Question 1.

(a). True

(b). False

(c) False

(d) True

(e). False.

Question 2:

(a): $\because$ $\dot{x}_1 = (r_1 - x_2)x_1$
$\dot{x}_2 = (r_2 - x_3)x_2$ and we have $x_1 = \bar{x}_1$
$\dot{x}_3 = u$

$\therefore$ let $\dot{x}_1 = 0 \Rightarrow x_2 = r_1$
let $\dot{x}_2 = 0 \Rightarrow x_3 = r_2$
let $\dot{x}_3 = 0 \Rightarrow u = 0$

$\therefore$ The equilibrium point is $x_1 = \bar{x}_1$, $x_2 = r_1$, $x_3 = r_2$, $u = 0$.

(b) $\because$ $\dot{x}_1 = (r_1 - x_2)x_1 = f_1(x_1, x_2, x_3, u)$

$\therefore$ $\frac{\partial f_1}{\partial x_1} = r_1 - x_2 = 0$ $\quad \frac{\partial f_1}{\partial x_2} = -x_1 = -\bar{x}_1$ $\quad \frac{\partial f_1}{\partial x_3} = \frac{\partial f_1}{\partial u} = 0.$

$\dot{x}_2 = (r_2 - x_3)x_2 = f_2(x_1, x_2, x_3, u)$

$\therefore$ $\frac{\partial f_2}{\partial x_1} = 0.$ $\quad \frac{\partial f_2}{\partial x_2} = r_2 - x_3 = 0$ $\quad \frac{\partial f_2}{\partial x_3} = -x_2 = -r_1$, $\quad \frac{\partial f_2}{\partial u} = 0.$

$\therefore$ The linearize model at the equilibrium point $x_1 = \bar{x}_1$, $x_2 = r_1$, $x_3 = r_2$; $u=0$

is:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\bar{x}_1 & 0 \\ 0 & 0 & -r_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c): $G(s) = C(sI - A)^{-1}B + D$ Here we have $C = (0\ 1\ 0)$ $A = \begin{pmatrix} 0 & -\bar{x}_1 & 0 \\ 0 & 0 & -r_1 \\ 0 & 0 & 0 \end{pmatrix}$
$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $D = 0$

$$\therefore G(s) = (0\ 1\ 0) \begin{pmatrix} s & \bar{x}_1 & 0 \\ 0 & s & r_1 \\ 0 & 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\because \begin{pmatrix} 1 & \bar{x}_1 & 0 \\ 0 & 1 & r_1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s}, & \frac{-\bar{x}_1}{s^2}, & \frac{\bar{x}_1 r_1}{s^3} \\ 0 & \frac{1}{s}, & -\frac{r_1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{pmatrix} \quad \therefore G(s) = \left(0, \frac{1}{s}, -\frac{r_1}{s^2}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= -\frac{r_1}{s^2}$$

$\therefore$ The TF of the Linearized state model from (b) is $-\frac{r_1}{s^2}$

Question 3:

Obviously, this is a Linear Time Invariant system.

$\therefore \quad X(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(d\tau)$

$\because \quad u(t) = 0$

$\therefore \quad x(t) = e^{A(t-t_0)} x(t_0).$

Now, let's compute e^{At}.

let $\det(\lambda I - A) = 0$, we have $\begin{vmatrix} \lambda & 0 \\ -2 & \lambda \end{vmatrix} = \lambda^2 = 0.$

$\therefore \lambda_1 = \lambda_2 = 0$

let $e^{\lambda t} = \beta_1 \lambda + \beta_0 \Rightarrow t e^{\lambda t} \doteq \beta_1$

$\therefore \quad \begin{cases} 1 = \beta_0 \\ t = \beta_1 \end{cases}$

$\therefore \quad e^{At} = \beta_1 A + \beta_0 I = \begin{bmatrix} 0 & 0 \\ 2t & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix}$

$\because \quad x(t) = e^{A(t-t_0)} x(t_0)$

$\therefore$ let $t_0 = 0$, $t=2$ we have $\quad x(2) = e^{2A} x(0)$

$\because \quad e^{2A} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$\therefore \quad x(2) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} x(0)$

$\therefore \quad x(0) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} x(2) = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\therefore \quad X(0) = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ when $u(t) = 0$ and $x(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Question 4:

Obviously, this is a continuous time, time invariant system.

$\because A = \begin{pmatrix} -20 & 20 \\ -20 & 20 \end{pmatrix}$ $\therefore$ let $\det(\lambda I - A) = 0$ we have

$$\begin{vmatrix} \lambda + 20 & -20 \\ 20 & \lambda - 20 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 400 + 400 = 0 \quad \therefore \lambda_1 = \lambda_2 = 0$$

$\because \text{rank}(\lambda I - A) = \text{rank}\left(\begin{pmatrix} 20 & -20 \\ 20 & -20 \end{pmatrix}\right) = 1$

$\therefore$ The jordan form of A will be $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Now, let's compute e^{Jt}.

let $\det(\lambda I - J) = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^2 = 0$

$\therefore \lambda_1 = 0 \quad \lambda_2 = 0$

$$\therefore \left.\begin{array}{l} e^{\lambda t} = \beta_1 \lambda + \beta_0 \\ t e^{\lambda t} = \beta_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \beta_0 = 1 \\ \beta_1 = t \end{array}\right.$$

$$\therefore e^{Jt} = \beta_1 J + \beta_0 I = t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t}, & te^{\lambda_1 t} \\ 0, & e^{\lambda_2 t} \end{pmatrix}$$

Let's consider $t^m e^{Re t}(\cos(Im t) + j\sin(Im t))$ for λ_1 and λ_2.

$\because$ For λ_1, we have $Re = 0$ and $m = 1 > 0$

$\therefore$ This balloon system is not Lyapunov stable or asymptotic stable. It is unstable.

Question 5:

(a): First, let's check the stability of this system.

$\because V(x) = x_1^2 + x_2^2.$

$\therefore \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$\because \dot{x}_1 = -\frac{x_2}{1+x_1^2} - 2x_1, \quad \dot{x}_2 = \frac{x_1}{1+x_1^2}$

$\therefore \dot{V}(x) = -\frac{2x_1x_2}{1+x_1^2} - 4x_1^2 + \frac{2x_1x_2}{1+x_1^2} = -4x_1^2$

Obviously. $V(x)$ and its partial derivatives are continuous and $V(x)$ is positive definite.

$\because \dot{V}(x) = -4x_1^2$

$\therefore \dot{V}(x)$ is negative. definite

$\therefore$ This system is asymptotically stable.

$\therefore$ This system is asymptotically stable at the equilibrium point.

(b): Let's find the equilibrium point.

Let $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Then we have $-\frac{x_2}{1+x_1^2} - 2x_1 = 0, \ \frac{x_1}{1+x_1^2} = 0$

$\therefore x_1 = 0$ and $x_2 = 0$ $\therefore$ The equilibrium point is $x_1 = 0, x_2 = 0$

Let $f_1(x_1, x_2) = -\frac{x_2}{1+x_1^2} - 2x_1 \qquad f_2(x_1, x_2) = \frac{x_1}{1+x_1^2}$

$\therefore \frac{\partial f_1}{\partial x_1}\Big|_{x_1=0,\, x_2=0} = -2 + \frac{x_2}{(1+x_1^2)^2} \cdot 2x_1\Big|_{x_1=0, x_2=0} = -2.$

$\frac{\partial f_1}{\partial x_2}\Big|_{x_1=0,\, x_2=0} = -\frac{1}{1+x_1^2}\Big|_{\substack{x_1=0\\x_2=0}} = -1.$

$\frac{\partial f_2}{\partial x_1} = \frac{(1+x_1^2) - x_1 \cdot 2x_1}{(1+x_1^2)^2}\Big|_{\substack{x_1=0\\x_2=0}} = \frac{1-x_1^2}{(1+x_1^2)^2}\Big|_{\substack{x_1=0\\x_2=0}} = 1.$

$\frac{\partial f_2}{\partial x_2} = 0$

$\therefore$ The Linearized system at equilibrium point $x_1=0, x_2=0$ is.

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-2 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

For the linearized system, we need to find the e^{Jt} of this system to check the stability.

$\because A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$ let $\det(\lambda I - A) = 0$ we have

$$\begin{vmatrix} \lambda+2 & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0 \quad \therefore \lambda_1 = \lambda_2 = -1.$$

$\because$ $\text{rank}(A-(-I)) = \text{rank}(A+I) = \text{rank}\left(\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right) = 1$

$\therefore$ The Jordan form of A is $J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

Now, let's compute e^{Jt}.

let $\det(\lambda I - J) = 0$, we have $\begin{vmatrix} \lambda+1 & -1 \\ 0 & \lambda+1 \end{vmatrix} = (\lambda+1)^2 = 0$

$\therefore \lambda_1 = \lambda_2 = -1$

$$\left.\begin{array}{l} e^{\lambda t} = \beta_1 \lambda + \beta_0 \\ t e^{\lambda t} = \beta_1 \end{array}\right\} \Rightarrow \begin{cases} e^{-t} = -\beta_1 + \beta_0 \\ \beta_1 = t e^{-t} \end{cases} \Rightarrow \begin{cases} \beta_1 = t e^{-t} \\ \beta_0 = (t+1) e^{-t} \end{cases}$$

$$\therefore e^{Jt} = \beta_1 J + \beta_0 I = \begin{pmatrix} -te^{-t} & te^{-t} \\ 0 & -te^{-t} \end{pmatrix} + \begin{pmatrix} (t+1)e^{-t} & 0 \\ 0 & (t+1)e^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

$\because$ For λ_1 and λ_2, we have $\text{Re} = -1 < 0$

$\therefore$ The linearized system is Asymptotic stable. and the original system is locally Asymptotic stable at the equilibrium point. $x_1 = 0,\ x_2 = 0$

Question 6:

First, let's find a state space representation of this system.

$\because G(s) = \begin{bmatrix} \frac{-2s-20}{s+11} \\ \frac{s+20}{3s+33} \end{bmatrix} \quad \therefore G(\infty) = \begin{bmatrix} -2 \\ \frac{1}{3} \end{bmatrix}$

$\therefore G_{sp} = G(s) - G(\infty) = \begin{pmatrix} \frac{2}{s+11} \\ \frac{9}{3(s+11)} \end{pmatrix} = \begin{pmatrix} \frac{2}{s+11} \\ \frac{3}{s+11} \end{pmatrix}$

$\therefore d(s) = s+11$

$\therefore G_{sp} = \frac{1}{s+11} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \therefore N_1(s) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad p=1. \Rightarrow I_p = 1$

$\therefore A = (-11) \qquad B = (1)$

$C = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad D = \begin{pmatrix} -2 \\ \frac{1}{3} \end{pmatrix}.$

$\therefore$ A state space representation for this system is

$\dot{x} = -11x + u$

$y = \begin{pmatrix} 2 \\ 3 \end{pmatrix} x + \begin{pmatrix} -2 \\ \frac{1}{3} \end{pmatrix} u$

Now, let's check the controllability and observability of this representation

$\because n = 1 \quad \therefore P = B = 1 \quad Q = C = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Obviously $\text{rank}(P) = 1 = n \quad \text{rank}(Q) = 1 = n$

$\therefore$ This representation is controllable and observable.

$\therefore$ The minimal realization for $G(s) = \begin{bmatrix} \frac{-2s-20}{s+11} \\ \frac{s+20}{3s+33} \end{bmatrix}$ is

$\dot{x} = -11x + u$

$y = \begin{pmatrix} 2 \\ 3 \end{pmatrix} x + \begin{pmatrix} -2 \\ \frac{1}{3} \end{pmatrix} u$

Question 7:

(a): Let's calculate the eigenvalues of A.

$\because A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\therefore \det(\lambda I - A) = \lambda(\lambda+10)(\lambda^2+\lambda+1)$

$\therefore \lambda_1 = 0 \quad \lambda_2 = -10, \quad \lambda_3 = \frac{-1+\sqrt{3}i}{2} \quad \lambda_4 = \frac{-1-\sqrt{3}i}{2}$.

Since this is a linearized model of the aircraft system, and we hav

$Re(\lambda_1) = 0$

$\therefore$ We don't know whether the original system (the aircraft) is locally asymptotically stable or not, locally stable or not.

(b): With just δr, $B = \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix}$

$\therefore P = (B, AB, A^2B, A^3B) = \begin{pmatrix} 0 & 0 & -1 & 11 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\therefore \text{rank}(P) = 4 = n$

$\therefore$ The aircraft is controllable with just δr.

(c): With just δa, $B = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore P = (B, AB, A^2B, A^3B) = \begin{pmatrix} 10 & -100 & 1000 & -10000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & -100 & 1000 \end{pmatrix}$

$\therefore \text{rank}(P) = 2 < 4 = n$

$\therefore$ It is impossible to control the aircraft using only the rudder angle δa.

(d): Obviously, $C = (1, 0, 0, 0)$

$\therefore Q = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \\ -1000 & -11 & -99 & 0 \end{pmatrix}$ $\quad \therefore \text{rank}(Q) = 3 < 4 = n$

$\therefore$ This system is not observable.