Exercise 1:

∵ the transfer function is

$$G(s) = \frac{s+3}{s^2+3s+2}.$$

∴ The controllable canonical form state representation is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (3, 1)\, x$$

Exercise 2:

Find G_{sp}: $\quad G(\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$\therefore G_{sp} = \hat{G}_1(s) - G(\infty) = \begin{bmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{1}{s+3} & \frac{-1}{s+1} \end{bmatrix}$$

$$\therefore d(s) = s(s+3)(s+1) = s^3 + 4s^2 + 3s$$

$$\Rightarrow G_{sp} = \frac{1}{s^3+4s^2+3s} \begin{bmatrix} (s+3)(s+1) & 2s(s+3) \\ s(s+1) & -s(s+3) \end{bmatrix} = \frac{1}{s^3+4s^2+3s} \begin{bmatrix} s^2+4s+3 & 2s^2+6s \\ s^2+s & -s^2-3s \end{bmatrix}$$

$$\therefore N_1(s) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \quad N_2(s) = \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix} \quad N_3(s) = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ The state space, realization for $\hat{G}_1(s)$ is $\{A, B, C, D\}$.

Exercise 3:

For the first system, we have.

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_1 = [2, 2] \quad D_1 = 0$$

$$G_1(s) = C_1(sI - A_1)^{-1} B_1 + D_1$$
$$= [2,2] \begin{pmatrix} s-2 & -1 \\ 0 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} s-2 & -1 \\ 0 & s-1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s-2} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{s-1} \end{pmatrix}$$

$$\therefore G_1(s) = \frac{2}{s-2}.$$

$P_1 = (B_1, A_1B_1) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad r(P_1) = 1 < 2 = n$

$\therefore$ This system isn't controllable.

$Q_1 = \begin{pmatrix} C_1 \\ C_1A_1 \end{pmatrix} = \begin{pmatrix} 2, 2 \\ 4, 4 \end{pmatrix} \quad r(Q_1) = 1 < 2 = n$

$\therefore$ This system isn't observable.

$\therefore$ This is not a minimal realization

For the second system, we have

$$A_2 = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad C_2 = [2, 0]. \quad D_2 = 0$$

$$G_2(s) = C_2(sI - A_2)^{-1} B_2 + D_2 = [2,0] \begin{pmatrix} s-2 & 0 \\ 1 & s+1 \end{pmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{s-2}$$

$\because G_1(s) = G_2(s) \quad \therefore$ These two systems have the same transfer function.

$P_2 = (B_2, A_2B_2) = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \quad r(P_2) = 2 = n.$

$\therefore$ This system is controllable

$Q_2 = \begin{pmatrix} C_2 \\ C_2A_2 \end{pmatrix} = \begin{pmatrix} 2, 0 \\ 4, 0 \end{pmatrix} \quad r(Q_2) = 1 < n$

$\therefore$ This system is not observable

$\therefore$ This is not a minimal realization.

Above all: These two systems are equivalent. They have the same TF, and they are not minimal realizations

Exercise 4:

a): Here we have.

$b_3 = 0 \quad b_2 = 0 \quad b_1 = 2 \quad b_0 = -4$

$a_2 = 0 \quad a_1 = -7 \quad a_0 = 6.$

∴ The standard controllable. realization is.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [-4, \ 2, \ 0] \, x$$

b): The standard observable realization is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} u$$

$$y = [1, 0, 0] \, x$$

C): The state space we got in (a) is already controllable.

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ -12 & 14 & -4 \end{bmatrix} \qquad r(Q) = 2 < 3 = n$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, Let $M^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C = [-4, 2, 0]$

$$\hat{A} = M^{-1} \cdot A \cdot M = \begin{bmatrix} 0 & 1 & 0 \\ 3 & -2 & 0 \\ -3 & 2 & 2 \end{bmatrix}$$

$$\hat{B} = M^{-1} B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

$$\hat{C} = CM = [-2, 0, 0].$$

∵ $r(Q) = 2$

∴ $A_{co} \in R^{2\times 2}$, $B_c \in R^{2\times 1}$ $C_c \in R^{1\times 2}$

∴ $A_{co} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$ $B_c = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $C_c = (-2, 0)$

∴ The state space

$\dot{x} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u$

$y = (-2, 0) \, x$.

is a minimal realization of $g(s)$.

Exercise 5:

Here we have $A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $C = [1, 1]$

$\therefore P = [B, AB] = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ $\therefore r(P) = 1 < 2 = n$

$\therefore A_c \in R^{1\times1}$ $B_c \in R^{1\times1}$ $C_c \in R^{1\times1}$.

Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then we have

$\hat{A} = M^{-1} A M = \begin{pmatrix} 3 & 4 \\ 0 & -5 \end{pmatrix}$

$\hat{B} = M^{-1} B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hat{C} = CM = (2, 1)$

$\therefore A_c = 3$ $B_c = 1$ $C_c = 2$

$\therefore$ The state space representation will be reduced to.

$\dot{x} = 3x + u$

$y = 2x.$

Here we have $Q = [2]$ $r(Q) = 1 = n$.

$\therefore$ The reduced state equation is observable.

Exercise 6:

First, let's get the transfer function of this system.

$$\because A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad C = [0\ 1\ 1\ 0\ 1]$$

$$\therefore G(s) = C(SI - A)^{-1} B.$$

$$SI - A = \begin{pmatrix} s-\lambda_1 & -1 & 0 & 0 & 0 \\ 0 & s-\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & s-\lambda_2 & -1 & 0 \\ 0 & 0 & 0 & s-\lambda_2 & -1 \\ 0 & 0 & 0 & 0 & s-\lambda_2 \end{pmatrix} \quad \therefore (SI-A)^{-1} = \begin{pmatrix} \frac{1}{s-\lambda_1} & \frac{1}{(s-\lambda_1)^2} & 0 & 0 & 0 \\ 0 & \frac{1}{s-\lambda_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s-\lambda_2} & \frac{1}{(s-\lambda_2)^2} & \frac{1}{(s-\lambda_2)^3} \\ 0 & 0 & 0 & \frac{1}{s-\lambda_2} & \frac{1}{(s-\lambda_2)^2} \\ 0 & 0 & 0 & 0 & \frac{1}{s-\lambda_2} \end{pmatrix}$$

$$\therefore G(s) = \frac{1}{s-\lambda_1} + \frac{1}{s-\lambda_2} = \frac{2s - (\lambda_1+\lambda_2)}{s^2 - (\lambda_1+\lambda_2)s + \lambda_1\lambda_2}$$

Now, we can get the controllable canonical form state representation of $G(s)$

Here we have. $b_2 = 0 \quad b_1 = 2 \quad b_0 = -(\lambda_1+\lambda_2)$

$$a_1 = -(\lambda_1+\lambda_2) \quad a_0 = s_1 s_2$$

$$\therefore \dot{x} = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = (-(\lambda_1+\lambda_2),\ 2)\ x.$$

For this state space representation, obviously, it is controllable. As for observable.

we have $Q = \begin{pmatrix} -(\lambda_1+\lambda_2) & 2 \\ -2\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{pmatrix}$

Now, we need to determine the rank of Q.

Here we want to multiple $2\lambda_1\lambda_2$ to the first row and $\lambda_1+\lambda_2$ to the second row. But before we do this, we need to make sure that $2\lambda_1\lambda_2 \neq 0$ and $\lambda_1+\lambda_2 \neq 0$ in order not to change the rank of Q. So know we assume $2\lambda_1\lambda_2 \neq 0$ and $\lambda_1+\lambda_2 \neq 0$, which equals to $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ and $\lambda_1 \neq -\lambda_2$.

$\therefore$ Now $Q = \begin{pmatrix} -(\lambda_1+\lambda_2) & 2 \\ -2\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{pmatrix} \Rightarrow \begin{pmatrix} -2(\lambda_1+\lambda_2)\lambda_1\lambda_2 & 4\lambda_1\lambda_2 \\ 0 & (\lambda_1-\lambda_2)^2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -(\lambda_1+\lambda_2) & 2 \\ 0 & (\lambda_1-\lambda_2)^2 \end{pmatrix}$.

Obviously $r(Q) \neq 0$. Since $\lambda_1+\lambda_2 \neq 0$, the only way to make $r(Q) = 1$ is to let $\lambda_1 = \lambda_2$. If $\lambda_1 \neq \lambda_2$, $r(Q) = 2$.

Now, Let's consider the condition when $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_1 = -\lambda_2$.

If $\lambda_1 = 0$, $Q \Rightarrow \begin{pmatrix} -\lambda_2 & 2 \\ 0 & \lambda_2 \end{pmatrix}$ the only way to make $r(Q) = 1$ is $\lambda_2 = 0 = \lambda_1$

This means $\lambda_1 = \lambda_2$ which has been discussed before

If $\lambda_2 = 0$, $Q \Rightarrow \begin{pmatrix} -\lambda_1 & 2 \\ 0 & \lambda_1 \end{pmatrix}$ the only way to make $r(Q) = 1$ is $\lambda_1 = 0 = \lambda_2$. This condition has been discussed before.

If $\lambda_1 = -\lambda_2$ $Q \Rightarrow \begin{pmatrix} 0 & 2 \\ -2\lambda_1\lambda_2 & 0 \end{pmatrix}$. the only way to make $r(Q) = 1$ is $\lambda_2 = 0$ or $\lambda_1 = 0$ $\because \lambda_1 = -\lambda_2$ $\therefore$ Only $\lambda_1 = \lambda_2 = 0$ can make $r(Q) = 1$

This condition has been discussed before.

Above all, only when $\lambda_1 = \lambda_2$, $r(Q) = 1$. In other conditions, $r(Q) = 2$

Condition 1: If $\lambda_1 \neq \lambda_2 \Rightarrow r(Q) = 2$, then the system is already observable.

$\therefore$ The state space representation $\dot{x} = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$. $(\lambda_1 \neq \lambda_2)$ is controllable

$y = (-(\lambda_1+\lambda_2), 2)x$ $(\lambda_1 \neq \lambda_2)$

and observable when $\lambda_1 \neq \lambda_2$

Condition 2: If $\lambda_1 = \lambda_2 \Rightarrow r(Q) = 1$ .. We need to use observable Decompos

$\because \lambda_1 = \lambda_2$, $Q = \begin{pmatrix} -2\lambda_1 & 2 \\ -2\lambda_1^2 & 2\lambda_1 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_1 & -1 \\ \lambda_1^2 & -\lambda_1 \end{pmatrix}$.

$\therefore$ We can Let $M^{-1} = \begin{pmatrix} \lambda_1 & -1 \\ 1 & 0 \end{pmatrix}$

$\because r(Q) = 1$

$\therefore A_{co} \in R^{1\times1}$ $B_{co} \in R^{1\times1}$ $C_{co} \in R^{1\times1}$.

$\hat{A} = M^{-1} \begin{pmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{pmatrix} M = \begin{pmatrix} \lambda_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\lambda_1^2 & 2\lambda_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \lambda_1 \end{pmatrix}$

$\hat{B} = M^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\hat{C} = (-(\lambda_1+\lambda_2), 2) M = (-2\lambda_1, 2) \begin{pmatrix} 0 & 1 \\ -1 & \lambda_1 \end{pmatrix}$

$\therefore \hat{A} = \begin{pmatrix} \lambda_1 & 0 \\ -1, & \lambda_1 \end{pmatrix}$ $\hat{B} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\hat{C} = (-2, 0)$

$\therefore A_{co} = \lambda_1$ $B_{co} = -1$ $C_{co} = -2$.

$\therefore$ The state space representation $\begin{matrix} \dot{x} = \lambda_1 x - u \quad (\lambda_1=\lambda_2) \\ y = -2x \quad (\lambda_1=\lambda_2) \end{matrix}$ is controllable and observable when $\lambda_1 = \lambda_2$.

Above all:

① When $\lambda_1 \neq \lambda_2$, the state space representation

$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (\lambda_1 \neq \lambda_2)$

$y = (-(\lambda_1+\lambda_2), 2] x \quad (\lambda_1 \neq \lambda_2)$ is controllable and observable.

② when $\lambda_1 = \lambda_2$, the state space representation.

$\dot{x} = \lambda_1 x - u \quad (\lambda_1 = \lambda_2)$

$y = -2x \quad (\lambda_1 = \lambda_2)$ is controllable and observable.

Exercise 7:

The transfor function is $\frac{\Theta(s)}{V(s)} = \frac{k_T}{(Ls+R)(Js^2+bs)} = \frac{k_T}{JLs^3+(bL+JR)s^2+bRs}$

$$= \frac{k_T/JL}{s^3+(bL+JR)/JL\, s^2+bRs/JL}$$

$\therefore b_3=0 \quad b_2=0 \quad b_1=0 \quad b_0=\frac{k_T}{JL}$

$a_2=\frac{bL+JR}{JL} \quad a_1=\frac{bR}{JL} \quad a_0=0.$

$\therefore$ The controllable canonical form of this system is

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{bR}{JL} & -\frac{(bL+JR)}{JL} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \left(\frac{k_T}{JL},\ 0,\ 0\right) x.$$