HW6 2023010747

Problem 6.1 1. $L_A = \left[[L_A(x_1)]_x \; [L_A(x_2)]_x \; [L_A(x_3)]_x \; [L_A(x_4)]_x \right]$

$L_A(x_1) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = 1\cdot x_1 + 0\cdot x_2 + 3\cdot x_3 + 0\cdot x_4$ $\quad$ $L_A(x_2) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 + 3\cdot x_4$

$L_A(x_3) = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} = 2\cdot x_1 + 0\cdot x_2 + 4\cdot x_3 + 0\cdot x_4$ $\quad$ $L_A(x_4) = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} = 0\cdot x_1 + 2\cdot x_2 + 0\cdot x_3 + 4\cdot x_4$

$$L_A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} I & 2I \\ 3I & 4I \end{bmatrix}$$

2. $R_A = \left[[R_A(x_1)]_x \; [R_A(x_2)]_x \; [R_A(x_3)]_x \; [R_A(x_4)]_x \right]$

$R_A(x_1) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = 1\cdot x_1 + 2\cdot x_2 + 0\cdot x_3 + 0\cdot x_4$ $\quad$ $R_A(x_2) = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = 3\cdot x_1 + 4\cdot x_2 + 0\cdot x_3 + 0\cdot x_4$

$R_A(x_3) = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 2\cdot x_4$ $\quad$ $R_A(x_4) = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = 0\cdot x_1 + 0\cdot x_2 + 3\cdot x_3 + 4\cdot x_4$

$$R_A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix}$$

3. $L_A R_A = \begin{bmatrix} I & 2I \\ 3I & 4I \end{bmatrix} \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} = \begin{bmatrix} A^T I & 2A^T I \\ 3A^T I & 4A^T I \end{bmatrix} = \begin{bmatrix} A^T & 2A^T \\ 3A^T & 4A^T \end{bmatrix}$

$R_A L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} I & 2I \\ 3I & 4I \end{bmatrix} = \begin{bmatrix} A^T I & 2A^T I \\ 3A^T I & 4A^T I \end{bmatrix} = \begin{bmatrix} A^T & 2A^T \\ 3A^T & 4A^T \end{bmatrix}$

4. associativity $(AX)A = A(XA)$

5. $\left\{ \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1\cdot\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0\cdot\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 2\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 1\cdot\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0\cdot\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - 1\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = -1\cdot\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + 2\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

therefore. the coordinate matrix is $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$

Problem 6.2

$$\begin{tikzcd}
 & V \arrow[rrrr, "L"] & & & & W & \\
1 \arrow[ur, "B_1"] \arrow[r, shift left, "I_{1\to2}"] & 2 \arrow[u, "B_2"'] \arrow[l, shift left, "I_{2\to1}"] \arrow[r, shift left, "I_{2\to3}"] & 3 \arrow[ul, "B_3"'] \arrow[l, shift left, "I_{3\to2}"] & & 1 \arrow[ur, "C_1"] \arrow[rr, shift left, "I_{1\to2}"] & & 2 \arrow[ul, "C_2"'] \arrow[ll, shift left, "I_{2\to1}"]
\end{tikzcd}$$

1. True 2. False. $L_{1\leftarrow1} = I_{1\leftarrow2} L_{2\leftarrow1}$, $L^{-1}_{1\leftarrow1} = L^{-1}_{2\leftarrow1} I^{-1}_{1\leftarrow2} = L^{-1}_{2\leftarrow1} I_{2\leftarrow1}$ 3. True 4. False $L_{2\leftarrow2} = I_{2\leftarrow1} I_{1\leftarrow1} L_{1\leftarrow2} \neq L_{2\leftarrow1} I_{1\leftarrow1} L_{1\leftarrow2}$ 5. True

Problem 6.3 1. $M_w(1) = 2+3i$

$M_w(i) = -3+2i$

$M_w = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$

2. for complex number $w = a+bi$

$M_w(1) = a+bi$ so $M_w = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$M_w(i) = -b+ai$

similarly $z = c+di$ $M_z = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ $\quad wz = ac-bd+(ad+bc)i$ $\quad M_{wz} = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix}$

$M_w M_z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -ad-bc \\ bc+ad & -bd+ac \end{bmatrix}$ so $M_w M_z = M_{wz}$

3. associativity for any complex number v $\quad w(zv) = (wz)v$

4. $\phi: a+bi \longrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then from sub-problem we can know that ϕ is bijective and linear

Problem 6.4 1. $0=1$ is in V, we put $0=1$ into the basis

but we can't express $x=0$. so we put $x=0$ into the basis

but we can't express $y=0$. so we put $y=0$ into the basis

but we can't express $z=0$. so we put $z=0$ into the basis

then. for arbitary $ax+by+cz=d$. we have $a\cdot(x=0)+b\cdot(y=0)+c\cdot(z=0)+d(0=1)$. coordinate is $\begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}$

so the basis is $\{0=1, x=0, y=0, z=0\}$. the dimension is 4

2. the solution set is $\emptyset$

since V_i $(i=1,2,\cdots,k)$ form a basis, there exists $a_1, a_2, \cdots, a_k$.

we have $(0=1) = a_1V_1 + a_2V_2 + \cdots + a_kV_k$

since V_i is satisfied. so $a_1V_1 + a_2V_2 + \cdots + a_kV_k$ is satisfied. but $0=1$ is not satisfied. which form a contradiction

3. $p=(x_0, y_0, z_0)$ for $v=(a_1x+b_1y+c_1z=d_1)$ and $w=(a_2x+b_2y+c_2z=d_2)$ in W

we have $a_1x_0+b_1y_0+c_1z_0=d_1$, $a_2x_0+b_2y_0+c_2z_0=d_2$

for $v+w=((a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z=d_1+d_2)$ we have $(a_1+a_2)x_0+(b_1+b_2)y_0+(c_1+c_2)z_0=d_1+d_2$

for $kv=(ka_1x+kb_1y+kc_1z=kd_1)$ we have $ka_1x_0+kb_1y_0+kc_1z_0=kd_1$

so $v+w$ and kv are also in W, W is a subspace

4. they are linearly independent

the first three are linearly independent and their spanning is $ax+by+cz=a+2b+3c$, which doesn't include $x+y+z=7$

5. they are linearly dependent

the span of $x=1$ is $kx=k$, can't express $x=2$.

the span of $x=1$, $x=2$ is $(k+r)x=k+2r$, if $k+r=1$. $k+2r=3$. we have $x=3$. thus $x=3$ is in the span

$\Downarrow$

$r=2$

$k=-1$

Problem 6.5 1. No. it's not linear

let's say the map of venction is R

$R(v)+R(w)=v+w-2C-2O_2+2CO_2$ $R(v+w)=v+w-C-O_2-CO_2$. so $R(v)+R(w)\neq R(v+w)$, it's not linear

2. No, they are linearly dependent, the dimension is 2

it's easy to see that $C+O_2=CO_2$, $2C+O_2=2CO$ are linearly independent since one has CO_2 while the other doesn't

the span of these two is like $(a+2b)C+(a+b)O_2=aCO_2+2bCO$

if $2CO + O_2 = 2CO_2$ can be expressed. then $\begin{cases} a+2b=0 \\ a+b=1 \\ -2b=2 \\ a=2 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=-1 \end{cases}$. we get a solution

so $C+O_2=CO_2$, $2C+O_2=2CO$. $2CO+O_2=2CO_2$ are linearly dependent. we only get 2 effective equations. the dimension of the span is 2

3. $\begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & 2 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow[r_3 \to r_3 + r_1]{r_4 \to r_4 - r_1} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[r_3 \to r_3 + r_2]{r_4 \to r_4 - \frac{r_2}{2}} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_1 \to -r_1]{r_2 \to \frac{r_2}{2}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \to r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad R = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_3 \to C_3 - 2C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_3 \to C_3 + C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$RMC = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ so the rank $r = M$ is 2

Problem 6.6 1. $B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$

$r(A) = r(B)$ because $\dim(\mathrm{ran}(A)) = \dim(\mathrm{ran}(B))$

2.

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[r_2 \to r_2 - 4r_1]{r_3 \to r_3 - 7r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{r_3 \to r_3 - 2r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2 \to -\frac{1}{3}r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \to r_1 - 2r_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ has two pivotal columns, so the rank of R is 2

3. $BR = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$

Problem 6.7 1. the rank is 3

because it has three independent rows

2. $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$

3. $Y = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

4. $XY = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \end{bmatrix}$

$M - XY = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$ is a rank 1 matrix

5. $M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$