Symmetrised partial derivatives of the Riemann tensor

Here we compute the symmetrised partial derivatoves $R^a{}_{(b\dot{c}d,\underline{e})}$ in terms of the symmetrised covariant derivatives $R^a{}_{(b\dot{c}d;\underline{e})}$. Note that the dot over an index indicates that that index does not take part in the symmetrisation.

We will use the algorithm described in section (10.3) of my lcb09-03 paper. Here we will make one small change of notation – the symbol D^a will replaced with A^a.

We have lots of space (and no annoying editors to appease with brevity) so I will take the liberty to expand slightly on what I wrote in the lcb0-03 paper.

Our starting point is the simple identity

$$\left(R^a{}_{cdb}B^b{}_a A^c A^d\right)_{;e} A^e = \left(R^a{}_{cdb}B^b{}_a A^c A^d\right)_{,e} A^e \tag{1}$$

This is true in all frames since the quantity inside the brackets is a scalar. We are free to make any choice we like for A^a and $B^a{}_b$ so let's choose A^a to be the tangent vector to any geodesic through the origin and choose the $B^a{}_b$ to be constants (i.e, all partial derivatives are zero). We will also use local Riemann normal coordinates and as a consequence, the A^a will also be constant along the integral curves of A (the geodesics in an RNC are always of the form $x^a(s) = sA^a$ for some affine parameter s on the geodesic). Let df/ds be the directional derivative of the function f along the geodesics defined by A^a and assume that s is the proper length along the geodesic (although any affine parameter would be sufficient).

Thus at the origin we have, by choice,

$$0 = B^a{}_{b,c} = B^a{}_{b,cd} = B^a{}_{b,cde} = \dots$$

$$0 = dA^a/ds = d^2A^a/ds^2 = d^3A^a/ds^3 = \dots$$

$$0 = A^a{}_{,b}A^b = \left(A^a{}_{,b}A^b\right)_{,c} A^c = \left(\left(A^a{}_{,b}A^b\right)_{,c} A^c\right)_{,d} A^d$$

$$0 = A^a{}_{;b}A^b = \left(A^a{}_{;b}A^b\right)_{;c} A^c = \left(\left(A^a{}_{;b}A^b\right)_{;c} A^c\right)_{;d} A^d$$

$$df/ds = f_{,a}A^a = f_{;a}A^a$$

$$d^2f/ds^2 = \left(f_{,a}A^a\right)_{,b} A^b = \left(f_{;a}A^a\right)_{;b} A^b$$

$$d^3f/ds^3 = \left(\left(f_{,a}A^a\right)_{,b} A^b\right)_{,c} A^c = \left(\left(f_{;a}A^a\right)_{;b} A^b\right)_{;c} A^c$$

I admit I've gone overboard here in writing out more than I need to but it's handy to have all of these equations laid bare in one convenient place.

Now put $f = R^p{}_{abq}B^q{}_pA^aA^b$. Then upon taking successive derivatives, while taking full advantage of the asummptions just noted, we can eaily see that

$$\left(R^a{}_{cdb}B^b{}_a\right)_{;\underline{e}}A^cA^dA^{\underline{e}} = \left(R^a{}_{cdb}\right)_{,\underline{e}}B^b{}_aA^cA^dA^{\underline{e}} \tag{2}$$

This is the equation that will be computed by the following Cadabra code. All of the compuations will be carried out on the left hand side (in the first version of the paper I swapped the left and righ hand sides).

We will need the successive covariant derivatives of B. The first covariant derivative is just

$$B^a{}_{b;c}A^c = \Gamma^a{}_{dc}B^d{}_bA^c - \Gamma^d{}_{bc}B^a{}_dA^c$$

The quantities on the left hand side are the components of a tensor so further covariant derivatives of the right hand side can be computed (despite the presence of the Γ's) by application of the usual rule for a covariant derivative of a mixed tensor.

Stage 1: Symmetrised partial derivatives of R

The first stage involves the expansion of the left side of (2). This leads to expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and $B^a{}_b$.

$$\begin{aligned}
\left(R^a{}_{cdb}\right)_{,e}B^b{}_aA^cA^dA^e &= -A^aA^bA^cB^d{}_e\nabla_aR_{bfcd}g^{ef} - A^aA^bA^cR_{afbd}\nabla_cB^d{}_eg^{ef}\\
\left(R^a{}_{cdb}\right)_{,ef}B^b{}_aA^cA^dA^eA^f &= -2A^aA^bA^cA^d\nabla_aB^e{}_f\nabla_bR_{cedg}g^{fg} - A^aA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^bA^cA^dR_{aebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
\left(R^a{}_{cdb}\right)_{,efg}B^b{}_aA^cA^dA^eA^fA^g &= -3A^aA^bA^cA^dA^e\nabla_aR_{bfch}\nabla_d\left(\nabla_eB^f{}_g\right)g^{hg} - 3A^aA^bA^cA^dA^e\nabla_aB^f{}_g\nabla_b\left(\nabla_cR_{dfeh}\right)g^{gh}\\
&\quad - A^aA^bA^cA^dA^eB^f{}_g\nabla_a\left(\nabla_b\left(\nabla_cR_{dfeh}\right)\right)g^{gh} - A^aA^bA^cA^dA^eR_{afbh}\nabla_c\left(\nabla_d\left(\nabla_eB^f{}_g\right)\right)g^{hg}
\end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

In this stage the symmetrized covariant derivatives of $B^a{}_b$ are computed in terms of its partial derivatives (which by choice are all zero) and the connection and its partial derivatives (which in general are not zero).

$$
\begin{aligned}
A^c\nabla_c\left(B^a{}_b\right) &= \Gamma^a{}_{pq}B^p{}_bA^q - \Gamma^p{}_{bq}B^a{}_pA^q\\
A^dA^c\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right) &= A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{cd}\Gamma^c{}_{pq}B^p{}_bA^dA^q - 2\Gamma^a{}_{cd}\Gamma^p{}_{bq}B^c{}_pA^dA^q + \Gamma^c{}_{bd}\Gamma^p{}_{cq}B^a{}_pA^dA^q\\
A^eA^dA^c\nabla_e\left(\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right)\right) &= A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{de}\Gamma^d{}_{pq}B^p{}_bA^eA^q + A^c\Gamma^a{}_{cd}\partial_e\Gamma^d{}_{pq}B^p{}_bA^eA^q\\
&\quad - 2A^c\partial_c\Gamma^a{}_{de}\Gamma^p{}_{bq}B^d{}_pA^eA^q - 2A^c\Gamma^a{}_{cd}\partial_e\Gamma^p{}_{bq}B^d{}_pA^eA^q + A^c\partial_c\Gamma^d{}_{be}\Gamma^p{}_{dq}B^a{}_pA^eA^q + A^c\Gamma^d{}_{bc}\partial_e\Gamma^p{}_{dq}B^a{}_pA^eA^q\\
&\quad + \Gamma^a{}_{ce}A^c\partial_f\Gamma^e{}_{pq}B^p{}_bA^fA^q - \Gamma^a{}_{ce}A^c\partial_f\Gamma^p{}_{bq}B^e{}_pA^fA^q + \Gamma^a{}_{cd}\Gamma^c{}_{ef}\Gamma^e{}_{pq}B^p{}_bA^dA^fA^q - 3\Gamma^a{}_{cd}\Gamma^e{}_{bf}\Gamma^c{}_{pq}B^p{}_eA^dA^fA^q\\
&\quad + 3\Gamma^a{}_{cd}\Gamma^e{}_{bf}\Gamma^p{}_{eq}B^c{}_pA^dA^fA^q - \Gamma^c{}_{be}A^e\partial_f\Gamma^a{}_{pq}B^p{}_cA^fA^q + \Gamma^c{}_{be}A^e\partial_f\Gamma^p{}_{cq}B^a{}_pA^fA^q - \Gamma^c{}_{bd}\Gamma^e{}_{cf}\Gamma^p{}_{eq}B^a{}_pA^dA^fA^q
\end{aligned}
$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

Here we impose the RNC condition (that $\Gamma = 0$ while $\partial\Gamma \neq 0$).

$$
\begin{aligned}
A^c\nabla_c\left(B^a{}_b\right) &= 0\\
A^dA^c\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right) &= A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q\\
A^eA^dA^c\nabla_e\left(\nabla_d\left(\nabla_c\left(B^a{}_b\right)\right)\right) &= A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q
\end{aligned}
$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

This stage uses the results from the second stage to eliminate the ∇B terms from the results of the first stage. This produces expressions for the symmetrized partial derivatives of R_{abcd} in terms of the symmetrized covariant derivatives of R_{abcd} and the partial derivatives of the connection. In this stage we also set the $B^a{}_b$ to equal 1.

$$
\begin{aligned}
\left(R^a{}_{cdb}\right)_{,e}A^cA^dA^e &= -A^cA^dA^e\nabla_cR_{dfeb}g^{af}\\
\left(R^a{}_{cdb}\right)_{,ef}A^cA^dA^eA^e &= A^cA^dA^eA^f\left(-\nabla_{cd}R_{ebfg}g^{ag} - R_{cgdh}\partial_e\Gamma^g{}_{bf}g^{ha} + R_{cbdg}\partial_e\Gamma^a{}_{hf}g^{gh}\right)\\
\left(R^a{}_{cdb}\right)_{,efg}A^cA^dA^eA^fA^g &= A^cA^dA^eA^fA^g\left(-3\nabla_cR_{dhei}\partial_f\Gamma^h{}_{bg}g^{ia} + 3\nabla_cR_{dbeh}\partial_f\Gamma^a{}_{ig}g^{hi} - \nabla_{cde}R_{fbgh}g^{ah} - R_{chdi}\partial_{ef}\Gamma^h{}_{bg}g^{ia} + R_{cbdh}\partial_{ef}\Gamma^a{}_{ig}g^{hi}\right)
\end{aligned}
$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

The fifth stage draws in results from `dGamma.tex` to replace the partial derivatives of Γ with partial derivatives of R_{abcd}.

$$
\begin{aligned}
\left(R^{a}{}_{cdb}\right)_{,e} A^c A^d A^e &= -A^c A^d A^e \nabla_c R_{dfeb} g^{af} \\
\left(R^{a}{}_{cdb}\right)_{,ef} A^c A^d A^e A^f &= -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R^{g}{}_{feb} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R^{a}{}_{feh} g^{gh} \\
\left(R^{a}{}_{cdb}\right)_{,efg} A^c A^d A^e A^f A^g &= -A^c A^d A^e A^f A^g R^{h}{}_{gfb} \nabla_c R_{dhei} g^{ia} + A^c A^d A^e A^f A^g R^{a}{}_{gfh} \nabla_c R_{dbei} g^{ih} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\
&\quad - \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \partial_f R^{h}{}_{geb} g^{ia} + \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \partial_f R^{a}{}_{gei} g^{hi}
\end{aligned}
$$

Stage 6: Replace partial derivs of R with covariant derivs of R

The final stage is to eliminate the ∂R by using earlier results. For example, in the equation for $\partial^3 R$ we see terms involving ∂R. These first order partial derivatives can be replaced with the expression previously computed for ∂R in terms of ∇R.

$$
\begin{aligned}
\left(R^{a}{}_{cdb}\right)_{,e} A^c A^d A^e &= A^c A^d A^e \nabla_c R_{bdef} g^{af} \\
\left(R^{a}{}_{cdb}\right)_{,ef} A^c A^d A^e A^e &= A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag} \\
\left(R^{a}{}_{cdb}\right)_{,efg} A^c A^d A^e A^f A^g &= -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}
\end{aligned}
$$

The end result are expressions for the symmetrized partial derivatives of R_{abcd} solely in terms of the symmetrized covariant derivatives of R_{abcd}.

Shared properties

```
import time

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

B^{a}_{b::Depends(\nabla{#}).
R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of R

```
def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d}   -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d}   -> g^{d e} R_{a b c e}$)
    unwrap      (obj)
    sort_product    (obj)
    rename_dummies (obj)
    return obj

# compute the symmetric covariant derivatives of R^{a}_{bcd} B^{d}_{a}

beg_stage_1 = time.time()

dRabcd00:=R^{a}_{b c d} B^{d}_{a} A^{b} A^{c}.         # cdb(dRabcd00.101,dRabcd00)

dRabcd01:=A^{a}\nabla_{a}{ @(dRabcd00) }.              # cdb(dRabcd01.101,dRabcd01)
distribute      (dRabcd01)                             # cdb(dRabcd01.102,dRabcd01)
product_rule    (dRabcd01)                             # cdb(dRabcd01.103,dRabcd01)
distribute      (dRabcd01)                             # cdb(dRabcd01.104,dRabcd01)
substitute      (dRabcd01,$\nabla_{a}{A^{b}} -> 0$)     # cdb(dRabcd01.105,dRabcd01)
substitute      (dRabcd01,$\nabla_{a}{g^{b c}} -> 0$)  # cdb(dRabcd01.106,dRabcd01)

sort_product    (dRabcd01)
rename_dummies (dRabcd01)
canonicalise    (dRabcd01)                             # cdb(dRabcd01.107,dRabcd01)
dRabcd01 = flatten_Rabcd (dRabcd01)                    # cdb(dRabcd01.108,dRabcd01)

dRabcd02:=A^{a}\nabla_{a}{ @(dRabcd01) }.              # cdb(dRabcd02.101,dRabcd02)
distribute      (dRabcd02)                             # cdb(dRabcd02.102,dRabcd02)
product_rule    (dRabcd02)                             # cdb(dRabcd02.103,dRabcd02)
distribute      (dRabcd02)                             # cdb(dRabcd02.104,dRabcd02)
substitute      (dRabcd02,$\nabla_{a}{A^{b}} -> 0$)     # cdb(dRabcd02.105,dRabcd02)
substitute      (dRabcd02,$\nabla_{a}{g^{b c}} -> 0$)  # cdb(dRabcd02.106,dRabcd02)

sort_product    (dRabcd02)
```

```
rename_dummies (dRabcd02)
canonicalise   (dRabcd02)                            # cdb(dRabcd02.107,dRabcd02)
dRabcd02 = flatten_Rabcd (dRabcd02)                  # cdb(dRabcd02.108,dRabcd02)

dRabcd03:=A^{a}\nabla_{a}{ @(dRabcd02) }.            # cdb(dRabcd03.101,dRabcd03)
distribute     (dRabcd03)                            # cdb(dRabcd03.102,dRabcd03)
product_rule   (dRabcd03)                            # cdb(dRabcd03.103,dRabcd03)
distribute     (dRabcd03)                            # cdb(dRabcd03.104,dRabcd03)
substitute     (dRabcd03,$\nabla_{a}{A^{b}} -> 0$)    # cdb(dRabcd03.105,dRabcd03)
substitute     (dRabcd03,$\nabla_{a}{g^{b c}} -> 0$)  # cdb(dRabcd03.106,dRabcd03)

sort_product   (dRabcd03)
rename_dummies (dRabcd03)
canonicalise   (dRabcd03)                            # cdb(dRabcd03.107,dRabcd03)
dRabcd03 = flatten_Rabcd (dRabcd03)                  # cdb(dRabcd03.108,dRabcd03)

dRabcd04:=A^{a}\nabla_{a}{ @(dRabcd03) }.
distribute     (dRabcd04)
product_rule   (dRabcd04)
distribute     (dRabcd04)
substitute     (dRabcd04,$\nabla_{a}{A^{b}} -> 0$)
substitute     (dRabcd04,$\nabla_{a}{g^{b c}} -> 0$)

sort_product   (dRabcd04)
rename_dummies (dRabcd04)
canonicalise   (dRabcd04)
dRabcd04 = flatten_Rabcd (dRabcd04)

dRabcd05:=A^{a}\nabla_{a}{ @(dRabcd04) }.
distribute     (dRabcd05)
product_rule   (dRabcd05)
distribute     (dRabcd05)
substitute     (dRabcd05,$\nabla_{a}{A^{b}} -> 0$)
substitute     (dRabcd05,$\nabla_{a}{g^{b c}} -> 0$)

sort_product   (dRabcd05)
rename_dummies (dRabcd05)
canonicalise   (dRabcd05)
```

```
dRabcd05 = flatten_Rabcd (dRabcd05)

def combine_nabla (obj):
    substitute (obj,$\nabla_{p}{\nabla_{q}{\nabla_{r}{\nabla_{s}{\nabla_{t}{A??}}}}}->\nabla_{p q r s t}{A??}$,repeat=True)
    substitute (obj,$\nabla_{p}{\nabla_{q}{\nabla_{r}{\nabla_{s}{A??}}}}->\nabla_{p q r s}{A??}$,repeat=True)
    substitute (obj,$\nabla_{p}{\nabla_{q}{\nabla_{r}{A??}}}->\nabla_{p q r}{A??}$,repeat=True)
    substitute (obj,$\nabla_{p}{\nabla_{q}{A??}}->\nabla_{p q}{A??}$,repeat=True)
    return obj

dRabcd01 = combine_nabla (dRabcd01)
dRabcd02 = combine_nabla (dRabcd02)
dRabcd03 = combine_nabla (dRabcd03)
dRabcd04 = combine_nabla (dRabcd04)
dRabcd05 = combine_nabla (dRabcd05)

end_stage_1 = time.time()
```

$$\texttt{dRabcd00.101} := R^{a}{}_{bcd}B^{d}{}_{a}A^{b}A^{c}$$

$$\texttt{dRabcd01.101} := A^{a}\nabla_{a}\left(R^{e}{}_{bcd}B^{d}{}_{e}A^{b}A^{c}\right)$$

$$\texttt{dRabcd01.102} := A^{a}\nabla_{a}\left(R^{e}{}_{bcd}B^{d}{}_{e}A^{b}A^{c}\right)$$

$$\texttt{dRabcd01.103} := A^{a}\left(\nabla_{a}R^{e}{}_{bcd}B^{d}{}_{e}A^{b}A^{c} + R^{e}{}_{bcd}\nabla_{a}B^{d}{}_{e}A^{b}A^{c} + R^{e}{}_{bcd}B^{d}{}_{e}\nabla_{a}A^{b}A^{c} + R^{e}{}_{bcd}B^{d}{}_{e}A^{b}\nabla_{a}A^{c}\right)$$

$$\texttt{dRabcd01.104} := A^{a}\nabla_{a}R^{e}{}_{bcd}B^{d}{}_{e}A^{b}A^{c} + A^{a}R^{e}{}_{bcd}\nabla_{a}B^{d}{}_{e}A^{b}A^{c} + A^{a}R^{e}{}_{bcd}B^{d}{}_{e}\nabla_{a}A^{b}A^{c} + A^{a}R^{e}{}_{bcd}B^{d}{}_{e}A^{b}\nabla_{a}A^{c}$$

$$\texttt{dRabcd01.105} := A^{a}\nabla_{a}R^{e}{}_{bcd}B^{d}{}_{e}A^{b}A^{c} + A^{a}R^{e}{}_{bcd}\nabla_{a}B^{d}{}_{e}A^{b}A^{c}$$

$$\texttt{dRabcd01.106} := A^{a}\nabla_{a}R^{e}{}_{bcd}B^{d}{}_{e}A^{b}A^{c} + A^{a}R^{e}{}_{bcd}\nabla_{a}B^{d}{}_{e}A^{b}A^{c}$$

$$\texttt{dRabcd01.107} := -A^{a}A^{b}A^{c}B^{d}{}_{e}\nabla_{a}R_{b}{}^{e}{}_{cd} - A^{a}A^{b}A^{c}R_{a}{}^{d}{}_{be}\nabla_{c}B^{e}{}_{d}$$

$$\texttt{dRabcd01.108} := -A^{a}A^{b}A^{c}B^{d}{}_{e}\nabla_{a}R_{bfcd}g^{ef} - A^{a}A^{b}A^{c}R_{afbd}\nabla_{c}B^{d}{}_{e}g^{ef}$$

$$\texttt{dRabcd02.101} := A^{a}\nabla_{a}\left(-A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} - A^{g}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef}\right)$$

$$\texttt{dRabcd02.102} := -A^{a}\nabla_{a}\left(A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef}\right) - A^{a}\nabla_{a}\left(A^{g}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef}\right)$$

$$\begin{aligned}\texttt{dRabcd02.103} := -A^{a}\big(&\nabla_{a}A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} + A^{g}\nabla_{a}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} + A^{g}A^{b}\nabla_{a}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} + A^{g}A^{b}A^{c}\nabla_{a}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef}\\ &+ A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{a}\left(\nabla_{g}R_{bfcd}\right)g^{ef} + A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}\nabla_{a}g^{ef}\big) - A^{a}\big(\nabla_{a}A^{g}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} + A^{g}\nabla_{a}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef}\\ &+ A^{g}A^{b}\nabla_{a}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} + A^{g}A^{b}A^{c}\nabla_{a}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} + A^{g}A^{b}A^{c}R_{gfbd}\nabla_{a}\left(\nabla_{c}B^{d}{}_{e}\right)g^{ef} + A^{g}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}\nabla_{a}g^{ef}\big)\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd02.104} := &-A^{a}\nabla_{a}A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} - A^{a}A^{g}\nabla_{a}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} - A^{a}A^{g}A^{b}\nabla_{a}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} - A^{a}A^{g}A^{b}A^{c}\nabla_{a}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef}\\ &- A^{a}A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{a}\left(\nabla_{g}R_{bfcd}\right)g^{ef} - A^{a}A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}\nabla_{a}g^{ef} - A^{a}\nabla_{a}A^{g}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} - A^{a}A^{g}\nabla_{a}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef}\\ &- A^{a}A^{g}A^{b}\nabla_{a}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} - A^{a}A^{g}A^{b}A^{c}\nabla_{a}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} - A^{a}A^{g}A^{b}A^{c}R_{gfbd}\nabla_{a}\left(\nabla_{c}B^{d}{}_{e}\right)g^{ef} - A^{a}A^{g}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}\nabla_{a}g^{ef}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd02.105} := &-A^{a}A^{g}A^{b}A^{c}\nabla_{a}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} - A^{a}A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{a}\left(\nabla_{g}R_{bfcd}\right)g^{ef} - A^{a}A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{g}R_{bfcd}\nabla_{a}g^{ef}\\ &- A^{a}A^{g}A^{b}A^{c}\nabla_{a}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} - A^{a}A^{g}A^{b}A^{c}R_{gfbd}\nabla_{a}\left(\nabla_{c}B^{d}{}_{e}\right)g^{ef} - A^{a}A^{g}A^{b}A^{c}R_{gfbd}\nabla_{c}B^{d}{}_{e}\nabla_{a}g^{ef}\end{aligned}$$

$$\texttt{dRabcd02.106} := -A^{a}A^{g}A^{b}A^{c}\nabla_{a}B^{d}{}_{e}\nabla_{g}R_{bfcd}g^{ef} - A^{a}A^{g}A^{b}A^{c}B^{d}{}_{e}\nabla_{a}\left(\nabla_{g}R_{bfcd}\right)g^{ef} - A^{a}A^{g}A^{b}A^{c}\nabla_{a}R_{gfbd}\nabla_{c}B^{d}{}_{e}g^{ef} - A^{a}A^{g}A^{b}A^{c}R_{gfbd}\nabla_{a}\left(\nabla_{c}B^{d}{}_{e}\right)g^{ef}$$

$$\texttt{dRabcd02.107} := -2A^{a}A^{b}A^{c}A^{d}\nabla_{a}B^{e}{}_{f}\nabla_{b}R_{cedg}g^{fg} - A^{a}A^{b}A^{c}A^{d}B^{e}{}_{f}\nabla_{a}\left(\nabla_{b}R_{cedg}\right)g^{fg} - A^{a}A^{b}A^{c}A^{d}R_{aebf}\nabla_{c}\left(\nabla_{d}B^{e}{}_{g}\right)g^{fg}$$

$$\texttt{dRabcd02.108} := -2A^{a}A^{b}A^{c}A^{d}\nabla_{a}B^{e}{}_{f}\nabla_{b}R_{cedg}g^{fg} - A^{a}A^{b}A^{c}A^{d}B^{e}{}_{f}\nabla_{a}\left(\nabla_{b}R_{cedg}\right)g^{fg} - A^{a}A^{b}A^{c}A^{d}R_{aebg}\nabla_{c}\left(\nabla_{d}B^{e}{}_{f}\right)g^{gf}$$

$$\texttt{dRabcd03.101} := A^a\nabla_a\left(-2A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - A^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\right)$$

$$\texttt{dRabcd03.102} := -2A^a\nabla_a\left(A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg}\right) - A^a\nabla_a\left(A^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\right) - A^a\nabla_a\left(A^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\right)$$

$$\begin{aligned}\texttt{dRabcd03.103} := -2A^a\Big(&\nabla_aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} + A^h\nabla_aA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} + A^hA^b\nabla_aA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg}\\
&+ A^hA^bA^c\nabla_aA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} + A^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} + A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg}\\
&+ A^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}\nabla_ag^{fg}\Big) - A^a\Big(\nabla_aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} + A^h\nabla_aA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&+ A^hA^b\nabla_aA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} + A^hA^bA^c\nabla_aA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} + A^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&+ A^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} + A^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)\nabla_ag^{fg}\Big) - A^a\Big(\nabla_aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&+ A^h\nabla_aA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} + A^hA^b\nabla_aA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} + A^hA^bA^c\nabla_aA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&+ A^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} + A^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf} + A^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)\nabla_ag^{gf}\Big)\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.104} :=& -2A^a\nabla_aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - 2A^aA^h\nabla_aA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^b\nabla_aA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg}\\
&- 2A^aA^hA^bA^c\nabla_aA^d\nabla_hB^e{}_f\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}\nabla_ag^{fg} - A^a\nabla_aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^h\nabla_aA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- A^aA^hA^b\nabla_aA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^c\nabla_aA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- A^aA^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} - A^aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)\nabla_ag^{fg} - A^a\nabla_aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&- A^aA^h\nabla_aA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^b\nabla_aA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^c\nabla_aA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf}\\
&- A^aA^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)\nabla_ag^{gf}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.105} :=& -2A^aA^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_bR_{cedg}\nabla_ag^{fg}\\
&- A^aA^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} - A^aA^hA^bA^cA^dB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)\nabla_ag^{fg}\\
&- A^aA^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)\nabla_ag^{gf}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.106} :=& -2A^aA^hA^bA^cA^d\nabla_a\left(\nabla_hB^e{}_f\right)\nabla_bR_{cedg}g^{fg} - 2A^aA^hA^bA^cA^d\nabla_hB^e{}_f\nabla_a\left(\nabla_bR_{cedg}\right)g^{fg} - A^aA^hA^bA^cA^d\nabla_aB^e{}_f\nabla_h\left(\nabla_bR_{cedg}\right)g^{fg}\\
&- A^aA^hA^bA^cA^dB^e{}_f\nabla_a\left(\nabla_h\left(\nabla_bR_{cedg}\right)\right)g^{fg} - A^aA^hA^bA^cA^d\nabla_aR_{hebg}\nabla_c\left(\nabla_dB^e{}_f\right)g^{gf} - A^aA^hA^bA^cA^dR_{hebg}\nabla_a\left(\nabla_c\left(\nabla_dB^e{}_f\right)\right)g^{gf}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.107} :=& -3A^aA^bA^cA^dA^e\nabla_aR_{bfcg}\nabla_d\left(\nabla_eB^f{}_h\right)g^{gh} - 3A^aA^bA^cA^dA^e\nabla_aB^f{}_g\nabla_b\left(\nabla_cR_{dfeh}\right)g^{gh}\\
&- A^aA^bA^cA^dA^eB^f{}_g\nabla_a\left(\nabla_b\left(\nabla_cR_{dfeh}\right)\right)g^{gh} - A^aA^bA^cA^dA^eR_{afbg}\nabla_c\left(\nabla_d\left(\nabla_eB^f{}_h\right)\right)g^{gh}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.108} :=& -3A^aA^bA^cA^dA^e\nabla_aR_{bfch}\nabla_d\left(\nabla_eB^f{}_g\right)g^{hg} - 3A^aA^bA^cA^dA^e\nabla_aB^f{}_g\nabla_b\left(\nabla_cR_{dfeh}\right)g^{gh}\\
&- A^aA^bA^cA^dA^eB^f{}_g\nabla_a\left(\nabla_b\left(\nabla_cR_{dfeh}\right)\right)g^{gh} - A^aA^bA^cA^dA^eR_{afbh}\nabla_c\left(\nabla_d\left(\nabla_eB^f{}_g\right)\right)g^{hg}\end{aligned}$$

Stage 2: Symmetrised covariant derivatives of B

```
# compute the covariant derivatives of B^{a}_{b}, note B^{a}_{b,c} is zero, by choice
# this method of computing covariant derivatives does not use auxillary fields

beg_stage_2 = time.time()

dBab00:=B^{a}_{b}.      # cdb(dBab00.201,dBab00)

dBab01:=A^{c}\partial_{c}{ @(dBab00) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
                                                      # cdb(dBab01.201,dBab01)
distribute   (dBab01)                                 # cdb(dBab01.202,dBab01)
product_rule (dBab01)                                 # cdb(dBab01.203,dBab01)
distribute   (dBab01)                                 # cdb(dBab01.204,dBab01)
substitute   (dBab01,$\partial_{a}{A^{b}} -> 0$)      # cdb(dBab01.205,dBab01)
substitute   (dBab01,$\partial_{a}{B^{b}_{c}} -> 0$)  # cdb(dBab01.206,dBab01)
substitute   (dBab01,$W^{a}_{b} -> @(dBab00)$)        # cdb(dBab01.207,dBab01)
distribute   (dBab01)                                 # cdb(dBab01.208,dBab01)
canonicalise (dBab01)                                 # cdb(dBab01.209,dBab01)

dBab02:=A^{c}\partial_{c}{ @(dBab01) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
                                                      # cdb(dBab02.201,dBab02)
distribute   (dBab02)                                 # cdb(dBab02.202,dBab02)
product_rule (dBab02)                                 # cdb(dBab02.203,dBab02)
distribute   (dBab02)                                 # cdb(dBab02.204,dBab02)
substitute   (dBab02,$\partial_{a}{A^{b}} -> 0$)      # cdb(dBab02.205,dBab02)
substitute   (dBab02,$\partial_{a}{B^{b}_{c}} -> 0$)  # cdb(dBab02.206,dBab02)
substitute   (dBab02,$W^{a}_{b} -> @(dBab01)$)        # cdb(dBab02.207,dBab02)
distribute   (dBab02)                                 # cdb(dBab02.208,dBab02)
canonicalise (dBab02)                                 # cdb(dBab02.209,dBab02)

dBab03:=A^{c}\partial_{c}{ @(dBab02) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
                                                      # cdb(dBab03.201,dBab03)
distribute   (dBab03)                                 # cdb(dBab03.202,dBab03)
product_rule (dBab03)                                 # cdb(dBab03.203,dBab03)
```

```
distribute   (dBab03)                             # cdb(dBab03.204,dBab03)
substitute   (dBab03,$\partial_{a}{A^{b}} -> 0$)      # cdb(dBab03.205,dBab03)
substitute   (dBab03,$\partial_{a}{B^{b}_{c}} -> 0$)  # cdb(dBab03.206,dBab03)
substitute   (dBab03,$W^{a}_{b} -> @(dBab02)$)        # cdb(dBab03.207,dBab03)
distribute   (dBab03)                             # cdb(dBab03.208,dBab03)
canonicalise (dBab03)                             # cdb(dBab03.209,dBab03)

dBab04:=A^{c}\partial_{c}{ @(dBab03) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
distribute   (dBab04)
product_rule (dBab04)
distribute   (dBab04)
substitute   (dBab04,$\partial_{a}{A^{b}} -> 0$)
substitute   (dBab04,$\partial_{a}{B^{b}_{c}} -> 0$)
substitute   (dBab04,$W^{a}_{b} -> @(dBab03)$)
distribute   (dBab04)
canonicalise (dBab04)

dBab05:=A^{c}\partial_{c}{ @(dBab04) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                       - \Gamma^{p}_{b q} W^{a}_{p} A^{q}.
distribute   (dBab05)
product_rule (dBab05)
distribute   (dBab05)
substitute   (dBab05,$\partial_{a}{A^{b}} -> 0$)
substitute   (dBab05,$\partial_{a}{B^{b}_{c}} -> 0$)
substitute   (dBab05,$W^{a}_{b} -> @(dBab04)$)
distribute   (dBab05)
canonicalise (dBab05)

end_stage_2 = time.time()
```

$$\texttt{dBab00.201} := B^a{}_b$$

$$\texttt{dBab01.201} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab01.202} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab01.203} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab01.204} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab01.205} := A^c\partial_c B^a{}_b + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab01.206} := \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab01.207} := \Gamma^a{}_{pq}B^p{}_b A^q - \Gamma^p{}_{bq}B^a{}_p A^q$$

$$\texttt{dBab01.208} := \Gamma^a{}_{pq}B^p{}_b A^q - \Gamma^p{}_{bq}B^a{}_p A^q$$

$$\texttt{dBab01.209} := \Gamma^a{}_{pq}B^p{}_b A^q - \Gamma^p{}_{bq}B^a{}_p A^q$$

$$\texttt{dBab02.201} := A^c\partial_c\left(\Gamma^a{}_{pq}B^p{}_b A^q - \Gamma^p{}_{bq}B^a{}_p A^q\right) + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab02.202} := A^c\partial_c\left(\Gamma^a{}_{pq}B^p{}_b A^q\right) - A^c\partial_c\left(\Gamma^p{}_{bq}B^a{}_p A^q\right) + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab02.203} := A^c\left(\partial_c\Gamma^a{}_{pq}B^p{}_b A^q + \Gamma^a{}_{pq}\partial_c B^p{}_b A^q + \Gamma^a{}_{pq}B^p{}_b\partial_c A^q\right) - A^c\left(\partial_c\Gamma^p{}_{bq}B^a{}_p A^q + \Gamma^p{}_{bq}\partial_c B^a{}_p A^q + \Gamma^p{}_{bq}B^a{}_p\partial_c A^q\right) + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab02.204} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_b A^q + A^c\Gamma^a{}_{pq}\partial_c B^p{}_b A^q + A^c\Gamma^a{}_{pq}B^p{}_b\partial_c A^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_p A^q - A^c\Gamma^p{}_{bq}\partial_c B^a{}_p A^q - A^c\Gamma^p{}_{bq}B^a{}_p\partial_c A^q + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab02.205} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_b A^q + A^c\Gamma^a{}_{pq}\partial_c B^p{}_b A^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_p A^q - A^c\Gamma^p{}_{bq}\partial_c B^a{}_p A^q + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab02.206} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_b A^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_p A^q + \Gamma^a{}_{pq}W^p{}_b A^q - \Gamma^p{}_{bq}W^a{}_p A^q$$

$$\texttt{dBab02.207} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_b A^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_p A^q + \Gamma^a{}_{pq}\left(\Gamma^p{}_{dc}B^d{}_b A^c - \Gamma^d{}_{bc}B^p{}_d A^c\right)A^q - \Gamma^p{}_{bq}\left(\Gamma^a{}_{dc}B^d{}_p A^c - \Gamma^d{}_{pc}B^a{}_d A^c\right)A^q$$

$$\texttt{dBab02.208} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_b A^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_p A^q + \Gamma^a{}_{pq}\Gamma^p{}_{dc}B^d{}_b A^c A^q - \Gamma^a{}_{pq}\Gamma^d{}_{bc}B^p{}_d A^c A^q - \Gamma^p{}_{bq}\Gamma^a{}_{dc}B^d{}_p A^c A^q + \Gamma^p{}_{bq}\Gamma^d{}_{pc}B^a{}_d A^c A^q$$

$$\texttt{dBab02.209} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_b A^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_p A^q + \Gamma^a{}_{cd}\Gamma^c{}_{pq}B^p{}_b A^d A^q - 2\Gamma^a{}_{cd}\Gamma^p{}_{bq}B^c{}_p A^d A^q + \Gamma^c{}_{bd}\Gamma^p{}_{cq}B^a{}_p A^d A^q$$

$$
\begin{aligned}
\texttt{dBab03.201} :={} & A^c\partial_c\left(A^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q - A^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q - 2\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q\right) + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.202} :={} & A^c\partial_c\left(A^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q\right) - A^c\partial_c\left(A^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q\right) + A^c\partial_c\left(\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q\right)\\
& - 2A^c\partial_c\left(\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q\right) + A^c\partial_c\left(\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q\right) + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.203} :={} & A^c\left(\partial_cA^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q + A^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q + A^e\partial_e\Gamma^a{}_{pq}\partial_cB^p{}_bA^q + A^e\partial_e\Gamma^a{}_{pq}B^p{}_b\partial_cA^q\right)\\
& - A^c\left(\partial_cA^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q + A^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^e\partial_e\Gamma^p{}_{bq}\partial_cB^a{}_pA^q + A^e\partial_e\Gamma^p{}_{bq}B^a{}_p\partial_cA^q\right)\\
& + A^c\left(\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + \Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}\partial_cB^p{}_bA^dA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_b\partial_cA^dA^q + \Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^d\partial_cA^q\right)\\
& - 2A^c\left(\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q + \Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + \Gamma^a{}_{ed}\Gamma^p{}_{bq}\partial_cB^e{}_pA^dA^q + \Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_p\partial_cA^dA^q + \Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^d\partial_cA^q\right)\\
& + A^c\left(\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}\partial_cB^a{}_pA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_p\partial_cA^dA^q + \Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^d\partial_cA^q\right)\\
& + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.204} :={} & A^c\partial_cA^e\partial_e\Gamma^a{}_{pq}B^p{}_bA^q + A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q + A^cA^e\partial_e\Gamma^a{}_{pq}\partial_cB^p{}_bA^q + A^cA^e\partial_e\Gamma^a{}_{pq}B^p{}_b\partial_cA^q - A^c\partial_cA^e\partial_e\Gamma^p{}_{bq}B^a{}_pA^q\\
& - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q - A^cA^e\partial_e\Gamma^p{}_{bq}\partial_cB^a{}_pA^q - A^cA^e\partial_e\Gamma^p{}_{bq}B^a{}_p\partial_cA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q\\
& + A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}\partial_cB^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_b\partial_cA^dA^q + A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^d\partial_cA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q\\
& - 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}\partial_cB^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_p\partial_cA^dA^q - 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^d\partial_cA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q\\
& + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}\partial_cB^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_p\partial_cA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^d\partial_cA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.205} :={} & A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q + A^cA^e\partial_e\Gamma^a{}_{pq}\partial_cB^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q - A^cA^e\partial_e\Gamma^p{}_{bq}\partial_cB^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q\\
& + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\Gamma^e{}_{pq}\partial_cB^p{}_bA^dA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\Gamma^p{}_{bq}\partial_cB^e{}_pA^dA^q\\
& + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\Gamma^p{}_{eq}\partial_cB^a{}_pA^dA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.206} :={} & A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q\\
& - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^a{}_{pq}W^p{}_bA^q - \Gamma^p{}_{bq}W^a{}_pA^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.207} :={} & A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q\\
& - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q\\
& + \Gamma^a{}_{pq}\left(A^c\partial_c\Gamma^p{}_{fe}B^f{}_bA^e - A^c\partial_c\Gamma^f{}_{be}B^p{}_fA^e + \Gamma^p{}_{cd}\Gamma^c{}_{fe}B^f{}_bA^dA^e - 2\Gamma^p{}_{cd}\Gamma^f{}_{be}B^c{}_fA^dA^e + \Gamma^c{}_{bd}\Gamma^f{}_{ce}B^p{}_fA^dA^e\right)A^q\\
& - \Gamma^p{}_{bq}\left(A^c\partial_c\Gamma^a{}_{fe}B^f{}_pA^e - A^c\partial_c\Gamma^f{}_{pe}B^a{}_fA^e + \Gamma^a{}_{cd}\Gamma^c{}_{fe}B^f{}_pA^dA^e - 2\Gamma^a{}_{cd}\Gamma^f{}_{pe}B^c{}_fA^dA^e + \Gamma^c{}_{pd}\Gamma^f{}_{ce}B^a{}_fA^dA^e\right)A^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.208} :={} & A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q + A^c\partial_c\Gamma^a{}_{ed}\Gamma^e{}_{pq}B^p{}_bA^dA^q + A^c\Gamma^a{}_{ed}\partial_c\Gamma^e{}_{pq}B^p{}_bA^dA^q - 2A^c\partial_c\Gamma^a{}_{ed}\Gamma^p{}_{bq}B^e{}_pA^dA^q\\
& - 2A^c\Gamma^a{}_{ed}\partial_c\Gamma^p{}_{bq}B^e{}_pA^dA^q + A^c\partial_c\Gamma^e{}_{bd}\Gamma^p{}_{eq}B^a{}_pA^dA^q + A^c\Gamma^e{}_{bd}\partial_c\Gamma^p{}_{eq}B^a{}_pA^dA^q + \Gamma^a{}_{pq}A^c\partial_c\Gamma^p{}_{fe}B^f{}_bA^eA^q - \Gamma^a{}_{pq}A^c\partial_c\Gamma^f{}_{be}B^p{}_fA^eA^q\\
& + \Gamma^a{}_{pq}\Gamma^p{}_{cd}\Gamma^c{}_{fe}B^f{}_bA^dA^eA^q - 2\Gamma^a{}_{pq}\Gamma^p{}_{cd}\Gamma^f{}_{be}B^c{}_fA^dA^eA^q + \Gamma^a{}_{pq}\Gamma^c{}_{bd}\Gamma^f{}_{ce}B^p{}_fA^dA^eA^q - \Gamma^p{}_{bq}A^c\partial_c\Gamma^a{}_{fe}B^f{}_pA^eA^q\\
& + \Gamma^p{}_{bq}A^c\partial_c\Gamma^f{}_{pe}B^a{}_fA^eA^q - \Gamma^p{}_{bq}\Gamma^a{}_{cd}\Gamma^c{}_{fe}B^f{}_pA^dA^eA^q + 2\Gamma^p{}_{bq}\Gamma^a{}_{cd}\Gamma^f{}_{pe}B^c{}_fA^dA^eA^q - \Gamma^p{}_{bq}\Gamma^c{}_{pd}\Gamma^f{}_{ce}B^a{}_fA^dA^eA^q
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dBab03.209} := {} & A^{c}A^{e}\partial_{ce}\Gamma^{a}{}_{pq}B^{p}{}_{b}A^{q} - A^{c}A^{e}\partial_{ce}\Gamma^{p}{}_{bq}B^{a}{}_{p}A^{q} + A^{c}\partial_{c}\Gamma^{a}{}_{de}\Gamma^{d}{}_{pq}B^{p}{}_{b}A^{e}A^{q} + A^{c}\Gamma^{a}{}_{cd}\partial_{e}\Gamma^{d}{}_{pq}B^{p}{}_{b}A^{e}A^{q} \\
& - 2A^{c}\partial_{c}\Gamma^{a}{}_{de}\Gamma^{p}{}_{bq}B^{d}{}_{p}A^{e}A^{q} - 2A^{c}\Gamma^{a}{}_{cd}\partial_{e}\Gamma^{p}{}_{bq}B^{d}{}_{p}A^{e}A^{q} + A^{c}\partial_{c}\Gamma^{d}{}_{be}\Gamma^{p}{}_{dq}B^{a}{}_{p}A^{e}A^{q} + A^{c}\Gamma^{d}{}_{bc}\partial_{e}\Gamma^{p}{}_{dq}B^{a}{}_{p}A^{e}A^{q} \\
& + \Gamma^{a}{}_{ce}A^{c}\partial_{f}\Gamma^{e}{}_{pq}B^{p}{}_{b}A^{f}A^{q} - \Gamma^{a}{}_{ce}A^{c}\partial_{f}\Gamma^{p}{}_{bq}B^{e}{}_{p}A^{f}A^{q} + \Gamma^{a}{}_{cd}\Gamma^{c}{}_{ef}\Gamma^{e}{}_{pq}B^{p}{}_{b}A^{d}A^{f}A^{q} - 3\Gamma^{a}{}_{cd}\Gamma^{e}{}_{bf}\Gamma^{c}{}_{pq}B^{p}{}_{e}A^{d}A^{f}A^{q} \\
& + 3\Gamma^{a}{}_{cd}\Gamma^{e}{}_{bf}\Gamma^{p}{}_{eq}B^{c}{}_{p}A^{d}A^{f}A^{q} - \Gamma^{c}{}_{be}A^{e}\partial_{f}\Gamma^{a}{}_{pq}B^{p}{}_{c}A^{f}A^{q} + \Gamma^{c}{}_{be}A^{e}\partial_{f}\Gamma^{p}{}_{cq}B^{a}{}_{p}A^{f}A^{q} - \Gamma^{c}{}_{bd}\Gamma^{e}{}_{cf}\Gamma^{p}{}_{eq}B^{a}{}_{p}A^{d}A^{f}A^{q}
\end{aligned}
$$

Stage 3: Impose the Riemann normal coordinate condition on covariant derivs of B

```
def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

# switch to RNC

beg_stage_3 = time.time()

dBab01 = impose_rnc (dBab01)   # cdb (dBab01.301,dBab01)
dBab02 = impose_rnc (dBab02)   # cdb (dBab02.301,dBab02)
dBab03 = impose_rnc (dBab03)   # cdb (dBab03.301,dBab03)
dBab04 = impose_rnc (dBab04)   # cdb (dBab04.301,dBab04)
dBab05 = impose_rnc (dBab05)   # cdb (dBab05.301,dBab05)

end_stage_3 = time.time()
```

$\texttt{dBab01.301} := 0$

$\texttt{dBab02.301} := A^c\partial_c\Gamma^a{}_{pq}B^p{}_bA^q - A^c\partial_c\Gamma^p{}_{bq}B^a{}_pA^q$

$\texttt{dBab03.301} := A^cA^e\partial_{ce}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^e\partial_{ce}\Gamma^p{}_{bq}B^a{}_pA^q$

$$\begin{aligned}\texttt{dBab04.301} := {} & A^cA^eA^g\partial_{ceg}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^eA^g\partial_{ceg}\Gamma^p{}_{bq}B^a{}_pA^q + 2A^cA^d\partial_c\Gamma^a{}_{de}\partial_g\Gamma^e{}_{pq}B^p{}_bA^gA^q - 4A^cA^d\partial_c\Gamma^a{}_{de}\partial_g\Gamma^p{}_{bq}B^e{}_pA^gA^q \\ & + 2A^cA^d\partial_c\Gamma^e{}_{bd}\partial_g\Gamma^p{}_{eq}B^a{}_pA^gA^q + A^c\partial_c\Gamma^a{}_{ef}A^e\partial_g\Gamma^f{}_{pq}B^p{}_bA^gA^q - 2A^c\partial_c\Gamma^a{}_{ef}A^e\partial_g\Gamma^p{}_{bq}B^f{}_pA^gA^q + A^c\partial_c\Gamma^e{}_{bf}A^f\partial_g\Gamma^p{}_{eq}B^a{}_pA^gA^q\end{aligned}$$

$$\begin{aligned}\texttt{dBab05.301} := {} & A^cA^eA^gA^i\partial_{cegi}\Gamma^a{}_{pq}B^p{}_bA^q - A^cA^eA^gA^i\partial_{cegi}\Gamma^p{}_{bq}B^a{}_pA^q + 3A^cA^dA^e\partial_{cd}\Gamma^a{}_{eg}\partial_i\Gamma^g{}_{pq}B^p{}_bA^iA^q + 3A^cA^dA^e\partial_c\Gamma^a{}_{dg}\partial_{ei}\Gamma^g{}_{pq}B^p{}_bA^iA^q \\ & - 6A^cA^dA^e\partial_{cd}\Gamma^a{}_{eg}\partial_i\Gamma^p{}_{bq}B^g{}_pA^iA^q - 6A^cA^dA^e\partial_c\Gamma^a{}_{dg}\partial_{ei}\Gamma^p{}_{bq}B^g{}_pA^iA^q + 3A^cA^dA^e\partial_{cd}\Gamma^g{}_{be}\partial_i\Gamma^p{}_{gq}B^a{}_pA^iA^q \\ & + 3A^cA^dA^e\partial_c\Gamma^g{}_{bd}\partial_{ei}\Gamma^p{}_{gq}B^a{}_pA^iA^q + A^cA^e\partial_{ce}\Gamma^a{}_{fg}A^f\partial_i\Gamma^g{}_{pq}B^p{}_bA^iA^q + 2A^cA^e\partial_c\Gamma^a{}_{ef}A^g\partial_{gi}\Gamma^f{}_{pq}B^p{}_bA^iA^q \\ & - 2A^cA^e\partial_{ce}\Gamma^a{}_{fg}A^f\partial_i\Gamma^p{}_{bq}B^g{}_pA^iA^q - 3A^cA^e\partial_c\Gamma^a{}_{ef}A^g\partial_{gi}\Gamma^p{}_{bq}B^f{}_pA^iA^q - A^cA^e\partial_c\Gamma^f{}_{be}A^g\partial_{gi}\Gamma^a{}_{pq}B^p{}_fA^iA^q \\ & + A^cA^e\partial_{ce}\Gamma^f{}_{bg}A^g\partial_i\Gamma^p{}_{fq}B^a{}_pA^iA^q + 2A^cA^e\partial_c\Gamma^f{}_{be}A^g\partial_{gi}\Gamma^p{}_{fq}B^a{}_pA^iA^q + A^c\partial_c\Gamma^a{}_{eg}A^eA^h\partial_{hi}\Gamma^g{}_{pq}B^p{}_bA^iA^q \\ & - A^c\partial_c\Gamma^a{}_{eg}A^eA^h\partial_{hi}\Gamma^p{}_{bq}B^g{}_pA^iA^q - A^c\partial_c\Gamma^e{}_{bg}A^gA^h\partial_{hi}\Gamma^a{}_{pq}B^p{}_eA^iA^q + A^c\partial_c\Gamma^e{}_{bg}A^gA^h\partial_{hi}\Gamma^p{}_{eq}B^a{}_pA^iA^q\end{aligned}$$

Stage 4: Replace covariant derivs of B with partial derivs of Γ

```
# substitute covariant derivs of B^{a}_{b} into covariant derivs of R^{a}_{bcd}B^{d}_{a}
# this produces expressions for the partial derivs of Rabcd its covariant derivs and partial derivs of Gamma
# the partial derivs of Gamma will be eliminted later by using results imported from dGamma.json

beg_stage_4 = time.time()

substitute (dRabcd01,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd01)
substitute (dRabcd02,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd02)
substitute (dRabcd03,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd03)
substitute (dRabcd04,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd04)
substitute (dRabcd05,$A^{c}\nabla_{c}{B^{a}_{b}} -> @(dBab01)$,repeat=True);   distribute (dRabcd05)

substitute (dRabcd02,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd02)
substitute (dRabcd03,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd03)
substitute (dRabcd04,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd04)
substitute (dRabcd05,$A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(dBab02)$,repeat=True);   distribute (dRabcd05)

substitute (dRabcd03,$A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(dBab03)$,repeat=True);   distribute (dRabcd03)
substitute (dRabcd04,$A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(dBab03)$,repeat=True);   distribute (dRabcd04)
substitute (dRabcd05,$A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(dBab03)$,repeat=True);   distribute (dRabcd05)

substitute (dRabcd04,$A^{c}A^{d}A^{e}A^{f}\nabla_{c d e f}{B^{a}_{b}} -> @(dBab04)$,repeat=True); distribute (dRabcd04)
substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}\nabla_{c d e f}{B^{a}_{b}} -> @(dBab04)$,repeat=True); distribute (dRabcd05)

substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{c d e f g}{B^{a}_{b}} -> @(dBab05)$,repeat=True); distribute (dRabcd05)

# no longer need B, so let's get rid of it

# two subtle tricks are used here
# 1) rename A and B as A002 and A001 before sort_product,
#    this ensures B will be to left of A after the sort
# 2) indices on B changed from B^{a}_{b} to B_{b}^{a},
#    this ensures that after factor_out B will have dummy indices B_{a}^{b}

def remove_Bab (obj):
    foo := @(obj).
```

```
    substitute     (foo,$A^{a}->A002^{a},B^{a}_{b}->A001_{b}^{a}$)  # need this to sort B to the left of A
    sort_product   (foo)
    rename_dummies (foo)
    factor_out     (foo,$A001^{a?}_{b?},A002^{c?}$)
    substitute     (foo,$A001_{a}^{b}->1,A002^{a}->A^{a}$)  # recover A and set B = 1, free indices now ^{a}_{b}
    return foo

dRabcd01 = remove_Bab (dRabcd01)   # cdb(dRabcd01.401,dRabcd01)
dRabcd02 = remove_Bab (dRabcd02)   # cdb(dRabcd02.401,dRabcd02)
dRabcd03 = remove_Bab (dRabcd03)   # cdb(dRabcd03.401,dRabcd03)
dRabcd04 = remove_Bab (dRabcd04)   # cdb(dRabcd04.401,dRabcd04)
dRabcd05 = remove_Bab (dRabcd05)   # cdb(dRabcd05.401,dRabcd05)

end_stage_4 = time.time()
```

$$\texttt{dRabcd01.401} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\texttt{dRabcd02.401} := A^c A^d A^e A^f \left(-\nabla_{cd} R_{ebfg} g^{ag} - R_{cgdh} \partial_e \Gamma^g{}_{bf} g^{ha} + R_{cbdg} \partial_e \Gamma^a{}_{hf} g^{gh}\right)$$

$$\texttt{dRabcd03.401} := A^c A^d A^e A^f A^g \left(-3\nabla_c R_{dhei} \partial_f \Gamma^h{}_{bg} g^{ia} + 3\nabla_c R_{dbeh} \partial_f \Gamma^a{}_{ig} g^{hi} - \nabla_{cde} R_{fbgh} g^{ah} - R_{chdi} \partial_{ef} \Gamma^h{}_{bg} g^{ia} + R_{cbdh} \partial_{ef} \Gamma^a{}_{ig} g^{hi}\right)$$

$$\begin{aligned}\texttt{dRabcd04.401} := A^c A^d A^e A^f A^g A^h &\left(-6\nabla_{de} R_{figj} \partial_c \Gamma^i{}_{bh} g^{aj} + 6\nabla_{de} R_{fbgi} \partial_c \Gamma^a{}_{jh} g^{ji} - 4\nabla_c R_{diej} \partial_{fg} \Gamma^i{}_{bh} g^{ja} + 4\nabla_c R_{dbei} \partial_{fg} \Gamma^a{}_{jh} g^{ij} - \nabla_{cdef} R_{gbhi} g^{ai}\right.\\ &\left. - R_{cidj} \partial_{efg} \Gamma^i{}_{bh} g^{ja} + R_{cbdi} \partial_{efg} \Gamma^a{}_{jh} g^{ij} - 3R_{cidj} \partial_e \Gamma^i{}_{fk} \partial_g \Gamma^k{}_{bh} g^{ja} + 6R_{cidj} \partial_e \Gamma^i{}_{fb} \partial_g \Gamma^a{}_{kh} g^{jk} - 3R_{cbdi} \partial_e \Gamma^j{}_{kf} \partial_g \Gamma^a{}_{jh} g^{ik}\right)\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd05.401} := A^c A^d A^e A^f A^g A^h A^i &\left(-10\nabla_{cd} R_{ejfk} \partial_{gh} \Gamma^j{}_{bi} g^{ka} + 10\nabla_{cd} R_{ebfj} \partial_{gh} \Gamma^a{}_{ki} g^{jk} - 10\nabla_{def} R_{gjhk} \partial_c \Gamma^j{}_{bi} g^{ak} + 10\nabla_{def} R_{gbhj} \partial_c \Gamma^a{}_{ki} g^{kj}\right.\\ & -5\nabla_c R_{djek} \partial_{fgh} \Gamma^j{}_{bi} g^{ka} + 5\nabla_c R_{dbej} \partial_{fgh} \Gamma^a{}_{ki} g^{jk} - 15\nabla_c R_{djek} \partial_f \Gamma^j{}_{gl} \partial_h \Gamma^l{}_{bi} g^{ka} + 30\nabla_c R_{djek} \partial_f \Gamma^j{}_{gb} \partial_h \Gamma^a{}_{li} g^{kl} - 15\nabla_c R_{dbej} \partial_f \Gamma^k{}_{lg} \partial_h \Gamma^a{}_{ki} g^{jl}\\ & - \nabla_{cdefg} R_{hbij} g^{aj} - R_{cjdk} \partial_{efgh} \Gamma^j{}_{bi} g^{ka} + R_{cbdj} \partial_{efgh} \Gamma^a{}_{ki} g^{jk} - 4R_{cjdk} \partial_h \Gamma^l{}_{bi} \partial_{ef} \Gamma^j{}_{gl} g^{ka} - 6R_{cjdk} \partial_e \Gamma^j{}_{fl} \partial_{gh} \Gamma^l{}_{bi} g^{ka}\\ &\left. + 8R_{cjdk} \partial_h \Gamma^a{}_{li} \partial_{ef} \Gamma^j{}_{gb} g^{kl} + 10R_{cjdk} \partial_e \Gamma^j{}_{fb} \partial_{gh} \Gamma^a{}_{li} g^{kl} - 4R_{cbdj} \partial_h \Gamma^a{}_{ki} \partial_{ef} \Gamma^k{}_{lg} g^{jl} - 6R_{cbdj} \partial_e \Gamma^k{}_{lf} \partial_{gh} \Gamma^a{}_{ki} g^{jl} + 2R_{cjdk} \partial_e \Gamma^a{}_{lf} \partial_{gh} \Gamma^j{}_{bi} g^{kl}\right)\end{aligned}$$

Stage 5: Replace partial derivs of Γ with partial derivs of R

```
import cdblib

beg_stage_5 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json')  # cdb(dGamma01.500,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json')  # cdb(dGamma02.500,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json')  # cdb(dGamma03.500,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json')  # cdb(dGamma04.500,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json')  # cdb(dGamma05.500,dGamma05)

distribute (dRabcd01)   # cdb(dRabcd01.500,dRabcd01)
distribute (dRabcd02)   # cdb(dRabcd02.500,dRabcd02)
distribute (dRabcd03)   # cdb(dRabcd03.500,dRabcd03)
distribute (dRabcd04)   # cdb(dRabcd04.500,dRabcd04)
distribute (dRabcd05)   # cdb(dRabcd05.500,dRabcd05)

# use dGamma to eliminate the partial derivs of Gamma
# this will introduces some lower order partial dervis of Rabcd on the rhs
# these extra partial derivs of Rabcd will be eliminated (later) by substiting lower order dRabcd into the higher order dRabcd

substitute (dRabcd02,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd02.501,dRabcd02)
substitute (dRabcd02,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd02.502,dRabcd02)
distribute (dRabcd02)                                                                                      # cdb(dRabcd02.503,dRabcd02)
sort_product   (dRabcd02)                                                                                  # cdb(dRabcd02.504,dRabcd02)
rename_dummies (dRabcd02)                                                                                  # cdb(dRabcd02.505,dRabcd02)


substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)         # cdb(dRabcd03.501,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)         # cdb(dRabcd03.502,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd03.503,dRabcd03)
substitute (dRabcd03,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd03.504,dRabcd03)
distribute (dRabcd03)                                                                                      # cdb(dRabcd03.505,dRabcd03)
sort_product   (dRabcd03)                                                                                  # cdb(dRabcd03.506,dRabcd03)
rename_dummies (dRabcd03)                                                                                  # cdb(dRabcd03.507,dRabcd03)

substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{d b}} -> @(dGamma03)$,repeat=True)  # cdb(dRabcd04.501,dRabcd04)
```

```
substitute (dRabcd04,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{b d}} -> @(dGamma03)$,repeat=True)  # cdb(dRabcd04.502,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)         # cdb(dRabcd04.503,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)         # cdb(dRabcd04.504,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd04.505,dRabcd04)
substitute (dRabcd04,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)                # cdb(dRabcd04.506,dRabcd04)
distribute (dRabcd04)                                                                                       # cdb(dRabcd04.507,dRabcd04)
sort_product   (dRabcd04)                                                                                   # cdb(dRabcd04.508,dRabcd04)
rename_dummies (dRabcd04)                                                                                   # cdb(dRabcd04.509,dRabcd04)

substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}{\Gamma^{a}_{d b}} -> @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}{\Gamma^{a}_{b d}} -> @(dGamma04)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{d b}} -> @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}A^{f}\partial_{c e f}{\Gamma^{a}_{b d}} -> @(dGamma03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{d b}} -> @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}A^{e}\partial_{c e}{\Gamma^{a}_{b d}} -> @(dGamma02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{d b}} -> @(dGamma01)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{b}\partial_{c}{\Gamma^{a}_{b d}} -> @(dGamma01)$,repeat=True)
distribute (dRabcd05)
sort_product   (dRabcd05)
rename_dummies (dRabcd05)

end_stage_5 = time.time()
```

$$\texttt{dRabcd01.500} := -A^cA^dA^e\nabla_cR_{dfeb}g^{af}$$

$$\texttt{dRabcd02.500} := -A^cA^dA^eA^f\nabla_{cd}R_{ebfg}g^{ag} - A^cA^dA^eA^fR_{cgdh}\partial_e\Gamma^g{}_{bf}g^{ha} + A^cA^dA^eA^fR_{cbdg}\partial_e\Gamma^a{}_{hf}g^{gh}$$

$$\texttt{dRabcd02.501} := -A^cA^dA^eA^f\nabla_{cd}R_{ebfg}g^{ag} - \frac{1}{3}A^cA^dA^fA^eR^g{}_{feb}R_{cgdh}g^{ha} + \frac{1}{3}A^cA^dA^fA^eR^a{}_{feh}R_{cbdg}g^{gh}$$

$$\texttt{dRabcd02.502} := -A^cA^dA^eA^f\nabla_{cd}R_{ebfg}g^{ag} - \frac{1}{3}A^cA^dA^fA^eR^g{}_{feb}R_{cgdh}g^{ha} + \frac{1}{3}A^cA^dA^fA^eR^a{}_{feh}R_{cbdg}g^{gh}$$

$$\texttt{dRabcd02.503} := -A^cA^dA^eA^f\nabla_{cd}R_{ebfg}g^{ag} - \frac{1}{3}A^cA^dA^fA^eR^g{}_{feb}R_{cgdh}g^{ha} + \frac{1}{3}A^cA^dA^fA^eR^a{}_{feh}R_{cbdg}g^{gh}$$

$$\texttt{dRabcd02.504} := -A^cA^dA^eA^f\nabla_{cd}R_{ebfg}g^{ag} - \frac{1}{3}A^cA^dA^eA^fR_{cgdh}R^g{}_{feb}g^{ha} + \frac{1}{3}A^cA^dA^eA^fR_{cbdg}R^a{}_{feh}g^{gh}$$

$$\texttt{dRabcd02.505} := -A^cA^dA^eA^f\nabla_{cd}R_{ebfg}g^{ag} - \frac{1}{3}A^cA^dA^eA^fR_{cgdh}R^g{}_{feb}g^{ha} + \frac{1}{3}A^cA^dA^eA^fR_{cbdg}R^a{}_{feh}g^{gh}$$

$$\begin{aligned}\texttt{dRabcd03.500} := &-3A^cA^dA^eA^fA^g\nabla_cR_{dhei}\partial_f\Gamma^h{}_{bg}g^{ia} + 3A^cA^dA^eA^fA^g\nabla_cR_{dbeh}\partial_f\Gamma^a{}_{ig}g^{hi}\\ &- A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - A^cA^dA^eA^fA^gR_{chdi}\partial_{ef}\Gamma^h{}_{bg}g^{ia} + A^cA^dA^eA^fA^gR_{cbdh}\partial_{ef}\Gamma^a{}_{ig}g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.501} := &-3A^cA^dA^eA^fA^g\nabla_cR_{dhei}\partial_f\Gamma^h{}_{bg}g^{ia} + 3A^cA^dA^eA^fA^g\nabla_cR_{dbeh}\partial_f\Gamma^a{}_{ig}g^{hi}\\ &- A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^h{}_{geb}R_{chdi}g^{ia} + \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^a{}_{gei}R_{cbdh}g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.502} := &-3A^cA^dA^eA^fA^g\nabla_cR_{dhei}\partial_f\Gamma^h{}_{bg}g^{ia} + 3A^cA^dA^eA^fA^g\nabla_cR_{dbeh}\partial_f\Gamma^a{}_{ig}g^{hi}\\ &- A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^h{}_{geb}R_{chdi}g^{ia} + \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^a{}_{gei}R_{cbdh}g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.503} := &-A^cA^dA^eA^gA^fR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^gA^fR^a{}_{gfi}\nabla_cR_{dbeh}g^{hi} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah}\\ &- \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^h{}_{geb}R_{chdi}g^{ia} + \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^a{}_{gei}R_{cbdh}g^{hi}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd03.504} := &-A^cA^dA^eA^gA^fR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^gA^fR^a{}_{gfi}\nabla_cR_{dbeh}g^{hi} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah}\\ &- \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^h{}_{geb}R_{chdi}g^{ia} + \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^a{}_{gei}R_{cbdh}g^{hi}\end{aligned}$$

$$\texttt{dRabcd03.505} := -A^cA^dA^eA^gA^fR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^gA^fR^a{}_{gfi}\nabla_cR_{dbeh}g^{hi} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^h{}_{geb}R_{chdi}g^{ia} + \frac{1}{2}A^cA^dA^fA^gA^e\partial_fR^a{}_{gei}R_{cbdh}g^{hi}$$

$$\texttt{dRabcd03.506} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfi}\nabla_cR_{dbeh}g^{hi} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^eA^fA^gR_{chdi}\partial_fR^h{}_{geb}g^{ia} + \frac{1}{2}A^cA^dA^eA^fA^gR_{cbdh}\partial_fR^a{}_{gei}g^{hi}$$

$$\texttt{dRabcd03.507} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfh}\nabla_cR_{dbei}g^{ih} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} - \frac{1}{2}A^cA^dA^eA^fA^gR_{chdi}\partial_fR^h{}_{geb}g^{ia} + \frac{1}{2}A^cA^dA^eA^fA^gR_{cbdh}\partial_fR^a{}_{gei}g^{hi}$$

$$\texttt{dRabcd04.500} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji} - 4A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_{fg}\Gamma^i{}_{bh}g^{ja} + 4A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_{fg}\Gamma^a{}_{jh}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - A^cA^dA^eA^fA^gA^hR_{cidj}\partial_{efg}\Gamma^i{}_{bh}g^{ja} + A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_{efg}\Gamma^a{}_{jh}g^{ij} - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$\texttt{dRabcd04.501} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji} - 4A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_{fg}\Gamma^i{}_{bh}g^{ja} + 4A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_{fg}\Gamma^a{}_{jh}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja} + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij} - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$\texttt{dRabcd04.502} := -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^i{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^a{}_{jh}g^{ji} - 4A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_{fg}\Gamma^i{}_{bh}g^{ja} + 4A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_{fg}\Gamma^a{}_{jh}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^i{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}\right)R_{cidj}g^{ja} + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^a{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}\right)R_{cbdi}g^{ij} - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fk}\partial_g\Gamma^k{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^i{}_{fb}\partial_g\Gamma^a{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^j{}_{kf}\partial_g\Gamma^a{}_{jh}g^{ik}$$

$$
\begin{aligned}
\texttt{dRabcd04.503} := {} & -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^{i}{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^{a}{}_{jh}g^{ji} \\
& - 2A^cA^dA^eA^gA^hA^f\partial_gR^{i}{}_{hfb}\nabla_cR_{diej}g^{ja} + 2A^cA^dA^eA^gA^hA^f\partial_gR^{a}{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} \\
& - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{i}{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{efk}R^{k}{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{egk}R^{k}{}_{hfb}\right)R_{cidj}g^{ja} \\
& + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{a}{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{efk}R^{k}{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{egk}R^{k}{}_{hfj}\right)R_{cbdi}g^{ij} \\
& - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^{i}{}_{fk}\partial_g\Gamma^{k}{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^{i}{}_{fb}\partial_g\Gamma^{a}{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^{j}{}_{kf}\partial_g\Gamma^{a}{}_{jh}g^{ik}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dRabcd04.504} := {} & -6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{figj}\partial_c\Gamma^{i}{}_{bh}g^{aj} + 6A^cA^dA^eA^fA^gA^h\nabla_{de}R_{fbgi}\partial_c\Gamma^{a}{}_{jh}g^{ji} \\
& - 2A^cA^dA^eA^gA^hA^f\partial_gR^{i}{}_{hfb}\nabla_cR_{diej}g^{ja} + 2A^cA^dA^eA^gA^hA^f\partial_gR^{a}{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} \\
& - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{i}{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{efk}R^{k}{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{egk}R^{k}{}_{hfb}\right)R_{cidj}g^{ja} \\
& + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{a}{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{efk}R^{k}{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{egk}R^{k}{}_{hfj}\right)R_{cbdi}g^{ij} \\
& - 3A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^{i}{}_{fk}\partial_g\Gamma^{k}{}_{bh}g^{ja} + 6A^cA^dA^eA^fA^gA^hR_{cidj}\partial_e\Gamma^{i}{}_{fb}\partial_g\Gamma^{a}{}_{kh}g^{jk} - 3A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_e\Gamma^{j}{}_{kf}\partial_g\Gamma^{a}{}_{jh}g^{ik}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dRabcd04.505} := {} & -2A^hA^cR^{i}{}_{hcb}A^dA^eA^fA^g\nabla_{de}R_{figj}g^{aj} + 2A^hA^cR^{a}{}_{hcj}A^dA^eA^fA^g\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^gA^hA^f\partial_gR^{i}{}_{hfb}\nabla_cR_{diej}g^{ja} \\
& + 2A^cA^dA^eA^gA^hA^f\partial_gR^{a}{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} \\
& - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{i}{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{efk}R^{k}{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{egk}R^{k}{}_{hfb}\right)R_{cidj}g^{ja} \\
& + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{a}{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{efk}R^{k}{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{egk}R^{k}{}_{hfj}\right)R_{cbdi}g^{ij} \\
& - A^cA^dA^eA^fA^hA^gR^{k}{}_{hgb}R_{cidj}\partial_e\Gamma^{i}{}_{fk}g^{ja} + 2A^cA^dA^eA^fA^hA^gR^{a}{}_{hgk}R_{cidj}\partial_e\Gamma^{i}{}_{fb}g^{jk} - \frac{1}{3}A^cA^dA^fA^eR^{j}{}_{fek}A^hA^gR^{a}{}_{hgj}R_{cbdi}g^{ik}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{dRabcd04.506} := {} & -2A^hA^cR^{i}{}_{hcb}A^dA^eA^fA^g\nabla_{de}R_{figj}g^{aj} + 2A^hA^cR^{a}{}_{hcj}A^dA^eA^fA^g\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^gA^hA^f\partial_gR^{i}{}_{hfb}\nabla_cR_{diej}g^{ja} \\
& + 2A^cA^dA^eA^gA^hA^f\partial_gR^{a}{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} \\
& - A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{i}{}_{heb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{efk}R^{k}{}_{hgb} - \frac{1}{15}A^hA^eA^fA^gR^{i}{}_{egk}R^{k}{}_{hfb}\right)R_{cidj}g^{ja} \\
& + A^cA^d\left(\frac{3}{5}A^hA^eA^fA^g\partial_{gf}R^{a}{}_{hej} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{efk}R^{k}{}_{hgj} - \frac{1}{15}A^hA^eA^fA^gR^{a}{}_{egk}R^{k}{}_{hfj}\right)R_{cbdi}g^{ij} \\
& - \frac{1}{3}A^cA^dA^fA^eR^{i}{}_{fek}A^hA^gR^{k}{}_{hgb}R_{cidj}g^{ja} + \frac{2}{3}A^cA^dA^fA^eR^{i}{}_{feb}A^hA^gR^{a}{}_{hgk}R_{cidj}g^{jk} - \frac{1}{3}A^cA^dA^fA^eR^{j}{}_{fek}A^hA^gR^{a}{}_{hgj}R_{cbdi}g^{ik}
\end{aligned}
$$

$$\begin{aligned}
\texttt{dRabcd04.507} := &-2A^hA^cR^i{}_{hcb}A^dA^eA^fA^g\nabla_{de}R_{figj}g^{aj} + 2A^hA^cR^a{}_{hcj}A^dA^eA^fA^g\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^gA^hA^f\partial_gR^i{}_{hfb}\nabla_cR_{diej}g^{ja}\\
&+2A^cA^dA^eA^gA^hA^f\partial_gR^a{}_{hfj}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - \frac{3}{5}A^cA^dA^hA^eA^fA^g\partial_{gf}R^i{}_{heb}R_{cidj}g^{ja}\\
&+\frac{1}{15}A^cA^dA^hA^eA^fA^gR^i{}_{efk}R^k{}_{hgb}R_{cidj}g^{ja} + \frac{1}{15}A^cA^dA^hA^eA^fA^gR^i{}_{egk}R^k{}_{hfb}R_{cidj}g^{ja} + \frac{3}{5}A^cA^dA^hA^eA^fA^g\partial_{gf}R^a{}_{hej}R_{cbdi}g^{ij}\\
&-\frac{1}{15}A^cA^dA^hA^eA^fA^gR^a{}_{efk}R^k{}_{hgj}R_{cbdi}g^{ij} - \frac{1}{15}A^cA^dA^hA^eA^fA^gR^a{}_{egk}R^k{}_{hfj}R_{cbdi}g^{ij}\\
&-\frac{1}{3}A^cA^dA^fA^eR^i{}_{fek}A^hA^gR^k{}_{hgb}R_{cidj}g^{ja} + \frac{2}{3}A^cA^dA^fA^eR^i{}_{feb}A^hA^gR^a{}_{hgk}R_{cidj}g^{jk} - \frac{1}{3}A^cA^dA^fA^eR^j{}_{fek}A^hA^gR^a{}_{hgj}R_{cbdi}g^{ik}
\end{aligned}$$

$$\begin{aligned}
\texttt{dRabcd04.508} := &-2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hcj}\nabla_{de}R_{fbgi}g^{ji} - 2A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_gR^i{}_{hfb}g^{ja}\\
&+2A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_gR^a{}_{hfj}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cidj}\partial_{gf}R^i{}_{heb}g^{ja}\\
&+\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{efk}R^k{}_{hgb}g^{ja} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{egk}R^k{}_{hfb}g^{ja} + \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_{gf}R^a{}_{hej}g^{ij}\\
&-\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{efk}R^k{}_{hgj}g^{ij} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{egk}R^k{}_{hfj}g^{ij}\\
&-\frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{fek}R^k{}_{hgb}g^{ja} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^a{}_{hgk}R^i{}_{feb}g^{jk} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{hgj}R^j{}_{fek}g^{ik}
\end{aligned}$$

$$\begin{aligned}
\texttt{dRabcd04.509} := &-2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij} - 2A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_gR^i{}_{hfb}g^{ja}\\
&+2A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_gR^a{}_{hfj}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} - \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cidj}\partial_{gf}R^i{}_{heb}g^{ja}\\
&+\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{efk}R^k{}_{hgb}g^{ja} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{egk}R^k{}_{hfb}g^{ja} + \frac{3}{5}A^cA^dA^eA^fA^gA^hR_{cbdi}\partial_{gf}R^a{}_{hej}g^{ij}\\
&-\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{efj}R^j{}_{hgk}g^{ik} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{egj}R^j{}_{hfk}g^{ik}\\
&-\frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{fek}R^k{}_{hgb}g^{ja} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^a{}_{hgk}R^i{}_{feb}g^{jk} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{hgj}R^j{}_{fek}g^{ik}
\end{aligned}$$

Stage 6: Replace partial derivs of R with covariant derivs of R

```
# now eliminate remaining partial derivs of Rabcd by substitution from the lower order dRabcd

# note that
#   dRabcd01 = R^a_{cdb,e} A^c A^d A^e
#   dRabcd02 = R^a_{cdb,ef} A^c A^d A^e A^f
#   dRabcd03 = R^a_{cdb,efg} A^c A^d A^e A^f A^g

# thus we can use
#   dRabcd01 to eliminate 1st partial derivs of R in dRabcd03, dRabcd04, etc.
#   dRabcd02 to eliminate 2nd partial derivs of R in dRabcd04, dRabcd05, etc.
#   dRabcd03 to eliminate 3rd partial derivs of R in dRabcd05, dRabcd06, etc.

beg_stage_6 = time.time()

substitute (dRabcd03,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} -> @(dRabcd01)$,repeat=True)         # cdb(dRabcd03.601,dRabcd03)
distribute (dRabcd03)                                                                             # cdb(dRabcd03.602,dRabcd03)

# note: dRabcd04 and dRabcd05 unused in this code (or any other code)

substitute (dRabcd04,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} -> @(dRabcd02)$,repeat=True)  # cdb(dRabcd04.601,dRabcd04)
substitute (dRabcd04,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} -> @(dRabcd01)$,repeat=True)         # cdb(dRabcd04.602,dRabcd04)
distribute (dRabcd04)                                                                             # cdb(dRabcd04.603,dRabcd04)

substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c d b}} -> @(dRabcd03)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} -> @(dRabcd02)$,repeat=True)
substitute (dRabcd05,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} -> @(dRabcd01)$,repeat=True)
distribute (dRabcd05)

end_stage_6 = time.time()
```

$$\texttt{dRabcd03.601} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfh}\nabla_cR_{dbei}g^{ih} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} + \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfb}g^{hj}R_{chdi}g^{ia} - \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfi}g^{aj}R_{cbdh}g^{hi}$$

$$\texttt{dRabcd03.602} := -A^cA^dA^eA^fA^gR^h{}_{gfb}\nabla_cR_{dhei}g^{ia} + A^cA^dA^eA^fA^gR^a{}_{gfh}\nabla_cR_{dbei}g^{ih} - A^cA^dA^eA^fA^g\nabla_{cde}R_{fbgh}g^{ah} + \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfb}g^{hj}R_{chdi}g^{ia} - \frac{1}{2}A^cA^dA^gA^eA^f\nabla_gR_{ejfi}g^{aj}R_{cbdh}g^{hi}$$

$$\begin{aligned}\texttt{dRabcd04.601} := &-2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij}\\ &-2A^cA^dA^eA^fA^gA^h\nabla_cR_{diej}\partial_gR^i{}_{hfb}g^{ja} + 2A^cA^dA^eA^fA^gA^h\nabla_cR_{dbei}\partial_gR^a{}_{hfj}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}\\ &-\frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gbfl}g^{il} - \frac{1}{3}A^hA^eA^gA^fR_{hlek}R^l{}_{fgb}g^{ki} + \frac{1}{3}A^hA^eA^gA^fR_{hbel}R^i{}_{fgk}g^{lk}\right)R_{cidj}g^{ja}\\ &+\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{efk}R^k{}_{hgb}g^{ja} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{egk}R^k{}_{hfb}g^{ja}\\ &+\frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gjfl}g^{al} - \frac{1}{3}A^hA^eA^gA^fR_{hlek}R^l{}_{fgj}g^{ka} + \frac{1}{3}A^hA^eA^gA^fR_{hjel}R^a{}_{fgk}g^{lk}\right)R_{cbdi}g^{ij}\\ &-\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{efj}R^j{}_{hgk}g^{ik} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{egj}R^j{}_{hfk}g^{ik}\\ &-\frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{fek}R^k{}_{hgb}g^{ja} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^a{}_{hgk}R^i{}_{feb}g^{jk} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{hgj}R^j{}_{fek}g^{ik}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd04.602} := &-2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij} + 2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgb}g^{ik}\nabla_cR_{diej}g^{ja}\\ &-2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgj}g^{ak}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai}\\ &-\frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gbfl}g^{il} - \frac{1}{3}A^hA^eA^gA^fR_{hlek}R^l{}_{fgb}g^{ki} + \frac{1}{3}A^hA^eA^gA^fR_{hbel}R^i{}_{fgk}g^{lk}\right)R_{cidj}g^{ja}\\ &+\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{efk}R^k{}_{hgb}g^{ja} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{egk}R^k{}_{hfb}g^{ja}\\ &+\frac{3}{5}A^cA^d\left(-A^hA^eA^gA^f\nabla_{he}R_{gjfl}g^{al} - \frac{1}{3}A^hA^eA^gA^fR_{hlek}R^l{}_{fgj}g^{ka} + \frac{1}{3}A^hA^eA^gA^fR_{hjel}R^a{}_{fgk}g^{lk}\right)R_{cbdi}g^{ij}\\ &-\frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{efj}R^j{}_{hgk}g^{ik} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{egj}R^j{}_{hfk}g^{ik}\\ &-\frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{fek}R^k{}_{hgb}g^{ja} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^a{}_{hgk}R^i{}_{feb}g^{jk} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{hgj}R^j{}_{fek}g^{ik}\end{aligned}$$

$$
\begin{aligned}
\texttt{dRabcd04.603} := {} & -2A^cA^dA^eA^fA^gA^hR^i{}_{hcb}\nabla_{de}R_{figj}g^{aj} + 2A^cA^dA^eA^fA^gA^hR^a{}_{hci}\nabla_{de}R_{fbgj}g^{ij} + 2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgb}g^{ik}\nabla_cR_{diej}g^{ja} \\
& - 2A^cA^dA^eA^hA^fA^g\nabla_hR_{fkgj}g^{ak}\nabla_cR_{dbei}g^{ij} - A^cA^dA^eA^fA^gA^h\nabla_{cdef}R_{gbhi}g^{ai} + \frac{3}{5}A^cA^dA^hA^eA^gA^f\nabla_{he}R_{gbfl}g^{il}R_{cidj}g^{ja} \\
& + \frac{1}{5}A^cA^dA^hA^eA^gA^fR_{hlek}R^l{}_{fgb}g^{ki}R_{cidj}g^{ja} - \frac{1}{5}A^cA^dA^hA^eA^gA^fR_{hbel}R^i{}_{fgk}g^{lk}R_{cidj}g^{ja} + \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{efk}R^k{}_{hgb}g^{ja} \\
& + \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{egk}R^k{}_{hfb}g^{ja} - \frac{3}{5}A^cA^dA^hA^eA^gA^f\nabla_{he}R_{gjfl}g^{al}R_{cbdi}g^{ij} - \frac{1}{5}A^cA^dA^hA^eA^gA^fR_{hlek}R^l{}_{fgj}g^{ka}R_{cbdi}g^{ij} \\
& + \frac{1}{5}A^cA^dA^hA^eA^gA^fR_{hjel}R^a{}_{fgk}g^{lk}R_{cbdi}g^{ij} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{efj}R^j{}_{hgk}g^{ik} - \frac{1}{15}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{egj}R^j{}_{hfk}g^{ik} \\
& - \frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^i{}_{fek}R^k{}_{hgb}g^{ja} + \frac{2}{3}A^cA^dA^eA^fA^gA^hR_{cidj}R^a{}_{hgk}R^i{}_{feb}g^{jk} - \frac{1}{3}A^cA^dA^eA^fA^gA^hR_{cbdi}R^a{}_{hgj}R^j{}_{fek}g^{ik}
\end{aligned}
$$

Stage 7: Reformatting

```
beg_stage_7 = time.time()

dRabcd01 = flatten_Rabcd (dRabcd01)  # cdb(dRabcd01.701,dRabcd01)
dRabcd02 = flatten_Rabcd (dRabcd02)  # cdb(dRabcd02.701,dRabcd02)
dRabcd03 = flatten_Rabcd (dRabcd03)  # cdb(dRabcd03.701,dRabcd03)
dRabcd04 = flatten_Rabcd (dRabcd04)  # cdb(dRabcd04.701,dRabcd04)
dRabcd05 = flatten_Rabcd (dRabcd05)  # cdb(dRabcd05.701,dRabcd05)

canonicalise (dRabcd01)   # cdb(dRabcd01.702,dRabcd01)
canonicalise (dRabcd02)   # cdb(dRabcd02.702,dRabcd02)
canonicalise (dRabcd03)   # cdb(dRabcd03.702,dRabcd03)
canonicalise (dRabcd04)   # cdb(dRabcd04.702,dRabcd04)
canonicalise (dRabcd05)   # cdb(dRabcd05.702,dRabcd05)

end_stage_7 = time.time()

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs\\hfill\\break".format(end_stage_4-beg_stage_4))
print ("Stage 5: {:7.1f} secs\\hfill\\break".format(end_stage_5-beg_stage_5))
print ("Stage 6: {:7.1f} secs\\hfill\\break".format(end_stage_6-beg_stage_6))
print ("Stage 7: {:7.1f} secs".format(end_stage_7-beg_stage_7))
# cdbEnd (timing)
```

$$\texttt{dRabcd01.701} := -A^c A^d A^e \nabla_c R_{dfeb} g^{af}$$

$$\texttt{dRabcd02.701} := -A^c A^d A^e A^f \nabla_{cd} R_{ebfg} g^{ag} - \frac{1}{3} A^c A^d A^e A^f R_{cgdh} R_{ifeb} g^{gi} g^{ha} + \frac{1}{3} A^c A^d A^e A^f R_{cbdg} R_{hfei} g^{ah} g^{gi}$$

$$\begin{aligned}\texttt{dRabcd03.701} := &-A^c A^d A^e A^f A^g R_{hgfb} \nabla_c R_{diej} g^{ih} g^{ja} + A^c A^d A^e A^f A^g R_{hgfi} \nabla_c R_{dbej} g^{ah} g^{ji} - A^c A^d A^e A^f A^g \nabla_{cde} R_{fbgh} g^{ah} \\ &+ \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_g R_{ejfb} g^{hj} g^{ia} - \frac{1}{2} A^c A^d A^e A^f A^g R_{cbdh} \nabla_g R_{eifj} g^{ai} g^{hj}\end{aligned}$$

$$\begin{aligned}\texttt{dRabcd04.701} := &-2 A^c A^d A^e A^f A^g A^h R_{ihcb} \nabla_{de} R_{fjgk} g^{ak} g^{ji} + 2 A^c A^d A^e A^f A^g A^h R_{ihcj} \nabla_{de} R_{fbgk} g^{ai} g^{jk} + 2 A^c A^d A^e A^f A^g A^h \nabla_c R_{diej} \nabla_h R_{fkgb} g^{ik} g^{ja} \\ &- 2 A^c A^d A^e A^f A^g A^h \nabla_c R_{dbei} \nabla_h R_{fjgk} g^{aj} g^{ik} - A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{gbhi} g^{ai} + \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{he} R_{gbfk} g^{ik} g^{ja} \\ &+ \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hkel} R_{mfgb} g^{ja} g^{li} g^{km} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cidj} R_{hbek} R_{lfgm} g^{il} g^{ja} g^{km} \\ &+ \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kefl} R_{mhgb} g^{ik} g^{ja} g^{lm} + \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kegl} R_{mhfb} g^{ik} g^{ja} g^{lm} \\ &- \frac{3}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} \nabla_{he} R_{gjfk} g^{ak} g^{ij} - \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{im} g^{ka} g^{jl} \\ &+ \frac{1}{5} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{hjek} R_{lfgm} g^{al} g^{ij} g^{km} - \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jefk} R_{lhgm} g^{aj} g^{im} g^{kl} \\ &- \frac{1}{15} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jegk} R_{lhfm} g^{aj} g^{im} g^{kl} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{kfel} R_{mhgb} g^{ik} g^{ja} g^{lm} \\ &+ \frac{2}{3} A^c A^d A^e A^f A^g A^h R_{cidj} R_{khgl} R_{mfeb} g^{ak} g^{im} g^{jl} - \frac{1}{3} A^c A^d A^e A^f A^g A^h R_{cbdi} R_{jhgk} R_{lfem} g^{aj} g^{im} g^{kl}\end{aligned}$$

$$\texttt{dRabcd01.702} := A^c A^d A^e \nabla_c R_{bdef} g^{af}$$

$$\texttt{dRabcd02.702} := A^c A^d A^e A^f \nabla_{cd} R_{befg} g^{ag}$$

$$\texttt{dRabcd03.702} := -\frac{1}{2} A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{figj} g^{ai} g^{hj} + \frac{1}{2} A^c A^d A^e A^f A^g R_{chdi} \nabla_e R_{bfgj} g^{ah} g^{ij} + A^c A^d A^e A^f A^g \nabla_{cde} R_{bfgh} g^{ah}$$

$$\texttt{dRabcd04.702} := -\frac{7}{5} A^c A^d A^e A^f A^g A^h R_{bcdi} \nabla_{ef} R_{gjhk} g^{aj} g^{ik} + \frac{7}{5} A^c A^d A^e A^f A^g A^h R_{cidj} \nabla_{ef} R_{bghk} g^{ai} g^{jk} + A^c A^d A^e A^f A^g A^h \nabla_{cdef} R_{bghi} g^{ai}$$

$$\begin{aligned}\texttt{dRabcd05.702} := &-2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{bdej} \nabla_{fg} R_{hkil} g^{ak} g^{jl} + 2 A^c A^d A^e A^f A^g A^h A^i \nabla_c R_{djek} \nabla_{fg} R_{bhil} g^{aj} g^{kl} \\ &- \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} \nabla_{efg} R_{hkil} g^{ak} g^{jl} + \frac{8}{3} A^c A^d A^e A^f A^g A^h A^i R_{cjdk} \nabla_{efg} R_{bhil} g^{aj} g^{kl} \\ &+ \frac{1}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{am} g^{jk} g^{ln} + A^c A^d A^e A^f A^g A^h A^i R_{cjdk} R_{elfm} \nabla_g R_{bhin} g^{aj} g^{kl} g^{mn} \\ &- \frac{4}{3} A^c A^d A^e A^f A^g A^h A^i R_{bcdj} R_{ekfl} \nabla_g R_{hmin} g^{ak} g^{jm} g^{ln} + A^c A^d A^e A^f A^g A^h A^i \nabla_{cdefg} R_{bhij} g^{aj}\end{aligned}$$

```
cdblib.create ('dRabcd.json')

cdblib.put ('dRabcd01',dRabcd01,'dRabcd.json')
cdblib.put ('dRabcd02',dRabcd02,'dRabcd.json')
cdblib.put ('dRabcd03',dRabcd03,'dRabcd.json')
cdblib.put ('dRabcd04',dRabcd04,'dRabcd.json')
cdblib.put ('dRabcd05',dRabcd05,'dRabcd.json')
```

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                            -> A001^{a}            $)
    substitute (obj,$ x^{a}                            -> A002^{a}            $)
    substitute (obj,$ g^{a b}                          -> A003^{a b}          $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}    -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}      -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}        -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}          -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                      -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                 -> A^{a}                           $)
    substitute (obj,$ A002^{a}                 -> x^{a}                           $)
    substitute (obj,$ A003^{a b}               -> g^{a b}                         $)
    substitute (obj,$ A004_{a b c d}           -> R_{a b c d}                     $)
    substitute (obj,$ A005_{a b c d e}         -> \nabla_{e}{R_{a b c d}}         $)
    substitute (obj,$ A006_{a b c d e f}       -> \nabla_{e f}{R_{a b c d}}       $)
    substitute (obj,$ A007_{a b c d e f g}     -> \nabla_{e f g}{R_{a b c d}}     $)
    substitute (obj,$ A008_{a b c d e f g h}   -> \nabla_{e f g h}{R_{a b c d}} $)

    return obj

def reformat (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    factor_out     (bah,$A^{a?}$)
    ans := @(bah).
    return ans

scaled1 = reformat (dRabcd01, 1)    # cdb(scaled1.601,scaled1)
scaled2 = reformat (dRabcd02, 1)    # cdb(scaled2.601,scaled2)
scaled3 = reformat (dRabcd03,-2)    # cdb(scaled3.601,scaled3)
scaled4 = reformat (dRabcd04,-5)    # cdb(scaled4.601,scaled4)
```

```
scaled5 = reformat (dRabcd05,-3)    # cdb(scaled5.601,scaled5)
```

Symmetrised partial derivatives of $R^a{}_{bcd}$

$$
\begin{aligned}
A^c A^d A^e R^a{}_{cdb,e} &= A^c A^d A^e g^{af} \nabla_c R_{bdef} \\
A^c A^d A^e A^f R^a{}_{cdb,ef} &= A^c A^d A^e A^f g^{ag} \nabla_{cd} R_{befg} \\
-2A^c A^d A^e A^f A^g R^a{}_{cdb,efg} &= A^c A^d A^e A^f A^g \left(g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - 2 g^{ah} \nabla_{cde} R_{bfgh} \right) \\
-5A^c A^d A^e A^f A^g A^h R^a{}_{cdb,efgh} &= A^c A^d A^e A^f A^g A^h \left(7 g^{ai} g^{jk} R_{bcdj} \nabla_{ef} R_{gihk} - 7 g^{ai} g^{jk} R_{cidj} \nabla_{ef} R_{bghk} - 5 g^{ai} \nabla_{cdef} R_{bghi} \right) \\
-3A^c A^d A^e A^f A^g A^h A^i R^a{}_{cdb,efghi} &= A^c A^d A^e A^f A^g A^h A^i \left(6 g^{aj} g^{kl} \nabla_c R_{bdek} \nabla_{fg} R_{hjil} - 6 g^{aj} g^{kl} \nabla_c R_{djek} \nabla_{fg} R_{bhil} + 8 g^{aj} g^{kl} R_{bcdk} \nabla_{efg} R_{hjil} \right. \\
&\quad - 8 g^{aj} g^{kl} R_{cjdk} \nabla_{efg} R_{bhil} - g^{aj} g^{kl} g^{mn} R_{bcdk} R_{elfm} \nabla_g R_{hjin} - 3 g^{aj} g^{kl} g^{mn} R_{cjdk} R_{elfm} \nabla_g R_{bhin} \\
&\quad \left. + 4 g^{aj} g^{kl} g^{mn} R_{bcdk} R_{ejfm} \nabla_g R_{hlin} - 3 g^{aj} \nabla_{cdefg} R_{bhij} \right)
\end{aligned}
$$

```
substitute (scaled1,$A^{a}->1$)
substitute (scaled2,$A^{a}->1$)
substitute (scaled3,$A^{a}->1$)
substitute (scaled4,$A^{a}->1$)
substitute (scaled5,$A^{a}->1$)

cdblib.create ('dRabcd.export')

# 6th order dRabcd, scaled
cdblib.put ('dRabcd61scaled',scaled1,'dRabcd.export')
cdblib.put ('dRabcd62scaled',scaled2,'dRabcd.export')
cdblib.put ('dRabcd63scaled',scaled3,'dRabcd.export')
cdblib.put ('dRabcd64scaled',scaled4,'dRabcd.export')
cdblib.put ('dRabcd65scaled',scaled5,'dRabcd.export')

checkpoint.append (scaled1)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)
```

Timing

Stage 1: 1.3 secs
Stage 2: 4.0 secs
Stage 3: 0.5 secs
Stage 4: 111.5 secs
Stage 5: 144.8 secs
Stage 6: 181.5 secs
Stage 7: 3.6 secs