Rich types, Tractable typing – Type Inference –

Yann Régis-Gianas
yrg@irif.fr

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Plan

Yann Régis-Gianas
yrg@irif.fr

What is type inference?

Write down your current answers to the following questions:

1. What is the type of a type inference engine?
2. What are the properties you expect from it?

Yann Régis-Gianas
yrg@irif.fr

What is type inference?

Informal definition

Type inference is a process that computes a type τ for a term t under some typing environment Γ if such a type exists. In other words, we are reading the judgment:

$$\Gamma \vdash t : \tau$$

where t and Γ are the inputs and τ is the output.

Hence, type inference determines if a term is typable.

Yann Régis-Gianas
yrg@irif.fr

Too vague!

Consider:

```
1  let f = fun x -> x + 1
```

Intuitively, a type inference engine must be able to process the term `x + 1` under an environment where the type of `x` is unknown.

Therefore, the typing environments used by a type inference engine differ slightly from the typing environments used by a type checker in the sense that the types bound to identifiers may contain pieces of unknown typing information. As soon as the syntax for types offers a notion of type variables, unknown typing information can be represented by free type variables.

This means that a free type variable occurring in a judgment may have two distinct roles: either it denotes a parameter of the typing derivation [1], or it denotes an unknown type to be instantiated by the inference algorithm. This distinction must be formally made.

[1] It is morally universally quantified at the meta-level.

Let us be more formal!

Let us write $\mathcal{V}$ to denote an infinite set of type variable identifiers from which we can generate fresh names. These type variables will denote the pieces of unknown typing information of the inference problem. The goal of type inference is to determine if these type variables can be assigned types to ensure the typability of the input term under the input environment.

Definition

A **type inference engine** is a partial function $\mathcal{I}$. This function expects $\mathcal{V}$, as well as a typing environment Γ and a term t. When defined, this function returns ϕ, an idempotent substitution whose domain is a finite subset of $\mathcal{V}$ and an inferred type τ.

Definition

A type inference engine $\mathcal{I}$ is **sound** if whenever $\mathcal{I}(\mathcal{V}, \Gamma, t) = (\mathcal{V}', \phi, \tau)$ then $\phi(\Gamma) \vdash t : \tau$.

The substitution ϕ witnesses the typability of t under Γ.

Parenthesis: α or $?\alpha$?

We could have followed another design choice by introducing a notion of **meta-variables**[2] to represent unknown types.

Yet, we will see that the strength of Hindley-Milner type system is to play with type variables to promote them from unknown types to parameters through the mechanism of **generalization**.

[2] As this is done in the Coq system for instance.

Yann Régis-Gianas
yrg@irif.fr

Parenthesis: Alternative definition

We could have devised an alternative definition in which only the term t is the input and Γ also has to be inferred in addition to the type. Such a definition has interesting benefits but is less standard and less studied.

Yann Régis-Gianas
yrg@irif.fr

Soundness is not enough

As soon as a term is typable, the type inference algorithm must be able to find a typing for this term.

Definition

A type inference engine is **complete** and **principal** if whenever there exists a substitution ϕ such that $\phi(\Gamma) \vdash t : \phi(\tau)$ holds, then there exists ϕ', ϕ'' and $\mathcal{V}$ disjoint from $\mathrm{FTV}(\Gamma, \tau)$ such that:

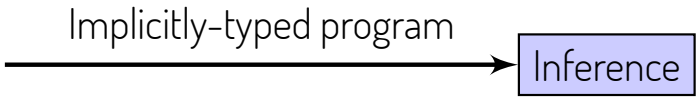
$$\begin{array}{rcll} \mathcal{I}(\mathcal{V}, \Gamma, t) & = & (\mathcal{V}', \phi'', \tau') & \\ \phi'(\tau') & = & \tau & \\ \phi(\alpha) & = & (\phi'' \circ \phi')(\alpha) & \forall \alpha \notin \mathcal{V} \end{array}$$

Yann Régis-Gianas
yrg@irif.fr

Type inference in a compiler

Type inference algorithms can be tricky to implement, in particular when we want them to be efficient. Fortunately, we have a safety net: the typechecker for the corresponding explicitly-typed language!

In practice, the so-called De Bruijn architecture is encouraged:



Yann Régis-Gianas
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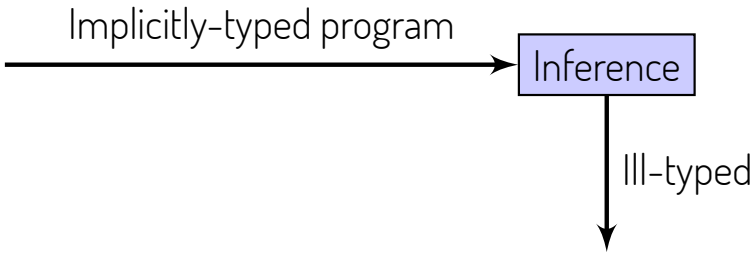

Yann Régis-Gianas
yrg@irif.fr

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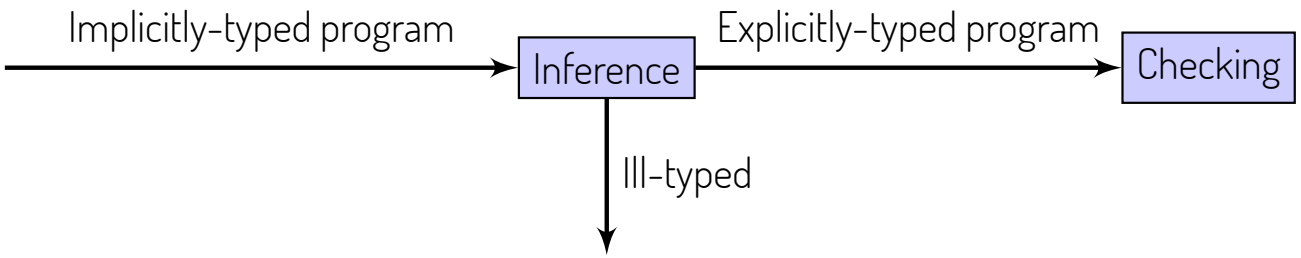

Yann Régis-Gianas
yrg@irif.fr

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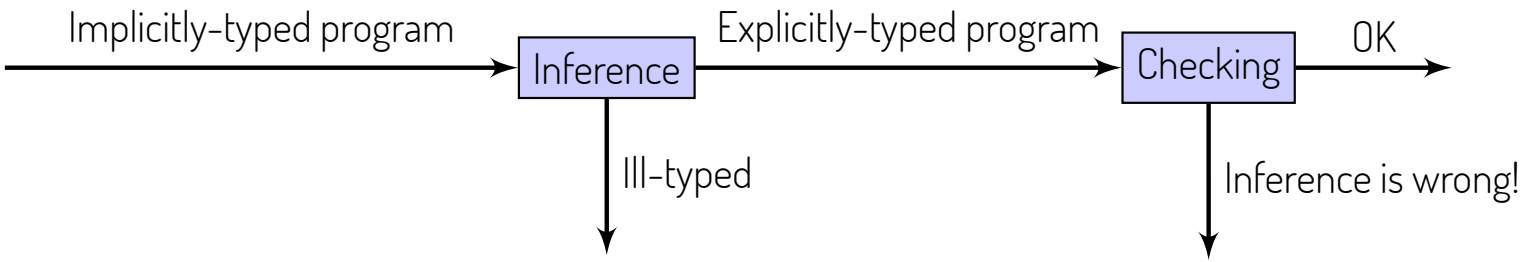

Yann Régis-Gianas
yrg@irif.fr

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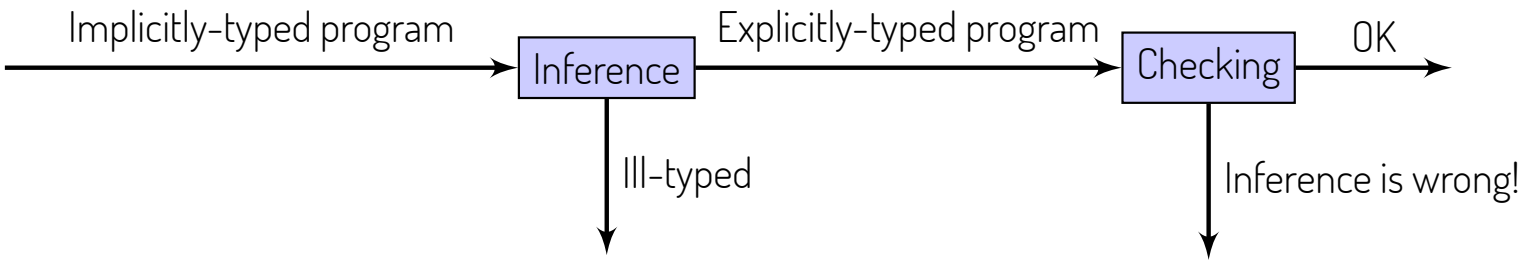

Elaboration

This extended type inference must **elaborate** an explicitly-typed program as a proof witness for the soundness of its answer. This proof is checked by an hopefully simpler, smaller and trustworthy program, the typechecker. Notice that the principality of the inferred program is not checked here.

Yann Régis-Gianas
yrg@irif.fr

From type inference to elaboration

Definition

A programming language is **implicitly typed** if its syntax allows the introduction of an identifier without declaring its type. The language is **explicitly typed** otherwise.

Let $\mathcal{L}_x$ be an explicitly typed language, and $\mathcal{L}_i$ be an implicitly typed language sharing the same type-erasure semantics and the same type algebra.

Definition

A **sound elaboration engine** is a partial function $\mathcal{E}$ such that:

$$\text{If } \mathcal{E}(\mathcal{V}, \Gamma, t) = (\mathcal{V}', \phi, \tau, t^\star) \text{ where } t \in \mathcal{L}_i$$
$$\text{then } t^\star \in \mathcal{L}_x \text{ and } \phi(\Gamma) \vdash t^\star : \phi(\tau)$$

Yann Régis-Gianas
yrg@irif.fr

This course

- ML, syntax, semantics and type system(s)
- Algorithm $\mathcal{W}$ in OCaml
- Constraint-based approach

Yann Régis-Gianas
yrg@irif.fr

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Yann Régis-Gianas
yrg@irif.fr

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Yann Régis-Gianas
yrg@irif.fr

MiniML with constants

Syntax for terms (`Syntax.ITerm.t`[3])

$$
\begin{array}{rcl}
t & ::= & x \\
& | & \lambda x.t \\
& | & t\ t \\
& | & \textbf{let}\ x = t\ \textbf{in}\ t
\end{array}
$$

Notice the absence of type annotations on bindings.

Operational semantics

We assume a call-by-value weak reduction semantics.

[3] See the companion code in the repository.

A stratified type algebra

Syntax for types (`Syntax.Type.t`)

$$\begin{array}{rcl} \tau & ::= & \alpha \\ & | & \varepsilon(\overline{\tau}) \\ \varepsilon & ::= & \rightarrow_2 | \ \mathbf{int}_0 \ | \ \ldots \end{array}$$

A type is a first-order term.

Syntax for type schemes (`Syntax.TypeScheme.t`)

$$\sigma \quad ::= \quad \forall \overline{\alpha}.\tau$$

- The $\forall$ binder quantifies over (mono)types.
- Quantification is **prenex** : it cannot appear everywhere as in F.
- This is **predicative** rank-1 parametric polymorphism.

Yann Régis-Gianas
yrg@irif.fr

Instance relation

Definition

The type τ' is an instance of the type scheme $\forall \overline{\alpha}.\tau$, written $\forall \overline{\alpha}.\tau \preceq \tau'$, if there exists $\overline{\tau}$ such that $[\overline{\alpha} \mapsto \overline{\tau}]\tau = \tau'$.

Yann Régis-Gianas
yrg@irif.fr

Type system

The type system of ML is defined by two typing judgments:

$$\Gamma \vdash t : \tau \quad \text{and} \quad \Gamma \vdash t : \forall \overline{\alpha}.\tau$$

where $\Gamma ::= \bullet \mid \Gamma(x : \sigma)$.

While the first judgment is a standard typing judgment, the second can be seen as a family of standard typing judgments, parameterized by the types $\overline{\alpha}$.

Going from the second judgment to the first is an instanciation. The other way around is a generalization:

$$\text{Inst}\ \frac{\Gamma \vdash t : \sigma \qquad \sigma \preceq \tau}{\Gamma \vdash t : \tau} \qquad \text{Gen}\ \frac{\Gamma \vdash t : \tau \qquad \overline{\alpha} \# \mathrm{FTV}(\Gamma)}{\Gamma \vdash t : \forall \overline{\alpha}.\tau}$$

Yann Régis-Gianas
yrg@irif.fr

Exercise

Exercise

Do you remember why the hypothesis

$$\overline{\alpha} \# \mathrm{FTV}(\Gamma)$$

is important in the Rule (Gen)?

Yann Régis-Gianas
yrg@irif.fr

Do you know ML?

Is

```
1  let f x = x in (f 0, f 'a')
```

equivalent to

```
1  (fun f -> (f 0, f 'a')) (fun x -> x)
```

?

Yann Régis-Gianas
yrg@irif.fr

Typing rules

In addition to (Gen), the rule (Var) can be used to introduce parameterized typing judgments:

$$\text{Var}\quad \frac{}{\Gamma \vdash x : \Gamma(x)}$$

Binding type schemes to variables is the role of (Let):

$$\text{Let}\quad \frac{\Gamma \vdash t : \sigma \qquad \Gamma, (x : \sigma) \vdash u : \tau}{\Gamma \vdash \textbf{let}\ x = t\ \textbf{in}\ u : \tau}$$

The rules for applications and abstractions are the same as for STLC:

$$\text{App}\quad \frac{\Gamma \vdash t : \tau_1 \to \tau_2 \qquad \Gamma \vdash u : \tau_1}{\Gamma \vdash t\, u : \tau_2} \qquad \text{Abs}\quad \frac{\Gamma, (x : \forall \emptyset . \tau_1) \vdash t : \tau_2}{\Gamma \vdash \lambda x . t : \tau_1 \to \tau_2}$$

Yann Régis-Gianas
yrg@irif.fr

Properties of this type system

(We defer the discussion about the soundness of this type system.)

Question 1

Given an environment Γ and a typable term t, is there a unique type τ such that $\Gamma \vdash t : \tau$?

Question 2

Given a derivable judgment $\Gamma \vdash t : \tau$, is there a unique typing derivation that has this conclusion?

Yann Régis-Gianas
yrg@irif.fr

Properties of this type system

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Question 1

Given an environment Γ and a typable term t, is there a unique type τ such that $\Gamma \vdash t : \tau$? **No!**

Question 2

Given a derivable judgment $\Gamma \vdash t : \tau$, is there a unique typing derivation that has this conclusion? **No! The rules are not syntax-directed.**

Properties of this type system

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Given an environment Γ and a typable term t, is there a unique type τ such that $\Gamma \vdash t : \tau$? **No!**

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Given a derivable judgment $\Gamma \vdash t : \tau$, is there a unique typing derivation that has this conclusion? **No! The rules are not syntax-directed.**

Question 1 will be tackled by the existence of principal type schemes. Let us deal with Question 2 for now.

Syntax directed typing rules

We define another type system enjoying uniqueness of typing derivation, similar to the previous one except that the rules (Gen), (Inst), (Var) et (Let) are replaced by [4]:

Let-Gen

$$\frac{\Gamma \vdash t : \tau_1 \quad \overline{\alpha} = \mathrm{FTV}(\tau_1) \setminus \mathrm{FTV}(\Gamma) \quad \Gamma, (x : \forall \overline{\alpha}.\tau_1) \vdash u : \tau_2}{\Gamma \vdash \textbf{let}\ x = t\ \textbf{in}\ u : \tau_2}$$

Var-Inst

$$\frac{\Gamma(x) \preceq \tau}{\Gamma \vdash x : \tau}$$

Since the system is now syntax-directed, does that mean that we have a type inference algorithm? or at least a type checking algorithm?

[4] We defer the proof of equivalence of the two type systems.

Yann Régis-Gianas
yrg@irif.fr

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Var-Inst

$$\frac{\Gamma(x) \preceq \tau}{\Gamma \vdash x : \tau}$$

Since the system is now syntax-directed, does that mean that we have a type inference algorithm? or at least a type checking algorithm? Unfortunately, no. There remain too many choices:

- What are the types of λ-bound identifiers?
- How much generalization is needed?

Typability in ML is a non local property.

[4] We defer the proof of equivalence of the two type systems.

ML typing constraints

The non locality of the typability problem is better handled by unification constraints. Historically, these constraints are implicitly generated and solved on-the-fly by an algorithm called $\mathcal{W}$.

More recent approaches split type inference into two phases: a constraint generation and the solving of these constraints.

We will now implement and prove $\mathcal{W}$, turn it into an elaboration algorithm and present a constraint-based approach to ML type inference.

Yann Régis-Gianas
yrg@irif.fr

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Yann Régis-Gianas
yrg@irif.fr

Good old $\mathcal{W}$ of 82

$$
\begin{array}{rcl}
\mathcal{W}(\mathcal{V}, \Gamma, x) & = & (\mathcal{V} \setminus \overline{\beta}, id, [\overline{\alpha} \mapsto \overline{\beta}]\tau) \\
\text{where} & & \Gamma(x) = \forall \overline{\alpha}.\tau \text{ and } \overline{\beta} \in \mathcal{V} \\
\\
\mathcal{W}(\mathcal{V}, \Gamma, \lambda x.t) & = & (\mathcal{V}', \phi, \phi(\alpha) \to \tau) \\
\text{where} & & \alpha \in \mathcal{V} \\
\text{and} & & \mathcal{W}(\mathcal{V} \setminus \alpha, \Gamma; (x : \alpha), t) = (\mathcal{V}', \phi, \tau) \\
\\
\mathcal{W}(\mathcal{V}, \Gamma, t\ u) & = & (\mathcal{V}'', \phi \circ \phi_t \circ \phi_u, \phi(\alpha)) \\
\text{where} & & \alpha \in \mathcal{V} \\
\text{and} & & \mathcal{W}(\mathcal{V} \setminus \alpha, \Gamma, u) = (\mathcal{V}', \phi_u, \tau_u) \\
\text{and} & & \mathcal{W}(\mathcal{V}', \phi_u(\Gamma), t) = (\mathcal{V}'', \phi_t, \tau_t) \\
\text{and} & & \phi = \mathsf{MGU}(\tau_t \stackrel{?}{=} \tau_u \to \alpha) \\
\\
\mathcal{W}(\mathcal{V}, \Gamma, \textbf{let } x = t \textbf{ in } u) & = & (\mathcal{V}'', \phi_2 \circ \phi_1, \tau_2) \\
\text{where} & & \mathcal{W}(\mathcal{V}, \Gamma, t) = (\mathcal{V}', \phi_1, \tau_1) \\
\text{and} & & \mathcal{W}(\mathcal{V}', \Gamma, (x : \sigma), u) = (\mathcal{V}'', \phi_2, \tau_2) \\
\text{and} & & \sigma = \mathsf{GEN}(\Gamma, \phi_1(\tau_1))
\end{array}
$$

Yann Régis-Gianas
yrg@irif.fr

Auxiliary functions

Let us write U for a first-order unification problem made of a conjunction of qualities between types.

$$\begin{array}{rcl}
\mathsf{MGU}(x \stackrel{?}{=} x \wedge U) & = & \mathsf{MGU}(U) \\
\mathsf{MGU}(\tau \stackrel{?}{=} x \wedge U) & = & \mathsf{MGU}(x \stackrel{?}{=} \tau \wedge U) \\
& \text{when} & \tau \text{ is not a variable} \\
\mathsf{MGU}(x \stackrel{?}{=} \tau \wedge U) & = & \mathsf{MGU}(U[x \mapsto \tau]) \circ [x \mapsto \tau] \\
& \text{when} & x \notin \mathsf{FTV}(\tau) \\
\mathsf{MGU}(\varepsilon_1(\overline{\tau}_1) \stackrel{?}{=} \varepsilon_2(\overline{\tau}_2) \wedge U) & = & \mathsf{MGU}(\overline{\tau}_1 \stackrel{?}{=} \overline{\tau}_2 \wedge U) \\
& \text{when} & \varepsilon_1 = \varepsilon_2 \\
\mathsf{MGU}(\top) & = & id
\end{array}$$

MGU is undefined for the other cases.

Besides, the generalization operation over types is defined as:

$$\mathsf{GEN}(\Gamma, \tau) = \forall(\mathsf{FTV}(\tau) \setminus \mathsf{FTV}(\Gamma)).\tau$$

Yann Régis-Gianas
yrg@irif.fr

Soundness

Theorem ($\mathcal{W}$ is sound)
If $\mathcal{W}(\mathcal{V}, \Gamma, t) = (\mathcal{V}', \phi, \tau)$ then $\phi(\Gamma) \vdash t : \tau$

Proof.
By induction over terms. □

Yann Régis-Gianas
yrg@irif.fr

Completeness and principality

Theorem ($\mathcal{W}$ is complete and computes principal types)

If there exists ϕ and τ such that $\phi(\Gamma) \vdash t : \tau$,
then there exists $\mathcal{V}', \phi', \tau'$ and ρ such that

$$\begin{array}{rcll} \mathcal{W}(\mathcal{V}, \Gamma, t) & = & (\mathcal{V}', \phi', \tau') & \\ \tau & = & \rho(\tau') & \\ \phi(\alpha) & = & \phi'(\rho(\alpha)) & \forall \alpha \notin \mathcal{V} \end{array}$$

Proof.

By induction over terms. □

Yann Régis-Gianas
yrg@irif.fr

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Proof.

By induction over terms. □

(In these proofs, the substitution manipulations are "tricky"!)

Yann Régis-Gianas
yrg@irif.fr

Back to Question 1

Question 1

Given an environment Γ and a typable term t, is there a unique type τ such that $\Gamma \vdash t : \tau$? **No!**

Yann Régis-Gianas
yrg@irif.fr

Back to Question 1

Question 1

Given an environment Γ and a typable term t, is there a unique type τ such that $\Gamma \vdash t : \tau$? **No!** But one can find a type that rules them all! $\mathcal{W}$ is a constructive proof of that fact!

Yann Régis-Gianas
yrg@irif.fr

Complexity of $\mathcal{W}$

Complexity

- ML typability is NP-hard and DEXPTIME-complete.
- Here is a typical example that requires an exponential time to type:

```
1  let f0 = fun x -> x in
2  let f1 = (f0, f0) in
3  let f2 = (f1, f1) in
4  ...
5  fN
```

- **But** under reasonable assumptions[5], the complexity is quasi-linear.

[5] In the wild, the depth of types are bounded!

Yann Régis-Gianas
yrg@irif.fr

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Yann Régis-Gianas
yrg@irif.fr

Do it yourself!

Programming exercise

1. Define the syntax of an explicitly typed version of ML.
2. Implement `Elaboration.algorithm_w` targeting the language you just defined.
3. Are you able to locate the binding site of every type variables that occur in the elaborated terms?
   (Let us name this question “Question 0”.)

Yann Régis-Gianas
yrg@irif.fr

eML, an explicitly typed ML (Syntax)

We reuse the syntax of System F for abstractions with respect to types and for type applications:

$$
\begin{array}{rcl}
M & ::= & x \\
& | & \lambda(x:\tau).M \\
& | & M\ M \\
& | & \Lambda\alpha.M \\
& | & M\tau \\
& | & \textbf{let}\ x:\sigma = M\ \textbf{in}\ M
\end{array}
$$

Notice that, contrary to System F, λ-bound identifiers are assigned a monomorphic type. As in ML, only **let**-bound identifiers can be polymorphic.

Yann Régis-Gianas
yrg@irif.fr

eML, an explicitly typed ML (Typing rules)

$$\frac{}{\Gamma \vdash x : \Gamma(x)} \qquad \frac{\Gamma, (x : \tau_1) \vdash M : \tau_2}{\Gamma \vdash \lambda(x : \tau_1).M : \tau_1 \to \tau_2}$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash M_2 : \tau_1}{\Gamma \vdash M_1\ M_2 : \tau_2}$$

$$\frac{\Gamma \vdash M_1 : \sigma_1 \qquad \Gamma, (x : \sigma_1) \vdash M_2 : \sigma_2}{\Gamma \vdash \textbf{let}\ x : \sigma_1 = M_1\ \textbf{in}\ M_2 : \sigma_2} \qquad \frac{\Gamma, \alpha \vdash M : \sigma}{\Gamma \vdash \Lambda\alpha.M : \forall\alpha.\sigma}$$

$$\frac{\Gamma \vdash M : \forall\alpha.\sigma}{\Gamma \vdash M\tau : \sigma[\alpha \mapsto \tau]}$$

Yann Régis-Gianas
yrg@irif.fr

Wait a second!

But this is not the language we define in the previous exercise, right?

Yann Régis-Gianas
yrg@irif.fr

xML, another explicitly typed ML

The language we defined as a target of $\mathcal{W}$ elaboration is:

$$
\begin{array}{rcl}
N & ::= & \Lambda\overline{\alpha}.Q \\
Q & ::= & x\overline{\tau} \\
 & \mid & \lambda(x:\tau).Q \\
 & \mid & Q\,Q \\
 & \mid & \textbf{let } x:\sigma = N \textbf{ in } Q
\end{array}
$$

Yann Régis-Gianas
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 & \mid & Q\,Q \\
 & \mid & \textbf{let } x:\sigma = N \textbf{ in } Q
\end{array}
$$

How do we relate eML and xML?

Yann Régis-Gianas
yrg@irif.fr

Normalization of eML typing derivations

Let us write

$$\Gamma \vdash M : \sigma \Rightarrow N$$

and

$$\Gamma \vdash M : \tau \Rightarrow Q$$

for two judgments that denote the normalization of the typing derivation of M as a typing derivation in xML.

More formally, we want the following properties to hold:

Lemma (Normalization preserves well-typedness)

If $\Gamma \vdash M : \sigma$ in eML and $\Gamma \vdash M : \sigma \Rightarrow N$ then $\Gamma \vdash N : \sigma$ in xML.
(Idem for the monomorphic case.)

Lemma (Well-formed typing derivations normalize)

If $\Gamma \vdash M : \sigma$ in eML then $\Gamma \vdash M : \sigma \Rightarrow N$.
(Idem for the monomorphic case.)

Yann Régis-Gianas
yrg@irif.fr

Do it yourself!

Formalization exercise

Define the rules for the previous two judgments.

Yann Régis-Gianas
yrg@irif.fr

Normalization rules

Let us start with

$$\Gamma \vdash M : \sigma \Rightarrow N$$

The syntax of N forces us to η-expand x:

$$\text{Norm-Var}\quad \frac{\Gamma(x) = \forall \overline{\alpha}.\tau}{\Gamma \vdash x : \forall \overline{\alpha}.\tau \Rightarrow \Lambda \overline{\alpha}.(x\,\overline{\alpha})}$$

The case for type abstraction is obvious:

$$\text{Norm-TAbs}\quad \frac{\Gamma, \alpha \vdash M : \sigma \Rightarrow N}{\Gamma \vdash \Lambda\alpha.M : \forall\alpha.\sigma \Rightarrow \Lambda\alpha.N}$$

Yann Régis-Gianas
yrg@irif.fr

Normalization rules (continued)

Type applications are reduced during the normalization so that the resulting term N is normalized with respect to strong ι-reduction:

$$\text{Norm-TApp}\quad \frac{\Gamma \vdash M : \forall\alpha.\sigma \Rightarrow \Lambda\alpha.N}{\Gamma \vdash M\tau : \sigma[\alpha \mapsto \tau] \Rightarrow N[\alpha \mapsto \tau]}$$

To comply with the syntax, the type abstractions coming from the right-hand-side of **let**-bindings must be extruded:

$$\text{Norm-Let}\quad \frac{\Gamma \vdash M_1 : \sigma \Rightarrow N_1 \qquad \begin{array}{c}\overline{\alpha} \# \sigma, N_1 \\ \Gamma, (x : \sigma) \vdash M_2 : \forall\overline{\alpha}.\tau \Rightarrow \Lambda\overline{\alpha}.Q\end{array}}{\Gamma \vdash \textbf{let}\ x : \sigma = M_1\ \textbf{in}\ M_2 : \forall\overline{\alpha}.\tau \Rightarrow \Lambda\overline{\alpha}.\textbf{let}\ x : \sigma = N_1\ \textbf{in}\ Q}$$

Yann Régis-Gianas
yrg@irif.fr

Normalization rules (continued)

Finally, the two rules for applications and λ-abstraction are straightforward since the two languages coincide on these constructions:

Norm-App

$$\frac{\Gamma \vdash M_1 : \tau_1 \to \tau_2 \Rightarrow Q_1 \qquad \Gamma \vdash M_2 : \tau_1 \Rightarrow Q_2}{\Gamma \vdash M_1 M_2 : \tau_2 \Rightarrow Q_1 Q_2}$$

Norm-Abs

$$\frac{\Gamma(x : \tau) \vdash M : \tau_2 \Rightarrow Q}{\Gamma \vdash \lambda(x : \tau).M : \tau_1 \to \tau_2 \Rightarrow \lambda(x : \tau).Q}$$

Yann Régis-Gianas
yrg@irif.fr

Back to Lemmas

Lemma (Well-formed typing derivations normalize)

If $\Gamma \vdash M : \sigma$ in eML then $\Gamma \vdash M : \sigma \Rightarrow N$.
(Idem for the monomorphic case.)

Proof.

Easy induction over typing derivations of eML. □

Lemma (Normalization preserves well-typedness)

If $\Gamma \vdash M : \sigma$ in eML and $\Gamma \vdash M : \sigma \Rightarrow N$ then $\Gamma \vdash N : \sigma$ in xML.
(Idem for the monomorphic case.)

Proof.

By induction over typing derivations of eML. □

Yann Régis-Gianas
yrg@irif.fr

Two birds with one stone!

Lemma

Type erasure preservation If $\Gamma \vdash M : \sigma \Rightarrow N$ then the type erasures of M and N are equal.

Proof.

Immediate by induction. □

Equivalence of two pairs of type systems for ML

- For any derivation of eML, there is an equivalent derivations of xML. (The other direction is obvious.) : We have a typechecker for explicitly-typed ML!
- If we remove the type annotations from the syntax, the proof can be transported to the (implicitly typed) ML type systems we have introduced earlier!
  (Question 2 is now solved.)

And what about "Question 0"?

Remember:

> Are you able to locate the binding site of every type variables that occur in the elaborated terms?

The unification type variables that have not been promoted to generalized type variables are still floating in the air. This is not really a problem: these variables can be seen as existentially quantified at the toplevel.

Yet, a cleaner treatment of these type variables consists in the introduction of an existential quantification over these (flexible) type variables at the level of terms:

$$t \quad ::= \quad \ldots \mid \exists \alpha.t$$

with a companion typing rule of the form:

$$\frac{\Gamma \vdash t : \tau[\alpha \mapsto \tau']}{\Gamma \vdash \exists \alpha.t : \tau}$$

An open question : the type soundness of ML

Many proofs are possible

- From scratch, by mimicking the proof for System F.
- By deducing the type soundness of eML from System F's. (See Didier Remy's course notes, section 4.6.3.)

Yann Régis-Gianas
yrg@irif.fr

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Yann Régis-Gianas
yrg@irif.fr

Syntax for typing constraints

We will now reformulate the type inference problem in ML using **typing constraints** written using the following syntax:

$$
\begin{array}{rcl}
C & ::= & \textbf{true} \\
& | & \textbf{false} \\
& | & C \wedge C \\
& | & \tau = \tau \\
& | & \exists \alpha . C \\
& | & \textbf{let } x = \lambda \alpha . C \textbf{ in } C \\
& | & x\tau
\end{array}
$$

Yann Régis-Gianas
yrg@irif.fr

Meaning of constraints

Consider the standard meaning of the unification constraints U where:

$$U ::= \textbf{true} \mid \textbf{false} \mid U \wedge U \mid \tau = \tau \mid \exists \alpha.U$$

then, we can define the meaning of a constraint C of the form:

$$\textbf{let } x = \lambda\alpha.C_1 \textbf{ in } C_2$$

by defining it as the meaning of the **let**-expanded constraint:

$$(\exists \alpha.C_1) \wedge C_2[x \mapsto \lambda\alpha.C_1]$$

If we call ζ this reduction, we have:

$$\frac{\psi \models U \qquad U \text{ is the } (\zeta, \beta)\text{-normal form of } C}{\psi \models C}$$

Yann Régis-Gianas
yrg@irif.fr

The key idea

So, what is the point of introducing **let** in constraints?

Yann Régis-Gianas
yrg@irif.fr

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So, what is the point of introducing **let** in constraints?

Lemma (Principal Type Scheme)

Every satisfiable constraint abstraction of the form $\lambda\alpha.C$ can be rewritten into an equivalent constraint of the form $\lambda\alpha.\exists\overline{\beta}.\alpha = \tau$.

Lemma (Satisfiable constraint abstractions are type schemes)

The type τ' satisfies the predicate $\lambda\alpha.\exists\overline{\beta}.\alpha = \tau$ iff $\forall\overline{\beta}.\tau \preceq \tau'$.

Yann Régis-Gianas
yrg@irif.fr

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Typing constraints capture the essence of ML type inference in a declarative way.

Yann Régis-Gianas
yrg@irif.fr

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Typing constraints capture the essence of ML type inference in a declarative way.

Exercise

Take some time to define the constraint $[\![t : \tau]\!]$ read as "t has type τ".

Constraint generation

The constraint $[\![t : \tau]\!]$ is read as "t has type τ" and is defined as:

$$
\begin{array}{rcl}
[\![x : \tau]\!] & = & x\,\tau \\
[\![t\,u : \tau]\!] & = & \exists\alpha.[\![t : \alpha \to \tau]\!] \wedge [\![u : \alpha]\!] \\
[\![\lambda x.t : \tau]\!] & = & \exists\beta\gamma.(\tau = \gamma \to \beta \wedge \\
 & & \qquad \textbf{let}\ x = (\lambda\alpha.\alpha = \gamma)\ \textbf{in}\ [\![t : \beta]\!]) \\
[\![\textbf{let}\ x = t\ \textbf{in}\ u : \tau]\!] & = & \textbf{let}\ x = \lambda\alpha.[\![t : \alpha]\!]\ \textbf{in}\ [\![u : \tau]\!]
\end{array}
$$

Yann Régis-Gianas
yrg@irif.fr

Soundness

Let us define "**def** Γ **in** C" by induction on Γ:

$$\begin{array}{rcl} \textbf{def } \bullet \textbf{ in } C & = & C \\ \textbf{def } \Gamma, (x : \forall \overline{\beta}.\tau) \textbf{ in } C & = & \textbf{def } \Gamma \textbf{ in let } x = \lambda\alpha.\exists \overline{\beta}.\alpha = \tau \textbf{ in } C \end{array}$$

Lemma (Soundness)

If the constraint **def** Γ **in** $[\![t : \tau]\!]$ is satisfiable then $\Gamma \vdash t : \tau$ holds.

Yann Régis-Gianas
yrg@irif.fr

Completeness

To reason about completeness, we need to extract the typing constraint of a specific typing derivation. This can be done using a (once again!) reformulation of ML type system as a **constrained type system** whose judgment:

$$C, \Gamma \vdash t : \tau$$

is read

> Under constraint C and environment Γ, the term t has type τ".

(A similar judgment for type schemes is introduced as well.)

Yann Régis-Gianas
yrg@irif.fr

Completeness and principality

Lemma (Completeness and principality)

If $C, \Gamma \vdash t : \tau$ holds then C entails **def** Γ **in** $[\![t : \tau]\!]$.

Yann Régis-Gianas
yrg@irif.fr

Advantages of the constraint-based approach over $\mathcal{W}$

By separating constraint generation and constraint solving, the constraint-based approach is more modular that the monolithic algorithm $\mathcal{W}$. This implies that:

- The constraint language remains small even if the programming language is extended with new constructions.
- Being declarative, the constraint language makes it easier to write **correct** generation rules.
- The optimizations of the type inference engine are clearly localized: they must occur in the constraint solver.

Besides, reasoning about the solving process or about the constraint generation using equivalence of constraints is "higher-level" than reasoning with substitutions.

Last but not least, we will later see that typing constraints lead us to a natural generalization of ML called HM(X) which makes it simpler to add some form of subtyping in ML.

Exercises

Formalization exercises

- Extend ML with algebraic data types (that is with data constructor applications and pattern-matching) and compare the amount of work needed to update $\mathcal{W}$ and the amount of work needed to update the constraint generation.

Yann Régis-Gianas
yrg@irif.fr

What about elaboration?

The constraint-based approach seems seducing as it abstracts the syntax of the source language to only focus on the type inference problem. But what if we want to use such an approach to implement an elaboration engine?

It is not totally true that the input source term is forgotten by the constraint-based inference engine since the syntax of the constraint follows the structure on the input source term. Would it be possible to preserve this structure through the solving process?

Yann Régis-Gianas
yrg@irif.fr

α-constraints

Constraint can actually be generalized into $\alpha-$constraints, constraints whose satisfiability is witnessed by a value of type α. In the case of an elaboration, this value will actually be the elaborated term.

The construction of this elaborated term is possible if the inferred types and the inferred type schemes can be "decoded" in the source syntax and if the structure of the elaborated term can be reconstructed bottom-up along the solving process. For the first condition, it suffices to ask the programmer to provide an appropriate decoding function. For the second condition, the syntax for constraints is extended with a new construct:

$$C ::= \ldots \mid \mathsf{fmap}\ \mathsf{f}\ C$$

which allows the programmer to incrementally construct the elaborated in a continuation-passing style.

(See François Pottier's paper in the reading list for more details.)

Yann Régis-Gianas
yrg@irif.fr

Do it yourself!

Programming exercise

1. Read François Pottier's paper.
2. Complete **`ConstraintBasedElaboration`** to implement ML type inference with **`inferno`**.

Yann Régis-Gianas
yrg@irif.fr

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Yann Régis-Gianas
yrg@irif.fr

Conclusion

What did you learn today?

- A type inference engine must be sound, complete and outputs principal types.
- Prefer elaboration engines over type inference engines.
- Type inference in ML is neat!

Is type inference a closed problem?

- Type inference in System F is undecidable.
- Type inference in presence of subtyping is subtle.
- Type inference in presence of GADTs is ugly.

Yann Régis-Gianas
yrg@irif.fr

Plan

Yann Régis-Gianas
yrg@irif.fr

About $\mathcal{W}$

- The original paper:
  Principal type-schemes for functional programs
  Luis Damas and Robin Milner
  POPL '82
- For a proof of $\mathcal{W}$:
  - The course notes of Xavier Leroy
  - A simple algorithm and proof for type inference
    Mitchell Wand
    FI'87
- About elaboration in ML:
  Course notes of Didier Remy (Section 4.6)

Yann Régis-Gianas
yrg@irif.fr

About constraint-based approaches

- About constraint abstractions:
  Constraint abstractions.
  Jörgen Gustavsson and Josef Svenningsson.
  SPDO'01
- About α-constraints:
  Hindley-Milner Elaboration in Applicative Style
  Francois Pottier
  ICFP'14
- About typing constraints for ML:
  The essence of ML type inference.
  François Pottier and Didier Rémy.
  Chapter 10 of ATAPL, 2005.

About efficient implementations of ML inference

- About first-order unification using Kaplan-Tarjan data structures:
  Résolution d'équations dans des langages d'ordre 1, 2, . . ., ω.
  Gérard Huet.
  PhD thesis, Université Paris 7, 1976.
- About using efficient generalization decision:
  Extending ML type system with a sorted equational theory.
  Didier Rémy.
  Technical Report 1766, INRIA, 1992.