Final Assignment - Contemporary Applied Mathematics

Description

For this assignment you will revisit and apply the topics from all units in the course. The main goals are to explore novel problems and connect prior techniques with the most recent topics.

LO Grading

This assignment has a significant weight ($8x$), so make sure to allot your time accordingly. Each exercise will be graded on the tagged LO(s).

HC Grading

Before you begin the problem set, review each prompt and determine a set of specific HCs you anticipate applying to each problem. After solving the problem, evaluate your application of the HCs in your set. Include an appendix in which you highlight 2 or 3 of your strongest HC applications. For each highlighted use, give the HC hashtag and a 1 to 2 sentence explanation that references the relevant problem and demonstrates the strength of your application. The total length of this appendix should be no more than 200 words. In addition to the HCs you tag, you will receive a single holistic score on **#quantProfessionalism**, and another on **#quantCommunication**.

Collaboration

We encourage you to collaborate and to discuss the problem sets with each other. However, it is key that you are able to apply the skills and concepts you learn in class and explain your approach on your own. For this reason, you should not be sharing **any** written work with any other groups outside of class and are required to write up all solutions independently. If you use any other external resources, make sure to cite them.

Submission

Consider these options for presenting your work neatly. You may choose to type your solutions using LaTeX, Google Documents or other software. You may also write on a graphic tablet and submit a PDF of your work. Hand-written solutions on paper are also acceptable, so long as they are scanned/photographed clearly and are fully legible. There exist scanning apps that will allow you to take photos of pages and create a PDF out of them (Office Lens, for example).

- Low scores will be assigned for illegible work and/or images of poor quality.
- All graphs should be included in your PDF file, with a relevant caption.
- The common conventions for writing an essay will apply. For example, if you use external resources, make sure to cite them using proper formatting.
- You should include all supporting material you used (e.g. code, computations) in an appendix or as a separate file.

Problems

1. (**#numimplementation**) *Fractals* - Newton's method may be extended to complex numbers, allowing us to compute complex roots of real functions. One way to do so is by allowing the initial guess, x_0, to be complex.

   (a) We know that the roots of $f(x) = x^2 + 1$ are equal to $x_1 = i$, $x_2 = -i$. Use your implementation of Newton's method to compute the roots of f with complex initial values $x_0 = a + bi$.

   (b) For which initial values x_0 does Newton's method converge to x_1? What about x_2? Are there initial values for which Newton's method fails to converge for this function?

   (c) Now, consider the function $g(x) = x^3 - 1$. What are all the complex roots of this function? Denote them x_1, x_2 and x_3.

   (d) Use Newton's methods to find all the roots of $g(x) = x^3 - 1$. For which initial values $x_0 = a + bi$ does Newton's method converge to x_1, x_2, and x_3? In particular, perform a thorough sweep of (a, b) pairs, with at least 100 values for a, 100 values for b. You may restrict your search to values in the square region $-1 \leq a \leq 1, -1 \leq b \leq 1$.

   (e) A nice way of visualizing your result for part (d) is to record which initial values lead to which root by using an RGB-color data structure. You may create three matrices initially consisting of all zeroes: R, G, and B. Each of these matrices will have size $n \times m$, where n, m are the number of initial a and b values considered.

   If a particular initial value $x_0 = a_k + ib_j$ leads to root x_1 (up to the tolerance of your Newton's method), you could denote that point in red by setting $R(j, k) = 1$. If the initial value leads to x_2, you can denote that point in green, with $G(j, k) = 1$, and finally if the initial value leads to x_3, you can denote that point in blue, with $B(j, k) = 1$.

   Finally, you can create a matrix combining all the information:

   ```
   A(:,:,1) = R; A(:,:,2) = G; A(:,:,3) = B;
   ```

   and show it using:

   `imshow(A)`.

   (f) (Optional) For more interesting patterns, find the roots of $h(x) = x^5 - 1$ analytically and using Newton's method. Repeat part (e) for this function and various initial conditions x_0 (you will have to come up with more color combinations to account for all roots).

2. (**#linalg**, **#diffeq**) *Strange Attractors* - While studying meteorological systems and convection patterns Ed Lorenz (1963) derived a three-dimensional model to describe convection rolls in the atmosphere, namely:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = xy - bz$$

where $\sigma, r, b > 0$ are physical parameters (σ is the Prandtl number, r the Rayleigh number, and b is related to the aspect ratio of convection rolls). These equations are known as Lorenz Equations.

(a) Use any implementation of RK-4 to solve the system above numerically for $x(t), y(t), z(t)$. Take $\sigma = 10$, $b = 8/3$, and initial condition $x(0) = 0$, $y(0) = 1$, $z(0) = 0$. Try different values of r in the range $20 \leq r \leq 30$.

(b) Discuss the behavior of each solution $x(t), y(t), z(t)$ for different values of r.

(c) You may visualize how the trajectory evolves over time by making a parametric plot of all three solutions in the $xyz-$plane[1]. The command in Matlab/Octave is `plot3(X,Y,Z)`.

(d) Now, fix $r = 28, \sigma = 10$, $b = 8/3$. Find and plot the solutions for the two cases below. You should find solutions for t in the range $[0, 100]$.

   i. $[x(0), y(0), z(0)] = [0, 1, 0]$

   ii. $[x(0), y(0), z(0)] = [0, 1.0000001, 0]$
   (Yes, that is six zeroes after the decimal place).

   Discuss how the solutions evolve over time. What happens with the small perturbation of the initial condition over time? (You may want to compute the distance between the two solutions as a function of time and plot it with a logarithmic scale)

[1] For certain values of r, we call this system a "strange attractor".

3. (**#numimplementation**,**#numanalysis**, **#diffeq**) *PDE Solver* - The goal of this problem is to implement a numerical solver for Laplace's Equation

$$u_{xx} + u_{yy} = 0$$

on the square region $0 \leq x \leq 1, 0 \leq y \leq 1$. In particular, we will assume the following boundary conditions:

$$u(0, y) = 0, u(1, y) = 0, u(x, 0) = 0, u(x, 1) = sin\,(2\pi x)$$

i.e. the temperature is zero at all boundaries except the top, where the temperature is the function $f(x) = \sin(2\pi x)$.

(a) Consider an $N \times N$ discretization grid, with indices $(1, 1)$ indicating the bottom left point, and (N, N) the top-right point. Write an equation for the temperature at all points on the boundaries. Denote them $u_{1,j}, u_{N,j}, u_{i,1}, u_{i,N}$ for the temperatures on the left, right, bottom and top boundaries respectively.

(b) Use a 5-point stencil to obtain an expression for u_{xx} and u_{yy} at all $(N-2) \times (N-2)$ internal points (that is, all points that are not on the boundaries). Your expression for the derivatives at the (i, j) point should thus depend on the values $u_{ij}, u_{i+1,j}, u_{i-1,j}, u_{i,j+1}, u_{i,j-1}$. Discuss the order of accuracy of your expressions.

(c) Plug in your expressions for u_{xx} and u_{yy} from part (b) into Laplace's equation. You now have a set of difference equations that we must solve for each u_{ij}.

(d) For simplicity, let us take the case $N = 4$. In this case, there are 2×2 internal points, with corresponding temperatures $u_{22}, u_{23}, u_{32}, u_{33}$. Create a vector with these four points, and then write the matrix equation that represents the discretized system.

(e) Use any method to solve your matrix equation to obtain the values of u at each of these four points.

(f) (Optional) Write a numerical solver for this problem for any value of N. The function `reshape` on Matlab/Octave may be useful.

4. (**#diffeq**)*Another Transform* - Another useful transform in the study of differential equations is the *Laplace Transform* (LT, which also happens to be your professor's initials). We denote it $\mathcal{L}\{f\}$, and define it as:

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{1}$$

   (a) Show that the Laplace Transform is a linear operator.

   (b) Does the Laplace Transform of a function $f(t)$ always exist? If so, why? If not, give a counterexample.

   (c) Compute the Laplace Transform of the step function $u(t)$, defined as:

$$u(t) = \begin{cases} 1, \; t \geq 0 \\ 0, \; \text{otherwise} \end{cases} \tag{2}$$

   (d) Denote the Laplace Transform of $y(t)$ as $Y(s)$. What is the Laplace Transform of $y'(t)$ in terms of Y? What other information do you need to compute these transforms?

   (e) Finally, consider the IVP:

$$y' + y = u(t), y(0) = 1,$$

   where $u(t)$ is the step function defined in part (c). Take the Laplace Transform of both sides of the equation and find an equation for $Y(s)$.

   (f) (Optional) Find the inverse Laplace Transform of your expression for $Y(s)$ to obtain the solution $y(t)$.

   (g) Discuss the advantages of using a Laplace Transform for solving differential equations.

5. *Individual Reflection* - Include a brief 200-400 word reflection (one for each team member) on how you have grown as a computational scientist this semester. Describe any topics/discussions that were particularly enlightening or confusing.