

# The determinant of the metric

Our game here is to compute (the leading terms) in  $\det g$  of the metric in RNC form

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{2}{45}x^c x^d x^e x^f R_{acd g} R_{bef h} g^{gh} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{aebf} \\ + \frac{1}{45}x^c x^d x^e x^f x^g R_{acd h} \nabla_e R_{bf g i} g^{hi} + \frac{1}{45}x^c x^d x^e x^f x^g R_{bcd h} \nabla_e R_{af g i} g^{hi} - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{afbg} + \mathcal{O}(\epsilon^5)$$

For the sake of simplicity let's assume that we are working in 3-dimensions. The following analysis is easily generalised to other dimensions (and the final answers for  $\det g$  and friends are unchanged).

Define  $\epsilon_{ijk}^{abc}$  by

$$\epsilon_{ijk}^{abc} = \delta_i^a \delta_j^b \delta_k^c - \delta_i^b \delta_j^a \delta_k^c + \delta_i^c \delta_j^a \delta_k^b - \delta_i^c \delta_j^b \delta_k^a + \delta_i^b \delta_j^c \delta_k^a - \delta_i^a \delta_j^c \delta_k^b \quad (1)$$

It is easy to see that  $\epsilon_{ijk}^{abc}$  is anti-symmetric in both its upper and lower indices. A trivial computation shows that for any  $3 \times 3$  square matrix  $M_{ab}$ ,

$$\epsilon_{123}^{abc} M_{1a} M_{2b} M_{3c} = (\delta_1^a \delta_2^b \delta_3^c - \delta_1^b \delta_2^a \delta_3^c + \delta_1^c \delta_2^a \delta_3^b - \delta_1^c \delta_2^b \delta_3^a + \delta_1^b \delta_2^c \delta_3^a - \delta_1^a \delta_2^c \delta_3^b) M_{1a} M_{2b} M_{3c} = \det M \quad (2)$$

This can be easily generalised to

$$\epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} = \begin{cases} \pm \det M & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The  $\pm$  sign in the above depends on the particular permutations of  $(ijk)$  and  $(pqr)$ . If both permutations are even or both odd then the sign is  $+1$  otherwise the sign is  $-1$ . The same arguments can also be applied to a matrix inverse  $N^{-1}$  leading to

$$\epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} = \begin{cases} \pm \det N^{-1} & \text{when } (ijk) \text{ and } (pqr) \text{ are permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Note that the  $\pm$  in this case will match exactly that for the case of  $\det M$ . Thus, multiplying both expressions and summing over all choices for  $(ijk)$  and  $(pqr)$  leads to

$$\sum_{\substack{(ijk) \\ (pqr)}} (\det N^{-1}) \det M = \epsilon_{uvw}^{ijk} N^{pu} N^{qv} M^{rw} \epsilon_{ijk}^{abc} M_{pa} M_{qb} M_{rc} \quad (5)$$

where the sum on the left hand side includes just those  $(ijk)$  and  $(prq)$  that are permutations of  $(123)$ . There are  $3!$  choices for  $(ijk)$  and  $3!$  choices for  $(prq)$  and thus the left hand side is easily reduced to  $(3!)^2 \det M / \det N$  where  $\det N = 1 / \det N^{-1}$ . For the right hand side notice that

$$\epsilon_{uvw}^{ijk} \epsilon_{ijk}^{abc} = 3! \epsilon_{uvw}^{abc} \quad (6)$$

which leads to

$$\det M = \frac{1}{3!} \det N \epsilon_{uvw}^{abc} M_{pa} M_{qb} M_{rc} N^{pu} N^{qv} N^{rw} \quad (7)$$

For our RNC metric we will set  $N^{ab} = g^{ab}$  and  $M_{ij} = g_{ij}(x)$ . Since  $g^{ab}$  is of the form  $\text{diag}(-1, 1, 1, 1)$  we have  $\det g = -1$  and thus

$$\det g(x) = -\frac{1}{3!} \epsilon_{ijk}^{abc} g_{pa}(x) g_{qb}(x) g_{rc}(x) g^{ip} g^{jq} g^{kr} \quad (8)$$

The  $\epsilon_{ijk}^{abc}$  can be constructed in Cadabra by applying the `asym` algorithm to the upper indices of  $\delta_i^a \delta_j^b \delta_k^c$ . Note that `asym` will include the  $1/3!$  coefficient as part of its output.

The following code computes  $-\det g$  rather than  $\det g$ .

**Note that Calzetta uses an opposite sign for  $R_{abcd}$  so when comparing my results against Calzetta do take note of this flipped sign in  $R_{abcd}$ .**

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices(position=independent).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Integer(1..2).

\nabla{#}::Derivative.

d{#}::KroneckerDelta.

g^{a b}::Symmetric.
g_{a b}::Symmetric.

R_{a b c d}::RiemannTensor.

x^{a}::Weight(label=numx,value=1).

def truncate (obj,n):

    ans = Ex(0)

    for i in range (0,n+1):
        foo := @(obj).
        bah = Ex("numx = " + str(i))
        keep_weight (foo, bah)
        ans = ans + foo

    return ans

import cdblib

g0ab = cdblib.get('g_ab_0','metric.json')
g1ab = cdblib.get('g_ab_1','metric.json')  # zero in RNC
g2ab = cdblib.get('g_ab_2','metric.json')
g3ab = cdblib.get('g_ab_3','metric.json')
g4ab = cdblib.get('g_ab_4','metric.json')
g5ab = cdblib.get('g_ab_5','metric.json')

gab := @(g0ab) + @(g1ab) + @(g2ab) + @(g3ab) + @(g4ab) + @(g5ab).  # cdb (gab.001,gab)
gxab := gx_{a b} -> @(gab).

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eps := d^{a}_{i} d^{b}_{j}.      # cdb(eps.001,eps)
asym (eps,$^{a},^{b}$)          # cdb(eps.002,eps) # includes a factor of 1/2!

# compute negative Ndetg rather than det g
Ndetg := @(eps) gx_{p a} gx_{q b} g^{i p} g^{j q}. # note 1/2! included in eps

substitute      (Ndetg,gxab)
distribute      (Ndetg)
Ndetg = truncate (Ndetg,5)                                     # cdb (Ndetg.001,Ndetg)
substitute      (Ndetg,$g^{a b} g_{b c} -> d^{a}_{c}$,repeat=True) # cdb (Ndetg.002,Ndetg)
eliminate_kronecker (Ndetg)                                    # cdb (Ndetg.003,Ndetg)
sort_product     (Ndetg)                                       # cdb (Ndetg.004,Ndetg)
rename_dummies   (Ndetg)                                       # cdb (Ndetg.005,Ndetg)
canonicalise     (Ndetg)                                       # cdb (Ndetg.006,Ndetg)

# introduce the Ricci tensor

substitute      (Ndetg,$R_{a b c d} g^{a c} -> R_{b d}$,repeat=True) # cdb (Ndetg.101,Ndetg)
substitute      (Ndetg,$\nabla_{a}\{R_{b c d e}\} g^{b d} -> \nabla_{a}\{R_{c e}\}$,repeat=True) # cdb (Ndetg.102,Ndetg)
substitute      (Ndetg,$\nabla_{a b}\{R_{c d e f}\} g^{c e} -> \nabla_{a b}\{R_{d f}\}$,repeat=True) # cdb (Ndetg.103,Ndetg)
substitute      (Ndetg,$\nabla_{a b c}\{R_{d e f g}\} g^{d f} -> \nabla_{a b c}\{R_{e g}\}$,repeat=True) # cdb (Ndetg.104,Ndetg)

# the following are based on sqrt-Ndetg.tex

sqrtNdetg := 1/2 + (1/2) @(Ndetg)
            - (1/8) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
            - (1/4) (1/18) R_{a b} \nabla_{c}\{R_{d e}\} x^{a} x^{b} x^{c} x^{d} x^{e}.
            # cdb (sqrtNdetg.001,sqrtNdetg)

sort_product     (sqrtNdetg)                                     # cdb (sqrtNdetg.002,sqrtNdetg)
rename_dummies   (sqrtNdetg)                                     # cdb (sqrtNdetg.003,sqrtNdetg)
canonicalise     (sqrtNdetg)                                     # cdb (sqrtNdetg.004,sqrtNdetg)

logNdetg := -1 + @(Ndetg)
            - (1/2) (1/9) R_{a b} R_{c d} x^{a} x^{b} x^{c} x^{d}
            - (1/18) R_{a b} \nabla_{c}\{R_{d e}\} x^{a} x^{b} x^{c} x^{d} x^{e}.
            # cdb (logNdetg.001,logNdetg)

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sort_product      (logNdetg)          # cdb (logNdetg.002,logNdetg)
rename_dummies    (logNdetg)          # cdb (logNdetg.003,logNdetg)
canonicalise      (logNdetg)          # cdb (logNdetg.004,logNdetg)
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$$\text{eps.001} := d^a_i d^b_j$$

$$\text{eps.002} := \frac{1}{2} d^a_i d^b_j - \frac{1}{2} d^b_i d^a_j$$

$$\begin{aligned} \text{Ndetg.001} := & \frac{1}{2} d^a_i d^b_j g_{pa} g_{qb} g^{ip} g^{jq} - \frac{1}{2} d^b_i d^a_j g_{pa} g_{qb} g^{ip} g^{jq} - \frac{1}{6} d^a_i d^b_j g_{pa} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{6} d^a_i d^b_j x^c x^d R_{pcad} g_{qb} g^{ip} g^{jq} + \frac{1}{6} d^b_i d^a_j g_{pa} x^l x^m R_{qlbm} g^{ip} g^{jq} \\ & + \frac{1}{6} d^b_i d^a_j x^c x^d R_{pcad} g_{qb} g^{ip} g^{jq} - \frac{1}{12} d^a_i d^b_j g_{pa} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} - \frac{1}{12} d^a_i d^b_j x^c x^d x^e \nabla_c R_{pdae} g_{qb} g^{ip} g^{jq} \\ & + \frac{1}{12} d^b_i d^a_j g_{pa} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} + \frac{1}{12} d^b_i d^a_j x^c x^d x^e \nabla_c R_{pdae} g_{qb} g^{ip} g^{jq} + \frac{1}{45} d^a_i d^b_j g_{pa} x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} g^{ip} g^{jq} \\ & - \frac{1}{40} d^a_i d^b_j g_{pa} x^l x^m x^n x^o \nabla_{lm} R_{qnbo} g^{ip} g^{jq} + \frac{1}{18} d^a_i d^b_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} + \frac{1}{45} d^a_i d^b_j x^c x^d x^e x^f R_{pcdg} R_{ae fh} g^{gh} g_{qb} g^{ip} g^{jq} \\ & - \frac{1}{40} d^a_i d^b_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g_{qb} g^{ip} g^{jq} - \frac{1}{45} d^b_i d^a_j g_{pa} x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} g^{ip} g^{jq} + \frac{1}{40} d^b_i d^a_j g_{pa} x^l x^m x^n x^o \nabla_{lm} R_{qnbo} g^{ip} g^{jq} \\ & - \frac{1}{18} d^b_i d^a_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{45} d^b_i d^a_j x^c x^d x^e x^f R_{pcdg} R_{ae fh} g^{gh} g_{qb} g^{ip} g^{jq} + \frac{1}{40} d^b_i d^a_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g_{qb} g^{ip} g^{jq} \\ & + \frac{1}{90} d^a_i d^b_j g_{pa} x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} g^{ip} g^{jq} + \frac{1}{90} d^a_i d^b_j g_{pa} x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} g^{ip} g^{jq} \\ & - \frac{1}{180} d^a_i d^b_j g_{pa} x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} g^{ip} g^{jq} + \frac{1}{36} d^a_i d^b_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} \\ & + \frac{1}{36} d^a_i d^b_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} + \frac{1}{90} d^a_i d^b_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g_{qb} g^{ip} g^{jq} \\ & + \frac{1}{90} d^a_i d^b_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g_{qb} g^{ip} g^{jq} - \frac{1}{180} d^a_i d^b_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g_{qb} g^{ip} g^{jq} \\ & - \frac{1}{90} d^b_i d^a_j g_{pa} x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} g^{ip} g^{jq} - \frac{1}{90} d^b_i d^a_j g_{pa} x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} g^{ip} g^{jq} \\ & + \frac{1}{180} d^b_i d^a_j g_{pa} x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} g^{ip} g^{jq} - \frac{1}{36} d^b_i d^a_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} \\ & - \frac{1}{36} d^b_i d^a_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{90} d^b_i d^a_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g_{qb} g^{ip} g^{jq} \\ & - \frac{1}{90} d^b_i d^a_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g_{qb} g^{ip} g^{jq} + \frac{1}{180} d^b_i d^a_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g_{qb} g^{ip} g^{jq} \end{aligned}$$

$$\begin{aligned}
\text{Ndetg.002} := & \frac{1}{2}d^a{}_i d^b{}_j d^i{}_a d^j{}_b - \frac{1}{2}d^b{}_i d^a{}_j d^i{}_a d^j{}_b - \frac{1}{6}d^a{}_i d^b{}_j x^l x^m R_{qlbm} d^i{}_a g^{jq} - \frac{1}{6}d^a{}_i d^b{}_j x^c x^d R_{pcad} g^{ip} d^j{}_b + \frac{1}{6}d^b{}_i d^a{}_j x^l x^m R_{qlbm} d^i{}_a g^{jq} \\
& + \frac{1}{6}d^b{}_i d^a{}_j x^c x^d R_{pcad} g^{ip} d^j{}_b - \frac{1}{12}d^a{}_i d^b{}_j x^l x^m x^n \nabla_l R_{qmbn} d^i{}_a g^{jq} - \frac{1}{12}d^a{}_i d^b{}_j x^c x^d x^e \nabla_c R_{pdae} g^{ip} d^j{}_b \\
& + \frac{1}{12}d^b{}_i d^a{}_j x^l x^m x^n \nabla_l R_{qmbn} d^i{}_a g^{jq} + \frac{1}{12}d^b{}_i d^a{}_j x^c x^d x^e \nabla_c R_{pdae} g^{ip} d^j{}_b + \frac{1}{45}d^a{}_i d^b{}_j x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} d^i{}_a g^{jq} \\
& - \frac{1}{40}d^a{}_i d^b{}_j x^l x^m x^n x^o \nabla_{lm} R_{qnbo} d^i{}_a g^{jq} + \frac{1}{18}d^a{}_i d^b{}_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} + \frac{1}{45}d^a{}_i d^b{}_j x^c x^d x^e x^f R_{pcdg} R_{aefh} g^{gh} g^{ip} d^j{}_b \\
& - \frac{1}{40}d^a{}_i d^b{}_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g^{ip} d^j{}_b - \frac{1}{45}d^b{}_i d^a{}_j x^l x^m x^n x^o R_{qlmr} R_{bnos} g^{rs} d^i{}_a g^{jq} + \frac{1}{40}d^b{}_i d^a{}_j x^l x^m x^n x^o \nabla_{lm} R_{qnbo} d^i{}_a g^{jq} \\
& - \frac{1}{18}d^b{}_i d^a{}_j x^c x^d R_{pcad} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{45}d^b{}_i d^a{}_j x^c x^d x^e x^f R_{pcdg} R_{aefh} g^{gh} g^{ip} d^j{}_b + \frac{1}{40}d^b{}_i d^a{}_j x^c x^d x^e x^f \nabla_{cd} R_{peaf} g^{ip} d^j{}_b \\
& + \frac{1}{90}d^a{}_i d^b{}_j x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} d^i{}_a g^{jq} + \frac{1}{90}d^a{}_i d^b{}_j x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} d^i{}_a g^{jq} \\
& - \frac{1}{180}d^a{}_i d^b{}_j x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} d^i{}_a g^{jq} + \frac{1}{36}d^a{}_i d^b{}_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} + \frac{1}{36}d^a{}_i d^b{}_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} \\
& + \frac{1}{90}d^a{}_i d^b{}_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g^{ip} d^j{}_b + \frac{1}{90}d^a{}_i d^b{}_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g^{ip} d^j{}_b \\
& - \frac{1}{180}d^a{}_i d^b{}_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g^{ip} d^j{}_b - \frac{1}{90}d^b{}_i d^a{}_j x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{bort} g^{st} d^i{}_a g^{jq} \\
& - \frac{1}{90}d^b{}_i d^a{}_j x^l x^m x^n x^o x^r R_{blms} \nabla_n R_{qort} g^{st} d^i{}_a g^{jq} + \frac{1}{180}d^b{}_i d^a{}_j x^l x^m x^n x^o x^r \nabla_{lmn} R_{qobr} d^i{}_a g^{jq} - \frac{1}{36}d^b{}_i d^a{}_j x^c x^d R_{pcad} x^l x^m x^n \nabla_l R_{qmbn} g^{ip} g^{jq} \\
& - \frac{1}{36}d^b{}_i d^a{}_j x^c x^d x^e \nabla_c R_{pdae} x^l x^m R_{qlbm} g^{ip} g^{jq} - \frac{1}{90}d^b{}_i d^a{}_j x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{afgk} g^{hk} g^{ip} d^j{}_b \\
& - \frac{1}{90}d^b{}_i d^a{}_j x^c x^d x^e x^f x^g R_{acdh} \nabla_e R_{pfgk} g^{hk} g^{ip} d^j{}_b + \frac{1}{180}d^b{}_i d^a{}_j x^c x^d x^e x^f x^g \nabla_{cde} R_{pfag} g^{ip} d^j{}_b
\end{aligned}$$

$$\begin{aligned}
\text{Ndetg.003} := & 1 - \frac{1}{6}x^l x^m R_{qljm} g^{jq} - \frac{1}{3}x^c x^d R_{pcid} g^{ip} + \frac{1}{6}x^c x^d R_{pcbd} g^{bp} - \frac{1}{12}x^l x^m x^n \nabla_l R_{qmjn} g^{jq} - \frac{1}{6}x^c x^d x^e \nabla_c R_{pdie} g^{ip} \\
& + \frac{1}{12}x^c x^d x^e \nabla_c R_{pdbe} g^{bp} + \frac{1}{45}x^l x^m x^n x^o R_{qlmr} R_{jnos} g^{rs} g^{jq} - \frac{1}{40}x^l x^m x^n x^o \nabla_{lm} R_{qnjo} g^{jq} + \frac{1}{18}x^c x^d R_{pcid} x^l x^m R_{qljm} g^{ip} g^{jq} \\
& + \frac{2}{45}x^c x^d x^e x^f R_{pcdg} R_{iefh} g^{gh} g^{ip} - \frac{1}{20}x^c x^d x^e x^f \nabla_{cd} R_{peif} g^{ip} - \frac{1}{18}x^c x^d R_{pcjd} x^l x^m R_{qlim} g^{ip} g^{jq} \\
& - \frac{1}{45}x^c x^d x^e x^f R_{pcdg} R_{befh} g^{gh} g^{bp} + \frac{1}{40}x^c x^d x^e x^f \nabla_{cd} R_{pebf} g^{bp} + \frac{1}{90}x^l x^m x^n x^o x^r R_{qlms} \nabla_n R_{jort} g^{st} g^{jq} \\
& + \frac{1}{90}x^l x^m x^n x^o x^r R_{jlms} \nabla_n R_{qort} g^{st} g^{jq} - \frac{1}{180}x^l x^m x^n x^o x^r \nabla_{lmn} R_{qojr} g^{jq} + \frac{1}{36}x^c x^d R_{pcid} x^l x^m x^n \nabla_l R_{qmjn} g^{ip} g^{jq} \\
& + \frac{1}{36}x^c x^d x^e \nabla_c R_{pdie} x^l x^m R_{qljm} g^{ip} g^{jq} + \frac{1}{45}x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{ifgk} g^{hk} g^{ip} + \frac{1}{45}x^c x^d x^e x^f x^g R_{icdh} \nabla_e R_{pfgk} g^{hk} g^{ip} \\
& - \frac{1}{90}x^c x^d x^e x^f x^g \nabla_{cde} R_{pfig} g^{ip} - \frac{1}{36}x^c x^d R_{pcjd} x^l x^m x^n \nabla_l R_{qmin} g^{ip} g^{jq} - \frac{1}{36}x^c x^d x^e \nabla_c R_{pdje} x^l x^m R_{qlim} g^{ip} g^{jq} \\
& - \frac{1}{90}x^c x^d x^e x^f x^g R_{pcdh} \nabla_e R_{bfgk} g^{hk} g^{bp} - \frac{1}{90}x^c x^d x^e x^f x^g R_{bcdh} \nabla_e R_{pfgk} g^{hk} g^{bp} + \frac{1}{180}x^c x^d x^e x^f x^g \nabla_{cde} R_{pfbg} g^{bp}
\end{aligned}$$

$$\begin{aligned}
\text{Ndetg.004} := & 1 - \frac{1}{6}R_{qljm} g^{jq} x^l x^m - \frac{1}{3}R_{pcid} g^{ip} x^c x^d + \frac{1}{6}R_{pcbd} g^{bp} x^c x^d - \frac{1}{12}\nabla_l R_{qmjn} g^{jq} x^l x^m x^n - \frac{1}{6}\nabla_c R_{pdie} g^{ip} x^c x^d x^e \\
& + \frac{1}{12}\nabla_c R_{pdbe} g^{bp} x^c x^d x^e + \frac{1}{45}R_{jnos} R_{qlmr} g^{jq} g^{rs} x^l x^m x^n x^o - \frac{1}{40}\nabla_{lm} R_{qnjo} g^{jq} x^l x^m x^n x^o + \frac{1}{18}R_{pcid} R_{qljm} g^{ip} g^{jq} x^c x^d x^l x^m \\
& + \frac{2}{45}R_{iefh} R_{pcdg} g^{gh} g^{ip} x^c x^d x^e x^f - \frac{1}{20}\nabla_{cd} R_{peif} g^{ip} x^c x^d x^e x^f - \frac{1}{18}R_{pcjd} R_{qlim} g^{ip} g^{jq} x^c x^d x^l x^m \\
& - \frac{1}{45}R_{befh} R_{pcdg} g^{bp} g^{gh} x^c x^d x^e x^f + \frac{1}{40}\nabla_{cd} R_{pebf} g^{bp} x^c x^d x^e x^f + \frac{1}{90}R_{qlms} \nabla_n R_{jort} g^{jq} g^{st} x^l x^m x^n x^o x^r \\
& + \frac{1}{90}R_{jlms} \nabla_n R_{qort} g^{jq} g^{st} x^l x^m x^n x^o x^r - \frac{1}{180}\nabla_{lmn} R_{qojr} g^{jq} x^l x^m x^n x^o x^r + \frac{1}{36}R_{pcid} \nabla_l R_{qmjn} g^{ip} g^{jq} x^c x^d x^l x^m x^n \\
& + \frac{1}{36}R_{qljm} \nabla_c R_{pdie} g^{ip} g^{jq} x^c x^d x^e x^l x^m + \frac{1}{45}R_{pcdh} \nabla_e R_{ifgk} g^{hk} g^{ip} x^c x^d x^e x^f x^g + \frac{1}{45}R_{icdh} \nabla_e R_{pfgk} g^{hk} g^{ip} x^c x^d x^e x^f x^g \\
& - \frac{1}{90}\nabla_{cde} R_{pfig} g^{ip} x^c x^d x^e x^f x^g - \frac{1}{36}R_{pcjd} \nabla_l R_{qmin} g^{ip} g^{jq} x^c x^d x^l x^m x^n - \frac{1}{36}R_{qlim} \nabla_c R_{pdje} g^{ip} g^{jq} x^c x^d x^e x^l x^m \\
& - \frac{1}{90}R_{pcdh} \nabla_e R_{bfgk} g^{bp} g^{hk} x^c x^d x^e x^f x^g - \frac{1}{90}R_{bcdh} \nabla_e R_{pfgk} g^{bp} g^{hk} x^c x^d x^e x^f x^g + \frac{1}{180}\nabla_{cde} R_{pfbg} g^{bp} x^c x^d x^e x^f x^g
\end{aligned}$$



$$\begin{aligned}
\text{Ndetg.005} &:= 1 - \frac{1}{3}R_{abcd}g^{ca}x^bx^d - \frac{1}{6}\nabla_e R_{abcd}g^{ca}x^ex^bx^d - \frac{1}{20}\nabla_{ef}R_{abcd}g^{ca}x^ex^fx^bx^d + \frac{1}{18}R_{abcd}R_{efgh}g^{ca}g^{ge}x^bx^dx^fx^h \\
&\quad + \frac{2}{45}R_{abcd}R_{efgh}g^{hd}g^{ae}x^fx^gx^bx^c - \frac{1}{18}R_{abcd}R_{efgh}g^{ga}g^{ce}x^bx^dx^fx^h - \frac{1}{90}\nabla_{efg}R_{abcd}g^{ca}x^ex^fx^gx^bx^d \\
&\quad + \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ca}g^{ge}x^bx^dx^ix^fx^h + \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ge}g^{ca}x^ix^fx^hx^bx^d + \frac{1}{45}R_{abcd}\nabla_i R_{efgh}g^{dh}g^{ea}x^bx^cx^ix^fx^g \\
&\quad + \frac{1}{45}R_{abcd}\nabla_i R_{efgh}g^{dh}g^{ae}x^bx^cx^ix^fx^g - \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ga}g^{ce}x^bx^dx^ix^fx^h - \frac{1}{36}R_{abcd}\nabla_i R_{efgh}g^{ce}g^{ga}x^ix^fx^hx^bx^d \\
\text{Ndetg.006} &:= 1 - \frac{1}{3}R_{abcd}g^{ac}x^bx^d - \frac{1}{6}\nabla_a R_{bcde}g^{bd}x^ax^cx^e - \frac{1}{20}\nabla_{ab}R_{cdef}g^{ce}x^ax^bx^dx^f + \frac{1}{18}R_{abcd}R_{efgh}g^{ac}g^{eg}x^bx^dx^fx^h - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{90}\nabla_{abc}R_{defg}g^{df}x^ax^bx^cx^ex^g + \frac{1}{18}R_{abcd}\nabla_e R_{fghi}g^{ac}g^{fh}x^bx^dx^ex^gx^i - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i \\
\text{Ndetg.104} &:= 1 - \frac{1}{3}R_{bd}x^bx^d - \frac{1}{6}\nabla_a R_{ce}x^ax^cx^e - \frac{1}{20}\nabla_{ab}R_{df}x^ax^bx^dx^f + \frac{1}{18}R_{bd}R_{fh}x^bx^dx^fx^h - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{90}\nabla_{abc}R_{eg}x^ax^bx^cx^ex^g + \frac{1}{18}R_{bd}\nabla_e R_{gi}x^bx^dx^ex^gx^i - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i \\
\text{sqrtNdetg.004} &:= 1 - \frac{1}{6}R_{ab}x^ax^b - \frac{1}{12}\nabla_a R_{bc}x^ax^bx^c - \frac{1}{40}\nabla_{ab}R_{cd}x^ax^bx^cx^d + \frac{1}{72}R_{ab}R_{cd}x^ax^bx^cx^d - \frac{1}{180}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{180}\nabla_{abc}R_{de}x^ax^bx^cx^dx^e + \frac{1}{72}R_{ab}\nabla_c R_{de}x^ax^bx^cx^dx^e - \frac{1}{180}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i \\
\text{logNdetg.004} &:= -\frac{1}{3}R_{ab}x^ax^b - \frac{1}{6}\nabla_a R_{bc}x^ax^bx^c - \frac{1}{20}\nabla_{ab}R_{cd}x^ax^bx^cx^d - \frac{1}{90}R_{abcd}R_{efgh}g^{ae}g^{cg}x^bx^dx^fx^h \\
&\quad - \frac{1}{90}\nabla_{abc}R_{de}x^ax^bx^cx^dx^e - \frac{1}{90}R_{abcd}\nabla_e R_{fghi}g^{af}g^{ch}x^bx^dx^ex^gx^i
\end{aligned}$$

```

# note: keeping numbering as is (out of order) to ensure R appears before \nabla R etc.
def product_sort (obj):
    substitute (obj,$ x^{a}                                -> A000^{a}                $)
    substitute (obj,$ g^{a b}                              -> A001^{a b}                $)
    substitute (obj,$ \nabla_{c d e f}\{R_{a b}\}           -> A007_{a b c d e f}      $)
    substitute (obj,$ \nabla_{c d e}\{R_{a b}\}              -> A006_{a b c d e}        $)
    substitute (obj,$ \nabla_{c d}\{R_{a b}\}                -> A005_{a b c d}          $)
    substitute (obj,$ \nabla_{c}\{R_{a b}\}                  -> A004_{a b c}            $)
    substitute (obj,$ \nabla_{e f g h}\{R_{a b c d}\}         -> A011_{a b c d e f g h}  $)
    substitute (obj,$ \nabla_{e f g}\{R_{a b c d}\}          -> A010_{a b c d e f g}    $)
    substitute (obj,$ \nabla_{e f}\{R_{a b c d}\}            -> A009_{a b c d e f}      $)
    substitute (obj,$ \nabla_{e}\{R_{a b c d}\}              -> A008_{a b c d e}        $)
    substitute (obj,$ R_{a b}                               -> A002_{a b}              $)
    substitute (obj,$ R_{a b c d}                           -> A003_{a b c d}          $)
    sort_product      (obj)
    rename_dummies    (obj)
    substitute (obj,$ A000^{a}                                -> x^{a}                $)
    substitute (obj,$ A001^{a b}                              -> g^{a b}              $)
    substitute (obj,$ A002_{a b}                              -> R_{a b}              $)
    substitute (obj,$ A003_{a b c d}                          -> R_{a b c d}          $)
    substitute (obj,$ A004_{a b c}                            -> \nabla_{c}\{R_{a b}\}    $)
    substitute (obj,$ A005_{a b c d}                          -> \nabla_{c d}\{R_{a b}\}   $)
    substitute (obj,$ A006_{a b c d e}                        -> \nabla_{c d e}\{R_{a b}\} $)
    substitute (obj,$ A007_{a b c d e f}                      -> \nabla_{c d e f}\{R_{a b}\} $)
    substitute (obj,$ A008_{a b c d e}                        -> \nabla_{e}\{R_{a b c d}\} $)
    substitute (obj,$ A009_{a b c d e f}                      -> \nabla_{e f}\{R_{a b c d}\} $)
    substitute (obj,$ A010_{a b c d e f g}                    -> \nabla_{e f g}\{R_{a b c d}\} $)
    substitute (obj,$ A011_{a b c d e f g h}                  -> \nabla_{e f g h}\{R_{a b c d}\} $)

    return obj

def get_term (obj,n):

    x^{a}::Weight(label=numx).

    foo := @(obj).
    bah  = Ex("numx = " + str(n))
    keep_weight (foo,bah)

```

```

    return foo

def reformat (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    bah = product_sort (bah)
    rename_dummies (bah)
    canonicalise (bah)
    sort_sum (bah)
    factor_out (bah,$x^{a?}$)
    ans := @(bah) / @(foo).
    return ans

def rescale (obj,scale):
    foo = Ex(str(scale))
    bah := @(foo) @(obj).
    distribute (bah)
    factor_out (bah,$x^{a?}$)
    return bah

# -----
# reformat Ndetg

Rterm0 = get_term (Ndetg,0)      # cdb(Rterm0.701,Rterm0)
Rterm1 = get_term (Ndetg,1)      # cdb(Rterm1.701,Rterm1)
Rterm2 = get_term (Ndetg,2)      # cdb(Rterm2.701,Rterm2)
Rterm3 = get_term (Ndetg,3)      # cdb(Rterm3.701,Rterm3)
Rterm4 = get_term (Ndetg,4)      # cdb(Rterm4.701,Rterm4)
Rterm5 = get_term (Ndetg,5)      # cdb(Rterm5.701,Rterm5)

Rterm0 = reformat (Rterm0, 1)    # cdb(Rterm0.702,Rterm0)
Rterm1 = reformat (Rterm1, 1)    # cdb(Rterm1.702,Rterm1)
Rterm2 = reformat (Rterm2, 3)    # cdb(Rterm2.702,Rterm2)
Rterm3 = reformat (Rterm3, 6)    # cdb(Rterm3.702,Rterm3)
Rterm4 = reformat (Rterm4,180)   # cdb(Rterm4.702,Rterm4)
Rterm5 = reformat (Rterm5, 90)   # cdb(Rterm5.702,Rterm5)

```

```

Ndetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (Ndetg.701,Ndetg)

# -----
# reformat sqrtNdetg

Rterm0 = get_term (sqrtNdetg,0)  # cdb(Rterm0.801,Rterm0)
Rterm1 = get_term (sqrtNdetg,1)  # cdb(Rterm1.801,Rterm1)
Rterm2 = get_term (sqrtNdetg,2)  # cdb(Rterm2.801,Rterm2)
Rterm3 = get_term (sqrtNdetg,3)  # cdb(Rterm3.801,Rterm3)
Rterm4 = get_term (sqrtNdetg,4)  # cdb(Rterm4.801,Rterm4)
Rterm5 = get_term (sqrtNdetg,5)  # cdb(Rterm5.801,Rterm5)

Rterm0 = reformat (Rterm0, 1)  # cdb(Rterm0.802,Rterm0)
Rterm1 = reformat (Rterm1, 1)  # cdb(Rterm1.802,Rterm1)
Rterm2 = reformat (Rterm2, 6)  # cdb(Rterm2.802,Rterm2)
Rterm3 = reformat (Rterm3, 12) # cdb(Rterm3.802,Rterm3)
Rterm4 = reformat (Rterm4,360) # cdb(Rterm4.802,Rterm4)
Rterm5 = reformat (Rterm5,360) # cdb(Rterm5.802,Rterm5)

sqrtNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5).  # cdb (sqrtNdetg.801,sqrtNdetg)

# -----
# reformat logNdetg

Rterm0 = get_term (logNdetg,0)  # cdb(Rterm0.901,Rterm0)
Rterm1 = get_term (logNdetg,1)  # cdb(Rterm1.901,Rterm1)
Rterm2 = get_term (logNdetg,2)  # cdb(Rterm2.901,Rterm2)
Rterm3 = get_term (logNdetg,3)  # cdb(Rterm3.901,Rterm3)
Rterm4 = get_term (logNdetg,4)  # cdb(Rterm4.901,Rterm4)
Rterm5 = get_term (logNdetg,5)  # cdb(Rterm5.901,Rterm5)

Rterm0 = reformat (Rterm0, 1)  # cdb(Rterm0.902,Rterm0)
Rterm1 = reformat (Rterm1, 1)  # cdb(Rterm1.902,Rterm1)
Rterm2 = reformat (Rterm2, 3)  # cdb(Rterm2.902,Rterm2)
Rterm3 = reformat (Rterm3, 6)  # cdb(Rterm3.902,Rterm3)
Rterm4 = reformat (Rterm4,180) # cdb(Rterm4.902,Rterm4)
Rterm5 = reformat (Rterm5, 90) # cdb(Rterm5.902,Rterm5)

```

```
logNdetg := @(Rterm0) + @(Rterm1) + @(Rterm2) + @(Rterm3) + @(Rterm4) + @(Rterm5). # cdb (logNdetg.901,logNdetg)
```

## The metric determinant in Riemann normal coordinates

$$\begin{aligned} -\det g(x) = & 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) \\ & + \frac{1}{90}x^a x^b x^c x^d x^e (-\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

## The volume element in Riemann normal coordinates

If  $-\det g(x)$  is non-negative then we also have

$$\begin{aligned} \sqrt{-\det g(x)} = & 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) \\ & + \frac{1}{360}x^a x^b x^c x^d x^e (-2\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - 2g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

## The log of -detg in Riemann normal coordinates

Apart from the signs, this matches exactly the expression given by Calzetta et al. (eq. A14)

$$\begin{aligned} \log(-\det g(x)) = & -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d (-9\nabla_{ab} R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cf dh}) \\ & + \frac{1}{90}x^a x^b x^c x^d x^e (-\nabla_{abc}R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}) + \mathcal{O}(\epsilon^6) \end{aligned}$$

```
cdblib.create ('detg2.export')

cdblib.put ('Ndetg',    Ndetg,    'detg2.export')
cdblib.put ('sqrtNdetg',sqrtNdetg,'detg2.export')
cdblib.put ('logNdetg', logNdetg, 'detg2.export')

checkpoint.append (Ndetg)
checkpoint.append (sqrtNdetg)
checkpoint.append (logNdetg)
```