Category $(\mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{C}_0, \mathbb{C}_1, \ldots \in \mathbb{CAT})$

- A class of objects $|\mathbb{C}|$ $(C, D, E, \ldots \in |\mathbb{C}|)$
- A class of morphisms $\mathbb{C}(C, D)$ from object C to object D
  $(f, g, h, \ldots \in \mathbb{C}(C, D))$
  $(f, g, h, \ldots \in C \to D)$

with:

- an identity morphism $\mathsf{id}_C \in \mathbb{C}(C, C)$
- a composition operation
  $- \circ - \in \mathbb{C}(D, E) \to \mathbb{C}(C, D) \to \mathbb{C}(C, E)$

such that:

- $\mathsf{id}_C \circ f = f \circ \mathsf{id}_D = f$ *(unity, left & right)*
  for any $f \in \mathbb{C}(C, D)$
- $(f \circ g) \circ h = f \circ (g \circ h)$ *(associativity)*
  for any $h \in \mathbb{C}(C, D)$, $g \in \mathbb{C}(D, E)$ and $f \in \mathbb{C}(E, F)$

Opposite category

The opposite category $\mathbb{C}^{\mathsf{op}}$ consists of:

- Objects $|\mathbb{C}^{\mathsf{op}}| = |\mathbb{C}|$
- Morphisms $\mathbb{C}^{\mathsf{op}}(C, D) = \mathbb{C}(D, C)$

Terminal object

- An object $\top \in |\mathbb{C}|$

such that for every

- object $X \in |\mathbb{C}|$

there exists a unique morphism $\mathsf{tt} \in \mathbb{C}(X, \top)$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists ! \mathsf{tt}"] \\ \top \end{tikzcd}$$

Cartesian product

Let $C, D \in |\mathbb{C}|$.

- An object $C \times D \in |\mathbb{C}|$
- A morphism $\pi_1 \in \mathbb{C}(C \times D, C)$
- A morphism $\pi_2 \in \mathbb{C}(C \times D, D)$

such that for every

- $X \in |\mathbb{C}|$
- $f \in \mathbb{C}(X, C)$
- $g \in \mathbb{C}(X, D)$

there exists a unique morphism $[f, g] \in \mathbb{C}(X, C \times D)$ verifying

$$\begin{tikzcd}
& X \arrow[d, dashed, "\exists ! [f, g]"] \arrow[ddl, bend right, "\forall f"'] \arrow[ddr, bend left, "\forall g"] & \\
& C \times D \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
C & & D
\end{tikzcd}$$

Exponential

Let $C \in |\mathbb{C}|$. Let $D \in |\mathbb{C}|$ with all Cartesian products.

- An object $C^D \in |\mathbb{C}|$
- A morphism $\mathsf{apply} \in \mathbb{C}(C^D \times D, C)$

such that for every

- $X \in |\mathbb{C}|$
- $f \in \mathbb{C}(X \times D, C)$

there exists a unique morphism $\lambda\, f \in \mathbb{C}(X, C^D)$ verifying

$$
\begin{tikzcd}
X \arrow[d, dashed, "\exists! \lambda\, f"'] \\
C^D
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times D \arrow[d, dashed, "{[\lambda\, f\, , \mathsf{id}_D]}"'] \arrow[dr, "\forall f"] & \\
C^D \times D \arrow[r, "\mathsf{apply}"'] & C
\end{tikzcd}
$$

Cartesian-closed category

A category $\mathbb{C}$ with

- A terminal object $\top$
- A Cartesian product $C \times D$ for any $C, D \in |\mathbb{C}|$
- An exponential object C^D for any $C, D \in |\mathbb{C}|$

Functor $(\mathcal{F}, \mathcal{G}, \mathcal{F}_0, \mathcal{F}_1, \ldots \in \mathbb{C} \Rightarrow \mathbb{D})$

- An action on objects $|\mathcal{F}| \in |\mathbb{C}| \to |\mathbb{D}|$
- An action on morphisms $\mathcal{F}^{\to} \in \mathbb{C}(C, C') \to \mathbb{D}(|\mathcal{F}|\,C, |\mathcal{F}|\,C')$

such that:

- $\mathcal{F}^{\to}\ \mathsf{id}_C = \mathsf{id}_{|\mathcal{F}|\ C}$ *(identity preservation)*
- $\mathcal{F}^{\to}\ (g \circ f) = (\mathcal{F}^{\to}\ g) \circ (\mathcal{F}^{\to}\ f)$ *(composition preservation)*

Adjunction

Let $\mathcal{F} \in \mathbb{D} \Rightarrow \mathbb{C}$.
$\mathcal{F}$ is a left adjoint if for every

- Object $C \in |\mathbb{C}|$

there exists

- An object $|\mathcal{G}|C \in |\mathbb{D}|$
- A morphism $\epsilon_C \in \mathbb{C}(|\mathcal{F}|(|\mathcal{G}|(C)), C)$

such that for every

- object $D \in |\mathbb{D}|$
- morphism $f \in \mathbb{C}(|\mathcal{F}|\ D, C)$

there exists a unique morphism $g \in \mathbb{D}(D, |\mathcal{G}|\ C)$ with

$$\begin{tikzcd}
D \arrow[d, dashed, "\exists! g"'] \\
|\mathcal{G}|(C)
\end{tikzcd}
\qquad
\begin{tikzcd}
|\mathcal{F}|\ D \arrow[d, dashed, "\mathcal{F}^{\rightarrow}\ g"'] \arrow[dr, "\forall f"] & \\
|\mathcal{F}|(|\mathcal{G}|(C)) \arrow[r, "\epsilon_C"'] & C
\end{tikzcd}$$

Natural transformation $(\varphi, \psi, \varphi_0, \varphi_1, \ldots \in \mathcal{F} \stackrel{.}{\Rightarrow} \mathcal{G})$

Let $\mathcal{F}, \mathcal{G} \in \mathbb{C} \Rightarrow \mathbb{D}$.

- A transformation $\varphi \in \forall C.\, \mathbb{D}(|\mathcal{F}|\ C, |\mathcal{G}|\ C)$

such that for every

- $k \in \mathbb{C}(C, D)$

we have

$$\begin{tikzcd}
|\mathcal{F}|\ C \arrow[r, "\varphi_C"] \arrow[d, "\mathcal{F}^{\rightarrow}\ k"'] & |\mathcal{G}|\ C \arrow[d, "\mathcal{G}^{\rightarrow}\ k"] \\
|\mathcal{F}|\ D \arrow[r, "\varphi_D"'] & |\mathcal{G}|\ D
\end{tikzcd}$$

Adjunction, take II

Let $\mathcal{F} \in \mathbb{D} \Rightarrow \mathbb{C}$ and $\mathcal{G} \in \mathbb{C} \Rightarrow \mathbb{D}$.

$\mathcal{F}$ is left adjoint to $\mathcal{G}$ ($\mathcal{F} \dashv \mathcal{G}$) if there exists

- a natural isomorphism $\varphi \in \mathbb{C}(|\mathcal{F}|\ D, C) \xrightarrow{\sim} \mathbb{D}(D, |\mathcal{G}|\ C)$

Conversely, $\mathcal{G}$ is right adjoint to $\mathcal{F}$.

Monad

- A functor $\mathcal{M} \in \mathbb{C} \Rightarrow \mathbb{C}$
- A natural transformation $\eta \in \forall C.\ C \to |\mathcal{M}|\ C$
- A natural transformation $\mu \in \forall C.\ |\mathcal{M}|(|\mathcal{M}|\ C) \to |\mathcal{M}|\ C$

such that

$$\begin{tikzcd}
|\mathcal{M}|^3\ C \arrow[r, "|\mathcal{M}|\mu_C"] \arrow[d, "\mu_{|\mathcal{M}|\ C}"'] & |\mathcal{M}|^2\ C \arrow[d, "\mu_C"] \\
|\mathcal{M}|^2\ C \arrow[r, "\mu_C"] & |\mathcal{M}|\ C
\end{tikzcd}
\qquad
\begin{tikzcd}
|\mathcal{M}|\ C \arrow[r, "\eta_{|\mathcal{M}|\ C}"] \arrow[d, "\eta_C"] \arrow[dr, equal] & |\mathcal{M}|^2\ C \arrow[d, "\mu_C"] \\
|\mathcal{M}|^2\ C \arrow[r, "\mu_C"] & |\mathcal{M}|\ C
\end{tikzcd}$$

Kleisli category

Let $\mathcal{M}$ be a monad over $\mathbb{C}$.

The Kleisli category $\mathbb{C}_{\mathcal{M}}$ consists of:

- Objects: $|\mathcal{M}|\ C$ for every $C \in |\mathbb{C}|$
- Morphisms: $\mathbb{C}_{\mathcal{M}}(C, C') = \mathbb{C}(C, |\mathcal{M}|\ C')$

Presheaf

A presheaf on a category $\mathbb{C}$ is a functor $\hat{\mathbb{C}} \in \mathbb{C} \Rightarrow \mathbb{SET}$.

In particular, it forms a functor category:

- Objects: presheaf functors
- Morphisms: natural transformations

Further reading

- "Conceptual Mathematics", Schanuel & Lawvere
- "An introduction to Category Theory", Simmons
- "Categories for Types", Crole
- "Categories for the Working Mathematician", Mac Lane
- "Sheaves in Geometry and Logic", Mac Lane & Moerdijk