

Making the stack explicit: the continuation-passing style transformation

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What if a program transformation could:

- ensure that every function call is a **tail call** and the **stack is explicit**, so the code is no longer really recursive, but **iterative**;
- make the evaluation order **explicit** in the code, so that it does not depend on the ambient strategy (CBN / CBV);
- eliminate the apparent **redundancy** between calls and returns, by exploiting solely function calls – **functions never return!**
- suggest extending the λ -calculus with **control operators**?

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- suggest extending the λ -calculus with **control operators**?

The **continuation-passing style** transformation does all this.

Motivation



D. Conversion to Continuation-Passing Style

This phase is the real meat of the compilation process. It is of interest primarily in that it transforms a program written in SCHEME into an equivalent program (the continuation-passing-style version, or CPS version), written in a language isomorphic to a subset of SCHEME with the property that interpreting it requires no control stack or other unbounded temporary storage and no decisions as to the order of evaluation of (non-trivial) subexpressions. The importance of these properties cannot be overemphasized. The fact that it is essentially a subset of SCHEME implies that its semantics are as clean, elegant, and well-understood as those of the original language. It is easy to build an

Steele, RABBIT: a compiler for SCHEME, 1978.

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From a direct-style interpreter down to an abstract machine

From recursive traversal down to iterative traversal with link inversion

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3 Soundness

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5 Madness in small steps

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A direct-style interpreter

Recall our environment-based interpreter for call-by-value λ -calculus:

```
let rec eval (e : cenv) (t : term) : cvalue =
  match t with
  | Var x ->
    lookup e x
  | Lam t ->
    Clo (t, e)
  | App (t1, t2) ->
    let cv1 = eval e t1 in
    let cv2 = eval e t2 in
    let Clo (u1, e') = cv1 in
    eval (cv2 :: e') u1
```

This is an OCaml transcription, without a fuel parameter.

A continuation-passing style interpreter

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Instead of [returning](#) a value,

```
let rec eval (e : cenv) (t : term) : cvalue =  
  ...
```

let's [pass](#) this value to a [continuation](#) that we get as an argument:

```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =  
  ...
```

[Exercise](#) (in class): write evalk. (See [EvalCBVExercise](#).)

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```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =
  match t with
  | Var x ->
    k (lookup e x)
  | Lam t ->
    k (Clo (t, e))
  | App (t1, t2) ->
    evalk e t1 (fun cv1 ->
      evalk e t2 (fun cv2 ->
        let Clo (u1, e') = cv1 in
        evalk (cv2 :: e') u1 k))
```

Instead of **returning** a value, **pass** it to k.

Instead of **sequencing** computations via **let**, **nest** continuations.

A continuation-passing style interpreter

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To run the interpreter, start it with the [identity](#) continuation:

```
let eval (e : cenv) (t : term) : cvalue =
    evalk e t (fun cv -> cv)
```

Correctness of the CPS interpreter

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The continuation-passing style interpreter is “obviously” correct.

Exercise: define eval_k in Coq (with fuel) and prove it equivalent to the direct-style interpreter: eval_k n e t k = k (eval n e t).

Properties of the interpreter

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What is special about this interpreter?

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What is special about this interpreter?

- Every call to evalk is a tail call.
- Every call to a continuation k is a tail call.

Tail calls

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A call $g\ x$ is a tail call if it is the “last thing” that the calling function does...

More formally,

$v ::= x \mid \lambda x. tt$	values
$tt ::=$	terms in tail position
v	
$nt\ nt$	– a tail call
$let\ nt\ in\ tt$	
$if\ nt\ then\ tt\ else\ tt$	
$nt ::=$	terms not in tail position
v	
$nt\ nt$	– not a tail call
$let\ nt\ in\ nt$	
$if\ nt\ then\ nt\ else\ nt$	

This can be understood as the description of a top-down computation that assigns a Boolean flag (“tail” or “non-tail”) to every subterm.

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OCaml allows us to [verify](#) that these are indeed tail calls:

```
let rec evalk (e : cenv) (t : term) (k : cvalue -> 'a) : 'a =
  match t with
  | Var x ->
    (k[@tailcall]) (lookup e x)
  | Lam t ->
    (k[@tailcall]) (Clo (t, e))
  | App (t1, t2) ->
    (evalk[@tailcall]) e t1 (fun cv1 ->
      (evalk[@tailcall]) e t2 (fun cv2 ->
        let Clo (u1, e') = cv1 in
        (evalk[@tailcall]) (cv2 :: e') u1 k))
```

A nice feature (though with somewhat ugly syntax).

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Tail calls are compiled by OCaml to **jumps**.

Thus, tail-recursive functions are compiled by OCaml to **loops**.

Steele, **Lambda: the ultimate GOTO**, 1977.

Thus, the CPS interpreter is not truly **recursive**: it is **iterative**.

It uses **constant space** on OCaml's implicit stack.

Wait! Does the interpreter really **not need a stack** any more?

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- Of course it **does** need a stack.

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Thus, the CPS interpreter is not truly **recursive**: it is **iterative**.

It uses **constant space** on OCaml's implicit stack.

Wait! Does the interpreter really **not need a stack** any more?

- Of course it **does** need a stack.
- The **continuation**, allocated in the OCaml heap, serves as a stack.

A defunctionalized CPS interpreter

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To better see the structure of the continuation,
let us **defunctionalize** the CPS interpreter.

Reynolds, **Definitional interpreters
for programming languages**, 1972 (1998).

Reynolds, **Definitional interpreters revisited**, 1998.

Defunctionalization (reminder)

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Steps:

- Identify the sites where closures are allocated, that is, where anonymous functions are built.
- Compute, at each site, the free variables of the anonymous function.
- Introduce an algebraic data type of closures.
- Transform the code:
 - replace anonymous functions with constructor applications,
 - replace function applications with calls to apply,
 - and define apply.

Exercise (in class): defunctionalize the CPS interpreter. ([EvalCBVExercise](#).)

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There are three sites where an anonymous continuation is built.

We name them and compute their free variables.

This leads to the following algebraic data type of continuations:

```
type kont =
| AppL of { e: cenv; t2: term; k: kont }
| AppR of { cvl: cvalue; k: kont }
| Init
```

What data structure is this?

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type kont =
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| AppR of { cvl: cvalue; k: kont }
| Init
```

What data structure is this? A [linked list](#). A heap-allocated stack.

In fact, it is a (call-by-value) [evaluation context](#):

$$E ::= E[] \ t_2[e] \ | \ E[v_1] \ | \ []$$

It is a [zipper](#), a path from the context's hole up to the root of a term.

Huet, [The Zipper](#), 1997.

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We transform the interpreter's main function:

```
let rec evalkd (e : cenv) (t : term) (k : kont) : cvalue =
  match t with
  | Var x ->
    apply k (lookup e x)
  | Lam t ->
    apply k (Clo (t, e))
  | App (t1, t2) ->
    evalkd e t1 (AppL { e; t2; k })
```

To evaluate $t_1 t_2$, the interpreter **pushes** information on the stack, then **jumps** straight to evaluating t_1 .

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apply interprets continuations as functions of values to values:

```
and apply (k : kont) (cv : cvalue) : cvalue =
  match k with
  | AppL { e; t2; k } ->
    let cv1 = cv in
    evalkd e t2 (AppR { cv1; k })
  | AppR { cv1; k } ->
    let cv2 = cv in
    let Clo (u1, e') = cv1 in
    evalkd (cv2 :: e') u1 k
  | Init ->
    cv
```

It **pops** the top stack frame and decides what to do, based on it.

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To run the interpreter, start it with the `identity` continuation:

```
let eval e t =
  evalkd e t Init
```

An abstract machine

We have reached an **abstract machine**, a simple **iterative** interpreter which maintains a few data structures:

- a **code** pointer: the term t ,
- an **environment** e ,
- a **stack**, or **continuation** k .

In fact, we have mechanically rediscovered the **CEK** machine.

Felleisen and Friedman,
Control operators, the SECD machine, and the λ -calculus, 1987.

Sig Ager, Biernacki, Danvy and Midtgård,
A Functional Correspondence between Evaluators
and Abstract Machines, 2003.

Re-discovering other abstract machines

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Exercise: start with a call-by-name interpreter and follow an analogous process to rediscover Krivine's machine.

The solution is in [EvalCBNCPS](#).

*There once was a man named Krivine
Who invented a wond'rous machine.
It pushed and it popped
On abstractions it stopped;
That lean mean machine from Krivine.*
— *Mitchell Wand*

Krivine, [A call-by-name lambda-calculus machine](#), (1985) 2007.

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A type of binary trees

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Consider a simple type of binary trees:

```
type tree =  
| Leaf  
| Node of { data: int; left: tree; right: tree }
```

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Suppose we wish to perform a postfix tree traversal:

```
let rec walk (t : tree) : unit =
  match t with
  | Leaf ->
    ()
  | Node { data; left; right } ->
    walk left;
    walk right;
    printf "%d\n" data
```

This is [recursive code in direct style](#).

Neither of the recursive calls is a tail call.

CPS traversal

Now suppose we wish to make the code [iterative](#). Swoop, CPS!

```
let rec walkk (t : tree) (k : unit -> 'a) : 'a =
  match t with
  | Leaf ->
    k()
  | Node { data; left; right } ->
    walkk left (fun () ->
      walkk right (fun () ->
        printf "%d\n" data;
        k())))

```

The traversal is initiated with an identity continuation:

```
let walk t =
  walkk t (fun t -> t)
```

CPS traversal, defunctionalized

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Next, we might wish to make the stack an explicit [data structure](#).

Swoop, defunctionalization!

The type of defunctionalized continuations:

```
type kont =
| Init
| GoneL of { data: int; tail: kont; right: tree }
| GoneR of { data: int; tail: kont }
```

CPS traversal, defunctionalized

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The main function is a loop that walks down the leftmost branch while **pushing** information onto the stack:

```
let rec walkkd (t : tree) (k : kont) : unit =
  match t with
  | Leaf ->
    apply k ()
  | Node { data; left; right } ->
    walkkd left (GoneL { data; tail = k; right })
```

Think of the stack as **Ariadne's thread**.

CPS traversal, defunctionalized

The apply function comes back up out of a child.

```
and apply k () =
  match k with
  | Init ->
    ()
  | GoneL { data; tail; right } ->
    walkkd right (GoneR { data; tail })
  | GoneR { data; tail } ->
    printf "%d\n" data;
    apply tail ()
```

It **pops** information off the stack so as to decide what to do.

When coming out of a left child, go down into its right sibling.

When coming out of a right child, go further up.

And now, for something a little
UNEXPECTED and WILD.

And now, for something a little
UNEXPECTED and WILD.
A CRAZY HACK.



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When we **allocate** a **GoneR** continuation,
we **drop** a **GoneL** continuation at the same time.

Indeed, here, continuations are **linear**. They are used exactly once.

```
| GoneL { data; tail; right } ->  
  walkkd right (GoneR { data; tail })
```

This suggests that the memory block could be **recycled** (re-used).

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When we **allocate** a **GoneL** continuation,
a **Node** goes **temporarily unused** at the same time.

This node won't be accessed until this **GoneL** frame
first is changed to **GoneR** then is popped off the stack.

```
| Node { data; left; right } ->  
    walkkd left (GoneL { data; tail = k; right })
```

This suggests that the memory block could be **recycled**, too,
provided we **restore** it when we are done with it.

A tree is a continuation is a tree

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In OCaml, the type of a memory block cannot be changed over time.

Thus, recycling tree nodes as stack frames, and vice-versa,
requires **trees** and **continuations** to have **the same type**.

Uh?

A tree is a continuation is a tree

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Could we **disguise** a continuation as a tree?

In other words, could a stack frame **fit** in a tree node?

```
type kont =
| Init
| GoneL of { data: int; tail: kont; right: tree }
| GoneR of { data: int; tail: kont }
```

```
type tree =
| Leaf
| Node of { data: int; left: tree; right: tree }
```

A tree is a continuation is a tree

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Could we **disguise** a continuation as a tree?

In other words, could a stack frame **fit** in a tree node?

```
type kont =
| Init
| Gonel of { data: int; tail: kont; right: tree }
| Goner of { data: int; tail: kont }
```

```
type tree =
| Leaf
| Node of { data: int; left: tree; right: tree }
```

Yes, kind of.

We just need **one extra bit** of storage per tree node,
so as to distinguish **Gonel** and **Goner**.

A tree is a continuation is a tree

Add one “status” bit per tree node. Make nodes [mutable](#).

```
type status = GoneL | GoneR
type mtree = Leaf | Node of {
    data: int;           mutable status: status;
    mutable left: mtree; mutable right: mtree
}
type mkont = mtree
```

Tree records and continuation records occupy the same space in memory.

Thus, a tree record can be turned into a continuation record, and back!

By convention, in a “tree” record, the status field is [GoneL](#).

In a “continuation” record,

- either status is [GoneL](#) and the left field stores tail;
- or status is [GoneR](#) and the right field stores tail.

CPS traversal with link inversion

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Instead of allocating a `GoneL` continuation,
we now [change](#) the tree record to a continuation record:

```
let rec walkkdi (t : mtree) (k : mkont) : unit =
  match t with
  | Leaf ->
    apply k t
  | Node ({ left; _ } as n) ->
    (* Change this tree to a [GoneL] continuation. *)
    assert (n.status = GoneL);
    n.left (* n.tail *) <- k;
    walkkdi left (t : mkont)
```

The `left` field is [overwritten](#), which is scary! We must restore it later.

We find that, in every call to `walkkdi t k` and `apply k t`,
`k` is the [parent](#) of `t` in the tree.

CPS traversal with link inversion

The rest of the code, in its horrific glory:

```
and apply (k : mkont) (child : mtree) : unit =
  match k with
  | Leaf -> ()
  | Node ({ status = GoneL; left = tail; right; _ } as n) ->
    n.status <- GoneR;          (* update continuation! *)
    n.left <- child;          (* restore orig. left child! *)
    n.right (* n.tail *) <- tail;
    walkkdi right k
  | Node ({ data; status = GoneR; right = tail; _ } as n) ->
    printf "%d\n" data;
    n.status <- GoneL;          (* change back to a tree! *)
    n.right <- child;          (* restore orig. right child! *)
    apply tail (k : mtree)
```

This code runs in **constant space**. Look Ma, no stack! (Uh?)

CPS traversal with link inversion

More accurately, the stack is stored [in the tree itself](#), by [reversing pointers](#).

This [hack](#) technique is known as [link inversion](#).

It was invented for use in garbage collectors, which must [traverse the heap](#) without requiring a huge stack.

We have re-discovered it via the idea of allocating continuations [in place](#).

Schorr and Waite, [An efficient machine-independent procedure for garbage collection in various list structures](#), 1967.

Hubert and Marché, [A case study of C source code verification: the Schorr-Waite algorithm](#), 2005.

Sobel and Friedman, [Recycling continuations](#), 1998.

CPS traversal with link inversion

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“Kids, do not try this at home”: this idea is **complicated** and **expensive**.

(The OCaml GC imposes a **write barrier**: write operations are slow.)

Exercise: Extend the code to deal with **graphs**, where there can be **sharing** and **cycles**. (Use a **mark** bit in every node.)

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Formulations of the CPS transformation

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There are many variants of the CPS transformation,
and sometimes many formulations of a single variant.

Let us begin with the simplest formulation: Fischer and Plotkin's.

Fischer, Lambda-Calculus Schemata, (1972) 1993.

Plotkin, Call-by-name, call-by-value and the λ -calculus, 1975.

Definition of the CBV CPS transformation

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A term is translated to a **function** of a continuation k to an answer.

$$[\![x]\!] =$$

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A term is translated to a **function** of a continuation k to an answer.

$$\llbracket x \rrbracket = \lambda k.$$

Definition of the CBV CPS transformation

A term is translated to a **function** of a continuation k to an answer.

$$\llbracket x \rrbracket = \lambda k. k x$$

$$\llbracket \lambda x. t \rrbracket =$$

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A term is translated to a **function** of a continuation k to an answer.

$$\llbracket x \rrbracket = \lambda k. k x$$

$$\llbracket \lambda x. t \rrbracket = \lambda k.$$

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A term is translated to a **function** of a continuation k to an answer.

$$[x] = \lambda k. k x$$

$$[\lambda x.t] = \lambda k. k (\lambda x.[t])$$

$$[t_1 t_2] = \lambda k.$$

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$$[t_1 t_2] = \lambda k. [t_1]$$

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A term is translated to a **function** of a continuation k to an answer.

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$$[t_1 t_2] = \lambda k. [t_1] (\lambda x_1.$$

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A term is translated to a **function** of a continuation k to an answer.

$$[x] = \lambda k. k x$$

$$[\lambda x. t] = \lambda k. k (\lambda x. [t])$$

$$[t_1 \ t_2] = \lambda k. [t_1] (\lambda x_1. [t_2]) (\lambda x_2.$$

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A term is translated to a function of a continuation k to an answer.

$$\llbracket x \rrbracket = \lambda k. k x$$

$$\llbracket \lambda x. t \rrbracket = \lambda k. k (\lambda x. \llbracket t \rrbracket)$$

$$\llbracket t_1 \ t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket (\lambda x_1. \llbracket t_2 \rrbracket) (\lambda x_2. x_1 \ x_2 \ k))$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket = \lambda k. \llbracket t_1 \rrbracket (\lambda x. \llbracket t_2 \rrbracket) k$$

A value $\lambda x. t$ is translated to a function of two arguments $\lambda x. \lambda k. \dots$

Definition of the CBV CPS transformation

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One avoids some redundancy by defining two mutually recursive functions, namely the translation of values (v) :

$$(x) = x$$

$$(\lambda x.t) = \lambda x. [t]$$

and the translation of terms $[t]$:

$$[v] = \lambda k. k (v)$$

$$[t_1 t_2] = \lambda k. [t_1] (\lambda x_1. [t_2]) (\lambda x_2. x_1 x_2 k))$$

$$[\text{let } x = t_1 \text{ in } t_2] = \lambda k. [t_1] (\lambda x. [t_2] k)$$

Indifference



In a transformed term, the right-hand side of every application is a value.

Therefore, its execution is indifferent to the choice of a call-by-name or call-by-value evaluation strategy.

In other words, evaluation order is fully explicit in a transformed term.

The transformation on the previous slide fixes a call-by-value strategy: it is the CBV CPS transformation.

It can serve as an encoding of call-by-value into call-by-name, thus answering a question raised in week 1.

Exercise (recommended): Define the CBN CPS transformation.

Stacklessness



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In a transformed term, **every call is a tail call**.

Therefore, reduction under a context is not required.

That is, execution **does not require a stack**.

We could (but won't) give a (small-step, substitution-based) semantics that takes **indifference** and **stacklessness** into account.

Exercise: Propose such a semantics. Prove that, when executing a CPS-transformed term, it is equivalent to the standard semantics.

Effect of the transformation of types

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How are **types** transformed?

A **value** of type T is translated to a value of type $\langle T \rangle$.

A **computation** of type T is translated to a value of type $\llbracket T \rrbracket$.

$$\langle \alpha \rangle = \alpha$$

$$\langle T_1 \rightarrow T_2 \rangle =$$

Effect of the transformation of types

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$$\langle \alpha \rangle = \alpha$$

$$\langle T_1 \rightarrow T_2 \rangle = \langle T_1 \rangle \rightarrow \llbracket T_2 \rrbracket$$

$$\llbracket T \rrbracket =$$

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How are **types** transformed?

A **value** of type T is translated to a value of type $\langle T \rangle$.

A **computation** of type T is translated to a value of type $[\![T]\!]$.

$$\langle \alpha \rangle = \alpha$$

$$\langle T_1 \rightarrow T_2 \rangle = \langle T_1 \rangle \rightarrow [\![T_2]\!]$$

$$[\![T]\!] = (\langle T \rangle \rightarrow A) \rightarrow A$$

The type A , known as the **answer** type, is arbitrary and fixed.

One may take A to be the **empty type** 0 . Then, $[\![T]\!]$ is $\neg\neg\langle T \rangle$. The CPS transformation is known in logic as the **double-negation translation**.

Exercise (recommended): state and prove Type Preservation.

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Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = ((\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Effect of the transformation of types – refined

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Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Every transformed term is in fact **answer-type polymorphic**:

$$\llbracket T \rrbracket = \forall A. (\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Furthermore,

Effect of the transformation of types – refined

Could the transformation of types be made **more precise** in some sense?

$$\llbracket T \rrbracket = ((\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Every transformed term is in fact **answer-type polymorphic**:

$$\llbracket T \rrbracket = \forall A. ((\llbracket T \rrbracket \rightarrow A) \rightarrow A$$

Furthermore, every transformed term invokes its continuation **once**:

$$\llbracket T \rrbracket = \forall A. ((\llbracket T \rrbracket \rightarrow A) \multimap A$$

However, these properties are violated in the presence of **control effects**.

Thielecke, **From control effects to typed continuation passing**, 2003.

Semantic preservation

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Plotkin (1975) proved semantic preservation,
based on a small-step simulation diagram.

This proof is complicated by the presence of administrative reductions.

A simpler approach is to use big-step semantics in the hypothesis:

Lemma (Semantic Preservation)

If $t \downarrow_{cbv} v$ and if w is a value, then $\llbracket t \rrbracket w \xrightarrow{^*_{cbv}} w(v)$.

One should prove, in addition, that divergence is preserved.

Exercise (recommended): Prove this lemma.

Administrative redexes

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The translation presented so far is naïve.

It produces many “administrative” β -redexes.

E.g., in an application of a variable to a variable:

$$\begin{aligned} \llbracket f \ x \rrbracket &= \lambda k. \llbracket f \rrbracket (\lambda x_1. \llbracket x \rrbracket (\lambda x_2. x_1 \ x_2 \ k)) \\ &= \lambda k. (\lambda k. k (\lfloor f \rfloor)) (\lambda x_1. (\lambda k. k (\lfloor x \rfloor)) (\lambda x_2. x_1 \ x_2 \ k)) \\ &= \lambda k. (\lambda k. k \ f) (\lambda x_1. (\lambda k. k \ x) (\lambda x_2. x_1 \ x_2 \ k)) \\ &\stackrel{=\beta}{=} \lambda k. (\lambda x_1. (\lambda k. k \ x) (\lambda x_2. x_1 \ x_2 \ k)) \ f \\ &\stackrel{=\beta}{=} \lambda k. (\lambda k. k \ x) (\lambda x_2. f \ x_2 \ k) \\ &\stackrel{=\beta}{=} \lambda k. (\lambda x_2. f \ x_2 \ k) \ x \\ &\stackrel{=\beta}{=} \lambda k. f \ x \ k \end{aligned}$$

This is inefficient: one function call is translated to five function calls!

Ways of eliminating administrative redexes

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Administrative redexes can be reduced **after** the CPS transformation.

- During the translation, mark each λ that corresponds to a source λ .
- After the translation, reduce every redex whose λ is unmarked.

Another idea is to reduce all “no-brainer” redexes. They include the admin. redexes and are size-decreasing. This can be done on the fly.

Davis, Meehan, Shivers, **No-brainer CPS conversion**, 2017.

Yet another approach is to define a “one-pass” CPS transformation that does not produce any administrative redexes in the first place...

Towards a one-pass transformation

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The first step is to make some of the abstractions and applications static.

They should take place at transformation time, not at runtime.

Instead of viewing $\llbracket t \rrbracket = \lambda k. \dots$ as a function of a term to a term,
let us view $\llbracket t \rrbracket \{ w \} = \dots$ as a function of a term and a value to a term.

$$\langle\!\langle x \rangle\!\rangle = x$$

$$\langle\!\langle \lambda x. t \rangle\!\rangle = \lambda x. \lambda k. \llbracket t \rrbracket \{ k \}$$

$$\llbracket v \rrbracket \{ w \} = w \langle\!\langle v \rangle\!\rangle$$

$$\llbracket t_1 \; t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x_1. \llbracket t_2 \rrbracket \{ \lambda x_2. x_1 \; x_2 \; w \} \}$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x. \llbracket t_2 \rrbracket \{ w \} \}$$

k denotes a variable; w denotes a value.

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This transformation produces [fewer administrative redexes](#):

$$\begin{aligned} \llbracket f x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 x_2 k \} \} \\ &= (\lambda x_1. (\lambda x_2. x_1 x_2 k) x) f \\ &=_{\beta} (\lambda x_2. f x_2 k) x \\ &=_{\beta} f x k \end{aligned}$$

The remaining administrative redexes arise from the equation

$$\llbracket v \rrbracket \{ w \} = w \llbracket v \rrbracket$$

in the case where the continuation w is a λ -abstraction.

How could we alter this equation?

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Define the [smart application](#) of a (continuation) value w to a value v :

$$\begin{aligned} x @_{\beta} v &= x v \\ (\lambda x.t) @_{\beta} v &= t[v/x] \end{aligned}$$

Note:

- A continuation w is always either a variable or a “transformation” λ , never a “source” λ , so the redex reduced by $w @_{\beta} v$ is [administrative](#).
- Provided every “transformation” λ uses its argument [linearly](#), $w @_{\beta} (v)$ does not duplicate (v) , so transformed terms remain [linear](#) in size.

A one-pass transformation

Change the translation of values. Make every “transformation” λ linear.

$$\langle\!\langle x\rangle\!\rangle = x$$

$$\langle\!\langle \lambda x. t \rangle\!\rangle = \lambda x. \lambda k. \llbracket t \rrbracket \{ k \}$$

$$\llbracket v \rrbracket \{ w \} = w @_{\beta} \langle\!\langle v \rangle\!\rangle$$

$$\llbracket t_1 \; t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x_1. \llbracket t_2 \rrbracket \{ \lambda x_2. x_1 \; x_2 \; w \} \}$$

$$\llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ w \} = \llbracket t_1 \rrbracket \{ \lambda x. \text{let } x = x \text{ in } \llbracket t_2 \rrbracket \{ w \} \}$$

This transformation produces no administrative redexes.

Dargaye and Leroy, Mechanized Verification
of CPS Transformations, 2007.

A one-pass transformation

Look Ma, no administrative redexes!

$$\begin{aligned} \llbracket f x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 x_2 k \} \} \\ &= (\lambda x_1. (\lambda x_2. x_1 x_2 k) @_\beta x) @_\beta f \\ &= (\lambda x_2. f x_2 k) @_\beta x \\ &= f x k \end{aligned}$$

A one-pass transformation

Look Ma, no administrative redexes!

$$\begin{aligned} \llbracket f x \rrbracket \{ k \} &= \llbracket f \rrbracket \{ \lambda x_1. \llbracket x \rrbracket \{ \lambda x_2. x_1 x_2 k \} \} \\ &= (\lambda x_1. (\lambda x_2. x_1 x_2 k) @_\beta x) @_\beta f \\ &= (\lambda x_2. f x_2 k) @_\beta x \\ &= f x k \end{aligned}$$

A drawback of Dargaye and Leroy's approach is that $\cdot @_\beta \cdot$ does not commute with substitutions, which causes a difficulty in the proof of semantic preservation.

This is repaired in the formulation shown next...



Higher-order versus first-order formulations

Danvy and Filinski (1992) first defined this one-pass transformation.

Their formulation was in a “higher-order” style.

Let me give a simpler, “first-order” presentation of their transformation.

Danvy and Filinski, *Representing control: a study of the CPS transformation*, 1992.

Pottier, *Revisiting the CPS transformation and its implementation*, 2017.

A first-order one-pass CPS transformation

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Let a continuation c be either a value w or a “transformation” λ :

$$c ::= w \mid mx.t$$

In $mx.t$, the term t must have exactly one occurrence of x .

Define [continuation application](#) $apply\ c\ v$ and [reification](#) $reify\ c$:

$$apply\ w\ v = w\ v \quad \text{-- an object-level application}$$

$$apply\ (mx.t)\ v = t[v/x] \quad \text{-- a meta-level substitution}$$

$$reify\ w = w \quad \text{-- a no-op}$$

$$reify\ (mx.t) = \lambda x.t$$

Reification converts a continuation to a term.

A first-order one-pass CPS transformation

Danvy and Filinski's transformation can then be presented as follows:

$$\begin{aligned}\langle x \rangle &= x \\ \langle \lambda x. t \rangle &= \lambda x. \lambda k. \llbracket t \rrbracket \{ k \}\end{aligned}$$

$$\begin{aligned}\llbracket v \rrbracket \{ c \} &= \text{apply } c \langle v \rangle \\ \llbracket t_1 \ t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ mx_1. \llbracket t_2 \rrbracket \{ mx_2. x_1 \ x_2 \ (\text{reify } c) \} \} \\ \llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ mx_1. \text{let } x = x_1 \text{ in } \llbracket t_2 \rrbracket \{ c \} \}\end{aligned}$$

It is close to Dargaye and Leroy's formulation, yet is better behaved: as we will see, it commutes with substitution.

Now, in de Bruijn style

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Let us use o and m as explicit injections:

$$c ::= \text{o } w \mid \text{m } t$$

m , like λ , is considered a binder.

Continuation application, reification, and substitution $c[\sigma]$ are as follows:

$$\begin{array}{lll} \text{apply } (\text{o } w) v = w v & \text{reify } (\text{o } w) = w & (\text{o } w)[\sigma] = \text{o } (w[\sigma]) \\ \text{apply } (\text{m } t) v = t[v/] & \text{reify } (\text{m } t) = \lambda t & (\text{m } t)[\sigma] = \text{m } (t[\uparrow \sigma]) \end{array}$$

See [CPSDefinition](#).

The CPS transformation in de Bruijn style

The transformation is formulated in de Bruijn style as follows:

$$\begin{aligned}\langle\!\langle x\rangle\!\rangle &= x \\ \langle\!\langle \lambda t \rangle\!\rangle &= \lambda\lambda(\llbracket \uparrow^1 t \rrbracket \{ o \ 0 \})\end{aligned}$$

$$\begin{aligned}\llbracket v \rrbracket \{ c \} &= \text{apply } c \langle\!\langle v \rangle\!\rangle \\ \llbracket t_1 \ t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ m \ \llbracket \uparrow^1 t_2 \rrbracket \{ m \ 1 \ 0 \ \uparrow^2 (\text{reify } c) \} \} \\ \llbracket \text{let } t_1 \ \text{in } t_2 \rrbracket \{ c \} &= \llbracket t_1 \rrbracket \{ m \ \text{let } 0 \ \text{in } \llbracket \uparrow^1 t_2 \rrbracket \{ \uparrow^2 c \} \}\end{aligned}$$

$\uparrow^i t$ is short for $t[+i]$. $\uparrow_1^1 t$ is short for $t[\uparrow (+1)]$.

\uparrow^1 means end-of-scope for variable 0.

The CPS transformation in de Bruijn style

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The transformation is formulated in de Bruijn style as follows:

$$\langle\!\langle x\rangle\!\rangle = x$$
$$\langle\!\langle \lambda t \rangle\!\rangle = \lambda \lambda (\langle\!\langle \uparrow^1 t \rangle\!\rangle \{ o \ 0 \})$$

$$\langle\!\langle v \rangle\!\rangle \{ c \} = \text{apply } c \langle\!\langle v \rangle\!\rangle$$
$$\langle\!\langle t_1 \ t_2 \rangle\!\rangle \{ c \} = \langle\!\langle t_1 \rangle\!\rangle \{ m \ \langle\!\langle \uparrow^1 t_2 \rangle\!\rangle \{ m \ 1 \ 0 \ \uparrow^2 (\text{reify } c) \} \}$$
$$\langle\!\langle \text{let } t_1 \ \text{in } t_2 \rangle\!\rangle \{ c \} = \langle\!\langle t_1 \rangle\!\rangle \{ m \ \text{let } 0 \ \text{in } \langle\!\langle \uparrow^1 t_2 \rangle\!\rangle \{ \uparrow^2 c \} \}$$

$\uparrow^i t$ is short for $t[+i]$. $\uparrow_1^1 t$ is short for $t[\uparrow (+1)]$.

\uparrow^1 means [end-of-scope](#) for variable 0.

\uparrow^2 means end-of-scope for variables 0 and 1.

\uparrow_1^1 means end-of-scope for variable 1.

Worse, Coq does not like this definition...



The CPS transformation in de Bruijn style

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The transformation is formulated in de Bruijn style as follows:

$$\langle\!\langle x\rangle\!\rangle = x$$
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$$\begin{aligned}\langle\!\langle v \rangle\!\rangle \{ c \} &= \text{apply } c \langle\!\langle v \rangle\!\rangle \\ \langle\!\langle t_1 \ t_2 \rangle\!\rangle \{ c \} &= \langle\!\langle t_1 \rangle\!\rangle \{ m \ \langle\!\langle \uparrow^1 t_2 \rangle\!\rangle \{ m \ 1 \ 0 \ \uparrow^2 (\text{reify } c) \} \} \\ \langle\!\langle \text{let } t_1 \ \text{in } t_2 \rangle\!\rangle \{ c \} &= \langle\!\langle t_1 \rangle\!\rangle \{ m \ \text{let } 0 \ \text{in } \langle\!\langle \uparrow^1 t_2 \rangle\!\rangle \{ \uparrow^2 c \} \}\end{aligned}$$

$\uparrow^i t$ is short for $t[+i]$. $\uparrow_1^1 t$ is short for $t[\uparrow (+1)]$.

\uparrow^1 means [end-of-scope](#) for variable 0.

\uparrow^2 means end-of-scope for variables 0 and 1.

\uparrow_1^1 means end-of-scope for variable 1.



Worse, Coq does not like this definition... because the recursive calls concern [renamed](#) subterms! Well-founded recursion on [size](#) is required.

See [CPSDefinition](#).

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We would like to prove this:

Lemma (Semantic Preservation)

If $t \downarrow_{cbv} v$, then $\llbracket t \rrbracket \{ m\ 0 \} \downarrow_{cbv} (v)$.

$m\ 0$ is the identity continuation: in nominal style, $mx.x$.

For an inductive proof, the statement must be generalized, as follows...

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Define $t \lesssim u$ as follows: for every value v , $u \downarrow_{cbv} v$ implies $t \downarrow_{cbv} v$.

Lemma (Big-step Simulation)

Suppose $\text{reify } c$ is a value. If $t \downarrow_{cbv} v$, then $\llbracket t \rrbracket \{ c \} \lesssim \text{apply } c (v)$.

Compare with [our earlier claim](#) concerning Plotkin's CPS transformation.

The proof is in [CPSCorrectnessBigStep](#).

Exercise: Replay the proof in Coq. Then erase it and redo it from scratch.

Exercise: Write a clear paper or \LaTeX proof and send it to me!

The proof requires two key lemmas, shown next...

Key Lemma 1: Substitution

Lemma (Substitution)

Let σ and σ' be value substitutions such that σ' is equal to $\sigma ; (\cdot)$. Then,

$$(\llbracket t \rrbracket \{ c \})[\sigma'] = \llbracket t[\sigma] \rrbracket \{ c[\sigma'] \}.$$

Lemma (Substitution—a special case)

Let v and w be values. Then,

$$(\llbracket t \rrbracket \{ \uparrow^2 c \ })[(\llbracket v \rrbracket) \cdot (\llbracket w \rrbracket) \cdot id] = \llbracket t[v \cdot w \cdot id] \rrbracket \{ c \}.$$

In nominal style: if $x, y \notin fv(c)$, then

$$(\llbracket t \rrbracket \{ c \ })[(\llbracket v \rrbracket)/x, (\llbracket w \rrbracket)/y] = \llbracket t[v/x, w/y] \rrbracket \{ c \}.$$

We push a substitution into the term, leaving the continuation untouched.
A target language substitution becomes a source language substitution.

See [CPSSubstitution](#).

Key Lemma 2: Ksubstition

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Lemma (Kubstitution)

Let θ and σ be substitutions such that $\theta ; \sigma$ is id. Then,

$$\llbracket (t[\theta]) \{ c \} \rrbracket [\sigma] = \llbracket t \rrbracket \{ c[\sigma] \}.$$

Lemma (Kubstitution—a special case)

For every value v , $(\llbracket \uparrow^1 t \rrbracket \{ c \})[v/] = \llbracket t \rrbracket \{ c[v/] \}$.

In nominal style: if $x \notin fv(t)$, then $(\llbracket t \rrbracket \{ c \})[v/x] = \llbracket t \rrbracket \{ c[v/x] \}$.

We push a substitution into the continuation, leaving the term untouched.
This is and remains a target language substitution.

See [CPSKubstitution](#).

Interlude: Enumerating λ -terms

Define the **size** of a term as follows: variables have size 0; λ -abstractions and applications contribute 1.

Step 1: In OCaml, implement an exhaustive **enumeration** of the λ -terms of size s and with at most n free variables. (Given as an exercise in week 1.)

```
(* Enumerate all variables between 0 and n excluded. *)
let var (n : int) (k : term -> unit) : unit = ...
(* Enumerate all manners of splitting an integer s. *)
let split (s : int) (k : int -> int -> unit) : unit = ...
(* Enumerate all terms of size s with at most n variables. *)
let term (s : int) (n : int) (k : term -> unit) : unit = ...
```

An enumerator is naturally written in CPS style!

Interlude: Testing Semantic Preservation

Step 2: In OCaml, implement the CPS transformation.

```
type continuation =
| O of term
| M of term
let rec cps (t : term) (c : continuation) : term = ...
```

Step 3: In OCaml, implement a test for the relation $\cdot \lesssim \cdot :$

```
let sim (t1 : term) (t2 : term) : bool = ...
```

Hint: Re-use the big-step interpreter of week 2. See [Lambda](#).

Step 4: Up to a certain size, search for a term that violates Semantic Preservation. There should be none!

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In a CPS-transformed program, the continuation is a first-class object.

Why not give programmers [access](#) to it?

That is, extend the source language with [control operators](#) that allow ([delimiting](#) and) [capturing](#) the current continuation.

Shift / reset

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An example is Danvy and Filinski's shift / reset (1990).

$$t ::= \dots | \langle t \rangle | \xi x.t$$

A "reset" $\langle t \rangle$ does nothing by itself: e.g., $\langle 42 \rangle$ reduces to 42.

A "shift" $\xi x.t$ captures the current evaluation context (up to and excluding the nearest reset), reifies it as a function, and binds the variable x to it.

Then it discards the evaluation context (up to and including the nearest reset) and executes t instead.

E.g., roughly,

$$\begin{aligned} & 1 + \langle 10 + \xi c.c (c 100) \rangle \\ \longrightarrow & 1 + (\text{let } c = \lambda x.(10 + x) \text{ in } c (c 100)) \\ \longrightarrow & 1 + (10 + (10 + 100)) \\ \longrightarrow & 121 \end{aligned}$$

Exercise: Give a small-step semantics to shift / reset.

CPS-transforming shift / reset

The naïve call-by-value CPS transformation is extended as follows:

$$\llbracket \langle t \rangle \rrbracket = \lambda k.$$

CPS-transforming shift / reset

The naïve call-by-value CPS transformation is extended as follows:

$$\begin{aligned}\llbracket \langle t \rangle \rrbracket &= \lambda k. k (\llbracket t \rrbracket (\lambda y. y)) \\ \llbracket \xi x. t \rrbracket &= \lambda k.\end{aligned}$$

CPS-transforming shift / reset

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The naïve call-by-value CPS transformation is extended as follows:

$$\begin{aligned}\llbracket \langle t \rangle \rrbracket &= \lambda k. k (\llbracket t \rrbracket (\lambda y. y)) \\ \llbracket \xi x. t \rrbracket &= \lambda k. \text{let } x = \lambda y. \lambda k'. k' (k y) \text{ in} \\ &\quad \llbracket t \rrbracket (\lambda y. y)\end{aligned}$$

Exercise (experimental!): Extend the proof of Semantic Preservation.

The target of the transformation is λ -calculus **without** shift / reset.

It is **no longer the case** that every call is a tail call, that the right-hand side of every application is a value, or that continuations are linearly used.

Thus, shift / reset allow reaching terms which previously lied **outside** the image of the CPS transformation. CPS lets us **think outside the box!**

Other control operators

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Many other control operators or control constructs can be explained and compiled away via CPS.

Exceptions can be compiled away by “double-barrelled CPS”, that is, by using two continuations.

Effect handlers can be compiled away via (type-directed, selective) CPS.

Rompf, Maier, Odersky, [Implementing first-class polymorphic delimited continuations by a type-directed selective CPS-transform](#), 2009.

Leijen, [Type-directed compilation of row-typed algebraic effects](#), 2017.

Monadic intermediate form

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If one just aims to make evaluation order explicit, CPS is overkill.
This transformation, too, achieves indifference:

$$\begin{aligned}\llbracket x \rrbracket &= x \\ \llbracket \lambda x. t \rrbracket &= \lambda x. \llbracket t \rrbracket \\ \llbracket t_1 \ t_2 \rrbracket &= \text{let } x_1 = \llbracket t_1 \rrbracket \text{ in} \\ &\quad \text{let } x_2 = \llbracket t_2 \rrbracket \text{ in} \\ &\quad \quad x_1 \ x_2 \\ \llbracket \text{let } x = t_1 \text{ in } t_2 \rrbracket &= \text{let } x = \llbracket t_1 \rrbracket \text{ in } \llbracket t_2 \rrbracket\end{aligned}$$

In a transformed term, the components of every application are values.

By further hoisting “*let*” out of the left-hand side of “*let*”,
one gets administrative normal form.

Flanagan, Sabry, Felleisen, The essence
of compiling with continuations, 1993 (2003).

The CPS monad

The CPS transformation is a special case of the monadic transformation.
See Dagand's lectures!

Some history



Continuations, and the CPS transformation, were independently discovered by many researchers during the 1960s.

John C. Reynolds, *The discoveries of continuations*, 1993.

Some history

The CPS transformation has been used in compilers.

Rabbit (Steele). SML/NJ.

Appel, *Compiling with Continuations*, 1992.

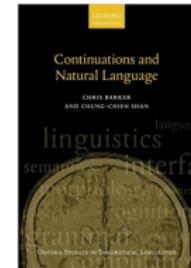
Today, heap-allocating the stack is considered too costly:

- bad locality;
- increased GC load;
- confuses the processor's built-in prediction of return addresses.

Yet, selective CPS transformations are used to compile effect handlers, and some compilers use CPS as an intermediate form before coming back to direct style.

Kennedy, *Compiling with continuations, continued*, 2007.

Some history



Can λ -calculus and continuations explain the structure of speech?

Chris Barker,
Continuations and the nature of quantification, 2002.

Chris Barker and Chung-Chieh Shan,
Continuations and Natural Language, 2014.

A few things to remember

Continuations rule!

- The CPS transformation achieves several remarkable effects:
 - making the stack explicit;
 - making evaluation order explicit;
 - suggesting/explaining control operators.
- It plays a fundamental role in prog. language theory and in logic.
- Continuation-passing is also a useful programming technique.

We have illustrated a few proof techniques:

- Another proof of semantic preservation.
- A small-step simulation diagram (see part 5).
- Testing, to refute a conjecture (see part 5).

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(Presented at MPRI 2.4 in 2017.)

Could we use a **small-step operational semantics**
in the proof that CPS is semantics-preserving?

Towards semantic preservation

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Let us consider the pure λ -calculus, without “let”.

Let us use de Bruijn notation.

The transformation is defined in [CPSDefinition](#).

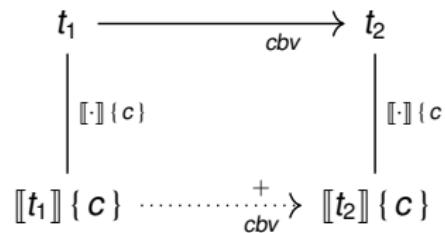
The proof of Simulation is in [CPSSimulationWithoutLet](#).

The key lemmas are in [CPSSpecialCases](#), [CPSSubstitution](#), [CPSKubstitution](#).

A small-step simulation diagram

We propose to use the **small-step substitution semantics** and to establish a **simulation diagram**.

One step by the source program is simulated in **one or more** steps by the transformed program:

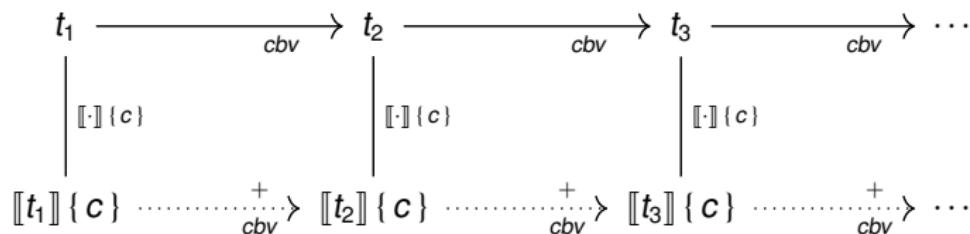


A solid arrow represents a **universal** quantification (a hypothesis).

A dashed arrow represents an **existential** quantification (a conclusion).

Consequences of the simulation diagram

There immediately follows that divergence is preserved.



The fact that each step is simulated by one or more steps is crucial.

(A proof by co-induction. See [Relations/infseq_simulation](#).)

Consequences of the simulation diagram

Obviously, **several** steps by the source program
are simulated in **several** steps by the transformed program:

$$[t_1] \xrightarrow[\text{cbv}]{\star} t_2$$

$\vdash [c] \quad \vdash [c]$

$$[t_1] \{ c \} \xrightarrow[\text{cbv}]{\star} [t_2] \{ c \}$$

(A proof by induction. See [Relations/star_diamond_left](#).)

Consequences of the simulation diagram

There follows that convergence to a value is preserved.

We use the identity continuation *done*, defined as $m\ 0$.

$$\begin{array}{ccc} t & \xrightarrow[\text{cbv}]{\star} & v \\ \left\| \cdot \right\| \{ \text{done} \} & & \left\| \cdot \right\| \{ \text{done} \} \\ \left[t \right] \{ \text{done} \} & \xrightarrow[\text{cbv}]{\star} & \left[v \right] \{ \text{done} \} \end{array}$$

By definition, $\left[v \right] \{ \text{done} \}$ is *apply done* (v), that is, (v) , therefore a value.

Thus, the CPS transformation is semantics-preserving.

The simulation lemma

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Here is the simulation statement again, this time in textual form:

Lemma (Simulation)

Assume $\text{reify } c \text{ is a value}$. Then $t_1 \rightarrow_{\text{cbv}} t_2$ implies $\llbracket t_1 \rrbracket \{ c \} \xrightarrow{+}_{\text{cbv}} \llbracket t_2 \rrbracket \{ c \}$.

Let us now do the proof.

Onscreen or in Coq? Both, probably.

See [CPSSimulationWithoutLet](#).

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Case: $(\lambda t) v \longrightarrow_{cbv} t[v/]$. We must show:

$$\llbracket (\lambda t) v \rrbracket \{ c \} \longrightarrow_{cbv}^+ \llbracket t[v/] \rrbracket \{ c \}$$

By the Value-Value Application lemma, the left-hand term is:

$$(\lambda t) (v) (\text{reify } c)$$

By definition of (λt) , this is:

$$(\lambda \lambda (\llbracket \uparrow^1 t \rrbracket \{ o 0 \})) (v) (\text{reify } c)$$

The transformed function is passed [an actual argument](#) (v) and [a continuation](#) $\text{reify } c$.

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$$(\lambda\lambda(\llbracket \uparrow^1 t \rrbracket \{ o \ 0 \})) \ (\!(v)\!) \ (reify \ c)$$

In two β -reduction steps, this term reduces to:

$$(\llbracket \uparrow^1 t \rrbracket \{ o \ 0 \}) \ [\uparrow ((\!(v)\!)/)] \ [reify \ c/]$$

We have two successive substitutions. This term could also be written using a single substitution that acts on variables 0 and 1:

$$(\llbracket \uparrow^1 t \rrbracket \{ o \ 0 \}) \ [reify \ c \cdot (\!(v)\!) \cdot id]$$

(We won't use this fact, though.)

We now wish to push the substitutions inside, one after the other.

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$$(\llbracket \uparrow^1 t \rrbracket \{ o 0 \}) \ [\uparrow (\langle v \rangle /)] \ [\text{reify } c /]$$

By the Substitution lemma, the substitution $\uparrow (\langle v \rangle /)$ acts on both **the term** $\uparrow^1 t$ and **the continuation** $o 0$.

However, $\uparrow (\langle v \rangle /)$ has no effect on variable 0.

Thus, the above term is:

$$(\llbracket (\uparrow^1 t) [\uparrow (v /)] \rrbracket \{ o 0 \}) \ [\text{reify } c /]$$

that is,

$$(\llbracket \uparrow^1 t[v/] \rrbracket \{ o 0 \}) \ [\text{reify } c /]$$

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$$(\llbracket \uparrow^1 t[v/] \rrbracket \{ o 0 \}) \ [reify c/]$$

By the Kuststitution lemma, the substitution *reify c/* acts **only on the continuation $o 0$, not on the term $t[v/]$** , because it cancels out with \uparrow^1 .

Thus, this term is:

$$\llbracket t[v/] \rrbracket \{ (o 0) [reify c/] \}$$

that is,

$$\llbracket t[v/] \rrbracket \{ o (reify c) \}$$

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We have now reached the term:

$$\llbracket t[v/] \rrbracket \{ o (reify c) \}$$

and the goal is to prove that it reduces (in zero or more steps) to:

$$\llbracket t[v/] \rrbracket \{ o c \}$$

This is the Magic Step lemma. This proof case is finished!

Here are the four key lemmas that we have used so far.

Lemma (Value-Value Application)

$$\llbracket v_1 \; v_2 \rrbracket \{ c \} = (\llbracket v_1 \rrbracket) (\llbracket v_2 \rrbracket) (\text{reify } c).$$

Lemma (Substitution)

Let σ and σ' be value substitutions such that σ' is equal to $\sigma ; \{\cdot\}$. Then,

$$(\llbracket t \rrbracket \{ c \})[\sigma'] = \llbracket t[\sigma] \rrbracket \{ c[\sigma'] \}.$$

Lemma (Kubstitution)

Let θ and σ be substitutions such that $\theta ; \sigma$ is id. Then,

$$\llbracket (t[\theta]) \{ c \} \rrbracket [\sigma] = \llbracket t \rrbracket \{ c[\sigma] \}.$$

Lemma (Magic Step)

$$\llbracket t \rrbracket \{ o (\text{reify } c) \} \xrightarrow[cbv]{} \llbracket t \rrbracket \{ c \}.$$

Proof of Simulation – cases AppL and AppR

Case: $t_1 \ u \longrightarrow_{cbv} t_2 \ u$, where $t_1 \longrightarrow_{cbv} t_2$.

We must show $\llbracket t_1 \ u \rrbracket \{ c \} \longrightarrow_{cbv}^+ \llbracket t_2 \ u \rrbracket \{ c \}$.

By definition of the CPS transformation, this is

$$\longrightarrow_{cbv}^+ \begin{array}{l} \llbracket t_1 \rrbracket \{ m \llbracket \uparrow^1 u \rrbracket \{ m 1 0 \uparrow^2 (\text{reify } c) \} \} \\ \llbracket t_2 \rrbracket \{ m \llbracket \uparrow^1 u \rrbracket \{ m 1 0 \uparrow^2 (\text{reify } c) \} \} \end{array}$$

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Case: $t_1 \ u \longrightarrow_{cbv} t_2 \ u$, where $t_1 \longrightarrow_{cbv} t_2$.

We must show $\llbracket t_1 \ u \rrbracket \{ c \} \xrightarrow{+}_{cbv} \llbracket t_2 \ u \rrbracket \{ c \}$.

By definition of the CPS transformation, this is

$$\xrightarrow{+}_{cbv} \begin{aligned} & \llbracket t_1 \rrbracket \{ m \llbracket \uparrow^1 u \rrbracket \{ m 1 0 \uparrow^2 (\text{reify } c) \} \} \\ & \llbracket t_2 \rrbracket \{ m \llbracket \uparrow^1 u \rrbracket \{ m 1 0 \uparrow^2 (\text{reify } c) \} \} \end{aligned}$$

Wow – the induction hypothesis applies directly to this goal!

Indeed, *reify* ($m \dots$) is a λ -abstraction, therefore a value.

This proof case is complete!

Case: $v \ u_1 \longrightarrow_{cbv} v \ u_2$, where $u_1 \longrightarrow_{cbv} u_2$.

Analogous to the previous case, using a Value-Term Application lemma.

We see in these proof cases that reduction under a context in the source program is translated to reduction at the root in the transformed program.

Simulation in the presence of *let* constructs

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In the presence of “*let*” constructs, Simulation breaks down.

Challenge: can you find a (minimal) counter-example?

Hint: Enlist a machine’s help. (See next two slides.)

Enumerating λ -terms

Define the **size** of a term as follows: variables have size 0; λ -abstractions and applications contribute 1.

Step 1: In OCaml, implement an exhaustive enumeration of the λ -terms of size s and with at most n free variables. (Given as an exercise in week 1.)

```
(* Enumerate all variables between 0 and n excluded. *)
let var (n : int) (k : term -> unit) : unit = ...
(* Enumerate all manners of splitting an integer s. *)
let split (s : int) (k : int -> int -> unit) : unit = ...
(* Enumerate all terms of size s with at most n variables. *)
let term (s : int) (n : int) (k : term -> unit) : unit = ...
```

An enumerator is naturally written in CPS style!

Testing Simulation

Step 2: In OCaml, implement the CPS transformation.

```
type continuation =
| O of term
| M of term
let cps (t : term) (c : continuation) : term = ...
```

Step 3: In OCaml, implement a test for the relation $\cdot \longrightarrow_{cbv}^* \cdot$:

```
let reduces (t1 : term) (t2 : term) : bool = ...
```

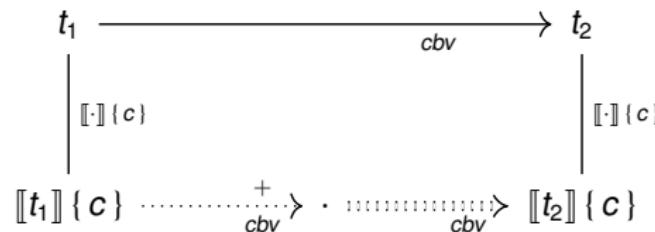
Hint: Re-use the auxiliary functions of week 2. See [Lambda](#).

Step 4: Find a term t_1 of minimal size that violates Simulation.

Solution: see [CPSCounterExample](#).

Fixing Simulation

In the presence of “let”, Simulation can be fixed as follows:



We allow one step of parallel call-by-value reduction \Rightarrow_{cbv} .

The proof of Simulation is more complex; see [CPSSimulation](#).

Parallel (call-by-value) reduction

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Parallel reduction allows reducing **all** (currently visible) redexes at once, including under “ λ ” and in the right-hand side of “ let ”.

PARALLEL β_V

$$\frac{t_1 \Rightarrow_{cbv} t_2 \quad v_1 \Rightarrow_{cbv} v_2}{(\lambda t_1) v_1 \Rightarrow_{cbv} t_2[v_2/]}$$

PARALLEL let_V

$$\frac{t_1 \Rightarrow_{cbv} t_2 \quad v_1 \Rightarrow_{cbv} v_2}{let v_1 \text{ in } t_1 \Rightarrow_{cbv} t_2[v_2/]} \qquad x \Rightarrow_{cbv} x$$

$$\frac{t_1 \Rightarrow_{cbv} t_2}{\lambda t_1 \Rightarrow_{cbv} \lambda t_2}$$

$$\frac{t_1 \Rightarrow_{cbv} t_2 \quad u_1 \Rightarrow_{cbv} u_2}{t_1 u_1 \Rightarrow_{cbv} t_2 u_2}$$

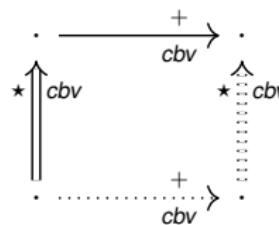
$$\frac{t_1 \Rightarrow_{cbv} t_2 \quad u_1 \Rightarrow_{cbv} u_2}{let t_1 \text{ in } u_1 \Rightarrow_{cbv} let t_2 \text{ in } u_2}$$

The ability to **reduce under a binder** is needed to fix Simulation.

Call-by-name parallel reduction is studied by [Takahashi \(1995\)](#).

[Crary \(2009\)](#) adapts these results to a call-by-value setting.

Well-behavedness of parallel reduction

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Lemma (Commutation)

$$(\Rightarrow_{cbv}^* ; \longrightarrow_{cbv}^+) \subseteq (\longrightarrow_{cbv}^+ ; \Rightarrow_{cbv}^*).$$

See [LambdaCalculusStandardization/pcbv_cbv_commutation](#).

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Lemma (Equiconvergence)

$$(\exists v, t \Rightarrow_{cbv}^* v) \iff (\exists v', t \longrightarrow_{cbv}^* v').$$

(The idea is, v' reduces to v via internal parallel reduction steps.)

See [LambdaCalculusStandardization/equiconvergence](#).

Consequences of Fixed Simulation

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There follows that divergence is preserved.

Indeed, from:

$$t \xrightarrow{cbv} \cdot \xrightarrow{cbv} \cdots$$

we get:

$$\llbracket t \rrbracket \{ c \} \xrightarrow[cbv]{+} \cdot \Rightarrow_{cbv} \cdot \xrightarrow[cbv]{+} \cdot \Rightarrow_{cbv} \cdots$$

which, by Commutation, yields:

$$\llbracket t \rrbracket \{ c \} \xrightarrow[cbv]{+} \cdot \xrightarrow[cbv]{+} \cdot \Rightarrow_{cbv}^{\star} \cdots$$

that is,

$$\llbracket t \rrbracket \{ c \} \xrightarrow[cbv]{\geq 2} \cdot \Rightarrow_{cbv}^{\star} \cdots$$

And so on. For an arbitrary $n \geq 0$, we have:

$$\llbracket t \rrbracket \{ c \} \xrightarrow[cbv]{\geq n} \cdot \Rightarrow_{cbv}^{\star} \cdots$$

Consequences of Fixed Simulation

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Convergence to a value is preserved, too.

Indeed, from:

$$t \xrightarrow{^n_{cbv}} v$$

we get, as on the previous slide:

$$\llbracket t \rrbracket \{ done \} \xrightarrow{\geq n}_{cbv} \cdot \Rightarrow_{cbv}^* (v)$$

and, by Equiconvergence:

$$\exists v' \quad \llbracket t \rrbracket \{ done \} \xrightarrow{\geq n}_{cbv} \cdot \xrightarrow{^n_{cbv}} v'$$

The CPS transformation remains semantics-preserving in the presence of “let” constructs (phew!).