

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

MPRI 2.4

Algebraic data types, existential types, and GADTs

François Pottier



2023

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker
Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Towards data types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Many data types can be built out of **sums** and **products** and a form of **recursion** at the level of types.

Binary sum $+$ and product \times , and their **neutral elements** 0 and 1, suffice.

- The **unit** type is 1.
- The **empty** type is 0.
- The **Boolean** type is $1 + 1$.
- The type \mathbb{N} of the natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
- The type $\mathbb{L}(X)$ of lists of elements of type X must satisfy

$$\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$$

Three technical approaches to data types

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

There are three main approaches to extending System F with data types:

- consider 0 , 1 , $+$, \times , and recursive types $\mu X.T$ as **primitive concepts** and encode all data types in terms of these concepts;
- consider **algebraic data types** as primitive and view sums, products, naturals, lists, etc., as instances of this general concept;
- introduce **no new primitive concept** and remark that **inductive types** can be encoded in System F .

In practice, the second approach is the most natural and user-friendly.

All three approaches, and their connections, are worth understanding.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

It is easy to add **pairs** and **projections** to the (call-by-value) λ -calculus.

$$\begin{aligned}
 t &::= \dots \mid (t, t) \mid \pi_i t && \text{where } i \in \{1, 2\} \\
 v &::= \dots \mid (v, v) \\
 E &::= \dots \mid (E, t) \mid (v, E) \mid \pi_i E
 \end{aligned}$$

One new reduction rule is needed: $\pi_i (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed:

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

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 E &::= \dots \mid (E, t) \mid (v, E) \mid \pi_i E
 \end{aligned}$$

One new reduction rule is needed: $\pi_i (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed: $T ::= \dots \mid T \times T$.

Two new typing rules are needed:

Binary products

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

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One new reduction rule is needed: $\pi_i (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed: $T ::= \dots \mid T \times T$.

Two new typing rules are needed:

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \qquad \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i}$$

Exercise: extend the proofs of Subject Reduction and Progress.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
Metatheory
Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

The **unit** type 1 can be viewed as a product type of arity 0.

It has an **introduction** form but no **elimination** form.

$$t ::= \dots \mid ()$$

$$v ::= \dots \mid ()$$

– no new evaluation context

No new reduction rule is needed.

A new type constructor is needed: $T ::= \dots \mid 1$.

One new typing rule is needed:

$$\Gamma \vdash () : 1$$

Binary sums

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Let us add **injections** and a **case analysis** to (call-by-value) λ -calculus.

$$\begin{aligned} t &::= \dots \mid \text{inj}_i t \mid \text{case } t \text{ of } t_1 \parallel t_2 && \text{where } i \in \{1, 2\} \\ v &::= \dots \mid \text{inj}_i v \\ E &::= \dots \mid \text{inj}_i E \mid \text{case } E \text{ of } t_1 \parallel t_2 \end{aligned}$$

One new reduction rule is needed: $\text{case } \text{inj}_i v \text{ of } t_1 \parallel t_2 \longrightarrow t_i v$.

In a case construct, the branches t_1 and t_2 should be functions.

A new type constructor is needed:

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

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One new reduction rule is needed: $\text{case } \text{inj}_i v \text{ of } t_1 \parallel t_2 \longrightarrow t_i v$.

In a case construct, the branches t_1 and t_2 should be functions.

A new type constructor is needed: $T ::= \dots \mid T + T$.

Two new typing rules are needed:

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

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$$\begin{aligned} t &::= \dots \mid \text{inj}_i t \mid \text{case } t \text{ of } t_1 \parallel t_2 && \text{where } i \in \{1, 2\} \\ v &::= \dots \mid \text{inj}_i v \\ E &::= \dots \mid \text{inj}_i E \mid \text{case } E \text{ of } t_1 \parallel t_2 \end{aligned}$$

One new reduction rule is needed: $\text{case } \text{inj}_i v \text{ of } t_1 \parallel t_2 \longrightarrow t_i v$.

In a case construct, the branches t_1 and t_2 should be functions.

A new type constructor is needed: $T ::= \dots \mid T + T$.

Two new typing rules are needed:

$$\frac{\Gamma \vdash t_i : T_i}{\Gamma \vdash \text{inj}_i t : T_1 + T_2} \qquad \frac{\Gamma \vdash t : T_1 + T_2 \quad \Gamma \vdash t_1 : T_1 \rightarrow T' \quad \Gamma \vdash t_2 : T_2 \rightarrow T'}{\Gamma \vdash \text{case } t \text{ of } t_1 \parallel t_2 : T'}$$

Exercise: extend the proofs of Subject Reduction and Progress.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Print & friends

Metatheory

The **empty** type can be viewed as a sum type of arity 0.

It has an **elimination** form but no **introduction** form.

$$\begin{aligned} t &::= \dots \mid \text{absurd } t \\ &\quad \text{— no new value} \\ E &::= \dots \mid \text{absurd } E \end{aligned}$$

No new reduction rule is needed. *absurd* *v* is stuck.

A new type constructor is needed: $T ::= \dots \mid 0$.

One new typing rule is needed:

$$\frac{\Gamma \vdash t : 0}{\Gamma \vdash \text{absurd } t : T'}$$

Exercise: extend the proof of Progress.

Approaches to recursive types

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory
Church
encoding

GADTs

Tagless
interpreters
Runtime type
descriptions
A well-typed
type-checker
Printf & friends
Metatheory

Recall what was said earlier about **recursive types**:

- Natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
- Lists must satisfy $\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$.

One approach is to extend the type system with **recursive types** $\mu X. T$. The type $\mu X. T$ and its unfolding $T[\mu X. T / X]$ must then be considered either **equal** or **related via explicit coercions**.

One can then define \mathbb{N} as $\mu X. 1 + X$ and $\mathbb{L}(X)$ as $\mu Y. 1 + X \times Y$.

A more pleasant approach is to just view \mathbb{N} and $\mathbb{L}(X)$ as **primitive types**. This is the topic of the next slides, and leads to **algebraic data types**.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker
Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

\mathbb{N} as a primitive type

Data types

Primitive sums
and products

Algebraic data types

Church
encodings

Existential types

Examples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

Consider λ -calculus with **injections** and **case analysis**.

Let us use inj_1 () to encode zero and $inj_2 v$ to encode the successor of v .

Introduce a new type constructor: $T ::= \dots \mid \mathbb{N}$.

Give three new typing rules:

$$\begin{array}{c}
\frac{\Gamma \vdash t_1 : 1}{\Gamma \vdash inj_1 t : \mathbb{N}} \qquad \frac{\Gamma \vdash t_2 : \mathbb{N}}{\Gamma \vdash inj_2 t : \mathbb{N}} \qquad \frac{\Gamma \vdash t : \mathbb{N} \quad \Gamma \vdash t_1 : 1 \rightarrow T' \quad \Gamma \vdash t_2 : \mathbb{N} \rightarrow T'}{\Gamma \vdash case\ t\ of\ t_1 \parallel t_2 : T'}
\end{array}$$

These are **exactly the typing rules proposed earlier for binary sums** where we have replaced $T_1 + T_2$ with \mathbb{N} , T_1 with 1, and T_2 with \mathbb{N} .

The types \mathbb{N} and $1 + \mathbb{N}$ are not equal, but they are **isomorphic**: one can write $in : 1 + \mathbb{N} \rightarrow \mathbb{N}$ and $out : \mathbb{N} \rightarrow 1 + \mathbb{N}$ such that $in \cdot out$ and $out \cdot in$ are $\beta\eta$ -equal to the identity.

$\mathbb{L}(X)$ as a primitive type

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

Consider again λ -calculus with **injections** and **case analysis**.

Let us use inj_1 () to encode [] and inj_2 (v_1, v_2) to encode $v_1 :: v_2$.

Introduce a new type constructor: $T ::= \dots \mid \mathbb{L}(T)$.

Give three new typing rules:

$$\frac{\Gamma \vdash t_1 : 1}{\Gamma \vdash inj_1 t : \mathbb{L}(T)} \qquad \frac{\Gamma \vdash t_2 : T \times \mathbb{L}(T)}{\Gamma \vdash inj_2 t : \mathbb{L}(T)}$$

$$\frac{\Gamma \vdash t : \mathbb{L}(T) \quad \Gamma \vdash t_1 : 1 \rightarrow T' \quad \Gamma \vdash t_2 : T \times \mathbb{L}(T) \rightarrow T'}{\Gamma \vdash \text{case } t \text{ of } t_1 \parallel t_2 : T'}$$

These are again **exactly the typing rules of binary sums** where we have replaced $T_1 + T_2$ with $\mathbb{L}(X)$, T_1 with 1, and T_2 with $X \times \mathbb{L}(X)$.

Again, the types $\mathbb{L}(X)$ and $1 + X \times \mathbb{L}(X)$ are isomorphic.

Algebraic data types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

Instead of offering a fixed set of primitive types such as \mathbb{N} and $\mathbb{L}(X)$, better **let the user define whatever custom types they need** using sums and products (of arbitrary arity) and recursion.

This idea gives rise to **algebraic data types**.

```
type      nat = Zero | Succ of nat
type 'a list = Nil   | Cons of 'a * 'a list
type 'a tree = Leaf  | Node of 'a tree * 'a * 'a tree
```

Named types, named data constructors, and **pattern matching** make algebraic data types extremely pleasant and safe to use.

Burstall, MacQueen, Sannella,
HOPE: An experimental applicative language, 1980.

Products and sums as algebraic data types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
Metatheory
Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

Sums and products can be viewed as algebraic data types.

```
type ('a, 'b) sum = Left of 'a | Right of 'b
type void = | (* zero constructors *)
type ('a, 'b) pair = Pair of 'a * 'b
type unit = Unit
```

Deconstructing the type void works as expected:

```
let absurd (type a) (x : void) : a =
  match x with _ -> . (* zero branches *)
```

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker
Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Encoding Booleans

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

The Boolean type $\mathbb{B} \simeq 1 + 1$ can be declared as an algebraic data type:

```
type bool = False | True
```

However, Booleans can also be **encoded** in pure λ -calculus.

A Boolean value is an “object with a case method”.

It can choose between two branches:

$$\begin{aligned}
 \mathbb{B} &\triangleq \forall X. (1 \rightarrow X) \rightarrow (1 \rightarrow X) \rightarrow X \\
 \textit{False} &\triangleq \lambda x_1. \lambda x_2. x_1 () \\
 \textit{True} &\triangleq \lambda x_1. \lambda x_2. x_2 () \\
 \textit{case } t \textit{ of } t_1 \parallel t_2 &\triangleq t \ t_1 \ t_2
 \end{aligned}$$

This is a **Scott encoding**, and also a **Church encoding**.

Exercise: reconstruct the omitted type abstractions and applications.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

More generally, the binary sum type $T_1 + T_2$ can be encoded as follows:

$$\begin{aligned}
 T_1 + T_2 &\triangleq \forall X. (T_1 \rightarrow X) \rightarrow (T_2 \rightarrow X) \rightarrow X \\
 \text{inj}_1 x &\triangleq \lambda x_1. \lambda x_2. x_1 x \\
 \text{inj}_2 x &\triangleq \lambda x_1. \lambda x_2. x_2 x \\
 \text{case } t \text{ of } t_1 \parallel t_2 &\triangleq t \ t_1 \ t_2
 \end{aligned}$$

The zero-ary sum type 0 can be encoded, too!

$$\begin{aligned}
 0 &\triangleq \forall X. X \\
 \text{absurd } t &\triangleq t
 \end{aligned}$$

Clearly this works for any number of branches.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

The binary product type $T_1 \times T_2$ can be encoded as follows:

$$\begin{aligned}
 T_1 \times T_2 &\triangleq \forall X. (T_1 \rightarrow T_2 \rightarrow X) \rightarrow X \\
 (x_1, x_2) &\triangleq \lambda k. k \ x_1 \ x_2 \\
 \pi_1 \ t &\triangleq t \ (\lambda x_1. \lambda x_2. x_1) \\
 \pi_2 \ t &\triangleq t \ (\lambda x_1. \lambda x_2. x_2)
 \end{aligned}$$

The zero-ary product type 1 can be encoded, too!

$$\begin{aligned}
 1 &\triangleq \forall X. X \rightarrow X \\
 () &\triangleq \lambda x. x
 \end{aligned}$$

Clearly this works for any number of tuple components.

Encoding natural integers

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Can we encode the recursive type $\mathbb{N} \simeq 1 + \mathbb{N}$ in the same way, à la Scott?

$$\mathbb{N} \triangleq \forall X. (1 \rightarrow X) \rightarrow (\mathbb{N} \rightarrow X) \rightarrow X$$

This doesn't work in System F , which doesn't have recursive types.

Here, [the Scott and Church encodings differ](#).

The Church encoding views a number as “an object with a *fold* method”.

$$\begin{aligned} \mathbb{N} &\triangleq \forall X. X \rightarrow (X \rightarrow X) \rightarrow X \\ \text{Zero} &\triangleq \lambda z. \lambda s. z \\ \text{Succ } x &\triangleq \lambda z. \lambda s. s (x \ z \ s) \end{aligned}$$

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrint & friends
Metatheory

The Church encoding views a list as “an object with a *fold* method”.

$$\begin{aligned}
 \mathbb{L}(Y) &\triangleq \forall X. X \rightarrow (Y \rightarrow X \rightarrow X) \rightarrow X \\
 [] &\triangleq \lambda n. \lambda c. n \\
 x :: xs &\triangleq \lambda n. \lambda c. c\ x\ (xs\ n\ c)
 \end{aligned}$$

The Church encoding works for all **inductive types**.

Girard, Taylor, Lafont, **Proofs and types**, 1990, §11.3–11.5.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends

Metatheory

- 1 Data types
 - Primitive sums and products
 - Algebraic data types
 - Church encodings

- 2 Existential types

Examples

Metatheory

Church encoding

- 3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Complex numbers are an **abstract concept**.

Outside of their implementation, how they are represented **should be irrelevant**, and one should not depend on implementation details.

In one section, Professor Descartes announced that a complex number was an ordered pair of reals [...].

In the other section, Professor Bessel announced that a complex number was an ordered pair of reals, the first of which was nonnegative [...].

An unfortunate mistake [...] caused the two sections to be interchanged.

Reynolds, **Types, Abstraction and Parametric Polymorphism**, 1983.

Complex numbers as an abstract type

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

In OCaml, one might implement complex numbers as an **abstract type**:

```
module Complex : sig
  type t
  val zero: t
  val one: t
  val add: t -> t -> t
  val mul: t -> t -> t
  val (=): t -> t -> bool
  (* etc. *)
end
```

Complex numbers as an existential type

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

In System F , this idea can be made precise via an **existential type**:

$$\text{Complex} : \exists X. \left\{ \begin{array}{l} \text{zero} : X \\ \text{add} : X \rightarrow X \rightarrow X \\ \text{mul} : X \rightarrow X \rightarrow X \\ \text{eq} : X \rightarrow X \rightarrow \text{bool} \\ \text{etc.} \end{array} \right\}$$

Mitchell and Plotkin, **Abstract types have existential type**, 1988.

Rossberg, Russo, Dreyer, **F-ing Modules**, 2014.

Streams as an existential type

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

Imagine we wish to define an abstract type of **streams**.

A stream is a **producer** of a sequence of elements,
out of which a **consumer** can **pull** elements on demand.

It is an “object” with a single method, *next*.

- a stream has a certain **current internal state**.
- *next* returns either nothing or a pair of an element and a new state.

A stream is analogous to a Java iterator, except it is **not mutable**.
Its current state is explicit.

$$S(X) \triangleq \exists S. \underbrace{(S \rightarrow 1 + X \times S)}_{\text{next}} \times \underbrace{S}_{\text{cur}}$$

Streams as an existential type

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrint & friends
Metatheory

$('a, 's)$ step corresponds to $1 + X \times S$:

```

type ('a, 's) step =
  | Done                                (* the stream is exhausted *)
  | Yield of 'a * 's      (* here is an element and a new state *)

```

OCaml views existential types as a special case of [algebraic data types](#):

```

type 'a stream =
  | Stream:
      (* The [next] method: *) ('s -> ('a, 's) step) *
      (* The current state: *) 's
      (* together form a stream: *) -> 'a stream

```

The data constructor **Stream** has [universal type](#): it is polymorphic in $'s$.

The producer chooses the type of the internal state;
the consumer must treat this type as abstract.

Converting a list to a stream

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

This conversion function is a nonrecursive **producer**:

```
let stream (xs : 'a list) : 'a stream =  
  let next xs =  
    match xs with  
    | [] -> Done  
    | x :: xs -> Yield (x, xs)  
  in  
  Stream (next, xs) (* packing an existential type *)
```

On the last line, what is the concrete type of states?

Converting a list to a stream

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

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  let next xs =  
    match xs with  
    | [] -> Done  
    | x :: xs -> Yield (x, xs)  
  in  
  Stream (next, xs) (* packing an existential type *)
```

On the last line, what is the concrete type of states?

It is 'a list.

Converting a stream to a list

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

This conversion function is a recursive **consumer**:

```
let unstream (Stream (next, s) : 'a stream) : 'a list =  
  let rec unfold s =  
    match next s with  
    | Done          -> []  
    | Yield (x, s) -> x :: unfold s  
  in  
  unfold s
```

The first line uses **pattern matching** to **unpack** an existential type.

What is the type of `unfold`?

Converting a stream to a list

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

This conversion function is a recursive **consumer**:

```
let unstream (Stream (next, s) : 'a stream) : 'a list =  
  let rec unfold s =  
    match next s with  
    | Done          -> []  
    | Yield (x, s) -> x :: unfold s  
  in  
  unfold s
```

The first line uses **pattern matching** to **unpack** an existential type.

What is the type of `unfold`?

It is `s -> 'a list`

where `s` is an abstract type introduced by unpacking at line 1.

Examples of stream producers

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends

Metatheory

How would you implement a singleton stream?

Examples of stream producers

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

How would you implement a singleton stream?

```
let return (x : 'a) : 'a stream =  
  let next s =  
    if s then Yield (x, false) else Done  
  in  
  Stream (next, true)           (* packing an existential type *)
```

On the last line, the concrete type of states is **bool**:
either we have already yielded an element, or we have not.

Exercise: Write interval of type **int** -> **int** -> **int** stream.

Exercise: Write append of type 'a stream -> 'a stream -> 'a stream.

An example consumer-and-producer

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

The map function on streams is also non-recursive:

```
let map (f : 'a -> 'b) (xs : 'a stream) : 'b stream =  
  let Stream (next, s) = xs in (* unpacking *)  
  let next s =  
    match next s with  
    | Done          -> Done  
    | Yield (x, s) -> Yield (f x, s)  
  in  
  Stream (next, s) (* packing *)
```

Streams as an existential type

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

This encoding of streams is used in practice.

In addition to **Done** and **Yield**, a third constructor **Skip** can be used, meaning “please ask again”

A consumer must ask, ask, ask until a non-**Skip** result is produced.

This allows most stream producers to be **nonrecursive** functions.

This makes optimization easier.

Coutts, Leshchinskiy, Stewart, **Stream fusion:
from lists to streams to nothing at all**, 2007.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends

Metatheory

- 1 Data types
 - Primitive sums and products
 - Algebraic data types
 - Church encodings

- 2 Existential types

Examples

Metatheory

Church encoding

- 3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

System F with existential types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

The syntax of types is extended with **existential types**:

$$T ::= \dots \mid \exists X. T$$

The syntax of terms is extended with **introduction** and **elimination** forms:

$$\begin{aligned} t &::= \dots \mid \text{pack } T, t \text{ as } \exists X. T \mid \text{let } X, x = \text{unpack } t \text{ in } t \\ v &::= \dots \mid \text{pack } T, v \text{ as } \exists X. T \\ E &::= \dots \mid \text{pack } T, E \text{ as } \exists X. T \mid \text{let } X, x = \text{unpack } E \text{ in } t \end{aligned}$$

A new reduction rule is introduced:

$$\text{let } X, x = \text{unpack } (\text{pack } T', v \text{ as } \exists X. T) \text{ in } t \longrightarrow t[v/x][T'/X]$$

∃-INTRO

$$\frac{\Gamma \vdash t : \tau}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X. T : \exists X. T}$$

 \exists -ELIM

$$\frac{\Gamma \vdash t_1 : \tau_1}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : \tau_2}$$

V-INTRO

$$\frac{\Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X. t : \forall X. T}$$

V-ELIM

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t \ T' : T[T'/X]}$$

36/115

$$\frac{\exists\text{-INTRO} \quad \Gamma \vdash t : T[T'/X]}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T} \quad \frac{\exists\text{-ELIM} \quad \Gamma \vdash t_1 : \exists X.T}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$
$$\frac{\begin{array}{c} \text{V-INTRO} \\ \Gamma; X \vdash t : T \quad X \# \Gamma \end{array}}{\Gamma \vdash \Lambda X. t : \forall X. T} \quad \frac{\begin{array}{c} \text{V-ELIM} \\ \Gamma \vdash t : \forall X. T \end{array}}{\Gamma \vdash t T' : T[T'/X]}$$

36/115

$$\frac{\exists\text{-INTRO} \quad \Gamma \vdash t : T[T'/X]}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T} \quad \frac{\exists\text{-ELIM} \quad \Gamma \vdash t_1 : \exists X.T}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$
$$\frac{\text{V-INTRO} \quad \Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X. t : \forall X. T} \quad \frac{\text{V-ELIM} \quad \Gamma \vdash t : \forall X. T}{\Gamma \vdash t T' : T[T'/X]}$$

36/115

∃-INTRO

$$\frac{\Gamma \vdash t : T[T'/X]}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T}$$

 \exists -ELIM

$$\frac{\Gamma \vdash t_1 : \exists X. T \quad \Gamma; X; x : T \vdash t_2 : T_2}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$

V-INTRO

$$\frac{\Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X. t : \forall X. T}$$

V-ELIM

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t T' : T[T'/X]}$$

36/115

System F with existential types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Print & friends

Metatheory

Two new typing rules are introduced:

$$\frac{\exists\text{-INTRO} \quad \Gamma \vdash t : T[T'/X]}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T}$$

$$\frac{\exists\text{-ELIM} \quad \begin{array}{l} \Gamma \vdash t_1 : \exists X.T \quad X \# \Gamma, T_2 \\ \Gamma; X; x : T \vdash t_2 : T_2 \end{array}}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$

For reference, recall the typing rules for universal types:

$$\frac{\forall\text{-INTRO} \quad \begin{array}{l} \Gamma; X \vdash t : T \quad X \# \Gamma \end{array}}{\Gamma \vdash \Lambda X.t : \forall X.T}$$

$$\frac{\forall\text{-ELIM} \quad \Gamma \vdash t : \forall X.T}{\Gamma \vdash t T' : T[T'/X]}$$

Exercise: extend the proofs of Subject Reduction and Progress.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Universal/existential duality

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

When a value has universal type $\forall X.T$,
the **producer** of this value must treat X as abstract
and the **consumer** can choose a type T' with which to instantiate X .

When a value has existential type $\exists X.T$,
the **producer** chooses a type T' with which to instantiate X
but the **consumer** must treat X as abstract.

When a value has existential type, **its consumer must be polymorphic**.

Church encoding of existential types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

Existential types can in fact be **encoded** in terms of universal types:

$$\begin{array}{lcl} \exists X. T & \triangleq & \forall Y. (\forall X. T \rightarrow Y) \rightarrow Y \\ \text{pack } T', v \text{ as } \exists X. T & \triangleq & \end{array}$$

Church encoding of existential types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Existential types can in fact be **encoded** in terms of universal types:

$$\begin{aligned}
 \exists X. T &\triangleq \forall Y. (\forall X. T \rightarrow Y) \rightarrow Y \\
 \text{pack } T', v \text{ as } \exists X. T &\triangleq \Lambda Y. \lambda k : (\forall X. T \rightarrow Y). k \ T' \ v \\
 \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2 &\triangleq
 \end{aligned}$$

Church encoding of existential types

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Existential types can in fact be **encoded** in terms of universal types:

$$\begin{aligned}
 \exists X. T &\triangleq \forall Y. (\forall X. T \rightarrow Y) \rightarrow Y \\
 \text{pack } T', v \text{ as } \exists X. T &\triangleq \Lambda Y. \lambda k : (\forall X. T \rightarrow Y). k \ T' \ v \\
 \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2 &\triangleq t_1 \ T_2 (\Lambda X. \lambda x : T \rightarrow T_2. t_2)
 \end{aligned}$$

This encoding validates the logical implication $\exists X. T \rightarrow \neg \forall X. \neg T$ where $\neg T$ is defined as $T \rightarrow 0$.

Exercise: check that this encoding validates the reduction rule and the typing rules proposed earlier for primitive existential types.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

- 1 Data types
 - Primitive sums and products
 - Algebraic data types
 - Church encodings

- 2 Existential types
 - Examples
 - Metatheory
 - Church encoding

- 3 GADTs
 - A well-typed tagless interpreter
 - Runtime type descriptions
 - A well-typed type-checker
 - Type-checking printf and friends
 - Metatheory

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Untyped expressions

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Consider a tiny language of expressions $t ::= k \mid (t, t) \mid \pi_i t :$

```
type expr =  
| EInt of int  
| EPair of expr * expr  
| EFst of expr  
| ESnd of expr
```

Expressions include integer constants, pairs, and projections.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

A straightforward interpreter for this language uses a type of all values:

```
type value =  
| VInt of int  
| VPair of value * value
```

This is an algebraic data type. Thus every value carries a [tag](#).

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

These tags are used in **runtime tests** that can cause **runtime errors**.

```
let as_pair (v : value) : value * value =  
  match v with  
  | VPair (v1, v2) ->  
    v1, v2  
  | _ ->  
    assert false (* runtime error! *)
```

An untyped interpreter

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrint & friends
Metatheory

Here, interpreting a pair projection operation involves a runtime test.

```
let rec eval (e : expr) : value =  
  match e with  
  | EInt x ->  
    VInt x  
  | EPair (e1, e2) ->  
    VPair (eval e1, eval e2)  
  | EFst e ->  
    fst (as_pair (eval e))  
  | ESnd e ->  
    snd (as_pair (eval e))
```

This is **necessary** because this interpreter accepts untyped expressions.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Let us impose a [simple type discipline](#) on expressions.

```

type _ expr =
| EInt   :      int ->      int expr
| EPair  : 'a expr * 'b expr -> ('a * 'b) expr
| EFst   :      ('a * 'b) expr ->      'a expr
| ESnd   :      ('a * 'b) expr ->      'b expr

```

This type definition encodes the following type discipline:

$$\begin{array}{c}
 \Gamma \vdash k : \text{int} \\
 \Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2 \\
 \hline
 \Gamma \vdash (t_1, t_2) : T_1 \times T_2
 \end{array}
 \qquad
 \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i}$$

A [meta-level](#) AST of type 'a **expr**
represents an [object-level](#) expression of type 'a.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Let us similarly impose a type discipline on values:

```
type _ value =  
| VInt   :          int ->      int value  
| VPair  : 'a value * 'b value -> ('a * 'b) value
```

Values are still tagged (for now), but runtime tests become unnecessary...

Look Ma, no runtime test!

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Only one branch is now necessary. A second branch would be *dead*.

```
let as_pair : type a b . (a * b) value -> a value * b value
= function
  | VPair (v1, v2) ->
      v1, v2
  (* In this branch, we would learn [a * b = int], *)
  (* which is contradictory. *)
  (* | _ -> . *)
```

In OCaml, destructing a GADT requires a type annotation in this style.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Evaluating an expression of type T yields a value of type T .

```

let rec eval : type a . a expr -> a value
= function
  | EInt x ->
    (* We learn [a = int] so returning [VInt _] is OK. *)
    VInt x
  | EPair (e1, e2) ->
    (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
    (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
    VPair (eval e1, eval e2)
  | EFst e ->
    fst (as_pair (eval e))
  | ESnd e ->
    snd (as_pair (eval e))

```

The type of the interpreter reflects the [subject reduction](#) property.
Type-checking it amounts to [checking the proof](#) of subject reduction!

A typed, tagless interpreter

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

Evaluating an expression of type T yields a **meta-level value** of type T .

```

let rec eval : type a . a expr -> a
= function
  | EInt x ->
    (* We learn [a = int] so returning an integer is OK. *)
    x (* no tagging! *)
  | EPair (e1, e2) ->
    (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
    (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
    (eval e1, eval e2) (* no tagging! *)
  | EFst e ->
    fst (eval e) (* no untagging! *)
  | ESnd e ->
    snd (eval e) (* no untagging! *)

```

The type of the interpreter reflects the **subject reduction** property.
Type-checking it amounts to **checking the proof** of subject reduction!

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Our tiny expressions are **closed**: the typing judgement is $\vdash t : T$.

When expressions involve variables, one needs a type $(\text{'g}, \text{'a}) \text{ expr}$ whose definition encodes the typing judgement $\Gamma \vdash t : T$.

This is reasonably easy if variables are encoded as de Bruijn indices.

Bird, Paterson, **de Bruijn notation as a nested datatype**, 1999.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker
Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Runtime type descriptions

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

A **value** of type `'a ty` is a runtime description of the **type** `'a`.

```
type 'a ty =  
| TyInt   : int ty  
| TySum   : 'a ty * 'b ty -> ('a, 'b) sum ty  
| TyPair  : 'a ty * 'b ty -> ('a * 'b) ty
```

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Although **inspecting a type** at runtime is impossible, as types are erased, **inspecting a runtime description of a type** is possible.

In other words, although the type $\forall X. X \rightarrow X$ has only **one** inhabitant, the type $\forall X. \text{Ty } X \rightarrow X \rightarrow X$ has **more than one**.

This let us write **polymorphic, type-directed** functions, an activity that is sometimes known as **generic programming**.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Here is a polymorphic, type-directed **conversion of a value to a string**.

```
let rec show : type a . a ty -> a -> string =  
  fun ty x ->  
    match ty with  
    | TyInt ->  
      string_of_int x  
    | TySum (ty1, ty2) ->  
      begin match x with  
      | Left x1 -> "left(" ^ show ty1 x1 ^ ")"  
      | Right x2 -> "right(" ^ show ty2 x2 ^ ")"  
      end  
    | TyPair (ty1, ty2) ->  
      let (x1, x2) = x in  
      "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

In each branch, **we learn something** about the type of x .

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends
Metatheory

It is more concise and looks better to deconstruct both arguments at once.

```
let rec show : type a . a ty -> a -> string =  
  fun ty x ->  
    match ty, x with  
    | TyInt, x ->  
      string_of_int x  
    | TySum (ty1, _), Left x1 ->  
      "left(" ^ show ty1 x1 ^ ")"  
    | TySum (_, ty2), Right x2 ->  
      "right(" ^ show ty2 x2 ^ ")"  
    | TyPair (ty1, ty2), (x1, x2) ->  
      "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

The OCaml type-checker reads patterns from left to right
so deconstructing (ty, x) works but deconstructing (x, ty) does not.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrint & friends
Metatheory

Here is a polymorphic, type-directed [equality test](#).

```

let rec equal : type a . a ty -> a -> a -> bool =
  fun ty x y ->
    match ty, x, y with
    | TyInt, x, y ->
      Int.equal x y
    | TySum (ty1, _), Left x1, Left y1 ->
      equal ty1 x1 y1
    | TySum (_, ty2), Right x2, Right y2 ->
      equal ty2 x2 y2
    | TySum _, Left _, Right _
    | TySum _, Right _, Left _ ->
      false
    | TyPair (ty1, ty2), (x1, x2), (y1, y2) ->
      equal ty1 x1 y1 && equal ty2 x2 y2
let rec equal : type a b . a ty -> b ty -> (a, b) eq =
  fun ty1 ty2 ->
    match ty1, ty2 with
    | TyInt, TyInt ->

```

Connections between GADTs and type classes

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

Eq and **Show** are typical examples of **type classes** in Haskell.

Upcoming lecture on type classes (PED).

Hinze, Jeuring, Löh,

Comparing Approaches to Generic Programming in Haskell, 2006.

Connections between GADTs and type classes

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker
Printf & friends
Metatheory

A **closed** set of type class instances can be compiled down to GADTs.

Pottier and Gauthier,

Polymorphic typed defunctionalization and concretization, 2006.

However a GADT describes a **closed** universe of **structural** types whereas type classes are **open-ended** and apply to **user-defined**, **nominal** types.

The Holy Grail is to propose a language where **a type of the representations of all types** (including itself!) can be defined.

Chapman, Dagand, McBride, Morris,
The Gentle Art of Levitation, 2010.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

- 1 Data types
 - Primitive sums and products
 - Algebraic data types
 - Church encodings

- 2 Existential types
 - Examples
 - Metatheory
 - Church encoding

- 3 GADTs
 - A well-typed tagless interpreter
 - Runtime type descriptions
 - A well-typed type-checker
 - Type-checking printf and friends
 - Metatheory

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

We have a type `'a` expr of well-typed expressions
and a type `'a` ty of runtime type descriptions.

Can we express a simple type **type inferencer** that accepts an untyped expression and **either fails or returns a typed expression**?

```
exception IllTyped
let rec infer : Raw.expr -> ???
= function
  | Raw.EInt i ->
    (TyInt, EInt i)
  | ...
```

What should its **result type** be?

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
Metatheory
Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrint & friends
Metatheory

A type inferencer

We need an **existential type** $\exists X. \text{Ty } X \times \text{Expr } X$.

```
type typed_expr =  
| Pack    : 'a ty * 'a expr -> typed_expr
```

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

We can now write the type inferencer:

```
let rec infer : Raw.expr -> typed_expr =  
  function  
  | Raw.EInt i ->  
    Pack (TyInt, EInt i)  
  | Raw.EFst e ->  
    let Pack (ty, e) = infer e in  
    begin match ty with  
    | TyPair (ty1, ty2) -> Pack (ty1, EFst e)  
    | -                    -> raise IllTyped  
    end
```

Exercise: write the two missing cases.

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
  if inferred = expected then  
    e  
  else  
    raise IllTyped
```

But **this code is not well-typed**. Why?

Data types

- Primitive sums and products

- Algebraic data types

- Church encodings

Existential types

- Examples

- Metatheory

- Church encoding

GADTs

- Tagless interpreters

- Runtime type descriptions

- A well-typed type-checker

- Printf & friends

- Metatheory

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
  if inferred = expected then  
    e  
  else  
    raise IllTyped
```

But **this code is not well-typed**. Why?

expected has type $a \text{ ty}$.

inferred has type $b \text{ ty}$

where b is an unknown type introduced by deconstructing **Pack**.

Data types

Primitive sums

and products

Algebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
  if inferred = expected then  
    e  
  else  
    raise IllTyped
```

But **this code is not well-typed**. Why?

expected has type $a \text{ ty}$.

inferred has type $b \text{ ty}$

where b is an unknown type introduced by deconstructing **Pack**.

They **cannot be compared** using homogeneous equality = .

Even if they could, e has type $b \text{ expr}$

whereas a result of type $a \text{ expr}$ is required.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrint & friends
Metatheory

The equality GADT

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

The solution involves the **type equality GADT**.

```
type (_, _) eq =  
  | Equal: ('a, 'a) eq
```

The type `('a, 'b) eq` has at most one inhabitant.

If it has one then this inhabitant must be **Equal**
and the types `'a` and `'b` must be the same.

A heterogenous type equality test

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

This lets us express a **heterogenous** type equality test:

```

let rec equal : type a b . a ty -> b ty -> (a, b) eq =
  fun ty1 ty2 ->
    match ty1, ty2 with
    | TyInt, TyInt ->
      Equal
    | TyPair (ty1a, ty1b), TyPair (ty2a, ty2b) ->
      let Equal = equal ty1a ty2a in
      let Equal = equal ty1b ty2b in
      Equal
    | -, - ->
      raise IllTyped

```

When `equal ty1 ty2` succeeds, we **learn** that the runtime type descriptions `ty1` and `ty2` describe the same static type.

Exercise: write the missing case.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

A type-checker

We can now write the type-checker:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
  let Equal = equal inferred expected in  
  e
```

Exercise: make sure that you understand why this code is well-typed.

Putting the pieces together

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Given an arbitrary untyped expression in our tiny language, we can now **infer** its type, **evaluate** it, and **show** its value, whatever its type may be.

```
let () =  
  let e = Raw.(EPair (EInt 42, EInt 0)) in  
  let Pack (ty, e) = infer e in  
  let v = eval e in  
  Printf.printf "%s\n%!" (show ty v)
```

The output in the REPL is:

```
(42, 0)
```

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends
Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

printf takes a “format string” followed with a number of arguments:

```
# open Printf;;  
# printf "%d * %s = %d\n" 2 "12" 24;;  
2 * 12 = 24  
- : unit = ()
```

The number and type of these arguments *depends* on the format string.

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

A format string is actually **not** a string: it is a **data structure**.

```
# open CamlinternalFormatBasics;;
# let desc : _ format6 = "%d * %s = %d\n";;
val desc :
  (int -> string -> int -> 'a, 'b, 'c, 'd, 'd, 'a) format6 =
  Format
    (Int (Int_d, No_padding, No_precision,
      String_literal (" * ",
        String (No_padding,
          String_literal (" = ",
            Int (Int_d, No_padding, No_precision,
              Char_literal ('\n', End_of_format))))),
      "%d * %s = %d\n")
```

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

This data structure has the shape of a list:

```
Int (Int_d, No_padding, No_precision,  
String_literal (" * ",  
String (No_padding,  
String_literal (" = ",  
Int (Int_d, No_padding, No_precision,  
Char_literal ('\n',  
End_of_format))))))
```

End_of_format is “nil”; the other constructors are “cons” constructors.

Int and **String** correspond to “holes” %d and %s.

String_literal and **Char_literal** correspond to literal pieces of string.

An algebraic data type of descriptors

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends

Metatheory

Can we define our own algebraic data type of formats, or **descriptors**?

```
type desc =  
  | Nil  
  | Lit of string * desc  
  | Int of desc
```


An algebraic data type of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Or, in this alternative syntax:

```
type desc =  
  | Nil : desc  
  | Lit : string * desc -> desc  
  | Int : desc -> desc
```

An algebraic data type of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Or, in this alternative syntax:

```
type desc =  
  | Nil : desc  
  | Lit : string * desc -> desc  
  | Int : desc -> desc
```

Now, please define `fprintf` so that `fprintf emit desc <args>`

- emits output via the function `emit : string -> unit`,
- obeys `desc`,
- expects arguments `<args>` whose number and type satisfy `desc`.

`fprintf` should have type `(string -> unit) -> desc -> ??? -> unit`.

Expressing the type of `fprintf`

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

The type `desc -> ??? -> unit` does not make sense.

The number of and type of the arguments `???` *depends* on the descriptor.

We seem to need a *dependent type* `(d: desc) -> shape d`

- where `shape` would be a function of descriptors to types,
- but OCaml does not have that.

Expressing the type of `fprintf`

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

The type `desc -> ??? -> unit` does not make sense.

The number of and type of the arguments `???` **depends** on the descriptor.

We seem to need a **dependent type** `(d: desc) -> shape d`

- where `shape` would be a function of descriptors to types,
- but OCaml does not have that.

Instead, let's use a plain function type `'shape desc -> 'shape`

- where the definition of `'shape desc` as a GADT encodes the correspondence between descriptors and shapes.

Descriptors form a **typed language** and `fprintf` is an **interpreter** for it!

A GADT of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

We want `fprintf : (string -> unit) -> 'a desc -> 'a`.

```

type desc =
  | Nil : desc
  | Lit : string * desc -> desc
  | Int : desc -> desc

```

We must turn the type `desc` into a GADT.

A GADT of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =  
  | Nil :                                ?? desc  
  | Lit : string * ?? desc ->          ?? desc  
  | Int :                               ?? desc ->    ?? desc
```

We parameterize the type `desc`.

A GADT of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =  
  | Nil : unit desc  
  | Lit : string * ?? desc -> ?? desc  
  | Int : ?? desc -> ?? desc
```

`Nil` requires no action; the corresponding shape is `unit`.

A GADT of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil      :                               unit desc
  | Lit      : string * 'a desc ->          'a desc
  | Int      : ?? desc ->                  ?? desc
```

`Lit` (`s`, `d`) requires printing `s` and interpreting `d`.

If `d` has shape `'a` then `Lit` (`s`, `d`) has shape `'a` as well.

A GADT of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

We want `fprintf : (string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil      : unit desc
  | Lit      : string * 'a desc -> 'a desc
  | Int      : 'a desc -> (int -> 'a) desc
```

`Int d` requires consuming an integer argument and interpreting `d`.

If `d` has shape `'a` then `Int d` has shape `int -> 'a`.

A GADT of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =  
  | Nil      : unit desc  
  | Lit      : string * 'a desc -> 'a desc  
  | Hole     : ('data -> string) * 'a desc -> ('data -> 'a) desc
```

We change the hole of type `int` with a hole of arbitrary type `'data`.

All that is needed is a conversion function of type `'data -> string`.

Implementing fprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ???
    | Lit (s, desc) ->
      ???
    | Hole (to_string, desc) ->
      ???
  in eval desc

```

Recall

```

| Nil : unit desc

```

Implementing fprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ??? (* We learn [a = unit]. *)
    | Lit (s, desc) ->
      ???
    | Hole (to_string, desc) ->
      ???

  in eval desc

```

Recall

```

| Nil : unit desc

```

Implementing fprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      () (* We learn [a = unit]. *)
    | Lit (s, desc) ->
      ???
    | Hole (to_string, desc) ->
      ???

  in eval desc

```

Recall

```

| Lit : string * 'a desc -> 'a desc

```

Implementing fprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      ??? (* We learn no new type equality. *)
    | Hole (to_string, desc) ->
      ???

  in eval desc

```

Recall

```

| Lit : string * 'a desc -> 'a desc

```

Implementing fprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        ()  
      | Lit (s, desc) ->  
        emit s; eval desc  
      | Hole (to_string, desc) ->  
        ???  
  in eval desc
```

Recall

```
| Hole : ('data -> string) * 'a desc -> ('data -> 'a) desc
```

Implementing fprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      ??? (* We learn [a = data -> b] *)
          (* [to_string : data -> string; desc : b desc] *)
  in eval desc

```

Recall

```

| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc

```


Implementing fprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

```

let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      fun x -> emit (to_string x); eval desc
      (* [x] has type [data]; [eval desc] has type [b] *)
  in eval desc
  (* and [data -> b] is [a] *)

```

Recall

```

| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc

```

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Using fprintf

Voilà! From fprintf, we get printf.

```
let printf desc =  
  let emit = print_string in  
  fprintf emit desc
```

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

To construct descriptors, some sugar is needed.

```
module Sugar = struct
  let nil = Nil
  let lit s desc = Lit (s, desc)
  let d desc = Hole (string_of_int, desc)
  let s desc = Hole (Fun.id, desc)
end
```

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

To construct descriptors, some sugar is needed.

```

module Sugar = struct
  let nil = Nil
  let lit s desc = Lit (s, desc)
  let d desc = Hole (string_of_int, desc)          (* %d *)
  let s desc = Hole (Fun.id, desc)                  (* %S *)
end

```

For example,

```

let desc =                                     (* "%d * %S = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil

```

Using `fprintf`

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

```
let desc = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf desc 2 "12" 24;;
2 * 12 = 24
```

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Can we implement `sprintf`, which returns a string?

```
let sprintf desc args =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b in  
  fprintf emit desc args;  
  Buffer.contents b
```

This is accepted but is **not** what we want.

Implementing sprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Can we implement `sprintf`, which returns a string?

```
let sprintf desc <arg ... arg> =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b in  
  fprintf emit desc <arg ... arg>;  
  Buffer.contents b
```

We want `sprintf` to accept [a variable number of arguments](#), not just one.

In fact, we cannot write the type of `sprintf`.

It is like the type of `fprintf` but should end in `string` instead of `unit`.

A more general type of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

We must equip ourselves with a **more general** type of descriptors.

```

type _ desc =
  | Nil : unit desc
  | Lit : string * 'a desc -> 'a desc
  | Hole : ('data -> string) * 'a desc -> ('data -> 'a ) desc

```


A more general type of descriptors

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

We must equip ourselves with a **more general** type of descriptors.

```
type (_, _) desc =  
  | Nil : ( 'r, 'r) desc  
  | Lit : string * ( 'a, 'r) desc -> ( 'a, 'r) desc  
  | Hole : ( 'data -> string) * ( 'a, 'r) desc ->  
           ( 'data -> 'a, 'r) desc
```

In the type ('a, 'r) desc,

- 'a is the **shape**, as before,
- 'r is the **eventual return type** of this shape.
 - it can be **unit** for `fprintf` and **string** for `sprintf`;
 - a descriptor can be polymorphic in 'r.

Implementing fprintf, again

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
types

Examples

Metatheory

Church
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

We can now give `fprintf` a **more general** type. We parameterize it with:

- `emit : string -> unit`
- `finished : unit -> r` — **new**
- `desc : (a, r) desc`

`fprintf emit finished desc` has type `a`.

`a` must in fact be a function type whose eventual return type is `r`.

`fprintf emit finished desc <args>` must eventually return a value of type `r`, which it obtains by calling `finished()`.

Implementing printf, again

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

```

let fprintf (type a r) emit (finished : unit -> r)
    (desc : (a, r) desc) : a =
  let rec eval : type a . (a, r) desc -> a =
    function
    | Nil ->
      (* We have [a = r] so [finished()] has type [a]. *)
      finished()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      fun x -> emit (to_string x); eval desc
  in eval desc

```

It is worth pointing out that `eval` involves [polymorphic recursion](#).

Implementing printf and sprintf

Data types

Primitive sums
and products
Algebraic data
types
Church
encodings

Existential
types

Examples
Metatheory
Church
encoding

GADTs

Tagless
interpreters
Runtime type
descriptions
A well-typed
type-checker
Printf & friends
Metatheory

We can now implement `printf` and `sprintf`, among other variations:

```
let printf desc =  
  let emit = print_string  
  and finished () = () in  
  fprintf emit finished desc  
  
let sprintf desc =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b  
  and finished () = Buffer.contents b in  
  fprintf emit finished desc
```

We get

```
val printf : ('a, unit) desc -> 'a  
val sprintf : ('a, string) desc -> 'a
```

Using printf and sprintf

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

```
let desc () = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf (desc()) 2 "12" 24;;
2 * 12 = 24
# let (s : string) = sprintf (desc()) 2 "12" 24;;
val s : string = "2 * 12 = 24\n"
```

Here, we make desc a (constant) function in order to work around the [value restriction](#). See upcoming lecture on mutable state (GS).

Danvy et al.'s approach

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Danvy, Keller and Puech (2015) view formats as **trees** instead of **lists**.

```
type (_, _) desc =
  | Lit   : string -> ('a, 'a) desc
  | Hole  : ('data -> string) -> ('data -> 'a, 'a) desc
  | Seq   : ('a, 'b) desc * ('b, 'c) desc -> ('a, 'c) desc
```

The type `('a, 'r) desc` has the same meaning as earlier.

Lit and **Hole** no longer play the role of list “cons” constructors.

Seq is a binary concatenation constructor, whose type says:

*If 'a is a multi-arrow type whose eventual return type is 'b and
if 'b is a multi-arrow type whose eventual return type is 'c then
'a is a multi-arrow type whose eventual return type is 'c.*

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Danvy et al. write `kprintf` in [continuation-passing style](#):

```
let rec kprintf
: type a r . (a, r) desc -> (string -> r) -> a =
  fun desc finished ->
    match desc with
    | Lit s ->
      finished s
    | Hole to_string ->
      fun x -> finished (to_string x)
    | Seq (desc1, desc2) ->
      kprintf desc1 @@ fun s1 ->
        kprintf desc2 @@ fun s2 ->
          finished (s1 ^ s2)
```

Exercise (easy): define `printf`, `sprintf`, and `fprintf` using `kprintf`.

Exercise (harder): define `fprintf` directly.

Do not use string concatenation `^`.

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential
types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends

Metatheory

1 Data types

Primitive sums and products

Algebraic data types

Church encodings

2 Existential types

Examples

Metatheory

Church encoding

3 GADTs

A well-typed tagless interpreter

Runtime type descriptions

A well-typed type-checker

Type-checking printf and friends

Metatheory

System F +GADTs

Data types

Primitive sums
and products
Algebraic data
types
Church
encodings

Existential types

Examples
Metatheory
Church
encoding

GADTs

Tagless
interpreters
Runtime type
descriptions
A well-typed
type-checker
Printf & friends
Metatheory

System F +GADTs was defined by Xi, Chen and Chen (2003).

Xi, Chen, Chen,
Guarded Recursive Datatype Constructors, 2003.

Pottier and Gauthier,
Polymorphic typed defunctionalization and concretization, 2006.

System F +GADTs: the typing judgement

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Printf & friends

Metatheory

Recall the typing judgement of System F :

$$\Gamma \vdash t : T$$

In System F +GADTs, must we change the shape of this judgement?

System F +GADTs: the typing judgement

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrintf & friends
Metatheory

Recall the typing judgement of System F :

$$\Gamma \vdash t : T$$

In System F +GADTs, must we change the shape of this judgement?

We must extend it with a conjunction of **equality hypotheses**.

$$\Gamma, C \vdash t : T$$

Equality constraints are given by $C, D ::= \text{True} \mid T = T \mid C \wedge C$.

System F +GADTs: type declarations

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checkerPrint & friends
Metatheory

We assume that a family of type constructors F is given.

- for simplicity, we assume they have arity 1.

We assume that a family of data constructors K is given.

- for simplicity, we assume they have arity 1.

We assume that each data constructor has a closed **type scheme**:

$$K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2$$

System F +GADTs: an auxiliary judgement

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

For readability, we introduce the auxiliary judgement

$$K \leq D \Rightarrow T_1 \rightarrow F T_2$$

whose definition is the following:

$$\frac{K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2}{K \leq D[\bar{T}/\bar{X}] \Rightarrow T_1[\bar{T}/\bar{X}] \rightarrow F T_2[\bar{T}/\bar{X}]}$$

System F +GADTs: the typing judgement

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

The typing rules of System F are unchanged. A constraint is transported.

$$\begin{array}{c}
 \text{VAR} \\
 \hline
 \Gamma, C \vdash x : \Gamma(x)
 \end{array}
 \qquad
 \begin{array}{c}
 \text{ABS} \\
 \hline
 \frac{\Gamma; x : T_1, C \vdash t : T_2}{\Gamma, C \vdash \lambda x. t : T_1 \rightarrow T_2}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{APP} \\
 \hline
 \frac{\Gamma, C \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma, C \vdash t_2 : T_1}{\Gamma, C \vdash t_1 \ t_2 : T_2}
 \end{array}$$

$$\begin{array}{c}
 \text{TABS} \\
 \hline
 \frac{\Gamma; X, C \vdash t : T \quad X \# \Gamma}{\Gamma, C \vdash \lambda X. t : \forall X. T}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{TAPP} \\
 \hline
 \frac{\Gamma, C \vdash t : \forall X. T}{\Gamma, C \vdash t \ T' : T[T'/X]}
 \end{array}$$

System F +GADTs: the typing judgement,
continued

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker

Print & friends

Metatheory

The typing rule for a **data constructor application** is straightforward:

$$\begin{array}{c}
 \text{DCon} \\
 K \leq D \Rightarrow T_1 \rightarrow F T_2 \\
 C \Vdash D \\
 \Gamma, C \vdash t : T_1 \\
 \hline
 \Gamma, C \vdash K t : F T_2
 \end{array}$$

We write $C \Vdash D$ when C entails D (see next slide).

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

System F +GADTs: entailment

Let ρ denote a total mapping of type variables to closed types.

We write $\rho \vdash C$ when ρ satisfies C :

$$\rho \vdash \text{True} \qquad \frac{\rho(T_1) = \rho(T_2)}{\rho \vdash T_1 = T_2} \qquad \frac{\rho \vdash C_1 \quad \rho \vdash C_2}{\rho \vdash C_1 \wedge C_2}$$

Entailment is then defined by:

$$\frac{\forall \rho. \rho \vdash C \Rightarrow \rho \vdash D}{C \Vdash D}$$

Entailment is decidable.

System F +GADTs: the typing judgement,
continued

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

Type-checking a **case analysis** construct is straightforward:

CASE

$$\frac{\begin{array}{c} \Gamma, C \vdash t : T_1 \\ \forall c \in \bar{c}. \quad \Gamma, C \vdash c : T_1 \rightarrow T_2 \\ \bar{c} \text{ is exhaustive} \end{array}}{\Gamma, C \vdash \text{case } t \text{ of } \bar{c} : T_2}$$

A **clause** takes the form $c ::= K \bar{X} x \mapsto t$.

\bar{c} is exhaustive if it contains a clause for every data constructor K .

System F +GADTs: the typing judgement,
continued

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Print & friends

Metatheory

When a clause is entered, **new constraints appear** locally.

CLAUSE

$$\begin{array}{c}
 K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2 \\
 (\Gamma; \bar{X}; x : T_1), (C \wedge D \wedge F T_2 = F T'_2) \vdash t : T' \\
 \bar{X} \# \Gamma, C, T'_2, T' \\
 \hline
 \Gamma, C \vdash K \bar{X} x \mapsto t : F T'_2 \rightarrow T'
 \end{array}$$

System F +GADTs: the typing judgement,
continued

Data types

Primitive sums
and productsAlgebraic data
typesChurch
encodingsExistential
typesExamples
MetatheoryChurch
encoding

GADTs

Tagless
interpretersRuntime type
descriptionsA well-typed
type-checker
Printf & friends

Metatheory

There remains to introduce a typing rule that **exploits** the hypothesis C :

$$\frac{\text{CONVERSION} \quad \Gamma, C \vdash t : T \quad C \Vdash T = T'}{\Gamma, C \vdash t : T'}$$

This rule is **not** syntax-directed.

One can imagine a variant of the system where conversion is explicit.
System FC is the core language of the Glasgow Haskell compiler.

Sulzmann, Chakravarty, Peyton Jones, Donnelly,
System F with Type Equality Coercions, 2007.

System F_{+} GADTs: type soundness

Data types

Primitive sums
and products

Algebraic data
types

Church
encodings

Existential types

Examples
Metatheory

Church
encoding

GADTs

Tagless
interpreters

Runtime type
descriptions

A well-typed
type-checker

Printf & friends

Metatheory

Exercise: write down the omitted details (e.g., the reduction rule for *case*), then prove Subject Reduction and Progress.