.000,eq4)

distribute        (eq4)             # cdb (eq4.001,eq4)
unwrap            (eq4)             # cdb (eq4.002,eq4)
product_rule      (eq4)             # cdb (eq4.003,eq4)
distribute        (eq4)             # cdb (eq4.004,eq4)
substitute        (eq4,dAds)        # cdb (eq4.005,eq4)
distribute        (eq4)             # cdb (eq4.006,eq4)
eq4 = truncateGam (eq4,5)           # cdb (eq4.007,eq4)
sort_product      (eq4)             # cdb (eq4.008,eq4)
rename_dummies    (eq4)             # cdb (eq4.009,eq4)
canonicalise      (eq4)             # cdb (eq4.010,eq4)

end_stage_1 = time.time()

```

$$\text{eq0.000} := \Gamma^d_{ab} A^a A^b$$

$$\text{eq1.000} := A^c \partial_c (\Gamma_{ab}^d A^a A^b)$$

$$\text{eq1.001} := A^c \partial_c (\Gamma_{ab}^d A^a A^b)$$

$$\text{eq1.002} := A^c \partial_c (\Gamma_{ab}^d A^a A^b)$$

$$\text{eq1.003} := A^c (\partial_c \Gamma_{ab}^d A^a A^b + \Gamma_{ab}^d \partial_c A^a A^b + \Gamma_{ab}^d A^a \partial_c A^b)$$

$$\text{eq1.004} := A^c \partial_c \Gamma_{ab}^d A^a A^b + A^c \Gamma_{ab}^d \partial_c A^a A^b + A^c \Gamma_{ab}^d A^a \partial_c A^b$$

$$\text{eq1.005} := A^c \partial_c \Gamma_{ab}^d A^a A^b - \Gamma_{ce}^a A^c A^e \Gamma_{ab}^d A^b - \Gamma_{ec}^b A^e A^c \Gamma_{ab}^d A^a$$

$$\text{eq1.006} := A^c \partial_c \Gamma_{ab}^d A^a A^b - \Gamma_{ce}^a A^c A^e \Gamma_{ab}^d A^b - \Gamma_{ec}^b A^e A^c \Gamma_{ab}^d A^a$$

$$\text{eq1.007} := A^c \partial_c \Gamma_{ab}^d A^a A^b - \Gamma_{ce}^a A^c A^e \Gamma_{ab}^d A^b - \Gamma_{ec}^b A^e A^c \Gamma_{ab}^d A^a$$

$$\text{eq1.008} := A^a A^b A^c \partial_c \Gamma_{ab}^d - A^b A^c A^e \Gamma_{ce}^a \Gamma_{ab}^d - A^a A^c A^e \Gamma_{ec}^b \Gamma_{ab}^d$$

$$\text{eq1.009} := A^a A^b A^c \partial_c \Gamma_{ab}^d - A^a A^b A^c \Gamma_{bc}^e \Gamma_{ea}^d - A^a A^b A^c \Gamma_{cb}^e \Gamma_{ae}^d$$

$$\text{eq1.010} := A^a A^b A^c \partial_a \Gamma_{bc}^d - 2A^a A^b A^c \Gamma_{ae}^d \Gamma_{bc}^e$$

$$\text{eq2.000} := A^c \partial_c (A^a A^b A^f \partial_a \Gamma_{bf}^d - 2A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e)$$

$$\text{eq2.001} := A^c \partial_c (A^a A^b A^f \partial_a \Gamma_{bf}^d) - 2A^c \partial_c (A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e)$$

$$\text{eq2.002} := A^c \partial_c (A^a A^b A^f \partial_a \Gamma_{bf}^d) - 2A^c \partial_c (A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e)$$

$$\begin{aligned} \text{eq2.003} := & A^c (\partial_c A^a A^b A^f \partial_a \Gamma_{bf}^d + A^a \partial_c A^b A^f \partial_a \Gamma_{bf}^d + A^a A^b \partial_c A^f \partial_a \Gamma_{bf}^d + A^a A^b A^f \partial_{ca} \Gamma_{bf}^d) \\ & - 2A^c (\partial_c A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e + A^a \partial_c A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e + A^a A^b \partial_c A^f \Gamma_{ae}^d \Gamma_{bf}^e + A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e + A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e) \end{aligned}$$

$$\begin{aligned} \text{eq2.004} := & A^c \partial_c A^a A^b A^f \partial_a \Gamma_{bf}^d + A^c A^a \partial_c A^b A^f \partial_a \Gamma_{bf}^d + A^c A^a A^b \partial_c A^f \partial_a \Gamma_{bf}^d + A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d - 2A^c \partial_c A^a A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e \\ & - 2A^c A^a \partial_c A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b \partial_c A^f \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.005} := & -\Gamma_{ce}^a A^c A^e A^b A^f \partial_a \Gamma_{bf}^d - \Gamma_{ec}^b A^e A^c A^a A^f \partial_a \Gamma_{bf}^d - \Gamma_{ce}^f A^c A^e A^a A^b \partial_a \Gamma_{bf}^d + A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d + 2\Gamma_{cg}^a A^c A^g A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e \\ & + 2\Gamma_{gc}^b A^g A^c A^a A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{cg}^f A^c A^g A^a A^b \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.006} := & -\Gamma_{ce}^a A^c A^e A^b A^f \partial_a \Gamma_{bf}^d - \Gamma_{ec}^b A^e A^c A^a A^f \partial_a \Gamma_{bf}^d - \Gamma_{ce}^f A^c A^e A^a A^b \partial_a \Gamma_{bf}^d + A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d + 2\Gamma_{cg}^a A^c A^g A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e \\ & + 2\Gamma_{gc}^b A^g A^c A^a A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{cg}^f A^c A^g A^a A^b \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.007} := & A^c A^a A^b A^f \partial_{ca} \Gamma_{bf}^d - \Gamma_{ce}^a A^c A^e A^b A^f \partial_a \Gamma_{bf}^d - \Gamma_{ec}^b A^e A^c A^a A^f \partial_a \Gamma_{bf}^d - \Gamma_{ce}^f A^c A^e A^a A^b \partial_a \Gamma_{bf}^d - 2A^c A^a A^b A^f \partial_c \Gamma_{ae}^d \Gamma_{bf}^e \\ & - 2A^c A^a A^b A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e + 2\Gamma_{cg}^a A^c A^g A^b A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{gc}^b A^g A^c A^a A^f \Gamma_{ae}^d \Gamma_{bf}^e + 2\Gamma_{cg}^f A^c A^g A^a A^b \Gamma_{ae}^d \Gamma_{bf}^e \end{aligned}$$

$$\begin{aligned} \text{eq2.008} := & A^a A^b A^c A^f \partial_{ca} \Gamma_{bf}^d - A^b A^c A^e A^f \Gamma_{ce}^a \partial_a \Gamma_{bf}^d - A^a A^c A^e A^f \Gamma_{ec}^b \partial_a \Gamma_{bf}^d - A^a A^b A^c A^e \Gamma_{ce}^f \partial_a \Gamma_{bf}^d - 2A^a A^b A^c A^f \Gamma_{bf}^e \partial_c \Gamma_{ae}^d \\ & - 2A^a A^b A^c A^f \Gamma_{ae}^d \partial_c \Gamma_{bf}^e + 2A^b A^c A^f A^g \Gamma_{cg}^a \Gamma_{ae}^d \Gamma_{bf}^e + 2A^a A^c A^f A^g \Gamma_{gc}^b \Gamma_{ae}^d \Gamma_{bf}^e + 2A^a A^b A^c A^g \Gamma_{ae}^d \Gamma_{bf}^e \Gamma_{cg}^f \end{aligned}$$

$$\begin{aligned} \text{eq2.009} := & A^a A^b A^c A^e \partial_{ca} \Gamma_{be}^d - A^a A^b A^c A^e \Gamma_{bc}^f \partial_f \Gamma_{ae}^d - A^a A^b A^c A^e \Gamma_{cb}^f \partial_a \Gamma_{fe}^d - A^a A^b A^c A^e \Gamma_{ce}^f \partial_a \Gamma_{bf}^d - 2A^a A^b A^c A^e \Gamma_{be}^f \partial_c \Gamma_{af}^d \\ & - 2A^a A^b A^c A^e \Gamma_{af}^d \partial_c \Gamma_{be}^f + 2A^a A^b A^c A^e \Gamma_{be}^f \Gamma_{fg}^d \Gamma_{ac}^g + 2A^a A^b A^c A^e \Gamma_{eb}^f \Gamma_{ag}^d \Gamma_{fc}^g + 2A^a A^b A^c A^e \Gamma_{af}^d \Gamma_{bg}^f \Gamma_{ce}^g \end{aligned}$$

$$\begin{aligned} \text{eq2.010} := & A^a A^b A^c A^e \partial_{ab} \Gamma_{ce}^d - A^a A^b A^c A^e \Gamma_{ab}^f \partial_f \Gamma_{ce}^d - 4A^a A^b A^c A^e \Gamma_{ab}^f \partial_c \Gamma_{ef}^d \\ & - 2A^a A^b A^c A^e \Gamma_{af}^d \partial_b \Gamma_{ce}^f + 2A^a A^b A^c A^e \Gamma_{fg}^d \Gamma_{ab}^f \Gamma_{ce}^g + 4A^a A^b A^c A^e \Gamma_{af}^d \Gamma_{bg}^f \Gamma_{ce}^g \end{aligned}$$

$$\begin{aligned}
\text{eq3.010} := & A^a A^b A^c A^e A^f \partial_{abc} \Gamma_{ef}^d - A^a A^b A^c A^e A^f \partial_g \Gamma_{ab}^d \partial_c \Gamma_{ef}^g - 6 A^a A^b A^c A^e A^f \partial_a \Gamma_{bg}^d \partial_c \Gamma_{ef}^g - 3 A^a A^b A^c A^e A^f \Gamma_{ab}^g \partial_{cg} \Gamma_{ef}^d \\
& - 6 A^a A^b A^c A^e A^f \Gamma_{ab}^g \partial_{ce} \Gamma_{fg}^d - 2 A^a A^b A^c A^e A^f \Gamma_{ag}^d \partial_{bc} \Gamma_{ef}^g + 2 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{cg}^h \partial_h \Gamma_{ef}^d + 6 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{ce}^h \partial_g \Gamma_{fh}^d \\
& + 12 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{cg}^h \partial_e \Gamma_{fh}^d + 6 A^a A^b A^c A^e A^f \Gamma_{ab}^g \Gamma_{ce}^h \partial_f \Gamma_{gh}^d + 6 A^a A^b A^c A^e A^f \Gamma_{gh}^g \Gamma_{ab}^g \partial_c \Gamma_{ef}^h \\
& + 2 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bc}^h \partial_h \Gamma_{ef}^g + 8 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bc}^h \partial_e \Gamma_{fh}^g + 4 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bh}^g \partial_c \Gamma_{ef}^h \\
& - 12 A^a A^b A^c A^e A^f \Gamma_{gh}^g \Gamma_{ab}^g \Gamma_{ci}^h \Gamma_{ef}^i - 4 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{hi}^g \Gamma_{bc}^h \Gamma_{ef}^i - 8 A^a A^b A^c A^e A^f \Gamma_{ag}^d \Gamma_{bh}^g \Gamma_{ci}^h \Gamma_{ef}^i
\end{aligned}$$

$$\begin{aligned}
\text{eq4.010} := & A^a A^b A^c A^e A^f A^g \partial_{abce} \Gamma_{fg}^d - 4A^a A^b A^c A^e A^f A^g \partial_a \Gamma_{bc}^h \partial_{eh} \Gamma_{fg}^d - A^a A^b A^c A^e A^f A^g \partial_h \Gamma_{ab}^d \partial_{ce} \Gamma_{fg}^h - 12A^a A^b A^c A^e A^f A^g \partial_a \Gamma_{bc}^h \partial_{ef} \Gamma_{gh}^d \\
& - 8A^a A^b A^c A^e A^f A^g \partial_a \Gamma_{bh}^d \partial_{ce} \Gamma_{fg}^h - 6A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_{ceh} \Gamma_{fg}^d - 8A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_{cef} \Gamma_{gh}^d \\
& + 8A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_i \Gamma_{ch}^d \partial_e \Gamma_{fg}^i + A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_i \Gamma_{ce}^d \partial_h \Gamma_{fg}^i + 4A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_i \Gamma_{ce}^d \partial_f \Gamma_{gh}^i \\
& + 12A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_h \Gamma_{ci}^d \partial_e \Gamma_{fg}^i + 24A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_c \Gamma_{hi}^d \partial_e \Gamma_{fg}^i + 8A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_c \Gamma_{ei}^d \partial_h \Gamma_{fg}^i \\
& + 32A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \partial_c \Gamma_{ei}^d \partial_f \Gamma_{gh}^i - 2A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \partial_{bce} \Gamma_{fg}^h + 2A^a A^b A^c A^e A^f A^g \Gamma_{ai}^h \partial_h \Gamma_{bc}^d \partial_e \Gamma_{fg}^i \\
& + 16A^a A^b A^c A^e A^f A^g \Gamma_{ai}^h \partial_b \Gamma_{ch}^d \partial_e \Gamma_{fg}^i + 6A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \partial_a \Gamma_{bc}^h \partial_e \Gamma_{fg}^i + 2A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \partial_b \Gamma_{ce}^h \partial_i \Gamma_{fg}^h \\
& + 12A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \partial_b \Gamma_{ci}^h \partial_e \Gamma_{fg}^i + 8A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ch}^i \partial_{ei} \Gamma_{fg}^d + 3A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \partial_{hi} \Gamma_{fg}^d \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \partial_{fh} \Gamma_{gi}^d + 24A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ch}^i \partial_{ef} \Gamma_{gi}^d + 12A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \partial_{fg} \Gamma_{hi}^d \\
& + 8A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{ab}^h \partial_{ce} \Gamma_{fg}^i + 6A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bc}^i \partial_{ei} \Gamma_{fg}^h + 12A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bc}^i \partial_{ef} \Gamma_{gi}^h \\
& + 4A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bi}^h \partial_{ce} \Gamma_{fg}^i - 4A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ch}^i \Gamma_{ei}^j \partial_j \Gamma_{fg}^d - 2A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{hi}^j \partial_j \Gamma_{fg}^d \\
& - 16A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fh}^j \partial_j \Gamma_{gi}^d - 24A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fh}^j \partial_i \Gamma_{gj}^d - 12A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fg}^j \partial_h \Gamma_{ij}^d \\
& - 32A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ch}^i \Gamma_{ei}^j \partial_f \Gamma_{gj}^d - 16A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{hi}^j \partial_f \Gamma_{gj}^d - 48A^a A^b A^c A^e A^f A^g \Gamma_{ab}^h \Gamma_{ce}^i \Gamma_{fh}^j \partial_g \Gamma_{ij}^d \\
& - 24A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{aj}^h \Gamma_{bc}^j \partial_e \Gamma_{fg}^i - 8A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{ce}^j \partial_j \Gamma_{fg}^i - 32A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{ce}^j \partial_f \Gamma_{gj}^i \\
& - 4A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ei}^j \partial_j \Gamma_{fg}^h - 12A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ef}^j \partial_i \Gamma_{gj}^h - 24A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ei}^j \partial_f \Gamma_{gj}^h \\
& - 12A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bc}^i \Gamma_{ef}^j \partial_g \Gamma_{ij}^h - 16A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{cj}^i \partial_e \Gamma_{fg}^j - 12A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{ij}^h \Gamma_{bc}^i \partial_e \Gamma_{fg}^j \\
& - 4A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{ce}^j \partial_j \Gamma_{fg}^i - 16A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{ce}^j \partial_f \Gamma_{gj}^i - 8A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{cj}^i \partial_e \Gamma_{fg}^j \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{aj}^h \Gamma_{bk}^i \Gamma_{ce}^j \Gamma_{fg}^k + 16A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{jk}^i \Gamma_{ce}^j \Gamma_{fg}^k + 32A^a A^b A^c A^e A^f A^g \Gamma_{hi}^d \Gamma_{ab}^h \Gamma_{cj}^i \Gamma_{ek}^j \Gamma_{fg}^k \\
& + 24A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{ij}^h \Gamma_{bc}^i \Gamma_{ek}^j \Gamma_{fg}^k + 8A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{jk}^i \Gamma_{ce}^j \Gamma_{fg}^k + 16A^a A^b A^c A^e A^f A^g \Gamma_{ah}^d \Gamma_{bi}^h \Gamma_{cj}^i \Gamma_{ek}^j \Gamma_{fg}^k
\end{aligned}$$

Stage 2: The fixed point scheme for y^a

```
{x^{a}}::Weight(label=eps,value=0).

{y00^{a},y10^{a},y20^{a},y30^{a},y40^{a}}::Weight(label=eps,value=0).
{y11^{a},y21^{a},y31^{a},y41^{a}}::Weight(label=eps,value=1).
{y22^{a},y32^{a},y42^{a}}::Weight(label=eps,value=2).
{y33^{a},y43^{a}}::Weight(label=eps,value=3).
{y44^{a}}::Weight(label=eps,value=4).

{Gam11^{a}_{b c}}::Weight(label=eps,value=1).
{Gam22^{a}_{b c d}}::Weight(label=eps,value=2).
{Gam33^{a}_{b c d e}}::Weight(label=eps,value=3).
{Gam44^{a}_{b c d e f}}::Weight(label=eps,value=4).
{Gam55^{a}_{b c d e f g}}::Weight(label=eps,value=5).

Gam11^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
Gam22^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
Gam33^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
Gam44^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
Gam55^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

y00{#}::LaTeXForm ("\\ny{00}").
y10{#}::LaTeXForm ("\\ny{10}").
y20{#}::LaTeXForm ("\\ny{20}").
y30{#}::LaTeXForm ("\\ny{30}").
y40{#}::LaTeXForm ("\\ny{40}").
y11{#}::LaTeXForm ("\\ny{11}").
y21{#}::LaTeXForm ("\\ny{21}").
y31{#}::LaTeXForm ("\\ny{31}").
y41{#}::LaTeXForm ("\\ny{41}").
y22{#}::LaTeXForm ("\\ny{22}").
y32{#}::LaTeXForm ("\\ny{32}").
y42{#}::LaTeXForm ("\\ny{42}").
y33{#}::LaTeXForm ("\\ny{33}").
y43{#}::LaTeXForm ("\\ny{43}").
y44{#}::LaTeXForm ("\\ny{44}").
```

```

Gam11{#}::LaTeXForm ("\\nGamma{11}").
Gam22{#}::LaTeXForm ("\\nGamma{22}").
Gam33{#}::LaTeXForm ("\\nGamma{33}").
Gam44{#}::LaTeXForm ("\\nGamma{44}").
Gam55{#}::LaTeXForm ("\\nGamma{55}").

def get_term (obj,n):

    foo := @(obj).
    bah = Ex("eps = " + str(n))
    distribute (foo)
    keep_weight (foo, bah)

    return foo

def truncateEps (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah = Ex("eps = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    return ans

def substitute_eps (obj):
    substitute (obj,epsy0)
    substitute (obj,epsy1)
    substitute (obj,epsy2)
    substitute (obj,epsy3)
    substitute (obj,epsy4)
    substitute (obj,epsGam1)
    substitute (obj,epsGam2)
    substitute (obj,epsGam3)
    substitute (obj,epsGam4)
    substitute (obj,epsGam5)

```

```

distribute      (obj)
obj = truncateEps (obj,4)
obj = product_sort (obj)
rename_dummies  (obj)
canonicalise     (obj)

return obj

def tidy (obj):
    obj = product_sort (obj)
    rename_dummies (obj)
    canonicalise    (obj)
    return obj

beg_stage_2 = time.time()

y0 := x^{a}.
y1 := x^{a} + (1/2) Gam^{a}_{b c} y0^{b} y0^{c}.
y2 := x^{a} + (1/2) Gam^{a}_{b c} y1^{b} y1^{c}
      + (1/6) Gam^{a}_{b c d} y0^{b} y0^{c} y0^{d}.
y3 := x^{a} + (1/2) Gam^{a}_{b c} y2^{b} y2^{c}
      + (1/6) Gam^{a}_{b c d} y1^{b} y1^{c} y1^{d}
      + (1/24) Gam^{a}_{b c d e} y0^{b} y0^{c} y0^{d} y0^{e}.
y4 := x^{a} + (1/2) Gam^{a}_{b c} y3^{b} y3^{c}
      + (1/6) Gam^{a}_{b c d} y2^{b} y2^{c} y2^{d}
      + (1/24) Gam^{a}_{b c d e} y1^{b} y1^{c} y1^{d} y1^{e}
      + (1/120) Gam^{a}_{b c d e f} y0^{b} y0^{c} y0^{d} y0^{e} y0^{f}.

# note that:
# y00 = y10 = y20 = y30 = y40
# y11 = y21 = y31 = y41
# y22 = y32 = y42
# y33 = y43
# y44

# expand each y in powers of eps

epsy0 := y0^{a} -> y00^{a}.

```

```

epsy1 := y1^{a} -> y10^{a}+y11^{a}.
epsy2 := y2^{a} -> y20^{a}+y21^{a}+y22^{a}.
epsy3 := y3^{a} -> y30^{a}+y31^{a}+y32^{a}+y33^{a}.
epsy4 := y4^{a} -> y40^{a}+y41^{a}+y42^{a}+y43^{a}+y44^{a}.

epsGam1 := Gam^{a}_{b c} -> Gam11^{a}_{b c}.
epsGam2 := Gam^{a}_{b c d} -> Gam22^{a}_{b c d}.
epsGam3 := Gam^{a}_{b c d e} -> Gam33^{a}_{b c d e}.
epsGam4 := Gam^{a}_{b c d e f} -> Gam44^{a}_{b c d e f}.
epsGam5 := Gam^{a}_{b c d e f g} -> Gam55^{a}_{b c d e f g}.

y0 = substitute_eps (y0)      # cdb (y0.001,y0)
y1 = substitute_eps (y1)      # cdb (y1.001,y1)
y2 = substitute_eps (y2)      # cdb (y2.001,y2)
y3 = substitute_eps (y3)      # cdb (y3.001,y3)
y4 = substitute_eps (y4)      # cdb (y4.001,y4)

defy0 := y0^{a} -> @(y0).
defy1 := y1^{a} -> @(y1).
defy2 := y2^{a} -> @(y2).
defy3 := y3^{a} -> @(y3).
defy4 := y4^{a} -> @(y4).

# -----
# y0

y00 := @(y0).                # cdb (y00.101,y00)

defy00 := y00^{a} -> @(y00).

# -----
# y1

substitute (y1,defy00)

distribute (y1)

y10 = get_term (y1,0)      # cdb (y10.101,y10)

```



```

y11 = get_term (y1,1)    # cdb (y11.101,y11)

defy10 := y10^{a} -> @(y10).
defy11 := y11^{a} -> @(y11).

# -----
# y2

substitute (y2,defy00)

substitute (y2,defy10)
substitute (y2,defy11)

distribute (y2)

y20 = get_term (y2,0)    # cdb (y20.101,y20)
y21 = get_term (y2,1)    # cdb (y21.101,y21)
y22 = get_term (y2,2)    # cdb (y22.101,y22)

y20 = tidy (y20)        # cdb (y20.201,y20)
y21 = tidy (y21)        # cdb (y21.201,y21)
y22 = tidy (y22)        # cdb (y22.201,y22)

defy20 := y20^{a} -> @(y20).
defy21 := y21^{a} -> @(y21).
defy22 := y22^{a} -> @(y22).

# -----
# y3

substitute (y3,defy00)

substitute (y3,defy10)
substitute (y3,defy11)

substitute (y3,defy20)
substitute (y3,defy21)
substitute (y3,defy22)

```

```

distribute (y3)

y30 = get_term (y3,0)    # cdb (y30.101,y30)
y31 = get_term (y3,1)    # cdb (y31.101,y31)
y32 = get_term (y3,2)    # cdb (y32.101,y32)
y33 = get_term (y3,3)    # cdb (y33.101,y33)

y30 = tidy (y30)        # cdb (y30.201,y30)
y31 = tidy (y31)        # cdb (y31.201,y31)
y32 = tidy (y32)        # cdb (y32.201,y32)
y33 = tidy (y33)        # cdb (y33.201,y33)

defy30 := y30^{a} -> @(y30).
defy31 := y31^{a} -> @(y31).
defy32 := y32^{a} -> @(y32).
defy33 := y33^{a} -> @(y33).

# -----
# y4

substitute (y4,defy00)

substitute (y4,defy10)
substitute (y4,defy11)

substitute (y4,defy20)
substitute (y4,defy21)
substitute (y4,defy22)

substitute (y4,defy30)
substitute (y4,defy31)
substitute (y4,defy32)
substitute (y4,defy33)

distribute (y4)

y40 = get_term (y4,0)    # cdb (y40.101,y40)

```

```

y41 = get_term (y4,1)    # cdb (y41.101,y41)
y42 = get_term (y4,2)    # cdb (y42.101,y42)
y43 = get_term (y4,3)    # cdb (y43.101,y43)
y44 = get_term (y4,4)    # cdb (y44.101,y44)

y40 = tidy (y40)        # cdb (y40.201,y40)
y41 = tidy (y41)        # cdb (y41.201,y41)
y42 = tidy (y42)        # cdb (y42.201,y42)
y43 = tidy (y43)        # cdb (y43.201,y43)
y44 = tidy (y44)        # cdb (y44.201,y44)

defy40 := y40^{a} -> @(y40).
defy41 := y41^{a} -> @(y41).
defy42 := y42^{a} -> @(y42).
defy43 := y43^{a} -> @(y43).
defy44 := y44^{a} -> @(y44).

end_stage_2 = time.time()

```

$$\text{y1.001} := x^a + \frac{1}{2} \Gamma^a{}_{bc}{}^{00} b^{00} y^c$$

$$\text{y2.001} := x^a + \frac{1}{2} \Gamma^a{}_{bc}{}^{10} b^{10} y^c + \Gamma^a{}_{bc}{}^{10} b^{11} y^c + \frac{1}{6} \Gamma^a{}_{bcd}{}^{00} b^{00} y^c y^d + \frac{1}{2} \Gamma^a{}_{bc}{}^{11} b^{11} y^c$$

$$\text{y3.001} := x^a + \frac{1}{2} \Gamma^a{}_{bc}{}^{20} b^{20} y^c + \Gamma^a{}_{bc}{}^{20} b^{21} y^c + \frac{1}{6} \Gamma^a{}_{bcd}{}^{10} b^{10} y^c y^d + \Gamma^a{}_{bc}{}^{20} b^{22} y^c + \frac{1}{2} \Gamma^a{}_{bc}{}^{21} b^{21} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{10} b^{10} y^c y^d + \frac{1}{24} \Gamma^a{}_{bcde}{}^{00} b^{00} y^c y^d y^e + \Gamma^a{}_{bc}{}^{21} b^{22} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{10} b^{11} y^c y^d$$

$$\begin{aligned} \text{y4.001} := & x^a + \frac{1}{2} \Gamma^a{}_{bc}{}^{30} b^{30} y^c + \Gamma^a{}_{bc}{}^{30} b^{31} y^c + \frac{1}{6} \Gamma^a{}_{bcd}{}^{20} b^{20} y^c y^d + \Gamma^a{}_{bc}{}^{30} b^{32} y^c + \frac{1}{2} \Gamma^a{}_{bc}{}^{31} b^{31} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{20} b^{20} y^c y^d + \frac{1}{24} \Gamma^a{}_{bcde}{}^{10} b^{10} y^c y^d y^e \\ & + \Gamma^a{}_{bc}{}^{30} b^{33} y^c + \Gamma^a{}_{bc}{}^{31} b^{32} y^c + \frac{1}{2} \Gamma^a{}_{bcd}{}^{20} b^{20} y^c y^d + \frac{1}{2} \Gamma^a{}_{bcd}{}^{21} b^{21} y^c y^d + \frac{1}{6} \Gamma^a{}_{bcde}{}^{10} b^{10} y^c y^d y^e + \frac{1}{120} \Gamma^a{}_{bcdef}{}^{00} b^{00} y^c y^d y^e y^f \end{aligned}$$

$$y_{10.101} := x^a$$

$$y_{11.101} := \frac{1}{2} \Gamma_{bc}^{11a} x^b x^c$$

$$y_{20.201} := x^a$$

$$y_{21.201} := \frac{1}{2} x^b x^c \Gamma_{bc}^{11a}$$

$$y_{22.201} := \frac{1}{2} x^b x^c x^d \Gamma_{be}^{11a} \Gamma_{cd}^{11e} + \frac{1}{6} x^b x^c x^d \Gamma_{bcd}^{22a}$$

$$y_{30.201} := x^a$$

$$y_{31.201} := \frac{1}{2} x^b x^c \Gamma_{bc}^{11a}$$

$$y_{32.201} := \frac{1}{2} x^b x^c x^d \Gamma_{be}^{11a} \Gamma_{cd}^{11e} + \frac{1}{6} x^b x^c x^d \Gamma_{bcd}^{22a}$$

$$y_{33.201} := \frac{1}{2} x^b x^c x^d x^e \Gamma_{bf}^{11a} \Gamma_{cg}^{11f} \Gamma_{de}^{11g} + \frac{1}{6} x^b x^c x^d x^e \Gamma_{bf}^{11a} \Gamma_{cde}^{22f} + \frac{1}{8} x^b x^c x^d x^e \Gamma_{fg}^{11a} \Gamma_{bc}^{11f} \Gamma_{de}^{11g} + \frac{1}{4} x^b x^c x^d x^e \Gamma_{bc}^{11f} \Gamma_{def}^{22a} + \frac{1}{24} x^b x^c x^d x^e \Gamma_{bcde}^{33a}$$

$$\text{y40.201} := x^a$$

$$\text{y41.201} := \frac{1}{2}x^b x^c \Gamma_{bc}^{11a}$$

$$\text{y42.201} := \frac{1}{2}x^b x^c x^d \Gamma_{be}^{11a} \Gamma_{cd}^{11e} + \frac{1}{6}x^b x^c x^d \Gamma_{bcd}^{22a}$$

$$\text{y43.201} := \frac{1}{2}x^b x^c x^d x^e \Gamma_{bf}^{11a} \Gamma_{cg}^{11f} \Gamma_{de}^{11g} + \frac{1}{6}x^b x^c x^d x^e \Gamma_{bf}^{11a} \Gamma_{cde}^{22f} + \frac{1}{8}x^b x^c x^d x^e \Gamma_{fg}^{11a} \Gamma_{bc}^{11f} \Gamma_{de}^{11g} + \frac{1}{4}x^b x^c x^d x^e \Gamma_{bc}^{11f} \Gamma_{def}^{22a} + \frac{1}{24}x^b x^c x^d x^e \Gamma_{bcde}^{33a}$$

$$\begin{aligned} \text{y44.201} := & \frac{1}{2}x^b x^c x^d x^e x^f \Gamma_{bg}^{11a} \Gamma_{ch}^{11g} \Gamma_{di}^{11h} \Gamma_{ef}^{11i} + \frac{1}{6}x^b x^c x^d x^e x^f \Gamma_{bg}^{11a} \Gamma_{ch}^{11g} \Gamma_{def}^{22h} + \frac{1}{8}x^b x^c x^d x^e x^f \Gamma_{bg}^{11a} \Gamma_{hi}^{11g} \Gamma_{cd}^{11h} \Gamma_{ef}^{11i} + \frac{1}{4}x^b x^c x^d x^e x^f \Gamma_{bg}^{11a} \Gamma_{cd}^{11h} \Gamma_{efh}^{22g} \\ & + \frac{1}{24}x^b x^c x^d x^e x^f \Gamma_{bg}^{11a} \Gamma_{cdef}^{33g} + \frac{1}{4}x^b x^c x^d x^e x^f \Gamma_{gh}^{11a} \Gamma_{bc}^{11g} \Gamma_{di}^{11h} \Gamma_{ef}^{11i} + \frac{1}{12}x^b x^c x^d x^e x^f \Gamma_{gh}^{11a} \Gamma_{bc}^{11g} \Gamma_{def}^{22h} + \frac{1}{4}x^b x^c x^d x^e x^f \Gamma_{bc}^{11g} \Gamma_{dgh}^{11h} \Gamma_{efh}^{22a} \\ & + \frac{1}{12}x^b x^c x^d x^e x^f \Gamma_{bcg}^{22a} \Gamma_{def}^{22g} + \frac{1}{8}x^b x^c x^d x^e x^f \Gamma_{bc}^{11g} \Gamma_{de}^{11h} \Gamma_{fgh}^{22a} + \frac{1}{12}x^b x^c x^d x^e x^f \Gamma_{bc}^{11g} \Gamma_{defg}^{33a} + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma_{bcdef}^{44a} \end{aligned}$$

Stage3: Introduce the generalised connections from Stage 1

```
def substitute_gam (obj):
    substitute (obj,defGam11)
    substitute (obj,defGam22)
    substitute (obj,defGam33)
    substitute (obj,defGam44)
    substitute (obj,defGam55)
    distribute (obj)

    return obj

beg_stage_3 = time.time()

Gam11 := @(eq0).
Gam22 := @(eq1).
Gam33 := @(eq2).
Gam44 := @(eq3).
Gam55 := @(eq4).

# peel off the  $A^{\{a\}}$ , must then symmetrise over revealed indices

substitute (Gam11,$A^{\{a\}}\rightarrow 1$)
substitute (Gam22,$A^{\{a\}}\rightarrow 1$)
substitute (Gam33,$A^{\{a\}}\rightarrow 1$)
substitute (Gam44,$A^{\{a\}}\rightarrow 1$)
substitute (Gam55,$A^{\{a\}}\rightarrow 1$)

# now symmetrise

sym (Gam11,$_{\{a\}},_{\{b\}}$)
sym (Gam22,$_{\{a\}},_{\{b\}},_{\{c\}}$)
sym (Gam33,$_{\{a\}},_{\{b\}},_{\{c\}},_{\{e\}}$)
sym (Gam44,$_{\{a\}},_{\{b\}},_{\{c\}},_{\{e\}},_{\{f\}}$)
sym (Gam55,$_{\{a\}},_{\{b\}},_{\{c\}},_{\{e\}},_{\{f\}},_{\{g\}}$)

defGam11 := Gam11^{\{d\}}_{\{a\} b} -> @(Gam11).
defGam22 := Gam22^{\{d\}}_{\{a\} b c} -> @(Gam22).
```

```

defGam33 := Gam33^{d}_{a b c e} -> @(Gam33).
defGam44 := Gam44^{d}_{a b c e f} -> @(Gam44).
defGam55 := Gam55^{d}_{a b c e f g} -> @(Gam55).

y31 = substitute_gam (y31)
y32 = substitute_gam (y32)
y33 = substitute_gam (y33)

y31 = tidy (y31)      # cdb (y31.301,y31)
y32 = tidy (y32)      # cdb (y32.301,y32)
y33 = tidy (y33)      # cdb (y33.301,y33)

y3 := @(y30) + @(y31) + @(y32) + @(y33).

y41 = substitute_gam (y41)
y42 = substitute_gam (y42)
y43 = substitute_gam (y43)
y44 = substitute_gam (y44)

y41 = tidy (y41)      # cdb (y41.301,y41)
y42 = tidy (y42)      # cdb (y42.301,y42)
y43 = tidy (y43)      # cdb (y43.301,y43)
y44 = tidy (y44)      # cdb (y44.301,y44)

y4 := @(y40) + @(y41) + @(y42) + @(y43) + @(y44).

end_stage_3 = time.time()

```


$$\text{y30.201} := x^a$$

$$\text{y31.301} := \frac{1}{2}x^b x^c \Gamma^a_{bc}$$

$$\text{y32.301} := \frac{1}{6}x^b x^c x^d \Gamma^a_{be} \Gamma^e_{cd} + \frac{1}{6}x^b x^c x^d \partial_b \Gamma^a_{cd}$$

$$\text{y33.301} := \frac{1}{12}x^b x^c x^d x^e \Gamma^a_{bf} \partial_c \Gamma^f_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^a_{fg} \Gamma^f_{bc} \Gamma^g_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^f_{bc} \partial_f \Gamma^a_{de} + \frac{1}{24}x^b x^c x^d x^e \partial_{bc} \Gamma^a_{de}$$

$$\text{y40.201} := x^a$$

$$\text{y41.301} := \frac{1}{2}x^b x^c \Gamma^a_{bc}$$

$$\text{y42.301} := \frac{1}{6}x^b x^c x^d \Gamma^a_{be} \Gamma^e_{cd} + \frac{1}{6}x^b x^c x^d \partial_b \Gamma^a_{cd}$$

$$\text{y43.301} := \frac{1}{12}x^b x^c x^d x^e \Gamma^a_{bf} \partial_c \Gamma^f_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^a_{fg} \Gamma^f_{bc} \Gamma^g_{de} + \frac{1}{24}x^b x^c x^d x^e \Gamma^f_{bc} \partial_f \Gamma^a_{de} + \frac{1}{24}x^b x^c x^d x^e \partial_{bc} \Gamma^a_{de}$$

$$\begin{aligned} \text{y44.301} := & -\frac{1}{90}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^g_{ch} \Gamma^h_{di} \Gamma^i_{ef} + \frac{1}{180}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^g_{ch} \partial_d \Gamma^h_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^g_{hi} \Gamma^h_{cd} \Gamma^i_{ef} \\ & - \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^h_{cd} \partial_e \Gamma^g_{fh} + \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^a_{bg} \Gamma^h_{cd} \partial_h \Gamma^g_{ef} + \frac{1}{40}x^b x^c x^d x^e x^f \Gamma^a_{bg} \partial_{cd} \Gamma^g_{ef} + \frac{1}{90}x^b x^c x^d x^e x^f \Gamma^a_{gh} \Gamma^g_{bc} \Gamma^h_{di} \Gamma^i_{ef} \\ & + \frac{13}{360}x^b x^c x^d x^e x^f \Gamma^a_{gh} \Gamma^g_{bc} \partial_d \Gamma^h_{ef} - \frac{1}{90}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{dg} \partial_e \Gamma^a_{fh} + \frac{1}{360}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{dg} \partial_h \Gamma^a_{ef} \\ & + \frac{1}{180}x^b x^c x^d x^e x^f \partial_b \Gamma^a_{cg} \partial_d \Gamma^g_{ef} + \frac{7}{360}x^b x^c x^d x^e x^f \partial_g \Gamma^a_{bc} \partial_d \Gamma^g_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{de} \partial_f \Gamma^a_{gh} \\ & + \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g_{bc} \Gamma^h_{de} \partial_g \Gamma^a_{fh} - \frac{1}{120}x^b x^c x^d x^e x^f \Gamma^g_{bc} \partial_{de} \Gamma^a_{fg} + \frac{1}{60}x^b x^c x^d x^e x^f \Gamma^g_{bc} \partial_{dg} \Gamma^a_{ef} + \frac{1}{120}x^b x^c x^d x^e x^f \partial_{bcd} \Gamma^a_{ef} \end{aligned}$$

Stage4: Reformatting and output

```
{x^{a}}>::Weight(label=numx).
\Gamma^{a}_{b c}>::TableauSymmetry(shape={2}, indices={1,2}).

def reformat (obj,scale):

    bah = Ex(str(scale))
    tmp := @(bah) @(obj).
    distribute      (tmp)
    tmp = product_sort (tmp)
    rename_dummies  (tmp)
    canonicalise    (tmp)
    factor_out      (tmp,$x^{a?}$)

    return tmp

def get_term (obj,n):

    tmp := @(obj).
    foo = Ex("numx = " + str(n))
    distribute      (tmp)
    keep_weight     (tmp, foo)

    return tmp

beg_stage_4 = time.time()

rnc := x^{a}
      + @(y41)
      + @(y42)
      + @(y43)
      + @(y44).

# substitute (rnc,$A^{a}->x^{a}$)

rnc1 = get_term (rnc,1)          # cdb (rnc1.001,rnc1)
rnc2 = get_term (rnc,2)          # cdb (rnc2.001,rnc2)
```

```

rnc3 = get_term (rnc,3)          # cdb (rnc3.001,rnc3)
rnc4 = get_term (rnc,4)          # cdb (rnc4.001,rnc4)
rnc5 = get_term (rnc,5)          # cdb (rnc5.001,rnc5)

scaled1 = reformat (rnc1,  1)    # cdb (scaled1.002,scaled1)
scaled2 = reformat (rnc2,  2)    # cdb (scaled2.002,scaled2)
scaled3 = reformat (rnc3,  6)    # cdb (scaled3.002,scaled3)
scaled4 = reformat (rnc4, 24)    # cdb (scaled4.002,scaled4)
scaled5 = reformat (rnc5, 360)   # cdb (scaled5.002,scaled5)

import cdblib

cdblib.create ('gen2rnc.json')

cdblib.put ('rnc',rnc,'gen2rnc.json')

cdblib.put ('rnc1',rnc1,'gen2rnc.json')
cdblib.put ('rnc2',rnc2,'gen2rnc.json')
cdblib.put ('rnc3',rnc3,'gen2rnc.json')
cdblib.put ('rnc4',rnc4,'gen2rnc.json')
cdblib.put ('rnc5',rnc5,'gen2rnc.json')

end_stage_4 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:.7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:.7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:.7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:.7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)

```

Timing

Stage 1: 6.9 secs

Stage 2: 1.9 secs

Stage 3: 285.7 secs

Stage 4: 0.9 secs

Convert from generic (x) to local RNC coords (y)

$$y^a = {}^0y^a + {}^1y^a + {}^2y^a + {}^3y^a + {}^4y^a$$

$${}^0y^a = x^a$$

$$2{}^1y^a = x^b x^c \Gamma^a_{bc}$$

$$6{}^2y^a = x^b x^c x^d (\Gamma^a_{be} \Gamma^e_{cd} + \partial_b \Gamma^a_{cd})$$

$$24{}^3y^a = x^b x^c x^d x^e (2\Gamma^a_{bf} \partial_c \Gamma^f_{de} + \Gamma^a_{fg} \Gamma^f_{bc} \Gamma^g_{de} + \Gamma^f_{bc} \partial_f \Gamma^a_{de} + \partial_{bc} \Gamma^a_{de})$$

$$\begin{aligned} 360{}^4y^a = & x^b x^c x^d x^e x^f (-4\Gamma^a_{bg} \Gamma^g_{ch} \Gamma^h_{di} \Gamma^i_{ef} + 2\Gamma^a_{bg} \Gamma^g_{ch} \partial_d \Gamma^h_{ef} + 3\Gamma^a_{bg} \Gamma^g_{hi} \Gamma^h_{cd} \Gamma^i_{ef} - 6\Gamma^a_{bg} \Gamma^h_{cd} \partial_e \Gamma^g_{fh} + 6\Gamma^a_{bg} \Gamma^h_{cd} \partial_h \Gamma^g_{ef} + 9\Gamma^a_{bg} \partial_{cd} \Gamma^g_{ef} \\ & + 4\Gamma^a_{gh} \Gamma^g_{bc} \Gamma^h_{di} \Gamma^i_{ef} + 13\Gamma^a_{gh} \Gamma^g_{bc} \partial_d \Gamma^h_{ef} - 4\Gamma^g_{bc} \Gamma^h_{dg} \partial_e \Gamma^a_{fh} + \Gamma^g_{bc} \Gamma^h_{dg} \partial_h \Gamma^a_{ef} + 2\partial_b \Gamma^a_{cg} \partial_d \Gamma^g_{ef} + 7\partial_g \Gamma^a_{bc} \partial_d \Gamma^g_{ef} + 3\Gamma^g_{bc} \Gamma^h_{de} \partial_f \Gamma^a_{gh} \\ & + 3\Gamma^g_{bc} \Gamma^h_{de} \partial_g \Gamma^a_{fh} - 3\Gamma^g_{bc} \partial_{de} \Gamma^a_{fg} + 6\Gamma^g_{bc} \partial_{dg} \Gamma^a_{ef} + 3\partial_{bcd} \Gamma^a_{ef}) \end{aligned}$$

```
cdblib.create ('gen2rnc.export')

# 6th order terms, scaled
cdblib.put ('rnc61scaled',scaled1,'gen2rnc.export')
cdblib.put ('rnc62scaled',scaled2,'gen2rnc.export')
cdblib.put ('rnc63scaled',scaled3,'gen2rnc.export')
cdblib.put ('rnc64scaled',scaled4,'gen2rnc.export')
cdblib.put ('rnc65scaled',scaled5,'gen2rnc.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```