

Deriving, transforming, optimizing programs

MPRI 2.4

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Let us be dreamers

An old dream:

- write high-level, abstract, modular code;
- let the compiler produce low-level, efficient code.

“Zero-cost abstraction”. (A C++/Rust slogan.)

(Pure) functional prog. languages should lend themselves well to this idea.

- No mutable state. Aliasing not a danger.
Syntactically obvious where each variable receives its value.
- Equational reasoning.
Programs denote values. Replace equals with equals.
- Simple, rich language.
Many transformations easily expressed as rewriting rules.

Perhaps not quite true (do need side effects in some form), but let's see.

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Equational reasoning

If two terms t_1 and t_2 are observationally equivalent,
and if we have reason to believe that t_2 is more efficient than t_1 ,

- or that this rewriting step will enable further optimizations,

then we can optimize a program by replacing t_1 with t_2 .

Equality

In a **pure & total** language, such as Coq, a term is **equal** to its value.

Two terms that have the same value are **equal**.

Equal terms are **interchangeable** – Leibniz's Principle.

Life in an ideal (mathematical) world. See [DemoEqReasoning](#).

Observational equivalence

Fix some notion of “success”, e.g. t succeeds iff t computes 42.

- Note that this notion depends on the evaluation strategy.

With respect to this notion of success, or “observation”,

t_1 and t_2 are observationally equivalent ($t_1 \simeq t_2$) iff,

for every (well-typed) context C ,

$C[t_1]$ succeeds if and only if $C[t_2]$ succeeds.

When is a rewriting step valid?

Is full β a valid law?

$$(\lambda x.t_2) \ t_1 \simeq t_2[t_1/x]$$

When is a rewriting step valid?

Is full β a valid law?

$$(\lambda x.t_2) \ t_1 \simeq t_2[t_1/x]$$

In a **pure & total** language, such as Coq, **yes**. Part of **definitional equality**.

Under **call-by-name**, even in the presence of non-termination, **yes**.

Under **call-by-value**, in the presence of non-termination or other side effects, **no**.

full β is invalid under call-by-value

Repeat after me:

full β is invalid under call-by-value

full β is invalid under call-by-value

full β is invalid under call-by-value

After 20+ years, I keep making this mistake from time to time!

$(\lambda x.t_2) t_1$ cannot be “simplified” to $t_2[t_1/x]$

let $x = t_1$ in t_2 cannot be “simplified” to $t_2[t_1/x]$

What about call-by-value? β_v

Under call-by-value, in the presence of side effects, full β is invalid.

One must restrict it to the case where t_1 is **pure**.

$$(\lambda x.t_2) t_1 \longrightarrow t_2[t_1/x] \quad \text{provided } t_1 \text{ is pure}$$

Roughly, a closed term t is pure if there exists a value v such that t reduces to v , independently of the store.

Whether a non-closed term t is closed depends on purity hypotheses about its free variables. E.g., is “ $f\ x$ ” pure? Yes, **IF** f has no side effects.

As a simple special case, one can use β_v , which is **valid**:

$$(\lambda x.t_2) v_1 \longrightarrow t_2[v_1/x]$$

This follows from the theory of parallel reduction.

See [LambdaCalculusStandardization/pcbv_adequacy](#).

When is a rewriting step profitable?

When it is valid, is full β a profitable optimization?

$$(\lambda x.t_2) \ t_1 \longrightarrow t_2[t_1/x]$$

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When is a rewriting step profitable?

When it is valid, is full β a profitable optimization?

$$(\lambda x. t_2) \ t_1 \longrightarrow t_2[t_1/x]$$

Under call-by-name, it is safe for time and space,
but can increase code size.

Under call-by-need, if x has multiple occurrences in t_2 , or if x occurs under a λ within t_2 , then the right-hand side risks repeating the computation of t_1 , wasting time and space. This danger exists even if t_1 is a value!

In short, this optimization step seems profitable when x is used “at most once” in t_2 , for a suitable definition of this notion.

Turner, Wadler, Mossin, Once upon a type, 1995.

Peyton Jones, Santos, A transformation-based optimiser for Haskell, 1997.

Summary so far

A proposed rewriting rule $t_1 \rightarrow t_2$ is **valid** if $t_1 \simeq t_2$ holds.

- This is influenced by the evaluation strategy, the presence or absence of side effects, and type hypotheses.

A proposed rewriting rule $t_1 \rightarrow t_2$ may or may not be **profitable**.

- This is influenced by many factors, including further optimizations and transformations.

let-reduction

So far, I have discussed full β versus β_v .

If the language has a primitive construct,
then an analogous discussion applies to “full let” versus let_v .

$$\begin{aligned} \text{let } x = t_1 \text{ in } t_2 &\longrightarrow t_2[t_1/x] \\ \text{let } x = v_1 \text{ in } t_2 &\longrightarrow t_2[v_1/x] \end{aligned}$$

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Is this optimization valid?

$$\lambda x.t \ x \simeq t \quad \text{provided } x \notin fv(t)$$

In a **pure & total** language, such as Coq, **yes**. Part of **definitional equality**.

Under **call-by-name**, in the presence of non-termination, I think it is...

Under **call-by-value**, in the presence of side effects, it definitely **isn't**.

When it is valid, is it profitable? **Possibly**. E.g., after a naïve CPS transformation, η -reduction turns $\lambda x.k \ x$ into k , which amounts to **tail call optimization**.

Yet η -reduction **can** be costly and η -expansion can be profitable. Tricky!

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What is inlining?

Inlining is the action of replacing a call to a known function with the suitably instantiated **body of this function**.

So, is **Inlining** just another name for β_v ?

$$(\lambda x.t_2) v_1 \longrightarrow t_2[v_1/x]$$

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No. Inlining can be more accurately described by several rewriting rules:

Looking up a definition: (IR1)

$$\text{let } x = v \text{ in } C[x] \rightarrow \text{let } x = v \text{ in } C[v] \quad \text{if } x \notin \text{bv}(C)$$

Eliminating dead code: (IR2)

$$\text{let } x = v \text{ in } t \rightarrow t \quad \text{if } x \notin \text{fv}(t)$$

Binding formals to actuals: (IR3)

$$(\lambda x. t_2) t_1 \rightarrow \text{let } x = t_1 \text{ in } t_2$$

These rules are **valid** under every strategy and in the face of side effects.

Rule IR1 works for every value v , not just λ -abstractions.

Rules IR1 and IR2 work for “let rec”, too!

Rule IR1 **duplicates** v and can cause non-termination at compile-time (!) or an explosion in code size.

Simplification rules

A few additional **simplification** rules are useful:

Eliminating an alias: (**SR1**)

$$\text{let } y = x \text{ in } C[y] \longrightarrow C[x]$$

Hoisting a binding: (**SR2**)

$$E[\text{let } x = t_1 \text{ in } t_2] \longrightarrow \text{let } x = t_1 \text{ in } E[t_2]$$

These rules are **valid** under every strategy and in the face of side effects.

Example

Consider this tiny example:

```
let succ x = x + 1
let even x = x mod 2 = 0
let test x = even (succ x)
```

This could be call-by-value (OCaml) or call-by-need (Haskell).

Example, continued

```
let succ x = x + 1
let even x = x mod 2 = 0
let test x = even (succ x)
```

Inlining `succ` and `even` (IR1, applied twice) yields:

```
let succ x = x + 1
let even x = x mod 2 = 0
let test x = (fun x -> x mod 2 = 0) ((fun x -> x + 1) x)
```

Example, continued

```
let succ x = x + 1
let even x = x mod 2 = 0
let test x = (fun x -> x mod 2 = 0) ((fun x -> x + 1) x)
```

Eliminating dead code (IR2, applied twice) yields:

```
let test x = (fun x -> x mod 2 = 0) ((fun x -> x + 1) x)
```

Example, continued

```
let test x = (fun x -> x mod 2 = 0) ((fun x -> x + 1) x)
```

Binding (IR3) yields:

```
let test x = (fun x -> x mod 2 = 0) (let x = x in x + 1)
```

Example, continued

```
let test x = (fun x -> x mod 2 = 0) (let x = x in x + 1)
```

Renaming (SR1) yields:

```
let test x =
  (fun x -> x mod 2 = 0) (x + 1)
```

Example, continued

```
let test x =
  (fun x -> x mod 2 = 0) (x + 1)
```

Binding (IR3) yields:

```
let test x =
  let x = x + 1 in
  x mod 2 = 0
```

Example, continued

```
let test x =
  let x = x + 1 in
  x mod 2 = 0
```

Optionally, one more application of IR1 & IR2 could yield:

```
let test x =
  (x + 1) mod 2 = 0
```

This would not improve the machine code that we get in the end, though.

Case of known constructor

IR3 is the simplification rule that actually **saves one step** of computation.

It is applicable when a **function** value is **eliminated**, that is, called.

What if a value of an **algebraic data type** is eliminated?

Case of known constructor

IR3 is the simplification rule that actually **saves one step** of computation.

It is applicable when a **function** value is **eliminated**, that is, called.

What if a value of an **algebraic data type** is eliminated?

A new rule is needed:

Case of known constructor: (IR4)

$\text{case inj}_i v \text{ of } x_1.t_1 \parallel x_2.t_2 \rightarrow \text{let } x_i = v \text{ in } t_i$

Example

Suppose Booleans are user-defined:

```
type bool = False | True
```

Now, consider this tiny example:

```
let not x = match x with False -> True | True -> False
let test x = not (not x)
```

Example, continued

```
let not x = match x with False -> True | True -> False
let test x = not (not x)
```

Inlining (IR1, applied twice) and dead code elimination (IR2) yield:

```
let test x =
  (fun x -> match x with False -> True | True -> False)
    ((fun x -> match x with False -> True | True -> False) x)
```

Binding (IR3) and renaming (SR1) yield:

```
let test x =
  (fun x -> match x with False -> True | True -> False)
    (match x with False -> True | True -> False)
```

Example, continued

```
let test x =
  (fun x -> match x with False -> True | True -> False)
  (match x with False -> True | True -> False)
```

Binding (IR3) yields:

```
let test x =
  let x = match x with False -> True | True -> False in
  match x with False -> True | True -> False
```

Example, continued

```
let test x =  
  let x = match x with False -> True | True -> False in  
    match x with False -> True | True -> False
```

Now, what? The rule β_v is not applicable here.

Example, continued

```
let test x =  
  let x = match x with False -> True | True -> False in  
    match x with False -> True | True -> False
```

Now, what? The rule β_v is not applicable here.

Under call-by-need, this let construct can be reduced:

```
let test x =  
  match  
    match x with False -> True | True -> False  
  with  
    False -> True | True -> False
```

We then seem to need a “case-of-case” simplification rule.

What happens under call-by-value, though?

E of case

Under call-by-value, one could argue that the right-hand side is pure and apply full β .

One can do better and directly apply a new rule:

E of case: (SR3)

$$E[\text{case } t \text{ of } x_1.t_1 \parallel x_2.t_2] \longrightarrow \text{case } t \text{ of } x_1.E[t_1] \parallel x_2.E[t_2]$$

This rule is valid under every strategy. I think.

It is known as a commuting conversion.

Case-of-case is a special case of it!

Exercise (recommended): Write the rule “case-of-case”.

Example, continued

```
let test x =
  let x = match x with False -> True | True -> False in
  match x with False -> True | True -> False
```

By E-of-case (SR3), we obtain:

```
let test x =
  match x with
  | False -> (
    let x = True in
    match x with False -> True | True -> False
  )
  | True -> (
    let x = False in
    match x with False -> True | True -> False
  )
```

Example, continued

```
let test x =
  match x with
  | False -> (
    let x = True in
    match x with False -> True | True -> False
  )
  | True -> (
    let x = False in
    match x with False -> True | True -> False
  )
```

Inlining (IR1, IR2) and case-of-known-constructor (IR4) yield:

```
let test x =
  match x with
  | False -> False
  | True -> True
```

Example, continued

```
let test x =  
  match x with  
  | False -> False  
  | True  -> True
```

Yet another simplification rule, η -reduction for sums, yields:

```
let test x = x
```



Case of case, improved

This rule **duplicates** the evaluation context:

$$\text{E of case: (SR3)} \\ E[\text{case } t \text{ of } x_1.t_1 \parallel x_2.t_2] \longrightarrow \text{case } t \text{ of } x_1.E[t_1] \parallel x_2.E[t_2]$$

This is potentially devastating!

E.g., suppose E is “case [] of $y_1.u_1 \parallel y_2.u_2$ ”:

$$\begin{aligned} &\text{Case of case: (SR3c)} \\ &\text{case (case } t \text{ of } x_1.t_1 \parallel x_2.t_2) \text{ of } y_1.u_1 \parallel y_2.u_2 \longrightarrow \\ &\quad \text{case } t \text{ of } x_1.(\text{case } t_1 \text{ of } y_1.u_1 \parallel y_2.u_2) \\ &\quad \parallel x_2.(\text{case } t_2 \text{ of } y_1.u_1 \parallel y_2.u_2) \end{aligned}$$

The branches u_1 and u_2 are duplicated! What to do?

Case of case, improved

A solution is to introduce **join points** to limit duplication.

Case of case, with join points: (**SR3cj**)
 $\text{case } (\text{case } t \text{ of } x_1.t_1 \parallel x_2.t_2) \text{ of } y_1.u_1 \parallel y_2.u_2 \rightarrow$

let $k_1 = \lambda y_1.u_1$ and $k_2 = \lambda y_2.u_2$ in
 $\text{case } t \text{ of } x_1.(\text{case } t_1 \text{ of } y_1.k_1\ y_1 \parallel y_2.k_2\ y_2)$
 $\parallel x_2.(\text{case } t_2 \text{ of } y_1.k_1\ y_1 \parallel y_2.k_2\ y_2)$

The names k_1 and k_2 can be thought of as **labels** to which one jumps.

We have intentionally **allowed** the outer case to be duplicated. The two copies scrutinize t_1 and t_2 , so further simplifications should be possible.

Example

Suppose the function `bor` implements Boolean disjunction. Consider this:

```
match bor b1 b2 with
| False -> <foo>
| True -> <bar>
```

Inlining yields:

```
match
  match b1 with False -> b2 | True -> True
with
| False -> <foo>
| True -> <bar>
```

Example, continued

```
match
  match b1 with False -> b2 | True -> True
  with
  | False -> <foo>
  | True -> <bar>
```

Applying rule SR3cj yields:

```
let foo () = <foo>
and bar () = <bar> in
match b1 with
| False -> (match b2 with False -> foo() | True -> bar())
| True -> (match True with False -> foo() | True -> bar())
```

Example, continued

```
let foo () = <foo>
and bar () = <bar> in
match b1 with
| False -> (match b2 with False -> foo() | True -> bar())
| True -> (match True with False -> foo() | True -> bar())
```

By case-of-known-constructor (IR4), we obtain:

```
let foo () = <foo>
and bar () = <bar> in
match b1 with
| False -> (match b2 with False -> foo() | True -> bar())
| True -> bar()
```

Example, continued

```
let foo () = <foo>
and bar () = <bar> in
match b1 with
| False -> (match b2 with False -> foo() | True -> bar())
| True  -> bar()
```

Because there is only one jump to `foo`, it can be inlined:

```
let bar () = <bar> in
match b1 with
| False -> (match b2 with False -> <foo> | True -> bar())
| True  -> bar()
```

Example, continued

`bar` is a “join point”, a local function that is meant to represent a code label.

It is always called via a tail call.

The idea is, it should not require a closure allocation.

```
let bar () = <bar> in
  match b1 with
    | False -> (match b2 with False -> <foo> | True -> bar())
    | True -> bar()
```

It must not be naïvely inlined: that would cause duplication again!

During further transformations, one should ensure that it remains a “join point” and is not inadvertently turned into a full-fledged first-class function.

Maurer, Ariola, Downen, Peyton Jones,
Compiling without continuations, 2017.

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Can we optimize this code?

```
match xs with
| []      -> []
| y :: ys ->
  match xs with
  | []      -> <foo>
  | z :: zs -> <bar>
```

The rules shown so far can simplify this **only** if there is a binding of the form `let xs = <value>` higher up. This is case-of-known-constructor.

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Can we optimize this code?

```
match xs with
| []      -> []
| y :: ys ->
  match xs with
  | []      -> <foo>
  | z :: zs -> <bar>
```

The rules shown so far can simplify this **only** if there is a binding of the form `let xs = <value>` higher up. This is **case-of-known-constructor**.

We could insert `let xs = y :: ys` at line 4,
but that would be potentially pessimizing.

Better **keep track of which equations are known** at each program point,
and improve **case-of-known-constructor** to exploit these equations.

See [Peyton Jones and Marlow](#), §6.3.

Inlining recursive functions

The rule IR1, as stated, does not allow inlining a function into itself.
This could be relaxed.

Inlining a recursive function into itself amounts to [loop unrolling](#).

Inlining a recursive function at its call site amounts to [loop peeling](#).

Summary

An old idea. Particularly **important** in very high-level languages.

It eliminates the function call overheard, and **enables other optimizations**.

The **danger** of inlining is an increase in **code size** and potential non-termination at compile time. This must be controlled via **heuristics** or via user annotations (**partial evaluation; staging**).

Aggressive inliners can be guided by **program analyses**.

Peyton Jones, Santos,
A transformation-based optimiser for Haskell, 1997.

Peyton Jones, Marlow,
Secrets of the Glasgow Haskell Compiler inliner, 2002.

Jagannathan and Wright, Flow-directed inlining, 1996.

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Example

Here is a reasonably elegant way of obtaining the last element of a list:

```
let rec last xs =
  match xs with
  | []       -> assert false
  | [x]      -> x
  | _ :: xs -> last (x :: xs)
```

Unfortunately, it is inefficient...

Example

Here is a reasonably elegant way of obtaining the last element of a list:

```
let rec last xs =
  match xs with
  | []          -> assert false
  | [x]         -> x
  | _ :: x :: xs -> last (x :: xs)
```

Unfortunately, it is inefficient...

- The cell `x1 :: xs` is re-allocated; CSE can recognize and avoid this.
- Two list cells are inspected to find that the third branch must be taken.

Every cell is tested `twice!` We `forget` information through the recursive call.

How would you remedy this (by hand)?

Example, hand-optimized

By hand, one might write this optimized code:

```
let rec last xs =
  match xs with
  | [] -> assert false
  | x :: xs -> last_cons x xs

and last_cons x xs =
  match xs with
  | [] -> x
  | x :: xs -> last_cons x xs
```

`last_cons` is a loop with two registers `x` and `xs`.

Keeping track of `x` does the trick. Each list cell is examined once.

Call-pattern specialization

Could a compiler do this automatically?

Inlining `last` into itself would amount to **loop unrolling** (i.e., doing two iterations at a time) but would **not** eliminate the problem entirely.

The problem lies in the call `last (x :: xs)`, where information is lost.

We must **specialize** `last` for this call pattern.

Example, optimized

The first step is to create a specialized function, `last_cons`.

```
let rec last xs =
  match xs with
  | []          -> assert false
  | [x]         -> x
  | _ :: x :: xs -> last (x :: xs)

and last_cons x xs =
  last (x :: xs)
```

The equation `last (x :: xs) = last_cons x xs` holds (obviously).

We **record** (remember) this equation for later use.

Example, optimized

The second step is to inline `last` into `last_cons`.

```
let rec last xs =
  match xs with
  | []          -> assert false
  | [x]         -> x
  | _ :: x :: xs -> last (x :: xs)

and last_cons x xs =
  let xs = x :: xs in
  match xs with
  | []          -> assert false
  | [x]         -> x
  | _ :: x :: xs -> last (x :: xs)
```

Example, optimized

By inlining `xs` and exploiting case-of-known-constructor, we get:

```
let rec last xs =
  match xs with
  | []          -> assert false
  | [x]         -> x
  | _ :: x :: xs -> last (x :: xs)

and last_cons x xs =
  match xs with
  | []          -> x
  | x :: xs     -> last (x :: xs)
```

What should be the last step?

Example, optimized

The last step is to replace `last (x :: xs)` with `last_cons x xs`.

There are two occurrences, one of which lies within `last_cons` itself:

```
let rec last xs =
  match xs with
  | [] -> assert false
  | [x] -> x
  | _ :: x :: xs -> last (x :: xs)

and last_cons x xs =
  match xs with
  | [] -> x
  | x :: xs -> last (x :: xs)
```

This **exploits an equation** that was recorded earlier.

We get the code that we would have written, with one iteration unrolled.

Danger!

The correctness of **exploiting an equation within itself** is nonobvious.

Recall this situation:

```
let rec last xs =
  match xs with
  | []          -> assert false
  | [x]         -> x
  | _ :: xs     -> last (x :: xs)

and last_cons x xs =
  last (x :: xs)
```

The equation `last (x :: xs) = last_cons x xs` holds (obviously).

There are **two** places where it can be used **right now...** What if we did so?

Danger!

We get a non-terminating version of the loop:

```
let rec last xs =
  match xs with
  | []          -> assert false
  | [x]         -> x
  | _ :: x :: xs -> last_cons x xs

and last_cons x xs =
  last_cons x xs
```

This “obviously correct” transformation is actually incorrect.

We have in fact rolled the loop so it jumps to itself after 0 iterations!

Exploiting $x = v$ within itself leads to $x = x$, which is nonsensical.

Summary

Call-pattern specialization is also known as constructor specialization.

It is simple, but runs a risk of generating uninteresting specializations and a risk of nontermination at compile-time. Heuristics are needed.

Peyton Jones, Call-pattern specialisation for Haskell programs, 2007.

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Deforestation

Programs expressed in a high-level style often build **intermediate data structures** (lists, trees, ...) which are immediately used and discarded.

They typically allow **communication** between a **producer** and a **consumer**.

Deforestation (Wadler, 1990) aims to get rid of them.

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Example

The composition of `filter` and `map` allocates an intermediate list.

As a direct attempt at deforestation, let us try and optimize it.

```
let bar p f xs =  
    List.filter p (List.map f xs)
```

Let us specialize for the call pattern `List.filter p (List.map f xs)...`

I am using an expression as a call pattern – this goes beyond GHC.

Example

After creating a specialized copy
and inlining `List.filter` and `List.map` into it, we get:

```
let filter_map p f xs =
  match
    match xs with
    | [] -> []
    | x :: xs -> f x :: List.map f xs
  with
  | [] -> []
  | x :: xs ->
    if p x then x :: List.filter p xs
    else List.filter p xs

let bar p f xs =
  filter_map p f xs
```

Example

Performing case-case conversion yields:

```
let filter_map p f xs =
  match xs with
  | [] -> []
  | x :: xs ->
    let x :: xs = f x :: List.map f xs in
    if p x then x :: List.filter p xs
    else List.filter p xs
```

Example

Deciding that $e1 :: e2$ is evaluated from left-to-right, we get:

```
let filter_map p f xs =
  match xs with
  | [] -> []
  | x :: xs ->
    let x = f x in
    let xs = List.map f xs in
    if p x then x :: List.filter p xs
    else List.filter p xs
```

Evaluation order is left undecided by OCaml.

Example

We **wisely** choose to inline `xs`, as it is used only once (in each branch):

```
let filter_map p f xs =
  match xs with
  | [] -> []
  | x :: xs ->
    let x = f x in
    if p x then x :: List.filter p (List.map f xs)
    else List.filter p (List.map f xs)
```

This is **full β** !

It is valid under call-by-need. (Assuming no side effects but divergence.)

It is **invalid** under call-by-value (with side effects), unless `f` is pure.

- `f` must not read or write mutable data, and must terminate.

The OCaml compiler won't do this!

Example

We now recognize the call pattern `List.filter p` (`List.map f xs`).

```
let rec filter_map p f xs =
  match xs with
  | [] -> []
  | x :: xs ->
    let x = f x in
    if p x then x :: filter_map p f xs
    else filter_map p f xs
```

We get the code that an OCaml programmer would write by hand.

No intermediate list! Successful deforestation.

Summary

The equation `List.filter p (List.map f xs) = filter_map p f xs`

- holds under call-by-need;
- holds under call-by-value (with side effects) if `f` is pure.

Pure languages offer greater potential for aggressive optimization!

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Idea 1: focus on lists

Focus on **lists**, a universal type for exchanging **sequences** of elements.

Some functions are list **producers**; some are list **consumers**.

Some, such as `filter` and `map`, are both. (Not a problem.)

Some, such as `zip` and `unzip` have two inputs or two outputs.

Composing these functions yields producer-consumer **pipelines**.

Idea 2: use a custom internal data format

Producers and consumers use lists as an [exchange](#) format.

They can work internally using a [different](#) data representation.

They are then wrapped in [conversions](#) to and from lists.

When a producer and consumer are composed,

- two conversions, to and from lists, should [cancel out](#),
- so there remains to optimize a composition at the [internal data type](#).

This is an instance of the [worker/wrapper transformation](#).

Gill and Hutton, [The worker/wrapper transformation](#), 2009.

Idea 3: avoid recursion

The internal data format should have a **nonrecursive** type, so that:

- Most producers and consumers are **not** recursive!
- At least one of the conversions, to and from lists, must be recursive.

Two approaches, based on two internal data formats, have been proposed:

- **shortcut deforestation**, based on **folds**;
- **stream fusion**, based on **streams**.

Gill, Launchbury, Peyton Jones,
A short cut to deforestation, 1993.

Coutts, Leshchinskiy, Stewart, **Stream fusion:
from lists to streams to nothing at all**, 2007.

The internal data format

In shortcut deforestation, a sequence is internally represented as a [fold](#).

A fold is a [function](#) that allows traversing the sequence.

```
type 'a fold =
  { fold: 'b. ('a -> 'b -> 'b) -> 'b -> 'b }
```

It is a producer which [pushes](#) elements towards a consumer.

This is the standard [Church encoding](#) of lists.

Gill et al.'s paper does not explicitly use the above polymorphic type.
I follow them.

Converting a list to a fold

This is OCaml's `List.fold_right`, with the last two parameters swapped:

```
let rec foldr c n xs =
  match xs with
  | []      -> n
  | x :: xs -> c x (foldr c n xs)
```

If `xs` is a list then `fun c n -> foldr c n xs` is the corresponding fold.

We could define:

```
let import (xs : 'a list) : 'a fold =
  { fold = fun c n -> foldr c n xs }
```

Converting a fold to a list

To convert a fold to a list, we apply it to “cons” and “nil”:

```
let build g =
  g (fun x xs -> x :: xs) []
```

We could define:

```
let export ({ fold } : 'a fold) : 'a list =
  build fold
```

Isomorphism

The idea is that we have an **isomorphism** between lists and (certain well-behaved) folds.

The following law holds:

- `export (import xs)` is observationally equivalent to `xs`.

The reverse law holds if `f` is **pure and terminating**:

- `import (export f)` is equivalent to `f`.

Naturally, the law that's needed when composing two components is...

Isomorphism

The idea is that we have an **isomorphism** between lists and (certain well-behaved) folds.

The following law holds:

- `export (import xs)` is observationally equivalent to `xs`.

The reverse law holds if `f` is **pure and terminating**:

- `import (export f)` is equivalent to `f`.

Naturally, the law that's needed when composing two components is...

...the second one.

Let's just pretend that it holds unconditionally.

Challenge: formalize `build/foldr` in Coq and establish the isomorphism.

Isomorphism

In Gill et al.'s paper, the second law is known as “the foldr/build rule”:

```
foldr c n (build g) = g c n
```

An example consumer-and-producer

In the list library, `map` is written as follows:

```
let map f xs =
  build (fun c n ->
    foldr (fun x xs -> c (f x) xs) n xs
  )
```

The list `xs` is imported using `foldr`, yielding a fold.

A new fold is then constructed on top of it.

This new fold is converted back to a list using `build`.

An example consumer-and-producer

Similarly, `filter` is written as follows:

```
let filter p xs =
  build (fun c n ->
    foldr (fun x xs -> if p x then c x xs else xs) n xs
  )
```

Back to (filter; map)

What happens when we compose filter and map?

```
let bar p f xs =
    filter p (map f xs)
```

Back to (filter; map)

Inlining filter and map yields:

```
let bar p f xs =
  build (fun c n ->
    foldr
      (fun x xs -> if p x then c x xs else xs)
      n
      (build (fun c n ->
        foldr (fun x xs -> c (f x) xs) n xs
      )))
  )
```

We recognize `foldr _ _ (build _)`.

Back to (filter; map)

Exploiting the equation `foldr c n (build g) = g c n` yields:

```
let bar p f xs =
  build (fun c n ->
    let c x xs = if p x then c x xs else xs in
    foldr (fun x xs -> c (f x) xs) n xs
  )
```

This is where we save an intermediate list.

Back to (filter; map)

Inlining c yields:

```
let bar p f xs =
  build (fun c n ->
    foldr (fun x xs ->
      let x = f x in
      if p x then c x xs else xs
    ) n xs
  )
```

This is where `filter` and `map` come into contact and combine.

Back to (filter; map)

We are essentially finished, but can work a little more.

Inlining build yields:

```
let bar p f xs =
  foldr (fun x xs ->
    let x = f x in
    if p x then x :: xs else xs
  ) [] xs
```

Back to (filter; map)

Call-pattern specialization for `foldr` yields:

```
let rec filter_map p f xs =
  match xs with
  | [] -> []
  | x :: xs ->
    let xs = filter_map p f xs in
    let x = f x in
    if p x then x :: xs else xs

let bar p f xs =
  filter_map p f xs
```

Assuming the language is pure,
or assuming `p` and `f` are pure, we can inline `xs`...

Back to (filter; map)

Inlining xs yields:

```
let rec filter_map p f xs =
  match xs with
  | [] -> []
  | x :: xs ->
    let x = f x in
    if p x then x :: filter_map p f xs
    else filter_map p f xs
```

We again get the code that an OCaml programmer would write by hand.

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The internal data format

In stream fusion, a sequence is internally represented as a **stream**.

A stream is a **function** that allows querying the sequence.

```
type 'a stream =
| S:
  (* If you have a pair of a producer function... *)
  ('s -> ('a, 's) step)
  (* ...and an initial state, *)
  * 's ->
  (* then you have a stream. *)
  'a stream
```

It is a producer from which a consumer can **pull** elements.

A typical object-oriented idiom, analogous to Java iterators, but not inherently mutable.

This is an **existential type**, very much like the type of closures in week 3.

The internal data format

Querying a stream produces a result of the following form:

```
type ('a, 's) step =
| Done          (* finished *)
| Yield of 'a * 's (* an element and a new state *)
| Skip of 's    (* just a new state - please ask again *)
```

The types `stream` and `step` are nonrecursive.

This, and the existence of `Skip`, allows most stream producers to be **nonrecursive** functions.

A consumer must ask, ask, ask until a non-`Skip` result is produced.

Converting a list to a stream

This conversion function is nonrecursive:

```
let stream (xs : 'a list) : 'a stream =
  let next xs =
    match xs with
    | [] -> Done
    | x :: xs -> Yield (x, xs)
  in
  S (next, xs)
```

Exercise: Here, what is the type `'s` of states?

Converting a list to a stream

The local function `next` is in fact closed, so one can also write:

```
let stream_next xs =
  match xs with
  | []      -> Done
  | x :: xs -> Yield (x, xs)

let stream (xs : 'a list) : 'a stream =
  S (stream_next, xs)
```

Converting a stream to a list

This is a recursive *consumer* function:

```
let unstream (S (next, s) : 'a stream) : 'a list =
  let rec unfold s =
    match next s with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
    | Skip s        -> unfold s
  in
  unfold s
```

Isomorphism

There is an **isomorphism** between lists and (certain) streams.

The following law holds:

- `unstream (stream xs)` is observationally equivalent to `xs`.

The reverse law holds if `str` is **pure and terminating**:

- `stream (unstream str)` is equivalent to `str`.

Again, we need the **second** law, known as “stream/unstream”.

Let's **pretend** that it holds unconditionally.

Examples of stream producers

How would you implement a singleton stream?

Examples of stream producers

How would you implement a singleton stream?

```
let return (x : 'a) : 'a stream =
  let next s =
    if s then Yield (x, false) else Done
  in
  S (next, true)
```

The type of `s` is `bool`: either we have already yielded an element, or we have not.

Each stream producer **freely chooses** its type of internal states.

Exercise: Write interval of type `int -> int -> int stream`.

Exercise: Write append of type `'a stream -> 'a stream -> 'a stream`.

An example consumer-and-producer

Here is `map` on streams, known as `S.map` in the following:

```
let map (f : 'a -> 'b) (S(next, s) : 'a stream) : 'b stream =
  let next s =
    match next s with
    | Done          -> Done
    | Yield (x, s) -> Yield (f x, s)
    | Skip s        -> Skip s
  in
  S (next, s)
```

Again, **not** a recursive function!

An example consumer-and-producer

Composing with conversions to and from streams yields `map` on lists:

```
let map (f : 'a -> 'b) (xs : 'a list) : 'b list =
  unstream (S.map f (stream xs))
```

An example consumer-and-producer

Here is filter on streams, known as `S.filter` in the following:

```
let filter (p : 'a -> bool) (S (next, s) : 'a stream) =
  let next s =
    match next s with
    | Done          -> Done
    | Yield (x, s) -> if p x then Yield (x, s) else Skip s
    | Skip s        -> Skip s
  in
  S (next, s)
```

Again, **not** a recursive function!

An example consumer-and-producer

Composing with conversions to and from streams yields `filter` on lists:

```
let filter (p : 'a -> bool) (xs : 'a list) : 'a list =
  unstream (S.filter p (stream xs))
```

Back to (filter; map)

What happens when we compose filter and map?

```
let bar p f xs =
  L.filter p (L.map f xs)
let bar p f xs =
  let next s =
    match s with
    | [] -> Done
    | x :: s ->
        let y = f x in if p y then Yield (y, s) else Skip s
  in
  let rec unfold s =
    match next s with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
    | Skip s         -> unfold s
  in
  unfold xs
let bar p f xs =
  let rec unfold s =
    match
      match s with
      | [] -> Done
```

Back to (filter; map)

Inline filter and map:

```
let bar p f xs =
  unstream (S.filter p (stream (
    unstream (S.map f (stream xs))
  )))
```

Back to (filter; map)

Use the stream/unstream rule:

```
let bar p f xs =
  unstream (S.filter p (S.map f (stream xs)))
```

S.filter and S.map come in contact.

Let's inline the hell out of this code!

Back to (filter; map)

Inline stream:

```
let bar p f xs =
  unstream (S.filter p (S.map f (S (stream_next, xs))))
```

Back to (filter; map)

Inline `S.map`:

```
let bar p f xs =
  let next s =
    match stream_next s with
    | Done          -> Done
    | Yield (x, s) -> Yield (f x, s)
    | Skip s        -> Skip s
  in
  unstream (S.filter p (S (next, xs)))
```

Back to (filter; map)

Inline stream_next:

```
let bar p f xs =
  let next s =
    match
      match s with
      | [] -> Done
      | x :: s -> Yield (x, s)
    with
    | Done          -> Done
    | Yield (x, s) -> Yield (f x, s)
    | Skip s        -> Skip s
  in
  unstream (S.filter p (S (next, xs)))
```

Back to (filter; map)

Perform case-of-case conversion, followed with case-of-constructor:

```
let bar p f xs =
  let next s =
    match s with
    | []      -> Done
    | x :: s -> Yield (f x, s)
  in
  unstream (S.filter p (S (next, xs)))
```

Back to (filter; map)

Inline `S.filter`:

```
let bar p f xs =
  let next s =
    match s with
    | [] -> Done
    | x :: s -> Yield (f x, s)
  in
  let next s =
    match next s with
    | Done          -> Done
    | Yield (x, s) -> if p x then Yield (x, s) else Skip s
    | Skip s        -> Skip s
  in
  unstream (S (next, xs))
```

Back to (filter; map)

Inline the first `next` function into the second one:

```
let bar p f xs =
  let next s =
    match
      match s with
      | [] -> Done
      | x :: s -> Yield (f x, s)
    with
    | Done          -> Done
    | Yield (x, s) -> if p x then Yield (x, s) else Skip s
    | Skip s        -> Skip s
  in
  unstream (S (next, xs))
```

Back to (filter; map)

Apply case-of-case and case-of-constructor again:

```
let bar p f xs =
  let next s =
    match s with
    | [] -> Done
    | x :: s ->
        let y = f x in if p y then Yield (y, s) else Skip s
    in
    unstream (S (next, xs))
```

Back to (filter; map)

Inline unstream:

```
let bar p f xs =
  let next s =
    match s with
    | []      -> Done
    | x :: s ->
        let y = f x in if p y then Yield (y, s) else Skip s
  in
  let rec unfold s =
    match next s with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
    | Skip s         -> unfold s
  in
  unfold xs
```

Back to (filter; map)

Inline next into unstream:

```
let bar p f xs =
  let rec unfold s =
    match
      match s with
      | [] -> Done
      | x :: s ->
          let y = f x in if p y then Yield (y, s) else Skip s
    with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
    | Skip s        -> unfold s
  in
  unfold xs
```

Back to (filter; map)

Apply case-of-case again, then a couple rules, then case-of-constructor:

```
let bar p f xs =
  let rec unfold s =
    match s with
    | []      -> []
    | x :: s ->
        let y = f x in
        if p y then y :: unfold s else unfold s
    in
  unfold xs
```

Exercise: Clarify which rewriting rules are used here.

Back to (filter; map)

(Optional.) Hoist `unfold` out. (This is λ -lifting.)

```
let rec unfold p f s =
  match s with
  | [] -> []
  | x :: s ->
    let y = f x in
    if p y then y :: unfold p f s
    else unfold p f s

let bar p f xs =
  unfold p f xs
```

We get the code that an OCaml programmer would write by hand.

No intermediate data structure! Successful deforestation again.

What's the point?

Why is **stream fusion** preferable to **shortcut deforestation**?

Shortcut deforestation cannot express `foldl` in a nice way.

Exercise: Implement `foldl` on streams, then on lists.

Exercise: Find out how `foldl (+) 0 (append xs ys)` is optimized.
You should reach a sequence of two loops – no memory allocation.

The way of the future?

Do not let the compiler's **heuristics** decide
which reductions and simplifications should take place at compile time.

Instead, give explicit **staging** annotations to distinguish
pipeline-construction-time computation and **pipeline-runtime** computation!

Relying on a general-purpose compiler for library optimization is slippery. [...] A compiler offers no guarantee that optimization will be successfully applied. [...] An innocuous change to a program [can] make it much slower.

Kiselyov, Biboudis, Palladinos, Smaragdakis,
Stream fusion, to completeness, 2017.

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② Inlining and simplification

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④ Deforestation

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⑤ Conclusion

Program derivation

Equational reasoning can be used not just by compilers, but also by **programmers**, by hand.

Starting from a simple, inefficient program, **derive** efficient code via a series of rewriting steps.

See my **blog post** on a derivation of Knuth-Morris-Pratt.

Supercompilation can do this, too!

Secher and Sørensen, **On Perfect Supercompilation**, 1999.

A few things to remember

- Equational reasoning can be a powerful means of transforming or deriving programs.
- λ -calculus-based (intermediate) languages allow expressing a wide range of program transformations and optimizations.
- Side effects (non-termination, mutable state...) complicate matters.