CS154 - Final Project

December 17, 2021

1 Fractals

1.1 Complex Newton's Method

[1]:
```
def newton(f, f_prime, x0, eps=1e-8, max_iter=1e5):
    x = x0
    for _ in range(int(max_iter)):
        y = f(x)
        z = f_prime(x)
        if z == 0 or abs(y) < eps:
            break
        x -= y/f_prime(x)
    return x, abs(y)

# our first function
def f(x):
    return x**2 + 1

def f_prime(x):
    return 2*x

newton(f, f_prime, 0.1 + 1j)
```

[1]:
```
((-3.0097253200397857e-11+1.0000000000712528j), 1.5469716888295374e-10)
```

So, ignoring the really small terms, this is saying that we achieve a root of the function at i.

1.2 Roots of $x^2 - 1$

[2]:
```
import numpy as np
import matplotlib.pyplot as plt

# how many points to sample along each axis
l = 100

def make_plot(f, f_prime):
    vals = []
    nums = np.linspace(-1, 1, l)
```

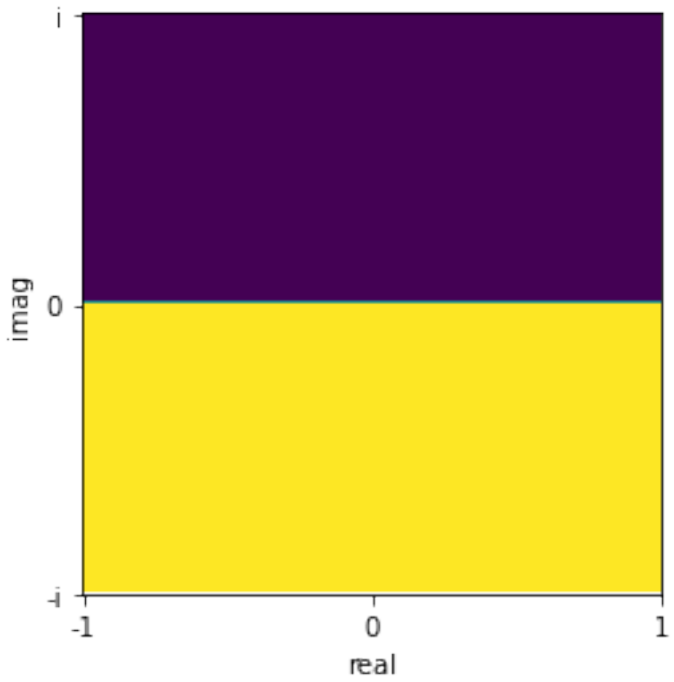
```
    # evaluate newton on grid
    for a in nums:
        row = []
        for b in nums:
            x0 = a + 1j*b
            row.append([newton(f, f_prime, x0)[0]])
        vals.append(row)

    # so the axes are pointing the right directions
    vals = np.transpose(np.array(vals).reshape(l, l))

    # plotting
    plt.imshow(vals.imag)
    plt.xlabel("real")
    plt.ylabel("imag")

    plt.xticks([0, l//2, l], [-1, 0, 1])
    plt.yticks([0, l//2, l], ["i", 0, "-i"])

make_plot(f, f_prime)
```



This plot shows which root Newton's method converges to for different seed values on the polynomial expression $x^2 + 1$. This expression has two roots, i (purple) and $-i$ (yellow). We can see that the method converges to whichever is closer. When the imaginary component is negative, the derivative

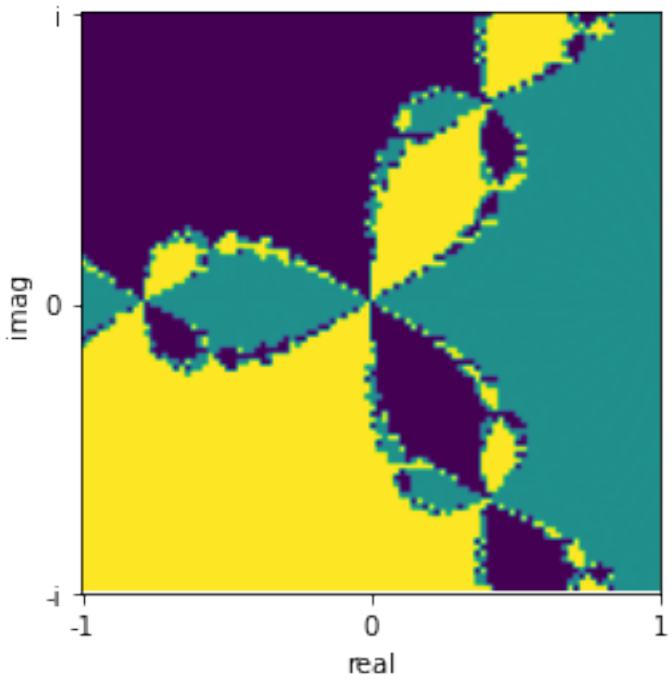
in the imaginary direction is negative, and the value is pushed to $-i$. Similarily, positive seeds are pushed to positive i.

When the imaginary part of the seed is zero (at the boundary of the two colors), the imaginary component of the derivative is also zero. So, the method will not converge to either root.

1.3 Roots of $x^3 - 1$

The polynomial expression $x^3 - 1$ has three roots: 1, $-i^{2/3}$, and $i^{4/3}$.

[3]:
```
make_plot(lambda x:x**3 - 1, lambda x:3*x**2)
```



This plot shows the root that Newton's method converges to for different seed values. Rather than using an RGB code, I opted to just plot the magnitude of the imaginary part of the root that is found. For these polynomials, this is enough to distinguish the roots with different colors. In this plot, purple represents the $-i^{2/3}$. Blue represents 1. Yellow represents $i^{4/3}$. We can see that, in a majority of cases, the seed will converge to the closest route. However, that is not always the case, and there are complicated patterns towards the center.

The function is cubic, so the only way for Newton's method to fail to converge is for it to get stuck at one of the nonzero optima of the function. Everywhere else, the derivative will be nonzero and pointed in the correct direction. So, the method will converge.

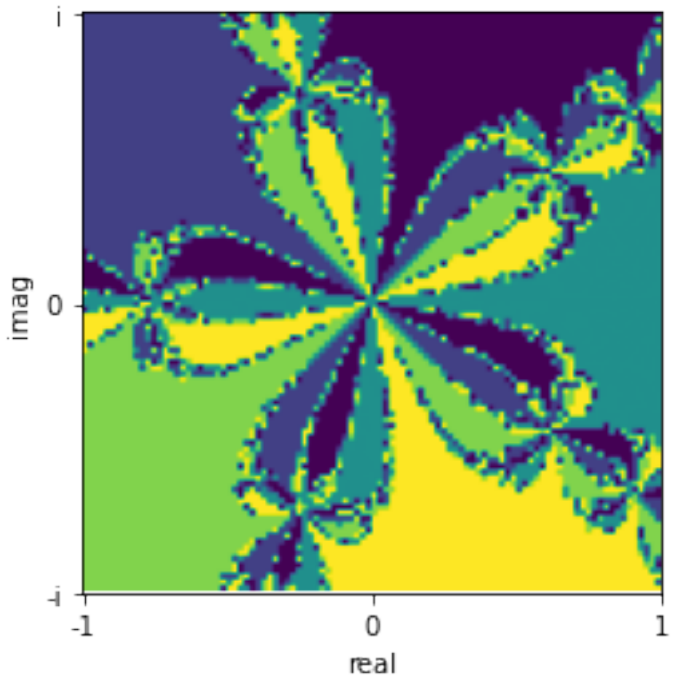
1.4 Roots of $x^5 - 1$

[4]:
```
import sympy as sp
x = sp.Symbol("x", complex=True)

[sp.N(i) for i in sp.solve(x**5 - 1)]
```

[4]:
```
[1.00000000000000,
 0.309016994374947 - 0.951056516295154*I,
 0.309016994374947 + 0.951056516295154*I,
 -0.809016994374947 - 0.587785252292473*I,
 -0.809016994374947 + 0.587785252292473*I]
```

[5]:
```
make_plot(lambda x:x**5 - 1, lambda x:5*x**4)
```



2 Strange Attractors

2.1 RK4 Implementation

[6]:
```
class Lorenz:
    """RK4 approximation to the Lorenz system"""
    def __init__(self, x0, y0, z0, sig, b, r, k=0.01):
        self.sig = sig
        self.b = b
```

```
        self.r = r
        self.k = k

        self.x = [x0]
        self.y = [y0]
        self.z = [z0]

    def plot_title(self):
        return f"Lorenz System ($\sigma$ = {round(self.sig, 2)}, b = {round(self.b, 2)}, r = {round(self.r, 2)})"

    def dxdt(self, x, y, z):
        return self.sig*(y-x)

    def dydt(self, x, y, z):
        return self.r*x - y - x*z

    def dzdt(self, x, y, z):
        return x*y - self.b*z

    def rk4(self, x, y, z):
        k1x = self.k*self.dxdt(x, y, z)
        k1y = self.k*self.dydt(x, y, z)
        k1z = self.k*self.dzdt(x, y, z)

        k2x = self.k*self.dxdt(x+k1x/2, y+k1y/2, z+k1z/2)
        k2y = self.k*self.dydt(x+k1x/2, y+k1y/2, z+k1z/2)
        k2z = self.k*self.dzdt(x+k1x/2, y+k1y/2, z+k1z/2)

        k3x = self.k*self.dxdt(x+k2x/2, y+k2y/2, z+k2z/2)
        k3y = self.k*self.dydt(x+k2x/2, y+k2y/2, z+k2z/2)
        k3z = self.k*self.dzdt(x+k2x/2, y+k2y/2, z+k2z/2)

        k4x = self.k*self.dxdt(x+k3x, y+k3y, z+k3z)
        k4y = self.k*self.dydt(x+k3x, y+k3y, z+k3z)
        k4z = self.k*self.dzdt(x+k3x, y+k3y, z+k3z)

        x_new = x + 1/6 * (k1x + 2*k2x + 2*k3x + k4x)
        y_new = y + 1/6 * (k1y + 2*k2y + 2*k3y + k4y)
        z_new = z + 1/6 * (k1z + 2*k2z + 2*k3z + k4z)

        return x_new, y_new, z_new

    def run(self, t_max):
        steps = int(t_max/self.k)
        for _ in range(steps):
            x_new, y_new, z_new = self.rk4(self.x[-1], self.y[-1], self.z[-1])
```

```
            self.x.append(x_new)
            self.y.append(y_new)
            self.z.append(z_new)

        self.x = np.array(self.x)
        self.y = np.array(self.y)
        self.z = np.array(self.z)
```

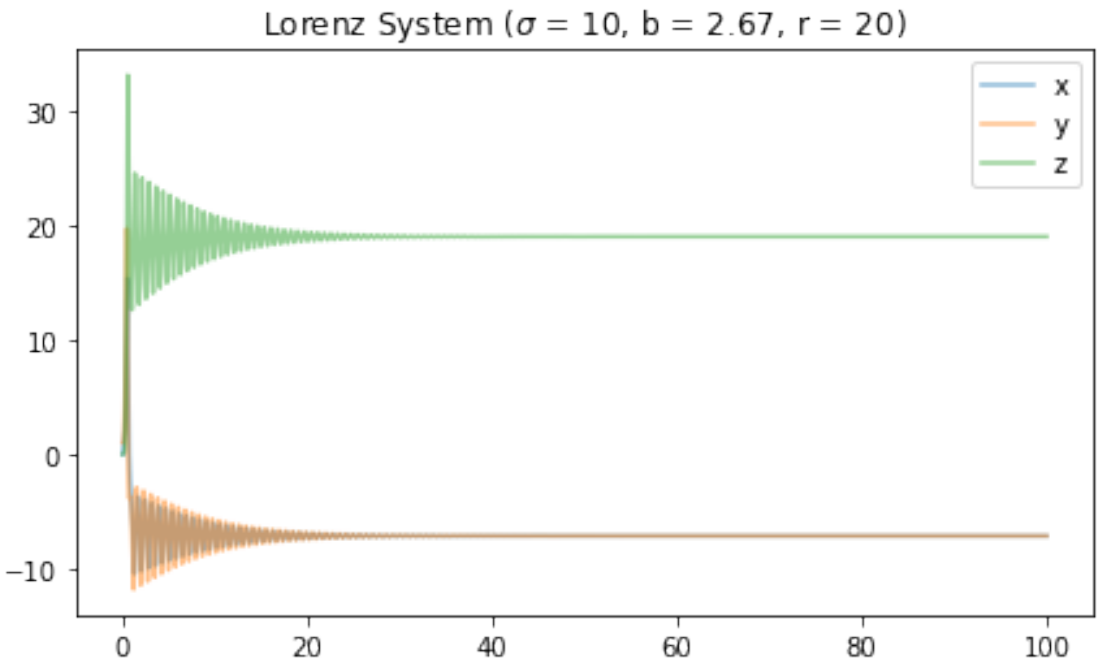
[7]:
```
def plot_2d(lorenz):
    """Plots each component of the Lorenz system as a function of time"""
    plt.figure(figsize=(7, 4))
    t = np.arange(0, lorenz.k*len(lorenz.x), lorenz.k)
    plt.plot(t, lorenz.x, label="x", alpha=0.5)
    plt.plot(t, lorenz.y, label="y", alpha=0.5)
    plt.plot(t, lorenz.z, label="z", alpha=0.5)
    plt.legend()
    plt.title(lorenz.plot_title())
    plt.show()
```

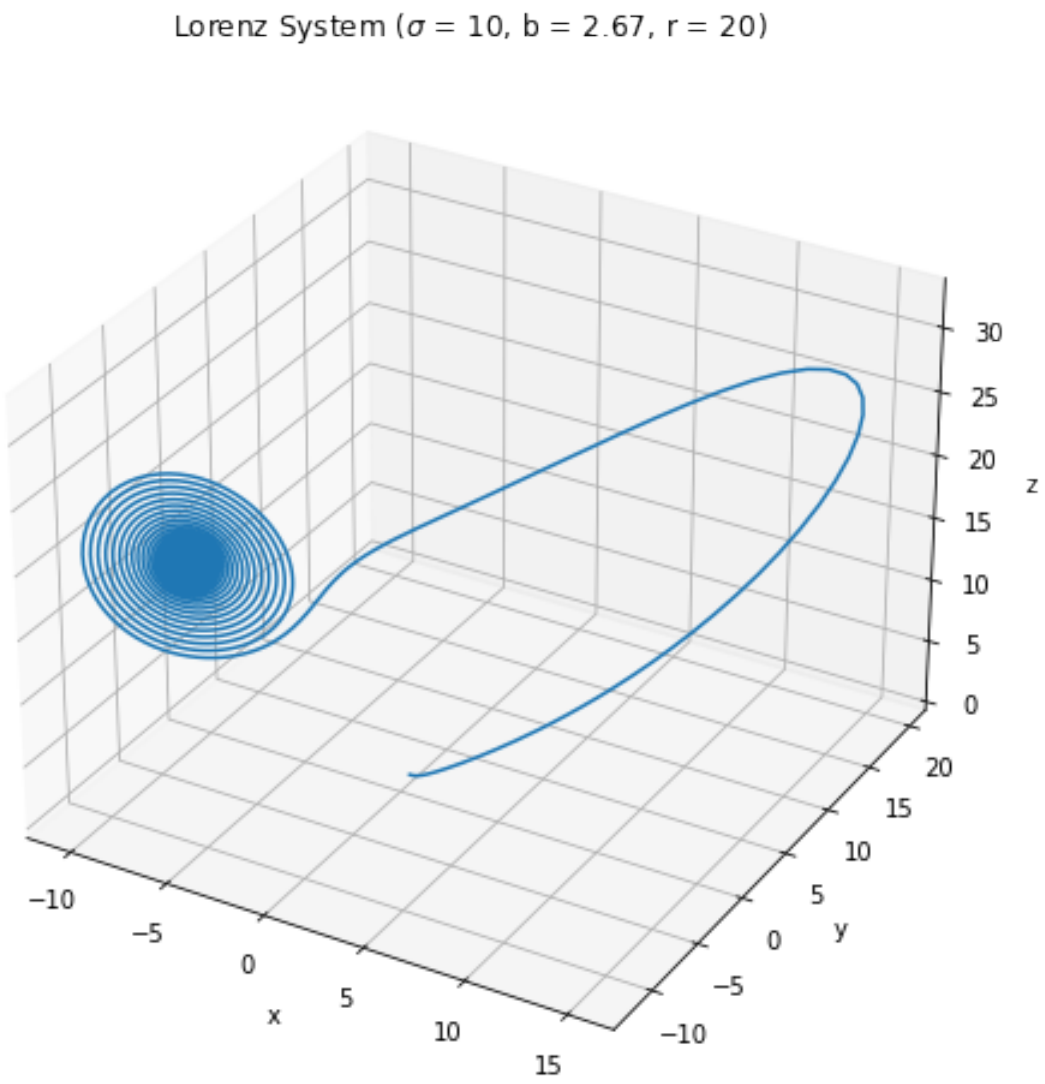
[8]:
```
def plot_3d(lorenz):
    """3d parametric plot of the Lorenz system"""
    fig = plt.figure(figsize=(8, 8))
    ax = fig.add_subplot(111, projection='3d')
    ax.plot(lorenz.x, lorenz.y, lorenz.z)

    ax.set_xlabel("x")
    ax.set_ylabel("y")
    ax.set_zlabel("z")
    ax.set_title(lorenz.plot_title())

    plt.show()
```

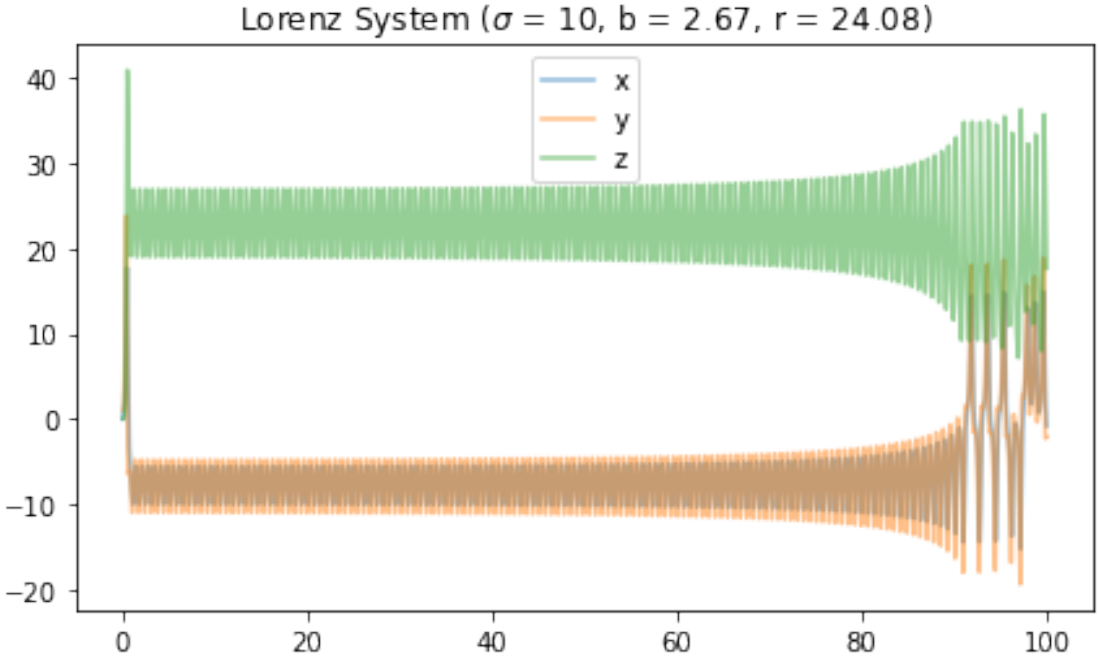
[9]:
```
sim20 = Lorenz(x0=0, y0=1, z0=0, sig=10, b=8/3, r=20)
sim20.run(100)
plot_2d(sim20)
plot_3d(sim20)
```

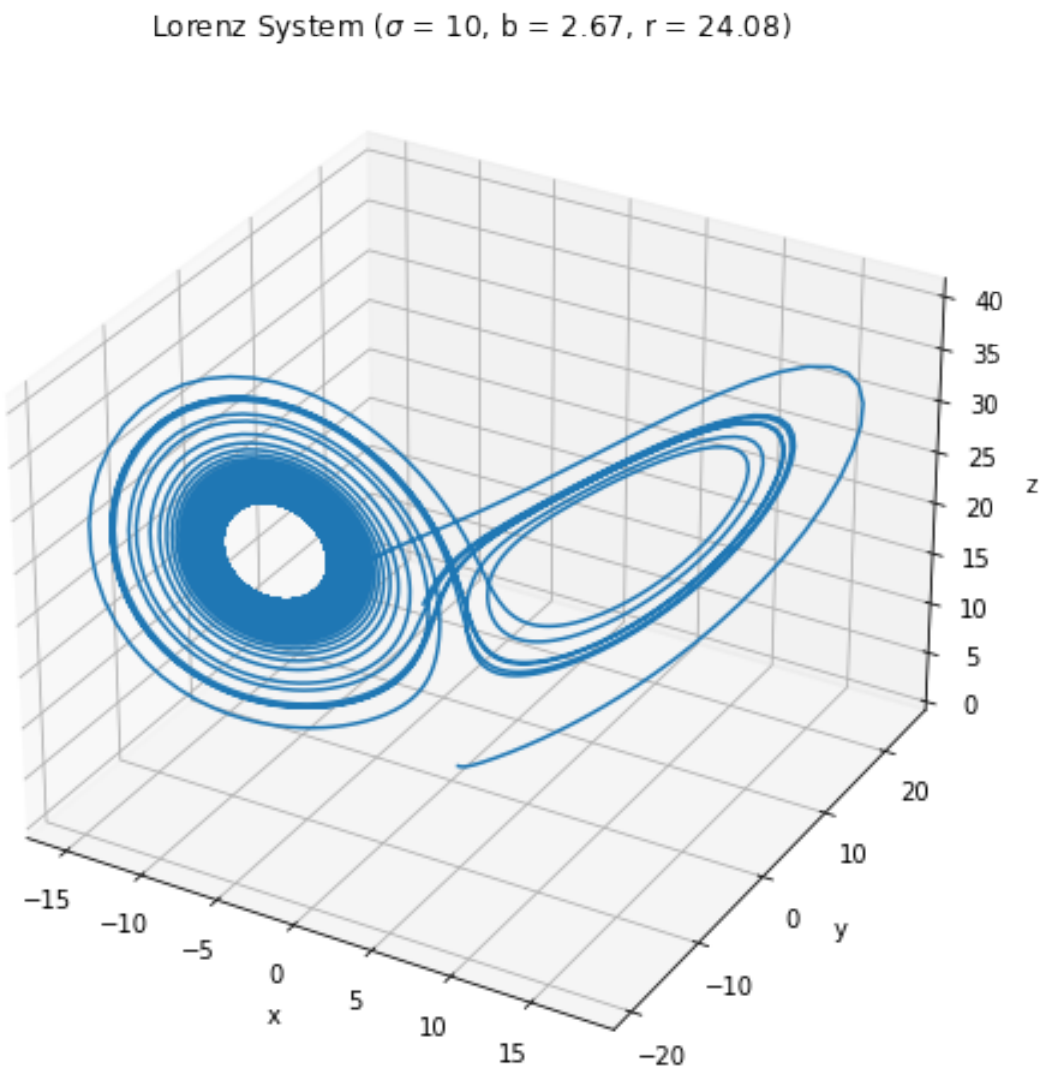

Lorenz System (σ = 10, b = 2.67, r = 20)
x
y
z
30
20
10
0
−10
0
20
40
60
80
100



We can see that when $r = 20$, the system approaches a point attractor relatively quickly. Then it never leaves this attractor.

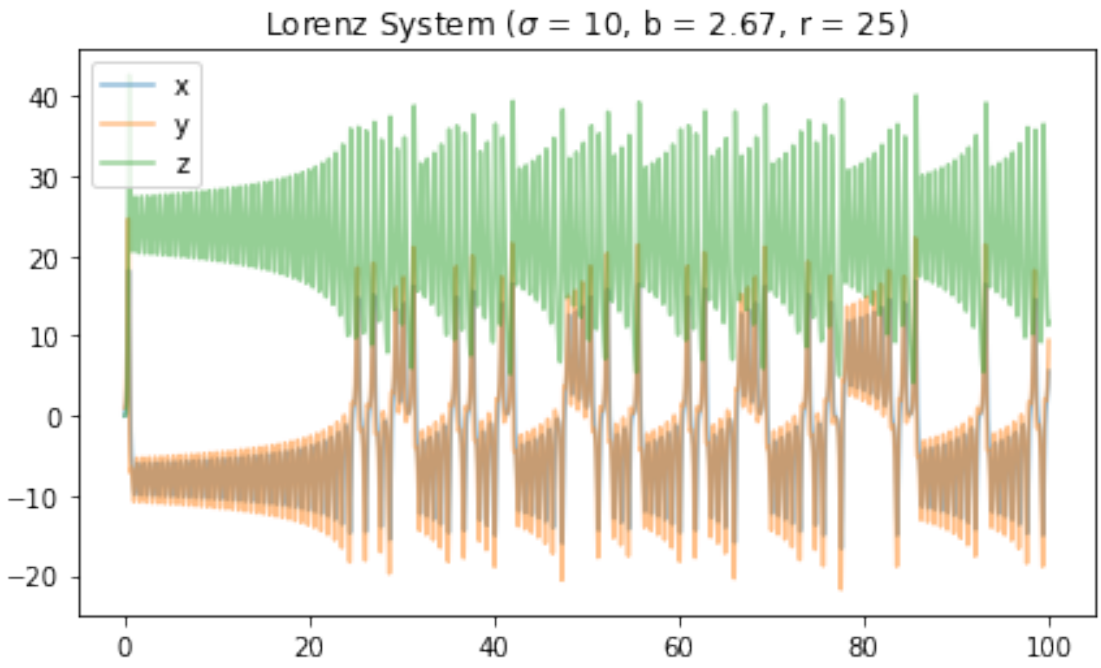
[10]:
```
sim22 = Lorenz(x0=0, y0=1, z0=0, sig=10, b=8/3, r=24.08)
sim22.run(100)
plot_2d(sim22)
plot_3d(sim22)
```

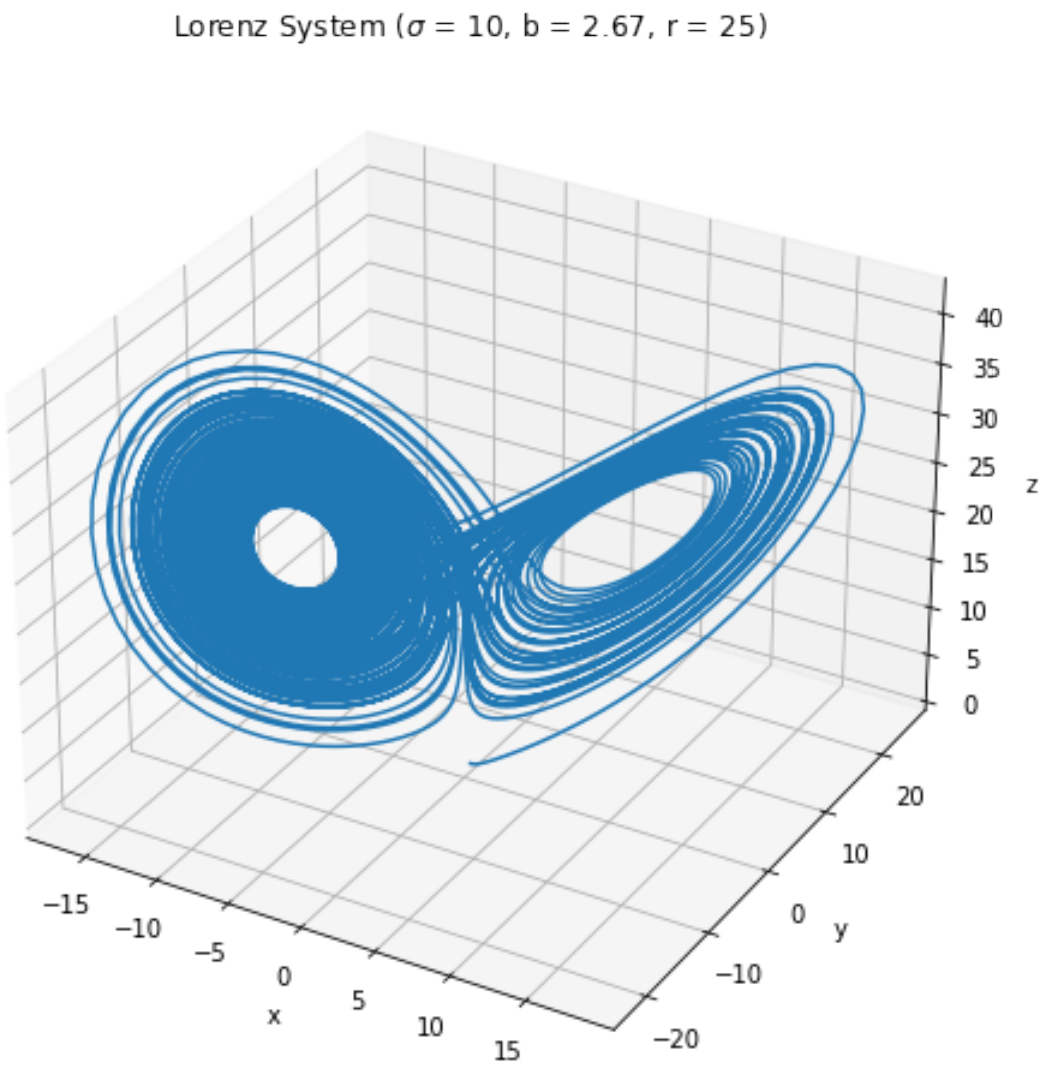

Lorenz System (σ = 10, b = 2.67, r = 24.08)
x
y
z
40
30
20
10
0
−10
−20
0
20
40
60
80
100



The system reaches a critical point near $r = 24.08$. We get the first look at a more complicated attractor. Note the first plot. Although the system looks like it might be approaching a point attractor for the first 50 timesteps or so, it is actually getting less and less stable. Around timestep 80, variation in all three variables explodes and we reach the more complicated Lorenz attractor.

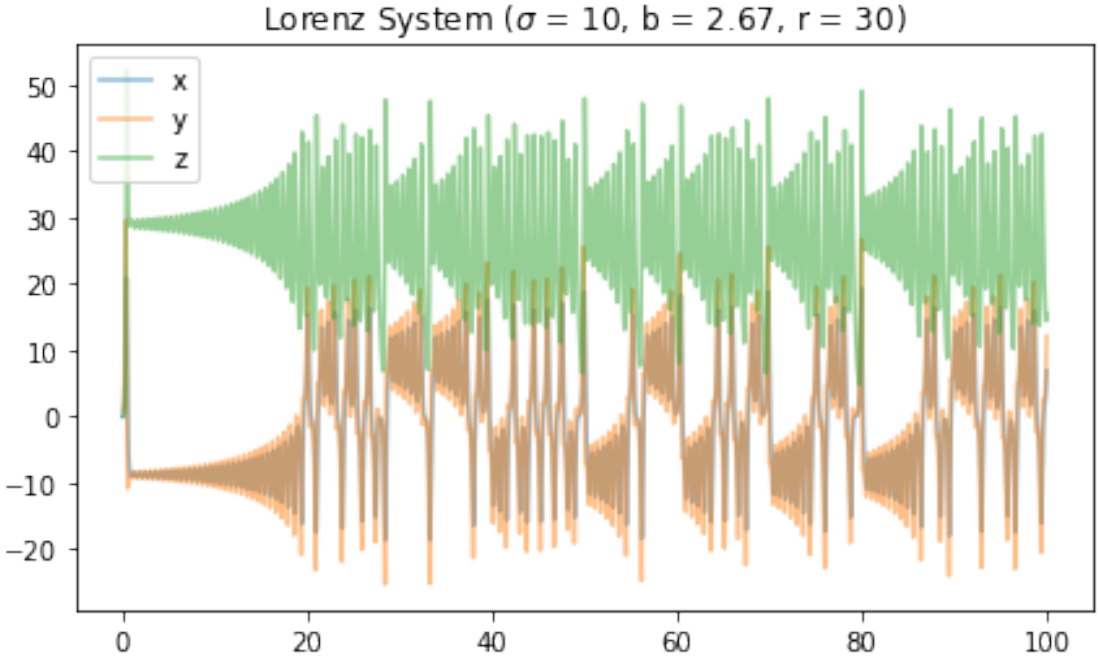
[11]:
```
sim25 = Lorenz(x0=0, y0=1, z0=0, sig=10, b=8/3, r=25)
sim25.run(100)
plot_2d(sim25)
plot_3d(sim25)
```

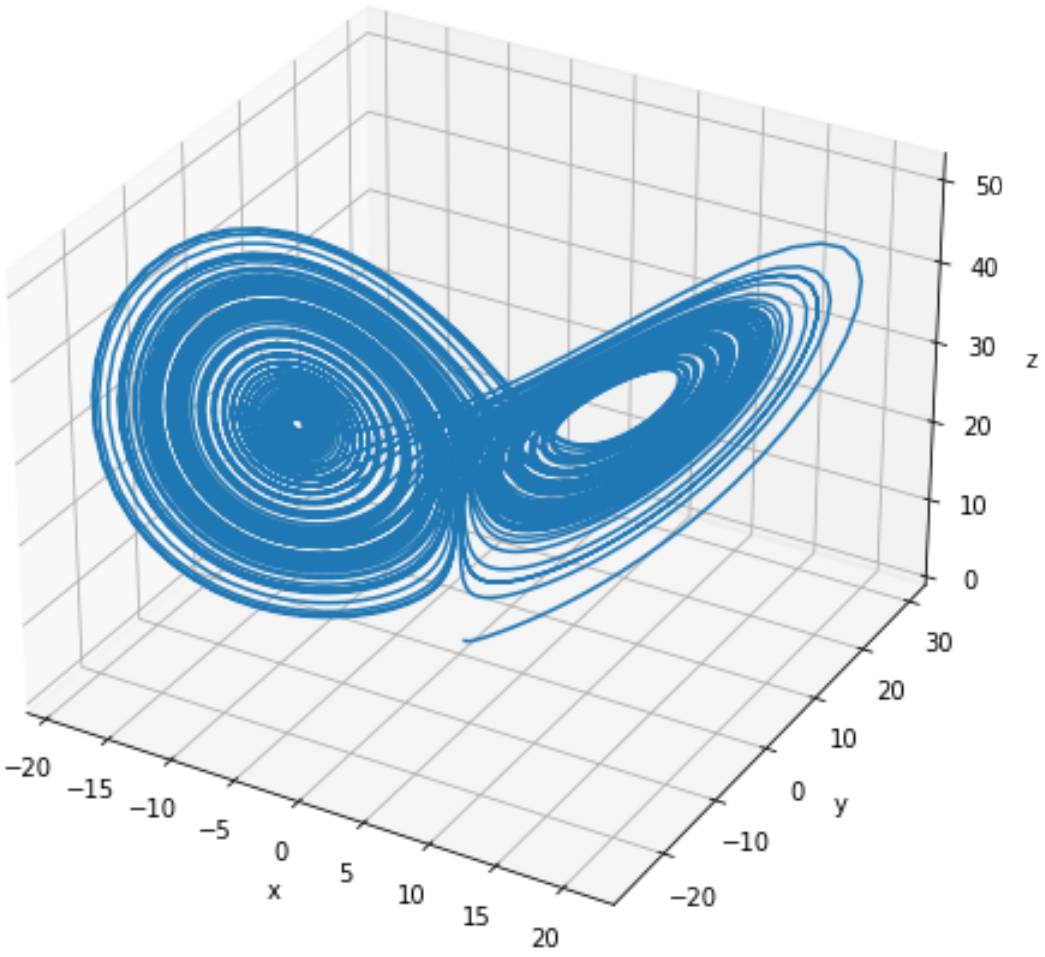

Lorenz System (σ = 10, b = 2.67, r = 25)
x
y
z
40
30
20
10
0
−10
−20
0
20
40
60
80
100



At $r = 25$, the circular path the system seems to follow starts out with a tiny circle which spirals outwards and gets larger, until it reaches a certain point at which it leaves its current path and follows a totally new circle, creating seemingly concentrix/spiralling figure-8s. Most of the time is spent in the first circle with only occasional loops in the new circle.

[12]:
```
sim30 = Lorenz(x0=0, y0=1, z0=0, sig=10, b=8/3, r=30)
sim30.run(100)
plot_2d(sim30)
plot_3d(sim30)
```


Lorenz System (σ = 10, b = 2.67, r = 30)
x
y
z
50
40
30
20
10
0
−10
−20
0
20
40
60
80
100



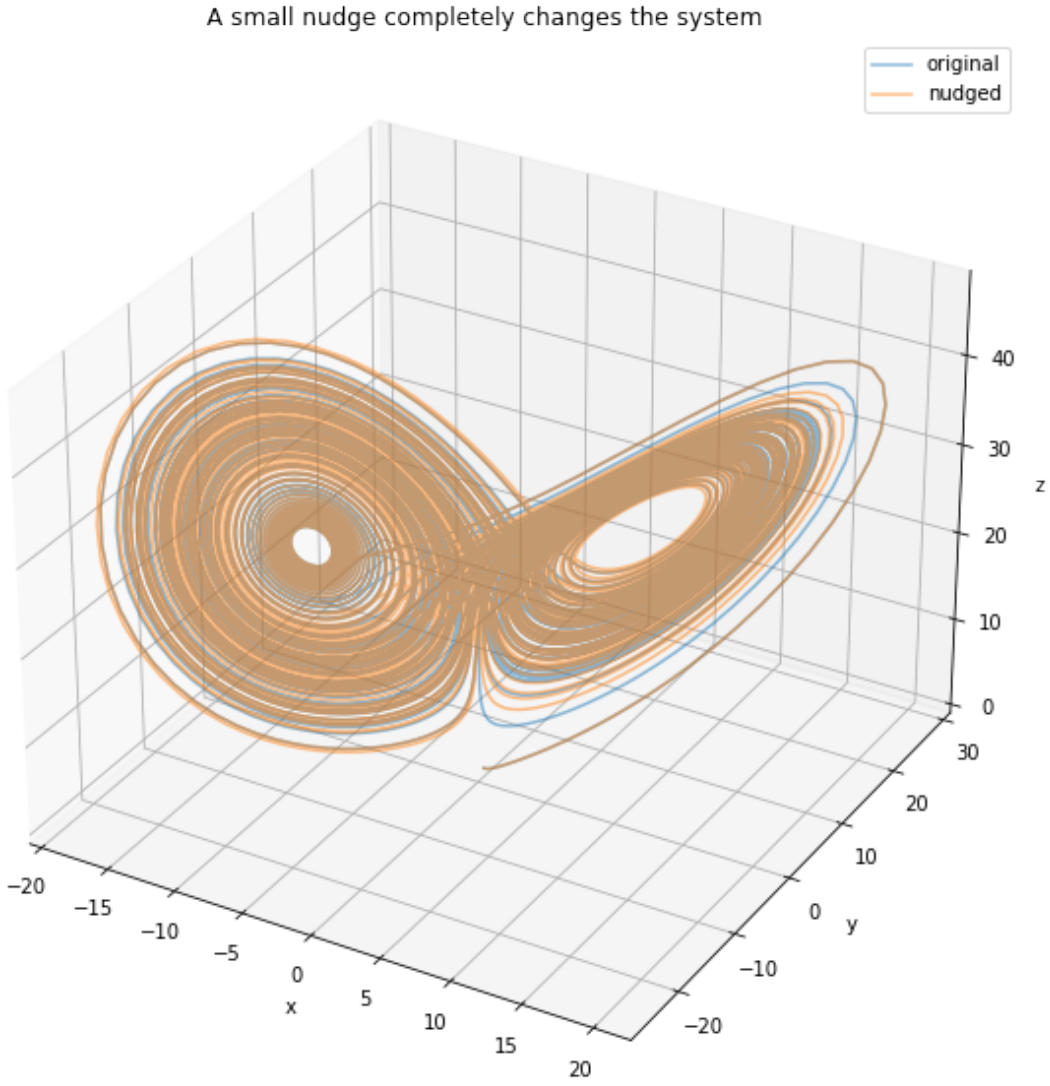
At $r = 30$, the system is comfortable settled into the Lorenz attractor. It reaches it quickly, around timestep 20, and it remains there. It distributes its time much more evenly between the two loops.

2.2 Chaos

[13]:
```
sim28 = Lorenz(x0=0, y0=1, z0=0, sig=10, b=8/3, r=28)
sim28.run(100)
sim28_nudged = Lorenz(x0=0, y0=1.0000001, z0=0, sig=10, b=8/3, r=28)
sim28_nudged.run(100)
```

[14]:
```
alpha = 0.5
fig = plt.figure(figsize=(10, 10))
ax = fig.add_subplot(111, projection='3d')
```

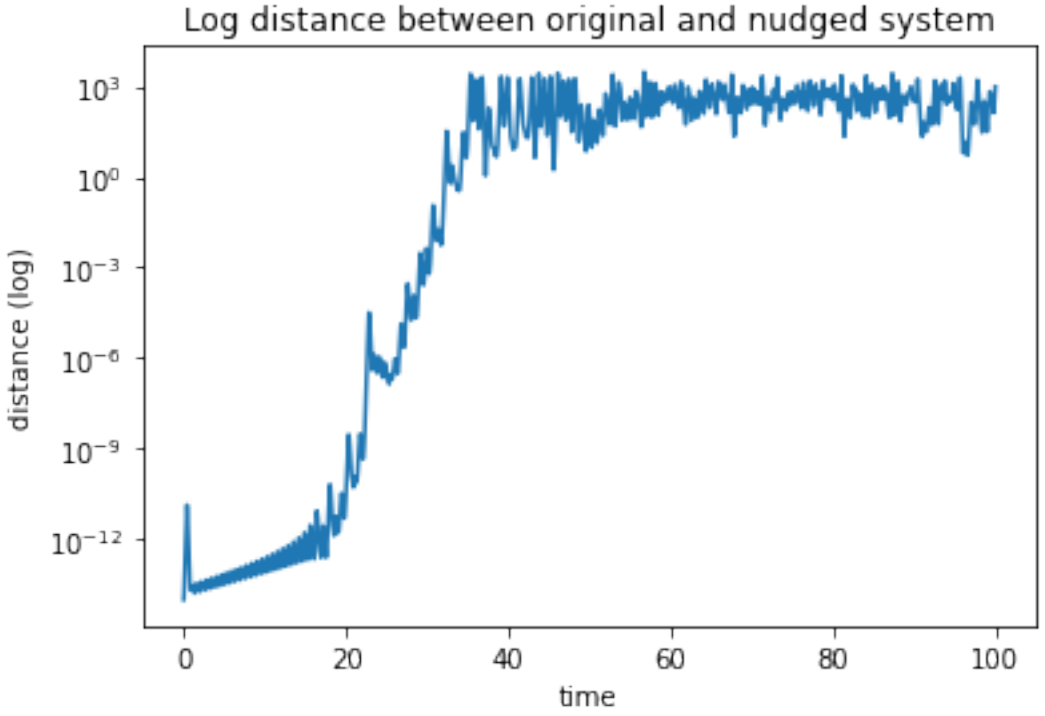
```
ax.plot(sim28.x, sim28.y, sim28.z, label="original", alpha=alpha)
ax.plot(sim28_nudged.x, sim28_nudged.y, sim28_nudged.z, label="nudged",␣
↪alpha=alpha)
ax.set_xlabel("x")
ax.set_ylabel("y")
ax.set_zlabel("z")
ax.set_title("A small nudge completely changes the system")
ax.legend()
plt.show()
```



[15]:
```
time = np.arange(0, sim28.k*len(sim28.x), sim28.k)
```

```
distance = (sim28.x - sim28_nudged.x)**2 + (sim28.y - sim28_nudged.y)**2 +␣
↪(sim28.z - sim28_nudged.z)**2
plt.plot(time, distance)

plt.title("Log distance between original and nudged system")
plt.yscale("log")
plt.ylabel("distance (log)")
plt.xlabel("time")
plt.show()
```



A *tiny* nudge to y results in significant divergence between the two systems fairly quickly. Something about the way the differential equation is set up causes very nearby (maybe even order of machine precision) intitial conditions to diverge from each other very quickly. However, they will still both be tracing the same loops. They are still both in the Lorenz attractor, and so they are qualitatively pretty similar. It would be hard to tell which one is which just based on the graph.

This is really interesting. We've discovered a system which maps all (lots of) initial conditions to qualitatively similar behavior, but quantitatively it quickly splits them far apart. This suggests that if we were studying a systm like this in nature, we might be able to make very good predictions about how the system would behave over time or in aggregate, but it might be very hard to make good predictions about how it will behave on the micro level.

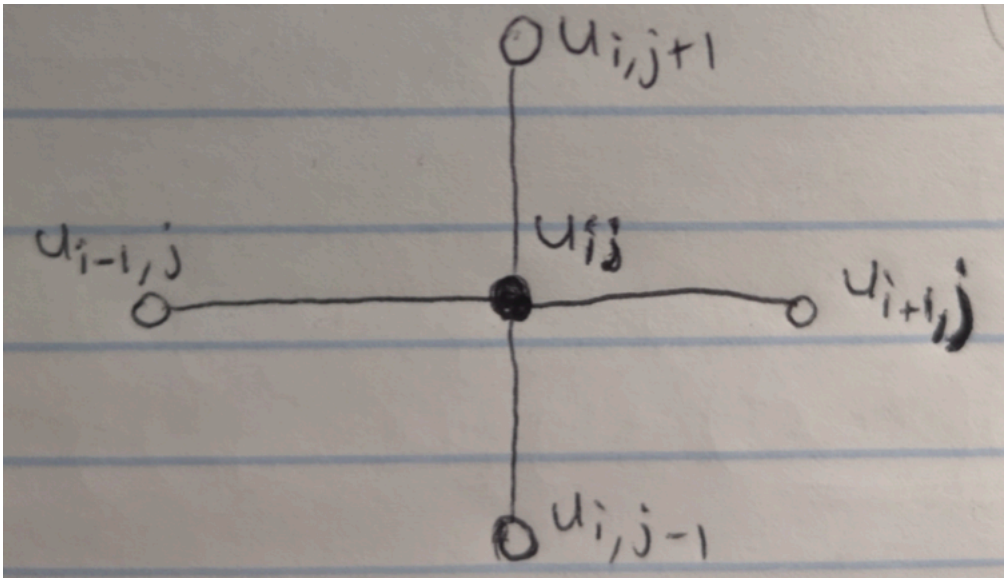
Cause and effect seem to be working very counterintuitively. Tiny changes cause big effects. Big changes can't change the final qualitative behavior of the system.

3 PDE Solver

3.1 Boundary Values

$$u_{1,j} = 0$$
$$u_{N,j} = 0$$
$$u_{i,1} = 0$$
$$u_{i,N} = sin(2\pi x)$$

3.2 Stencil



From the stencil, we see that we need to use the subsequent and previous points, i+1 & i-1 and j+1 & j-1 respectively, to calculate the 2nd derivative with respect to x and y at the point $u_{i,j}$. For this, we need two 2nd order central difference approximations, one for x and one for y.

$$u_{xx(i,j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + O(\Delta x^2)$$

$$u_{yy(i,j)} = \frac{u_{i,j+1}, -2u_{i,j} + u_{i,j-1}}{\Delta y^2} + O(\Delta y^2)$$

Because we are taking the second difference of both the x and y values at the point $u_{i,j}$, i.e. we are finding the slope on both sides of the point (in both directions) and finding the change in the slope per change in distance, the Taylor series approximation truncates the terms after the x^2 term. Since the manipulation of the Taylor series was done by adding the two series for $u_{i-1,j}$ and $u_{i+1,j}$ for u_{xx}, and $u_{i,j-1}$ and $u_{i,j+1}$ for u_{yy}, in both cases, the odd powers of x and y cancelled each other out meaning only the even powers, i.e. δx^2 and δx^4, etc. remained. Since we are trying to solve for the 2nd derivative, divide both sides by δx^2 which leaves a δx^2 in the numerator which gets truncated, meaning our errors for x and y are $O(\Delta x^2)$ and $O(\Delta y^2)$, respectively.

So, our error for the whole approximation is $O(\Delta x^2) + O(\Delta y^2)$. The two errors will sum because the expressions sum when substituted into the Laplace equation.

3.3 Substitution

Let $\Delta x = \Delta y = d$. Then we can substitute into Laplace's equation, yielding,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{d^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{d^2} = 0$$

Multiply both sides by d^2.

$$4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} = 0$$

3.4 Matrix Equation of the Discretized System

When we substitute directly, we get:

$$4u_{2,2} - u_{1,2} - u_{3,2} - u_{2,1} - u_{2,3} = 0$$

$$4u_{2,3} - u_{1,3} - u_{3,3} - u_{2,2} - u_{2,1} = 0$$

$$4u_{3,2} - u_{2,2} - u_{4,2} - u_{3,1} - u_{3,3} = 0$$

$$4u_{3,3} - u_{2,3} - u_{4,3} - u_{3,2} - u_{3,4} = 0$$

We can use the boundary conditions to cancel a number of these terms, yielding,

$$4u_{2,2} - u_{3,2} - u_{2,3} = 0$$

$$4u_{2,3} - u_{3,3} - u_{2,2} = 0$$

$$4u_{3,2} - u_{2,2} - u_{3,3} = \frac{\sqrt{3}}{2}$$

$$4u_{3,3} - u_{2,3} - u_{3,2} = -\frac{\sqrt{3}}{2}$$

We put the u's into the matrix in this order: $< u_{2,2}, u_{2,3}, u_{3,2}, u_{3,3} >$. Then, we can solve the systems equations using numpy as follows.

[16]:
```
A = np.array([
    [4, -1, -1, 0],
    [-1, 4, 0, -1],
    [-1, 0, 4, -1],
    [0, -1, -1, 4]
```

```
])

b = np.array([0, 0, 3**0.5/2, -3**0.5/2])

np.linalg.solve(A, b)
```

[16]:
```
array([ 0.03608439, -0.03608439,  0.18042196, -0.18042196])
```

So,

$$< u_{2,2}, u_{2,3}, u_{3,2}, u_{3,3} >=< 0.03608439, -0.03608439, 0.18042196, -0.18042196 >$$

4 Another Transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty \frac{f(t)}{e^{st}} dt$$

4.1 Linearity

$$\mathcal{L}\{f+g\}(s) = \int_0^\infty \frac{f(t)+g(t)}{e^{st}} dt$$

Distribute the integral,

$$\int_0^\infty \frac{f(t)}{e^{st}} dt + \int_0^\infty \frac{g(t)}{e^{st}} dt$$

Which we recognize,

$$\mathcal{L}\{f\}(s) + \mathcal{L}\{g\}(s)$$

4.2 Existence

The transform will only exist if that infinite integral exists. Sometimes, it doesn't. For example, let $f(x) = e^x$.

$$\mathcal{L}\{f\}(s) = \int_0^\infty \frac{f(t)}{e^{st}} dt$$

$$\mathcal{L}\{f\}(s) = \int_0^\infty \frac{e^t}{e^{st}} dt$$

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^s dt$$

The e^s is always positive. So, the integral diverges.

The integral will also not exist if the function is not piecewise continuous. So, we will not be able to take the Laplace transform of the following piecewise function g.

$$g(x) = \begin{cases} 1 & \text{x is rational} \\ 0 & \text{otherwise} \end{cases}$$

4.3 Step Function

Let $u(x)$ represent the step function, such that,

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

We can get an expression for the Laplace transform.

$$\mathcal{L}\{u\}(s) = \int_0^\infty \frac{u(t)}{e^{st}} dt$$

Since $u(t)$ is always 1 on the domain of the integral, we can replace it with a one.

$$\mathcal{L}\{u\}(s) = \int_0^\infty e^{-st} dt$$

$$= \frac{1}{-s} \int_0^\infty -se^{-st} dt$$

$$= \frac{1}{-s} \Big[e^{-st} \Big]_0^\infty$$

$$= \frac{1}{-s} \lim_{t \to \infty} (e^{-st} - 1)$$

When s is negative, the sign of the exponent will be positive, and the whole thing will go to infinity. When s is zero, the exponent will be zero and so the exp expression will be 1. So, the whole expression will be 0. When s is positive, the sign if the exponent will be negative and the exp goes away. We just end up with a -1 from the limit and a $1/s$ from the entire expression.

$$\mathcal{L}\{u\}(s) = \begin{cases} \infty & x < 0 \\ 1 & s = 0 \\ 1/s & s > 0 \end{cases}$$

4.4 Laplace of Derivative

Let,

$$\mathcal{L}\{y\}(s) = Y$$

Now, we compute the Laplace transform of y'.

$$\mathcal{L}\{y'\}(s) = \int_0^\infty \frac{y'(t)}{e^{st}} dt$$

Integrate by parts

$$\begin{array}{ll} v = y & dv = y' \\ u = e^{-st} & du = -se^{-st} \end{array}$$

$$\mathcal{L}\{y'\}(s) = \left[ye^{-st}\right]_0^\infty - \int_0^\infty -se^{-st} y dt$$

$$\mathcal{L}\{y'\}(s) = \left[ye^{-st}\right]_0^\infty + s\int_0^\infty e^{-st} y dt$$

$$\mathcal{L}\{y'\}(s) = \left[ye^{-st}\right]_0^\infty + sY$$

At $s = \infty$, the left term will just be zero, because there is a negative s in the exponent. At $s = 0$, the exponential term will be one. So, the whole left hand term just evaluates to $-y(0)$.

$$\mathcal{L}\{y'\}(s) = sY - y(0)$$

So, when we differentiate the function, we multiply the Laplace transform by s and subtract the initial condition.

4.5 Solving the IVP

Let $u(t)$ represent the step function. Assume y is a function of t. We want to solve the IVP,

$$y' + y = u$$

With initial condition $y(0) = 1$

Laplace both sides:

$$\mathcal{L}(y' + y) = \mathcal{L}u$$

By linearity,

$$\mathcal{L}y' + \mathcal{L}y = \mathcal{L}u$$

Use our formula for Laplace of derivative, continuing to use Y to represent the Laplace transform of y.

$$(sY - y(0)) + Y = \mathcal{L}u$$

From the initial condition, we know that $y(0) = 1$.

$$(s+1)Y - 1 = \mathcal{L}u$$

$$Y = \frac{\mathcal{L}u + 1}{s+1}$$

At every time greater than zero, we can substitute $1/s$ in for $\mathcal{L}u$, yielding,

$$Y(s) = \frac{1/s + 1}{s+1}$$

4.6 Using the Laplace Tranform

The Laplace transform is nice because it gives us a way to replace differentiation with multiplication. Derivatives on y become multiplication on Y. In addition, it's linear. So, a linear combination of derivatives on y is equivalent to a polynomial of Y. Laplace takes in calculus and gives us algebra.

5 Reflection

Leo

I got better at coding this semester, especially at using numpy and scipy. I also feel like I know a lot more about the theory around scientific computing. I have a better understanding of how these work under the hood (from CS154), and I'm a lot more familiar with the documentation. I think this was probably the biggest gain in terms of things I will use in the future.

I still feel pretty shaky on many of the mathier parts, especially around differential equations. I never really understood integrating factors, and I think I would need help to push through any of the harder analytical problems. I do feel equipped to use numerical solvers though, and I think I know enough of the theory to be useful. The ideas around bifurcations, steady states, and systems of linear equations are all fairly simple conceptually once you understand them, and I think I will remember them. For example, in CS166, I had to do a theoretical approximation to a simulation we were writing, so I used a linear system of differential equations like we learned about in CS154.

Sean

I have grown as a computational scientist this semester in that I have learnt to try thinking about the world in terms of systems that change over time which can potentially be modelled with a set of differential equations. I have gone from struggling to get code to work for some assignments to figuring out the entirety of the code for my half of this assignment. I have learnt how to think about defining a function both from the "top-down" (explicitly defining it for all values in the

domain) and from the "bottom-up" (defining how the slope changes at each time step, and using those slopes to approximate the subsequent values).

The most interesting discussions were applying some of the linear algebra concepts to solving differential equations. I really enjoyed the course last semester and it was fun seeing it used again. I thoroughly enjoyed the Fourier Series and Fourier Transform sections and I will now always think about functions in terms of their decomposition into sinusoids.

Most of the maths went over my head when trying to do the readings, and even for the majority of the class, but sometimes there were moments that cleared up previous and current confusions. Getting the code to work was always the most difficult part for me for nearly every topic.

6 HC Applications

#responsibility: We both knew that finals week was going to be stressful so we wanted to be proactive and finish this assignment well in advance to have one less thing to worry about. Leo finished the majority of his section in the week before finals, Sean finished his section on Tuesday, and we were ready to submit by Wednesday evening, more than 48 hours before the deadline.

#systemdynamics: In the Lorenz system visualisation, we show how the equations describe a system that moves between two attractor states, depending on the values of some of the constants, specifically r in this case. For $r = 20$, there was a single point attractor. This was true up until a critical threshold for r at about $r = 24.08$. From this point on, the system entered a complicated attractor in 3d space.

#complexcausality: We discuss the interplay of cause and effect in the Lorenz system. We discuss how, while small nudges to initial conditions can cause large quantitative differences in predictions, large changes cannot influence the aggregate behavior of the system. We talk about how this is very spooky.