Preprint: arXiv:0903.2087
Journal: *Class. Quantum. Grav.* **26** 175017 (2009)

Riemann Normal Coordinate expansions using Cadabra.

Leo Brewin

School of Mathematical Sciences
Monash University, 3800
Australia

12-Aug-2009 (v1)
10-Dec-2019 (v2)

Abstract

Riemann normal coordinate expansions of the metric and other geometrical quantities, including the geodesic arc-length, will be presented. All of the results are given to fifth-order in the curvature and were obtained using the computer algebra package Cadabra.

Version 2

The first version of this paper was written in 2009. This second version (written in 2019) differs from the first in a number of important ways. First, the discussion on Cadabra has been completely rewritten to account for the revised syntax introduced in the latest version of Cadabra (itself updated in 2016)[1]. Second, and more importantly, a significant computational error in the first version (of this paper) has been corrected. Though the algorithms given in the first version were correct their implementation in Cadabra (by this author) was incorrect. This lead to errors first in the quartic and higher order terms in the generalised connections and then, down the line, to computations built upon the generalised connections (e.g., the initial and boundary value problem and the arc-length equations). The Cadabra codes have been corrected and thoroughly cross-checked. A number of minor typographical errors have also been corrected.

1 Introduction

In a previous paper [1] a series of simple examples were used to demonstrate how the computer algebra program Cadabra ([2], [3], [4], [5]) could be employed to do the kinds of tensor computations often encountered in General Relativity. The examples were deliberately chosen to be sufficiently simple as to allow the reader to appreciate the basic functionality of Cadabra. This left open the question of how well Cadabra might perform on more challenging computations. The purpose of this paper is to address that point by showing how Cadabra can be used to express various geometrical objects, such as the metric, the connection and the geodesic arc-length, in terms of Riemann normal coordinates (also known as geodesic coordinates). This is a standard computation in differential geometry that leads to a series expansion for the various geometrical objects in powers of the curvature and its derivatives. The first few terms

[1]The Cadabra results in this paper are based on Cadabra 2.2.7 (build 2268.ba747e0b49 dated 2019-12-01)

are rather easy to compute by hand but further progress, to higher order terms, quickly becomes prohibitively difficult. Thus some computer tool, in this case Cadabra, is essential when computing these higher order terms.

The first part of this paper will be concerned with purely mathematical and algorithmic issues including a definition of Riemann normal coordinates, how they can be constructed (from generic coordinates) and what properties they confer upon geometrical objects (such as the metric, the connection etc.). In the second part the attention will shift to issues specific to Cadabra. This will be followed by full details of the 5th-order expansion of the metric and other objects.

It should be emphasised that the primary objective of this paper is to demonstrate that Cadabra is a mature program that can easily handle an otherwise prohibitive computation. Some of the Cadabra results for the metric do replicate results by other authors (e.g., equation (11.1) is consistent with that given in [6], [7], [8], and [9] while some results are original (in particular, the solution for the two-point boundary value problem, (11.20)).

Before turning the discussion to the definition and construction of Riemann normal coordinates, a simple coordinate transformation will be introduced. This will help in making clear what is meant by the statement that an expression is an expansion to a given order.

2 Conformal coordinates

Each algorithm given later in this paper yields polynomial approximations to particular geometric quantities (e.g., the metric). Higher order approximations are obtained by recursive application of the algorithms.

The primary goal of this section is to provide meaning to the statement that the polynomial S_ϵ is an expansion of S up to and including terms of order $\mathcal{O}(\epsilon^n)$. The key to this definition is the use of a conformal transformation of the original metric.

Consider some neighbourhood of O and let ϵ be a typical length scale for O (for example, ϵ might be the length of the longest geodesic that passes through O and confined by the neighbourhood). Construct any regular set of coordinates x^a (i.e., such that the metric components are non-singular) in the neighbourhood of O and let the coordinates of O be $x^a_\star$. The word *patch* will be used to denote the neighbourhood of O in which these coordinates are defined. Now define a new set of coordinates y^a by

$$x^a = x^a_\star + \epsilon y^a$$

and thus

$$ds^2 = g_{ab}(x)dx^a dx^b = \epsilon^2 g_{ab}(x_\star + \epsilon y)dy^a dy^b$$

The conformal metric $d\tilde{s}$ can now be defined by $d\tilde{s} = ds/\epsilon^2$. This leads to

$$d\tilde{s}^2 = g_{ab}(x_\star + \epsilon y)dy^a dy^b = \tilde{g}_{ab}(y,\epsilon)dy^a dy^b$$

and

$$\tilde{g}_{ab} = g_{ab}\ , \qquad \tilde{g}_{ab,c} = \epsilon g_{ab,c}\ , \qquad \tilde{g}_{ab,cd} = \epsilon^2 g_{ab,cd} \qquad \text{at } O$$

where the partial derivatives on the left are with respect to y and those on the right are with respect to x. Since $g_{ab}(x_\star)$ does not depend on ϵ it is easy to see that

$$\tilde{g}_{ab,i_1 i_2 i_3\cdots i_n} = \mathcal{O}(\epsilon^n) \qquad \text{at } O$$

From this it follows, by simple inspection of the standard equations, that

$$\tilde{\Gamma}^a{}_{bc,i_1 i_2 i_3 \cdots i_n} = \mathcal{O}\left(\epsilon^{n+1}\right) \qquad \text{at } O$$

$$\tilde{R}^a{}_{bcd,i_1 i_2 i_3 \cdots i_n} = \mathcal{O}\left(\epsilon^{n+2}\right) \qquad \text{at } O$$

There are now two ways to look at the patch. It can be viewed as a patch of length scale ϵ with a curvature independent of ϵ. Or it can be viewed as a patch of fixed size but with a curvature that depends on ϵ (and where the limit $\epsilon \to 0$ corresponds to flat space). This later view is useful since it ensures that the series expansions around flat space can be made convergent (by choosing a sufficiently small ϵ).

These conformal coordinates will be used for the remainder of this paper. As there is no longer any reason to distinguish between x^a and y^a, the y^a will be replaced with x^a. The x^a will now be treated as generic coordinates (but keep in mind that there is an underlying conformal transformation built into these coordinates).

Finally, the statement that S_ϵ is an order $\mathcal{O}\left(\epsilon^n\right)$ expansion of S will be taken to mean that n is the smallest positive integer for which

$$0 < \lim_{\epsilon \to 0} \frac{|S - S_\epsilon|}{\epsilon^{n+1}} < M$$

for some finite positive M i.e., if S were expanded as a Taylor series in ϵ around $\epsilon = 0$ then S and S_ϵ would differ by terms proportional to ϵ^{n+1}.

3 Riemann Normal Coordinates

The basic idea behind Riemann normal coordinates is to use the geodesics through a given point to define the coordinates for nearby points. Let the given point be O (this will be the origin of the Riemann normal frame) and consider some nearby point P. If P is sufficiently close to O then there exists a unique geodesic joining O to P. Let v^a be the components of the unit tangent vector to this geodesic at O and let s be the geodesic arc length measured from O to P. Then the Riemann normal coordinates x^a of P relative to O are defined to be $x^a = sv^a$. These coordinates are well defined provided the geodesics do not cross (which can always be ensured by choosing the neighbourhood of O to be sufficiently small). For more mathematical details see ([10], [11], [12]) and in particular the elegant exposition by Gray ([13], [14]).

One trivial consequence of this definition is that all geodesics through O are of the form $x^a(s) = sv^a$ and that the v^a are constant along each geodesic. This implies, by direct substitution into the geodesic equation, that $\Gamma^c{}_{ab} = 0$ at O which in turn implies that $g_{ab,c} = 0$ at O. Suppose now that the metric can be written as a Taylor series in x^a about O. In that series only the zero, second and higher derivatives of the g_{ab} would appear. Thus the leading terms of the metric can be expressed as a sum of a constant part (the leading term) plus a curvature part (from the second and higher derivative terms). Similar expansions arise for other geometrical quantities (e.g., geodesics, arc length) in terms of a flat space part plus a curvature contribution.

Suppose, as is almost always the case, that the coordinates x^a are *not* in Riemann normal form. How might they be transformed to a local set of Riemann normal coordinates? A first attempt might be to make direct appeal to the basic definition, namely, $y^a = sv^a$. Unfortunately this does raise an immediate problem. The quantities v^a are rarely known explicitly but must instead be computed by solving a two-point boundary value problem. This is non-trivial but it can be dealt with in a number of ways. The approach adopted here is to iterate over a sequence of initial value problems to provide a sequence of approximations to the solution of the original boundary value problem. Here is a brief outline of the process – this serves as an introduction to the next two sections where the full details will be given.

First, consider any method for computing the coordinates $x^a(s)$ of a typical geodesic that originates from O. This solution of the typical initial value problem will depend on two integration constants, x^a and $\dot{x}^a$, being the respective values of $x^a(s)$ and dx^a/ds at $s = 0$. Next, embed this first step in an iterative scheme (e.g., a fixed-point scheme) to produce successive approximations to $\dot{x}^a$ so that the geodesic passes through not only O but also P. The final step, assuming the sequence converges, is to set v^a to be $\dot{x}^a$ and finally $y^a = sv^a$.

The next two sub-sections provide further details on how the initial and boundary value problems were solved.

3.1 The initial value problem

The aim here is to obtain a Taylor series, about the point O, for the solution of the geodesic equation

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a/ds = \dot{x}^a$ at $s = 0$.

First choose $s = 0$ at O and then write the Taylor series for $x^a(s)$ as

$$x^a(s) = x^a\big|_{s=0} + s\,\frac{dx^a}{ds}\bigg|_{s=0} + \sum_{n=2}^{\infty}\frac{s^n}{n!}\,\frac{d^nx^a}{ds^n}\bigg|_{s=0}$$

The second and higher derivatives can be obtained by successive differentiation of the geodesic equation leading to

$$x^a(s) = x^a + s\dot{x}^a - \sum_{n=2}^{\infty}\frac{s^n}{n!}\Gamma^a{}_{i_1i_2i_3\cdots i_n}\dot{x}^{i_1}\dot{x}^{i_2}\dot{x}^{i_3}\cdots\dot{x}^{i_n} \tag{3.1}$$

where the $\Gamma^a{}_{i_1i_2i_3\cdots i_n}$, known as *generalised connections*, are defined recursively by

$$\Gamma^a{}_{i_1i_2i_3\cdots i_n} = \Gamma^a{}_{(i_1i_2i_3\cdots i_{n-1},i_n)} - n\Gamma^a{}_{p(i_2i_3\cdots i_{n-1}}\Gamma^p{}_{i_1i_n)} \tag{3.2}$$

Note that the use of round brackets (...) denotes total symmetrisation over the included indices (see Appendix A for more details).

A convenient shorthand for equation (3.2) in terms of covariant derivatives can be obtained by ignoring (in this context alone) the single upper index. This leads to the compact notation

$$\Gamma^a{}_{i_1i_2i_3\cdots i_n} = \Gamma^a{}_{(i_1i_2;i_3i_4i_5\cdots i_n)} \tag{3.3}$$

3.2 The boundary value problem

The question here is: How can $\dot{x}^a$ be chosen so that the geodesic passes through not only O but also P? The geodesic, by construction, already passes through O so that leaves P as the main focus of attention. Suppose that the coordinates of P are $x^a + \Delta x^a$ where x^a are the coordinates of O. Let s_P be the geodesic distance from O to P. The challenge now is to solve

$$\Delta x^a = s_P \dot{x}^a - \sum_{n=2}^{\infty} \frac{s_P^n}{n!} \Gamma^a{}_{i_1 i_2 i_3 \cdots i_n} \dot{x}^{i_1} \dot{x}^{i_2} \dot{x}^{i_3} \cdots \dot{x}^{i_n}$$

for $\dot{x}^a$ in terms of Δx^a and the generalised connections.

Put $y^a = s_P \dot{x}^a$ (this introduces the Riemann normal coordinates) and re-arrange the equation into the form

$$y^a = \Delta x^a + \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^a{}_{i_1 i_2 i_3 \cdots i_n} y^{i_1} y^{i_2} y^{i_3} \cdots y^{i_n} \tag{3.4}$$

In a small variation to the plan given above, the plan will be modified slightly to solve this equation for y^a (rather than $\dot{x}^a$) by constructing a sequence of approximations y^a_m to y^a. The y^a will, through the generalised connections, depend on the Riemann curvature and its derivatives (at O). So in principle the y^a could be expanded as power series in ϵ (i.e., as a power series in the curvatures). The y^a_m will be chosen to be the Taylor polynomial of y^a to order ϵ^m. That is, y^a_m is a polynomial in the curvatures (and its derivatives) up to and including terms of order $\mathcal{O}(\epsilon^m)$. The y^a_m can computed by truncating both sides of (3.4) to terms no higher than $\mathcal{O}(\epsilon^m)$. But note that the $\Gamma^a{}_{i_1 i_2 i_3 \cdots i_n}$ are of order $\mathcal{O}(\epsilon^{n-1})$. The upshot is that the infinite series may be truncated at $n = m$ while still retaining all terms up to and including ϵ^m. This leads to

$$y^a_m = \Delta x^a + T^m_\epsilon \left(\sum_{n=2}^{n=m} \frac{1}{n!} \Gamma^a{}_{i_1 i_2 i_3 \cdots i_n} y^{i_1}_m y^{i_2}_m y^{i_3}_m \cdots y^{i_n}_m \right)$$

where T^m_ϵ is a simple truncation operator (it deletes all terms of order $\mathcal{O}(\epsilon^{m+1})$ or higher). This is a marginal improvement on (3.4) (at least it is a finite series) but it is still a non-linear equation for y^a_m. But fortunately improvements are readily made. Notice, once again, that $\Gamma^a{}_{i_1 i_2 i_3 \cdots i_n} = \mathcal{O}(\epsilon^{n-1})$ and this allows the use of lower order estimates for y^a in the product terms on the right hand side. This leads to

$$y^a_m = \Delta x^a + T^m_\epsilon \left(\sum_{n=2}^{n=m} \frac{1}{n!} \Gamma^a{}_{i_1 i_2 i_3 \cdots i_n} y^{i_1}_{m-n+1} y^{i_2}_{m-n+1} y^{i_3}_{m-n+1} \cdots y^{i_n}_{m-n+1} \right) \tag{3.5}$$

This shows that y^a_m appears only on the left hand side and thus this equation can be used to recursively compute y^a_p for $p = 2, 3, 4, \cdots$. Here are the first few y^a_m. Starting with the lowest order approximation,

$$y^a_0 = \Delta x^a$$

and as there are no ϵ^1 terms in (3.4) this can also be written as

$$y^a_1 = y^a_0 = \Delta x^a$$

Now set $m = 2$ in equation (3.5) to obtain

$$\begin{aligned} y^a_2 &= \Delta x^a + T^2_\epsilon \left(\frac{1}{2!} \Gamma^a{}_{i_1 i_2} y^{i_1}_1 y^{i_2}_1 \right) \\ &= \Delta x^a + \frac{1}{2} \Gamma^a{}_{i_1 i_2} \Delta x^{i_1}_1 \Delta x^{i_2}_1 \end{aligned}$$

and once more, with $m = 3$, with the result

$$
\begin{aligned}
y_3^a &= \Delta x^a + T_\epsilon^3 \left(\frac{1}{2!} \Gamma^a{}_{i_1 i_2} y_2^{i_1} y_2^{i_2} + \frac{1}{3!} \Gamma^a{}_{i_1 i_2 i_3} y_1^{i_1} y_1^{i_2} y_1^{i_3} \right) \\
&= \Delta x^a + \frac{1}{2} \Gamma^a{}_{i_1 i_2} \Delta x^{i_1} \Delta x^{i_2} + \frac{1}{6} \left(\Gamma^a{}_{b i_1} \Gamma^b{}_{i_2 i_3} + \Gamma^a{}_{i_1 i_2, i_3} \right) \Delta x^{i_1} \Delta x^{i_2} \Delta x^{i_3}
\end{aligned}
$$

This process may seem simple but looks can be deceiving – the higher order y_m^a contain a profusion of terms that, when computed by hand, are largely unmanageable beyond $m \approx 7$. At this point there is little option but, if higher order terms are needed, to defer the computations to a suitable computer algebra package. This point will be discussed in more detail in section (10).

This completes the first objective – to find a way to transform from any non-singular set of coordinates to a local set of Riemann normal coordinates. The question to ask now is – What form does the metric (and other objects) take in these coordinates? This is the subject of the second next section. But before doing so, here is a short digression to introduce some new notation.

4 Notation

It has already been noted that the recursive nature of equations like (3.2) imposes a significant computational cost with each iteration. A less obvious side effect is that number of tensor indices on each term will also increase with each iteration. The equation for the generalised connections, (3.2), is a case in point. This contains an index list of the form $i_1 i_2 i_3 \cdots i_n$. Lists such as these are tedious to write and are prone to transcription error (when in the hands of humans). It thus makes sense to use a notation that is easy to read and write while also not detracting from the meaning in the expression. The proposal is that a sequence of indices such as $i_1 i_2 i_3 \cdots i_n$ be replaced with a single index of the form $\underline{i}$. In this notation the equation for the generalised connections (3.2) would be written as

$$
\Gamma^a{}_{b\underline{c}d} = \Gamma^a{}_{(b\underline{c},d)} - (n+1)\Gamma^a{}_{p(\underline{c}}\Gamma^p{}_{bd)}
$$

where $\underline{c}$ contains $n > 0$ indices. There will be cases where the number of hidden indices needs to be made clear. In such cases one of two options can be taken – either state explicitly the number of hidden indices in words (as in the above example) or by including the number as a subscript as in this example

$$
\Gamma^a{}_{b\underline{c}_n d} = \Gamma^a{}_{(b\underline{c}_n,d)} - (n+1)\Gamma^a{}_{p(\underline{c}_n}\Gamma^p{}_{bd)}
$$

This notation can be extended to handle expressions such as

$$
\Gamma^a{}_{bc,i_1 i_2 i_3 \cdots i_n} A^{i_1} A^{i_2} A^{i_3} \cdots A^{i_n}
$$

which would be written in condensed form as

$$
\Gamma^a{}_{bc,\underline{d}} A^{\cdot \underline{d}}
$$

Note carefully the dot to the left of the $\underline{d}$ superscript. Its purpose is to avoid an ambiguity that would arise in using the condensed notation for an expression such as

$$
\Gamma^a{}_{bc,i_1 i_2 i_3 \cdots i_n} A^{i_1 i_2 i_3 \cdots i_n}
$$

which, following the earlier discussion, would be written as

$$\Gamma^a{}_{bc,\underline{d}} A^{\underline{d}}$$

These simple changes bring some degree of normalcy to the printed form but those gains rapidly pale into insignificance when displaying the fifth order expansions generated by Cadabra (in section (11)). Cadabra's output does not employ the familiar use of brackets to denote symmetrisation over an index list. Instead, it uses the fully expanded form which, on paper, can lead to dramatic explosion in otherwise similar looking terms. There seems little point in printing terms that differ only by swapping pairs of indices. Thus the convention used here will be that symmetrisation over an index list on the right hand side of an equation will be inferred from any explicit use of brackets for symmetrisation written on the left hand side. Thus an equation like

$$A_{(ab)} = B_a C_b$$

will be taken to mean

$$A_{(ab)} = \frac{1}{2!}\left(B_a C_b + B_b C_a\right)$$

This convention will only be applied to results generated by Cadabra. The left hand side will only contain a single term with just one pair of brackets.

The final notational device concerns cases where an index needs to be excluded from symmetrisation. The normal practise is to enclose the index in a pair of vertical lines. This involves two characters whereas a single dot over the chosen index can serve the same purpose. Thus $(ab\dot{c}d\dot{e}fg)$ will denote symmetrisation over only a, b, d, f and g. In the standard notation this would have been written as $(ab|c|d|e|fg)$.

5 The metric in Riemann normal form

In the preceding section the generic and Riemann normal coordinates distinguished by using the symbols x^a and y^a respectively. Now, for notational convenience and to accord with convention, the Riemann normal coordinates will be denoted x^a while y^a will be stripped of any special meaning.

The aim of this section is to express the metric in Riemann normal form. This will take the form of an infinite series in powers of the curvature and its derivatives. Start by writing out the Taylor series for the metric around $x^a = 0$

$$g_{ab}(x) = g_{ab} + \sum_{n=1}^{\infty} \frac{1}{n!}\, g_{ab,\underline{c}}\, x^{\cdot\underline{c}}$$

where $\underline{c}$ contains n indices and g_{ab} are constants (e.g. $g_{ab} = \mathrm{diag}(1,1,1,\cdots)$).

The present task is to express the partial derivatives of the metric in terms of the Riemann tensor. From the standard definition of a metric compatible connection, a series of partial derivatives leads to

$$g_{ab,c\underline{d}} = \left(g_{ae}\Gamma^e{}_{bc} + g_{eb}\Gamma^e{}_{ac}\right)_{,\underline{d}}$$

and since $g_{ab,c\underline{d}}$ is totally symmetric in $c\underline{d}$ it is easy to see that

$$g_{ab,c\underline{d}} = \left(g_{ae}\Gamma^e{}_{b(c}\right)_{,\underline{d})} + \left(g_{eb}\Gamma^e{}_{a(c}\right)_{,\underline{d})}$$

Two points should be noted, first, the connection appears only in the form ${\Gamma^a}_{b(c,\underline{d})}$, second, the left hand side contains derivatives one order higher than in the corresponding terms on the right hand side. The upshot is that this equation can be used to recursively compute all of the metric derivatives solely in terms of the ${\Gamma^a}_{b(c,\underline{d})}$ and the constants g_{ab}. In this way the above Taylor series for the metric can be expressed solely in terms of the connection and its derivatives. But more can be done – the derivatives of the connection must surely tie in with the curvatures. Thus attention turns to the standard definition for the curvature, which after a series of derivatives, can be written in the form

$$
{R^a}_{(bc\dot{d},\underline{e})} = {\Gamma^a}_{d(bc,\underline{e})} - {\Gamma^a}_{(bc,\underline{e})d} + \left({\Gamma^a}_{i(c}{\Gamma^i}_{b\dot{d})}\right)_{,\underline{e})} - \left({\Gamma^a}_{id}{\Gamma^i}_{(bc)}\right)_{,\underline{e})} \tag{5.1}
$$

(Note that the ${\Gamma^a}_{dbc}$ in ${\Gamma^a}_{d(bc,\underline{e})}$ is the first of the generalised connections discussed earlier in section (3.1)). Can this result be used to eliminate the connection and its derivatives from the metric? Yes, but only after specialising to the Riemann normal coordinates.

Recall that, in Riemann normal coordinates, all geodesics through O are of the form

$$
x^a(s) = sv^a
$$

which upon substitution into the geodesic equations leads to

$$
0 = {\Gamma^a}_{(b\underline{c})} \qquad \text{at } O
$$

It follows, by recursion on equation (3.3), that

$$
0 = {\Gamma^a}_{(bc,\underline{d})} \qquad \text{at } O \tag{5.2}
$$

Recall that ${\Gamma^a}_{bc,\underline{e}d}$ is separately symmetric in its first pair of indices and in the remaining $(n+1)$ lower indices (assuming $\underline{e}$ contains n indices). Thus using equation (A.1) it follows that

$$
0 = (n+3){\Gamma^a}_{(bc,\underline{e}d)} = 2{\Gamma^a}_{d(b,c\underline{e})} + (n+1){\Gamma^a}_{(bc,\underline{e})d}
$$

This result can be used to eliminate the ${\Gamma^a}_{(bc,\underline{e})d}$ term in equation (5.1) for the derivatives of the curvature. The result, after a minor shuffling of terms is

$$
(n+3){\Gamma^a}_{d(b,c\underline{e})} = (n+1)\left({R^a}_{(bc\dot{d},\underline{e})} - \left({\Gamma^a}_{i(c}{\Gamma^i}_{b\dot{d})}\right)_{,\underline{e})} + \left({\Gamma^a}_{id}{\Gamma^i}_{(bc)}\right)_{,\underline{e})}\right)
$$

(the reason for rearranging the terms will become clear in a moment). Note also that the last term in the previous equation can be eliminated by equation (5.2) and the product rule.

In summary, the equations of interest are

$$
g_{ab}(x) = g_{ab} + \sum_{n=1}^{\infty} \frac{1}{n!}\, g_{ab,\underline{c}}\, x^{\cdot\underline{c}} \tag{5.3}
$$

$$
g_{ab,c\underline{d}} = \left(g_{ae}{\Gamma^e}_{b(c}\right)_{,\underline{d})} + \left(g_{eb}{\Gamma^e}_{a(c}\right)_{,\underline{d})} \tag{5.4}
$$

$$
(n+3){\Gamma^a}_{d(b,c\underline{e})} = (n+1)\left({R^a}_{(bc\dot{d},\underline{e})} - \left({\Gamma^a}_{i(c}{\Gamma^i}_{b\dot{d})}\right)_{,\underline{e})}\right) \tag{5.5}
$$

These equations could be used as follows. First, equation (5.5) is used to recursively compute the ${\Gamma^a}_{b(c,\underline{e}d)}$ in terms of the Riemann tensor and its partial derivatives (this was the reason behind the shuffling of terms noted above). Note that $\underline{e}$ in equation (5.5) contains n hidden indices. The ${\Gamma^a}_{b(c,\underline{d})}$ are then substituted into (5.4) which in turn is used to recursively express all of the $g_{ab,\underline{c}}$ in terms of the Riemann tensor and its partial derivatives. When the dust settles the result is a finite series expansion for the metric in terms of the Riemann tensor and its partial derivatives. The result, to 3rd-order in the curvature, is

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \partial_c R_{adbe} + \mathcal{O}\left(\epsilon^4\right)$$

Though this result meets the stated aim – to express the metric in terms of the curvatures and its derivatives – there is a small problem. The partial derivatives make subsequent raising and lowering of indices very tedious. A far better result would be one that employs covariant derivatives. Fortunately the work required to achieve that is not too onerous. The details are described in the remainder of this section.

It is not hard to see that a series of covariant derivatives of the Riemann tensor, would lead to an equation of the form, in any coordinate frame,

$$R^a{}_{(bc\dot{d};\underline{e})} = R^a{}_{(bc\dot{d},\underline{e})} + Q^a{}_{(bc\dot{d}\underline{e})}$$

where $Q^a{}_{(bc\dot{d}\underline{e})}$ is a function of the $\Gamma^a{}_{bc}$, the $R^a{}_{bcd}$ and their partial derivatives. If this is going to sit nicely with the algorithm given above then it will be necessary to show, in the Riemann normal frame, that this equation only contains connection terms of the form $\Gamma^p{}_{q(r,\underline{s})}$. Fortunately this is rather easy to do.

Each term of the form $\Gamma^p{}_{qr,\underline{s}}$ in Q arose during one round of covariant differentiation. Thus *at least one* of the indices q, r and *all* of the indices in $\underline{s}$ must be drawn from the index list $\underline{e}$. If both q and r are contained in $\underline{e}$ then the term is of the form $\Gamma^p{}_{(qr,\underline{s})}$ and thus will vanish when specialised to the Riemann normal frame. This completes the proof. By re-arranging the above equation into the following form

$$R^a{}_{(bc\dot{d},\underline{e})} = R^a{}_{(bc\dot{d};\underline{e})} - Q^a{}_{(bc\dot{d}\underline{e})} \tag{5.6}$$

it can be used to recursively compute all of the partial derivatives of the curvatures in terms of their covariant derivatives. The $Q^a{}_{(bc\dot{d}\underline{e})}$ will contain lower order derivatives of the curvatures and partial derivatives of the connection all of which can be eliminated (in favour of covariant derivatives) using previously computed results. For the first two derivatives it turns out that the partial and covariant derivatives are equal (as expected) but differences do appear in higher order derivatives. These differences will be apparent when the fifth order results are given later in section (11).

6 The inverse metric in Riemann normal form

Most of the hard work is done and it is now time to develop algorithms for Riemann normal expansions for other interesting quantities, in this instance the inverse metric $g^{ab}(x)$. The previous section used $0 = g_{ab;\underline{c}}$ as a starting point to express the metric in terms of the curvatures. On this occasion, for the inverse metric, the starting point will be $0 = g^{ab}{}_{;\underline{c}}$. Then, following a path similar to that used in the previous section, leads to the following equations

$$g^{ab}(x) = g^{ab} + \sum_{n=1}^{\infty} \frac{1}{n!}\, g^{ab}{}_{,\underline{c}}\, x^{\cdot\underline{c}} \tag{6.1}$$

$$g^{ab}{}_{,c\underline{d}} = -\left(g^{ae}\Gamma^b{}_{e(c}\right)_{,\underline{d})} - \left(g^{eb}\Gamma^a{}_{e(c}\right)_{,\underline{d})} \tag{6.2}$$

$$(n+3)\Gamma^a{}_{d(b,c\underline{e})} = (n+1)\left(R^a{}_{(bc\dot{d},\underline{e})} - \left(\Gamma^a{}_{i(c}\Gamma^i{}_{bd}\right)_{,\underline{e})}\right) \tag{5.5}$$

These equations can be used to construct the series expansion for $g^{ab}(x)$, which to 3rd-order is

$$g^{ab}(x) = g^{ab} - \frac{1}{3}x^c x^d R^a{}_{cd}{}^b - \frac{1}{6}x^c x^d x^e \partial_c R^a{}_{de}{}^b + \mathcal{O}\left(\epsilon^4\right)$$

7 Generalised connections

In section (3.1) the generalised connections $\Gamma^a{}_{b\underline{cd}}$ were shown to arise from successive differentiation of the geodesic equation and that they can be computed recursively using

$$\Gamma^a{}_{b\underline{c}d} = \Gamma^a{}_{(b\underline{c},d)} - (n+1)\Gamma^a{}_{p(\underline{c}}\Gamma^p{}_{bd)} \qquad (3.2)$$

where the index list $\underline{c}$ contains $n > 0$ indices.

Here are the first three generalised connections

$$\Gamma^a_{(bc)}(x) = \frac{2}{3}x^d g^{ae} R_{bdce} + \frac{1}{12}x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) + \mathcal{O}\left(\epsilon^4\right)$$

$$\Gamma^a_{(bcd)}(x) = \frac{1}{2}x^e g^{af}\nabla_b R_{cedf} + \mathcal{O}\left(\epsilon^4\right)$$

$$\Gamma^a_{(bcde)}(x) = \mathcal{O}\left(\epsilon^4\right)$$

8 Geodesics

The discussion so far has been framed mostly around the set of geodesics that pass though the point O. There are of course many geodesics within the patch of O that do not pass through O. The challenge now is to construct any geodesic within the patch of O. The approach followed here will be similar to that taken previously in which solutions of the geodesic initial value problem are used to solve the geodesic boundary value problem.

8.1 Geodesic initial value problem

Consider a point P distinct from O. At P it is reasonable to assume that the generalised connections do not vanish (which is generally true, the exception being flat space). Thus the coordinates x^a in the neighbourhood of P do not constitute a Riemann normal frame relative at P. But as P lies in the patch for O it follows (by definition of the patch) that the metric is non-singular at P and thus a new set of Riemann normal coordinates, y^a, with P as the origin, can always be constructed.

This problem has been discussed once before, in section (3.2). Using equation (3.1) and the generalised connections from section (7) leads to

$$\begin{aligned} x^a(s) = x^a + s\dot{x}^a + \frac{1}{24}s^2\dot{x}^b\dot{x}^c\Big(&-8x^d g^{ae}R_{bdce} - x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right)\Big) \\ &+ \frac{1}{12}s^3\dot{x}^b\dot{x}^c\dot{x}^d - x^e g^{af}\nabla_b R_{cedf} + \mathcal{O}\left(s^4, \epsilon^4\right) \end{aligned}$$

8.2 Geodesic boundary value problem

Consider now the case of three distinct points O, P and Q. The goal in this section is to compute the geodesic that passes through P and Q. Start by using equation (3.5) for the generalised connections from section (7) to obtain

$$
\begin{aligned}
x^a(\lambda) = x^a &+ \lambda Dx^a \\
&- \frac{1}{24}\left(\lambda - \lambda^2\right) Dx^b Dx^c \left(8x^d g^{ae} R_{dbce} + x^d x^e \left(2g^{af}\nabla_b R_{dcef} + 4g^{af}\nabla_d R_{ebcf} - g^{af}\nabla_f R_{dbec}\right)\right) \\
&- \frac{1}{12}\left(\lambda - \lambda^3\right) x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} + \mathcal{O}\left(\epsilon^4\right)
\end{aligned}
$$

where $\lambda = s/L_{PQ}$ is the scaled geodesic distance from $P(\lambda = 0)$ to $Q(\lambda = 1)$.

9 Geodesic arc-length

Given the explicit expressions for the metric and the geodesic that joins the points P and Q, the length of that geodesic can be computed by way of the integral

$$
L_{PQ} = \int_P^Q \left(g_{ab}(x)\frac{dx^a}{ds}\frac{dx^b}{ds}\right)^{1/2} ds
$$

Up to this point the parameter s has been taken to be the proper distance along the geodesic. However, after careful inspection of the geodesic path (3.1) it is clear that any uniform scaling of s is allowed. Thus it is possible to re-scale s so that $s = 0$ at P and $s = 1$ at Q (of course, the parameter s no longer measures proper distance so it may be better to use a different symbol, say λ). Furthermore, a standard result states that the integrand is constant along the geodesic and can thus be evaluated at any point such as P. The integral is now trivial to evaluate with the result

$$
L^2_{PQ} = g_{ab}(x)\frac{dx^a}{ds}\frac{dx^b}{ds}\bigg|_P
$$

which, using previous results, leads to

$$
\begin{aligned}
L^2_{PQ} = g_{ab}Dx^a Dx^b &- \frac{1}{3}x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12}x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} \\
&- \frac{1}{6}x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} + \mathcal{O}\left(\epsilon^4\right)
\end{aligned}
$$

A simple calculation shows that this result can also be written in the slightly more suggestive form

$$
L^2_{PQ} = g_{ab}(\bar{x})Dx^a Dx^b + \mathcal{O}\left(\epsilon^4\right)
$$

where $\bar{x}^a = (x^a_P + x^a_Q)/2$ is the *coordinate* mid-point of the geodesic. This does raise the question – Can higher order estimates for L^2 (see equation 11.21) be obtained by sampling the metric at suitablly chosen points on (or near) the geodesic? That question will be left for another occassion.

10 Cadabra

Though the above equations seem simple they do impose a significant computational cost – their recursive structure quickly leads to expressions that are too hard to manage (for humans) beyond the first few iterations. The computations are best left to computer programs specifically designed for tenor computations. One such program is Cadabra – a `C++` program that reads plain text files to perform the various tensor computations. It was initially designed for computations in high energy physics but it also very well suited to tensor computations in general.

Cadabra's syntax is a hybrid of LaTeX to express tensor expressions, Python to coordinate the computations and some unique Cadabra syntax to describe properties of various objects (e.g., index sets, symmetries, commutation rules etc.). A number of simple examples, including a detailed discussion of Cadabra's syntax with particular emphasis for use in General Relativity, can be found on the GitHub repository [15]. This includes full Cadabra sources for all the examples and exercises (with solutions) covering elementary topics (e.g., verifying $\nabla g = 0$ for a metric connection) through to more advanced topics (e.g., deriving the BSSN equations from the ADM equations). This material was based on an earlier paper [1] (written for an ealier version of Cadabra).

For those readers seeking a flavour of what Cadabra code looks like (before rushing off to learn Cadabra in detail) here are two short examples. The first shows how Cadabra can be used to verify that $\nabla g = 0$ when ∇ is the metric compatible covariant derivative. The second example shows how the truncation operator T^m_ϵ introduced in section (3.2) can be implemented in Cadabra.

10.1 The metric connection

This simple example (adapted from the earlier paper [1]) verifies that $\nabla g = 0$ when ∇ is the metric compatible covariant derivative.

```
# Define some properties

{a,b,c,d,e,f,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices.

g_{a b}::Metric.
g_{a}^{b}::KroneckerDelta.

\partial_{#}::PartialDerivative.

# Define a rule for the Christoffel symbol

Gamma := \Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
                                             + \partial_{c}{g_{b d}}
                                             - \partial_{d}{g_{b c}} );

# Define the covariant derivative of the metric

cderiv := \partial_{c}{g_{a b}} - g_{a d}\Gamma^{d}_{b c}
                                - g_{d b}\Gamma^{d}_{a c};

```

```
21 # Do the computations
22
23 substitute          (cderiv, Gamma);
24 distribute          (cderiv);
25 eliminate_metric    (cderiv);
26 eliminate_kronecker (cderiv);
27 canonicalise        (cderiv);
```

The code is rather easy to follow. The first few lines defines a set of indices (line 3) and some objects with particular properties (lines 5 to 8). This is followed by two rules for constructing a connection (line 12) and the covariant derivative of g_{ab} (line 18). The body of the calculations can be seen in the last few lines (23 to 27) where various operations (known as *algorithms* in Cadabra's lexicon) are applied. As expected, the above code returns a value of zero for the object `cderiv`.

A few points are worth noting. Cadabra uses a hybrid syntax of Python and LaTeX. Tensor equations are defined and results are returned using a subset of LaTeX while computations (such as substitutions, simplifications etc.) are expressed using a Python syntax. This use of Python and LaTeX makes for an easy entry into Cadabra programming as most users would already have familiarity with Python and LaTeX. Using LaTeX to define and record tensors means that it is very easy to carry results from one Cadabra notebook to another or even to other LaTeX documents – all of the Cadabra output shown in this paper were imported without change from the Cadabra output generated by other documents.

10.2 Truncation of polynomials

In section (3.2) a truncation operator T^m_ϵ was introduced. The question here is – How might that operator be implemented in Cadabra?

Suppose you are asked to extract the leading terms from an expression such as

$$P^a(x) = c^a + c^a_b x^b + c^a_{bc} x^b x^c + c^a_{bcd} x^b x^c x^d + c^a_{bcde} x^b x^c x^d x^e$$

One approach (there are others, e.g., emulating a truncated Taylor series) is to use Cadabra's `::Weight` property and the `keep_weight` algorithm. The idea is to assign weights to nominated objects (through the `::Weight` property) and then extract terms matching a chosen weight (using the `keep_weight` algorithm).

Here is a small piece of Cadabra code that does the job.

```
def truncate (obj,n):

    # define the weight and give it a label
    x^{a}::Weight(label=\epsilon).

    # start with an empty espression
    ans = Ex(0)

    # loop over selected terms in the source
    for i in range (0,n+1):

       foo := @(obj).
       bah  = Ex("\epsilon = " + str(i))
```

```
14
15          # extract a single term
16          keep_weight (foo, bah)
17
18          # update the running sum
19          ans = ans + foo
20
21      # all done, return final answer
22      return ans
23
24  # the quartic polynomial
25  quarticPoly :=   c^{a}
26                 + c^{a}_{b} x^{b}
27                 + c^{a}_{b c} x^{b} x^{c}
28                 + c^{a}_{b c d} x^{b} x^{c} x^{d}
29                 + c^{a}_{b c d e} x^{b} x^{c} x^{d} x^{e}.
30
31  # truncate to cubic terms
32  cubicPoly = truncate (quarticPoly,3)
```

The first thing to note is the Python function `truncate`. This simply does as its name suggests – it truncates an object to a certain order. How does it do the job? The first line to note is line 4. This identifies x^a as the target to carry the weights (and is given the label `\epsilon` to distinguish it from other targets declared by other instances of `::Weight`). Cadabra now sees the polynomial $P^a(x)$ as if it had been written as

$$P^a(x) = c^a + c^a_b x^b \epsilon + c^a_{bc} x^b x^c \epsilon^2 + c^a_{bcd} x^b x^c x^d \epsilon^3 + c^a_{bcde} x^b x^c x^d x^e \epsilon^4$$

The Python `for loop` then extracts the requested terms from the source returning the final truncated answer in `ans`. The final result is exactly as expected – the leading cubic part of the original quartic polynomial. This function and variations on it (e.g., extracting a single term) are used extensively in all of the Cadabra codes used in this paper.

11 Expansions to fifth order

All of the $\mathcal{O}(\epsilon^6)$ Cadabra programs were not overly demanding on computational resources, taking between 0.5 second and 12 minutes to run and requiring between 40 and 360 Mbyte of memory (on a Mac Pro (2013) running macOS 10.14.4).

The Cadabra codes and several support scripts are available from the author's GitHub site https://github.com/leo-brewin/riemann-normal-coords.

Some of the following expansions can be compared directly with results obtained by traditional methods. In particular the expansions for the metric and its inverse, equations (11.1) and (11.2) agree exactly with those given in equations (17) and (18) respectively of [6]. Also, equation (11.4) agrees with that given in equation (A12) of [16] (though showing that they are equivalent does require some work using the first and second Bianchi identities).

Note that in the following output any symmetrisation of indices on the right hand side should be inferred from the explicit symmetrisation (the round brackets) on the left hand side (as per the discussion in section 4).

The metric

$$g_{(ab)}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f \left(8g^{gh}R_{acdg}R_{befh} - 9\nabla_{cd}R_{aebf}\right)$$
$$+ \frac{1}{90}x^c x^d x^e x^f x^g \left(2g^{hi}R_{acdh}\nabla_e R_{bfgi} + 2g^{hi}R_{bcdh}\nabla_e R_{afgi} - \nabla_{cde}R_{afbg}\right) + \mathcal{O}\left(\epsilon^6\right) \quad (11.1)$$

The inverse metric

$$g^{(ab)}(x) = g^{ab} + \frac{1}{3}x^c x^d g^{ae}g^{bf}R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af}g^{bg}\nabla_c R_{dfeg}$$
$$+ \frac{1}{60}x^c x^d x^e x^f \left(4g^{ag}g^{bh}g^{ij}R_{cgdi}R_{ehfj} + 3g^{ag}g^{bh}\nabla_{cd}R_{egfh}\right)$$
$$+ \frac{1}{90}x^c x^d x^e x^f x^g \left(3g^{ah}g^{bi}g^{jk}R_{chdj}\nabla_e R_{figk} + 3g^{ah}g^{bi}g^{jk}R_{cidj}\nabla_e R_{fhgk} + g^{ah}g^{bi}\nabla_{cde}R_{fhgi}\right)$$
$$+ \mathcal{O}\left(\epsilon^6\right) \quad (11.2)$$

The metric determinant

$$-g(x) = 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc}$$
$$+ \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right)$$
$$+ \mathcal{O}\left(\epsilon^6\right) \quad (11.3a)$$

$$\sqrt{-g(x)} = 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc}$$
$$+ \frac{1}{360}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{360}x^a x^b x^c x^d x^e \left(-2\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - 2g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right)$$
$$+ \mathcal{O}\left(\epsilon^6\right) \quad (11.3b)$$

$$\log\left(-g(x)\right) = -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right) \quad (11.3c)$$

Generalised connections

The following results were obtained by recursive application of equation (3.2).

$$
\begin{aligned}
\Gamma^a_{(bc)}(x) = {}& \frac{2}{3}x^d g^{ae} R_{bdce} + \frac{1}{12}x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) \\
& + \frac{1}{360}x^d x^e x^f \left(64 g^{ag} g^{hi} R_{bdch} R_{egfi} - 32 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdeh} R_{cifg}\right. \\
& + 18 g^{ag}\nabla_{bd} R_{cefg} + 18 g^{ag}\nabla_{db} R_{cefg} + 36 g^{ag}\nabla_{de} R_{bfcg} - 16 g^{ag} g^{hi} R_{bdeh} R_{cfgi} \\
& \left. + 9 g^{ag}\nabla_{gd} R_{becf} + 9 g^{ag}\nabla_{dg} R_{becf}\right) + \frac{1}{180}x^d x^e x^f x^g \left(16 g^{ah} g^{ij} R_{bdci}\nabla_e R_{fhgj}\right. \\
& + 6 g^{ah} g^{ij} R_{dhei}\nabla_b R_{cfgj} + 16 g^{ah} g^{ij} R_{dhei}\nabla_f R_{bgcj} + 5 g^{ah} g^{ij} R_{dhei}\nabla_j R_{bfcg} \\
& - 8 g^{ah} g^{ij} R_{bhdi}\nabla_e R_{cfgj} - 4 g^{ah} g^{ij} R_{bidh}\nabla_e R_{cfgj} - 4 g^{ah} g^{ij} R_{bdei}\nabla_c R_{fhgj} \\
& - 8 g^{ah} g^{ij} R_{bdei}\nabla_f R_{chgj} - 4 g^{ah} g^{ij} R_{bdei}\nabla_f R_{cjgh} + 2 g^{ah}\nabla_{bde} R_{cfgh} + 2 g^{ah}\nabla_{dbe} R_{cfgh} \\
& + 2 g^{ah}\nabla_{deb} R_{cfgh} + 4 g^{ah}\nabla_{def} R_{bgch} - 4 g^{ah} g^{ij} R_{bdhi}\nabla_e R_{cfgj} - 4 g^{ah} g^{ij} R_{bdei}\nabla_h R_{cfgj} \\
& \left. - 4 g^{ah} g^{ij} R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde} R_{bfcg} + g^{ah}\nabla_{dhe} R_{bfcg} + g^{ah}\nabla_{deh} R_{bfcg}\right) \\
& + \mathcal{O}\left(\epsilon^6\right)
\end{aligned} \tag{11.4}
$$

$$
\begin{aligned}
\Gamma^a_{(bcd)}(x) = {}& \frac{1}{2}x^e g^{af}\nabla_b R_{cedf} + \frac{1}{120}x^e x^f \left(64 g^{ag} g^{hi} R_{bech} R_{dgfi} + 16 g^{ag} g^{hi} R_{bech} R_{difg}\right. \\
& - 16 g^{ag} g^{hi} R_{befh} R_{cgdi} + 12 g^{ag}\nabla_{bc} R_{defg} + 18 g^{ag}\nabla_{be} R_{cfdg} + 18 g^{ag}\nabla_{eb} R_{cfdg} \\
& \left. + 48 g^{ag} g^{hi} R_{bech} R_{dfgi} + 3 g^{ag}\nabla_{gb} R_{cedf} + 3 g^{ag}\nabla_{bg} R_{cedf}\right) \\
& + \frac{1}{180}x^e x^f x^g \left(32 g^{ah} g^{ij} R_{beci}\nabla_d R_{fhgj} + 48 g^{ah} g^{ij} R_{beci}\nabla_f R_{dhgj} + 12 g^{ah} g^{ij} R_{beci}\nabla_f R_{djgh}\right. \\
& + 18 g^{ah} g^{ij} R_{bhei}\nabla_c R_{dfgj} + 2 g^{ah} g^{ij} R_{bieh}\nabla_c R_{dfgj} + 22 g^{ah} g^{ij} R_{ehfi}\nabla_b R_{cgdj} \\
& + 48 g^{ah} g^{ij} R_{bhei}\nabla_f R_{cgdj} + 12 g^{ah} g^{ij} R_{bieh}\nabla_f R_{cgdj} + 15 g^{ah} g^{ij} R_{bhei}\nabla_j R_{cfdg} \\
& + 5 g^{ah} g^{ij} R_{bieh}\nabla_j R_{cfdg} - 12 g^{ah} g^{ij} R_{bhci}\nabla_e R_{dfgj} - 12 g^{ah} g^{ij} R_{befi}\nabla_c R_{dhgj} \\
& - 8 g^{ah} g^{ij} R_{befi}\nabla_c R_{djgh} - 12 g^{ah} g^{ij} R_{befi}\nabla_g R_{chdj} + 4 g^{ah}\nabla_{bce} R_{dfgh} + 4 g^{ah}\nabla_{bec} R_{dfgh} \\
& + 6 g^{ah}\nabla_{bef} R_{cgdh} + 4 g^{ah}\nabla_{ebc} R_{dfgh} + 6 g^{ah}\nabla_{ebf} R_{cgdh} + 6 g^{ah}\nabla_{efb} R_{cgdh} \\
& + 16 g^{ah} g^{ij} R_{behi}\nabla_c R_{dfgj} + 36 g^{ah} g^{ij} R_{behi}\nabla_f R_{cgdj} + 16 g^{ah} g^{ij} R_{beci}\nabla_h R_{dfgj} \\
& - 4 g^{ah} g^{ij} R_{befi}\nabla_h R_{cgdj} + 36 g^{ah} g^{ij} R_{beci}\nabla_f R_{dghj} - 4 g^{ah} g^{ij} R_{befi}\nabla_c R_{dghj} \\
& + g^{ah}\nabla_{hbe} R_{cfdg} + g^{ah}\nabla_{heb} R_{cfdg} + g^{ah}\nabla_{bhe} R_{cfdg} + g^{ah}\nabla_{ehb} R_{cfdg} + g^{ah}\nabla_{beh} R_{cfdg} \\
& \left. + g^{ah}\nabla_{ebh} R_{cfdg} - 20 g^{ah} g^{ij} R_{beci}\nabla_j R_{dfgh} + 10 g^{ah} g^{ij} R_{behi}\nabla_j R_{cfdg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned} \tag{11.5}
$$

$$\begin{aligned}
\Gamma^a_{(bcde)}(x) = {} & \frac{1}{15}x^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right)\\
& + \frac{1}{90}x^f x^g \left(64g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej}\right.\\
& \qquad + 4g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} + 44g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej}\\
& \qquad + 24g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} - 16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej}\\
& \qquad + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh}\\
& \qquad + 26g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 6g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj}\\
& \qquad \left. + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_j R_{dgeh}\right)\\
& + \mathcal{O}\left(\epsilon^6\right)
\end{aligned} \tag{11.6}$$

$$\Gamma^a_{(bcdef)}(x) = \frac{1}{3}x^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_d R_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right) + \mathcal{O}\left(\epsilon^6\right) \tag{11.7}$$

Symmetrised partial derivatives of the connection.

The following results were obtained from equation (5.5)

$$3\Gamma^a{}_{d(b,c)} = R^a{}_{bcd} \tag{11.8}$$

$$6\Gamma^a{}_{d(b,ce)} = 3\partial_e R^a{}_{bcd} \tag{11.9}$$

$$15\Gamma^a{}_{d(b,cef)} = 9\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed} \tag{11.10}$$

$$\begin{aligned}
9\Gamma^a{}_{d(b,cefg)} = {} & 6\partial_{gfe}R^a{}_{bcd} - R^h{}_{bgd}\partial_e R^a{}_{cfh} - R^h{}_{bfd}\partial_e R^a{}_{cgh} - R^a{}_{ceh}\partial_f R^h{}_{bgd} - R^h{}_{bed}\partial_f R^a{}_{cgh}\\
& - R^a{}_{cfh}\partial_e R^h{}_{bgd} - R^a{}_{cgh}\partial_e R^h{}_{bfd}
\end{aligned} \tag{11.11}$$

$$\begin{aligned}
252\Gamma^a{}_{d(b,cefgh)} = {} & 180\partial_{hgfe}R^a{}_{bcd} - 36R^i{}_{bgd}\partial_{he}R^a{}_{cfi} + 4R^a{}_{fei}R^i{}_{chj}R^j{}_{bgd} + 4R^a{}_{fhi}R^i{}_{cej}R^j{}_{bgd}\\
& - 72R^i{}_{bfd}\partial_{he}R^a{}_{cgi} + 8R^a{}_{gei}R^i{}_{chj}R^j{}_{bfd} + 8R^a{}_{ghi}R^i{}_{cej}R^j{}_{bfd} - 45\partial_e R^a{}_{cfi}\partial_g R^i{}_{bhd}\\
& - 45\partial_e R^a{}_{cgi}\partial_f R^i{}_{bhd} - 45\partial_e R^a{}_{chi}\partial_f R^i{}_{bgd} - 36R^a{}_{cei}\partial_{hf}R^i{}_{bgd} + 4R^a{}_{cei}R^i{}_{gfj}R^j{}_{bhd}\\
& + 4R^a{}_{cei}R^i{}_{ghj}R^j{}_{bfd} - 36R^i{}_{bed}\partial_{hf}R^a{}_{cgi} + 4R^a{}_{gfi}R^i{}_{chj}R^j{}_{bed} + 4R^a{}_{ghi}R^i{}_{cfj}R^j{}_{bed}\\
& - 45\partial_f R^a{}_{cgi}\partial_e R^i{}_{bhd} - 45\partial_f R^a{}_{chi}\partial_e R^i{}_{bgd} - 72R^a{}_{cfi}\partial_{he}R^i{}_{bgd} + 8R^a{}_{cfi}R^i{}_{gej}R^j{}_{bhd}\\
& + 8R^a{}_{cfi}R^i{}_{ghj}R^j{}_{bed} - 45\partial_g R^a{}_{chi}\partial_e R^i{}_{bfd} - 36R^a{}_{cgi}\partial_{he}R^i{}_{bfd} + 4R^a{}_{cgi}R^i{}_{fej}R^j{}_{bhd}\\
& + 4R^a{}_{cgi}R^i{}_{fhj}R^j{}_{bed}
\end{aligned} \tag{11.12}$$

Symmetrised partial derivatives of theRiemann curvature tensor.

The following results were obtained from equation (5.6). Note that by inspection of these results and those above it is easy to verify the claim made in section (5) that the connection terms that arise in ${R^a}_{(bc\dot{d};\underline{e})}$ are always of the form ${\Gamma^p}_{q(r,\underline{s})}$.

$$
\begin{aligned}
{R^a}_{(cd\dot{b},e)} &= g^{af}\nabla_c R_{bdef} && (11.13)\\
{R^a}_{(cd\dot{b},ef)} &= g^{ag}\nabla_{cd} R_{befg} && (11.14)\\
-2{R^a}_{(cd\dot{b},efg)} &= g^{ah}g^{ij}R_{bcdi}\nabla_e R_{fhgj} - g^{ah}g^{ij}R_{chdi}\nabla_e R_{bfgj} - 2g^{ah}\nabla_{cde}R_{bfgh} && (11.15)\\
-5{R^a}_{(cd\dot{b},efgh)} &= 7g^{ai}g^{jk}R_{bcdj}\nabla_{ef}R_{gihk} - 7g^{ai}g^{jk}R_{cidj}\nabla_{ef}R_{bghk} - 5g^{ai}\nabla_{cdef}R_{bghi} && (11.16)\\
-3{R^a}_{(cd\dot{b},efghi)} &= 6g^{aj}g^{kl}\nabla_c R_{bdek}\nabla_{fg}R_{hjil} - 6g^{aj}g^{kl}\nabla_c R_{djek}\nabla_{fg}R_{bhil} + 8g^{aj}g^{kl}R_{bcdk}\nabla_{efg}R_{hjil}\\
&\quad - 8g^{aj}g^{kl}R_{cjdk}\nabla_{efg}R_{bhil} - g^{aj}g^{kl}g^{mn}R_{bcdk}R_{elfm}\nabla_g R_{hjin}\\
&\quad - 3g^{aj}g^{kl}g^{mn}R_{cjdk}R_{elfm}\nabla_g R_{bhin} + 4g^{aj}g^{kl}g^{mn}R_{bcdk}R_{ejfm}\nabla_g R_{hlin}\\
&\quad - 3g^{aj}\nabla_{cdefg}R_{bhij} && (11.17)
\end{aligned}
$$

Riemann normal coordinates

$$
y^a = \overset{0}{y}{}^a + \overset{1}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a \qquad (11.18)
$$

$$
\begin{aligned}
\overset{0}{y}{}^a &= x^a && (11.18a)\\
2\overset{1}{y}{}^a &= x^b x^c {\Gamma^a}_{bc} && (11.18b)\\
6\overset{2}{y}{}^a &= x^b x^c x^d \left({\Gamma^a}_{be}{\Gamma^e}_{cd} + \partial_b {\Gamma^a}_{cd}\right) && (11.18c)\\
24\overset{3}{y}{}^a &= x^b x^c x^d x^e \left(2{\Gamma^a}_{bf}\partial_c{\Gamma^f}_{de} + {\Gamma^a}_{fg}{\Gamma^f}_{bc}{\Gamma^g}_{de} + {\Gamma^f}_{bc}\partial_f{\Gamma^a}_{de} + \partial_{bc}{\Gamma^a}_{de}\right) && (11.18d)\\
360\overset{4}{y}{}^a &= x^b x^c x^d x^e x^f \big(-4{\Gamma^a}_{bg}{\Gamma^g}_{ch}{\Gamma^h}_{di}{\Gamma^i}_{ef} + 2{\Gamma^a}_{bg}{\Gamma^g}_{ch}\partial_d{\Gamma^h}_{ef} + 3{\Gamma^a}_{bg}{\Gamma^g}_{hi}{\Gamma^h}_{cd}{\Gamma^i}_{ef}\\
&\quad - 6{\Gamma^a}_{bg}{\Gamma^h}_{cd}\partial_e{\Gamma^g}_{fh} + 6{\Gamma^a}_{bg}{\Gamma^h}_{cd}\partial_h{\Gamma^g}_{ef} + 9{\Gamma^a}_{bg}\partial_{cd}{\Gamma^g}_{ef} + 4{\Gamma^a}_{gh}{\Gamma^g}_{bc}{\Gamma^h}_{di}{\Gamma^i}_{ef}\\
&\quad + 13{\Gamma^a}_{gh}{\Gamma^g}_{bc}\partial_d{\Gamma^h}_{ef} - 4{\Gamma^g}_{bc}{\Gamma^h}_{dg}\partial_e{\Gamma^a}_{fh} + {\Gamma^g}_{bc}{\Gamma^h}_{dg}\partial_h{\Gamma^a}_{ef} + 2\partial_b{\Gamma^a}_{cg}\partial_d{\Gamma^g}_{ef}\\
&\quad + 7\partial_g{\Gamma^a}_{bc}\partial_d{\Gamma^g}_{ef} + 3{\Gamma^g}_{bc}{\Gamma^h}_{de}\partial_f{\Gamma^a}_{gh} + 3{\Gamma^g}_{bc}{\Gamma^h}_{de}\partial_g{\Gamma^a}_{fh} - 3{\Gamma^g}_{bc}\partial_{de}{\Gamma^a}_{fg} + 6{\Gamma^g}_{bc}\partial_{dg}{\Gamma^a}_{ef}\\
&\quad + 3\partial_{bcd}{\Gamma^a}_{ef}\big) && (11.18e)
\end{aligned}
$$

Geodesic IVP

$$
\begin{aligned}
x^a(s) &= x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A^a_{bc} + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A^a_{bcd} + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A^a_{bcde}\\
&\quad + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A^a_{bcdef} + \mathcal{O}\left(s^6, \epsilon^6\right) && (11.19)
\end{aligned}
$$

$$
\begin{aligned}
360A^a_{bc} = &-240x^d g^{ae} R_{bdce} - 30x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) \\
&- x^d x^e x^f \left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg}\right. \\
&\qquad + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} \\
&\qquad \left. + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\right) - 2x^d x^e x^f x^g \left(16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj}\right. \\
&\qquad + 6g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} \\
&\qquad - 8g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} \\
&\qquad - 8g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh} \\
&\qquad + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} \\
&\qquad \left. - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\right)
\end{aligned} \tag{11.19a}
$$

$$
\begin{aligned}
360A^a_{bcd} = &-180x^e g^{af}\nabla_b R_{cedf} - 3x^e x^f \left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg}\right. \\
&\qquad - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg} + 18g^{ag}\nabla_{eb}R_{cfdg} \\
&\qquad \left. + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right) \\
&- 2x^e x^f x^g \left(32g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh}\right. \\
&\qquad + 18g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} + 2g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} + 22g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} \\
&\qquad + 48g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} \\
&\qquad + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} - 12g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} \\
&\qquad - 8g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh} \\
&\qquad + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} \\
&\qquad + 16g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 16g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} \\
&\qquad - 4g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} \\
&\qquad + g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} \\
&\qquad \left. - 20g^{ah}g^{ij}R_{beci}\nabla_j R_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg}\right)
\end{aligned} \tag{11.19b}
$$

$$
\begin{aligned}
90A^a_{bcde} = &-6x^f \left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) \\
&- x^f x^g \left(64g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej}\right. \\
&\qquad + 4g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} + 44g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} \\
&\qquad + 24g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} - 16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} \\
&\qquad + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} \\
&\qquad + 26g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 6g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} \\
&\qquad \left. + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_j R_{dgeh}\right)
\end{aligned} \tag{11.19c}
$$

$$
3A^a_{bcdef} = -x^g \left(3g^{ah}g^{ij}R_{bgci}\nabla_d R_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right) \tag{11.19d}
$$

Geodesic BVP

$$
x^a(s) = x^a + sDx^a + (s-s^2)x^a_2 + (s-s^3)x^a_3 + (s-s^4)x^a_4 + (s-s^5)x^a_5 + \mathcal{O}\left(s^6, \epsilon^6\right) \tag{11.20}
$$

$$
x^a_2 = \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \overset{4}{x}{}^a_2 + \overset{5}{x}{}^a_2 + \mathcal{O}\left(\epsilon^6\right) \tag{11.20a}
$$

$$
x^a_3 = \overset{3}{x}{}^a_3 + \overset{4}{x}{}^a_3 + \overset{5}{x}{}^a_3 + \mathcal{O}\left(\epsilon^6\right) \tag{11.20b}
$$

$$
x^a_4 = \overset{4}{x}{}^a_4 + \overset{5}{x}{}^a_4 + \mathcal{O}\left(\epsilon^6\right) \tag{11.20c}
$$

$$
x^a_5 = \overset{5}{x}{}^a_5 + \mathcal{O}\left(\epsilon^6\right) \tag{11.20d}
$$

$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde} \tag{11.20a.1}$$

$$-24\overset{3}{x}{}^a_2 = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right) \tag{11.20a.2}$$

$$\begin{aligned}
-720\overset{4}{x}{}^a_2 = {} & x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) \\
& + x^b x^c x^d Dx^e Dx^f \big(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{bech}R_{dgfi} \\
& + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} \\
& - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\big)
\end{aligned} \tag{11.20a.3}$$

$$\begin{aligned}
-360\overset{5}{x}{}^a_2 = {} & x^b x^c x^d Dx^e Dx^f Dx^g \big(10g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 20g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} \\
& - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} - 20g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} \\
& + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} \\
& + 20g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} - 20g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 10g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj}\big) \\
& + x^b x^c Dx^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 10g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj}\right) \\
& + x^b x^c x^d x^e Dx^f Dx^g \big(16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} \\
& + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} - 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} \\
& - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} \\
& - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 2g^{ah}\nabla_{fbc}R_{dgeh} + 2g^{ah}\nabla_{bfc}R_{dgeh} + 2g^{ah}\nabla_{bcf}R_{dgeh} \\
& + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} \\
& + 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} - g^{ah}\nabla_{hbc}R_{dfeg} - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\big)
\end{aligned} \tag{11.20a.4}$$

$$-12\overset{3}{x}{}^a_3 = x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} \tag{11.20b.1}$$

$$\begin{aligned}
-720\overset{4}{x}{}^a_3 = {} & x^b x^c Dx^d Dx^e Dx^f \big(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} \\
& + 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} - 48g^{ag}g^{hi}R_{bdeh}R_{cfgi} \\
& - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\big)
\end{aligned} \tag{11.20b.2}$$

$$
\begin{aligned}
-1080\overset{5}{x}{}^a_3 = x^b x^c Dx^d Dx^e Dx^f Dx^g \big(&30g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} - 30g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} \\
&- 30g^{ah}g^{ij}R_{bdei}\nabla_j R_{cfgh}\big) + x^b x^c x^d Dx^e Dx^f Dx^g \big(32g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} \\
&+ 48g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} + 18g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} \\
&+ 2g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 22g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} + 48g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} \\
&+ 12g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} - 15g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} - 5g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} \\
&- 12g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} - 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} - 8g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} \\
&- 12g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 4g^{ah}\nabla_{efb}R_{cgdh} + 4g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{ebc}R_{dfgh} \\
&+ 4g^{ah}\nabla_{bef}R_{cgdh} + 6g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bce}R_{dfgh} - 16g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} \\
&- 36g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 16g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 4g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} \\
&- 36g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + 4g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - g^{ah}\nabla_{heb}R_{cfdg} - g^{ah}\nabla_{hbe}R_{cfdg} \\
&- g^{ah}\nabla_{ehb}R_{cfdg} - g^{ah}\nabla_{bhe}R_{cfdg} - g^{ah}\nabla_{ebh}R_{cfdg} - g^{ah}\nabla_{beh}R_{cfdg} \\
&+ 20g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 10g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg}\big) \qquad (11.20b.3)
\end{aligned}
$$

$$
-180\overset{4}{x}{}^a_4 = x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right) \qquad (11.20c.1)
$$

$$
\begin{aligned}
-2160\overset{5}{x}{}^a_4 = x^b x^c Dx^d Dx^e Dx^f Dx^g \big(&64g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} + 18g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} \\
&+ 24g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} + 4g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 44g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} \\
&+ 18g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj} + 24g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} - 10g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} \\
&- 16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}\nabla_{def}R_{bgch} + 8g^{ah}\nabla_{deb}R_{cfgh} + 8g^{ah}\nabla_{dbe}R_{cfgh} \\
&+ 8g^{ah}\nabla_{bde}R_{cfgh} - 26g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 6g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} \\
&- 46g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} - g^{ah}\nabla_{hde}R_{bfcg} - g^{ah}\nabla_{dhe}R_{bfcg} - g^{ah}\nabla_{deh}R_{bfcg} \\
&+ 40g^{ah}g^{ij}R_{bdei}\nabla_j R_{cfgh}\big) \qquad (11.20c.2)
\end{aligned}
$$

$$
\begin{aligned}
-360\overset{5}{x}{}^a_5 = x^b Dx^c Dx^d Dx^e Dx^f Dx^g \big(&3g^{ah}g^{ij}R_{bcdi}\nabla_e R_{fhgj} + 3g^{ah}g^{ij}R_{chdi}\nabla_e R_{bfgj} \\
&+ g^{ah}\nabla_{cde}R_{bfgh}\big) \qquad (11.20d.1)
\end{aligned}
$$

Geodesic arc-length

$$
(\Delta s)^2 = \overset{0}{\Delta} + \overset{2}{\Delta} + \overset{3}{\Delta} + \overset{4}{\Delta} + \overset{5}{\Delta} + \mathcal{O}\left(\epsilon^6\right) \qquad (11.21)
$$

$$
\overset{0}{\Delta} = g_{ab}Dx^a Dx^b \qquad (11.21a)
$$

$$
3\overset{2}{\Delta} = -x^a x^b Dx^c Dx^d R_{acbd} \qquad (11.21b)
$$

$$
12\overset{3}{\Delta} = -x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - 2x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \qquad (11.21c)
$$

$$
\begin{aligned}
360\overset{4}{\Delta} = {} & x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8g^{gh}R_{acdg}R_{befh} - 6\nabla_{cd}R_{aebf}\right) \\
&+ x^a x^b x^c Dx^d Dx^e Dx^f \left(16g^{gh}R_{adbg}R_{cefh} - 9\nabla_{da}R_{becf} - 9\nabla_{ad}R_{becf}\right) \\
&+ x^a x^b x^c x^d Dx^e Dx^f \left(16g^{gh}R_{aebg}R_{cfdh} - 18\nabla_{ab}R_{cedf}\right) \qquad (11.21d)
\end{aligned}
$$

$$
\begin{aligned}
1080\overset{5}{\Delta} = {}& x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4g^{hi}R_{adeh}\nabla_f R_{bgci} - 24g^{hi}R_{adeh}\nabla_b R_{cfgi} + 10g^{hi}R_{adeh}\nabla_i R_{bfcg}\right.\\
&\qquad\left. + 16g^{hi}R_{adbh}\nabla_e R_{cfgi} - 4\nabla_{dea}R_{bfcg} - 4\nabla_{dae}R_{bfcg} - 4\nabla_{ade}R_{bfcg}\right)\\
&+ x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18g^{hi}R_{acdh}\nabla_e R_{bfgi} - 3\nabla_{cde}R_{afbg}\right)\\
&+ x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24g^{hi}R_{aefh}\nabla_b R_{cgdi} + 24g^{hi}R_{aebh}\nabla_f R_{cgdi} + 24g^{hi}R_{aebh}\nabla_c R_{dfgi}\right.\\
&\qquad\left. - 6\nabla_{eab}R_{cfdg} - 6\nabla_{aeb}R_{cfdg} - 6\nabla_{abe}R_{cfdg}\right)\\
&+ x^a x^b x^c x^d x^e Dx^f Dx^g \left(48g^{hi}R_{afbh}\nabla_c R_{dgei} - 12\nabla_{abc}R_{dfeg}\right) \qquad (11.21e)
\end{aligned}
$$

12 Source

All of the Cadabra files used in preparing this paper are available at the GitHub repository https://github.com/leo-brewin/riemann-normal-coords

13 Acknowledgements

I am very grateful to Kasper Peeters for his many helpful suggestions. Any errors, omissions or inaccuracies in regard to Cadabra are entirely my fault.

Appendix A. Symmetrisation of tensors

The totally symmetric part of a tensor $A_{i_1 i_2 i_3 \cdots i_n}$ is commonly defined by

$$A_{(i_1 i_2 i_3 \cdots i_n)} = \frac{1}{n!}\left(A_{i_1 i_2 i_3 \cdots i_n} + A_{i_1 i_2 i_3 \cdots i_n} + A_{i_1 i_2 i_3 \cdots i_n} + \cdots\right)$$

where the sum on the right hand side includes every permutation of the indices of $i_1 i_2 i_3 \cdots i_n$. If the tensor $A_{i_1 i_2 i_3 \cdots i_n}$ happens to be symmetric in every pair of indices then

$$A_{(i_1 i_2 i_3 \cdots i_n)} = A_{i_1 i_2 i_3 \cdots i_n}$$

From the above definition it is very easy to establish the following theorems

$$
\begin{aligned}
A_{(i_1 i_2 i_3 \cdots (j_1 j_2 j_3 \cdots j_m) \cdots i_n)} &= A_{(i_1 i_2 i_3 \cdots j_1 j_2 j_3 \cdots j_m \cdots i_n)}\\
A_{(i_1 i_2 i_3 \cdots i_n} B_{j_1 j_2 j_3 \cdots j_m)} &= A_{((i_1 i_2 i_3 \cdots i_n)} B_{(j_1 j_2 j_3 \cdots j_m))}\\
nA_{(i_1 i_2 i_3 i_4 \cdots i_n)} &= A_{i_1 (i_2 i_3 i_4 \cdots i_n)} + A_{i_2 (i_1 i_3 i_4 \cdots i_n)} + A_{i_3 (i_1 i_2 i_4 \cdots i_n)} + \cdots\\
&\qquad + A_{i_n (i_1 i_2 i_3 \cdots i_{n-1})}\\
nA_{(i_1 i_2 i_3 i_4 \cdots i_n)} &= A_{(i_2 i_3 i_4 \cdots i_n) i_1} + A_{(i_1 i_3 i_4 \cdots i_n) i_2} + A_{(i_1 i_2 i_4 \cdots i_n) i_3} + \cdots\\
&\qquad + A_{(i_1 i_2 i_3 \cdots i_{n-1}) i_n}
\end{aligned}
$$

Suppose now that $A_{i_1 i_2 i_3 \cdots i_n} = A_{(i_1 i_2 i_3 \cdots i_n)}$, that is, $A_{i_1 i_2 i_3 \cdots i_n}$ is totally symmetric. Then for any B_j

$$
\begin{aligned}
(n+1)A_{(i_1 i_2 i_3 \cdots i_n} B_{j)} &= A_{j i_2 i_3 \cdots i_n} B_{i_1} + A_{i_1 j i_3 \cdots i_n} B_{i_2} + A_{i_1 i_2 j \cdots i_n} B_{i_3}\\
&\qquad + \cdots + A_{i_1 i_2 i_3 \cdots i_{n-1} j} B_{i_n}
\end{aligned}
$$

and

$$(n+1)A_{(i_1 i_2 i_3 \cdots i_n, j)} = A_{j i_2 i_3 \cdots i_n, i_1} + A_{i_1 j i_3 \cdots i_n, i_2} + A_{i_1 i_2 j \cdots i_n, i_3} + \cdots + A_{i_1 i_2 i_3 \cdots i_{n-1} j, i_n} + A_{i_1 i_2 i_3 \cdots i_n, j}$$

All of the above are very easy to prove but one result which requires just a little more thought is the following.

Suppose $A_{i_1 i_2 j_3 j_4 j_5 \cdots j_n}$ is symmetric in the pair $i_1 i_2$ and symmetric in all the indices $j_3 j_4 j_5 \cdots j_n$. That is, it is symmetric under the interchange of any pair of i's and any pair of j's but it is *not* necessarily symmetric when any i is swapped with any j. What can be said about $A_{(i_1 i_2 j_3 j_4 j_5 \cdots j_n)}$? Here is the result

$$nA_{(i_1 i_2 i_3 \cdots i_n)} = 2A_{i_n (i_1 i_2 i_3 \cdots i_{n-1})} + (n-2)A_{(i_1 i_2 i_3 \cdots i_{n-1}) i_n} \tag{A.1}$$

The proof is very easy. Begin by writing out $n!\, A_{(i_1 i_2 i_3 \cdots i_n)}$ in full. Then partition the terms into two disjoint sets, one set in which i_n appears in one of the first two index slots, the other set in which i_n appears in any of the remaining $n-2$ slots. The terms in the first set are exactly those that define $A_{i_n (i_1 i_2 i_3 \cdots i_{n-1})}$ while those in the second set define $A_{(i_1 i_2 i_3 \cdots i_{n-1}) i_n}$. The above equation follows by simply counting the number of terms in each set ($2(n-1)!$ and $(n-2)(n-1)!$ respectively) and the simple observation that $n!\, A_{(i_1 i_2 i_3 \cdots i_n)}$ equals the sum of the terms from both sets.

Finally note that if $Q = A_{i_1 i_2 i_3 \cdots i_n} x^{i_1} x^{i_2} x^{i_3} \cdots x^{i_n}$ then

$$Q_{,i_1 i_2 i_3 \cdots i_n} = n!\, A_{(i_1 i_2 i_3 \cdots i_n)}$$

$$Q = A_{(i_1 i_2 i_3 \cdots i_n)} x^{i_1} x^{i_2} x^{i_3} \cdots x^{i_n}$$

References

[1] Leo Brewin. "A brief introduction to Cadabra: a tool for tensor computations in General Relativity". In: *Computer Physics Communications* 181 (2010), pp. 489–498. eprint: arXiv:0903.2085. URL: http://users.monash.edu.au/~leo/research/papers/files/lcb09-02.html.

[2] Kasper Peeters. "Introducing Cadabra: a symbolic computer algebra system for field theory problems". In: (2007). eprint: arXiv:hep-th/0701238.

[3] Kasper Peeters. "Cadabra2: computer algebra for field theory revisited." In: *Journal of Open Source Software.* 3.32 (2018), p. 1118.

[4] Kasper Peeters. *The Cadabra v2.x home page.* 2017. URL: https://cadabra.science/index.html.

[5] Kasper Peeters. *The Cadabra2 GitHub reporistory.* 2017. URL: https://github.com/kpeeters/cadabra2.

[6] Agapitos Hatzinikitas. "A note on Riemann normal coordinates". In: (2000). eprint: arXiv:hep-th/0001078v1.

[7] Kiyoshi Higashijima, Etsuko Itou, and Muneto Nitta. "Normal Coordinates in Kahler Manifolds and the Background Field Method". In: *Prog.Theor.Phys.* 108 (2002), pp. 185–202. eprint: arXiv:hep-th/0203081v3.

[8] Yoshiyuki Yamashita. “Computer Calculation of Tensors in Riemann Normal Coordinates”. In: *General Relativity and Gravitation* 16.2 (1984), pp. 99–110.

[9] Uwe Müller, Christian Schubert, and Anton E.M. van de Ven. “A Closed Formula for the Riemann Normal Coordinate Expansion”. In: *Gen.Rel.Grav* 31 (1999), pp. 1759–1768. eprint: arXiv:gr-qc/9712092v2.

[10] S.S. Chern, W.H. Chen, and K.S. Lam. *Lectures on Differential Geometry*. World Scientific, Singapore, 2000.

[11] Isaac Chavel. *Riemannian Geometry. A modern introduction, 2nd ed.* Cambridge University Press, Cambridge., 2006.

[12] Luther Pfahler Eisenhart. *Riemannian Geometry*. Princeton University Press, Princeton, 1926.

[13] Alfred Gray. “The volume of a small geodesic ball of a Riemannian manifold”. In: *Michigan.Math.J.* 20 (1973), pp. 329–344.

[14] T.J. Willmore. *Riemannian Geometry*. Oxford University Press, Oxford, 1996.

[15] Leo Brewin. *A tutorial on Cadabra*. URL: https://github.com/leo-brewin/cadabra-tutorial (visited on 2019).

[16] E. Calzetta, S. Habib, and B.L. Hu. “Quantum kinetic field theory in curved spacetime: Covariant Wigner function and Liouville-Vlasov equations”. In: *Phys. Rev. D* 37.10 (1988), pp. 2901–2919.