This code verifies that successive $\overset{n}{y}{}^a$ introduce a new leading power of ϵ while leaving the lower order terms unchanged. That is

$$\overset{n+1}{y}{}^a - \overset{n}{y}{}^a = \mathcal{O}\left(\epsilon^{n+1}\right) \tag{1}$$

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

# Dx{#}::LaTeXForm{"{\Dx}"}.  # LCB: currently causes a bug, it kills ::KeepWeight for Dx
```

```
def product_sort (obj):
    substitute (obj,$ A^{a}                              -> A001^{a}             $)
    substitute (obj,$ x^{a}                              -> A002^{a}             $)
    substitute (obj,$ g^{a b}                            -> A003^{a b}           $)
    substitute (obj,$ R_{a b c d}                        -> A004_{a b c d}       $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A005_{a b c d e}     $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A006_{a b c d e f}   $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A007_{a b c d e f g} $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A008_{a b c d e f g h} $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                -> A^{a}                          $)
    substitute (obj,$ A002^{a}                -> x^{a}                          $)
    substitute (obj,$ A003^{a b}              -> g^{a b}                        $)
    substitute (obj,$ A004_{a b c d}          -> R_{a b c d}                    $)
    substitute (obj,$ A005_{a b c d e}        -> \nabla_{e}{R_{a b c d}}        $)
    substitute (obj,$ A006_{a b c d e f}      -> \nabla_{e f}{R_{a b c d}}      $)
    substitute (obj,$ A007_{a b c d e f g}    -> \nabla_{e f g}{R_{a b c d}}    $)
    substitute (obj,$ A008_{a b c d e f g h}  -> \nabla_{e f g h}{R_{a b c d}} $)

    return obj

# now check that y(n+1) - y(n) = Order \eps^{n+1}

import cdblib

y2 = cdblib.get ('y2','../geodesic-bvp.json')
y3 = cdblib.get ('y3','../geodesic-bvp.json')
y4 = cdblib.get ('y4','../geodesic-bvp.json')
y5 = cdblib.get ('y5','../geodesic-bvp.json')

diff32 := @(y3) - @(y2).
diff43 := @(y4) - @(y3).
diff54 := @(y5) - @(y4).

diff32 = product_sort (diff32)
rename_dummies        (diff32)
```

```
canonicalise          (diff32)         # cdb (diff32.001,diff32)

diff43 = product_sort (diff43)
rename_dummies        (diff43)
canonicalise          (diff43)         # cdb (diff43.001,diff43)

diff54 = product_sort (diff54)
rename_dummies        (diff54)
canonicalise          (diff54)         # cdb (diff54.001,diff54)

def truncateR (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
    substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
    substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    ans = Ex(0)

    for i in range (0,n+1):
       foo := @(tmp).
       bah = Ex("numR = " + str(i))
       keep_weight (foo, bah)
       ans = ans + foo

    substitute (ans, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (ans, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
```

```
    substitute (ans, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (ans, $Q_{a b c d} -> R_{a b c d}$)

    return ans

diff32 = truncateR (diff32,2)        # cdb (diff32.002,diff32)
diff43 = truncateR (diff43,3)        # cdb (diff43.002,diff43)
diff54 = truncateR (diff54,4)        # cdb (diff54.002,diff54)
```

Verify order of $\overset{n}{y}{}^{a}$

If things have gone to plan then we should see that $\overset{n+1}{y}{}^{a} - \overset{n}{y}{}^{a} = \mathcal{O}\left(\epsilon^{n+1}\right)$. And we do. Good show.

$$
\begin{aligned}
\overset{3}{y}{}^{a} - \overset{2}{y}{}^{a} = {} & -\frac{1}{12}x^b x^c g^{ad} Dx^e Dx^f \nabla_e R_{bdcf} + \frac{1}{6}x^b x^c g^{ad} Dx^e Dx^f \nabla_b R_{cedf} + \frac{1}{24}x^b x^c g^{ad} Dx^e Dx^f \nabla_d R_{becf} + \frac{1}{12}x^b g^{ac} Dx^d Dx^e Dx^f \nabla_d R_{becf} \\
\overset{4}{y}{}^{a} - \overset{3}{y}{}^{a} = {} & \frac{2}{45}x^b x^c g^{ad} g^{ef} R_{bgde} R_{chfi} Dx^g Dx^h Dx^i + \frac{1}{45}x^b x^c g^{ad} g^{ef} R_{bedg} R_{chfi} Dx^g Dx^h Dx^i + \frac{4}{45}x^b x^c x^d g^{ae} g^{fg} R_{becf} R_{dhgi} Dx^h Dx^i \\
& - \frac{2}{45}x^b x^c x^d g^{ae} g^{fg} R_{bfch} R_{dgei} Dx^h Dx^i - \frac{1}{45}x^b x^c x^d g^{ae} g^{fg} R_{befh} R_{cgdi} Dx^h Dx^i - \frac{1}{40}x^b x^c x^d g^{ae} Dx^f Dx^g \nabla_{fb} R_{cedg} \\
& - \frac{1}{40}x^b x^c x^d g^{ae} Dx^f Dx^g \nabla_{bf} R_{cedg} + \frac{1}{20}x^b x^c x^d g^{ae} Dx^f Dx^g \nabla_{bc} R_{dfeg} - \frac{1}{45}x^b x^c x^d g^{ae} g^{fg} R_{bfch} R_{dieg} Dx^h Dx^i \\
& + \frac{1}{80}x^b x^c x^d g^{ae} Dx^f Dx^g \nabla_{eb} R_{cfdg} + \frac{1}{80}x^b x^c x^d g^{ae} Dx^f Dx^g \nabla_{be} R_{cfdg} - \frac{1}{45}x^b x^c g^{ad} g^{ef} R_{bdeg} R_{chfi} Dx^g Dx^h Dx^i \\
& + \frac{1}{45}x^b x^c g^{ad} g^{ef} R_{becg} R_{dhfi} Dx^g Dx^h Dx^i - \frac{1}{60}x^b x^c g^{ad} Dx^e Dx^f Dx^g \nabla_{ef} R_{bdcg} + \frac{1}{40}x^b x^c g^{ad} Dx^e Dx^f Dx^g \nabla_{eb} R_{cfdg} \\
& + \frac{1}{40}x^b x^c g^{ad} Dx^e Dx^f Dx^g \nabla_{be} R_{cfdg} + \frac{1}{240}x^b x^c g^{ad} Dx^e Dx^f Dx^g \nabla_{de} R_{bfcg} + \frac{1}{240}x^b x^c g^{ad} Dx^e Dx^f Dx^g \nabla_{ed} R_{bfcg} \\
& + \frac{1}{45}x^b g^{ac} g^{de} R_{bfdg} R_{chei} Dx^f Dx^g Dx^h Dx^i + \frac{1}{60}x^b g^{ac} Dx^d Dx^e Dx^f Dx^g \nabla_{de} R_{bfcg}
\end{aligned}
$$

$$\overset{2}{T}\left(\overset{3}{y}{}^{a} - \overset{2}{y}{}^{a}\right) = 0$$

$$\overset{3}{T}\left(\overset{4}{y}{}^{a} - \overset{3}{y}{}^{a}\right) = 0$$

$$\overset{4}{T}\left(\overset{5}{y}{}^{a} - \overset{4}{y}{}^{a}\right) = 0$$