MPRI 2.4

From operational semantics to interpreters

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The λ-calculus

The formal model that underlies all functional programming languages.

Abstract syntax:

$$t, u ::= x \mid \lambda x.t \mid t\ t \qquad \text{(terms)}$$

Reduction:

$$(\lambda x.t)\ u \longrightarrow t[u/x] \qquad (\beta)$$

Mnemonic: read $t[u/x]$ as "t, where u is substituted for x".

Landin, Correspondence betw. ALGOL 60 and Church's λ-notation, 1965.

> *"It seems possible that the correspondence might form the basis of a formal description of the semantics of Algo 60."*

From the λ-calculus to a functional programming language

Start from the λ-calculus, and follow several steps:

- Fix a reduction strategy (today).
- Develop efficient execution mechanisms (today).
- Enrich the language with primitive data types and operations, recursion, algebraic data structures, and so on.
- Define a static type system (next week).

Landin, The next 700 programming languages, 1966.

"Most programming languages are partly a way of expressing things in terms of other things and partly a basic set of given things."

Operational semantics

An operational semantics describes the actions of a machine,
in the simplest possible manner / at the most abstract level.

Plotkin, A Structural Approach to Operational Semantics, 1981, (2004).

Plotkin, The Origins of Structural Operational Semantics, 2004.

Plotkin: — *It is only through having an operational semantics that the [λ-calculus can] be viewed as a programming language.*

Scott: — *Why call it operational semantics? What is operational about it?*

Denotational semantics

Scott preferred denotational semantics, where the meaning of a program is a mathematical function of an input to an output.

Benton, Kennedy, Varming,
Some Domain Theory and Denotational Semantics in Coq, 2009.

Benton, Birkedal, Kennedy, Varming, Formalizing domains, ultrametric spaces and semantics of programming languages, 2010.

Dockins, Formalized, Effective Domain Theory in Coq, 2014.

The call-by-value strategy

Values form a subset of terms:

$$\begin{array}{rcll} t, u & ::= & x \mid \lambda x.t \mid t\ t & \text{(terms)} \\ v & ::= & x \mid \lambda x.t & \text{(values)} \end{array}$$

A value represents the result of a computation.

The call-by-value reduction relation $t \longrightarrow_{\text{cbv}} t'$ is inductively defined:

$$\beta_v \quad \frac{}{(\lambda x.t)\ v \longrightarrow_{\text{cbv}} t[v/x]} \qquad \text{AppL} \quad \frac{t \longrightarrow_{\text{cbv}} t'}{t\ u \longrightarrow_{\text{cbv}} t'\ u} \qquad \text{AppVR} \quad \frac{u \longrightarrow_{\text{cbv}} u'}{v\ u \longrightarrow_{\text{cbv}} v\ u'}$$

This is known as a small-step operational semantics.

Example

This is a proof (a.k.a. derivation) that one reduction step is permitted:

$$\dfrac{\dfrac{\dfrac{x[1/x] = 1}{(\lambda x.x)\ 1 \longrightarrow_{\text{cbv}} 1}\ \beta_v}{(\lambda x.\lambda y.y\ x)\ ((\lambda x.x)\ 1) \longrightarrow_{\text{cbv}} (\lambda x.\lambda y.y\ x)\ 1}\ \text{AppR}}{(\lambda x.\lambda y.y\ x)\ ((\lambda x.x)\ 1)\ (\lambda x.x) \longrightarrow_{\text{cbv}} (\lambda x.\lambda y.y\ x)\ 1\ (\lambda x.x)}\ \text{AppL}$$

Features of call-by-value reduction

- Weak reduction. One cannot reduce under a λ-abstraction.

$$\xcancel{\dfrac{t \longrightarrow_{\text{cbv}} t'}{\lambda x.t \longrightarrow_{\text{cbv}} \lambda x.t'}}$$

  Thus, values do not reduce.
  Also, we are interested in reducing closed terms only.

- Call-by-value. An actual argument is reduced to a value before it is passed to a function.

$$(\lambda x.t)\ v \longrightarrow_{\text{cbv}} t[v/x] \qquad \xcancel{(\lambda x.t)\ (u_1\ u_2) \longrightarrow_{\text{cbv}} t[u_1\ u_2/x]}$$

Features of call-by-value reduction

- Left-to-right. In an application $t\ u$, the term t must be reduced to a value before u can be reduced at all.

$$\textsc{AppVR}\quad \frac{u \longrightarrow_{\text{cbv}} u'}{v\ u \longrightarrow_{\text{cbv}} v\ u'}$$

- Determinism. For every term t, there is at most one term t' such that $t \longrightarrow_{\text{cbv}} t'$ holds.

Reduction sequences

Sequences of reduction steps describe the behavior of a term.

The following three situations are mutually exclusive:

- Termination: $t \longrightarrow_{\text{cbv}} t_1 \longrightarrow_{\text{cbv}} t_2 \longrightarrow_{\text{cbv}} \ldots \longrightarrow_{\text{cbv}} v$
  The value v is the result of evaluating t.
  The term t converges to v.
- Divergence: $t \longrightarrow_{\text{cbv}} t_1 \longrightarrow_{\text{cbv}} t_2 \longrightarrow_{\text{cbv}} \ldots \longrightarrow_{\text{cbv}} t_n \longrightarrow_{\text{cbv}} \ldots$
  The sequence of reductions is infinite.
  The term t diverges.
- Error: $t \longrightarrow_{\text{cbv}} t_1 \longrightarrow_{\text{cbv}} t_2 \longrightarrow_{\text{cbv}} \ldots \longrightarrow_{\text{cbv}} t_n \not\longrightarrow_{\text{cbv}} \cdot$
  where t_n is not a value, yet does not reduce: t_n is stuck.
  The term t goes wrong. This is a runtime error.

A strong type system rules out errors (Milner, 1978).

Some type systems rule out both errors and divergence.

Examples of reduction sequences

Termination:

$$\begin{aligned}(\lambda x.\lambda y.y\ x)\ ((\lambda x.x)\ 1)\ (\lambda x.x) &\longrightarrow_{\text{cbv}} (\lambda x.\lambda y.y\ x)\ 1\ (\lambda x.x)\\ &\longrightarrow_{\text{cbv}} (\lambda y.y\ 1)\ (\lambda x.x)\\ &\longrightarrow_{\text{cbv}} (\lambda x.x)\ 1\\ &\longrightarrow_{\text{cbv}} 1\end{aligned}$$

Divergence:

$$(\lambda x.x\ x)\ (\lambda x.x\ x) \longrightarrow_{\text{cbv}} (\lambda x.x\ x)\ (\lambda x.x\ x) \longrightarrow_{\text{cbv}} \ldots$$

Error:

$$(\lambda x.x\ x)\ 2 \longrightarrow_{\text{cbv}} 2\ 2 \not\longrightarrow_{\text{cbv}}.$$

The active redex is highlighted in red.

An alternative style: evaluation contexts

First, define head reduction:

$$\frac{}{(\lambda x.t)\ v \longrightarrow^{\text{head}}_{\text{cbv}} t[v/x]}\ \beta_v$$

Then, define reduction as head reduction under an evaluation context:

$$\frac{t \longrightarrow^{\text{head}}_{\text{cbv}} t'}{E[t] \longrightarrow_{\text{cbv}} E[t']}\ \textsc{Ctx}$$

where evaluation contexts E are defined by $E ::= [\,] \mid E\ u \mid v\ E$.

Wright and Felleisen, A syntactic approach to type soundness, 1992.

Unique decomposition

In this alternative style, the determinism of the reduction relation follows from a unique decomposition lemma:

Lemma (Unique Decomposition)

For every term t, there exists at most one pair (E, u) such that

- $t = E[u]$
- $\exists u' \quad u \longrightarrow^{head}_{cbv} u'$.

The call-by-name strategy

The call-by-name reduction relation $t \longrightarrow_{\text{cbn}} t'$ is defined as follows:

$$\frac{}{(\lambda x.t)\; u \longrightarrow_{\text{cbn}} t[u/x]}\;\beta \qquad \frac{t \longrightarrow_{\text{cbn}} t'}{t\; u \longrightarrow_{\text{cbn}} t'\; u}\;\textsc{AppL}$$

The unevaluated actual argument is passed to the function.

It is later reduced if / when / every time the function demands its value.

An example reduction sequence

$$\begin{array}{lll}
(\lambda x.\lambda y.y\ x)\ ((\lambda x.x)\ 1)\ (\lambda x.x) & \longrightarrow_{\text{cbn}} & (\lambda y.y\ ((\lambda x.x)\ 1))\ (\lambda x.x) \\
& \longrightarrow_{\text{cbn}} & (\lambda x.x)\ ((\lambda x.x)\ 1) \\
& \longrightarrow_{\text{cbn}} & (\lambda x.x)\ 1 \\
& \longrightarrow_{\text{cbn}} & 1
\end{array}$$

Call-by-value versus call-by-name

If t terminates under CBV, then it also terminates under CBN (⋆).

The converse is false:

$$\begin{array}{lll}(\lambda x.1)\ \omega & \longrightarrow_{\mathsf{cbn}} & 1 \\ (\lambda x.1)\ \omega & \longrightarrow^{\infty}_{\mathsf{cbv}} & \end{array}$$

where $\omega = (\lambda x.x\ x)\ (\lambda x.x\ x)$ diverges under both strategies.

Call-by-value can perform fewer reduction steps:
$(\lambda x.\ x + x)\ t$ evaluates t once under CBV, twice under CBN.

Call-by-name can perform fewer reduction steps:
$(\lambda x.\ 1)\ t$ evaluates t once under CBV, not at all under CBN.

(⋆) In fact, the standardization theorem implies that
if t can be reduced to a value via any strategy,
then it can be reduced to a value via CBN.
See Takahashi (1995).

Encoding call-by-name in a CBV language

Use thunks: functions $\lambda\_.u$ whose purpose is to delay the evaluation of u.

$$\begin{array}{rcl} [\![x]\!] & = & x\ () \\ [\![\lambda x.t]\!] & = & \lambda x.[\![t]\!] \\ [\![t\ u]\!] & = & [\![t]\!]\ (\lambda\_.[\![u]\!]) \end{array}$$

Exercise: Can you state that this encoding is correct? Can you prove it?
— 2017 exam! (paper assignment and solution) (Coq solution)

Encoding call-by-name in a CBV language

In a simply-typed setting, this transformation is type-preserving: that is,

$$\Gamma \vdash t : T \quad \text{implies} \quad [\![\Gamma]\!] \vdash [\![t]\!] : [\![T]\!].$$

The translation of types is defined by

$$[\![T_1 \to T_2]\!] = \mathit{thunk}\ [\![T_1]\!] \to [\![T_2]\!]$$

where *thunk* T is $\mathit{unit} \to T$.

The translation of type environments is as follows:
$[\![x_1 : T_1; \ldots; x_n : T_n]\!]$ stands for $x_1 : \mathit{thunk}\ [\![T_1]\!]; \ldots; x_n : \mathit{thunk}\ [\![T_n]\!]$.

Encoding call-by-value in a CBN language

The reverse encoding is somewhat more involved.

The call-by-value continuation-passing style (CPS) transformation, studied later on in this course, achieves such an encoding.

Call-by-push-value

Levy: — *The existence of two separate paradigms is troubling*.

Levy proposes call-by-push-value,
a lower-level calculus into which both CBV and CBN can be encoded,
thus avoiding a certain amount of duplication between their theories.

Levy, Call-by-Push-Value: A Subsuming Paradigm, 1999.

Forster et al., Call-By-Push-Value in Coq: Operational, Equational, and Denotational Theory, 2018.

Call-by-need

Call-by-need, a.k.a. lazy evaluation, eliminates the main inefficiency of call-by-name (namely, repeated computation) by introducing memoization.

Its description via an operational semantics involves:

- either mutable state and sharing (Ariola and Felleisen, 1997; Maraist, Odersky, Wadler, 1998);
- or nondeterminism: “call-by-need is clairvoyant call-by-value” (Hackett and Hutton, 2019).

It is used in Haskell, where it encourages a modular style of programming.

Hughes, Why functional programming matters, 1990.

Also see Harper’s and Augustsson’s blog posts on laziness.

Newton-Raphson iteration (after Hughes)

This is pseudo-Haskell code. The colon `:` is “cons”.

An approximation of the square root of `n` can be computed as follows:

```
next n x = (x + n / x) / 2
repeat f a = a : (repeat f (f a))
within eps (a : b : rest) =
  if abs (a - b) <= eps then b
  else within eps (b : rest)
sqrt a0 eps n =
  within eps (repeat (next n) a0)
```

`repeat (next n) a0` is a producer of an infinite stream of numbers.

Its type is just “list of numbers” – look Ma, no iterators à la Java!

The consumer `within eps` decides how many elements to demand.

The two are programmed independently.

Encoding call-by-need in a CBV language

Call-by-need can be encoded into CBV by using memoizing thunks:

$$\begin{array}{rcl} [\![x]\!] & = & \mathit{force}\ x \\ [\![\lambda x.t]\!] & = & \lambda x.[\![t]\!] \\ [\![t\ u]\!] & = & [\![t]\!]\ (\mathit{suspend}\ (\lambda\_.[\![u]\!])) \end{array}$$

Such a thunk evalutes u when first forced,
then memoizes the result,
so no computation is required if the thunk is forced again.

Thunks can be thought of as an abstract type with this API or signature:

```
type 'a thunk
val suspend: (unit -> 'a) -> 'a thunk
val force: 'a thunk -> 'a
```

Encoding call-by-need in a CBV language

Exercise: implement the thunk API in OCaml. (Solution.)

In reality, this exercise is unnecessary, as OCaml has built-in thunks:

- "*suspend* $(\lambda\_.u)$" is written `lazy u`.
- "*force x*" is written `Lazy.force x`.

Exercise: port Newton-Raphson iteration to OCaml.
Make sure that each element is computed at most once
and no more elements than necessary are computed.
Write tests to verify these properties. (Solution.)

A naïve interpreter

An interpreter executes a program (represented by its AST).

Let us write one, in OCaml, by paraphrasing the small-step semantics.

Abstract syntax

This is the abstract syntax of the λ-calculus:

```
type var = int (* a de Bruijn index *)
type term =
  | Var of var
  | Lam of (* bind: *) term
  | App of term * term
```

For example, the term $\lambda x.x$ is represented as follows:

```
let id =
  Lam (Var 0)
```

Renaming

`lift_ i k` represents the renaming $\Uparrow^i (+k)$.

```
let rec lift_ i k (t : term) : term =
  match t with
  | Var x ->
      if x < i then t else Var (x + k)
  | Lam t ->
      Lam (lift_ (i + 1) k t)
  | App (t1, t2) ->
      App (lift_ i k t1, lift_ i k t2)

let lift k t =
  lift_ 0 k t
```

Thus, `lift k` represents $+k$. (This renaming adds k to every variable.)

It is used when one moves the term t down into k binders. (Next slide.)

Substitution

`subst_ i sigma` represents the substitution $\Uparrow^i \sigma$.

```
let rec subst_ i (sigma : var -> term) (t : term) : term =
  match t with
  | Var x ->
      if x < i then t else lift i (sigma (x - i))
  | Lam t ->
      Lam (subst_ (i + 1) sigma t)
  | App (t1, t2) ->
      App (subst_ i sigma t1, subst_ i sigma t2)

let subst sigma t =
  subst_ 0 sigma t
```

Thus, `subst sigma` represents σ.

Substitution

A substitution is encoded as a total function of variables to terms.

```
let singleton (u : term) : var -> term =
  function 0 -> u | x -> Var (x - 1)
```

`singleton u` represents the substitution $u \cdot id$.

Recognizing values

It is easy to test whether a term is a value:

```
let is_value = function
  | Var _ | Lam _ -> true
  | App _         -> false
```

Performing one step of reduction

A direct transcription of Plotkin's definition of call-by-value reduction:

```
let rec step (t : term) : term option =
  match t with
  | Lam _ | Var _ -> None
  (* Plotkin's BetaV *)
  | App (Lam t, v) when is_value v ->
      Some (subst (singleton v) t)
  (* Plotkin's AppL *)
  | App (t, u) when not (is_value t) ->
      in_context (fun t' -> App (t', u)) (step t)
  (* Plotkin's AppVR *)
  | App (v, u) when is_value v ->
      in_context (fun u' -> App (v, u')) (step u)
  (* All cases covered already, but OCaml cannot see it. *)
  | App (_, _) ->
      assert false
```

We have guarded AppL so that AppL and AppVR are mutually exclusive.

Performing one step of reduction

`in_context` is just the `map` combinator of the type `_ option`.

```
let in_context f ox =
  match ox with
  | None ->   None
  | Some x -> Some (f x)
```

Performing many steps of reduction

To evaluate a term, one performs as many reduction steps as possible:

```
let rec eval (t : term) : term =
  match step t with
  | None ->
      t
  | Some t' ->
      eval t'
```

The function call `eval t` either diverges or returns an irreducible term, which must be either a value or stuck.

Sources of inefficiency

Unfortunately, this is a terribly inefficient way of interpreting programs.

At each reduction step, one must:

- Find the next redex, that is, decompose the term t as $E[\lambda(x.u)\ v]$.
- Perform the substitution $u[v/x]$.
- Construct the term $E[u[v/x]]$.

The time required to do this is not O(1). Why?

There seem to be two main sources of inefficiency:

- We keep forgetting the current evaluation context, only to discover it again at the next reduction step.
- We perform costly substitutions.

Towards an alternative to small steps

A reduction sequence from an application $t_1\ t_2$ to a final value v always has the form:

$$t_1\ t_2 \longrightarrow^{\star}_{\mathrm{cbv}} (\lambda x.u_1)\ t_2 \longrightarrow^{\star}_{\mathrm{cbv}} (\lambda x.u_1)\ v_2 \longrightarrow_{\mathrm{cbv}} u_1[v_2/x] \longrightarrow^{\star}_{\mathrm{cbv}} v$$

where $t_1 \longrightarrow^{\star}_{\mathrm{cbv}} \lambda x.u_1$ and $t_2 \longrightarrow^{\star}_{\mathrm{cbv}} v_2$. That is,

Evaluate operator; evaluate operand; call; continue execution.

Idea: define a “big-step” relation $t \downarrow_{\mathrm{cbv}} v$,
which relates a term directly with the final outcome v of its evaluation, and whose definition reflects the above structure.

Natural semantics, a.k.a. big-step semantics

The relation $t \downarrow_{\text{cbv}} v$ means that evaluating t terminates and produces v.

Here is its definition, for call-by-value:

$$\textsc{BigCbvValue}\quad \frac{}{v \downarrow_{\text{cbv}} v} \qquad\qquad \textsc{BigCbvApp}\quad \frac{t_1 \downarrow_{\text{cbv}} \lambda x.u_1 \qquad t_2 \downarrow_{\text{cbv}} v_2 \qquad u_1[v_2/x] \downarrow_{\text{cbv}} v}{t_1\; t_2 \downarrow_{\text{cbv}} v}$$

Exercise: define $\downarrow_{\text{cbn}}$.

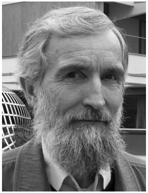
Example

Let us write $\downarrow$ for $\downarrow_{\text{cbv}}$, and “$v \downarrow \cdot$ ” for “$v \downarrow v$”.

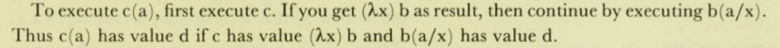
$$\dfrac{\dfrac{\lambda x.\lambda y.y\;x \downarrow \cdot \qquad \dfrac{\begin{array}{c}\lambda x.x \downarrow \cdot \\ 1 \downarrow \cdot \\ 1 \downarrow \cdot\end{array}}{(\lambda x.x)\;1 \downarrow 1} \qquad \lambda y.y\;1 \downarrow \cdot}{(\lambda x.\lambda y.y\;x)\;((\lambda x.x)\;1) \downarrow \lambda y.y\;1} \qquad \lambda x.x \downarrow \cdot \qquad \dfrac{\begin{array}{c}\lambda x.x \downarrow \cdot \\ 1 \downarrow \cdot \\ 1 \downarrow \cdot\end{array}}{(\lambda x.x)\;1 \downarrow 1}}{(\lambda x.\lambda y.y\;x)\;((\lambda x.x)\;1)\;(\lambda x.x) \downarrow 1}$$

Whereas a proof of $t \longrightarrow_{\text{cbv}} t'$ has linear structure,
a proof of $t \downarrow_{\text{cbv}} v$ has tree structure.

Some history



Martin-Löf uses big-step semantics, in English:



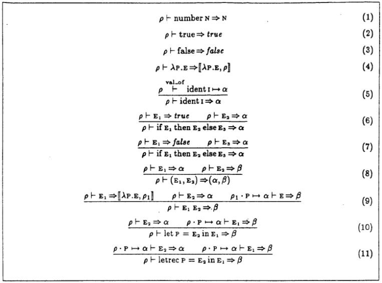
To execute c(a), first execute c. If you get (λx) b as result, then continue by executing b(a/x). Thus c(a) has value d if c has value (λx) b and b(a/x) has value d.

He proposes type theory (1975) as a very high-level programming language in which both programs and specifications can be written.

Per Martin-Löf,
Constructive Mathematics and Computer Programming, 1984.

Some history

Kahn promotes big-step operational semantics:



$$\rho \vdash \mathrm{number}\ \mathrm{N} \Rightarrow \mathrm{N} \tag{1}$$

$$\rho \vdash \mathrm{true} \Rightarrow \mathit{true} \tag{2}$$

$$\rho \vdash \mathrm{false} \Rightarrow \mathit{false} \tag{3}$$

$$\rho \vdash \lambda \mathrm{P.E} \Rightarrow [\![\lambda \mathrm{P.E}, \rho]\!] \tag{4}$$

$$\frac{\rho \overset{\mathrm{val\_of}}{\vdash} \mathrm{ident}\ \mathrm{I} \mapsto \alpha}{\rho \vdash \mathrm{ident}\ \mathrm{I} \Rightarrow \alpha} \tag{5}$$

$$\frac{\rho \vdash \mathrm{E}_1 \Rightarrow \mathit{true} \qquad \rho \vdash \mathrm{E}_2 \Rightarrow \alpha}{\rho \vdash \mathrm{if}\ \mathrm{E}_1\ \mathrm{then}\ \mathrm{E}_2\ \mathrm{else}\ \mathrm{E}_3 \Rightarrow \alpha} \tag{6}$$

$$\frac{\rho \vdash \mathrm{E}_1 \Rightarrow \mathit{false} \qquad \rho \vdash \mathrm{E}_3 \Rightarrow \alpha}{\rho \vdash \mathrm{if}\ \mathrm{E}_1\ \mathrm{then}\ \mathrm{E}_2\ \mathrm{else}\ \mathrm{E}_3 \Rightarrow \alpha} \tag{7}$$

$$\frac{\rho \vdash \mathrm{E}_1 \Rightarrow \alpha \qquad \rho \vdash \mathrm{E}_2 \Rightarrow \beta}{\rho \vdash (\mathrm{E}_1, \mathrm{E}_2) \Rightarrow (\alpha, \beta)} \tag{8}$$

$$\frac{\rho \vdash \mathrm{E}_1 \Rightarrow [\![\lambda \mathrm{P.E}, \rho_1]\!] \qquad \rho \vdash \mathrm{E}_2 \Rightarrow \alpha \qquad \rho_1 \cdot \mathrm{P} \mapsto \alpha \vdash \mathrm{E} \Rightarrow \beta}{\rho \vdash \mathrm{E}_1\ \mathrm{E}_2 \Rightarrow \beta} \tag{9}$$

$$\frac{\rho \vdash \mathrm{E}_2 \Rightarrow \alpha \qquad \rho \cdot \mathrm{P} \mapsto \alpha \vdash \mathrm{E}_1 \Rightarrow \beta}{\rho \vdash \mathrm{let}\ \mathrm{P} = \mathrm{E}_2\ \mathrm{in}\ \mathrm{E}_1 \Rightarrow \beta} \tag{10}$$

$$\frac{\rho \cdot \mathrm{P} \mapsto \alpha \vdash \mathrm{E}_2 \Rightarrow \alpha \qquad \rho \cdot \mathrm{P} \mapsto \alpha \vdash \mathrm{E}_1 \Rightarrow \beta}{\rho \vdash \mathrm{letrec}\ \mathrm{P} = \mathrm{E}_2\ \mathrm{in}\ \mathrm{E}_1 \Rightarrow \beta} \tag{11}$$

Figure 2. The dynamic semantics of mini-ML

He gives a big-step operational semantics of MiniML, a static type system, and a compilation scheme towards the CAM.

Gilles Kahn, Natural semantics, 1987.

A big-step interpreter

The call `eval t` attempts to compute a value v such that $t \downarrow_{\text{cbv}} v$ holds.

```
exception RuntimeError
let rec eval (t : term) : term =
  match t with
  | Lam _ | Var _ -> t
  | App (t1, t2) ->
      let v1 = eval t1 in
      let v2 = eval t2 in
      match v1 with
      | Lam u1 -> eval (subst (singleton v2) u1)
      | _      -> raise RuntimeError
```

If `eval` terminates normally, then it obviously returns a value;
but it can also fail to terminate or terminate with a runtime error. (Why?)

This interpreter does not forget and rediscover the evaluation context.
The context is now implicit in the interpreter's stack!

We could prove this interpreter correct, but will first optimize it further.

Equivalence between small-step and big-step semantics

Lemma (From big-step to small-step)

If $t \downarrow_{cbv} v$, then $t \longrightarrow^{\star}_{cbv} v$.

Proof.

By induction on the derivation of $t \downarrow_{\text{cbv}} v$.
Case BigCbvValue. We have $t = v$. The result is immediate.
Case BigCbvApp. t is $t_1\ t_2$, and we have three subderivations:

$$t_1 \downarrow_{\text{cbv}} \lambda x.u_1 \qquad t_2 \downarrow_{\text{cbv}} v_2 \qquad u_1[v_2/x] \downarrow_{\text{cbv}} v$$

Applying the ind. hyp. to them yields three reduction sequences:

$$t_1 \longrightarrow^{\star}_{\text{cbv}} \lambda x.u_1 \qquad t_2 \longrightarrow^{\star}_{\text{cbv}} v_2 \qquad u_1[v_2/x] \longrightarrow^{\star}_{\text{cbv}} v$$

By reducing under an evaluation context and by chaining, we obtain:

$$t_1\ t_2 \longrightarrow^{\star}_{\text{cbv}} (\lambda x.u_1)\ t_2 \longrightarrow^{\star}_{\text{cbv}} (\lambda x.u_1)\ v_2 \longrightarrow_{\text{cbv}} u_1[v_2/x] \longrightarrow^{\star}_{\text{cbv}} v$$

See `LambdaCalculusBigStep/bigcbv_star_cbv`. □

Equivalence between small-step and big-step semantics

Lemma (From small-step to big-step, preliminary)

If $t_1 \longrightarrow_{cbv} t_2$ and $t_2 \downarrow_{cbv} v$, then $t_1 \downarrow_{cbv} v$.

Proof (Sketch).

By induction on the first hypothesis and case analysis on the second hypothesis. See `LambdaCalculusBigStep/cbv_bigcbv_bigcbv`. □

Lemma (From small-step to big-step)

If $t \longrightarrow^{\star}_{cbv} v$, then $t \downarrow_{cbv} v$.

Proof.

By induction on the first hypothesis, using $v \downarrow_{\text{cbv}} v$ in the base case and the above lemma in the inductive case.
See `LambdaCalculusBigStep/star_cbv_bigcbv`. □

An alternative to naïve substitution

A basic need is to record that x is bound to v while evaluating a term t.

So far, we have used an eager substitution, $t[v/x]$, but:

- This is inefficient.
- This does not respect the separation between immutable code and mutable data imposed by current hardware and operating systems.

Idea: instead of applying the substitution [v/x] to the code,
record the binding $x \mapsto v$ in a data structure, known as an environment.

An environment is a finite map of variables to (closed) values.

A first attempt

Let us try and define a new big-step evaluation judgement, $e \vdash t \downarrow_{\text{cbv}} v$.

(previous definition)

$$\textsc{BigCbvValue}\quad \frac{}{v \downarrow_{\text{cbv}} v}$$

$$\textsc{BigCbvApp}\quad \frac{t_1 \downarrow_{\text{cbv}} \lambda x.u_1 \qquad t_2 \downarrow_{\text{cbv}} v_2 \qquad u_1[v_2/x] \downarrow_{\text{cbv}} v}{t_1\ t_2 \downarrow_{\text{cbv}} v}$$

(attempt at a new definition)

$$\textsc{EBigCbvVar}\quad \frac{e(x) = v}{e \vdash x \downarrow_{\text{cbv}} v} \qquad \textsc{EBigCbvLam}\quad \frac{}{e \vdash \lambda x.t \downarrow_{\text{cbv}} \lambda x.t}$$

$$\textsc{EBigCbvApp}\quad \frac{e \vdash t_1 \downarrow_{\text{cbv}} \lambda x.u_1 \qquad e \vdash t_2 \downarrow_{\text{cbv}} v_2 \qquad e[x \mapsto v_2] \vdash u_1 \downarrow_{\text{cbv}} v}{e \vdash t_1\ t_2 \downarrow_{\text{cbv}} v}$$

What is wrong with this definition?

In $t \downarrow_{\text{cbv}} v$, both t and v are closed.

In $e \vdash t \downarrow_{\text{cbv}} v$, we expect $fv(t) \subseteq dom(e)$. What about v? Is it closed?

What about the values stored in e? Are they closed? ...

Lexical scoping versus dynamic scoping

What value should the following OCaml code produce?

```
let x = 42 in
let f = fun () -> x in
let x = "oops" in
f()
```

Well,

- The answer is `42`. This is lexical scoping. This is λ-calculus.
- The answer is not `"oops"`. That would be dynamic scoping.

Thus, the free variables of a λ-abstraction must be evaluated:

- in the environment that exists at the function's creation site,
- not in the environment that exists at the function's call site.

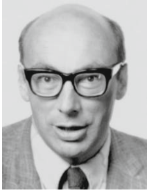
A failed attempt

Thus, our first attempt is wrong:

- It implements dynamic scoping instead of lexical scoping.
- If $e \vdash t \downarrow_{\text{cbv}} v$ and $fv(t) \subseteq dom(e)$ then we would expect that v is closed and $t[e] \downarrow_{\text{cbv}} v$ holds — but that is not the case.
- The candidate rule EBigCbvLam obviously violates this property. It fails to record the environment that exists at function creation time.

How can we fix the problem?

Closures



The result of evaluating a λ-abstraction $\lambda x.t$, where $fv(\lambda x.t)$ may be nonempty, should not be $\lambda x.t$.

It should be a closure $\langle \lambda x.t \mid e \rangle$,

- that is, a pair of a λ-abstraction and an environment,
- in other words, a pair of a code pointer and a pointer to a heap-allocated data structure.

Landin, The Mechanical Evaluation of Expressions, 1964.

Closures and environments

The abstract syntax of closures is:

$$c ::= \langle \lambda x.t \mid e \rangle$$

We expect the evaluation of a term to produce a closure:

$$e \vdash t \downarrow_{\text{cbv}} c$$

Because evaluating x produces $e(x)$,
an environment must be a finite map of variables to closures:

$$e ::= [] \mid e[x \mapsto c]$$

Thus, the syntaxes of closures and environments are mutually inductive.

A big-step semantics with environments

Evaluating a λ-abstraction produces a newly allocated closure.

$$\textsc{EBigCbvVar}\quad \frac{e(x) = c}{e \vdash x \downarrow_{\text{cbv}} c}$$

$$\textsc{EBigCbvLam}\quad \frac{fv(\lambda x.t) \subseteq dom(e)}{e \vdash \lambda x.t \downarrow_{\text{cbv}} \langle \lambda x.t \mid e \rangle}$$

$$\textsc{EBigCbvApp}\quad \frac{\begin{array}{c} e \vdash t_1 \downarrow_{\text{cbv}} \langle \lambda x.u_1 \mid e' \rangle \\ e \vdash t_2 \downarrow_{\text{cbv}} c_2 \\ e'[x \mapsto c_2] \vdash u_1 \downarrow_{\text{cbv}} c \end{array}}{e \vdash t_1\ t_2 \downarrow_{\text{cbv}} c}$$

Invoking a closure causes the closure's code to be evaluated in the closure's environment, extended with a binding of formal to actual.

Equivalence between big-step semantics without and with environments

How can we relate the judgements $t \downarrow_{\text{cbv}} v$ and $e \vdash t \downarrow_{\text{cbv}} c$?

What lemma should we state?

Assuming t is closed, we would like to prove that

$$t \downarrow_{\text{cbv}} v$$

holds if and only if

$$[] \vdash t \downarrow_{\text{cbv}} c$$

holds for some closure c such that c represents v in a certain sense.

Decoding closures

c represents v can be defined as $\lceil c \rceil = v$, where $\lceil c \rceil$ is defined by:

$$\lceil \langle \lambda x.t \mid e \rangle \rceil \quad = \quad (\lambda x.t)[\lceil e \rceil]$$

and where the substitution $\lceil e \rceil$ maps every variable x in $dom(e)$ to $\lceil e(x) \rceil$.

($\lceil c \rceil$ and $\lceil e \rceil$ are mutually inductively defined.)

Equivalence between big-step semantics without and with environments

One implication is easily established:

Lemma (Soundness of the environment semantics)

$e \vdash t \downarrow_{cbv} c$ implies $t[\lceil e \rceil] \downarrow_{cbv} \lceil c \rceil$.

Proof (Sketch).

By induction on the hypothesis.
See `LambdaCalculusBigStep/ebigcbv_bigcbv`. □

In particular, $[] \vdash t \downarrow_{\text{cbv}} c$ implies $t \downarrow_{\text{cbv}} \lceil c \rceil$.

Equivalence between big-step semantics without and with environments

The reverse implication requires a more complex statement:

Lemma (Completeness of the environment semantics)

If $t[\lceil e \rceil] \downarrow_{cbv} v$, where $fv(t) \subseteq dom(e)$ and e is well-formed, then there exists c such that $e \vdash t \downarrow_{cbv} c$ and $\lceil c \rceil = v$.

Proof (Sketch).

By induction on the first hypothesis and by case analysis on t.
See `LambdaCalculusBigStep/bigcbv_ebigcbv`. □

In particular, if t is closed, then $t \downarrow_{\text{cbv}} v$ implies $[] \vdash t \downarrow_{\text{cbv}} c$, for some closure c such that $\lceil c \rceil = v$.

Equivalence between big-step semantics without and with environments

The notion of well-formedness on the previous slide is inductively defined:

$$\frac{\begin{array}{c} fv(\lambda x.t) \subseteq dom(e) \\ e \text{ is well-formed} \end{array}}{\langle \lambda x.t \mid e \rangle \text{ is well-formed}} \qquad \frac{\forall x, x \in dom(e) \Rightarrow e(x) \text{ is well-formed}}{e \text{ is well-formed}}$$

Lemma (Well-formedness is an invariant)

If $e \vdash t \downarrow_{cbv} c$ holds and e is well-formed, then c is well-formed.

Proof.

See `LambdaCalculusBigStep/ebigcbv_wf_cvalue`. □

This property is exploited in the proof of the previous lemma.

From big-step semantics to interpreter, again

The big-step semantics $e \vdash t \downarrow_{\text{cbv}} c$ is a 3-place relation.

We now wish to define a (partial) function of two arguments e and t.

We could do this in OCaml, as we did earlier today.

Let us do it in Coq and prove this interpreter correct and complete!

See `LambdaCalculusInterpreter`.

Syntax

The syntax of terms (in de Bruijn's representation) is as before.

The syntax of closures and environments is as shown earlier:

```
Inductive cvalue :=
| Clo: {bind term} -> list cvalue -> cvalue.

Definition cenv :=
  list cvalue.
```

A first attempt

```
Fail Fixpoint interpret (e : cenv) (t : term) : cvalue :=
  match t with
  | Var x =>
      nth x e dummy_cvalue
        (* dummy is used when x is out of range *)
  | Lam t =>
      Clo t e
  | App t1 t2 =>
      let cv1 := interpret e t1 in
      let cv2 := interpret e t2 in
      match cv1 with Clo u1 e' =>
        interpret (cv2 :: e') u1
      end
  end.
```

Why is this definition rejected by Coq?

A standard trick: fuel

We parameterize the interpreter with a maximum recursive call depth *n*.

```
Fixpoint interpret (n : nat) e t : option cvalue :=
  match n with
  | 0   => None (* not enough fuel *)
  | S n =>
      match t with
      | Var x     => Some (nth x e dummy_cvalue)
      | Lam t     => Some (Clo t e)
      | App t1 t2 =>
          interpret n e t1 >>= fun cv1 =>
          interpret n e t2 >>= fun cv2 =>
          match cv1 with Clo u1 e' =>
            interpret n (cv2 :: e') u1
          end
  end end.
```

The interpreter can now fail, therefore has return type `option cvalue`.

Equivalence between the big-step semantics and the interpreter

If the interpreter produces a result, then it is a correct result.

Lemma (Soundness of the interpreter)

If interpret $n\ e\ t = Some\ c$ and $fv(t) \subseteq dom(e)$ and e is well-formed then $e \vdash t \downarrow_{cbv} c$ holds.

Proof (Sketch).

By induction on n, by case analysis on t, and by inspection of the first hypothesis. See `LambdaCalculusInterpreter/interpret_ebigcbv`. □

An interpreter that always returns *None* would satisfy this lemma, hence the need for a completeness statement...

Equivalence between the big-step semantics and the interpreter

If the evaluation of t is supposed to produce c, then, given sufficient fuel, the interpreter returns c.

Lemma (Completeness of the interpreter)

If $e \vdash t \downarrow_{cbv} c$, then there exists n such that $interpret\ n\ e\ t = Some\ c$.

Proof (Sketch).

By induction on the hypothesis, exploiting the fact that *interpret* is monotonic in n, that is, $n_1 \leq n_2$ implies $interpret\ n_1\ e\ t \preceq interpret\ n_2\ e\ t$, where the "definedness" partial order $\preceq$ is generated by $None \preceq Some\ c$. See `LambdaCalculusInterpreter/ebigcbv_interpret`. □

Summary

If t is closed and v is a value, then the following are equivalent:

$t \longrightarrow^{\star}_{\text{cbv}} v$	small-step substitution semantics
$t \downarrow_{\text{cbv}} v$	big-step substitution semantics
$\exists c \left\{ \begin{array}{l} [] \vdash t \downarrow_{\text{cbv}} c \\ \lceil c \rceil = v \end{array} \right.$	big-step environment semantics
$\exists c \exists n \left\{ \begin{array}{l} \mathit{interpret}\ n\ []\ t = \mathit{Some}\ c \\ \lceil c \rceil = v \end{array} \right.$	interpreter

Cost model

We have represented environments as lists. Extension costs $O(1)$, but lookup has complexity $O(n)$, where n is the number of variables in scope.

A better approach is to represent the environment as an n-tuple. Then,

- evaluating a variable costs $O(1)$;
- evaluating a λ-abstraction costs $O(n)$;
- evaluating a function call costs $O(1)$.

n can be considered $O(1)$ as it depends only on the program's text, not on the input data.

This simple cost model is implemented by the OCaml compiler.

Digression: the cost of garbage collection

The previous slide does not discuss the cost of garbage collection.

Let H be the total heap size.

Let R be the total size of the live objects. Thus, $R \leq H$.

Assuming a copying collector, one collection costs $O(R)$.

Collection takes place when the heap is full, so frees up $H - R$ words.

Thus, the amortized cost of collection, per freed-up word, is

$$\frac{O(R)}{H - R}$$

Under the hypothesis $\frac{R}{H} \leq \frac{1}{2}$, this cost is $O(1)$. That is,

> *Provided the heap is not allowed to become more than half full, freeing up an object takes constant (amortized) time.*

Full closures versus minimal closures

In reality, this interpreter has one subtle but serious inefficiency.

When a closure $\langle \lambda x.t \mid e \rangle$ is allocated,
the entire environment e is stored in it,
even though $fv(\lambda x.t)$ may be a strict subset of the domain of e.

We store data that the closure will never need. This is a space leak!

To fix this, one should store a trimmed-down environment in the closure.

Exercise: state and prove that, if x does not occur free in t, then the evaluation of t in an environment e does not depend on the value $e(x)$.

Exercise: define an optimized interpreter where, at a closure allocation, every unneeded value in e is replaced with a dummy value. Prove it equivalent to the simpler interpreter.

A few things to remember

An efficient interpreter uses environments and closures, not substitutions.

- It can (easily) be proved correct and complete!

There are several styles of operational semantics.

- They can (easily) be proved equivalent!