# MPRI FUN

# Algebraic data types and existential types

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2025–2026

# Towards data types

Many data types can be built out of sums and products and a form of recursion at the level of types.

Binary sum $+$ and product $\times$, and their neutral elements 0 and 1, suffice.

- The unit type is 1.
- The empty type is 0.
- The Boolean type is $1 + 1$.
- The type $\mathbb{N}$ of the natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
- The type $\mathbb{L}(X)$ of lists of elements of type $X$ must satisfy

$$\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$$

# Three technical approaches to data types

There are three main approaches to extending System $F$ with data types:

- consider 0, 1, $+$, $\times$, and recursive types $\mu X.T$ as primitive concepts and encode all data types in terms of these concepts;
- consider algebraic data types as primitive and view sums, products, naturals, lists, etc., as instances of this general concept;
- introduce no new primitive concept and remark that inductive types can be encoded in System $F$.

In practice, the second approach is the most natural and user-friendly.

All three approaches, and their connections, are worth understanding.

# Binary products

It is easy to add pairs and projections to the (call-by-value) $\lambda$-calculus.

$$
\begin{array}{rcll}
t & ::= & \ldots \mid (t, t) \mid \pi_i\ t & \qquad \text{where } i \in \{1, 2\} \\
v & ::= & \ldots \mid (v, v) & \\
E & ::= & \ldots \mid (E, t) \mid (v, E) \mid \pi_i\ E &
\end{array}
$$

One new reduction rule is needed: $\pi_i\ (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed:

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A new type constructor is needed: $T ::= \ldots \mid T \times T$.

Two new typing rules are needed:

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Two new typing rules are needed:

$$\frac{\Gamma \vdash t_1 : T_1 \qquad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \qquad \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i\ t : T_i}$$

Exercise: extend the proofs of Subject Reduction and Progress.

Variation: introduce the elimination form *let* $(x_1, x_2) = t$ *in* $t$.

## Unit

The unit type 1 can be viewed as a product type of arity 0.

It has an introduction form but no elimination form.

$$\begin{array}{lcl} t & ::= & \ldots \mid () \\ v & ::= & \ldots \mid () \end{array}$$
– no new evaluation context

No new reduction rule is needed.

A new type constructor is needed: $T ::= \ldots \mid 1$.

One new typing rule is needed:

$$\Gamma \vdash () : 1$$

Variation: introduce the elimination form $let\ () = t\ in\ t$.

# Binary sums

Let us add injections and a case analysis to (call-by-value) $\lambda$-calculus.

$$\begin{array}{rcll} t & ::= & \ldots \mid \mathit{inj}_i\ t \mid \mathit{case}\ t\ \mathit{of}\ t_1 \parallel t_2 & \qquad \text{where } i \in \{1,2\} \\ v & ::= & \ldots \mid \mathit{inj}_i\ v & \\ E & ::= & \ldots \mid \mathit{inj}_i\ E \mid \mathit{case}\ E\ \mathit{of}\ t_1 \parallel t_2 & \end{array}$$

One new reduction rule is needed: $\mathit{case}\ \mathit{inj}_i\ v\ \mathit{of}\ t_1 \parallel t_2 \longrightarrow t_i\ v$.

In a *case* construct, the branches $t_1$ and $t_2$ should be functions.

A new type constructor is needed:

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A new type constructor is needed: $T ::= \ldots \mid T + T$.

Two new typing rules are needed:

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A new type constructor is needed: $T ::= \ldots \mid T + T$.

Two new typing rules are needed:

$$\frac{\Gamma \vdash t : T_i}{\Gamma \vdash inj_i\ t : T_1 + T_2} \qquad \frac{\begin{array}{c}\Gamma \vdash t : T_1 + T_2 \\ \Gamma \vdash t_1 : T_1 \rightarrow T' \qquad \Gamma \vdash t_2 : T_2 \rightarrow T'\end{array}}{\Gamma \vdash case\ t\ of\ t_1 \parallel t_2 : T'}$$

Exercise: extend the proofs of Subject Reduction and Progress.

## Void

The empty type can be viewed as a sum type of arity 0.

It has an elimination form but no introduction form.

$$\begin{array}{rcl} t & ::= & \ldots \mid \mathit{absurd}\ t \\ & & \text{– no new value} \\ E & ::= & \ldots \mid \mathit{absurd}\ E \end{array}$$

No new reduction rule is needed. *absurd v* is stuck.

A new type constructor is needed: $T ::= \ldots \mid 0$.

One new typing rule is needed:

$$\frac{\Gamma \vdash t : 0}{\Gamma \vdash \mathit{absurd}\ t : T'}$$

Exercise: extend the proof of Progress.

# Approaches to recursive types

Recall what was said earlier about recursive types:

- Natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
  *A natural number is either zero*
  *or the successor of a natural number.*
- Lists must satisfy $\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$.
  *A list is either the empty list*
  *or a pair of an element and a list.*

The types $\mathbb{N}$ and $\mathbb{L}(X)$ appear to satisfy recursive equations.

What is $\simeq$? How can the types $\mathbb{N}$ and $\mathbb{L}(X)$ be defined?

# Approaches to recursive types

Several answers are possible.

1. Equi-recursive types. Interpret $\simeq$ as equality. A type is a possibly infinite tree. The notation $\mu X.T$ describes such a tree.
2. Structural iso-recursive types. Interpret $\simeq$ as isomorphism. A type is a finite tree. The syntax of types is extended with a general form of recursive type, $\mu X.T$.
3. Nominal iso-recursive types. Interpret $\simeq$ as isomorphism. A type is a finite tree. The syntax of types is extended with user-defined types such as $\mathbb{N}$, $\mathbb{L}(X)$, or (more generally) algebraic data types.

# Approach 1: equi-recursive types

Suppose we want $\mathbb{N} = 1 + \mathbb{N}$ and $\mathbb{L}(X) = 1 + X \times \mathbb{L}(X)$.

Then, a type must be a possibly infinite tree.

```
CoInductive ty :=
| TyVar (x : var)
| TyFun (A B : ty).
```

Here is an example of an infinite tree:

```
CoFixpoint arrows :=
  TyFun arrows arrows.
```

On paper, this type is usually written $\mu X.X \rightarrow X$.

$\mu$ is not a constructor in the syntax of types.

The equality $\textit{arrows} = \textit{arrows} \rightarrow \textit{arrows}$ is true.

In Coq, a suitable notion of extensional equality of types must be co-inductively defined.

# Approach 1: equi-recursive types

In this approach, assuming we have sum and product types,

- $\mathbb{N}$ can be defined as a notation for $\mu X. 1 + X$,
- $\mathbb{L}(X)$ can be defined as a notation for $\mu Y.1 + X \times Y$.

In this approach,

- $inj_1$ () has type $\mathbb{N}$, and also has type $\mathbb{L}(\mathbb{N})$.

This works in theory, but is not very pleasant in practice.

# Approach 1: equi-recursive types

In this approach, only the nature of types changes, from finite trees to possibly infinite trees.

The typing rules of the simply-typed $\lambda$-calculus, or of System *F*, are unchanged.

The proof of type soundness is unchanged.

Exercise: on paper or in Coq, extend the simply-typed $\lambda$-calculus with equi-recursive types, and update the proof of type soundness, where needed. Prove that every (pure, closed) $\lambda$-term has type $\mu X. X \rightarrow X$.

# Approach 1: equi-recursive types

In this approach, many nonsensical terms become well-typed.

```
ocaml -rectypes
# let f x = [x] :: x;;
val f : (('a list as 'b) list as 'a) -> 'b list = <fun>
```

OCaml infers that `f` has type $A \rightarrow \mathbb{L}(B)$
where $\mathbb{L}(B) = A$ and $\mathbb{L}(A) = B$.

This type is in fact equal to *lists* $\rightarrow$ *lists*,
where $\mathit{lists} = \mu X.\mathbb{L}(X) = \mathbb{L}(\mathit{lists}) = \mathbb{L}(\mathbb{L}(\ldots))$.

```
# type lists = ('a list as 'a);;
type lists = 'a list as 'a
# let f (x : lists) : lists = [x] :: x;;
val f : lists -> lists = <fun>
```

This downside explains why this approach is not used in practice.

# Approach 2: structural iso-recursive types

Suppose we want types to remain finite trees.

We extend the syntax of types: $T ::= \ldots \mid \mu X.T$.

We extend the syntax of terms with introduction and elimination forms:

$$\begin{array}{rcl} t & ::= & \ldots \mid \mathit{fold}_{\mu X.T}\ t \mid \mathit{unfold}_{\mu X.T}\ t \\ v & ::= & \ldots \mid \mathit{fold}_{\mu X.T}\ v \\ E & ::= & \ldots \mid \mathit{fold}_{\mu X.T}\ E \mid \mathit{unfold}_{\mu X.T}\ E \end{array}$$

Their operational semantics is simple:

$$\mathit{unfold}_{\mu X.T}\ (\mathit{fold}_{\mu X.T}\ v) \quad \longrightarrow \quad v$$

Two new typing rules are introduced:

$$\frac{\Gamma \vdash t : T[\mu X.T/X]}{\Gamma \vdash \mathit{fold}_{\mu X.T}\ t : \mu X.T} \qquad \frac{\Gamma \vdash t : \mu X.T}{\Gamma \vdash \mathit{unfold}_{\mu X.T}\ t : T[\mu X.T/X]}$$

# Approach 2: structural iso-recursive types

$\mathit{fold}_{\mu X.T}$ and $\mathit{unfold}_{\mu X.T}$ are coercions between the types $\mu X.T$ and $T[\mu X.T/X]$.
They are mutual inverses.

These types are said to be isomorphic:

$$\mu X.T \simeq T[\mu X.T/X]$$

Exercise: on paper or in Coq, extend the simply-typed $\lambda$-calculus with iso-recursive types. Update the proof of type soundness where needed.

# Approach 2: structural iso-recursive types

In this approach, as in the previous approach,

- $\mathbb{N}$ can be defined as a notation for $\mu X.1 + X$,
- $\mathbb{L}(X)$ can be defined as a notation for $\mu Y.1 + X \times Y$.

In this approach,

- $inj_1\ ()$ has type $1 + \mathbb{N}$, and also has type $1 + \mathbb{N} \times \mathbb{L}(\mathbb{N})$,
- $fold_{\mathbb{N}}\ (inj_1\ ())$ has type $\mathbb{N}$.
- $fold_{\mathbb{L}(\mathbb{N})}\ (inj_1\ ())$ has type $\mathbb{L}(\mathbb{N})$.

This works in theory, but is not very pleasant in practice.

# Approach 3: nominal iso-recursive types

Let us view $\mathbb{N}$ as a primitive type: $T ::= \ldots \mid \mathbb{N}$.

Give new typing rules—two introduction rules and an elimination rule:

$$\frac{\Gamma \vdash t : 1}{\Gamma \vdash inj_1\ t : \mathbb{N}} \qquad \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash inj_2\ t : \mathbb{N}} \qquad \frac{\begin{array}{c}\Gamma \vdash t : \mathbb{N} \\ \Gamma \vdash t_1 : 1 \rightarrow T' \qquad \Gamma \vdash t_2 : \mathbb{N} \rightarrow T'\end{array}}{\Gamma \vdash case\ t\ of\ t_1 \parallel t_2 : T'}$$

These are exactly the typing rules proposed earlier for binary sums where we have replaced $T_1 + T_2$ with $\mathbb{N}$, $T_1$ with 1, and $T_2$ with $\mathbb{N}$.

We have $\mathbb{N} \simeq 1 + \mathbb{N}$: one can write $in : 1 + \mathbb{N} \rightarrow \mathbb{N}$ and $out : \mathbb{N} \rightarrow 1 + \mathbb{N}$ such that $in \cdot out \equiv_{\beta\eta} out \cdot in \equiv_{\beta\eta} id$. This is an iso-recursive approach.

In this approach, there is no $\mu$ syntax or $\mu$ notation.
$\mathbb{N}$ is viewed as the name of a basic type.

$\mathbb{N}$ is an abstract type with construction and deconstruction operations.

# Approach 3: nominal iso-recursive types

Let us view $\mathbb{L}(X)$ as a primitive type constructor: $T ::= \ldots \mid \mathbb{L}(T)$.

Give new typing rules—two introduction rules and an elimination rule:

$$\frac{\Gamma \vdash t : 1}{\Gamma \vdash \mathit{inj}_1\ t : \mathbb{L}(T)} \qquad \frac{\Gamma \vdash t : T \times \mathbb{L}(T)}{\Gamma \vdash \mathit{inj}_2\ t : \mathbb{L}(T)}$$

$$\frac{\begin{array}{c}\Gamma \vdash t : \mathbb{L}(T) \\ \Gamma \vdash t_1 : 1 \to T' \qquad \Gamma \vdash t_2 : T \times \mathbb{L}(T) \to T'\end{array}}{\Gamma \vdash \mathit{case}\ t\ \mathit{of}\ t_1 \parallel t_2 : T'}$$

These are again exactly the typing rules of binary sums where we have replaced $T_1 + T_2$ with $\mathbb{L}(X)$, $T_1$ with 1, and $T_2$ with $X \times \mathbb{L}(X)$.

We have $\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$.

$\mathbb{L}$ is viewed as the name of a basic type constructor.

$\mathbb{L}(X)$ is an abstract type with construction and deconstruction operations.

# Algebraic data types

Instead of offering a fixed set of primitive types such as $\mathbb{N}$ and $\mathbb{L}(X)$, let users define whatever custom types they need using sums and products (of arbitrary arity) and recursion.

This idea gives rise to algebraic data types.

```
type     nat = Zero | Succ of nat
type 'a list = Nil  | Cons of 'a * 'a list
type 'a tree = Leaf | Node of 'a tree * 'a * 'a tree
```

# Algebraic data types

It is now easy to construct data:

```
let one : nat = Succ Zero
```

and to deconstruct data:

```
let predecessor (n : nat) : nat =
  match n with
  | Zero -> Zero
  | Succ n -> n
```

OCaml also offers a more concise function definition form:

```
let predecessor :  nat -> nat =
  function Zero -> Zero | Succ n -> n
```

# Algebraic data types

Pattern matching allows deconstructing data in depth.

This is an implementation of rotations of binary trees in Standard ML:

```
fun n (v, l, r) =
  T(v, 1 + size l + size r, l, r)
fun single_L (a, x, T(b, _, y, z)) =
  n(b, n(a, x, y), z)
fun double_L (a, x, T(c, _, T(b, _, y1, y2), z)) =
  n(b, n(a, x, y1), n(c, y2, z))
```

It is concise!

That said, it is not perfect. Adopting the convention `(l, v, r)` would make it much easier to read and debug.

Adams,
Efficient sets—a balancing act, 1993.

# Algebraic data types

Named types, named data constructors, and pattern matching make algebraic data types extremely pleasant and safe to use.

> be instantiated to any type. We suspect that a great many errors are caused by the complications introduced when encoding data in terms of the commonly-supplied low-level types; the provision of a simple and powerful facility for defining types should greatly simplify the programmer's task.

Burstall, MacQueen, Sannella,
HOPE: An experimental applicative language, 1980.

# Products and sums as algebraic data types

Sums and products can be viewed as algebraic data types.

```
type ('a, 'b) sum  = Left of 'a | Right of 'b
type void = |                          (* zero constructors *)
type ('a, 'b) pair = Pair of 'a * 'b
type unit = ()
```

Deconstructing the type `void` works as expected:

```
let absurd (type a) (x : void) : a =
  match x with _ -> .                        (* zero branches *)
```

# An isomorphism

The types $\mathbb{N}$ and $1 + \mathbb{N}$ are not equal, but they are isomorphic.

```
let in_ : (unit, nat) sum -> nat =
  function Left () -> Zero | Right n -> Succ n
let out : nat -> (unit, nat) sum =
  function Zero -> Left () | Succ n -> Right n
```

Algebraic data types are a form of nominal iso-recursive types.

# An alternative syntax

In the usual syntax, the type of lists is declared as follows:

```
type 'a list =
  | Nil
  | Cons of 'a * 'a list
```

In the alternative syntax, the type of each data constructor is given:

```
type _ list =
  | Nil  :                    'a list
  | Cons : 'a * 'a list -> 'a list
```

The result type of each constructor is `'a list`.

Each constructor is polymorphic in `'a`. This is implicit.

# Analogy with inductive types

Coq has inductive types, which seem similar to algebraic data types.

It offers similar syntaxes:

```
Inductive list (A : Type) : Type :=
| Nil
| Cons (x : A) (xs : list A).

Inductive list (A : Type) : Type :=
| Nil  :                    list A
| Cons : A -> list A -> list A.
```

However, inductive types must be strictly positive.

# Analogy with inductive types

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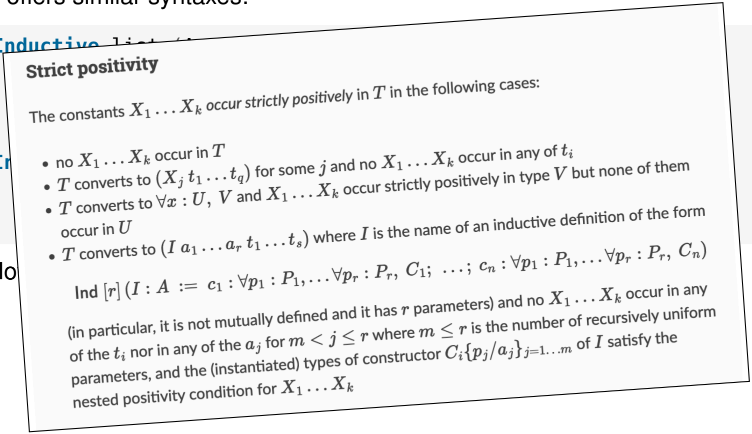
```
Inductive list (
|
|

In
|
|
```

Ho



**Strict positivity**

The constants $X_1 \dots X_k$ *occur strictly positively* in $T$ in the following cases:

- no $X_1 \dots X_k$ occur in $T$
- $T$ converts to $(X_j\, t_1 \dots t_q)$ for some $j$ and no $X_1 \dots X_k$ occur in any of $t_i$
- $T$ converts to $\forall x : U,\ V$ and $X_1 \dots X_k$ occur strictly positively in type $V$ but none of them occur in $U$
- $T$ converts to $(I\, a_1 \dots a_r\, t_1 \dots t_s)$ where $I$ is the name of an inductive definition of the form

$$\textsf{Ind}\ [r]\ (I : A\ :=\ c_1 : \forall p_1 : P_1, \dots \forall p_r : P_r,\ C_1;\ \dots;\ c_n : \forall p_1 : P_1, \dots \forall p_r : P_r,\ C_n)$$

(in particular, it is not mutually defined and it has $r$ parameters) and no $X_1 \dots X_k$ occur in any of the $t_i$ nor in any of the $a_j$ for $m < j \leq r$ where $m \leq r$ is the number of recursively uniform parameters, and the (instantiated) types of constructor $C_i\{p_j/a_j\}_{j=1\dots m}$ of $I$ satisfy the nested positivity condition for $X_1 \dots X_k$

# Analogy with inductive types

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Inductive list (A : Type) : Type :=
| Nil  :                list A
| Cons : A -> list A -> list A.
```

However, inductive types must be strictly positive. In each argument of each constructor, `list` must not appear in the left of an arrow.

# Algebraic data types are recursive types

Algebraic data types are unrestricted: they are true recursive types.

This breaks strong normalization.

```
type term =
  T of (term -> term) (* not strictly positive! *)

let app (t : term) (u : term) : term =
  match t with T t -> t u

let delta : term =
  T (fun x -> app x x)

let omega : term =
  app delta delta     (* diverges! *)
```

`app delta delta` reduces to itself in one step.

# Algebraic data types are recursive types

In Haskell, $\mu$ itself can be defined as an algebraic data type:

```
data Fix f =        -- the algebraic data type Fix f
  Fix (f (Fix f))   -- has one constructor, also named Fix
```

The parameter `f` has kind $\star \to \star$. It is itself a parameterized type.

If a non-recursive type of lists is defined as follows,

```
data ListF a self = Nil | Cons a self
```

then `Fix (ListF a)` is a recursive type of lists.

# Encoding Booleans

The Boolean type $\mathbb{B} \simeq 1 + 1$ can be declared as an algebraic data type:

```
type bool = False | True
```

However, Booleans can also be encoded in pure $\lambda$-calculus.

A Boolean value is an “object with a *case* method”.
It can choose between two branches:

$$\begin{array}{rcl} \mathbb{B} & \triangleq & \forall X.\,(1 \to X) \to (1 \to X) \to X \\ \mathit{False} & \triangleq & \lambda x_1.\,\lambda x_2.\,x_1\ () \\ \mathit{True} & \triangleq & \lambda x_1.\,\lambda x_2.\,x_2\ () \\ \mathit{case}\ t\ \mathit{of}\ t_1 \parallel t_2 & \triangleq & t\ t_1\ t_2 \end{array}$$

This is a Scott encoding, and also a Church encoding.

Exercise: reconstruct the omitted type abstractions and applications.

# Encoding sums

More generally, the binary sum type $T_1 + T_2$ can be encoded as follows:

$$\begin{array}{rcl} T_1 + T_2 & \triangleq & \forall X.\,(T_1 \rightarrow X) \rightarrow (T_2 \rightarrow X) \rightarrow X \\ \mathit{inj}_1\ x & \triangleq & \lambda x_1.\lambda x_2.x_1\ x \\ \mathit{inj}_2\ x & \triangleq & \lambda x_1.\lambda x_2.x_2\ x \\ \mathit{case}\ t\ \mathit{of}\ t_1 \parallel t_2 & \triangleq & t\ t_1\ t_2 \end{array}$$

The zero-ary sum type 0 can be encoded, too!

$$\begin{array}{rcl} 0 & \triangleq & \forall X.\,X \\ \mathit{absurd}\ t & \triangleq & t \end{array}$$

Clearly this works for any number of branches.

# Encoding products

The binary product type $T_1 \times T_2$ can be encoded as follows:

$$\begin{array}{rcl} T_1 \times T_2 & \triangleq & \forall X.\, (T_1 \to T_2 \to X) \to X \\ (x_1, x_2) & \triangleq & \lambda k.\, k\ x_1\ x_2 \\ \pi_1\ t & \triangleq & t\ (\lambda x_1.\, \lambda x_2.\, x_1) \\ \pi_2\ t & \triangleq & t\ (\lambda x_1.\, \lambda x_2.\, x_2) \end{array}$$

The zero-ary product type 1 can be encoded, too!

$$\begin{array}{rcl} 1 & \triangleq & \forall X.\, X \to X \\ () & \triangleq & \lambda x.x \end{array}$$

Clearly this works for any number of tuple components.

# Encoding natural integers

Can we encode the recursive type $\mathbb{N} \simeq 1 + \mathbb{N}$ in the same way, à la Scott?

$$\mathbb{N} \quad \triangleq \quad \forall X. (1 \to X) \to (\mathbb{N} \to X) \to X$$

This doesn't work in System *F*, which doesn't have recursive types.

Here, the Scott and Church encodings differ.

The Church encoding views a number as "an object with a *fold* method".

$$\begin{array}{rcl} \mathbb{N} & \triangleq & \forall X. X \to (X \to X) \to X \\ \mathit{Zero} & \triangleq & \lambda z. \lambda s. z \\ \mathit{Succ}\, x & \triangleq & \lambda z. \lambda s. s\ (x\ z\ s) \end{array}$$

# Encoding lists

The Church encoding views a list as “an object with a *fold* method”.

$$\begin{array}{rcl}
\mathbb{L}(Y) & \triangleq & \forall X.\, X \rightarrow (Y \rightarrow X \rightarrow X) \rightarrow X \\
[] & \triangleq & \lambda n.\, \lambda c.\, n \\
x :: xs & \triangleq & \lambda n.\, \lambda c.\, c\; x\; (xs\; n\; c)
\end{array}$$

The Church encoding works for all inductive types.

Girard, Taylor, Lafont, Proofs and types, 1990, §11.3–11.5.

# Motivation

Complex numbers are an abstract concept.

Outside of their implementation, how they are represented should be irrelevant, and one should not depend on implementation details.

> *In one section, Professor Descartes announced that a complex number was an ordered pair of reals [...].*
>
> *In the other section, Professor Bessel announced that a complex number was an ordered pair of reals, the first of which was nonnegative [...].*
>
> *An unfortunate mistake [...] caused the two sections to be interchanged.*

Reynolds, Types, Abstraction and Parametric Polymorphism, 1983.

# Complex numbers as an abstract type

In OCaml, one might implement complex numbers as an abstract type:

```
module Complex : sig
  type t
  val zero: t
  val one: t
  val add: t -> t -> t
  val mul: t -> t -> t
  val (=): t -> t -> bool
  (* etc. *)
end
```

# Complex numbers as an existential type

In System $F$, this idea can be made precise via an existential type:

$$Complex : \exists X. \left\{ \begin{array}{r} zero : X \\ add : X \to X \to X \\ mul : X \to X \to X \\ eq : X \to X \to bool \\ etc. \end{array} \right\}$$

Mitchell and Plotkin, Abstract types have existential type, 1988.

Rossberg, Russo, Dreyer, F-ing Modules, 2014.

# Streams as an existential type

Imagine we wish to define an abstract type of streams.

A stream is a producer of a sequence of elements,
out of which a consumer can pull elements on demand.

It is an "object" with a single method, *next*.

- a stream has a certain current internal state.
- *next* returns either nothing or a pair of an element and a new state.

A stream is analogous to a Java iterator, except it is not mutable.
Its current state is explicit.

$$\mathit{Stream}(X) \simeq \exists S.\ \ (\underbrace{S \to 1 + X \times S}_{\mathit{next}}) \times \underbrace{S}_{\mathit{cur}}$$

## Streams as an existential type

How do we translate this equation in OCaml?

$$Stream(X) \;\simeq\; \exists S.\quad (S \to 1 + X \times S) \times S$$

We first define the sum type $1 + X \times S$ as an algebraic data type:

$$Step\ X\ S \simeq 1 + X \times S$$

so the equation becomes:

$$Stream(X) \;\simeq\; \exists S.\quad (S \to Step\ X\ S) \times S$$

Then we define this existential type as an algebraic data type with one data constructor whose type is

$$\forall S.\quad (S \to Step\ X\ S) \times S \to Stream(X)$$

# Streams as an existential type

`('a, 's) step` corresponds to $\mathit{Step}\ X\ S$ and is isomorphic to $1 + X \times S$:

```
type ('a, 's) step =
  | Done                        (* the stream is exhausted *)
  | Yield of 'a * 's    (* here is an element and a new state *)
```

An existential type can be defined as an algebraic data type:

```
type 'a stream =
  | Stream:
              (* The [next] method: *) ('s -> ('a, 's) step) *
              (* The current state: *)  's
          (* together form a stream: *) -> 'a stream
```

The data constructor `Stream` has universal type: it is polymorphic in `'s`.

The producer chooses the type of the internal state;
the consumer must treat this type as abstract.

# Converting a list to a stream

This conversion function is a nonrecursive producer:

```
let stream (xs : 'a list) : 'a stream =
  let next xs =
    match xs with
    |       [] -> Done
    | x :: xs -> Yield (x, xs)
  in
  Stream (next, xs)            (* packing an existential type *)
```

On the last line, what is the concrete type of states?

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On the last line, what is the concrete type of states?

It is `'a list`.

# Converting a stream to a list

This conversion function is a recursive consumer:

```
let unstream (Stream (next, s) : 'a stream) : 'a list =
  let rec unfold s =
    match next s with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
  in
  unfold s
```

The first line uses pattern matching to unpack an existential type.

What is the type of `unfold`?

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The first line uses pattern matching to unpack an existential type.

What is the type of `unfold`?

It is `s -> 'a list`
where `s` is an abstract type introduced by unpacking at line 1.

# Examples of stream producers

How would you implement a singleton stream?

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```
let return (x : 'a) : 'a stream =
  let next s =
    if s then Yield (x, false) else Done
  in
  Stream (next, true)          (* packing an existential type *)
```

On the last line, the concrete type of states is **`bool`**:
either we have already yielded an element, or we have not.

Exercise: Write `interval` of type `int -> int -> int stream`.

Exercise: Write `append` of type `'a stream -> 'a stream -> 'a stream`.

# An example consumer-and-producer

The `map` function on streams is also non-recursive:

```
let map (f : 'a -> 'b) (xs : 'a stream) : 'b stream =
  let Stream (next, s) = xs in                      (* unpacking *)
  let next s =
    match next s with
    | Done          -> Done
    | Yield (x, s) -> Yield (f x, s)
  in
  Stream (next, s)                                    (* packing *)
```

# Existential types enforce abstraction

When a stream is unpacked, a fresh unknown type `'s` is introduced.

Unpacking two distinct streams gives rise to two distinct types:

```
let wrong (xs1 : 'a stream) (xs2 : 'a stream) =
  match xs1, xs2 with
  | Stream (next1, s1), Stream (next2, s2) ->
      next1 s2
```

```
Error: This expression has type $Stream_'s1
       but an expression was expected of type $Stream_'s
```

Fortunately, the "next" function of stream 1
cannot be applied to the internal state of stream 2.

## Streams as an existential type

This encoding of streams is used in practice.

In addition to `Done` and `Yield`, a third constructor `Skip` can be used, meaning “please ask again”.

A consumer must ask, ask, ask until a non-`Skip` result is produced.

This allows most stream producers to be nonrecursive functions.
This makes optimization easier.

Coutts, Leshchinskiy, Stewart, Stream fusion: from lists to streams to nothing at all, 2007.

# Thoughts about encodings

The Church encoding of lists encodes

- (finite) lists as a universal type,
- a producer object with a *fold* method;
- the producer is in control and "pushes" data towards the consumer.

The streams that I have just presented encodes

- (possibly infinite) lists as an existential type,
- a producer object with a *next* method;
- the consumer is in control and "pulls" data from the producer.

Neither encoding uses a recursive type.

Both involve procedural abstraction,
that is, exploiting the function type as an abstract type.

# System $F$ with existential types

The syntax of types is extended with existential types:

$$T ::= \ldots \mid \exists X.\ T$$

The syntax of terms is extended with introduction and elimination forms:

$$\begin{array}{rl} t ::= \ldots \mid \mathit{pack}\ T, t\ \ \mathit{as}\ \exists X.T \mid \mathit{let}\ X, x = \mathit{unpack}\ t\ \mathit{in}\ t \\ v ::= \ldots \mid \mathit{pack}\ T, v\ \ \mathit{as}\ \exists X.T \\ E ::= \ldots \mid \mathit{pack}\ T, E\ \ \mathit{as}\ \exists X.T \mid \mathit{let}\ X, x = \mathit{unpack}\ E\ \mathit{in}\ t \end{array}$$

A new reduction rule is introduced:

$$\begin{array}{r} \mathit{let}\ X, x = \mathit{unpack}\ (\mathit{pack}\ T', v\ \mathit{as}\ \exists X.T)\ \mathit{in}\ t \quad \longrightarrow \\ t[v/x][T'/X] \end{array}$$

Note: "*unpack t*" is not a term. Only "*let* ... *unpack* ... *in* ..." is a term.

# System $F$ with existential types

Two new typing rules are introduced:

$$\exists\text{-INTRO}\quad \frac{\Gamma \vdash t :}{\Gamma \vdash pack\ T', t\ as\ \exists X.T : \exists X.T}$$

$$\exists\text{-ELIM}\quad \frac{\Gamma \vdash t_1 :}{\Gamma \vdash let\ X, x = unpack\ t_1\ in\ t_2 : T_2}$$

For reference, recall the typing rules for universal types:

$$\forall\text{-INTRO}\quad \frac{\Gamma; X \vdash t : T}{\Gamma \vdash \Lambda X.t : \forall X.T}$$

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Exercise: extend the proofs of Subject Reduction and Progress.

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Exercise: extend the proofs of Subject Reduction and Progress.

# Universal/existential duality

When a value has universal type $\forall X.T$,
the producer of this value must treat $X$ as abstract
and the consumer can choose a type $T'$ with which to instantiate $X$.

When a value has existential type $\exists X.T$,
the producer chooses a type $T'$ with which to instantiate $X$
but the consumer must treat $X$ as abstract.

When a value has existential type, its consumer must be polymorphic.

# Church encoding of existential types

Existential types can in fact be encoded in terms of universal types:

$$\exists X.T \quad \triangleq \quad \forall Y.\, (\forall X.\, T \to Y) \to Y$$

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As the wizard was studying the black box, suddenly the box spoke:

> *I hold a $T$, but I cannot give it to you,*
> *because I cannot reveal $X$.*

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*I hold a $T$, but I cannot give it to you,*
*because I cannot reveal $X$.*

*What do you want to use it for?*

*Tell me how you wish to transform a $T$ into a $Y$,*
*in a way that works for every $X$.*
*Then I will give you a $Y$.*

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\mathit{let}\ X, x = \mathit{unpack}\ t_1\ \mathit{in}\ t_2 : T_2 & \triangleq & t_1\ T_2\ (\Lambda X.\,\lambda x : T \to T_2.\,t_2)
\end{array}
$$

This encoding validates the logical implication $\exists X.T \to \neg\forall X.\neg T$ where $\neg T$ is defined as $T \to 0$.

Exercise: check that this encoding validates the reduction rule and the typing rules proposed earlier for primitive existential types.