

MPRI 2.4

From operational semantics to interpreters

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The λ -calculus

The formal model that underlies all functional programming languages.

Abstract syntax:

$$t, u ::= x \mid \lambda x. t \mid t \ t \quad (\text{terms})$$

Reduction:

$$(\lambda x. t) \ u \longrightarrow t[u/x] \quad (\beta)$$

Mnemonic: read $t[u/x]$ as “ t , where u is substituted for x ”.

Landin, **Correspondence betw. ALGOL 60 and Church's λ -notation**, 1965.

“It seems possible that the correspondence might form the basis of a formal description of the semantics of Algol 60.”

From the λ -calculus to a functional programming language

Reduction
strategies

Call-by-value

Call-by-name

Call-by-need

Efficient
execution
mechanisms

A naïve
interpreter

Natural
semantics

Environments
and closures

An efficient
interpreter

Digression

Start from the λ -calculus, and follow several steps:

- Fix a reduction strategy (today).
- Develop efficient execution mechanisms (today).
- Enrich the language with primitive data types and operations, recursion, algebraic data structures, and so on.
- Define a static type system (next week).

Landin, *The next 700 programming languages*, 1966.

"Most programming languages are partly a way of expressing things in terms of other things and partly a basic set of given things."

Operational semantics

An operational semantics describes **the actions of a machine**, in the simplest possible manner / at the most abstract level.

Plotkin, **A Structural Approach to Operational Semantics**, 1981, (2004).

Plotkin, **The Origins of Structural Operational Semantics**, 2004.

Plotkin: — *It is only through having an operational semantics that the [λ -calculus can] be viewed as a programming language.*

Denotational semantics

Scott: — *Why call it operational semantics? What is operational about it?*

Scott preferred **denotational** semantics, where the meaning of a program is a mathematical function of an input to an output.

Benton, Kennedy, Varming,

Some Domain Theory and Denotational Semantics in Coq, 2009.

Benton, Birkedal, Kennedy, Varming, **Formalizing domains, ultrametric spaces and semantics of programming languages**, 2010.

Dockins, **Formalized, Effective Domain Theory in Coq**, 2014.

The call-by-value strategy

Values form a subset of terms:

$$\begin{array}{lll} t, u ::= x \mid \lambda x. t \mid t \; t & \text{(terms)} \\ v ::= \lambda x. t & \text{(values)} \end{array}$$

A value represents the result of a computation.

The call-by-value reduction relation $t \rightarrow_{cbv} t'$ is inductively defined:

$$\frac{\beta_v}{(\lambda x. t) v \rightarrow_{cbv} t[v/x]} \qquad \frac{A_{\text{PP}} L}{t \rightarrow_{cbv} t'} \qquad \frac{A_{\text{PP}} R}{u \rightarrow_{cbv} u'}$$

$$\frac{}{t \; u \rightarrow_{cbv} t' \; u} \qquad \frac{}{v \; u \rightarrow_{cbv} v \; u'}$$

This is known as a small-step operational semantics.

Example

This is a proof (a.k.a. derivation) that one reduction step is permitted:

$$\frac{\frac{x[1/x] = 1}{(\lambda x.x) 1 \xrightarrow{cbv} 1} \beta_v}{\frac{(\lambda x.\lambda y.y x) ((\lambda x.x) 1) \xrightarrow{cbv} (\lambda x.\lambda y.y x) 1}{(\lambda x.\lambda y.y x) ((\lambda x.x) 1) (\lambda x.x) \xrightarrow{cbv} (\lambda x.\lambda y.y x) 1 (\lambda x.x)}} \text{APP}_R \quad \text{APP}_L$$

Features of call-by-value reduction

- Weak reduction. One cannot reduce under a λ -abstraction.

$$\cancel{\frac{t \rightarrow_{cbv} t'}{\lambda x.t \rightarrow_{cbv} \lambda x.t'}}$$

Thus, values do not reduce.

Also, we are interested in reducing closed terms only.

- Call-by-value. An actual argument is reduced to a value before it is passed to a function.

$$(\lambda x.t) v \rightarrow_{cbv} t[v/x]$$

$$(\lambda x.t)(u_1 u_2) \rightarrow_{cbv} t[u_1 u_2/x]$$

Features of call-by-value reduction

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Call-by-name

Call-by-need

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Digression

- **Left-to-right.** In an application $t u$, the term t must be reduced to a value before u can be reduced at all.

$$\text{APPVR} \quad \frac{u \longrightarrow_{cbv} u'}{\textcolor{blue}{V} u \longrightarrow_{cbv} \textcolor{blue}{V} u'}$$

- **Determinism.** For every term t , there is at most one term t' such that $t \longrightarrow_{cbv} t'$ holds.

Reduction sequences

Sequences of reduction steps describe the behavior of a term.

The following three situations are mutually exclusive:

- **Termination:** $t \xrightarrow{\text{cbv}} t_1 \xrightarrow{\text{cbv}} t_2 \xrightarrow{\text{cbv}} \dots \xrightarrow{\text{cbv}} v$
 The value v is the result of evaluating t .
 The term t converges to v .
- **Divergence:** $t \xrightarrow{\text{cbv}} t_1 \xrightarrow{\text{cbv}} t_2 \xrightarrow{\text{cbv}} \dots \xrightarrow{\text{cbv}} t_n \xrightarrow{\text{cbv}} \dots$
 The sequence of reductions is infinite.
 The term t diverges.
- **Error:** $t \xrightarrow{\text{cbv}} t_1 \xrightarrow{\text{cbv}} t_2 \xrightarrow{\text{cbv}} \dots \xrightarrow{\text{cbv}} t_n \not\xrightarrow{\text{cbv}} \cdot$
 where t_n is not a value, yet does not reduce: t_n is stuck.
 The term t goes wrong. This is a runtime error.

A strong type system rules out errors (Milner, 1978).

Some type systems rule out both errors and divergence.

Examples of reduction sequences

Termination:

$$\begin{aligned}
 (\lambda x. \lambda y. y \ x) ((\lambda x. x) \ 1) (\lambda x. x) &\xrightarrow{\textit{cbv}} (\lambda x. \lambda y. y \ x) \ 1 \ (\lambda x. x) \\
 &\xrightarrow{\textit{cbv}} (\lambda y. y \ 1) (\lambda x. x) \\
 &\xrightarrow{\textit{cbv}} (\lambda x. x) \ 1 \\
 &\xrightarrow{\textit{cbv}} 1
 \end{aligned}$$

Divergence:

$$(\lambda x. x \ x) (\lambda x. x \ x) \xrightarrow{\textit{cbv}} (\lambda x. x \ x) (\lambda x. x \ x) \xrightarrow{\textit{cbv}} \dots$$

Error:

$$(\lambda x. x \ x) \ 2 \xrightarrow{\textit{cbv}} 2 \ 2 \not\xrightarrow{\textit{cbv}} .$$

The active redex is highlighted in red.

An alternative style: evaluation contexts

APP L and APPVR can be combined as follows:

$$\frac{\beta_v}{(\lambda x.t) v \longrightarrow_{cbv}^{\text{head}} t[v/x]} \qquad \frac{\text{C}_\text{TX}}{t \longrightarrow_{cbv}^{\text{head}} t' \quad E[t] \longrightarrow_{cbv} E[t']}$$

Head reduction $\longrightarrow_{cbv}^{\text{head}}$ allows reduction at the root.

Reduction \longrightarrow_{cbv} allows reduction under an evaluation context E .

Evaluation contexts E are defined by $E ::= [] \mid E u \mid v E$.

$E[t]$ is the result of plugging the term t in the hole of the context E .

Wright and Felleisen, A syntactic approach to type soundness, 1992.

Unique decomposition

In this alternative style, the determinism of the reduction relation follows from a [unique decomposition](#) lemma:

Lemma (Unique Decomposition)

For every term t , there exists at most one pair (E, u) such that

- $t = E[u]$
- $\exists u' \quad u \xrightarrow[cbv]{\text{head}} u'$.

The call-by-name strategy

The **call-by-name** reduction relation $t \rightarrow_{cbn} t'$ is defined as follows:

$$\frac{\beta}{(\lambda x.t) u \rightarrow_{cbn} t[u/x]} \qquad \frac{APPL}{t \rightarrow_{cbn} t' \quad t u \rightarrow_{cbn} t' u}$$

The **unevaluated** actual argument is passed to the function.

It is later reduced if / when / every time the function **demands** its value.

An example reduction sequence

$$\begin{array}{lcl} (\lambda x. \lambda y. y\,x) ((\lambda x. x)\,1) (\lambda x. x) & \xrightarrow{\text{cbn}} & (\lambda y. y\,((\lambda x. x)\,1))\,(\lambda x. x) \\ & \xrightarrow{\text{cbn}} & (\lambda x. x)\,((\lambda x. x)\,1) \\ & \xrightarrow{\text{cbn}} & (\lambda x. x)\,1 \\ & \xrightarrow{\text{cbn}} & 1 \end{array}$$

Call-by-value versus call-by-name

If t terminates under CBV, then it also terminates under CBN (*).

The converse is false:

$$\begin{array}{lcl} (\lambda x.1) \omega & \longrightarrow_{cbn} & 1 \\ (\lambda x.1) \omega & \longrightarrow_{cbv}^{\infty} & \end{array}$$

where $\omega = (\lambda x.x\ x)\ (\lambda x.x\ x)$ diverges under both strategies.

Call-by-value can perform fewer reduction steps:

$(\lambda x. x + x)\ t$ evaluates t once under CBV, twice under CBN.

Call-by-name can perform fewer reduction steps:

$(\lambda x. 1)\ t$ evaluates t once under CBV, not at all under CBN.

(*) In fact, the standardization theorem implies that if t can be reduced to a value via any strategy, then it can be reduced to a value via CBN.
See Takahashi (1995).

Encoding call-by-name in a CBV language

Use **thunks**: functions $\lambda_.u$ whose purpose is to delay the evaluation of u .

$$\begin{aligned} \llbracket x \rrbracket &= x() \\ \llbracket \lambda x.t \rrbracket &= \lambda x. \llbracket t \rrbracket \\ \llbracket t u \rrbracket &= \llbracket t \rrbracket (\lambda_. \llbracket u \rrbracket) \end{aligned}$$

Exercise: Can you **state** that this encoding is correct? Can you **prove it**?
 — 2017 exam! (paper assignment and solution) (Coq solution)

Encoding call-by-name in a CBV language

In a simply-typed setting, this transformation is **type-preserving**: that is,

$$\Gamma \vdash t : T \text{ implies } \llbracket \Gamma \rrbracket \vdash \llbracket t \rrbracket : \llbracket T \rrbracket.$$

The translation of types is defined by

$$\llbracket T_1 \rightarrow T_2 \rrbracket = \textit{thunk} \llbracket T_1 \rrbracket \rightarrow \llbracket T_2 \rrbracket$$

where *thunk* T is $\text{unit} \rightarrow T$.

The translation of type environments is as follows:

$\llbracket x_1 : T_1; \dots; x_n : T_n \rrbracket$ stands for $x_1 : \text{thunk} \llbracket T_1 \rrbracket; \dots; x_n : \text{thunk} \llbracket T_n \rrbracket$.

Encoding call-by-value in a CBN language

The reverse encoding is somewhat more involved.

The call-by-value continuation-passing style (CPS) transformation, studied later on in this course, achieves such an encoding.

Call-by-push-value

Levy: — *The existence of two separate paradigms is troubling.*

Levy proposes **call-by-push-value**,
a lower-level calculus into which both CBV and CBN can be encoded,
thus avoiding a certain amount of duplication between their theories.

Levy, **Call-by-Push-Value: A Subsuming Paradigm**, 1999.

Forster et al., **Call-By-Push-Value in Coq:
Operational, Equational, and Denotational Theory**, 2018.

Call-by-need

Call-by-need, a.k.a. [lazy evaluation](#), eliminates the main inefficiency of call-by-name (namely, repeated computation) by introducing [memoization](#).

Its description via an operational semantics involves:

- either [mutable state and sharing](#) ([Ariola and Felleisen, 1997; Maraist, Odersky, Wadler, 1998](#));
- or [nondeterminism](#): “call-by-need is clairvoyant call-by-value” ([Hackett and Hutton, 2019](#)).

It is used in Haskell, where it encourages a [modular style](#) of programming.

Hughes, [Why functional programming matters](#), 1990.

Also see [Harper’s](#) and [Augustsson’s](#) blog posts on laziness.

Newton-Raphson iteration (after Hughes)

This is pseudo-Haskell code. The colon : is “cons”.

An approximation of the square root of n can be computed as follows:

```
next n x = (x + n / x) / 2
repeat f a = a : (repeat f (f a))
within eps (a : b : rest) =
  if abs (a - b) <= eps then b
  else within eps (b : rest)
sqrt a0 eps n =
  within eps (repeat (next n) a0)
```

`repeat (next n) a0` is a **producer** of an infinite stream of numbers.

Its type is just “list of numbers” – look Ma, **no iterators à la Java!**

The **consumer** `within eps` decides how many elements to demand.

The two are programmed **independently**.

Encoding call-by-need in a CBV language

Call-by-need can be encoded into CBV by using [memoizing thunks](#):

$$\begin{aligned} \llbracket x \rrbracket &= \text{force } x \\ \llbracket \lambda x.t \rrbracket &= \lambda x. \llbracket t \rrbracket \\ \llbracket t u \rrbracket &= \llbracket t \rrbracket (\text{suspend } (\lambda_. \llbracket u \rrbracket)) \end{aligned}$$

Such a thunk evaluates u when [first](#) forced,
then memoizes the result,
so no computation is required if the thunk is forced [again](#).

Thunks can be thought of as an abstract type with this API or signature:

```
type 'a thunk
val suspend: (unit -> 'a) -> 'a thunk
val force: 'a thunk -> 'a
```

Encoding call-by-need in a CBV language

Exercise: implement the thunk API in OCaml. ([Solution.](#))

In reality, this exercise is unnecessary, as OCaml has built-in thunks:

- “*suspend* ($\lambda_. u$)” is written `lazy u`.
- “*force x*” is written `Lazy.force x`.

Exercise: port Newton-Raphson iteration to OCaml.

Make sure that each element is computed at most once
and no more elements than necessary are computed.

Write tests to verify these properties. ([Solution.](#))

A naïve interpreter

An **interpreter** executes a program (represented by its AST).

Let us write one, in OCaml, by paraphrasing the small-step semantics.

Abstract syntax

This is the abstract syntax of the λ -calculus:

```
type var = int (* a de Bruijn index *)
type term =
| Var of var
| Lam of (* bind: *) term
| App of term * term
```

For example, the term $\lambda x.x$ is represented as follows:

```
let id =
  Lam (Var 0)
```

Renaming

`lift_ i k` represents the renaming $\uparrow^i (+k)$.

```
let rec lift_ i k (t : term) : term =
  match t with
  | Var x ->
    if x < i then t else Var (x + k)
  | Lam t ->
    Lam (lift_ (i + 1) k t)
  | App (t1, t2) ->
    App (lift_ i k t1, lift_ i k t2)

let lift k t =
  lift_ 0 k t
```

Thus, `lift k` represents $+k$. (This renaming adds k to every variable.)

It is used when one moves the term t down into k binders. (Next slide.)

Substitution

`subst_ i sigma` represents the substitution $\uparrow^i \sigma$.

```
let rec subst_ i (sigma : var -> term) (t : term) : term =
  match t with
  | Var x ->
    if x < i then t else lift i (sigma (x - i))
  | Lam t ->
    Lam (subst_ (i + 1) sigma t)
  | App (t1, t2) ->
    App (subst_ i sigma t1, subst_ i sigma t2)

let subst sigma t =
  subst_ 0 sigma t
```

Thus, `subst sigma` represents σ .

Substitution

A substitution is encoded as a total function of variables to terms.

```
let singleton (u : term) : var -> term =
    function 0 -> u | x -> Var (x - 1)
```

`singleton u` represents the substitution $u \cdot id$.

Recognizing values

It is easy to test whether a term is a value:

```
let is_value = function
| Var _ -> assert false (* we work with closed terms only *)
| Lam _ -> true
| App _ -> false
```

Performing one step of reduction

A direct transcription of Plotkin's definition of call-by-value reduction:

```
let rec step (t : term) : term option =
  match t with
  | Lam _ | Var _ -> fail
  | App (Lam t, v) when is_value v -> (* Plotkin's BetaV *)
    return (subst (singleton v) t)
  | App (t, u) when not (is_value t) -> (* Plotkin's AppL *)
    let* t' = step t in
    return (App (t', u))
  | App (v, u) when is_value v -> (* Plotkin's AppVR *)
    let* u' = step u in
    return (App (v, u'))
  | App (_, _) -> (* All cases covered already *)
    assert false (* but OCaml cannot see it. *)
```

We have guarded **AppL** so that **AppL** and **AppVR** are mutually exclusive.

Performing one step of reduction

fail, return, bind are the basic operations of the option monad.

```
let fail : 'a option =
  None
let return (x : 'a) : 'a option =
  Some x
let bind (ox : 'a option) (f : 'a -> 'b option) : 'b option =
  match ox with
  | None -> None
  | Some x -> f x
let (let*) = bind
```

The binding operator `let*` is sugar for `bind`.

(See upcoming lecture by PED.)

Performing many steps of reduction

Reduction
strategies

Call-by-value

Call-by-name

Call-by-need

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Digression

To evaluate a term, one performs as many reduction steps as possible:

```
let rec eval (t : term) : term =
  match step t with
  | None ->
    t
  | Some t' ->
    eval t'
```

The function call `eval t` either diverges or returns an irreducible term, which must be either a value or stuck.

Sources of inefficiency

Unfortunately, this is a terribly **inefficient** way of interpreting programs.

At each reduction step, one must:

- Find the next redex, that is, decompose the term t as $E[\lambda(x.u) v]$.
- Perform the substitution $u[v/x]$.
- Construct the term $E[u[v/x]]$.

The time required to do this is **not** $O(1)$. Why?

There seem to be two main sources of inefficiency:

- We keep **forgetting** the current evaluation context, only to **discover** it again at the next reduction step.
- We perform costly substitutions.

Towards an alternative to small steps

A reduction sequence from an application $t_1 t_2$ to a final value v always has the form:

$$t_1 t_2 \xrightarrow{^*_{cbv}} (\lambda x. u_1) t_2 \xrightarrow{^*_{cbv}} (\lambda x. u_1) v_2 \xrightarrow{_{cbv}} u_1[v_2/x] \xrightarrow{^*_{cbv}} v$$

where $t_1 \xrightarrow{^*_{cbv}} \lambda x. u_1$ and $t_2 \xrightarrow{^*_{cbv}} v_2$. That is,

Evaluate operator; evaluate operand; call; continue execution.

Idea: define a “big-step” relation $t \downarrow_{cbv} v$, which relates a term directly with the **final outcome** v of its evaluation, and whose definition reflects the above structure.

Natural semantics, a.k.a. big-step semantics

The relation $t \downarrow_{cbv} v$ means that evaluating t terminates and produces v .

Here is its definition, for call-by-value:

$$\frac{\text{BigCBVVALUE}}{v \downarrow_{cbv} v} \qquad \frac{\text{BigCBVAPP}}{t_1 \downarrow_{cbv} \lambda x. u_1 \quad t_2 \downarrow_{cbv} v_2 \quad u_1[v_2/x] \downarrow_{cbv} v}{t_1 \ t_2 \downarrow_{cbv} v}$$

Exercise: define \downarrow_{cbn} .

Example

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Digression

Let us write \downarrow for \downarrow_{cbv} , and “ $v \downarrow \cdot$ ” for “ $v \downarrow v$ ”.

$$\frac{\begin{array}{c} \lambda x.x \downarrow \cdot \\ 1 \downarrow \cdot \\ 1 \downarrow \cdot \\ \hline \lambda x.\lambda y.y \ x \downarrow \cdot \quad \frac{\lambda x.x \downarrow \cdot}{(\lambda x.x) \ 1 \downarrow 1} \quad \lambda y.y \ 1 \downarrow \cdot \end{array}}{(\lambda x.\lambda y.y \ x) ((\lambda x.x) \ 1) \downarrow \lambda y.y \ 1} \quad \frac{\lambda x.x \downarrow \cdot}{\lambda x.x \downarrow (\lambda x.x) \downarrow 1} \quad \frac{\lambda x.x \downarrow \cdot}{(\lambda x.x) \ 1 \downarrow 1}$$

Whereas a proof of $t \rightarrow_{cbv} t'$ has linear structure,
a proof of $t \downarrow_{cbv} v$ has tree structure.

Some history



Martin-Löf uses big-step semantics, in English:

To execute $c(a)$, first execute c . If you get $(\lambda x) b$ as result, then continue by executing $b(a/x)$.
Thus $c(a)$ has value d if c has value $(\lambda x) b$ and $b(a/x)$ has value d .

He proposes type theory (1975) as a very high-level programming language in which both **programs** and **specifications** can be written.

Per Martin-Löf,
Constructive Mathematics and Computer Programming, 1984.

Some history

Kahn promotes big-step operational semantics:

$\rho \vdash \text{number } n \Rightarrow n$	(1)
$\rho \vdash \text{true} \Rightarrow \text{true}$	(2)
$\rho \vdash \text{false} \Rightarrow \text{false}$	(3)
$\rho \vdash \lambda P.E \Rightarrow [\lambda P.E, \rho]$	(4)
$\frac{\text{val}, \text{def}}{\rho \vdash \text{ident} i \Rightarrow \alpha}$	(5)
$\frac{\rho \vdash E_1 \Rightarrow \text{true} \quad \rho \vdash E_2 \Rightarrow \alpha}{\rho \vdash \text{if } E_1 \text{ then } E_2 \text{ else } E_3 \Rightarrow \alpha}$	(6)
$\frac{\rho \vdash E_1 \Rightarrow \text{false} \quad \rho \vdash E_2 \Rightarrow \alpha}{\rho \vdash \text{if } E_1 \text{ then } E_3 \text{ else } E_2 \Rightarrow \alpha}$	(7)
$\frac{\rho \vdash E_1 \Rightarrow \alpha \quad \rho \vdash E_2 \Rightarrow \beta}{\rho \vdash (E_1, E_2) \Rightarrow (\alpha, \beta)}$	(8)
$\frac{\rho \vdash E_1 \Rightarrow [\lambda P.E, \rho] \quad \rho \vdash E_2 \Rightarrow \alpha \quad \rho_1 : P \mapsto \alpha \vdash E \Rightarrow \beta}{\rho \vdash E_2 \Rightarrow \beta}$	(9)
$\frac{\rho \vdash E_2 \Rightarrow \alpha \quad \rho : P \mapsto \alpha \vdash E_1 \Rightarrow \beta}{\rho \vdash \text{let } P = E_2 \text{ in } E_1 \Rightarrow \beta}$	(10)
$\frac{\rho : P \mapsto \alpha \vdash E_1 \Rightarrow \alpha \quad \rho : P \mapsto \alpha \vdash E_1 \Rightarrow \beta}{\rho \vdash \text{letrec } P = E_2 \text{ in } E_1 \Rightarrow \beta}$	(11)



Figure 2. The dynamic semantics of mini-ML

He gives a big-step operational semantics of MiniML, a static type system, and a compilation scheme towards the CAM.

Gilles Kahn, **Natural semantics**, 1987.

A big-step interpreter

The call `eval t` attempts to compute a value v such that $t \downarrow_{cbv} v$ holds.

```
exception RuntimeError
let rec eval (t : term) : term =
  match t with
  | Lam _ | Var _ -> t
  | App (t1, t2) ->
    let v1 = eval t1 in
    let v2 = eval t2 in
    match v1 with
    | Lam u1 -> eval (subst (singleton v2) u1)
    | _ -> raise RuntimeError
```

If `eval` terminates normally, then it **obviously** returns a value;
but it can also fail to terminate or terminate with a runtime error. (Why?)

This interpreter does not **forget** and **rediscover** the evaluation context.
The context is now **implicit** in the interpreter's **stack**!

We **could** prove this interpreter correct, but will first optimize it further.

Equivalence between small-step and big-step semantics

Lemma (From big-step to small-step)

If $t \downarrow_{\text{cbv}} v$, then $t \xrightarrow{\star}_{\text{cbv}} v$.

Proof.

By induction on the derivation of $t \downarrow_{\text{cbv}} v$.

Case **BigCvValue**. We have $t = v$. The result is immediate.

Case **BigCvApp**. t is $t_1 t_2$, and we have three subderivations:

$$t_1 \downarrow_{\text{cbv}} \lambda x. u_1 \quad t_2 \downarrow_{\text{cbv}} v_2 \quad u_1[v_2/x] \downarrow_{\text{cbv}} v$$

Applying the ind. hyp. to them yields three reduction sequences:

$$t_1 \xrightarrow{\star}_{\text{cbv}} \lambda x. u_1 \quad t_2 \xrightarrow{\star}_{\text{cbv}} v_2 \quad u_1[v_2/x] \xrightarrow{\star}_{\text{cbv}} v$$

By reducing under an evaluation context and by chaining, we obtain:

$$t_1 t_2 \xrightarrow{\star}_{\text{cbv}} (\lambda x. u_1) t_2 \xrightarrow{\star}_{\text{cbv}} (\lambda x. u_1) v_2 \xrightarrow{\text{cbv}} u_1[v_2/x] \xrightarrow{\star}_{\text{cbv}} v$$

See [LambdaCalculusBigStep/bigcbv_star_cbv](#).



Equivalence between small-step and big-step semantics

Reduction strategies

Call-by-value

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Natural semantics

Environments and closures

An efficient interpreter

Digression

Lemma (From small-step to big-step, preliminary)

If $t_1 \rightarrow_{\text{cbv}} t_2$ and $t_2 \downarrow_{\text{cbv}} v$, then $t_1 \downarrow_{\text{cbv}} v$.

Proof (Sketch).

By induction on the first hypothesis and case analysis on the second hypothesis. See [LambdaCalculusBigStep/cbv_bigcbv_bigcbv](#). □

Lemma (From small-step to big-step)

If $t \xrightarrow{\star}_{\text{cbv}} v$, then $t \downarrow_{\text{cbv}} v$.

Proof.

By induction on the first hypothesis, using $v \downarrow_{\text{cbv}} v$ in the base case and the above lemma in the inductive case.

See [LambdaCalculusBigStep/star_cbv_bigcbv](#). □

Limitations of big-step semantics

The judgement $t \downarrow_{cbv} v$ describes a terminating computation.

This judgement does not allow saying that “ t diverges” or “ t crashes”.

One can define these two extra judgements in big-step style,
but this requires many rules and seems intuitively redundant.

Charguéraud, [Pretty-Big-Step Semantics](#), 2012.

Dagnino, [A meta-theory for big-step semantics](#), 2022.

An alternative to naïve substitution

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Digression

A basic need is to **record** that x is bound to v while evaluating a term t .

So far, we have used an eager substitution, $t[v/x]$, but:

- This is inefficient.
- This does not respect the separation between immutable **code** and mutable **data** imposed by current hardware and operating systems.

Idea: instead of applying the substitution $[v/x]$ to the code, record the binding $x \mapsto v$ in a data structure, known as an **environment**.

An environment is a **finite map** of variables to (closed) values.

A first attempt

Let us **try** and define a new big-step evaluation judgement, $e \vdash t \downarrow_{cbv} v$.

(previous definition)

 BIGCBVVALUE

$$\frac{}{v \downarrow_{cbv} v}$$

 BIGCBVAPP

$$\frac{\begin{array}{c} t_1 \downarrow_{cbv} \lambda x. u_1 \\ t_2 \downarrow_{cbv} v_2 \\ u_1[v_2/x] \downarrow_{cbv} v \end{array}}{t_1 t_2 \downarrow_{cbv} v}$$

(attempt at a new definition)

 EBIGCBVVAR

$$e(x) = v$$

$$\frac{}{e \vdash x \downarrow_{cbv} v}$$

 EBIGCBVLAM

$$\frac{}{e \vdash \lambda x. t \downarrow_{cbv} \lambda x. t}$$

 EBIGCBVAPP

$$e \vdash t_1 \downarrow_{cbv} \lambda x. u_1$$

$$e \vdash t_2 \downarrow_{cbv} v_2$$

$$\frac{e[x \mapsto v_2] \vdash u_1 \downarrow_{cbv} v}{e \vdash t_1 t_2 \downarrow_{cbv} v}$$

What is **wrong** with this definition?

In $t \downarrow_{cbv} v$, both t and v are closed.

In $e \vdash t \downarrow_{cbv} v$, we expect $fv(t) \subseteq \text{dom}(e)$. What about v ? Is it closed?

... the values stored in e ? Are they closed? ...

Lexical scoping versus dynamic scoping

What value should the following OCaml code produce?

```
let x = 42 in
let f = fun () -> x in
let x = "oops" in
f()
```

Well,

- The answer is 42. This is lexical scoping. This is λ -calculus.
- The answer is not "oops". That would be dynamic scoping.

Thus, the free variables of a λ -abstraction must be evaluated:

- in the environment that exists at the function's creation site,
- not in the environment that exists at the function's call site.

A failed attempt

Thus, our first attempt is wrong:

- It implements **dynamic scoping** instead of lexical scoping.
- If $e \vdash t \downarrow_{cbv} v$ and $fv(t) \subseteq dom(e)$ then we would expect that v is closed and $t[e] \downarrow_{cbv} v$ holds — but that is **not** the case.
- The candidate rule **EBIGCBV_{LAM}** obviously **violates** this property.
It fails to **record the environment** that exists at function creation time.

How can we **fix** the problem?

Closures



The result of evaluating a λ -abstraction $\lambda x.t$, where $fv(\lambda x.t)$ may be nonempty, should **not** be $\lambda x.t$.

It should be a **closure** $\langle \lambda x.t \mid e \rangle$,

- that is, a **pair** of a λ -abstraction and an environment,
- in other words, a pair of a **code** pointer and a pointer to a heap-allocated **data** structure.

Landin, *The Mechanical Evaluation of Expressions*, 1964.

Closures and environments

The abstract syntax of closures is:

$$c ::= \langle \lambda x.t \mid e \rangle$$

We expect the evaluation of a term to produce a closure:

$$e \vdash t \downarrow_{cbv} c$$

Because evaluating x produces $e(x)$,
an environment must be a finite map of variables to closures:

$$e ::= [] \mid e[x \mapsto c]$$

Thus, the syntaxes of closures and environments are mutually inductive.

A big-step semantics with environments

Evaluating a λ -abstraction produces a newly allocated closure.

$$\frac{\text{EBigC�VVAR}}{e(x) = c}$$

$$\frac{}{e \vdash x \downarrow_{\text{cbv}} c}$$

$$\frac{\text{EBigC�VLAM}}{fv(\lambda x.t) \subseteq \text{dom}(e)}$$

$$\frac{}{e \vdash \lambda x.t \downarrow_{\text{cbv}} \langle \lambda x.t \mid e \rangle}$$

$$\frac{\text{EBigC�VAPP}}{e \vdash t_1 \downarrow_{\text{cbv}} \langle \lambda x.u_1 \mid e' \rangle}$$

$$\frac{e \vdash t_2 \downarrow_{\text{cbv}} c_2}{e'[x \mapsto c_2] \vdash u_1 \downarrow_{\text{cbv}} c}$$

$$\frac{}{e \vdash t_1 t_2 \downarrow_{\text{cbv}} c}$$

Invoking a closure causes the closure's code to be evaluated in the closure's environment, extended with a binding of formal to actual.

Equivalence between big-step semantics without and with environments

How can we relate the judgements $t \downarrow_{cbv} v$ and $e \vdash t \downarrow_{cbv} c$?

What lemma should we state?

Assuming t is closed, we would like to prove that

$$t \downarrow_{cbv} v$$

holds if and only if

$$\boxed{} \vdash t \downarrow_{cbv} c$$

holds for some closure c such that c represents v in a certain sense.

Decoding closures

c represents v can be defined as $\llbracket c \rrbracket = v$, where $\llbracket c \rrbracket$ is defined by:

$$\llbracket (\lambda x.t \mid e) \rrbracket = (\lambda x.t)[\llbracket e \rrbracket]$$

and where the substitution $\llbracket e \rrbracket$ maps every variable x in $\text{dom}(e)$ to $\llbracket e(x) \rrbracket$.
($\llbracket c \rrbracket$ and $\llbracket e \rrbracket$ are mutually inductively defined.)

Equivalence between big-step semantics without and with environments

One implication is easily established:

Lemma (Soundness of the environment semantics)

$e \vdash t \downarrow_{cbv} c$ implies $t[[e]] \downarrow_{cbv} [c]$.

Proof (Sketch).

By induction on the hypothesis.

See [LambdaCalculusBigStep/ebigcbv_bigcbv](#). □

In particular, $[] \vdash t \downarrow_{cbv} c$ implies $t \downarrow_{cbv} [c]$.

Equivalence between big-step semantics without and with environments

The reverse implication requires a more complex statement:

Lemma (Completeness of the environment semantics)

If $t[\Gamma e] \downarrow_{cbv} v$, where $\text{fv}(t) \subseteq \text{dom}(e)$ and e is well-formed, then there exists c such that $e \vdash t \downarrow_{cbv} c$ and $[c] = v$.

Proof (Sketch).

By induction on the first hypothesis and by case analysis on t .

See [LambdaCalculusBigStep/bigcbv_ebigcbv](#).

□

In particular, if t is closed, then $t \downarrow_{cbv} v$ implies $[] \vdash t \downarrow_{cbv} c$, for some closure c such that $[c] = v$.

Equivalence between big-step semantics without and with environments

The notion of **well-formedness** on the previous slide is inductively defined:

$$\frac{\begin{array}{c} \text{fv}(\lambda x.t) \subseteq \text{dom}(e) \\ e \text{ is well-formed} \end{array}}{\langle \lambda x.t \mid e \rangle \text{ is well-formed}} \qquad \frac{\forall x, x \in \text{dom}(e) \Rightarrow e(x) \text{ is well-formed}}{e \text{ is well-formed}}$$

Lemma (Well-formedness is an invariant)

If $e \vdash t \downarrow_{\text{cbv}} c$ holds and e is well-formed, then c is well-formed.

Proof.

See [LambdaCalculusBigStep/ebigcbv_wf_cvalue](#).



This property is exploited in the proof of the previous lemma.

From big-step semantics to interpreter, again

The big-step semantics $e \vdash t \downarrow_{cbv} c$ is a 3-place relation.

We now wish to define a (partial) function of two arguments e and t .

We **could** do this in OCaml, as we did earlier today.

Let us do it in Coq and prove this interpreter correct and complete!

See [LambdaCalculusInterpreter](#).

Syntax

The syntax of terms (in de Bruijn's representation) is as before.

The syntax of closures and environments is as shown earlier:

```
Inductive cvalue :=  
| Clo: {bind term} -> list cvalue -> cvalue.
```

```
Definition cenv :=  
list cvalue.
```

A first attempt

```
Fail Fixpoint interpret (e : cenv) (t : term) : cvalue :=
  match t with
  | Var x =>
    nth x e dummy_cvalue
    (* dummy is used when x is out of range *)
  | Lam t =>
    Clo t e
  | App t1 t2 =>
    let cv1 := interpret e t1 in
    let cv2 := interpret e t2 in
    match cv1 with Clo u1 e' =>
      interpret (cv2 :: e') u1
    end
  end.
```

Why is this definition **rejected** by Coq?

A standard trick: fuel

We parameterize the interpreter with a maximum recursive call depth n .

```
Fixpoint interpret (n : nat) e t : option cvalue :=
  match n with
  | 0    => None (* not enough fuel *)
  | S n =>
    match t with
    | Var x      => Some (nth x e dummy_cvalue)
    | Lam t      => Some (Clo t e)
    | App t1 t2 =>
        interpret n e t1 >>= fun cv1 =>
        interpret n e t2 >>= fun cv2 =>
        match cv1 with Clo u1 e' =>
          interpret n (cv2 :: e') u1
        end
    end
  end end.
```

The interpreter can now fail, therefore has return type `option cvalue`.

Equivalence between the big-step semantics and the interpreter

If the interpreter produces a result, then it is a correct result.

Lemma (Soundness of the interpreter)

If $\text{interpret } n \ e \ t = \text{Some } c$ and $\text{fv}(t) \subseteq \text{dom}(e)$ and e is well-formed then $e \vdash t \downarrow_{\text{cbv}} c$ holds.

Proof (Sketch).

By induction on n , by case analysis on t , and by inspection of the first hypothesis. See [LambdaCalculusInterpreter/interpret_ebigcbv](#). □

An interpreter that always returns *None* would satisfy this lemma, hence the need for a completeness statement...

Equivalence between the big-step semantics and the interpreter

If the evaluation of t is supposed to produce c , then, given sufficient fuel, the interpreter returns c .

Lemma (Completeness of the interpreter)

If $e \vdash t \downarrow_{cbv} c$, then there exists n such that $\text{interpret } n \ e \ t = \text{Some } c$.

Proof (Sketch).

By induction on the hypothesis, exploiting the fact that *interpret* is monotonic in n , that is, $n_1 \leq n_2$ implies $\text{interpret } n_1 \ e \ t \leq \text{interpret } n_2 \ e \ t$, where the “definedness” partial order \leq is generated by $\text{None} \leq \text{Some } c$. See [LambdaCalculusInterpreter/ebigcbv_interpret](#). □

Summary

If t is closed and v is a value, then the following are equivalent:

$$t \xrightarrow{cbv}^* v$$

small-step substitution semantics

$$t \downarrow_{cbv} v$$

big-step substitution semantics

$$\exists c \left\{ \begin{array}{l} [] \vdash t \downarrow_{cbv} c \\ \llbracket c \rrbracket = v \end{array} \right.$$

big-step environment semantics

$$\exists c \exists n \left\{ \begin{array}{l} \text{interpret } n [] t = \text{Some } c \\ \llbracket c \rrbracket = v \end{array} \right.$$

interpreter

A few things to remember

An efficient interpreter uses environments and closures, not substitutions.

- It can (easily) be proved correct and complete!

There are several styles of operational semantics.

- They can (easily) be proved equivalent!

Cost model

We have represented environments as [lists](#). Extension costs $O(1)$, but lookup has complexity $O(n)$, where n is the number of variables in scope.

A better approach is to represent the environment as an n -tuple. Then,

- evaluating a variable costs $O(1)$;
- evaluating a λ -abstraction costs $O(n)$;
- evaluating a function call costs $O(1)$.

n can be considered $O(1)$ as it depends only on the program's text, not on the input data.

This [simple cost model](#) is implemented by the OCaml compiler.

The cost of garbage collection

The previous slide does not discuss the cost of garbage collection.

Let H be the total heap size.

Let R be the total size of the *live* objects. Thus, $R \leq H$.

Assuming a copying collector, one collection costs $O(R)$.

Collection takes place when the heap is full, so frees up $H - R$ words.

Thus, the *amortized* cost of collection, per freed-up word, is

$$\frac{O(R)}{H - R}$$

Under the hypothesis $\frac{R}{H} \leq \frac{1}{2}$, this cost is $O(1)$. That is,

*Provided the heap is not allowed to become more than half full,
freeing up an object takes constant (amortized) time.*

Full closures versus minimal closures

In reality, this interpreter has one subtle but serious inefficiency.

When a closure $\langle \lambda x.t \mid e \rangle$ is allocated,
the entire environment e is stored in it,
even though $fv(\lambda x.t)$ may be a **strict subset** of the domain of e .

We store data that the closure will never need. This is a **space leak!**

To fix this, one should store **a trimmed-down environment** in the closure.

Exercise: state and prove that, if x does not occur free in t , then the evaluation of t in an environment e does not depend on the value $e(x)$.

Exercise: define an optimized interpreter where, at a closure allocation, every unneeded value in e is replaced with a dummy value. Prove it equivalent to the simpler interpreter.