

# The generalised connections

The generalised connections may be computed recursively using

$$\Gamma^a_{bcd} = \Gamma^a_{(b\bar{c},d)} - (n+1)\Gamma^a_{p(\bar{c}}\Gamma^p_{bd)} \quad (1)$$

where  $\bar{c}$  contains  $n > 0$  indices. The sequence begins with the standard metric compatible connection

$$\Gamma^d_{ab} = \frac{1}{2}g^{dc}(g_{cb,a} + g_{ac,b} - g_{ab,c}) \quad (2)$$

Here we will use the results of `metric.tex` and `metric-inv.tex` to compute the metric connection  $\Gamma^d_{ab}$ . But since the  $g_{ab}$  and  $g^{ab}$  provided by those codes are truncated at a particular order in the curvatures (and thus are only approximations to the  $g_{ab}$  and  $g^{ab}$ ) similar truncations will arise in the  $\Gamma^a_{bcd}$ .

Approximations will be denoted by the addition of an overbar to an object. In this notation the metric  $g$  can be written as

$$g = \bar{g} + \mathcal{O}(\epsilon^n) \quad (3)$$

in which  $\bar{g}$  is the truncated polynomial approximation to  $g$  and  $\mathcal{O}(\epsilon^n)$  is the error term (containing terms no smaller than  $\epsilon^n$ ). The polynomial structure of  $\bar{g}$  can be expressed as

$$\bar{g} = \bar{g}^0 + \bar{g}^1 + \bar{g}^2 + \cdots + \bar{g}^m \quad (4)$$

in which each terms like  $\bar{g}^m$  contains only terms of order  $m$ . This notation will be applied to other quantities in particular the generalised connections.

The notation  $\mathcal{O}(\epsilon^n)$  denotes terms in the curvatures that are of order  $\epsilon^n$ . What does this actually mean? Each term in  $R$  is of order  $\epsilon^2$  while each derivative of  $R$  carries an extra power of  $\epsilon$ . Thus  $R \cdot R = \mathcal{O}(\epsilon^4)$ ,  $R \cdot R \cdot \nabla R = \mathcal{O}(\epsilon^7)$  and  $R \cdot R \cdot \nabla^2 R = \mathcal{O}(\epsilon^8)$ .

We will also adopt the convention that an object is said to be an  $\mathcal{O}(\epsilon^m)$  approximation when the corresponding error term is  $\mathcal{O}(\epsilon^{m+1})$ .

Consider the  $\mathcal{O}(\epsilon^m)$  approximation of the generalised connection, namely,

$$\bar{\Gamma}^a_{b\bar{c}_n d} = \bar{\Gamma}^a_{b\bar{c}_n d}^0 + \bar{\Gamma}^a_{b\bar{c}_n d}^1 + \bar{\Gamma}^a_{b\bar{c}_n d}^2 + \cdots + \bar{\Gamma}^a_{b\bar{c}_n d}^m \quad (5)$$

where  $\bar{c}_n$  denotes a set of indices such as  $c_1 c_2 c_3 \dots c_n$ .

The first thing to note is that

$$0 = \bar{\Gamma}^{1+n a}_{(b\bar{c}_n, d)} \quad (6)$$

There are two proofs of this claim. For the first proof, note (by inspection) that the order  $\mathcal{O}(\epsilon^p)$  approximation for  $\bar{\Gamma}^a_{b\bar{c}_n d}$  is a polynomial in  $x$  of degree  $p - n - 1$ . Thus  $\bar{\Gamma}^{1+n a}_{(b\bar{c}_n, d)}$  is a polynomial in  $x$  of degree zero, i.e., a constant. However, we know that all generalised connections vanish at the origin of the RNC frame. Thus this constant must be zero. The second proof makes explicit use of the first (and second?) Bianchi identity, that is  $0 = R_{a(bcd)}$ . The term  $\bar{\Gamma}^{1+n a}_{(b\bar{c}_n, d)}$  will itself consist of a sum of terms built from combinations of  $x$ ,  $R$ ,  $\nabla R$  etc. The  $x^a$  will always appear in a contraction with one of the indices on  $R_{abcd}$  or one of its derivatives. Consider any one of these terms, denoted by  $A$ , and assume for the moment that  $1 + n$  is an even number, say  $1 + n = 2p$ . The indices  $(b\bar{c}_n, d)$  must somehow be assigned to the factors that comprise  $A$ . Our aim is to show that at least one  $R$  factor in  $A$  will receive 3 of these indices and thus by the Bianchi identities will be zero. If there are too many  $R$  factors then the Bianchi identities will not come into play. So how many  $R$  factors can we expect? Since  $A$  is a term in an  $\mathcal{O}(\epsilon^{(n+1)})$  approximation there can be no more than  $(n + 1)/2 = p$  Riemann factors. There will be at least one  $x$  term contracted with one of the  $p$  Riemann factors. However, we have  $n + 2 = 2p + 1$  indices to distribute amongst the  $x$  term and  $p$  Riemann factors. One of the indices is a derivative index and will have nett effect of transferring that index from  $x$  to one of the Riemann factors. The remaining  $2p$  indices must be distributed amongst the  $p$  Riemann factors. It is not possible to avoid assigning three indices to at least one of the Riemann factors. Thus, by the Bianchi identity, this  $A$  term must vanish. Similar arguments can be applied to the other cases where the  $A$  terms consists of products of  $R$  and its derivatives and in the case where  $n + 1$  is an odd number. The analysis always comes down to the distribution of the indices  $(b\bar{c}_n, d)$  amongst the factors of a typical  $A$  term. In all cases the Bianchi identity will enter the play and force  $A$  to be zero.

A corollary of the second proof is that for all  $m < n + 2$

$$0 = \bar{\Gamma}^a_{b\bar{c}_n d} \quad (7)$$

The proof follows exactly that of the second proof given above.

We can use the above results to streamline the computation of the generalised connections. We begin with the formal expression for the  $\mathcal{O}(\epsilon^m)$  approximations

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} + \bar{\Gamma}^a_{bc} + \bar{\Gamma}^a_{bc} + \cdots + \bar{\Gamma}^a_{bc} \quad (8)$$

$$\Gamma^a_{b\bar{c}} = \bar{\Gamma}^{n+1 a}_{b\bar{c}} + \bar{\Gamma}^{n+2 a}_{b\bar{c}} + \bar{\Gamma}^{n+3 a}_{b\bar{c}} + \cdots + \bar{\Gamma}^m a_{b\bar{c}} \quad (9)$$

$$\Gamma^a_{b\bar{c}d} = \bar{\Gamma}^{n+2 a}_{b\bar{c}d} + \bar{\Gamma}^{n+3 a}_{b\bar{c}d} + \bar{\Gamma}^{n+4 a}_{b\bar{c}d} + \cdots + \bar{\Gamma}^m a_{b\bar{c}d} \quad (10)$$

These can be substituted into equation (1) with the result

$$\Gamma^a_{b\bar{c}d} = \bar{\Gamma}^{n+1 a}_{(b\bar{c}, d)} + \bar{\Gamma}^{n+2 a}_{(b\bar{c}, d)} + \bar{\Gamma}^{n+3 a}_{(b\bar{c}, d)} + \cdots + \bar{\Gamma}^m a_{(b\bar{c}, d)} - (n + 1) \left( \bar{\Gamma}^{n+1 a}_{p\bar{c}} + \bar{\Gamma}^{n+2 a}_{p\bar{c}} + \bar{\Gamma}^{n+3 a}_{p\bar{c}} + \cdots + \bar{\Gamma}^m a_{p\bar{c}} \right) \left( \bar{\Gamma}^2_{bd} + \bar{\Gamma}^3_{bd} + \bar{\Gamma}^4_{bd} + \cdots + \bar{\Gamma}^p_{bd} \right) \quad (11)$$

where it is understood that in expanding the pair of bracketed terms in the last result the terms should be symmetrised over  $b\bar{c}d$  and also truncated to terms of order  $\mathcal{O}(\epsilon^m)$ . Note that the first term on the right hand side of this equation vanishes by way of the results described above.

Comparing the order  $m$  terms in equation (10) and (11) leads to the following equation

$$\bar{\Gamma}_{b\bar{c}d}^a = \bar{\Gamma}_{(b\bar{c},d)}^a - (n+1) \left( \bar{\Gamma}_{p(\bar{c}}^{m-2} \bar{\Gamma}_{bd)}^2 + \bar{\Gamma}_{p(\bar{c}}^{m-3} \bar{\Gamma}_{bd)}^3 + \bar{\Gamma}_{p(\bar{c}}^{m-4} \bar{\Gamma}_{bd)}^4 + \cdots + \bar{\Gamma}_{p(\bar{c}}^{n+1} \bar{\Gamma}_{bd)}^{m-n-1} \right) \quad (12)$$

This one equation is all that is needed to compute all of the  $\bar{\Gamma}_{b\bar{c}d}^p$  for  $p = 3, 4, 5, \dots, m$  given just the  $\bar{\Gamma}_{bd}^p$  for  $p = 2, 3, 4, \dots, m$ . For example, suppose  $m = 5$  and suppose that we are given  $\bar{\Gamma}_{bd}^p$  for  $p = 2, 3, 4, 5$ . Then with  $n = 1$  we can use equation (12) to compute in turn,  $\bar{\Gamma}_{bc_1d}^p$  for  $p = 3, 4, 5$ . Then with  $n = 2$  we compute  $\bar{\Gamma}_{bc_1c_2d}^p$  for  $p = 4, 5$  and finally with  $n = 3$  we compute  $\bar{\Gamma}_{bc_1c_2c_3d}^p$  for  $p = 5$ . There are no terms like  $\bar{\Gamma}_{bc_1c_2c_3c_4d}^p$  for  $p \leq 5$  due to the corollary given earlier.

The explicit computations for  $m = 5$  are as follows.

For  $n = 1$ ,

$$\bar{\Gamma}^a_{bc_1d} = \bar{\Gamma}^a_{(bc_1,d)} \quad (13)$$

$$\bar{\Gamma}^a_{bc_1d} = \bar{\Gamma}^a_{(bc_1,d)} - 2\bar{\Gamma}^a_{p(c_1}\bar{\Gamma}^p_{bd)} \quad (14)$$

$$\bar{\Gamma}^a_{bc_1d} = \bar{\Gamma}^a_{(bc_1,d)} - 2\bar{\Gamma}^a_{p(c_1}\bar{\Gamma}^p_{bd)} - 2\bar{\Gamma}^a_{p(c_1}\bar{\Gamma}^p_{bd)} \quad (15)$$

For  $n = 2$ ,

$$\bar{\Gamma}^a_{bc_1c_2d} = \bar{\Gamma}^a_{(bc_1c_2,d)} \quad (16)$$

$$\bar{\Gamma}^a_{bc_1c_2d} = \bar{\Gamma}^a_{(bc_1c_2,d)} - 3\bar{\Gamma}^a_{p(c_1c_2}\bar{\Gamma}^p_{bd)} \quad (17)$$

For  $n = 3$ ,

$$\bar{\Gamma}^a_{bc_1c_2c_3d} = \bar{\Gamma}^a_{(bc_1c_2c_3,d)} \quad (18)$$

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,c1,c2,c3,c4,c5,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

x^{a}::Depends(D{#}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

import cdblib

term0 = cdblib.get ('GammaRterm0','connection.json')
term1 = cdblib.get ('GammaRterm2','connection.json')
term2 = cdblib.get ('GammaRterm3','connection.json')
term3 = cdblib.get ('GammaRterm4','connection.json')
term4 = cdblib.get ('GammaRterm5','connection.json')

# LCB: these terms were not computed in connection.tex so set them to zero

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#      maybe in the future I will compute down to term6.

term5 := 0.
term6 := 0.

# genGmn : m = eps order of Rabcd terms
#          n = number of c indices

# -----
# rules for building the genGmn

# note: after applying each rule, must symmetrise over (b c1 c2 ... cn d)

# n = 0

genG20 := genG2^{a}_{b d}.
genG30 := genG3^{a}_{b d}.
genG40 := genG4^{a}_{b d}.
genG50 := genG5^{a}_{b d}.

defG20 := genG2^{d}_{a b} -> @(term1).
defG30 := genG3^{d}_{a b} -> @(term2).
defG40 := genG4^{d}_{a b} -> @(term3).
defG50 := genG5^{d}_{a b} -> @(term4).

# LCB: rncGamma in connection.json limited to "term4" (ie. to 4th order in x)
#      so can only compute genG3*, genG4* and genG5* (at this stage)
#      but it doesn't hurt to provide the definitions for genG6*, genG7* etc. we just won't use them (at this atage)

defG60 := genG6^{d}_{a b} -> @(term5).
defG70 := genG7^{d}_{a b} -> @(term6).

# n = 1

defG31 := genG3^{a}_{b c1 d} -> D_{d}{genG3^{a}_{b c1}}.

defG41 := genG4^{a}_{b c1 d} -> D_{d}{genG4^{a}_{b c1}}
      - 2 genG2^{a}_{p c1} genG2^{p}_{b d}.

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defG51 := genG5^{a}_{b c1 d} -> D_{d}{genG5^{a}_{b c1}}
      - 2 genG3^{a}_{p c1} genG2^{p}_{b d}
      - 2 genG2^{a}_{p c1} genG3^{p}_{b d}.

defG61 := genG6^{a}_{b c1 d} -> D_{d}{genG6^{a}_{b c1}}
      - 2 genG4^{a}_{p c1} genG2^{p}_{b d}
      - 2 genG3^{a}_{p c1} genG3^{p}_{b d}
      - 2 genG3^{a}_{p c1} genG4^{p}_{b d}.

defG71 := genG7^{a}_{b c1 d} -> D_{d}{genG7^{a}_{b c1}}
      - 2 genG5^{a}_{p c1} genG2^{p}_{b d}
      - 2 genG4^{a}_{p c1} genG3^{p}_{b d}
      - 2 genG3^{a}_{p c1} genG4^{p}_{b d}
      - 2 genG2^{a}_{p c1} genG5^{p}_{b d}.

# n = 2

defG42 := genG4^{a}_{b c1 c2 d} -> D_{d}{genG4^{a}_{b c1 c2}}.

defG52 := genG5^{a}_{b c1 c2 d} -> D_{d}{genG5^{a}_{b c1 c2}}
      - 3 genG3^{a}_{p c1 c2} genG2^{p}_{b d}.

defG62 := genG6^{a}_{b c1 c2 d} -> D_{d}{genG6^{a}_{b c1 c2}}
      - 3 genG4^{a}_{p c1 c2} genG2^{p}_{b d}
      - 3 genG3^{a}_{p c1 c2} genG3^{p}_{b d}.

defG72 := genG7^{a}_{b c1 c2 d} -> D_{d}{genG7^{a}_{b c1 c2}}
      - 3 genG5^{a}_{p c1 c2} genG2^{p}_{b d}
      - 3 genG4^{a}_{p c1 c2} genG3^{p}_{b d}
      - 3 genG3^{a}_{p c1 c2} genG4^{p}_{b d}.

# n = 3

defG53 := genG5^{a}_{b c1 c2 c3 d} -> D_{d}{genG5^{a}_{b c1 c2 c3}}.

defG63 := genG6^{a}_{b c1 c2 c3 d} -> D_{d}{genG6^{a}_{b c1 c2 c3}}
      - 4 genG3^{a}_{p c1 c2 c3} genG3^{p}_{b d}.

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defG73 := genG7^{a}_{b c1 c2 c3 d} -> D_{d}{genG7^{a}_{b c1 c2 c3}}
      - 4 genG4^{a}_{p c1 c2 c3} genG3^{p}_{b d}
      - 4 genG3^{a}_{p c1 c2 c3} genG4^{p}_{b d}.

# n = 4

defG64 := genG6^{a}_{b c1 c2 c3 c4 d} -> D_{d}{genG6^{a}_{b c1 c2 c3 c4}}.

defG74 := genG7^{a}_{b c1 c2 c3 c4 d} -> D_{d}{genG7^{a}_{b c1 c2 c3 c4}}
      - 5 genG5^{a}_{p c1 c2 c3 c4} genG2^{p}_{b d}.

# n = 5

defG75 := genG7^{a}_{b c1 c2 c3 c4 c5 d} -> D_{d}{genG7^{a}_{b c1 c2 c3 c4 c5}}.

# -----
# build the genGmn
# =====
# n = 1

genG31 := genG3^{a}_{b c1 d}. # cdb (genG31.000,genG31)
genG41 := genG4^{a}_{b c1 d}. # cdb (genG41.000,genG41)
genG51 := genG5^{a}_{b c1 d}.
# genG61 := genG6^{a}_{b c1 d}.
# genG71 := genG7^{a}_{b c1 d}.

# -----
substitute (genG20,defG20) # cdb (genG20.001,genG20)
substitute (genG30,defG30) # cdb (genG30.001,genG30)
substitute (genG40,defG40) # cdb (genG40.001,genG40)
substitute (genG50,defG50) # cdb (genG50.001,genG50)

# -----
substitute (genG31,defG31) # cdb (genG31.001,genG31)
substitute (genG31,defG30) # cdb (genG31.002,genG31)

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distribute      (genG31)                # cdb (genG31.002,genG31)
unwrap          (genG31)                # cdb (genG31.003,genG31)
product_rule    (genG31)                # cdb (genG31.004,genG31)
distribute      (genG31)                # cdb (genG31.005,genG31)
substitute      (genG31,$D_{a}{x^b}->\delta_{a}^{b}$) # cdb (genG31.006,genG31)
eliminate_kronecker (genG31)          # cdb (genG31.007,genG31)
sym             (genG31,$_{b},_{c1},_{d}$)
sort_product    (genG31)                # cdb (genG31.008,genG31)
rename_dummies  (genG31)                # cdb (genG31.009,genG31)
canonicalise    (genG31)                # cdb (genG31.010,genG31)

# -----
substitute      (genG41,defG41)          # cdb (genG41.001,genG41)
substitute      (genG41,defG40)          # cdb (genG41.002,genG41)
substitute      (genG41,defG20,repeat=True) # cdb (genG41.003,genG41)

distribute      (genG41)                # cdb (genG41.004,genG41)
unwrap          (genG41)                # cdb (genG41.005,genG41)
product_rule    (genG41)                # cdb (genG41.006,genG41)
distribute      (genG41)                # cdb (genG41.007,genG41)
substitute      (genG41,$D_{a}{x^b}->\delta_{a}^{b}$) # cdb (genG41.008,genG41)
eliminate_kronecker (genG41)          # cdb (genG41.009,genG41)
sym             (genG41,$_{b},_{c1},_{d}$)
sort_product    (genG41)                # cdb (genG41.010,genG41)
rename_dummies  (genG41)                # cdb (genG41.011,genG41)
canonicalise    (genG41)                # cdb (genG41.012,genG41)

# -----
substitute      (genG51,defG51)
substitute      (genG51,defG50)
substitute      (genG51,defG30,repeat=True)
substitute      (genG51,defG20,repeat=True)

distribute      (genG51)
unwrap          (genG51)
product_rule    (genG51)
distribute      (genG51)
substitute      (genG51,$D_{a}{x^b}->\delta_{a}^{b}$)

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```

eliminate_kronecker (genG51)
sym                (genG51,${b}, ${c1}, ${d}$)
sort_product       (genG51)
rename_dummies     (genG51)
canonicalise       (genG51)

# update the rules

defG31 := genG3^{a}_{b c1 d} -> @(genG31).
defG41 := genG4^{a}_{b c1 d} -> @(genG41).
defG51 := genG5^{a}_{b c1 d} -> @(genG51).

# =====
# n = 2

genG42 := genG4^{a}_{b c1 c2 d}.
genG52 := genG5^{a}_{b c1 c2 d}.
# genG62 := genG6^{a}_{b c1 c2 d}.
# genG72 := genG7^{a}_{b c1 c2 d}.

# -----
substitute      (genG42,defG42)
substitute      (genG42,defG41)

distribute      (genG42)
unwrap          (genG42)
product_rule     (genG42)
distribute      (genG42)
substitute      (genG42,$D_{a}{x^b}->\delta_{a}^{b}$)
eliminate_kronecker (genG42)
sym             (genG42,${b}, ${c1}, ${c2}, ${d}$)
sort_product     (genG42)
rename_dummies   (genG42)
canonicalise     (genG42)

# -----
substitute      (genG52,defG52)
substitute      (genG52,defG51)

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```

substitute      (genG52,defG31,repeat=True)
substitute      (genG52,defG20,repeat=True)

distribute      (genG52)
unwrap          (genG52)
product_rule    (genG52)
distribute      (genG52)
substitute      (genG52,$D_{a}{x^b}->\delta_{a}^{b}$)
eliminate_kronecker (genG52)
sym             (genG52,$_ {b}, _{c1}, _{c2}, _{d}$)
sort_product    (genG52)
rename_dummies  (genG52)
canonicalise    (genG52)                                # cdb (genG52.001,genG52)

# update the rules

defG42 := genG4^{a}_{b c1 c2 d} -> @(genG42).
defG52 := genG5^{a}_{b c1 c2 d} -> @(genG52).

# =====
# n = 3

genG53 := genG5^{a}_{b c1 c2 c3 d}.
# genG63 := genG6^{a}_{b c1 c2 c3 d}.
# genG73 := genG7^{a}_{b c1 c2 c3 d}.

# -----
substitute      (genG53,defG53)
substitute      (genG53,defG52)

distribute      (genG53)
unwrap          (genG53)
product_rule    (genG53)
distribute      (genG53)
substitute      (genG53,$D_{a}{x^b}->\delta_{a}^{b}$)
eliminate_kronecker (genG53)
sym             (genG53,$_ {b}, _{c1}, _{c2}, _{c3}, _{d}$)
sort_product    (genG53)

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```
rename_dummies (genG53)
canonicalise    (genG53)                # cdb (genG53.001,genG53)

# update the rules

defG53 := genG5^{\{a\}_{b\ c1\ c2\ c3\ d}} -> @(genG53).
```

$$\text{genG31.000} := \text{gen}G_3^a{}_{bc_1d}$$

$$\text{genG31.001} := D_d(\text{gen}G_3^a{}_{bc_1})$$

$$\text{genG31.002} := \frac{1}{12}D_d(x^c x^e \nabla_b R_{c_1 c e f} g^{af}) + \frac{1}{6}D_d(x^c x^e \nabla_c R_{b f c_1 e} g^{af}) + \frac{1}{12}D_d(x^c x^e \nabla_{c_1} R_{b c e f} g^{af}) + \frac{1}{6}D_d(x^c x^e \nabla_c R_{b e c_1 f} g^{af}) + \frac{1}{12}D_d(x^c x^e \nabla_f R_{b c c_1 e} g^{af})$$

$$\text{genG31.003} := \frac{1}{12}\nabla_b R_{c_1 c e f} g^{af} D_d(x^c x^e) + \frac{1}{6}\nabla_c R_{b f c_1 e} g^{af} D_d(x^c x^e) + \frac{1}{12}\nabla_{c_1} R_{b c e f} g^{af} D_d(x^c x^e) + \frac{1}{6}\nabla_c R_{b e c_1 f} g^{af} D_d(x^c x^e) + \frac{1}{12}\nabla_f R_{b c c_1 e} g^{af} D_d(x^c x^e)$$

$$\begin{aligned} \text{genG31.004} := & \frac{1}{12}\nabla_b R_{c_1 c e f} g^{af} (D_d x^c x^e + x^c D_d x^e) + \frac{1}{6}\nabla_c R_{b f c_1 e} g^{af} (D_d x^c x^e + x^c D_d x^e) + \frac{1}{12}\nabla_{c_1} R_{b c e f} g^{af} (D_d x^c x^e + x^c D_d x^e) \\ & + \frac{1}{6}\nabla_c R_{b e c_1 f} g^{af} (D_d x^c x^e + x^c D_d x^e) + \frac{1}{12}\nabla_f R_{b c c_1 e} g^{af} (D_d x^c x^e + x^c D_d x^e) \end{aligned}$$

$$\begin{aligned} \text{genG31.005} := & \frac{1}{12}\nabla_b R_{c_1 c e f} g^{af} D_d x^c x^e + \frac{1}{12}\nabla_b R_{c_1 c e f} g^{af} x^c D_d x^e + \frac{1}{6}\nabla_c R_{b f c_1 e} g^{af} D_d x^c x^e + \frac{1}{6}\nabla_c R_{b f c_1 e} g^{af} x^c D_d x^e + \frac{1}{12}\nabla_{c_1} R_{b c e f} g^{af} D_d x^c x^e \\ & + \frac{1}{12}\nabla_{c_1} R_{b c e f} g^{af} x^c D_d x^e + \frac{1}{6}\nabla_c R_{b e c_1 f} g^{af} D_d x^c x^e + \frac{1}{6}\nabla_c R_{b e c_1 f} g^{af} x^c D_d x^e + \frac{1}{12}\nabla_f R_{b c c_1 e} g^{af} D_d x^c x^e + \frac{1}{12}\nabla_f R_{b c c_1 e} g^{af} x^c D_d x^e \end{aligned}$$

$$\begin{aligned} \text{genG31.006} := & \frac{1}{12}\nabla_b R_{c_1 c e f} g^{af} \delta_d^c x^e + \frac{1}{12}\nabla_b R_{c_1 c e f} g^{af} x^c \delta_d^e + \frac{1}{6}\nabla_c R_{b f c_1 e} g^{af} \delta_d^c x^e + \frac{1}{6}\nabla_c R_{b f c_1 e} g^{af} x^c \delta_d^e + \frac{1}{12}\nabla_{c_1} R_{b c e f} g^{af} \delta_d^c x^e \\ & + \frac{1}{12}\nabla_{c_1} R_{b c e f} g^{af} x^c \delta_d^e + \frac{1}{6}\nabla_c R_{b e c_1 f} g^{af} \delta_d^c x^e + \frac{1}{6}\nabla_c R_{b e c_1 f} g^{af} x^c \delta_d^e + \frac{1}{12}\nabla_f R_{b c c_1 e} g^{af} \delta_d^c x^e + \frac{1}{12}\nabla_f R_{b c c_1 e} g^{af} x^c \delta_d^e \end{aligned}$$

$$\begin{aligned} \text{genG31.007} := & \frac{1}{12}\nabla_b R_{c_1 d e f} g^{af} x^e + \frac{1}{12}\nabla_b R_{c_1 c d f} g^{af} x^c + \frac{1}{6}\nabla_d R_{b f c_1 e} g^{af} x^e + \frac{1}{6}\nabla_c R_{b f c_1 d} g^{af} x^c + \frac{1}{12}\nabla_{c_1} R_{b d e f} g^{af} x^e \\ & + \frac{1}{12}\nabla_{c_1} R_{b c d f} g^{af} x^c + \frac{1}{6}\nabla_d R_{b e c_1 f} g^{af} x^e + \frac{1}{6}\nabla_c R_{b d c_1 f} g^{af} x^c + \frac{1}{12}\nabla_f R_{b d c_1 e} g^{af} x^e + \frac{1}{12}\nabla_f R_{b c c_1 d} g^{af} x^c \end{aligned}$$

$$\begin{aligned}
\text{genG31.008} := & \frac{1}{36} \nabla_b R_{c_1 def} g^{af} x^e + \frac{1}{36} \nabla_b R_{dc_1 ef} g^{af} x^e + \frac{1}{36} \nabla_{c_1} R_{bdef} g^{af} x^e + \frac{1}{36} \nabla_{c_1} R_{dbef} g^{af} x^e + \frac{1}{36} \nabla_d R_{bc_1 ef} g^{af} x^e + \frac{1}{36} \nabla_d R_{c_1 bef} g^{af} x^e \\
& + \frac{1}{36} \nabla_b R_{c_1 cdf} g^{af} x^c + \frac{1}{36} \nabla_b R_{dcc_1 f} g^{af} x^c + \frac{1}{36} \nabla_{c_1} R_{bcd f} g^{af} x^c + \frac{1}{36} \nabla_{c_1} R_{dc b f} g^{af} x^c + \frac{1}{36} \nabla_d R_{bcc_1 f} g^{af} x^c + \frac{1}{36} \nabla_d R_{c_1 cb f} g^{af} x^c \\
& + \frac{1}{36} \nabla_d R_{bf c_1 e} g^{af} x^e + \frac{1}{36} \nabla_{c_1} R_{bf de} g^{af} x^e + \frac{1}{36} \nabla_d R_{c_1 f be} g^{af} x^e + \frac{1}{36} \nabla_b R_{c_1 f de} g^{af} x^e + \frac{1}{36} \nabla_{c_1} R_{df be} g^{af} x^e + \frac{1}{36} \nabla_b R_{df c_1 e} g^{af} x^e \\
& + \frac{1}{6} \nabla_c \left( \frac{1}{6} R_{bf c_1 d} + \frac{1}{6} R_{bf dc_1} + \frac{1}{6} R_{c_1 f bd} + \frac{1}{6} R_{c_1 f db} + \frac{1}{6} R_{df bc_1} + \frac{1}{6} R_{df c_1 b} \right) g^{af} x^c + \frac{1}{36} \nabla_d R_{bec_1 f} g^{af} x^e + \frac{1}{36} \nabla_{c_1} R_{bed f} g^{af} x^e + \frac{1}{36} \nabla_d R_{c_1 ebf} g^{af} x^e \\
& + \frac{1}{36} \nabla_b R_{c_1 edf} g^{af} x^e + \frac{1}{36} \nabla_{c_1} R_{debf} g^{af} x^e + \frac{1}{36} \nabla_b R_{dec_1 f} g^{af} x^e + \frac{1}{6} \nabla_c \left( \frac{1}{6} R_{bdc_1 f} + \frac{1}{6} R_{bc_1 df} + \frac{1}{6} R_{c_1 dbf} + \frac{1}{6} R_{c_1 bdf} + \frac{1}{6} R_{dc_1 bf} + \frac{1}{6} R_{dbc_1 f} \right) g^{af} x^c \\
& + \frac{1}{12} \nabla_f \left( \frac{1}{6} R_{bdc_1 e} + \frac{1}{6} R_{bc_1 de} + \frac{1}{6} R_{c_1 dbe} + \frac{1}{6} R_{c_1 bde} + \frac{1}{6} R_{dc_1 be} + \frac{1}{6} R_{dbc_1 e} \right) g^{af} x^e \\
& + \frac{1}{12} \nabla_f \left( \frac{1}{6} R_{bcc_1 d} + \frac{1}{6} R_{bcd c_1} + \frac{1}{6} R_{c_1 cbd} + \frac{1}{6} R_{c_1 cdb} + \frac{1}{6} R_{dcb c_1} + \frac{1}{6} R_{dcc_1 b} \right) g^{af} x^c \\
\\
\text{genG31.009} := & \frac{1}{36} \nabla_b R_{c_1 dce} g^{ae} x^c + \frac{1}{36} \nabla_b R_{dc_1 ce} g^{ae} x^c + \frac{1}{36} \nabla_{c_1} R_{bdce} g^{ae} x^c + \frac{1}{36} \nabla_{c_1} R_{dbce} g^{ae} x^c + \frac{1}{36} \nabla_d R_{bc_1 ce} g^{ae} x^c + \frac{1}{36} \nabla_d R_{c_1 bce} g^{ae} x^c \\
& + \frac{1}{18} \nabla_b R_{c_1 cde} g^{ae} x^c + \frac{1}{18} \nabla_b R_{dcc_1 e} g^{ae} x^c + \frac{1}{18} \nabla_{c_1} R_{bcde} g^{ae} x^c + \frac{1}{18} \nabla_{c_1} R_{dcbe} g^{ae} x^c + \frac{1}{18} \nabla_d R_{bcc_1 e} g^{ae} x^c \\
& + \frac{1}{18} \nabla_d R_{c_1 cbe} g^{ae} x^c + \frac{1}{36} \nabla_d R_{bcc_1 e} g^{ac} x^e + \frac{1}{36} \nabla_{c_1} R_{bcde} g^{ac} x^e + \frac{1}{36} \nabla_d R_{c_1 cbe} g^{ac} x^e + \frac{1}{36} \nabla_b R_{c_1 cde} g^{ac} x^e + \frac{1}{36} \nabla_{c_1} R_{dcbe} g^{ac} x^e \\
& + \frac{1}{36} \nabla_b R_{dcc_1 e} g^{ac} x^e + \frac{1}{6} \nabla_e \left( \frac{1}{6} R_{bcc_1 d} + \frac{1}{6} R_{bcd c_1} + \frac{1}{6} R_{c_1 cbd} + \frac{1}{6} R_{c_1 cdb} + \frac{1}{6} R_{dcb c_1} + \frac{1}{6} R_{dcc_1 b} \right) g^{ac} x^e \\
& + \frac{1}{6} \nabla_e \left( \frac{1}{6} R_{bdc_1 c} + \frac{1}{6} R_{bc_1 dc} + \frac{1}{6} R_{c_1 dbc} + \frac{1}{6} R_{c_1 bdc} + \frac{1}{6} R_{dc_1 bc} + \frac{1}{6} R_{dbc_1 c} \right) g^{ac} x^e \\
& + \frac{1}{12} \nabla_e \left( \frac{1}{6} R_{bdc_1 c} + \frac{1}{6} R_{bc_1 dc} + \frac{1}{6} R_{c_1 dbc} + \frac{1}{6} R_{c_1 bdc} + \frac{1}{6} R_{dc_1 bc} + \frac{1}{6} R_{dbc_1 c} \right) g^{ae} x^c \\
& + \frac{1}{12} \nabla_e \left( \frac{1}{6} R_{bcc_1 d} + \frac{1}{6} R_{bcd c_1} + \frac{1}{6} R_{c_1 cbd} + \frac{1}{6} R_{c_1 cdb} + \frac{1}{6} R_{dcb c_1} + \frac{1}{6} R_{dcc_1 b} \right) g^{ae} x^c \\
\\
\text{genG31.010} := & \frac{1}{12} \nabla_b R_{c_1 cde} g^{ae} x^c + \frac{1}{12} \nabla_b R_{c_1 cde} g^{ac} x^e + \frac{1}{12} \nabla_{c_1} R_{bcde} g^{ae} x^c + \frac{1}{12} \nabla_{c_1} R_{bcde} g^{ac} x^e + \frac{1}{12} \nabla_d R_{bcc_1 e} g^{ae} x^c + \frac{1}{12} \nabla_d R_{bcc_1 e} g^{ac} x^e
\end{aligned}$$

$$\text{genG41.000} := \text{gen}G_4^a{}_{bc_1d}$$

$$\text{genG41.001} := D_d \left( \text{gen}G_4^a{}_{bc_1} \right) - 2\text{gen}G_2^a{}_{pc_1}\text{gen}G_2^p{}_{bd}$$

$$\begin{aligned} \text{genG41.002} := D_d \left( x^c x^e x^f \left( \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} \right) \right) - 2\text{gen}G_2^a{}_{pc_1}\text{gen}G_2^p{}_{bd} \end{aligned}$$

$$\begin{aligned} \text{genG41.003} := D_d \left( x^c x^e x^f \left( \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} \right. \right. \\ \left. \left. + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} \right) \right) \\ - 2x^c \left( \frac{1}{3} R_{pec_1c} g^{ae} + \frac{1}{3} R_{pcc_1e} g^{ae} \right) x^f \left( \frac{1}{3} R_{bgdf} g^{pg} + \frac{1}{3} R_{bfdg} g^{pg} \right) \end{aligned}$$

$$\begin{aligned} \text{genG41.004} := \frac{4}{45} D_d \left( x^c x^e x^f R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} \right) + \frac{4}{45} D_d \left( x^c x^e x^f R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} \right) - \frac{2}{45} D_d \left( x^c x^e x^f R_{bgch} R_{c_1efi} g^{ag} g^{hi} \right) \\ - \frac{1}{45} D_d \left( x^c x^e x^f R_{bgch} R_{c_1efi} g^{ah} g^{gi} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{bc} R_{c_1efg} g^{ag} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{cb} R_{c_1efg} g^{ag} \right) + \frac{1}{20} D_d \left( x^c x^e x^f \nabla_{ce} R_{bgc_1f} g^{ag} \right) \\ - \frac{2}{45} D_d \left( x^c x^e x^f R_{bceg} R_{c_1hfi} g^{ah} g^{gi} \right) - \frac{1}{45} D_d \left( x^c x^e x^f R_{bceg} R_{c_1hfi} g^{ai} g^{gh} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{c_1c} R_{befg} g^{ag} \right) \\ + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{cc_1} R_{befg} g^{ag} \right) + \frac{1}{20} D_d \left( x^c x^e x^f \nabla_{ce} R_{bfc_1g} g^{ag} \right) - \frac{1}{45} D_d \left( x^c x^e x^f R_{bcgh} R_{c_1efi} g^{ag} g^{hi} \right) \\ - \frac{1}{45} D_d \left( x^c x^e x^f R_{bceg} R_{c_1fhi} g^{ah} g^{gi} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{gc} R_{bec_1f} g^{ag} \right) + \frac{1}{40} D_d \left( x^c x^e x^f \nabla_{cg} R_{bec_1f} g^{ag} \right) \\ - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bfdg} g^{pg} \end{aligned}$$

$$\begin{aligned}
\text{genG41.005} := & \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} D_d (x^c x^e x^f) + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} D_d (x^c x^e x^f) - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} D_d (x^c x^e x^f) \\
& - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} D_d (x^c x^e x^f) + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} D_d (x^c x^e x^f) \\
& - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} D_d (x^c x^e x^f) - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} D_d (x^c x^e x^f) \\
& + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} D_d (x^c x^e x^f) + \frac{1}{20} \nabla_{ce} R_{bf c_1g} g^{ag} D_d (x^c x^e x^f) - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} D_d (x^c x^e x^f) \\
& - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} D_d (x^c x^e x^f) + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} D_d (x^c x^e x^f) \\
& - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bfdg} g^{pg} \\
\\
\text{genG41.006} := & \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{20} \nabla_{ce} R_{bf c_1g} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} (D_d x^c x^e x^f + x^c D_d x^e x^f + x^c x^e D_d x^f) \\
& - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$



$$\begin{aligned}
\text{genG41.007} := & \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} D_d x^c x^e x^f + \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} x^c D_d x^e x^f + \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} x^c x^e D_d x^f + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} D_d x^c x^e x^f \\
& + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} x^c D_d x^e x^f + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} x^c x^e D_d x^f - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} D_d x^c x^e x^f - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} x^c D_d x^e x^f \\
& - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} x^c x^e D_d x^f - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} D_d x^c x^e x^f - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} x^c D_d x^e x^f - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} x^c x^e D_d x^f \\
& + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} x^c x^e D_d x^f + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} D_d x^c x^e x^f \\
& + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} x^c x^e D_d x^f + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} D_d x^c x^e x^f + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} x^c D_d x^e x^f \\
& + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} x^c x^e D_d x^f - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} D_d x^c x^e x^f - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} x^c D_d x^e x^f - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} x^c x^e D_d x^f \\
& - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} D_d x^c x^e x^f - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} x^c D_d x^e x^f - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} x^c x^e D_d x^f + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} D_d x^c x^e x^f \\
& + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} x^c x^e D_d x^f + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} x^c D_d x^e x^f \\
& + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} x^c x^e D_d x^f + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} D_d x^c x^e x^f + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} x^c D_d x^e x^f + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} x^c x^e D_d x^f \\
& - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} D_d x^c x^e x^f - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} x^c D_d x^e x^f - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} x^c x^e D_d x^f - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} D_d x^c x^e x^f \\
& - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} x^c D_d x^e x^f - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} x^c x^e D_d x^f + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} x^c D_d x^e x^f \\
& + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} x^c x^e D_d x^f + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} D_d x^c x^e x^f + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} x^c D_d x^e x^f + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} x^c x^e D_d x^f \\
& - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$

$$\begin{aligned}
\text{genG41.008} := & \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} \delta_d^c x^e x^f + \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} x^c \delta_d^e x^f + \frac{4}{45} R_{bgc_1c} R_{ehfi} g^{ah} g^{gi} x^c x^e \delta_d^f + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} \delta_d^c x^e x^f \\
& + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} x^c \delta_d^e x^f + \frac{4}{45} R_{bcc_1g} R_{ehfi} g^{ah} g^{gi} x^c x^e \delta_d^f - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} \delta_d^c x^e x^f - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} x^c \delta_d^e x^f \\
& - \frac{2}{45} R_{bgch} R_{c_1efi} g^{ag} g^{hi} x^c x^e \delta_d^f - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} \delta_d^c x^e x^f - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} x^c \delta_d^e x^f - \frac{1}{45} R_{bgch} R_{c_1efi} g^{ah} g^{gi} x^c x^e \delta_d^f \\
& + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{bc} R_{c_1efg} g^{ag} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} \delta_d^c x^e x^f \\
& + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{cb} R_{c_1efg} g^{ag} x^c x^e \delta_d^f + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} \delta_d^c x^e x^f + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} x^c \delta_d^e x^f + \frac{1}{20} \nabla_{ce} R_{bgc_1f} g^{ag} x^c x^e \delta_d^f \\
& - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} \delta_d^c x^e x^f - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} x^c \delta_d^e x^f - \frac{2}{45} R_{bceg} R_{c_1hfi} g^{ah} g^{gi} x^c x^e \delta_d^f - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} \delta_d^c x^e x^f \\
& - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} x^c \delta_d^e x^f - \frac{1}{45} R_{bceg} R_{c_1hfi} g^{ai} g^{gh} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} x^c \delta_d^e x^f \\
& + \frac{1}{40} \nabla_{c_1c} R_{befg} g^{ag} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{cc_1} R_{befg} g^{ag} x^c x^e \delta_d^f + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} \delta_d^c x^e x^f \\
& + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} x^c \delta_d^e x^f + \frac{1}{20} \nabla_{ce} R_{bfc_1g} g^{ag} x^c x^e \delta_d^f - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} \delta_d^c x^e x^f - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} x^c \delta_d^e x^f \\
& - \frac{1}{45} R_{bcgh} R_{c_1efi} g^{ag} g^{hi} x^c x^e \delta_d^f - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} \delta_d^c x^e x^f - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} x^c \delta_d^e x^f - \frac{1}{45} R_{bceg} R_{c_1fhi} g^{ah} g^{gi} x^c x^e \delta_d^f \\
& + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} x^c \delta_d^e x^f + \frac{1}{40} \nabla_{gc} R_{bec_1f} g^{ag} x^c x^e \delta_d^f + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} \delta_d^c x^e x^f + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} x^c \delta_d^e x^f \\
& + \frac{1}{40} \nabla_{cg} R_{bec_1f} g^{ag} x^c x^e \delta_d^f - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$

$$\begin{aligned}
\text{genG41.009} := & \frac{4}{45} R_{bgc_1d} R_{ehfi} g^{ah} g^{gi} x^e x^f + \frac{4}{45} R_{bgc_1c} R_{dhfi} g^{ah} g^{gi} x^c x^f + \frac{4}{45} R_{bgc_1c} R_{ehdi} g^{ah} g^{gi} x^c x^e + \frac{4}{45} R_{bdc_1g} R_{ehfi} g^{ah} g^{gi} x^e x^f \\
& + \frac{4}{45} R_{bcc_1g} R_{dhfi} g^{ah} g^{gi} x^c x^f + \frac{4}{45} R_{bcc_1g} R_{ehdi} g^{ah} g^{gi} x^c x^e - \frac{2}{45} R_{bgdh} R_{c_1efi} g^{ag} g^{hi} x^e x^f - \frac{2}{45} R_{bgch} R_{c_1dfi} g^{ag} g^{hi} x^c x^f \\
& - \frac{2}{45} R_{bgch} R_{c_1edi} g^{ag} g^{hi} x^c x^e - \frac{1}{45} R_{bgdh} R_{c_1efi} g^{ah} g^{gi} x^e x^f - \frac{1}{45} R_{bgch} R_{c_1dfi} g^{ah} g^{gi} x^c x^f - \frac{1}{45} R_{bgch} R_{c_1edi} g^{ah} g^{gi} x^c x^e + \frac{1}{40} \nabla_{bd} R_{c_1efg} g^{ag} x^e x^f \\
& + \frac{1}{40} \nabla_{bc} R_{c_1dfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{bc} R_{c_1edg} g^{ag} x^c x^e + \frac{1}{40} \nabla_{db} R_{c_1efg} g^{ag} x^e x^f + \frac{1}{40} \nabla_{cb} R_{c_1dfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{cb} R_{c_1edg} g^{ag} x^c x^e \\
& + \frac{1}{20} \nabla_{de} R_{bgc_1f} g^{ag} x^e x^f + \frac{1}{20} \nabla_{cd} R_{bgc_1f} g^{ag} x^c x^f + \frac{1}{20} \nabla_{ce} R_{bgc_1d} g^{ag} x^c x^e - \frac{2}{45} R_{bdeg} R_{c_1hfi} g^{ah} g^{gi} x^e x^f - \frac{2}{45} R_{bcdg} R_{c_1hfi} g^{ah} g^{gi} x^c x^f \\
& - \frac{2}{45} R_{bceg} R_{c_1hdi} g^{ah} g^{gi} x^c x^e - \frac{1}{45} R_{bdeg} R_{c_1hfi} g^{ai} g^{gh} x^e x^f - \frac{1}{45} R_{bcdg} R_{c_1hfi} g^{ai} g^{gh} x^c x^f - \frac{1}{45} R_{bceg} R_{c_1hdi} g^{ai} g^{gh} x^c x^e + \frac{1}{40} \nabla_{c_1d} R_{befg} g^{ag} x^e x^f \\
& + \frac{1}{40} \nabla_{c_1c} R_{bdfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{c_1c} R_{bedg} g^{ag} x^c x^e + \frac{1}{40} \nabla_{dc_1} R_{befg} g^{ag} x^e x^f + \frac{1}{40} \nabla_{cc_1} R_{bdfg} g^{ag} x^c x^f + \frac{1}{40} \nabla_{cc_1} R_{bedg} g^{ag} x^c x^e \\
& + \frac{1}{20} \nabla_{de} R_{bfc_1g} g^{ag} x^e x^f + \frac{1}{20} \nabla_{cd} R_{bfc_1g} g^{ag} x^c x^f + \frac{1}{20} \nabla_{ce} R_{bdc_1g} g^{ag} x^c x^e - \frac{1}{45} R_{bdgh} R_{c_1efi} g^{ag} g^{hi} x^e x^f - \frac{1}{45} R_{bcgh} R_{c_1dfi} g^{ag} g^{hi} x^c x^f \\
& - \frac{1}{45} R_{bcgh} R_{c_1edi} g^{ag} g^{hi} x^c x^e - \frac{1}{45} R_{bdeg} R_{c_1fhi} g^{ah} g^{gi} x^e x^f - \frac{1}{45} R_{bcdg} R_{c_1fhi} g^{ah} g^{gi} x^c x^f - \frac{1}{45} R_{bceg} R_{c_1dhi} g^{ah} g^{gi} x^c x^e \\
& + \frac{1}{40} \nabla_{gd} R_{bec_1f} g^{ag} x^e x^f + \frac{1}{40} \nabla_{gc} R_{bdc_1f} g^{ag} x^c x^f + \frac{1}{40} \nabla_{gc} R_{bec_1d} g^{ag} x^c x^e + \frac{1}{40} \nabla_{dg} R_{bec_1f} g^{ag} x^e x^f + \frac{1}{40} \nabla_{cg} R_{bdc_1f} g^{ag} x^c x^f \\
& + \frac{1}{40} \nabla_{cg} R_{bec_1d} g^{ag} x^c x^e - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pec_1c} g^{ae} x^f R_{bfdg} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bgdf} g^{pg} - \frac{2}{9} x^c R_{pcc_1e} g^{ae} x^f R_{bfdg} g^{pg}
\end{aligned}$$

$$\begin{aligned}
\text{genG41.012} := & -\frac{4}{45}R_{bcc_1e}R_{dfgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcde}R_{c_1fgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcc_1e}R_{dfgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c_1gdh}g^{ac}g^{eg}x^fx^h \\
& - \frac{4}{45}R_{bcde}R_{c_1fgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c_1gdh}g^{ac}g^{eh}x^fx^g - \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ag}g^{cf}x^ex^h - \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ag}g^{cf}x^ex^h \\
& - \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ae}g^{cg}x^fx^h - \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ae}g^{ch}x^fx^g \\
& + \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ag}g^{eh}x^cx^f + \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ae}g^{cg}x^fx^h \\
& + \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ah}g^{eg}x^cx^f + \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ae}g^{cg}x^fx^h - \frac{1}{60}\nabla_{bd}R_{c_1cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{bc_1}R_{dcef}g^{ae}x^cx^f \\
& - \frac{1}{60}\nabla_{c_1d}R_{bcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c_1b}R_{dcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{dc_1}R_{bcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{db}R_{c_1cef}g^{ae}x^cx^f + \frac{1}{40}\nabla_{bc}R_{c_1edf}g^{af}x^cx^e \\
& + \frac{1}{40}\nabla_{bc}R_{c_1edf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{c_1c}R_{bedf}g^{af}x^cx^e + \frac{1}{40}\nabla_{c_1c}R_{bedf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{dc}R_{bec_1f}g^{af}x^cx^e + \frac{1}{40}\nabla_{dc}R_{bec_1f}g^{ae}x^cx^f \\
& + \frac{1}{40}\nabla_{cb}R_{c_1edf}g^{af}x^cx^e + \frac{1}{40}\nabla_{cb}R_{c_1edf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cc_1}R_{bedf}g^{af}x^cx^e + \frac{1}{40}\nabla_{cc_1}R_{bedf}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cd}R_{bec_1f}g^{af}x^cx^e \\
& + \frac{1}{40}\nabla_{cd}R_{bec_1f}g^{ae}x^cx^f + \frac{1}{15}R_{bcef}R_{c_1gdh}g^{ae}g^{fh}x^cx^g + \frac{1}{15}R_{bcef}R_{c_1gdh}g^{ae}g^{fg}x^cx^h + \frac{1}{15}R_{bcde}R_{c_1fgh}g^{ag}g^{eh}x^cx^f \\
& + \frac{1}{15}R_{bcde}R_{c_1fgh}g^{ag}g^{ch}x^ex^f + \frac{1}{15}R_{bcc_1e}R_{dfgh}g^{ag}g^{eh}x^cx^f + \frac{1}{15}R_{bcc_1e}R_{dfgh}g^{ag}g^{ch}x^ex^f + \frac{1}{120}\nabla_{cd}R_{bec_1f}g^{ac}x^ex^f \\
& + \frac{1}{120}\nabla_{cc_1}R_{bedf}g^{ac}x^ex^f + \frac{1}{120}\nabla_{cb}R_{c_1edf}g^{ac}x^ex^f + \frac{1}{120}\nabla_{dc}R_{bec_1f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{c_1c}R_{bedf}g^{ac}x^ex^f + \frac{1}{120}\nabla_{bc}R_{c_1edf}g^{ac}x^ex^f
\end{aligned}$$

$$\text{genG42.000} := \text{gen}G_4^a{}_{bc_1c_2d}$$

$$\text{genG42.001} := D_d \left( \text{gen}G_4^a{}_{bc_1c_2} \right)$$

$$\begin{aligned} \text{genG42.002} := D_d \Bigg( & -\frac{4}{45}R_{bcc_1e}R_{c_2fgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcc_2e}R_{c_1fgh}g^{af}g^{cg}x^ex^h - \frac{4}{45}R_{bcc_1e}R_{c_2fgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c_1gc_2h}g^{ac}g^{eg}x^fx^h \\ & - \frac{4}{45}R_{bcc_2e}R_{c_1fgh}g^{af}g^{eg}x^cx^h - \frac{4}{45}R_{bcef}R_{c_1gc_2h}g^{ac}g^{eh}x^fx^g - \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{cf}x^ex^h - \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{cf}x^ex^h \\ & - \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ae}g^{cg}x^fx^h - \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ef}x^cx^h - \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ae}g^{ch}x^fx^g \\ & + \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ac}g^{eg}x^fx^h + \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ag}g^{eh}x^cx^f + \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ae}g^{cg}x^fx^h \\ & + \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ah}g^{eg}x^cx^f + \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ae}g^{cg}x^fx^h - \frac{1}{60}\nabla_{bc_2}R_{c_1cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{bc_1}R_{c_2cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c_1c_2}R_{bcef}g^{ae}x^cx^f \\ & - \frac{1}{60}\nabla_{c_1b}R_{c_2cef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c_2c_1}R_{bcef}g^{ae}x^cx^f - \frac{1}{60}\nabla_{c_2b}R_{c_1cef}g^{ae}x^cx^f + \frac{1}{40}\nabla_{bc}R_{c_1ec_2f}g^{af}x^cx^e + \frac{1}{40}\nabla_{bc}R_{c_1ec_2f}g^{ae}x^cx^f \\ & + \frac{1}{40}\nabla_{c_1c}R_{bec_2f}g^{af}x^cx^e + \frac{1}{40}\nabla_{c_1c}R_{bec_2f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{c_2c}R_{bec_1f}g^{af}x^cx^e + \frac{1}{40}\nabla_{c_2c}R_{bec_1f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cb}R_{c_1ec_2f}g^{af}x^cx^e \\ & + \frac{1}{40}\nabla_{cb}R_{c_1ec_2f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cc_1}R_{bec_2f}g^{af}x^cx^e + \frac{1}{40}\nabla_{cc_1}R_{bec_2f}g^{ae}x^cx^f + \frac{1}{40}\nabla_{cc_2}R_{bec_1f}g^{af}x^cx^e + \frac{1}{40}\nabla_{cc_2}R_{bec_1f}g^{ae}x^cx^f \\ & + \frac{1}{15}R_{bcef}R_{c_1gc_2h}g^{ae}g^{fh}x^cx^g + \frac{1}{15}R_{bcef}R_{c_1gc_2h}g^{ae}g^{fg}x^cx^h + \frac{1}{15}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{eh}x^cx^f + \frac{1}{15}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ch}x^ex^f \\ & + \frac{1}{15}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{eh}x^cx^f + \frac{1}{15}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ch}x^ex^f + \frac{1}{120}\nabla_{cc_2}R_{bec_1f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{cc_1}R_{bec_2f}g^{ac}x^ex^f \\ & + \frac{1}{120}\nabla_{cb}R_{c_1ec_2f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{c_2c}R_{bec_1f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{c_1c}R_{bec_2f}g^{ac}x^ex^f + \frac{1}{120}\nabla_{bc}R_{c_1ec_2f}g^{ac}x^ex^f \Bigg) \end{aligned}$$

$$\begin{aligned}
\text{genG42.003} := & -\frac{4}{45}D_d(R_{bcc_1e}R_{c_2fgh}g^{af}g^{cg}x^ex^h) - \frac{4}{45}D_d(R_{bcc_2e}R_{c_1fgh}g^{af}g^{cg}x^ex^h) - \frac{4}{45}D_d(R_{bcc_1e}R_{c_2fgh}g^{af}g^{eg}x^cx^h) \\
& - \frac{4}{45}D_d(R_{bcef}R_{c_1gc_2h}g^{ac}g^{eg}x^fx^h) - \frac{4}{45}D_d(R_{bcc_2e}R_{c_1fgh}g^{af}g^{eg}x^cx^h) - \frac{4}{45}D_d(R_{bcef}R_{c_1gc_2h}g^{ac}g^{eh}x^fx^g) \\
& - \frac{1}{45}D_d(R_{bcc_1e}R_{c_2fgh}g^{ag}g^{cf}x^ex^h) - \frac{1}{45}D_d(R_{bcc_2e}R_{c_1fgh}g^{ag}g^{cf}x^ex^h) - \frac{1}{45}D_d(R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ef}x^cx^h) \\
& - \frac{1}{45}D_d(R_{bcef}R_{c_1gc_2h}g^{ae}g^{cg}x^fx^h) - \frac{1}{45}D_d(R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ef}x^cx^h) - \frac{1}{45}D_d(R_{bcef}R_{c_1gc_2h}g^{ae}g^{ch}x^fx^g) \\
& + \frac{1}{45}D_d(R_{bcc_2e}R_{c_1fgh}g^{ac}g^{eg}x^fx^h) + \frac{1}{45}D_d(R_{bcc_1e}R_{c_2fgh}g^{ac}g^{eg}x^fx^h) + \frac{1}{45}D_d(R_{bcef}R_{c_1gc_2h}g^{ag}g^{eh}x^cx^f) \\
& + \frac{1}{45}D_d(R_{bcc_1e}R_{c_2fgh}g^{ae}g^{cg}x^fx^h) + \frac{1}{45}D_d(R_{bcef}R_{c_1gc_2h}g^{ah}g^{eg}x^cx^f) + \frac{1}{45}D_d(R_{bcc_2e}R_{c_1fgh}g^{ae}g^{cg}x^fx^h) \\
& - \frac{1}{60}D_d(\nabla_{bc_2}R_{c_1cef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{bc_1}R_{c_2cef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{c_1c_2}R_{bcef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{c_1b}R_{c_2cef}g^{ae}x^cx^f) \\
& - \frac{1}{60}D_d(\nabla_{c_2c_1}R_{bcef}g^{ae}x^cx^f) - \frac{1}{60}D_d(\nabla_{c_2b}R_{c_1cef}g^{ae}x^cx^f) + \frac{1}{40}D_d(\nabla_{bc}R_{c_1ec_2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{bc}R_{c_1ec_2f}g^{ae}x^cx^f) \\
& + \frac{1}{40}D_d(\nabla_{c_1c}R_{bec_2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{c_1c}R_{bec_2f}g^{ae}x^cx^f) + \frac{1}{40}D_d(\nabla_{c_2c}R_{bec_1f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{c_2c}R_{bec_1f}g^{ae}x^cx^f) \\
& + \frac{1}{40}D_d(\nabla_{cb}R_{c_1ec_2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{cb}R_{c_1ec_2f}g^{ae}x^cx^f) + \frac{1}{40}D_d(\nabla_{cc_1}R_{bec_2f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{cc_1}R_{bec_2f}g^{ae}x^cx^f) \\
& + \frac{1}{40}D_d(\nabla_{cc_2}R_{bec_1f}g^{af}x^cx^e) + \frac{1}{40}D_d(\nabla_{cc_2}R_{bec_1f}g^{ae}x^cx^f) + \frac{1}{15}D_d(R_{bcef}R_{c_1gc_2h}g^{ae}g^{fh}x^cx^g) + \frac{1}{15}D_d(R_{bcef}R_{c_1gc_2h}g^{ae}g^{fg}x^cx^h) \\
& + \frac{1}{15}D_d(R_{bcc_2e}R_{c_1fgh}g^{ag}g^{eh}x^cx^f) + \frac{1}{15}D_d(R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ch}x^ex^f) + \frac{1}{15}D_d(R_{bcc_1e}R_{c_2fgh}g^{ag}g^{eh}x^cx^f) \\
& + \frac{1}{15}D_d(R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ch}x^ex^f) + \frac{1}{120}D_d(\nabla_{cc_2}R_{bec_1f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{cc_1}R_{bec_2f}g^{ac}x^ex^f) \\
& + \frac{1}{120}D_d(\nabla_{cb}R_{c_1ec_2f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{c_2c}R_{bec_1f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{c_1c}R_{bec_2f}g^{ac}x^ex^f) + \frac{1}{120}D_d(\nabla_{bc}R_{c_1ec_2f}g^{ac}x^ex^f)
\end{aligned}$$

$$\begin{aligned}
\text{genG42.004} := & -\frac{4}{45}R_{bcc_1e}R_{c_2fgh}g^{af}g^{cg}D_d(x^ex^h) - \frac{4}{45}R_{bcc_2e}R_{c_1fgh}g^{af}g^{cg}D_d(x^ex^h) - \frac{4}{45}R_{bcc_1e}R_{c_2fgh}g^{af}g^{eg}D_d(x^cx^h) \\
& - \frac{4}{45}R_{bcef}R_{c_1gc_2h}g^{ac}g^{eg}D_d(x^fx^h) - \frac{4}{45}R_{bcc_2e}R_{c_1fgh}g^{af}g^{eg}D_d(x^cx^h) - \frac{4}{45}R_{bcef}R_{c_1gc_2h}g^{ac}g^{eh}D_d(x^fx^g) \\
& - \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{cf}D_d(x^ex^h) - \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{cf}D_d(x^ex^h) - \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ef}D_d(x^cx^h) \\
& - \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ae}g^{cg}D_d(x^fx^h) - \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ef}D_d(x^cx^h) - \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ae}g^{ch}D_d(x^fx^g) \\
& + \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ac}g^{eg}D_d(x^fx^h) + \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ac}g^{eg}D_d(x^fx^h) + \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ag}g^{eh}D_d(x^cx^f) \\
& + \frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ae}g^{cg}D_d(x^fx^h) + \frac{1}{45}R_{bcef}R_{c_1gc_2h}g^{ah}g^{eg}D_d(x^cx^f) + \frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ae}g^{cg}D_d(x^fx^h) \\
& - \frac{1}{60}\nabla_{bc_2}R_{c_1cef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{bc_1}R_{c_2cef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{c_1c_2}R_{bcef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{c_1b}R_{c_2cef}g^{ae}D_d(x^cx^f) \\
& - \frac{1}{60}\nabla_{c_2c_1}R_{bcef}g^{ae}D_d(x^cx^f) - \frac{1}{60}\nabla_{c_2b}R_{c_1cef}g^{ae}D_d(x^cx^f) + \frac{1}{40}\nabla_{bc}R_{c_1ec_2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{bc}R_{c_1ec_2f}g^{ae}D_d(x^cx^f) \\
& + \frac{1}{40}\nabla_{c_1c}R_{bec_2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{c_1c}R_{bec_2f}g^{ae}D_d(x^cx^f) + \frac{1}{40}\nabla_{c_2c}R_{bec_1f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{c_2c}R_{bec_1f}g^{ae}D_d(x^cx^f) \\
& + \frac{1}{40}\nabla_{cb}R_{c_1ec_2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{cb}R_{c_1ec_2f}g^{ae}D_d(x^cx^f) + \frac{1}{40}\nabla_{cc_1}R_{bec_2f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{cc_1}R_{bec_2f}g^{ae}D_d(x^cx^f) \\
& + \frac{1}{40}\nabla_{cc_2}R_{bec_1f}g^{af}D_d(x^cx^e) + \frac{1}{40}\nabla_{cc_2}R_{bec_1f}g^{ae}D_d(x^cx^f) + \frac{1}{15}R_{bcef}R_{c_1gc_2h}g^{ae}g^{fh}D_d(x^cx^g) + \frac{1}{15}R_{bcef}R_{c_1gc_2h}g^{ae}g^{fg}D_d(x^cx^h) \\
& + \frac{1}{15}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{eh}D_d(x^cx^f) + \frac{1}{15}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ch}D_d(x^ex^f) + \frac{1}{15}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{eh}D_d(x^cx^f) \\
& + \frac{1}{15}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ch}D_d(x^ex^f) + \frac{1}{120}\nabla_{cc_2}R_{bec_1f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{cc_1}R_{bec_2f}g^{ac}D_d(x^ex^f) \\
& + \frac{1}{120}\nabla_{cb}R_{c_1ec_2f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{c_2c}R_{bec_1f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{c_1c}R_{bec_2f}g^{ac}D_d(x^ex^f) + \frac{1}{120}\nabla_{bc}R_{c_1ec_2f}g^{ac}D_d(x^ex^f)
\end{aligned}$$

$$\begin{aligned}
\text{genG42.005} := & -\frac{4}{45}R_{bcc_1e}R_{c_2fgh}g^{af}g^{cg}(D_dx^ex^h+x^eD_dx^h)-\frac{4}{45}R_{bcc_2e}R_{c_1fgh}g^{af}g^{cg}(D_dx^ex^h+x^eD_dx^h)-\frac{4}{45}R_{bcc_1e}R_{c_2fgh}g^{af}g^{eg}(D_dx^cx^h+x^cD_dx^h) \\
& -\frac{4}{45}R_{bce_f}R_{c_1gc_2h}g^{ac}g^{eg}(D_dx^fx^h+x^fD_dx^h)-\frac{4}{45}R_{bcc_2e}R_{c_1fgh}g^{af}g^{eg}(D_dx^cx^h+x^cD_dx^h)-\frac{4}{45}R_{bce_f}R_{c_1gc_2h}g^{ac}g^{eh}(D_dx^fx^g+x^fD_dx^g) \\
& -\frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{cf}(D_dx^ex^h+x^eD_dx^h)-\frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{cf}(D_dx^ex^h+x^eD_dx^h)-\frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ef}(D_dx^cx^h+x^cD_dx^h) \\
& -\frac{1}{45}R_{bce_f}R_{c_1gc_2h}g^{ae}g^{cg}(D_dx^fx^h+x^fD_dx^h)-\frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ef}(D_dx^cx^h+x^cD_dx^h)-\frac{1}{45}R_{bce_f}R_{c_1gc_2h}g^{ae}g^{ch}(D_dx^fx^g+x^fD_dx^g) \\
& +\frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ac}g^{eg}(D_dx^fx^h+x^fD_dx^h)+\frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ac}g^{eg}(D_dx^fx^h+x^fD_dx^h)+\frac{1}{45}R_{bce_f}R_{c_1gc_2h}g^{ag}g^{eh}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{45}R_{bcc_1e}R_{c_2fgh}g^{ae}g^{cg}(D_dx^fx^h+x^fD_dx^h)+\frac{1}{45}R_{bce_f}R_{c_1gc_2h}g^{ah}g^{eg}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{45}R_{bcc_2e}R_{c_1fgh}g^{ae}g^{cg}(D_dx^fx^h+x^fD_dx^h) \\
& -\frac{1}{60}\nabla_{bc_2}R_{c_1ce_f}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{bc_1}R_{c_2ce_f}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{c_1c_2}R_{bce_f}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& -\frac{1}{60}\nabla_{c_1b}R_{c_2ce_f}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{c_2c_1}R_{bce_f}g^{ae}(D_dx^cx^f+x^cD_dx^f)-\frac{1}{60}\nabla_{c_2b}R_{c_1ce_f}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{40}\nabla_{bc}R_{c_1ec_2f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{bc}R_{c_1ec_2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{c_1c}R_{bec_2f}g^{af}(D_dx^cx^e+x^cD_dx^e) \\
& +\frac{1}{40}\nabla_{c_1c}R_{bec_2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{c_2c}R_{bec_1f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{c_2c}R_{bec_1f}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{40}\nabla_{cb}R_{c_1ec_2f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{cb}R_{c_1ec_2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{cc_1}R_{bec_2f}g^{af}(D_dx^cx^e+x^cD_dx^e) \\
& +\frac{1}{40}\nabla_{cc_1}R_{bec_2f}g^{ae}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{40}\nabla_{cc_2}R_{bec_1f}g^{af}(D_dx^cx^e+x^cD_dx^e)+\frac{1}{40}\nabla_{cc_2}R_{bec_1f}g^{ae}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{15}R_{bce_f}R_{c_1gc_2h}g^{ae}g^{fh}(D_dx^cx^g+x^cD_dx^g)+\frac{1}{15}R_{bce_f}R_{c_1gc_2h}g^{ae}g^{fg}(D_dx^cx^h+x^cD_dx^h)+\frac{1}{15}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{eh}(D_dx^cx^f+x^cD_dx^f) \\
& +\frac{1}{15}R_{bcc_2e}R_{c_1fgh}g^{ag}g^{ch}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{15}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{eh}(D_dx^cx^f+x^cD_dx^f)+\frac{1}{15}R_{bcc_1e}R_{c_2fgh}g^{ag}g^{ch}(D_dx^ex^f+x^eD_dx^f) \\
& +\frac{1}{120}\nabla_{cc_2}R_{bec_1f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{cc_1}R_{bec_2f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{cb}R_{c_1ec_2f}g^{ac}(D_dx^ex^f+x^eD_dx^f) \\
& +\frac{1}{120}\nabla_{c_2c}R_{bec_1f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{c_1c}R_{bec_2f}g^{ac}(D_dx^ex^f+x^eD_dx^f)+\frac{1}{120}\nabla_{bc}R_{c_1ec_2f}g^{ac}(D_dx^ex^f+x^eD_dx^f)
\end{aligned}$$



$$\begin{aligned}
\text{genG42.011} := & \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{af} g^{cg} x^e + \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{ag} g^{cf} x^e + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{af} g^{cg} x^e + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{ag} g^{cf} x^e + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{af} g^{cg} x^e \\
& + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{ag} g^{cf} x^e + \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{af} g^{eg} x^c + \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{ag} g^{ef} x^c + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{ac} g^{ef} x^g + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{ae} g^{cf} x^g \\
& + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{ac} g^{ef} x^g + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{ae} g^{cf} x^g + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{af} g^{eg} x^c + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{ag} g^{ef} x^c + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{ac} g^{eg} x^f \\
& + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{ae} g^{cg} x^f + \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{ac} g^{ef} x^g + \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{ae} g^{cf} x^g + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{af} g^{eg} x^c + \frac{1}{45} R_{bcde} R_{c_1fc_2g} g^{ag} g^{ef} x^c \\
& + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{ac} g^{eg} x^f + \frac{1}{45} R_{bcc_2e} R_{c_1fdg} g^{ae} g^{cg} x^f + \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{ac} g^{eg} x^f + \frac{1}{45} R_{bcc_1e} R_{c_2fdg} g^{ae} g^{cg} x^f + \frac{1}{60} \nabla_{bc_2} R_{c_1cde} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{bd} R_{c_1cc_2e} g^{ae} x^c + \frac{1}{60} \nabla_{bc_1} R_{c_2cde} g^{ae} x^c + \frac{1}{60} \nabla_{bd} R_{c_1cc_2e} g^{ac} x^e + \frac{1}{60} \nabla_{bc_1} R_{c_2cde} g^{ac} x^e + \frac{1}{60} \nabla_{bc_2} R_{c_1cde} g^{ac} x^e + \frac{1}{60} \nabla_{c_1c_2} R_{bcde} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{c_1d} R_{bcc_2e} g^{ae} x^c + \frac{1}{60} \nabla_{c_1b} R_{c_2cde} g^{ae} x^c + \frac{1}{60} \nabla_{c_1d} R_{bcc_2e} g^{ac} x^e + \frac{1}{60} \nabla_{c_1b} R_{c_2cde} g^{ac} x^e + \frac{1}{60} \nabla_{c_1c_2} R_{bcde} g^{ac} x^e + \frac{1}{60} \nabla_{c_2c_1} R_{bcde} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{c_2d} R_{bcc_1e} g^{ae} x^c + \frac{1}{60} \nabla_{c_2b} R_{c_1cde} g^{ae} x^c + \frac{1}{60} \nabla_{c_2d} R_{bcc_1e} g^{ac} x^e + \frac{1}{60} \nabla_{c_2b} R_{c_1cde} g^{ac} x^e + \frac{1}{60} \nabla_{c_2c_1} R_{bcde} g^{ac} x^e + \frac{1}{60} \nabla_{dc_1} R_{bcc_2e} g^{ae} x^c \\
& + \frac{1}{60} \nabla_{dc_2} R_{bcc_1e} g^{ae} x^c + \frac{1}{60} \nabla_{db} R_{c_1cc_2e} g^{ae} x^c + \frac{1}{60} \nabla_{dc_2} R_{bcc_1e} g^{ac} x^e + \frac{1}{60} \nabla_{db} R_{c_1cc_2e} g^{ac} x^e + \frac{1}{60} \nabla_{dc_1} R_{bcc_2e} g^{ac} x^e
\end{aligned}$$

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                -> A001^{a}                $)
    substitute (obj,$ x^{a}                -> A002^{a}                $)
    substitute (obj,$ g^{a b}              -> A003^{a b}              $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\} -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}   -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}      -> A006_{a b c d e f}    $)
    substitute (obj,$ \nabla_e\{R_{a b c d}\}         -> A005_{a b c d e}     $)
    substitute (obj,$ R_{a b c d}           -> A004_{a b c d}     $)
    sort_product (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}              -> A^{a}              $)
    substitute (obj,$ A002^{a}              -> x^{a}              $)
    substitute (obj,$ A003^{a b}            -> g^{a b}            $)
    substitute (obj,$ A004_{a b c d}        -> R_{a b c d}        $)
    substitute (obj,$ A005_{a b c d e}      -> \nabla_{e}\{R_{a b c d}\} $)
    substitute (obj,$ A006_{a b c d e f}    -> \nabla_{e f}\{R_{a b c d}\} $)
    substitute (obj,$ A007_{a b c d e f g}  -> \nabla_{e f g}\{R_{a b c d}\} $)
    substitute (obj,$ A008_{a b c d e f g h} -> \nabla_{e f g h}\{R_{a b c d}\} $)

    return obj

# -----
symG20 := @(genG20) A^{b} A^{d}.                # cdb (symG20.100,symG20)

distribute (symG20)                            # cdb (symG20.101,symG20)
symG20 = product_sort (symG20)                 # cdb (symG20.102,symG20)
rename_dummies (symG20)                       # cdb (symG20.103,symG20)
canonicalise (symG20)                         # cdb (symG20.104,symG20)

# -----
symG30 := @(genG30) A^{b} A^{d}.                # cdb (symG30.100,symG30)

distribute (symG30)                            # cdb (symG30.101,symG30)
symG30 = product_sort (symG30)                 # cdb (symG30.102,symG30)
rename_dummies (symG30)                       # cdb (symG30.103,symG30)
canonicalise (symG30)                         # cdb (symG30.104,symG30)

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# -----
symG40 := @(genG40) A{b} A{d}. # cdb (symG40.100,symG40)

distribute (symG40) # cdb (symG40.101,symG40)
symG40 = product_sort (symG40) # cdb (symG40.102,symG40)
rename_dummies (symG40) # cdb (symG40.103,symG40)
canonicalise (symG40) # cdb (symG40.104,symG40)

# -----
symG50 := @(genG50) A{b} A{d}. # cdb (symG50.100,symG50)

distribute (symG50) # cdb (symG50.101,symG50)
symG50 = product_sort (symG50) # cdb (symG50.102,symG50)
rename_dummies (symG50) # cdb (symG50.103,symG50)
canonicalise (symG50) # cdb (symG50.104,symG50)

# -----
symG31 := @(genG31) A{b} A{c1} A{d}. # cdb (symG31.100,symG31)

distribute (symG31) # cdb (symG31.101,symG31)
symG31 = product_sort (symG31) # cdb (symG31.102,symG31)
rename_dummies (symG31) # cdb (symG31.103,symG31)
canonicalise (symG31) # cdb (symG31.104,symG31)

# -----
symG41 := @(genG41) A{b} A{c1} A{d}. # cdb (symG41.100,symG41)

distribute (symG41) # cdb (symG41.101,symG41)
symG41 = product_sort (symG41) # cdb (symG41.102,symG41)
rename_dummies (symG41) # cdb (symG41.103,symG41)
canonicalise (symG41) # cdb (symG41.104,symG41)

# -----
symG51 := @(genG51) A{b} A{c1} A{d}. # cdb (symG51.100,symG51)

distribute (symG51) # cdb (symG51.101,symG51)
symG51 = product_sort (symG51) # cdb (symG51.102,symG51)

```

```

rename_dummies      (symG51)                # cdb (symG51.103,symG51)
canonicalise        (symG51)                # cdb (symG51.104,symG51)

# -----
symG42 := @(genG42) A^{b} A^{c1} A^{c2} A^{d}.      # cdb (symG42.100,symG42)

distribute          (symG42)                # cdb (symG42.101,symG42)
symG42 = product_sort (symG42)              # cdb (symG42.102,symG42)
rename_dummies      (symG42)                # cdb (symG42.103,symG42)
canonicalise        (symG42)                # cdb (symG42.104,symG42)

# -----
symG52 := @(genG52) A^{b} A^{c1} A^{c2} A^{d}.      # cdb (symG52.100,symG52)

distribute          (symG52)                # cdb (symG52.101,symG52)
symG52 = product_sort (symG52)              # cdb (symG52.102,symG52)
rename_dummies      (symG52)                # cdb (symG52.103,symG52)
canonicalise        (symG52)                # cdb (symG52.104,symG52)

# -----
symG53 := @(genG53) A^{b} A^{c1} A^{c2} A^{c3} A^{d}.  # cdb (symG53.100,symG53)

distribute          (symG53)                # cdb (symG53.101,symG53)
symG53 = product_sort (symG53)              # cdb (symG53.102,symG53)
rename_dummies      (symG53)                # cdb (symG53.103,symG53)
canonicalise        (symG53)                # cdb (symG53.104,symG53)

```

$$\begin{aligned}
\text{symG31.100} &:= \left( \frac{1}{12} \nabla_b R_{c_1 c d e} g^{a e} x^c + \frac{1}{12} \nabla_b R_{c_1 c d e} g^{a c} x^e + \frac{1}{12} \nabla_{c_1} R_{b c d e} g^{a e} x^c + \frac{1}{12} \nabla_{c_1} R_{b c d e} g^{a c} x^e + \frac{1}{12} \nabla_d R_{b c c_1 e} g^{a e} x^c + \frac{1}{12} \nabla_d R_{b c c_1 e} g^{a c} x^e \right) A^b A^{c_1} A^d \\
\text{symG31.101} &:= \frac{1}{12} \nabla_b R_{c_1 c d e} g^{a e} x^c A^b A^{c_1} A^d + \frac{1}{12} \nabla_b R_{c_1 c d e} g^{a c} x^e A^b A^{c_1} A^d + \frac{1}{12} \nabla_{c_1} R_{b c d e} g^{a e} x^c A^b A^{c_1} A^d \\
&\quad + \frac{1}{12} \nabla_{c_1} R_{b c d e} g^{a c} x^e A^b A^{c_1} A^d + \frac{1}{12} \nabla_d R_{b c c_1 e} g^{a e} x^c A^b A^{c_1} A^d + \frac{1}{12} \nabla_d R_{b c c_1 e} g^{a c} x^e A^b A^{c_1} A^d \\
\text{symG31.102} &:= \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_b R_{c e d f} + \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_b R_{c f d e} + \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_c R_{b e d f} \\
&\quad + \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_c R_{b f d e} + \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_d R_{b e c f} + \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_d R_{b f c e} \\
\text{symG31.103} &:= \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_b R_{c e d f} + \frac{1}{12} A^b A^c A^d x^f g^{a e} \nabla_b R_{c e d f} + \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_c R_{b e d f} \\
&\quad + \frac{1}{12} A^b A^c A^d x^f g^{a e} \nabla_c R_{b e d f} + \frac{1}{12} A^b A^c A^d x^e g^{a f} \nabla_d R_{b e c f} + \frac{1}{12} A^b A^c A^d x^f g^{a e} \nabla_d R_{b e c f} \\
\text{symG31.104} &:= \frac{1}{2} A^b A^c A^d x^e g^{a f} \nabla_b R_{c e d f}
\end{aligned}$$

$$\begin{aligned}
\text{symG41.100} := & \left( -\frac{4}{45} R_{bcc_1e} R_{dfgh} g^{af} g^{cg} x^e x^h - \frac{4}{45} R_{bcde} R_{c_1fgh} g^{af} g^{cg} x^e x^h - \frac{4}{45} R_{bcc_1e} R_{dfgh} g^{af} g^{eg} x^c x^h - \frac{4}{45} R_{bcef} R_{c_1gdh} g^{ac} g^{eg} x^f x^h \right. \\
& - \frac{4}{45} R_{bcde} R_{c_1fgh} g^{af} g^{eg} x^c x^h - \frac{4}{45} R_{bcef} R_{c_1gdh} g^{ac} g^{eh} x^f x^g - \frac{1}{45} R_{bcc_1e} R_{dfgh} g^{ag} g^{cf} x^e x^h - \frac{1}{45} R_{bcde} R_{c_1fgh} g^{ag} g^{cf} x^e x^h \\
& - \frac{1}{45} R_{bcc_1e} R_{dfgh} g^{ag} g^{ef} x^c x^h - \frac{1}{45} R_{bcef} R_{c_1gdh} g^{ae} g^{cg} x^f x^h - \frac{1}{45} R_{bcde} R_{c_1fgh} g^{ag} g^{ef} x^c x^h - \frac{1}{45} R_{bcef} R_{c_1gdh} g^{ae} g^{ch} x^f x^g \\
& + \frac{1}{45} R_{bcde} R_{c_1fgh} g^{ac} g^{eg} x^f x^h + \frac{1}{45} R_{bcc_1e} R_{dfgh} g^{ac} g^{eg} x^f x^h + \frac{1}{45} R_{bcef} R_{c_1gdh} g^{ag} g^{eh} x^c x^f + \frac{1}{45} R_{bcc_1e} R_{dfgh} g^{ae} g^{cg} x^f x^h \\
& + \frac{1}{45} R_{bcef} R_{c_1gdh} g^{ah} g^{eg} x^c x^f + \frac{1}{45} R_{bcde} R_{c_1fgh} g^{ae} g^{cg} x^f x^h - \frac{1}{60} \nabla_{bd} R_{c_1cef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{bc_1} R_{dcef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{c_1d} R_{bcef} g^{ae} x^c x^f \\
& - \frac{1}{60} \nabla_{c_1b} R_{dcef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{dc_1} R_{bcef} g^{ae} x^c x^f - \frac{1}{60} \nabla_{db} R_{c_1cef} g^{ae} x^c x^f + \frac{1}{40} \nabla_{bc} R_{c_1edf} g^{af} x^c x^e + \frac{1}{40} \nabla_{bc} R_{c_1edf} g^{ae} x^c x^f \\
& + \frac{1}{40} \nabla_{c_1c} R_{bedf} g^{af} x^c x^e + \frac{1}{40} \nabla_{c_1c} R_{bedf} g^{ae} x^c x^f + \frac{1}{40} \nabla_{dc} R_{bec_1f} g^{af} x^c x^e + \frac{1}{40} \nabla_{dc} R_{bec_1f} g^{ae} x^c x^f + \frac{1}{40} \nabla_{cb} R_{c_1edf} g^{af} x^c x^e \\
& + \frac{1}{40} \nabla_{cb} R_{c_1edf} g^{ae} x^c x^f + \frac{1}{40} \nabla_{cc_1} R_{bedf} g^{af} x^c x^e + \frac{1}{40} \nabla_{cc_1} R_{bedf} g^{ae} x^c x^f + \frac{1}{40} \nabla_{cd} R_{bec_1f} g^{af} x^c x^e + \frac{1}{40} \nabla_{cd} R_{bec_1f} g^{ae} x^c x^f \\
& + \frac{1}{15} R_{bcef} R_{c_1gdh} g^{ae} g^{fh} x^c x^g + \frac{1}{15} R_{bcef} R_{c_1gdh} g^{ae} g^{fg} x^c x^h + \frac{1}{15} R_{bcde} R_{c_1fgh} g^{ag} g^{eh} x^c x^f + \frac{1}{15} R_{bcde} R_{c_1fgh} g^{ag} g^{ch} x^e x^f \\
& + \frac{1}{15} R_{bcc_1e} R_{dfgh} g^{ag} g^{eh} x^c x^f + \frac{1}{15} R_{bcc_1e} R_{dfgh} g^{ag} g^{ch} x^e x^f + \frac{1}{120} \nabla_{cd} R_{bec_1f} g^{ac} x^e x^f + \frac{1}{120} \nabla_{cc_1} R_{bedf} g^{ac} x^e x^f + \frac{1}{120} \nabla_{cb} R_{c_1edf} g^{ac} x^e x^f \\
& \left. + \frac{1}{120} \nabla_{dc} R_{bec_1f} g^{ac} x^e x^f + \frac{1}{120} \nabla_{c_1c} R_{bedf} g^{ac} x^e x^f + \frac{1}{120} \nabla_{bc} R_{c_1edf} g^{ac} x^e x^f \right) A^b A^{c_1} A^d
\end{aligned}$$

$$\begin{aligned}
\text{symG41.101} := & -\frac{4}{45}R_{bcc_1e}R_{dfgh}g^{af}g^{cg}x^ex^hA^bA^{c_1}A^d - \frac{4}{45}R_{bcde}R_{c_1fgh}g^{af}g^{cg}x^ex^hA^bA^{c_1}A^d - \frac{4}{45}R_{bcc_1e}R_{dfgh}g^{af}g^{eg}x^cx^hA^bA^{c_1}A^d \\
& - \frac{4}{45}R_{bcef}R_{c_1gdh}g^{ac}g^{eg}x^fx^hA^bA^{c_1}A^d - \frac{4}{45}R_{bcde}R_{c_1fgh}g^{af}g^{eg}x^cx^hA^bA^{c_1}A^d - \frac{4}{45}R_{bcef}R_{c_1gdh}g^{ac}g^{eh}x^fx^gA^bA^{c_1}A^d \\
& - \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ag}g^{cf}x^ex^hA^bA^{c_1}A^d - \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ag}g^{cf}x^ex^hA^bA^{c_1}A^d - \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ag}g^{ef}x^cx^hA^bA^{c_1}A^d \\
& - \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ae}g^{cg}x^fx^hA^bA^{c_1}A^d - \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ag}g^{ef}x^cx^hA^bA^{c_1}A^d - \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ae}g^{ch}x^fx^gA^bA^{c_1}A^d \\
& + \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ac}g^{eg}x^fx^hA^bA^{c_1}A^d + \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ac}g^{eg}x^fx^hA^bA^{c_1}A^d + \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ag}g^{eh}x^cx^fA^bA^{c_1}A^d \\
& + \frac{1}{45}R_{bcc_1e}R_{dfgh}g^{ae}g^{cg}x^fx^hA^bA^{c_1}A^d + \frac{1}{45}R_{bcef}R_{c_1gdh}g^{ah}g^{eg}x^cx^fA^bA^{c_1}A^d + \frac{1}{45}R_{bcde}R_{c_1fgh}g^{ae}g^{cg}x^fx^hA^bA^{c_1}A^d \\
& - \frac{1}{60}\nabla_{bd}R_{c_1cef}g^{ae}x^cx^fA^bA^{c_1}A^d - \frac{1}{60}\nabla_{bc_1}R_{dcef}g^{ae}x^cx^fA^bA^{c_1}A^d - \frac{1}{60}\nabla_{c_1d}R_{bcef}g^{ae}x^cx^fA^bA^{c_1}A^d \\
& - \frac{1}{60}\nabla_{c_1b}R_{dcef}g^{ae}x^cx^fA^bA^{c_1}A^d - \frac{1}{60}\nabla_{dc_1}R_{bcef}g^{ae}x^cx^fA^bA^{c_1}A^d - \frac{1}{60}\nabla_{db}R_{c_1cef}g^{ae}x^cx^fA^bA^{c_1}A^d \\
& + \frac{1}{40}\nabla_{bc}R_{c_1edf}g^{af}x^cx^eA^bA^{c_1}A^d + \frac{1}{40}\nabla_{bc}R_{c_1edf}g^{ae}x^cx^fA^bA^{c_1}A^d + \frac{1}{40}\nabla_{c_1c}R_{bedf}g^{af}x^cx^eA^bA^{c_1}A^d + \frac{1}{40}\nabla_{c_1c}R_{bedf}g^{ae}x^cx^fA^bA^{c_1}A^d \\
& + \frac{1}{40}\nabla_{dc}R_{bec_1f}g^{af}x^cx^eA^bA^{c_1}A^d + \frac{1}{40}\nabla_{dc}R_{bec_1f}g^{ae}x^cx^fA^bA^{c_1}A^d + \frac{1}{40}\nabla_{cb}R_{c_1edf}g^{af}x^cx^eA^bA^{c_1}A^d + \frac{1}{40}\nabla_{cb}R_{c_1edf}g^{ae}x^cx^fA^bA^{c_1}A^d \\
& + \frac{1}{40}\nabla_{cc_1}R_{bedf}g^{af}x^cx^eA^bA^{c_1}A^d + \frac{1}{40}\nabla_{cc_1}R_{bedf}g^{ae}x^cx^fA^bA^{c_1}A^d + \frac{1}{40}\nabla_{cd}R_{bec_1f}g^{af}x^cx^eA^bA^{c_1}A^d + \frac{1}{40}\nabla_{cd}R_{bec_1f}g^{ae}x^cx^fA^bA^{c_1}A^d \\
& + \frac{1}{15}R_{bcef}R_{c_1gdh}g^{ae}g^{fh}x^cx^gA^bA^{c_1}A^d + \frac{1}{15}R_{bcef}R_{c_1gdh}g^{ae}g^{fg}x^cx^hA^bA^{c_1}A^d + \frac{1}{15}R_{bcde}R_{c_1fgh}g^{ag}g^{eh}x^cx^fA^bA^{c_1}A^d \\
& + \frac{1}{15}R_{bcde}R_{c_1fgh}g^{ag}g^{ch}x^ex^fA^bA^{c_1}A^d + \frac{1}{15}R_{bcc_1e}R_{dfgh}g^{ag}g^{eh}x^cx^fA^bA^{c_1}A^d + \frac{1}{15}R_{bcc_1e}R_{dfgh}g^{ag}g^{ch}x^ex^fA^bA^{c_1}A^d \\
& + \frac{1}{120}\nabla_{cd}R_{bec_1f}g^{ac}x^ex^fA^bA^{c_1}A^d + \frac{1}{120}\nabla_{cc_1}R_{bedf}g^{ac}x^ex^fA^bA^{c_1}A^d + \frac{1}{120}\nabla_{cb}R_{c_1edf}g^{ac}x^ex^fA^bA^{c_1}A^d \\
& + \frac{1}{120}\nabla_{dc}R_{bec_1f}g^{ac}x^ex^fA^bA^{c_1}A^d + \frac{1}{120}\nabla_{c_1c}R_{bedf}g^{ac}x^ex^fA^bA^{c_1}A^d + \frac{1}{120}\nabla_{bc}R_{c_1edf}g^{ac}x^ex^fA^bA^{c_1}A^d
\end{aligned}$$

$$\begin{aligned}
\text{symG41.102} := & -\frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhce}R_{dgif} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhde}R_{cgif} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bech}R_{dgif} \\
& - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bghe}R_{cidf} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bedh}R_{cgif} - \frac{4}{45}A^bA^cA^dA^eA^fA^gA^hR_{bghe}R_{cfdi} \\
& - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhce}R_{digf} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhde}R_{cgif} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bech}R_{digf} \\
& - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhge}R_{cidf} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bedh}R_{cgif} - \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhge}R_{cfdi} \\
& + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bgdh}R_{ceif} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bgch}R_{deif} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{behf}R_{cgdi} \\
& + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhcg}R_{deif} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{behf}R_{cidg} + \frac{1}{45}A^bA^cA^dA^eA^fA^gA^hR_{bhdg}R_{ceif} \\
& - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{bd}R_{ceg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{bc}R_{deg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{cd}R_{beg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{cb}R_{deg f} \\
& - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{dc}R_{beg f} - \frac{1}{60}A^bA^cA^dA^eA^fA^g\nabla_{db}R_{ceg f} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{be}R_{cf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{be}R_{cg df} \\
& + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ce}R_{bf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ce}R_{bg df} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{de}R_{bf cg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{de}R_{bg cf} \\
& + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{eb}R_{cf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{eb}R_{cg df} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ec}R_{bf dg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ec}R_{bg df} \\
& + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ed}R_{bf cg} + \frac{1}{40}A^bA^cA^dA^eA^fA^g\nabla_{ed}R_{bg cf} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{begh}R_{cf di} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{begh}R_{cidf} \\
& + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bedh}R_{cf gi} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bhde}R_{cf gi} + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bech}R_{df gi} \\
& + \frac{1}{15}A^bA^cA^dA^eA^fA^gA^hR_{bhce}R_{df gi} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{gd}R_{becf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{gc}R_{bedf} \\
& + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{gb}R_{cedf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{dg}R_{becf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{cg}R_{bedf} + \frac{1}{120}A^bA^cA^dA^eA^fA^g\nabla_{bg}R_{cedf}
\end{aligned}$$



$$\begin{aligned}
\text{symG41.103} := & -\frac{4}{45}A^bA^cA^dA^fx^ig^{ag}g^{eh}R_{becf}R_{dghi} - \frac{4}{45}A^bA^cA^dA^fx^ig^{ag}g^{eh}R_{bedf}R_{cghi} - \frac{4}{45}A^bA^cA^dA^ex^ig^{ag}g^{fh}R_{becf}R_{dghi} \\
& - \frac{4}{45}A^bA^cA^dA^gx^ig^{ae}g^{fh}R_{befg}R_{chdi} - \frac{4}{45}A^bA^cA^dA^ex^ig^{ag}g^{fh}R_{bedf}R_{cghi} - \frac{4}{45}A^bA^cA^dA^gx^hg^{ae}g^{fi}R_{befg}R_{chdi} \\
& - \frac{1}{45}A^bA^cA^dA^fx^ig^{ah}g^{eg}R_{becf}R_{dghi} - \frac{1}{45}A^bA^cA^dA^fx^ig^{ah}g^{eg}R_{bedf}R_{cghi} - \frac{1}{45}A^bA^cA^dA^ex^ig^{ah}g^{fg}R_{becf}R_{dghi} \\
& - \frac{1}{45}A^bA^cA^dA^gx^ig^{af}g^{eh}R_{befg}R_{chdi} - \frac{1}{45}A^bA^cA^dA^ex^ig^{ah}g^{fg}R_{bedf}R_{cghi} - \frac{1}{45}A^bA^cA^dA^gx^hg^{af}g^{ei}R_{befg}R_{chdi} \\
& + \frac{1}{45}A^bA^cA^dA^gx^ig^{ae}g^{fh}R_{bedf}R_{cghi} + \frac{1}{45}A^bA^cA^dA^gx^ig^{ae}g^{fh}R_{becf}R_{dghi} + \frac{1}{45}A^bA^cA^dA^ex^ig^{ah}g^{fi}R_{befg}R_{chdi} \\
& + \frac{1}{45}A^bA^cA^dA^gx^ig^{af}g^{eh}R_{becf}R_{dghi} + \frac{1}{45}A^bA^cA^dA^ex^ig^{ah}g^{fi}R_{befg}R_{chdi} + \frac{1}{45}A^bA^cA^dA^gx^ig^{af}g^{eh}R_{bedf}R_{cghi} \\
& - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{bd}R_{cefg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{bc}R_{defg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{cd}R_{befg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{cb}R_{defg} \\
& - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{dc}R_{befg} - \frac{1}{60}A^bA^cA^dA^ex^ig^{af}\nabla_{db}R_{cefg} + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{bg}R_{cedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{bg}R_{cedf} \\
& + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{cg}R_{bedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{cg}R_{bedf} + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{dg}R_{becf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{dg}R_{becf} \\
& + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{gc}R_{bedf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{gc}R_{bedf} \\
& + \frac{1}{40}A^bA^cA^dA^gx^eg^{af}\nabla_{gd}R_{becf} + \frac{1}{40}A^bA^cA^dA^gx^fg^{ae}\nabla_{gd}R_{becf} + \frac{1}{15}A^bA^cA^dA^ex^hg^{af}g^{gi}R_{befg}R_{chdi} + \frac{1}{15}A^bA^cA^dA^ex^ig^{af}g^{gh}R_{befg}R_{chdi} \\
& + \frac{1}{15}A^bA^cA^dA^ex^ig^{ah}g^{fi}R_{bedf}R_{cghi} + \frac{1}{15}A^bA^cA^dA^fx^hg^{ah}g^{ei}R_{bedf}R_{cghi} + \frac{1}{15}A^bA^cA^dA^ex^ig^{ah}g^{fi}R_{becf}R_{dghi} \\
& + \frac{1}{15}A^bA^cA^dA^fx^hg^{ah}g^{ei}R_{becf}R_{dghi} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{gd}R_{becf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{gc}R_{bedf} \\
& + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{gb}R_{cedf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{dg}R_{becf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{cg}R_{bedf} + \frac{1}{120}A^bA^cA^dA^ex^fg^{ag}\nabla_{bg}R_{cedf} \\
\text{symG41.104} := & \frac{8}{15}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{bech}R_{dghi} + \frac{2}{15}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{bech}R_{difg} - \frac{2}{15}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{befh}R_{cgdi} \\
& + \frac{1}{10}A^bA^cA^dA^ex^fg^{ag}\nabla_{bc}R_{defg} + \frac{3}{20}A^bA^cA^dA^ex^fg^{ag}\nabla_{be}R_{cfdg} + \frac{3}{20}A^bA^cA^dA^ex^fg^{ag}\nabla_{eb}R_{cfdg} \\
& + \frac{2}{5}A^bA^cA^dA^ex^fg^{ag}g^{hi}R_{bech}R_{dfgi} + \frac{1}{40}A^bA^cA^dA^ex^fg^{ag}\nabla_{gb}R_{cedf} + \frac{1}{40}A^bA^cA^dA^ex^fg^{ag}\nabla_{bg}R_{cedf}
\end{aligned}$$

$$\begin{aligned}
\text{symG51.104} := & \frac{8}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + \frac{4}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} \\
& + \frac{1}{10} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} + \frac{1}{90} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + \frac{11}{90} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} \\
& + \frac{4}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\
& + \frac{1}{12} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} + \frac{1}{36} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\
& - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - \frac{2}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} \\
& - \frac{1}{15} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bce} R_{dfgh} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bec} R_{dfgh} \\
& + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bef} R_{cgdh} + \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebc} R_{dfgh} + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebf} R_{cgdh} \\
& + \frac{1}{30} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{efb} R_{cgdh} + \frac{4}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} + \frac{1}{5} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\
& + \frac{4}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + \frac{1}{5} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} \\
& - \frac{1}{45} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{hbe} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{heb} R_{cfdg} \\
& + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{bhe} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ehb} R_{cfdg} + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{beh} R_{cfdg} \\
& + \frac{1}{180} A^b A^c A^d x^e x^f x^g g^{ah} \nabla_{ebh} R_{cfdg} - \frac{1}{9} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} + \frac{1}{18} A^b A^c A^d x^e x^f x^g g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}
\end{aligned}$$

$$\text{symG42.104} := \frac{8}{15} A^b A^c A^d A^e x^f g^{ag} g^{hi} R_{bfch} R_{dgei} + \frac{2}{5} A^b A^c A^d A^e x^f g^{ag} \nabla_{bc} R_{dfeg}$$

$$\begin{aligned} \text{symG52.104} := & \frac{32}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + \frac{1}{5} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + \frac{4}{15} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} \\ & + \frac{2}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} + \frac{22}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} \\ & + \frac{1}{5} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + \frac{4}{15} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + \frac{1}{9} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} \\ & - \frac{8}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} + \frac{1}{15} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bcd} R_{efgh} + \frac{4}{45} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bcf} R_{dgeh} \\ & + \frac{4}{45} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bfc} R_{dgeh} + \frac{4}{45} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{fbc} R_{dgeh} + \frac{13}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} \\ & + \frac{1}{15} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgej} + \frac{23}{45} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + \frac{1}{90} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{hbc} R_{dfeg} \\ & + \frac{1}{90} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bhc} R_{dfeg} + \frac{1}{90} A^b A^c A^d A^e x^f x^g g^{ah} \nabla_{bch} R_{dfeg} - \frac{4}{9} A^b A^c A^d A^e x^f x^g g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh} \end{aligned}$$

$$\text{symG53.104} := A^b A^c A^d A^e A^f x^g g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + A^b A^c A^d A^e A^f x^g g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + \frac{1}{3} A^b A^c A^d A^e A^f x^g g^{ah} \nabla_{bcd} R_{egfh}$$

```

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    factor_out (bah,$A^{a?},x^{b?}$)
    ans := @(bah) / @(foo).
    return ans

fooG20 = reformat (symG20,3)
fooG30 = reformat (symG30,12)
fooG40 = reformat (symG40,360)
fooG50 = reformat (symG50,180)

fooG31 = reformat (symG31,2)
fooG41 = reformat (symG41,120)
fooG51 = reformat (symG51,180)

fooG42 = reformat (symG42,15)
fooG52 = reformat (symG52,90)

fooG53 = reformat (symG53,3)

genGamma0 := @(fooG20) + @(fooG30) + @(fooG40) + @(fooG50). # cdb (genGamma0.000,genGamma0)
genGamma1 := @(fooG31) + @(fooG41) + @(fooG51). # cdb (genGamma1.000,genGamma1)
genGamma2 := @(fooG42) + @(fooG52). # cdb (genGamma2.000,genGamma2)
genGamma3 := @(fooG53). # cdb (genGamma3.000,genGamma3)

cdblib.create ('genGamma.json')

cdblib.put ('genGamma0',genGamma0,'genGamma.json')
cdblib.put ('genGamma1',genGamma1,'genGamma.json')
cdblib.put ('genGamma2',genGamma2,'genGamma.json')
cdblib.put ('genGamma3',genGamma3,'genGamma.json')

cdblib.put ('genGamma01',fooG20,'genGamma.json')
cdblib.put ('genGamma02',fooG30,'genGamma.json')
cdblib.put ('genGamma03',fooG40,'genGamma.json')
cdblib.put ('genGamma04',fooG50,'genGamma.json')

```

```
cdblib.put ('genGamma11',fooG31,'genGamma.json')
cdblib.put ('genGamma12',fooG41,'genGamma.json')
cdblib.put ('genGamma13',fooG51,'genGamma.json')

cdblib.put ('genGamma21',fooG42,'genGamma.json')
cdblib.put ('genGamma22',fooG52,'genGamma.json')

cdblib.put ('genGamma31',fooG53,'genGamma.json')
```

# The generalised connection in Riemann normal coordinates

$$\begin{aligned}
A^b A^c \Gamma_{bc}^a(x) &= \frac{2}{3} A^b A^c x^d g^{ae} R_{bdce} + \frac{1}{12} A^b A^c x^d x^e (2g^{af} \nabla_b R_{cdef} + 4g^{af} \nabla_d R_{becf} + g^{af} \nabla_f R_{bdce}) \\
&\quad + \frac{1}{360} A^b A^c x^d x^e x^f (64g^{ag} g^{hi} R_{bdch} R_{egfi} - 32g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag} g^{hi} R_{bdeh} R_{cifg} + 18g^{ag} \nabla_{bd} R_{cefg} + 18g^{ag} \nabla_{db} R_{cefg} + 36g^{ag} \nabla_{de} R_{bfcg} \\
&\quad - 16g^{ag} g^{hi} R_{bdeh} R_{cfig} + 9g^{ag} \nabla_{gd} R_{becf} + 9g^{ag} \nabla_{dg} R_{becf}) + \frac{1}{180} A^b A^c x^d x^e x^f x^g (16g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} \\
&\quad + 16g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 8g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} \\
&\quad - 8g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 2g^{ah} \nabla_{bde} R_{cfgh} + 2g^{ah} \nabla_{dbe} R_{cfgh} + 2g^{ah} \nabla_{deb} R_{cfgh} + 4g^{ah} \nabla_{def} R_{bgch} \\
&\quad - 4g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} - 4g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} + g^{ah} \nabla_{hde} R_{bfcg} + g^{ah} \nabla_{dhe} R_{bfcg} + g^{ah} \nabla_{deh} R_{bfcg}) \\
A^b A^c A^d \Gamma_{bcd}^a(x) &= \frac{1}{2} A^b A^c A^d x^e g^{af} \nabla_b R_{cedf} + \frac{1}{120} A^b A^c A^d x^e x^f (64g^{ag} g^{hi} R_{bech} R_{dgfi} + 16g^{ag} g^{hi} R_{bech} R_{difg} - 16g^{ag} g^{hi} R_{befh} R_{cgdi} + 12g^{ag} \nabla_{bc} R_{defg} \\
&\quad + 18g^{ag} \nabla_{be} R_{cfdg} + 18g^{ag} \nabla_{eb} R_{cfdg} + 48g^{ag} g^{hi} R_{bech} R_{dfgi} + 3g^{ag} \nabla_{gb} R_{cedf} + 3g^{ag} \nabla_{bg} R_{cedf}) \\
&\quad + \frac{1}{180} A^b A^c A^d x^e x^f x^g (32g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} + 48g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} + 12g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} + 18g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
&\quad + 2g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 22g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} \\
&\quad + 5g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} - 12g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} - 12g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} - 8g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} - 12g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} \\
&\quad + 4g^{ah} \nabla_{bce} R_{dfgh} + 4g^{ah} \nabla_{bec} R_{dfgh} + 6g^{ah} \nabla_{bef} R_{cgdh} + 4g^{ah} \nabla_{ebc} R_{dfgh} + 6g^{ah} \nabla_{ebf} R_{cgdh} + 6g^{ah} \nabla_{efb} R_{cgdh} + 16g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} \\
&\quad + 36g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} + 16g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 4g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 36g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - 4g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} \\
&\quad + g^{ah} \nabla_{hbe} R_{cfdg} + g^{ah} \nabla_{heb} R_{cfdg} + g^{ah} \nabla_{bhe} R_{cfdg} + g^{ah} \nabla_{ehb} R_{cfdg} + g^{ah} \nabla_{beh} R_{cfdg} + g^{ah} \nabla_{ebh} R_{cfdg} - 20g^{ah} g^{ij} R_{beci} \nabla_j R_{dfgh} \\
&\quad + 10g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}) \\
A^b A^c A^d A^e \Gamma_{bcde}^a(x) &= \frac{1}{15} A^b A^c A^d A^e x^f (8g^{ag} g^{hi} R_{bfch} R_{dgei} + 6g^{ag} \nabla_{bc} R_{dfeg}) \\
&\quad + \frac{1}{90} A^b A^c A^d A^e x^f x^g (64g^{ah} g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah} g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24g^{ah} g^{ij} R_{bfci} \nabla_g R_{dhej} + 4g^{ah} g^{ij} R_{bhci} \nabla_d R_{efgj} \\
&\quad + 44g^{ah} g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah} g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah} g^{ij} R_{bhci} \nabla_f R_{dgej} + 10g^{ah} g^{ij} R_{bhci} \nabla_j R_{dfeg} - 16g^{ah} g^{ij} R_{bfgi} \nabla_c R_{dhej} \\
&\quad + 6g^{ah} \nabla_{bcd} R_{efgh} + 8g^{ah} \nabla_{bcf} R_{dgeh} + 8g^{ah} \nabla_{bfc} R_{dgeh} + 8g^{ah} \nabla_{fbc} R_{dgeh} + 26g^{ah} g^{ij} R_{bfhi} \nabla_c R_{dgej} + 6g^{ah} g^{ij} R_{bfci} \nabla_h R_{dgeh} \\
&\quad + 46g^{ah} g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah} \nabla_{hbc} R_{dfeg} + g^{ah} \nabla_{bhc} R_{dfeg} + g^{ah} \nabla_{bch} R_{dfeg} - 40g^{ah} g^{ij} R_{bfci} \nabla_j R_{dgeh}) \\
A^b A^c A^d A^e A^f \Gamma_{bcdef}^a(x) &= \frac{1}{3} A^b A^c A^d A^e A^f x^g (3g^{ah} g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3g^{ah} g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah} \nabla_{bcd} R_{egfh})
\end{aligned}$$

```
scaledGamma0 := 360 @(genGamma0). # cdb (scaledGamma0.001,scaledGamma0)
scaledGamma1 := 360 @(genGamma1). # cdb (scaledGamma1.001,scaledGamma1)
scaledGamma2 := 90  @(genGamma2). # cdb (scaledGamma2.001,scaledGamma2)
scaledGamma3 := 3   @(genGamma3). # cdb (scaledGamma3.001,scaledGamma3)
```



# The generalised connection in Riemann normal coordinates

This is the same as the previous page but with a small change in the format to avoid fractions.

$$\begin{aligned}
360A^bA^c\Gamma_{bc}^a(x) = & 240A^bA^cx^d g^{ae} R_{bdce} + 30A^bA^cx^d x^e (2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}) \\
& + A^bA^cx^d x^e x^f (64g^{ag}g^{hi} R_{bdch} R_{egfi} - 32g^{ag}g^{hi} R_{bdeh} R_{cgfi} - 16g^{ag}g^{hi} R_{bdeh} R_{cifg} + 18g^{ag}\nabla_{bd} R_{cefg} \\
& + 18g^{ag}\nabla_{db} R_{cefg} + 36g^{ag}\nabla_{de} R_{bfeg} - 16g^{ag}g^{hi} R_{bdeh} R_{cfig} + 9g^{ag}\nabla_{gd} R_{becf} + 9g^{ag}\nabla_{dg} R_{becf}) \\
& + 2A^bA^cx^d x^e x^f x^g (16g^{ah}g^{ij} R_{bdci} \nabla_e R_{fhgj} + 6g^{ah}g^{ij} R_{dhei} \nabla_b R_{cfig} + 16g^{ah}g^{ij} R_{dhei} \nabla_f R_{bgcj} + 5g^{ah}g^{ij} R_{dhei} \nabla_j R_{bfeg} \\
& - 8g^{ah}g^{ij} R_{bhdi} \nabla_e R_{cfig} - 4g^{ah}g^{ij} R_{bidh} \nabla_e R_{cfig} - 4g^{ah}g^{ij} R_{bdei} \nabla_c R_{fhgj} - 8g^{ah}g^{ij} R_{bdei} \nabla_f R_{chgj} - 4g^{ah}g^{ij} R_{bdei} \nabla_f R_{cjgh} \\
& + 2g^{ah}\nabla_{bde} R_{cfig} + 2g^{ah}\nabla_{dbe} R_{cfig} + 2g^{ah}\nabla_{deb} R_{cfig} + 4g^{ah}\nabla_{def} R_{bgch} - 4g^{ah}g^{ij} R_{bdhi} \nabla_e R_{cfig} - 4g^{ah}g^{ij} R_{bdei} \nabla_h R_{cfig} \\
& - 4g^{ah}g^{ij} R_{bdei} \nabla_f R_{chgj} + g^{ah}\nabla_{hde} R_{bfeg} + g^{ah}\nabla_{dhe} R_{bfeg} + g^{ah}\nabla_{deh} R_{bfeg})
\end{aligned}$$

$$\begin{aligned}
360A^bA^cA^d\Gamma_{bcd}^a(x) = & 180A^bA^cA^d x^e g^{af}\nabla_b R_{cedf} + 3A^bA^cA^d x^e x^f (64g^{ag}g^{hi} R_{bech} R_{dghi} + 16g^{ag}g^{hi} R_{bech} R_{difg} - 16g^{ag}g^{hi} R_{befh} R_{cgdi} \\
& + 12g^{ag}\nabla_{bc} R_{defg} + 18g^{ag}\nabla_{be} R_{cfdg} + 18g^{ag}\nabla_{eb} R_{cfdg} + 48g^{ag}g^{hi} R_{bech} R_{dfgi} + 3g^{ag}\nabla_{gb} R_{cedf} + 3g^{ag}\nabla_{bg} R_{cedf}) \\
& + 2A^bA^cA^d x^e x^f x^g (32g^{ah}g^{ij} R_{beci} \nabla_d R_{fhgj} + 48g^{ah}g^{ij} R_{beci} \nabla_f R_{dhgj} + 12g^{ah}g^{ij} R_{beci} \nabla_f R_{djgh} + 18g^{ah}g^{ij} R_{bhei} \nabla_c R_{dfgj} \\
& + 2g^{ah}g^{ij} R_{bieh} \nabla_c R_{dfgj} + 22g^{ah}g^{ij} R_{ehfi} \nabla_b R_{cgdj} + 48g^{ah}g^{ij} R_{bhei} \nabla_f R_{cgdj} + 12g^{ah}g^{ij} R_{bieh} \nabla_f R_{cgdj} + 15g^{ah}g^{ij} R_{bhei} \nabla_j R_{cfdg} \\
& + 5g^{ah}g^{ij} R_{bieh} \nabla_j R_{cfdg} - 12g^{ah}g^{ij} R_{bhci} \nabla_e R_{dfgj} - 12g^{ah}g^{ij} R_{befi} \nabla_c R_{dhgj} - 8g^{ah}g^{ij} R_{befi} \nabla_c R_{djgh} - 12g^{ah}g^{ij} R_{befi} \nabla_g R_{chdj} \\
& + 4g^{ah}\nabla_{bce} R_{dfgh} + 4g^{ah}\nabla_{bec} R_{dfgh} + 6g^{ah}\nabla_{bef} R_{cgdh} + 4g^{ah}\nabla_{ebc} R_{dfgh} + 6g^{ah}\nabla_{ebf} R_{cgdh} + 6g^{ah}\nabla_{efb} R_{cgdh} \\
& + 16g^{ah}g^{ij} R_{behi} \nabla_c R_{dfgj} + 36g^{ah}g^{ij} R_{behi} \nabla_f R_{cgdj} + 16g^{ah}g^{ij} R_{beci} \nabla_h R_{dfgj} - 4g^{ah}g^{ij} R_{befi} \nabla_h R_{cgdj} + 36g^{ah}g^{ij} R_{beci} \nabla_f R_{dghj} \\
& - 4g^{ah}g^{ij} R_{befi} \nabla_c R_{dghj} + g^{ah}\nabla_{hbe} R_{cfdg} + g^{ah}\nabla_{heb} R_{cfdg} + g^{ah}\nabla_{bhe} R_{cfdg} + g^{ah}\nabla_{ehb} R_{cfdg} + g^{ah}\nabla_{beh} R_{cfdg} + g^{ah}\nabla_{ebh} R_{cfdg} \\
& - 20g^{ah}g^{ij} R_{beci} \nabla_j R_{dfgh} + 10g^{ah}g^{ij} R_{behi} \nabla_j R_{cfdg})
\end{aligned}$$

$$\begin{aligned}
90A^bA^cA^dA^e\Gamma_{bcde}^a(x) = & 6A^bA^cA^dA^e x^f (8g^{ag}g^{hi} R_{bfch} R_{dgei} + 6g^{ag}\nabla_{bc} R_{dfeg}) \\
& + A^bA^cA^dA^e x^f x^g (64g^{ah}g^{ij} R_{bfci} \nabla_d R_{ehgj} + 18g^{ah}g^{ij} R_{bfci} \nabla_d R_{ejgh} + 24g^{ah}g^{ij} R_{bfci} \nabla_g R_{dhej} + 4g^{ah}g^{ij} R_{bhci} \nabla_d R_{efgj} \\
& + 44g^{ah}g^{ij} R_{bhfi} \nabla_c R_{dgej} + 18g^{ah}g^{ij} R_{bifh} \nabla_c R_{dgej} + 24g^{ah}g^{ij} R_{bhci} \nabla_f R_{dgej} + 10g^{ah}g^{ij} R_{bhci} \nabla_j R_{dfeg} \\
& - 16g^{ah}g^{ij} R_{bfgi} \nabla_c R_{dhej} + 6g^{ah}\nabla_{bcd} R_{efgh} + 8g^{ah}\nabla_{bcf} R_{dgeh} + 8g^{ah}\nabla_{bfc} R_{dgeh} + 8g^{ah}\nabla_{fbc} R_{dgeh} + 26g^{ah}g^{ij} R_{bfhi} \nabla_c R_{dgej} \\
& + 6g^{ah}g^{ij} R_{bfci} \nabla_h R_{dgej} + 46g^{ah}g^{ij} R_{bfci} \nabla_d R_{eghj} + g^{ah}\nabla_{hbc} R_{dfeg} + g^{ah}\nabla_{bhc} R_{dfeg} + g^{ah}\nabla_{bch} R_{dfeg} - 40g^{ah}g^{ij} R_{bfci} \nabla_j R_{dgeh})
\end{aligned}$$

$$3A^bA^cA^dA^eA^f\Gamma_{bcdef}^a(x) = A^bA^cA^dA^eA^f x^g (3g^{ah}g^{ij} R_{bgci} \nabla_d R_{ehfj} + 3g^{ah}g^{ij} R_{bhci} \nabla_d R_{egfj} + g^{ah}\nabla_{bcd} R_{egfh})$$

```

deriv01:=B^{a}:

deriv02:=-\Gamma^{a}_{[b c]} B^{b} B^{c}:          # cdb (deriv02.100,deriv02)

deriv03:=\nabla{@(deriv02)}.          # cdb (deriv03.100,deriv03)
distribute      (deriv03)
product_rule     (deriv03)          # cdb (deriv03.101,deriv03)
substitute       (deriv03,$\nabla{B^{a}}->{@(deriv02)}) # cdb (deriv03.102,deriv03)
substitute       (deriv03,$\nabla{\Gamma^{m}_{[s t]}}->B^{d}\partial_{d}\{\Gamma^{m}_{[s t]}\}) # cdb (deriv03.103,deriv03)
sort_product     (deriv03)          # cdb (deriv03.104,deriv03)
rename_dummies   (deriv03)          # cdb (deriv03.105,deriv03)
canonicalise     (deriv03)          # cdb (deriv03.106,deriv03)

deriv04:=\nabla{@(deriv03)}.          # cdb (deriv04.100,deriv04)
distribute      (deriv04)
product_rule     (deriv04)          # cdb (deriv04.101,deriv04)
substitute       (deriv04,$\nabla{B^{a}}->{@(deriv02)}) # cdb (deriv04.102,deriv04)
substitute       (deriv04,$\nabla{\Gamma^{m}_{[s t]}}->B^{d}\partial_{d}\{\Gamma^{m}_{[s t]}\}) # cdb (deriv04.103,deriv04)
substitute       (deriv04,$\nabla{\partial_{[e}\{\Gamma^{m}_{[s t]}\}}->B^{d}\partial_{d e}\{\Gamma^{m}_{[s t]}\}) # cdb (deriv04.104,deriv04)
sort_product     (deriv04)          # cdb (deriv04.105,deriv04)
rename_dummies   (deriv04)          # cdb (deriv04.106,deriv04)
canonicalise     (deriv04)          # cdb (deriv04.107,deriv04)

pderiv02 := -{@(deriv02)}.          # cdb (pderiv02.100,pderiv02)
factor_out      (pderiv02, $B^{a?}$) # cdb (pderiv02.101,pderiv02)
substitute      (pderiv02, $B^{a} -> 1$) # cdb (pderiv02.102,pderiv02)

pderiv03 := -{@(deriv03)}.          # cdb (pderiv03.100,pderiv03)
factor_out      (pderiv03, $B^{a?}$) # cdb (pderiv03.101,pderiv03)
substitute      (pderiv03, $B^{a} -> 1$) # cdb (pderiv03.102,pderiv03)

pderiv04 := -{@(deriv04)}.          # cdb (pderiv04.100,pderiv04)
factor_out      (pderiv04, $B^{a?}$) # cdb (pderiv04.101,pderiv04)
substitute      (pderiv04, $B^{a} -> 1$) # cdb (pderiv04.102,pderiv04)

```

## The generalised connection in generic coordinates (for the paper section 7)

$$\Gamma_{(bc)}^a(x) = \Gamma_{bc}^a \tag{pderiv02.102}$$

$$\Gamma_{(bcd)}^a(x) = \partial_b \Gamma_{cd}^a - 2\Gamma_{be}^a \Gamma_{cd}^e \tag{pderiv03.102}$$

$$\Gamma_{(bcde)}^a(x) = -\Gamma_{bc}^f \partial_f \Gamma_{de}^a - 4\Gamma_{bc}^f \partial_d \Gamma_{ef}^a + \partial_{bc} \Gamma_{de}^a + 2\Gamma_{fg}^a \Gamma_{bc}^f \Gamma_{de}^g + 4\Gamma_{bf}^a \Gamma_{cg}^f \Gamma_{de}^g - 2\Gamma_{bf}^a \partial_c \Gamma_{de}^f \tag{pderiv04.102}$$

```

tmp0 := @(fooG20) + @(fooG30).
tmp1 := @(fooG31).

alt0 := @(genGamma0).
alt1 := @(genGamma1).
alt2 := @(genGamma2).
alt3 := @(genGamma3).

alt0scaled := @(scaledGamma0).
alt1scaled := @(scaledGamma1).
alt2scaled := @(scaledGamma2).
alt3scaled := @(scaledGamma3).

substitute (tmp0, $A^{a}->1$)
substitute (tmp1, $A^{a}->1$)

substitute (alt0, $A^{a}->1$)
substitute (alt1, $A^{a}->1$)
substitute (alt2, $A^{a}->1$)
substitute (alt3, $A^{a}->1$)

substitute (alt0scaled, $A^{a}->1$)
substitute (alt1scaled, $A^{a}->1$)
substitute (alt2scaled, $A^{a}->1$)
substitute (alt3scaled, $A^{a}->1$)

cdblib.create ('genGamma.export')

# 4th order gen gamma
cdblib.put ('gen_gamma_0_4th',tmp0,'genGamma.export')
cdblib.put ('gen_gamma_1_4th',tmp1,'genGamma.export')

# 6th order gen gamma
cdblib.put ('gen_gamma_0',alt0,'genGamma.export')
cdblib.put ('gen_gamma_1',alt1,'genGamma.export')
cdblib.put ('gen_gamma_2',alt2,'genGamma.export')
cdblib.put ('gen_gamma_3',alt3,'genGamma.export')

```

```

# 6th order gen gamma scaled
cdblib.put ('gen_gamma_0_scaled',alt0scaled,'genGamma.export')
cdblib.put ('gen_gamma_1_scaled',alt1scaled,'genGamma.export')
cdblib.put ('gen_gamma_2_scaled',alt2scaled,'genGamma.export')
cdblib.put ('gen_gamma_3_scaled',alt3scaled,'genGamma.export')

# gen gamma in terms of partial derivs of Gamma^{a}_{bc}
cdblib.put ('gen_gamma_pderiv0',pderiv02,'genGamma.export')
cdblib.put ('gen_gamma_pderiv1',pderiv03,'genGamma.export')
cdblib.put ('gen_gamma_pderiv2',pderiv04,'genGamma.export')

checkpoint.append (tmp0)
checkpoint.append (tmp1)

checkpoint.append (alt0)
checkpoint.append (alt1)
checkpoint.append (alt2)
checkpoint.append (alt3)

checkpoint.append (alt0scaled)
checkpoint.append (alt1scaled)
checkpoint.append (alt2scaled)
checkpoint.append (alt3scaled)

checkpoint.append (pderiv02)
checkpoint.append (pderiv03)
checkpoint.append (pderiv04)

```