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MPRI 2.4

Algebraic data types, existential types, and GADTs

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Towards data types

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Many data types can be built out of **sums** and **products** and a form of **recursion** at the level of types.

Binary sum $+$ and product \times , and their **neutral elements** 0 and 1, suffice.

- The **unit** type is 1.
- The **empty** type is 0.
- The **Boolean** type is $1 + 1$.
- The type \mathbb{N} of the natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
- The type $\mathbb{L}(X)$ of lists of elements of type X must satisfy

$$\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$$

Three technical approaches to data types

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There are three main approaches to extending System F with data types:

- consider 0 , 1 , $+$, \times , and recursive types $\mu X.T$ as **primitive concepts** and encode all data types in terms of these concepts;
- consider **algebraic data types** as primitive and view sums, products, naturals, lists, etc., as instances of this general concept;
- introduce **no new primitive concept** and remark that **inductive types** can be encoded in System F .

In practice, the second approach is the most natural and user-friendly.

All three approaches, and their connections, are worth understanding.

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It is easy to add **pairs** and **projections** to the (call-by-value) λ -calculus.

$$\begin{aligned} t &::= \dots \mid (t, t) \mid \pi_i t && \text{where } i \in \{1, 2\} \\ v &::= \dots \mid (v, v) \\ E &::= \dots \mid (E, t) \mid (v, E) \mid \pi_i E \end{aligned}$$

One new reduction rule is needed: $\pi_i (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed: $T ::= \dots \mid T \times T$.

Two new typing rules are needed:

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \qquad \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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The **unit** type 1 can be viewed as a product type of arity 0.

It has an **introduction** form but no **elimination** form.

$$\begin{aligned} t &::= \dots | () \\ v &::= \dots | () \\ &\text{-- no new evaluation context} \end{aligned}$$

No new reduction rule is needed.

A new type constructor is needed: $T ::= \dots | 1$.

One new typing rule is needed:

$$\Gamma \vdash () : 1$$

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Let us add **injections** and a **case analysis** to (call-by-value) λ -calculus.

$$\begin{aligned} t &::= \dots \mid inj_i t \mid case\ t\ of\ t_1 \parallel t_2 && \text{where } i \in \{1, 2\} \\ v &::= \dots \mid inj_i v \\ E &::= \dots \mid inj_i E \mid case\ E\ of\ t_1 \parallel t_2 \end{aligned}$$

One new reduction rule is needed: $case\ inj_i\ v\ of\ t_1 \parallel t_2 \longrightarrow t_i\ v$.

In a *case* construct, the branches t_1 and t_2 should be functions.

A new type constructor is needed: $T ::= \dots \mid T + T$.

Two new typing rules are needed:

$$\frac{\Gamma \vdash t_i : T_i}{\Gamma \vdash inj_i\ t : T_1 + T_2} \qquad \frac{\Gamma \vdash t : T_1 + T_2 \quad \Gamma \vdash t_1 : T_1 \rightarrow T' \quad \Gamma \vdash t_2 : T_2 \rightarrow T'}{\Gamma \vdash case\ t\ of\ t_1 \parallel t_2 : T'}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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The **empty** type can be viewed as a sum type of arity 0.

It has an **elimination** form but no **introduction** form.

$$t ::= \dots \mid \text{absurd } t$$

– no new value

$$E ::= \dots \mid \text{absurd } E$$

No new reduction rule is needed. *absurd* v is stuck.

A new type constructor is needed: $T ::= \dots \mid 0$.

One new typing rule is needed:

$$\frac{\Gamma \vdash t : 0}{\Gamma \vdash \text{absurd } t : T'}$$

Exercise: extend the proof of Progress.

Approaches to recursive types

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Recall what was said earlier about **recursive types**:

- Natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
- Lists must satisfy $\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$.

One approach is to extend the type system with **recursive types** $\mu X. T$. The type $\mu X. T$ and its unfolding $T[\mu X. T / X]$ must then be considered either **equal** or **related via explicit coercions**.

One can then define \mathbb{N} as $\mu X. 1 + X$ and $\mathbb{L}(X)$ as $\mu Y. 1 + X \times Y$.

A more pleasant approach is to just view \mathbb{N} and $\mathbb{L}(X)$ as **primitive types**. This is the topic of the next slides, and leads to **algebraic data types**.

\mathbb{N} as a primitive type

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Consider λ -calculus with **injections** and **case analysis**.

Let us use $inj_1 ()$ to encode zero and $inj_2 v$ to encode the successor of v .

Introduce a new type constructor: $T ::= \dots \mid \mathbb{N}$.

Give three new typing rules:

$$\frac{\Gamma \vdash t_1 : 1}{\Gamma \vdash inj_1 t : \mathbb{N}} \quad \frac{\Gamma \vdash t_2 : \mathbb{N}}{\Gamma \vdash inj_2 t : \mathbb{N}} \quad \frac{\Gamma \vdash t : \mathbb{N} \quad \Gamma \vdash t_1 : 1 \rightarrow T' \quad \Gamma \vdash t_2 : \mathbb{N} \rightarrow T'}{\Gamma \vdash \text{case } t \text{ of } t_1 \parallel t_2 : T'}$$

These are **exactly the typing rules proposed earlier for binary sums** where we have replaced $T_1 + T_2$ with \mathbb{N} , T_1 with 1 , and T_2 with \mathbb{N} .

The types \mathbb{N} and $1 + \mathbb{N}$ are not equal, but they are **isomorphic**: one can write $in : 1 + \mathbb{N} \rightarrow \mathbb{N}$ and $out : \mathbb{N} \rightarrow 1 + \mathbb{N}$ such that $in \cdot out$ and $out \cdot in$ are $\beta\eta$ -equal to the identity.

$\mathbb{L}(X)$ as a primitive type

Consider again λ -calculus with **injections** and **case analysis**.

Let us use $inj_1 ()$ to encode $[]$ and $inj_2 (v_1, v_2)$ to encode $v_1 :: v_2$.

Introduce a new type constructor: $T ::= \dots \mid \mathbb{L}(T)$.

Give three new typing rules:

$$\frac{\Gamma \vdash t_1 : 1}{\Gamma \vdash inj_1 t : \mathbb{L}(T)} \qquad \frac{\Gamma \vdash t_2 : T \times \mathbb{L}(T)}{\Gamma \vdash inj_2 t : \mathbb{L}(T)}$$

$$\frac{\Gamma \vdash t : \mathbb{L}(T) \quad \Gamma \vdash t_1 : 1 \rightarrow T' \quad \Gamma \vdash t_2 : T \times \mathbb{L}(T) \rightarrow T'}{\Gamma \vdash \text{case } t \text{ of } t_1 \parallel t_2 : T'}$$

These are again **exactly the typing rules of binary sums** where we have replaced $T_1 + T_2$ with $\mathbb{L}(X)$, T_1 with 1 , and T_2 with $X \times \mathbb{L}(X)$.

Again, the types $\mathbb{L}(X)$ and $1 + X \times \mathbb{L}(X)$ are isomorphic.

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Instead of offering a fixed set of primitive types such as \mathbb{N} and $\mathbb{L}(X)$, better **let the user define whatever custom types they need** using sums and products (of arbitrary arity) and recursion.

This idea gives rise to **algebraic data types**.

```
type      nat = Zero | Succ of nat
type 'a list = Nil   | Cons of 'a * 'a list
type 'a tree = Leaf  | Node of 'a tree * 'a * 'a tree
```

Named types, named data constructors, and **pattern matching** make algebraic data types extremely pleasant and safe to use.

Burstall, MacQueen, Sannella,
HOPE: An experimental applicative language, 1980.

Products and sums as algebraic data types

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Sums and products can be viewed as algebraic data types.

```
type ('a, 'b) sum = Left of 'a | Right of 'b
type void = |
type ('a, 'b) pair = Pair of 'a * 'b
type unit = Unit
```

Deconstructing the type void works as expected:

```
let absurd (type a) (x : void) : a =
  match x with _ -> .
```

Encoding Booleans

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The Boolean type $\mathbb{B} \simeq 1 + 1$ can be declared as an algebraic data type:

```
type bool = False | True
```

However, Booleans can also be **encoded** in pure λ -calculus.

A Boolean value is an “object with a *case* method”.

It can choose between two branches:

$$\begin{aligned}
 \mathbb{B} &\triangleq \forall X. (1 \rightarrow X) \rightarrow (1 \rightarrow X) \rightarrow X \\
 \text{False} &\triangleq \lambda x_1. \lambda x_2. x_1 () \\
 \text{True} &\triangleq \lambda x_1. \lambda x_2. x_2 () \\
 \text{case } t \text{ of } t_1 \parallel t_2 &\triangleq t \ t_1 \ t_2
 \end{aligned}$$

This is a **Scott encoding**, and also a **Church encoding**.

Exercise: reconstruct the omitted type abstractions and applications.

Encoding sums

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More generally, the binary sum type $T_1 + T_2$ can be encoded as follows:

$$\begin{aligned}
 T_1 + T_2 &\triangleq \forall X. (T_1 \rightarrow X) \rightarrow (T_2 \rightarrow X) \rightarrow X \\
 \text{inj}_1 x &\triangleq \lambda x_1. \lambda x_2. x_1 x \\
 \text{inj}_2 x &\triangleq \lambda x_1. \lambda x_2. x_2 x \\
 \text{case } t \text{ of } t_1 \parallel t_2 &\triangleq t \ t_1 \ t_2
 \end{aligned}$$

The zero-ary sum type 0 can be encoded, too!

$$\begin{aligned}
 0 &\triangleq \forall X. X \\
 \text{absurd } t &\triangleq t
 \end{aligned}$$

Clearly this works for any number of branches.

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The binary product type $T_1 \times T_2$ can be encoded as follows:

$$\begin{aligned} T_1 \times T_2 &\triangleq \forall X. (T_1 \rightarrow T_2 \rightarrow X) \rightarrow X \\ (x_1, x_2) &\triangleq \lambda k. k \ x_1 \ x_2 \\ \pi_1 \ t &\triangleq t \ (\lambda x_1. \lambda x_2. x_1) \\ \pi_2 \ t &\triangleq t \ (\lambda x_1. \lambda x_2. x_2) \end{aligned}$$

The zero-ary product type 1 can be encoded, too!

$$\begin{aligned} 1 &\triangleq \forall X. X \rightarrow X \\ () &\triangleq \lambda x. x \end{aligned}$$

Clearly this works for any number of tuple components.

Encoding natural integers

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Can we encode the recursive type $\mathbb{N} \simeq 1 + \mathbb{N}$ in the same way, à la Scott?

$$\mathbb{N} \triangleq \forall X. (1 \rightarrow X) \rightarrow (\mathbb{N} \rightarrow X) \rightarrow X$$

This doesn't work in System F , which doesn't have recursive types.

Here, [the Scott and Church encodings differ](#).

The Church encoding views a number as “an object with a *fold* method”.

$$\begin{aligned} \mathbb{N} &\triangleq \forall X. X \rightarrow (X \rightarrow X) \rightarrow X \\ \text{Zero} &\triangleq \lambda z. \lambda s. z \\ \text{Succ } x &\triangleq \lambda z. \lambda s. s \ (x \ z \ s) \end{aligned}$$

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The Church encoding views a list as “an object with a *fold* method”.

$$\begin{aligned}
 \mathbb{L}(Y) &\triangleq \forall X. X \rightarrow (Y \rightarrow X \rightarrow X) \rightarrow X \\
 [] &\triangleq \lambda n. \lambda c. n \\
 x :: xs &\triangleq \lambda n. \lambda c. c \ x \ (xs \ n \ c)
 \end{aligned}$$

The Church encoding works for all **inductive types**.

Girard, Taylor, Lafont, **Proofs and types**, 1990, §11.3–11.5.

Motivation

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Complex numbers are an **abstract concept**.

Outside of their implementation, how they are represented **should be irrelevant**, and one should not depend on implementation details.

In one section, Professor Descartes announced that a complex number was an ordered pair of reals [...].

In the other section, Professor Bessel announced that a complex number was an ordered pair of reals, the first of which was nonnegative [...].

An unfortunate mistake [...] caused the two sections to be interchanged.

Reynolds, **Types, Abstraction and Parametric Polymorphism**, 1983.

Complex numbers as an abstract type

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In OCaml, one might implement complex numbers as an **abstract type**:

```

module Complex : sig
  type t
  val zero: t
  val one: t
  val add: t -> t -> t
  val mul: t -> t -> t
  val (=): t -> t -> bool
  (* etc. *)
end

```

Complex numbers as an existential type

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In System F , this idea can be made precise via an **existential type**:

$$\text{Complex} : \exists X. \left\{ \begin{array}{l} \text{zero} : X \\ \text{add} : X \rightarrow X \rightarrow X \\ \text{mul} : X \rightarrow X \rightarrow X \\ \text{eq} : X \rightarrow X \rightarrow \text{bool} \\ \text{etc.} \end{array} \right\}$$

Mitchell and Plotkin, **Abstract types have existential type**, 1988.

Rossberg, Russo, Dreyer, **F-ing Modules**, 2014.

Streams as an existential type

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Imagine we wish to define an abstract type of **streams**.

A stream is a **producer** of a sequence of elements,
out of which a **consumer** can **pull** elements on demand.

It is an “object” with a single method, *next*.

- a stream has a certain **current internal state**.
- *next* returns either nothing or a pair of an element and a new state.

A stream is analogous to a Java iterator, except it is **not mutable**.
Its current state is explicit.

$$S(X) \triangleq \exists S. \underbrace{(S \rightarrow 1 + X \times S)}_{\text{next}} \times \underbrace{S}_{\text{cur}}$$

Streams as an existential type

('a, 's) step corresponds to $1 + X \times S$:

```
type ('a, 's) step =
  | Done                                     (* the stream is exhausted *)
  | Yield of 'a * 's                       (* here is an element and a new state *)
```

OCaml views existential types as a special case of **algebraic data types**:

```
type 'a stream =
  | Stream:
      (* The [next] method: *) ('s -> ('a, 's) step) *
      (* The current state: *) 's
      (* together form a stream: *) -> 'a stream
```

The data constructor **Stream** has **universal type**: it is polymorphic in 's.

The producer chooses the type of the internal state;
the consumer must treat this type as abstract.

Converting a list to a stream

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This conversion function is a nonrecursive **producer**:

```
let stream (xs : 'a list) : 'a stream =
  let next xs =
    match xs with
    | [] -> Done
    | x :: xs -> Yield (x, xs)
  in
  Stream (next, xs) (* packing an existential type *)
```

On the last line, what is the concrete type of states?

It is 'a **list**.

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This conversion function is a recursive **consumer**:

```
let unstream (Stream (next, s) : 'a stream) : 'a list =
  let rec unfold s =
    match next s with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
  in
  unfold s
```

The first line uses **pattern matching** to **unpack** an existential type.

What is the type of `unfold`?

It is `s -> 'a list`

where `s` is an abstract type introduced by unpacking at line 1.

Examples of stream producers

How would you implement a singleton stream?

```
let return (x : 'a) : 'a stream =
  let next s =
    if s then Yield (x, false) else Done
  in
  Stream (next, true)           (* packing an existential type *)
```

On the last line, the concrete type of states is **bool**:
either we have already yielded an element, or we have not.

Exercise: Write interval of type **int** -> **int** -> **int** stream.

Exercise: Write append of type 'a stream -> 'a stream -> 'a stream.

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An example consumer-and-producer

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The map function on streams is also non-recursive:

```

let map (f : 'a -> 'b) (xs : 'a stream) : 'b stream =
  let Stream (next, s) = xs in                                (* unpacking *)
  let next s =
    match next s with
    | Done          -> Done
    | Yield (x, s) -> Yield (f x, s)
  in
  Stream (next, s)                                           (* packing *)

```

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This encoding of streams is used in practice.

In addition to **Done** and **Yield**, a third constructor **Skip** can be used, meaning “please ask again”

A consumer must ask, ask, ask until a non-**Skip** result is produced.

This allows most stream producers to be **nonrecursive** functions.

This makes optimization easier.

Coutts, Leshchinskiy, Stewart, **Stream fusion:
from lists to streams to nothing at all**, 2007.

System F with existential types

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The syntax of types is extended with **existential types**:

$$T ::= \dots \mid \exists X. T$$

The syntax of terms is extended with **introduction** and **elimination** forms:

$$\begin{aligned} t &::= \dots \mid \text{pack } T, t \text{ as } \exists X. T \mid \text{let } X, x = \text{unpack } t \text{ in } t \\ v &::= \dots \mid \text{pack } T, v \text{ as } \exists X. T \\ E &::= \dots \mid \text{pack } T, E \text{ as } \exists X. T \mid \text{let } X, x = \text{unpack } E \text{ in } t \end{aligned}$$

A new reduction rule is introduced:

$$\text{let } X, x = \text{unpack } (\text{pack } T', v \text{ as } \exists X. T) \text{ in } t \longrightarrow t[v/x][T'/X]$$

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∃-INTRO

$$\frac{\Gamma \vdash t : T[T'/X]}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T}$$

 \exists -ELIM

$$\frac{\Gamma \vdash t_1 : \exists X. T \quad X \# \Gamma, T_2 \quad \Gamma; X; x : T \vdash t_2 : T_2}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$

V-INTRO

$$\frac{\Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X. t : \forall X. T}$$

 \vee -ELIM

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t \ T' : T[T'/X]}$$

Exercise: extend the proofs of Subject Reduction and Progress.

Universal/existential duality

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When a value has universal type $\forall X.T$,
the **producer** of this value must treat X as abstract
and the **consumer** can choose a type T' with which to instantiate X .

When a value has existential type $\exists X.T$,
the **producer** chooses a type T' with which to instantiate X
but the **consumer** must treat X as abstract.

When a value has existential type, **its consumer must be polymorphic**.

Church encoding of existential types

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Existential types can in fact be **encoded** in terms of universal types:

$$\begin{aligned}
 \exists X. T &\triangleq \forall Y. (\forall X. T \rightarrow Y) \rightarrow Y \\
 \text{pack } T', v \text{ as } \exists X. T &\triangleq \Lambda Y. \lambda k : (\forall X. T \rightarrow Y). k \ T' \ v \\
 \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2 &\triangleq t_1 \ T_2 (\Lambda X. \lambda x : T \rightarrow T_2. t_2)
 \end{aligned}$$

This encoding validates the logical implication $\exists X. T \rightarrow \neg \forall X. \neg T$ where $\neg T$ is defined as $T \rightarrow 0$.

Exercise: check that this encoding validates the reduction rule and the typing rules proposed earlier for primitive existential types.

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Consider a tiny language of expressions $t ::= k \mid (t, t) \mid \pi_i t$:

```
type expr =
| EInt of int
| EPair of expr * expr
| EFst of expr
| ESnd of expr
```

Expressions include integer constants, pairs, and projections.

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A straightforward interpreter for this language uses a type of all values:

```
type value =  
| VInt of int  
| VPair of value * value
```

This is an algebraic data type. Thus every value carries a **tag**.

Runtime tests

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These tags are used in **runtime tests** that can cause **runtime errors**.

```
let as_pair (v : value) : value * value =
  match v with
  | VPair (v1, v2) ->
    v1, v2
  | _ ->
    assert false (* runtime error! *)
```

An untyped interpreter

Here, interpreting a pair projection operation involves a runtime test.

```
let rec eval (e : expr) : value =
  match e with
  | EInt x ->
      VInt x
  | EPair (e1, e2) ->
      VPair (eval e1, eval e2)
  | EFst e ->
      fst (as_pair (eval e))
  | ESnd e ->
      snd (as_pair (eval e))
```

This is **necessary** because this interpreter accepts untyped expressions.

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Let us impose a **simple type discipline** on expressions.

```

type _ expr =
| EInt   :      int ->      int expr
| EPair  : 'a expr * 'b expr -> ('a * 'b) expr
| EFst   :      ('a * 'b) expr ->      'a expr
| ESnd   :      ('a * 'b) expr ->      'b expr

```

This type definition encodes the following type discipline:

$$\Gamma \vdash k : \text{int} \qquad \frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \qquad \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i}$$

A **meta-level** AST of type `'a expr`
represents an **object-level** expression of type `'a`.

Typed values

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Let us similarly impose a type discipline on values:

```
type _ value =
| VInt   :                int ->      int value
| VPair  : 'a value * 'b value -> ('a * 'b) value
```

Values are still tagged (for now), but runtime tests become unnecessary...

Look Ma, no runtime test!

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Only one branch is now necessary. A second branch would be **dead**.

```
let as_pair : type a b . (a * b) value -> a value * b value
= function
  | VPair (v1, v2) ->
    v1, v2
  (* In this branch, we would learn [a * b = int], *)
  (* which is contradictory. *)
  (* | _ -> . *)
```

In OCaml, destructing a GADT requires a type annotation in this style.

A typed interpreter

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Evaluating an expression of type T yields a value of type T .

```

let rec eval : type a . a expr -> a value
= function
  | EInt x ->
    (* We learn [a = int] so returning [VInt _] is OK. *)
    VInt x
  | EPair (e1, e2) ->
    (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
    (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
    VPair (eval e1, eval e2)
  | EFst e ->
    fst (as_pair (eval e))
  | ESnd e ->
    snd (as_pair (eval e))

```

The type of the interpreter reflects the **subject reduction** property.
 It amounts to **checking the proof** of subject reduction!

A typed, tagless interpreter

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Evaluating an expression of type T yields a **meta-level value** of type T .

```

let rec eval : type a . a expr -> a
= function
  | EInt x ->
    (* We learn [a = int] so returning an integer is OK. *)
    x (* no tagging! *)
  | EPair (e1, e2) ->
    (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
    (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
    (eval e1, eval e2) (* no tagging! *)
  | EFst e ->
    fst (eval e) (* no untagging! *)
  | ESnd e ->
    snd (eval e) (* no untagging! *)

```

The type of the interpreter reflects the **subject reduction** property.
 it amounts to **checking the proof** of subject reduction!

Going further

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Our tiny expressions are **closed**: the typing judgement is $\vdash t : T$.

When expressions involve variables, one needs a type $(\text{'g}, \text{'a}) \text{ expr}$ whose definition encodes the typing judgement $\Gamma \vdash t : T$.

This is reasonably easy if variables are encoded as de Bruijn indices.

Bird, Paterson, **de Bruijn notation as a nested datatype**, 1999.

Runtime type descriptions

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Metatheory

A **value** of type `'a ty` is a runtime description of the **type** `'a`.

```
type 'a ty =
| TyInt   : int ty
| TySum   : 'a ty * 'b ty -> ('a, 'b) sum ty
| TyPair  : 'a ty * 'b ty -> ('a * 'b) ty
```

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Although **inspecting a type** at runtime is impossible, as types are erased, **inspecting a runtime description of a type** is possible.

In other words, although the type $\forall X. X \rightarrow X$ has only **one** inhabitant, the type $\forall X. \text{Ty } X \rightarrow X \rightarrow X$ has **more than one**.

This let us write **polymorphic, type-directed** functions, an activity that is sometimes known as **generic programming**.

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Here is a polymorphic, type-directed **conversion of a value to a string**.

```
let rec show : type a . a ty -> a -> string =
  fun ty x ->
    match ty with
    | TyInt ->
      string_of_int x
    | TySum (ty1, ty2) ->
      begin match x with
      | Left x1 -> "left(" ^ show ty1 x1 ^ ")"
      | Right x2 -> "right(" ^ show ty2 x2 ^ ")"
      end
    | TyPair (ty1, ty2) ->
      let (x1, x2) = x in
      "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

In each branch, **we learn something** about the type of x .

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It is more concise and looks better to deconstruct both arguments at once.

```
let rec show : type a . a ty -> a -> string =
  fun ty x ->
    match ty, x with
    | TyInt, x ->
      string_of_int x
    | TySum (ty1, _), Left x1 ->
      "left(" ^ show ty1 x1 ^ ")"
    | TySum (_, ty2), Right x2 ->
      "right(" ^ show ty2 x2 ^ ")"
    | TyPair (ty1, ty2), (x1, x2) ->
      "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

The OCaml type-checker reads patterns from left to right
so deconstructing (ty, x) works but deconstructing (x, ty) does not.

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Here is a polymorphic, type-directed **equality test**.

```

let rec equal : type a . a ty -> a -> a -> bool =
  fun ty x y ->
    match ty, x, y with
    | TyInt, x, y ->
      Int.equal x y
    | TySum (ty1, _), Left x1, Left y1 ->
      equal ty1 x1 y1
    | TySum (_, ty2), Right x2, Right y2 ->
      equal ty2 x2 y2
    | TySum _, Left _, Right _
    | TySum _, Right _, Left _ ->
      false
    | TyPair (ty1, ty2), (x1, x2), (y1, y2) ->
      equal ty1 x1 y1 && equal ty2 x2 y2
let rec equal : type a b . a ty -> b ty -> (a, b) eq =
  fun ty x y ->

```

Connections between GADTs and type classes

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Eq and **Show** are typical examples of **type classes** in Haskell.

Upcoming lecture on type classes (PED).

Hinze, Jeuring, Löh,

Comparing Approaches to Generic Programming in Haskell, 2006.

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A **closed** set of type class instances can be compiled down to GADTs.

Pottier and Gauthier,

Polymorphic typed defunctionalization and concretization, 2006.

However a GADT describes a **closed** universe of **structural** types whereas type classes are **open-ended** and apply to **user-defined**, **nominal** types.

The Holy Grail is to propose a language where **a type of the representations of all types** (including itself!) can be defined.

Chapman, Dagand, McBride, Morris,

The Gentle Art of Levitation, 2010.

A type inferencer

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We have a type `'a` expr of well-typed expressions
and a type `'a` ty of runtime type descriptions.

Can we express a simple type **type inferencer** that accepts an untyped
expression and **either fails or returns a typed expression**?

```
exception IllTyped
let rec infer : Raw.expr -> ???
= function
  | Raw.EInt i ->
    (TyInt, EInt i)
  | ...
```

What should its **result type** be?

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We need an **existential type** $\exists X. \text{Ty } X \times \text{Expr } X$.

```
type typed_expr =  
| Pack      : 'a ty * 'a expr -> typed_expr
```

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We can now write the type inferencer:

```

let rec infer : Raw.expr -> typed_expr =
function
  | Raw.EInt i ->
    Pack (TyInt, EInt i)
  | Raw.EFst e ->
    let Pack (ty, e) = infer e in
    begin match ty with
    | TyPair (ty1, ty2) -> Pack (ty1, EFst e)
    | _ -> raise IllTyped
    end

```

Exercise: write the two missing cases.

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =
  let Pack (inferred, e) = infer e in
  if inferred = expected then
    e
  else
    raise IllTyped
```

But **this code is not well-typed**. Why?

expected has type $a \text{ ty}$.

inferred has type $b \text{ ty}$

where b is an unknown type introduced by deconstructing **Pack**.

They **cannot be compared** using homogeneous equality $=$.

Even if they could, e has type $b \text{ expr}$

whereas a result of type $a \text{ expr}$ is required.

The equality GADT

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The solution involves the **type equality GADT**.

```
type (_, _) eq =  
  | Equal: ('a, 'a) eq
```

The type `('a, 'b) eq` has at most one inhabitant.

If it has one then this inhabitant must be **Equal**
and the types `'a` and `'b` must be the same.

A heterogeneous type equality test

This lets us express a **heterogenous** type equality test:

```
let rec equal : type a b . a ty -> b ty -> (a, b) eq =
  fun ty1 ty2 ->
    match ty1, ty2 with
    | TyInt, TyInt ->
      Equal
    | TyPair (ty1a, ty1b), TyPair (ty2a, ty2b) ->
      let Equal = equal ty1a ty2a in
      let Equal = equal ty1b ty2b in
      Equal
    | -, - ->
      raise IllTyped
```

When `equal ty1 ty2` succeeds, we **learn** that the runtime type descriptions `ty1` and `ty2` describe the same static type.

Exercise: write the missing case.

A type-checker

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We can now write the type-checker:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
  let Equal = equal inferred expected in  
  e
```

Exercise: make sure that you understand why this code is well-typed.

Putting the pieces together

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Given an arbitrary untyped expression in our tiny language, we can now **infer** its type, **evaluate** it, and **show** its value, whatever its type may be.

```
let () =
  let e = Raw.(EPair (EInt 42, EInt 0)) in
  let Pack (ty, e) = infer e in
  let v = eval e in
  Printf.printf "%s\n!" (show ty v)
```

The output in the REPL is:

```
(42, 0)
```


Printf in OCaml

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printf takes a “format string” followed with a number of arguments:

```
# open Printf;;  
# printf "%d * %s = %d\n" 2 "12" 24;;  
2 * 12 = 24  
- : unit = ()
```

The number and type of these arguments *depends* on the format string.

Printf in OCaml

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A format string is actually **not** a string: it is a **data structure**.

```
# open CamlinternalFormatBasics;;
# let desc : _ format6 = "%d * %s = %d\n";;
val desc :
  (int -> string -> int -> 'a, 'b, 'c, 'd, 'd, 'a) format6 =
  Format
    (Int (Int_d, No_padding, No_precision,
      String_literal (" * ",
        String (No_padding,
          String_literal (" = ",
            Int (Int_d, No_padding, No_precision,
              Char_literal ('\n', End_of_format)))))),
      "%d * %s = %d\n")
```

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This data structure has the shape of a list:

```
Int (Int_d, No_padding, No_precision,
String_literal ( " * ",
String (No_padding,
String_literal ( " = ",
Int (Int_d, No_padding, No_precision,
Char_literal ( '\n',
End_of_format ) ) ) ) ) )
```

End_of_format is “nil”; the other constructors are “cons” constructors.

Int and **String** correspond to “holes” %d and %s.

String_literal and **Char_literal** correspond to literal pieces of string.

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Metatheory

Can we define our own algebraic data type of formats, or **descriptors**?

```
type desc =  
  | Nil  
  | Lit of string * desc  
  | Int of desc
```

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Or, in this alternative syntax:

```
type desc =
  | Nil : desc
  | Lit : string * desc -> desc
  | Int : desc -> desc
```

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Or, in this alternative syntax:

```
type desc =
  | Nil  :          desc
  | Lit  : string * desc -> desc
  | Int  : desc      -> desc
```

Now, please define `fprintf` so that `fprintf` emit `desc <args>`

- emits output via the function `emit : string -> unit`,
- obeys `desc`,
- expects arguments `<args>` whose number and type satisfy `desc`.

`fprintf` should have type `(string -> unit) -> desc -> ??? -> unit`.

Expressing the type of printf

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The type desc $\rightarrow ??? \rightarrow \text{unit}$ does not make sense.

The number of and type of the arguments ??? **depends** on the descriptor.

We seem to need a **dependent type** $(d : \text{desc}) \rightarrow \text{shape } d$

- where shape would be a function of descriptors to types,
- but OCaml does not have that.

Instead, let's use a plain function type $\text{'shape } \text{desc} \rightarrow \text{'shape}$

- where the definition of $\text{'shape } \text{desc}$ as a GADT encodes the correspondence between descriptors and shapes.

Descriptors form a **typed language** and printf is an **interpreter** for it!

A GADT of descriptors

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We want `fprintf` : (`string` -> `unit`) -> 'a desc -> 'a.

```

type    desc =
| Nil   :                               desc
| Lit   :      string *                 desc -> desc
| Int   :                               desc -> desc

```

We must turn the type `desc` into a GADT.

A GADT of descriptors

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We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```

type _ desc =
  | Nil      :                               ?? desc
  | Lit      :      string * ?? desc ->      ?? desc
  | Int      :                               ?? desc ->      ?? desc

```

We parameterize the type `desc`.

A GADT of descriptors

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We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```

type _ desc =
  | Nil      :                               unit desc
  | Lit      :      string * ?? desc ->      ?? desc
  | Int      :                ?? desc ->      ?? desc

```

`Nil` requires no action; the corresponding shape is `unit`.

A GADT of descriptors

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We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```

type _ desc =
  | Nil      :                               unit desc
  | Lit      :      string * 'a desc ->      'a desc
  | Int      :                ?? desc ->      ?? desc

```

`Lit` (`s`, `d`) requires printing `s` and interpreting `d`.

If `d` has shape `'a` then `Lit` (`s`, `d`) has shape `'a` as well.

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Metatheory

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```

type _ desc =
  | Nil   :                               unit desc
  | Lit   :          string * 'a desc ->    'a desc
  | Int   :          'a desc ->    (int -> 'a) desc

```

`Int d` requires consuming an integer argument and interpreting `d`.

If `d` has shape `'a` then `Int d` has shape `int -> 'a`.

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Metatheory

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil      :                               unit desc
  | Lit      :          string * 'a desc ->      'a desc
  | Hole     : ('data -> string) * 'a desc -> ('data -> 'a) desc
```

We change the hole of type `int` with a hole of arbitrary type `'data`.

All that is needed is a conversion function of type `'data -> string`.

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Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ???
    | Lit (s, desc) ->
      ???
    | Hole (to_string, desc) ->
      ???
  in eval desc

```

Recall

```

| Nil : unit desc

```

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Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ??? (* We learn [a = unit]. *)
    | Lit (s, desc) ->
      ???
    | Hole (to_string, desc) ->
      ???

  in eval desc

```

Recall

```

| Nil : unit desc

```

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```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      () (* We learn [a = unit]. *)
    | Lit (s, desc) ->
      ???
    | Hole (to_string, desc) ->
      ???

  in eval desc

```

Recall

```

| Lit : string * 'a desc -> 'a desc

```


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```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      ??? (* We learn no new type equality. *)
    | Hole (to_string, desc) ->
      ???

  in eval desc

```

Recall

```

| Lit : string * 'a desc -> 'a desc

```

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```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      ???
  in eval desc

```

Recall

```

| Hole : ('data -> string) * 'a desc -> ('data -> 'a) desc

```

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```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      ??? (* We learn [a = data -> b] *)
          (* [to_string : data -> string; desc : b desc] *)
  in eval desc

```

Recall

```

| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc

```

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```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      fun x -> emit (to_string x); eval desc
      (* [x] has type [data]; [eval desc] has type [b] *)
  in eval desc
  (* and [data -> b] is [a] *)

```

Recall

```

| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc

```

Using fprintf

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Voilà! From fprintf, we get printf.

```
let printf desc =  
  let emit = print_string in  
  fprintf emit desc
```

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Metatheory

To construct descriptors, some sugar is needed.

```
module Sugar = struct
  let nil = Nil
  let lit s desc = Lit (s, desc)
  let d desc = Hole (string_of_int, desc)
  let s desc = Hole (Fun.id, desc)
end
```

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Metatheory

To construct descriptors, some sugar is needed.

```

module Sugar = struct
  let nil = Nil
  let lit s desc = Lit (s, desc)
  let d desc = Hole (string_of_int, desc)          (* %d *)
  let s desc = Hole (Fun.id, desc)                  (* %S *)
end

```

For example,

```

let desc =                                     (* "%d * %S = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil

```

Using `fprintf`

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```
let desc = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf desc 2 "12" 24;;
2 * 12 = 24
```


Implementing sprintf

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Metatheory

Can we implement sprintf, which returns a string?

```
let sprintf desc args =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b in  
  fprintf emit desc args;  
  Buffer.contents b
```

This is accepted but is **not** what we want.

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Metatheory

Can we implement `sprintf`, which returns a string?

```
let sprintf desc <arg ... arg> =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b in  
  fprintf emit desc <arg ... arg>;  
  Buffer.contents b
```

We want `sprintf` to accept a variable number of arguments, not just one.

In fact, we cannot write the type of `sprintf`.

It is like the type of `fprintf` but should end in `string` instead of `unit`.

A more general type of descriptors

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Metatheory

We must equip ourselves with a **more general** type of descriptors.

```
type _ desc =
  | Nil   :                                     unit desc
  | Lit   :          string * 'a             desc ->      'a desc
  | Hole  : ('data -> string) * 'a           desc ->      ('data -> 'a      ) desc
```

A more general type of descriptors

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Metatheory

We must equip ourselves with a **more general** type of descriptors.

```
type (_, _) desc =
  | Nil : ('r, 'r) desc
  | Lit : string * ('a, 'r) desc -> ('a, 'r) desc
  | Hole : ('data -> string) * ('a, 'r) desc ->
    ('data -> 'a, 'r) desc
```

In the type `('a, 'r) desc`,

- `'a` is the **shape**, as before,
- `'r` is the **eventual return type** of this shape.
 - it can be **unit** for `fprintf` and **string** for `sprintf`;
 - a descriptor can be polymorphic in `'r`.

Implementing fprintf, again

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We can now give fprintf a **more general** type. We parameterize it with:

- `emit : string -> unit`
- `finished : unit -> r` — **new**
- `desc : (a, r) desc`

fprintf emit finished desc has type a.

a must in fact be a function type whose eventual return type is r.

fprintf emit finished desc <args> must eventually return
a value of type r, which it obtains by calling finished().

Implementing printf, again

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```

let fprintf (type a r) emit (finished : unit -> r)
    (desc : (a, r) desc) : a =
  let rec eval : type a . (a, r) desc -> a =
    function
    | Nil ->
      (* We have [a = r] so [finished()] has type [a]. *)
      finished()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      fun x -> emit (to_string x); eval desc
  in eval desc

```

It is worth pointing out that eval involves **polymorphic recursion**.

Implementing printf and sprintf

We can now implement `printf` and `sprintf`, among other variations:

```
let printf desc =
  let emit = print_string
  and finished () = () in
  fprintf emit finished desc

let sprintf desc =
  let b = Buffer.create 128 in
  let emit = Buffer.add_string b
  and finished () = Buffer.contents b in
  fprintf emit finished desc
```

We get

```
val printf : ('a, unit) desc -> 'a
val sprintf : ('a, string) desc -> 'a
```

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```
let desc () = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf (desc()) 2 "12" 24;;
2 * 12 = 24
# let (s : string) = sprintf (desc()) 2 "12" 24;;
val s : string = "2 * 12 = 24\n"
```

Here, we make desc a (constant) function in order to work around the [value restriction](#). See upcoming lecture on mutable state (GS).

Danvy et al.'s approach

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Danvy, Keller and Puech (2015) view formats as **trees** instead of **lists**.

```
type (_, _) desc =
  | Lit  : string -> ('a, 'a) desc
  | Hole : ('data -> string) -> ('data -> 'a, 'a) desc
  | Seq  : ('a, 'b) desc * ('b, 'c) desc -> ('a, 'c) desc
```

The type ('a, 'r) desc has the same meaning as earlier.

Lit and **Hole** no longer play the role of list “cons” constructors.

Seq is a binary concatenation constructor, whose type says:

*If 'a is a multi-arrow type whose eventual return type is 'b and
if 'b is a multi-arrow type whose eventual return type is 'c then
'a is a multi-arrow type whose eventual return type is 'c.*

Danvy et al.'s approach

Danvy et al. write `kprintf` in [continuation-passing style](#):

```

let rec kprintf
: type a r . (a, r) desc -> (string -> r) -> a =
  fun desc finished ->
    match desc with
    | Lit s ->
      finished s
    | Hole to_string ->
      fun x -> finished (to_string x)
    | Seq (desc1, desc2) ->
      kprintf desc1 @@ fun s1 ->
      kprintf desc2 @@ fun s2 ->
      finished (s1 ^ s2)

```

Exercise (easy): define `printf`, `sprintf`, and `fprintf` using `kprintf`.

Exercise (harder): define `fprintf` directly.

Do not use string concatenation `^`.

System F +GADTs

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System F +GADTs was defined by Xi, Chen and Chen (2003).

Xi, Chen, Chen,
Guarded Recursive Datatype Constructors, 2003.

Pottier and Gauthier,
Polymorphic typed defunctionalization and concretization, 2006.

System F +GADTs: the typing judgement

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Metatheory

Recall the typing judgement of System F :

$$\Gamma \vdash t : T$$

In System F +GADTs, must we change the shape of this judgement?We must extend it with a conjunction of **equality hypotheses**.

$$\Gamma, C \vdash t : T$$

Equality constraints are given by $C, D ::= \text{True} \mid T = T \mid C \wedge C$.

System F +GADTs: type declarations

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Metatheory

We assume that a family of type constructors F is given.

- for simplicity, we assume they have arity 1.

We assume that a family of data constructors K is given.

- for simplicity, we assume they have arity 1.

We assume that each data constructor has a closed **type scheme**:

$$K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2$$

System F +GADTs: an auxiliary judgement

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Metatheory

For readability, we introduce the auxiliary judgement

$$K \leq D \Rightarrow T_1 \rightarrow F T_2$$

whose definition is the following:

$$\frac{K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2}{K \leq D[\bar{T}/\bar{X}] \Rightarrow T_1[\bar{T}/\bar{X}] \rightarrow F T_2[\bar{T}/\bar{X}]}$$

System F +GADTs: the typing judgement

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Metatheory

The typing rules of System F are unchanged. A constraint is transported.

$$\begin{array}{c}
 \text{VAR} \\
 \Gamma, C \vdash x : \Gamma(x)
 \end{array}
 \qquad
 \begin{array}{c}
 \text{ABS} \\
 \dfrac{\Gamma; x : T_1, C \vdash t : T_2}{\Gamma, C \vdash \lambda x. t : T_1 \rightarrow T_2}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{APP} \\
 \dfrac{\Gamma, C \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma, C \vdash t_2 : T_1}{\Gamma, C \vdash t_1 t_2 : T_2}
 \end{array}$$

$$\begin{array}{c}
 \text{TABS} \\
 \dfrac{\Gamma; X, C \vdash t : T \quad X \# \Gamma}{\Gamma, C \vdash \Lambda X. t : \forall X. T}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{TAPP} \\
 \dfrac{\Gamma, C \vdash t : \forall X. T}{\Gamma, C \vdash t T' : T[T'/X]}
 \end{array}$$

System F +GADTs: the typing judgement,
continued

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Metatheory

The typing rule for a **data constructor application** is straightforward:

$$\text{DCon} \quad \frac{\begin{array}{c} K \leq D \Rightarrow T_1 \rightarrow F T_2 \\ C \Vdash D \\ \Gamma, C \vdash t : T_1 \end{array}}{\Gamma, C \vdash K t : F T_2}$$

We write $C \Vdash D$ when C **entails** D (see next slide).

System F +GADTs: entailment

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Metatheory

Let ρ denote a total mapping of type variables to closed types.

We write $\rho \vdash C$ when ρ satisfies C :

$$\rho \vdash \text{True} \qquad \frac{\rho(T_1) = \rho(T_2)}{\rho \vdash T_1 = T_2} \qquad \frac{\rho \vdash C_1 \quad \rho \vdash C_2}{\rho \vdash C_1 \wedge C_2}$$

Entailment is then defined by:

$$\frac{\forall \rho. \rho \vdash C \Rightarrow \rho \vdash D}{C \Vdash D}$$

Entailment is decidable.

System F +GADTs: the typing judgement,
continued

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Metatheory

Type-checking a **case analysis** construct is straightforward:

CASE

$$\frac{\begin{array}{l} \Gamma, C \vdash t : T_1 \\ \forall c \in \bar{c}. \quad \Gamma, C \vdash c : T_1 \rightarrow T_2 \\ \bar{c} \text{ is exhaustive} \end{array}}{\Gamma, C \vdash \text{case } t \text{ of } \bar{c} : T_2}$$

A **clause** takes the form $c ::= K \bar{X} x \mapsto t$.

\bar{c} is exhaustive if it contains a clause for every data constructor K .

System F +GADTs: the typing judgement,
continued

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When a clause is entered, **new constraints appear** locally.

CLAUSE

$$\begin{array}{c}
 K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2 \\
 (\Gamma; \bar{X}; x : T_1), (C \wedge D \wedge F T_2 = F T'_2) \vdash t : T' \\
 \bar{X} \# \Gamma, C, T'_2, T' \\
 \hline
 \Gamma, C \vdash K \bar{X} x \mapsto t : F T'_2 \rightarrow T'
 \end{array}$$

System F +GADTs: the typing judgement, continued

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Metatheory

There remains to introduce a typing rule that **exploits** the hypothesis C :

$$\text{CONVERSION} \quad \frac{\Gamma, C \vdash t : T \quad C \Vdash T = T'}{\Gamma, C \vdash t : T'}$$

This rule is **not** syntax-directed.

One can imagine a variant of the system where conversion is explicit.
System FC is the core language of the Glasgow Haskell compiler.

Sulzmann, Chakravarty, Peyton Jones, Donnelly,
System F with Type Equality Coercions, 2007.

System F +GADTs: type soundness

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Exercise: write down the omitted details (e.g., the reduction rule for *case*), then prove Subject Reduction and Progress.