The connection

Here we use the output from `metric.tex` and `metric-inv.tex` to compute the metric connection Γ^{d}_{ab}. We use the standard metric compatible connection

$$\Gamma^{d}_{ab} = \frac{1}{2} g^{dc} \left(g_{cb,a} + g_{ac,b} - g_{ab,c} \right) \tag{1}$$

Since `metric.tex` and `metric-inv.tex` generate truncated expressions for g_{ab} and g^{ab} a similar truncation must be applied to this computation of Γ^{d}_{ab}. The naive choice is to truncate Γ^{d}_{ab} *after* it has been fully evaluated on the truncated expersions for g_{ab} and g^{ab}. This will work but it wastes time and memory (big time).

A better approach is to truncate Γ^{d}_{ab} during its construction. That is, we take careful note of how the terms in the finite series for g_{ab} and g^{ab} combine to produce the terms of a particular order in the expansion of Γ^{d}_{ab}.

Suppose g_{ab} and g^{ab} are known to say fourth order. We can write each of these as follows

$$g_{ab} = \overset{0}{g}_{ab} + \overset{1}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} \tag{2}$$

$$g^{ab} = \overset{0}{g}{}^{ab} + \overset{1}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} \tag{3}$$

where $\overset{n}{g}$ denotes a term of order $\mathcal{O}\left(\epsilon^{n}\right)$. A similar expansion applies for Γ^{d}_{ab}, that is

$$\Gamma^{d}_{ab} = \overset{0}{\Gamma}{}^{d}_{ab} + \overset{1}{\Gamma}{}^{d}_{ab} + \overset{2}{\Gamma}{}^{d}_{ab} + \overset{3}{\Gamma}{}^{d}_{ab} + \overset{4}{\Gamma}{}^{d}_{ab} \tag{4}$$

After substituting these formal expansions into the equation (1) and then matching corresponing terms we obtain

$$\overset{n}{\Gamma}{}^{d}_{ab} = \frac{1}{2} \sum_{i=0}^{i=n} \overset{i}{g}{}^{dc} \left(\overset{n-i}{g}{}_{cb,a} + \overset{n-i}{g}{}_{ac,b} - \overset{n-i}{g}{}_{ab,c} \right) \tag{5}$$

We use this equation to compute the successive terms in Γ^{d}_{ab}.

```
{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

D{#}::Derivative.
\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.
\delta^{a}_{b}::KroneckerDelta.
\delta_{a}^{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

x^{a}::Depends(D{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).

import cdblib

gab = cdblib.get ('g_ab','metric.json')      # cdb(gab.000,gab)
iab = cdblib.get ('g^ab','metric-inv.json')  # cdb(iab.000,iab)

defgab := g_{a b} -> @(gab).
defiab := g^{a b} -> @(iab).

dgab := D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}.  # cdb(dgab.001,dgab)

substitute   (dgab,defgab)

distribute   (dgab)                # cdb(dgab.002,dgab)
unwrap       (dgab)                # cdb(dgab.003,dgab)
product_rule (dgab)                # cdb(dgab.004,dgab)
distribute   (dgab)                # cdb(dgab.005,dgab)
substitute   (dgab,$D_{a}{x^{b}}->\delta^{b}_{a}$,repeat=True)  # cdb(dgab.006,dgab)
```

```
eliminate_kronecker (dgab)       # cdb(dgab.007,dgab)
sort_product    (dgab)           # cdb(dgab.008,dgab)
rename_dummies  (dgab)           # cdb(dgab.009,dgab)
canonicalise    (dgab)           # cdb(dgab.010,dgab)
```

$$\texttt{gab.000} := g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45}x^c x^d x^e x^f R_{acdg} R_{befh} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf}$$
$$+ \frac{1}{45}x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg}$$

$$\texttt{dgab.001} := D_a g_{cb} + D_b g_{ac} - D_c g_{ab}$$

$$\texttt{dgab.002} := D_a g_{cb} - \frac{1}{3}D_a\left(x^j x^d R_{cjbd}\right) - \frac{1}{6}D_a\left(x^j x^d x^e \nabla_j R_{cdbe}\right) + \frac{2}{45}D_a\left(x^j x^d x^e x^f R_{cjdg} R_{befh} g^{gh}\right) - \frac{1}{20}D_a\left(x^j x^d x^e x^f \nabla_{jd} R_{cebf}\right)$$
$$+ \frac{1}{45}D_a\left(x^j x^d x^e x^f x^g R_{cjdh} \nabla_e R_{bfgi} g^{hi}\right) + \frac{1}{45}D_a\left(x^j x^d x^e x^f x^g R_{bjdh} \nabla_e R_{cfgi} g^{hi}\right) - \frac{1}{90}D_a\left(x^j x^d x^e x^f x^g \nabla_{jde} R_{cfbg}\right)$$
$$+ D_b g_{ac} - \frac{1}{3}D_b\left(x^j x^d R_{ajcd}\right) - \frac{1}{6}D_b\left(x^j x^d x^e \nabla_j R_{adce}\right) + \frac{2}{45}D_b\left(x^j x^d x^e x^f R_{ajdg} R_{cefh} g^{gh}\right) - \frac{1}{20}D_b\left(x^j x^d x^e x^f \nabla_{jd} R_{aecf}\right)$$
$$+ \frac{1}{45}D_b\left(x^j x^d x^e x^f x^g R_{ajdh} \nabla_e R_{cfgi} g^{hi}\right) + \frac{1}{45}D_b\left(x^j x^d x^e x^f x^g R_{cjdh} \nabla_e R_{afgi} g^{hi}\right) - \frac{1}{90}D_b\left(x^j x^d x^e x^f x^g \nabla_{jde} R_{afcg}\right)$$
$$- D_c g_{ab} + \frac{1}{3}D_c\left(x^j x^d R_{ajbd}\right) + \frac{1}{6}D_c\left(x^j x^d x^e \nabla_j R_{adbe}\right) - \frac{2}{45}D_c\left(x^j x^d x^e x^f R_{ajdg} R_{befh} g^{gh}\right) + \frac{1}{20}D_c\left(x^j x^d x^e x^f \nabla_{jd} R_{aebf}\right)$$
$$- \frac{1}{45}D_c\left(x^j x^d x^e x^f x^g R_{ajdh} \nabla_e R_{bfgi} g^{hi}\right) - \frac{1}{45}D_c\left(x^j x^d x^e x^f x^g R_{bjdh} \nabla_e R_{afgi} g^{hi}\right) + \frac{1}{90}D_c\left(x^j x^d x^e x^f x^g \nabla_{jde} R_{afbg}\right)$$

$$
\begin{aligned}
\texttt{dgab.010} := {} & \frac{2}{3} R_{acbd} x^{d} - \frac{1}{6} \nabla_{a} R_{bdce} x^{d} x^{e} + \frac{1}{3} \nabla_{d} R_{acbe} x^{d} x^{e} - \frac{4}{45} R_{acde} R_{bfgh} g^{dg} x^{e} x^{f} x^{h} - \frac{2}{45} R_{adce} R_{bfgh} g^{dg} x^{e} x^{f} x^{h} - \frac{2}{45} R_{adbe} R_{cfgh} g^{dg} x^{e} x^{f} x^{h} \\
& - \frac{1}{20} \nabla_{ad} R_{becf} x^{d} x^{e} x^{f} - \frac{1}{20} \nabla_{da} R_{becf} x^{d} x^{e} x^{f} + \frac{1}{10} \nabla_{de} R_{acbf} x^{d} x^{e} x^{f} - \frac{2}{45} R_{acde} \nabla_{f} R_{bghi} g^{dh} x^{e} x^{f} x^{g} x^{i} - \frac{1}{45} R_{adce} \nabla_{f} R_{bghi} g^{dh} x^{e} x^{f} x^{g} x^{i} \\
& + \frac{1}{45} R_{cdef} \nabla_{a} R_{bghi} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{1}{45} R_{cdef} \nabla_{g} R_{ahbi} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{1}{45} R_{adbe} \nabla_{f} R_{cghi} g^{dh} x^{e} x^{f} x^{g} x^{i} + \frac{1}{45} R_{bdef} \nabla_{a} R_{cghi} g^{eh} x^{d} x^{f} x^{g} x^{i} \\
& - \frac{2}{45} R_{bdef} \nabla_{g} R_{achi} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{1}{45} R_{bdef} \nabla_{g} R_{ahci} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{1}{90} \nabla_{ade} R_{bfcg} x^{d} x^{e} x^{f} x^{g} - \frac{1}{90} \nabla_{dae} R_{bfcg} x^{d} x^{e} x^{f} x^{g} \\
& - \frac{1}{90} \nabla_{dea} R_{bfcg} x^{d} x^{e} x^{f} x^{g} + \frac{1}{45} \nabla_{def} R_{acbg} x^{d} x^{e} x^{f} x^{g} + \frac{2}{3} R_{adbc} x^{d} - \frac{1}{6} \nabla_{b} R_{adce} x^{d} x^{e} + \frac{1}{3} \nabla_{d} R_{aebc} x^{d} x^{e} - \frac{2}{45} R_{adbe} R_{cfgh} g^{eg} x^{d} x^{f} x^{h} \\
& - \frac{4}{45} R_{adef} R_{bcgh} g^{eg} x^{d} x^{f} x^{h} - \frac{2}{45} R_{adef} R_{bgch} g^{eg} x^{d} x^{f} x^{h} - \frac{1}{20} \nabla_{bd} R_{aecf} x^{d} x^{e} x^{f} - \frac{1}{20} \nabla_{db} R_{aecf} x^{d} x^{e} x^{f} + \frac{1}{10} \nabla_{de} R_{afbc} x^{d} x^{e} x^{f} \\
& - \frac{1}{45} R_{adbe} \nabla_{f} R_{cghi} g^{eh} x^{d} x^{f} x^{g} x^{i} + \frac{1}{45} R_{adef} \nabla_{b} R_{cghi} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{2}{45} R_{adef} \nabla_{g} R_{bchi} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{1}{45} R_{adef} \nabla_{g} R_{bhci} g^{eh} x^{d} x^{f} x^{g} x^{i} \\
& - \frac{2}{45} R_{bcde} \nabla_{f} R_{aghi} g^{dh} x^{e} x^{f} x^{g} x^{i} - \frac{1}{45} R_{bdce} \nabla_{f} R_{aghi} g^{dh} x^{e} x^{f} x^{g} x^{i} + \frac{1}{45} R_{cdef} \nabla_{b} R_{aghi} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{1}{45} R_{cdef} \nabla_{g} R_{ahbi} g^{ei} x^{d} x^{f} x^{g} x^{h} \\
& - \frac{1}{90} \nabla_{bde} R_{afcg} x^{d} x^{e} x^{f} x^{g} - \frac{1}{90} \nabla_{dbe} R_{afcg} x^{d} x^{e} x^{f} x^{g} - \frac{1}{90} \nabla_{deb} R_{afcg} x^{d} x^{e} x^{f} x^{g} + \frac{1}{45} \nabla_{def} R_{agbc} x^{d} x^{e} x^{f} x^{g} + \frac{1}{6} \nabla_{c} R_{adbe} x^{d} x^{e} \\
& + \frac{2}{45} R_{adce} R_{bfgh} g^{eg} x^{d} x^{f} x^{h} + \frac{2}{45} R_{adef} R_{bgch} g^{eh} x^{d} x^{f} x^{g} + \frac{1}{20} \nabla_{cd} R_{aebf} x^{d} x^{e} x^{f} + \frac{1}{20} \nabla_{dc} R_{aebf} x^{d} x^{e} x^{f} + \frac{1}{45} R_{adce} \nabla_{f} R_{bghi} g^{eh} x^{d} x^{f} x^{g} x^{i} \\
& - \frac{1}{45} R_{adef} \nabla_{c} R_{bghi} g^{eh} x^{d} x^{f} x^{g} x^{i} + \frac{1}{45} R_{adef} \nabla_{g} R_{bhci} g^{ei} x^{d} x^{f} x^{g} x^{h} + \frac{1}{45} R_{bdce} \nabla_{f} R_{aghi} g^{eh} x^{d} x^{f} x^{g} x^{i} - \frac{1}{45} R_{bdef} \nabla_{c} R_{aghi} g^{eh} x^{d} x^{f} x^{g} x^{i} \\
& + \frac{1}{45} R_{bdef} \nabla_{g} R_{ahci} g^{ei} x^{d} x^{f} x^{g} x^{h} + \frac{1}{90} \nabla_{cde} R_{afbg} x^{d} x^{e} x^{f} x^{g} + \frac{1}{90} \nabla_{dce} R_{afbg} x^{d} x^{e} x^{f} x^{g} + \frac{1}{90} \nabla_{dec} R_{afbg} x^{d} x^{e} x^{f} x^{g}
\end{aligned}
$$

```
# Note:
# Computing Gamma directly by (1/2) iab dgab and *then* truncating to lower order
# is not optimal. We only want the leading oder terms (to 4th order in x). But the direct
# calculation would compute *all* terms before the truncation. This does work but it
# is slower than the following code.
#
# The better approach (as adopted in this code) is to extract all of the terms of iab
# and dgab then construct the leading order terms of Gamma (to fifth order) term by term.

def get_Rterm (obj,n):

# I would like to assign different weights to \nabla_{a}, \nabla_{a b}, \nabla_{a b c} etc. but no matter
# what I do it appears that Cadabra assigns the same weight to all of these regardless of the number of subscripts.
# It seems that the weight is assigned to the symbol \nabla alone. So I'm forced to use the following substitution trick.

    Q_{a b c d}::Weight(label=numR,value=2).
    Q_{a b c d e}::Weight(label=numR,value=3).
    Q_{a b c d e f}::Weight(label=numR,value=4).
    Q_{a b c d e f g}::Weight(label=numR,value=5).

    tmp := @(obj).

    distribute (tmp)

    substitute (tmp, $\nabla_{e f g}{R_{a b c d}} -> Q_{a b c d e f g}$)
    substitute (tmp, $\nabla_{e f}{R_{a b c d}} -> Q_{a b c d e f}$)
    substitute (tmp, $\nabla_{e}{R_{a b c d}} -> Q_{a b c d e}$)
    substitute (tmp, $R_{a b c d} -> Q_{a b c d}$)

    foo := @(tmp).
    bah = Ex("numR = " + str(n))
    keep_weight (foo, bah)

    substitute (foo, $Q_{a b c d e f g} -> \nabla_{e f g}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e f} -> \nabla_{e f}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d e} -> \nabla_{e}{R_{a b c d}}$)
    substitute (foo, $Q_{a b c d} -> R_{a b c d}$)
```

```
    return foo

# terms of the curvature expansion of dg_{ab}

dgab00 = get_Rterm (dgab,0)   # cdb(dgab00.105,dgab00)  # zero
dgab01 = get_Rterm (dgab,1)   # cdb(dgab01.105,dgab01)  # zero
dgab02 = get_Rterm (dgab,2)   # cdb(dgab02.105,dgab02)
dgab03 = get_Rterm (dgab,3)   # cdb(dgab03.105,dgab03)
dgab04 = get_Rterm (dgab,4)   # cdb(dgab04.105,dgab04)
dgab05 = get_Rterm (dgab,5)   # cdb(dgab05.105,dgab05)

# Convert free indices on iab from ^{a b} to ^{d c}
# This ensures we can later build products like @(iab) @(dgab) knowing that the indices are correctly ordered.
# Without this step we would be using free indices ^{a b} and _{a b c}. Thus the product @(iab) @(dgab) would
# have just one free index _{c}. This is clearly wrong.

tmp := @(iab) \delta_{a}^{d} \delta_{b}^{c}.

distribute     (tmp)
eliminate_kronecker (tmp)
sort_product   (tmp)
rename_dummies (tmp)
canonicalise   (tmp)

idc := @(tmp).

# terms of the curvature expansion of g^{ab}

idc00 = get_Rterm (idc,0)   # cdb(idc00.105,idc00)
idc01 = get_Rterm (idc,1)   # cdb(idc01.105,idc01)  # zero
idc02 = get_Rterm (idc,2)   # cdb(idc02.105,idc02)
idc03 = get_Rterm (idc,3)   # cdb(idc03.105,idc03)
idc04 = get_Rterm (idc,4)   # cdb(idc04.105,idc04)
idc05 = get_Rterm (idc,5)   # cdb(idc05.105,idc05)
```

$\texttt{dgab00.105} := 0$

$\texttt{dgab01.105} := 0$

$$\texttt{dgab02.105} := \frac{2}{3}R_{acbd}x^d + \frac{2}{3}R_{adbc}x^d$$

$$\texttt{dgab03.105} := -\frac{1}{6}\nabla_a R_{bdce}x^d x^e + \frac{1}{3}\nabla_d R_{acbe}x^d x^e - \frac{1}{6}\nabla_b R_{adce}x^d x^e + \frac{1}{3}\nabla_d R_{aebc}x^d x^e + \frac{1}{6}\nabla_c R_{adbe}x^d x^e$$

$$\begin{aligned}\texttt{dgab04.105} := &-\frac{4}{45}R_{acde}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adce}R_{bfgh}g^{dg}x^e x^f x^h - \frac{2}{45}R_{adbe}R_{cfgh}g^{dg}x^e x^f x^h - \frac{1}{20}\nabla_{ad}R_{becf}x^d x^e x^f \\
&- \frac{1}{20}\nabla_{da}R_{becf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{acbf}x^d x^e x^f - \frac{2}{45}R_{adbe}R_{cfgh}g^{eg}x^d x^f x^h - \frac{4}{45}R_{adef}R_{bcgh}g^{eg}x^d x^f x^h \\
&- \frac{2}{45}R_{adef}R_{bgch}g^{eg}x^d x^f x^h - \frac{1}{20}\nabla_{bd}R_{aecf}x^d x^e x^f - \frac{1}{20}\nabla_{db}R_{aecf}x^d x^e x^f + \frac{1}{10}\nabla_{de}R_{afbc}x^d x^e x^f \\
&+ \frac{2}{45}R_{adce}R_{bfgh}g^{eg}x^d x^f x^h + \frac{2}{45}R_{adef}R_{bgch}g^{eh}x^d x^f x^g + \frac{1}{20}\nabla_{cd}R_{aebf}x^d x^e x^f + \frac{1}{20}\nabla_{dc}R_{aebf}x^d x^e x^f\end{aligned}$$

$$\begin{aligned}\texttt{dgab05.105} := &-\frac{2}{45}R_{acde}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i - \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_a R_{bghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{eh}x^d x^f x^g x^i \\
&- \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{bdef}\nabla_a R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bdef}\nabla_g R_{achi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{eh}x^d x^f x^g x^i \\
&- \frac{1}{90}\nabla_{ade}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dae}R_{bfcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{dea}R_{bfcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{acbg}x^d x^e x^f x^g - \frac{1}{45}R_{adbe}\nabla_f R_{cghi}g^{eh}x^d x^f x^g x^i \\
&+ \frac{1}{45}R_{adef}\nabla_b R_{cghi}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{adef}\nabla_g R_{bchi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{eh}x^d x^f x^g x^i - \frac{2}{45}R_{bcde}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i \\
&- \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{dh}x^e x^f x^g x^i + \frac{1}{45}R_{cdef}\nabla_b R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{cdef}\nabla_g R_{ahbi}g^{ei}x^d x^f x^g x^h - \frac{1}{90}\nabla_{bde}R_{afcg}x^d x^e x^f x^g \\
&- \frac{1}{90}\nabla_{dbe}R_{afcg}x^d x^e x^f x^g - \frac{1}{90}\nabla_{deb}R_{afcg}x^d x^e x^f x^g + \frac{1}{45}\nabla_{def}R_{agbc}x^d x^e x^f x^g + \frac{1}{45}R_{adce}\nabla_f R_{bghi}g^{eh}x^d x^f x^g x^i \\
&- \frac{1}{45}R_{adef}\nabla_c R_{bghi}g^{eh}x^d x^f x^g x^i + \frac{1}{45}R_{adef}\nabla_g R_{bhci}g^{ei}x^d x^f x^g x^h + \frac{1}{45}R_{bdce}\nabla_f R_{aghi}g^{eh}x^d x^f x^g x^i - \frac{1}{45}R_{bdef}\nabla_c R_{aghi}g^{eh}x^d x^f x^g x^i \\
&+ \frac{1}{45}R_{bdef}\nabla_g R_{ahci}g^{ei}x^d x^f x^g x^h + \frac{1}{90}\nabla_{cde}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dce}R_{afbg}x^d x^e x^f x^g + \frac{1}{90}\nabla_{dec}R_{afbg}x^d x^e x^f x^g\end{aligned}$$

$$\texttt{idc00.105} := g^{cd}$$

$$\texttt{idc01.105} := 0$$

$$\texttt{idc02.105} := \frac{1}{3} R_{abef} g^{ca} g^{de} x^b x^f$$

$$\texttt{idc03.105} := \frac{1}{6} \nabla_a R_{befg} g^{cb} g^{df} x^a x^e x^g$$

$$\texttt{idc04.105} := \frac{1}{15} R_{abef} R_{ghij} g^{ca} g^{dg} g^{ei} x^b x^f x^h x^j + \frac{1}{20} \nabla_{ab} R_{efgh} g^{ce} g^{dg} x^a x^b x^f x^h$$

$$\texttt{idc05.105} := \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ch} g^{da} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{30} R_{abef} \nabla_g R_{hijk} g^{ca} g^{dh} g^{ej} x^b x^f x^g x^i x^k + \frac{1}{90} \nabla_{abe} R_{fghi} g^{cf} g^{dh} x^a x^b x^e x^g x^i$$

```
# idc  = g^{d c}
# dgab = D_{a}{g_{c b}} + D_{b}{g_{a c}} - D_{c}{g_{a b}}

# terms of the curvature expansion of \Gamma^{d}_{a b}

# term0 := (1/2)  @(idc00) @(dgab00).
# term1 := (1/2) (@(idc01) @(dgab00) + @(idc00) @(dgab01)).
# term2 := (1/2) (@(idc02) @(dgab00) + @(idc01) @(dgab01) + @(idc00) @(dgab02)).
# term3 := (1/2) (@(idc03) @(dgab00) + @(idc02) @(dgab01) + @(idc01) @(dgab02) + @(idc00) @(dgab03)).
# term4 := (1/2) (@(idc04) @(dgab00) + @(idc03) @(dgab01) + @(idc02) @(dgab02) + @(idc01) @(dgab03) + @(idc00) @(dgab04)).
# term5 := (1/2) (@(idc05) @(dgab00) + @(idc04) @(dgab01) + @(idc03) @(dgab02) + @(idc02) @(dgab03) + @(idc01) @(dgab04) + @(idc00) @(d

# simplidied version of the above after noting dgab00 = dgab01 = 0

term0 := 0.
term1 := 0.
term2 := (1/2) (@(idc00) @(dgab02)).
term3 := (1/2) (@(idc01) @(dgab02) + @(idc00) @(dgab03)).
term4 := (1/2) (@(idc02) @(dgab02) + @(idc01) @(dgab03) + @(idc00) @(dgab04)).
term5 := (1/2) (@(idc03) @(dgab02) + @(idc02) @(dgab03) + @(idc01) @(dgab04) + @(idc00) @(dgab05)).

def tidy_terms (obj):
    substitute     (obj,$x^{a}->AA^{a}$,repeat=True)  # will force AA to the left of all terms
    distribute     (obj)
    sort_product   (obj)
    rename_dummies (obj)
    canonicalise   (obj)
    substitute     (obj,$AA^{a}->x^{a}$,repeat=True)  # replace AA with x
    factor_out     (obj,$x^{a?}$)

    return obj

term0 = tidy_terms (term0)   # cdb(term0.201,term0)  # zero
term1 = tidy_terms (term1)   # cdb(term1.201,term1)  # zero
term2 = tidy_terms (term2)   # cdb(term2.201,term2)
term3 = tidy_terms (term3)   # cdb(term3.201,term3)
term4 = tidy_terms (term4)   # cdb(term4.201,term4)
term5 = tidy_terms (term5)   # cdb(term5.201,term5)
```

```
Gamma := @(term0) + @(term1) + @(term2) + @(term3) + @(term4) + @(term5). # cdb(Gamma.200,Gamma)
```

$$\texttt{term0.201} := 0$$

$$\texttt{term1.201} := 0$$

$$\texttt{term2.201} := x^c \left(\frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de} \right)$$

$$\texttt{term3.201} := x^c x^e \left(\frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df} \right)$$

$$\begin{aligned}\texttt{term4.201} := x^c x^e x^f \Big(& \frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \\ & + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \\ & - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg} \Big)\end{aligned}$$

$$\begin{aligned}\texttt{term5.201} := x^c x^e x^f x^g \Big(& \frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} \\ & + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} \\ & - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cea} R_{bfgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} \\ & - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} \\ & + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} \\ & - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{agij} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} \\ & + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh} \Big)\end{aligned}$$

$$
\begin{aligned}
\texttt{Gamma.200} := {} & x^c \left(\frac{1}{3} R_{aebc} g^{de} + \frac{1}{3} R_{acbe} g^{de} \right) + x^c x^e \left(\frac{1}{12} \nabla_a R_{bcef} g^{df} + \frac{1}{6} \nabla_c R_{afbe} g^{df} + \frac{1}{12} \nabla_b R_{acef} g^{df} + \frac{1}{6} \nabla_c R_{aebf} g^{df} + \frac{1}{12} \nabla_f R_{acbe} g^{df} \right) \\
& + x^c x^e x^f \left(\frac{4}{45} R_{agbc} R_{ehfi} g^{dh} g^{gi} + \frac{4}{45} R_{acbg} R_{ehfi} g^{dh} g^{gi} - \frac{2}{45} R_{agch} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{agch} R_{befi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{ac} R_{befg} g^{dg} + \frac{1}{40} \nabla_{ca} R_{befg} g^{dg} \right. \\
& + \frac{1}{20} \nabla_{ce} R_{agbf} g^{dg} - \frac{2}{45} R_{aceg} R_{bhfi} g^{dh} g^{gi} - \frac{1}{45} R_{aceg} R_{bhfi} g^{di} g^{gh} + \frac{1}{40} \nabla_{bc} R_{aefg} g^{dg} + \frac{1}{40} \nabla_{cb} R_{aefg} g^{dg} + \frac{1}{20} \nabla_{ce} R_{afbg} g^{dg} \\
& \left. - \frac{1}{45} R_{acgh} R_{befi} g^{dg} g^{hi} - \frac{1}{45} R_{aceg} R_{bfhi} g^{dh} g^{gi} + \frac{1}{40} \nabla_{gc} R_{aebf} g^{dg} + \frac{1}{40} \nabla_{cg} R_{aebf} g^{dg} \right) \\
& + x^c x^e x^f x^g \left(\frac{2}{45} R_{ahbc} \nabla_e R_{figj} g^{di} g^{hj} + \frac{2}{45} R_{acbh} \nabla_e R_{figj} g^{di} g^{hj} + \frac{1}{60} R_{chei} \nabla_a R_{bfgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{ajbg} g^{dh} g^{ij} \right. \\
& + \frac{1}{60} R_{chei} \nabla_b R_{afgj} g^{dh} g^{ij} + \frac{2}{45} R_{chei} \nabla_f R_{agbj} g^{dh} g^{ij} + \frac{1}{36} R_{chei} \nabla_j R_{afbg} g^{dh} g^{ij} - \frac{1}{45} R_{ahci} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{ahci} \nabla_e R_{bfgj} g^{di} g^{hj} \\
& - \frac{1}{90} R_{bceh} \nabla_a R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{bceh} \nabla_f R_{aigj} g^{di} g^{hj} - \frac{1}{90} R_{bceh} \nabla_f R_{aigj} g^{dj} g^{hi} + \frac{1}{180} \nabla_{ace} R_{bfgh} g^{dh} + \frac{1}{180} \nabla_{cae} R_{bfgh} g^{dh} \\
& + \frac{1}{180} \nabla_{cea} R_{bfgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{ahbg} g^{dh} - \frac{1}{90} R_{aceh} \nabla_b R_{figj} g^{di} g^{hj} - \frac{1}{45} R_{aceh} \nabla_f R_{bigj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bigj} g^{dj} g^{hi} \\
& - \frac{1}{45} R_{bhci} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bhci} \nabla_e R_{afgj} g^{di} g^{hj} + \frac{1}{180} \nabla_{bce} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{cbe} R_{afgh} g^{dh} + \frac{1}{180} \nabla_{ceb} R_{afgh} g^{dh} + \frac{1}{90} \nabla_{cef} R_{agbh} g^{dh} \\
& - \frac{1}{90} R_{achi} \nabla_e R_{bfgj} g^{dh} g^{ij} - \frac{1}{90} R_{aceh} \nabla_i R_{bfgj} g^{di} g^{hj} - \frac{1}{90} R_{aceh} \nabla_f R_{bgij} g^{di} g^{hj} - \frac{1}{90} R_{bchi} \nabla_e R_{afgj} g^{dh} g^{ij} - \frac{1}{90} R_{bceh} \nabla_i R_{afgj} g^{di} g^{hj} \\
& \left. - \frac{1}{90} R_{bceh} \nabla_f R_{agij} g^{di} g^{hj} + \frac{1}{180} \nabla_{hce} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{che} R_{afbg} g^{dh} + \frac{1}{180} \nabla_{ceh} R_{afbg} g^{dh} \right)
\end{aligned}
$$

```
cdblib.create ('connection.json')

cdblib.put ('Gamma',Gamma,'connection.json')

cdblib.put ('GammaRterm0',term0,'connection.json')
cdblib.put ('GammaRterm1',term1,'connection.json')
cdblib.put ('GammaRterm2',term2,'connection.json')
cdblib.put ('GammaRterm3',term3,'connection.json')
cdblib.put ('GammaRterm4',term4,'connection.json')
cdblib.put ('GammaRterm5',term5,'connection.json')

checkpoint.append (term0)
checkpoint.append (term1)
checkpoint.append (term2)
checkpoint.append (term3)
checkpoint.append (term4)
checkpoint.append (term5)
```

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                              -> A001^{a}              $)
    substitute (obj,$ x^{a}                              -> A002^{a}              $)
    substitute (obj,$ g^{a b}                            -> A003^{a b}            $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A008_{a b c d e f g h} $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A007_{a b c d e f g}   $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A006_{a b c d e f}     $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A005_{a b c d e}       $)
    substitute (obj,$ R_{a b c d}                        -> A004_{a b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                    -> A^{a}                          $)
    substitute (obj,$ A002^{a}                    -> x^{a}                          $)
    substitute (obj,$ A003^{a b}                  -> g^{a b}                        $)
    substitute (obj,$ A008_{a b c d e f g h}      -> \nabla_{e f g h}{R_{a b c d}} $)
    substitute (obj,$ A007_{a b c d e f g}        -> \nabla_{e f g}{R_{a b c d}}   $)
    substitute (obj,$ A006_{a b c d e f}          -> \nabla_{e f}{R_{a b c d}}     $)
    substitute (obj,$ A005_{a b c d e}            -> \nabla_{e}{R_{a b c d}}       $)
    substitute (obj,$ A004_{a b c d}              -> R_{a b c d}                    $)

    return obj

def reformat (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute     (bah)
   bah = product_sort (bah)
   rename_dummies (bah)
   canonicalise   (bah)
   factor_out     (bah,$A^{a?},x^{b?}$)
   ans := @(bah) / @(foo).
   return ans

def rescale (obj,scale):
   foo  = Ex(str(scale))
   bah := @(foo) @(obj).
   distribute  (bah)
```

```
   factor_out  (bah,$A^{a?},x^{b?}$)
   return bah

Rterm2 := @(term2) A^{a} A^{b}.
Rterm3 := @(term3) A^{a} A^{b}.
Rterm4 := @(term4) A^{a} A^{b}.
Rterm5 := @(term5) A^{a} A^{b}.

Rterm2 = reformat (Rterm2,  3)    # cdb(Rterm2.301,Rterm2)
Rterm3 = reformat (Rterm3, 12)    # cdb(Rterm3.301,Rterm3)
Rterm4 = reformat (Rterm4,360)    # cdb(Rterm4.301,Rterm4)
Rterm5 = reformat (Rterm5,180)    # cdb(Rterm5.301,Rterm5)

Gamma  := @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Gamma.301,Gamma)
Scaled := 360 @(Gamma).                                   # cdb (Scaled.301,Scaled)

scaled2 = rescale (Rterm2,   3)   # cdb (scaled2.301,scaled2)
scaled3 = rescale (Rterm3,  12)   # cdb (scaled3.301,scaled3)
scaled4 = rescale (Rterm4, 360)   # cdb (scaled4.301,scaled4)
scaled5 = rescale (Rterm5, 180)   # cdb (scaled5.301,scaled5)
```

The connection in Riemann normal coordinates

$$A^aA^b\Gamma^d_{ab} = \frac{2}{3}A^aA^bx^cg^{de}R_{acbe} + \frac{1}{12}A^aA^bx^cx^e\left(2g^{df}\nabla_aR_{bcef} + 4g^{df}\nabla_cR_{aebf} + g^{df}\nabla_fR_{acbe}\right) + \frac{1}{360}A^aA^bx^cx^ex^f\left(64g^{dg}g^{hi}R_{acbh}R_{egfi} - 32g^{dg}g^{hi}R_{aceh}R_{bgfi}\right.$$
$$- 16g^{dg}g^{hi}R_{aceh}R_{bifg} + 18g^{dg}\nabla_{ac}R_{befg} + 18g^{dg}\nabla_{ca}R_{befg} + 36g^{dg}\nabla_{ce}R_{afbg} - 16g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9g^{dg}\nabla_{gc}R_{aebf} + \left.9g^{dg}\nabla_{cg}R_{aebf}\right)$$
$$+ \frac{1}{180}A^aA^bx^cx^ex^fx^g\left(16g^{dh}g^{ij}R_{acbi}\nabla_eR_{fhgj} + 6g^{dh}g^{ij}R_{chei}\nabla_aR_{bfgj} + 16g^{dh}g^{ij}R_{chei}\nabla_fR_{agbj} + 5g^{dh}g^{ij}R_{chei}\nabla_jR_{afbg}\right.$$
$$- 8g^{dh}g^{ij}R_{ahci}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{aich}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_bR_{fhgj} - 8g^{dh}g^{ij}R_{acei}\nabla_fR_{bhgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bjgh} + 2g^{dh}\nabla_{ace}R_{bfgh}$$
$$+ 2g^{dh}\nabla_{cae}R_{bfgh} + 2g^{dh}\nabla_{cea}R_{bfgh} + 4g^{dh}\nabla_{cef}R_{agbh} - 4g^{dh}g^{ij}R_{achi}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_hR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bghj}$$
$$\left.+ g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + g^{dh}\nabla_{ceh}R_{afbg}\right)$$

$$360A^aA^b\Gamma^d_{ab} = 240A^aA^bx^cg^{de}R_{acbe} + 30A^aA^bx^cx^e\left(2g^{df}\nabla_aR_{bcef} + 4g^{df}\nabla_cR_{aebf} + g^{df}\nabla_fR_{acbe}\right) + A^aA^bx^cx^ex^f\left(64g^{dg}g^{hi}R_{acbh}R_{egfi} - 32g^{dg}g^{hi}R_{aceh}R_{bgfi}\right.$$
$$- 16g^{dg}g^{hi}R_{aceh}R_{bifg} + 18g^{dg}\nabla_{ac}R_{befg} + 18g^{dg}\nabla_{ca}R_{befg} + 36g^{dg}\nabla_{ce}R_{afbg} - 16g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9g^{dg}\nabla_{gc}R_{aebf} + \left.9g^{dg}\nabla_{cg}R_{aebf}\right)$$
$$+ 2A^aA^bx^cx^ex^fx^g\left(16g^{dh}g^{ij}R_{acbi}\nabla_eR_{fhgj} + 6g^{dh}g^{ij}R_{chei}\nabla_aR_{bfgj} + 16g^{dh}g^{ij}R_{chei}\nabla_fR_{agbj} + 5g^{dh}g^{ij}R_{chei}\nabla_jR_{afbg}\right.$$
$$- 8g^{dh}g^{ij}R_{ahci}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{aich}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_bR_{fhgj} - 8g^{dh}g^{ij}R_{acei}\nabla_fR_{bhgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bjgh}$$
$$+ 2g^{dh}\nabla_{ace}R_{bfgh} + 2g^{dh}\nabla_{cae}R_{bfgh} + 2g^{dh}\nabla_{cea}R_{bfgh} + 4g^{dh}\nabla_{cef}R_{agbh} - 4g^{dh}g^{ij}R_{achi}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_hR_{bfgj}$$
$$\left.- 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bghj} + g^{dh}\nabla_{hce}R_{afbg} + g^{dh}\nabla_{che}R_{afbg} + g^{dh}\nabla_{ceh}R_{afbg}\right)$$

Curvature expansion of the connection

$$A^aA^b\Gamma^d_{ab} = A^aA^b\overset{2}{\Gamma}{}^d{}_{ab} + A^aA^b\overset{3}{\Gamma}{}^d{}_{ab} + A^aA^b\overset{4}{\Gamma}{}^d{}_{ab} + A^aA^b\overset{5}{\Gamma}{}^d{}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned}
3A^aA^b\overset{2}{\Gamma}{}^d_{ab} &= 2A^aA^bx^cg^{de}R_{acbe}\\
12A^aA^b\overset{3}{\Gamma}{}^d_{ab} &= A^aA^bx^cx^e\left(2g^{df}\nabla_aR_{bcef} + 4g^{df}\nabla_cR_{aebf} + g^{df}\nabla_fR_{acbe}\right)\\
360A^aA^b\overset{4}{\Gamma}{}^d_{ab} &= A^aA^bx^cx^ex^f\left(64g^{dg}g^{hi}R_{acbh}R_{egfi} - 32g^{dg}g^{hi}R_{aceh}R_{bgfi} - 16g^{dg}g^{hi}R_{aceh}R_{bifg} + 18g^{dg}\nabla_{ac}R_{befg} + 18g^{dg}\nabla_{ca}R_{befg} + 36g^{dg}\nabla_{ce}R_{afbg}\right.\\
&\qquad\left. - 16g^{dg}g^{hi}R_{aceh}R_{bfgi} + 9g^{dg}\nabla_{gc}R_{aebf} + 9g^{dg}\nabla_{cg}R_{aebf}\right)\\
180A^aA^b\overset{5}{\Gamma}{}^d_{ab} &= A^aA^bx^cx^ex^fx^g\left(16g^{dh}g^{ij}R_{acbi}\nabla_eR_{fhgj} + 6g^{dh}g^{ij}R_{chei}\nabla_aR_{bfgj} + 16g^{dh}g^{ij}R_{chei}\nabla_fR_{agbj} + 5g^{dh}g^{ij}R_{chei}\nabla_jR_{afbg} - 8g^{dh}g^{ij}R_{ahci}\nabla_eR_{bfgj}\right.\\
&\qquad - 4g^{dh}g^{ij}R_{aich}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_bR_{fhgj} - 8g^{dh}g^{ij}R_{acei}\nabla_fR_{bhgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bjgh} + 2g^{dh}\nabla_{ace}R_{bfgh} + 2g^{dh}\nabla_{cae}R_{bfgh}\\
&\qquad + 2g^{dh}\nabla_{cea}R_{bfgh} + 4g^{dh}\nabla_{cef}R_{agbh} - 4g^{dh}g^{ij}R_{achi}\nabla_eR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_hR_{bfgj} - 4g^{dh}g^{ij}R_{acei}\nabla_fR_{bghj} + g^{dh}\nabla_{hce}R_{afbg}\\
&\qquad\left. + g^{dh}\nabla_{che}R_{afbg} + g^{dh}\nabla_{ceh}R_{afbg}\right)
\end{aligned}$$