19 Jan 2024

The metric tensor in Riemann normal coordinates

In this notebook we compute the recursive sequences

$$g_{ab,d\underline{e}} = \left(g_{cb}\Gamma^{c}{}_{a(d}\right)_{,\underline{e})} + \left(g_{ac}\Gamma^{c}{}_{b(d}\right)_{,\underline{e})} \tag{1}$$

$$(n+3)\Gamma^{a}{}_{d(b,c\underline{e})} = (n+1)\left(R^{a}{}_{(bcd,\underline{e})} - \left(\Gamma^{a}{}_{f(c}\Gamma^{f}{}_{bd}\right)_{,\underline{e})}\right) \tag{2}$$

for $n = 1, 2, 3, \cdots$. Note in these equations that the (extended) index $\underline{e}$ contains n normal indices.

We then construct a Taylor series for the metric using

$$\begin{aligned} g_{ab}(x) &= g_{ab} + g_{ab,c}x^{c} + \frac{1}{2!}g_{ab,cd}x^{c}x^{d} + \frac{1}{3!}g_{ab,cde}x^{c}x^{d}x^{e} + \cdots \\ &= g_{ab} + \sum_{n=1}^{\infty} \frac{1}{n!}\, g_{ab,\underline{c}}\, x^{\cdot\underline{c}} \end{aligned}$$

Stage 1: Symmetrised partial derivatives of g_{ab}

In this stage, equation (1) is used to express the symmetrised partial derivatives of the metric in terms of the symmetrised partial derivatives of the connection.

$$\begin{aligned} g_{ab,c}A^{c} &= 0 \\ g_{ab,cd}A^{c}A^{d} &= g_{cb}\partial_{e}\Gamma^{c}{}_{ad}A^{d}A^{e} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}A^{d}A^{e} \\ g_{ab,cde}A^{c}A^{d}A^{e} &= g_{cb}\partial_{fe}\Gamma^{c}{}_{ad}A^{d}A^{e}A^{f} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}A^{d}A^{e}A^{f} \end{aligned}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

Now we use the results from `dGamma` to replace derivatives of Γ with partial derivatives of R. These were computed in `dGamma` using equation (2) above.

$$
\begin{aligned}
g_{ab,c}A^c &= 0 \\
g_{ab,cd}A^cA^d &= \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb} \\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}
\end{aligned}
$$

Stage 3: Replace partial derivs of R with covariant derivs of R

Next we use the results from `dRabcd` to replace the partial derivatives of R with covariant deriavtives.

$$
\begin{aligned}
g_{ab,c}A^c &= 0 \\
g_{ab,cd}A^cA^d &= -\frac{2}{3}A^cA^dR_{acbd} \\
g_{ab,cde}A^cA^dA^e &= \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}
\end{aligned}
$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

Each of the above expressions constitutues one term in the Taylor series for the metric. We also make the trivial change $A \to x$. Then we do some trivial reformatting.

$$
\begin{aligned}
g_{ab}(x) &= g_{ab} + g_{ab,c}x^c + \frac{1}{2!}g_{ab,cd}x^cx^d + \frac{1}{3!}g_{ab,cde}x^cx^dx^e + \mathcal{O}\left(\epsilon^4\right) \\
&= g_{ab} - \frac{1}{3}x^cx^dR_{acbd} - \frac{1}{6}x^cx^dx^e\nabla_cR_{adbe} + \mathcal{O}\left(\epsilon^4\right)
\end{aligned}
$$

Shared properties

```
import time

def flatten_Rabcd (obj):
    substitute (obj,$R^{a}_{b c d}   -> g^{a e} R_{e b c d}$)
    substitute (obj,$R_{a}^{b}_{c d} -> g^{b e} R_{a e c d}$)
    substitute (obj,$R_{a b}^{c}_{b} -> g^{c e} R_{a b e d}$)
    substitute (obj,$R_{a b c}^{d}   -> g^{d e} R_{a b c e}$)
    unwrap     (obj)
    return obj

def impose_rnc (obj):
    # hide the derivatives of Gamma
    substitute (obj,$\partial_{d}{\Gamma^{a}_{b c}} -> zzz_{d}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e}{\Gamma^{a}_{b c}} -> zzz_{d e}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f}{\Gamma^{a}_{b c}} -> zzz_{d e f}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g}{\Gamma^{a}_{b c}} -> zzz_{d e f g}^{a}_{b c}$,repeat=True)
    substitute (obj,$\partial_{d e f g h}{\Gamma^{a}_{b c}} -> zzz_{d e f g h}^{a}_{b c}$,repeat=True)
    # set Gamma to zero
    substitute (obj,$\Gamma^{a}_{b c} -> 0$,repeat=True)
    # recover the derivatives Gamma
    substitute (obj,$zzz_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}$,repeat=True)
    substitute (obj,$zzz_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}$,repeat=True)
    return obj

def get_xterm (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah  = Ex("numx = " + str(n))
    keep_weight (foo,bah)

    return foo
```

```
# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ A^{a}                              -> A001^{a}                 $)
    substitute (obj,$ x^{a}                              -> A002^{a}                 $)
    substitute (obj,$ g_{a b}                            -> A003_{a b}               $)
    substitute (obj,$ g^{a b}                            -> A004^{a b}               $)
    substitute (obj,$ \nabla_{e f g h}{R_{a b c d}}      -> A010_{a b c d e f g h}   $)
    substitute (obj,$ \nabla_{e f g}{R_{a b c d}}        -> A009_{a b c d e f g}     $)
    substitute (obj,$ \nabla_{e f}{R_{a b c d}}          -> A008_{a b c d e f}       $)
    substitute (obj,$ \nabla_{e}{R_{a b c d}}            -> A007_{a b c d e}         $)
    substitute (obj,$ \partial_{e f g h}{R_{a b c d}}    -> A014_{a b c d e f g h}   $)
    substitute (obj,$ \partial_{e f g}{R_{a b c d}}      -> A013_{a b c d e f g}     $)
    substitute (obj,$ \partial_{e f}{R_{a b c d}}        -> A012_{a b c d e f}       $)
    substitute (obj,$ \partial_{e}{R_{a b c d}}          -> A011_{a b c d e}         $)
    substitute (obj,$ \partial_{e f g h}{R^{a}_{b c d}} -> A018^{a}_{b c d e f g h} $)
    substitute (obj,$ \partial_{e f g}{R^{a}_{b c d}}   -> A017^{a}_{b c d e f g}   $)
    substitute (obj,$ \partial_{e f}{R^{a}_{b c d}}     -> A016^{a}_{b c d e f}     $)
    substitute (obj,$ \partial_{e}{R^{a}_{b c d}}       -> A015^{a}_{b c d e}       $)
    substitute (obj,$ R_{a b c d}                        -> A005_{a b c d}           $)
    substitute (obj,$ R^{a}_{b c d}                      -> A006^{a}_{b c d}         $)
    sort_product   (obj)
    rename_dummies (obj)
    substitute (obj,$ A001^{a}                   -> A^{a}                             $)
    substitute (obj,$ A002^{a}                   -> x^{a}                             $)
    substitute (obj,$ A003_{a b}                 -> g_{a b}                           $)
    substitute (obj,$ A004^{a b}                 -> g^{a b}                           $)
    substitute (obj,$ A005_{a b c d}             -> R_{a b c d}                       $)
    substitute (obj,$ A006^{a}_{b c d}           -> R^{a}_{b c d}                     $)
    substitute (obj,$ A007_{a b c d e}           -> \nabla_{e}{R_{a b c d}}           $)
    substitute (obj,$ A008_{a b c d e f}         -> \nabla_{e f}{R_{a b c d}}         $)
    substitute (obj,$ A009_{a b c d e f g}       -> \nabla_{e f g}{R_{a b c d}}       $)
    substitute (obj,$ A010_{a b c d e f g h}     -> \nabla_{e f g h}{R_{a b c d}}     $)
    substitute (obj,$ A011_{a b c d e}           -> \partial_{e}{R_{a b c d}}         $)
    substitute (obj,$ A012_{a b c d e f}         -> \partial_{e f}{R_{a b c d}}       $)
    substitute (obj,$ A013_{a b c d e f g}       -> \partial_{e f g}{R_{a b c d}}     $)
    substitute (obj,$ A014_{a b c d e f g h}     -> \partial_{e f g h}{R_{a b c d}}   $)
    substitute (obj,$ A015^{a}_{b c d e}         -> \partial_{e}{R^{a}_{b c d}}       $)
```

```
    substitute (obj,$ A016^{a}_{b c d e f}      -> \partial_{e f}{R^{a}_{b c d}}     $)
    substitute (obj,$ A017^{a}_{b c d e f g}    -> \partial_{e f g}{R^{a}_{b c d}}   $)
    substitute (obj,$ A018^{a}_{b c d e f g h}  -> \partial_{e f g h}{R^{a}_{b c d}} $)

    return obj

def reformat_xterm (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute     (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise   (bah)
    factor_out     (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale_xterm (obj,scale):
    foo  = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute  (bah)
    factor_out  (bah,$x^{a?}$)
    return bah

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

\nabla{#}::Derivative.
\partial{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::InverseMetric.
g_{a}^{b}::KroneckerDelta.
g^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R_{a b c}^{d}::RiemannTensor.
```

```
\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g_{a b}::Depends(\partial{#}).
R_{a b c d}::Depends(\partial{#}).
R^{a}_{b c d}::Depends(\partial{#}).
\Gamma^{a}_{b c}::Depends(\partial{#}).

R_{a b c d}::Depends(\nabla{#}).
R^{a}_{b c d}::Depends(\nabla{#}).
```

Stage 1: Symmetrised partial derivatives of g_{ab}

```
beg_stage_1 = time.time()

# symmetrised partial derivatives of g_{ab}

gab00:=g_{a b}.                                            # cdb (gab00.101,gab00)

gab01:=g_{c b}\Gamma^{c}_{a d} + g_{a c}\Gamma^{c}_{b d}.  # cdb (gab01.101,gab01)

gab02:=\partial_{e}{ @(gab01) }.                           # cdb (gab02.101,gab02)
distribute   (gab02)                                       # cdb (gab02.102,gab02)
product_rule (gab02)                                       # cdb (gab02.103,gab02)
substitute   (gab02, $\partial_{d}{g_{a b}} -> @(gab01)$)  # cdb (gab02.104,gab02)
distribute   (gab02)                                       # cdb (gab02.105,gab02)

gab03:=\partial_{f}{ @(gab02) }.                           # cdb (gab03.101,gab03)
distribute   (gab03)                                       # cdb (gab03.102,gab03)
product_rule (gab03)                                       # cdb (gab03.103,gab03)
substitute   (gab03, $\partial_{d}{g_{a b}} -> @(gab01)$)  # cdb (gab03.104,gab03)
distribute   (gab03)                                       # cdb (gab03.105,gab03)

gab04:=\partial_{g}{ @(gab03) }.                           # cdb (gab04.101,gab04)
distribute   (gab04)                                       # cdb (gab04.102,gab04)
product_rule (gab04)                                       # cdb (gab04.103,gab04)
substitute   (gab04, $\partial_{d}{g_{a b}} -> @(gab01)$)  # cdb (gab04.104,gab04)
distribute   (gab04)                                       # cdb (gab04.105,gab04)

gab05:=\partial_{h}{ @(gab04) }.                           # cdb (gab05.101,gab05)
distribute   (gab05)                                       # cdb (gab05.102,gab05)
product_rule (gab05)                                       # cdb (gab05.103,gab05)
substitute   (gab05, $\partial_{d}{g_{a b}} -> @(gab01)$)  # cdb (gab05.104,gab05)
distribute   (gab05)                                       # cdb (gab05.105,gab05)

gab00 = impose_rnc (gab00)   # cdb (gab00.102,gab00)
gab01 = impose_rnc (gab01)   # cdb (gab01.102,gab01)
gab02 = impose_rnc (gab02)   # cdb (gab02.106,gab02)
gab03 = impose_rnc (gab03)   # cdb (gab03.106,gab03)
```

```
gab04 = impose_rnc (gab04)   # cdb (gab04.106,gab04)
gab05 = impose_rnc (gab05)   # cdb (gab05.106,gab05)
```

$$\texttt{gab00.101} := g_{ab}$$

$$\texttt{gab00.102} := g_{ab}$$

$$\texttt{gab01.101} := g_{cb}\Gamma^{c}{}_{ad} + g_{ac}\Gamma^{c}{}_{bd}$$

$$\texttt{gab01.102} := 0$$

$$\texttt{gab02.101} := \partial_{e}\left(g_{cb}\Gamma^{c}{}_{ad} + g_{ac}\Gamma^{c}{}_{bd}\right)$$

$$\texttt{gab02.102} := \partial_{e}\left(g_{cb}\Gamma^{c}{}_{ad}\right) + \partial_{e}\left(g_{ac}\Gamma^{c}{}_{bd}\right)$$

$$\texttt{gab02.103} := \partial_{e}g_{cb}\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + \partial_{e}g_{ac}\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab02.104} := \left(g_{fb}\Gamma^{f}{}_{ce} + g_{cf}\Gamma^{f}{}_{be}\right)\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + \left(g_{fc}\Gamma^{f}{}_{ae} + g_{af}\Gamma^{f}{}_{ce}\right)\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab02.105} := g_{fb}\Gamma^{f}{}_{ce}\Gamma^{c}{}_{ad} + g_{cf}\Gamma^{f}{}_{be}\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{fc}\Gamma^{f}{}_{ae}\Gamma^{c}{}_{bd} + g_{af}\Gamma^{f}{}_{ce}\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab02.106} := g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}$$

$$\texttt{gab03.101} := \partial_{f}\left(g_{gb}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{cg}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{gc}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}\right)$$

$$\texttt{gab03.102} := \partial_{f}\left(g_{gb}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad}\right) + \partial_{f}\left(g_{cg}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad}\right) + \partial_{f}\left(g_{cb}\partial_{e}\Gamma^{c}{}_{ad}\right) + \partial_{f}\left(g_{gc}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd}\right) + \partial_{f}\left(g_{ag}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd}\right) + \partial_{f}\left(g_{ac}\partial_{e}\Gamma^{c}{}_{bd}\right)$$

$$\begin{aligned}\texttt{gab03.103} := {} & \partial_{f}g_{gb}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + \partial_{f}g_{cg}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\partial_{f}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\Gamma^{g}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + \partial_{f}g_{cb}\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{fe}\Gamma^{c}{}_{ad}\\ & + \partial_{f}g_{gc}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\partial_{f}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + \partial_{f}g_{ag}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + \partial_{f}g_{ac}\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.104} := {} & \left(g_{hb}\Gamma^{h}{}_{gf} + g_{gh}\Gamma^{h}{}_{bf}\right)\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + \left(g_{hg}\Gamma^{h}{}_{cf} + g_{ch}\Gamma^{h}{}_{gf}\right)\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\partial_{f}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad}\\ & + g_{cg}\Gamma^{g}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + \left(g_{gb}\Gamma^{g}{}_{cf} + g_{cg}\Gamma^{g}{}_{bf}\right)\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + \left(g_{hc}\Gamma^{h}{}_{gf} + g_{gh}\Gamma^{h}{}_{cf}\right)\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\partial_{f}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd}\\ & + \left(g_{hg}\Gamma^{h}{}_{af} + g_{ah}\Gamma^{h}{}_{gf}\right)\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + \left(g_{gc}\Gamma^{g}{}_{af} + g_{ag}\Gamma^{g}{}_{cf}\right)\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}\end{aligned}$$

$$\begin{aligned}\texttt{gab03.105} := {} & g_{hb}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gh}\Gamma^{h}{}_{bf}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{hg}\Gamma^{h}{}_{cf}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad} + g_{cg}\partial_{f}\Gamma^{g}{}_{be}\Gamma^{c}{}_{ad}\\ & + g_{cg}\Gamma^{g}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{gb}\Gamma^{g}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{cg}\Gamma^{g}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + g_{hc}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gh}\Gamma^{h}{}_{cf}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\partial_{f}\Gamma^{g}{}_{ae}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd}\\ & + g_{hg}\Gamma^{h}{}_{af}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{gf}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\partial_{f}\Gamma^{g}{}_{ce}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{gc}\Gamma^{g}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ag}\Gamma^{g}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}\end{aligned}$$

$$\texttt{gab03.106} := g_{cb}\partial_{fe}\Gamma^{c}{}_{ad} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}$$

$$\texttt{gab04.101} := \partial_g \left(g_{hb}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ib}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ci}\partial_f\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ci}\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ci}\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} + g_{cb}\partial_{fe}\Gamma^c{}_{ad} + g_{hc}\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\partial_f\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ic}\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ac}\partial_{fe}\Gamma^c{}_{bd}\right)$$

$$\texttt{gab04.102} := \partial_g \left(g_{hb}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ih}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ib}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ib}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}\right) + \partial_g \left(g_{hi}\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ch}\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ci}\partial_f\Gamma^i{}_{be}\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ci}\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ib}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}\right) + \partial_g \left(g_{ci}\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}\right) + \partial_g \left(g_{cb}\partial_{fe}\Gamma^c{}_{ad}\right) + \partial_g \left(g_{hc}\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ih}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ic}\partial_f\Gamma^i{}_{ae}\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ic}\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}\right) + \partial_g \left(g_{hi}\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ah}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ai}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ai}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ic}\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ai}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}\right) + \partial_g \left(g_{ac}\partial_{fe}\Gamma^c{}_{bd}\right)$$

$$\texttt{gab04.103} := \partial_g g_{hb}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{hb}\Gamma^h{}_{if}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + \partial_g g_{ih}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\partial_g\Gamma^h{}_{bf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bf}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ih}\Gamma^h{}_{bf}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + \partial_g g_{ib}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ib}\partial_{gf}\Gamma^i{}_{ce}\Gamma^c{}_{ad} + g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad} + \partial_g g_{ib}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{ce}\partial_{gf}\Gamma^c{}_{ad} + \partial_g g_{hi}\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\partial_g\Gamma^h{}_{cf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cf}\partial_g\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{hi}\Gamma^h{}_{cf}\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + \partial_g g_{ch}\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ch}\Gamma^h{}_{if}\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + \partial_g g_{ci}\partial_f\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ci}\partial_{gf}\Gamma^i{}_{be}\Gamma^c{}_{ad} + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad} + \partial_g g_{ci}\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad} + g_{ci}\Gamma^i{}_{be}\partial_{gf}\Gamma^c{}_{ad} + \partial_g g_{ib}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad} + g_{ib}\Gamma^i{}_{cf}\partial_{ge}\Gamma^c{}_{ad} + \partial_g g_{ci}\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad} + g_{ci}\Gamma^i{}_{bf}\partial_{ge}\Gamma^c{}_{ad} + \partial_g g_{cb}\partial_{fe}\Gamma^c{}_{ad} + g_{cb}\partial_{gfe}\Gamma^c{}_{ad} + \partial_g g_{hc}\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hc}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{hc}\Gamma^h{}_{if}\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + \partial_g g_{ih}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\partial_g\Gamma^h{}_{cf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cf}\partial_g\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ih}\Gamma^h{}_{cf}\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + \partial_g g_{ic}\partial_f\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\partial_{gf}\Gamma^i{}_{ae}\Gamma^c{}_{bd} + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd} + \partial_g g_{ic}\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd} + g_{ic}\Gamma^i{}_{ae}\partial_{gf}\Gamma^c{}_{bd} + \partial_g g_{hi}\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\partial_g\Gamma^h{}_{af}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{af}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{hi}\Gamma^h{}_{af}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + \partial_g g_{ah}\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\partial_g\Gamma^h{}_{if}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{if}\partial_g\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ah}\Gamma^h{}_{if}\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + \partial_g g_{ai}\partial_f\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\partial_{gf}\Gamma^i{}_{ce}\Gamma^c{}_{bd} + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd} + \partial_g g_{ai}\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{ce}\partial_{gf}\Gamma^c{}_{bd} + \partial_g g_{ic}\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd} + g_{ic}\Gamma^i{}_{af}\partial_{ge}\Gamma^c{}_{bd} + \partial_g g_{ai}\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd} + g_{ai}\Gamma^i{}_{cf}\partial_{ge}\Gamma^c{}_{bd} + \partial_g g_{ac}\partial_{fe}\Gamma^c{}_{bd} + g_{ac}\partial_{gfe}\Gamma^c{}_{bd}$$

$$
\begin{aligned}
\texttt{gab04.104} := {} & \left(g_{jb}\Gamma^{j}{}_{hg} + g_{hj}\Gamma^{j}{}_{bg}\right)\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + \left(g_{jh}\Gamma^{j}{}_{ig} + g_{ij}\Gamma^{j}{}_{hg}\right)\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} \\
& + g_{ih}\partial_{g}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bf}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + \left(g_{hb}\Gamma^{h}{}_{ig} + g_{ih}\Gamma^{h}{}_{bg}\right)\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ib}\partial_{gf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ib}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} \\
& + \left(g_{hb}\Gamma^{h}{}_{ig} + g_{ih}\Gamma^{h}{}_{bg}\right)\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{ib}\Gamma^{i}{}_{ce}\partial_{gf}\Gamma^{c}{}_{ad} + \left(g_{ji}\Gamma^{j}{}_{hg} + g_{hj}\Gamma^{j}{}_{ig}\right)\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\partial_{g}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} \\
& + g_{hi}\Gamma^{h}{}_{cf}\partial_{g}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + \left(g_{jh}\Gamma^{j}{}_{cg} + g_{cj}\Gamma^{j}{}_{hg}\right)\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} \\
& + g_{ch}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + \left(g_{hi}\Gamma^{h}{}_{cg} + g_{ch}\Gamma^{h}{}_{ig}\right)\partial_{f}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ci}\partial_{gf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ci}\partial_{f}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + \left(g_{hi}\Gamma^{h}{}_{cg} + g_{ch}\Gamma^{h}{}_{ig}\right)\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} \\
& + g_{ci}\partial_{g}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{ci}\Gamma^{i}{}_{be}\partial_{gf}\Gamma^{c}{}_{ad} + \left(g_{hb}\Gamma^{h}{}_{ig} + g_{ih}\Gamma^{h}{}_{bg}\right)\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ib}\Gamma^{i}{}_{cf}\partial_{ge}\Gamma^{c}{}_{ad} + \left(g_{hi}\Gamma^{h}{}_{cg} + g_{ch}\Gamma^{h}{}_{ig}\right)\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} \\
& + g_{ci}\partial_{g}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ci}\Gamma^{i}{}_{bf}\partial_{ge}\Gamma^{c}{}_{ad} + \left(g_{hb}\Gamma^{h}{}_{cg} + g_{ch}\Gamma^{h}{}_{bg}\right)\partial_{fe}\Gamma^{c}{}_{ad} + g_{cb}\partial_{gfe}\Gamma^{c}{}_{ad} + \left(g_{jc}\Gamma^{j}{}_{hg} + g_{hj}\Gamma^{j}{}_{cg}\right)\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{hc}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} \\
& + g_{hc}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{hc}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + \left(g_{jh}\Gamma^{j}{}_{ig} + g_{ij}\Gamma^{j}{}_{hg}\right)\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\partial_{g}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cf}\partial_{g}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} \\
& + \left(g_{hc}\Gamma^{h}{}_{ig} + g_{ih}\Gamma^{h}{}_{cg}\right)\partial_{f}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ic}\partial_{gf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ic}\partial_{f}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + \left(g_{hc}\Gamma^{h}{}_{ig} + g_{ih}\Gamma^{h}{}_{cg}\right)\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\partial_{g}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\Gamma^{i}{}_{ae}\partial_{gf}\Gamma^{c}{}_{bd} \\
& + \left(g_{ji}\Gamma^{j}{}_{hg} + g_{hj}\Gamma^{j}{}_{ig}\right)\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hi}\partial_{g}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hi}\Gamma^{h}{}_{af}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hi}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + \left(g_{jh}\Gamma^{j}{}_{ag} + g_{aj}\Gamma^{j}{}_{hg}\right)\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} \\
& + g_{ah}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + \left(g_{hi}\Gamma^{h}{}_{ag} + g_{ah}\Gamma^{h}{}_{ig}\right)\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\partial_{gf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} \\
& + \left(g_{hi}\Gamma^{h}{}_{ag} + g_{ah}\Gamma^{h}{}_{ig}\right)\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{ce}\partial_{gf}\Gamma^{c}{}_{bd} + \left(g_{hc}\Gamma^{h}{}_{ig} + g_{ih}\Gamma^{h}{}_{cg}\right)\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ic}\partial_{g}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} \\
& + g_{ic}\Gamma^{i}{}_{af}\partial_{ge}\Gamma^{c}{}_{bd} + \left(g_{hi}\Gamma^{h}{}_{ag} + g_{ah}\Gamma^{h}{}_{ig}\right)\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{cf}\partial_{ge}\Gamma^{c}{}_{bd} + \left(g_{hc}\Gamma^{h}{}_{ag} + g_{ah}\Gamma^{h}{}_{cg}\right)\partial_{fe}\Gamma^{c}{}_{bd} + g_{ac}\partial_{gfe}\Gamma^{c}{}_{bd}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.105} := {} & g_{jb}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hj}\Gamma^{j}{}_{bg}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + g_{jh}\Gamma^{j}{}_{ig}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} \\
& + g_{ij}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\partial_{g}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bf}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bf}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{ig}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bg}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} \\
& + g_{ib}\partial_{gf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{ad} + g_{ib}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{ih}\Gamma^{h}{}_{bg}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{ib}\Gamma^{i}{}_{ce}\partial_{gf}\Gamma^{c}{}_{ad} + g_{ji}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} \\
& + g_{hj}\Gamma^{j}{}_{ig}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\partial_{g}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cf}\partial_{g}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + g_{jh}\Gamma^{j}{}_{cg}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{cj}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} \\
& + g_{ch}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{if}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cg}\partial_{f}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{ig}\partial_{f}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} + g_{ci}\partial_{gf}\Gamma^{i}{}_{be}\Gamma^{c}{}_{ad} \\
& + g_{ci}\partial_{f}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cg}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{ci}\partial_{g}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{ci}\Gamma^{i}{}_{be}\partial_{gf}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} \\
& + g_{ih}\Gamma^{h}{}_{bg}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ib}\Gamma^{i}{}_{cf}\partial_{ge}\Gamma^{c}{}_{ad} + g_{hi}\Gamma^{h}{}_{cg}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ci}\partial_{g}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} \\
& + g_{ci}\Gamma^{i}{}_{bf}\partial_{ge}\Gamma^{c}{}_{ad} + g_{hb}\Gamma^{h}{}_{cg}\partial_{fe}\Gamma^{c}{}_{ad} + g_{ch}\Gamma^{h}{}_{bg}\partial_{fe}\Gamma^{c}{}_{ad} + g_{cb}\partial_{gfe}\Gamma^{c}{}_{ad} + g_{jc}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{hj}\Gamma^{j}{}_{cg}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{hc}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} \\
& + g_{hc}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{hc}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + g_{jh}\Gamma^{j}{}_{ig}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ij}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\partial_{g}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cf}\partial_{g}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} \\
& + g_{ih}\Gamma^{h}{}_{cf}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + g_{hc}\Gamma^{h}{}_{ig}\partial_{f}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cg}\partial_{f}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ic}\partial_{gf}\Gamma^{i}{}_{ae}\Gamma^{c}{}_{bd} + g_{ic}\partial_{f}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + g_{hc}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} \\
& + g_{ih}\Gamma^{h}{}_{cg}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\partial_{g}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\Gamma^{i}{}_{ae}\partial_{gf}\Gamma^{c}{}_{bd} + g_{ji}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hj}\Gamma^{j}{}_{ig}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hi}\partial_{g}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} \\
& + g_{hi}\Gamma^{h}{}_{af}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{hi}\Gamma^{h}{}_{af}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + g_{jh}\Gamma^{j}{}_{ag}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{aj}\Gamma^{j}{}_{hg}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\partial_{g}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{if}\partial_{g}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} \\
& + g_{ah}\Gamma^{h}{}_{if}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + g_{hi}\Gamma^{h}{}_{ag}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{ig}\partial_{f}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\partial_{gf}\Gamma^{i}{}_{ce}\Gamma^{c}{}_{bd} + g_{ai}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + g_{hi}\Gamma^{h}{}_{ag}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} \\
& + g_{ah}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{ce}\partial_{gf}\Gamma^{c}{}_{bd} + g_{hc}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ih}\Gamma^{h}{}_{cg}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ic}\partial_{g}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ic}\Gamma^{i}{}_{af}\partial_{ge}\Gamma^{c}{}_{bd} \\
& + g_{hi}\Gamma^{h}{}_{ag}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{ig}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\Gamma^{i}{}_{cf}\partial_{ge}\Gamma^{c}{}_{bd} + g_{hc}\Gamma^{h}{}_{ag}\partial_{fe}\Gamma^{c}{}_{bd} + g_{ah}\Gamma^{h}{}_{cg}\partial_{fe}\Gamma^{c}{}_{bd} + g_{ac}\partial_{gfe}\Gamma^{c}{}_{bd}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{gab04.106} := {} & g_{ib}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{ad} + g_{ci}\partial_{f}\Gamma^{i}{}_{be}\partial_{g}\Gamma^{c}{}_{ad} + g_{ci}\partial_{g}\Gamma^{i}{}_{be}\partial_{f}\Gamma^{c}{}_{ad} + g_{ib}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{ad} + g_{ci}\partial_{g}\Gamma^{i}{}_{bf}\partial_{e}\Gamma^{c}{}_{ad} + g_{cb}\partial_{gfe}\Gamma^{c}{}_{ad} \\
& + g_{ic}\partial_{f}\Gamma^{i}{}_{ae}\partial_{g}\Gamma^{c}{}_{bd} + g_{ic}\partial_{g}\Gamma^{i}{}_{ae}\partial_{f}\Gamma^{c}{}_{bd} + g_{ai}\partial_{f}\Gamma^{i}{}_{ce}\partial_{g}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{ce}\partial_{f}\Gamma^{c}{}_{bd} + g_{ic}\partial_{g}\Gamma^{i}{}_{af}\partial_{e}\Gamma^{c}{}_{bd} + g_{ai}\partial_{g}\Gamma^{i}{}_{cf}\partial_{e}\Gamma^{c}{}_{bd} + g_{ac}\partial_{gfe}\Gamma^{c}{}_{bd}
\end{aligned}
$$

```
# prepare first six terms in the Taylor series expansion of g_{ab}(x)

term0:= @(gab00).
distribute (term0)                              # cdb(term0.200,term0)

term1:= @(gab01) A^d.
distribute (term1)                              # cdb(term1.200,term1)

term2:= @(gab02) A^d A^e.
distribute (term2)                              # cdb(term2.200,term2)

term3:= @(gab03) A^d A^e A^f.
distribute (term3)                              # cdb(term3.200,term3)

term4:= @(gab04) A^d A^e A^f A^g.
distribute (term4)                              # cdb(term4.200,term4)

term5:= @(gab05) A^d A^e A^f A^g A^h.
distribute (term5)                              # cdb(term5.200,term5)

end_stage_1 = time.time()
```

$$\texttt{term0.200} := g_{ab}$$

$$\texttt{term1.200} := 0$$

$$\texttt{term2.200} := g_{cb}\partial_{e}\Gamma^{c}{}_{ad}A^{d}A^{e} + g_{ac}\partial_{e}\Gamma^{c}{}_{bd}A^{d}A^{e}$$

$$\texttt{term3.200} := g_{cb}\partial_{fe}\Gamma^{c}{}_{ad}A^{d}A^{e}A^{f} + g_{ac}\partial_{fe}\Gamma^{c}{}_{bd}A^{d}A^{e}A^{f}$$

Stage 2: Replace derivatives of Γ with partial derivs of R

```
import cdblib

beg_stage_2 = time.time()

dGamma01 = cdblib.get ('dGamma01','dGamma.json')  # cdb(dGamma01.300,dGamma01)
dGamma02 = cdblib.get ('dGamma02','dGamma.json')  # cdb(dGamma02.300,dGamma02)
dGamma03 = cdblib.get ('dGamma03','dGamma.json')  # cdb(dGamma03.300,dGamma03)
dGamma04 = cdblib.get ('dGamma04','dGamma.json')  # cdb(dGamma04.300,dGamma04)
dGamma05 = cdblib.get ('dGamma05','dGamma.json')  # cdb(dGamma05.300,dGamma05)

# replace partial derivs of \Gamma with products and derivs of Riemann tensor

substitute (term2,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term2.301,term2)
substitute (term2,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term2.302,term2)
distribute (term2)                                                                                          # cdb(term2.303,term2)

substitute (term3,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term3.301,term3)
substitute (term3,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term3.302,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term3.303,term3)
substitute (term3,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term3.304,term3)
distribute (term3)                                                                                          # cdb(term3.305,term3)

substitute (term4,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term4.301,term4)
substitute (term4,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term4.302,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term4.303,term4)
substitute (term4,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term4.304,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term4.305,term4)
substitute (term4,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)                    # cdb(term4.306,term4)
distribute (term4)                                                                                          # cdb(term4.307,term4)

substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.301,term5)
substitute (term5,$\partial_{c e f g}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f}A^{g} -> @(dGamma04)$,repeat=True)  # cdb(term5.302,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term5.303,term5)
substitute (term5,$\partial_{c e f}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e}A^{f} -> @(dGamma03)$,repeat=True)      # cdb(term5.304,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{d b}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term5.305,term5)
substitute (term5,$\partial_{c e}{\Gamma^{a}_{b d}}A^{c}A^{b}A^{e} -> @(dGamma02)$,repeat=True)             # cdb(term5.306,term5)
```

```
substitute (term5,$\partial_{c}{\Gamma^{a}_{b d}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)          # cdb(term5.307,term5)
substitute (term5,$\partial_{c}{\Gamma^{a}_{d b}}A^{c}A^{b} -> @(dGamma01)$,repeat=True)          # cdb(term5.308,term5)
distribute (term5)                                                                                 # cdb(term5.309,term5)

end_stage_2 = time.time()

# ---------------------------------------------------------------------------------------------
# this block of Xterms only produces formatted output, it's not part of the main computation
# ---------------------------------------------------------------------------------------------

# the metric in terms of partial derivatives of Rabcd

metric:=@(term0)
      + (1/1) @(term1)  # zero
      + (1/2) @(term2)
      + (1/6) @(term3)
      + (1/24) @(term4)
      + (1/120) @(term5).  # cdb(metric.301,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.302,metric)

# reformat and tidy up

Xterm0 := @(term0).
Xterm1 := (1/1) @(term1).
Xterm2 := (1/2) @(term2).
Xterm3 := (1/6) @(term3).
Xterm4 := (1/24) @(term4).
Xterm5 := (1/120) @(term5).

substitute (Xterm0,$A^{a} -> x^{a}$)
substitute (Xterm1,$A^{a} -> x^{a}$)
substitute (Xterm2,$A^{a} -> x^{a}$)
substitute (Xterm3,$A^{a} -> x^{a}$)
substitute (Xterm4,$A^{a} -> x^{a}$)
substitute (Xterm5,$A^{a} -> x^{a}$)

substitute (Xterm2,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm2.301,Xterm2)
```

```
substitute (Xterm3,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{a b} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm5.301,Xterm5)

substitute (Xterm2,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm2.301,Xterm2)
substitute (Xterm3,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm3.301,Xterm3)
substitute (Xterm4,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm4.301,Xterm4)
substitute (Xterm5,$g_{b a} \partial_{c}{R^{b}_{d e f}} -> \partial_{c}{R_{a d e f}}$)  # cdb(Xterm5.301,Xterm5)

eliminate_metric (Xterm2)  # cdb(Xterm2.302,Xterm2)
eliminate_metric (Xterm3)  # cdb(Xterm3.302,Xterm3)
eliminate_metric (Xterm4)  # cdb(Xterm4.302,Xterm4)
eliminate_metric (Xterm5)  # cdb(Xterm5.302,Xterm5)

sort_product     (Xterm2)  # cdb(Xterm2.303,Xterm2)
sort_product     (Xterm3)  # cdb(Xterm3.303,Xterm3)
sort_product     (Xterm4)  # cdb(Xterm4.303,Xterm4)
sort_product     (Xterm5)  # cdb(Xterm5.303,Xterm5)

rename_dummies   (Xterm2)  # cdb(Xterm2.304,Xterm2)
rename_dummies   (Xterm3)  # cdb(Xterm3.304,Xterm3)
rename_dummies   (Xterm4)  # cdb(Xterm4.304,Xterm4)
rename_dummies   (Xterm5)  # cdb(Xterm5.304,Xterm5)

canonicalise     (Xterm2)  # cdb(Xterm2.305,Xterm2)
canonicalise     (Xterm3)  # cdb(Xterm3.305,Xterm3)
canonicalise     (Xterm4)  # cdb(Xterm4.305,Xterm4)
canonicalise     (Xterm5)  # cdb(Xterm5.305,Xterm5)

# push upper index to the left
def tidy_Rabcd (obj):
    substitute (obj,$R_{a b c}^{d} -> - R^{d}_{c a b}$)
    substitute (obj,$R_{a b}^{c}_{d} -> R^{c}_{d a b}$)
    substitute (obj,$R_{a}^{b}_{c d} -> - R^{b}_{a c d}$)
    return obj

Xterm0 = tidy_Rabcd (Xterm0)  # cdb(Xterm0.666,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2)  # cdb(Xterm2.666,Xterm2)
```

```
Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.666,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.666,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.666,Xterm5)

Xterm0 = reformat_xterm (Xterm0,  1)    # cdb(Xterm0.301,Xterm0)
Xterm2 = reformat_xterm (Xterm2,  3)    # cdb(Xterm2.301,Xterm2)
Xterm3 = reformat_xterm (Xterm3,  6)    # cdb(Xterm3.301,Xterm3)
Xterm4 = reformat_xterm (Xterm4,360)    # cdb(Xterm4.301,Xterm4)
Xterm5 = reformat_xterm (Xterm5,180)    # cdb(Xterm5.301,Xterm5)

# canonicalise from reformat_xterm will slide upper index from left hand side
# so now we slide the upper index back to the left

Xterm0 = tidy_Rabcd (Xterm0)  # cdb(Xterm0.667,Xterm0)
Xterm2 = tidy_Rabcd (Xterm2)  # cdb(Xterm2.667,Xterm2)
Xterm3 = tidy_Rabcd (Xterm3)  # cdb(Xterm3.667,Xterm3)
Xterm4 = tidy_Rabcd (Xterm4)  # cdb(Xterm4.667,Xterm4)
Xterm5 = tidy_Rabcd (Xterm5)  # cdb(Xterm5.667,Xterm5)

# metric to 3rd, 4th, 5th and 6th order terms in powers of x^a

Metric3 := @(Xterm0) + @(Xterm2).                                     # cdb (Metric3.301,Metric3)
Metric4 := @(Xterm0) + @(Xterm2) + @(Xterm3).                         # cdb (Metric4.301,Metric4)
Metric5 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).             # cdb (Metric5.301,Metric5)
Metric6 := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (Metric6.301,Metric6)

# ----------------------------------------------------------------------------------
# end of format block
# ----------------------------------------------------------------------------------
```

$$\texttt{term2.301} := g_{cb}\partial_e\Gamma^c{}_{ad}A^dA^e + g_{ac}\partial_e\Gamma^c{}_{bd}A^dA^e$$

$$\texttt{term2.302} := \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb}$$

$$\texttt{term2.303} := \frac{1}{3}g_{cb}A^dA^eR^c{}_{dea} + \frac{1}{3}g_{ac}A^dA^eR^c{}_{deb}$$

$$\texttt{term3.301} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.302} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.303} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.304} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\texttt{term3.305} := \frac{1}{2}g_{cb}A^eA^dA^f\partial_eR^c{}_{dfa} + \frac{1}{2}g_{ac}A^eA^dA^f\partial_eR^c{}_{dfb}$$

$$\begin{aligned}\texttt{term4.301} := {}& g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g\\
&+ g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g\\
&+ g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g\\
&+ g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g\\
&+ g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)\end{aligned}$$

$$\begin{aligned}\texttt{term4.302} := {}& g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g\\
&+ g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g\\
&+ g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g\\
&+ g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g\\
&+ g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)\end{aligned}$$

$$
\begin{aligned}
\texttt{term4.303} := {} & g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g \\
& + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g \\
& + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g \\
& + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g \\
& + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.304} := {} & g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g \\
& + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g \\
& + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g \\
& + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g \\
& + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.305} := {} & g_{ib}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_f\Gamma^i{}_{be}\partial_g\Gamma^c{}_{ad}A^dA^eA^fA^g \\
& + g_{ci}\partial_g\Gamma^i{}_{be}\partial_f\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ib}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g + g_{ci}\partial_g\Gamma^i{}_{bf}\partial_e\Gamma^c{}_{ad}A^dA^eA^fA^g \\
& + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + g_{ic}\partial_f\Gamma^i{}_{ae}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g \\
& + g_{ic}\partial_g\Gamma^i{}_{ae}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_f\Gamma^i{}_{ce}\partial_g\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ai}\partial_g\Gamma^i{}_{ce}\partial_f\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ic}\partial_g\Gamma^i{}_{af}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g \\
& + g_{ai}\partial_g\Gamma^i{}_{cf}\partial_e\Gamma^c{}_{bd}A^dA^eA^fA^g + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.306} := {} & \frac{1}{9}g_{ib}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i{}_{efb}A^dA^gR^c{}_{dga} \\
& + \frac{1}{9}g_{ci}A^eA^gR^i{}_{egb}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ib}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i{}_{fgb}A^dA^eR^c{}_{dea} \\
& + g_{cb}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dga} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa}\right) + \frac{1}{9}g_{ic}A^eA^fR^i{}_{efa}A^dA^gR^c{}_{dgb} \\
& + \frac{1}{9}g_{ic}A^eA^gR^i{}_{ega}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ai}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i{}_{fga}A^dA^eR^c{}_{deb} \\
& + \frac{1}{9}g_{ai}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{deb} + g_{ac}\left(\frac{3}{5}A^dA^gA^fA^e\partial_{ef}R^c{}_{dgb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\right)
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.307} := {}& \frac{1}{9} g_{ib} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ib} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfa} + \frac{1}{9} g_{ci} A^e A^f R^i{}_{efb} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ci} A^e A^g R^i{}_{egb} A^d A^f R^c{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{dea} + \frac{1}{9} g_{ci} A^f A^g R^i{}_{fgb} A^d A^e R^c{}_{dea} + \frac{3}{5} g_{cb} A^d A^g A^f A^e \partial_{ef} R^c{}_{dga} - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{dea} \\
& - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} + \frac{1}{9} g_{ic} A^e A^f R^i{}_{efa} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ic} A^e A^g R^i{}_{ega} A^d A^f R^c{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ai} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfb} + \frac{1}{9} g_{ic} A^f A^g R^i{}_{fga} A^d A^e R^c{}_{deb} + \frac{1}{9} g_{ai} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{deb} \\
& + \frac{3}{5} g_{ac} A^d A^g A^f A^e \partial_{ef} R^c{}_{dgb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{deb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb}
\end{aligned}
$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe}$$

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f \left(-3R_{bcdg}R^g{}_{fae} - 13R_{acdg}R^g{}_{fbe} - 9g_{bg}\partial_{cd}R^g{}_{fae} - 9g_{ag}\partial_{cd}R^g{}_{fbe}\right)$$

$$\begin{aligned} g_{ab}(x) = g_{ab} &- \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \frac{1}{360}x^c x^d x^e x^f \left(-3R_{bcdg}R^g{}_{fae} - 13R_{acdg}R^g{}_{fbe} - 9g_{bg}\partial_{cd}R^g{}_{fae} - 9g_{ag}\partial_{cd}R^g{}_{fbe}\right) \\ &+ \frac{1}{180}x^c x^d x^e x^f x^g \left(-3R^h{}_{dac}\partial_e R_{bfgh} - R_{bcdh}\partial_e R^h{}_{gaf} - 3R^h{}_{dbc}\partial_e R_{afgh} - g_{bh}\partial_{cde}R^h{}_{gaf} - R_{acdh}\partial_e R^h{}_{gbf} - g_{ah}\partial_{cde}R^h{}_{gbf}\right) \end{aligned}$$

Stage 3: Replace partial derivs of R with covariant derivs of R

```
beg_stage_3 = time.time()

# now convert partial derivs of Rabcd to covariant derivs

dRabcd01 = cdblib.get ('dRabcd01','dRabcd.json')  # cdb(dRabcd01.400,dRabcd01)
dRabcd02 = cdblib.get ('dRabcd02','dRabcd.json')  # cdb(dRabcd02.400,dRabcd02)
dRabcd03 = cdblib.get ('dRabcd03','dRabcd.json')  # cdb(dRabcd03.400,dRabcd03)

# term1 & term2 need no special care, just a bit of tidying

eliminate_metric (term1)   # cdb(term1.401,term1)
sort_product     (term1)   # cdb(term1.402,term1)
rename_dummies   (term1)   # cdb(term1.403,term1)
canonicalise     (term1)   # cdb(term1.404,term1)

eliminate_metric (term2)   # cdb(term2.401,term2)
sort_product     (term2)   # cdb(term2.402,term2)
rename_dummies   (term2)   # cdb(term2.403,term2)
canonicalise     (term2)   # cdb(term2.404,term2)

# replace partial derivatives of Riemann tensor in term3, term4 etc. with covariant derivatives of Rabcd

tmp01 := @(dRabcd01).      # cdb(tmp01.403,tmp01)
tmp02 := @(dRabcd02).      # cdb(tmp02.403,tmp02)
tmp03 := @(dRabcd03).      # cdb(tmp03.403,tmp03)

substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)          # cdb(term3.401,term3)
substitute (term3,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)          # cdb(term3.402,term3)
distribute (term3)                                                                                  # cdb(term3.403,term3)

substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)  # cdb(term4.401,term4)
substitute (term4,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)  # cdb(term4.402,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)          # cdb(term4.403,term4)
substitute (term4,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)          # cdb(term4.404,term4)
distribute (term4)                                                                                  # cdb(term4.405,term4)
```

```
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c d b}} ->   @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}{R^{a}_{c b d}} -> - @(tmp03)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c d b}} ->   @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}A^{f}\partial_{e f}{R^{a}_{c b d}} -> - @(tmp02)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c d b}} ->   @(tmp01)$,repeat=True)
substitute (term5,$A^{c}A^{d}A^{e}\partial_{e}{R^{a}_{c b d}} -> - @(tmp01)$,repeat=True)
distribute (term5)

end_stage_3 = time.time()
```

$$\texttt{tmp01.403} := A^{c}A^{d}A^{e}\nabla_{c}R_{bdef}g^{af}$$

$$\texttt{tmp02.403} := A^{c}A^{d}A^{e}A^{f}\nabla_{cd}R_{befg}g^{ag}$$

$$\texttt{tmp03.403} := -\frac{1}{2}A^{c}A^{d}A^{e}A^{f}A^{g}R_{bcdh}\nabla_{e}R_{figj}g^{ai}g^{hj} + \frac{1}{2}A^{c}A^{d}A^{e}A^{f}A^{g}R_{chdi}\nabla_{e}R_{bfgj}g^{ah}g^{ij} + A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{cde}R_{bfgh}g^{ah}$$

$$\texttt{term1.401} := 0$$
$$\texttt{term1.402} := 0$$
$$\texttt{term1.403} := 0$$
$$\texttt{term1.404} := 0$$

$$\texttt{term2.401} := \frac{1}{3}A^dA^eR_{bdea} + \frac{1}{3}A^dA^eR_{adeb}$$
$$\texttt{term2.402} := \frac{1}{3}A^dA^eR_{bdea} + \frac{1}{3}A^dA^eR_{adeb}$$
$$\texttt{term2.403} := \frac{1}{3}A^cA^dR_{bcda} + \frac{1}{3}A^cA^dR_{acdb}$$
$$\texttt{term2.404} := -\frac{2}{3}A^cA^dR_{acbd}$$

$$\texttt{term3.401} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$
$$\texttt{term3.402} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$
$$\texttt{term3.403} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$

$$\begin{aligned}\texttt{term4.401} := {} & \frac{1}{9}g_{ib}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ib}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfa} + \frac{1}{9}g_{ci}A^eA^fR^i{}_{efb}A^dA^gR^c{}_{dga} + \frac{1}{9}g_{ci}A^eA^gR^i{}_{egb}A^dA^fR^c{}_{dfa} \\ & + \frac{1}{9}g_{ib}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{dea} + \frac{1}{9}g_{ci}A^fA^gR^i{}_{fgb}A^dA^eR^c{}_{dea} + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} \\ & - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{dea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfa} + \frac{1}{9}g_{ic}A^eA^fR^i{}_{efa}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ic}A^eA^gR^i{}_{ega}A^dA^fR^c{}_{dfb} \\ & + \frac{1}{9}g_{ai}A^eA^fR^i{}_{efc}A^dA^gR^c{}_{dgb} + \frac{1}{9}g_{ai}A^eA^gR^i{}_{egc}A^dA^fR^c{}_{dfb} + \frac{1}{9}g_{ic}A^fA^gR^i{}_{fga}A^dA^eR^c{}_{deb} + \frac{1}{9}g_{ai}A^fA^gR^i{}_{fgc}A^dA^eR^c{}_{deb} \\ & + \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c{}_{gfh}R^h{}_{deb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eR^c{}_{geh}R^h{}_{dfb}\end{aligned}$$

$$
\begin{aligned}
\texttt{term4.402} :=\ & \frac{1}{9} g_{ib} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ib} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfa} + \frac{1}{9} g_{ci} A^e A^f R^i{}_{efb} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ci} A^e A^g R^i{}_{egb} A^d A^f R^c{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{dea} + \frac{1}{9} g_{ci} A^f A^g R^i{}_{fgb} A^d A^e R^c{}_{dea} + \frac{3}{5} g_{cb} A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{dea} - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} + \frac{1}{9} g_{ic} A^e A^f R^i{}_{efa} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ic} A^e A^g R^i{}_{ega} A^d A^f R^c{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ai} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfb} + \frac{1}{9} g_{ic} A^f A^g R^i{}_{fga} A^d A^e R^c{}_{deb} + \frac{1}{9} g_{ai} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{deb} \\
& + \frac{3}{5} g_{ac} A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{deb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.403} :=\ & \frac{1}{9} g_{ib} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ib} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfa} + \frac{1}{9} g_{ci} A^e A^f R^i{}_{efb} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ci} A^e A^g R^i{}_{egb} A^d A^f R^c{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{dea} + \frac{1}{9} g_{ci} A^f A^g R^i{}_{fgb} A^d A^e R^c{}_{dea} + \frac{3}{5} g_{cb} A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{dea} - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} + \frac{1}{9} g_{ic} A^e A^f R^i{}_{efa} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ic} A^e A^g R^i{}_{ega} A^d A^f R^c{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ai} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfb} + \frac{1}{9} g_{ic} A^f A^g R^i{}_{fga} A^d A^e R^c{}_{deb} + \frac{1}{9} g_{ai} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{deb} \\
& + \frac{3}{5} g_{ac} A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{deb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.404} :=\ & \frac{1}{9} g_{ib} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ib} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfa} + \frac{1}{9} g_{ci} A^e A^f R^i{}_{efb} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ci} A^e A^g R^i{}_{egb} A^d A^f R^c{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{dea} + \frac{1}{9} g_{ci} A^f A^g R^i{}_{fgb} A^d A^e R^c{}_{dea} + \frac{3}{5} g_{cb} A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{dea} - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} + \frac{1}{9} g_{ic} A^e A^f R^i{}_{efa} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ic} A^e A^g R^i{}_{ega} A^d A^f R^c{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ai} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfb} + \frac{1}{9} g_{ic} A^f A^g R^i{}_{fga} A^d A^e R^c{}_{deb} + \frac{1}{9} g_{ai} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{deb} \\
& + \frac{3}{5} g_{ac} A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{deb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{term4.405} := {} & \frac{1}{9} g_{ib} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ib} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfa} + \frac{1}{9} g_{ci} A^e A^f R^i{}_{efb} A^d A^g R^c{}_{dga} + \frac{1}{9} g_{ci} A^e A^g R^i{}_{egb} A^d A^f R^c{}_{dfa} \\
& + \frac{1}{9} g_{ib} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{dea} + \frac{1}{9} g_{ci} A^f A^g R^i{}_{fgb} A^d A^e R^c{}_{dea} + \frac{3}{5} g_{cb} A^d A^g A^e A^f \nabla_{dg} R_{aefh} g^{ch} \\
& - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{dea} - \frac{1}{15} g_{cb} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfa} + \frac{1}{9} g_{ic} A^e A^f R^i{}_{efa} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ic} A^e A^g R^i{}_{ega} A^d A^f R^c{}_{dfb} \\
& + \frac{1}{9} g_{ai} A^e A^f R^i{}_{efc} A^d A^g R^c{}_{dgb} + \frac{1}{9} g_{ai} A^e A^g R^i{}_{egc} A^d A^f R^c{}_{dfb} + \frac{1}{9} g_{ic} A^f A^g R^i{}_{fga} A^d A^e R^c{}_{deb} + \frac{1}{9} g_{ai} A^f A^g R^i{}_{fgc} A^d A^e R^c{}_{deb} \\
& + \frac{3}{5} g_{ac} A^d A^g A^e A^f \nabla_{dg} R_{befh} g^{ch} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{gfh} R^h{}_{deb} - \frac{1}{15} g_{ac} A^d A^g A^f A^e R^c{}_{geh} R^h{}_{dfb}
\end{aligned}
$$

Stage 4: Build the Taylor series for g_{ab}, reformatting and output

```
beg_stage_4 = time.time()
# final housekeeping

term1 = flatten_Rabcd (term1)          # cdb(term1.501,term1)
term2 = flatten_Rabcd (term2)          # cdb(term2.501,term2)
term3 = flatten_Rabcd (term3)          # cdb(term3.501,term3)
term4 = flatten_Rabcd (term4)          # cdb(term4.501,term4)
term5 = flatten_Rabcd (term5)          # cdb(term5.501,term5)

eliminate_metric (term1)
eliminate_metric (term2)
eliminate_metric (term3)
eliminate_metric (term4)
eliminate_metric (term5)

eliminate_kronecker (term1)
eliminate_kronecker (term2)
eliminate_kronecker (term3)
eliminate_kronecker (term4)
eliminate_kronecker (term5)

sort_product (term1)
sort_product (term2)
sort_product (term3)
sort_product (term4)
sort_product (term5)

rename_dummies (term1)
rename_dummies (term2)
rename_dummies (term3)
rename_dummies (term4)
rename_dummies (term5)

canonicalise (term1)                   # cdb(term1.502,term1)
canonicalise (term2)                   # cdb(term2.502,term2)
canonicalise (term3)                   # cdb(term3.502,term3)
```

```
canonicalise (term4)                    # cdb(term4.502,term4)
canonicalise (term5)                    # cdb(term5.502,term5)

# this is out final answer

metric:=@(term0)
      + (1/1) @(term1)
      + (1/2) @(term2)
      + (1/6) @(term3)
      + (1/24) @(term4)
      + (1/120) @(term5).               # cdb(metric.501,metric)

substitute (metric,$A^{a} -> x^{a}$)  # cdb (metric.502,metric)

cdblib.create ('metric.json')

cdblib.put ('g_ab',metric,'metric.json')

# extract the terms of the metric in powers of x

term0 = get_xterm (metric,0)            # cdb(term0.503,term0)
term1 = get_xterm (metric,1)            # cdb(term1.503,term1)
term2 = get_xterm (metric,2)            # cdb(term2.503,term2)
term3 = get_xterm (metric,3)            # cdb(term3.503,term3)
term4 = get_xterm (metric,4)            # cdb(term4.503,term4)
term5 = get_xterm (metric,5)            # cdb(term5.503,term5)

cdblib.put ('g_ab_0',term0,'metric.json')
cdblib.put ('g_ab_1',term1,'metric.json')
cdblib.put ('g_ab_2',term2,'metric.json')
cdblib.put ('g_ab_3',term3,'metric.json')
cdblib.put ('g_ab_4',term4,'metric.json')
cdblib.put ('g_ab_5',term5,'metric.json')

# this version of "metric" is used only in the commentary at the start of this notebook

metric4:=@(term0) + @(term1) + @(term2) + @(term3).  # cdb(metric4.501,metric4)
```

$$\texttt{term2.501} := -\frac{2}{3}A^cA^dR_{acbd}$$

$$\texttt{term2.502} := -\frac{2}{3}A^cA^dR_{acbd}$$

$$\texttt{term3.501} := \frac{1}{2}g_{cb}A^dA^fA^e\nabla_dR_{afeg}g^{cg} + \frac{1}{2}g_{ac}A^dA^fA^e\nabla_dR_{bfeg}g^{cg}$$

$$\texttt{term3.502} := -A^cA^dA^e\nabla_cR_{adbe}$$

$$\begin{aligned}\texttt{term4.501} := {} & \frac{1}{9}g_{ib}A^eA^fg^{ih}R_{hefc}A^dA^gg^{cj}R_{jdga} + \frac{1}{9}g_{ib}A^eA^gg^{ih}R_{hegc}A^dA^fg^{cj}R_{jdfa} + \frac{1}{9}g_{ci}A^eA^fg^{ih}R_{hefb}A^dA^gg^{cj}R_{jdga}\\ & + \frac{1}{9}g_{ci}A^eA^gg^{ih}R_{hegb}A^dA^fg^{cj}R_{jdfa} + \frac{1}{9}g_{ib}A^fA^gg^{ih}R_{hfgc}A^dA^eg^{cj}R_{jdea} + \frac{1}{9}g_{ci}A^fA^gg^{ih}R_{hfgb}A^dA^eg^{cj}R_{jdea}\\ & + \frac{3}{5}g_{cb}A^dA^gA^eA^f\nabla_{dg}R_{aefh}g^{ch} - \frac{1}{15}g_{cb}A^dA^gA^fA^eg^{ci}R_{igfh}g^{hj}R_{jdea} - \frac{1}{15}g_{cb}A^dA^gA^fA^eg^{ci}R_{igeh}g^{hj}R_{jdfa}\\ & + \frac{1}{9}g_{ic}A^eA^fg^{ih}R_{hefa}A^dA^gg^{cj}R_{jdgb} + \frac{1}{9}g_{ic}A^eA^gg^{ih}R_{hega}A^dA^fg^{cj}R_{jdfb} + \frac{1}{9}g_{ai}A^eA^fg^{ih}R_{hefc}A^dA^gg^{cj}R_{jdgb}\\ & + \frac{1}{9}g_{ai}A^eA^gg^{ih}R_{hegc}A^dA^fg^{cj}R_{jdfb} + \frac{1}{9}g_{ic}A^fA^gg^{ih}R_{hfga}A^dA^eg^{cj}R_{jdeb} + \frac{1}{9}g_{ai}A^fA^gg^{ih}R_{hfgc}A^dA^eg^{cj}R_{jdeb}\\ & + \frac{3}{5}g_{ac}A^dA^gA^eA^f\nabla_{dg}R_{befh}g^{ch} - \frac{1}{15}g_{ac}A^dA^gA^fA^eg^{ci}R_{igfh}g^{hj}R_{jdeb} - \frac{1}{15}g_{ac}A^dA^gA^fA^eg^{ci}R_{igeh}g^{hj}R_{jdfb}\end{aligned}$$

$$\texttt{term4.502} := \frac{16}{15}A^cA^dA^eA^fR_{acdg}R_{befh}g^{gh} - \frac{6}{5}A^cA^dA^eA^f\nabla_{cd}R_{aebf}$$

$$
\begin{aligned}
\texttt{term5.501} :=\ & \frac{1}{6} g_{ib} A^e A^g A^f \nabla_e R_{cgfj} g^{ij} A^d A^h g^{ck} R_{kdha} + \frac{1}{6} g_{ib} A^e A^h A^f \nabla_e R_{chfj} g^{ij} A^d A^g g^{ck} R_{kdga} + \frac{1}{6} g_{ib} A^e A^f g^{ik} R_{kefc} A^d A^h A^g \nabla_d R_{ahgj} g^{cj} \\
& + \frac{1}{6} g_{ib} A^e A^h A^g \nabla_e R_{chgj} g^{ij} A^d A^f g^{ck} R_{kdfa} + \frac{1}{6} g_{ib} A^e A^g g^{ik} R_{kegc} A^d A^h A^f \nabla_d R_{ahfj} g^{cj} + \frac{1}{6} g_{ib} A^e A^h g^{ik} R_{kehc} A^d A^g A^f \nabla_d R_{agfj} g^{cj} \\
& + \frac{1}{6} g_{ci} A^e A^g A^f \nabla_e R_{bgfj} g^{ij} A^d A^h g^{ck} R_{kdha} + \frac{1}{6} g_{ci} A^e A^h A^f \nabla_e R_{bhfj} g^{ij} A^d A^g g^{ck} R_{kdga} + \frac{1}{6} g_{ci} A^e A^f g^{ik} R_{kefb} A^d A^h A^g \nabla_d R_{ahgj} g^{cj} \\
& + \frac{1}{6} g_{ci} A^e A^h A^g \nabla_e R_{bhgj} g^{ij} A^d A^f g^{ck} R_{kdfa} + \frac{1}{6} g_{ci} A^e A^g g^{ik} R_{kegb} A^d A^h A^f \nabla_d R_{ahfj} g^{cj} + \frac{1}{6} g_{ci} A^e A^h g^{ik} R_{kehb} A^d A^g A^f \nabla_d R_{agfj} g^{cj} \\
& + \frac{1}{6} g_{ib} A^f A^h A^g \nabla_f R_{chgj} g^{ij} A^d A^e g^{ck} R_{kdea} + \frac{1}{6} g_{ib} A^f A^g g^{ik} R_{kfgc} A^d A^h A^e \nabla_d R_{ahej} g^{cj} + \frac{1}{6} g_{ib} A^f A^h g^{ik} R_{kfhc} A^d A^g A^e \nabla_d R_{agej} g^{cj} \\
& + \frac{1}{6} g_{ci} A^f A^h A^g \nabla_f R_{bhgj} g^{ij} A^d A^e g^{ck} R_{kdea} + \frac{1}{6} g_{ci} A^f A^g g^{ik} R_{kfgb} A^d A^h A^e \nabla_d R_{ahej} g^{cj} + \frac{1}{6} g_{ci} A^f A^h g^{ik} R_{kfhb} A^d A^g A^e \nabla_d R_{agej} g^{cj} \\
& + \frac{1}{6} g_{kb} A^g A^h g^{kj} R_{jghc} A^d A^f A^e \nabla_d R_{afei} g^{ci} + \frac{1}{6} g_{ck} A^g A^h g^{kj} R_{jghb} A^d A^f A^e \nabla_d R_{afei} g^{ci} - \frac{1}{3} g_{cb} A^d A^h A^e A^f A^g R_{adhk} \nabla_e R_{figj} g^{ci} g^{kj} \\
& + \frac{1}{3} g_{cb} A^d A^h A^e A^f A^g R_{dkhi} \nabla_e R_{afgj} g^{ck} g^{ij} + \frac{2}{3} g_{cb} A^d A^h A^e A^f A^g \nabla_{dhe} R_{afgk} g^{ck} - \frac{1}{9} g_{cb} A^d A^h A^f A^g \nabla_h R_{ifgj} g^{cj} A^e g^{ik} R_{kdea} \\
& - \frac{1}{9} g_{cb} A^d A^h A^e A^g \nabla_h R_{iegj} g^{cj} A^f g^{ik} R_{kdfa} - \frac{1}{9} g_{cb} A^d A^e A^f \nabla_d R_{aefj} g^{ij} A^h A^g g^{ck} R_{khgi} - \frac{1}{9} g_{cb} A^d A^h A^e A^f \nabla_h R_{iefj} g^{cj} A^g g^{ik} R_{kdga} \\
& - \frac{1}{9} g_{cb} A^d A^e A^g \nabla_d R_{aegj} g^{ij} A^h A^f g^{ck} R_{khfi} - \frac{1}{9} g_{cb} A^d A^f A^g \nabla_d R_{afgj} g^{ij} A^h A^e g^{ck} R_{khei} + \frac{1}{6} g_{ic} A^e A^g A^f \nabla_e R_{agfj} g^{ij} A^d A^h g^{ck} R_{kdhb} \\
& + \frac{1}{6} g_{ic} A^e A^h A^f \nabla_e R_{ahfj} g^{ij} A^d A^g g^{ck} R_{kdgb} + \frac{1}{6} g_{ic} A^e A^f g^{ik} R_{kefa} A^d A^h A^g \nabla_d R_{bhgj} g^{cj} + \frac{1}{6} g_{ic} A^e A^h A^g \nabla_e R_{ahgj} g^{ij} A^d A^f g^{ck} R_{kdfb} \\
& + \frac{1}{6} g_{ic} A^e A^g g^{ik} R_{kega} A^d A^h A^f \nabla_d R_{bhfj} g^{cj} + \frac{1}{6} g_{ic} A^e A^h g^{ik} R_{keha} A^d A^g A^f \nabla_d R_{bgfj} g^{cj} + \frac{1}{6} g_{ai} A^e A^g A^f \nabla_e R_{cgfj} g^{ij} A^d A^h g^{ck} R_{kdhb} \\
& + \frac{1}{6} g_{ai} A^e A^h A^f \nabla_e R_{chfj} g^{ij} A^d A^g g^{ck} R_{kdgb} + \frac{1}{6} g_{ai} A^e A^f g^{ik} R_{kefc} A^d A^h A^g \nabla_d R_{bhgj} g^{cj} + \frac{1}{6} g_{ai} A^e A^h A^g \nabla_e R_{chgj} g^{ij} A^d A^f g^{ck} R_{kdfb} \\
& + \frac{1}{6} g_{ai} A^e A^g g^{ik} R_{kegc} A^d A^h A^f \nabla_d R_{bhfj} g^{cj} + \frac{1}{6} g_{ai} A^e A^h g^{ik} R_{kehc} A^d A^g A^f \nabla_d R_{bgfj} g^{cj} + \frac{1}{6} g_{ic} A^f A^h A^g \nabla_f R_{ahgj} g^{ij} A^d A^e g^{ck} R_{kdeb} \\
& + \frac{1}{6} g_{ic} A^f A^g g^{ik} R_{kfga} A^d A^h A^e \nabla_d R_{bhej} g^{cj} + \frac{1}{6} g_{ic} A^f A^h g^{ik} R_{kfha} A^d A^g A^e \nabla_d R_{bgej} g^{cj} + \frac{1}{6} g_{ai} A^f A^h A^g \nabla_f R_{chgj} g^{ij} A^d A^e g^{ck} R_{kdeb} \\
& + \frac{1}{6} g_{ai} A^f A^g g^{ik} R_{kfgc} A^d A^h A^e \nabla_d R_{bhej} g^{cj} + \frac{1}{6} g_{ai} A^f A^h g^{ik} R_{kfhc} A^d A^g A^e \nabla_d R_{bgej} g^{cj} + \frac{1}{6} g_{kc} A^g A^h g^{kj} R_{jgha} A^d A^f A^e \nabla_d R_{bfei} g^{ci} \\
& + \frac{1}{6} g_{ak} A^g A^h g^{kj} R_{jghc} A^d A^f A^e \nabla_d R_{bfei} g^{ci} - \frac{1}{3} g_{ac} A^d A^h A^e A^f A^g R_{bdhk} \nabla_e R_{figj} g^{ci} g^{kj} + \frac{1}{3} g_{ac} A^d A^h A^e A^f A^g R_{dkhi} \nabla_e R_{bfgj} g^{ck} g^{ij} \\
& + \frac{2}{3} g_{ac} A^d A^h A^e A^f A^g \nabla_{dhe} R_{bfgk} g^{ck} - \frac{1}{9} g_{ac} A^d A^h A^f A^g \nabla_h R_{ifgj} g^{cj} A^e g^{ik} R_{kdeb} - \frac{1}{9} g_{ac} A^d A^h A^e A^g \nabla_h R_{iegj} g^{cj} A^f g^{ik} R_{kdfb} \\
& - \frac{1}{9} g_{ac} A^d A^e A^f \nabla_d R_{befj} g^{ij} A^h A^g g^{ck} R_{khgi} - \frac{1}{9} g_{ac} A^d A^h A^e A^f \nabla_h R_{iefj} g^{cj} A^g g^{ik} R_{kdgb} \\
& - \frac{1}{9} g_{ac} A^d A^e A^g \nabla_d R_{begj} g^{ij} A^h A^f g^{ck} R_{khfi} - \frac{1}{9} g_{ac} A^d A^f A^g \nabla_d R_{bfgj} g^{ij} A^h A^e g^{ck} R_{khei}
\end{aligned}
$$

$$\texttt{term5.502} := \frac{8}{3}A^c A^d A^e A^f A^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{8}{3}A^c A^d A^e A^f A^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{4}{3}A^c A^d A^e A^f A^g \nabla_{cde} R_{afbg}$$

$$\texttt{metric.501} \coloneqq g_{ab} - \frac{1}{3}A^{c}A^{d}R_{acbd} - \frac{1}{6}A^{c}A^{d}A^{e}\nabla_{c}R_{adbe} + \frac{2}{45}A^{c}A^{d}A^{e}A^{f}R_{acdg}R_{befh}g^{gh} - \frac{1}{20}A^{c}A^{d}A^{e}A^{f}\nabla_{cd}R_{aebf}$$
$$+ \frac{1}{45}A^{c}A^{d}A^{e}A^{f}A^{g}R_{acdh}\nabla_{e}R_{bfgi}g^{hi} + \frac{1}{45}A^{c}A^{d}A^{e}A^{f}A^{g}R_{bcdh}\nabla_{e}R_{afgi}g^{hi} - \frac{1}{90}A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{cde}R_{afbg}$$

$$\texttt{metric.502} \coloneqq g_{ab} - \frac{1}{3}x^{c}x^{d}R_{acbd} - \frac{1}{6}x^{c}x^{d}x^{e}\nabla_{c}R_{adbe} + \frac{2}{45}x^{c}x^{d}x^{e}x^{f}R_{acdg}R_{befh}g^{gh} - \frac{1}{20}x^{c}x^{d}x^{e}x^{f}\nabla_{cd}R_{aebf}$$
$$+ \frac{1}{45}x^{c}x^{d}x^{e}x^{f}x^{g}R_{acdh}\nabla_{e}R_{bfgi}g^{hi} + \frac{1}{45}x^{c}x^{d}x^{e}x^{f}x^{g}R_{bcdh}\nabla_{e}R_{afgi}g^{hi} - \frac{1}{90}x^{c}x^{d}x^{e}x^{f}x^{g}\nabla_{cde}R_{afbg}$$

$\texttt{term0.503} := g_{ab}$

$\texttt{term1.503} := 0$

$$\texttt{term2.503} := -\frac{1}{3}x^c x^d R_{acbd}$$

$$\texttt{term3.503} := -\frac{1}{6}x^c x^d x^e \nabla_c R_{adbe}$$

$$\texttt{term4.503} := \frac{2}{45}x^c x^d x^e x^f R_{acdg} R_{befh} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf}$$

$$\texttt{term5.503} := \frac{1}{45}x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{bfgi} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{afgi} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg}$$

```
Xterm0 := @(term0).
Xterm1 := @(term1).  # zero
Xterm2 := @(term2).
Xterm3 := @(term3).
Xterm4 := @(term4).
Xterm5 := @(term5).

Xterm0 = reformat_xterm (Xterm0,  1)    # cdb(Xterm0.601,Xterm0)
Xterm2 = reformat_xterm (Xterm2,  3)    # cdb(Xterm2.601,Xterm2)
Xterm3 = reformat_xterm (Xterm3,  6)    # cdb(Xterm3.601,Xterm3)
Xterm4 = reformat_xterm (Xterm4,180)    # cdb(Xterm4.601,Xterm4)
Xterm5 = reformat_xterm (Xterm5, 90)    # cdb(Xterm5.601,Xterm5)

gab3   := @(Xterm0) + @(Xterm2).                                    # cdb (gab3.601,gab3)
gab4   := @(Xterm0) + @(Xterm2) + @(Xterm3).                        # cdb (gab4.601,gab4)
gab5   := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4).            # cdb (gab5.601,gab5)
gab6   := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (gab6.601,gab6)

Metric := @(Xterm0) + @(Xterm2) + @(Xterm3) + @(Xterm4) + @(Xterm5).  # cdb (Metric.601,Metric)

scaled0 = rescale_xterm (Xterm0,  1)    # cdb(scaled0.601,scaled0)
scaled2 = rescale_xterm (Xterm2,  3)    # cdb(scaled2.601,scaled2)
scaled3 = rescale_xterm (Xterm3,  6)    # cdb(scaled3.601,scaled3)
scaled4 = rescale_xterm (Xterm4,180)    # cdb(scaled4.601,scaled4)
scaled5 = rescale_xterm (Xterm5, 90)    # cdb(scaled5.601,scaled5)

end_stage_4 = time.time()
```

The metric in Riemann normal coordinates

$$
\begin{aligned}
g_{ab}(x) = g_{ab} &- \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f \left(8g^{gh} R_{acdg} R_{befh} - 9\nabla_{cd} R_{aebf}\right) \\
&+ \frac{1}{90}x^c x^d x^e x^f x^g \left(2g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

Curvature expansion of the metric

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned}
\overset{0}{g}_{ab} &= g_{ab} \\
3\overset{2}{g}_{ab} &= -x^c x^d R_{acbd} \\
6\overset{3}{g}_{ab} &= -x^c x^d x^e \nabla_c R_{adbe} \\
180\overset{4}{g}_{ab} &= x^c x^d x^e x^f \left(8 g^{gh} R_{acdg} R_{befh} - 9 \nabla_{cd} R_{aebf}\right) \\
90\overset{5}{g}_{ab} &= x^c x^d x^e x^f x^g \left(2 g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2 g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right)
\end{aligned}$$

```
cdblib.create ('metric.export')

cdblib.put ('g_ab_3',Metric3,'metric.export')  # R and \partial R
cdblib.put ('g_ab_4',Metric4,'metric.export')
cdblib.put ('g_ab_5',Metric5,'metric.export')
cdblib.put ('g_ab_6',Metric6,'metric.export')

cdblib.put ('g_ab',  Metric, 'metric.export')  # R and \nabla R

cdblib.put ('g_ab_scaled0',scaled0,'metric.export')
cdblib.put ('g_ab_scaled2',scaled2,'metric.export')
cdblib.put ('g_ab_scaled3',scaled3,'metric.export')
cdblib.put ('g_ab_scaled4',scaled4,'metric.export')
cdblib.put ('g_ab_scaled5',scaled5,'metric.export')

checkpoint.append (Metric4)
checkpoint.append (Metric6)

checkpoint.append (Metric)

checkpoint.append (scaled0)
checkpoint.append (scaled2)
checkpoint.append (scaled3)
checkpoint.append (scaled4)
checkpoint.append (scaled5)

# cdbBeg (timing)
print ("Stage 1: {:7.1f} secs\\hfill\\break".format(end_stage_1-beg_stage_1))
print ("Stage 2: {:7.1f} secs\\hfill\\break".format(end_stage_2-beg_stage_2))
print ("Stage 3: {:7.1f} secs\\hfill\\break".format(end_stage_3-beg_stage_3))
print ("Stage 4: {:7.1f} secs".format(end_stage_4-beg_stage_4))
# cdbEnd (timing)
```

Timing

Stage 1: 2.6 secs
Stage 2: 0.9 secs
Stage 3: 56.2 secs
Stage 4: 1.9 secs