

MPRI 2.4  
System F

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Types

Type safety

Polymorphism

Erasure

System F

Trivia

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Types

Type safety

Polymorphism

Erasure

System F

Trivia

## 1 Types

## 2 Type safety

## 3 Polymorphism

## 4 Erasure

## 5 System F

## 6 Trivia

# What is a type?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

A type is a concise, formal description of the behavior of a program fragment.

For instance, in OCaml, the following are types:

- int  
*an integer*
- $\text{int} \rightarrow \text{bool}$   
*a function that maps an integer argument to a Boolean result*
- $(\text{int} \rightarrow \text{bool}) \rightarrow (\text{int list} \rightarrow \text{int list})$   
*a function that maps an integer predicate to an integer list transformer*

## Sound, static type-checking

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Types must be **sound**: the behavior of a program must obey its type.

- an expression of type `int` must actually produce an integer value, if it terminates.

We want to **type-check** programs and **reject ill-typed programs**.

We want to do so **at compile time**, not at runtime.

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Benefits

Types serve as machine-checked documentation.

Types provide a safety guarantee.

– *Well-typed expressions do not go wrong.*

Milner, *A Theory of Type Polymorphism in Programming*, 1978.

Types enable modularity and abstraction,  
thereby enabling separate compilation and increasing robustness.

– *Type structure is a syntactic discipline  
for enforcing levels of abstraction.*

Reynolds, *Types, Abstraction and Parametric Polymorphism*, 1983.

Types enable potentially greater efficiency.

## Types all the way down

Types make sense in low-level programming languages, too:  
even assembly language can be typed!

Morrisett et al., From System F to Typed Assembly Language, 1999.

In a type-preserving compiler, every intermediate language is typed,  
and every compilation phase maps typed programs to typed programs.

- Preserving types helps understand a transformation,
- helps debug it,
- and can pave the way to a semantics preservation proof.

Chlipala, A certified type-preserving compiler  
from lambda calculus to assembly language, 2007.

## Downsides

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Types are descriptions of programs,  
so annotating programs with types can lead to redundancy.

- There is a need for a certain degree of type inference.

Types restrict expressiveness.

- A sound, decidable type system must reject some safe programs.

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Typed or untyped?

Reynolds nicely sums up a long and rather acrimonious debate:

- *One side claims that untyped languages preclude compile-time error checking and are succinct to the point of unintelligibility, while the other side claims that typed languages preclude a variety of powerful programming techniques and are verbose to the point of unintelligibility.*

Reynolds, [Three Approaches to Type Structure](#), 1985.

## Typed, with ever richer types

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In fact, Reynolds settles the debate:

– *From the theorist's point of view, both sides are right, and their arguments are the motivation for seeking type systems that are more flexible and succinct than those of existing typed languages.*

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## 1 Types

## 2 Type safety

## 3 Polymorphism

## 4 Erasure

## 5 System F

## 6 Trivia

# The simply-typed $\lambda$ -calculus

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Let us first review the simply-typed  $\lambda$ -calculus  
and a simple syntactic proof of its type soundness.

For Coq versions of these definitions and proofs, see [STLCDefinition](#),  
[STLCDefinitions](#), [STLCTypeSoundnessComplete](#) (and upcoming lecture).

# Terms and dynamic semantics

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

The terms are the pure  $\lambda$ -terms:  $t ::= x \mid \lambda x.t \mid t t.$

The reduction relation  $\cdot \rightarrow \cdot$  can be defined as follows:

$$\begin{array}{rcl} (\lambda x.t) v & \rightarrow & t[v/x] \\ E[t] & \rightarrow & E[t'] \quad \text{if } t \rightarrow t' \end{array} \quad \begin{array}{l} (\beta_v) \\ (\text{context}) \end{array}$$

where values and evaluation contexts are defined by:

$$\begin{array}{rcl} v & ::= & \lambda x.t \\ E & ::= & [] \mid E t \mid v E \end{array}$$

## Simple types

Types

Type safety

Polymorphism

Erasure

System F

Trivia

The syntax of types, in its simplest form,  
includes type variables and function types:

$$T ::= X \mid T \rightarrow T$$

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## The type system

The type system is a 3-place predicate.

A [typing judgement](#) takes the form:

$$\Gamma \vdash t : T$$

A [type environment](#)  $\Gamma$  is a finite sequence of bindings of variables to types:

$$\Gamma ::= \emptyset \mid \Gamma; x : T$$

It can also be viewed as a partial function of variables to types.

# The type system

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

The typing judgement is inductively defined:

$$\frac{\text{V}_\text{AR}}{\Gamma \vdash x : \Gamma(x)} \quad \frac{\text{A}_\text{BS}}{\Gamma; x : T_1 \vdash t : T_2} \quad \frac{\text{A}_\text{PP}}{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 \ t_2 : T_2}$$

It is [syntax-directed](#).

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Stating type soundness

What is a formal statement of Milner's slogan?

– *Well-typed expressions do not go wrong.*

Milner, *A Theory of Type Polymorphism in Programming*, 1978.

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Stating type soundness

What is a formal statement of Milner's slogan?

– *Well-typed expressions do not go wrong.*

Milner, *A Theory of Type Polymorphism in Programming*, 1978.

A well-typed, closed program must converge or diverge. It cannot crash.

Theorem (Type Soundness)

If  $\emptyset \vdash t : T$  then either  $\exists v, t \rightarrow^* v$     or     $t \rightarrow^\omega$ .

# Establishing type soundness

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

Type soundness follows from two properties:

## Theorem (Subject reduction)

*Reduction preserves types:*

$\emptyset \vdash t : T$  and  $t \rightarrow t'$  imply  $\emptyset \vdash t' : T$ .

## Theorem (Progress)

A well-typed, irreducible term is a value:

if  $\emptyset \vdash t : T$  and  $t \not\rightarrow$ , then  $t$  is a value.

Well-typedness is an invariant that implies absence of crashes.

Wright and Felleisen, A Syntactic Approach to Type Soundness, 1994.

## Establishing subject reduction

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Subject reduction is proved by induction over the hypothesis  $t \rightarrow t'$ .

There are two cases, corresponding to  $(\beta_v)$  and  $(context)$ .

## Establishing subject reduction

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of  $(\beta_v)$ , the first hypothesis is

$$\emptyset \vdash (\lambda x.t) v : T_2$$

and the goal is

$$\emptyset \vdash t[v/x] : T_2$$

How do we proceed?

## Establishing subject reduction

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We decompose the first hypothesis.

Because the type system is syntax-directed, the derivation of the first hypothesis must be of this form, for some type  $T_1$ :

$$\text{A}_{\text{PP}} \frac{\text{A}_{\text{BS}} \frac{x : T_1 \vdash t : T_2}{\emptyset \vdash \lambda x.t : T_1 \rightarrow T_2} \quad \emptyset \vdash v : T_1}{\emptyset \vdash (\lambda x.t) v : T_2}$$

The goal is still

$$\emptyset \vdash t[v/x] : T_2$$

Where next?

# Establishing subject reduction

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

We need a simple lemma:

## Lemma (Value substitution)

*Replacing a formal parameter with a type-compatible actual argument preserves types:*

$$x : T_1 \vdash t : T_2 \quad \text{and} \quad \emptyset \vdash v : T_1 \quad \text{imply} \quad \emptyset \vdash t[v/x] : T_2.$$

How do we prove this lemma?

## Establishing subject reduction

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

The lemma must be generalized so it can be proven by [induction](#) over a typing judgement for  $t$ :

### [Lemma \(Value substitution\)](#)

$x : T_1, \Gamma \vdash t : T_2 \text{ and } x \notin \text{dom}(\Gamma) \text{ and } \emptyset \vdash v : T_1 \text{ imply } \Gamma \vdash t[v/x] : T_2.$

The proof is straightforward.

At variables, one must argue that  $\emptyset \vdash v : T_1$  implies  $\Gamma \vdash v : T_1$  ([weakening](#)).

This closes the case of  $(\beta_v)$ .

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Establishing subject reduction

In the case of rule (*context*), the first hypothesis is

$$\emptyset \vdash E[t] : T$$

The second hypothesis is

$$t \longrightarrow t'$$

where, by the induction hypothesis, this reduction preserves types.

The goal is

$$\emptyset \vdash E[t'] : T$$

How do we proceed?

## Establishing subject reduction

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Type-checking is compositional. For the judgement  $\emptyset \vdash E[t] : T$  to hold, only the type of the subterm in the hole matters, not its exact form.

# Establishing subject reduction

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Type-checking is compositional. For the judgement  $\emptyset \vdash E[t] : T$  to hold, only the type of the subterm in the hole matters, not its exact form.

## Lemma (Compositionality)

Assume  $\emptyset \vdash E[t] : T$ . Then, there exists a type  $T'$  such that:

- $\emptyset \vdash t : T'$ ,
- for every term  $t'$ ,  $\emptyset \vdash t' : T'$  implies  $\emptyset \vdash E[t'] : T$ .

Using this lemma, the (context) case of subject reduction is immediate.

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Recall the statement of Progress:

*if  $\emptyset \vdash t : T$  and  $t \not\rightarrow$ , then  $t$  is a value.*

This can be reformulated in a positive way:

*if  $\emptyset \vdash t : T$  then  $t \rightarrow \cdot$  or  $t$  is a value.*

How can we prove this?

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Establishing progress

Progress is proved by induction over the term  $t$  or over the hypothesis  $\emptyset \vdash t : T$ .

Thus, there is one case per construct in the syntax of terms.

In the pure  $\lambda$ -calculus, there are just three cases:

- variable;
- $\lambda$ -abstraction;
- application.

Two of these are immediate...

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

The case of variables cannot occur: a variable is not closed.

The case of  $\lambda$ -abstractions is immediate: a  $\lambda$ -abstraction is a value.

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of applications, the goal is:

*if  $\emptyset \vdash t_1 t_2 : T$  then  $t_1 t_2 \longrightarrow \cdot$  or  $t_1 t_2$  is a value.*

This goal can be simplified:

# Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of applications, the goal is:

*if  $\emptyset \vdash t_1 t_2 : T$  then  $t_1 t_2 \rightarrow \cdot$  or  $t_1 t_2$  is a value.*

This goal can be simplified:

*if  $\emptyset \vdash t_1 t_2 : T$  then  $t_1 t_2 \rightarrow \cdot \cdot$ .*

Indeed, an application is never a value.

How do we proceed?

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

The goal is

$$\textit{if } \emptyset \vdash t_1 \ t_2 : T \textit{ then } t_1 \ t_2 \longrightarrow \dots$$

# Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

The goal is

$$\textit{if } \emptyset \vdash t_1 \ t_2 : T \textit{ then } t_1 \ t_2 \longrightarrow \dots$$

By **inversion** of the type-checking rule for applications, we must have  $\emptyset \vdash t_1 : T_1 \rightarrow T_2$  and  $\emptyset \vdash t_2 : T_1$  for some types  $T_1, T_2$ .

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

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By the induction hypothesis,  $t_1$  must be reducible or a value  $v_1$ .

## Establishing progress

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

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By the [induction hypothesis](#),  $t_1$  must be reducible or a value  $v_1$ .

If  $t_1$  is reducible, then, because  $[]\ t_2$  is an evaluation context,  $t_1\ t_2$  is reducible as well, and we are done. So, assume  $t_1$  is  $v_1$ .

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

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By the **induction hypothesis**,  $t_2$  must be reducible or a value  $v_2$ .

# Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

The goal is

$$\text{if } \emptyset \vdash t_1 \ t_2 : T \text{ then } t_1 \ t_2 \longrightarrow \dots$$

By **inversion** of the type-checking rule for applications, we must have  $\emptyset \vdash t_1 : T_1 \rightarrow T_2$  and  $\emptyset \vdash t_2 : T_1$  for some types  $T_1, T_2$ .

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By the **induction hypothesis**,  $t_2$  must be reducible or a value  $v_2$ .

If  $t_2$  is reducible, then, because  $v_1$   $[]$  is an evaluation context,  $v_1 \ t_2$  is reducible as well, and we are done. So, assume  $t_2$  is  $v_2$ .

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

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If  $t_1$  is reducible, then, because  $[]$   $t_2$  is an evaluation context,  $t_1 \ t_2$  is reducible as well, and we are done. So, assume  $t_1$  is  $v_1$ .

By the [induction hypothesis](#),  $t_2$  must be reducible or a value  $v_2$ .

If  $t_2$  is reducible, then, because  $v_1$   $[]$  is an evaluation context,  $v_1 \ t_2$  is reducible as well, and we are done. So, assume  $t_2$  is  $v_2$ .

$\emptyset \vdash v_1 : T_1 \rightarrow T_2$  implies that  $v_1$  must be a  $\lambda$ -abstraction ([see next slide](#)). So  $v_1 \ v_2$  is a  $\beta$ -redex: it is reducible. We are done.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

# Classification of values

We have appealed to the following property:

## Lemma (Classification)

Assume  $\emptyset \vdash v : T$ . Then,

- if  $T$  is an arrow type, then  $v$  is a  $\lambda$ -abstraction;
- ...

In pure  $\lambda$ -calculus, this result is trivial. In a richer type system, this lemma claims that [the head constructor of the type](#) conveys information about [the head constructor of the value](#).

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## 1 Types

## 2 Type safety

## 3 Polymorphism

## 4 Erasure

## 5 System F

## 6 Trivia

## What is polymorphism?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Polymorphism is the possibility that a term may

- simultaneously admit several distinct types
- or be able to operate at several distinct types.

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Flavors of polymorphism

Strachey distinguishes

- parametric polymorphism (universal types; today);
- ad hoc polymorphism  
(e.g., overloaded arithmetic operations; upcoming lecture by PED).

Strachey, *Fundamental Concepts in Programming Languages*, 1967.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Why polymorphism?

Polymorphism seems [indispensable](#): a comparison-based sorting function should be applicable to lists of integers, lists of Booleans, etc.

In short, it should have polymorphic type:

$$\forall X. (X \rightarrow X \rightarrow \text{bool}) \rightarrow X \text{ list} \rightarrow X \text{ list}$$

whose [instances](#) are the monomorphic types:

$$(\text{int} \rightarrow \text{int} \rightarrow \text{bool}) \rightarrow \text{int list} \rightarrow \text{int list}$$

$$(\text{bool} \rightarrow \text{bool} \rightarrow \text{bool}) \rightarrow \text{bool list} \rightarrow \text{bool list}$$

...

## Why polymorphism?

Without polymorphism, the only ways of achieving this effect would be:

- to manually duplicate the list sorting function at every type ([no-no!](#));
- to use subtyping and claim that the function can sort [heterogeneous](#) lists:

$$(\top \rightarrow \top \rightarrow \text{bool}) \rightarrow \top \text{ list} \rightarrow \top \text{ list}$$

The type  $\top$  is the type of all values, and the supertype of all types.

This leads to a [loss of information](#). To recover this information, a [downcast](#) operation is required.

This approach is common in C and was followed in Java prior to 5.

## Polymorphism seems almost free

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Some polymorphism is already implicitly present in simply-typed  $\lambda$ -calculus.

The term  $\lambda fxy.(f\ x, f\ y)$  admits a principal type:

$$(X_1 \rightarrow X_2) \rightarrow X_1 \rightarrow X_1 \rightarrow X_2 \times X_2$$

By saying that this term admits a polymorphic type,

$$\forall X_1 X_2. (X_1 \rightarrow X_2) \rightarrow X_1 \rightarrow X_1 \rightarrow X_2 \times X_2$$

we make polymorphism internal to the type system.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Towards type abstraction

Polymorphism is a step on the road towards [type abstraction](#).

If a sorting function has a polymorphic type:

$$\forall X. (X \rightarrow X \rightarrow \text{bool}) \rightarrow X \text{ list} \rightarrow X \text{ list}$$

then it [knows nothing](#) about  $X$  so it must manipulate elements [abstractly](#).

It can move them, copy them, pass them to the comparison function,  
but cannot directly inspect their structure.

Inside the sorting function,  $X$  is an [abstract type](#).

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

A type can reveal a lot of information about the terms that inhabit it.  
For instance, in a pure and total language, the polymorphic type

$$\forall X. X \rightarrow X$$

has **only one inhabitant**, namely the identity.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Parametricity

Similarly, the type of a polymorphic sorting function:

$$\forall X. (X \rightarrow X \rightarrow \text{bool}) \rightarrow X \text{ list} \rightarrow X \text{ list}$$

reveals a “free theorem” about its behavior: roughly, the outcome of sorting depends only on the outcomes of comparisons:

*For all types  $X_1$  and  $X_2$ ,  
for every binary relation  $R$  between  $X_1$  and  $X_2$ ,  
if  $\text{cmp}$  maps related arguments to identical Boolean results,  
then  $\text{sort cmp}$  maps related lists to related lists.*

See the lecture on binary logical relations and parametricity (GS).

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## 1 Types

## 2 Type safety

## 3 Polymorphism

## 4 Erasure

## 5 System F

## 6 Trivia

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## System *F*: types

The polymorphic  $\lambda$ -calculus, also known as [System F](#), was independently defined by [Girard \(1972\)](#) and Reynolds (1974).

Reynolds, [Towards a theory of type structure](#), 1974.

Compared to the simply-typed  $\lambda$ -calculus, the syntax of [types](#) is extended with [universal types](#):

$$T ::= \dots | \forall X. T$$

## System *F*: terms

Types

Type safety

Polymorphism

Erasure

System F

Trivia

How should the syntax and semantics of [terms](#) be extended?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## System F: terms

How should the syntax and semantics of [terms](#) be extended?

Several answers are possible:

- not at all!

*“Curry style”: keep type information implicit.*

*Types do not exist at runtime and do not influence computation.*

## System F: terms

How should the syntax and semantics of terms be extended?

Several answers are possible:

- not at all!

*“Curry style”: keep type information implicit.*

*Types do not exist at runtime and do not influence computation.*

- with type abstractions and type applications:  $t ::= \dots | \lambda X.t | t\ T$  and with a new reduction rule:

$$(\lambda X.t) \ T \longrightarrow t[T/X]$$

*“Church style”: explicitly show where universal quantifiers are introduced and eliminated.*

Does “Church style” imply that type abstractions and applications necessarily influence computation?

## Type erasure

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Must type abstractions and applications influence computation?

## Type erasure

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

Must type abstractions and applications influence computation?

Several answers are again possible:

- No. Define  $v ::= \dots | \lambda X.v$  and  $E ::= \dots | \lambda X.E | E T$ .  
Then a **type erasure** property holds: the program **with or without** type abstractions and applications has the same behavior.  
*Types need not exist at runtime and do not influence computation.*

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

Must type abstractions and applications influence computation?

Several answers are again possible:

- **No.** Define  $v ::= \dots | \lambda X.v$  and  $E ::= \dots | \lambda X.E | E T$ .  
Then a **type erasure** property holds: the program **with or without** type abstractions and applications has the same behavior.  
*Types need not exist at runtime and do not influence computation.*
- **Yes.** Define  $v ::= \dots | \lambda X.t$  and  $E ::= \dots | E T$ .  
Then type erasure does not hold:  
e.g.,  $\lambda X.\omega$  is a value but its erasure  $\omega$  diverges.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## The type erasure property

A typed programming language has [the type erasure property](#) if:

$$\text{behaviors}(t) = \text{behaviors}(\lceil t \rceil)$$

where the function  $\lceil \cdot \rceil$  erases all type annotations and  $\text{behaviors}(t)$  is the set of the observable behaviors of the (closed) term  $t$ .

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Philosophy of type erasure

Type erasure means that **types** need not exist at runtime.

Type erasure supports the idea that **untyped terms** have well-defined behavior and that **types** are post hoc descriptions of pre-existing behavior.

Some researchers disagree. They argue that **only typed terms** should have a meaning and/or that **one should let types influence reduction**.

The two views can be reconciled. Instead of letting “**types exist at runtime**”, one can **erase types** and use **type descriptions** (values) at runtime. (See upcoming lecture on GADTs.)

## What style?

A formal definition of a typed language must choose:

- between **implicit or explicit** type information in terms;
  - Here, limited technical impact; mainly a matter of stylistic preference.
  - In richer calculi (dependent types), this can make a big difference.
- whether **type erasure** is desired.
  - I like the idea that types need not exist at runtime.

For a presentation of System *F* in implicit style, see upcoming lecture on Syntactic type soundness for System *F* in Coq.

Today, I use a presentation in **explicit** style, **with** a type erasure property.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## System *F* in explicit style

The syntax and operational semantics are as follows:

$$\begin{array}{lll} t & ::= & \dots \mid \Lambda X.t \mid t\ T & \text{terms} \\ v & ::= & \dots \mid \Lambda X.v & \text{values} \\ E & ::= & \dots \mid \Lambda X.E \mid E\ T & \text{evaluation contexts} \\ (\Lambda X.t)\ T & \longrightarrow & t[T/X] & \text{a new reduction rule} \end{array}$$

Reduction is non-deterministic. This will not be a problem.

Before going further, let us check that the [type erasure](#) property holds.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## A type erasure function

The [erasure](#) function  $\lceil \cdot \rceil$  maps a term to a term:

$$\begin{array}{rcl} \lceil x \rceil & = & x \\ \lceil \lambda x. t \rceil & = & \lambda x. \lceil t \rceil \\ \lceil t_1 \ t_2 \rceil & = & \lceil t_1 \rceil \lceil t_2 \rceil \\ \lceil \Lambda X. t \rceil & = & \lceil t \rceil \\ \lceil t \ T \rceil & = & \lceil t \rceil \end{array}$$

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## A type erasure function

The [erasure](#) function  $\lceil \cdot \rceil$  maps a term to a term:

$$\begin{aligned}\lceil x \rceil &= x \\ \lceil \lambda x. t \rceil &= \lambda x. \lceil t \rceil \\ \lceil t_1 \ t_2 \rceil &= \lceil t_1 \rceil \lceil t_2 \rceil \\ \lceil \Lambda X. t \rceil &= \lceil t \rceil \\ \lceil t \ T \rceil &= \lceil t \rceil\end{aligned}$$

What statement(s) do we wish to prove about this function?

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## A type erasure function

The [erasure](#) function  $\lceil \cdot \rceil$  maps a term to a term:

$$\begin{aligned}\lceil x \rceil &= x \\ \lceil \lambda x. t \rceil &= \lambda x. \lceil t \rceil \\ \lceil t_1 \ t_2 \rceil &= \lceil t_1 \rceil \lceil t_2 \rceil \\ \lceil \Lambda X. t \rceil &= \lceil t \rceil \\ \lceil t \ T \rceil &= \lceil t \rceil\end{aligned}$$

What statement(s) do we wish to prove about this function?

We wish to state that computing [with type annotations](#) is “the same” as computing [without them](#).

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Simulation, take 1

Here is a first attempt at a simulation statement:

**Lemma (Simulation)**

*If  $t \rightarrow t'$  then  $\lceil t \rceil \rightarrow \lceil t' \rceil$ .*

Is this true?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Simulation, take 1

Here is a first attempt at a simulation statement:

**Lemma (Simulation)**

*If  $t \rightarrow t'$  then  $\lceil t \rceil \rightarrow \lceil t' \rceil$ .*

Is this true?

No. We must allow stuttering, that is, zero steps on the right-hand side.

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Simulation, take 2

Here is a second simulation statement:

**Lemma (Simulation)**

*If  $t \rightarrow t'$  then  $\lceil t \rceil \rightarrow^? \lceil t' \rceil$ .*

Is this true?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Simulation, take 2

Here is a second simulation statement:

### Lemma (Simulation)

*If  $t \rightarrow t'$  then  $\lceil t \rceil \rightarrow^? \lceil t' \rceil$ .*

Is this true?

Yes. The proof is by induction over  $t \rightarrow t'$ .

Is this enough?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Are we happy with (just) this simulation statement?

Is this enough?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Are we happy with (just) this simulation statement?

Does it really mean that  $t$  and  $\lceil t \rceil$  compute “the same thing”?

## Is this enough?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Are we happy with (just) this simulation statement?

Does it really mean that  $t$  and  $\lceil t \rceil$  compute “the same thing”?

If we had posited  $\lceil t \rceil \triangleq \lambda x.x$  then it would still hold!

## Is this enough?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Are we happy with (just) this simulation statement?

Does it really mean that  $t$  and  $\lceil t \rceil$  compute “the same thing”?

If we had posited  $\lceil t \rceil \triangleq \lambda x.x$  then it would still hold!

We must also show that  $\lceil \cdot \rceil$  preserves the **observable behavior** of a term.

## Is this enough?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Are we happy with (just) this simulation statement?

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If we had posited  $\lceil t \rceil \triangleq \lambda x.x$  then it would still hold!

We must also show that  $\lceil \cdot \rceil$  preserves the **observable behavior** of a term.

The three possible observable behaviors of a (closed) term are:

## Is this enough?

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Are we happy with (just) this simulation statement?

Does it really mean that  $t$  and  $\lceil t \rceil$  compute “the same thing”?

If we had posited  $\lceil t \rceil \triangleq \lambda x.x$  then it would still hold!

We must also show that  $\lceil \cdot \rceil$  preserves the **observable behavior** of a term.

The three possible observable behaviors of a (closed) term are:

- to **converge** (to reduce to a value),
- to **diverge** (to reduce forever),
- and to **go wrong** (to reduce to a stuck term).

## Preservation of values

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

We must check:

**Lemma (Erasure of a value)**

*For every value  $v$ ,  $\lceil v \rceil$  is a value.*

Recall the definition of values:  $v ::= \lambda x.t \mid \Lambda X.v$ .

## Preservation of values

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We must check:

**Lemma (Erasure of a value)**

*For every value  $v$ ,  $\lceil v \rceil$  is a value.*

Recall the definition of values:  $v ::= \lambda x.t \mid \Lambda X.v$ .

The proof (by induction on  $v$ ) is easy.

## Preservation of divergence

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We must check:

Lemma (Erasure of a divergent computation)

*If  $t$  diverges then  $\lceil t \rceil$  diverges.*

Is this true?

## Preservation of divergence

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We must check:

Lemma (Erasure of a divergent computation)

*If  $t$  diverges then  $\lceil t \rceil$  diverges.*

Is this true?

Recall the statement of Simulation: *if  $t \rightarrow t'$  then  $\lceil t \rceil \rightarrow^? \lceil t' \rceil$ .*

## Preservation of divergence

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We must check:

**Lemma (Erasure of a divergent computation)**

*If  $t$  diverges then  $\lceil t \rceil$  diverges.*

Is this true?

Recall the statement of Simulation: *if  $t \rightarrow t'$  then  $\lceil t \rceil \rightarrow^? \lceil t' \rceil$ .*

This does **not** allow proving that  $t \rightarrow^\omega$  implies  $\lceil t \rceil \rightarrow^\omega$ .

## Preservation of divergence

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We must check:

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*If  $t$  diverges then  $\lceil t \rceil$  diverges.*

Is this true?

Recall the statement of Simulation: *if  $t \rightarrow t'$  then  $\lceil t \rceil \rightarrow^? \lceil t' \rceil$ .*

This does **not** allow proving that  $t \rightarrow^\omega$  implies  $\lceil t \rceil \rightarrow^\omega$ .

We must find a way of proving that  $\lceil t \rceil$  cannot **stutter forever**.

How?

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Simulation, take 3

Here is a final simulation statement:

### Lemma (Simulation)

If  $t \rightarrow t'$  then

- either  $\lceil t \rceil \rightarrow \lceil t' \rceil$
- or  $\lceil t \rceil = \lceil t' \rceil$  and  $\text{size}(t) > \text{size}(t')$

where *size* maps terms into  $\mathbb{N}$ .

The proof is by induction over  $t \rightarrow t'$ .

# Preservation of divergence

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Lemma

If  $t \rightarrow^\omega$  diverges then  $\lceil t \rceil \rightarrow \dots$

More precisely, there exists  $t'$  such that  $t'$  diverges and  $\lceil t \rceil \rightarrow \lceil t' \rceil$ .

## Proof.

If  $\lceil t \rceil$  could not make one step then (by Simulation) we would have an infinite sequence  $t_1 \rightarrow t_2 \rightarrow \dots$  with  $\text{size}(t_1) > \text{size}(t_2) > \dots$ . □

## Lemma (Erasure of a divergent computation)

If  $t$  diverges then  $\lceil t \rceil$  diverges.

## Proof.

By infinite iteration of the previous lemma. □

# Preservation of going-wrongness

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Can we prove this?

**Lemma**

*If  $t$  goes wrong then  $\lceil t \rceil$  goes wrong.*

# Preservation of going-wrongness

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Can we prove this?

**Lemma**

*If  $t$  goes wrong then  $\lceil t \rceil$  goes wrong.*

No. This statement is false.

# Preservation of going-wrongness

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Can we prove this?

## Lemma

*If  $t$  goes wrong then  $\lceil t \rceil$  goes wrong.*

No. This statement is false.

$(\lambda X.\lambda x.x) 0$  is stuck, yet its erasure  $(\lambda x.x) 0$  is not stuck.

We won't need this statement anyway  
because we care about well-typed terms only.

## Preservation of observable behavior

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Let  $\downarrow$ ,  $\nearrow$ , and  $\perp$  stand for the three possible observable behaviors: convergence, divergence, and going wrong.

Let  $\text{behaviors}(t)$  stand for the set

$$\begin{aligned} & \{ \downarrow \mid \exists v, t \xrightarrow{*} v \} \cup \\ & \{ \nearrow \mid t \xrightarrow{\omega} \} \cup \\ & \{ \perp \mid \exists t', t \xrightarrow{*} t' \wedge t' \text{ is stuck} \} \end{aligned}$$

By putting together the previous results, we get:

**Lemma (Forward preservation of observable behavior)**  
if  $t$  cannot go wrong then  $\text{behaviors}(t) \subseteq \text{behaviors}(\lceil t \rceil)$ .

## Preservation of observable behavior

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We have just proved:

**Lemma (Forward preservation of observable behavior)**

*if  $t$  cannot go wrong then  $\text{behaviors}(t) \subseteq \text{behaviors}(\lceil t \rceil)$ .*

# Preservation of observable behavior

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We have just proved:

**Lemma (Forward preservation of observable behavior)**

*if  $t$  cannot go wrong then  $\text{behaviors}(t) \subseteq \text{behaviors}(\lceil t \rceil)$ .*

Because erased terms have **deterministic** semantics,  
 $\text{behaviors}(\lceil t \rceil)$  must be a singleton set.

## Preservation of observable behavior

Types

Type safety

Polymorphism

Erasure

System F

Trivia

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 $\text{behaviors}(\lceil t \rceil)$  must be a singleton set.

Because every term has some behavior,  
 $\text{behaviors}(t)$  must be a nonempty set.

# Preservation of observable behavior

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We have just proved:

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Thus, we have:

**Lemma (Preservation of observable behavior)**

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# Preservation of observable behavior

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We have just proved:

**Lemma (Forward preservation of observable behavior)**

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Because every term has some behavior,  
 $\text{behaviors}(t)$  must be a nonempty set.

Thus, we have:

**Lemma (Preservation of observable behavior)**

*if  $t$  cannot go wrong then  $\text{behaviors}(t) = \text{behaviors}(\lceil t \rceil)$ .*

**Corollary (Preservation of safety)**

*if  $t$  cannot go wrong then  $\lceil t \rceil$  cannot go wrong.*

## Type erasure

Types

Type safety

Polymorphism

Erasure

System F

Trivia

We have just proved a type erasure property.

One can

- use type-annotated terms when defining the type discipline and proving type soundness,
- erase type annotations when executing terms,
- and all will be well,  
provided ill-typed terms are rejected up front.

# Type erasure without determinism

Types

Type safety

Polymorphism

Erasure

System F

Trivia

What if the semantics is not deterministic?

# Type erasure without determinism

Types

Type safety

Polymorphism

Erasure

System F

Trivia

What if the semantics is not deterministic?

A **backward simulation** statement seems necessary:

## Lemma

If  $\lceil t \rceil = u$  and  $u \rightarrow u'$  and  $\emptyset \vdash t : T$  then  
there exists  $t'$  such that  $t \rightarrow t'$  and

- either  $\lceil t' \rceil = u'$
- or  $\lceil t' \rceil = u$  and  $\text{size}(t) > \text{size}(t')$ .

# Type erasure without determinism

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

What if the semantics is not deterministic?

A [backward simulation](#) statement seems necessary:

## Lemma

*If  $\lceil t \rceil = u$  and  $u \rightarrow u'$  and  $\emptyset \vdash t : T$  then  
there exists  $t'$  such that  $t \rightarrow t'$  and*

- either  $\lceil t' \rceil = u'$
- or  $\lceil t' \rceil = u$  and  $\text{size}(t) > \text{size}(t')$ .

From this, we get:

## Lemma (Backward preservation of observable behavior)

*If  $\emptyset \vdash t : T$  then  $\text{behaviors}(\lceil t \rceil) \subseteq \text{behaviors}(t)$ .*

Thus, if  $t$  is well-typed then  $t$  and  $\lceil t \rceil$  have the same behaviors.

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## Caveat

I have not formalized the proofs about erasure in Coq.

I would need terms that contain term binders  $\lambda$  and type binders  $\Lambda$ , which AutoSubst 1 does not support.

AutoSubst 2 supports this, but I have not tried it.

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## 1 Types

## 2 Type safety

## 3 Polymorphism

## 4 Erasure

## 5 System F

## 6 Trivia

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## System F: terms

The terms include type abstractions and type applications:

$$t ::= x \mid \lambda x. t \mid t \ t \mid \Lambda X. t \mid t \ T$$

## System F: dynamic semantics

Types

Type safety

Polymorphism

Erasure

System F

Trivia

The reduction rules are:

$$\begin{array}{lll} (\lambda x.t) v & \longrightarrow & t[v/x] \\ (\Lambda X.t) T & \longrightarrow & t[T/X] \\ E[t] & \longrightarrow & E[t'] \quad \text{if } t \longrightarrow t' \end{array} \quad \begin{array}{c} (\beta_v) \\ (\iota) \\ (\text{context}) \end{array}$$

where values and evaluation contexts are defined by:

$$\begin{array}{ll} v & ::= \lambda x.t \mid \Lambda X.v \\ E & ::= [] \mid E\ t \mid v\ E \mid \Lambda X.E \mid E\ T \end{array}$$

## System *F*: types and type environments

Types

Type safety

Polymorphism

Erasure

System F

Trivia

The types include universal types:

$$T ::= X \mid T \rightarrow T \mid \forall X. T$$

Type environments bind both term variables and type variables:

$$\Gamma ::= \emptyset \mid \Gamma; x : T \mid \Gamma; X$$

They still act as partial functions of variables  $x$  to types  $T$ .

A “runtime type environment” binds just type variables:  $\Delta ::= \emptyset \mid \Delta; X$ .  
This notion is used in some lemmas.

## Hygiene

Whenever  $\Gamma; X$  is constructed, e.g. in the typing rule  $\text{TAbs}$ , we require  $X \# T$  ("X is fresh for  $T$ "), which stands for  $X \notin \text{ftv}(T)$  ("X does not occur free in  $T$ ").

This ensures that X **does not shadow** older variables in  $\Gamma$ .

Furthermore, we allow **renaming** type variables bound by  $\Lambda$  or  $\forall$ .

These statements make sense when type variables are **named**. If type variables are **de Bruijn indices** then shadowing is avoided by **lifting** types in suitable places: see **SystemFDefinitions**.

## System *F*: the typing judgement

The typing judgement is inductively defined:

$$\begin{array}{c}
 \text{V}_\text{AR} \quad \text{A}_\text{BS} \quad \text{A}_\text{PP} \\
 \Gamma \vdash x : \Gamma(x) \quad \frac{\Gamma; x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x. t : T_1 \rightarrow T_2} \quad \frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 \ t_2 : T_2} \\
 \\ 
 \text{T A}_\text{BS} \quad \text{T A}_\text{PP} \\
 \frac{\Gamma; X \vdash t : T \quad X \not\in \Gamma}{\Gamma \vdash \Lambda X. t : \forall X. T} \quad \frac{\Gamma \vdash t : \forall X. T}{\Gamma \vdash t \ T' : T[T'/X]}
 \end{array}$$

It is **syntax-directed** thanks to explicit type abstractions and applications.

Polymorphism is **impredicative**: a type variable denotes an arbitrary type.

## Establishing type soundness

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Type soundness again follows from subject reduction and progress.

### Theorem (Subject reduction)

*Reduction preserves types:*

$\Delta \vdash t : T$  and  $t \rightarrow t'$  imply  $\Delta \vdash t' : T$ .

### Theorem (Progress)

A well-typed, irreducible term is a value:

if  $\Delta \vdash t : T$  and  $t \not\rightarrow$ , then  $t$  is a value.

## Establishing subject reduction

Subject reduction is still proved by induction over  $t \rightarrow t'$ .

As before, there is one case per reduction rule, so now three cases.

- the case of  $(\beta_v)$  is unchanged  
*because the type system is syntax-directed.*  
A derivation of  $\Delta \vdash (\lambda x.t) v : T_2$  must use the rules  $A_{\text{PP}}$  and  $A_{\text{BS}}$ .
- the case of  $(\text{context})$  is unchanged.
- the case of  $(\iota)$  is new (see next slides).

In a Curry-style presentation, the case of  $(\iota)$  would not exist at all, but the case of  $(\beta_v)$  would be more complex: see [SystemFTypeSoundnessComplete](#).

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Establishing subject reduction

In the case of  $(\iota)$ , the first hypothesis is

$$\Delta \vdash (\Lambda X.t) \ T : T_2$$

and the goal is

$$\Delta \vdash t[T/X] : T_2$$

How do we proceed?

## Establishing subject reduction

We decompose the first hypothesis.

Because the type system is syntax-directed, the derivation of the first hypothesis must be of this form:

$$\text{TA}_{\text{PP}} \frac{\text{TA}_{\text{BS}} \frac{\Delta; X \vdash t : T_1 \quad X \# \Delta}{\Delta \vdash \Lambda X.t : \forall X.T_1} \quad T_1[T/X] = T_2}{\Delta \vdash (\Lambda X.t) T : T_2}$$

The goal is still

$$\Delta \vdash t[T/X] : T_2$$

Where next?

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Establishing subject reduction

We need a simple lemma:

### Lemma (Type substitution)

*Replacing a type variable  $X$  with an arbitrary type  $T$  preserves types:*

$\Delta; X; \Gamma \vdash t : T_1$  and  $X \notin \text{btv}(\Gamma)$  imply  $\Delta; \Gamma \vdash t[T/X] : T_1[T/X]$ .

This lemma is [the essence of parametric polymorphism](#).

Its proof is straightforward.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Establishing progress

Recall the statement of Progress:

*if  $\Delta \vdash t : T$  then  $t \longrightarrow \cdot$  or  $t$  is a value.*

As before, progress is proved by induction over the hypothesis  $\Delta \vdash t : T$ .

There is one case per typing rule:

- the cases of  $\text{VAR}$ ,  $\text{ABS}$ ,  $\text{APP}$  are unchanged.
- the cases of  $\text{TABS}$  and  $\text{TAPP}$  are new.

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of TAbs, the judgement  $\Delta \vdash \lambda X.t : \forall X.T$  follows from  $\Delta; X \vdash t : T$  and the goal is

$\lambda X.t \longrightarrow \cdot \text{ or } \lambda X.t \text{ is a value.}$

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of TAbs, the judgement  $\Delta \vdash \Lambda X.t : \forall X.T$  follows from  $\Delta; X \vdash t : T$  and the goal is

$\Lambda X.t \rightarrow \cdot$  or  $\Lambda X.t$  is a value.

The induction hypothesis assures us that  $t \rightarrow \cdot$  or  $t$  is a value.

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of TAbs, the judgement  $\Delta \vdash \Lambda X.t : \forall X.T$  follows from  $\Delta; X \vdash t : T$  and the goal is

$\Lambda X.t \rightarrow \cdot \text{ or } \Lambda X.t \text{ is a value.}$

The induction hypothesis assures us that  $t \rightarrow \cdot$  or  $t$  is a value.

- in the first case, the left-hand disjunct of the goal holds, because  $\Lambda X.[]$  is an evaluation context.
- in the second case, the right-hand disjunct of the goal holds, because  $\Lambda X.v$  is a value.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Establishing progress

In the case of  $\text{TA}_{\text{PP}}$ , the judgement  $\Delta \vdash t : T[T_2/X]$  follows from  $\Delta \vdash t : \forall X.T$  and the goal is

$t : T_2 \longrightarrow \cdot$  or  $t : T_2$  is a value.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Establishing progress

In the case of TAPP, the judgement  $\Delta \vdash t : T[T_2/X]$  follows from  $\Delta \vdash t : \forall X.T$  and the goal is

$t : T_2 \longrightarrow \cdot \quad \text{or} \quad t : T_2 \text{ is a value.}$

As  $t : T_2$  is not a value, this goal can be simplified to:

$t : T_2 \longrightarrow \cdot \quad \cdot$

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of TAPP, the judgement  $\Delta \vdash t : T[T_2/X]$  follows from  $\Delta \vdash t : \forall X.T$  and the goal is

$t : T_2 \longrightarrow \cdot \quad \text{or} \quad t : T_2 \text{ is a value.}$

As  $t : T_2$  is not a value, this goal can be simplified to:

$t : T_2 \longrightarrow \cdot \quad \cdot$

The induction hypothesis assures us that  $t \longrightarrow \cdot \quad \text{or} \quad t \text{ is a value.}$

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of TAPP, the judgement  $\Delta \vdash t : T[T_2/X]$  follows from  $\Delta \vdash t : \forall X.T$  and the goal is

$t : T_2 \longrightarrow \cdot \quad \text{or} \quad t : T_2 \text{ is a value.}$

As  $t : T_2$  is not a value, this goal can be simplified to:

$t : T_2 \longrightarrow \cdot \quad \cdot$

The induction hypothesis assures us that  $t \longrightarrow \cdot \quad \text{or} \quad t \text{ is a value.}$

- in the first case, the goal holds because  $[] T_2$  is an evaluation context.

## Establishing progress

Types

Type safety

Polymorphism

Erasure

System F

Trivia

In the case of TAPP, the judgement  $\Delta \vdash t : T[T_2/X]$  follows from  $\Delta \vdash t : \forall X.T$  and the goal is

$t : T_2 \longrightarrow \cdot \quad \text{or} \quad t : T_2 \text{ is a value.}$

As  $t : T_2$  is not a value, this goal can be simplified to:

$t : T_2 \longrightarrow \cdot \quad \cdot$

The induction hypothesis assures us that  $t \longrightarrow \cdot \quad \text{or} \quad t \text{ is a value.}$

- in the first case, the goal holds because  $[] T_2$  is an evaluation context.
- in the second case, because  $t$  is a value and has a universal type,  $t$  must be of the form  $\Lambda X.v$ , so  $(\iota)$  fires, and the goal holds.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Classification of values

We have again appealed to a classification lemma:

### Lemma (Classification)

Assume  $\Delta \vdash v : T$ . Then,

- if  $T$  is an arrow type, then  $v$  is a  $\lambda$ -abstraction;
- if  $T$  is a universal type, then  $v$  is a  $\Lambda$ -abstraction.

This lemma remains simple because we use a syntax-directed presentation of System  $F$ .

In Curry style, a more complex inversion lemma is needed:  
see [SystemFTypeSoundnessComplete](#).

Types

Type safety

Polymorphism

Erasure

System F

Trivia

## 1 Types

## 2 Type safety

## 3 Polymorphism

## 4 Erasure

## 5 System F

## 6 Trivia

## Strong normalization

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Strong reduction allows  $\beta$ -reduction everywhere, including under  $\lambda$  and  $\Lambda$ .

### Theorem (Strong normalization)

*If  $\Gamma \vdash t : T$  then every strong reduction sequence out of  $t$  is finite.*

This result, due to Girard (1972), is more accessibly described in the textbook **Proofs and Types** by Girard, Lafont and Taylor (1990).

The proof uses logical relations (see upcoming lecture by GS).

Through the Curry-Howard isomorphism, System  $F$  is also a logic, known as second-order logic.

$\emptyset \vdash t : T$  means that  $t$  is a proof of the proposition  $T$ .

Strong normalization implies that second-order logic is consistent: there is no proof of  $\forall X.X$ .

## The type inference problem

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Let  $u$  be a (closed) unannotated term.

Provided we decorate it with type abstractions and applications,  
is it well-typed? Can one find a type for it?

## The type inference problem

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Let  $u$  be a (closed) unannotated term.

Provided we decorate it with type abstractions and applications,  
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Can one find  $t$  and  $T$  such that  $\llbracket t \rrbracket = u$  and  $\emptyset \vdash t : T$ ?

# The type inference problem

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Let  $u$  be a (closed) unannotated term.

Provided we decorate it with type abstractions and applications,  
is it well-typed? Can one find a type for it?

Can one find  $t$  and  $T$  such that  $\lceil t \rceil = u$  and  $\emptyset \vdash t : T$ ?

Wells (1999) proves that this problem is undecidable.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## An example

Consider the unannotated term  $\lambda fxy.(f\ x, f\ y)$ .

Here is one way of decorating it:

$$\lambda X_1.\lambda X_2.\lambda f: X_1 \rightarrow X_2.\lambda x: X_1.\lambda y: X_1.(f\ x, f\ y)$$

For readability, we have also annotated every  $\lambda$  binder with its type.

This term admits the polymorphic type:

$$\forall X_1.\forall X_2.(X_1 \rightarrow X_2) \rightarrow X_1 \rightarrow X_1 \rightarrow X_2 \times X_2$$

This is the type that would be [inferred](#) by OCaml.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## An example

This untyped term can also be decorated in a [different way](#):

$$\Lambda X_1. \Lambda X_2. \lambda f: \forall X. X \rightarrow X. \lambda x: X_1. \lambda y: X_2. (f\ X_1\ x, f\ X_2\ y)$$

This term admits the polymorphic type:

$$\forall X_1. \forall X_2. (\forall X. X \rightarrow X) \rightarrow X_1 \rightarrow X_2 \rightarrow X_1 \times X_2$$

This begs a question...

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Incomparable types in System *F*

Is one of these two types “more general” than the other?  
And if so, in what sense?

$$\begin{aligned} \forall X_1. \forall X_2. (X_1 \rightarrow X_2) \rightarrow X_1 \rightarrow X_1 \rightarrow X_2 \times X_2 \\ \forall X_1. \forall X_2. (\forall X. X \rightarrow X) \rightarrow X_1 \rightarrow X_2 \rightarrow X_1 \times X_2 \end{aligned}$$

One requires  $x$  and  $y$  to admit a common type,  
while the other requires  $f$  to be polymorphic.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Incomparable types in System *F*

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One requires  $x$  and  $y$  to admit a common type,  
while the other requires  $f$  to be polymorphic.

Neither can be “more general than” the other,  
for any reasonable definition of the relation “more general than”,  
because each has an inhabitant that does not inhabit the other.

Exercise: find these inhabitants!

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Absence of principal types

I believe that the unannotated term  $\lambda fxy.(f\,x,f\,y)$  does **not** admit a type that is more general than the previous two types.

In other words, System *F* **does not have principal types**.

But, to clarify this statement, I should define when  $T_1$  is **more general** than  $T_2$ , which we usually write  $T_1 \leq T_2$ .

We also say that  $T_2$  is an **instance** of  $T_1$ .

# System *F* in Curry style

This is System *F* with implicit type abstractions and applications:

$$\frac{\text{V}_{\text{AR}}}{\Gamma \vdash x : \Gamma(x)}$$
     $\frac{\text{A}_{\text{BS}}}{\begin{array}{c} \Gamma; x : T_1 \vdash t : T_2 \\ \Gamma \vdash \lambda x. t : T_1 \rightarrow T_2 \end{array}}$     
$$\frac{\text{A}_{\text{PP}}}{\begin{array}{c} \Gamma \vdash t_1 : T_1 \rightarrow T_2 \\ \Gamma \vdash t_2 : T_1 \\ \hline \Gamma \vdash t_1 t_2 : T_2 \end{array}}$$
  
  
$$\frac{\text{T}_{\text{Abs}}}{\begin{array}{c} \Gamma; X \vdash t : T \\ X \# \Gamma \end{array}}$$
    
$$\frac{\text{T}_{\text{App}}}{\begin{array}{c} \Gamma \vdash t : \forall X. T \\ \hline \Gamma \vdash t : T[T'/X] \end{array}}$$

The rules  $\text{T}_{\text{Abs}}$  and  $\text{T}_{\text{App}}$  are not syntax-directed.

## System *F* in Curry style

And here is an equivalent presentation with a subtyping rule **SUB**:

$$\frac{\text{V}_{\text{AR}} \quad \Gamma \vdash x : \Gamma(x)}{\Gamma \vdash x : \Gamma(x)}$$
$$\frac{\text{A}_{\text{BS}} \quad \Gamma; x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x. t : T_1 \rightarrow T_2}$$
$$\frac{\text{A}_{\text{PP}} \quad \begin{array}{c} \Gamma \vdash t_1 : T_1 \rightarrow T_2 \\ \Gamma \vdash t_2 : T_1 \end{array}}{\Gamma \vdash t_1 \ t_2 : T_2}$$
$$\frac{\text{T}_{\text{ABS}} \quad \begin{array}{c} \Gamma; X \vdash t : T \\ X \# \Gamma \end{array}}{\Gamma \vdash t : \forall X. T}$$
$$\frac{\text{S}_{\text{UB}} \quad \begin{array}{c} \Gamma \vdash t : T \\ T \leq T' \end{array}}{\Gamma \vdash t : T'}$$

where  $T \leq T'$  is defined on the next slide.

[Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

## Subtyping in System *F*

Subtyping is defined by

$$\text{INST} \quad \forall X.T \leq T[T'/X]$$

$$\text{GEN} \quad \frac{X \# T}{T \leq \forall X.T}$$

$$\text{TRANSITIVITY} \quad \frac{T_1 \leq T_2 \quad T_2 \leq T_3}{T_1 \leq T_3}$$

Exercise: check that these two presentations of System *F* in Curry style are indeed equivalent.

A richer notion of subtyping: System  $F_\eta$ 

Types

Type safety

Polymorphism

Erasure

System F

Trivia

Mitchell (1988) defines System  $F_\eta$ , a more powerful variant of System  $F$ , based on a richer subtyping relation:

$$\text{INST} \quad \forall X. T \leq T[T'/X]$$

$$\text{GEN} \quad \frac{}{T \leq \forall X. T}$$

$$\text{TRANSITIVITY} \quad \frac{T_1 \leq T_2 \quad T_2 \leq T_3}{T_1 \leq T_3}$$

$$\begin{aligned} \text{DISTRIBUTIVITY} \\ \forall \bar{X}. (T_1 \rightarrow T_2) \\ \leq (\forall \bar{X}. T_1) \rightarrow (\forall \bar{X}. T_2) \end{aligned}$$

$$\text{CONGRUENCE-}\rightarrow \quad \frac{T_2 \leq T_1 \quad T'_1 \leq T'_2}{T_1 \rightarrow T'_1 \leq T_2 \rightarrow T'_2}$$

$$\text{CONGRUENCE-}\vee \quad \frac{T_1 \leq T_2}{\forall X. T_1 \leq \forall X. T_2}$$

Clearly  $\Gamma \vdash_F t : T$  implies  $\Gamma \vdash_{F_\eta} t : T$ .

Conversely  $\Gamma \vdash_{F_\eta} t : T$  implies  $\Gamma \vdash_F t' : T$  for some  $t'$  such that  $t \equiv_\eta t'$ .

Exercise: prove this claim!

Therefore System  $F_\eta$  is the closure of System  $F$  under  $\eta$ -equality.

## Type inference in System $F_\eta$

Types

Type safety

Polymorphism

Erasure

System F

Trivia

One might hope that type inference is easier in System  $F_\eta$  than in System  $F$ .

Type inference in System  $F_\eta$ [Types](#)[Type safety](#)[Polymorphism](#)[Erasure](#)[System F](#)[Trivia](#)

One might hope that type inference is easier in System  $F_\eta$  than in System  $F$ .

Unfortunately [Tiuryn and Urzyczyn \(1995\)](#) prove that in System  $F_\eta$  even just the subtyping problem is undecidable.

This implies that typability in System  $F_\eta$  is undecidable (Wells, 1996).

[Chrzaszcz \(1998\)](#) proves that even without [DISTRIBUTIVITY](#) the subtyping problem of System  $F_\eta$  is undecidable.