

Making the stack explicit: the continuation-passing style transformation

MPRI 2.4

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What if a program transformation could:

- ensure that every function call is a **tail call** and the **stack** is **explicit**, so the code is no longer really recursive, but **iterative**;
- make the evaluation order **explicit** in the code, so that it does not depend on the ambient strategy (CBN / CBV);
- eliminate the apparent **redundancy** between calls and returns, by exploiting solely function calls – **functions never return!**
- suggest extending the λ -calculus with **control operators**?

Motivation

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- suggest extending the λ -calculus with **control operators**?

The **continuation-passing style** transformation does all this.

Motivation



D. Conversion to Continuation-Passing Style

This phase is the real meat of the compilation process. It is of interest primarily in that it transforms a program written in SCHEME into an equivalent program (the continuation-passing-style version, or CPS version), written in a language isomorphic to a subset of SCHEME with the property that interpreting it requires no control stack or other unbounded temporary storage and no decisions as to the order of evaluation of (non-trivial) subexpressions. The importance of these properties cannot be overemphasized. The fact that it is essentially a subset of SCHEME implies that its semantics are as clean, elegant, and well-understood as those of the original language. It is easy to build an

Steele, **RABBIT: a compiler for SCHEME**, 1978.

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From recursive traversal down to iterative traversal with link inversion

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3 Soundness

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5 Madness in small steps

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A direct-style interpreter

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Recall our environment-based interpreter for call-by-value λ -calculus:

```
let rec eval (e : cenv) (t : term) : cvalue =  
  match t with  
  | Var x ->  
    lookup e x  
  | Lam t ->  
    Clo (t, e)  
  | App (t1, t2) ->  
    let cv1 = eval e t1 in  
    let cv2 = eval e t2 in  
    let Clo (u1, e') = cv1 in  
    eval (cv2 :: e') u1
```

This is an OCaml transcription, without a fuel parameter.

A continuation-passing style interpreter

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Instead of **returning** a value,

```
let rec eval (e : cenv) (t : term) : cvalue =  
  ...
```

let's **pass** this value to a **continuation** that we get as an argument:

```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =  
  ...
```

Exercise (in class): write evalk. (See [EvalCBVExercise](#).)

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```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =  
  match t with  
  | Var x ->  
    k (lookup e x)  
  | Lam t ->  
    k (Clo (t, e))  
  | App (t1, t2) ->  
    evalk e t1 (fun cv1 ->  
      evalk e t2 (fun cv2 ->  
        let Clo (u1, e') = cv1 in  
        evalk (cv2 :: e') u1 k))
```

Instead of **returning** a value, **pass** it to k.

Instead of **sequencing** computations via **let**, **nest** continuations.

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To run the interpreter, start it with the [identity](#) continuation:

```
let eval (e : cenv) (t : term) : cvalue =  
  evalk e t (fun cv -> cv)
```

Correctness of the CPS interpreter

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The continuation-passing style interpreter is “obviously” correct.

Exercise: define `evalk` in Coq (with fuel) and prove it equivalent to the direct-style interpreter: `evalk n e t k = k (eval n e t)`.

Properties of the interpreter

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What is special about this interpreter?

Properties of the interpreter

What is special about this interpreter?

- Every call to `evalk` is a **tail call**.
- Every call to a continuation `k` is a **tail call**.

A call $g\ x$ is a tail call if it is the “last thing” that the calling function does...

More formally,

$v ::= x \mid \lambda x. tt$	values
$tt ::=$	terms in tail position
$ \ v$	
$ \ nt\ nt$	– a tail call
$ \ let\ nt\ in\ tt$	
$ \ if\ nt\ then\ tt\ else\ tt$	
$nt ::=$	terms not in tail position
$ \ v$	
$ \ nt\ nt$	– not a tail call
$ \ let\ nt\ in\ nt$	
$ \ if\ nt\ then\ nt\ else\ nt$	

This can be understood as the description of a top-down computation that assigns a Boolean flag (“tail” or “non-tail”) to every subterm.

OCaml allows us to **verify** that these are indeed tail calls:

```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =  
  match t with  
  | Var x ->  
    (k[@tailcall]) (lookup e x)  
  | Lam t ->  
    (k[@tailcall]) (Clo (t, e))  
  | App (t1, t2) ->  
    (evalk[@tailcall]) e t1 (fun cv1 ->  
      (evalk[@tailcall]) e t2 (fun cv2 ->  
        let Clo (u1, e') = cv1 in  
        (evalk[@tailcall]) (cv2 :: e') u1 k))
```

A nice feature (though with somewhat ugly syntax).

Properties of the interpreter

Tail calls are compiled by OCaml to **jumps**.

Thus, tail-recursive functions are compiled by OCaml to **loops**.

Steele, **Lambda: the ultimate GOTO**, 1977.

Thus, the CPS interpreter is not truly **recursive**: it is **iterative**.

It uses **constant space** on OCaml's implicit stack.

Wait! Does the interpreter really **not need a stack** any more?

Properties of the interpreter

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Wait! Does the interpreter really **not need a stack** any more?

- Of course it **does** need a stack.

Properties of the interpreter

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Wait! Does the interpreter really **not need a stack** any more?

- Of course it **does** need a stack.
- The **continuation**, allocated in the OCaml heap, serves as a stack.

A defunctionalized CPS interpreter

To better see the structure of the continuation,
let us **defunctionalize** the CPS interpreter.

Reynolds, **Definitional interpreters**
for programming languages, 1972 (1998).

Reynolds, **Definitional interpreters revisited**, 1998.

Defunctionalization (reminder)

Steps:

- Identify the sites where closures are allocated, that is, where anonymous functions are built.
- Compute, at each site, the free variables of the anonymous function.
- Introduce an algebraic data type of closures.
- Transform the code:
 - replace anonymous functions with constructor applications,
 - replace function applications with calls to `apply`,
 - and define `apply`.

Exercise (in class): defunctionalize the CPS interpreter. ([EvalCBVExercise.](#))

A defunctionalized CPS interpreter

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There are three sites where an anonymous continuation is built.

We name them and compute their free variables.

This leads to the following algebraic data type of continuations:

```
type kont =  
  | AppL of { e: cenv; t2: term; k: kont }  
  | AppR of {          cv1: cvalue; k: kont }  
  | Init
```

What data structure is this?

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  | Init
```

What data structure is this? A [linked list](#). A heap-allocated stack.

In fact, it is a (call-by-value) [evaluation context](#):

$$E ::= E[\] \ t_2[e] \mid E[v_1 \ \] \mid \ []$$

It is a [zipper](#), a path from the context's hole up to the root of a term.

Huet, [The Zipper](#), 1997.

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We transform the interpreter's main function:

```
let rec evalkd (e : cenv) (t : term) (k : kont) : cvalue =  
  match t with  
  | Var x ->  
    apply k (lookup e x)  
  | Lam t ->  
    apply k (Clo (t, e))  
  | App (t1, t2) ->  
    evalkd e t1 (AppL { e; t2; k })
```

To evaluate t_1 t_2 , the interpreter **pushes** information on the stack, then **jumps** straight to evaluating t_1 .

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apply interprets continuations as functions of values to values:

```
and apply (k : kont) (cv : cvalue) : cvalue =  
  match k with  
  | AppL { e; t2; k } ->  
    let cv1 = cv in  
    evalkd e t2 (AppR { cv1; k })  
  | AppR { cv1; k } ->  
    let cv2 = cv in  
    let Clo (u1, e') = cv1 in  
    evalkd (cv2 :: e') u1 k  
  | Init ->  
    cv
```

It **pops** the top stack frame and decides what to do, based on it.

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To run the interpreter, start it with the [identity](#) continuation:

```
let eval e t =  
  evalkd e t Init
```

An abstract machine

We have reached an **abstract machine**, a simple **iterative** interpreter which maintains a few data structures:

- a **code** pointer: the term t ,
- an **environment** e ,
- a stack, or **continuation** k .

In fact, we have mechanically rediscovered the **CEK** machine.

Felleisen and Friedman,
Control operators, the SECD machine, and the λ -calculus, 1987.

Sig Ager, Biernacki, Danvy and Midtgaard,
**A Functional Correspondence between Evaluators
and Abstract Machines**, 2003.

Re-discovering other abstract machines

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Exercise: start with a **call-by-name** interpreter and follow an analogous process to rediscover Krivine's machine.

The solution is in **EvalCBNCPS**.

There once was a man named Krivine

Who invented a wond'rous machine.

It pushed and it popped

On abstractions it stopped;

That lean mean machine from Krivine.

— **Mitchell Wand**

Krivine, **A call-by-name lambda-calculus machine**, (1985) 2007.

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A type of binary trees

Consider a simple type of binary trees:

```
type tree =  
  | Leaf  
  | Node of { data: int; left: tree; right: tree }
```

Direct-style traversal

Suppose we wish to perform a postfix tree traversal:

```
let rec walk (t : tree) : unit =  
  match t with  
  | Leaf ->  
    ()  
  | Node { data; left; right } ->  
    walk left;  
    walk right;  
    printf "%d\n" data
```

This is **recursive** code in **direct style**.

Neither of the recursive calls is a tail call.

Now suppose we wish to make the code [iterative](#). Swoop, CPS!

```
let rec walkk (t : tree) (k : unit -> 'a) : 'a =  
  match t with  
  | Leaf ->  
    k()  
  | Node { data; left; right } ->  
    walkk left (fun () ->  
      walkk right (fun () ->  
        printf "%d\n" data;  
        k()))
```

The traversal is initiated with an identity continuation:

```
let walk t =  
  walkk t (fun t -> t)
```

CPS traversal, defunctionalized

Next, we might wish to make the stack an explicit [data structure](#).

Swoop, defunctionalization!

The type of defunctionalized continuations:

```
type kont =  
  | Init  
  | GoneL of { data: int; tail: kont; right: tree }  
  | GoneR of { data: int; tail: kont }
```


CPS traversal, defunctionalized

The main function is a loop that **walks down the leftmost branch** while **pushing** information onto the stack:

```
let rec walkkd (t : tree) (k : kont) : unit =  
  match t with  
  | Leaf ->  
    apply k ()  
  | Node { data; left; right } ->  
    walkkd left (GoneL { data; tail = k; right })
```

Think of the stack as **Ariadne's thread**.

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The apply function comes back up out of a child.

```
and apply k () =  
  match k with  
  | Init ->  
    ()  
  | GoneL { data; tail; right } ->  
    walkkd right (GoneR { data; tail })  
  | GoneR { data; tail } ->  
    printf "%d\n" data;  
    apply tail ()
```

It pops information off the stack so as to decide what to do.

When coming out of a left child, go down into its right sibling.

When coming out of a right child, go further up.

And now, for something a little
UNEXPECTED and WILD.

And now, for something a little
UNEXPECTED and WILD.
A CRAZY HACK.



Recycling

When we **allocate** a **GoneR** continuation,
we **drop** a **GoneL** continuation at the same time.

Indeed, here, continuations are **linear**. They are used exactly once.

```
| GoneL { data; tail; right } ->  
  walkkd right ( GoneR { data; tail } )
```

This suggests that the memory block could be **recycled** (re-used).

More recycling

When we **allocate** a **GoneL** continuation,
a **Node** goes **temporarily unused** at the same time.

This node won't be accessed until this **GoneL** frame
first is changed to **GoneR** then is popped off the stack.

```
| Node { data; left; right } ->  
  walkkd left ( GoneL { data; tail = k; right } )
```

This suggests that the memory block could be **recycled**, too,
provided we **restore** it when we are done with it.

A tree is a continuation is a tree

In OCaml, the type of a memory block **cannot** be changed over time.

Thus, recycling tree nodes as stack frames, and vice-versa, requires **trees** and **continuations** to have **the same type**.

Uh?

A tree is a continuation is a tree

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Could we **disguise** a continuation as a tree?

In other words, could a stack frame **fit** in a tree node?

```
type kont =  
  | Init  
  | GoneL of { data: int; tail: kont; right: tree }  
  | GoneR of { data: int;                tail: kont }
```

```
type tree =  
  | Leaf  
  | Node of { data: int; left: tree; right: tree }
```


A tree is a continuation is a tree

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In other words, could a stack frame **fit** in a tree node?

```
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  | Init  
  | GoneL of { data: int; tail: kont; right: tree }  
  | GoneR of { data: int;                tail: kont }
```

```
type tree =  
  | Leaf  
  | Node of { data: int; left: tree; right: tree }
```

Yes, kind of.

We just need **one extra bit** of storage per tree node,
so as to distinguish **GoneL** and **GoneR**.

A tree is a continuation is a tree

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Add one “status” bit per tree node. Make nodes **mutable**.

```
type status = GoneL | GoneR
type mtree  = Leaf | Node of {
  data: int;          mutable status: status;
  mutable left: mtree; mutable right: mtree
}
type mkont = mtree
```

Tree records and continuation records occupy **the same space** in memory.

Thus, a tree record can be turned into a continuation record, and back!

By convention, in a “tree” record, the status field is **GoneL**.

In a “continuation” record,

- **either** status is **GoneL** and the left field stores tail;
- **or** status is **GoneR** and the right field stores tail.

CPS traversal with link inversion

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Instead of allocating a **GoneL** continuation,
we now **change** the tree record to a continuation record:

```
let rec walkkdi (t : mtree) (k : mkont) : unit =  
  match t with  
  | Leaf ->  
    apply k t  
  | Node ({ left; _ } as n) ->  
    (* Change this tree to a [GoneL] continuation. *)  
    assert (n.status = GoneL);  
    n.left (* n.tail *) <- k;  
    walkkdi left (t : mkont)
```

The `left` field is **overwritten**, which is scary! We must **restore** it later.

We find that, in every call to `walkkdi t k` and `apply k t`,
`k` is the **parent** of `t` in the tree.

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The rest of the code, in its horrific glory:

```
and apply (k : mkont) (child : mtree) : unit =  
  match k with  
  | Leaf -> ()  
  | Node ({ status = GoneL; left = tail; right; _ } as n) ->  
    n.status <- GoneR;      (* update continuation! *)  
    n.left <- child;        (* restore orig. left child! *)  
    n.right (* n.tail *) <- tail;  
    walkkdi right k  
  | Node ({ data; status = GoneR; right = tail; _ } as n) ->  
    printf "%d\n" data;  
    n.status <- GoneL;      (* change back to a tree! *)  
    n.right <- child;        (* restore orig. right child! *)  
    apply tail (k : mtree)
```

This code runs in **constant space**. Look Ma, no stack! (Uh?)

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More accurately, the stack is stored **in the tree** itself, by **reversing pointers**.

This ~~hack~~ technique is known as **link inversion**.

It was invented for use in garbage collectors, which must **traverse the heap** without requiring a huge stack.

We have re-discovered it via the idea of allocating continuations **in place**.

Schorr and Waite, **An efficient machine-independent procedure for garbage collection in various list structures**, 1967.

Hubert and Marché, **A case study of C source code verification: the Schorr-Waite algorithm**, 2005.

Sobel and Friedman, **Recycling continuations**, 1998.

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“Kids, do not try this at home”: this idea is **complicated** and **expensive**.

(The OCaml GC imposes a **write barrier**: write operations are slow.)

Exercise: Extend the code to deal with **graphs**, where there can be **sharing** and **cycles**. (Use a **mark** bit in every node.)

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Formulations of the CPS transformation

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There are **many** variants of the CPS transformation,
and sometimes **many** formulations of a single variant.

Let us begin with the simplest formulation: Fischer and Plotkin's.

Fischer, **Lambda-Calculus Schemata**, (1972) 1993.

Plotkin, **Call-by-name, call-by-value and the λ -calculus**, 1975.

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A term is translated to a **function** of a continuation k to an answer.

$$\llbracket x \rrbracket =$$

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$$\llbracket x \rrbracket = \lambda k.$$

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A term is translated to a **function** of a continuation k to an answer.

$$\llbracket x \rrbracket = \lambda k. k \ x$$

$$\llbracket \lambda x. t \rrbracket =$$

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A term is translated to a **function** of a continuation k to an answer.

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$$\llbracket \lambda x. t \rrbracket = \lambda k.$$

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$$\llbracket x \rrbracket = \lambda k. k \ x$$

$$\llbracket \lambda x. t \rrbracket = \lambda k. k \ (\lambda x. \llbracket t \rrbracket)$$

$$\llbracket t_1 \ t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket \ (\lambda k. \llbracket t_2 \rrbracket)$$

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$$\llbracket t_1 \ t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket \ (\lambda x_1. \llbracket t_2 \rrbracket \ (\lambda x_2. x_1 \ x_2 \ k))$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket \ (\lambda x. \llbracket t_2 \rrbracket \ k)$$

A **value** $\lambda x. t$ is translated to a function of **two** arguments $\lambda x. \lambda k. \dots$

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One avoids some redundancy by defining **two mutually recursive functions**, namely the translation of values $\langle v \rangle$:

$$\langle x \rangle = x$$

$$\langle \lambda x. t \rangle = \lambda x. \llbracket t \rrbracket$$

and the translation of terms $\llbracket t \rrbracket$:

$$\llbracket v \rrbracket = \lambda k. k \langle v \rangle$$

$$\llbracket t_1 \ t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket (\lambda x_1. \llbracket t_2 \rrbracket (\lambda x_2. x_1 \ x_2 \ k))$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket (\lambda x. \llbracket t_2 \rrbracket k)$$

Indifference



In a transformed term, **the right-hand side of every application** is a **value**.

Therefore, its execution is **indifferent** to the choice of a call-by-name or call-by-value evaluation strategy.

In other words, **evaluation order** is fully **explicit** in a transformed term.

The transformation on the previous slide fixes a call-by-value strategy: it is the **CBV CPS transformation**.

It can serve as an **encoding** of call-by-value into call-by-name, thus answering a question raised in week 1.

Exercise (recommended): Define the CBN CPS transformation.

Stacklessness



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In a transformed term, **every call is a tail call**.

Therefore, reduction under a context is not required.

That is, execution **does not require a stack**.

We could (but won't) give a (small-step, substitution-based) semantics that takes **indifference** and **stacklessness** into account.

Exercise: Propose such a semantics. Prove that, when executing a CPS-transformed term, it is equivalent to the standard semantics.

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How are **types** transformed?

A **value** of type T is translated to a value of type $\llbracket T \rrbracket$.

A **computation** of type T is translated to a value of type $\llbracket T \rrbracket$.

$$\llbracket \alpha \rrbracket = \alpha$$

$$\llbracket T_1 \rightarrow T_2 \rrbracket =$$

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$$\llbracket T \rrbracket =$$

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A **computation** of type T is translated to a value of type $\llbracket T \rrbracket$.

$$\langle \alpha \rangle = \alpha$$

$$\langle T_1 \rightarrow T_2 \rangle = \langle T_1 \rangle \rightarrow \langle T_2 \rangle$$

$$\llbracket T \rrbracket = (\langle T \rangle \rightarrow A) \rightarrow A$$

The type A , known as the **answer** type, is arbitrary and fixed.

One may take A to be the **empty type** 0 . Then, $\llbracket T \rrbracket$ is $\neg\neg\langle T \rangle$. The CPS transformation is known in logic as the **double-negation translation**.

Exercise (recommended): state and prove Type Preservation.

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refined

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Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

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Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Every transformed term is in fact **answer-type polymorphic**:

$$\llbracket T \rrbracket = \forall A. (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Furthermore,

Effect of the transformation of types – refined

Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Every transformed term is in fact **answer-type polymorphic**:

$$\llbracket T \rrbracket = \forall A. (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Furthermore, every transformed term invokes its continuation **once**:

$$\llbracket T \rrbracket = \forall A. (\llbracket T \rrbracket \rightarrow A) \multimap A$$

However, these properties are violated in the presence of **control effects**.

Thielecke, **From control effects to typed continuation passing**, 2003.

Semantic preservation

Plotkin (1975) proved semantic preservation,
based on a [small-step simulation diagram](#).

This proof is complicated by the presence of administrative reductions.

A simpler approach is to use big-step semantics in the hypothesis:

Lemma (Semantic Preservation)

If $t \Downarrow_{cbv} v$ and if w is a value, then $\llbracket t \rrbracket w \longrightarrow_{cbv}^ w \langle v \rangle$.*

One should prove, in addition, that divergence is preserved.

[Exercise](#) (recommended): Prove this lemma.

Administrative redexes

The translation presented so far is naïve.

It produces many “administrative” β -redexes.

E.g., in an application of a variable to a variable:

$$\begin{aligned}
 \llbracket f \ x \rrbracket &= \lambda k. \llbracket f \rrbracket (\lambda x_1. \llbracket x \rrbracket (\lambda x_2. x_1 \ x_2 \ k)) \\
 &= \lambda k. (\lambda k. k \ \llbracket f \rrbracket) (\lambda x_1. (\lambda k. k \ \llbracket x \rrbracket) (\lambda x_2. x_1 \ x_2 \ k)) \\
 &= \lambda k. (\lambda k. k \ f) (\lambda x_1. (\lambda k. k \ x) (\lambda x_2. x_1 \ x_2 \ k)) \\
 &=_{\beta} \lambda k. (\lambda x_1. (\lambda k. k \ x) (\lambda x_2. x_1 \ x_2 \ k)) \ f \\
 &=_{\beta} \lambda k. (\lambda k. k \ x) (\lambda x_2. f \ x_2 \ k) \\
 &=_{\beta} \lambda k. (\lambda x_2. f \ x_2 \ k) \ x \\
 &=_{\beta} \lambda k. f \ x \ k
 \end{aligned}$$

This is inefficient: **one** function call is translated to **five** function calls!

Ways of eliminating administrative redexes

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Administrative redexes can be reduced **after** the CPS transformation.

- During the translation, mark each λ that corresponds to a source λ .
- After the translation, reduce every redex whose λ is unmarked.

Another idea is to reduce all “**no-brainer**” redexes. They include the admin. redexes and are size-decreasing. This can be done on the fly.

Davis, Meehan, Shivers, **No-brainer CPS conversion**, 2017.

Yet another approach is to define a “**one-pass**” CPS transformation that does not produce any administrative redexes in the first place...

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The first step is to make some of the abstractions and applications **static**.

They should take place at **transformation time**, not at **runtime**.

Instead of viewing $\llbracket t \rrbracket = \lambda k. \dots$ as a function of a term to a term, let us view $\llbracket t \rrbracket \{ w \} = \dots$ as a function of a term and a value to a term.

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda x. t \rrbracket = \lambda x. \lambda k. \llbracket t \rrbracket \{ k \}$$

$$\llbracket v \rrbracket \{ w \} = w \llbracket v \rrbracket$$

$$\llbracket t_1 \ t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x_1. \llbracket t_2 \rrbracket \{ \lambda x_2. x_1 \ x_2 \ w \} \}$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x. \llbracket t_2 \rrbracket \{ w \} \}$$

k denotes a **variable**; w denotes a **value**.

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This transformation produces **fewer administrative redexes**:

$$\begin{aligned}
 \llbracket f \ x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 \ x_2 \ k \} \} \\
 &= (\lambda x_1. (\lambda x_2. x_1 \ x_2 \ k) \ x) \ f \\
 &=_{\beta} (\lambda x_2. f \ x_2 \ k) \ x \\
 &=_{\beta} f \ x \ k
 \end{aligned}$$

The remaining administrative redexes arise from the equation

$$\llbracket v \rrbracket \{ w \} = w \ (v)$$

in the case where the continuation w is a λ -abstraction.

How could we alter this equation?

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Define the **smart application** of a (continuation) value w to a value v :

$$\begin{aligned}x @_{\beta} v &= x v \\(\lambda x. t) @_{\beta} v &= t[v/x]\end{aligned}$$

Note:

- A continuation w is always either a variable or a “transformation” λ , never a “source” λ , so the redex reduced by $w @_{\beta} v$ is **administrative**.
- Provided every “transformation” λ uses its argument **linearly**, $w @_{\beta} \langle v \rangle$ does not duplicate $\langle v \rangle$, so transformed terms remain **linear** in size.

A one-pass transformation

Change the translation of values. Make every “transformation” λ linear.

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda x. t \rrbracket = \lambda x. \lambda k. \llbracket t \rrbracket \{ k \}$$

$$\llbracket v \rrbracket \{ w \} = w @_{\beta} \llbracket v \rrbracket$$

$$\llbracket t_1 \ t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x_1. \llbracket t_2 \rrbracket \{ \lambda x_2. x_1 \ x_2 \ w \} \}$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x. \text{let } x = x \text{ in } \llbracket t_2 \rrbracket \{ w \} \}$$

This transformation produces **no administrative redexes**.

Dargaye and Leroy, **Mechanized Verification
of CPS Transformations**, 2007.

A one-pass transformation

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Look Ma, **no administrative redexes!**

$$\begin{aligned}\llbracket f \ x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 \ x_2 \ k \} \} \\ &= (\lambda x_1. (\lambda x_2. x_1 \ x_2 \ k) @_{\beta} x) @_{\beta} f \\ &= (\lambda x_2. f \ x_2 \ k) @_{\beta} x \\ &= f \ x \ k\end{aligned}$$

A one-pass transformation

Look Ma, **no administrative redexes!**

$$\begin{aligned}
 \llbracket f \ x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 \ x_2 \ k \} \} \\
 &= (\lambda x_1. (\lambda x_2. x_1 \ x_2 \ k) @_{\beta} x) @_{\beta} f \\
 &= (\lambda x_2. f \ x_2 \ k) @_{\beta} x \\
 &= f \ x \ k
 \end{aligned}$$

A drawback of Dargaye and Leroy's approach is that $\cdot @_{\beta} \cdot$ **does not commute** with substitutions, which causes a difficulty in the proof of semantic preservation.

This is repaired in the formulation shown next...



Higher-order versus first-order formulations

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Danvy and Filinski (1992) first defined this one-pass transformation.

Their formulation was in a “higher-order” style.

Let me give a simpler, “first-order” presentation of their transformation.

Danvy and Filinski, **Representing control:
a study of the CPS transformation**, 1992.

Pottier, **Revisiting the CPS transformation
and its implementation**, 2017.

A first-order one-pass CPS transformation

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Let a continuation c be either a value w or a “transformation” λ :

$$c ::= w \mid mx.t$$

In $mx.t$, the term t must have exactly one occurrence of x .

Define **continuation application** $apply\ c\ v$ and **reification** $reify\ c$:

$apply\ w\ v = w\ v$	– an object-level application
$apply\ (mx.t)\ v = t[v/x]$	– a meta-level substitution
$reify\ w = w$	– a no-op
$reify\ (mx.t) = \lambda x.t$	

Reification converts a continuation to a term.

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Danvy and Filinski's transformation can then be presented as follows:

$$\begin{aligned}
 \llbracket x \rrbracket &= x \\
 \llbracket \lambda x. t \rrbracket &= \lambda x. \lambda k. \llbracket t \rrbracket \{ k \} \\
 \llbracket v \rrbracket \{ c \} &= \text{apply } c \llbracket v \rrbracket \\
 \llbracket t_1 \ t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ mx_1. \llbracket t_2 \rrbracket \{ mx_2. x_1 \ x_2 \ (\text{reify } c) \} \} \\
 \llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ mx_1. \text{let } x = x_1 \text{ in } \llbracket t_2 \rrbracket \{ c \} \}
 \end{aligned}$$

It is close to Dargaye and Leroy's formulation, yet is **better behaved**:
as we will see, it commutes with substitution.

Now, in de Bruijn style



Let us use o and m as explicit injections:

$$c ::= o\ w \mid m\ t$$

m , like λ , is considered a binder.

Continuation application, reification, and substitution $c[\sigma]$ are as follows:

$$\begin{array}{lll} \text{apply } (o\ w)\ v = w\ v & \text{reify } (o\ w) = w & (o\ w)[\sigma] = o\ (w[\sigma]) \\ \text{apply } (m\ t)\ v = t[v/] & \text{reify } (m\ t) = \lambda t & (m\ t)[\sigma] = m\ (t[\uparrow\sigma]) \end{array}$$

See [CPSDefinition](#).

The CPS transformation in de Bruijn style

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The transformation is formulated in de Bruijn style as follows:

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda t \rrbracket = \lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{ \textcolor{blue}{o} 0 \})$$

$$\llbracket v \rrbracket \{ c \} = \textit{apply } c \llbracket v \rrbracket$$

$$\llbracket t_1 \ t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \textcolor{blue}{m} \llbracket \uparrow^1 t_2 \rrbracket \{ \textcolor{blue}{m} 1 0 \uparrow^2 (\textit{reify } c) \} \}$$

$$\llbracket \textit{let } t_1 \textit{ in } t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \textcolor{blue}{m} \textit{let } 0 \textit{ in } \llbracket \uparrow_1^1 t_2 \rrbracket \{ \uparrow^2 c \} \}$$

$\uparrow^i t$ is short for $t[+i]$. $\uparrow_1^1 t$ is short for $t[\uparrow(+1)]$.

\uparrow^1 means **end-of-scope** for variable 0.

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The transformation is formulated in de Bruijn style as follows:

$$\llbracket x \rrbracket = x$$

$$\llbracket \lambda t \rrbracket = \lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{ \textcolor{blue}{o} 0 \})$$

$$\llbracket v \rrbracket \{ c \} = \textit{apply } c \llbracket v \rrbracket$$

$$\llbracket t_1 t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \textcolor{blue}{m} \llbracket \uparrow^1 t_2 \rrbracket \{ \textcolor{blue}{m} 1 0 \uparrow^2 (\textit{reify } c) \} \}$$

$$\llbracket \textit{let } t_1 \textit{ in } t_2 \rrbracket \{ c \} = \llbracket t_1 \rrbracket \{ \textcolor{blue}{m} \textit{let } 0 \textit{ in } \llbracket \uparrow_1^1 t_2 \rrbracket \{ \uparrow^2 c \} \}$$

$\uparrow^i t$ is short for $t[+i]$. $\uparrow_1^1 t$ is short for $t[\uparrow(+1)]$.

\uparrow^1 means **end-of-scope** for variable 0.

\uparrow^2 means end-of-scope for variables 0 and 1.

\uparrow_1^1 means end-of-scope for variable 1.

Worse, Coq does not like this definition...



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The transformation is formulated in de Bruijn style as follows:

$$\begin{aligned}
 \llbracket x \rrbracket &= x \\
 \llbracket \lambda t \rrbracket &= \lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{ o \ 0 \}) \\
 \llbracket v \rrbracket \{ c \} &= \text{apply } c \ (\llbracket v \rrbracket) \\
 \llbracket t_1 \ t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ m \ \llbracket \uparrow^1 t_2 \rrbracket \{ m \ 1 \ 0 \ \uparrow^2 (\text{reify } c) \} \} \\
 \llbracket \text{let } t_1 \text{ in } t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ m \ \text{let } 0 \text{ in } \llbracket \uparrow_1^1 t_2 \rrbracket \{ \uparrow^2 c \} \}
 \end{aligned}$$

$\uparrow^i t$ is short for $t[+i]$. $\uparrow_1^1 t$ is short for $t[\uparrow(+1)]$.

\uparrow^1 means **end-of-scope** for variable 0.

\uparrow^2 means end-of-scope for variables 0 and 1.

\uparrow_1^1 means end-of-scope for variable 1.



Worse, Coq does not like this definition... because the recursive calls concern **renamed** subterms! Well-founded recursion on **size** is required.

See **CPSDefinition**.

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Semantic Preservation

We would like to prove this:

Lemma (Semantic Preservation)

If $t \downarrow_{cbv} v$, then $\llbracket t \rrbracket \{ m\ 0 \} \downarrow_{cbv} \langle v \rangle$.

$m\ 0$ is the **identity continuation**: in nominal style, $m\ x.x$.

For an inductive proof, the statement must be generalized, as follows...

Semantic Preservation

Define $t \lesssim u$ as follows: for every value v , $u \downarrow_{cbv} v$ implies $t \downarrow_{cbv} v$.

Lemma (Big-step Simulation)

Suppose reify c is a value. If $t \downarrow_{cbv} v$, then $\llbracket t \rrbracket \{c\} \lesssim \text{apply } c \ (v)$.

Compare with **our earlier claim** concerning Plotkin's CPS transformation.

The proof is in **CPSCorrectnessBigStep**.

Exercise: Replay the proof in Coq. Then erase it and redo it from scratch.

Exercise: Write a clear paper or \LaTeX proof and send it to me!

The proof requires two key lemmas, shown next...

Key Lemma 1: Substitution

Lemma (Substitution)

Let σ and σ' be value substitutions such that σ' is equal to σ ; (\cdot) . Then,

$$(\llbracket t \rrbracket \{ c \})[\sigma'] = \llbracket t[\sigma] \rrbracket \{ c[\sigma'] \}.$$

Lemma (Substitution—a special case)

Let v and w be values. Then,

$$(\llbracket t \rrbracket \{ \uparrow^2 c \})[(v) \cdot (w) \cdot id] = \llbracket t[v \cdot w \cdot id] \rrbracket \{ c \}.$$

In nominal style: if $x, y \notin fv(c)$, then

$$(\llbracket t \rrbracket \{ c \})[(v)/x, (w)/y] = \llbracket t[v/x, w/y] \rrbracket \{ c \}.$$

We push a substitution into the term, leaving the continuation untouched.
A target language substitution becomes a source language substitution.

See [CPSSubstitution](#).

Key Lemma 2: Kubstitution

Lemma (Kubstitution)

Let θ and σ be substitutions such that $\theta ; \sigma$ is id. Then,

$$\llbracket (t[\theta]) \{ c \} \rrbracket [\sigma] = \llbracket t \{ c[\sigma] \} \rrbracket.$$

Lemma (Kubstitution—a special case)

For every value v , $(\llbracket \uparrow^1 t \{ c \} \rrbracket [v/] = \llbracket t \{ c[v/] \} \rrbracket$.

In nominal style: if $x \notin \text{fv}(t)$, then $(\llbracket t \{ c \} \rrbracket [v/x] = \llbracket t \{ c[v/x] \} \rrbracket$.

We push a substitution into the continuation, leaving the term untouched.

This is and remains a target language substitution.

See [CPSKubstitution](#).

Interlude: Enumerating λ -terms

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Define the **size** of a term as follows: variables have size 0; λ -abstractions and applications contribute 1.

Step 1: In OCaml, implement an exhaustive **enumeration** of the λ -terms of size s and with at most n free variables. (Given as an exercise in week 1.)

```
(* Enumerate all variables between 0 and n excluded. *)  
let var (n : int) (k : term -> unit) : unit = ...  
(* Enumerate all manners of splitting an integer s. *)  
let split (s : int) (k : int -> int -> unit) : unit = ...  
(* Enumerate all terms of size s with at most n variables. *)  
let term (s : int) (n : int) (k : term -> unit) : unit = ...
```

An enumerator is naturally written in CPS style!

Interlude: Testing Semantic Preservation

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Step 2: In OCaml, implement the CPS transformation.

```
type continuation =  
  | O of term  
  | M of term  
let rec cps (t : term) (c : continuation) : term = ...
```

Step 3: In OCaml, implement a test for the relation $\cdot \lesssim \cdot$:

```
let sim (t1 : term) (t2 : term) : bool = ...
```

Hint: Re-use the big-step interpreter of week 2. See [Lambda](#).

Step 4: Up to a certain size, search for a term that violates Semantic Preservation. There should be none!

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Control operators

In a CPS-transformed program, the continuation is a first-class object.

Why not give programmers **access** to it?

That is, extend the source language with **control operators** that allow (**delimiting** and) **capturing** the current continuation.

An example is Danvy and Filinski's shift / reset (1990).

$$t ::= \dots \mid \langle t \rangle \mid \xi x. t$$

A “reset” $\langle t \rangle$ does nothing by itself: e.g., $\langle 42 \rangle$ reduces to 42.

A “shift” $\xi x. t$ captures the current evaluation context (up to and excluding the nearest reset), reifies it as a function, and binds the variable x to it.

Then it discards the evaluation context (up to and including the nearest reset) and executes t instead.

E.g., roughly,

$$\begin{aligned} & 1 + \langle 10 + \xi c. c (c \ 100) \rangle \\ \longrightarrow & 1 + (\text{let } c = \lambda x. (10 + x) \text{ in } c (c \ 100)) \\ \longrightarrow & 1 + (10 + (10 + 100)) \\ \longrightarrow & 121 \end{aligned}$$

Exercise: Give a small-step semantics to shift / reset.

CPS-transforming shift / reset

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The naïve call-by-value CPS transformation is extended as follows:

$$\llbracket \langle t \rangle \rrbracket = \lambda k.$$

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The naïve call-by-value CPS transformation is extended as follows:

$$\begin{aligned}\llbracket \langle t \rangle \rrbracket &= \lambda k. k (\llbracket t \rrbracket (\lambda y. y)) \\ \llbracket \xi x. t \rrbracket &= \lambda k.\end{aligned}$$

CPS-transforming shift / reset

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The naïve call-by-value CPS transformation is extended as follows:

$$\begin{aligned}\llbracket \langle t \rangle \rrbracket &= \lambda k. k (\llbracket t \rrbracket (\lambda y. y)) \\ \llbracket \xi x. t \rrbracket &= \lambda k. \text{let } x = \lambda y. \lambda k'. k' (k y) \text{ in} \\ &\quad \llbracket t \rrbracket (\lambda y. y)\end{aligned}$$

Exercise (experimental!): Extend the proof of Semantic Preservation.

The target of the transformation is λ -calculus **without** shift / reset.

It is **no longer the case** that every call is a tail call, that the right-hand side of every application is a value, or that continuations are linearly used.

Thus, shift / reset allow reaching terms which previously lied **outside** the image of the CPS transformation. CPS lets us **think outside the box**!

Other control operators

Many other control operators or control constructs can be **explained** and **compiled away** via CPS.

Exceptions can be compiled away by “double-barrelled CPS”, that is, by using **two** continuations.

Effect handlers can be compiled away via (type-directed, selective) CPS.

Rompf, Maier, Odersky, **Implementing first-class polymorphic delimited continuations by a type-directed selective CPS-transform**, 2009.

Leijen, **Type-directed compilation of row-typed algebraic effects**, 2017.

Monadic intermediate form

If one just aims to make evaluation order explicit, CPS is **overkill**.

This transformation, too, achieves **indifference**:

$$\begin{aligned}
 \llbracket x \rrbracket &= x \\
 \llbracket \lambda x. t \rrbracket &= \lambda x. \llbracket t \rrbracket \\
 \llbracket t_1 \ t_2 \rrbracket &= \text{let } x_1 = \llbracket t_1 \rrbracket \text{ in} \\
 &\quad \text{let } x_2 = \llbracket t_2 \rrbracket \text{ in} \\
 &\quad x_1 \ x_2 \\
 \llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket &= \text{let } x = \llbracket t_1 \rrbracket \text{ in } \llbracket t_2 \rrbracket
 \end{aligned}$$

In a transformed term, **the components of every application are values**.

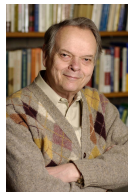
By further hoisting “*let*” out of the left-hand side of “*let*”, one gets **administrative normal form**.

Flanagan, Sabry, Felleisen, **The essence of compiling with continuations**, 1993 (2003).

The CPS monad

The CPS transformation is a special case of the [monadic transformation](#).
See Dagand's lectures!

Some history



Continuations, and the CPS transformation, were independently discovered by many researchers during the 1960s.

John C. Reynolds, *The discoveries of continuations*, 1993.

Some history

The CPS transformation has been used in compilers.

Rabbit (Steele). SML/NJ.

Appel, *Compiling with Continuations*, 1992.

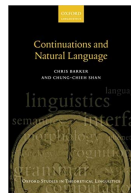
Today, heap-allocating the stack is considered *too costly*:

- bad locality;
- increased GC load;
- confuses the processor's built-in prediction of return addresses.

Yet, *selective* CPS transformations are used to compile effect handlers, and some compilers use CPS as an *intermediate form* before coming back to direct style.

Kennedy, *Compiling with continuations, continued*, 2007.

Some history



Can λ -calculus and continuations explain the structure of speech?

Chris Barker,
Continuations and the nature of quantification, 2002.

Chris Barker and Chung-Chieh Shan,
Continuations and Natural Language, 2014.

A few things to remember

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Continuations rule!

- The CPS transformation achieves several remarkable effects:
 - making [the stack](#) explicit;
 - making [evaluation order](#) explicit;
 - suggesting/explaining [control operators](#).
- It plays a [fundamental role](#) in prog. language theory and in logic.
- Continuation-passing is also a useful [programming technique](#).

We have illustrated a few proof techniques:

- Another proof of semantic preservation.
- A small-step [simulation](#) diagram (see part [5](#)).
- [Testing](#), to refute a conjecture (see part [5](#)).

1 Examples

From a direct-style interpreter down to an abstract machine

From recursive traversal down to iterative traversal with link inversion

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5 Madness in small steps

Madness Soundness in small steps

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(Presented at MPRI 2.4 in 2017.)

Could we use a **small-step operational semantics**
in the proof that CPS is semantics-preserving?

Towards semantic preservation

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Let us consider the pure λ -calculus, without “*let*”.

Let us use de Bruijn notation.

The transformation is defined in [CPSSDefinition](#).

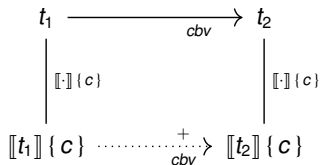
The proof of Simulation is in [CPSSimulationWithoutLet](#).

The key lemmas are in [CPSSpecialCases](#), [CPSSubstitution](#), [CPSKubstitution](#).

A small-step simulation diagram

We propose to use the **small-step substitution** semantics and to establish a **simulation** diagram.

One step by the source program is simulated in **one or more** steps by the transformed program:



A solid arrow represents a **universal** quantification (a hypothesis).

A dashed arrow represents an **existential** quantification (a conclusion).

Consequences of the simulation diagram

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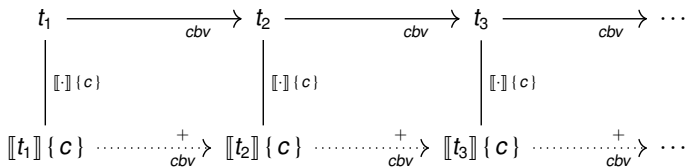
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There immediately follows that **divergence** is preserved.



The fact that each step is simulated by **one or more** steps is crucial.

(A proof by co-induction. See [Relations/infseq_simulation.](#))

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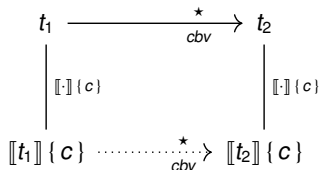
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Obviously, **several** steps by the source program
are simulated in **several** steps by the transformed program:



(A proof by induction. See [Relations/star_diamond_left](#).)

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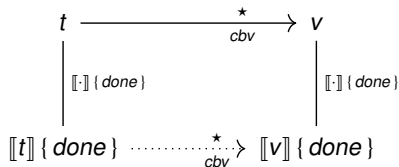
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There follows that **convergence to a value** is preserved.

We use the identity continuation *done*, defined as $m\ 0$.



By definition, $\llbracket v \rrbracket \{ \text{done} \}$ is *apply done* $\langle v \rangle$, that is, $\langle v \rangle$, therefore **a value**.

Thus, the CPS transformation is **semantics-preserving**.

The simulation lemma

Here is the simulation statement again, this time in textual form:

Lemma (Simulation)

Assume $\text{reify } c$ is a value. Then $t_1 \longrightarrow_{cbv} t_2$ implies $\llbracket t_1 \rrbracket \{c\} \longrightarrow_{cbv}^+ \llbracket t_2 \rrbracket \{c\}$.

Let us now do the proof.

Onscreen or in Coq? Both, probably.

See `CPSSimulationWithoutLet`.

Proof of Simulation – case β_v

Case: $(\lambda t) v \longrightarrow_{cbv} t[v/]$. We must show:

$$\llbracket (\lambda t) v \rrbracket \{c\} \longrightarrow_{cbv}^+ \llbracket t[v/] \rrbracket \{c\}$$

By the Value-Value Application lemma, the left-hand term is:

$$\llbracket \lambda t \rrbracket \llbracket v \rrbracket (reify\ c)$$

By definition of $\llbracket \lambda t \rrbracket$, this is:

$$(\lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{o\ 0\})) \llbracket v \rrbracket (reify\ c)$$

The transformed function is passed **an actual argument** $\llbracket v \rrbracket$
and **a continuation** $reify\ c$.

Proof of Simulation – case β_v

$$(\lambda\lambda(\llbracket \uparrow^1 t \rrbracket \{o\ 0\})) \langle v \rangle (\text{reify } c)$$

In two β -reduction steps, this term reduces to:

$$(\llbracket \uparrow^1 t \rrbracket \{o\ 0\}) \llbracket \uparrow (\langle v \rangle /) \rrbracket \llbracket \text{reify } c / \rrbracket$$

We have **two successive substitutions**. This term could also be written using a single substitution that acts on variables 0 and 1:

$$(\llbracket \uparrow^1 t \rrbracket \{o\ 0\}) \llbracket \text{reify } c \cdot \langle v \rangle \cdot \text{id} \rrbracket$$

(We won't use this fact, though.)

We now wish to **push** the substitutions inside, one after the other.

Proof of Simulation – case β_v

$$(\llbracket \uparrow^1 t \rrbracket \{o\ 0\}) \ [\uparrow (\langle v \rangle /)] \ [reify\ c /]$$

By the Substitution lemma, the substitution $\uparrow (\langle v \rangle /)$ acts on both the term $\uparrow^1 t$ and the continuation $o\ 0$.

However, $\uparrow (\langle v \rangle /)$ has no effect on variable 0 .

Thus, the above term is:

$$(\llbracket (\uparrow^1 t) [\uparrow (v /)] \rrbracket \{o\ 0\}) \ [reify\ c /]$$

that is,

$$(\llbracket \uparrow^1 t[v /] \rrbracket \{o\ 0\}) \ [reify\ c /]$$

Proof of Simulation – case β_v

$$(\llbracket \uparrow^1 t[v/] \rrbracket \{ o 0 \}) [reify\ c/]$$

By the Substitution lemma, the substitution *reify c/* acts **only on the continuation** *o 0*, **not on the term** *t[v/]*, because it cancels out with \uparrow^1 .

Thus, this term is:

$$\llbracket t[v/] \rrbracket \{ (o 0)[reify\ c/] \}$$

that is,

$$\llbracket t[v/] \rrbracket \{ o (reify\ c) \}$$

Proof of Simulation – case β_v

We have now reached the term:

$$\llbracket t[v/] \rrbracket \{ o(\text{reify } c) \}$$

and the goal is to prove that it reduces (in zero or more steps) to:

$$\llbracket t[v/] \rrbracket \{ o c \}$$

This is the Magic Step lemma. This proof case is finished!

Here are the four key lemmas that we have used so far.

Lemma (Value-Value Application)

$$\llbracket v_1 \ v_2 \rrbracket \{ c \} = \llbracket v_1 \rrbracket \llbracket v_2 \rrbracket \text{ (reify } c \text{)}.$$

Lemma (Substitution)

Let σ and σ' be value substitutions such that σ' is equal to σ ; $\langle \cdot \rangle$. Then,

$$(\llbracket t \rrbracket \{ c \})[\sigma'] = \llbracket t[\sigma] \rrbracket \{ c[\sigma'] \}.$$

Lemma (Kubstitution)

Let θ and σ be substitutions such that θ ; σ is id. Then,

$$\llbracket (t[\theta]) \rrbracket \{ c \}[\sigma] = \llbracket t \rrbracket \{ c[\sigma] \}.$$

Lemma (Magic Step)

$$\llbracket t \rrbracket \{ o \text{ (reify } c \text{)} \} \longrightarrow_{cbv}^? \llbracket t \rrbracket \{ c \}.$$

Proof of Simulation – cases AppL and AppR

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Case: $t_1 \ u \longrightarrow_{cbv} t_2 \ u$, where $t_1 \longrightarrow_{cbv} t_2$.

We must show $\llbracket t_1 \ u \rrbracket \{c\} \longrightarrow_{cbv}^+ \llbracket t_2 \ u \rrbracket \{c\}$.

By definition of the CPS transformation, this is

$$\longrightarrow_{cbv}^+ \begin{array}{l} \llbracket t_1 \rrbracket \{m \ \llbracket \uparrow^1 u \rrbracket \{m \ 1 \ 0 \ \uparrow^2 (reify \ c)\} \} \\ \llbracket t_2 \rrbracket \{m \ \llbracket \uparrow^1 u \rrbracket \{m \ 1 \ 0 \ \uparrow^2 (reify \ c)\} \} \end{array}$$

Proof of Simulation – cases AppL and AppR

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Case: $t_1 \ u \longrightarrow_{cbv} t_2 \ u$, where $t_1 \longrightarrow_{cbv} t_2$.

We must show $\llbracket t_1 \ u \rrbracket \{c\} \longrightarrow_{cbv}^+ \llbracket t_2 \ u \rrbracket \{c\}$.

By definition of the CPS transformation, this is

$$\longrightarrow_{cbv}^+ \begin{array}{l} \llbracket t_1 \rrbracket \{m \ \uparrow^1 u\} \{m \ 1 \ 0 \ \uparrow^2 (reify \ c)\} \} \\ \llbracket t_2 \rrbracket \{m \ \uparrow^1 u\} \{m \ 1 \ 0 \ \uparrow^2 (reify \ c)\} \} \end{array}$$

Wow – the **induction hypothesis applies** directly to this goal!

Indeed, $reify \ (m \ \dots)$ is a λ -abstraction, therefore a value.

This proof case is complete!

Case: $v \ u_1 \longrightarrow_{cbv} v \ u_2$, where $u_1 \longrightarrow_{cbv} u_2$.

Analogous to the previous case, using a Value-Term Application lemma.

We see in these proof cases that **reduction under a context** in the source program is translated to **reduction at the root** in the transformed program.

Simulation in the presence of *let* constructs

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In the presence of “*let*” constructs, Simulation breaks down.

Challenge: can you find a (minimal) counter-example?

Hint: Enlist a machine’s help. (See next two slides.)

Enumerating λ -terms

Define the **size** of a term as follows: variables have size 0; λ -abstractions and applications contribute 1.

Step 1: In OCaml, implement an exhaustive **enumeration** of the λ -terms of size s and with at most n free variables. (Given as an exercise in week 1.)

```
(* Enumerate all variables between 0 and n excluded. *)  
let var (n : int) (k : term -> unit) : unit = ...  
(* Enumerate all manners of splitting an integer s. *)  
let split (s : int) (k : int -> int -> unit) : unit = ...  
(* Enumerate all terms of size s with at most n variables. *)  
let term (s : int) (n : int) (k : term -> unit) : unit = ...
```

An enumerator is naturally written in CPS style!

Step 2: In OCaml, implement the CPS transformation.

```
type continuation =  
  | O of term  
  | M of term  
let cps (t : term) (c : continuation) : term = ...
```

Step 3: In OCaml, implement a test for the relation $\cdot \longrightarrow_{cbv}^* \cdot$:

```
let reduces (t1 : term) (t2 : term) : bool = ...
```

Hint: Re-use the auxiliary functions of week 2. See [Lambda](#).

Step 4: Find a term t_1 of minimal size that violates Simulation.

Solution: see [CPSCounterExample](#).

Fixing Simulation

In the presence of “*let*”, Simulation can be fixed as follows:

$$\begin{array}{ccc}
 t_1 & \xrightarrow{\quad cbv \quad} & t_2 \\
 \left| \begin{array}{c} \llbracket \cdot \rrbracket \{c\} \\ \hline \llbracket t_1 \rrbracket \{c\} \end{array} \right. & & \left| \begin{array}{c} \llbracket \cdot \rrbracket \{c\} \\ \hline \llbracket t_2 \rrbracket \{c\} \end{array} \right. \\
 & \begin{array}{c} \cdots \xrightarrow[\quad cbv \quad]{+} \cdot \cdots \xrightarrow[\quad cbv \quad]{} \end{array} &
 \end{array}$$

We allow one step of **parallel call-by-value reduction** \Rightarrow_{cbv} .

The proof of Simulation is more complex; see [CPSSimulation](#).

Parallel (call-by-value) reduction

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Parallel reduction allows reducing **all** (currently visible) redexes at once, including under “ λ ” and in the right-hand side of “*let*”.

$$\begin{array}{c}
 \text{PARALLEL } \beta_v \\
 \frac{t_1 \Rightarrow_{cbv} t_2 \quad v_1 \Rightarrow_{cbv} v_2}{(\lambda t_1) v_1 \Rightarrow_{cbv} t_2[v_2/]}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{PARALLEL } let_v \\
 \frac{t_1 \Rightarrow_{cbv} t_2 \quad v_1 \Rightarrow_{cbv} v_2}{let\ v_1\ in\ t_1 \Rightarrow_{cbv} t_2[v_2/]}
 \end{array}
 \qquad
 X \Rightarrow_{cbv} X$$

$$\frac{t_1 \Rightarrow_{cbv} t_2}{\lambda t_1 \Rightarrow_{cbv} \lambda t_2}
 \qquad
 \frac{t_1 \Rightarrow_{cbv} t_2 \quad u_1 \Rightarrow_{cbv} u_2}{t_1\ u_1 \Rightarrow_{cbv} t_2\ u_2}
 \qquad
 \frac{t_1 \Rightarrow_{cbv} t_2 \quad u_1 \Rightarrow_{cbv} u_2}{let\ t_1\ in\ u_1 \Rightarrow_{cbv} let\ t_2\ in\ u_2}$$

The ability to **reduce under a binder** is needed to fix Simulation.

Call-by-name parallel reduction is studied by **Takahashi (1995)**.

Crary (2009) adapts these results to a call-by-value setting.

Well-behavedness of parallel reduction

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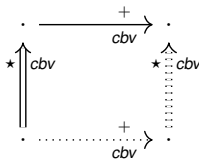
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Lemma (Commutation)

$$(\Rightarrow_{cbv}^* ; \longrightarrow_{cbv}^+) \subseteq (\longrightarrow_{cbv}^+ ; \Rightarrow_{cbv}^*).$$

See [LambdaCalculusStandardization/pcbv_cbv_commutation](#).

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Lemma (Equiconvergence)

$$(\exists v, t \Rightarrow_{cbv}^* v) \iff (\exists v', t \longrightarrow_{cbv}^* v').$$

(The idea is, v' reduces to v via [internal](#) parallel reduction steps.)

See [LambdaCalculusStandardization/equiconvergence](#).

Consequences of Fixed Simulation

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There follows that **divergence** is preserved.

Indeed, from:

$$t \longrightarrow_{cbv} \cdot \longrightarrow_{cbv} \cdots$$

we get:

$$\llbracket t \rrbracket \{c\} \longrightarrow_{cbv}^+ \cdot \Rightarrow_{cbv} \cdot \longrightarrow_{cbv}^+ \cdot \Rightarrow_{cbv} \cdots$$

which, by Commutation, yields:

$$\llbracket t \rrbracket \{c\} \longrightarrow_{cbv}^+ \cdot \xrightarrow{\text{blue}}_{cbv}^+ \cdot \Rightarrow_{cbv}^* \cdot \Rightarrow_{cbv} \cdots$$

that is,

$$\llbracket t \rrbracket \{c\} \longrightarrow_{cbv}^{\geq 2} \cdot \Rightarrow_{cbv}^* \cdots$$

And so on. For an arbitrary $n \geq 0$, we have:

$$\llbracket t \rrbracket \{c\} \longrightarrow_{cbv}^{\geq n} \cdot \Rightarrow_{cbv}^* \cdots$$

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Convergence to a value is preserved, too.

Indeed, from:

$$t \longrightarrow_{cbv}^n v$$

we get, as on the previous slide:

$$\llbracket t \rrbracket \{ done \} \longrightarrow_{cbv}^{\geq n} \cdot \Rightarrow_{cbv}^{\star} (\llbracket v \rrbracket)$$

and, by Equiconvergence:

$$\exists v' \quad \llbracket t \rrbracket \{ done \} \longrightarrow_{cbv}^{\geq n} \cdot \longrightarrow_{cbv}^{\star} v'$$

The CPS transformation remains **semantics-preserving** in the presence of “*let*” constructs (phew!).