Exercise 1:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C = (1\ 2\ 1) \quad n=3.$$

Controllability

$\therefore\ P = (B,\ AB,\ A^2B)$

$$AB = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$A^2B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore\ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \quad \therefore\ rank(P) = 3 = n$$

$\therefore$ This system is controllable.

Observability

$$Q = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

$$CA = (1\ 2\ 1)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} = (-1\ -2\ -1)$$

$$CA^2 = (1\ 2\ 1)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} = (1\ 2\ 1)$$

$$\therefore\ Q = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \therefore\ rank(Q) = 1 < 3 = n$$

$\therefore$ The system is not observable.

Exercise 2:

Controllability:

It is easy to know here we have two distinct λ_i, $\lambda_1=2$, $\lambda_2=1$.

$\hat{B}^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ $\therefore$ $rank(\hat{B}^2) = 3$.

$\hat{B}^1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\therefore$ $rank(\hat{B}^1) = 2$.

$\therefore$ modes $\lambda = 2, 1$ are controllable.

Observability

$\hat{C}^2 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 0 \end{pmatrix}$ $\therefore$ $rank(\hat{C}^2) = 2$ not observable

$\hat{C}^1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\therefore$ $rank(\hat{C}^1) = 2$

mode: $\lambda = 2$ not observable, mode $\lambda = 1$ observable.

Exercise: 3:

In Homework 2 exercise 2, we have.

$$\dot{x} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$\Rightarrow \dot{x} = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$\therefore A = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n = 2$$

$$\therefore P = (B, AB) = \begin{pmatrix} 1 & -0.1 \\ 0 & 0.1 \end{pmatrix} \Rightarrow \text{rank}(P) = 2 = n$$

$\therefore$ The system is controllable.

For the new system, we have

$$\dot{x} = \begin{pmatrix} -\alpha & 0 \\ \alpha & -\beta \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$\dot{x} = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$A = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad n = 2$$

$$\therefore P = (B, AB) = \begin{pmatrix} 0 & 0 \\ 1 & -0.2 \end{pmatrix} \Rightarrow \text{rank}(P) = 1 < 2 = n$$

$\therefore$ The system is not controllable.

I think the reason is, that we put inlet pipe above tank 1 in the old system. If we change the input u, we can control the amount of water in tank 1 and 2. But in the new system, the inlet pipe can only control the amount of water in tank 2. It is impossible to control the water in tank 1. That's why the old system is controllable but the new one is not.

Exercise 4:

1. a: $A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 6 \end{pmatrix}$

$\therefore -3z = 6 \Rightarrow z = -2$

$y + 2z = -3 \Rightarrow y = 1$

$x + y + z = 3 \Rightarrow x = 4.$

$\therefore$ the solution is $\begin{cases} x = 4 \\ y = 1 \\ z = -2. \end{cases}$

b: $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

$\therefore z = 2$

$y + z = 1 \Rightarrow y = -1$

$x + 2y - z = 1 \Rightarrow x = 5$

$\therefore$ the solution is $\begin{cases} x = 5 \\ y = -1 \\ z = 2. \end{cases}$

c: $A = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 1 & 3 & -2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

$\because 0 \cdot x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \neq -1$

$\therefore$ This system of linear equations doesn't have a solution.

2: Here we have $Ax = y$ with $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix}$ $y = \begin{pmatrix} 3 \\ 13 \\ 14 \end{pmatrix}$.

First let $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

such that $A = LU$.

$\therefore$ we have $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix}$.

$\therefore$ $u_{11} = 1$ $\quad u_{12} = 2$ $\quad u_{13} = 4$.

$l_{21} = 3$ $\quad u_{22} = 2$ $\quad u_{23} = 2$

$l_{31} = 2$. $\quad l_{32} = 1$. $\quad u_{33} = 3$.

$\therefore$ $L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

Now, we need to solve $Lz = y$. Here $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

$\therefore$ $Lz = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ 3z_1+z_2 \\ 2z_1+z_2+z_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 13 \\ 14 \end{pmatrix}$

$\therefore$ $z_1 = 3$ $\quad z_2 = 4$ $\quad z_3 = 4$.

$\therefore$ $z = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$

Now let's solve $Ux = z$.

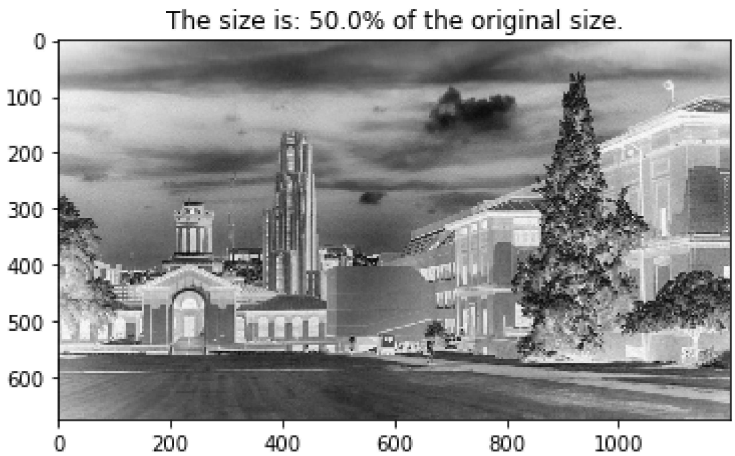
$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1+2x_2+4x_3 \\ 2x_2+2x_3 \\ 3x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$

$\therefore$ $x_3 = \frac{4}{3}$ $\quad x_2 = \frac{2}{3}$ $\quad x_1 = -\frac{11}{3}$

$\therefore$ the solution is $\begin{pmatrix} -\frac{11}{3} \\ \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$

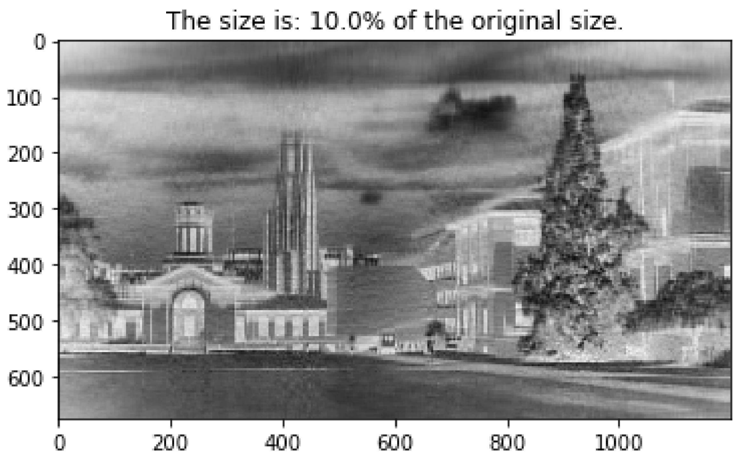
In [1]:

```
# This is the code I use for Exercise 5.
import numpy as np
import matplotlib.pyplot as plt
from PIL import Image
import math

def rebuild_img(u,sigma,v,num):
    m=len(u)
    n=len(v)
    a=np.zeros((m,n))

    k=0

    while k <= num:
        uk=u[:,k].reshape(m,1)
        vk=v[k].reshape(1,n)
        a+=sigma[k]*np.dot(uk,vk)
        k+=1


    return a.astype('float32')
# This is the code to compress the given image to 50% of its original size.
im = plt.imread("D:/CMU_Grayscale.png") # This is the place where I store the image.
rate = 0.5
num = (int) (im.shape[0] * im.shape[1] * rate / (im.shape[0] + im.shape[1]))

u,sigma,v=np.linalg.svd(im)
after_compressed = rebuild_img(u,sigma,v,num)

plt.title("The size is: " + str(rate * 100) + "% of the original size.")
plt.imshow(after_compressed, plt.cm.gray_r)
plt.show()
```



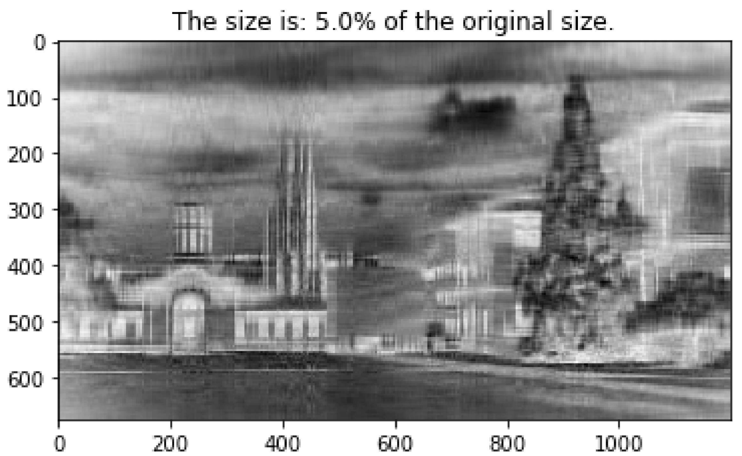
In [2]:

```
# This is the code to compress the given image to 10% of its original size.
im = plt.imread("D:/CMU_Grayscale.png") # This is the place where I store the image.
rate = 0.1
num = (int) (im.shape[0] * im.shape[1] * rate / (im.shape[0] + im.shape[1]))


u,sigma,v=np.linalg.svd(im)
after_compressed = rebuild_img(u,sigma,v,num)

plt.title("The size is: " + str(rate * 100) + "% of the original size.")
plt.imshow(after_compressed, plt.cm.gray_r)
plt.show()
```



In [3]:

```
# This is the code to compress the given image to 5% of its original size.
im = plt.imread("D:/CMU_Grayscale.png") # This is the place where I store the image.
rate = 0.05
num = (int) (im.shape[0] * im.shape[1] * rate / (im.shape[0] + im.shape[1]))

u,sigma,v=np.linalg.svd(im)
after_compressed = rebuild_img(u,sigma,v,num)

plt.title("The size is: " + str(rate * 100) + "% of the original size.")
plt.imshow(after_compressed, plt.cm.gray_r)
plt.show()
```



Exercise 6:

First, let's compute the P and Q for this system.

$$A = \begin{pmatrix} -3 & 3 \\ r & -4 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = (1, 1)$$

$$\therefore P = (B, AB) = \begin{pmatrix} 1 & -3 \\ 0 & r \end{pmatrix}$$

$$Q = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1, & 1 \\ r-3, & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2-r \end{pmatrix}$$

1: If we want to make this system controllable but not observable, then $\text{rank}(P) = 2 = n$ $\quad \text{rank}(Q) < 2 = n$

$\because \text{rank}(P) = 2 \Rightarrow r \neq 0. \quad \text{rank}(Q) < 2 \Rightarrow 2 - r = 0$

$\therefore r = 2.$

Let $r = 2$, then we have $P = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} \Rightarrow \text{rank}(P) = 2.$

$$Q = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(Q) = 1 < 2 = n$$

$\therefore$ When $r = 2$, this system is controllable but not observable.

2. If we want to make this system observable, but not controllable, then $\text{rank}(P) < 2 = n$, $\text{rank}(Q) = 2 = n$.

$\therefore \text{rank}(P) < 2 \Rightarrow r = 0 \qquad \text{rank}(Q) = 2 \Rightarrow 2 - r \neq 0 \Rightarrow r \neq 2$

$\therefore r = 0.$

Let $r = 0$, then we have $P = \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}$

It is easy to know that $\text{rank}(P) = 1 < 2 = n$

$\text{rank}(Q) = 2 = n$

$\therefore$ When $r = 0$, this system is observable but not controllable.

Exercise 7:

1. Let x_1 denote the left-most LED. then the state equations of this system can be written as below.

$$X(k+1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} X(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u(t).$$

$$X(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{pmatrix}$$

2: In the system above, we have $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\therefore P = (B, AB, A^2B, A^3B, A^4B)$

$AB = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $A^2B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $A^3B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $A^4B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ obviously $\text{rank}(P) = 5 = n$.

$\therefore$ This system is controllable.

I think the reason why this system is controllable is that we can use the input to adjust every LEDs' brighteness. Although there may be a time delay of our adjustion, the required brighenesses can be set finally.