Identification of Joint Interventional Distributions in Recursive Semi-Markovian Causal Models *

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Abstract

This paper is concerned with estimating the effects of actions from causal assumptions, represented concisely as a directed graph, and statistical knowledge, given as a probability distribution. We provide a necessary and sufficient graphical condition for the cases when the causal effect of an arbitrary set of variables on another arbitrary set can be determined uniquely from the available information, as well as an algorithm which computes the effect whenever this condition holds. Furthermore, we use our results to prove completeness of do-calculus [Pearl, 1995], and a version of an identification algorithm in [Tian, 2002] for the same identification problem. Finally, we derive a complete characterization of semi-Markovian models in which all causal effects are identifiable.

Introduction

This paper deals with computing effects of actions in domains specified as $causal\ diagrams$, or graphs with directed and bidirected edges. Vertices in such graphs correspond to variables of interest, directed edges correspond to potential direct causal relationships between variables, and bidirected edges correspond to 'hidden common causes,' or spurious dependencies between variables [Pearl, 1995], [Pearl, 2000]. Aside from causal knowledge encoded by these graphs, we also have statistical knowledge in the form of a joint probability distribution over observable variables, which we will denote by P.

An action on a variable set $\mathbf{X}$ in a causal domain consists of forcing $\mathbf{X}$ to particular values $\mathbf{x}$, regardless of the values $\mathbf{X}$ would have taken prior to the intervention. This action, denoted $do(\mathbf{x})$ in [Pearl, 2000], changes the original joint distribution P over observables into a new $interventional\ distribution$ denoted $P_{\mathbf{x}}$. The marginal distribution $P_{\mathbf{x}}(\mathbf{Y})$ of a variable set $\mathbf{Y}$ obtained from $P_{\mathbf{x}}$ will be our notion of effect of action $do(\mathbf{x})$ on $\mathbf{Y}$.

Our task is to characterize cases when $P_{\mathbf{x}}(\mathbf{Y})$ can be determined uniquely from P, or $identified$ in a given graph G. It is well known that in Markovian models, those causal domains whose graphs do not contain bidirected edges, all effects are identifiable [Pearl, 2000]. If our model contains 'hidden common causes,' that is if the model is semi-Markovian, the situation is less clear.

Consider the causal diagrams in Fig. 1 (a) and (b) which might represent a situation in diagnostic medicine. For instance, nodes W_1, W_2 are afflictions of a pregnant mother and her unborn child, respectively. X is a toxin produced in the mother's body as a result of the illness, which could be artificially lowered by a treatment. Y_1, Y_2 stand for the survival of mother and child. Bidirected arcs represent confounding factors for this situation not explicitly named in the model, but affecting the outcome. We are interested in computing the effect of lowering X on Y_1, Y_2 without actually performing the potentially dangerous treatment. In our framework this corresponds to computing $P_x(Y_1, Y_2)$ from $P(X, W_1, W_2, Y_1, Y_2)$. The subtlety of this problem can be illustrated by noting that in Fig. 1 (a), the effect is identifiable, while in Fig. 1 (b), it is not.

Multiple sufficient conditions for identifiability in the semi-Markovian case are known [Spirtes, Glymour, & Scheines, 1993], [Pearl & Robins, 1995], [Pearl, 1995], [Kuroki & Miyakawa, 1999]. A summary of these results can be found in [Pearl, 2000]. Most work in this area has generally taken advantage of the fact that certain properties of the causal diagram reflect properties of P, and is phrased in the language of graph theory. For example, the back-door criterion [Pearl, 2000], states that if there exists a set $\mathbf{Z}$ of non-descendants of $\mathbf{X}$ that 'blocks' certain paths in the graph from $\mathbf{X}$ to $\mathbf{Y}$, then $P_{\mathbf{x}}(\mathbf{Y}) = \sum_{\mathbf{z}} P(\mathbf{Y}|\mathbf{z}, \mathbf{x})P(\mathbf{z})$.

Results in [Pearl, 1995], [Halpern, 2000] take a different approach, and provide sound rules which are used to manipulate the expression corresponding to the effect algebraically. These rules are then applied until the resulting expression can be computed from P.

Though the axioms in [Halpern, 2000] were shown to be complete, the practical applicability of the result to identifiability is limited, since it does not provide a closed form criterion for the cases when effects are not identifiable, nor a closed form algorithm for expressing effects in terms of P when they are identifiable. Instead, one must rely on finding a good proof strategy and hope the effect expression is reduced to something derivable from P.

Recently, a number of necessity results for identifiability have been proven. One such result [Tian & Pearl, 2002]

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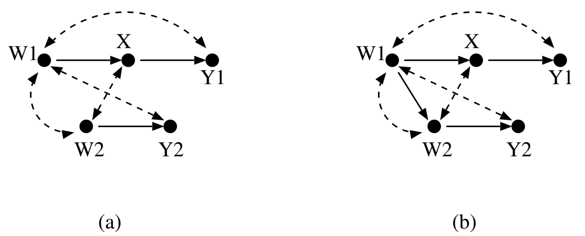

Figure 1: (a) Graph G. (b) Graph G'.
W_1, W_2 - illnesses of pregnant mother and unborn child, X - toxin-lowering treatment, Y_1, Y_2 - survival of the patients

states that P_x is identifiable if and only if there is no path consisting entirely of bidirected arcs from X to a child of X. The authors have also been made aware of a paper currently in review [Huang & Valtorta, 2006] which shows a modified version of an algorithm found in [Tian, 2002] is complete for identifying $P_{\mathbf{x}}(\mathbf{y})$, where $\mathbf{X}$, $\mathbf{Y}$ are sets. One of the contributions of this paper is a simpler proof of the same result, using non-positive distributions. The results in this paper were independently derived.

In this paper, we offer a complete solution to the problem of identifying $P_{\mathbf{x}}(\mathbf{y})$ in semi-Markovian models. Using a graphical structure called a *hedge*, we construct a sound and complete algorithm for identifying $P_{\mathbf{x}}(\mathbf{y})$ from P. The algorithm returns either an expression derivable from P or a hedge which witnesses the non-identifiability of the effect. We also show that steps of our algorithm correspond to sequences of applications of rules of do-calculus [Pearl, 1995], thus proving completeness of do-calculus for the same identification problem. Furthermore, we show a version of Tian's algorithm [Tian, 2002] is also complete and thus equivalent to ours. Finally, we derive a complete characterization of models in which all effects are identifiable.

Notation and Definitions

In this section we reproduce the technical definitions needed for the rest of the paper, and introduce common non-identifying graph structures. We will denote variables by capital letters, and their values by small letters. Similarly, sets of variables will be denoted by bold capital letters, and sets of values by bold small letters. We will use some graph-theoretic abbreviations: $Pa(\mathbf{Y})_G$, $An(\mathbf{Y})_G$, and $De(\mathbf{Y})_G$ will denote the set of (observable) parents, ancestors, and descendants of the node set $\mathbf{Y}$ in G, respectively. The lowercase versions of the above kinship sets will denote corresponding sets of values. We will omit the graph subscript if the graph in question is assumed or obvious. We will denote the set $\{X \in G | De(X)_G = \emptyset\}$ as the *root set* of G. For a given node V_i in a graph G and a topological ordering π of nodes in G, we denote $V_\pi^{(i-1)}$ to be the set of observable nodes preceding V_i in π. A topological ordering π of G is a total order where no node can be greater than its descendant in G.

Having fixed our notation, we can proceed to formalize the notions discussed in the previous section. A probabilistic causal model is a tuple $M = \langle \mathbf{U}, \mathbf{V}, \mathbf{F}, P(\mathbf{U}) \rangle$, where $\mathbf{V}$ is a set of observable variables, $\mathbf{U}$ is a set of unobservable variables distributed according to $P(\mathbf{U})$, and $\mathbf{F}$ is a set of functions. Each variable $V \in \mathbf{V}$ has a corresponding function $f_V \in \mathbf{F}$ that determines the value of V in terms of other variables in $\mathbf{V}$ and $\mathbf{U}$.

The induced graph G of a causal model M contains a node for every element in $\mathbf{V}$, a directed edge between nodes X and Y if f_Y possibly uses the values of X directly to determine the value of Y, and a bidirected edge between nodes X and Y if f_X and f_Y both possibly use the value of some variable in $\mathbf{U}$ to determine their respective values. In this paper we consider *recursive* causal models, those models which induce acyclic graphs.

For the purposes of this paper, we assume all variable domains are finite, and $P(\mathbf{U}) = \prod_i P(U_i)$. The distribution on $\mathbf{V}$ induced by $P(\mathbf{U})$ and $\mathbf{F}$ will be denoted $P(\mathbf{V})$.

Sometimes it is assumed $P(\mathbf{V})$ is a positive distribution. In this paper we do not make this assumption. Thus, we must make sure that for every distribution $P(\mathbf{W}|\mathbf{Z})$ that we consider, $P(\mathbf{Z})$ must be positive. This can be achieved by making sure to sum over events with positive probability only. Furthermore, for any action $do(\mathbf{x})$ that we consider, it must be the case that $P(\mathbf{x}|Pa(\mathbf{X})_G \setminus \mathbf{X}) > 0$ otherwise the distribution $P_{\mathbf{x}}(\mathbf{V})$ is not well defined [Pearl, 2000].

In any causal model there is a relationship between its induced graph G and P, where $P(v_1, ..., v_n, u_1, ..., u_k) = \prod_i P(v_i | pa^*(V_i)_G) \prod_j P(u_j)$, and $Pa^*(.)_G$ also includes unobservable parents [Pearl, 2000]. Whenever this relationship holds, we say that G is an I-map (independence map) of P. The I-map relationship allows us to link independence properties of P to G by using the following well known notion of path separation [Pearl, 1988].

Definition 1 (d-separation) *A path p in G is said to be d-separated by a set $\mathbf{Z}$ if and only if either*

1 *p contains a chain $I \to M \to J$ or fork $I \leftarrow M \to J$, such that $M \in \mathbf{Z}$, or*

2 *p contains an inverted fork $I \to M \leftarrow J$ such that $De(M)_G \cap \mathbf{Z} = \emptyset$.*

Two sets $\mathbf{X}, \mathbf{Y}$ are said to be d-separated given $\mathbf{Z}$ in G if all paths from $\mathbf{X}$ to $\mathbf{Y}$ in G are d-separated by $\mathbf{Z}$. The following well known theorem links d-separation of vertex sets in an I-map G with the independence of corresponding variable sets in P.

Theorem 1 *If sets $\mathbf{X}$ and $\mathbf{Y}$ are d-separated by $\mathbf{Z}$ in G, then $\mathbf{X}$ is independent of $\mathbf{Y}$ given $\mathbf{Z}$ in every P for which G is an I-map. We will abbreviate this statement of d-separation, and corresponding independence by $(\mathbf{X} \perp\!\!\!\perp \mathbf{Y} | \mathbf{Z})_G$, following the notation in [Dawid, 1979].*

A path that is not d-separated is said to be $d-connected$. A path starting from a node X with an arrow pointing to X is called a $back-door\ path$ from X. A path consisting entirely of bidirected arcs is called a $bidirected\ path$.

In the framework of causal models, actions are modifications of functional relationships. Each action $do(\mathbf{x})$ on a causal model M produces a new model $M_{\mathbf{x}} = \langle \mathbf{U}, \mathbf{V}, \mathbf{F}_{\mathbf{x}}, P(\mathbf{U}) \rangle$, where $\mathbf{F}_{\mathbf{x}}$, is obtained by replacing $f_X \in$

$\mathbf{F}$ for every $X \in \mathbf{X}$ with a new function that outputs a constant value x given by $do(\mathbf{x})$.

Since subscripts are used to denote submodels, we will use numeric superscripts to enumerate models (e.g. M^1). For a model M^i, we will often denote its associated probability distributions as P^i rather than P.

We can now define formally the notion of identifiability of interventions from observational distributions.

Definition 2 (Causal Effect Identifiability) *The causal effect of an action $do(\boldsymbol{x})$ on a set of variables $\boldsymbol{Y}$ such that $\boldsymbol{Y} \cap \boldsymbol{X} = \emptyset$ is said to be identifiable from P in G if $P_{\boldsymbol{x}}(\boldsymbol{Y})$ is (uniquely) computable from $P(\boldsymbol{V})$ in any causal model which induces G.*

The following lemma establishes the conventional technique used to prove non-identifiability in a given G.

Lemma 1 *Let $\boldsymbol{X}, \boldsymbol{Y}$ be two sets of variables. Assume there exist two causal models M^1 and M^2 with the same induced graph G such that $P^1(\boldsymbol{V}) = P^2(\boldsymbol{V})$, $P^1(\boldsymbol{x}|Pa(\boldsymbol{X})_G \backslash \boldsymbol{X}) > 0$, and $P^1_{\boldsymbol{x}}(\boldsymbol{Y}) \neq P^2_{\boldsymbol{x}}(\boldsymbol{Y})$. Then $P_{\boldsymbol{x}}(\mathbf{y})$ is not identifiable in G.*

Proof: No function from P to $P_{\mathbf{x}}(\mathbf{y})$ can exist by assumption, let alone a computable function. □

The simplest example of a non-identifiable graph structure is the so called 'bow arc' graph, see Fig. 2 (a). Although it is well known that $P_x(Y)$ is not identifiable in this graph, we give a simple proof here which will serve as a seed of a similar proof for more general graph structures.

Theorem 2 *$P_x(Y)$ is not identifiable in the bow arc graph.*

Proof: We construct two causal models M^1 and M^2 such that $P^1(X, Y) = P^2(X, Y)$, and $P^1_x(Y) \neq P^2_x(Y)$. The two models agree on the following: all 3 variables are boolean, U is a fair coin, and $f_X(u) = u$. Let $\oplus$ denote the exclusive or (XOR) function. Then the value of Y is determined by the function $u \oplus x$ in M^1, while Y is set to 0 in M^2. Then $P^1(Y = 0) = P^2(Y = 0) = 1$, $P^1(X = 0) = P^2(X = 0) = 0.5$. Therefore, $P^1(X, Y) = P^2(X, Y)$, while $P^2_x(Y = 0) = 1 \neq P^1_x(Y = 0) = 0.5$.

Note that while P is non-positive, it is straightforward to modify the proof for the positive case by letting f_Y functions in both models return 1 half the time, and the values outlined above half the time. □

A number of other specific graphs have been shown to contain unidentifiable effects. For instance, in all graphs in Fig. 2, taken from [Pearl, 2000], $P_x(Y)$ is not identifiable.

Throughout the paper, we will make use of the 3 rules of do-calculus [Pearl, 1995].

- Rule 1: $P_{\mathbf{x}}(\mathbf{y}|\mathbf{z}, \mathbf{w}) = P_{\mathbf{x}}(\mathbf{y}|\mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z}|\mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{x}}}}$
- Rule 2: $P_{\mathbf{x},\mathbf{z}}(\mathbf{y}|\mathbf{w}) = P_{\mathbf{x}}(\mathbf{y}|\mathbf{z}, \mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z}|\mathbf{X}, \mathbf{W})_{G_{\overline{\mathbf{x}},\underline{\mathbf{z}}}}$
- Rule 3: $P_{\mathbf{x},\mathbf{z}}(\mathbf{y}|\mathbf{w}) = P_{\mathbf{x}}(\mathbf{y}|\mathbf{w})$ if $(\mathbf{Y} \perp\!\!\!\perp \mathbf{Z}|\mathbf{X}, \mathbf{W})_{G_{\overline{x},\overline{z(\mathbf{w})}}}$

where $Z(\mathbf{W}) = \mathbf{Z} \setminus An(\mathbf{W})_{G_{\overline{\mathbf{X}}}}$.

These rules allow insertion and deletion of interventions and observational evidence into and from distributions, using probabilistic independencies implied by the causal graph due to Theorem 1. Here $G_{\overline{\mathbf{x}}\underline{\mathbf{z}}}$ is taken to mean the graph obtained from G by removing arrows pointing to $\mathbf{X}$ and arrows leaving $\mathbf{Z}$.

C-Trees and Direct Effects

Sets of nodes interconnected by bidirected paths turned out to be an important notion for identifiability and have been studied at length in [Tian, 2002] under the name of C-$components$.

Definition 3 (C-component) *Let G be a semi-Markovian graph such that a subset of its bidirected arcs forms a spanning tree over all vertices in G. Then G is a C-component (confounded component).*

If G is not a C-component, it can be uniquely partitioned into a set $C(G)$ of subgraphs, each a maximal C-component. An important result states that for any set $\mathbf{C}$ which is a C-component, in a causal model M with graph G, $P_{\mathbf{v}\setminus\mathbf{c}}(\mathbf{C})$ is identifiable [Tian, 2002]. The quantity $P_{\mathbf{v}\setminus\mathbf{c}}(\mathbf{C})$ will also be denoted as $Q[\mathbf{C}]$. For the purposes of this paper, C-components are important because a distribution P in a semi-Markovian graph G factorizes such that each product term corresponds to a C-component. For instance, the graphs shown in Fig. 2 (b) and (c), both have 2 C-components: $\{X, Z\}$ and $\{Y\}$. Thus, the corresponding distribution factorizes as $P(x, z, y) = Q[\{x, z\}]Q[\{y\}] = P_y(x, z)P_{x,z}(y)$. It is this factorization which will ultimately allow us to decompose the identification problem into smaller subproblems, and thus construct an identification algorithm.

We now consider a special kind of C-component which generalizes the unidentifiable bow arc graph from the previous section.

Definition 4 (C-tree) *Let G be a semi-Markovian graph such that G is a C-component, all observable nodes have at most one child, and there is a node Y such that $An(Y)_G = G$. Then G is a Y-rooted C-tree (confounded tree).*

The graphs in Fig. 2 (a) (d) (e) (f) and (h) are Y-rooted C-trees.

There is a relationship between C-trees and interventional distributions of the form $P_{pa(Y)}(Y)$. Such distributions are known as $direct\ effects$, and correspond to the influence of a variable X on its child Y along some edge, where the variables $Pa(Y) \setminus \{X\}$ are fixed to some reference values.

Direct effects are of great importance in the legal domain, where one is often concerned with whether a given party was directly responsible for damages, as well as medicine, where elucidating the direct effect of medication, or disease on the human body in a given context is crucial. See [Pearl, 2000],[Pearl, 2001] for a more complete discussion of direct effects. The absence of Y-rooted C-trees in G means the direct effect on Y is identifiable.

Lemma 2 *Let M be a causal model with graph G. Then for any node Y, the direct effect $P_{pa(Y)}(Y)$ is identifiable if there is no subgraph of G which forms a Y-rooted C-tree.*

Proof: From [Tian, 2002], we know that whenever there is no subgraph G' of G, such that all nodes in G' are ancestors of Y, and G' is a C-component, $P_{pa(Y)}(Y)$ is identifiable. This entails the lemma. □

Theorem 2 suggests that C-trees are troublesome structures for the purposes of identification of direct effects. In

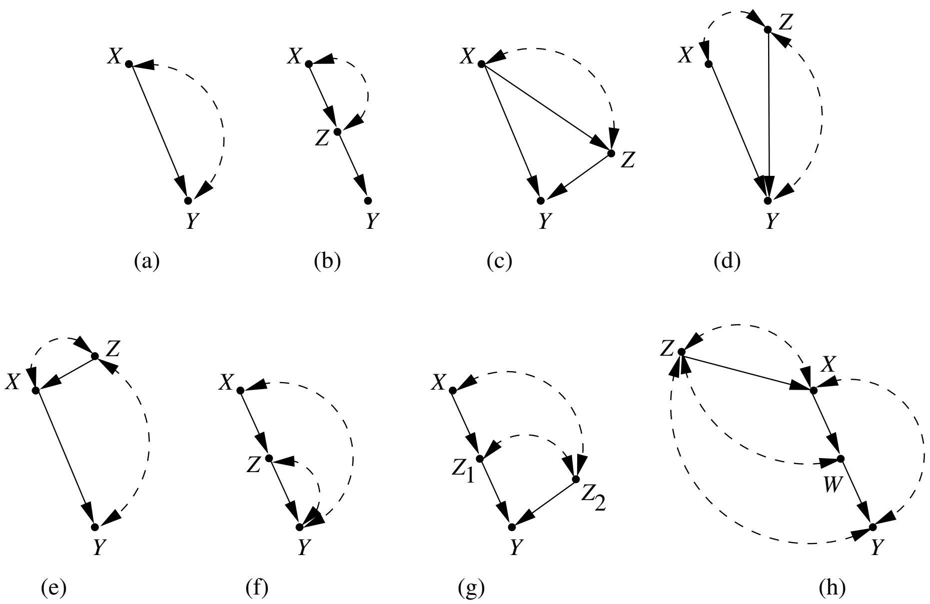

Figure 2: Graphs where $P_x(Y)$ is not identifiable.

fact, our investigation revealed that Y-rooted C-trees are troublesome for any effect on Y, as the following theorem shows.

Theorem 3 *Let G be a Y-rooted C-tree. Then the effect of any set of nodes $\mathbf{X}$ in G on Y is not identifiable if $Y \notin \mathbf{X}$.*

Proof: We generalize the proof for the bow arc graph. We construct two models with binary nodes. In the first model, the value of all observable nodes is set to the bit parity (sum modulo 2) of the parent values. In the second model, the same is true for all nodes except Y, with the latter being set to 0 explicitly. All $\mathbf{U}$ nodes in both models are fair coins. Since G is a tree, and since every $U \in \mathbf{U}$ has exactly two children in G, every $U \in \mathbf{U}$ has exactly two distinct downward paths to Y in G. It's then easy to establish that Y counts the bit parity of every node in $\mathbf{U}$ twice in the first model. But this implies $P^1(Y = 1) = 0$.

Because bidirected arcs form a spanning tree over observable nodes in G, for any set of nodes $\mathbf{X}$ such that $Y \notin \mathbf{X}$, there exists $U \in \mathbf{U}$ with one child in $An(\mathbf{X})_G$ and one child in $G \setminus An(\mathbf{X})_G$. Thus $P^1_{\mathbf{x}}(Y = 1) > 0$, but $P^2_{\mathbf{x}}(Y = 1) = 0$. It is straightforward to generalize this proof for the positive $P(\mathbf{V})$ in the same way as in Theorem 2. $\square$

While this theorem closes the identification problem for direct effects, the problem of identifying general effects on a single variable Y is more subtle, as the following corollary shows.

Corollary 1 *Let G be a semi-Markovian graph, let $\mathbf{X}$ and $\mathbf{Y}$ be disjoint sets of variables. If there exists $W \in An(\mathbf{Y})_{G_{\underline{\mathbf{x}}}}$ such that there exists a W-rooted C-tree which contains any variables in $\mathbf{X}$, then $P_{\mathbf{x}}(\mathbf{Y})$ is not identifiable.*

Proof: Fix a W-rooted C-tree T, and a path p from W to $Y \in \mathbf{Y}$, where $W \in An(Y)_{G_{\underline{\mathbf{x}}}}$. Consider the graph $p \cup T$. Note that in this graph $P_{\mathbf{x}}(Y) = \sum_w P_{\mathbf{x}}(w)P(Y|w)$. It is now easy to construct $P(Y|W)$ in such a way that the mapping from $P_{\mathbf{x}}(W)$ to $P_{\mathbf{x}}(Y)$ is one to one, while making sure P is positive. $\square$

This corollary implies that the effect of $do(\mathbf{x})$ on a given singleton Y can be non-identifiable even if Y is nowhere near a C-tree, as long as the effect of $do(\mathbf{x})$ on a set of ancestors of Y is non-identifiable. Therefore identifying effects on a single variable is not really any easier than the general problem of identifying effects on multiple variables. We consider this general problem in the next section.

Finally, we note that the last two results relied on existence of a C-tree without giving an explicit algorithm for constructing one. In the remainder of the paper we will give an algorithm which, among other things, will construct the necessary C-tree, if it exists.

C-Forests, Hedges, and Non-Identifiability

The previous section established a powerful necessary condition for the identification of effects on a single variable. It is the natural next step to ask whether a similar condition exists for effects on multiple variables. We start by considering a multi-root generalization of a C-tree.

Definition 5 (C-forest) *Let G be a semi-Markovian graph, where $\mathbf{Y}$ is the root set. Then G is a $\mathbf{Y}$-rooted $C\text{-}forest$ (confounded forest) if G is a C-component, and all observable nodes have at most one child.*

We will show that just as there is a close relationship between C-trees and direct effects, there is a close connection between C-forests and general effects of the form $P_{\mathbf{x}}(\mathbf{Y})$, where $\mathbf{X}$ and $\mathbf{Y}$ are sets of variables. To explicate this connection, we introduce a special structure formed by a pair of C-forests that will feature prominently in the remainder of the paper.

Definition 6 (hedge) *Let $\mathbf{X}, \mathbf{Y}$ be disjoint sets of variables in G. Let F, F' be $\mathbf{R}$-rooted C-forests such that $F \cap \mathbf{X} \neq \emptyset$, $F' \cap \mathbf{X} = \emptyset$, $F' \subseteq F$, and $\mathbf{R} \subset An(\mathbf{Y})_{G_{\overline{\mathbf{x}}}}$. Then F and F' form a hedge for $P_{\mathbf{x}}(\mathbf{y})$ in G.*

The mental picture for a hedge is as follows. We start with a C-forest F'. Then, F' 'grows' new branches, while retaining the same root set, and becomes F. Finally, we 'trim the hedge,' by performing the action $do(\mathbf{x})$ which has the effect of removing some incoming arrows in $F \setminus F'$, the 'newly grown' portion of the hedge. It's easy to check that every graph in Fig. 2 contains a pair of C-forests that form a hedge for $P_x(Y)$.

The graph in Fig. 1 (a) does not contain C-forests forming a hedge for $P_x(Y_1, Y_2)$, while the graph in Fig. 1 (b) does: if e is the edge between W_1 and X, then $F = G \setminus \{e\}$, and $F' = F \setminus \{X\}$. Note that for the special case of C-trees, F is the C-tree itself, and F' is the singleton root Y. This last observation suggests the next result as a generalization of Theorem 3.

Theorem 4 *Assume there exist $\mathbf{R}$-rooted C-forests F, F' that form a hedge for $P_{\mathbf{x}}(\mathbf{y})$ in G. Then $P_{\mathbf{x}}(\mathbf{y})$ is not identifiable in G.*

Proof: We first show $P_{\mathbf{x}}(\mathbf{r})$ is not identifiable in F. As before, we construct two models with binary nodes. In M^1 every variable in F is equal to the bit parity of its parents. In M^2 the same is true, except all nodes in F' disregard the parent values in $F \setminus F'$. All $\mathbf{U}$ are fair coins in both models.

As was the case with C-trees, for any C-forest F, every $U \in \mathbf{U} \cap F$ has exactly two downward paths to $\mathbf{R}$. It is now easy to establish that in M^1, $\mathbf{R}$ counts the bit parity of every node in $\mathbf{U}^1$ twice, while in M^2, $\mathbf{R}$ counts the bit parity of every node in $\mathbf{U}^2 \cap F'$ twice. Thus, in both models with no interventions, the bit parity of $\mathbf{R}$ is even.

Next, fix two distinct instantiations of $\mathbf{U}$ that differ by values of $\mathbf{U}^*$. Consider the topmost node $W \in F$ with an odd number of parents in $\mathbf{U}^*$ (which exists because bidirected edges in F form a spanning tree). Then flipping the values of $\mathbf{U}^*$ once will flip the value W once. Thus the function from $\mathbf{U}$ to $\mathbf{V}$ induced by a C-forest F in M^1 and M^2 is one to one.

The above results, coupled with the fact that in a C-forest, $|\mathbf{U}| + 1 = |\mathbf{V}|$ implies that any assignment where $\sum \mathbf{r} \pmod 2 = 0$ is equally likely, and all other node assignments are impossible in both F and F'. Since the two models agree on all functions and distributions in $F \setminus F'$, $\sum_{f'} P^1 = \sum_{f'} P^2$. It follows that the observational distributions are the same in both models. Furthermore, $\sum_{\mathbf{r}} P^1(\mathbf{V})$ is a positive distribution, thus $P^1(\mathbf{x}|Pa(\mathbf{X})_G \setminus \mathbf{X}) > 0$ for any $\mathbf{x}$.

As before, we can find $U \in \mathbf{U}$ with one child in $An(\mathbf{X})_F$, and one child in $F \setminus An(\mathbf{X})_F$, which implies $P_{\mathbf{x}}(1 = \sum \mathbf{r} \pmod 2)) > 0$ in M^1, but not M^2. Since $P_{\mathbf{x}}(\mathbf{r})$ is not identifiable in G, and $\mathbf{R} \subset An(\mathbf{Y})_{G_{\overline{\mathbf{x}}}}$, we can construct $P(\mathbf{Y}|\mathbf{R})$ to be a one to one mapping between $P_{\mathbf{x}}(\mathbf{r})$ and $P_{\mathbf{x}}(\mathbf{y})$, as we did in Corollary 1.

For instance, let $\mathbf{Y}'$ be the minimal subset of $\mathbf{Y}$ such that $\mathbf{R} \subseteq An(\mathbf{Y}')_{G_{\overline{\mathbf{x}}}}$. Then let all nodes in $G' = An(\mathbf{Y}')_{G_{\overline{\mathbf{x}}}} \setminus An(\mathbf{R})$ be equal to the bit parity of the parents. Without loss of generality, assume every node in G' has at most one child. Then every $R \in \mathbf{R}$ has a unique downward path to $\mathbf{Y}'$, which means the bit parities of $\mathbf{R}$ and $\mathbf{Y}'$ are the same. This implies the result. $\square$

Hedges generalize not only the C-tree condition, but also the complete condition for identification of P_x from P in [Tian & Pearl, 2002] which states that if Y is a child of X and there a bidirected path from X to Y then (and only then) P_x is not identifiable. Let G consist of X, Y and the nodes $W_1, ..., W_k$ on the bidirected path from X to Y. It is not difficult to check that G and $G \setminus \{X\}$ form a hedge for $P_x(Y, W_1, ..., W_k)$.

Since hedges generalize two complete conditions for special cases of the identification problem, it might be reasonable to suppose that a complete characterization of identifiability might involve hedges in some way. To prove this supposition, we would need to construct an algorithm which identifies any effect lacking a hedge. This algorithm is the subject of the next section.

function **ID**($\mathbf{y}$, $\mathbf{x}$, P, G)
INPUT: $\mathbf{x}$,$\mathbf{y}$ value assignments, P a probability distribution, G a causal diagram (an I-map of P).
OUTPUT: Expression for $P_{\mathbf{x}}(\mathbf{y})$ in terms of P or **FAIL**(F,F').

1 if $\mathbf{x} = \emptyset$, return $\sum_{\mathbf{v} \setminus \mathbf{y}} P(\mathbf{v})$.

2 if $\mathbf{V} \neq An(\mathbf{Y})_G$,
return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap An(\mathbf{Y})_G, P(An(\mathbf{Y})), An(\mathbf{Y})_G)$.

3 let $\mathbf{W} = (\mathbf{V} \setminus \mathbf{X}) \setminus An(\mathbf{Y})_{G_{\overline{\mathbf{x}}}}$.
if $\mathbf{W} \neq \emptyset$, return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cup \mathbf{w}, P, G)$.

4 if $C(G \setminus \mathbf{X}) = \{S_1, ..., S_k\}$,
return $\sum_{\mathbf{v} \setminus (\mathbf{y} \cup \mathbf{x})} \prod_i \mathbf{ID}(s_i, \mathbf{v} \setminus s_i, P, G)$.

if $C(G \setminus \mathbf{X}) = \{S\}$,

5 if $C(G) = \{G\}$, throw $\mathbf{FAIL}(G, S)$.

6 if $S \in C(G)$, return $\sum_{s \setminus \mathbf{y}} \prod_{V_i \in S} P(v_i | v_\pi^{(i-1)})$.

7 if $(\exists S') S \subset S' \in C(G)$, return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap S',$
$\prod_{V_i \in S'} P(V_i | V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S'), S')$.

Figure 3: A complete identification algorithm. **FAIL** propagates through recursive calls like an exception, and returns F, F' which form the hedge which witnesses non-identifiability of $P_{\mathbf{x}}(\mathbf{y})$. π is some topological ordering of nodes in G.

A Complete Identification Algorithm

Given the characterization of unidentifiable effects in the previous section, we can attempt to solve the identification problem in all other cases, and hope for completeness. To do this we construct an algorithm that systematically takes advantage of the properties of C-components to decompose the identification problem into smaller subproblems until either the entire expression is identified, or we run into the problematic hedge structure. This algorithm, called **ID**, is shown in Fig. 3.

Before showing the soundness and completeness proper-

(a) (b)

Figure 4: Subgraphs of G used for identifying $P_x(y_1, y_2)$.

ties of **ID**, we give the following example of the operation of the algorithm. Consider the graph G in Fig. 1 (a), where we want to identify $P_x(y_1, y_2)$ from $P(\mathbf{V})$.

We know that $G = An(\{Y_1, Y_2\})_G, C(G \setminus \{X\}) = \{G \setminus \{X\}\}$, and $\mathbf{W} = \{W_1\}$. Thus, we invoke line 3 and attempt to identify $P_{x,w}(y_1, y_2)$. Now $C(G \setminus \{X, W\}) = \{\{Y_1\}, \{W_2\}, \{Y_2\}\}$, so we invoke line 4. Thus the original problem reduces to identifying $\sum_{w_2} P_{x,w_1,w_2,y_2}(y_1) P_{w_1,x,y_1,y_2}(w_2) P_{x,w_1,w_2,y_1}(y_2)$.

Solving for the second expression, we trigger line 2, noting that we can ignore nodes which are not ancestors of W_2. Thus, $P_{w_1,x,y_1,y_2}(w_2) = P(w_2)$. Similarly, we ignore non-ancestors of Y_2 in the third expression to obtain $P_{x,w_1,w_2,y_1}(y_2) = P_{w_2}(y_2)$. We conclude at line 6, to obtain $P_{w_2}(y_2) = P(y_2|w_2)$.

Solving for the first expression, we first trigger line 2 also, obtaining $P_{x,w_1,w_2,y_2}(y_1) = P_{x,w_1}(y_1)$. The corresponding G is shown in Fig. 4 (a). Next, we trigger line 7, reducing the problem to computing $P_{w_1}(y_1)$ from $P(Y_1|X, W_1)P(W_1)$. The corresponding G is shown in Fig. 4 (b). Finally, we trigger line 2, obtaining $P_{w_1}(y_1) = \sum_{w_1} P(y_1|x, w_1)P(w_1)$. Putting everything together, we obtain: $P_x(y_1, y_2) = \sum_{w_2} P(y_2|w_2)P(w_2) \sum_{w_1} P(y_1|x, w_1)P(w_1)$.

As we showed before, the very same effect $P_x(y_1, y_2)$ in a very similar graph G' shown in Fig. 1 (b) is not identifiable due to the presence of C-forests forming a hedge.

We now prove that **ID** terminates and is sound.

Lemma 3 *ID always terminates.*

Proof: At any call on line 7, $(\exists X \in \mathbf{X}) X \notin S'$, else the failure condition on line 5 would have been triggered. Thus any recursive call to **ID** reduces the size of either the set $\mathbf{X}$ or the set $\mathbf{V} \setminus \mathbf{X}$. Since both of these sets are finite, and their union forms $\mathbf{V}$, **ID** must terminate. □

To show soundness, we need a number of utility lemmas justifying various lines of the algorithm. Though some of these results are already known, we will reprove them here using do-calculus to entail the results in the next section. When we refer to do-calculus we will just refer to rule numbers (e.g. 'by rule 2'). Throughout the proofs we will fix some topological ordering π of observable nodes in G. First, we must show that an effect of the form $P_{\mathbf{x}}(\mathbf{y})$ decomposes according to the set of C-components of the graph $G \setminus \mathbf{X}$.

Lemma 4 *Let M be a causal model with graph G. Let $\boldsymbol{y}, \boldsymbol{x}$ be value assignments. Let $C(G \setminus \boldsymbol{X}) = \{S_1, ..., S_k\}$. Then $P_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{\boldsymbol{v} \setminus (\boldsymbol{y} \cup \boldsymbol{x})} \prod_i P_{\boldsymbol{v} \setminus s_i}(s_i)$.*

Proof: Assume $\mathbf{X} = \emptyset$, and let $A_i = An(S_i)_G \setminus S_i$. Then

$$\prod_i P_{\mathbf{v} \setminus s_i}(s_i) = \prod_i P_{a_i}(s_i) = \prod_i \prod_{V_j \in S_i} P_{a_i}(v_j | v_\pi^{(j-1)} \setminus a_i)$$

$$= \prod_i \prod_{V_j \in S_i} P(v_j | v_\pi^{(j-1)}) = \prod_i P(v_i | v_\pi^{(i-1)}) = P(\mathbf{v})$$

The first identity is by rule 3, the second is by chain rule of probability. To prove the third identity, we consider two cases. If $A \in A_i \setminus V_\pi^{(j-1)}$, we can eliminate the intervention on A from the expression $P_{a_i}(v_j | v_\pi^{(j-1)})$ by rule 3, since $(V_j \perp\!\!\!\perp A | V_\pi^{(j-1)})_{G_{\overline{a_i}}}$.

If $A \in A_i \cap V_\pi^{(j-1)}$, consider any back-door path from A_i to V_j. Any such path with a node not in $V_\pi^{(j-1)}$ will be d-separated because, due to recursiveness, it must contain a blocked collider. Further, this path must contain bidirected arcs only, since all nodes on this path are conditioned or fixed. Because $A_i \cap S_i = \emptyset$, all such paths are d-separated. The identity now follows from rule 2.

The last two identities are just grouping of terms, and application of chain rule. The same factorization applies to the submodel $M_{\mathbf{x}}$ which induces the graph $G \setminus \mathbf{X}$, which implies the result. □

The next lemma shows that to identify the effect on $\mathbf{Y}$, it is sufficient to restrict our attention to the ancestor set of $\mathbf{Y}$, thereby ensuring the soundness of line 2.

Lemma 5 *Let $\boldsymbol{X}' = \boldsymbol{X} \cap An(\boldsymbol{Y})_G$. Then $P_{\boldsymbol{x}}(\boldsymbol{y})$ obtained from P in G is equal to $P'_{\boldsymbol{x}'}(\boldsymbol{y})$ obtained from $P' = P(An(\boldsymbol{Y}))$ in $An(\boldsymbol{Y})_G$.*

Proof: Let $\mathbf{W} = \mathbf{V} \setminus An(\mathbf{Y})_G$. Then the submodel $M_{\mathbf{w}}$ induces the graph $G \setminus \mathbf{W} = An(\mathbf{Y})_G$, and its distribution is $P' = P_{\mathbf{w}}(An(\mathbf{Y})) = P(An(\mathbf{Y}))$ by rule 3. Now $P_{\mathbf{x}}(\mathbf{y}) = P_{\mathbf{x}'}(\mathbf{y}) = P_{\mathbf{x}',\mathbf{w}}(\mathbf{y}) = P'_{\mathbf{x}'}(\mathbf{y})$ by rule 3. □

Next, we use do-calculus to show that introducing additional interventions in line 3 is sound as well.

Lemma 6 *Let $\boldsymbol{W} = (\boldsymbol{V} \setminus \boldsymbol{X}) \setminus An(\boldsymbol{Y})_{G_{\overline{\boldsymbol{x}}}}$. Then $P_{\boldsymbol{x}}(\boldsymbol{y}) = P_{\boldsymbol{x},\boldsymbol{w}}(\boldsymbol{y})$, where $\boldsymbol{w}$ are arbitrary values of $\boldsymbol{W}$.*

Proof: Note that by assumption, $\mathbf{Y} \perp\!\!\!\perp \mathbf{W} | \mathbf{X}$ in $G_{\overline{\mathbf{x}},\overline{\mathbf{w}}}$. The conclusion follows by rule 3. □

Next, we must ensure the validity of the positive base case on line 6.

Lemma 7 *When the conditions of line 6 are satisfied, $P_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{s \setminus \boldsymbol{y}} \prod_{V_i \in S} P(v_i | v_\pi^{(i-1)})$.*

Proof: If line 6 preconditions are met, then G local to that recursive call is partitioned into S and $\mathbf{X}$, and there are no bidirected arcs from $\mathbf{X}$ to S. The conclusion now follows from the proof of Lemma 4. □

Finally, we show the soundness of the last recursive call.

Lemma 8 *Whenever the conditions of the last recursive call of **ID** are satisfied, $P_{\boldsymbol{x}}$ obtained from P in the graph G is equal to $P'_{\boldsymbol{x} \cap S'}$ obtained from $P' = \prod_{V_i \in S'} P(V_i | V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S')$ in the graph S'.*

Proof: It is easy to see that when the last recursive call executes, $\mathbf{X}$ and S partition G, and $\mathbf{X} \subset An(S)_G$. This implies that the submodel $M_{\mathbf{x} \setminus S'}$ induces the graph $G \setminus (\mathbf{X} \setminus S') = S'$. The distribution $P_{\mathbf{x} \setminus S'}$ of $M_{\mathbf{x} \setminus S'}$ is equal to P' by the proof of Lemma 4. It now follows that $P_{\mathbf{x}} = P_{\mathbf{x} \cap S', \mathbf{x} \setminus S'} = P'_{\mathbf{x} \cap S'}$. □

We can now show the soundness of **ID**.

Theorem 5 (soundness) *Whenever **ID** returns an expression for $P_{\mathbf{x}}(\mathbf{y})$, it is correct.*

Proof: If $\mathbf{x} = \emptyset$, the desired effect can be obtained from P by marginalization, thus this base case is clearly correct. The soundness of all other lines except the failing line 5 has already been established. $\square$

Finally, we can characterize the relationship between hedges and the inability of **ID** to identify an effect.

Theorem 6 *Assume **ID** fails to identify $P_{\mathbf{x}}(\mathbf{y})$ (executes line 5). Then there exist $\mathbf{X}' \subseteq \mathbf{X}$, $\mathbf{Y}' \subseteq \mathbf{Y}$ such that the graph pair G, S returned by the fail condition of **ID** contain as edge subgraphs C-forests F, F' that form a hedge for $P_{\mathbf{x}'}(\mathbf{y}')$.*

Proof: Consider line 5, and G and $\mathbf{y}$ local to that recursive call. Let $\mathbf{R}$ be the root set of G. Since G is a single C-component, it is possible to remove a set of directed arrows from G while preserving the root set $\mathbf{R}$ such that the resulting graph F is an $\mathbf{R}$-rooted C-forest.

Moreover, since $F' = F \cap S$ is closed under descendants, and since only single directed arrows were removed from S to obtain F', F' is also a C-forest. $F' \cap \mathbf{X} = \emptyset$, and $F \cap \mathbf{X} \neq \emptyset$ by construction. $\mathbf{R} \subseteq An(\mathbf{Y})_{G_{\overline{\mathbf{x}}}}$ by lines 2 and 3 of the algorithm. It's also clear that $\mathbf{y}, \mathbf{x}$ local to the recursive call in question are subsets of the original input. $\square$

Corollary 2 (completeness) ***ID** is complete.*

Proof: By the previous theorem, if **ID** fails, then $P_{\mathbf{x}'}(\mathbf{y}')$ is not identifiable in a subgraph $H = An(\mathbf{Y})_G \cap De(F)_G$ of G. Moreover, $\mathbf{X} \cap H = \mathbf{X}'$, by construction of H. As such, it is easy to extend the counterexamples in Theorem 6 with variables independent of H, with the resulting models inducing G, and witnessing the unidentifiability of $P_{\mathbf{x}}(\mathbf{y})$. $\square$

The following is now immediate.

Corollary 3 (hedge criterion) *$P_{\mathbf{x}}(\mathbf{y})$ is identifiable from P in G if and only if there does not exist a hedge for $P_{\mathbf{x}'}(\mathbf{y}')$ in G, for any $\mathbf{X}' \subseteq \mathbf{X}$ and $\mathbf{Y}' \subseteq \mathbf{Y}$.*

So far we have not only established completeness, but also fully characterized graphically all situations where distributions of the form $P_{\mathbf{x}}(\mathbf{y})$ are identifiable. We can use these results to derive a characterization of identifiable models, that is, causal models where all effects are identifiable.

Corollary 4 (model identification) *Let G be a semi-Markovian causal diagram. Then all causal effects are identifiable in G if and only if G does not contain a node X connected to its child Y by a bidirected path.*

Proof: If F, F' are C-forests which form a hedge for some effect, there must be a variable $X \in F$, which is an ancestor of another variable $Y \in F$. Thus, if no X exists with a child Y in the same C-component, then no hedge can exist in G, and **ID** never reaches the fail condition. Thus all effects are identifiable. Otherwise, P_x is not identifiable by [Tian & Pearl, 2002]. $\square$

The complete algorithm presented in this section can be viewed as a marriage of graphical and algebraic approaches to identifiability. **ID** manipulates the first three arguments algebraically, in a manner similar to do-calculus – not a coincidental similarity as the following section will show. At the same time, if we ignore the third argument, **ID** can be viewed as a purely graphical algorithm which, given an effect suspected of being non-identifiable, constructs the problematic hedge structure witnessing this property.

function **c-identify**(C, T, Q[T])
INPUT: $C \subseteq T$, both are C-components, $Q[T]$ a probability distribution
OUTPUT: Expression for $Q[C]$ in terms of $Q[T]$ or **FAIL**

Let $A = An(C)_T$.

1 If $A = C$, return $\sum_{T \setminus C} P$

2 If $A = T$, return **FAIL**

3 If $C \subset A \subset T$, there exists a C-component T' such that $C \subset T' \subset A$.
return **c-identify**$(C, T', Q[T'])$
($Q[T']$ is known to be computable from $\sum_{T \setminus A} Q[T]$)

function **identify**(**y**, **x**, P, G)
INPUT: **x**,**y** value assignments, P a probability distribution, G a causal diagram.
OUTPUT: Expression for $P_{\mathbf{x}}(\mathbf{y})$ in terms of P or **FAIL**.

1 Let $D = An(\mathbf{Y})_{G_{\underline{\mathbf{x}}}}$.

2 Assume $C(D) = \{D_1, ..., D_k\}$, $C(G) = \{C_1, ..., C_m\}$.

3 return $\sum_{D \setminus S} \prod_i$ **c-identify**$(D_i, C_{D_i}, Q[C_{D_i}])$,
where $(\forall i) D_i \subseteq C_{D_i}$

Figure 5: An identification algorithm modified from [Tian, 2002]

Connections to Existing Identification Algorithms

In the previous section we established that **ID** is a sound and complete algorithm for all effects of the form $P_{\mathbf{x}}(\mathbf{y})$. It is natural to ask whether this result can be used to show completeness of earlier algorithms conjectured to be complete.

First we consider do-calculus, which can be viewed as a declarative identification algorithm, with its completeness remaining an open question. We show that the steps of the algorithm **ID** correspond to sequences of standard probabilistic manipulations, and applications of rules of do-calculus, which entails completeness of do-calculus for identifying unconditional effects.

Theorem 7 *The rules of do-calculus, together with standard probability manipulations are complete for determining identifiability of all effects of the form $P_{\mathbf{x}}(\mathbf{y})$.*

Proof: We must show that all operations corresponding to lines of **ID** correspond to sequences of standard probability manipulations and applications of the rules of do-calculus. These manipulations are done either on the effect expression $P_{\mathbf{x}}(\mathbf{y})$, or the observational distribution P, until the algorithm either fails, or the two expressions 'meet' by producing a single chain of manipulations.

Line 1 is just standard probability operations. Line 5 is a fail condition. The proof that lines 2, 3, 4, 6, and 7 correspond to sequences of do-calculus manipulations follows from Lemmas 5, 6, 4, 7, and 8 respectively. □

Next, we consider a version of an identification algorithm due to Tian, shown in Fig. 5. The soundness of this algorithm has already been addressed elsewhere, so we turn to the matter of completeness.

Theorem 8 *Assume* ***identify*** *fails to identify* $P_{\mathbf{x}}(\mathbf{y})$*. Then there exist C-forests* F, F' *forming a hedge for* $P_{\mathbf{x}'}(\mathbf{y}')$*, where* $\mathbf{X}' \subseteq \mathbf{X}$*,* $\mathbf{Y}' \subseteq \mathbf{Y}$*.*

Proof: Assume **c-identify** fails. Consider C-components C, T local to the failed recursive call. Let $\mathbf{R}$ be the root set of C. Because $T = An(C)_T$, $\mathbf{R}$ is also a root set of T. As in the proof of Theorem 6, we can remove a set of directed arrows from C and T while preserving $\mathbf{R}$ as the root set such that the resulting edge subgraphs are C-forests. By line 1 of **identify**, $C, T \subseteq An(\mathbf{Y})_{G_{\overline{\mathbf{x}}}}$.

Finally, because **c-identify** will always succeed if $D_i = C_{D_i}$, it must be the case that $D_i \subset C_{D_i}$. But this implies $\mathbf{X} \cap C = \emptyset$, $\mathbf{X} \cap T \neq \emptyset$. Thus, edge subgraphs of C, and T are C-forests forming a hedge for $P_{\mathbf{x}'}(\mathbf{y}')$, where $\mathbf{X}' \subseteq \mathbf{X}$, $\mathbf{Y}' \subseteq \mathbf{Y}$. □

Corollary 5 ***identify*** *is complete.*

Proof: This is implied by Theorem 8, and Corollary 3. □

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Conclusions

We have provided a complete characterization of cases when joint interventional distributions are identifiable in semi-Markovian models. Using a graphical structure called the hedge, we were able to construct a sound and complete algorithm for this identification problem, prove completeness of two existing algorithms, and derive a complete description of semi-Markovian models in which all effects are identifiable.

The natural open question stemming from this work is whether the algorithm presented can lead to the identification of $conditional\ interventional\ distributions$ of the form $P_{\mathbf{x}}(\mathbf{y}|\mathbf{z})$. Another remaining question is whether the results in this paper could prove helpful for identifying general counterfactual expressions such as those invoked in natural direct and indirect effects [Pearl, 2001], and path-specific effects [Avin, Shpitser, & Pearl, 2005].

References

[1] Avin, C.; Shpitser, I.; and Pearl, J. 2005. Identifiability of path-specific effects. In *International Joint Conference on Artificial Intelligence*, volume 19, 357–363.

[2] Dawid, A. P. 1979. Conditional independence in statistical theory. *Journal of the Royal Statistical Society* 41:1–31.

[3] Halpern, J. 2000. Axiomatizing causal reasoning. *Journal of A.I. Research* 317–337.

[4] Huang, Y., and Valtorta, M. 2006. On the completeness of an identifiability algorithm for semi-markovian models. Technical Report TR-2006-01, Computer Science and Engineering Department, University of South Carolina.

[5] Kuroki, M., and Miyakawa, M. 1999. Identifiability criteria for causal effects of joint interventions. *Journal of Japan Statistical Society* 29:105–117.

[6] Pearl, J., and Robins, J. M. 1995. Probabilistic evaluation of sequential plans from causal models with hidden variables. In *Uncertainty in Artificial Intelligence*, volume 11, 444–453.

[7] Pearl, J. 1988. *Probabilistic Reasoning in Intelligent Systems*. Morgan and Kaufmann, San Mateo.

[8] Pearl, J. 1995. Causal diagrams for empirical research. *Biometrika* 82(4):669–709.

[9] Pearl, J. 2000. *Causality: Models, Reasoning, and Inference*. Cambridge University Press.

[10] Pearl, J. 2001. Direct and indirect effects. In *Proceedings of UAI-01*, 411–420.

[11] Spirtes, P.; Glymour, C.; and Scheines, R. 1993. *Causation, Prediction, and Search*. Springer Verlag, New York.

[12] Tian, J., and Pearl, J. 2002. A general identification condition for causal effects. In *Eighteenth National Conference on Artificial Intelligence*, 567–573.

[13] Tian, J. 2002. *Studies in Causal Reasoning and Learning*. Ph.D. Dissertation, Department of Computer Science, University of California, Los Angeles.