HW11 2023010747

Problem 11.1 1. $X=\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$

$XB=\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ N & NM \end{bmatrix}=\begin{bmatrix} X_2N & X_2NM \\ X_4N & X_4NM \end{bmatrix}$

$AX=\begin{bmatrix} MN & 0 \\ N & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}=\begin{bmatrix} MNX_1 & MNX_2 \\ NX_1 & NX_2 \end{bmatrix}$

$XB=AX \Rightarrow \begin{cases} X_2N=MNX_1 \\ X_4N=NX_1 \\ X_2NM=MNX_2 \\ X_4NM=NX_2 \end{cases}$

let $X_2=M$. $X_1=I$ $X_4=I$. and we can satisfy

$X=\begin{bmatrix} I & M \\ X_3 & I \end{bmatrix}$ let $X_3=0$. then $X=\begin{bmatrix} I & M \\ 0 & I \end{bmatrix}$ is also invertible

2. $A=\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ $B=\begin{bmatrix} & -i \\ i & \end{bmatrix}$

$X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left.\begin{matrix} AX=\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} c & d \\ a & b \end{bmatrix} \\ XB=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} & -i \\ i & \end{bmatrix}=\begin{bmatrix} bi & -ai \\ di & -ci \end{bmatrix} \end{matrix}\right\} \Rightarrow \begin{cases} c=bi \\ d=-ai \\ di=a \\ b=-ci \end{cases} \Leftrightarrow \begin{cases} c=bi \\ a=di \end{cases}$$

. let $d=-i$. $a=1$. $b=c=0$. $X=\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$ is also invertible

3. $A=\begin{bmatrix} 1 & 2 & 3 \\ & 1 & 3 \\ & & 1 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$

$$\left.\begin{matrix} AX=\begin{bmatrix} 1 & 2 & 3 \\ & 1 & 3 \\ & & 1 \end{bmatrix}\begin{bmatrix} g & a & b \\ d & h & c \\ e & f & i \end{bmatrix}=\begin{bmatrix} g+2d+3e & a+2+3f & b+2c+3i \\ d+3e & h+3f & c+3i \\ e & f & i \end{bmatrix} \\ XB=\begin{bmatrix} g & a & b \\ d & h & c \\ e & f & i \end{bmatrix}\begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}=\begin{bmatrix} g & g+a & a+b \\ d & d+h & h+c \\ e & e+f & f+i \end{bmatrix} \end{matrix}\right\} \Rightarrow \begin{cases} d=e=f=0 \\ g=2 \\ h=3i \\ a=2c+3i \end{cases}$$

so we can pick $X=\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and X is invertible

4. $A=\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ $B=\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}$ $X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left.\begin{matrix} \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 4a-2c & 4b-2d \\ 3a-c & 3b-d \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}=\begin{bmatrix} 2a & b \\ 2c & d \end{bmatrix} \end{matrix}\right\} \Rightarrow \begin{cases} a=c \\ 3b=2d \end{cases}$$

let $a=c=1$. $b=2$. $d=3$ $X=\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and is invertible

5. $A=\begin{bmatrix} 4M & -2M \\ 3M & -M \end{bmatrix}$ $B=\begin{bmatrix} 2M & \\ & M \end{bmatrix}$

$X=\begin{bmatrix} I & 2I \\ I & 3I \end{bmatrix}$ and X is invertible

6. $A=\begin{bmatrix} M & -N \\ N & M \end{bmatrix}$ $B=\begin{bmatrix} M+iN & \\ & M-iN \end{bmatrix}$

$X=\begin{bmatrix} I & I \\ -iI & iI \end{bmatrix}$ and X is invertible

7. $A=\begin{bmatrix} 2I_2 & M \\ & 3I_3 \end{bmatrix}$ $B=\begin{bmatrix} 2I_2 & \\ & 3I_3 \end{bmatrix}$

$X=\begin{bmatrix} I_2 & MI_3 \\ 0 & I_3 \end{bmatrix}$ and X is invertible

Problem 11.2 1. let $B=[\lambda]$ and we know that $AX=\lambda X$

so X is an eigenvector of A

2. if $v\in Ran(X)$ $Xw=v$

$AXw=Av \Rightarrow XBw=Av \Rightarrow X(Bw)=Av$

so $Av\in Ran(X)$

3. we use mathematical induction to prove $A^nX=XB^n$

$A^1X=XB^1$ is true. suppose $A^kX=XB^k$ is true

then $A^{k+1}X=A(A^kX)=AXB^k=(AX)B^k=(XB)B^k=XB^{k+1}$ is true

therefore we know $A^nX=XB^n$ is true for all $n\in\mathbb{Z}^+$

since polynomials are just linear combinations of powers

thus $p(A)X=Xp(B)$

4. if B is diagonalizable. $B=X_BD_BX_B^{-1}$ $p_A(B)=X_Bp_A(D_B)X_B^{-1}$

since A. B has no common eigenvalue. thus $p_A(D_B)$ diagonal entries are non-zero thus $p_A(B)$ is invertible

if B is not diagonalizable. we just pick diagonalizable matrices B_n such that $B=\lim_{n\to\infty}B_n$

so $P_A(B) = \lim_{n\to+\infty} P_A(B_n)$ is also invertible

$P_A(A)X = XP_A(B)$. from Cayley-Hamilton Theorem. $P_A(A)=0$

thus $XP_A(B)=0$. since $P_A(B)$ is invertible. so $X=0$

Problem 11.3 1. for arbitrary X, Y.

$L(X+Y) = A(X+Y) - (X+Y)B = AX - XB + AY - YB = L(X) + L(Y)$

$L(kX) = AkX - kXB = k(AX-XB) = kL(X)$

so L is a linear map

2. if A, B has no eigenvalue

then if $L(X)=0$. from Problem 11.2.4. $L(X)=0 \Leftrightarrow AX=XB \Leftrightarrow X=0$

so $\mathrm{Ker}(L) = \{0\}$. therefore it's trivial

3. if $AX_1 - X_1B = C$ and $AX_2 - X_2B = C$

then $A(X_1 - X_2) = (X_1 - X_2)B \Rightarrow X_1 - X_2 = 0 \Rightarrow X_1 = X_2$.

so we know that L is injective

since $L: V \to V$. so L is bijective

so the solution to $AX - XB = C$ exists and unique

Problem 11.4 1. if B is not diagonal. lets say $B = \begin{bmatrix} \ddots & t \\ & \end{bmatrix}$ has $t \neq 0$ on (i,j) entry

$e_i^T A B e_j = a_i t$ $\quad e_i^T B A e_j = a_j t$

Since A has distinct diagonal entries. $a_i \neq a_j$. so (i,j) entry of AB and BA is not the same, thus $AB \neq BA$

therefore. B must be diagonal

2. A has distinct eigenvalues. so A can be diagonalizable.

$A = XDX^{-1}$, and D has distinct diagonal entries

since $AB = BA$. so $XDX^{-1}B = BXDX^{-1} \Rightarrow DX^{-1}BX = X^{-1}BXD$

from previous problem. we know that $X^{-1}BX$ is also diagonal

so both $X^{-1}AX$ and $X^{-1}BX$ are diagonal

3. $A = XDX^{-1}$ $\quad B = XD'X^{-1}$ $\quad$ A has distinct eigenvalues

X is the combination (concat) of all eigenvectors of A and $B = XD'X^{-1}$

thus all eigenvectors of A are eigenvectors of B

4. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ B has eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ that is not eigenvector of A

5. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and B is not diagonalizable

6. $Av = \lambda v \Rightarrow BAv = B\lambda v \Rightarrow ABv = \lambda Bv \Rightarrow A(Bv) = \lambda(Bv)$

thus Bv is also an eigenvector of A

Problem 11.5 1. if λ is an eigenvalue of A

then $\lambda^k = 0. \Rightarrow \lambda = 0$

so if A is $n \times n$. then A has eigenvalue 0 with n algebraic multiplicity

$\det(A) = 0^n = 0$. so A is non-invertible

2. $A^k = 0 \Leftrightarrow$ all eigenvalues of A are $0 \Leftrightarrow P_A(x) = (-1)^n A^n$

if $P_A(x) = (-1)^n A^n$. since $P_A(A) = 0$, thus $(-1)^n A^n = 0 \Rightarrow A^n = 0$

if $A^n = 0$. $\lambda^n = 0 \Rightarrow \lambda = 0$. so all eigenvalues are 0

so $A^k = 0 \Leftrightarrow$ all eigenvalues of A are $0 \Leftrightarrow A^n = 0$

3. $A = UTU^{-1}$ let D be the diagonal matrix taking all the diagonal entries of T

$T - D$ is an upper triangular matrix and diagonal entries are 0, so all eigenvalues are 0

$(U(T-D)U^{-1})^k = U(T-D)^k U^{-1} = 0 \Rightarrow U(T-D)U^{-1}$ is nilpotent.

And UDU^{-1} is a normal matrix, so $A = UDU^{-1} + U(T-D)U^{-1}$ is the sum of a normal matrix and a nilpotent matrix