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MPRI 2.4

Algebraic data types, existential types, and GADTs

François Pottier



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Many data types can be built out of **sums** and **products** and a form of **recursion** at the level of types.

Binary sum $+$ and product \times , and their **neutral elements** 0 and 1, suffice.

- The **unit** type is 1.
- The **empty** type is 0.
- The **Boolean** type is $1 + 1$.
- The type \mathbb{N} of the natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
- The type $\mathbb{L}(X)$ of lists of elements of type X must satisfy

$$\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$$

Three technical approaches to data types

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There are three main approaches to extending System F with data types:

- consider 0, 1, +, \times , and recursive types $\mu X.T$ as primitive concepts and encode all data types in terms of these concepts;
- consider algebraic data types as primitive and view sums, products, naturals, lists, etc., as instances of this general concept;
- introduce no new primitive concept and remark that inductive types can be encoded in System F .

In practice, the second approach is the most natural and user-friendly.

All three approaches, and their connections, are worth understanding.

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It is easy to add pairs and projections to the (call-by-value) λ -calculus.

$$\begin{aligned} t &::= \dots | (t, t) | \pi_i t && \text{where } i \in \{1, 2\} \\ v &::= \dots | (v, v) \\ E &::= \dots | (E, t) | (v, E) | \pi_i E \end{aligned}$$

One new reduction rule is needed: $\pi_i (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed:

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One new reduction rule is needed: $\pi_i (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed: $T ::= \dots | T \times T$.

Two new typing rules are needed:

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$$\begin{array}{ll} t ::= \dots | (t, t) | \pi_i t & \text{where } i \in \{1, 2\} \\ v ::= \dots | (v, v) \\ E ::= \dots | (E, t) | (v, E) | \pi_i E \end{array}$$

One new reduction rule is needed: $\pi_i (v_1, v_2) \longrightarrow v_i$.

A new type constructor is needed: $T ::= \dots | T \times T$.

Two new typing rules are needed:

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \qquad \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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The **unit** type 1 can be viewed as a product type of arity 0.

It has an **introduction** form but no **elimination** form.

$$\begin{aligned} t &::= \dots | () \\ v &::= \dots | () \\ &\quad - \text{no new evaluation context} \end{aligned}$$

No new reduction rule is needed.

A new type constructor is needed: $T ::= \dots | 1$.

One new typing rule is needed:

$$\Gamma \vdash () : 1$$

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Let us add **injections** and a **case analysis** to (call-by-value) λ -calculus.

$$\begin{aligned} t &::= \dots \mid \text{inj}_i t \mid \text{case } t \text{ of } t_1 \parallel t_2 && \text{where } i \in \{1, 2\} \\ v &::= \dots \mid \text{inj}_i v \\ E &::= \dots \mid \text{inj}_i E \mid \text{case } E \text{ of } t_1 \parallel t_2 \end{aligned}$$

One new reduction rule is needed: $\text{case } \text{inj}_i v \text{ of } t_1 \parallel t_2 \longrightarrow t_i v$.

In a **case** construct, the branches t_1 and t_2 should be functions.

A new type constructor is needed:

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Let us add **injections** and a **case analysis** to (call-by-value) λ -calculus.

$$\begin{aligned} t &::= \dots \mid \text{inj}_i t \mid \text{case } t \text{ of } t_1 \parallel t_2 && \text{where } i \in \{1, 2\} \\ v &::= \dots \mid \text{inj}_i v \\ E &::= \dots \mid \text{inj}_i E \mid \text{case } E \text{ of } t_1 \parallel t_2 \end{aligned}$$

One new reduction rule is needed: $\text{case } \text{inj}_i v \text{ of } t_1 \parallel t_2 \longrightarrow t_i v$.

In a **case** construct, the branches t_1 and t_2 should be functions.

A new type constructor is needed: $T ::= \dots \mid T + T$.

Two new typing rules are needed:

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Let us add **injections** and a **case analysis** to (call-by-value) λ -calculus.

$$\begin{array}{lll} t & ::= & \dots \mid inj_i t \mid \text{case } t \text{ of } t_1 \parallel t_2 & \text{where } i \in \{1, 2\} \\ v & ::= & \dots \mid inj_i v \\ E & ::= & \dots \mid inj_i E \mid \text{case } E \text{ of } t_1 \parallel t_2 \end{array}$$

One new reduction rule is needed: $\text{case } inj_i v \text{ of } t_1 \parallel t_2 \longrightarrow t_i v$.

In a **case** construct, the branches t_1 and t_2 should be functions.

A new type constructor is needed: $T ::= \dots \mid T + T$.

Two new typing rules are needed:

$$\frac{\Gamma \vdash t : T_i}{\Gamma \vdash inj_i t : T_1 + T_2} \qquad \frac{\Gamma \vdash t : T_1 + T_2 \quad \Gamma \vdash t_1 : T_1 \rightarrow T' \quad \Gamma \vdash t_2 : T_2 \rightarrow T'}{\Gamma \vdash \text{case } t \text{ of } t_1 \parallel t_2 : T'}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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The **empty** type can be viewed as a sum type of arity 0.

It has an **elimination** form but no **introduction** form.

$$\begin{aligned} t &::= \dots \mid \text{absurd } t \\ &\quad - \text{no new value} \\ E &::= \dots \mid \text{absurd } E \end{aligned}$$

No new reduction rule is needed. *absurd v* is stuck.

A new type constructor is needed: $T ::= \dots \mid 0$.

One new typing rule is needed:

$$\frac{\Gamma \vdash t : 0}{\Gamma \vdash \text{absurd } t : T'}$$

Exercise: extend the proof of Progress.

Approaches to recursive types

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Recall what was said earlier about **recursive types**:

- Natural numbers must satisfy $\mathbb{N} \simeq 1 + \mathbb{N}$.
- Lists must satisfy $\mathbb{L}(X) \simeq 1 + X \times \mathbb{L}(X)$.

One approach is to extend the type system with **recursive types** $\mu X.T$.
The type $\mu X.T$ and its unfolding $T[\mu X.T/X]$ must then be considered
either **equal** or **related via explicit coercions**.

One can then define \mathbb{N} as $\mu X. 1 + X$ and $\mathbb{L}(X)$ as $\mu Y. 1 + X \times Y$.

A more pleasant approach is to just view \mathbb{N} and $\mathbb{L}(X)$ as **primitive types**.
This is the topic of the next slides, and leads to **algebraic data types**.

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Consider λ -calculus with injections and case analysis.

Let us use $inj_1()$ to encode zero and $inj_2 v$ to encode the successor of v .

Introduce a new type constructor: $T ::= \dots | \mathbb{N}$.

Give three new typing rules:

$$\frac{\Gamma \vdash t : 1}{\Gamma \vdash inj_1 t : \mathbb{N}} \quad \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash inj_2 t : \mathbb{N}} \quad \frac{\Gamma \vdash t_1 : 1 \rightarrow T' \quad \Gamma \vdash t_2 : \mathbb{N} \rightarrow T'}{\Gamma \vdash \text{case } t \text{ of } t_1 \parallel t_2 : T'}$$

These are exactly the typing rules proposed earlier for binary sums where we have replaced $T_1 + T_2$ with \mathbb{N} , T_1 with 1, and T_2 with \mathbb{N} .

The types \mathbb{N} and $1 + \mathbb{N}$ are not equal, but they are isomorphic: one can write $in : 1 + \mathbb{N} \rightarrow \mathbb{N}$ and $out : \mathbb{N} \rightarrow 1 + \mathbb{N}$ such that $in \cdot out$ and $out \cdot in$ are $\beta\eta$ -equal to the identity.

$\mathbb{L}(X)$ as a primitive type

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Consider again λ -calculus with [injections](#) and [case analysis](#).

Let us use $inj_1()$ to encode $[]$ and $inj_2(v_1, v_2)$ to encode $v_1 :: v_2$.

Introduce a new type constructor: $T ::= \dots | \mathbb{L}(T)$.

Give three new typing rules:

$$\frac{\Gamma \vdash t : 1}{\Gamma \vdash inj_1 t : \mathbb{L}(T)} \qquad \frac{\Gamma \vdash t : T \times \mathbb{L}(T)}{\Gamma \vdash inj_2 t : \mathbb{L}(T)}$$

$$\frac{\begin{array}{c} \Gamma \vdash t : \mathbb{L}(T) \\ \Gamma \vdash t_1 : 1 \rightarrow T' \\ \Gamma \vdash t_2 : T \times \mathbb{L}(T) \rightarrow T' \end{array}}{\Gamma \vdash \text{case } t \text{ of } t_1 \parallel t_2 : T'}$$

These are again [exactly the typing rules of binary sums](#) where we have replaced $T_1 + T_2$ with $\mathbb{L}(X)$, T_1 with 1 , and T_2 with $X \times \mathbb{L}(X)$.

Again, the types $\mathbb{L}(X)$ and $1 + X \times \mathbb{L}(X)$ are isomorphic.

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Instead of offering a fixed set of primitive types such as \mathbb{N} and $\mathbb{L}(X)$, better let the user define whatever custom types they need using sums and products (of arbitrary arity) and recursion.

This idea gives rise to algebraic data types.

```
type nat = Zero | Succ of nat
type 'a list = Nil | Cons of 'a * 'a list
type 'a tree = Leaf | Node of 'a tree * 'a * 'a tree
```

Named types, named data constructors, and pattern matching make algebraic data types extremely pleasant and safe to use.

Burstall, MacQueen, Sannella,
HOPE: An experimental applicative language, 1980.

Products and sums as algebraic data types

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Sums and products can be viewed as algebraic data types.

```
type ('a, 'b) sum = Left of 'a | Right of 'b
type void = | (* zero constructors *)
type ('a, 'b) pair = Pair of 'a * 'b
type unit = Unit
```

Deconstructing the type void works as expected:

```
let absurd (type a) (x : void) : a =
  match x with _ -> . (* zero branches *)
```

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Encoding Booleans

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The Boolean type $\mathbb{B} \simeq 1 + 1$ can be declared as an algebraic data type:

```
type bool = False | True
```

However, Booleans can also be [encoded](#) in pure λ -calculus.

A Boolean value is an “object with a *case* method”.

It can choose between two branches:

$$\begin{aligned}\mathbb{B} &\triangleq \forall X. (1 \rightarrow X) \rightarrow (1 \rightarrow X) \rightarrow X \\ \textit{False} &\triangleq \lambda x_1. \lambda x_2. x_1 () \\ \textit{True} &\triangleq \lambda x_1. \lambda x_2. x_2 () \\ \textit{case } t \textit{ of } t_1 \parallel t_2 &\triangleq t\ t_1\ t_2\end{aligned}$$

This is a [Scott encoding](#), and also a [Church encoding](#).

Exercise: reconstruct the omitted type abstractions and applications.

Encoding sums

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More generally, the binary sum type $T_1 + T_2$ can be encoded as follows:

$$\begin{aligned} T_1 + T_2 &\triangleq \forall X. (T_1 \rightarrow X) \rightarrow (T_2 \rightarrow X) \rightarrow X \\ inj_1\ x &\triangleq \lambda x_1. \lambda x_2. x_1\ x \\ inj_2\ x &\triangleq \lambda x_1. \lambda x_2. x_2\ x \\ \text{case } t \text{ of } t_1 \parallel t_2 &\triangleq t\ t_1\ t_2 \end{aligned}$$

The zero-ary sum type 0 can be encoded, too!

$$\begin{aligned} 0 &\triangleq \forall X. X \\ absurd\ t &\triangleq t \end{aligned}$$

Clearly this works for any number of branches.

Encoding products

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The binary product type $T_1 \times T_2$ can be encoded as follows:

$$\begin{aligned}T_1 \times T_2 &\triangleq \forall X. (T_1 \rightarrow T_2 \rightarrow X) \rightarrow X \\(x_1, x_2) &\triangleq \lambda k. k\ x_1\ x_2 \\ \pi_1\ t &\triangleq t\ (\lambda x_1. \lambda x_2. x_1) \\ \pi_2\ t &\triangleq t\ (\lambda x_1. \lambda x_2. x_2)\end{aligned}$$

The zero-ary product type 1 can be encoded, too!

$$\begin{aligned}1 &\triangleq \forall X. X \rightarrow X \\ () &\triangleq \lambda x. x\end{aligned}$$

Clearly this works for any number of tuple components.

Encoding natural integers

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Can we encode the recursive type $\mathbb{N} \simeq 1 + \mathbb{N}$ in the same way, à la Scott?

$$\mathbb{N} \triangleq \forall X. (1 \rightarrow X) \rightarrow (\mathbb{N} \rightarrow X) \rightarrow X$$

This doesn't work in System F, which doesn't have recursive types.

Here, the Scott and Church encodings differ.

The Church encoding views a number as “an object with a *fold* method”.

$$\begin{aligned}\mathbb{N} &\triangleq \forall X. X \rightarrow (X \rightarrow X) \rightarrow X \\ \textit{Zero} &\triangleq \lambda z. \lambda s. z \\ \textit{Succ } x &\triangleq \lambda z. \lambda s. s (x z s)\end{aligned}$$

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The Church encoding views a list as “an object with a *fold* method”.

$$\begin{aligned}\mathbb{L}(Y) &\triangleq \forall X. X \rightarrow (Y \rightarrow X \rightarrow X) \rightarrow X \\ [] &\triangleq \lambda n. \lambda c. n \\ x :: xs &\triangleq \lambda n. \lambda c. c x (xs\ n\ c)\end{aligned}$$

The Church encoding works for all [inductive types](#).

Girard, Taylor, Lafont, [Proofs and types](#), 1990, §11.3–11.5.

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Complex numbers are an abstract concept.

Outside of their implementation, how they are represented should be irrelevant, and one should not depend on implementation details.

In one section, Professor Descartes announced that a complex number was an ordered pair of reals [...].

In the other section, Professor Bessel announced that a complex number was an ordered pair of reals, the first of which was nonnegative [...].

An unfortunate mistake [...] caused the two sections to be interchanged.

Reynolds, *Types, Abstraction and Parametric Polymorphism*, 1983.

Complex numbers as an abstract type

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In OCaml, one might implement complex numbers as an [abstract type](#):

```
module Complex : sig
  type t
  val zero: t
  val one: t
  val add: t -> t -> t
  val mul: t -> t -> t
  val (=): t -> t -> bool
  (* etc. *)
end
```

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In System F , this idea can be made precise via an existential type:

$$\text{Complex} : \exists X. \left\{ \begin{array}{l} \text{zero} : X \\ \text{add} : X \rightarrow X \rightarrow X \\ \text{mul} : X \rightarrow X \rightarrow X \\ \text{eq} : X \rightarrow X \rightarrow \text{bool} \\ \text{etc.} \end{array} \right\}$$

Mitchell and Plotkin, [Abstract types have existential type](#), 1988.

Rossberg, Russo, Dreyer, [F-ing Modules](#), 2014.

Streams as an existential type

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Imagine we wish to define an abstract type of streams.

A stream is a producer of a sequence of elements,
out of which a consumer can pull elements on demand.

It is an “object” with a single method, *next*.

- a stream has a certain current internal state.
- *next* returns either nothing or a pair of an element and a new state.

A stream is analogous to a Java iterator, except it is not mutable.
Its current state is explicit.

$$\mathbb{S}(X) \triangleq \exists S. \underbrace{(S \rightarrow 1 + X \times S)}_{\textit{next}} \times \underbrace{S}_{\textit{cur}}$$

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$('a, 's)$ step corresponds to $1 + X \times S$:

```
type ('a, 's) step =
| Done (* the stream is exhausted *)
| Yield of 'a * 's (* here is an element and a new state *)
```

OCaml views existential types as a special case of [algebraic data types](#):

```
type 'a stream =
| Stream:
    (* The [next] method: *) ('s -> ('a, 's) step) *
    (* The current state: *) 's
    (* together form a stream: *) -> 'a stream
```

The data constructor **Stream** has [universal type](#): it is polymorphic in ' s '.

The producer chooses the type of the internal state;
the consumer must treat this type as abstract.

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This conversion function is a nonrecursive [producer](#):

```
let stream (xs : 'a list) : 'a stream =
  let next xs =
    match xs with
    | [] -> Done
    | x :: xs -> Yield (x, xs)
  in
  Stream (next, xs)           (* packing an existential type *)
```

On the last line, what is the concrete type of states?

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This conversion function is a nonrecursive [producer](#):

```
let stream (xs : 'a list) : 'a stream =
  let next xs =
    match xs with
    | [] -> Done
    | x :: xs -> Yield (x, xs)
  in
  Stream (next, xs)           (* packing an existential type *)
```

On the last line, what is the concrete type of states?

It is '[a list](#)'.

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This conversion function is a recursive consumer:

```
let unstream (Stream (next, s) : 'a stream) : 'a list =
  let rec unfold s =
    match next s with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
  in
  unfold s
```

The first line uses pattern matching to unpack an existential type.

What is the type of unfold?

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This conversion function is a recursive consumer:

```
let unstream (Stream (next, s) : 'a stream) : 'a list =
  let rec unfold s =
    match next s with
    | Done          -> []
    | Yield (x, s) -> x :: unfold s
  in
  unfold s
```

The first line uses pattern matching to unpack an existential type.

What is the type of unfold?

It is $s \rightarrow 'a \text{ list}$

where s is an abstract type introduced by unpacking at line 1.

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How would you implement a singleton stream?

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How would you implement a singleton stream?

```
let return (x : 'a) : 'a stream =
  let next s =
    if s then Yield (x, false) else Done
  in
  Stream (next, true)           (* packing an existential type *)
```

On the last line, the concrete type of states is `bool`:
either we have already yielded an element, or we have not.

Exercise: Write interval of type `int -> int -> int` stream.

Exercise: Write append of type `'a stream -> 'a stream -> 'a stream`.

An example consumer-and-producer

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The `map` function on streams is also non-recursive:

```
let map (f : 'a -> 'b) (xs : 'a stream) : 'b stream =
  let Stream (next, s) = xs in                                (* unpacking *)
    let next s =
      match next s with
      | Done          -> Done
      | Yield (x, s) -> Yield (f x, s)
    in
    Stream (next, s)                                         (* packing *)
```

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This encoding of streams is used in practice.

In addition to `Done` and `Yield`, a third constructor `Skip` can be used,
meaning “please ask again”

A consumer must ask, ask, ask until a non-`Skip` result is produced.

This allows most stream producers to be `nonrecursive` functions.

This makes optimization easier.

Coutts, Leshchinskiy, Stewart, `Stream fusion:
from lists to streams to nothing at all`, 2007.

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The syntax of types is extended with existential types:

$$T ::= \dots \mid \exists X. T$$

The syntax of terms is extended with introduction and elimination forms:

$$t ::= \dots \mid \text{pack } T, t \text{ as } \exists X. T \mid \text{let } X, x = \text{unpack } t \text{ in } t$$

$$v ::= \dots \mid \text{pack } T, v \text{ as } \exists X. T$$

$$E ::= \dots \mid \text{pack } T, E \text{ as } \exists X. T \mid \text{let } X, x = \text{unpack } E \text{ in } t$$

A new reduction rule is introduced:

$$\text{let } X, x = \text{unpack} (\text{pack } T', v \text{ as } \exists X. T) \text{ in } t \longrightarrow t[v/x][T'/X]$$

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Two new typing rules are introduced:

$\exists\text{-INTRO}$

$\Gamma \vdash t :$

$$\frac{}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T}$$

$\exists\text{-ELIM}$

$\Gamma \vdash t_1 :$

$$\frac{}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$

For reference, recall the typing rules for universal types:

$\forall\text{-INTRO}$

$$\frac{\Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X.t : \forall X.T}$$

$\forall\text{-ELIM}$

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t T' : T[T'/X]}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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Two new typing rules are introduced:

$\exists\text{-INTRO}$

$$\frac{}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T} \quad \Gamma \vdash t : T[T'/X]$$

$\exists\text{-ELIM}$

$$\frac{\Gamma \vdash t_1 : \exists X.x = \text{unpack } t_1 \text{ in } t_2 : T_2}{\Gamma \vdash t_2 : T_2}$$

For reference, recall the typing rules for universal types:

$\forall\text{-INTRO}$

$$\frac{\Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X.t : \forall X.T}$$

$\forall\text{-ELIM}$

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t T' : T[T'/X]}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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Two new typing rules are introduced:

$\exists\text{-INTRO}$

$$\frac{}{\Gamma \vdash t : T[T'/X]} \quad \Gamma \vdash \text{pack } T', t \text{ as } \exists X. T : \exists X. T$$

$\exists\text{-ELIM}$

$$\frac{\Gamma \vdash t_1 : \exists X. T}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$

For reference, recall the typing rules for universal types:

$\forall\text{-INTRO}$

$$\frac{\Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X. t : \forall X. T}$$

$\forall\text{-ELIM}$

$$\frac{\Gamma \vdash t : \forall X. T}{\Gamma \vdash t T' : T[T'/X]}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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System F with existential types

Two new typing rules are introduced:

 $\exists\text{-INTRO}$

$$\frac{\Gamma \vdash t : T[T'/X]}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T}$$

 $\exists\text{-ELIM}$

$$\frac{\begin{array}{c} \Gamma \vdash t_1 : \exists X.T \\ \Gamma; X; x : T \vdash t_2 : T_2 \end{array}}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$

For reference, recall the typing rules for universal types:

 $\forall\text{-INTRO}$

$$\frac{\Gamma; X \vdash t : T \quad X \# \Gamma}{\Gamma \vdash \Lambda X.t : \forall X.T}$$

 $\forall\text{-ELIM}$

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t T' : T[T'/X]}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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System F with existential types

Two new typing rules are introduced:

 $\exists\text{-INTRO}$

$$\Gamma \vdash t : T[T'/X]$$

$$\frac{}{\Gamma \vdash \text{pack } T', t \text{ as } \exists X.T : \exists X.T}$$

 $\exists\text{-ELIM}$

$$\Gamma \vdash t_1 : \exists X.T \quad X \# \Gamma, T_2$$

$$\Gamma; X; x : T \vdash t_2 : T_2$$

$$\frac{\Gamma \vdash t_1 : \exists X.T \quad X \# \Gamma, T_2}{\Gamma \vdash \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2}$$

For reference, recall the typing rules for universal types:

 $\forall\text{-INTRO}$

$$\Gamma; X \vdash t : T \quad X \# \Gamma$$

$$\frac{}{\Gamma \vdash \Lambda X.t : \forall X.T}$$

 $\forall\text{-ELIM}$

$$\Gamma \vdash t : \forall X.T$$

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t : T[T'/X]}$$

Exercise: extend the proofs of Subject Reduction and Progress.

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Universal/existential duality

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Metatheory

When a value has universal type $\forall X.T$,
the **producer** of this value must treat X as abstract
and the **consumer** can choose a type T' with which to instantiate X .

When a value has existential type $\exists X.T$,
the **producer** chooses a type T' with which to instantiate X
but the **consumer** must treat X as abstract.

When a value has existential type, its **consumer** must be polymorphic.

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Existential types can in fact be **encoded** in terms of universal types:

$$\begin{aligned}\exists X.T &\triangleq \forall Y. (\forall X. T \rightarrow Y) \rightarrow Y \\ \text{pack } T', v \text{ as } \exists X.T &\triangleq\end{aligned}$$

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Metatheory

Existential types can in fact be **encoded** in terms of universal types:

$$\begin{aligned}\exists X. T &\triangleq \forall Y. (\forall X. T \rightarrow Y) \rightarrow Y \\ \text{pack } T', v \text{ as } \exists X. T &\triangleq \Lambda Y. \lambda k : (\forall X. T \rightarrow Y). k \ T' \ v \\ \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2 &\triangleq\end{aligned}$$

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Metatheory

Existential types can in fact be **encoded** in terms of universal types:

$$\begin{aligned}\exists X.T &\triangleq \forall Y. (\forall X. T \rightarrow Y) \rightarrow Y \\ \text{pack } T', v \text{ as } \exists X.T &\triangleq \Lambda Y. \lambda k : (\forall X. T \rightarrow Y). k\ T'\ v \\ \text{let } X, x = \text{unpack } t_1 \text{ in } t_2 : T_2 &\triangleq t_1\ T_2\ (\Lambda X. \lambda x : T \rightarrow T_2. t_2)\end{aligned}$$

This encoding validates the logical implication $\exists X.T \rightarrow \neg\forall X.\neg T$
where $\neg T$ is defined as $T \rightarrow 0$.

Exercise: check that this encoding validates the reduction rule
and the typing rules proposed earlier for primitive existential types.

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Consider a tiny language of expressions $t ::= k \mid (t, t) \mid \pi_i t :$

```
type expr =  
| EInt of int  
| EPair of expr * expr  
| EFst of expr  
| ESnd of expr
```

Expressions include integer constants, pairs, and projections.

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A straightforward interpreter for this language uses a type of all values:

```
type value =  
| VInt of int  
| VPair of value * value
```

This is an algebraic data type. Thus every value carries a tag.

Runtime tests

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These tags are used in **runtime tests** that can cause **runtime errors**.

```
let as_pair (v : value) : value * value =
  match v with
  | VPair (v1, v2) ->
    v1, v2
  | _ ->
    assert false (* runtime error! *)
```

An untyped interpreter

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Here, interpreting a pair projection operation involves a runtime test.

```
let rec eval (e : expr) : value =
  match e with
  | EInt x ->
    VInt x
  | EPair (e1, e2) ->
    VPair (eval e1, eval e2)
  | EFst e ->
    fst (as_pair (eval e))
  | ESnd e ->
    snd (as_pair (eval e))
```

This is **necessary** because this interpreter accepts untyped expressions.

Typed expressions

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Let us impose a simple type discipline on expressions.

```
type _ expr =  

| EInt : int -> int expr  

| EPair : 'a expr * 'b expr -> ('a * 'b) expr  

| EFst : ('a * 'b) expr -> 'a expr  

| ESnd : ('a * 'b) expr -> 'b expr
```

This type definition encodes the following type discipline:

$$\frac{\Gamma \vdash k : \text{int}}{\Gamma \vdash k : \text{int}} \quad \frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash (t_1, t_2) : T_1 \times T_2} \quad \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i}$$

A meta-level AST of type '`a` expr
represents an object-level expression of type '`a`.

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Let us similarly impose a type discipline on values:

```
type _ value =
| VInt :           int ->      int value
| VPair : 'a value * 'b value -> ('a * 'b) value
```

Values are still tagged (for now), but runtime tests become unnecessary...

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Look Ma, no runtime test!

Only one branch is now necessary. A second branch would be **dead**.

```
let as_pair : type a b . (a * b) value -> a value * b value
= function
| VPair (v1, v2) ->
  v1, v2
(* In this branch, we would learn [a * b = int], *)
(* which is contradictory. *)
(* | _ -> . *)
```

In OCaml, destructing a GADT requires a type annotation in this style.

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Evaluating an expression of type T yields a value of type T .

```
let rec eval : type a . a expr -> a value
= function
| EInt x ->
    (* We learn [a = int] so returning [VInt_] is OK. *)
    VInt x
| EPair (e1, e2) ->
    (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
    (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
    VPair (eval e1, eval e2)
| EFst e ->
    fst (as_pair (eval e))
| ESnd e ->
    snd (as_pair (eval e))
```

The type of the interpreter reflects the subject reduction property.
Type-checking it amounts to checking the proof of subject reduction!

A typed, tagless interpreter

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Metatheory

Evaluating an expression of type T yields a [meta-level value](#) of type T .

```
let rec eval : type a . a expr -> a
= function
| EInt x ->
  (* We learn [a = int] so returning an integer is OK. *)
  x           (* no tagging! *)
| EPair (e1, e2) ->
  (* For some types [a1] and [a2], we learn [a = a1 * a2] *)
  (* and we can assume [e1 : a1 expr] and [e2 : a2 expr]. *)
  (eval e1, eval e2) (* no tagging! *)
| EFst e ->
  fst (eval e)      (* no untagging! *)
| ESnd e ->
  snd (eval e)      (* no untagging! *)
```

The type of the interpreter reflects the [subject reduction](#) property.
Type-checking it amounts to [checking the proof](#) of subject reduction!

Going further

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Our tiny expressions are **closed**: the typing judgement is $\vdash t : T$.

When expressions involve variables, one needs a type ('g, 'a) expr whose definition encodes the typing judgement $\Gamma \vdash t : T$.

This is reasonably easy if variables are encoded as de Bruijn indices.

Bird, Paterson, **de Bruijn notation as a nested datatype**, 1999.

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A value of type `'a ty` is a runtime description of the type `'a`.

```
type 'a ty =
| TyInt : int ty
| TySum : 'a ty * 'b ty -> ('a, 'b) sum ty
| TyPair : 'a ty * 'b ty -> ('a * 'b) ty
```

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Although inspecting a type at runtime is impossible, as types are erased, inspecting a runtime description of a type is possible.

In other words, although the type $\forall X. X \rightarrow X$ has only one inhabitant, the type $\forall X. \text{Ty } X \rightarrow X \rightarrow X$ has more than one.

This let us write polymorphic, type-directed functions, an activity that is sometimes known as generic programming.

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Here is a polymorphic, type-directed conversion of a value to a string.

```
let rec show : type a . a ty -> a -> string =
  fun ty x =>
    match ty with
    | TyInt ->
        string_of_int x
    | TySum (ty1, ty2) ->
        begin match x with
        | Left x1 -> "left(" ^ show ty1 x1 ^ ")"
        | Right x2 -> "right(" ^ show ty2 x2 ^ ")"
        end
    | TyPair (ty1, ty2) ->
        let (x1, x2) = x in
        "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

In each branch, we learn something about the type of x.

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It is more concise and looks better to deconstruct both arguments at once.

```
let rec show : type a . a ty -> a -> string =
  fun ty x ->
    match ty, x with
    | TyInt, x ->
        string_of_int x
    | TySum (ty1, _), Left x1 ->
        "left(" ^ show ty1 x1 ^ ")"
    | TySum (_, ty2), Right x2 ->
        "right(" ^ show ty2 x2 ^ ")"
    | TyPair (ty1, ty2), (x1, x2) ->
        "(" ^ show ty1 x1 ^ ", " ^ show ty2 x2 ^ ")"
```

The OCaml type-checker reads patterns from left to right
so deconstructing (ty, x) works but deconstructing (x, ty) does not.

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Here is a polymorphic, type-directed equality test.

```
let rec equal : type a . a ty -> a -> a -> bool =
  fun ty x y =>
    match ty, x, y with
    | TyInt, x, y ->
        Int.equal x y
    | TySum (ty1, _), Left x1, Left y1 ->
        equal ty1 x1 y1
    | TySum (_, ty2), Right x2, Right y2 ->
        equal ty2 x2 y2
    | TySum _, Left _, Right _ -
    | TySum _, Right _, Left _ ->
        false
    | TyPair (ty1, ty2), (x1, x2), (y1, y2) ->
        equal ty1 x1 y1 && equal ty2 x2 y2
```

Connections between GADTs and type classes

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Eq and **Show** are typical examples of **type classes** in Haskell.

Upcoming lecture on type classes (PED).

Hinze, Jeuring, Löh,

Comparing Approaches to Generic Programming in Haskell, 2006.

Connections between GADTs and type classes

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A closed set of type class instances can be compiled down to GADTs.

Pottier and Gauthier,
Polymorphic typed defunctionalization and concretization, 2006.

However a GADT describes a closed universe of structural types whereas type classes are open-ended and apply to user-defined, nominal types.

The Holy Grail is to propose a language where a type of the representations of all types (including itself!) can be defined.

Chapman, Dagand, McBride, Morris,
The Gentle Art of Levitation, 2010.

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We have a type '`a` expr' of well-typed expressions
and a type '`a` ty' of runtime type descriptions.

Can we express a simple type inferencer that accepts an untyped
expression and either fails or returns a typed expression?

```
exception IllTyped
let rec infer : Raw.expr -> ????
= function
| Raw.EInt i ->
  (TyInt, EInt i)
| ...
```

What should its result type be?

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A type inferencer

We need an existential type $\exists X. \text{Ty } X \times \text{Expr } X$.

```
type typed_expr =  
| Pack : 'a ty * 'a expr -> typed_expr
```

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A type inferencer

We can now write the type inferencer:

```
let rec infer : Raw.expr -> typed_expr =
  function
  | Raw.EInt i ->
    Pack (TyInt, EInt i)
  | Raw.EFst e ->
    let Pack (ty, e) = infer e in
    begin match ty with
    | TyPair (ty1, ty2) -> Pack (ty1, EFst e)
    | _                      -> raise IllTyped
    end
```

Exercise: write the two missing cases.

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
    if inferred = expected then  
      e  
    else  
      raise IllTyped
```

But **this code is not well-typed. Why?**

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A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
    if inferred = expected then  
      e  
    else  
      raise IllTyped
```

But **this code is not well-typed**. Why?

expected has type a ty.

inferred has type b ty

where b is an unknown type introduced by deconstructing **Pack**.

A type-checker

Can we **check** whether an expression has a certain expected type?

We would like to write something like this:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
    if inferred = expected then  
      e  
    else  
      raise IllTyped
```

But **this code is not well-typed**. Why?

expected has type a ty.

inferred has type b ty

where b is an unknown type introduced by deconstructing **Pack**.

They **cannot be compared** using homogeneous equality = .

Even if they could, e has type b expr

whereas a result of type a expr is required.

The equality GADT

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The solution involves the type equality GADT.

```
type (_ , _) eq =  
| Equal : ('a , 'a) eq
```

The type ('a , 'b) eq has at most one inhabitant.

If it has one then this inhabitant must be Equal
and the types 'a and 'b must be the same.

A heterogenous type equality test

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This lets us express a heterogenous type equality test:

```
let rec equal : type a b . a ty -> b ty -> (a, b) eq =
  fun ty1 ty2 ->
    match ty1, ty2 with
    | TyInt, TyInt ->
        Equal
    | TyPair (ty1a, ty1b), TyPair (ty2a, ty2b) ->
        let Equal = equal ty1a ty2a in
        let Equal = equal ty1b ty2b in
        Equal
    | _, _ ->
        raise IllTyped
```

When `equal ty1 ty2` succeeds, we learn that the runtime type descriptions `ty1` and `ty2` describe the same static type.

Exercise: write the missing case.

A type-checker

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We can now write the type-checker:

```
let check (type a) (e : Raw.expr) (expected : a ty) : a expr =  
  let Pack (inferred, e) = infer e in  
    let Equal = equal inferred expected in  
      e
```

Exercise: make sure that you understand why this code is well-typed.

Putting the pieces together

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Given an arbitrary untyped expression in our tiny language,
we can now **infer** its type, **evaluate** it, and **show** its value,
whatever its type may be.

```
let () =
  let e = Raw.(EPair (EInt 42, EInt 0)) in
  let Pack (ty, e) = infer e in
  let v = eval e in
  Printf.printf "%s\n%" (show ty v)
```

The output in the REPL is:

```
(42, 0)
```

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printf takes a “format string” followed with a number of arguments:

```
# open Printf;;
# printf "%d * %s = %d\n" 2 "12" 24;;
2 * 12 = 24
- : unit = ()
```

The number and type of these arguments depends on the format string.

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A format string is actually **not** a string: it is a **data structure**.

```
# open CamlinternalFormatBasics;;
# let desc : _ format6 = "%d * %s = %d\n";;
val desc :
  (int -> string -> int -> 'a, 'b, 'c, 'd, 'd, 'a) format6 =
Format
  (Int (Int_d, No_padding, No_precision,
        String_literal (" * ",
                         String (No_padding,
                                  String_literal (" = ",
                                                 Int (Int_d, No_padding, No_precision,
                                                       Char_literal ('\n', End_of_format))))),
        "%d * %s = %d\n")
```

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This data structure has the shape of a list:

```
Int (Int_d, No_padding, No_precision,  
String_literal (" * ",  
String (No_padding,  
String_literal (" = ",  
Int (Int_d, No_padding, No_precision,  
Char_literal ('\n',  
End_of_format))))))
```

End_of_format is “nil”; the other constructors are “cons” constructors.

Int and **String** correspond to “holes” %d and %s.

String_literal and **Char_literal** correspond to literal pieces of string.

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Can we define our own algebraic data type of formats, or [descriptors](#)?

```
type desc =
| Nil
| Lit of string * desc
| Int of desc
```

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Or, in this alternative syntax:

```
type desc =
| Nil : desc
| Lit : string * desc -> desc
| Int : desc -> desc
```

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Or, in this alternative syntax:

```
type desc =
| Nil : desc
| Lit : string * desc -> desc
| Int : desc -> desc
```

Now, please define `fprintf` so that `fprintf emit desc <args>`

- emits output via the function `emit : string -> unit`,
- obeys `desc`,
- expects arguments `<args>` whose number and type satisfy `desc`.

`fprintf` should have type `(string -> unit) -> desc -> ??? -> unit`.

Expressing the type of `fprintf`

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Metatheory

The type `desc -> ??? -> unit` does not make sense.

The number of and type of the arguments `???` depends on the descriptor.

We seem to need a dependent type `(d: desc) -> shape d`

- where shape would be a function of descriptors to types,
- but OCaml does not have that.

Expressing the type of `fprintf`

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Metatheory

The type `desc -> ??? -> unit` does not make sense.

The number of and type of the arguments `???` depends on the descriptor.

We seem to need a dependent type `(d: desc) -> shape d`

- where `shape` would be a function of descriptors to types,
- but OCaml does not have that.

Instead, let's use a plain function type `'shape desc -> 'shape`

- where the definition of `'shape desc` as a GADT encodes the correspondence between descriptors and shapes.

Descriptors form a typed language and `fprintf` is an interpreter for it!

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We want `fprintf : (string -> unit) -> 'a desc -> 'a.`

```
type desc =
| Nil : desc
| Lit : string * desc -> desc
| Int : desc -> desc
```

We must turn the type `desc` into a GADT.

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We want `fprintf : (string -> unit) -> 'a desc -> 'a.`

```
type _ desc =
| Nil      : ?? desc
| Lit      : string * ?? desc -> ?? desc
| Int      : ?? desc -> ?? desc
```

We parameterize the type `desc`.

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We want `fprintf : (string -> unit) -> 'a desc -> 'a.`

```
type _ desc =
| Nil : unit desc
| Lit : string * ?? desc -> ?? desc
| Int : ?? desc -> ?? desc
```

`Nil` requires no action; the corresponding shape is `unit`.

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We want `fprintf : (string -> unit) -> 'a desc -> 'a.`

```
type _ desc =
| Nil : unit desc
| Lit : string * 'a desc -> 'a desc
| Int : ?? desc -> ?? desc
```

`Lit` (`s`, `d`) requires printing `s` and interpreting `d`.

If `d` has shape `'a` then `Lit` (`s`, `d`) has shape `'a` as well.

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We want `fprintf : (string -> unit) -> 'a desc -> 'a.`

```
type _ desc =
| Nil :                                     unit desc
| Lit : string * 'a desc ->                 'a desc
| Int : 'a desc -> (int -> 'a) desc
```

`Int d` requires consuming an integer argument and interpreting `d`.

If `d` has shape `'a` then `Int d` has shape `int -> 'a`.

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We want `fprintf : (string -> unit) -> 'a desc -> 'a.`

```
type _ desc =
| Nil : unit desc
| Lit : string * 'a desc -> 'a desc
| Hole : ('data -> string) * 'a desc -> ('data -> 'a) desc
```

We change the hole of type `int` with a hole of arbitrary type `'data`.

All that is needed is a conversion function of type `'data -> string`.

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```
let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ???
    | Lit (s, desc) ->
      ???
    | Hole (to_string, desc) ->
      ???

in eval desc
```

Recall

```
| Nil   :                                     unit desc
```

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```
let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
        ??? (* We learn [a = unit]. *)
    | Lit (s, desc) ->
        ???
    | Hole (to_string, desc) ->
        ???

in eval desc
```

Recall

```
| Nil   :                                unit desc
```

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```
let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
        () (* We learn [a = unit]. *)
    | Lit (s, desc) ->
        ???
    | Hole (to_string, desc) ->
        ???

in eval desc
```

Recall

```
| Lit  :           string * 'a desc ->          'a desc
```

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```
let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
      ()
    | Lit (s, desc) ->
      ??? (* We learn no new type equality. *)
    | Hole (to_string, desc) ->
      ???
in eval desc
```

Recall

```
| Lit : string * 'a desc -> 'a desc
```

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```
let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
        ()
    | Lit (s, desc) ->
        emit s; eval desc
    | Hole (to_string, desc) ->
        ???  
  
in eval desc
```

Recall

```
| Hole : ('data -> string) * 'a desc -> ('data -> 'a) desc
```

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```
let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
        ()
    | Lit (s, desc) ->
        emit s; eval desc
    | Hole (to_string, desc) ->
        ??? (* We learn [a = data -> b] *)
        (* [to_string : data -> string; desc : b desc] *)
  in eval desc
```

Recall

```
| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc
```

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Metatheory

```
let fprintf (type a) emit (desc : a desc) : a =
  let rec eval : type a . a desc -> a =
    function
    | Nil ->
        ()
    | Lit (s, desc) ->
        emit s; eval desc
    | Hole (to_string, desc) ->
        fun x -> emit (to_string x); eval desc
        (* [x] has type [data]; [eval desc] has type [b] *)
  in eval desc
          (* and [data -> b] is [a] *)
```

Recall

```
| Hole : ('data -> string) * 'b desc -> ('data -> 'b) desc
```

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Voilà! From `fprintf`, we get `printf`.

```
let printf desc =
  let emit = print_string in
    fprintf emit desc
```

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Metatheory

To construct descriptors, some sugar is needed.

```
module Sugar = struct
    let nil = Nil
    let lit s desc = Lit (s, desc)
    let d desc = Hole (string_of_int, desc)
    let s desc = Hole (Fun.id, desc)
end
```

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Metatheory

To construct descriptors, some sugar is needed.

```
module Sugar = struct
    let nil = Nil
    let lit s desc = Lit (s, desc)
    let d desc = Hole (string_of_int, desc)          (* %d *)
    let s desc = Hole (Fun.id, desc)                 (* %s *)
end
```

For example,

```
let desc =
    d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
(* "%d * %s = %d\n" *)
```

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Metatheory

```
let desc = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf desc 2 "12" 24;;
2 * 12 = 24
```

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Can we implement sprintf, which returns a string?

```
let sprintf desc args =
  let b = Buffer.create 128 in
  let emit = Buffer.add_string b in
  fprintf emit desc args;
  Buffer.contents b
```

This is accepted but is **not** what we want.

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Metatheory

Can we implement sprintf, which returns a string?

```
let sprintf desc <arg ... arg> =
  let b = Buffer.create 128 in
  let emit = Buffer.add_string b in
  fprintf emit desc <arg ... arg>;
  Buffer.contents b
```

We want sprintf to accept a variable number of arguments, not just one.

In fact, we cannot write the type of sprintf.

It is like the type of fprintf but should end in `string` instead of `unit`.

A more general type of descriptors

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We must equip ourselves with a [more general type of descriptors](#).

```
type desc =
| Nil : unit desc
| Lit : string * 'a desc
| Hole : ('data -> string) * 'a desc
```

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Metatheory

We must equip ourselves with a more general type of descriptors.

```
type ('_, '_) desc =
| Nil : ('r, 'r) desc
| Lit : string * ('a, 'r) desc -> ('a, 'r) desc
| Hole : ('data -> string) * ('a, 'r) desc ->
          ('data -> 'a, 'r) desc
```

In the type ('a, 'r) desc,

- 'a is the **shape**, as before,
- 'r is the **eventual return type** of this shape.
 - it can be **unit** for `fprintf` and **string** for `sprintf`;
 - a descriptor can be polymorphic in 'r.

Implementing fprintf, again

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We can now give `fprintf` a more general type. We parameterize it with:

- `emit : string -> unit`
- `finished : unit -> r` — new
- `desc : (a, r) desc`

`fprintf emit finished desc` has type `a`.

`a` must in fact be a function type whose eventual return type is `r`.

`fprintf emit finished desc <args>` must eventually return
a value of type `r`, which it obtains by calling `finished()`.

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```
let sprintf (type a r) emit (finished : unit -> r)
    (desc : (a, r) desc) : a =
  let rec eval : type a . (a, r) desc -> a =
    function
    | Nil ->
        (* We have [a = r] so [finished()] has type [a]. *)
        finished()
    | Lit (s, desc) ->
        emit s; eval desc
    | Hole (to_string, desc) ->
        fun x -> emit (to_string x); eval desc
  in eval desc
```

It is worth pointing out that eval involves polymorphic recursion.

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We can now implement printf and sprintf, among other variations:

```
let printf desc =
  let emit = print_string
  and finished () = () in
  fprintf emit finished desc

let sprintf desc =
  let b = Buffer.create 128 in
  let emit = Buffer.add_string b
  and finished () = Buffer.contents b in
  fprintf emit finished desc
```

We get

```
val printf : ('a, unit) desc -> 'a
val sprintf : ('a, string) desc -> 'a
```

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```
let desc () = (* "%d * %s = %d\n" *)
  d @@ lit " * " @@ s @@ lit " = " @@ d @@ lit "\n" @@ nil
```

Try this in the OCaml REPL (read-eval-print-loop):

```
# let () = printf (desc()) 2 "12" 24;;
2 * 12 = 24
# let (s : string) = sprintf (desc()) 2 "12" 24;;
val s : string = "2 * 12 = 24\n"
```

Here, we make `desc` a (constant) function in order to work around the **value restriction**. See upcoming lecture on mutable state (GS).

Danvy et al.'s approach

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Danvy, Keller and Puech (2015) view formats as trees instead of lists.

```
type (_ , _ ) desc =  
| Lit : string -> ('a, 'a) desc  
| Hole : ('data -> string) -> ('data -> 'a, 'a) desc  
| Seq : ('a, 'b) desc * ('b, 'c) desc -> ('a, 'c) desc
```

The type ('a, 'r) desc has the same meaning as earlier.

Lit and **Hole** no longer play the role of list “cons” constructors.

Seq is a binary concatenation constructor, whose type says:

*If 'a is a multi-arrow type whose eventual return type is 'b and
if 'b is a multi-arrow type whose eventual return type is 'c then
'a is a multi-arrow type whose eventual return type is 'c.*

Danvy et al.'s approach

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Danvy et al. write kprintf in continuation-passing style:

```
let rec kprintf
: type a r . (a, r) desc -> (string -> r) -> a =
  fun desc finished ->
    match desc with
    | Lit s ->
      finished s
    | Hole to_string ->
      fun x -> finished (to_string x)
    | Seq (desc1, desc2) ->
      kprintf desc1 @@ fun s1 ->
      kprintf desc2 @@ fun s2 ->
      finished (s1 ^ s2)
```

Exercise (easy): define printf, sprintf, and fprintf using kprintf.

Exercise (harder): define fprintf directly.

Do not use string concatenation \wedge .

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System *F*+GADTs was defined by Xi, Chen and Chen (2003).

Xi, Chen, Chen,

Guarded Recursive Datatype Constructors, 2003.

Pottier and Gauthier,

Polymorphic typed defunctionalization and concretization, 2006.

System F +GADTs: the typing judgement

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Recall the typing judgement of System F :

$$\Gamma \vdash t : T$$

In System F +GADTs, must we change the shape of this judgement?

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Recall the typing judgement of System F :

$$\Gamma \vdash t : T$$

In System F +GADTs, must we change the shape of this judgement?

We must extend it with a conjunction of **equality hypotheses**.

$$\Gamma, C \vdash t : T$$

Equality constraints are given by $C, D ::= \text{True} \mid \text{False} \mid T = T \mid C \wedge C$.

System F +GADTs: type declarations

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We assume that a family of type constructors F is given.

- for simplicity, we assume they have arity 1.

The syntax of types includes applications of type constructors:

$$T := X \mid T \rightarrow T \mid T + T \mid T \times T \mid 0 \mid 1 \mid F\ T$$

We assume that a family of data constructors K is given.

- for simplicity, we assume they have arity 1.

We assume that each data constructor has a closed [type scheme](#):

$$K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F\ T_2$$

System F +GADTs: an auxiliary judgement

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For readability, we introduce the auxiliary judgement

$$K \leq D \Rightarrow T_1 \rightarrow F T_2$$

whose definition is the following:

$$\frac{K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2}{K \leq D[\bar{T}/\bar{X}] \Rightarrow T_1[\bar{T}/\bar{X}] \rightarrow F T_2[\bar{T}/\bar{X}]}$$

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The typing rules of System F are unchanged. A constraint is transported.

$$\text{VAR} \quad \frac{}{\Gamma, C \vdash x : \Gamma(x)}$$

$$\text{ABS} \quad \frac{\Gamma; x : T_1, C \vdash t : T_2}{\Gamma, C \vdash \lambda x. t : T_1 \rightarrow T_2}$$

$$\text{APP} \quad \frac{\begin{array}{c} \Gamma, C \vdash t_1 : T_1 \rightarrow T_2 \\ \Gamma, C \vdash t_2 : T_1 \end{array}}{\Gamma, C \vdash t_1 \ t_2 : T_2}$$

$$\text{TABS} \quad \frac{\Gamma; X, C \vdash t : T \quad X \# \Gamma}{\Gamma, C \vdash \lambda X. t : \forall X. T}$$

$$\text{TAPP} \quad \frac{\Gamma, C \vdash t : \forall X. T}{\Gamma, C \vdash t \ T' : T[T'/X]}$$

System F +GADTs: the typing judgement, continued

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The typing rule for a **data constructor application** is straightforward:

$$\frac{\begin{array}{c} \text{DCon} \\ K \leq D \Rightarrow T_1 \rightarrow F T_2 \\ C \Vdash D \\ \Gamma, C \vdash t : T_1 \end{array}}{\Gamma, C \vdash K t : F T_2}$$

We write $C \Vdash D$ when C entails D (see next slide).

System F +GADTs: entailment

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Let ρ denote a total mapping of type variables to closed types.

We write $\rho \vdash C$ when ρ satisfies C :

$$\frac{\rho \vdash \text{True}}{\rho(T_1) = \rho(T_2)} \qquad \frac{\rho \vdash C_1 \quad \rho \vdash C_2}{\rho \vdash C_1 \wedge C_2}$$

Entailment is then defined by:

$$\frac{\forall \rho. \rho \vdash C \Rightarrow \rho \vdash D}{C \Vdash D}$$

Entailment is decidable.

System F +GADTs: the typing judgement, continued

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Type-checking a case analysis construct is straightforward:

CASE

$$\frac{\begin{array}{c} \Gamma, C \vdash t : T_1 \\ \forall c \in \bar{c}. \quad \Gamma, C \vdash c : T_1 \rightarrow T_2 \\ \bar{c} \text{ is exhaustive} \end{array}}{\Gamma, C \vdash \text{case } t \text{ of } \bar{c} : T_2}$$

A clause takes the form $c ::= K \bar{X} x \mapsto t$.

\bar{c} is exhaustive if it contains a clause for every data constructor K .

System F +GADTs: the typing judgement, continued

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When a clause is entered, new constraints appear locally.

CLAUSE

$$\frac{\begin{array}{c} K : \forall \bar{X}. D \Rightarrow T_1 \rightarrow F T_2 \\ (\Gamma; \bar{X}; x : T_1), (C \wedge D \wedge F T_2 = F' T'_2) \vdash t : T' \\ \bar{X} \# \Gamma, C, T'_2, T' \end{array}}{\Gamma, C \vdash K \bar{X} x \mapsto t : F' T'_2 \rightarrow T'}$$

System *F*+GADTs: the typing judgement, continued

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There remains to introduce a typing rule that **exploits** the hypothesis *C*:

$$\frac{\text{CONVERSION} \quad \Gamma, C \vdash t : T \quad C \Vdash T = T'}{\Gamma, C \vdash t : T'}$$

This rule is **not** syntax-directed.

One can imagine a variant of the system where conversion is explicit.
System *FC* is the core language of the Glasgow Haskell compiler.

Sulzmann, Chakravarty, Peyton Jones, Donnelly,
System F with Type Equality Coercions, 2007.

System F+GADTs: type soundness

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Exercise: write down the omitted details (e.g., the reduction rule for *case*),
then prove Subject Reduction and Progress.