

MPRI 2.4

Operational semantics and reduction strategies

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The λ -calculus

The formal model that underlies all functional programming languages.

Abstract syntax:

$$t, u ::= x \mid \lambda x. t \mid t \ t \quad (\text{terms})$$

Reduction:

$$(\lambda x. t) \ u \longrightarrow t[u/x] \quad (\beta)$$

Mnemonic: read $t[u/x]$ as “ t , where u is substituted for x ”.

Landin, [Correspondence betw. ALGOL 60 and Church's \$\lambda\$ -notation](#), 1965.

From the λ -calculus to a functional programming language

Start from the λ -calculus, and follow several steps:

- Fix a **reduction strategy** (today).
- Develop **efficient execution mechanisms** (next week).
- **Enrich the language** with primitive data types and operations, recursion, algebraic data structures, and so on (next week).
- Define a static **type system** (Rémy's lectures).

1 Reduction strategies

Operational semantics

Plotkin: — *It is only through having an operational semantics that the λ -calculus can be viewed as a programming language.*

Scott: — *Why call it operational semantics? What is operational about it?*

An operational semantics describes **the actions of a machine**,
in the simplest possible manner / at the most abstract level.

Plotkin, **A Structural Approach to Operational Semantics**, 1981, (2004).

Plotkin, **The Origins of Structural Operational Semantics**, 2004.

The call-by-value strategy

Values form a subset of terms:

$$\begin{array}{ll} t, u & ::= x \mid \lambda x. t \mid t t & \text{(terms)} \\ v & ::= x \mid \lambda x. t & \text{(values)} \end{array}$$

A value represents the **result** of a computation.

The **call-by-value** reduction relation $t \longrightarrow_{\text{cbv}} t'$ is inductively defined:

$$\begin{array}{c} \beta_v \\ \hline (\lambda x. t) \textcolor{red}{v} \longrightarrow_{\text{cbv}} t[\textcolor{red}{v}/x] \end{array} \qquad \begin{array}{c} \text{APP L} \\ t \longrightarrow_{\text{cbv}} t' \\ \hline t u \longrightarrow_{\text{cbv}} t' u \end{array} \qquad \begin{array}{c} \text{APP VR} \\ u \longrightarrow_{\text{cbv}} u' \\ \hline \textcolor{red}{v} u \longrightarrow_{\text{cbv}} \textcolor{red}{v} u' \end{array}$$

This is known as a **small-step** operational semantics.

Example

This is a proof (a.k.a. derivation) that **one** reduction step is permitted:

$$\frac{\displaystyle \frac{\displaystyle \frac{x[1/x] = 1}{(\lambda x.x) \ 1 \longrightarrow_{\text{cbv}} 1} \beta_v}{(\lambda x.\lambda y.y \ x) \ ((\lambda x.x) \ 1) \longrightarrow_{\text{cbv}} (\lambda x.\lambda y.y \ x) \ 1} \text{APP R}}{(\lambda x.\lambda y.y \ x) \ ((\lambda x.x) \ 1) \ (\lambda x.x) \longrightarrow_{\text{cbv}} (\lambda x.\lambda y.y \ x) \ 1 \ (\lambda x.x)} \text{APP L}$$

Features of call-by-value reduction

- **Weak reduction.** One cannot reduce under a λ -abstraction.

$$\frac{t \rightarrow_{\text{cbv}} t'}{\lambda x. t \rightarrow_{\text{cbv}} \lambda x. t'}$$

Thus, **values do not reduce**.

Also, we are interested in reducing **closed terms** only.

- **Call-by-value.** An actual argument is reduced to a value **before** it is passed to a function.

$$(\lambda x. t) \text{ } \textcolor{red}{v} \rightarrow_{\text{cbv}} t[\textcolor{red}{v}/x]$$

$$(\lambda x. t) (u_1 \ u_2) \rightarrow_{\text{cbv}} t[u_1 \ u_2/x]$$

Features of call-by-value reduction

- **Left-to-right.** In an application $t\ u$, the term t must be reduced to a value before u can be reduced at all.

$$\text{APPVR} \quad \frac{u \longrightarrow_{\text{cbv}} u'}{\textcolor{red}{V}\ u \longrightarrow_{\text{cbv}} \textcolor{red}{V}\ u'}$$

- **Determinism.** For every term t , there is at most one term t' such that $t \longrightarrow_{\text{cbv}} t'$ holds.

Reduction sequences

Sequences of reduction steps describe the behavior of a term.

The three following situations are mutually exclusive:

- **Termination:** $t \longrightarrow_{\text{cbv}} t_1 \longrightarrow_{\text{cbv}} t_2 \longrightarrow_{\text{cbv}} \dots \longrightarrow_{\text{cbv}} v$
The value v is the result of evaluating t .
The term t **converges** to v .
- **Divergence:** $t \longrightarrow_{\text{cbv}} t_1 \longrightarrow_{\text{cbv}} t_2 \longrightarrow_{\text{cbv}} \dots \longrightarrow_{\text{cbv}} t_n \longrightarrow_{\text{cbv}} \dots$
The sequence of reductions is infinite.
The term t **diverges**.
- **Error:** $t \longrightarrow_{\text{cbv}} t_1 \longrightarrow_{\text{cbv}} t_2 \longrightarrow_{\text{cbv}} \dots \longrightarrow_{\text{cbv}} t_n \not\longrightarrow_{\text{cbv}} \cdot$
where t_n is not a value, yet does not reduce: t_n is **stuck**.
The term t **goes wrong**. This is a **runtime error**.

A strong **type system** rules out errors (**Milner, 1978**).

Some type systems rule out both errors and divergence.

Examples of reduction sequences

Termination:

$$\begin{aligned}
 (\lambda x. \lambda y. y \ x) ((\lambda x. x) \ 1) (\lambda x. x) &\longrightarrow_{\text{cbv}} (\lambda x. \lambda y. y \ x) \ 1 (\lambda x. x) \\
 &\longrightarrow_{\text{cbv}} (\lambda y. y \ 1) (\lambda x. x) \\
 &\longrightarrow_{\text{cbv}} (\lambda x. x) \ 1 \\
 &\longrightarrow_{\text{cbv}} 1
 \end{aligned}$$

Divergence:

$$(\lambda x. x \ x) (\lambda x. x \ x) \longrightarrow_{\text{cbv}} (\lambda x. x \ x) (\lambda x. x \ x) \longrightarrow_{\text{cbv}} \dots$$

Error:

$$(\lambda x. x \ x) \ 2 \longrightarrow_{\text{cbv}} 2 \ 2 \not\rightarrow_{\text{cbv}} \cdot$$

The active redex is highlighted in red.

An alternative style: evaluation contexts

First, define **head reduction**:

$$\frac{\beta_v}{(\lambda x.t) \ v \longrightarrow_{\text{cbv}}^{\text{head}} t[v/x]}$$

Then, define **reduction** as head reduction under an evaluation context:

$$\frac{\text{Ctx} \quad t \longrightarrow_{\text{cbv}}^{\text{head}} t'}{E[t] \longrightarrow_{\text{cbv}} E[t']}$$

where evaluation contexts E are defined by $E ::= [] \mid E \ u \mid v \ E$.

Wright and Felleisen, **A syntactic approach to type soundness**, 1992.

Unique decomposition

In this alternative style, the determinism of the reduction relation follows from a **unique decomposition** lemma:

Lemma (Unique Decomposition)

For every term t , there exists at most one pair (E, u) such that $t = E[u]$ and $u \longrightarrow_{cbv}^{head} \cdot$.

The call-by-name strategy

The **call-by-name** reduction relation $t \longrightarrow_{\text{cbn}} t'$ is defined as follows:

$$\frac{\beta}{(\lambda x.t) \textcolor{red}{u} \longrightarrow_{\text{cbn}} t[\textcolor{red}{u}/x]} \qquad \frac{\text{APPL} \quad t \longrightarrow_{\text{cbn}} t'}{t \textcolor{red}{u} \longrightarrow_{\text{cbn}} t' \textcolor{red}{u}}$$

The **unevaluated** actual argument is passed to the function.

It is later reduced if / when / every time the function **demands** its value.

An example reduction sequence

$$\begin{aligned} (\lambda x. \lambda y. y \ x) ((\lambda x. x) \ 1) (\lambda x. x) &\longrightarrow_{\text{cbn}} (\lambda y. y \ ((\lambda x. x) \ 1)) (\lambda x. x) \\ &\longrightarrow_{\text{cbn}} (\lambda x. x) ((\lambda x. x) \ 1) \\ &\longrightarrow_{\text{cbn}} (\lambda x. x) \ 1 \\ &\longrightarrow_{\text{cbn}} 1 \end{aligned}$$

Call-by-value versus call-by-name

If t terminates under CBV, then it also terminates under CBN (*).

The converse is **false**:

$$\begin{aligned}(\lambda x.1) \omega &\longrightarrow_{\text{cbn}} 1 \\(\lambda x.1) \omega &\longrightarrow_{\text{cbv}}^{\infty}\end{aligned}$$

where $\omega = (\lambda x.x x) (\lambda x.x x)$ diverges under both strategies.

Call-by-value can perform fewer reduction steps:

$(\lambda x. x + x) t$ evaluates t once under CBV, **twice** under CBN.

Call-by-name can perform fewer reduction steps:

$(\lambda x. 1) t$ evaluates t once under CBV, **not at all** under CBN.

- (*) In fact, the **standardization** theorem implies that
if t can be reduced to a value via any strategy,
then it can be reduced to a value via CBN.
See **Takahashi (1995)**.

Encoding call-by-name in a CBV language

Use **thunks**: functions $\lambda_ . u$ whose purpose is to delay the evaluation of u .

$$\begin{aligned}\llbracket x \rrbracket &= x () \\ \llbracket \lambda x . t \rrbracket &= \lambda x . \llbracket t \rrbracket \\ \llbracket t \ u \rrbracket &= \llbracket t \rrbracket (\lambda_ . \llbracket u \rrbracket)\end{aligned}$$

Exercise: Can you **state** that this encoding is correct? Can you **prove** it?
— 2017 exam! (**paper assignment and solution**) (**Coq solution**)

Encoding call-by-name in a CBV language

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The translation of types is defined by

$$\llbracket T_1 \rightarrow T_2 \rrbracket = \text{thunk } \llbracket T_1 \rrbracket \rightarrow \llbracket T_2 \rrbracket$$

where $\text{thunk } T$ is $\text{unit} \rightarrow T$.

The translation of type environments is as follows:

$\llbracket x_1 : T_1; \dots; x_n : T_n \rrbracket$ stands for $x_1 : \text{thunk } \llbracket T_1 \rrbracket; \dots; x_n : \text{thunk } \llbracket T_n \rrbracket$.

Encoding call-by-value in a CBN language

This is somewhat more involved.

The call-by-value [continuation-passing style](#) (CPS) transformation, studied later on in this course, achieves this.

Call-by-need

Call-by-need, also known as **lazy evaluation**, eliminates the main inefficiency of call-by-name (namely, possibly repeated computation) by introducing **memoization**.

It, too, can be defined via an operational semantics (**Ariola and Felleisen, 1997**; **Maraist, Odersky, Wadler, 1998**).

It is used in Haskell, where it encourages a **modular style** of programming.

Hughes, **Why functional programming matters**, 1990.

Also see **Harper's** and **Augustsson's** blog posts on laziness.

Newton-Raphson iteration (after Hughes)

This is pseudo-Haskell code. The colon `:` is “cons”.

An approximation of a square root can be computed as follows:

```
next n x = (x + n / x) / 2
repeat f a = a : (repeat f (f a))
within eps (a : b : rest) =
  if abs (a - b) <= eps then b
  else within eps (b : rest)
sqrt a0 eps n =
  within eps (repeat (next n) a0)
```

`repeat (next n) a0` is a **producer** of an infinite stream of numbers.

Its type is just “list of numbers” – look Ma, **no iterators**!

The **consumer** `within eps` decides how many elements to demand.

The two are programmed **independently**.

Encoding call-by-need in a CBV language

Call-by-need can be encoded into CBV by using **memoizing thunks**:

$$\begin{aligned}\llbracket x \rrbracket &= \text{force } x \\ \llbracket \lambda x. t \rrbracket &= \lambda x. \llbracket t \rrbracket \\ \llbracket t \ u \rrbracket &= \llbracket t \rrbracket (\text{suspend } (\lambda_. \llbracket u \rrbracket))\end{aligned}$$

Such a thunk evaluates u when **first** forced,
then memoizes the result,
so no computation is required if the thunk is forced **again**.

Thunks can be thought of as an abstract type with this API or signature:

```
type 'a thunk
val suspend: (unit -> 'a) -> 'a thunk
val force: 'a thunk -> 'a
```


Encoding call-by-need in a CBV language

Exercise: implement the thunk API in OCaml. (**Solution.**)

In reality, this exercise is unnecessary, as OCaml has built-in thunks:

- “suspend $(\lambda_.u)$ ” is written **lazy** u .
- “force x ” is written **Lazy**.force x .

Exercise: port Newton-Raphson iteration to OCaml.

Make sure that **each element is computed at most once** and **no more elements than necessary** are computed.

Write tests to verify these properties. (**Solution.**)