

# Making the stack explicit: the continuation-passing style transformation

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# Motivation

What if a program transformation could:

- ensure that every function call is a **tail call** and the **stack** is **explicit**, so the code is no longer really recursive, but **iterative**;
- make the evaluation order **explicit** in the code, so that it does not depend on the ambient strategy (CBN / CBV);
- eliminate the apparent **redundancy** between calls and returns, by exploiting solely function calls – **functions never return!**
- suggest extending the  $\lambda$ -calculus with **control operators**?

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- eliminate the apparent **redundancy** between calls and returns, by exploiting solely function calls – **functions never return!**
- suggest extending the  $\lambda$ -calculus with **control operators**?

The **continuation-passing style** transformation does all this.

## Motivation



### D. Conversion to Continuation-Passing Style

This phase is the real meat of the compilation process. It is of interest primarily in that it transforms a program written in SCHEME into an equivalent program (the continuation-passing-style version, or CPS version), written in a language isomorphic to a subset of SCHEME with the property that interpreting it requires no control stack or other unbounded temporary storage and no decisions as to the order of evaluation of (non-trivial) subexpressions. The importance of these properties cannot be overemphasized. The fact that it is essentially a subset of SCHEME implies that its semantics are as clean, elegant, and well-understood as those of the original language. It is easy to build an

Steele, **RABBIT: a compiler for SCHEME**, 1978.

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From a direct-style interpreter down to an abstract machine

From recursive traversal down to iterative traversal with link inversion

## 2 Formulations

## 3 Soundness

## 4 Remarks

## 5 Madness in small steps

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## A direct-style interpreter

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Recall our environment-based interpreter for call-by-value  $\lambda$ -calculus:

```
let rec eval (e : cenv) (t : term) : cvalue =  
  match t with  
  | Var x ->  
    lookup e x  
  | Lam t ->  
    Clo (t, e)  
  | App (t1, t2) ->  
    let cv1 = eval e t1 in  
    let cv2 = eval e t2 in  
    let Clo (u1, e') = cv1 in  
    eval (cv2 :: e') u1
```

This is an OCaml transcription, without a fuel parameter.

# A continuation-passing style interpreter

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Instead of **returning** a value,

```
let rec eval (e : cenv) (t : term) : cvalue =  
  ...
```

let's **pass** this value to a **continuation** that we get as an argument:

```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =  
  ...
```

**Exercise** (in class): write evalk. (See **EvalCBVExercise**.)



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```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =  
  match t with  
  | Var x ->  
    k (lookup e x)  
  | Lam t ->  
    k (Clo (t, e))  
  | App (t1, t2) ->  
    evalk e t1 (fun cv1 ->  
      evalk e t2 (fun cv2 ->  
        let Clo (u1, e') = cv1 in  
        evalk (cv2 :: e') u1 k))
```

Instead of **returning** a value, **pass** it to `k`.

Instead of **sequencing** computations via `let`, **nest** continuations.

# A continuation-passing style interpreter

To run the interpreter, start it with the `identity` continuation:

```
let eval (e : cenv) (t : term) : cvalue =  
  evalk e t (fun cv -> cv)
```

## Correctness of the CPS interpreter

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The continuation-passing style interpreter is “obviously” correct.

**Exercise:** define `evalk` in Coq (with fuel) and prove it equivalent to the direct-style interpreter: `evalk n e t k = k (eval n e t)`.

## Properties of the interpreter

What is special about this interpreter?

## Properties of the interpreter

What is special about this interpreter?

- Every call to `evalk` is a **tail call**.
- Every call to a continuation `k` is a **tail call**.

A call  $g\ x$  is a tail call if it is the “last thing” that the calling function does...

More formally,

$$v ::= x \mid \lambda x. tt$$

values

$$tt ::=$$

terms in tail position

$$\mid v$$
$$\mid nt\ nt$$

– a tail call

$$\mid \text{let } nt \text{ in } tt$$
$$\mid \text{if } nt \text{ then } tt \text{ else } tt$$
$$nt ::=$$

terms not in tail position

$$\mid v$$
$$\mid nt\ nt$$

– an ordinary call

$$\mid \text{let } nt \text{ in } nt$$
$$\mid \text{if } nt \text{ then } nt \text{ else } nt$$

OCaml allows us to [verify](#) that these are indeed tail calls:

```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =  
  match t with  
  | Var x ->  
    (k[@tailcall]) (lookup e x)  
  | Lam t ->  
    (k[@tailcall]) (Clo (t, e))  
  | App (t1, t2) ->  
    (evalk[@tailcall]) e t1 (fun cv1 ->  
      (evalk[@tailcall]) e t2 (fun cv2 ->  
        let Clo (u1, e') = cv1 in  
        (evalk[@tailcall]) (cv2 :: e') u1 k))
```

A nice feature (though with somewhat ugly syntax).

## Properties of the interpreter

Tail calls are compiled by OCaml to **jumps**.

Thus, tail-recursive functions are compiled by OCaml to **loops**.

Steele, **Lambda: the ultimate GOTO**, 1977.

Thus, the CPS interpreter is not truly **recursive**: it is **iterative**.

It uses **constant space** on OCaml's implicit stack.

Wait! Does the interpreter really **not need a stack** any more?



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Wait! Does the interpreter really **not need a stack** any more?

- Of course it **does** need a stack.

## Properties of the interpreter

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Wait! Does the interpreter really **not need a stack** any more?

- Of course it **does** need a stack.
- The **continuation**, allocated in the OCaml heap, serves as a stack.

## A defunctionalized CPS interpreter

To better see the structure of the continuation,  
let us **defunctionalize** the CPS interpreter.

Reynolds, **Definitional interpreters**  
**for programming languages**, 1972 (1998).

Reynolds, **Definitional interpreters revisited**, 1998.

## Defunctionalization (reminder)

### Steps:

- Identify the sites where closures are allocated, that is, where anonymous functions are built.
- Compute, at each site, the free variables of the anonymous function.
- Introduce an algebraic data type of closures.
- Transform the code:
  - replace anonymous functions with constructor applications,
  - replace function applications with calls to `apply`,
  - and define `apply`.

**Exercise** (in class): defunctionalize the CPS interpreter. ([EvalCBVExercise.](#))

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There are three sites where an anonymous continuation is built.

We name them and compute their free variables.

This leads to the following algebraic data type of continuations:

```
type kont =  
  | AppL of { e: cenv; t2: term; k: kont }  
  | AppR of {          cv1: cvalue; k: kont }  
  | Init
```

What data structure is this?

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  | Init
```

What data structure is this? A [linked list](#). A heap-allocated stack.

In fact, it is a (call-by-value) [evaluation context](#):

$$E ::= E \ t_2[e] \mid v_1 \ E \mid []$$

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We transform the interpreter's main function:

```
let rec evalkd (e : cenv) (t : term) (k : kont) : cvalue =  
  match t with  
  | Var x ->  
    apply k (lookup e x)  
  | Lam t ->  
    apply k (Clo (t, e))  
  | App (t1, t2) ->  
    evalkd e t1 (AppL { e; t2; k })
```

To evaluate  $t_1\ t_2$ , the interpreter **pushes** information on the stack, then **jumps** straight to evaluating  $t_1$ .

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`apply` interprets continuations as functions of values to values:

```
and apply (k : kont) (cv : cvalue) : cvalue =  
  match k with  
  | AppL { e; t2; k } ->  
    let cv1 = cv in  
    evalkd e t2 (AppR { cv1; k })  
  | AppR { cv1; k } ->  
    let cv2 = cv in  
    let Clo (u1, e') = cv1 in  
    evalkd (cv2 :: e') u1 k  
  | Init ->  
    cv
```

It `pops` the top stack frame and decides what to do, based on it.



# A defunctionalized CPS interpreter

To run the interpreter, start it with the [identity](#) continuation:

```
let eval e t =  
  evalkd e t Init
```

## An abstract machine

We have reached an **abstract machine**, a simple **iterative** interpreter which maintains a few data structures:

- a **code** pointer: the term  $t$ ,
- an **environment**  $e$ ,
- a stack, or **continuation**  $k$ .

In fact, we have mechanically rediscovered the **CEK** machine.

Felleisen and Friedman,  
**Control operators, the SECD machine, and the  $\lambda$ -calculus**, 1987.

Sig Ager, Biernacki, Danvy and Midtgaard,  
**A Functional Correspondence between Evaluators  
and Abstract Machines**, 2003.

## Re-discovering other abstract machines

**Exercise:** start with a **call-by-name** interpreter and follow an analogous process to rediscover Krivine's machine.

The solution is in **EvalCBNCPS**.

*There once was a man named Krivine  
Who invented a wond'rous machine.  
It pushed and it popped  
On abstractions it stopped;  
That lean mean machine from Krivine.*

— **Mitchell Wand**

Krivine, **A call-by-name lambda-calculus machine**, (1985) 2007.

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## A type of binary trees

Consider a simple type of binary trees:

```
type tree =  
  | Leaf  
  | Node of { data: int; left: tree; right: tree }
```

## Direct-style traversal

Suppose we wish to perform a postfix tree traversal:

```
let rec walk (t : tree) : unit =  
  match t with  
  | Leaf ->  
    ()  
  | Node { data; left; right } ->  
    walk left;  
    walk right;  
    printf "%d\n" data
```

This is **recursive** code in **direct style**.

Neither of the recursive calls is a tail call.

Now suppose we wish to make the code [iterative](#). Swoop, CPS!

```
let rec walkk (t : tree) (k : unit -> 'a) : 'a =  
  match t with  
  | Leaf ->  
    k()  
  | Node { data; left; right } ->  
    walkk left (fun () ->  
      walkk right (fun () ->  
        printf "%d\n" data;  
        k()))
```

The traversal is initiated with an identity continuation:

```
let walk t =  
  walkk t (fun t -> t)
```

## CPS traversal, defunctionalized

Next, we might wish to make the stack an explicit [data structure](#).

Swoop, defunctionalization!

The type of defunctionalized continuations:

```
type kont =  
  | Init  
  | GoneL of { data: int; tail: kont; right: tree }  
  | GoneR of { data: int; tail: kont }
```



## CPS traversal, defunctionalized

The main function is a loop that **walks down the leftmost branch** while **pushing** information onto the stack:

```
let rec walkkd (t : tree) (k : kont) : unit =  
  match t with  
  | Leaf ->  
    apply k ()  
  | Node { data; left; right } ->  
    walkkd left (GoneL { data; tail = k; right })
```

Think of the stack as **Ariadne's thread**.

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The apply function comes back up out of a child.

```
and apply k () =  
  match k with  
  | Init ->  
    ()  
  | GoneL { data; tail; right } ->  
    walkkd right (GoneR { data; tail })  
  | GoneR { data; tail } ->  
    printf "%d\n" data;  
    apply tail ()
```

It **pops** information off the stack so as to decide what to do.

When coming out of a left child, go down into its right sibling.

When coming out of a right child, go further up.

And now, for something a little  
**UNEXPECTED** and **WILD**.

And now, for something a little  
UNEXPECTED and WILD.  
**A CRAZY HACK.**



# Recycling

When we **allocate** a **GoneR** continuation,  
we **drop** a **GoneL** continuation at the same time.

Indeed, here, continuations are **linear**. They are used exactly once.

```
| GoneL { data; tail; right } ->  
  walkkd right (GoneR { data; tail })
```

This suggests that the memory block could be **recycled** (re-used).

## More recycling

When we **allocate** a **GoneL** continuation,  
a **Node** goes **temporarily unused** at the same time.

This node won't be accessed until this **GoneL** frame  
first is changed to **GoneR** then is popped off the stack.

```
| Node { data; left; right } ->  
  walkkd left (GoneL { data; tail = k; right })
```

This suggests that the memory block could be **recycled**, too,  
provided we **restore** it when we are done with it.

## A tree is a continuation is a tree

In OCaml, the type of a memory block **cannot** be changed over time.

Thus, recycling tree nodes as stack frames, and vice-versa, requires **trees** and **continuations** to have **the same type**.

Uh?

## A tree is a continuation is a tree

Could we **disguise** a continuation as a tree?

In other words, could a stack frame **fit** in a tree node?

```
type kont =  
  | Init  
  | GoneL of { data: int; tail: kont; right: tree }  
  | GoneR of { data: int;                tail: kont }
```

```
type tree =  
  | Leaf  
  | Node of { data: int; left: tree; right: tree }
```



## A tree is a continuation is a tree

Could we **disguise** a continuation as a tree?

In other words, could a stack frame **fit** in a tree node?

```
type kont =  
  | Init  
  | GoneL of { data: int; tail: kont; right: tree }  
  | GoneR of { data: int;                tail: kont }
```

```
type tree =  
  | Leaf  
  | Node of { data: int; left: tree; right: tree }
```

Yes, kind of.

We just need **one extra bit** of storage per tree node,  
so as to distinguish **GoneL** and **GoneR**.

## A tree is a continuation is a tree

Add one “status” bit per tree node. Make nodes **mutable**.

```
type status = GoneL | GoneR
type mtree  = Leaf | Node of {
  data: int;          mutable status: status;
  mutable left: mtree; mutable right: mtree
}
type mkont = mtree
```

Tree records and continuation records occupy **the same space** in memory.

Thus, a tree record can be turned into a continuation record, and back!

By convention, in a “tree” record, the status field is **GoneL**.

In a “continuation” record,

- **either** status is **GoneL** and the left field stores tail;
- **or** status is **GoneR** and the right field stores tail.

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Instead of allocating a **GoneL** continuation,  
we now **change** the tree record to a continuation record:

```
let rec walkkdi (t : mtree) (k : mkont) : unit =  
  match t with  
  | Leaf ->  
    apply k t  
  | Node ({ left; _ } as n) ->  
    (* Change this tree to a [GoneL] continuation. *)  
    assert (n.status = GoneL);  
    n.left (* n.tail *) <- k;  
    walkkdi left (t : mkont)
```

The `left` field is **overwritten**, which is scary! We must **restore** it later.

We find that, in every call to `walkkdi t k` and `apply k t`,  
`k` is the **parent** of `t` in the tree.

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The rest of the code, in its horrific glory:

```
and apply (k : mkont) (child : mtree) : unit =  
  match k with  
  | Leaf -> ()  
  | Node ({ status = GoneL; left = tail; right; _ } as n) ->  
    n.status <- GoneR;           (* update continuation! *)  
    n.left <- child;             (* restore orig. left child! *)  
    n.right (* n.tail *) <- tail;  
    walkkdi right k  
  | Node ({ data; status = GoneR; right = tail; _ } as n) ->  
    printf "%d\n" data;  
    n.status <- GoneL;           (* change back to a tree! *)  
    n.right <- child;            (* restore orig. right child! *)  
    apply tail (k : mtree)
```

This code runs in **constant space**. Look Ma, no stack! (Uh?)

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More accurately, the stack is stored **in the tree** itself, by **reversing pointers**.

This hack technique is known as **link inversion**.

It was invented for use in garbage collectors, which must **traverse the heap** without requiring a huge stack.

We have re-discovered it via the idea of allocating continuations **in place**.

Schorr and Waite, **An efficient machine-independent procedure for garbage collection in various list structures**, 1967.

Hubert and Marché, **A case study of C source code verification: the Schorr-Waite algorithm**, 2005.

Sobel and Friedman, **Recycling continuations**, 1998.

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“Kids, do not try this at home”: this idea is **complicated** and **expensive**.

(The OCaml GC imposes a **write barrier**: write operations are slow.)

**Exercise**: Extend the code to deal with **graphs**, where there can be **sharing** and **cycles**. (Use a **mark** bit in every node.)

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# Formulations of the CPS transformation

There are **many** variants of the CPS transformation,  
and sometimes **many** formulations of a single variant.

Let us begin with the simplest formulation: Fischer and Plotkin's.

Fischer, **Lambda-Calculus Schemata**, (1972) 1993.

Plotkin, **Call-by-name, call-by-value and the  $\lambda$ -calculus**, 1975.



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A term is translated to a **function** of a continuation  $k$  to an answer.

$$\llbracket x \rrbracket =$$

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$$\llbracket x \rrbracket = \lambda k.$$

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A term is translated to a **function** of a continuation  $k$  to an answer.

$$\llbracket x \rrbracket = \lambda k. k \ x$$

$$\llbracket \lambda x. t \rrbracket =$$

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$$\llbracket x \rrbracket = \lambda k. k \ x$$

$$\llbracket \lambda x. t \rrbracket = \lambda k. k \ (\lambda x. \llbracket t \rrbracket)$$

$$\llbracket t_1 \ t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket \ (\llbracket t_2 \rrbracket)$$

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$$\llbracket t_1 \ t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket \ (\lambda x_1. \llbracket t_2 \rrbracket \ (\lambda x_2. x_1 \ x_2 \ k))$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket \ (\lambda x. \llbracket t_2 \rrbracket \ k)$$

A function  $\lambda x. t$  is translated to a function of **two** arguments  $\lambda x. \lambda k..$

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One avoids some redundancy by defining **two mutually recursive functions**, namely the translation of values  $\llbracket v \rrbracket$ :

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda x. t \rrbracket = \lambda x. \llbracket t \rrbracket$$

and the translation of terms  $\llbracket t \rrbracket$ :

$$\llbracket v \rrbracket = \lambda k. k \llbracket v \rrbracket$$

$$\llbracket t_1 \ t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket (\lambda x_1. \llbracket t_2 \rrbracket (\lambda x_2. x_1 \ x_2 \ k))$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket (\lambda x. \llbracket t_2 \rrbracket k)$$

# Indifference



In a transformed term, **the right-hand side of every application** is a **value**.

Therefore, its execution is **indifferent** to the choice of a call-by-name or call-by-value evaluation strategy.

In other words, **evaluation order** is fully **explicit** in a transformed term.

The transformation on the previous slide fixes a call-by-value strategy: it is the **CBV CPS transformation**.

It can serve as an **encoding** of call-by-value into call-by-name, thus answering a question raised in week 1.

**Exercise** (recommended): Define the CBN CPS transformation.

# Stacklessness



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In a transformed term, **every call is a tail call**.

Therefore, reduction under a context is not required.

That is, execution **does not require a stack**.

We could (but won't) give a (small-step, substitution-based) semantics that takes **indifference** and **stacklessness** into account.

**Exercise:** Propose such a semantics. Prove that, when executing a CPS-transformed term, it is equivalent to the standard semantics.

## Effect of the transformation of types

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How are **types** transformed?

A **value** of type  $T$  is translated to a value of type  $\llbracket T \rrbracket$ .

A **computation** of type  $T$  is translated to a value of type  $\llbracket T \rrbracket$ .

$$\llbracket \alpha \rrbracket = \alpha$$

$$\llbracket T_1 \rightarrow T_2 \rrbracket =$$

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How are **types** transformed?

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A **computation** of type  $T$  is translated to a value of type  $\llbracket T \rrbracket$ .

$$\langle \alpha \rangle = \alpha$$

$$\langle T_1 \rightarrow T_2 \rangle = \langle T_1 \rangle \rightarrow \llbracket T_2 \rrbracket$$

$$\llbracket T \rrbracket =$$

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How are **types** transformed?

A **value** of type  $T$  is translated to a value of type  $\langle T \rangle$ .

A **computation** of type  $T$  is translated to a value of type  $\llbracket T \rrbracket$ .

$$\langle \alpha \rangle = \alpha$$

$$\langle T_1 \rightarrow T_2 \rangle = \langle T_1 \rangle \rightarrow \llbracket T_2 \rrbracket$$

$$\llbracket T \rrbracket = (\langle T \rangle \rightarrow A) \rightarrow A$$

The type  $A$ , known as the **answer** type, is arbitrary and fixed.

One may take  $A$  to be the **empty type**  $0$ . Then,  $\llbracket T \rrbracket$  is  $\neg\neg\langle T \rangle$ . The CPS transformation is known in logic as the **double-negation translation**.

**Exercise** (recommended): state and prove Type Preservation.



## Effect of the transformation of types – refined

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Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = ((\llbracket T \rrbracket \rightarrow A) \rightarrow A)$$

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Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = ((\llbracket T \rrbracket \rightarrow A) \rightarrow A)$$

Every transformed term is in fact **answer-type polymorphic**:

$$\llbracket T \rrbracket = \forall A. (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Furthermore,

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Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = ((\llbracket T \rrbracket \rightarrow A) \rightarrow A)$$

Every transformed term is in fact **answer-type polymorphic**:

$$\llbracket T \rrbracket = \forall A. (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Furthermore, every transformed term invokes its continuation **once**:

$$\llbracket T \rrbracket = \forall A. (\llbracket T \rrbracket \rightarrow A) \multimap A$$

However, these properties are violated in the presence of **control effects**.

Thielecke, **From control effects to typed continuation passing**, 2003.

## Administrative redexes

The translation presented so far is naïve.

It produces many “administrative”  $\beta$ -redexes.

E.g., in an application of a variable to a variable:

$$\begin{aligned}
 \llbracket f \ x \rrbracket &= \lambda k. \llbracket f \rrbracket (\lambda x_1. \llbracket x \rrbracket (\lambda x_2. x_1 \ x_2 \ k)) \\
 &= \lambda k. (\lambda k. k \ \llbracket f \rrbracket) (\lambda x_1. (\lambda k. k \ \llbracket x \rrbracket) (\lambda x_2. x_1 \ x_2 \ k)) \\
 &= \lambda k. (\lambda k. k \ f) (\lambda x_1. (\lambda k. k \ x) (\lambda x_2. x_1 \ x_2 \ k)) \\
 &=_{\beta} \lambda k. (\lambda x_1. (\lambda k. k \ x) (\lambda x_2. x_1 \ x_2 \ k)) \ f \\
 &=_{\beta} \lambda k. (\lambda k. k \ x) (\lambda x_2. f \ x_2 \ k) \\
 &=_{\beta} \lambda k. (\lambda x_2. f \ x_2 \ k) \ x \\
 &=_{\beta} \lambda k. f \ x \ k
 \end{aligned}$$

This is inefficient: **one** function call is translated to **five** function calls!

## Semantic preservation

Plotkin (1975) proved semantic preservation, based on a **small-step simulation diagram**.

This proof is complicated by the presence of administrative reductions.

A simpler approach is to use big-step semantics in the hypothesis:

### Lemma (Semantic Preservation)

*If  $t \Downarrow_{cbv} v$  and if  $w$  is a value, then  $\llbracket t \rrbracket w \longrightarrow_{cbv}^* w \langle v \rangle$ .*

One should prove, in addition, that divergence is preserved.

**Exercise** (recommended): Prove this lemma.

## Ways of eliminating administrative redexes

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Administrative redexes can be reduced **after** the CPS transformation.

- During the translation, mark each  $\lambda$  that corresponds to a source  $\lambda$ .
- After the translation, reduce every redex whose  $\lambda$  is unmarked.

Another idea is to reduce all “**no-brainer**” redexes. They include the admin. redexes and are size-decreasing. This can be done on the fly.

Davis, Meehan, Shivers, **No-brainer CPS conversion**, 2017.

Yet another approach is to define a “**one-pass**” CPS transformation that does not produce any administrative redexes in the first place...

## Towards a one-pass transformation

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The first step is to make some of the abstractions and applications **static**.

They should take place at **transformation time**, not at **runtime**.

Instead of viewing  $\llbracket t \rrbracket = \lambda k. \dots$  as a function of a term to a term, let us view  $\llbracket t \rrbracket \{ w \} = \dots$  as a function of a term and a value to a term.

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda x. t \rrbracket = \lambda x. \lambda k. \llbracket t \rrbracket \{ k \}$$

$$\llbracket v \rrbracket \{ w \} = w \llbracket v \rrbracket$$

$$\llbracket t_1 \ t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x_1. \llbracket t_2 \rrbracket \{ \lambda x_2. x_1 \ x_2 \ w \} \}$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x. \llbracket t_2 \rrbracket \{ w \} \}$$

$k$  denotes a **variable**;  $w$  denotes a **value**.

## Towards a one-pass transformation

This transformation produces **fewer administrative redexes**:

$$\begin{aligned}
 \llbracket f \ x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 \ x_2 \ k \} \} \\
 &= (\lambda x_1. (\lambda x_2. x_1 \ x_2 \ k) \ x) \ f \\
 &=_{\beta} (\lambda x_2. f \ x_2 \ k) \ x \\
 &=_{\beta} f \ x \ k
 \end{aligned}$$

The remaining administrative redexes arise from the equation

$$\llbracket v \rrbracket \{ w \} = w \ (v)$$

in the case where the continuation  $w$  is a  $\lambda$ -abstraction.

How could we alter this equation?



## Towards a one-pass transformation

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Define the **smart application** of a (continuation) value  $w$  to a value  $v$ :

$$\begin{aligned}x @_{\beta} v &= x \ v \\(\lambda x.t) @_{\beta} v &= t[v/x]\end{aligned}$$

Note:

- A continuation  $w$  is always either a variable or a “transformation”  $\lambda$ , never a “source”  $\lambda$ , so the redex reduced by  $w @_{\beta} v$  is **administrative**.
- Provided every “transformation”  $\lambda$  uses its argument **linearly**,  $w @_{\beta} \langle v \rangle$  does not duplicate  $\langle v \rangle$ , so transformed terms remain **linear** in size.

## A one-pass transformation

Change the translation of values. Make every “transformation”  $\lambda$  linear.

$$\langle x \rangle = x$$

$$\langle \lambda x. t \rangle = \lambda x. \lambda k. \llbracket t \rrbracket \{ k \}$$

$$\llbracket v \rrbracket \{ w \} = w @_{\beta} \langle v \rangle$$

$$\llbracket t_1 \ t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x_1. \llbracket t_2 \rrbracket \{ \lambda x_2. x_1 \ x_2 \ w \} \}$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x. \text{let } x = x \text{ in } \llbracket t_2 \rrbracket \{ w \} \}$$

This transformation produces **no administrative redexes**.

Dargaye and Leroy, **Mechanized Verification  
of CPS Transformations**, 2007.

## A one-pass transformation

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Look Ma, **no administrative redexes!**

$$\begin{aligned}\llbracket f \ x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 \ x_2 \ k \} \} \\ &= (\lambda x_1. (\lambda x_2. x_1 \ x_2 \ k) @_{\beta} x) @_{\beta} f \\ &= (\lambda x_2. f \ x_2 \ k) @_{\beta} x \\ &= f \ x \ k\end{aligned}$$

## A one-pass transformation

Look Ma, **no administrative redexes!**

$$\begin{aligned}
 \llbracket f \ x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 \ x_2 \ k \} \} \\
 &= (\lambda x_1. (\lambda x_2. x_1 \ x_2 \ k) @_{\beta} x) @_{\beta} f \\
 &= (\lambda x_2. f \ x_2 \ k) @_{\beta} x \\
 &= f \ x \ k
 \end{aligned}$$

A drawback of Dargaye and Leroy's approach is that  $\cdot @_{\beta} \cdot$  **does not commute** with substitutions, which causes a difficulty in the proof of semantic preservation.

This is repaired in the formulation shown next...



## Higher-order versus first-order formulations

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Danvy and Filinski (1992) first defined this one-pass transformation.

Their formulation was in a “higher-order” style.

Let me give a simpler, “first-order” presentation of their transformation.

Danvy and Filinski, **Representing control:  
a study of the CPS transformation**, 1992.

Pottier, **Revisiting the CPS transformation  
and its implementation**, 2017.

## A first-order one-pass CPS transformation

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Let a continuation  $c$  be either a value  $w$  or a “transformation”  $\lambda$ :

$$c ::= w \mid \text{mx}.t$$

In  $\text{mx}.t$ , the term  $t$  must have exactly one occurrence of  $x$ .

Define **continuation application**  $\text{apply } c \ v$  and **reification**  $\text{reify } c$ :

$\text{apply } w \ v = w \ v$	– an object-level application
$\text{apply } (\text{mx}.t) \ v = t[v/x]$	– a meta-level substitution
$\text{reify } w = w$	– a no-op
$\text{reify } (\text{mx}.t) = \lambda x.t$	

## A first-order one-pass CPS transformation

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Danvy and Filinski's transformation can then be presented as follows:

$$\begin{aligned}
 \llbracket x \rrbracket &= x \\
 \llbracket \lambda x. t \rrbracket &= \lambda x. \lambda k. \llbracket t \rrbracket \{ k \} \\
 \llbracket v \rrbracket \{ c \} &= \text{apply } c \llbracket v \rrbracket \\
 \llbracket t_1 \ t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ \text{mx}_1. \llbracket t_2 \rrbracket \{ \text{mx}_2. x_1 \ x_2 \ (\text{reify } c) \} \} \\
 \llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ \text{mx}_1. \text{let } x = x_1 \text{ in } \llbracket t_2 \rrbracket \{ c \} \}
 \end{aligned}$$

It is close to Dargaye and Leroy's formulation, yet is **better behaved**: **as we will see**, it commutes with substitution.

## Now, in de Bruijn style



Let us use **o** and **m** as explicit injections:

$$c ::= \mathbf{o} \ w \mid \mathbf{m} \ t$$

**m**, like  $\lambda$ , is considered a binder.

Continuation application, reification, and substitution  $c[\sigma]$  are as follows:

$$\begin{array}{lll} \text{apply } (\mathbf{o} \ w) \ v = w \ v & \text{reify } (\mathbf{o} \ w) = w & (\mathbf{o} \ w)[\sigma] = \mathbf{o} \ (w[\sigma]) \\ \text{apply } (\mathbf{m} \ t) \ v = t[v/] & \text{reify } (\mathbf{m} \ t) = \lambda t & (\mathbf{m} \ t)[\sigma] = \mathbf{m} \ (t[\uparrow \sigma]) \end{array}$$

See **CPSDefinition**.



## The CPS transformation in de Bruijn style

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The transformation is formulated in de Bruijn style as follows:

$$\langle x \rangle = x$$

$$\langle \lambda t \rangle = \lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{ \text{ } \circ \ 0 \} )$$

$$\llbracket v \rrbracket \{ c \} = \text{apply } c \langle v \rangle$$

$$\llbracket t_1 \ t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \text{ } \text{ } \uparrow^1 t_2 \rrbracket \{ \text{ } \text{ } \text{ } 1 \ 0 \ \uparrow^2 (\text{reify } c) \} \}$$

$$\llbracket \text{let } t_1 \text{ in } t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \text{ } \text{ } \text{ } \text{let } 0 \text{ in } \llbracket \uparrow_1^1 t_2 \rrbracket \{ \uparrow^2 c \} \}$$

$\uparrow^i t$  is short for  $t[+i]$ .  $\uparrow_1^1 t$  is short for  $t[\uparrow(+1)]$ .

$\uparrow^1$  means **end-of-scope** for variable 0.

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The transformation is formulated in de Bruijn style as follows:

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda t \rrbracket = \lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{ \text{red } 0 \})$$

$$\llbracket v \rrbracket \{ c \} = \text{apply } c \llbracket v \rrbracket$$

$$\llbracket t_1 \ t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \text{red } \llbracket \uparrow^1 t_2 \rrbracket \{ \text{red } 1 \ 0 \ \uparrow^2 (\text{reify } c) \} \}$$

$$\llbracket \text{let } t_1 \text{ in } t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \text{red let } 0 \text{ in } \llbracket \uparrow_1^1 t_2 \rrbracket \{ \uparrow^2 c \} \}$$

$\uparrow^i t$  is short for  $t[+i]$ .  $\uparrow_1^1 t$  is short for  $t[\uparrow(+1)]$ .

$\uparrow^1$  means **end-of-scope** for variable 0.

$\uparrow^2$  means end-of-scope for variables 0 and 1.

$\uparrow_1^1$  means end-of-scope for variable 1.

Worse, Coq does not like this definition...



## The CPS transformation in de Bruijn style

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The transformation is formulated in de Bruijn style as follows:

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda t \rrbracket = \lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{ \text{red } 0 \})$$

$$\llbracket v \rrbracket \{ c \} = \text{apply } c \llbracket v \rrbracket$$

$$\llbracket t_1 \ t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \text{red } \llbracket \uparrow^1 t_2 \rrbracket \{ \text{red } 1 \ 0 \ \uparrow^2 (\text{reify } c) \} \}$$

$$\llbracket \text{let } t_1 \text{ in } t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \text{red } \text{let } 0 \text{ in } \llbracket \uparrow_1^1 t_2 \rrbracket \{ \uparrow^2 c \} \}$$

$\uparrow^i t$  is short for  $t[+i]$ .  $\uparrow_1^1 t$  is short for  $t[\uparrow(+1)]$ .

$\uparrow^1$  means **end-of-scope** for variable 0.

$\uparrow^2$  means end-of-scope for variables 0 and 1.

$\uparrow_1^1$  means end-of-scope for variable 1.



Worse, Coq does not like this definition... because the recursive calls concern **renamed** subterms! Well-founded recursion on **size** is required.

See **CPSDefinition**.

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# Semantic Preservation

We would like to prove this:

## Lemma (Semantic Preservation)

*If  $t \downarrow_{cbv} v$ , then  $\llbracket t \rrbracket \{ m\ 0 \} \downarrow_{cbv} \langle v \rangle$ .*

$m\ 0$  is the **identity continuation**: in nominal style,  $m\ x.x$ .

For an inductive proof, the statement must be generalized, as follows...

# Semantic Preservation

Define  $t \lesssim u$  as follows: for every value  $v$ ,  $u \downarrow_{\text{cbv}} v$  implies  $t \downarrow_{\text{cbv}} v$ .

## Lemma (Big-step Simulation)

*Suppose  $\text{reify } c$  is a value. If  $t \downarrow_{\text{cbv}} v$ , then  $\llbracket t \rrbracket \{c\} \lesssim \text{apply } c \ (\llbracket v \rrbracket)$ .*

Compare with **our earlier claim** concerning Plotkin's CPS transformation.

The proof is in **CPSCorrectnessBigStep**.

**Exercise:** Replay the proof in Coq. Then erase it and redo it from scratch.

**Exercise:** Write a clear paper or  $\text{\LaTeX}$  proof and send it to me!

The proof requires two key lemmas, shown next...

## Key Lemma 1: Substitution

## Lemma (Substitution)

Let  $\sigma$  and  $\sigma'$  be value substitutions such that  $\sigma'$  is equal to  $\sigma$ ;  $\langle \cdot \rangle$ . Then,

$$(\llbracket t \rrbracket \{ c \})[\sigma'] = \llbracket t[\sigma] \rrbracket \{ c[\sigma'] \}.$$

## Lemma (Substitution—a special case)

Let  $v$  and  $w$  be values. Then,

$$(\llbracket t \rrbracket \{ \uparrow^2 c \})[\langle v \rangle \cdot \langle w \rangle \cdot id] = \llbracket t[v \cdot w \cdot id] \rrbracket \{ c \}.$$

In nominal style: if  $x, y \notin fv(c)$ , then

$$(\llbracket t \rrbracket \{ c \})[\langle v \rangle/x, \langle w \rangle/y] = \llbracket t[v/x, w/y] \rrbracket \{ c \}.$$

We push a substitution into the term, leaving the continuation untouched.  
A target language substitution becomes a source language substitution.

See [CPSSubstitution](#).

## Key Lemma 2: Kubstitution

### Lemma (Kubstitution)

*Let  $\theta$  and  $\sigma$  be substitutions such that  $\theta ; \sigma$  is id. Then,*

$$\llbracket (t[\theta]) \{ c \} \rrbracket [\sigma] = \llbracket t \{ c[\sigma] \} \rrbracket.$$

### Lemma (Kubstitution—a special case)

*For every value  $v$ ,  $(\llbracket \uparrow^1 t \{ c \} \rrbracket [v/]) = \llbracket t \{ c[v/] \} \rrbracket$ .*

In nominal style: if  $x \notin \text{fv}(t)$ , then  $(\llbracket t \{ c \} \rrbracket [v/x]) = \llbracket t \{ c[v/x] \} \rrbracket$ .

We push a substitution into the continuation, leaving the term untouched.  
This is and remains a target language substitution.

See [CPSKubstitution](#).



## Interlude: Enumerating $\lambda$ -terms

Define the **size** of a term as follows: variables have size 0;  $\lambda$ -abstractions and applications contribute 1.

**Step 1:** In OCaml, implement an exhaustive **enumeration** of the  $\lambda$ -terms of size  $s$  and with at most  $n$  free variables. (Given as an exercise in week 1.)

```
(* Enumerate all variables between 0 and n excluded. *)
let var (n : int) (k : term -> unit) : unit = ...

(* Enumerate all manners of splitting an integer s. *)
let split (s : int) (k : int -> int -> unit) : unit = ...

(* Enumerate all terms of size s with at most n variables. *)
let term (s : int) (n : int) (k : term -> unit) : unit = ...
```

An enumerator is naturally written in CPS style!

## Interlude: Testing Semantic Preservation

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**Step 2:** In OCaml, implement the CPS transformation.

```
type continuation =  
  | O of term  
  | M of term  
let rec cps (t : term) (c : continuation) : term = ...
```

**Step 3:** In OCaml, implement a test for the relation  $\cdot \lesssim \cdot$ :

```
let sim (t1 : term) (t2 : term) : bool = ...
```

Hint: Re-use the big-step interpreter of week 2. See [Lambda](#).

**Step 4:** Up to a certain size, search for a term that violates Semantic Preservation. There should be none!

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## Control operators

In a CPS-transformed program, the continuation is a first-class object.

Why not give programmers **access** to it?

That is, extend the source language with **control operators** that allow (**delimiting** and) **capturing** the current continuation.

An example is Danvy and Filinski's shift / reset (1990).

$$t ::= \dots \mid \langle t \rangle \mid \xi x.t$$

A “reset”  $\langle t \rangle$  does nothing by itself: e.g.,  $\langle 42 \rangle$  reduces to 42.

A “shift”  $\xi x.t$  captures the current evaluation context (up to and excluding the nearest reset), reifies it as a function, and binds the variable  $x$  to it.

Then it discards the evaluation context (up to and including the nearest reset) and executes  $t$  instead.

E.g., roughly,

$$\begin{aligned} & 1 + \langle 10 + \xi c.c \ (c \ 100) \rangle \\ \longrightarrow & 1 + (\text{let } c = \lambda x.(10 + x) \text{ in } c \ (c \ 100)) \\ \longrightarrow & 1 + (10 + (10 + 100)) \\ \longrightarrow & 121 \end{aligned}$$

**Exercise:** Give a small-step semantics to shift / reset.

## CPS-transforming shift / reset

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The naïve call-by-value CPS transformation is extended as follows:

$$\llbracket \langle t \rangle \rrbracket = \lambda k.$$

## CPS-transforming shift / reset

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The naïve call-by-value CPS transformation is extended as follows:

$$\begin{aligned}\llbracket \langle t \rangle \rrbracket &= \lambda k. k \ (\llbracket t \rrbracket \ (\lambda y. y)) \\ \llbracket \xi x. t \rrbracket &= \lambda k.\end{aligned}$$

## CPS-transforming shift / reset

The naïve call-by-value CPS transformation is extended as follows:

$$\begin{aligned}\llbracket \langle t \rangle \rrbracket &= \lambda k. k (\llbracket t \rrbracket (\lambda y. y)) \\ \llbracket \xi x. t \rrbracket &= \lambda k. \text{let } x = \lambda y. \lambda k'. k' (k \ y) \text{ in} \\ &\quad \llbracket t \rrbracket (\lambda y. y)\end{aligned}$$

**Exercise** (experimental!): Extend the proof of Semantic Preservation.

The target of the transformation is  $\lambda$ -calculus **without** shift / reset.

It is **no longer the case** that every call is a tail call, that the right-hand side of every application is a value, or that continuations are linearly used.

Thus, shift / reset allow reaching terms which previously lied **outside** the image of the CPS transformation. CPS lets us **think outside the box**!



## Other control operators

Many other control operators or control constructs can be **explained** and **compiled away** via CPS.

**Exceptions** can be compiled away by “double-barrelled CPS”, that is, by using **two** continuations.

**Effect handlers** can be compiled away via (type-directed, selective) CPS.

Rompf, Maier, Odersky, **Implementing first-class polymorphic delimited continuations by a type-directed selective CPS-transform**, 2009.

Leijen, **Type-directed compilation of row-typed algebraic effects**, 2017.

See Régis-Gianas’ lectures!

## Monadic intermediate form

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If one just aims to make evaluation order explicit, CPS is **overkill**.

This transformation, too, achieves **indifference**:

$$\begin{aligned}\llbracket x \rrbracket &= x \\ \llbracket \lambda x. t \rrbracket &= \lambda x. \llbracket t \rrbracket \\ \llbracket t_1 \ t_2 \rrbracket &= \text{let } x_1 = \llbracket t_1 \rrbracket \text{ in} \\ &\quad \text{let } x_2 = \llbracket t_2 \rrbracket \text{ in} \\ &\quad x_1 \ x_2 \\ \llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket &= \text{let } x = \llbracket t_1 \rrbracket \text{ in } \llbracket t_2 \rrbracket\end{aligned}$$

In a transformed term, **the components of every application are values**.

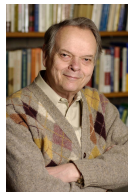
By further hoisting “let” out of the left-hand side of “let”,  
one gets **administrative normal form**.

Flanagan, Sabry, Felleisen, **The essence  
of compiling with continuations**, 1993 (2003).

## The CPS monad

The CPS transformation is a special case of the [monadic transformation](#).  
See Dagand's lectures!

## Some history



Continuations, and the CPS transformation, were independently discovered by many researchers during the 1960s.

John C. Reynolds, *The discoveries of continuations*, 1993.

## Some history

The CPS transformation has been used in compilers.

Rabbit (Steele). SML/NJ.

Appel, *Compiling with Continuations*, 1992.

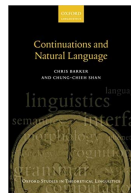
Today, heap-allocating the stack is considered *too costly*:

- bad locality;
- increased GC load;
- confuses the processor's built-in prediction of return addresses.

Yet, *selective* CPS transformations are used to compile effect handlers, and some compilers use CPS as an *intermediate form* before coming back to direct style.

Kennedy, *Compiling with continuations, continued*, 2007.

## Some history



Can  $\lambda$ -calculus and continuations explain the structure of speech?

Chris Barker,  
**Continuations and the nature of quantification**, 2002.

Chris Barker and Chung-Chieh Shan,  
**Continuations and Natural Language**, 2014.

## A few things to remember

### Continuations rule!

- The CPS transformation achieves several remarkable effects:
  - making **the stack** explicit;
  - making **evaluation order** explicit;
  - suggesting/explaining **control operators**.
- It plays a **fundamental role** in prog. language theory and in logic.
- Continuation-passing is also a useful **programming technique**.

We have illustrated a few proof techniques:

- Another proof of semantic preservation.
- A small-step **simulation** diagram (see part **5**).
- **Testing**, to refute a conjecture (see part **5**).

## 1 Examples

From a direct-style interpreter down to an abstract machine

From recursive traversal down to iterative traversal with link inversion

## 2 Formulations

## 3 Soundness

## 4 Remarks

## 5 Madness in small steps



## Madness Soundness in small steps

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(Presented at MPRI 2.4 in 2017.)

Could we use a **small-step operational semantics**  
in the proof that CPS is semantics-preserving?

## Towards semantic preservation

Let us consider the pure  $\lambda$ -calculus, without “let”.

Let us use de Bruijn notation.

The transformation is defined in [CPSDefinition](#).

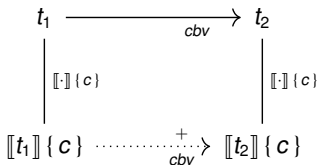
The proof of Simulation is in [CPSSimulationWithoutLet](#).

The key lemmas are in [CPSSpecialCases](#), [CPSSubstitution](#), [CPSKubstitution](#).

## A small-step simulation diagram

We propose to use the **small-step substitution** semantics and to establish a **simulation** diagram.

**One** step by the source program is simulated in **one or more** steps by the transformed program:



A solid arrow represents a **universal** quantification (a hypothesis).

A dashed arrow represents an **existential** quantification (a conclusion).

## Consequences of the simulation diagram

Examples

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Traversal

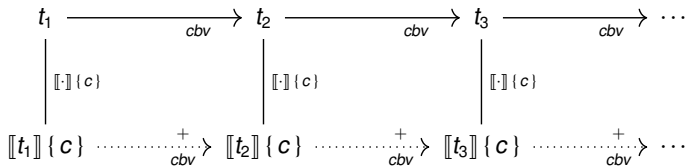
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There immediately follows that **divergence** is preserved.



The fact that each step is simulated by **one or more** steps is crucial.

(A proof by co-induction. See [Relations/infseq\\_simulation](#).)

## Consequences of the simulation diagram

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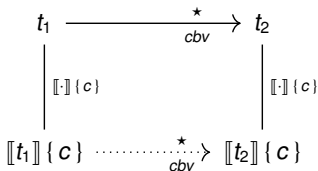
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Obviously, **several** steps by the source program  
are simulated in **several** steps by the transformed program:



(A proof by induction. See [Relations/star\\_diamond\\_left](#).)

## Consequences of the simulation diagram

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There follows that **convergence to a value** is preserved.

We use the identity continuation *done*, defined as  $m\ 0$ .

$$\begin{array}{ccc}
 t & \xrightarrow[\text{cbv}]{\star} & v \\
 \left| \begin{array}{c} \llbracket \cdot \rrbracket \{ \text{done} \} \end{array} \right. & & \left| \begin{array}{c} \llbracket \cdot \rrbracket \{ \text{done} \} \end{array} \right. \\
 \llbracket t \rrbracket \{ \text{done} \} & \xrightarrow[\text{cbv}]{\star} & \llbracket v \rrbracket \{ \text{done} \}
 \end{array}$$

By definition,  $\llbracket v \rrbracket \{ \text{done} \}$  is *apply done*  $\langle v \rangle$ , that is,  $\langle v \rangle$ , therefore **a value**.

Thus, the CPS transformation is **semantics-preserving**.

## The simulation lemma

Here is the simulation statement again, this time in textual form:

### Lemma (Simulation)

*Assume reify  $c$  is a value. Then  $t_1 \longrightarrow_{cbv} t_2$  implies  $\llbracket t_1 \rrbracket \{c\} \longrightarrow_{cbv}^+ \llbracket t_2 \rrbracket \{c\}$ .*

Let us now do the proof.

Onscreen or in Coq? Both, probably.

See `CPSSimulationWithoutLet`.

Proof of Simulation – case  $\beta_v$ 

**Case:**  $(\lambda t) v \longrightarrow_{\text{cbv}} t[v/]$ . We must show:

$$\llbracket (\lambda t) v \rrbracket \{c\} \longrightarrow_{\text{cbv}}^+ \llbracket t[v/] \rrbracket \{c\}$$

By the Value-Value Application lemma, the left-hand term is:

$$\langle \lambda t \rangle \langle v \rangle (\text{reify } c)$$

By definition of  $\langle \lambda t \rangle$ , this is:

$$(\lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{o\ 0\})) \langle v \rangle (\text{reify } c)$$

The transformed function is passed **an actual argument**  $\langle v \rangle$   
and **a continuation**  $\text{reify } c$ .



Proof of Simulation – case  $\beta_v$ 

$$(\lambda\lambda(\llbracket \uparrow^1 t \rrbracket \{o\ 0\})) \langle v \rangle (\text{reify } c)$$

In two  $\beta$ -reduction steps, this term reduces to:

$$(\llbracket \uparrow^1 t \rrbracket \{o\ 0\}) \llbracket \uparrow (\langle v \rangle /) \rrbracket [\text{reify } c /]$$

We have **two successive substitutions**. This term could also be written using a single substitution that acts on variables 0 and 1:

$$(\llbracket \uparrow^1 t \rrbracket \{o\ 0\}) [\text{reify } c \cdot \langle v \rangle \cdot \text{id}]$$

(We won't use this fact, though.)

We now wish to **push** the substitutions inside, one after the other.

Proof of Simulation – case  $\beta_v$ 

$$(\llbracket \uparrow^1 t \rrbracket \{o\ 0\}) \llbracket \uparrow (\llbracket v \rrbracket /) \rrbracket \text{[reify } c /]$$

By the Substitution lemma, the substitution  $\uparrow (\llbracket v \rrbracket /)$  acts on both **the term**  $\uparrow^1 t$  and **the continuation**  $o\ 0$ .

However,  $\uparrow (\llbracket v \rrbracket /)$  has no effect on variable 0.

Thus, the above term is:

$$(\llbracket (\uparrow^1 t) \llbracket \uparrow (v /) \rrbracket \rrbracket \{o\ 0\}) \text{[reify } c /]$$

that is,

$$(\llbracket \uparrow^1 t \llbracket v / \rrbracket \rrbracket \{o\ 0\}) \text{[reify } c /]$$

Proof of Simulation – case  $\beta_v$ 

$$(\llbracket \uparrow^1 t[v/] \rrbracket \{ o 0 \}) [reify\ c/]$$

By the Substitution lemma, the substitution *reify c/* acts **only on the continuation** *o 0*, **not on the term** *t[v/]*, because it cancels out with  $\uparrow^1$ .

Thus, this term is:

$$\llbracket t[v/] \rrbracket \{ (o 0)[reify\ c/] \}$$

that is,

$$\llbracket t[v/] \rrbracket \{ o (reify\ c) \}$$

Proof of Simulation – case  $\beta_v$ 

We have now reached the term:

$$\llbracket t[v/] \rrbracket \{ o(\textit{reify } c) \}$$

and the goal is to prove that it reduces (in zero or more steps) to:

$$\llbracket t[v/] \rrbracket \{ o c \}$$

This is the Magic Step lemma. This proof case is finished!

Here are the four key lemmas that we have used so far.

### Lemma (Value-Value Application)

$$\llbracket v_1 \ v_2 \rrbracket \{ c \} = \llbracket v_1 \rrbracket \llbracket v_2 \rrbracket (reify \ c).$$

### Lemma (Substitution)

Let  $\sigma$  and  $\sigma'$  be value substitutions such that  $\sigma'$  is equal to  $\sigma ; \langle \cdot \rangle$ . Then,

$$(\llbracket t \rrbracket \{ c \})[\sigma'] = \llbracket t[\sigma] \rrbracket \{ c[\sigma'] \}.$$

### Lemma (Kubstitution)

Let  $\theta$  and  $\sigma$  be substitutions such that  $\theta ; \sigma$  is *id*. Then,

$$\llbracket (t[\theta]) \rrbracket \{ c \})[\sigma] = \llbracket t \rrbracket \{ c[\sigma] \}.$$

### Lemma (Magic Step)

$$\llbracket t \rrbracket \{ o \ (reify \ c) \} \longrightarrow_{cbv}^? \llbracket t \rrbracket \{ c \}.$$

## Proof of Simulation – cases AppL and AppR

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**Case:**  $t_1 \ u \longrightarrow_{\text{cbv}} t_2 \ u$ , where  $t_1 \longrightarrow_{\text{cbv}} t_2$ .

We must show  $\llbracket t_1 \ u \rrbracket \{c\} \longrightarrow_{\text{cbv}}^+ \llbracket t_2 \ u \rrbracket \{c\}$ .

By definition of the CPS transformation, this is

$$\longrightarrow_{\text{cbv}}^+ \begin{array}{l} \llbracket t_1 \rrbracket \{m \llbracket \uparrow^1 u \rrbracket \{m \ 1 \ 0 \ \uparrow^2 (reify \ c)\} \} \} \\ \llbracket t_2 \rrbracket \{m \llbracket \uparrow^1 u \rrbracket \{m \ 1 \ 0 \ \uparrow^2 (reify \ c)\} \} \} \end{array}$$

## Proof of Simulation – cases AppL and AppR

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**Case:**  $t_1 \ u \longrightarrow_{\text{cbv}} t_2 \ u$ , where  $t_1 \longrightarrow_{\text{cbv}} t_2$ .

We must show  $\llbracket t_1 \ u \rrbracket \{c\} \longrightarrow_{\text{cbv}}^+ \llbracket t_2 \ u \rrbracket \{c\}$ .

By definition of the CPS transformation, this is

$$\longrightarrow_{\text{cbv}}^+ \begin{array}{l} \llbracket t_1 \rrbracket \{m \ \llbracket \uparrow^1 u \rrbracket \{m \ 1 \ 0 \ \uparrow^2 (reify \ c) \} \} \} \\ \llbracket t_2 \rrbracket \{m \ \llbracket \uparrow^1 u \rrbracket \{m \ 1 \ 0 \ \uparrow^2 (reify \ c) \} \} \} \end{array}$$

Wow – the **induction hypothesis** applies directly to this goal!

Indeed, *reify* (m ...) is a  $\lambda$ -abstraction, therefore a value.

This proof case is complete!

**Case:**  $v \ u_1 \longrightarrow_{\text{cbv}} v \ u_2$ , where  $u_1 \longrightarrow_{\text{cbv}} u_2$ .

Analogous to the previous case, using a Value-Term Application lemma.

We see in these proof cases that **reduction under a context** in the source program is translated to **reduction at the root** in the transformed program.

## Simulation in the presence of let constructs

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In the presence of “let” constructs, Simulation breaks down.

**Challenge:** can you find a (minimal) counter-example?

Hint: Enlist a machine’s help. (See next two slides.)



Enumerating  $\lambda$ -terms

Define the **size** of a term as follows: variables have size 0;  $\lambda$ -abstractions and applications contribute 1.

**Step 1:** In OCaml, implement an exhaustive **enumeration** of the  $\lambda$ -terms of size  $s$  and with at most  $n$  free variables. (Given as an exercise in week 1.)

```
(* Enumerate all variables between 0 and n excluded. *)  
let var (n : int) (k : term -> unit) : unit = ...  
(* Enumerate all manners of splitting an integer s. *)  
let split (s : int) (k : int -> int -> unit) : unit = ...  
(* Enumerate all terms of size s with at most n variables. *)  
let term (s : int) (n : int) (k : term -> unit) : unit = ...
```

An enumerator is naturally written in CPS style!

## Testing Simulation

Step 2: In OCaml, implement the CPS transformation.

```
type continuation =  
  | O of term  
  | M of term  
let cps (t : term) (c : continuation) : term = ...
```

Step 3: In OCaml, implement a test for the relation  $\cdot \longrightarrow_{\text{cbv}}^* \cdot$ :

```
let reduces (t1 : term) (t2 : term) : bool = ...
```

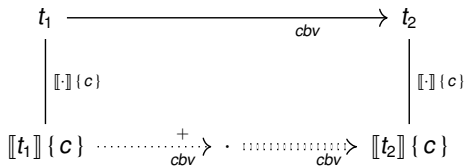
Hint: Re-use the auxiliary functions of week 2. See [Lambda](#).

Step 4: Find a term  $t_1$  of minimal size that violates Simulation.

Solution: see [CPSCounterExample](#).

## Fixing Simulation

In the presence of “let”, Simulation can be fixed as follows:



We allow one step of **parallel call-by-value reduction**  $\Rightarrow_{cbv}$ .

The proof of Simulation is more complex; see **CPSSimulation**.

## Parallel (call-by-value) reduction

Parallel reduction allows reducing **all** (currently visible) redexes at once, including under “ $\lambda$ ” and in the right-hand side of “let”.

PARALLEL  $\beta_v$

$$\frac{t_1 \Rightarrow_{\text{cbv}} t_2 \quad v_1 \Rightarrow_{\text{cbv}} v_2}{(\lambda t_1) v_1 \Rightarrow_{\text{cbv}} t_2[v_2/]}$$

PARALLEL  $\text{let}_v$

$$\frac{t_1 \Rightarrow_{\text{cbv}} t_2 \quad v_1 \Rightarrow_{\text{cbv}} v_2}{\text{let } v_1 \text{ in } t_1 \Rightarrow_{\text{cbv}} t_2[v_2/]}$$

$$X \Rightarrow_{\text{cbv}} X$$

$$\frac{t_1 \Rightarrow_{\text{cbv}} t_2}{\lambda t_1 \Rightarrow_{\text{cbv}} \lambda t_2}$$

$$\frac{t_1 \Rightarrow_{\text{cbv}} t_2 \quad u_1 \Rightarrow_{\text{cbv}} u_2}{t_1 u_1 \Rightarrow_{\text{cbv}} t_2 u_2}$$

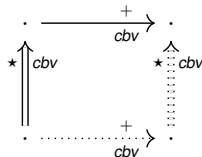
$$\frac{t_1 \Rightarrow_{\text{cbv}} t_2 \quad u_1 \Rightarrow_{\text{cbv}} u_2}{\text{let } t_1 \text{ in } u_1 \Rightarrow_{\text{cbv}} \text{let } t_2 \text{ in } u_2}$$

The ability to **reduce under a binder** is needed to fix Simulation.

Call-by-name parallel reduction is studied by **Takahashi (1995)**.

**Crary (2009)** adapts these results to a call-by-value setting.

## Well-behavedness of parallel reduction



## Lemma (Commutation)

$$(\Rightarrow_{cbv}^* ; \longrightarrow_{cbv}^+) \subseteq (\longrightarrow_{cbv}^+ ; \Rightarrow_{cbv}^*).$$

See [LambdaCalculusStandardization/pcbv\\_cbv\\_commutation](#).

## Well-behavedness of parallel reduction

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## Lemma (Equiconvergence)

$$(\exists v, t \Rightarrow_{cbv}^* v) \iff (\exists v', t \longrightarrow_{cbv}^* v').$$

(The idea is,  $v'$  reduces to  $v$  via [internal](#) parallel reduction steps.)

See [LambdaCalculusStandardization/equiconvergence](#).

## Consequences of Fixed Simulation

There follows that **divergence** is preserved.

Indeed, from:

$$t \longrightarrow_{\text{cbv}} \cdot \longrightarrow_{\text{cbv}} \cdots$$

we get:

$$\llbracket t \rrbracket \{ c \} \longrightarrow_{\text{cbv}}^+ \cdot \Rightarrow_{\text{cbv}} \cdot \longrightarrow_{\text{cbv}}^+ \cdot \Rightarrow_{\text{cbv}} \cdots$$

which, by Commutation, yields:

$$\llbracket t \rrbracket \{ c \} \longrightarrow_{\text{cbv}}^+ \cdot \xrightarrow{\text{cbv}}^+ \cdot \Rightarrow_{\text{cbv}}^* \cdot \Rightarrow_{\text{cbv}} \cdots$$

that is,

$$\llbracket t \rrbracket \{ c \} \longrightarrow_{\text{cbv}}^{\geq 2} \cdot \Rightarrow_{\text{cbv}}^* \cdots$$

And so on. For an arbitrary  $n \geq 0$ , we have:

$$\llbracket t \rrbracket \{ c \} \longrightarrow_{\text{cbv}}^{\geq n} \cdot \Rightarrow_{\text{cbv}}^* \cdots$$

## Consequences of Fixed Simulation

Convergence to a value is preserved, too.

Indeed, from:

$$t \longrightarrow_{\text{cbv}}^n v$$

we get, as on the previous slide:

$$\llbracket t \rrbracket \{ \text{done} \} \longrightarrow_{\text{cbv}}^{\geq n} \cdot \Rightarrow_{\text{cbv}}^{\star} (\langle v \rangle)$$

and, by Equiconvergence:

$$\exists v' \quad \llbracket t \rrbracket \{ \text{done} \} \longrightarrow_{\text{cbv}}^{\geq n} \cdot \longrightarrow_{\text{cbv}}^{\star} v'$$

The CPS transformation remains **semantics-preserving** in the presence of “let” constructs (phew!).