Notes

The convention for the curvature used in these notes conforms to that of Misner-Thorne-Wheeler (MTW, eq. 11.12) , namely

$$V^a{}_{;bc} - V^a{}_{;cb} = -R^a{}_{dbc} V^d$$

Also, note the following shorthand for mixed covariant derivatives

$$\begin{aligned}
\nabla_a \left(\nabla_b\right) &= \nabla_{ab} \\
\nabla_a \left(\nabla_b \left(\nabla_c\right)\right) &= \nabla_{abc} \\
\nabla_a \left(\nabla_b \left(\nabla_c \left(\nabla_d\right)\right)\right) &= \nabla_{abcd}
\end{aligned}$$

and so on.

See for example the Python function `combine_nabla` in `cadabra/dRabcd.tex`.

In terms of ∇ the above MTW definition of $R^a{}_{bcd}$ can written as

$$\left(\nabla_{cb} - \nabla_{bc}\right) V^a = -R^a{}_{dbc} V^d$$

Symmetrisation

In the following pages there will be frequent constructions of the form

$$\begin{aligned}
3A^b A^c \Gamma^a{}_{d(b,c)} &= A^b A^c R^a{}_{bcd} \\
6A^b A^c A^e \Gamma^a{}_{d(b,ce)} &= 3A^b A^c A^e \partial_e R^a{}_{bcd} \\
15A^b A^c A^e A^f \Gamma^a{}_{d(b,cef)} &= A^b A^c A^e A^f \left(9\partial_{fe} R^a{}_{bcd} - R^a{}_{ceg} R^g{}_{bfd} - R^a{}_{cfg} R^g{}_{bed}\right)
\end{aligned}$$

The vector A^a has no special meaning. Its purpose is to indicate that the associciated tensor is symmetric over a selection of its indices. If the A^a were not included then the right hand side would either need to be spelt out in full or some other device would be needed to denote the symmetries. The symmetrisation brackets are included on the left hand side though they are redundant (in the presence of the A^a).

The metric in RNC

$$g_{ab}(x) = g_{ab} - \frac{1}{3}x^c x^d R_{acbd} - \frac{1}{6}x^c x^d x^e \nabla_c R_{adbe} + \frac{1}{180}x^c x^d x^e x^f \left(8g^{gh} R_{acdg} R_{befh} - 9\nabla_{cd} R_{aebf}\right)$$
$$+ \frac{1}{90}x^c x^d x^e x^f x^g \left(2g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right) + \mathcal{O}\left(\epsilon^6\right)$$

Curvature expansion of the metric

$$g_{ab}(x) = \overset{0}{g}_{ab} + \overset{2}{g}_{ab} + \overset{3}{g}_{ab} + \overset{4}{g}_{ab} + \overset{5}{g}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{g}_{ab} = g_{ab}$$

$$3\overset{2}{g}_{ab} = -x^c x^d R_{acbd}$$

$$6\overset{3}{g}_{ab} = -x^c x^d x^e \nabla_c R_{adbe}$$

$$180\overset{4}{g}_{ab} = x^c x^d x^e x^f \left(8g^{gh} R_{acdg} R_{befh} - 9\nabla_{cd} R_{aebf}\right)$$

$$90\overset{5}{g}_{ab} = x^c x^d x^e x^f x^g \left(2g^{hi} R_{acdh} \nabla_e R_{bfgi} + 2g^{hi} R_{bcdh} \nabla_e R_{afgi} - \nabla_{cde} R_{afbg}\right)$$

The inverse metric in RNC

$$
\begin{aligned}
g^{ab}(x) = g^{ab} &+ \frac{1}{3}x^c x^d g^{ae} g^{bf} R_{cedf} + \frac{1}{6}x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} + \frac{1}{60}x^c x^d x^e x^f \left(4 g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3 g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right) \\
&+ \frac{1}{90}x^c x^d x^e x^f x^g \left(3 g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3 g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

Curvature expansion of the inverse metric

$$
g^{ab}(x) = \overset{0}{g}{}^{ab} + \overset{2}{g}{}^{ab} + \overset{3}{g}{}^{ab} + \overset{4}{g}{}^{ab} + \overset{5}{g}{}^{ab} + \mathcal{O}\left(\epsilon^6\right)
$$

$$
\begin{aligned}
\overset{0}{g}{}^{ab} &= g^{ab} \\
3\overset{2}{g}{}^{ab} &= x^c x^d g^{ae} g^{bf} R_{cedf} \\
6\overset{3}{g}{}^{ab} &= x^c x^d x^e g^{af} g^{bg} \nabla_c R_{dfeg} \\
60\overset{4}{g}{}^{ab} &= x^c x^d x^e x^f \left(4 g^{ag} g^{bh} g^{ij} R_{cgdi} R_{ehfj} + 3 g^{ag} g^{bh} \nabla_{cd} R_{egfh}\right) \\
90\overset{5}{g}{}^{ab} &= x^c x^d x^e x^f x^g \left(3 g^{ah} g^{bi} g^{jk} R_{chdj} \nabla_e R_{figk} + 3 g^{ah} g^{bi} g^{jk} R_{cidj} \nabla_e R_{fhgk} + g^{ah} g^{bi} \nabla_{cde} R_{fhgi}\right)
\end{aligned}
$$

The metric determinant in RNC

$$-\det g(x) = 1 - \frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 10R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)$$

The metric Jacobian in RNC

$$\sqrt{-\det g(x)} = 1 - \frac{1}{6}x^a x^b R_{ab} - \frac{1}{12}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{360}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} + 5R_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{360}x^a x^b x^c x^d x^e \left(-2\nabla_{abc}R_{de} + 5R_{ab}\nabla_c R_{de} - 2g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)$$

The log of detg in RNC

$$\log\left(-\det g(x)\right) = -\frac{1}{3}x^a x^b R_{ab} - \frac{1}{6}x^a x^b x^c \nabla_a R_{bc} + \frac{1}{180}x^a x^b x^c x^d \left(-9\nabla_{ab}R_{cd} - 2g^{ef}g^{gh}R_{aebg}R_{cfdh}\right)$$
$$+ \frac{1}{90}x^a x^b x^c x^d x^e \left(-\nabla_{abc}R_{de} - g^{fg}g^{hi}R_{afbh}\nabla_c R_{dgei}\right) + \mathcal{O}\left(\epsilon^6\right)$$

The connection in RNC

$$
\begin{aligned}
A^a A^b \Gamma^d_{ab} = \frac{2}{3} A^a A^b x^c g^{de} R_{acbe} + \frac{1}{12} A^a A^b x^c x^e \left(2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}\right) + \frac{1}{360} A^a A^b x^c x^e x^f \big(64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} \\
- 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}\big) \\
+ \frac{1}{180} A^a A^b x^c x^e x^f x^g \big(16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} \\
- 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bfgh} \\
+ 2g^{dh} \nabla_{cae} R_{bfgh} + 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} \\
+ g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg}\big)
\end{aligned}
$$

$$
\begin{aligned}
360 A^a A^b \Gamma^d_{ab} = 240 A^a A^b x^c g^{de} R_{acbe} + 30 A^a A^b x^c x^e \left(2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}\right) + A^a A^b x^c x^e x^f \big(64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} \\
- 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} - 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}\big) \\
+ 2 A^a A^b x^c x^e x^f x^g \big(16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} \\
- 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} \\
+ 2g^{dh} \nabla_{ace} R_{bfgh} + 2g^{dh} \nabla_{cae} R_{bfgh} + 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} \\
- 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} + g^{dh} \nabla_{hce} R_{afbg} + g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg}\big)
\end{aligned}
$$

Curvature expansion of the connection

$$A^a A^b \Gamma^d_{ab} = A^a A^b \overset{2}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{3}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{4}{\Gamma}{}^d{}_{ab} + A^a A^b \overset{5}{\Gamma}{}^d{}_{ab} + \mathcal{O}\left(\epsilon^6\right)$$

$$3A^a A^b \overset{2}{\Gamma}{}^d_{ab} = 2A^a A^b x^c g^{de} R_{acbe}$$

$$12A^a A^b \overset{3}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e \left(2g^{df} \nabla_a R_{bcef} + 4g^{df} \nabla_c R_{aebf} + g^{df} \nabla_f R_{acbe}\right)$$

$$\begin{aligned} 360A^a A^b \overset{4}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f \big(&64g^{dg} g^{hi} R_{acbh} R_{egfi} - 32g^{dg} g^{hi} R_{aceh} R_{bgfi} - 16g^{dg} g^{hi} R_{aceh} R_{bifg} + 18g^{dg} \nabla_{ac} R_{befg} + 18g^{dg} \nabla_{ca} R_{befg} + 36g^{dg} \nabla_{ce} R_{afbg} \\ &- 16g^{dg} g^{hi} R_{aceh} R_{bfgi} + 9g^{dg} \nabla_{gc} R_{aebf} + 9g^{dg} \nabla_{cg} R_{aebf}\big) \end{aligned}$$

$$\begin{aligned} 180A^a A^b \overset{5}{\Gamma}{}^d_{ab} = A^a A^b x^c x^e x^f x^g \big(&16g^{dh} g^{ij} R_{acbi} \nabla_e R_{fhgj} + 6g^{dh} g^{ij} R_{chei} \nabla_a R_{bfgj} + 16g^{dh} g^{ij} R_{chei} \nabla_f R_{agbj} + 5g^{dh} g^{ij} R_{chei} \nabla_j R_{afbg} - 8g^{dh} g^{ij} R_{ahci} \nabla_e R_{bfgj} \\ &- 4g^{dh} g^{ij} R_{aich} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_b R_{fhgj} - 8g^{dh} g^{ij} R_{acei} \nabla_f R_{bhgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bjgh} + 2g^{dh} \nabla_{ace} R_{bfgh} + 2g^{dh} \nabla_{cae} R_{bfgh} \\ &+ 2g^{dh} \nabla_{cea} R_{bfgh} + 4g^{dh} \nabla_{cef} R_{agbh} - 4g^{dh} g^{ij} R_{achi} \nabla_e R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_h R_{bfgj} - 4g^{dh} g^{ij} R_{acei} \nabla_f R_{bghj} + g^{dh} \nabla_{hce} R_{afbg} \\ &+ g^{dh} \nabla_{che} R_{afbg} + g^{dh} \nabla_{ceh} R_{afbg}\big) \end{aligned}$$

Symmetrised partial derivatives of the connection

$$
\begin{aligned}
3A^bA^c\Gamma^a{}_{d(b,c)} &= A^bA^cR^a{}_{bcd} \\
6A^bA^cA^e\Gamma^a{}_{d(b,ce)} &= 3A^bA^cA^e\partial_eR^a{}_{bcd} \\
15A^bA^cA^eA^f\Gamma^a{}_{d(b,cef)} &= A^bA^cA^eA^f\left(9\partial_{fe}R^a{}_{bcd} - R^a{}_{ceg}R^g{}_{bfd} - R^a{}_{cfg}R^g{}_{bed}\right) \\
9A^bA^cA^eA^fA^g\Gamma^a{}_{d(b,cefg)} &= A^bA^cA^eA^fA^g\Big(6\partial_{gfe}R^a{}_{bcd} - R^h{}_{bgd}\partial_eR^a{}_{cfh} - R^h{}_{bfd}\partial_eR^a{}_{cgh} - R^a{}_{ceh}\partial_fR^h{}_{bgd} - R^h{}_{bed}\partial_fR^a{}_{cgh} - R^a{}_{cfh}\partial_eR^h{}_{bgd} - R^a{}_{cgh}\partial_eR^h{}_{bfd}\Big) \\
252A^bA^cA^eA^fA^gA^h\Gamma^a{}_{d(b,cefgh)} &= A^bA^cA^eA^fA^gA^h\Big(180\partial_{hgfe}R^a{}_{bcd} - 36R^i{}_{bgd}\partial_{he}R^a{}_{cfi} + 4R^a{}_{fei}R^i{}_{chj}R^j{}_{bgd} + 4R^a{}_{fhi}R^i{}_{cej}R^j{}_{bgd} - 72R^i{}_{bfd}\partial_{he}R^a{}_{cgi} \\
&\quad + 8R^a{}_{gei}R^i{}_{chj}R^j{}_{bfd} + 8R^a{}_{ghi}R^i{}_{cej}R^j{}_{bfd} - 45\partial_eR^a{}_{cfi}\partial_gR^i{}_{bhd} - 45\partial_eR^a{}_{cgi}\partial_fR^i{}_{bhd} - 45\partial_eR^a{}_{chi}\partial_fR^i{}_{bgd} \\
&\quad - 36R^a{}_{cei}\partial_{hf}R^i{}_{bgd} + 4R^a{}_{cei}R^i{}_{gfj}R^j{}_{bhd} + 4R^a{}_{cei}R^i{}_{ghj}R^j{}_{bfd} - 36R^i{}_{bed}\partial_{hf}R^a{}_{cgi} + 4R^a{}_{gfi}R^i{}_{chj}R^j{}_{bed} \\
&\quad + 4R^a{}_{ghi}R^i{}_{cfj}R^j{}_{bed} - 45\partial_fR^a{}_{cgi}\partial_eR^i{}_{bhd} - 45\partial_fR^a{}_{chi}\partial_eR^i{}_{bgd} - 72R^a{}_{cfi}\partial_{he}R^i{}_{bgd} + 8R^a{}_{cfi}R^i{}_{gej}R^j{}_{bhd} \\
&\quad + 8R^a{}_{cfi}R^i{}_{ghj}R^j{}_{bed} - 45\partial_gR^a{}_{chi}\partial_eR^i{}_{bfd} - 36R^a{}_{cgi}\partial_{he}R^i{}_{bfd} + 4R^a{}_{cgi}R^i{}_{fej}R^j{}_{bhd} + 4R^a{}_{cgi}R^i{}_{fhj}R^j{}_{bed}\Big)
\end{aligned}
$$

Symmetrised partial derivatives of $R^a{}_{bcd}$

$$
\begin{aligned}
A^c A^d A^e R^a{}_{cdb,e} &= A^c A^d A^e g^{af} \nabla_c R_{bdef} \\
A^c A^d A^e A^f R^a{}_{cdb,ef} &= A^c A^d A^e A^f g^{ag} \nabla_{cd} R_{befg} \\
-2 A^c A^d A^e A^f A^g R^a{}_{cdb,efg} &= A^c A^d A^e A^f A^g \left(g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} - g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} - 2 g^{ah} \nabla_{cde} R_{bfgh} \right) \\
-5 A^c A^d A^e A^f A^g A^h R^a{}_{cdb,efgh} &= A^c A^d A^e A^f A^g A^h \left(7 g^{ai} g^{jk} R_{bcdj} \nabla_{ef} R_{gihk} - 7 g^{ai} g^{jk} R_{cidj} \nabla_{ef} R_{bghk} - 5 g^{ai} \nabla_{cdef} R_{bghi} \right) \\
-3 A^c A^d A^e A^f A^g A^h A^i R^a{}_{cdb,efghi} &= A^c A^d A^e A^f A^g A^h A^i \left(6 g^{aj} g^{kl} \nabla_c R_{bdek} \nabla_{fg} R_{hjil} - 6 g^{aj} g^{kl} \nabla_c R_{djek} \nabla_{fg} R_{bhil} + 8 g^{aj} g^{kl} R_{bcdk} \nabla_{efg} R_{hjil} \right. \\
&\quad - 8 g^{aj} g^{kl} R_{cjdk} \nabla_{efg} R_{bhil} - g^{aj} g^{kl} g^{mn} R_{bcdk} R_{elfm} \nabla_g R_{hjin} - 3 g^{aj} g^{kl} g^{mn} R_{cjdk} R_{elfm} \nabla_g R_{bhin} \\
&\quad \left. + 4 g^{aj} g^{kl} g^{mn} R_{bcdk} R_{ejfm} \nabla_g R_{hlin} - 3 g^{aj} \nabla_{cdefg} R_{bhij} \right)
\end{aligned}
$$

The generalised connection in RNC

$$
\begin{aligned}
A^bA^c\Gamma^a_{bc} = {}& \frac{2}{3}A^bA^cx^dg^{ae}R_{bdce} + \frac{1}{12}A^bA^cx^dx^e\left(2g^{af}\nabla_bR_{cdef} + 4g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right) + \frac{1}{360}A^bA^cx^dx^ex^f\big(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi} \\
& - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\big) \\
& + \frac{1}{180}A^bA^cx^dx^ex^fx^g\big(16g^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg} \\
& - 8g^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh} + 2g^{ah}\nabla_{bde}R_{cfgh} \\
& + 2g^{ah}\nabla_{dbe}R_{cfgh} + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} \\
& + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\big)
\end{aligned}
$$

$$
\begin{aligned}
A^bA^cA^d\Gamma^a_{bcd} = {}& \frac{1}{2}A^bA^cA^dx^eg^{af}\nabla_bR_{cedf} + \frac{1}{120}A^bA^cA^dx^ex^f\big(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} \\
& + 18g^{ag}\nabla_{be}R_{cfdg} + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\big) \\
& + \frac{1}{180}A^bA^cA^dx^ex^fx^g\big(32g^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj} \\
& + 2g^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} + 22g^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg} \\
& + 5g^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} - 12g^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_gR_{chdj} \\
& + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} + 16g^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj} \\
& + 36g^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + 16g^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_fR_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} \\
& + g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20g^{ah}g^{ij}R_{beci}\nabla_jR_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}\big)
\end{aligned}
$$

$$
\begin{aligned}
A^bA^cA^dA^e\Gamma^a_{bcde} = {}& \frac{1}{15}A^bA^cA^dA^ex^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) \\
& + \frac{1}{90}A^bA^cA^dA^ex^fx^g\big(64g^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} + 4g^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj} \\
& + 44g^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg} - 16g^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej} \\
& + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej} + 6g^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej} \\
& + 46g^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\big)
\end{aligned}
$$

$$
A^bA^cA^dA^eA^f\Gamma^a_{bcdef} = \frac{1}{3}A^bA^cA^dA^eA^fx^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)
$$

The generalised connection in RNC

This is the same as the previous page but with a small change in the format to avoid fractions.

$$\begin{aligned}
360A^bA^c\Gamma^a_{bc} &= 240A^bA^cx^dg^{ae}R_{bdce} + 30A^bA^cx^dx^e\left(2g^{af}\nabla_bR_{cdef} + 4g^{af}\nabla_dR_{becf} + g^{af}\nabla_fR_{bdce}\right) \\
&\quad + A^bA^cx^dx^ex^f\big(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} \\
&\qquad + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\big) \\
&\quad + 2A^bA^cx^dx^ex^fx^g\big(16g^{ah}g^{ij}R_{bdci}\nabla_eR_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_bR_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_fR_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_jR_{bfcg} \\
&\qquad - 8g^{ah}g^{ij}R_{bhdi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bidh}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_cR_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_fR_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cjgh} \\
&\qquad + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh} + 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_eR_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_hR_{cfgj} \\
&\qquad - 4g^{ah}g^{ij}R_{bdei}\nabla_fR_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg} + g^{ah}\nabla_{deh}R_{bfcg}\big)
\end{aligned}$$

$$\begin{aligned}
360A^bA^cA^d\Gamma^a_{bcd} &= 180A^bA^cA^dx^eg^{af}\nabla_bR_{cedf} + 3A^bA^cA^dx^ex^f\big(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} \\
&\qquad + 18g^{ag}\nabla_{be}R_{cfdg} + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\big) \\
&\quad + 2A^bA^cA^dx^ex^fx^g\big(32g^{ah}g^{ij}R_{beci}\nabla_dR_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_fR_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_fR_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_cR_{dfgj} \\
&\qquad + 2g^{ah}g^{ij}R_{bieh}\nabla_cR_{dfgj} + 22g^{ah}g^{ij}R_{ehfi}\nabla_bR_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_fR_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_fR_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_jR_{cfdg} \\
&\qquad + 5g^{ah}g^{ij}R_{bieh}\nabla_jR_{cfdg} - 12g^{ah}g^{ij}R_{bhci}\nabla_eR_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_cR_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_cR_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_gR_{chdj} \\
&\qquad + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} \\
&\qquad + 16g^{ah}g^{ij}R_{behi}\nabla_cR_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_fR_{cgdj} + 16g^{ah}g^{ij}R_{beci}\nabla_hR_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_hR_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_fR_{dghj} \\
&\qquad - 4g^{ah}g^{ij}R_{befi}\nabla_cR_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg} + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} \\
&\qquad - 20g^{ah}g^{ij}R_{beci}\nabla_jR_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_jR_{cfdg}\big)
\end{aligned}$$

$$\begin{aligned}
90A^bA^cA^dA^e\Gamma^a_{bcde} &= 6A^bA^cA^dA^ex^f\left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) \\
&\quad + A^bA^cA^dA^ex^fx^g\big(64g^{ah}g^{ij}R_{bfci}\nabla_dR_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_dR_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_gR_{dhej} + 4g^{ah}g^{ij}R_{bhci}\nabla_dR_{efgj} \\
&\qquad + 44g^{ah}g^{ij}R_{bhfi}\nabla_cR_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_cR_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_fR_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_jR_{dfeg} - 16g^{ah}g^{ij}R_{bfgi}\nabla_cR_{dhej} \\
&\qquad + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_cR_{dgej} + 6g^{ah}g^{ij}R_{bfci}\nabla_hR_{dgej} \\
&\qquad + 46g^{ah}g^{ij}R_{bfci}\nabla_dR_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_jR_{dgeh}\big)
\end{aligned}$$

$$3A^bA^cA^dA^eA^f\Gamma^a_{bcdef} = A^bA^cA^dA^eA^fx^g\left(3g^{ah}g^{ij}R_{bgci}\nabla_dR_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_dR_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

Convert from generic (x) to local RNC coords (y)

$$y^a = \overset{0}{y}{}^a + \overset{1}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a$$

$$\begin{aligned}
\overset{0}{y}{}^a &= x^a \\
2\overset{1}{y}{}^a &= x^b x^c \Gamma^a{}_{bc} \\
6\overset{2}{y}{}^a &= x^b x^c x^d \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} + \partial_b \Gamma^a{}_{cd}\right) \\
24\overset{3}{y}{}^a &= x^b x^c x^d x^e \left(2\Gamma^a{}_{bf}\partial_c\Gamma^f{}_{de} + \Gamma^a{}_{fg}\Gamma^f{}_{bc}\Gamma^g{}_{de} + \Gamma^f{}_{bc}\partial_f\Gamma^a{}_{de} + \partial_{bc}\Gamma^a{}_{de}\right) \\
360\overset{4}{y}{}^a &= x^b x^c x^d x^e x^f \Big(-4\Gamma^a{}_{bg}\Gamma^g{}_{ch}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 2\Gamma^a{}_{bg}\Gamma^g{}_{ch}\partial_d\Gamma^h{}_{ef} + 3\Gamma^a{}_{bg}\Gamma^g{}_{hi}\Gamma^h{}_{cd}\Gamma^i{}_{ef} - 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_e\Gamma^g{}_{fh} + 6\Gamma^a{}_{bg}\Gamma^h{}_{cd}\partial_h\Gamma^g{}_{ef} + 9\Gamma^a{}_{bg}\partial_{cd}\Gamma^g{}_{ef} \\
&\quad + 4\Gamma^a{}_{gh}\Gamma^g{}_{bc}\Gamma^h{}_{di}\Gamma^i{}_{ef} + 13\Gamma^a{}_{gh}\Gamma^g{}_{bc}\partial_d\Gamma^h{}_{ef} - 4\Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_e\Gamma^a{}_{fh} + \Gamma^g{}_{bc}\Gamma^h{}_{dg}\partial_h\Gamma^a{}_{ef} + 2\partial_b\Gamma^a{}_{cg}\partial_d\Gamma^g{}_{ef} + 7\partial_g\Gamma^a{}_{bc}\partial_d\Gamma^g{}_{ef} + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_f\Gamma^a{}_{gh} \\
&\quad + 3\Gamma^g{}_{bc}\Gamma^h{}_{de}\partial_g\Gamma^a{}_{fh} - 3\Gamma^g{}_{bc}\partial_{de}\Gamma^a{}_{fg} + 6\Gamma^g{}_{bc}\partial_{dg}\Gamma^a{}_{ef} + 3\partial_{bcd}\Gamma^a{}_{ef}\Big)
\end{aligned}$$

The geodesic ivp

$$x^a(s) = x^a + s\dot{x}^a + \frac{s^2}{2!}\dot{x}^b\dot{x}^c A^a_{bc} + \frac{s^3}{3!}\dot{x}^b\dot{x}^c\dot{x}^d A^a_{bcd} + \frac{s^4}{4!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e A^a_{bcde} + \frac{s^5}{5!}\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f A^a_{bcdef} + \cdots$$

$$\begin{aligned}
360A^a_{bc} = {}& -240x^d g^{ae} R_{bdce} - 30x^d x^e \left(2g^{af}\nabla_b R_{cdef} + 4g^{af}\nabla_d R_{becf} + g^{af}\nabla_f R_{bdce}\right) - x^d x^e x^f \left(64g^{ag}g^{hi}R_{bdch}R_{egfi} - 32g^{ag}g^{hi}R_{bdeh}R_{cgfi}\right.\\
&\left. - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 18g^{ag}\nabla_{bd}R_{cefg} + 18g^{ag}\nabla_{db}R_{cefg} + 36g^{ag}\nabla_{de}R_{bfcg} - 16g^{ag}g^{hi}R_{bdeh}R_{cfgi} + 9g^{ag}\nabla_{gd}R_{becf} + 9g^{ag}\nabla_{dg}R_{becf}\right)\\
&- 2x^d x^e x^f x^g \left(16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} + 16g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 5g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 8g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj}\right.\\
&- 4g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} - 8g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} + 2g^{ah}\nabla_{bde}R_{cfgh} + 2g^{ah}\nabla_{dbe}R_{cfgh}\\
&+ 2g^{ah}\nabla_{deb}R_{cfgh} + 4g^{ah}\nabla_{def}R_{bgch} - 4g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + g^{ah}\nabla_{hde}R_{bfcg} + g^{ah}\nabla_{dhe}R_{bfcg}\\
&\left. + g^{ah}\nabla_{deh}R_{bfcg}\right)
\end{aligned}$$

$$\begin{aligned}
360A^a_{bcd} = {}& -180x^e g^{af}\nabla_b R_{cedf} - 3x^e x^f \left(64g^{ag}g^{hi}R_{bech}R_{dgfi} + 16g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{befh}R_{cgdi} + 12g^{ag}\nabla_{bc}R_{defg} + 18g^{ag}\nabla_{be}R_{cfdg}\right.\\
&\left. + 18g^{ag}\nabla_{eb}R_{cfdg} + 48g^{ag}g^{hi}R_{bech}R_{dfgi} + 3g^{ag}\nabla_{gb}R_{cedf} + 3g^{ag}\nabla_{bg}R_{cedf}\right)\\
&- 2x^e x^f x^g \left(32g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 48g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} + 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} + 18g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} + 2g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj}\right.\\
&+ 22g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} + 48g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} + 15g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg}\\
&- 12g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} - 8g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} - 12g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} + 4g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}\nabla_{bec}R_{dfgh}\\
&+ 6g^{ah}\nabla_{bef}R_{cgdh} + 4g^{ah}\nabla_{ebc}R_{dfgh} + 6g^{ah}\nabla_{ebf}R_{cgdh} + 6g^{ah}\nabla_{efb}R_{cgdh} + 16g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} + 36g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj}\\
&+ 16g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} - 4g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} + 36g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - 4g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} + g^{ah}\nabla_{hbe}R_{cfdg} + g^{ah}\nabla_{heb}R_{cfdg}\\
&\left. + g^{ah}\nabla_{bhe}R_{cfdg} + g^{ah}\nabla_{ehb}R_{cfdg} + g^{ah}\nabla_{beh}R_{cfdg} + g^{ah}\nabla_{ebh}R_{cfdg} - 20g^{ah}g^{ij}R_{beci}\nabla_j R_{dfgh} + 10g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg}\right)
\end{aligned}$$

$$\begin{aligned}
90A^a_{bcde} = {}& -6x^f \left(8g^{ag}g^{hi}R_{bfch}R_{dgei} + 6g^{ag}\nabla_{bc}R_{dfeg}\right) - x^f x^g \left(64g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 18g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} + 24g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej}\right.\\
&+ 4g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} + 44g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} + 18g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} + 24g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 10g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg}\\
&- 16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}\nabla_{bcd}R_{efgh} + 8g^{ah}\nabla_{bcf}R_{dgeh} + 8g^{ah}\nabla_{bfc}R_{dgeh} + 8g^{ah}\nabla_{fbc}R_{dgeh} + 26g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej}\\
&\left. + 6g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 46g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} + g^{ah}\nabla_{hbc}R_{dfeg} + g^{ah}\nabla_{bhc}R_{dfeg} + g^{ah}\nabla_{bch}R_{dfeg} - 40g^{ah}g^{ij}R_{bfci}\nabla_j R_{dgeh}\right)
\end{aligned}$$

$$3A^a_{bcdef} = -x^g \left(3g^{ah}g^{ij}R_{bgci}\nabla_d R_{ehfj} + 3g^{ah}g^{ij}R_{bhci}\nabla_d R_{egfj} + g^{ah}\nabla_{bcd}R_{egfh}\right)$$

Geodesic boundary value problem to terms linear in R

$$x^a(s) = x^a + sDx^a - \frac{1}{3}\left(s - s^2\right) x^b Dx^c Dx^d g^{ae} R_{bcde} + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2) x^a_2 + \mathcal{O}\left(s^3, \epsilon^3\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \mathcal{O}\left(\epsilon^3\right)$$

$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

Geodesic boundary value problem to terms linear in ∇R

$$x^a(s) = x^a + sDx^a + \left(s - s^2\right)\left(-\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\right)$$
$$- \frac{1}{12}\left(s - s^3\right) x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} + \mathcal{O}\left(s^4, \epsilon^4\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2)x^a_2 + (s - s^3)x^a_3 + \mathcal{O}\left(s^4, \epsilon^4\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \mathcal{O}\left(\epsilon^4\right)$$
$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$
$$-24\overset{3}{x}{}^a_2 = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)$$

$$x^a_3 = \overset{3}{x}{}^a_3 + \mathcal{O}\left(\epsilon^4\right)$$
$$-12\overset{3}{x}{}^a_3 = x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef}$$

Geodesic boundary value problem to terms linear in $\nabla^2 R$

$$x^a(s) = x^a + sDx^a + \left(s - s^2\right)\left(-\frac{1}{3}x^b Dx^c Dx^d g^{ae} R_{bcde} - \frac{1}{24}x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\right.$$
$$- \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) - \frac{1}{720}x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right.$$
$$\left.\left. - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right)\right)$$
$$+ \left(s - s^3\right)\left(-\frac{1}{12}x^b Dx^c Dx^d Dx^e g^{af}\nabla_c R_{bdef} - \frac{1}{720}x^b x^c Dx^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi}\right.\right.$$
$$\left.\left. + 12g^{ag}\nabla_{de}R_{bfcg} + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} - 48g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\right)\right)$$
$$- \frac{1}{180}\left(s - s^4\right)x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right) + \mathcal{O}\left(s^5, \epsilon^5\right)$$

$$x^a(s) = x^a + sDx^a + (s - s^2)x^a_2 + (s - s^3)x^a_3 + (s - s^4)x^a_4 + \mathcal{O}\left(s^5, \epsilon^5\right)$$

$$x^a_2 = \overset{2}{x}{}^a_2 + \overset{3}{x}{}^a_2 + \overset{4}{x}{}^a_2 + \mathcal{O}\left(\epsilon^5\right)$$

$$-3\overset{2}{x}{}^a_2 = x^b Dx^c Dx^d g^{ae} R_{bcde}$$

$$-24\overset{3}{x}{}^a_2 = x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)$$

$$-720\overset{4}{x}{}^a_2 = x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right.$$
$$\left. - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right)$$

$$x_3^a = \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-12\overset{3}{x}{}_3^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$-720\overset{4}{x}{}_3^a = x^b x^c Dx^d Dx^e Dx^f \left(64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg} - 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\right)$$

$$x_4^a = \overset{4}{x}{}_4^a + \mathcal{O}\left(\epsilon^5\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4 g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3 g^{ag} \nabla_{cd} R_{befg}\right)$$

Geodesic boundary value problem to terms linear in $\nabla^3 R$

The geodesic that connects the points with RNC coordinates x^a and $x^a + Dx^a$ is described, for $0 \leq s \leq 1$, by

$$x^a(s) = x^a + sDx^a + (s - s^2)x_2^a + (s - s^3)x_3^a + (s - s^4)x_4^a + (s - s^5)x_5^a + \mathcal{O}\left(s^6, \epsilon^6\right)$$

$$\begin{aligned}
x_2^a &= \overset{2}{x}{}_2^a + \overset{3}{x}{}_2^a + \overset{4}{x}{}_2^a + \overset{5}{x}{}_2^a + \mathcal{O}\left(\epsilon^6\right) \\
-3\overset{2}{x}{}_2^a &= x^b Dx^c Dx^d g^{ae} R_{bcde} \\
-24\overset{3}{x}{}_2^a &= x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right) \\
-720\overset{4}{x}{}_2^a &= x^b x^c Dx^d Dx^e Dx^f \left(80g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 80g^{ag}g^{hi}R_{bdeh}R_{cifg}\right) + x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg}\right. \\
&\quad \left. - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg} + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right) \\
-360\overset{5}{x}{}_2^a &= x^b x^c x^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 20g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj}\right. \\
&\quad - 20g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} + 5g^{ah}g^{ij}R_{bieh}\nabla_j R_{cfdg} - 10g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 20g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} \\
&\quad \left. - 20g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 10g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj}\right) + x^b x^c Dx^d Dx^e Dx^f Dx^g \left(10g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 10g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj}\right) \\
&\quad + x^b x^c x^d x^e Dx^f Dx^g \left(16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg}\right. \\
&\quad - 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 2g^{ah}\nabla_{fbc}R_{dgeh} \\
&\quad + 2g^{ah}\nabla_{bfc}R_{dgeh} + 2g^{ah}\nabla_{bcf}R_{dgeh} + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} \\
&\quad \left. - g^{ah}\nabla_{hbc}R_{dfeg} - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\right)
\end{aligned}$$

$$x_3^a = \overset{3}{x}{}_3^a + \overset{4}{x}{}_3^a + \overset{5}{x}{}_3^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-12\overset{3}{x}{}_3^a = x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef}$$

$$\begin{aligned}-720\overset{4}{x}{}_3^a = x^b x^c Dx^d Dx^e Dx^f \big(&64 g^{ag} g^{hi} R_{bdeh} R_{cifg} + 16 g^{ag} g^{hi} R_{bdeh} R_{cgfi} - 16 g^{ag} g^{hi} R_{bdch} R_{egfi} + 12 g^{ag} \nabla_{de} R_{bfcg} + 18 g^{ag} \nabla_{db} R_{cefg} + 18 g^{ag} \nabla_{bd} R_{cefg} \\ &- 48 g^{ag} g^{hi} R_{bdeh} R_{cfgi} - 3 g^{ag} \nabla_{gd} R_{becf} - 3 g^{ag} \nabla_{dg} R_{becf}\big)\end{aligned}$$

$$\begin{aligned}-1080\overset{5}{x}{}_3^a = {} & x^b x^c Dx^d Dx^e Dx^f Dx^g \left(30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - 30 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} - 30 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\right) \\ &+ x^b x^c x^d Dx^e Dx^f Dx^g \big(32 g^{ah} g^{ij} R_{befi} \nabla_g R_{chdj} + 48 g^{ah} g^{ij} R_{befi} \nabla_c R_{djgh} + 12 g^{ah} g^{ij} R_{befi} \nabla_c R_{dhgj} + 18 g^{ah} g^{ij} R_{bieh} \nabla_f R_{cgdj} \\ &+ 2 g^{ah} g^{ij} R_{bhei} \nabla_f R_{cgdj} + 22 g^{ah} g^{ij} R_{bhci} \nabla_e R_{dfgj} + 48 g^{ah} g^{ij} R_{bieh} \nabla_c R_{dfgj} + 12 g^{ah} g^{ij} R_{bhei} \nabla_c R_{dfgj} - 15 g^{ah} g^{ij} R_{bieh} \nabla_j R_{cfdg} \\ &- 5 g^{ah} g^{ij} R_{bhei} \nabla_j R_{cfdg} - 12 g^{ah} g^{ij} R_{ehfi} \nabla_b R_{cgdj} - 12 g^{ah} g^{ij} R_{beci} \nabla_f R_{djgh} - 8 g^{ah} g^{ij} R_{beci} \nabla_f R_{dhgj} - 12 g^{ah} g^{ij} R_{beci} \nabla_d R_{fhgj} \\ &+ 4 g^{ah} \nabla_{efb} R_{cgdh} + 4 g^{ah} \nabla_{ebf} R_{cgdh} + 6 g^{ah} \nabla_{ebc} R_{dfgh} + 4 g^{ah} \nabla_{bef} R_{cgdh} + 6 g^{ah} \nabla_{bec} R_{dfgh} + 6 g^{ah} \nabla_{bce} R_{dfgh} - 16 g^{ah} g^{ij} R_{behi} \nabla_f R_{cgdj} \\ &- 36 g^{ah} g^{ij} R_{behi} \nabla_c R_{dfgj} - 16 g^{ah} g^{ij} R_{befi} \nabla_h R_{cgdj} + 4 g^{ah} g^{ij} R_{beci} \nabla_h R_{dfgj} - 36 g^{ah} g^{ij} R_{befi} \nabla_c R_{dghj} + 4 g^{ah} g^{ij} R_{beci} \nabla_f R_{dghj} - g^{ah} \nabla_{heb} R_{cfdg} \\ &- g^{ah} \nabla_{hbe} R_{cfdg} - g^{ah} \nabla_{ehb} R_{cfdg} - g^{ah} \nabla_{bhe} R_{cfdg} - g^{ah} \nabla_{ebh} R_{cfdg} - g^{ah} \nabla_{beh} R_{cfdg} + 20 g^{ah} g^{ij} R_{befi} \nabla_j R_{cgdh} + 10 g^{ah} g^{ij} R_{behi} \nabla_j R_{cfdg}\big)\end{aligned}$$

$$x_4^a = \overset{4}{x}{}_4^a + \overset{5}{x}{}_4^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-180\overset{4}{x}{}_4^a = x^b Dx^c Dx^d Dx^e Dx^f \left(4 g^{ag} g^{hi} R_{bcdh} R_{egfi} + 3 g^{ag} \nabla_{cd} R_{befg}\right)$$

$$\begin{aligned}-2160\overset{5}{x}{}_4^a = x^b x^c Dx^d Dx^e Dx^f Dx^g \big(&64 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cjgh} + 18 g^{ah} g^{ij} R_{bdei} \nabla_f R_{chgj} + 24 g^{ah} g^{ij} R_{bdei} \nabla_c R_{fhgj} + 4 g^{ah} g^{ij} R_{dhei} \nabla_f R_{bgcj} \\ &+ 44 g^{ah} g^{ij} R_{bidh} \nabla_e R_{cfgj} + 18 g^{ah} g^{ij} R_{bhdi} \nabla_e R_{cfgj} + 24 g^{ah} g^{ij} R_{dhei} \nabla_b R_{cfgj} - 10 g^{ah} g^{ij} R_{dhei} \nabla_j R_{bfcg} - 16 g^{ah} g^{ij} R_{bdci} \nabla_e R_{fhgj} \\ &+ 6 g^{ah} \nabla_{def} R_{bgch} + 8 g^{ah} \nabla_{deb} R_{cfgh} + 8 g^{ah} \nabla_{dbe} R_{cfgh} + 8 g^{ah} \nabla_{bde} R_{cfgh} - 26 g^{ah} g^{ij} R_{bdhi} \nabla_e R_{cfgj} - 6 g^{ah} g^{ij} R_{bdei} \nabla_h R_{cfgj} \\ &- 46 g^{ah} g^{ij} R_{bdei} \nabla_f R_{cghj} - g^{ah} \nabla_{hde} R_{bfcg} - g^{ah} \nabla_{dhe} R_{bfcg} - g^{ah} \nabla_{deh} R_{bfcg} + 40 g^{ah} g^{ij} R_{bdei} \nabla_j R_{cfgh}\big)\end{aligned}$$

$$x_5^a = \overset{5}{x}{}_5^a + \mathcal{O}\left(\epsilon^6\right)$$

$$-360\overset{5}{x}{}_5^a = x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(3 g^{ah} g^{ij} R_{bcdi} \nabla_e R_{fhgj} + 3 g^{ah} g^{ij} R_{chdi} \nabla_e R_{bfgj} + g^{ah} \nabla_{cde} R_{bfgh}\right)$$

Geodesic arc-length

$$
\begin{aligned}
(\Delta s)^2 = {} & g_{ab} Dx^a Dx^b - \frac{1}{3} x^a x^b Dx^c Dx^d R_{acbd} - \frac{1}{12} x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - \frac{1}{6} x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce} \\
& + \frac{1}{360} x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8 g^{gh} R_{acdg} R_{befh} - 6 \nabla_{cd} R_{aebf}\right) + \frac{1}{360} x^a x^b x^c Dx^d Dx^e Dx^f \left(16 g^{gh} R_{adbg} R_{cefh} - 9 \nabla_{da} R_{becf} - 9 \nabla_{ad} R_{becf}\right) \\
& + \frac{1}{360} x^a x^b x^c x^d Dx^e Dx^f \left(16 g^{gh} R_{aebg} R_{cfdh} - 18 \nabla_{ab} R_{cedf}\right) + \frac{1}{1080} x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4 g^{hi} R_{adeh} \nabla_f R_{bgci} - 24 g^{hi} R_{adeh} \nabla_b R_{cfgi} \right. \\
& \qquad \left. + 10 g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16 g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4 \nabla_{dea} R_{bfcg} - 4 \nabla_{dae} R_{bfcg} - 4 \nabla_{ade} R_{bfcg}\right) \\
& + \frac{1}{1080} x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18 g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3 \nabla_{cde} R_{afbg}\right) \\
& + \frac{1}{1080} x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24 g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24 g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6 \nabla_{eab} R_{cfdg} - 6 \nabla_{aeb} R_{cfdg} - 6 \nabla_{abe} R_{cfdg}\right) \\
& + \frac{1}{1080} x^a x^b x^c x^d x^e Dx^f Dx^g \left(48 g^{hi} R_{afbh} \nabla_c R_{dgei} - 12 \nabla_{abc} R_{dfeg}\right) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

Geodesic arc-length curvature expansion

$$(\Delta s)^2 = \overset{0}{\Delta} + \overset{2}{\Delta} + \overset{3}{\Delta} + \overset{4}{\Delta} + \overset{5}{\Delta} + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{\Delta} = g_{ab} Dx^a Dx^b$$

$$3\overset{2}{\Delta} = -x^a x^b Dx^c Dx^d R_{acbd}$$

$$12\overset{3}{\Delta} = -x^a x^b Dx^c Dx^d Dx^e \nabla_c R_{adbe} - 2x^a x^b x^c Dx^d Dx^e \nabla_a R_{bdce}$$

$$\begin{aligned} 360\overset{4}{\Delta} = {} & x^a x^b Dx^c Dx^d Dx^e Dx^f \left(-8g^{gh} R_{acdg} R_{befh} - 6\nabla_{cd} R_{aebf}\right) + x^a x^b x^c Dx^d Dx^e Dx^f \left(16g^{gh} R_{adbg} R_{cefh} - 9\nabla_{da} R_{becf} - 9\nabla_{ad} R_{becf}\right) \\ & + x^a x^b x^c x^d Dx^e Dx^f \left(16g^{gh} R_{aebg} R_{cfdh} - 18\nabla_{ab} R_{cedf}\right) \end{aligned}$$

$$\begin{aligned} 1080\overset{5}{\Delta} = {} & x^a x^b x^c Dx^d Dx^e Dx^f Dx^g \left(-4g^{hi} R_{adeh} \nabla_f R_{bgci} - 24g^{hi} R_{adeh} \nabla_b R_{cfgi} + 10g^{hi} R_{adeh} \nabla_i R_{bfcg} + 16g^{hi} R_{adbh} \nabla_e R_{cfgi} - 4\nabla_{dea} R_{bfcg} - 4\nabla_{dae} R_{bfcg} \right. \\ & \left. - 4\nabla_{ade} R_{bfcg}\right) + x^a x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(-18g^{hi} R_{acdh} \nabla_e R_{bfgi} - 3\nabla_{cde} R_{afbg}\right) \\ & + x^a x^b x^c x^d Dx^e Dx^f Dx^g \left(24g^{hi} R_{aefh} \nabla_b R_{cgdi} + 24g^{hi} R_{aebh} \nabla_f R_{cgdi} + 24g^{hi} R_{aebh} \nabla_c R_{dfgi} - 6\nabla_{eab} R_{cfdg} - 6\nabla_{aeb} R_{cfdg} - 6\nabla_{abe} R_{cfdg}\right) \\ & + x^a x^b x^c x^d x^e Dx^f Dx^g \left(48g^{hi} R_{afbh} \nabla_c R_{dgei} - 12\nabla_{abc} R_{dfeg}\right) \end{aligned}$$

Tranformation between two RNC frames

$$y^a = \overset{0}{y}{}^a + \overset{2}{y}{}^a + \overset{3}{y}{}^a + \overset{4}{y}{}^a + \overset{5}{y}{}^a + \mathcal{O}\left(\epsilon^6\right)$$

$$\overset{0}{y}{}^a = Dx^a$$

$$\begin{aligned}\overset{2}{y}{}^a &= \overset{2}{y}{}^a_1 \\ 3\overset{2}{y}{}^a_1 &= -x^b Dx^c Dx^d g^{ae} R_{bcde}\end{aligned}$$

$$\begin{aligned}\overset{3}{y}{}^a &= \overset{3}{y}{}^a_1 + \overset{3}{y}{}^a_2 \\ -12\overset{3}{y}{}^a_1 &= x^b Dx^c Dx^d Dx^e g^{af} \nabla_c R_{bdef} \\ -24\overset{3}{y}{}^a_2 &= x^b x^c Dx^d Dx^e \left(2g^{af}\nabla_d R_{becf} + 4g^{af}\nabla_b R_{cdef} - g^{af}\nabla_f R_{bdce}\right)\end{aligned}$$

$$\begin{aligned}\overset{4}{y}{}^a &= \overset{4}{y}{}^a_1 + \overset{4}{y}{}^a_2 + \overset{4}{y}{}^a_3 \\ -180\overset{4}{y}{}^a_1 &= x^b Dx^c Dx^d Dx^e Dx^f \left(4g^{ag}g^{hi}R_{bcdh}R_{egfi} + 3g^{ag}\nabla_{cd}R_{befg}\right) \\ -720\overset{4}{y}{}^a_2 &= x^b x^c Dx^d Dx^e Dx^f \left(32g^{ag}g^{hi}R_{bdeh}R_{cfgi} - 16g^{ag}g^{hi}R_{bdeh}R_{cifg} + 16g^{ag}g^{hi}R_{bdeh}R_{cgfi} - 16g^{ag}g^{hi}R_{bdch}R_{egfi} + 12g^{ag}\nabla_{de}R_{bfcg}\right. \\ &\qquad \left. + 18g^{ag}\nabla_{db}R_{cefg} + 18g^{ag}\nabla_{bd}R_{cefg} - 3g^{ag}\nabla_{gd}R_{becf} - 3g^{ag}\nabla_{dg}R_{becf}\right) \\ -720\overset{4}{y}{}^a_3 &= x^b x^c x^d Dx^e Dx^f \left(64g^{ag}g^{hi}R_{befh}R_{cgdi} - 32g^{ag}g^{hi}R_{bech}R_{difg} - 16g^{ag}g^{hi}R_{bech}R_{dgfi} + 18g^{ag}\nabla_{eb}R_{cfdg} + 18g^{ag}\nabla_{be}R_{cfdg} + 36g^{ag}\nabla_{bc}R_{defg}\right. \\ &\qquad \left. + 16g^{ag}g^{hi}R_{bech}R_{dfgi} - 9g^{ag}\nabla_{gb}R_{cedf} - 9g^{ag}\nabla_{bg}R_{cedf}\right)\end{aligned}$$

$$
\begin{aligned}
\overset{5}{y}{}^{a} &= \overset{5}{y}{}^{a}_{1} + \overset{5}{y}{}^{a}_{2} + \overset{5}{y}{}^{a}_{3} + \overset{5}{y}{}^{a}_{4} \\
-360\overset{5}{y}{}^{a}_{1} &= x^b Dx^c Dx^d Dx^e Dx^f Dx^g \left(3g^{ah}g^{ij}R_{bcdi}\nabla_e R_{fhgj} + 3g^{ah}g^{ij}R_{chdi}\nabla_e R_{bfgj} + g^{ah}\nabla_{cde}R_{bfgh}\right) \\
-2160\overset{5}{y}{}^{a}_{2} &= x^b x^c Dx^d Dx^e Dx^f Dx^g \big(34g^{ah}g^{ij}R_{bdhi}\nabla_e R_{cfgj} - 16g^{ah}g^{ij}R_{bidh}\nabla_e R_{cfgj} + 14g^{ah}g^{ij}R_{bdei}\nabla_f R_{cghj} + 4g^{ah}g^{ij}R_{bdei}\nabla_f R_{cjgh} \\
&\quad - 20g^{ah}g^{ij}R_{bdei}\nabla_j R_{cfgh} + 18g^{ah}g^{ij}R_{bdei}\nabla_f R_{chgj} + 24g^{ah}g^{ij}R_{bdei}\nabla_c R_{fhgj} + 4g^{ah}g^{ij}R_{dhei}\nabla_f R_{bgcj} + 18g^{ah}g^{ij}R_{bhdi}\nabla_e R_{cfgj} \\
&\quad + 24g^{ah}g^{ij}R_{dhei}\nabla_b R_{cfgj} - 10g^{ah}g^{ij}R_{dhei}\nabla_j R_{bfcg} - 16g^{ah}g^{ij}R_{bdci}\nabla_e R_{fhgj} + 6g^{ah}\nabla_{def}R_{bgch} + 8g^{ah}\nabla_{deb}R_{cfgh} + 8g^{ah}\nabla_{dbe}R_{cfgh} \\
&\quad + 8g^{ah}\nabla_{bde}R_{cfgh} - 6g^{ah}g^{ij}R_{bdei}\nabla_h R_{cfgj} - g^{ah}\nabla_{hde}R_{bfcg} - g^{ah}\nabla_{dhe}R_{bfcg} - g^{ah}\nabla_{deh}R_{bfcg}\big) \\
-1080\overset{5}{y}{}^{a}_{3} &= x^b x^c x^d Dx^e Dx^f Dx^g \big(14g^{ah}g^{ij}R_{behi}\nabla_f R_{cgdj} + 24g^{ah}g^{ij}R_{behi}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{behi}\nabla_j R_{cfdg} - 12g^{ah}g^{ij}R_{bieh}\nabla_f R_{cgdj} - 12g^{ah}g^{ij}R_{bieh}\nabla_c R_{dfgj} \\
&\quad + 2g^{ah}g^{ij}R_{befi}\nabla_g R_{chdj} - 10g^{ah}g^{ij}R_{befi}\nabla_j R_{cgdh} + 24g^{ah}g^{ij}R_{befi}\nabla_c R_{dghj} - 12g^{ah}g^{ij}R_{befi}\nabla_c R_{djgh} + 14g^{ah}g^{ij}R_{befi}\nabla_h R_{cgdj} \\
&\quad + 12g^{ah}g^{ij}R_{befi}\nabla_c R_{dhgj} + 2g^{ah}g^{ij}R_{bhei}\nabla_f R_{cgdj} + 22g^{ah}g^{ij}R_{bhci}\nabla_e R_{dfgj} + 12g^{ah}g^{ij}R_{bhei}\nabla_c R_{dfgj} - 5g^{ah}g^{ij}R_{bhei}\nabla_j R_{cfdg} \\
&\quad - 12g^{ah}g^{ij}R_{ehfi}\nabla_b R_{cgdj} - 12g^{ah}g^{ij}R_{beci}\nabla_f R_{djgh} - 8g^{ah}g^{ij}R_{beci}\nabla_f R_{dhgj} - 12g^{ah}g^{ij}R_{beci}\nabla_d R_{fhgj} + 4g^{ah}\nabla_{efb}R_{cgdh} + 4g^{ah}\nabla_{ebf}R_{cgdh} \\
&\quad + 6g^{ah}\nabla_{ebc}R_{dfgh} + 4g^{ah}\nabla_{bef}R_{cgdh} + 6g^{ah}\nabla_{bec}R_{dfgh} + 6g^{ah}\nabla_{bce}R_{dfgh} + 4g^{ah}g^{ij}R_{beci}\nabla_h R_{dfgj} + 4g^{ah}g^{ij}R_{beci}\nabla_f R_{dghj} - g^{ah}\nabla_{heb}R_{cfdg} \\
&\quad - g^{ah}\nabla_{hbe}R_{cfdg} - g^{ah}\nabla_{ehb}R_{cfdg} - g^{ah}\nabla_{bhe}R_{cfdg} - g^{ah}\nabla_{ebh}R_{cfdg} - g^{ah}\nabla_{beh}R_{cfdg}\big) \\
-360\overset{5}{y}{}^{a}_{4} &= x^b x^c x^d x^e Dx^f Dx^g \big(16g^{ah}g^{ij}R_{bfgi}\nabla_c R_{dhej} + 6g^{ah}g^{ij}R_{bhci}\nabla_f R_{dgej} + 16g^{ah}g^{ij}R_{bhci}\nabla_d R_{efgj} - 5g^{ah}g^{ij}R_{bhci}\nabla_j R_{dfeg} - 8g^{ah}g^{ij}R_{bifh}\nabla_c R_{dgej} \\
&\quad - 4g^{ah}g^{ij}R_{bhfi}\nabla_c R_{dgej} - 4g^{ah}g^{ij}R_{bfci}\nabla_g R_{dhej} - 8g^{ah}g^{ij}R_{bfci}\nabla_d R_{ejgh} - 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{ehgj} + 2g^{ah}\nabla_{fbc}R_{dgeh} + 2g^{ah}\nabla_{bfc}R_{dgeh} \\
&\quad + 2g^{ah}\nabla_{bcf}R_{dgeh} + 4g^{ah}\nabla_{bcd}R_{efgh} + 4g^{ah}g^{ij}R_{bfhi}\nabla_c R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_h R_{dgej} + 4g^{ah}g^{ij}R_{bfci}\nabla_d R_{eghj} - g^{ah}\nabla_{hbc}R_{dfeg} \\
&\quad - g^{ah}\nabla_{bhc}R_{dfeg} - g^{ah}\nabla_{bch}R_{dfeg}\big)
\end{aligned}
$$