MAPPING SPACES OF 1-TRUNCATED ANIMATED RINGS AND TRANSMUTED GERBES

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ABSTRACT. In this note, we expose a technique to compute the groupoid of maps between two 1-truncated animated rings when presented as quasi-ideals. In particular we compute the groupoid of maps from a discrete ring to a 1-truncated animated ring presented as a quasi-ideal.

This analysis is motivated by their prominence in [BhFG], and what is exposed here is meant to explain in more detail some arguments that can be found in Bhatt's notes. In particular we show an explicit form of the groupoid of points of the so called "transmuted stacks". Also we explain why these objects have a natural gerbe structure.

CONTENTS

Add formal cases (in section 2 and 4) add trivial gerbe argument in section 4

1. COMPUTING MAPPING SPACES OF 1-TRUNCATED ANIMATED RINGS

Fix a base discrete ring k in what follows. Recall the following about quasi-ideals.

Definition 1.1 (k-quasi-ideals)**.** A k-quasi ideal is a triple $C = (A, I, d)$ where A is a k-algebra I is an A-module and $d\colon I \to A$ is an A-module map such that for all $i, j \in I$ the equality

$$d(j) \cdot i = d(i) \cdot j$$

holds in I. In the context of quasi-ideals, we will denote the scalar multiplication from the A-module structure on I using the "$\cdot$" symbol.

A morphism of k-quasi-ideals is a commuting diagram

$$\begin{tikzcd} I_1 \arrow[r, "d_1"] \arrow[d, "\psi"] & A_1 \arrow[d, "\varphi"] \\ I_2 \arrow[r, "d_2"] & A_2 \end{tikzcd}$$

such that φ is a k-algebra morphism and for all $a \in A_1$ and $i \in I_1$ we have

$$\psi(a \cdot i) = \varphi(a) \cdot \psi(i).$$

The above just means that ψ is an A_1-module morphism where I_2 is an A_1-module from $\varphi\colon A_1 \to A_2$. We define

$$\pi_1(C) = \ker(d) \quad \pi_0(C) = \mathrm{coker}(d).$$

We say that a map of quasi-ideal is a *quasi-isomorphism* if it induces an isomorphisms on π_1 and π_0.

Remark 1.2. If $C = (I \xrightarrow{d} A)$ is a quasi-ideal, then $\pi_0(C) = \mathrm{coker}(d)$ is a k-algebra and $\pi_1(C) = \ker(d)$ is canonically a $\pi_0(C)$-module. Indeed if $k \in \ker(d)$ and $i \in I$ then $d(i) \cdot k = d(k) \cdot i = 0$.

Example 1.3. An example of quasi-ideals is the inclusion of an ideal $(I, A, \subset)$ in any k-algebra. An orthogonal example is $(B, N, 0)$ for any k-algebra B and B-module N. We call such examples *formal* k-quasi-ideals.

Remark 1.4. A morphism of quasi-ideals from $(\psi, \varphi)\colon (I, A, \subset) \to (B, N, 0)$ with notation as in Example 1.3 will satisfy for any $i, j \in I$ that

$$\psi(ij) = \psi(i \cdot j) = \varphi(i) \cdot j = 0 \cdot j = 0.$$

In other words in this case ψ factors through I/I^2.

Remark 1.5. The 1-category of quasi-ideals is equivalent to the 1-category of groupoids in k-algebras, see [Dri21, Section 3]. We point out how to see a quasi-ideal as a k-algebra groupoid.

If (A, I, d) is a k-quasi-ideal then the corresponding groupoid k-algebra has A as a k-algebra of objects, and $A \oplus_d I$ as the k-algebra of morphisms where the multiplication is given by

$$(a, i)(b, j) = (ab, b \cdot i + a \cdot j + d(i) \cdot j).$$

The identity morphism at $a \in A$ being $(a, 0)$, the source of (a, i) being a and the target of (a, i) being $a + d(i)$.

In [Dri21, Section 3], it is shown that this functor is an equivalence of categories from the 1-category of k-quasi-ideals to the 1-category of groupoids in k-algebras. It sends a quasi-isomorphism to an equivalence of groupoids.

To see how to extend this to a (2,1)-categorical equivalence, see the proof of Proposition 1.12 and Remark 1.10.

Definition 1.6 (Homotopies between morphisms of k-quasi-ideals.)**.** Consider two k-quasi-ideals $C_1 = (I_1, A_1, d_1)$ and $C_2 = (I_2, A_2, d_2)$. Consider two morphisms $(\psi_i, \varphi_i)\colon (I_1, A_1, d_1) \to (I_2, A_2, d_2)$ for $i = 1, 2$

$$\begin{tikzcd} I_1 \arrow[r, "d_1"] \arrow[d, "\psi_i"] & A_1 \arrow[d, "\varphi_i"] \\ I_2 \arrow[r, "d_2"] & A_2 \end{tikzcd}$$

An homotopy from (ψ_1, φ_1) to (ψ_2, φ_2) is an additive module map

$$h\colon A_1 \to I_2$$

such that

(1) $hd_1 = \psi_2 - \psi_1$
(2) $d_2 h = \varphi_2 - \varphi_1$

(3) for every $a, b \in A_1$ we have

$$h(ab) = \varphi_1(a) \cdot h(b) + \varphi_1(b) \cdot h(a) + d_2(h(a))h(b) = \varphi_1(b) \cdot h(a) + \varphi_2(a) \cdot h(b).$$

(4) For every $\lambda \in k$ we have $h(\lambda) = 0$.

Note most importantly that if $(\psi_1, \varphi_1) = (\psi_2, \varphi_2)$ then $hd = dh = 0$ implying together with the multiplicative axioms that

$$h\colon \pi_0(C_1) \to \pi_1(C_2)$$

is a k-derivation.

Remark 1.7. We want to point out that the "derivation like axiom" in Definition 1.6 can be inferred precisely using the groupoid in k-algebras perspective: namely if we write what should be a natural transformation there, we ultimately get to this axiom from the form of the k-algebra structure on $A \oplus_d I$. Therefore this give a formal mathematical meaning to "a derivation is a loop of a morphism". More precisely a derivation is exactly an automorphism of a morphism of derived schemes.

Definition 1.8. Let $C_1 = (A_1, I_1, d_1)$ and $C_2 = (A_2, I_2, d_2)$ be k-quasi ideals. We define a groupoid

$$\mathrm{Map}^{\mathrm{hom}}(C_1, C_2)$$

whose objects are maps of quasi-ideals as defined in Definition 1.1 and morphisms are homotopies between maps as defined in Definition 1.6. The composition of homotopies is given by the sum of homotopies, the zero homotopy being the identity.

Remark 1.9. Beware, as usual in homological algebra this will not compute the correct homotopy type, unless C_1 or C_2 have good "lifing properties". This is the content of Proposition 1.12.

Remark 1.10. This remark is preliminary content for Lemma 1.11 and Proposition 1.12. The hurried reader can skip this remark and take Proposition 1.12 as a fact.

We point out to the vocabulary of "animation" introduced in [CS23, Section 5.1.4]. Say $\mathrm{Pol}_{k,\mathrm{fin}}$ designates the full subcategory of k-algebras consisting of k-polynominal algebras on finitely many variables. Then the (2,1)-category of 1-truncated animated k-algebras

$$\mathrm{Ani}(k\text{-Alg})_{\leq 1} = \mathrm{Fun}_{\times}(\mathrm{Pol}_{k,\mathrm{fin}}, \mathrm{Grpd})$$

sits fully faithfully in

$$\mathrm{Ani}(k\text{-Alg}) = \mathrm{Fun}_{\times}(\mathrm{Pol}_{k,\mathrm{fin}}, \mathrm{Ani}).$$

In this note, we are interested in the (2,1)-category $\mathrm{Ani}(k\text{-Alg})_{\leq 1}$ of *1-truncated animated k-algebras.*

In [Dri21, Proposition 3.2.4, Section 3], it is shown that we have an essentially surjective functor from the 1-category of k-quasi-ideals to the (2,1)-category of 1-truncated animated k-algebras. It means that we can represent any 1-truncated animated k-algebra as a quasi-ideal.

In [Dri21, Section 4] it is explained that given two k-quasi ideals $C_1 = (A_1, I_1, d_1)$ and $C_2 = (A_2, I_2, d_2)$, then the mapping groupoid

$$\mathrm{Map}_{\mathrm{Ani}(k\text{-Alg})}(C_1, C_2)$$

can be realized as the groupoid of *butterflies*

$$\begin{tikzcd}
I_1 \arrow[dd, "d_1"'] \arrow[dr, "g_1"] & & I_2 \arrow[dl, "g_2"'] \arrow[dd, "d_2"] \\
& B \arrow[dl, "f_1"] \arrow[dr, "f_2"'] & \\
A_1 & & A_2
\end{tikzcd}$$

where

(1) B is a k-algebra, f_1 and f_2 are k-algebra maps,
(2) g_1 and g_2 are module maps where I_1 and I_2 are seen as B-modules by the maps f_1 and f_2,
(3) The SE-NW sequence is a complex and the NE-SW sequence is exact.

A morphism between butterflies with center being B_1 and B_2 respectively is a k-algebra morphism $B_1 \to B_2$ commuting to the arrows from the butterfly data. Note that by the five-lemma applied to NE-SW we see that any morphism of butterflies is an isomorphism.

We now construct a functor

$$\mathrm{Map}^{\mathrm{hom}}(C_1, C_2) \to \mathrm{Map}_{\mathrm{Ani}(k\text{-Alg})}(C_1, C_2)$$

where the target is to be interpreted as the above defined groupoid of butterflies. Given a map

$$\begin{tikzcd}
I_1 \arrow[r, "d_1"] \arrow[d, "\psi"] & A_1 \arrow[d, "\varphi"] \\
I_2 \arrow[r, "d_2"] & A_2
\end{tikzcd}$$

we define the butterfly

$$\begin{tikzcd}
I_1 \arrow[dd, "d_1"'] \arrow[dr, "{(d_1, -\psi)}"] & & I_2 \arrow[dl, "\supset"'] \arrow[dd, "d_2"] \\
& A_1 \oplus_{d_2} I_{2,\varphi} \arrow[dl, "proj_1"] \arrow[dr, "\varphi + d_2"'] & \\
A_1 & & A_2
\end{tikzcd}$$

where A_1 acts on I_2 using $\varphi\colon A_1 \to A_2$, that's what we meant by $I_{2,\varphi}$.

Note that the NW-SE sequence has a canonical k-algebra splitting.

Now, given an homotopy $h\colon A_1 \to I_2$ between to maps, one checks that $(a, i) \mapsto (a, h(a) + i)$ defines a butterfly isomorphism between the corresponding butterflies – which concludes the definition of the functor.

Lemma 1.11. *Say that $C_1 = (I_1 \xrightarrow{d_1} A_1)$ and $C_2 = (I_2 \xrightarrow{d_2} A_2)$ are k-quasi ideals, The functor defined above in Remark 1.10*

$$\mathrm{Map}^{\mathrm{hom}}(C_1, C_2) \to \mathrm{Map}_{\mathrm{Ani}(k\text{-Alg})}(C_1, C_2)$$

is fully-faithful.

Proof. Say we have two morphisms (φ_1, ψ_1) and (φ_2, ψ_2) between C_1 and C_2. A morphism between the associated butterflies is a k-algebra map

$$A_1 \oplus_{d_2} I_{2,\varphi_1} \to A_1 \oplus_{d_2} I_{2,\varphi_2}$$

with commutes to all other maps present in the butterflies. In particular because it should commute to the projection to A_1 and the inclusion of I_2 we see it has to be of the form

$$(a, i) \mapsto (a, h(a) + i)$$

where $h\colon A_1 \to I_2$ is some additive map such that $h(\lambda) = 0$ for all $\lambda \in k$. The commutation with both wings of the butterfly gives that $hd_1 = \psi_2 - \psi_1$ and $d_2 h = \varphi_2 - \varphi_1$. That it has to preserve multiplication gives the rule that for all $a, b \in A_1$ we have

$$h(ab) = \varphi_1(a) \cdot h(b) + \varphi_1(b) \cdot h(a) + d_2(h(a))h(b) = \varphi_1(b) \cdot h(a) + \varphi_2(a) \cdot h(b).$$

In other words, we see that the data of a morphism between the butterflies is exactly the data of an homotopy between the morphism. □

Proposition 1.12. *Say that $C_1 = (I_1 \xrightarrow{d_1} A_1)$ and $C_2 = (I_2 \xrightarrow{d_2} A_2)$ are k-quasi ideals, and suppose furthermore that A_1 is a polynomial k-algebra. Then the functor defined above in Remark 1.10*

$$\mathrm{Map}^{\mathrm{hom}}(C_1, C_2) \to \mathrm{Map}_{\mathrm{Ani}(k\text{-}\mathrm{Alg})}(C_1, C_2)$$

is an equivalence.

Proof. Note that because A_1 is supposed to be a k-algebra, we always have a k-algebra splitting $s\colon A_1 \to A_1 \oplus_{d_2} I_2$ for any butterfly

$$\begin{tikzcd}
I_1 \arrow[dd, "d_1"'] \arrow[dr, "g_1"] & & I_2 \arrow[dl, "g_2"'] \arrow[dd, "d_2"] \\
& B \arrow[dl, "f_1"] \arrow[dr, "f_2"'] & \\
A_1 \arrow[ur, bend left, "s"] & & A_2
\end{tikzcd}$$

But then one can check that

$$A_1 \oplus I_2 \xrightarrow{s+g_2} B$$

gives a butterfly isomorphism to the butterfly coming from

$$\begin{tikzcd}
I_1 \arrow[r, "d_1"] \arrow[d, "g_2^{-1}(sd_1 - g_1)"'] & A_1 \arrow[d, "f_2 s"] \\
I_2 \arrow[r, "d_2"] & A_2
\end{tikzcd}$$

which shows essential surjectivity. □

Remark 1.13. Say that $C_2 = (I_2 \subset A_2)$ where I_2 is an ideal of A_2. This is to be treated as a discrete (meaning concentrated in degree zero) algebra, because $\pi_1(C_2) = 0$. A map from an animated ring to a usual discrete ring is just the data of the induced map on π_0. We check this using the perspective of Proposition 1.12 to serve as an exercise in manipulating quasi-ideals.

In the above case a self map of a morphism $C_1 \to C_2$ is a derivation

$$h\colon \pi_0(C_1) \to \pi_1(C_2) = 0$$

we see that the groupoid

$$\mathrm{Map}^{\mathrm{hom}}(C_1, C_2)$$

is in fact just equivalent to the set of it's connected components. Moreover, note that if we suppose that $C_1 = (I_1 \xrightarrow{d_1} A_1)$ where A_1 is a polynomial, there will always be a lift

$$\begin{tikzcd}
I_1 \arrow[d, "d_1"] & I_2 \arrow[d, "\subset"] \\
A_1 \arrow[r, dashed] \arrow[d] & A_2 \arrow[d] \\
\pi_0(C_1) \arrow[r] & \pi_0(C_2)
\end{tikzcd}$$

Moreover, because it is a lift of the map on π_0, the composition $I_1 \to A_1 \to A_2$ will factor through I_2. Also, one can checks that the difference of two lifts will define an homotopy $A_1 \to I_2$ between the lifts. All in all we have shown that we have an equivalence

$$\mathrm{Map}_{\mathrm{Ani}(k\text{-Alg})}(C_1, C_2) \to \mathrm{Hom}_{k\text{-Alg}}(\pi_0(C_1), \pi_0(C_2)),$$

as expected.

2. 1-truncated animated rings and groupoids of ring extensions

The following theorem describes explicitly the groupoid of maps from a ring to a 1-truncated animated ring presented as quasi-ideal as the groupoid of some ring extension problem.

In particular this gives some explicit description of points of the stacky geometrizations from [BhFG].

Theorem 2.1 (1-truncated animated rings classify extension problems)**.** *Let $C = (I \xrightarrow{d} A)$ be a k-quasi-ideal. Let R be a k-algebra. Then we have the following identifications of groupoids*

$$\mathrm{Map}_{\mathrm{Ani}(k-\mathrm{Alg})}(R, C) = \left\{ \begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\iota"] \arrow[dr, "d"'] & E \arrow[r] \arrow[d, "\sigma"] & R \arrow[r] & 0 \\ & & A & & \end{tikzcd} \;\middle|\; \begin{array}{l} E \text{ is a } k\text{-algebra such that for all} \\ e \in E \text{ and } i \in I \quad \iota(\sigma(e)\cdot i) = e\iota(i) \\ \text{and } \sigma\colon E \to A \text{ is a } k\text{-algebra map} \\ \text{commuting as displayed.} \end{array} \right\}$$

where the groupoid structure on the right is to be understood as objects being the displayed data and isomorphisms being maps $E \to E'$ commuting with the aforementioned data.

For any extension E as displayed, the group of automorphisms of E naturally identifies to

$$\mathrm{Hom}_{R\text{-Mod}}(\Omega_{R|k}, \pi_1(C)_\eta)$$

where $\eta\colon R \to \pi_0(C)$ is the map induced by σ on the quotient by I and $\pi_1(C)_\eta$ denotes the R-module structure coming from this map.

Proof. We define a functor from left to right. We take a polynomnial k-algebra surjection $P \to R$ and and denote the kernel by J. By Proposition 1.12, the left hand side is computed as the groupoid of quasi-ideals morphisms

$$\begin{tikzcd} J \arrow[r, "\psi"] \arrow[d, "\subset"'] & I \arrow[d, "d"] \\ P \arrow[r, "\varphi"] & A \end{tikzcd}$$

and homotopy between maps. Given such a map define E as the pushout

$$\begin{tikzcd} J \arrow[r, "\subset"] \arrow[d, "\psi"'] & P \arrow[d] \\ I \arrow[r] & E \end{tikzcd}$$

this is a k-algebra seen as the quotient of $P \oplus_{\varphi,d} I$ where I is seen as a P-module by $\varphi\colon P \to A$, and the multiplication of elements in I is given by the $ij := d(i) \cdot j$. We define $\sigma\colon E \to A$ to be the map induced by $\varphi + d$. Everything is therefore setup so that E fits in a diagram as displayed on the right hand side.

We now define the (contravariant) functor on the morphisms. Given two different maps of k-quasi-ideals

$$\begin{tikzcd} J \arrow[r, "\psi_1"] \arrow[d, "\subset"'] & I \arrow[d, "d"] \\ P \arrow[r, "\varphi_1"] & A \end{tikzcd} \qquad \begin{tikzcd} J \arrow[r, "\psi_2"] \arrow[d, "\subset"'] & I \arrow[d, "d"] \\ P \arrow[r, "\varphi_2"] & A \end{tikzcd}$$

and an homotopy h from (φ_1, ψ_1) to (φ_2, ψ_2). One can check that the map $(p, j) \mapsto (p, j + h(p))$ defines a k-algebra isomorphism from $P \oplus_{\varphi_2,d} I \to P \oplus_{\varphi_1,d}$, which inverse is given by the same recipe with $-h$.

We now conclude the proof by proving that this functor is fully-faithful. Consider a k-algebra isomorphism

$$P \oplus_{\varphi_2,d} I/(j, -\psi_2(j)) \to P \oplus_{\varphi_1,d} I/(j, -\psi_1(j))$$

commuting to the data. In particular this isomorphism should commute to the inclusion of I, implying that $(0, i) \mapsto (0, i)$. Note the following about the composition

$$P \to P \oplus_{\varphi_2,d} I/(j, -\psi_2(j)) \to P \oplus_{\varphi_1,d} I/(j, -\psi_1(j)).$$

First, this is a k-algebra morphism. Also, because the isomorphism commutes with the quotient to R, it implies that there is some $h(p) \in I$ such that $p \mapsto (p, h(p))$ under the above composition. Because of the form of the equivalence relation on the quotient, we see that this $h(p)$ is actually uniquely defined.

One therefore sees that the isomorphism comes from a k-algebra map of the form $(p, i) \mapsto (p, i + h(p))$ between

$$P \oplus_{\varphi_2,d} I \to P \oplus_{\varphi_1,d} I$$

that passes to the quotient. Passing to the quotient implies $d_{|J} = \psi_2 - \psi_1$. Commuting with the map to A gives $dh = \varphi_2 - \varphi_1$. The fact that the above gives a k-algebra morphism gives the "derivation like" axiom. $\square$

Remark 2.2. In this remark we observe some trivial cases of the above Theorem 2.1.

(1) First note that the case where the quasi-ideal is of the form
$$0 \to A$$
then the above groupoid is just the set of k-algebra maps from $R \to A$ as expected.

(2) When the quasi-ideal is of the form
$$I \subset A$$
so an honest ideal, then we see from Remark 1.13 that the groupoid of extensions form Theorem 2.1 form a groupoid which is equivalent to the set of maps $R \to A/I$. In other words, two such diagrams will be uniquely isomorphic if and only if they induce the same map on $E/I \to A/I$.

From the second point of Remark 2.2, we see that the interesting cases of groupoids appears when d is not injective – the most extreme case being when $d = 0$. In the next Corollary 2.3 we explain the case where the quasi-ideal is *formal*, meaning of the form
$$N \xrightarrow{0} A.$$

Recall that if N is an R-module, then a *k-algebra square-zero extension of R by N* is a k-algebra E together with a surjective map $E \to R$ which kernel is a square-zero ideal and an identification of N with this ideal as R-modules. We abbreviate the above data in the following notation by
$$0 \to N \to E \to R \to 0 \quad \text{sq. zero}$$
Note that if we have $d = 0$ in Theorem 2.1, then the condition on ι will imply that
$$\iota(j)\iota(i) = \iota(\sigma(\iota(j) \cdot i)) = \iota(0 \cdot i) = 0,$$
because of $\sigma\iota = 0$. Another consequence is that σ is therefore the same data as a map from $E/I = R \to A$. Therefore we get the following Corollary 2.3.

Corollary 2.3 (Deformation theory understood as maps of derived rings)**.** *Let R, A be k-algebras and N an A-module. Write $A \oplus N[1]$ for the k-quasi-ideal $N \xrightarrow{0} A$.*

Then
$$\mathrm{Map}_{\mathrm{Ani}(k-\mathrm{Alg})}(R, A\oplus_0 N[1]) = \left\{ \begin{array}{c|c} \begin{tikzcd} 0 \arrow[r] & N_\eta \arrow[r] & E \arrow[r] & R \arrow[r] \arrow[d, "\eta"] & 0 \text{ sq. zero} \\ & & & A & \end{tikzcd} & \begin{array}{l} R \to A \text{ is a} \\ k\text{-algebra map.} \end{array} \end{array} \right\}$$
where N_η means that the R-module structure is given by $\eta\colon R \to A$. In other words, the fiber over η by the natural map $\mathrm{Map}_{\mathrm{Ani}(k-\mathrm{Alg})}(R, A \oplus_0 N[1]) \to \mathrm{Hom}_{k\text{-}\mathrm{Alg}}(R, A)$ is the groupoid of square-zero extensions
$$0 \to N_\eta \to E \to R \to 0 \quad \text{sq. zero}$$
which can be expressed using deformation theory as
$$\mathrm{Map}_{\mathrm{Ani}(R\text{-}\mathrm{Mod})}(L_{R|k}, N_\eta[1]).$$

Remark 2.4. If R is smooth, then $L_{R|k} = \Omega_{R|k}[0]$ and this last module is finite projective. We denote it's dual by $T_{X|k}$.

Therefore the group of automorphisms mentioned in Theorem 2.1 rewrites as
$$T_{X|k} \otimes_k \pi_1(C)_\eta.$$

Also, the groupoid mentionned at the end of Corollary 2.3 rewrites as

$$B(T_{X|k} \otimes_k N_\eta).$$

3. Formal smoothness for 1-truncated animated rings

We now prove a formal smoothness property for 1-truncated animated rings.

Proposition 3.1. *Let R be a formally smooth k-algebra. Let C be a 1-truncated animated k-algebra. Let $\eta\colon R \to \pi_0(C)$ a k-algebra morphism. Then the groupoid of lifts*

$$\begin{tikzcd} & C \arrow[d] \\ R \arrow[ur, "\eta'"] \arrow[r, "\eta"'] & \pi_0(C) \end{tikzcd}$$

is equivalent to the R-module groupoid

$$\mathrm{Hom}_R(\Omega^1_{R|k}, \pi_1(C)_\eta)[1].$$

Remark 3.2. Proposition 3.1 is a form of formal smoothness because in particular it tells that *there is a* lift. It is more precise in saying that all lifts are homotopic, and that the automorphism group of lifts is given by $\mathrm{Hom}_R(\Omega^1_{R|k}, \pi_1(C)_\eta)$. Note that this last fact was already known from Theorem 2.1.

Proof of Proposition 3.1. In this proof we use a foundational fact in derived ring theory –namely that any Postinkov truncation is a square-zero extension. This is [HA, Corollary 7.4.1.28] for the case of connective E_∞-rings – but the same hold for animated rings. In particular the following holds for C a 1-truncated animated k-algebra;

Claim. *There is a unique (up to homotopy) derivation $\epsilon\colon L_{\pi_0(C)|k} \to \pi_1(C)[2]$ such that*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \pi_0(C) \arrow[d, "d_\epsilon"] \\ \pi_0(C) \arrow[r, "d_0"] & \pi_0(C)_{\oplus 0}\pi_1(C)[2] \end{tikzcd}$$

is a cartesian square of animated k-algebras.

Therefore, we see that the groupoid of morphisms $R \to C$ such that $R \to C \to \pi_0(C)$ is η is the path space between 0 and $\epsilon\eta$ in the 2-truncated anima

$$\mathrm{Map}_{\mathrm{Ani}(R\text{-Mod})}(L_{R|k}, \pi_1(C)[2]) = \mathrm{Der}_k(R, \pi_1(C)[2])$$

Because R is formally smooth, we have that $L\Omega^1_{R|k} = \Omega_{R|k}[0]$ and that $\Omega_{R|k}$ is projective. Therefore

$$\begin{aligned} \pi_0(\mathrm{Map}_{\mathrm{Ani}(R\text{-Mod})}(L_{R|k}, \pi_1(C)[2])) &= \mathrm{Ext}^2_R(\Omega_{R|k}, \pi_1(C)) = 0 \\ \pi_1(\mathrm{Map}_{\mathrm{Ani}(R\text{-Mod})}(L_{R|k}, \pi_1(C)[2])) &= \mathrm{Ext}^1_R(\Omega_{R|k}, \pi_1(C)) = 0 \\ \pi_2(\mathrm{Map}_{\mathrm{Ani}(R\text{-Mod})}(L_{R|k}, \pi_1(C)[2])) &= \mathrm{Hom}_R(\Omega^1_{R|k}, \pi_1(C)) \end{aligned}$$

and $\pi_n = 0$ for $n \geq 3$.

This implies the claim on the path space. Indeed, the vanishing of π_0 tells that the path space is non-empty because the mapping space is connected. The vanishing of π_1 implies that the mapping space is simply connected, or in other words that there is a unique path up to homotopy between two maps. Therefore the path-space is connected. The form of π_2 tells that group of automorphisms of one path is isomorphic to $\mathrm{Hom}_R(\Omega^1_{R|k}, \pi_1(C))$ which concludes.

$\square$

Remark 3.3. Proposition 3.1 tells that we have a pullback diagram

$$\begin{tikzcd} \mathrm{Hom}_R(\Omega^1_{R|k}, \pi_1(C)_\eta)[1] \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Ani}(k\text{-Alg})}(R, C) \arrow[d] \\ * \arrow[r, "\eta"] & \mathrm{Hom}_{k\text{-Alg}}(R, \pi_0(C)) \end{tikzcd}$$

4. Transmuted gerbes

In this section we fix a smooth k-scheme X. Whenever we say "sheaf" it is with respect to some sub-canonical topology τ on $k\text{-Alg}^{\mathrm{op}}$ that may be taken to the reader's discretion.

Setup 4.1. In this section we work with ring sheaf (for example a ring scheme)

$$\underline{A} \colon k\text{-Alg}^{\mathrm{op}} \to k\text{-Alg}$$

together with a $\underline{A}$ module sheaf (for example an $\underline{A}$-module scheme)

$$\underline{I} \colon k\text{-Alg}^{\mathrm{op}} \to k\text{-Alg}$$

together with a map

$$d \colon \underline{I} \to \underline{A}$$

which turn the above into a k-quasi-ideal sheaf. We denote by

$$\mathbb{G}_a^{\mathcal{T}} \colon k\text{-Alg}^{\mathrm{op}} \to \mathrm{Ani}(k\text{-Alg})_{\leq 1}$$

the associated 1-truncated animated k-algebra sheaf. This is a "ring" stack where here stack has the usual meaning of being valued in groupoids.

Suppose moreover that

$$\mathrm{H}^{>0}_\tau(\underline{I}) = 0$$

so that the ring sheaf $\pi_0(\mathbb{G}_a^{\mathcal{T}})$ can be computed as the presheaf cokernel

$$\mathrm{coker}(d \colon \underline{I} \to \underline{A}).$$

We denote the ring sheaf $\pi_0(\mathbb{G}_a^{\mathcal{T}})$ by

$$\mathbb{G}_a^{T} \colon k\text{-Alg}^{\mathrm{op}} \to k\text{-Alg}$$

Definition 4.2. Let X be a sheaf on $k\text{-Alg}^{\mathrm{op}}$. Define the *transmutation* $X^{\mathcal{T}}$ to be the (groupoid valued) functor

$$X^{\mathcal{T}} \colon k\text{-Alg}^{\mathrm{op}} \to \mathrm{Ani}_{\leq 1}$$

with

$$X^{\mathcal{T}}(S) = X(\mathbb{G}_a^{\mathcal{T}}(S))$$

where S is any k-algebra. We also define similarly a (set valued) functor

$$X^{T}(S) = X(\mathbb{G}_a^{T}(S)).$$

Remark 4.3. Note that from the natural map
$$\mathbb{G}_a^{\mathcal{T}} \to \pi_0(\mathbb{G}_a^{\mathcal{T}}) = \mathbb{G}_a^T$$
we get a natural map
$$X^{\mathcal{T}} \to X^T.$$

In the case where $X = \mathrm{Spec}(R)$ is affine, we define a functor of R-modules over X^T
$$T_{X|k} \otimes \pi_1(\mathbb{G}_a^{\mathcal{T}}) \to X^T$$
with values on $S \in k\text{-Alg}$ being a pair of $\eta \in X^T(S) = \mathrm{Hom}_{k\text{-Alg}}(R, \mathbb{G}_a^T(S))$ and an element in
$$\mathrm{Hom}_{R\text{-Mod}}(\Omega^1_{R|k}, \pi_1(\mathbb{G}_a^{\mathcal{T}}(S))_\eta) = \mathrm{Der}_k(R, \pi_1(\mathbb{G}_a^{\mathcal{T}}(S))_\eta)$$
where $\pi_1(\mathbb{G}_a^{\mathcal{T}}(S))_\eta$, which is naturally a $\mathbb{G}_a^T(S)$-module, has a structure of R-module by $\eta\colon R \to \mathbb{G}_a^T(S)$

Proposition 4.4. *Let $X = \mathrm{Spec}(R)$ be a smooth k-algebra. Then*
$$X^{\mathcal{T}} \to X^T$$
is a $B(T_{X|k} \otimes \pi_1(\mathbb{G}_a^{\mathcal{T}}))$-gerbe.

Proof. Fibers of
$$X^{\mathcal{T}}(S) \to X^T(S)$$
or in other words, fibers of
$$\mathrm{Map}_{\mathrm{Ani}(k\text{-Alg})}(R, \mathbb{G}_a^{\mathcal{T}}(S)) \to \mathrm{Hom}_{k\text{-Alg}}(R, \mathbb{G}_a^T(S))$$
are according to the analysis made in Proposition 3.1 and Theorem 2.1 isomorphic to
$$\mathrm{Hom}_{R\text{-Mod}}(\Omega^1_{R|k}, \pi_1(\mathbb{G}_a^{\mathcal{T}}(S))_\eta).$$
Therefore the claim follows by definition of $B(T_{X|k} \otimes \pi_1(\mathbb{G}_a^{\mathcal{T}}))$.

□

Remark 4.5. Typically in situations from [BhFG], functors $X^{\mathcal{T}}$ and X^T are sheaves for some topologies, and the associations $X \mapsto X^{\mathcal{T}}$ and $X \mapsto X^T$ preserves coverings with respect to some topologies. In this last case, one can get variants of Proposition 4.4 for smooth schemes or any smooth widget such that one can get result by descent from the affine case.

We now inspect the particular case where the quasi-ideal from the beginning of the section is of the form
$$\underline{N} \xrightarrow{0} \underline{A}.$$
We denote this quasi-ideal sheaf as
$$\underline{A} \oplus_0 \underline{N}[1].$$

Corollary 4.6. *Let $X = \mathrm{Spec}(R)$ be a smooth k-algebra. Say that*
$$\mathbb{G}_a^{\mathcal{T}} = \underline{A} \oplus_0 \underline{N}[1].$$
Then
$$X^{\mathcal{T}} \to X^T$$
is a trivial $B(T_{X|k} \otimes \pi_1(\mathbb{G}_a^{\mathcal{T}}))$-gerbe.

Proof. Follows from Corollary 2.3 and Remark 2.4. $\square$

Remark 4.7. Note that in this case, we have a splitting $\mathbb{G}_a^T \to \mathbb{G}_a^{\mathcal{T}}$ given by the natural map $\underline{A} \to \underline{A} \oplus_0 \underline{I}$. So we have a natural splitting of the map of functors $X^{\mathcal{T}} \to X^T$. This observation together with Proposition 4.4 also gives Corollary 4.6.

Appendix A. Animated vs E_∞

In this appendix we show that the (2,1)-category of 1-truncated animated rings fully faithfully embeds in the (2,1)-category of 1-truncated E_∞-$H\mathbb{Z}$-algebras. This section is not needed for the previous sections of this note. It just tells the curious fact the mapping spaces computed in Section 1 and Section 2 do not depend on the foundations of derived geometry we choose to work with. This is a specifity of the 1-truncated case. We thank Maxime Ramzy for sharing this proof with us. This explains Bhatt's footnote in [BhFG, p. 23].

Lemma A.1. *The functor*

$$\mathrm{Ani}(\mathrm{Ring})_{\leq 1} \to \mathrm{CAlg}_{H\mathbb{Z}\leq 1}$$

is fully faithful.

Proof. By colimit generation, it is enough to show that for any 1-truncated animated ring R,

$$\mathrm{Map}_{\mathrm{Ani(Ring)}}(\mathbb{Z}[t], R) \to \mathrm{Map}_{\mathrm{CAlg}_{H\mathbb{Z}}}(\mathbb{Z}[t], R)$$

is an equivalence. Note that if $\mathbb{Z}\{t\}$ denotes the free E_∞-ring on one element, the following composition (coming from $\mathbb{Z}\{t\} \to \mathbb{Z}[t]$)

$$R = \mathrm{Map}_{\mathrm{Ani(Ring)}}(\mathbb{Z}[t], R) \to \mathrm{Map}_{\mathrm{CAlg}_{H\mathbb{Z}}}(\mathbb{Z}[t], R) \to \mathrm{Map}_{\mathrm{CAlg}_{H\mathbb{Z}}}(\mathbb{Z}\{t\}, R) = R$$

is always an equivalence. Therefore the first map is fully-faithful. The second map is also surjective on π_0. If we show that it is also injective, then it would mean that the first map is essentially surjective, concluding.

As R is 1-truncated, it suffices to show that

$$\tau_{\leq 1}\mathbb{Z}\{t\} \to \mathbb{Z}[t]$$

is a 1-epimorphism. Note that this last morphism is the image by $\tau_{\leq 1}\mathbb{Z}\{-\}$ of the map of E_∞-monoids

$$\mathrm{Fin} \to \mathbb{N} = \pi_0(\mathrm{Fin})$$

where Fin is the groupoid of finite sets and bijections (the free E_∞-monoid on one element) and $\mathbb{N}$ the discrete free monoid on one element. Note that these two are 1-truncated anima.

Claim. *If X is a 1-truncated anima, then $X \to \pi_0(X)$ is an epimorphism in the category of 1-truncated anima (=groupoids).*

Proof. It suffices to show that if X is connected then $X \to *$ is an epimorphism, meaning that the pushout $(* \leftarrow X \to *)$ is equivalent to $*$. But the suspension of a connected anima is always simply connected (for example by the theorem of Van-Kampen to compute π_1 of the suspension). The claim therefore follows. $\square$

Now as E_∞-monoids embed faithfully in anima it follows that $\mathrm{Fin} \to \pi_0(\mathrm{Fin}) = \mathbb{N}$ is a 1-epimorphism. The functor $\tau_{\leq 1}\mathbb{Z}\{-\}$ being a left adjoint, it preserves epimorphisms implying our original claim. $\square$

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