Homework 4

Ding Zhao
24-677 Special Topics: Linear Control Systems

Due: Oct 6, 2020, 9:50 am. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Canonical forms* ***(10 points)***

Consider the system given by:

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Find the controllable canonical form state representation.

Solution:

By inspection, $n = 2$ (the highest exponent of s), therefore $a_1 = 3, a_0 = 2$ $b_2 = 0, b_1 = 1$ and $b_0 = 3$.
This is achieved by comparing the coefficients with the companion matrix form.
The controllable canonical form is given by:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} b_0 - b_2 a_0 & b_1 - b_2 a_1 \end{bmatrix} x + \begin{bmatrix} b_2 \end{bmatrix} u\end{aligned}$$

We can write the state space model as follows:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u\end{aligned}$$

Exercise 2. *Realization matrix form of realizable MIMO system* ***(15 points)***

Find a state-space realization for

$$\hat{G}_1(s) = \begin{bmatrix} \dfrac{1}{s} & \dfrac{s+3}{s+1} \\ \dfrac{1}{s+3} & \dfrac{s}{s+1} \end{bmatrix}$$

Solution:

A state space realization for the above system is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & -\alpha_3 I_p \\ I_p & 0 & 0 \\ 0 & I_p & 0 \end{bmatrix} x + \begin{bmatrix} I_p \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y &= \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

Exercise 3. *Minimum Realizations* ***(20 points)***

Are the two state equations

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 2 & 2 \end{bmatrix} x
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \\
y &= \begin{bmatrix} 2 & 0 \end{bmatrix} x
\end{aligned}
$$

equivalent, i.e. do they have the same transfer function? Are they minimal realizations?

Solution:

For the first state equations we have

$$
\begin{aligned}
G(s) &= C(SI - A)^{-1}B \\
&= \begin{bmatrix} 2 & 2 \end{bmatrix} \frac{1}{(s-2)(s-1)} \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{2(s-1)}{(s-2)(s-1)} \\
&= \frac{2}{(s-2)} \\
&= \frac{2(s+1)}{s^2 - s - 2}
\end{aligned}
$$

Note that this is a realization of $\frac{2s+2}{s^2 - s - 2}$, and is not a minimal realization, as the minimal realization would result in $\frac{2}{s-2}$ and not $\frac{2(s+1)}{(s-2)(s+1)}$. In addition, the controllability matrix of (A, B) does not have full rank.

For case 2 we have

$$
\begin{aligned}
G(s) &= C(SI - A)^{-1}B \\
&= \begin{bmatrix} 2 & 0 \end{bmatrix} \frac{1}{(s-2)(s+1)} \begin{bmatrix} s+1 & 0 \\ -1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \frac{2(s+1)}{(s-2)(s+1)} \\
&= \frac{2}{(s-2)} \\
&= \frac{2(s+1)}{s^2 - s - 2}
\end{aligned}
$$

Once again, this is not a minimal realization, and the observability matrix does not have full rank.
The transfer functions are the same, but are not minimal realizations.

Exercise 4. *Realization* ***(15 points)***

Consider the following transfer function

$$g(s) = \frac{2s - 4}{s^3 - 7s + 6}$$

(a) Determine the standard controllable realization. **(5 points)**

(b) Determine the standard observable realization. **(5 points)**

(c) Determine a minimal realization. **(5 points)**

Solution:
The denominator of $g(s)$ is:

$$\Delta(s) = s^3 - 7s + 6$$

and thus, $n = 3, a_2 = 0, a_1 = -7, a_0 = 6$. And,

$$b_0 = -4, b_1 = 2, b_2 = 0.$$

Note also that $D = 0$, since $g(s)$ is strictly proper.

(a) The controllable canonical form is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 - b_n a_0 & b_1 - b_n a_1 & \cdots & b_{n-1} - b_n a_{n-1} \end{bmatrix} x + b_n u$$

From the strictly proper transfer function $g(s)$, the standard controllable realization can be immediately obtained as

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} -4 & 2 & 0 \end{bmatrix}, D = 0$$

(b) From the strictly proper transfer function $g(s)$, the standard controllable realization can be immediately obtained as

$$\dot{x} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & \cdots & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} b_{n-1} - b_n a_{n-1} \\ \vdots \\ b_1 - b_n a_1 \\ b_0 - b_n a_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x + b_n u$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D = 0$$

This realization is observable by construction.

(c) Let us simplify the transfer function:

$$g(s) = \frac{2s-4}{s^3 - 7s + 6} = \frac{2(s-2)}{(s-2)(s-1)(s+3)} = \frac{2}{(s-1)(s+3)}$$

$$g(s) = \frac{2}{(s-1)(s+3)}$$

and let us determine the controllable canonical form:

$$\chi(s) = (s-1)(s+3) = s^2 + 2s - 3 \rightarrow a_1 = 2, a_0 = -3$$

and

$$b_0 = 2, b_1 = 0$$

So the controllable canonical form is:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 \end{bmatrix}, D = 0$$

Is this realization minimal? It is controllable by construction, and so we have only to check if it is observable:

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

which is clearly full rank. Therefore, this realization is minimal.

Exercise 5. *Controllable decomposition* ***(10 points)***

Reduce the state equation

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x\end{aligned}$$

to a controllable form. Is the reduced state equation observable?

Solution:

The controllability matrix $P = [B \quad AB] = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ has rank 1. The model matrix is chosen as $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\hat{A} = M^{-1}AM = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{C} = CM = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

The transformed state equation is therefore given by

$$\begin{aligned}\dot{\hat{x}} &= \left[\begin{array}{c|c} 3 & 4 \\ \hline 0 & -5 \end{array}\right] \hat{x} + \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right] u \\ y &= \left[\begin{array}{c|c} 2 & 1 \end{array}\right] \hat{x}\end{aligned}$$

The controllable reduced form is therefore

$$\begin{aligned}\dot{\hat{x}} &= 3\hat{x} + 1u \\ y &= 2\hat{x}\end{aligned}$$

The reduced equation is observable

Exercise 6. *kalman decomposition **(10 points)***

Decompose the state equation

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix} x$$

to a form that is both controllable and observable.

Solution:

Note that the given $A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$ is in Jordan form. To find the controllability matrix $P = [B\ AB\ A^2B\ A^3B\ A^4B]$, we first write the generalized form for A^kB

$$A^kB = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 & 0 \\ 0 & \lambda_1^k & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^k & k\lambda_2^{k-1} & \frac{k(k-1)}{2}\lambda_2^{k-2} \\ 0 & 0 & 0 & \lambda_2^k & k\lambda_2^{k-1} \\ 0 & 0 & 0 & 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k\lambda_1^{k-1} \\ \lambda_1^k \\ \lambda_2^k \\ 0 \\ 0 \end{bmatrix}$$

The controllability matrix $P = \begin{bmatrix} 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 \\ 1 & \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ 1 & \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(P) = 3 = n_c.$

Given A is already in Jordan form, the given state equation can be treated as equivalent to the modal decomposed form. Writing the state equation to identify controllable and uncontrollable parts, we get

$$\dot{x} = \left[\begin{array}{ccc|cc} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ \hline 0 \\ 0 \end{array}\right] u$$
$$y = \left[\begin{array}{ccc|cc} 0 & 1 & 1 & 0 & 1 \end{array}\right] x$$

The reduced controllable equation is therefore

$$\dot{x_c} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x_c$$

The observability matrix for the reduced form is $Q = \begin{bmatrix} C_c \\ C_c A_c \\ C_c A_c^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & \lambda_1^2 & \lambda_2^2 \end{bmatrix} \Rightarrow rank(Q) = 2 = n_O$

Re-arranging the controllable reduced form to identify the observable and unobservable parts, we get

$$\dot{x_c} = \left[\begin{array}{cc|c} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 1 & \lambda_1 \end{array}\right] x_c + \left[\begin{array}{c} 1 \\ 1 \\ \hline 0 \end{array}\right] u$$
$$y = \left[\begin{array}{cc|c} 1 & 1 & 0 \end{array}\right] x_c$$

Therefore, the reduced state equation which is both controllable and observable is

$$\dot{x_{co}} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} x_{co} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x_{co}$$

Exercise 7. *Controllable Canonical Form* ***(20 points)***

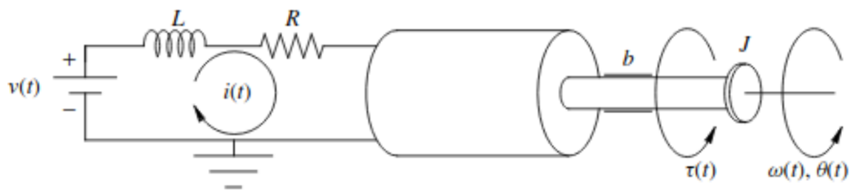

Figure 1: An electromechanical system

The dynamic model of this system can be derived in three segments: a circuit model, electromechanical coupling, and a rotational mechanical model. For the circuit model, Kirchoff's voltage law yields a first order differential equation relating the armature current to the armature voltage; that is,

$$L\frac{di(t)}{dt} + Ri(t) = v(t) \tag{1}$$

Motor torque is modeled as being proportional to the armature current, so the electromechanical coupling equation is

$$\tau(t) = k_T i(t) \tag{2}$$

where k_T is the motor torque constant. For the rotational mechanical model, Euler's rotational law results in the following second-order differential equation relating the motor shaft angle $\theta(t)$ to the input torque $\tau(t)$.

$$J\ddot{\theta}(t) + b\dot{\theta}(t) = \tau(t) \tag{3}$$

Converting the ODEs into transfer functions and multiplying them together, we eliminate the intermediate variables to get the overall transfer function:

$$\frac{\Theta(s)}{V(s)} = \frac{k_T}{(Ls + R)(Js^2 + bs)} \tag{4}$$

Write the controllable canonical form of this system.

Solution:
The transfer function of this system is:

$$\frac{\Theta(s)}{V(s)} = \frac{k_T}{LJs^3 + (RJ + Lb)s^2 + Rbs}$$

Dividing the numerator and denominator by LJ which is the coefficient of the highest order of s, we get,

$$\frac{\Theta(s)}{V(s)} = \frac{k_T/LJ}{s^3 + (R/L + b/J)s^2 + (Rb/LJ)s}$$

The denominator of $g(s)$ is:

$$\chi(s) = s^3 + (R/L + b/J)s^2 + (Rb/LJ)s$$

and thus, $n = 3, a_0 = 0, a_1 = Rb/LJ, a_2 = R/L + b/J$. And,

$$b_0 = k_T/LJ.$$

Note also that $D = 0$, since $g(s)$ is strictly proper.
The controllable canonical form is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 - b_n a_0 & b_1 - b_n a_1 & \cdots & b_{n-1} - b_n a_{n-1} \end{bmatrix} x + b_n u$$

From the strictly proper transfer function $g(s)$, the standard controllable realization can be immediately obtained as

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -Rb/LJ & -(R/L + b/J) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} k_T/LJ & 0 & 0 \end{bmatrix}, D = 0$$