Exercise 1:

1. This is a linear and time invariant system.

Let $u_3(t) = \alpha u_1(t) + \beta u_2(t)$ where α and β are constant.

Then we have $y_3(t) = 0 = y_2(t) = y_1(t)$

$\therefore y_3(t) = \alpha y_1(t) + \beta y_2(t)$

$\therefore$ This is a linear system.

Let $u_2(t) = u_1(t-\tau)$. where τ is constant.

then $y_2(t) = 0 = y_1(t) = y_1(t-\tau)$

$\therefore$ This is a time invariant system.

2. This is a non-linear and time-invariant system.

Let $u_3(t) = \alpha u_1(t) + \beta u_2(t)$. where α and β are constant.

Then we have $y_3(t) = u_3^3(t) = (\alpha u_1(t) + \beta u_2(t))^3 \neq \alpha u_1^3(t) + \beta u_2^3(t) = \alpha y_1(t) + \beta y_2(t)$

$\therefore$ This is a non-linear system.

Let $u_2(t) = u_1(t-\tau)$ where τ is constant.

then $y_2(t) = u_2^3(t) = u_1^3(t-\tau) = y_1(t-\tau)$.

$\therefore$ This is a time invariant system.

3. This is a linear and time varying system.

Let $u_3(t) = \alpha u_1(t) + \beta u_2(t)$ where α and β are constant.

Then $u_3(3t) = \alpha u_1(3t) + \beta u_2(3t)$.

$y_3(t) = u_3(3t) = \alpha u_1(3t) + \beta u_2(3t) = \alpha y_1(t) + \beta y_2(t)$.

$\therefore$ This is a linear system.

Let $u_2(t) = u_1(t-\tau)$ where τ is constant.

Then $u_2(3t) = u_1(3t-\tau)$

$y_2(t) = u_2(3t) = u_1(3t-\tau) \neq y_1(t-\tau) = u_1(3t-3\tau)$.

$\therefore$ This is a time varying system.

4. This is a linear and time varying system.

Let $u_3(t) = \alpha u_1(t) + \beta u_2(t)$

$\therefore\ y_3(t) = e^{-t} u_3(t-T) = e^{-t}\alpha\, u_1(t-T) + e^{-t}\beta\, u_2(t-T)$

$= \alpha\, y_1(t) + \beta\, y_2(t)$

$\therefore$ This is a linear system.

Let $u_2(t) = u_1(t-\tau)$

$\therefore\ y_2(t) = e^{-t} u_2(t-T) = e^{-t} u_1(t-\tau-T)$.

Since $y_1(t-\tau) = e^{-t+\tau} u_1(t-\tau-T)$.

$\therefore\ y_2(t) \neq y_1(t-\tau)$

$\therefore$ This is a time varying system.

5. This is a linear and time varying system.

When $t \leq 0$, we have already known this is a linear system.

when $t \geq 0$ Let $u_3(t) = \alpha u_1(t) + \beta u_2(t)$

$\therefore\ y_3(t) = u_3(t) = \alpha u_1(t) + \beta u_2(t) = \alpha\, y_1(t) + \beta\, y_2(t)$

$\therefore$ when $t \geq 0$, this is a linear system.

Above all, this is a linear system.

As for time varying, considering $u_2(t) = u_1(t-\tau)$ where $t-\tau \leq 0$ and $t \geq 0$

We have $y_2(t) = u_2(t) = u_1(t-\tau)$.

Since $t-\tau \leq 0$, $y_1(t) = 0$

$\therefore\ y_2(t) \neq y_1(t)$

$\therefore$ This is a time varying system.

Exercise 2:

1. It is easy to know $S_1(k) = G_{11} P_1(k)$.

$q_1(k) = \sigma^2 + G_{12} P_2(k) + G_{13} P_3(k)$.

$$\therefore S_1(k) = \frac{S_1(k)}{q_1(k)} = \frac{G_{11} P_1(k)}{\sigma^2 + G_{12} P_2(k) + G_{13} P_3(k)}$$

$$\therefore P_1(k+1) = P_1(k) \frac{\alpha\gamma}{S_1(k)} = \frac{\alpha\gamma(\sigma^2 + G_{12} P_2(k) + G_{13} P_3(k))}{G_{11}}.$$

Simularly, we have

$$P_2(k+1) = \alpha\gamma(\sigma^2 + G_{21} P_1(k) + G_{23} P_3(k)) / G_{22}$$

$$P_3(k+1) = \alpha\gamma(\sigma^2 + G_{31} P_1(k) + G_{32} P_2(k)) / G_{33}.$$

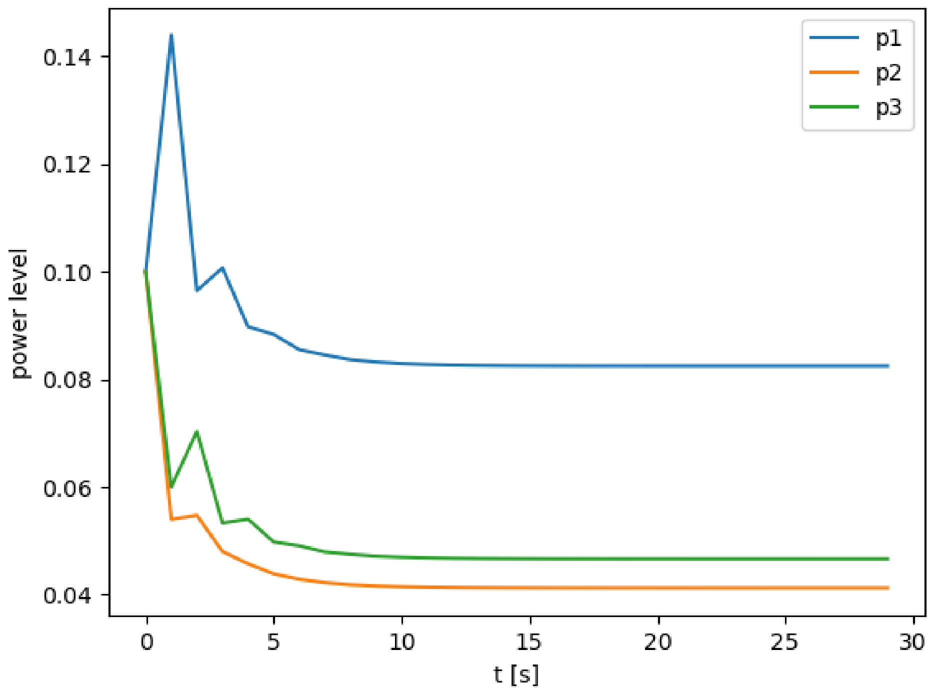
$$\therefore P(k+1) = \begin{bmatrix} 0, & \alpha\gamma\frac{G_{12}}{G_{11}}, & \alpha\gamma\frac{G_{13}}{G_{11}} \\ \alpha\gamma\frac{G_{21}}{G_{22}}, & 0, & \alpha\gamma\frac{G_{23}}{G_{22}} \\ \alpha\gamma\frac{G_{31}}{G_{33}}, & \alpha\gamma\frac{G_{32}}{G_{33}}, & 0 \end{bmatrix} P(k) + \begin{bmatrix} \frac{\alpha\gamma}{G_{11}} \\ \frac{\alpha\gamma}{G_{22}} \\ \frac{\alpha\gamma}{G_{33}} \end{bmatrix} \sigma^2$$

$$\therefore A = \begin{bmatrix} 0, & \alpha\gamma\frac{G_{12}}{G_{11}}, & \alpha\gamma\frac{G_{13}}{G_{11}} \\ \alpha\gamma\frac{G_{21}}{G_{22}}, & 0, & \alpha\gamma\frac{G_{23}}{G_{22}} \\ \alpha\gamma\frac{G_{31}}{G_{33}}, & \alpha\gamma\frac{G_{32}}{G_{33}}, & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\alpha\gamma}{G_{11}} \\ \frac{\alpha\gamma}{G_{22}} \\ \frac{\alpha\gamma}{G_{33}} \end{bmatrix}$$

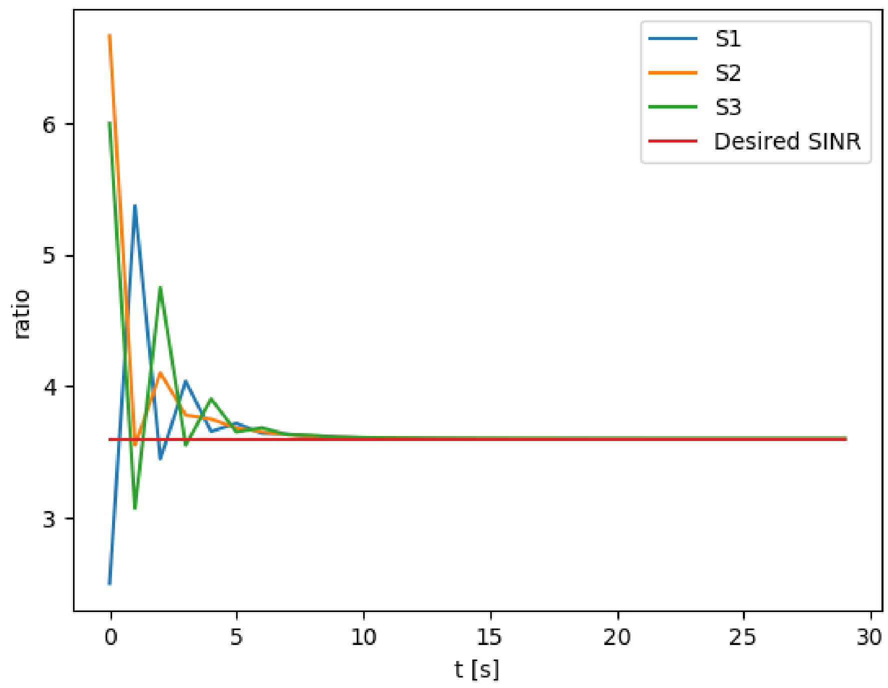
Exercise 2

2: (1) When γ =3 and initial condition is $p_1 = p_2 = p_3 = 0.1$.

This is the plot of p1, p2 and p3 with t



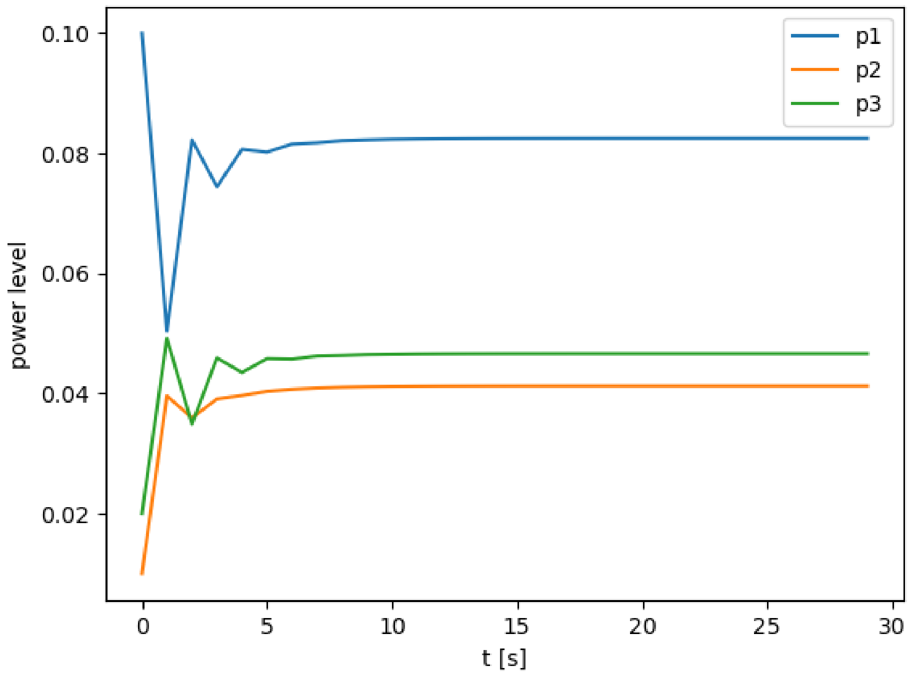
This is the plot of S1, S2, S3 and desired SINR with t



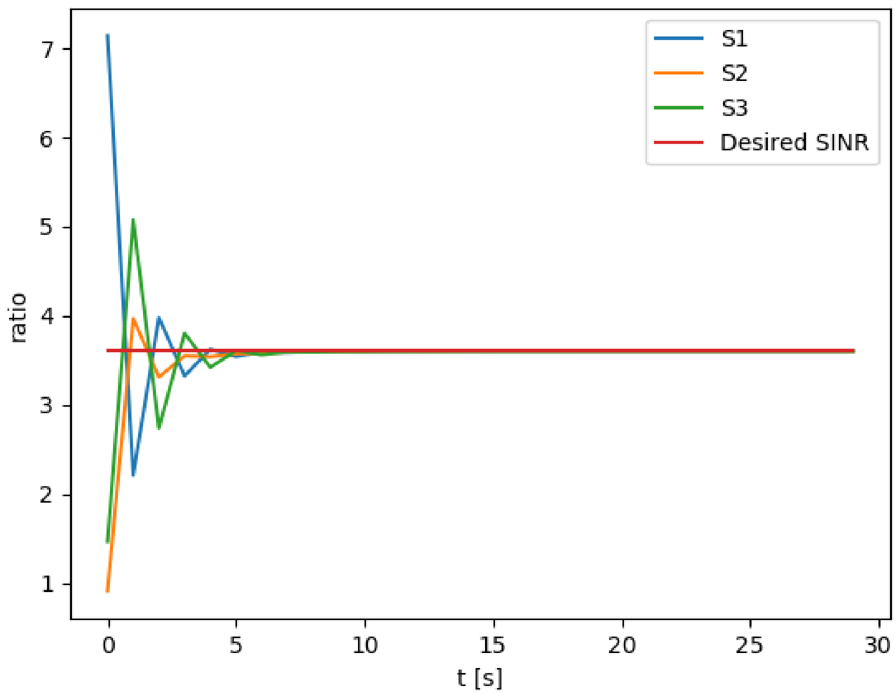
As for the final values of S1, S2 and S3, they are really close to the Desired SINR. So, I think the controller achieve the goal to force $S_i(t) \to \alpha\gamma$.

(2) When γ =3 and initial condition is p_1 =0.1, p_2 =0.01, p_3 = 0.02

This is the plot of p1, p2 and p3 with t

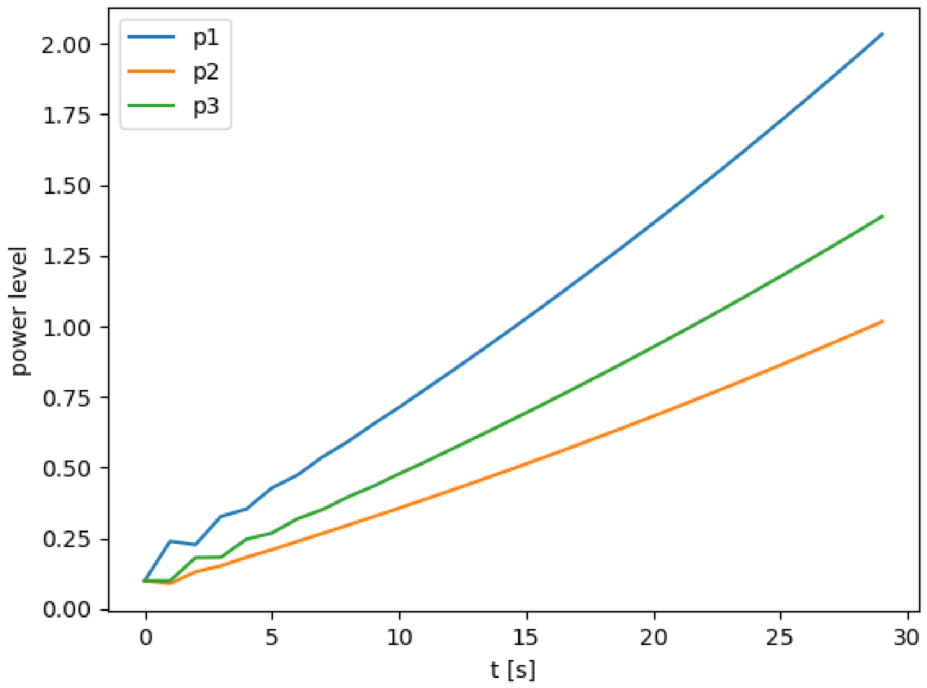


This is the plot of S1, S2, S3 and desired SINR with t

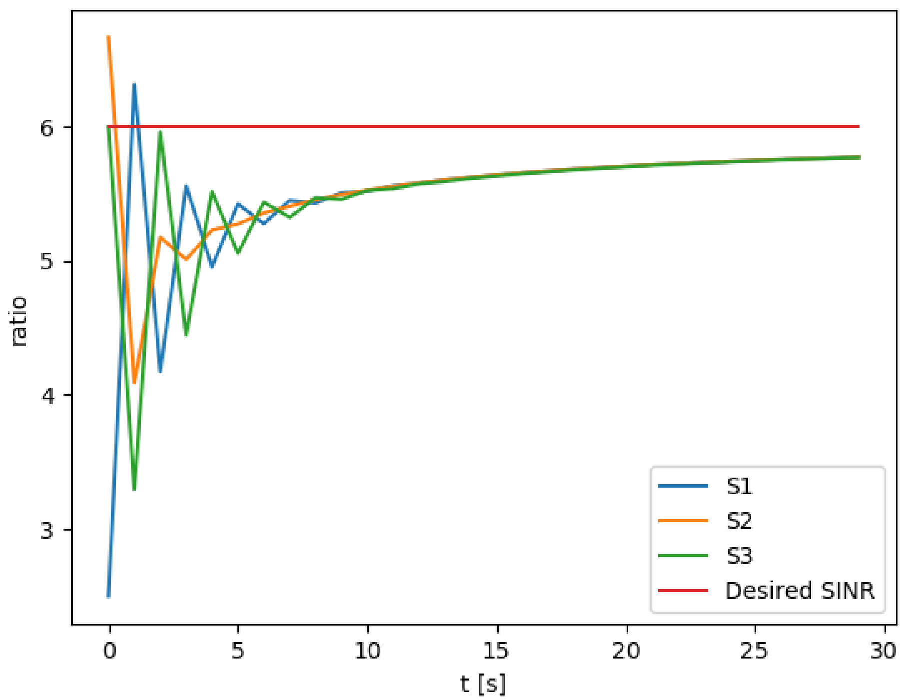


As for the final values of S1, S2 and S3, they are really close to the Desired SINR. So, I think the controller achieve the goal to force $S_i(t) \to \alpha\gamma$.

(3) When $\gamma = 5$ and initial condition is $p_1 = p_2 = p_3 = 0.1$.
This is the plot of p1, p2 and p3 with t

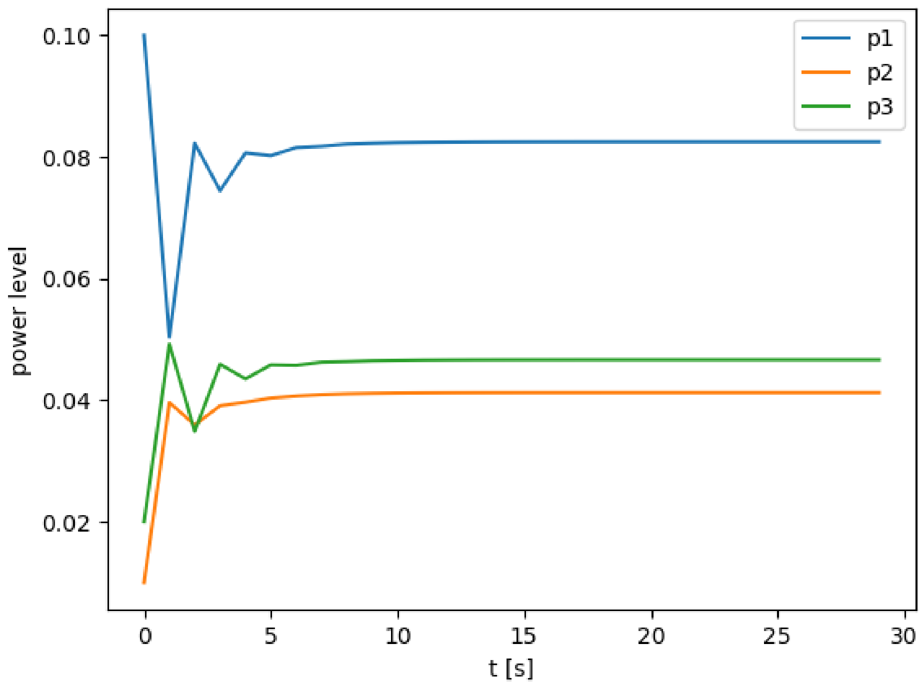


This is the plot of S1, S2, S3 and desired SINR with t

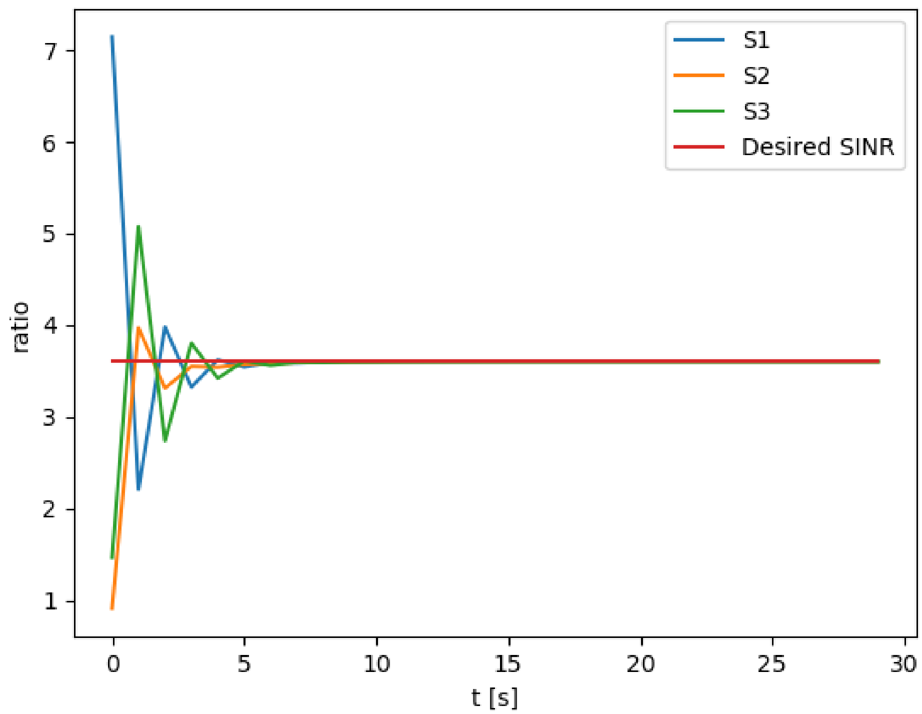


As for the final values of S1, S2 and S3, they are close to the Desired SINR. So, I think the controller achieve the goal to force $S_i(t) \to \alpha\gamma$.

(4) When γ =5 and initial condition is p_1 =0.1, p_2 =0.01, p_3 = 0.02
This is the plot of p1, p2 and p3 with t



This is the plot of S1, S2, S3 and desired SINR with t



As for the final values of S1, S2 and S3, they are really close to the Desired SINR. So, I think the controller achieve the goal to force $S_i(t) \rightarrow \alpha\gamma$.

Exercise 3.

$\because \ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 = 0$

$\therefore \ddot{y} = -(1+y)\dot{y} + (2-0.5y^2)y$

$\therefore \dot{Y} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -(1+y)\dot{y} + (2-0.5y^2)y \end{bmatrix}$

let $\dot{Y} = 0$, $\therefore \dot{y} = 0$ and $(2-0.5y^2)\cdot y = 0$.

$\therefore y = \pm 2$ or $y = 0$.

When $\dot{y} = 0$, $y = 0$. $\ddot{y} = (-1-y)|_{y=0,\ \dot{y}=0}\ \dot{y} + (-\dot{y} + 2 - 1.5y^2)|_{y=0,\ \dot{y}=0}\ y$

$= -\dot{y} + 2y$.

when $\dot{y} = 0$, $y = 2$ $\ddot{y} = -3\dot{y} - 4y$

When $\dot{y} = 0$, $y = -2$ $\ddot{y} = \dot{y} - 4y$.

Exercise 4:

$$\because \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g(\frac{D}{x_1(t)+D})^2 + \frac{\ln(u)}{m} \end{bmatrix}$$

let $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = 0$.

$\therefore \quad x_2(t) = 0$

$$(\frac{D}{x_1(t)+D})^2 g = \frac{\ln(u)}{m}$$

$$x_1(t) = \sqrt{\frac{mg}{\ln(u)}} D - D \quad \text{or} \quad x_1(t) = -(\sqrt{\frac{mg}{\ln(u)}} D + D).$$

$\therefore$ the equilibrium state (x_1^*, x_2^*) are.

$$(\sqrt{\frac{mg}{\ln(u)}} D - D, 0) \quad , \quad (-(\sqrt{\frac{mg}{\ln(u)}} D + D), 0).$$

let $f_1 = x_2(t) \qquad f_2 = -g(\frac{D}{x_1(t)+D})^2 + \frac{\ln(u)}{m}$

$$\therefore \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{x_1^*, x_2^*} & \frac{\partial f_1}{\partial x_2}\Big|_{x_1^*, x_2^*} \\ \frac{\partial f_2}{\partial x_1}\Big|_{x_1^*, x_2^*} & \frac{\partial f_2}{\partial x_2}\Big|_{x_1^*, x_2^*} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0\big|_{x_1^*, x_2^*} & 1\big|_{x_1^*, x_2^*} \\ \frac{2gD^2}{(x_1(t)+D)^3}\Big|_{x_1^*, x_2^*} & 0\big|_{x_1^*, x_2^*} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$\therefore$ At $(\sqrt{\frac{mg}{\ln(u)}} D - D, 0)$ $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{2g}{D\cdot(\frac{mg}{\ln(u)})^{\frac{3}{2}}} x_1(t) \end{bmatrix}$

At $(-(\sqrt{\frac{mg}{\ln(u)}} D + D), 0)$ $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{2g}{D(\frac{mg}{\ln(u)})^{\frac{3}{2}}} x_1(t) \end{bmatrix}$

Exercise 5:

1. $\because$ $r(t) \equiv p$ and $\theta(t) = wt$, $u_1 = 0$, $u_2 = 0$

$\therefore$ We have $\ddot{r}(t) = 0 = \dot{r}(t)$ $\ddot{\theta}(t) = 0$ $\dot{\theta}(t) = w$.

$$\therefore \quad 0 = p w^2 - \frac{k}{p^2}$$

$$\therefore \quad k = p^3 w^2 .$$

2. Since when $u_1 = u_2 = 0$, we have $r(t) \equiv p$ $\theta(t) = wt$

$\therefore$ We can do linearization at point $u_1 = 0 = u_2$, $r = p$ $\dot{\theta} = w$. $\dot{r} = 0$

Let $f_1(u_1, u_2, \dot{\theta}, r, \dot{r}) = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$

$f_2(u_1, u_2, \dot{\theta}, r, \dot{r}) = -2\frac{\dot{\theta}}{r}\dot{r} + \frac{u_2}{r}$

$$\therefore \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}, & \frac{\partial f_1}{\partial u_2}, & \frac{\partial f_1}{\partial \dot{\theta}}, & \frac{\partial f_1}{\partial r}, & \frac{\partial f_1}{\partial \dot{r}} \\ \frac{\partial f_2}{\partial u_1}, & \frac{\partial f_2}{\partial u_2}, & \frac{\partial f_2}{\partial \dot{\theta}}, & \frac{\partial f_2}{\partial r}, & \frac{\partial f_2}{\partial \dot{r}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dot{\theta} \\ r \\ \dot{r} \end{bmatrix}$$

$$= \begin{bmatrix} 1, & 0, & 2r\dot{\theta}, & \dot{\theta}^2 + \frac{2k}{r^3}, & 0 \\ 0, & \frac{1}{r}, & \frac{-2\dot{r}}{r}, & \frac{2\dot{\theta}\dot{r}}{r^2} - \frac{u_2}{r^2}, & \frac{-2\dot{\theta}}{r} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dot{\theta} \\ r \\ \dot{r} \end{bmatrix}$$

Let $u_1 = u_2 = 0$ $\dot{\theta} = w$ $r = p$ $\dot{r} = 0$, we can get

$$\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1, & 0, & 2pw, & w^2 + \frac{2k}{p^3}, & 0 \\ 0, & \frac{1}{p}, & 0, & 0, & -\frac{2w}{p} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dot{\theta} \\ r \\ \dot{r} \end{bmatrix}$$

$\because k = p^3 w^2$

$$\therefore \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u_1 + 2pw\dot{\theta} + 3w^2 r \\ \frac{u_2}{p} - \frac{2w}{p}\dot{r} \end{bmatrix}$$