

MPRI 2.4

GADTs

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Untyped expressions

Consider a tiny language of expressions $t ::= k \mid (t, t) \mid \pi_i t$:

```
type expr =  
| EInt of int  
| EPair of expr * expr  
| EFst of expr  
| ESnd of expr
```

Expressions include integer constants, pairs, and projections.

Untyped values

A straightforward interpreter for this language uses a type of all values:

```
type value =  
| VInt of int  
| VPair of value * value
```

This is an algebraic data type. Thus every value carries a [tag](#).

Runtime tests

These tags are used in **runtime tests** that can cause **runtime errors**.

```
let as_pair (v : value) : value * value =  
  match v with  
  | VPair (v1, v2) ->  
    v1, v2  
  | _ ->  
    assert false (* runtime error! *)
```

An untyped interpreter

Here, interpreting a pair projection operation involves a runtime test.

```
let rec eval (e : expr) : value =
  match e with
  | EInt x ->
    VInt x
  | EPair (e1, e2) ->
    VPair (eval e1, eval e2)
  | EFst e ->
    fst (as_pair (eval e))
  | ESnd e ->
    snd (as_pair (eval e))
```

This is **necessary** because this interpreter accepts untyped expressions.

Typed expressions

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Let us impose a **simple type discipline** on expressions.

```

type _ expr =
| EInt   :          int ->          int expr
| EPair  : 'a expr * 'b expr -> ('a * 'b) expr
| EFst   :      ('a * 'b) expr ->      'a expr
| ESnd   :      ('a * 'b) expr ->      'b expr

```

This type definition encodes the following type discipline:

$$\begin{array}{c}
 \Gamma \vdash k : \text{int} \qquad \frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \qquad \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i}
 \end{array}$$

A **meta-level** abstract syntax tree (AST) of type 'a **expr** represents an **object-level** expression of type 'a.

Typed values

Let us similarly impose a type discipline on values:

```
type _ value =  
| VInt   :          int ->      int value  
| VPair  : 'a value * 'b value -> ('a * 'b) value
```

Values are still tagged (for now), but runtime tests become unnecessary...

Look Ma, no runtime test!

Only one branch is now necessary. A second branch would be **dead**.

```
let as_pair : type a b . (a * b) value -> a value * b value
= function
  | VPair (v1, v2) ->
    v1, v2
  (* In this branch, we would learn [a * b = int], *)
  (* which is contradictory. *)
  (* | _ -> . *)
```

In OCaml, destructing a GADT requires a type annotation in this style.

A typed interpreter

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Evaluating an expression of type T yields a value of type T .

```
let rec eval : type a . a expr -> a value
= function
| EInt x ->
    (* We learn [a = int] so returning [VInt _] is OK. *)
    VInt x
| EPair (e1, e2) ->
    (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
    (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
    VPair (eval e1, eval e2)
| EFst e ->
    fst (as_pair (eval e))
| ESnd e ->
    snd (as_pair (eval e))
```

The type of the interpreter reflects the [subject reduction](#) property.
Type-checking it amounts to [checking the proof](#) of subject reduction!

A typed, tagless interpreter

Evaluating an expression of type T yields a **meta-level value** of type T .

```
let rec eval : type a . a expr -> a
= function
| EInt x ->
    (* We learn [a = int] so returning an integer is OK. *)
    x                                     (* no tagging! *)
| EPair (e1, e2) ->
    (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
    (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
    (eval e1, eval e2) (* no tagging! *)
| EFst e ->
    fst (eval e)                (* no untagging! *)
| ESnd e ->
    snd (eval e)                (* no untagging! *)
```

The type of the interpreter reflects the **subject reduction** property.
Type-checking it amounts to **checking the proof** of subject reduction!

Polymorphic recursion

`eval` involves **polymorphic recursion**:
the fact that `eval` is polymorphic
is exploited in the definition of `eval` itself.

For example, when applied to an expression of type `a` `expr`,
`eval` calls itself recursively with an expression of type `a1` `expr`.

Polymorphic recursion is naturally available in System F (with *let rec*).
In OCaml, Haskell, etc., a type annotation is required.

Mycroft, **Polymorphic type schemes and recursive definitions**, 1984.

Going further

Our tiny expressions are **closed**: the typing judgement is $\vdash t : T$.

When expressions involve variables, one needs a type $(\text{'g'}, \text{'a'}) \text{ expr}$ whose definition encodes the typing judgement $\Gamma \vdash t : T$.

This is reasonably easy if variables are encoded as de Bruijn indices.

Bird, Paterson, **de Bruijn notation as a nested datatype**, 1999.

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A **value** of type `'a ty` is a runtime description of the **type** `'a`.

```
type _ ty =  
| TyInt   : int ty  
| TySum   : 'a ty * 'b ty -> ('a, 'b) sum ty  
| TyPair  : 'a ty * 'b ty -> ('a * 'b) ty
```

The binary sum type `('a, 'b) sum` is defined as follows:

```
type ('a, 'b) sum = Left of 'a | Right of 'b
```

It is also available in the standard library module **Either**.

An example of a runtime type description

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```
let example : (int * int) ty =  
  TyPair (TyInt, TyInt)
```

The value `example` is a runtime description of the type `int * int`.

This value has no other type.

This type has no other inhabitant: `(int * int) ty` is a [singleton type](#).

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Although **inspecting a type** at runtime is impossible, as types are erased, **inspecting a runtime description of a type** is possible.

In other words, although the type $\forall X. X \rightarrow X$ has only **one** inhabitant, the type $\forall X. \text{Ty } X \rightarrow X \rightarrow X$ has **more than one**.

This lets us write **polymorphic, type-directed** functions, an activity that is sometimes known as **generic programming**.

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Here is a polymorphic, type-directed [conversion of a value to a string](#).

```
let rec show : type a . a ty -> a -> string =  
  fun ty x ->  
    match ty with  
    | TyInt ->  
      string_of_int x  
    | TySum (ty1, ty2) ->  
      begin match x with  
      | Left x1 -> "left(" ^ show ty1 x1 ^ ")"  
      | Right x2 -> "right(" ^ show ty2 x2 ^ ")"  
      end  
    | TyPair (ty1, ty2) ->  
      let (x1, x2) = x in  
      "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

In each branch, [we learn something](#) about the type of x .

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It is more concise and looks better to deconstruct both arguments at once.

```
let rec show : type a . a ty -> a -> string =  
  fun ty x ->  
    match ty, x with  
    | TyInt, x ->  
      string_of_int x  
    | TySum (ty1, _), Left x1 ->  
      "left(" ^ show ty1 x1 ^ ")"  
    | TySum (_, ty2), Right x2 ->  
      "right(" ^ show ty2 x2 ^ ")"  
    | TyPair (ty1, ty2), (x1, x2) ->  
      "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

The OCaml type-checker reads patterns from left to right
so deconstructing (ty, x) works but deconstructing (x, ty) does not.

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Here is a polymorphic, type-directed **equality test**.

```
let rec equal : type a . a ty -> a -> a -> bool =
  fun ty x y ->
    match ty, x, y with
    | TyInt, x, y ->
      Int.equal x y
    | TySum (ty1, _), Left x1, Left y1 ->
      equal ty1 x1 y1
    | TySum (_, ty2), Right x2, Right y2 ->
      equal ty2 x2 y2
    | TySum _, Left _, Right _
    | TySum _, Right _, Left _ ->
      false
    | TyPair (ty1, ty2), (x1, x2), (y1, y2) ->
      equal ty1 x1 y1 && equal ty2 x2 y2
```

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Connections between GADTs and type classes

Eq and **Show** are typical examples of **type classes** in Haskell.

Here, a somewhat similar effect is achieved using GADTs.

Upcoming lecture on type classes (PED).

Hinze, Jeuring, Löh,
Comparing Approaches to Generic Programming in Haskell, 2006.

Connections between GADTs and type classes

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A **fixed** set of type class instances can be compiled down to a GADT.

If a Haskell program contains three instances of the class **Show**,
for integers, products, and sums,
then compiling type classes to GADTs
would produce (roughly) the function `show` of the previous slide.

Pottier and Gauthier,
Polymorphic typed defunctionalization and concretization, 2006.

A fixed set of **functions** can also be compiled down to a GADT.

– See above paper and next week's lecture!

Connections between GADTs and type classes

A limitation of this compilation scheme is that
a GADT describes a **fixed** universe of types
whereas type classes are **open-ended**.

The Holy Grail is to propose a language where **a type of the
representations of all types** (including itself!) can be defined.

Chapman, Dagand, McBride, Morris,
The Gentle Art of Levitation, 2010.

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Untyped expressions

We have a type of raw (untyped) expressions:

```
module Raw = struct
  type expr =
    | EInt   of int
    | EPair  of expr * expr
    | EFst   of expr
    | ESnd   of expr
end
```

This is an ordinary algebraic data type.

Typed expressions and type descriptions

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We also have a type `'a expr` of well-typed expressions:

```
type _ expr =
| EInt   :          int ->      int expr
| ...
```

and a type `'a ty` of runtime type descriptions:

```
type _ ty =
| TyInt   :          int ty
| ...
```

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Conclusion

Can we write a simple **type inferencer**
that accepts an untyped expression
and **either fails or returns a typed expression?**

```
exception IllTyped
let rec infer : Raw.expr -> ???
= function
  | Raw.EInt i ->
    (TyInt, EInt i)
  | ...
```

What should its **result type** be?

A type inferencer

We need an **existential type** $\exists X. \text{Ty } X \times \text{Expr } X$.

```
type typed_expr =  
| TypedExpr : 'a ty * 'a expr -> typed_expr
```

A type inferencer

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Conclusion

We can now write the type inferencer:

```
let rec infer : Raw.expr -> typed_expr =  
  function  
  | Raw.EInt i ->  
    TypedExpr (TyInt, EInt i)  
  | Raw.EFst e ->  
    let TypedExpr (ty, e) = infer e in  
    begin match ty with  
    | TyPair (ty1, ty2) -> TypedExpr (ty1, EFst e)  
    | _ -> raise IllTyped  
    end
```

Exercise: write the two missing cases (**EPair** and **ESnd**).

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let TypedExpr (inferred, e) = infer e in  
  if inferred = expected then  
    e  
  else  
    raise IllTyped
```

But **this code is not well-typed**. Why?

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let TypedExpr (inferred, e) = infer e in  
  if inferred = expected then  
    e  
  else  
    raise IllTyped
```

But **this code is not well-typed**. Why?

expected has type $a \text{ ty}$.

inferred has type $b \text{ ty}$

where b is an unknown type introduced by deconstructing **TypedExpr**.

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =
  let TypedExpr (inferred, e) = infer e in
  if inferred = expected then
    e
  else
    raise IllTyped
```

But **this code is not well-typed**. Why?

expected has type $a \text{ ty}$.

inferred has type $b \text{ ty}$

where b is an unknown type introduced by deconstructing **TypedExpr**.

They **cannot be compared** using homogeneous equality = .

Even if they could, e has type $b \text{ expr}$

whereas a result of type $a \text{ expr}$ is required.

The equality GADT

The solution involves the **type equality GADT**.

```
type (_, _) eq =  
  | Equal: ('a, 'a) eq
```

The type `('a, 'b) eq` has at most one inhabitant.

If it has one then this inhabitant must be **Equal**
and the types `'a` and `'b` must be the same.

For example, the type `(int, int) eq` has one inhabitant, namely **Equal**.
The type `(int, bool) eq` has no inhabitant.

The equality GADT

```
type (_, _) eq =  
| Equal: ('a, 'a) eq
```

The data constructor **Equal** has polymorphic type:

$$\forall \alpha. (\alpha, \alpha) \text{ eq}$$

which can also be understood as a **constrained** polymorphic type:

$$\forall \alpha \beta. (\alpha = \beta) \Rightarrow (\alpha, \beta) \text{ eq}$$

Any color the customer wants, as long as it's black. – Henry Ford

A heterogeneous type equality test

This lets us express a **heterogeneous** type equality test:

```
let rec equal : type a b . a ty -> b ty -> (a, b) eq =
  fun ty1 ty2 ->
    match ty1, ty2 with
    | TyInt, TyInt ->
      Equal
    | TyPair (ty1a, ty1b), TyPair (ty2a, ty2b) ->
      let Equal = equal ty1a ty2a in
      let Equal = equal ty1b ty2b in
      Equal
    | -, - ->
      raise IllTyped
```

When `equal ty1 ty2` succeeds, we **learn** that the runtime type descriptions `ty1` and `ty2` describe the same static type.

Exercise: write the missing case.

A type-checker

We can now write the type-checker:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let TypedExpr (inferred, e) = infer e in  
  let Equal = equal inferred expected in  
  e
```

Exercise: make sure that you understand why this code is well-typed.

Putting the pieces together

Given an arbitrary untyped expression in our tiny language, we can now **infer** its type, **evaluate** it, and **show** its value, whatever its type may be.

```
let () =  
  let e = Raw.(EPair (EInt 42, EInt 0)) in  
  let TypedExpr (ty, e) = infer e in  
  let v = eval e in  
  Printf.printf "%S\n%!" (show ty v)
```

This program prints:

```
(42, 0)
```

Putting the pieces together

Here is
a second example
(of a different type!):

```
let () =  
  let e = Raw.(EFst (EPair (EInt 42, EInt 0))) in  
  let TypedExpr (ty, e) = infer e in  
  let v = eval e in  
  Printf.printf "%s\n%!" (show ty v)
```

This program prints:

42

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Printf in OCaml

printf takes a “format string” followed with a number of arguments:

```
# open Printf;;  
# printf "%d * %s = %d\n" 2 "12" 24;;  
2 * 12 = 24  
- : unit = ()
```

The number and type of these arguments *depends* on the format string.

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A format string is actually **not** a string: it is a **data structure**.

```
# open CamlinternalFormatBasics;;
# let desc : _ format6 = "%d * %s = %d\n";;
val desc :
  (int -> string -> int -> 'a, 'b, 'c, 'd, 'd, 'a) format6 =
  Format
    (Int (Int_d, No_padding, No_precision,
      String_literal (" * ",
        String (No_padding,
          String_literal (" = ",
            Int (Int_d, No_padding, No_precision,
              Char_literal ('\n', End_of_format))))),
      "%d * %s = %d\n")
```

Printf in OCaml

This data structure has the shape of a list:

```
Int (Int_d, No_padding, No_precision,  
String_literal (" * ",  
String (No_padding,  
String_literal (" = ",  
Int (Int_d, No_padding, No_precision,  
Char_literal ('\n',  
End_of_format))))))
```

End_of_format is “nil”; the other constructors are “cons” constructors.

Int and **String** correspond to “holes” %d and %s.

String_literal and **Char_literal** correspond to literal pieces of string.

An algebraic data type of descriptors

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Can we define our own algebraic data type of formats, or **descriptors**?

```
type desc =  
  | Nil  
  | Lit of string * desc  
  | Int of desc
```

An algebraic data type of descriptors

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Or, in this alternative syntax:

```
type desc =  
  | Nil : desc  
  | Lit : string * desc -> desc  
  | Int : desc -> desc
```

An algebraic data type of descriptors

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Or, in this alternative syntax:

```
type desc =  
  | Nil  : desc  
  | Lit  : string * desc -> desc  
  | Int  : desc -> desc
```

Now, please define `fprintf` so that `fprintf` emit `desc <args>`

- emits output via the function `emit : string -> unit`,
- obeys `desc`,
- expects arguments `<args>` whose number and type satisfy `desc`.

`fprintf` should have type `(string -> unit) -> desc -> ??? -> unit`.

Expressing the type of `fprintf`

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The type `desc -> ??? -> unit` does not make sense.

The number of and type of the arguments `???` **depends** on the descriptor.

We seem to need a **dependent type** `(d: desc) -> shape d`

- where `shape` would be a function of descriptors to types,
- but OCaml does not have that.

Expressing the type of `fprintf`

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The type `desc -> ??? -> unit` does not make sense.

The number of and type of the arguments `???` depends on the descriptor.

We seem to need a dependent type `(d: desc) -> shape d`

- where `shape` would be a function of descriptors to types,
- but OCaml does not have that.

Instead, let's use a plain function type `'shape desc -> 'shape`

- where the definition of `'shape desc` as a GADT encodes the correspondence between descriptors and shapes.

Descriptors form a typed language and `fprintf` is an interpreter for it!

A GADT of descriptors

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type desc =
  | Nil : desc
  | Lit : string * desc -> desc
  | Int : desc -> desc
```

We must turn the type `desc` into a GADT.

A GADT of descriptors

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =  
  | Nil :                                ?? desc  
  | Lit :          string * ?? desc ->    ?? desc  
  | Int :                                ?? desc ->    ?? desc
```

We parameterize the type `desc`.

A GADT of descriptors

We want `fprintf : (string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil   :                               unit desc
  | Lit   :          string * ?? desc ->    ?? desc
  | Int   :                               ?? desc ->    ?? desc
```

`Nil` requires no action; the corresponding shape is `unit`.

A GADT of descriptors

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We want `fprintf : (string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil   :                               unit desc
  | Lit   :          string * 'a desc ->    'a desc
  | Int   :                               ?? desc ->    ?? desc
```

`Lit` (`s`, `d`) requires printing `s` and interpreting `d`.

If `d` has shape `'a` then `Lit` (`s`, `d`) has shape `'a` as well.

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We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil   :                               unit desc
  | Lit   :          string * 'a desc ->      'a desc
  | Int   :          'a desc ->    (int -> 'a) desc
```

`Int d` requires consuming an integer argument and interpreting `d`.

If `d` has shape `'a` then `Int d` has shape `int -> 'a`.

A GADT of descriptors

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We want `fprintf : (string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil   :                               unit desc
  | Lit   :          string * 'a desc ->      'a desc
  | Int   :          'a desc ->    (int -> 'a) desc
```

`Int d` requires consuming an integer argument and interpreting `d`.

If `d` has shape `'a` then `Int d` has shape `int -> 'a`.

We can in fact replace `Int` with a more general constructor `Hole`...

A GADT of descriptors

We want `fprintf` : `(string -> unit) -> 'a desc -> 'a`.

```
type _ desc =
  | Nil      : unit desc
  | Lit      : string * 'a desc -> 'a desc
  | Hole     : ('data -> string) * 'a desc -> ('data -> 'a) desc
```

We now allow a hole of arbitrary type `'data`.

We require a conversion function of type `'data -> string`.

Implementing fprintf

```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        ???  
      | Lit (s, desc) ->  
        ???  
      | Hole (to_string, desc) ->  
        ???  
  
  in eval desc
```

Recall

```
| Nil : unit desc
```

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```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        ??? (* We learn [a = unit]. *)  
      | Lit (s, desc) ->  
        ???  
      | Hole (to_string, desc) ->  
        ???  
  
  in eval desc
```

Recall

```
| Nil : unit desc
```

Implementing fprintf

```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        () (* We learn [a = unit]. *)  
      | Lit (s, desc) ->  
        ???  
      | Hole (to_string, desc) ->  
        ???  
  
  in eval desc
```

Recall

```
| Lit : string * 'a desc -> 'a desc
```

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```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        ()  
      | Lit (s, desc) ->  
        ??? (* We learn no new type equality. *)  
      | Hole (to_string, desc) ->  
        ???  
  
  in eval desc
```

Recall

```
| Lit : string * 'a desc -> 'a desc
```

Implementing fprintf

```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        ()  
      | Lit (s, desc) ->  
        emit s; eval desc  
      | Hole (to_string, desc) ->  
        ???  
  
  in eval desc
```

Recall

```
| Hole : ('data -> string) * 'a desc -> ('data -> 'a) desc
```

Implementing fprintf

```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        ()  
      | Lit (s, desc) ->  
        emit s; eval desc  
      | Hole (to_string, desc) ->  
        ??? (* We learn [a = data -> b] *)  
             (* [to_string : data -> string; desc : b desc] *)  
  in eval desc
```

Recall

```
| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc
```

Implementing fprintf

```
let fprintf (type a) emit (desc : a desc) : a =  
  let rec eval : type a . a desc -> a =  
    function  
      | Nil ->  
        ()  
      | Lit (s, desc) ->  
        emit s; eval desc  
      | Hole (to_string, desc) ->  
        fun x -> emit (to_string x); eval desc  
        (* [x] has type [data]; [eval desc] has type [b] *)  
  in eval desc (* and [data -> b] is [a] *)
```

Recall

```
| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc
```

Using fprintf

Voilà! From fprintf, we get printf.

```
let printf desc =  
  let emit = print_string in  
  fprintf emit desc
```

Its type is 'a desc -> 'a.

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Conclusion

To construct descriptors, some sugar is needed.

```
module Sugar = struct
  let nil = Nil
  let lit s desc = Lit (s, desc)
  let d desc = Hole (string_of_int, desc)
  let s desc = Hole (Fun.id, desc)
end
```

Using fprintf

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Conclusion

To construct descriptors, some sugar is needed.

```
module Sugar = struct
  let nil = Nil
  let lit s desc = Lit (s, desc)
  let d desc = Hole (string_of_int, desc)          (* %d *)
  let s desc = Hole (Fun.id, desc)                 (* %S *)
end
```

For example,

```
let desc = (* "%d * %S = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

`@@` is OCaml's low-priority **application operator**.

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Using fprintf

```
let desc = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf desc 2 "12" 24;;
2 * 12 = 24
```

Implementing sprintf

Can we implement sprintf, which returns a string?

```
let naive_sprintf desc args =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b in  
  fprintf emit desc args;  
  Buffer.contents b
```

This is accepted but is **not** what we want.

Implementing sprintf

Can we implement sprintf, which returns a string?

```
let naive_sprintf desc args =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b in  
  fprintf emit desc args;  
  Buffer.contents b
```

This is accepted but is **not** what we want.

Its (inferred) type is `('a -> 'b) desc -> 'a -> string`.

Implementing sprintf

Can we implement sprintf, which returns a string?

```
let naive_sprintf desc args =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b in  
  fprintf emit desc args;  
  Buffer.contents b
```

This is accepted but is **not** what we want.

Its (inferred) type is `('a -> 'b) desc -> 'a -> string`.

We want sprintf to accept **a variable number of arguments**, not just one.

Implementing printf

In fact, we cannot write the desired type of `sprintf`.

Whereas `fprintf` has type `desc -> ??? -> unit`
which we have encoded as `'a desc -> 'a`,
we want `sprintf` to have type `desc -> ??? -> string`.

How can this be expressed?

A more general type of descriptors

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Conclusion

We must equip ourselves with a **more general** type of descriptors.

```
type _ desc =
  | Nil : unit desc
  | Lit : string * 'a desc -> 'a desc
  | Hole : ('data -> string) * 'a desc -> ('data -> 'a ) desc
```

A more general type of descriptors

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Conclusion

We must equip ourselves with a **more general** type of descriptors.

```
type (_, _) desc =
  | Nil : ('r, 'r) desc
  | Lit : string * ('a, 'r) desc -> ('a, 'r) desc
  | Hole : ('data -> string) * ('a, 'r) desc ->
            ('data -> 'a, 'r) desc
```

In the type `('a, 'r) desc`,

- `'a` is the **shape**, as before,
- `'r` is the **eventual return type** of this shape.
 - it can be **unit** for `fprintf` and **string** for `sprintf`;
 - a descriptor can be polymorphic in `'r`.

Implementing fprintf, again

We can now give fprintf a **more general** type. We parameterize it with:

- `emit : string -> unit`
- `finished : unit -> r` — **new**
- `desc : (a, r) desc`

`fprintf emit finished desc` has type `a`.

`a` must in fact be a function type whose eventual return type is `r`.

`fprintf emit finished desc <args>` must eventually return a value of type `r`, which it obtains by calling `finished()`.

Implementing fprintf, again

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```
let fprintf (type a r) emit (finished : unit -> r)
      (desc : (a, r) desc) : a =
  let rec eval : type a . (a, r) desc -> a =
    function
    | Nil ->
      (* We have [a = r] so [finished()] has type [a]. *)
      finished()
    | Lit (s, desc) ->
      emit s; eval desc
    | Hole (to_string, desc) ->
      fun x -> emit (to_string x); eval desc
  in eval desc
```

Implementing printf and sprintf

We can now implement printf and sprintf, among other variations:

```
let printf desc =  
  let emit = print_string  
  and finished () = () in  
  fprintf emit finished desc  
  
let sprintf desc =  
  let b = Buffer.create 128 in  
  let emit = Buffer.add_string b  
  and finished () = Buffer.contents b in  
  fprintf emit finished desc
```

We get

```
val printf : ('a, unit) desc -> 'a  
val sprintf : ('a, string) desc -> 'a
```

Using printf and sprintf

```
let desc () = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf (desc()) 2 "12" 24;;
2 * 12 = 24
# let (s : string) = sprintf (desc()) 2 "12" 24;;
val s : string = "2 * 12 = 24\n"
```

Here, we make desc a (constant) function in order to work around the [value restriction](#). See upcoming lecture on mutable state (GS).

Danvy et al.'s approach

Danvy, Keller and Puech (2015) view formats as **trees** instead of **lists**.

```
type (_, _) desc =
  | Lit   : string -> ('a, 'a) desc
  | Hole  : ('data -> string) -> ('data -> 'a, 'a) desc
  | Seq   : ('a, 'b) desc * ('b, 'c) desc -> ('a, 'c) desc
```

The type `('a, 'r) desc` has the same meaning as earlier.

Lit and **Hole** no longer play the role of list “cons” constructors.

Seq is a binary concatenation constructor, whose type says:

*If 'a is a multi-arrow type whose eventual return type is 'b and
if 'b is a multi-arrow type whose eventual return type is 'c then
'a is a multi-arrow type whose eventual return type is 'c.*

Danvy et al.'s approach

Danvy et al. write `kprintf` in [continuation-passing style](#):

```
let rec kprintf
: type a r . (a, r) desc -> (string -> r) -> a =
  fun desc finished ->
    match desc with
    | Lit s ->
      finished s
    | Hole to_string ->
      fun x -> finished (to_string x)
    | Seq (desc1, desc2) ->
      kprintf desc1 @@ fun s1 ->
        kprintf desc2 @@ fun s2 ->
          finished (s1 ^ s2)
```

Exercise (easy): define `printf`, `sprintf`, and `fprintf` using `kprintf`.

Exercise (harder): define `fprintf` directly.

Do not use string concatenation `^`.

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System F +GADTs

System F +GADTs was defined by Xi, Chen and Chen (2003).

Xi, Chen, Chen,
Guarded Recursive Datatype Constructors, 2003.

Pottier and Gauthier,
Polymorphic typed defunctionalization and concretization, 2006.

System F +GADTs: terms

The syntax of terms is extended with constructor applications and case analysis constructs:

$$\begin{array}{lcl} t & ::= & \dots \\ & | & K\ t \\ & | & \text{case } t \text{ of } \bar{c} \\ c & ::= & K\ \bar{X}\ x \mapsto t \end{array}$$

Each branch in a case construct is a **clause** c .

System F_{+} GADTs: the typing judgement

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Recall the typing judgement of System F :

$$\Gamma \vdash t : T$$

In System F_{+} GADTs, must we change the shape of this judgement?

System F +GADTs: the typing judgement

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Recall the typing judgement of System F :

$$\Gamma \vdash t : T$$

In System F +GADTs, must we change the shape of this judgement?

We must extend it with a conjunction of **equality hypotheses**.

$$\Gamma \mid C \vdash t : T$$

This means: under Γ , **assuming the constraint C is true**, t has type T .

Equality constraints are given by $C, D ::= \text{True} \mid \text{False} \mid T = T \mid C \wedge C$.

In a well-formed judgement, every variable or type variable in C, t, T is introduced by Γ .

System F +GADTs: type declarations

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We assume that a family of type constructors F is given.

- for simplicity, we assume they have arity 1.

The syntax of types includes applications of type constructors:

$$T := X \mid T \rightarrow T \mid T + T \mid T \times T \mid 0 \mid 1 \mid F\ T$$

We assume that a family of data constructors K is given.

- for simplicity, we assume they have arity 1.

We assume that each data constructor has a closed **type scheme**:

$$K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F\ T_2$$

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For example, the equality GADT *eq* could be a type constructor:

$$F ::= \dots \mid eq$$

Equal could be a data constructor:

$$K ::= \dots \mid Equal$$

whose type scheme would be:

$$Equal : \forall \alpha \beta. \alpha = \beta \Rightarrow unit \rightarrow eq (\alpha \times \beta)$$

This is a **constrained type scheme** – remember Henry Ford.

System F +GADTs: an auxiliary judgement

For readability, we introduce the auxiliary judgement

$$K \leq D \Rightarrow T_1 \rightarrow F T_2$$

whose definition is the following:

$$\frac{K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2}{K \leq D[\bar{T}/\bar{X}] \Rightarrow T_1[\bar{T}/\bar{X}] \rightarrow F T_2[\bar{T}/\bar{X}]}$$

This judgement means that $D \Rightarrow T_1 \rightarrow F T_2$ is
a **valid monomorphic constrained type** for K .

For example, we have

$$\frac{Equal : \forall \alpha \beta. \alpha = \beta \Rightarrow unit \rightarrow eq (\alpha \times \beta)}{Equal \leq int = int \Rightarrow unit \rightarrow eq (int \times int)}$$

System F +GADTs: the typing judgement

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Conclusion

The shape of the typing judgement is now $\Gamma \mid C \vdash t : T$.

This means: under Γ , **assuming the constraint C is true**, t has type T .

$$\begin{array}{c}
 \text{VAR} \\
 \Gamma \mid C \vdash x : \Gamma(x)
 \end{array}
 \qquad
 \begin{array}{c}
 \text{ABS} \\
 \dfrac{\Gamma; x : T_1 \mid C \vdash t : T_2}{\Gamma \mid C \vdash \lambda x. t : T_1 \rightarrow T_2}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{APP} \\
 \dfrac{\Gamma \mid C \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \mid C \vdash t_2 : T_1}{\Gamma \mid C \vdash t_1 t_2 : T_2}
 \end{array}$$

$$\begin{array}{c}
 \text{TAbs} \\
 \dfrac{\Gamma; X \mid C \vdash t : T \quad X \# C}{\Gamma \mid C \vdash \lambda X. t : \forall X. T}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{TApp} \\
 \dfrac{\Gamma \mid C \vdash t : \forall X. T}{\Gamma \mid C \vdash t T' : T[T'/X]}
 \end{array}$$

The rules of System F are unchanged, except a constraint is transported.

System F +GADTs: the typing judgement, continued

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The typing rule for a **data constructor application** is straightforward:

$$\text{DCon} \quad \frac{\begin{array}{c} K \leq D \Rightarrow T_1 \rightarrow F T_2 \\ C \Vdash D \\ \Gamma \mid C \vdash t : T_1 \end{array}}{\Gamma \mid C \vdash K t : F T_2}$$

We write $C \Vdash D$ when **C entails D** (see next slide).

For example, we have

$$\frac{\begin{array}{c} \text{Equal} \leq \text{int} = \text{int} \Rightarrow \text{unit} \rightarrow \text{eq} (\text{int} \times \text{int}) \\ \text{True} \Vdash \text{int} = \text{int} \quad \Gamma \mid \text{True} \vdash () : \text{unit} \end{array}}{\Gamma \mid \text{True} \vdash \text{Equal} () : \text{eq} (\text{int} \times \text{int})}$$

On the other hand, $\Gamma \mid \text{True} \vdash \text{Equal} () : \text{eq} (\text{int} \times \text{bool})$ **cannot** be proved because $\text{True} \Vdash \text{int} = \text{bool}$ does not hold.

System F_{+} GADTs: entailment

Let ρ denote a total mapping of type variables to closed types.

We write $\rho \vdash C$ to mean that ρ satisfies C or ρ is a solution of C :

$$\rho \vdash \text{True} \qquad \frac{\rho(T_1) = \rho(T_2)}{\rho \vdash T_1 = T_2} \qquad \frac{\rho \vdash C_1 \quad \rho \vdash C_2}{\rho \vdash C_1 \wedge C_2}$$

The entailment $C \Vdash D$ holds if every solution of C is also a solution of D :

$$\frac{\forall \rho. \rho \vdash C \Rightarrow \rho \vdash D}{C \Vdash D}$$

Entailment is decidable.

System F_{+} GADTs: entailment

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For example,

$True$	entails	$int = int$	
$\alpha = int$	entails	$int = \alpha$	
$\alpha = int \wedge \alpha = \beta$	entails	$\beta = int$	
$\alpha \times \beta = int \times bool$	entails	$\alpha = int \wedge \beta = bool$	
$\alpha \times \beta = int \times bool$	entails	$\alpha \times \alpha = int \times int$	
$int = bool$	entails	$False$	
$F \ T = F' \ T'$	entails	$False$	if $F \neq F'$
$False$	entails	$T = T'$	

System F +GADTs: the typing judgement, continued

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Type-checking a **case analysis** construct is straightforward:

$$\text{CASE} \quad \frac{\begin{array}{c} \Gamma \mid C \vdash t : T_1 \\ \forall c \in \bar{c}. \quad \Gamma \mid C \vdash c : T_1 \rightarrow T_2 \\ \bar{c} \text{ is exhaustive} \end{array}}{\Gamma \mid C \vdash \text{case } t \text{ of } \bar{c} : T_2}$$

A **clause** takes the form $c ::= K \bar{X} x \mapsto t$.

\bar{c} is exhaustive if it contains a clause for every data constructor K .

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When a clause is entered, **new constraints appear** locally.

CLAUSE

$$\frac{\begin{array}{c} K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2 \\ (\Gamma; \bar{X}; x : T_1) \mid (C \wedge D \wedge F T_2 = F' T'_2) \vdash t : T' \\ \bar{X} \# C, T'_2, T' \end{array}}{\Gamma \mid C \vdash K \bar{X} x \mapsto t : F' T'_2 \rightarrow T'}$$

For example,

$$\frac{\begin{array}{c} \text{Equal} : \forall \alpha \beta. \alpha = \beta \Rightarrow \text{unit} \rightarrow \text{eq} (\alpha \times \beta) \\ (\Gamma; \alpha; \beta; x : \text{unit}) \mid (\text{True} \wedge \alpha = \beta \wedge \text{eq} (\alpha \times \beta) = \text{eq} (T \times U)) \vdash t : T' \end{array}}{\Gamma \mid \text{True} \vdash \text{Equal } \alpha \beta x \mapsto t : \text{eq} (T \times U) \rightarrow T'}$$

so the branch t is type-checked under the assumption $\text{eq} (\alpha \times \beta) = \text{eq} (T \times U)$, which entails $T = U$.

The type-checker **can assume** $T = U$ while type-checking t .

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Conclusion

There remains to introduce a typing rule that **exploits** the hypothesis C :

$$\text{CONVERSION} \quad \frac{\Gamma \mid C \vdash t : T \quad C \Vdash T = T'}{\Gamma \mid C \vdash t : T'}$$

This rule is **not** syntax-directed.

One can imagine a variant of the system where conversion is explicit.
System FC is the core language of the Glasgow Haskell compiler.

Sulzmann, Chakravarty, Peyton Jones, Donnelly,
System F with Type Equality Coercions, 2007.

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Another rule can exploit the hypothesis C when this hypothesis is *False*:

$$\frac{\text{CONTRADICTION} \quad C \Vdash \text{False}}{\Gamma \mid C \vdash t : T}$$

Ex falso, quod libet.

In particular, if $C \Vdash \text{False}$ then $\Gamma \mid C \vdash \text{absurd} : T$.

In OCaml, *absurd* is written `.` and is used to indicate a dead branch.

System F_{+} GADTs: type soundness

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Exercise: write down the omitted details (e.g., the reduction rule for *case*), then prove Subject Reduction and Progress.

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An affair to remember

The fundamental feature of GADTs is to let a **value** serve as a witness of an equality between two **types**.

Therefore, a **runtime test** (a case analysis) **can reveal type information**, even though types do not exist at runtime: **type erasure** is still possible.

GADTs allow simulating some uses of **dependent types**.

Without GADTs, one could live with ordinary algebraic data types: more **tags**, more (redundant) **runtime tests**, more **dead branches**.

With GADTs, one can gain space, time, elegance and static assurance.

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Beyond GADTs

GADTs allow reasoning about equalities between types, but the idea can be generalized by allowing constraints to be **formulae in a more expressive logic**, as long as entailment remains decidable.

For instance, Xi's "Dependent ML" exploits Presburger arithmetic formulae to reason about integers, with applications to array bounds checking, balanced binary trees, etc.