NOTES ON DERIVED STACKS

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Contents

1. Introduction

This note was written as a supporting material for a talk for a seminar at EPFL in autumn 2025, studying [Bu+25].

In the first part we introduce elements of ∞-categories, trying to motivate the latter for applications in algebraic geometry.

In the main part of the note, we introduce *derived algebras* and the *cotangent complex*. A notable part of the exposition is the calculation of the cotangent complex of a derived zero locus of a section of a vector bundle in Example 3.29.

As we will explain in *Conventions*, we focus our attention on what happens over $\mathbb{C}$. But this note is written in a way that most of the propositions (everything except Proposition 3.4) hold over an arbitrary base.

At the end, we make use of the language of ∞-categories to define *derived algebraic stacks*, a derived enhancement of usual (higher) stacks.

Conventions. Everything happens over $\mathbb{C}$. Namely in what follows a vector space is a $\mathbb{C}$-vector space, an algebra is a $\mathbb{C}$-algebra, any (affine) scheme is over $\mathbb{C}$, etc. We will denote by Alg the category of $\mathbb{C}$-algebras, Aff the category of $\mathbb{C}$-affine schemes, Sch the category of $\mathbb{C}$-schemes.

By *chain complex*, we mean a $\mathbb{N}$-homologically graded chain complex of $\mathbb{C}$-vector spaces. Some would use term *connective complex of $\mathbb{C}$-vector spaces.* We use homological notation for complexes (differentials go down!). Homology groups of a chain complex will be denoted by the topological notation $\pi_i (= H_i)$ as homotopy groups. We also the topological convention to call *fiber sequences*

$$A \to B \to C$$

in a stable ∞-category whenever we have a pullback(=pushout) diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

Whenever we such a fiber sequence, there is an associated *long exact sequence of homotopy groups.*[1]

2. ∞-categories

The language of ∞-categories is an enlargement of the language of categories where the homotopical mechanism is built in. For example, if one considers complexes of abelian groups, or complexes of coherent sheaves on a scheme, then there is a notion of homotopy between maps, and one sees that the right notion of "being the same" for complexes is when two complexes are "quasi-isomorphic" a notion which ultimately relies on the homotopical logic present in complexes. One can then consider morphisms of complexes only up to quasi-isomorphisms, meaning precisely that one can invert all quasi-isomorphisms to form a new category. But this process is too crude in some aspects.

- For example, one learns that the even though the cone $\mathrm{Cone}(\varphi)$ of a morphism of complexes

  $$\varphi\colon A_\bullet \to B_\bullet$$

  should be thought of as a "quotient", it does not possess the universal property of the quotient in this new category where we inverted quasi-isomorphisms. Namely it is not same to give a map $\mathrm{Cone}(\varphi) \to K_\bullet$ in this category and a map $B_\bullet \to K_\bullet$ which is zero when precomposing by φ.

  However, one sees by the very construction of the cone in homological algebra that, giving a map of complexes $B_\bullet \to \mathrm{Cone}(\varphi)$ is exactly the same as the data of a map of complexes $\psi\colon B_\bullet \to K_\bullet$ *together with a homotopy* between $\psi\varphi$ and the zero map. Here we see that a correction to the universality of the cone would be to "remember the data of an homotopy". What we learn here is the following principle of higher categories: it is unreasonable to "only" ask that two things are homotopic. Instead one should ask that they are homotopic and we *remember* an homotopy witnessing their identification.
- Say X is a scheme covered by affine schemes U_i. Then the data of a quasi-coherent sheaf on X is the same as the data of collection of $\mathcal{O}_X(U_i)$-modules together with compatible (*i.e.* respecting the cocycle conditions) identifications on the intersections. Say we want to mimic this so-called *descent* process but for complexes of quasi-coherent sheaves in lieu and place of quasi-coherent sheaves.

[1] These are the ∞-categorical version of "distinguished triangles" in triangulated categories.

What would be appropriate would be to apply the aforementioned "higher philosophy": let's take a collection of complexes of modules on affines, and quasi-isomorphisms on intersections satisfying the cocycle conditions. So here, instead of asking only that $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, we should *remember* an homotopy witnessing that these two maps are "equal" in the derived category. But then we have homotopies h_{ijk} that become part of the data and one would want again that they are compatible in some appropriate sense when going to quadruple intersections. And again, this should be understood as the extra data of an homotopy (of homotopies) which witness this higher compatibility. This process can potentially go on, and on, and on...

Even though this seems very complicated, this is the correct thing to do if one wants to have descent for complexes.

- From the above, we see that what would be needed is some categorical language where morphisms in a category are not only sets, but are "homotopy types". Namely a mathematical entity where it makes sense to have points, morphisms between points, morphisms between morphisms, and so on, in order to be able to remember all this extra data. Simplicial sets are a combinatorial tool that do the perfect job for that.

We claim that the language of simplicial sets is adequate to achieve the above. If so, ordinary category theory should be a special case. So we first explain how to understand a category as a simplicial set. Namely, given a simplicial set (X_n), one should think of X_0 as the set of objects, X_1 as the set of morphisms, X_2 as the set of homotopies between morphisms, etc.

References and proofs for Propositions and definitions below are found in [Lur25, Tag 002L].

Proposition 2.1. *Let $\mathcal{C}$ be a small category. Then $N(\mathcal{C})$ the simplicial set defined by*

$$N(\mathcal{C})_n = \mathrm{Fun}(\Delta_n, \mathcal{C})$$

which is the set of tuples of n-composable morphisms gives a fully-faithful embedding $\mathrm{Cat} \to \mathrm{sSet}$.

One can characterize the image of this embedding as those simplicial sets S such that for every pair of integers $0 < i < n$ and $\sigma_0 \colon \Lambda^n_i \to S$, then there is a unique extension $\sigma \colon \Delta^n \to S$.

The condition in the characterization essentially tells that there is a unique associative way to compose morphisms. The simplicial set Λ^n_i is the sub-simplcial set of the boundary of Δ^n where we removed the face opposite to the vertex i. For example, we have

$$\begin{tikzcd} & 1 \arrow[d, dotted] \\ 0 \arrow[ur] \arrow[r] \arrow[ur, phantom, "\Lambda^2_0" near end, shift right=3] & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[d] \\ 0 \arrow[ur] \arrow[r, dashed] \arrow[ur, phantom, "\Lambda^2_1" near end, shift right=3] & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[d] \\ 0 \arrow[ur, dotted] \arrow[r] \arrow[ur, phantom, "\Lambda^2_2" near end, shift right=3] & 2 \end{tikzcd}$$

and so we see that the condition for $n = 2$ amounts that for any pair of composable morphisms, there is a unique composition. Note that asking that one would have extension for outer horns, so when $i = 0$ and $i = n$ would mean that every morphisms has a unique inverse – for example the unique extension property for Λ^2_0 would be given f and g, that there is a unique h such that $hf = g$. One can therefore also characterize groupoids as simplicial sets.

Proposition 2.2. *Those simplicial sets S such that for every pair of integers $0 \leq i \leq n$ and $\sigma_0 \colon \Lambda^n_i \to S$, then there is a unique extension $\sigma \colon \Delta^n \to S$, form a category which is equivalent to the category of groupoids.*

Simplicial sets are also known to model homotopy types. The incarnation of homotopy types would be as the fundamental ∞-groupoid of a space, namely something with points, paths (=morphisms) between points, homotopies between paths (=morphisms between morphisms), etc. This fits in the above framework.

Definition 2.3 (∞-groupoid)**.** A simplicial set S such that for every pair of integers $0 \leq i \leq n$ and $\sigma_0 \colon \Lambda^n_i \to S$, then there is an extension $\sigma \colon \Delta^n \to S$, is called an *$\infty$-groupoid* or an *homotopy type* or a *Kan complex* or a *space*, or an *anima*.

Here, instead of asking that there is a unique composition of two composable morphisms (f, g) we are asking that there is one composition h, and the data of the extension $\sigma \colon \Delta^2 \to S$ is to be interpreted as the data of an homotopy witnessing that $f \circ g \simeq h$. Also, in this case we mimicked the groupoid condition so that every morphism, and higher morphism was asked to be invertible. One can instead mimic the condition for categories and get the following.

Definition 2.4 (∞-categories)**.** A simplicial set S such that for every pair of integers $0 < i < n$ and $\sigma_0 \colon \Lambda^n_i \to S$, then there is an extension $\sigma \colon \Delta^n \to S$, is called an *$\infty$-category*. Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, then we denote the simplicial set of morphisms between $\mathcal{C}$ and $\mathcal{D}$ by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. This simplicial set is again an ∞-category, that we call the ∞-category of functors between $\mathcal{C}$ and $\mathcal{D}$. Given $c, c' \in \mathcal{C}$, then the simplicial set

$$\mathrm{Map}_{\mathcal{C}}(c, c') = \{c\} \times_{\mathrm{Fun}(\{0\}, \mathcal{C})} \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathrm{Fun}(\{1\}, \mathcal{C})} \{c'\}$$

is an ∞-groupoid, and is called the mapping anima of between c and c'.

Example 2.5. One may think of a ∞-category as a category enriched over ∞-groupoids due to the *homotopy coherent nerve construction* [Lur25, Tag 00LH]. Therefore we present two ∞-categories as categories enriched over ∞-groupoids in what follows.

(1) *The ∞-category* Ani *of ∞-groupoids.* This category plays the role that the category of sets plays in ordinary category theory. Because the simplicial set of morphisms between two ∞-groupoids is again an ∞-groupoid, ∞-groupoids form an ∞-category that we denote by Ani.
(2) *The ∞-category* dMod_A *of derived A-modules.* Let $A \in \mathrm{Alg}$ be an algebra. Consider $\mathrm{Ch}_{A,\geq 0}$ the category of chain complexes of A-modules. From topology, and the homology construction, we know how to associate a chain complex to a simplicial vector space. This construction actually goes also the other way.

Proposition 2.6 (Dold-Kan, [Lur25, Tag 00QQ])**.** *There is an equivalence of categories called the* Dold-Kan correspondence

$$\mathrm{DK} \colon \mathrm{Ch}_{A,\geq 0} \to \mathrm{sMod}_A$$

between chain complexes and simplicial A-modules.

Therefore we can enrich the category of chain complexes in ∞-groupoids by defining the mapping homotopy type between two chain complexes A and B by

$$\mathrm{Map}_{\mathrm{sVec}}(\mathrm{DK}(A), \mathrm{DK}(B)).$$

This forms an ∞-category that we denote by dVec.

Remark 2.7. Let us precise a bit the nature of example (2). Usually one considers chain complexes of A-modules and wants to invert quasi-isomorphisms. Actually, one can get to the ∞-category mentioned in (2) using the same idea. Namely, instead of inverting morphisms in the usual "1-categorical sense" one can invert morphisms in the ∞-categorical sense ([Lur25, Tag 01M4]) to get an equivalent result to the one explained above. The fact the two constructions agree essentially boils down to the key fact that DK sends quasi-isomorphisms to homotopy equivalences.

A key feature of ∞-category theory is that every construction that one may think possible in ordinary category theory has an analogue in ∞-category theory. For example, (co)limits, adjoints, sheaves, etc. For sheaves one important feature is that this language allows to access cohomology more directly than the usual constructions.

In ordinary category, limits and colimits are ultimately determined by limits and colimits of sets using Yoneda's lemma. In the next example we explain what is the pullback in the category of ∞-groupoids.

Example 2.8 ([Lur25, Tag 010B])**.** Let

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

be a diagram of ∞-groupoids. The pullback of this diagram in Ani is given by the simplicial set

$$X \times \times_{\mathrm{Fun}(\{0\},Z)} \mathrm{Fun}(\Delta^1, Z) \times_{\mathrm{Fun}(\{1\},Z)} Y$$

To get a grasp on this, note that an object of this ∞-groupoid is a triple consisting of

(1) An object of $x \in X$,
(2) an object of $y \in Y$,
(3) an isomorphism between $f(x)$ and $g(y)$ in Z.

As for a morphism between such triples, it is the data of

(1) a morphism $x \to x'$ in X
(2) a morphism $y \to y'$ in Y,
(3) the data of a "commuting square"

$$\begin{tikzcd} f(x) \arrow[r] \arrow[d] & f(x') \arrow[d] \\ g(y) \arrow[r] & g(y') \end{tikzcd}$$

meaning the data of two vertical morphisms as depicted, one diagonal morphism $f(x) \to g(y')$, and two homotopies witnessing that the composition on the sides of the square are homotopic with the diagonal morphism.

3. Derived algebras and derived stacks

In this section, the goal is to explain the notion of *a derived algebraic stack*. In order to do this, we will add the "derived adjective" progressively to objects of increasing complexity.

3.1. **Automorphisms and tangent vectors.** The first step is to look at so called "derived algebras". This will form an ∞-category that we denote by dAlg. Derived algebras bear the same relation to algebras as chain complexes of vector spaces bear to vector spaces. They are a generalization of algebras where there is no "underlying vector space". Instead, there is an underlying chain complex of vector spaces.

As the opposite of the category of algebras correspond to affine schemes, the opposite of the ∞-category of derived algebras will correspond to the ∞-category of "derived affine schemes". Let's do a thought experiment. Consider

$$\iota_0 \colon \operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(\mathbb{C}[t])$$

the inclusion of the origin in the affine line. Say we are computing the pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \operatorname{Spec}(\mathbb{C}) \arrow[d] \\ \operatorname{Spec}(\mathbb{C}) \arrow[r] & \operatorname{Spec}(\mathbb{C}[t]) \end{tikzcd}$$

but not in the category of schemes, but in the (for now putative and to explain) ∞-category of derived schemes. Note that in the category of schemes, this pullback is the origin itself: it is the self-intersection of the origin with the origin... but the derived intersection will differ! Derived schemes can ultimately be seen as a combinatorial way to record redundancy in algebraic equations [Lur04, Section 1.1.1], and here derived pullback will remember that we are intersecting the origin with itself by "adding" a self-automorphism.

As explained above, an homotopy pullback should not only record the data of maps that commute, but maps that commute up to homotopy *and the data of an homotopy witnessing it.* In this example, there should be only one choice of maps: namely, the constant map to the point. So here the pullback should be one "point", with some extra data of automorphisms of the composite map $F \to \operatorname{Spec}(\mathbb{C}[t])$.

To compute the above fiber product is equivalent to computing the pushout of algebras

$$\begin{tikzcd} \mathbb{C}[t] \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C} \arrow[r] & P \end{tikzcd}$$

but in the ∞-category of derived algebras. Because, as mentioned above, derived algebras bear the same relation to algebras as chains complexes to vector spaces, we might as well use the *derived* *tensor product* of dVec because the ordinary pushout would be computed with the ordinary tensor product. So,

$$\mathbb{C} \otimes^L_{\mathbb{C}[t]} \mathbb{C} = \mathbb{C}[t]/(t) \otimes_{\mathbb{C}[t]} \left(\mathbb{C}[t] \xrightarrow{\cdot t} \mathbb{C}[t]\right) = \mathbb{C} \xrightarrow{0} \mathbb{C}.$$

We denote this chain complex by

$$\mathbb{C} \oplus \mathbb{C}[1].$$

We may even look at a "ring structure" on this complex which is given by

$$(\alpha_1, \alpha_2) \cdot (\beta_1, \beta_2) = (\alpha_1\beta_1, \alpha_1\beta_2 + \beta_1\alpha_2).$$

Now we should have a groupoid of morphisms

$$\mathrm{dAlg}(\mathbb{C}[t], \mathbb{C} \oplus \mathbb{C}[1])$$

with one point, corresponding to $\mathrm{ev}_0 \colon f(t) \mapsto f(0)$ landing in degree zero, and this map should have automorphisms which correspond to self homotopies of this map. Considering ring maps

$$\mathbb{C}[t] \to \mathbb{C} \oplus \mathbb{C}[1]$$

of the form $f(t) \mapsto (f(0), d(f))$ with respect to the ring structure explained above, bearing in mind that the data of d which lands in degree 1 is the data of an automorphism of the evaluation map we get a natural identification

$$\mathrm{Aut}(\mathrm{ev}_0) = \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[t], \mathbb{C}) = \left(T_{\mathbb{A}^1_{\mathbb{C}}, \mathbb{C}}\right)_0$$

the fiber of the tangent space of $\mathbb{A}^1$ at the origin.

What do we get from this discussion: *the natural notion of automorphisms of a morphism of algebras that we deduce from homological algebra indicates it is the same as a tangent vector/derivation.*

We state the following as a generalization of the above discussion.

Proposition 3.1. *Let $A \in \mathrm{dAlg}$ a 1-truncated algebra, meaning that $\pi_n(A) = 0$ if $n > 1$. Let R be a discrete algebra. Let $\varphi \colon \mathrm{Spec}(A) \to \mathrm{Spec}(R)$, meaning a morphism $R \to A$ in dAlg. Then*

$$\mathrm{Aut}(\varphi) = \mathrm{Der}_{\mathbb{C}}(R, \pi_1(A)) = \mathrm{Hom}_R(\Omega^1_{R|\mathbb{C}}, \pi_1(A)) = T_{R|\mathbb{C}} \otimes_R \pi_1(A).$$

where the last equality is either taken as a definition or holds as a property in the case when A is smooth.

3.2. **Commutative differential graded algebras.** In this section, we explain more concretely what is the ∞-category dAlg. Namely, as dVec is more concretely chain complexes, there is also such a description as complexes for derived algebras.

Definition 3.2 (Commutative differential graded algebras)**.** A *commutative differential graded algebras* or in short, *cdga* is a chain complex of vector spaces $(C_\bullet, d)$ such that

$$\bigoplus_{n \geq 0} C_n$$

is a graded ring which satisfies the following "commutativity axiom"

$$ab = (-1)^{|a||b|} ba$$

and a graded Leibniz rule

$$d(ab) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b).$$

Remark 3.3. Given any cdga C, we have that $\pi_0(C) = H_0(C)$ inherits the structure of an algebra and $\pi_i(C) = H_i(C)$ for $i \in \mathbb{N}$ has a natural structure of $\pi_0(C)$-module.

We briefly explain why cdga's form an ∞-category using the following proposition.

Proposition 3.4. *The Dold-Kan correspondence (Proposition 2.6) refines to an equivalence of categories*

$$\mathrm{DK}\colon \mathrm{cdga} \to \mathrm{sAlg}$$

The mapping homotopy type between two cdgas A and B can be therefore defined as the simplicial set

$$\mathrm{Map}_{\mathrm{sAlg}}(\mathrm{DK}(A), \mathrm{DK}(B)).$$

One can also obtain the ∞-category above by inverting ∞-categorically the 1-category of cdga's at quasi-isomorphism of cdgas.

Remark 3.5 (Subtle points away from characteristics zero)**.** Here being in characteristic zero is crucially important for the above. Over $\mathbb{Z}$ or $\mathbb{F}_p$ for a prime number p, the above does not hold. One has to work directly with simplicial algebras for derived geometry and cdga's are not anymore well-behaved.

If $A \in \mathrm{dAlg}$ we denote by dAlg_A the ∞-category of A-algebras. This is the ∞-category of maps $A \to B$ where $B \in \mathrm{dAlg}$.

As in homological algebra, it is important to distinguished a class of objects which behaves better with respect to "derived functors" and homotopies.

Definition 3.6. Let $A \in \mathrm{Alg}$ be a discrete algebra. A *quasi-free* A-cdga is cdga such that the underlying graded ring is of the form

$$\bigwedge M$$

where $\bigwedge$ denotes the alternating algebra=(free graded commuative algebra) and M is finite projective A-module.

Let $B \in \mathrm{dAlg}_A$. We say that $B' \to B$ is *quasi-free replacement* if B' is quasi-free and the map is quasi-isomorphism.

Proposition 3.7. *Let $A \in \mathrm{Alg}$ be a discrete algebra and $B, C \in \mathrm{dAlg}_A$. Let $B' \to B$ be a quasi free-replacement. Then*

$$B \otimes_A^L C = B' \otimes_A C$$

where on the right hand side, the tensor product is the usual tensor product of cdgas, i.e the underlying tensor product of chain complexes equipped with the canonical structure of cdga in the tensor product.

As there is a notion of modules over an algebra, there is a notion of derived modules over a derived algebra $A \in \mathrm{dAlg}$.

Definition 3.8 (Differential graded module)**.** Let $A = (A_\bullet, d_A)$ be a cdga. A differential graded A-module is a $\mathbb{Z}$-chain complex $M = (M_\bullet, d_M)$ such that

$$\bigoplus_{n\in\mathbb{Z}} M_n$$

is a graded module over the graded ring $(\bigoplus_{n\in\mathbb{N}} A_n)$, and that for every homogeneous element $a \in A$ and $m \in M$ we have the following compatibility with the differential

$$d_M(am) = d_A(a)m + (-1)^{|a|} a d_M(m).$$

Differential graded A-modules also form an ∞-category, that we denote by $\mathcal{D}(A)$.

In the case when $\pi_i(M) = 0$ for $i < 0$, we say that M is *connective.* We will denote the ∞-category of connective A-modules by dMod_A.

Remark 3.9 (Stable derived categories $\mathcal{D}(A)$)**.** There is also a way to incorporate unbounded chain complexes, where as only connective ones are in dMod_A, to get a so called *stable* ∞*-category*, which may be thought as the ∞-categorical version of triangulated categories. We will denote this category by $\mathcal{D}(A)$. Objects here are $\mathbb{Z}$-chain complexes that are dg-A-modules, and the ∞-category associated is the one coming from usual homotopical algebra one these objects.

Notably, because we can represent any element of $\mathcal{D}(A)$ as a chain complex, we can define a *t-structure* [HA, Section 1.2.1] on this category by

$$\mathcal{D}(A)_{\geq 0} = \{M \in \mathcal{D}(A) \mid \pi_i(M) = 0 \quad i < 0\}$$

$$\mathcal{D}(A)_{\leq 0} = \{M \in \mathcal{D}(A) \mid \pi_i(M) = 0 \quad i > 0\}$$

We have $\mathcal{D}(A)_{\geq 0} = \mathrm{dMod}_A$ and $\mathcal{D}(A)^\heartsuit$ is the category of discrete $\pi_0(A)$-modules.

The notion of *perfect complex* is easily characterized in this language.

Definition 3.10 (Perfect complexes, [Lur04, Section 2.4])**.** An object of $\mathcal{D}(A)$ is *finitely presented* it if can be obtained in finitely many steps from A direct sums and pushouts (=pullbacks=exact triangles). An object is called *perfect* if it is a retract of a finitely presented object. A perfect object is a compact object in the sense of [HTT, Section 5.3.4].

Remark 3.11. If A is discrete, then an object of $\mathcal{D}(A)$ is perfect if and only it is isomorphic to a bounded complex of finite projective modules.

3.3. **Cotangent complex.** In this section, we will define an object which bears a crucial place in the theory and applications, the *cotangent complex.* This object was originally introduced in Illusie's thesis [Ill72].

The cotangent complex has a great importance in the theory of derived algebras. Namely, it can be made precise that to compute a mapping space in derived algebras amounts to specifying an algebra map on π_0 and compatible higher derivations in an inductive process [HA, Remark 7.4.1.29], and the latter is entirely controlled by the cotangent complex.

In order to introduce this object, we will first introduce derived analogues of dual numbers.

Example 3.12. Let $A \to B$ be a cdga. Let $M \in \mathrm{dMod}_B$ be a connective dg-module over B.

Then we consider the *square zero extension of B by M*

$$B \oplus M \in \mathrm{dAlg}_B$$

which as a complex is defined as the direct sum of complexes and the algebra structure is defined on homogeneous elements as B acting on itself and on M the obvious way.

For example if $A = B = M = \mathbb{C}$ we get the dual numbers $\mathbb{C} \oplus \mathbb{C} = \mathbb{C}[t]/(t^2)$.

If A, B and M are discrete, recall that A-linear sections of the projection $B \oplus M \to B$ are in one to one correspondence to A-linear derivations $B \to M$. This is expressed as the pullback

$$\begin{tikzcd} \mathrm{Der}_A(B, M) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Alg}_B}(B, B \oplus M) \arrow[d] \\ * \arrow[r, "\mathrm{id}"] & \mathrm{Hom}_{\mathrm{Alg}_B}(B, B) \end{tikzcd}$$

Namely a section of the projection is of the form $b \mapsto (b, d(b))$ for some map d, and the fact that this has to be a ring morphism precisely spits out the derivations axioms.

We can now define derivations in the context of derived algebras.

Definition 3.13 (Higher derivations)**.** Let $A \to B$ be a cdga morphism. Let M be a dg-module over B. The homotopy type of A-linear derivations from B to M is the pullback

$$\begin{tikzcd} \mathrm{dDer}_A(B, M) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dAlg}_B}(B, B \oplus M) \arrow[d] \\ * \arrow[r, "\mathrm{id}"] & \mathrm{Map}_{\mathrm{dAlg}_B}(B, B) \end{tikzcd}$$

Like in the discrete case, we have the following.

Proposition 3.14 (Cotangent complex)**.** *Let $A \to B$ be a cdga morphism. Then there is a complex in dMod_B unique up to unique isomorphism in the sense that it satisfies the following universal property,*

$$\mathrm{Map}_{\mathrm{dMod}_B}(L_{B|A}, M) = \mathrm{dDer}_A(B, M).$$

For every morphisms $A \to B \to C$ in dAlg we have a fiber sequence in $\mathcal{D}(C)$

$$L_{B|A} \otimes_B^L C \to L_{C|A} \to L_{C|B}$$

If A is a discrete algebra then we have

$$\pi_0(L_{B|A}) = \Omega_{\pi_0(B)|A}.$$

Moreover if $A \in \mathrm{Alg}$ is discrete and $B \in \mathrm{Alg}_A$ is a smooth A-algebra, then

$$L_{B|A} = \Omega_{B|A}[0].$$

The cotangent complex is subject to the following base change formulas, that one checks using the universal property from Proposition 3.14.

Proposition 3.15. *Let $R \in \mathrm{dAlg}$. Suppose that we have a pushout diagram in dAlg_R of the following form.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

(1) There is a natural isomorphism

$$D \otimes_C L_{C|A} \to L_{D|B}.$$

(2) We have a pushout diagram

$$\begin{tikzcd} D \otimes_A L_{A|R} \arrow[r] \arrow[d] & D \otimes_B L_{B|R} \arrow[d] \\ D \otimes_C L_{C|R} \arrow[r] & L_{D|R} \end{tikzcd}$$

Equivalently, we have a fiber sequence

$$D \otimes_A L_{A|R} \to D \otimes_B L_{B|R} \oplus D \otimes_C L_{C|R} \to L_{D|R}$$

Remark 3.16. About the last point of Proposition 3.15 – whenever there is a pushout diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "g'"] \\ z \arrow[r, "f'"'] & t \end{tikzcd}$$

in a stable ∞-category, then we have a fiber sequence

$$x \xrightarrow{(f,-g)} y \oplus z \xrightarrow{f'+g'} t.$$

Example 3.17. Let R be discrete R-algebra and let P be a projective R-module. Consider the natural surjection

$$S_P := \mathrm{Sym}_R(P^\vee) \to R$$

which corresponds geometrically to the zero section of the vector bundle $\mathbb{V}(P)$ over $X = \mathrm{Spec}(R)$. Let's compute $L_{R|S_P}$. Note that the composite $R \to S_P \to R$ is the identity. Therefore using the fiber sequence of cotangent complexes we get that

$$L_{S_P|R} \otimes_{S_P} R \to L_{R|R} \to L_{R|S_P}$$

so we get that

$$L_{R|S_P} = L_{S_P|R} \otimes_{S_P} R[1].$$

But as $R \to S_P$ is smooth, and that $\Omega^1_{S_P|R} = S_P \otimes_R P^\vee$ we get in the end that

$$L_{R|S_P} = P^\vee[1].$$

Definition 3.18 (Properties of morphisms)**.** Let $A \to B$ be a morphism in dAlg such that $\pi_0(A) \to \pi_0(B)$ is finite type. We then say that

(1) B is smooth over A if $L_{B|A}$ is of tor-amplitude $[0,0]$ in $\mathcal{D}(B)$,
(2) B is étale over A if $L_{B|A} = 0$.

The following is important for applications of the theory.

Proposition 3.19 (Perfect obstruction theory)**.** *Let $A \in \mathrm{dAlg}$ be a derived algebra which can be written as a finite colimit of algebras of the form $\mathbb{C}[x]$ (a finitely presented derived algebra). Then $L_{A|\mathbb{C}}$ is perfect and the natural map*

$$\pi_0(A) \otimes_A L_{A|\mathbb{C}} \to L_{\pi_0(A)|\mathbb{C}}$$

is an isomorphism and a surjection at π_1.

Proof. The fact that $L_{A|\mathbb{C}}$ is perfect follows from the fact that the cotangent complex construction $L_{-|\mathbb{C}}$ sends finitely presented algebras to finitely presented modules. Also, it is a general fact that $L_{\pi_0(A)|A}$ is 2-connective (vanishing π_0 and π_1).[2] Now the last claim follows using the fiber sequence for $\mathbb{C} \to A \to \pi_0(A)$ □

3.4. **Derived loci and derived critical loci.** In this section, we compute key examples of derived schemes. Namely we expand on the example of Section 3.1. So it is better to first give the following definition.

Definition 3.20 (Derived affine schemes)**.** The ∞-category of derived affine schemes is defined as the opposite of the ∞-category of derived algebras $\mathrm{dAlg}^{\mathrm{op}}$. We denote it by dAff.

Definition 3.21. Let $X = \mathrm{Spec}(A) \in \mathrm{dAff}$. We call the *truncation* of X the affine scheme $\mathrm{Spec}(\pi_0(A))$ and we denote it by X_{cl}. There is a canonical map

$$X_{\mathrm{cl}} \to X.$$

Remark 3.22. General derived schemes will be defined in the next section when introducing sheaves. The notion of pullback of derived affine schemes is the same as the notion of pullback of derived schemes, because derived affine schemes and derived schemes belong in a similar adjunction that affine schemes and schemes do.

First, let R be a discrete algebra and let $f \in R$ be any element, and denote $X = \mathrm{Spec}(R)$. Say, in ordinary scheme theory, that one wants to compute the locus $\{f = 0\}$ of X. A way to do this is to look at the pullback in schemes

$$\begin{tikzcd} V(f) \arrow[r] \arrow[d] & X \arrow[d, "0"] \\ X \arrow[r, "f"'] & \mathbb{A}^1_R \end{tikzcd}$$

which is just $\mathrm{Spec}(R/(f))$. Now that we have access to the ∞-category of derived schemes, one might as well ask what is the intersection in this new category.

We first note that a quasi-free resolution of R seen as an $R[t]$-algebra with the evaluation at zero is given by

[2] This is because the 1-truncation of $A \to \pi_0(A)$ is a 1-epimorphism, essentially because of Van-Kampen's theorem in topology.

$$(R[t] \xrightarrow{\cdot t} R[t]) \to R.$$

Therefore using Proposition 3.7, one sees that the algebra of the derived intersection is given by

$$R/^L(f) := R \xrightarrow{f} R$$

Let's list some cases.

(1) If f is a non-zero divisor, then this cdga is quasi-isomorphic to $R/(f)$, which equals in any case $\pi_0(R/^L(f))$.
(2) Otherwise we have some non-trivial π_1, namely

$$\pi_1(R/^L(f)))\{g \in R \mid gf = 0\} = \mathrm{Tor}^1_R(R, R/(f)).$$

(3) As examples, we can take $f = 0$ and get a straightforward generalization of the self intersection from Section 3.1.
(4) Say $R = \mathbb{C}[x, y]/(xy^2)$. Then for the derived loci of $y = 0$ we will have

$$\pi_1(R/^L(y)) = (xy)/(xy^2).$$

We generalize the above to the zero locus of a section of a vector bundle. So let P be a vector bundle of finite rank (=finite projective module) on an algebra R. And fix $s \in P$ an element. Denote by $S_P = \mathrm{Sym}_R(P^\vee)$.

We define a cdga structure on the complex of S_P-modules

$$S_P \otimes_R \bigwedge^{\bullet} P^\vee$$

Namely, say that

$$e = \sum_i \alpha_i \otimes x_i \in P^\vee \otimes P$$

is the canonical element. Then we define a differential by

$$S_P \otimes_R \bigwedge^{n+1} P^\vee \to S_P \otimes_R \bigwedge^{n} P^\vee$$

$$d(a \otimes \beta_1 \wedge \cdots \beta_{n+1}) \mapsto \sum_i a\alpha_i \otimes \sum_k (-1)^k \beta_k(x_i) \beta_1 \wedge \cdots \hat{\beta}_k \wedge \cdots \wedge \beta_{n+1}.$$

We denote this cdga by $\mathrm{Kos}(P)$ and call it the *Koszul complex of P.*

We have the following proposition.

Proposition 3.23. *The natural S_P-cdga map*

$$\mathrm{Kos}(P) \to R$$

is quasi-free resolution of R seen as an S_P-algebra sending $P^\vee$ in degree 1 to zero.

Therefore, we arrive at the following definition.

Definition 3.24 (Derived loci)**.** Let R be a discrete algebra P a finite projective module and $s \in P$. Then we call the *derived zero locus* of s the pullback in the ∞-category of derived schemes

$$\begin{tikzcd} V_d(s) \arrow[r] \arrow[d] & X \arrow[d, "0"] \\ X \arrow[r, "s"'] & \mathbb{V}(P) \end{tikzcd}$$

By the above, the algebra of functions is given by

$$\mathrm{Kos}(P) \otimes_{S_P} R$$

where here R is seen as an R-algebra by $\mathrm{ev}_s \colon S_P \to R$ which is given by evaluation at s.

We denote this cdga by $\mathrm{Kos}(P, s)$. This is the complex $\bigwedge P^\vee$ with differential

$$d_s \colon \bigwedge^{n+1} P^\vee \to \bigwedge^{n} P^\vee$$

given by

$$\beta_1 \wedge \cdots \wedge \beta_{n+1} \mapsto \sum_k (-1)^k \beta_k(s) \beta_1 \wedge \cdots \hat{\beta_k} \wedge \cdots \wedge \beta_{n+1}.$$

Remark 3.25. The truncation of $V_d(s)$ is the usual zero locus $V(s)$.

Example 3.26 (Derived zero locus of a vector field)**.** Let X be a smooth scheme and $v \in T_{X|\mathbb{C}}$ be a global vector field. Then the derived algebra of functions of $V_d(v)$ is given by

$$(\bigwedge \Omega_{X|\mathbb{C}}, \iota_v)$$

where ι_v is the *interior product with respect to* v from differential geometry.

Definition 3.27 (Derived critical loci)**.** Let $X = \mathrm{Spec}(R)$ be a smooth affine scheme over $\mathbb{C}$ and $f \in \mathcal{O}_X(X)$ a function. Consider $df \in \Omega^1_{X|\mathbb{C}}$. We call $V_d(df)$ the *derived critical locus of* f.

Example 3.28. For a derived critical loci, the derived algebra of functions on $V_d(df)$ is represented by a cdga which has as an underlying graded ring $\bigwedge T_{X|\mathbb{C}}$ the *algebra of multivectors* from differential geometry. The differential comes from applying the differential 1-form df to those tangent multivectors.

Example 3.29. In this example, we compute the cotangent complex of a derived locus. So let R be a smooth algebra, P a finite projective module, and $s \in P$. Denote as above $S_P = \mathrm{Sym}_R(P^\vee)$ the functions on the vector bundle $\mathbb{V}(P)$. The goal is to compute

$$L_{V_d(s)|\mathbb{C}}.$$

First, as by definition we have a pushout square in dAlg

$$\begin{tikzcd} S_p \arrow[r, "s"] \arrow[d, "0"'] & R \arrow[d] \\ R \arrow[r] & \mathrm{Kos}(P, s) \end{tikzcd}$$

So we use Proposition 3.15 to get a fiber sequence of the following form.

$$\mathrm{Kos}(P,s)\otimes_{S_P} L_{S_p|\mathbb{C}} \to \mathrm{Kos}(P,s)\otimes_R L_{R|\mathbb{C}} \oplus \mathrm{Kos}(P,s)\otimes_R L_{R|\mathbb{C}} \to L_{\mathrm{Kos}(P,s)|\mathbb{C}} \tag{1}$$

Therefore to compute $L_{\mathrm{Kos}(P,s)|\mathbb{C}}$, we need to understand the complexes and the maps $L_{S_P|\mathbb{C}} \to L_{R|\mathbb{C}}$ induced by the zero section and s respectively. First, because $\mathbb{C} \to R$ is smooth and so is $R \to S_P$ we have that

$$L_{R|\mathbb{C}} = \Omega_{R|\mathbb{C}}[0] \quad L_{S_P|\mathbb{C}} = \Omega_{S_P|\mathbb{C}}[0]$$

Also, using the fiber sequence from $\mathbb{C} \to R \to S_P$ one gets a split exact sequence (because these are projective modules concentrated in degree zero)

$$0 \to S_P \otimes_R \Omega_{R|\mathbb{C}} \to \Omega_{S_P|\mathbb{C}} \to \Omega_{S_p|R} \to 0$$

But $\Omega_{S_p|R} = S_P \otimes_R P^\vee$. Therefore we have

$$\Omega_{S_P|\mathbb{C}} = S_P \otimes_R (P^\vee \oplus \Omega_{R|\mathbb{C}})$$

Inserting this in Equation (1) we get that the first two terms are

$$\mathrm{Kos}(P,s)\otimes_R (P^\vee \oplus \Omega_{R|\mathbb{C}}) \to \mathrm{Kos}(P,s)\otimes_R (\Omega_{R|\mathbb{C}} \oplus \Omega_{R|\mathbb{C}}).$$

Using matrix notation, this map is given by the tensor by $\mathrm{Kos}(P,s)\otimes_R$ of the R-linear map

$$P^\vee \oplus \Omega_{R|\mathbb{C}} \to \Omega_{R|\mathbb{C}} \oplus \Omega_{R|\mathbb{C}}$$

given by

$$\begin{pmatrix} 0 & ds \\ \mathrm{id} & \mathrm{id} \end{pmatrix}$$

where ds denotes the map taking $\varphi \in P^\vee$ and sending it to $d(\varphi(s))$.

This complex is quasi-isomorphic to the following one

$$P^\vee \xrightarrow{ds} \Omega_{R|\mathbb{C}}.$$

In the end we conclude that $L_{\mathrm{Kos}(P,s)|\mathbb{C}}$ is the complex

$$\mathrm{Kos}(P,s)\otimes_R (P^\vee \xrightarrow{ds} \Omega_{R|\mathbb{C}}).$$

Example 3.30. We specialize the previous example to the case of a derived critical locus. Namely, take $f \in \mathcal{O}_X(X)$ where $X = \mathrm{Spec}(R)$ is a smooth affine scheme. Considering $\mathrm{Kos}(\Omega_{R|\mathbb{C}}, df)$, we get that $L_{\mathrm{Kos}(\Omega_{R|\mathbb{C}},df)|\mathbb{C}}$ is obtained by tensoring the complex of finite projective R-modules concentrated in degree 0 and 1

$$T_{X|\mathbb{C}} \xrightarrow{D^2(f)} \Omega_{X|\mathbb{C}} \tag{2}$$

where $D^2(f)$ denotes the map sending a tangent vector v to $d(df(v))$. In other words, étale locally this map is given by the *Hessian* of the function f, *i.e.* the matrix of second derivatives of f.

Note that the symmetry of the Hessian implies that the map from Equation (2) is self-dual.

3.5. **Derived stacks.** In what follows, we will define derived schemes, and derived algebraic stacks. A prerequisite is the notion of sheaf of ∞-groupoids/anima.

Definition 3.31. Let $\mathcal{C}$ be an ∞-category equipped with a structure of site τ.

(1) The ∞-category of presheaves of anima on $\mathcal{C}$ is $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani})$.
(2) We say that a presheaf $\mathcal{F}$ is a sheaf if for every covering $(C_i \to C)$ then if $C_\bullet$ denote the Čech nerve of these maps, then
$$\mathcal{F}(C) = \varprojlim_{n \in \Delta^{\mathrm{op}}} F(C_n)$$

The last condition can seem mysterious so we provide an example. Say we are working on the site of topological space and $X = \bigcup_i U_i$ is a covering by open sets. Then the Čech nerve is

$$\cdots \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \; \bigsqcup U_{ijk} \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; \bigsqcup U_{ij} \; \substack{\longrightarrow\\\longrightarrow} \; \bigsqcup U_i$$

If the sheaf is set valued, then the limit in the definition is equivalent to the usual equalizer. When it is groupoid valued, it is equivalent to the usual stack condition.

In general it states that a morphism to $\mathcal{F}(C)$ is the same as a collection of morphism to $\mathcal{F}(C_i)$, together with homotopies that witness that the maps agree on theses intersections, but these homotopies have to satisfy a compatibility condition on triple intersections (cocycle condition), a compatibility which is expressed by homotopies, and then these homotopies have to satisfy a compatibility condition on quadruple intersections, and so on. This is the meaning of the sheaf condition.

Definition 3.32 (Étale topology)**.** We say that an étale cover is a finite collection of maps $(A \to A_i)$ in dAlg such that each map is étale, and $(\pi_0(A) \to \pi_0(A_i))$ is a faithfully flat cover. We consider the étale topology on dAlg in what follows.

Definition 3.33 (Derived stacks)**.** A *derived stack* $\mathcal{F}$ is a sheaf of anima on $\mathrm{dAlg}^{\mathrm{op}}$ for the étale topology. We denote this ∞-category dSt. We denote by $\mathrm{Spec}(A)$ the sheaf represented by some $A \in \mathrm{dAlg}$.

For any $\mathcal{F} \in \mathrm{dSt}$ we can consider the composition
$$\mathrm{Alg}^{\mathrm{op}} \to \mathrm{dAlg}^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathrm{Ani}\,.$$
This defines a sheaf of anima on Alg for the étale topology, that we denote by $\mathcal{F}_{\mathrm{cl}}$.

Example 3.34. If $\mathcal{F} = \mathrm{Spec}(A)$, then $\mathcal{F}_{\mathrm{cl}} = \mathrm{Spec}(\pi_0(A))$ which is set valued.

Definition 3.35 (Derived scheme and schematic maps)**.** A derived scheme X is a derived stack such that there is an epimorphism
$$\bigsqcup_i U_i \to X$$
where U_i is affine, and the map $U_i \to X$ is an open immersion.[3]

[3]We say that a map of sheaves $\mathcal{F} \to \mathcal{G}$ is an open immersion if for every map $\mathrm{Spec}(A) \to \mathcal{G}$, with pullback $\mathcal{F}_A \to \mathrm{Spec}(A)$, then map $\mathcal{F}_{A,\mathrm{cl}} \to \mathrm{Spec}(A)_{\mathrm{cl}}$ is an open immersion of schemes.

We say that a morphism of derived stacks $\mathcal{F} \to \mathcal{G}$ is *schematic* if for every map $\mathrm{Spec}(A) \to \mathcal{G}$ the pullback $\mathcal{F}_A \to \mathrm{Spec}(A)$ is a derived scheme.

Definition 3.36 (Smooth morphisms)**.** A map of derived stacks $\mathcal{F} \to \mathcal{G}$ is smooth if for every map $\mathrm{Spec}(A) \to \mathcal{G}$ the pullback $\mathcal{F}_A \to \mathrm{Spec}(A)$ is a derived scheme smooth over $\mathrm{Spec}(A)$. This means that $\mathcal{F}_A$ can be covered by affine opens such that restriction correspond to smooth maps of derived algebras.

Definition 3.37 (Derived algebraic stacks)**.** This is an inductive definition. It starts with *derived algebraic spaces* then goes to derived 1-stack, derived 2-stacks, etc.

(0) We say that a derived stack X is a *derived algebraic spaces* or a 0-*Artin stack* if $X \to X \times X$ is a schematic monomorphism and that there exists an étale surjection $U \to X$ where U is a derived scheme. We say that a morphism $\mathcal{F} \to \mathcal{G}$ is *0-Artin* if for every map $\mathrm{Spec}(A) \to \mathcal{G}$ the pullback $\mathcal{F}_A \to \mathrm{Spec}(A)$ is a derived algebraic space.

(1) We say that X is a 1-Artin derived stack if its diagonal is 0-Artin and there exists a smooth surjection $U \to X$ where U is a derived scheme. Same method of definition for a morphism which is 1-Artin.

(n) ($n > 1$) We say that X is a n-Artin stack if its diagonal is $(n-1)$-Artin and that there exists a smooth surjecion $U \to X$ from a derived scheme. Same method of definition for a morphism which is n-Artin.

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