Exercise 1:

a): This system is a DIT system, and $A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$

$\therefore \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 \\ 0.5 & \lambda-0.5 \end{vmatrix} = (\lambda-1)(\lambda-0.5) = 0$

$\therefore \lambda_1 = 1 \quad \lambda_2 = 0.5.$

$\therefore$ It is easy to know the jordan form of A is

$J = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$

$\therefore J^k = \begin{pmatrix} 1^k & 0 \\ 0 & 0.5^k \end{pmatrix}$

for $\lambda_i = 0.5$, we have $r_i = 0.5 < 1$

for $\lambda_i = 1$, we have $r_i = 1$ and $m = 0$.

$\therefore$ This system is Lyapunov stable but not asymptotic stable.

b): This system is a LIT system and $A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

$\therefore \det(\lambda I - A) = \begin{vmatrix} \lambda+7 & 2 & -6 \\ -2 & \lambda+3 & 2 \\ 2 & 2 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda+5)(\lambda+3) = 0$

$\therefore \lambda_1 = -1 \quad \lambda_2 = -3 \quad \lambda_3 = -5.$

$\because A \in R^{3\times3}$

$\therefore$ It is easy to know that the jordan form of A is $J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

$\therefore e^{Jt} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-5t} \end{pmatrix}$

It is easy to find out that $\forall \lambda_i$, we have $Re < 0$.

$\therefore$ This system is asymptotic stable and also Lyapunov stable.

Exercise 2:

First Let's use K-D to decompose this system. $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ $C = (1,1,1)$ $n=3$

$$P = (B, AB, A^2B) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore r(P) = 2$

$\therefore M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ $M^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$

$\therefore \hat{A} = M^{-1}AM = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow A^c = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

$\hat{B} = M^{-1}B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow B_c = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$\hat{C} = CM = (3\ 0 . 1) \Rightarrow C_c = (3, 0)$

$\therefore$ This system can be decomposed to the controllable system below.

$$\boxed{\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} u \\ y &= (3, 0)\, x. \end{aligned}}$$

Now, Let's check the observability of this decomposed system.

$Q = \begin{pmatrix} C_c \\ C_c A_c \end{pmatrix} = \begin{pmatrix} 3, & 0 \\ 0, & 0 \end{pmatrix} \Rightarrow r(Q) = 1 < 2 = n$

$\therefore$ **This decomposed system is not observable.**

Since this decomposed system is controllable, obviously **this decomposed system must be stabilizable.**

As for the detectability. Since this system is not observable, we need to check whether it is Lyapunov stable or not.

$\because A_c = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ $\therefore \lambda_1 = 0$ $\lambda_2 = -1$

$\therefore$ The jordan of A_c is $J = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

$\therefore e^{Jt} = \begin{pmatrix} e^{0 \cdot t} & 0 \\ 0 & e^{-t} \end{pmatrix}$ for $\lambda_1 = 0$, we have $m=0$; for $\lambda_2 = -1$, we have $Re<0$

$\therefore$ This system is Lyapunov stable **$\therefore$ This decomposed system is detectable**

Exercise 3:

$\because$ The state variables are $z = (\theta, \dot{x}, \dot{y}, \dot{\theta})^T$

$\therefore$ $\dot{z} = \begin{pmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix}$ let $\dot{z} = 0$, then we can get the

equilibrium solution $\dot{\tilde{\theta}}(t) = 0$, $\tilde{\dot{x}}(t), \tilde{\dot{y}}(t), \tilde{\dot{\theta}}(t) = 0$, $\tilde{u}_1(t) = mg$; $\tilde{u}_2(t) = 0$.

$\because$ $m\ddot{x} = -u_1 \sin\theta + \epsilon u_2 \cos\theta$

$\therefore$ $\ddot{x} = f_1(\theta, \dot{x}, \dot{y}, \dot{\theta}, u_1, u_2) = \dfrac{-u_1 \sin\theta + \epsilon u_2 \cos\theta}{m}$

$\therefore$ $\dfrac{\partial f_1}{\partial \theta} = -\dfrac{u_1}{m} \cos\theta \cdot \dot{\theta} - \dfrac{\epsilon u_2}{m} \sin\theta\, \dot{\theta} = 0.$

$\dfrac{\partial f_1}{\partial \dot{x}} = 0$, $\dfrac{\partial f_1}{\partial \dot{y}} = 0$, $\dfrac{\partial f_1}{\partial \dot{\theta}} = 0$

$\dfrac{\partial f_1}{\partial u_1} = -\dfrac{\sin\theta}{m} = 0$ $\dfrac{\partial f_1}{\partial u_2} = \dfrac{\epsilon \cos\theta}{m} = \dfrac{\epsilon}{m}.$

$\because$ $m\ddot{y} = u_1 \cos\theta + \epsilon u_2 \sin\theta - mg$

$\therefore$ $\ddot{y} = \dfrac{u_1}{m} \cos\theta + \dfrac{\epsilon u_2}{m} \sin\theta - g = f_2(\theta, \dot{x}, \dot{y}, \dot{\theta}, u_1, u_2)$

$\therefore$ $\dfrac{\partial f_2}{\partial \theta} = -\dfrac{u_1}{m} \sin\theta\, \dot{\theta} + \dfrac{\epsilon u_2}{m} \cos\theta \cdot \dot{\theta} = 0$, $\dfrac{\partial f_2}{\partial \dot{x}} = 0$, $\dfrac{\partial f_2}{\partial \dot{y}} = 0$, $\dfrac{\partial f_2}{\partial \dot{\theta}} = 0$

$\dfrac{\partial f_2}{\partial u_1} = \dfrac{\cos\theta}{m} = \dfrac{1}{m}$ $\dfrac{\partial f_2}{\partial u_2} = \dfrac{\epsilon}{m} \sin\theta = 0$

$\because$ $J\ddot{\theta} = u_2$ $\therefore$ $\ddot{\theta} = f_3(\theta, \dot{x}, \dot{y}, \dot{\theta}, u_1, u_2) = \dfrac{u_2}{J}.$

$\dfrac{\partial f_3}{\partial \theta} = \dfrac{\partial f_3}{\partial \dot{x}} = \dfrac{\partial f_3}{\partial \dot{y}} = \dfrac{\partial f_3}{\partial \dot{\theta}} = \dfrac{\partial f_3}{\partial u_1} = 0$ $\dfrac{\partial f_3}{\partial u_2} = \dfrac{1}{J}$

$\therefore$ The linearized model around the equilibrium solution is

$$\dot{z} = \begin{pmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\because$ $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\therefore$ It is easy to know that the eigenvalues of A are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

$\because$ $A - \lambda I = A \Rightarrow \text{rank}(A) = 1$

$\therefore$ the dimension of null space of A is 3.

$\therefore$ the jordan form of A is $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Let's consider the matrix $J_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. the eigen values are $\lambda_1 = \lambda_2 = 0$

$\therefore$ $e^{\lambda t} = \beta_1 \lambda + \beta_0 \Rightarrow \begin{cases} 1 = \beta_0 \\ t = \beta_1 \end{cases}$

$te^{\lambda t} = \beta_1$

$\therefore$ $e^{J_{11} t} = t J_{11} + \beta_0 I = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$\therefore$ $e^{J} = \begin{pmatrix} e^{0t} & t \cdot e^{0t} & 0 & 0 \\ 0 & e^{0t} & 0 & 0 \\ 0 & 0 & e^{0t} & 0 \\ 0 & 0 & 0 & e^{0t} \end{pmatrix} = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\because$ for the term t, we have λ_i's $Re = 0$ and the power of t is $1 > 0$

$\therefore$ This linearized model system is not Lyapunov stable.

But for the original model, it is hard to say whether the original model is stable or unstable.

Exercise 4:

Using $V(x) = x_1^2 + x_2^2$. obviously. $V(x) > 0 \quad \forall x \neq 0$.

$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$ $\therefore$ obviously $V(x)$ and its partial derivatives are continuous. and $V(x)$ is positive definite.

$\because$ $\dot{x}_1 = ax_1$ $\quad \dot{x}_2 = x_1 - x_2$

$\therefore$ $\dot{V}(x) = 2ax_1^2 + 2x_1x_2 - 2x_2^2$.

$\because$ This system needs to be asymptotically stable

$\therefore$ $\dot{V}(x)$ must be negative definite.

$\therefore$ for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \dot{V}(x) = 2ax_1^2 + 2x_1x_2 - 2x_2^2 < 0$.

$\Rightarrow \quad ax_1^2 + x_1x_2 - x_2^2 < 0$.

$\Rightarrow \quad (a+\frac{1}{2})x_1^2 - \frac{1}{2}x_1^2 + x_1x_2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_2^2 < 0$

$\Rightarrow \quad (a+\frac{1}{2})x_1^2 - \frac{1}{2}(x_1 - x_2)^2 - \frac{1}{2}x_2^2 < 0$

In order to achieve this inequation for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we need to make sure. $a + \frac{1}{2} \leq 0$.

$\therefore$ $a \leq -\frac{1}{2}$

$\therefore$ When $a \leq -\frac{1}{2}$ the given system is asymptotically stable.

Exercise 5:

a): First, we need to linearize this model.

Let $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, we can get the equilibrium solution is $x_2 - x_1 x_2^2 = 0$ $-x_1^3 = 0$

$\therefore x_1 = 0, \quad x_2 = 0.$

$\because \dot{x}_1 = f_1(x_1, x_2) = -x_1 x_2^2 + x_2.$

$\therefore \frac{\partial f_1}{\partial x_1} = -x_2^2 = 0. \qquad \frac{\partial f_1}{\partial x_2} = -2x_1 x_2 + 1 = 1$

$\because \dot{x}_2 = f_2(x_1, x_2) = -x_1^3.$

$\therefore \frac{\partial f_2}{\partial x_1} = -3x_1^2 = 0 \qquad \frac{\partial f_2}{\partial x_2} = 0.$

$$\therefore \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\det(\lambda I - A) = 0 \Rightarrow \lambda_1 = \lambda_2 = 0.$

It is easy to know the Jordan form of A is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$$\begin{cases} e^{\lambda t} = \beta_1 \lambda + \beta_0 \\ t e^{\lambda t} = \beta_1 \end{cases} \Rightarrow \begin{cases} 1 = \beta_0 \\ \beta_1 = t \end{cases}$$

$$\therefore e^{Jt} = tJ + I = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$\because$ for λ_1 and λ_2, we have $Re = 0$ and $m = 1 > 0$

$\therefore$ The approximated linearized system is unstable.

But for the original system, it is hard to say whether this system is Lyapunov stable, since $Re(\lambda_1) = Re(\lambda_2) = 0$

b) $\dot{V}(x_1, x_2) = 4x_1^3\dot{x}_1 + 4x_2 \cdot \dot{x}_2$

Obviously, $V(x_1, x_2)$ and $\dot{V}(x_1, x_2)$ are continuous.

$\because V(x_1, x_2) = x_1^4 + 2x_2^2$, $\therefore$ for all $x \neq 0$ and $x(0)=0$ we have $V(x_1, x_2) > 0$

$\therefore V(x_1, x_2)$ is positive definite.

$\because \dot{x}_1 = x_2 - x_1 x_2^2 \qquad \dot{x}_2 = -x_1^3$

$$\therefore \dot{V}(x_1, x_2) = 4x_1^3(x_2 - x_1 x_2^2) + 4x_2(-x_1^3)$$

$$= 4x_1^3 x_2 - 4x_1^4 x_2^2 - 4x_1^3 x_2$$

$$= -4x_1^4 x_2^2.$$

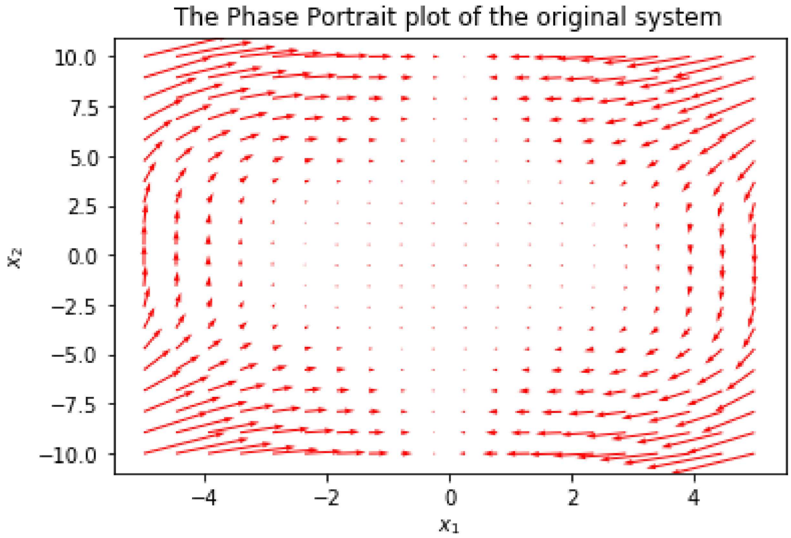
Obviously, for all $x \neq 0$ and $x(0)=0$, we have $\dot{V}(x_1, x_2) < 0$.

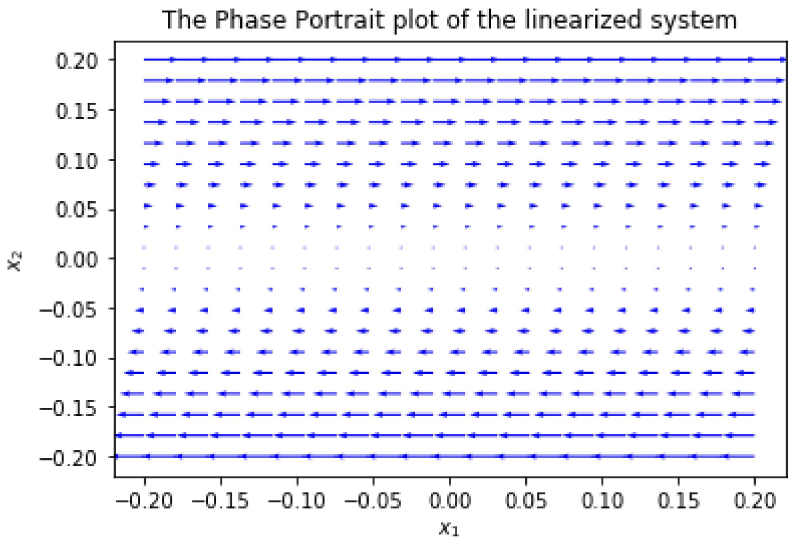
$\therefore \dot{V}(x_1, x_2)$ is negative definite.

$\therefore$ This system is asymptotically stable.

In [1]:

```
# This is the code I use for HW5 Exercise5 question c
import numpy as np
import matplotlib.pyplot as plt

def f_original(X):
    x1, x2 = X
    return [x2 - x1 * x2 * x2, -x1 * x1 * x1]

def f_linearized(X):
    x1, x2 = X
    return [x2, 0]

x1_original = np.linspace(-5, 5, 20)
x2_original = np.linspace(-10, 10, 20)

x1_linearized = np.linspace(-0.2, 0.2, 20)
x2_linearized = np.linspace(-0.2, 0.2, 20)

X1_original, X2_original = np.meshgrid(x1_original, x2_original)
X1_linearized, X2_linearized = np.meshgrid(x1_linearized, x2_linearized)

u_original, v_original = np.zeros(X1_original.shape), np.zeros(X2_original.shape)
u_linearized, v_linearized = np.zeros(X1_linearized.shape), np.zeros(X2_linearized.shape)

NI_original, NJ_original = X1_original.shape
NI_linearized, NJ_linearized = X1_linearized.shape

for i in range(NI_original):
    for j in range(NJ_original):
        x1 = X1_original[i, j]
        x2 = X2_original[i, j]
        y_original = f_original([x1, x2])
        u_original[i,j] = y_original[0]
        v_original[i,j] = y_original[1]

for i in range(NI_linearized):
    for j in range( NJ_linearized):
        x1 = X1_original[i, j]
        x2 = X2_original[i, j]
        y_linearized = f_linearized([x1, x2])
        u_linearized[i,j] = y_linearized[0]
        v_linearized[i,j] = y_linearized[1]

plt.figure()
Q1 = plt.quiver(X1_original, X2_original, u_original, v_original, color='r')
plt.title("The Phase Portrait plot of the original system")
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.show()


plt.figure()
Q2 = plt.quiver(X1_linearized, X2_linearized, u_linearized, v_linearized, color='b')
plt.title("The Phase Portrait plot of the linearized system")
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.show()
```

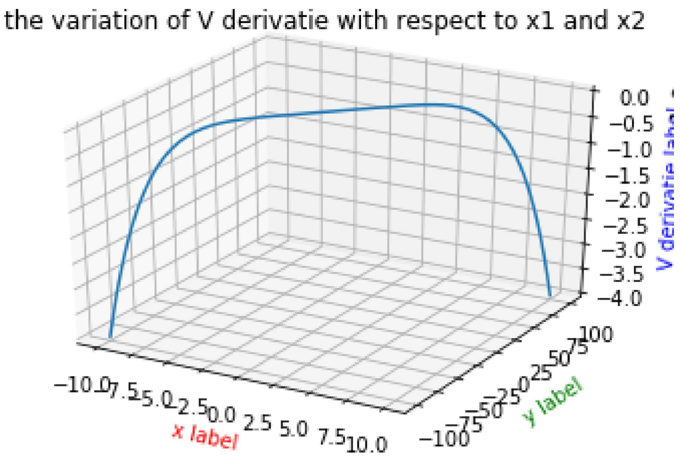

The Phase Portrait plot of the original system
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
x_2
−4
−2
0
2
4
x_1


The Phase Portrait plot of the linearized system
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
x_2
−0.20
−0.15
−0.10
−0.05
0.00
0.05
0.10
0.15
0.20
x_1

In [ ]:

In [1]:

```
# This is the code I use for HW5 exercise5 question d.
import matplotlib.pyplot as plt
import numpy as np

from mpl_toolkits.mplot3d import Axes3D

fig = plt.figure()
ax = fig.gca(projection='3d')

x1 = np.linspace(-10, 10, 1000)
x2 = np.linspace(-100, 100, 1000)
x1Square = np.multiply(x1, x1)
x1Biquadrate = np.multiply(x1Square, x1Square)
x2Square = np.multiply(x2, x2)
z = -4 * np.multiply(x1Biquadrate, x2Square)

ax.plot(x1, x2, z)
plt.title("the variation of V derivatie with respect to x1 and x2")
ax.set_xlabel('x label', color='r')
ax.set_ylabel('y label', color='g')
ax.set_zlabel('V derivatie label', color='b')
plt.show()
```



In [ ]:

Exercise 6:

a): This is a Discrete time Linear Time invariant system and

$A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $C = [5, 5]$ $D = 0$

$\therefore G_D(z) = C(zI-A)^{-1}B + D = (5, 5)\begin{pmatrix} z-1 & 0 \\ 0.5 & z-0.5 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \frac{5}{(z-1)} - \frac{5}{(2z-1)(z-1)} - \frac{10}{2z-1}$$

$$= \frac{0}{(z-1)(2z-1)}$$

The poles of this system are. $z_1 = 1$ $z_2 = 0.5$.

Since $|z_1| = 1$ which means that this pole is on the unit circle. $\therefore$ This system is BIBO unstable.

b): This is a CTLTI system and $A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$

$C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ $D = 0$

$\therefore G_C(s) = C(sI-A)^{-1}B + D = \begin{pmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$

$$\therefore G_C(s) = \begin{pmatrix} \frac{2}{(s+1)(s+3)}, & \frac{0}{(s+1)(s+3)(s+5)} \\ \frac{1}{s+3}, & \frac{0}{(s+1)(s+3)(s+5)} \end{pmatrix}$$

$\therefore$ The poles of $G_{c11}(s)$ are $s=-1$, $s=-3$; $G_{c12}(s)$ are $s=-1$, $s=-3$, $s=-5$

$G_{c21}(s)$ is $s=-3$; $G_{c22}(s)$ are $s=-1$, $s=-3$, $s=-5$.

$\because$ They all have negative real part.

$\therefore$ This system is BIBO stable.

Exercise 7:

1. $\because u_1 = T_{ci}$, $u_2 = T_{Hi}$, $y_1 = T_C$, $y_2 = T_H$, $f_C = f_H = 0.1\ (m^3/min)$

$\beta = 0.2\ (m^3/min)$ $V_H = V_C = 1\ m^3$.

$$\therefore \frac{dy_1}{dt} = 0.1(u_1 - y_1) + 0.2(y_2 - y_1) \quad (1).$$

$$\frac{dy_2}{dt} = 0.1(u_2 - y_2) + 0.2(y_2 - y_1) \quad (2).$$

Here we use $x_1 = y_1 = T_C$ $x_2 = y_2 = T_H$ as state space variables

$$\therefore \dot{x}_1 = 0.1(u_1 - x_1) + 0.2(x_2 - x_1) = -0.3x_1 + 0.2x_2 + 0.1u_1$$

$$\dot{x}_2 = 0.1(u_2 - x_2) + 0.2(x_2 - x_1) = -0.2x_1 + 0.1x_2 + 0.1u_2.$$

$$\therefore \dot{X} = \begin{pmatrix} -0.3 & 0.2 \\ -0.2 & 0.1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Above is the state space and output equations for this system.

2. Since absence of any input, we can get that.

$$x(t) = e^{A(t-t_0)}x(t_0)$$

$$y(t) = Ce^{A(t-t_0)}x(t_0).$$

$\because A = \begin{pmatrix} -0.3 & 0.2 \\ -0.2 & 0.1 \end{pmatrix}$ $\therefore \det(\lambda I - A) = 0 \Rightarrow \lambda_1 = \lambda_2 = -0.1.$

$$\therefore \left.\begin{aligned} e^{\lambda t} &= \beta_1 \lambda + \beta_0 \\ te^{\lambda t} &= \beta_1. \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} e^{-0.1t} &= -0.1\beta_1 + \beta_0 \\ te^{-0.1t} &= \beta_1. \end{aligned}\right.$$

$$\therefore \beta_1 = te^{-0.1t} \qquad \beta_0 = (1+0.1t)e^{-0.1t}.$$

$$\therefore e^{At} = te^{-0.1t}A + (1+0.1t)e^{-0.1t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1-0.2t)e^{-0.1t}, & 0.2te^{-0.1t} \\ -0.2te^{-0.1t}, & (1+0.2t)e^{-0.1t} \end{pmatrix}$$

Let $t_0 = 0$, we have

$$y(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{At} x(0) = e^{At} y(0) = \begin{pmatrix} (1-0.2t)e^{-0.1t}, & 0.2t e^{-0.1t} \\ -0.2t\, e^{-0.1t}, & (1+0.2t)e^{-0.1t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$$

$$\therefore y_1(t) = (1-0.2t)e^{-0.1t} y_1(0) + 0.2t e^{-0.1t} y_2(0).$$

$$y_2(t) = -0.2t\, e^{-0.1t} y_1(0) + (1+0.2t)e^{-0.1t} y_2(0).$$

Here $y_1(0)$ and $y_2(0)$ are the initial values of y_1 and y_2.

3. $\because$ This system is a CT LTI system and we have $A = \begin{pmatrix} -0.3 & 0.2 \\ -0.2 & 0.1 \end{pmatrix}$

$B = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $D = 0$

$$\therefore G_c(s) = C(sI - A)^{-1} B + D$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s+0.3 & -0.2 \\ 0.2 & s-0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{10s-1}{(10s+1)^2}, & \frac{2}{(10s+1)^2} \\ \frac{-2}{(10s+1)^2}, & \frac{10s+3}{(10s+1)^2} \end{pmatrix}$$

$\therefore$ The poles of $G_{c11}(s)$ are $s_1 = s_2 = -0.1$

$G_{c12}(s)$ are $s_1 = s_2 = -0.1$

$G_{c21}(s)$ are $s_1 = s_2 = -0.1$

$G_{c22}(s)$ are $s_1 = s_2 = -0.1$

$\because$ They all have negative real part.

$\therefore$ This system is BIBO stable.