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线性代数作业9

1. (1) $P=\begin{bmatrix}0&4&0\\1&8&0\\-\frac14&4&1\end{bmatrix}$ $J=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$

$\frac{dx(t)}{dt}=Ax(t)$ 的解为 $Pe^{Jt}\vec{c}$

即为 $c_1e^t\begin{pmatrix}1\\2\\1\end{pmatrix}+c_2e^t\left(\begin{pmatrix}0\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\1\end{pmatrix}\right)+c_3e^t\begin{pmatrix}0\\4\\-1\end{pmatrix}$

(2) $P=\begin{bmatrix}-1&-\frac12&0\\1&0&-1\\0&0&1\end{bmatrix}$ $J=\begin{bmatrix}2&1&0\\0&2&0\\0&0&4\end{bmatrix}$

解为 $Pe^{Jt}\vec{c}$

即为 $c_1e^{4t}\begin{pmatrix}0\\-1\\1\end{pmatrix}+c_2\begin{pmatrix}-1\\1\\0\end{pmatrix}e^{2t}+c_3\left(t\begin{pmatrix}-1\\1\\0\end{pmatrix}+\begin{pmatrix}-\frac12\\0\\0\end{pmatrix}\right)e^{2t}$

2. $e^{iA}=I+(iA)+\frac{(iA)^2}{2}+\frac{1}{3!}(iA)^3+\cdots$

$e^{-iA}=I+(-iA)+\frac{(-iA)^2}{2}+\frac{1}{3!}(-iA)^3+\cdots$

因此 $\frac12(e^{iA}+e^{-iA})=I+\frac{(iA)^2}{2}+\frac{(iA)^4}{4!}+\frac{(iA)^6}{6!}+\cdots=I-\frac{A^2}{2}+\frac{A^4}{4!}-\frac{A^6}{6!}+\cdots=\cos A$

$\frac{1}{2i}(e^{iA}-e^{-iA})=\frac1i\left(iA+\frac{(iA)^3}{3!}+\frac{(iA)^5}{5!}+\cdots\right)=A-\frac{A^3}{3!}+\frac{A^5}{5!}-\cdots=\sin A$

$\sin^2A+\cos^2A=\frac14(e^{2iA}+2I+e^{-2iA})-\frac14(e^{2iA}-2I+e^{-2iA})=I$

3. $f(z)=\sin(e^z)$ $f(cI)=\sin(e^{cI})$. 又 $f(cI)=\begin{pmatrix}\sin e^c&&\\&\ddots&\\&&\sin e^c\end{pmatrix}=(\sin e^c)I \Rightarrow \sin(e^{cI})=(\sin e^c)I$

令 $f(z)=\cos(e^z)$ 同理 知 $\cos(e^{cI})=(\cos e^c)I$

4. 验知 $(a\vec{\alpha}+b\vec{\gamma},\vec{\beta})=a(\vec{\alpha},\vec{\beta})+b(\vec{\gamma},\vec{\beta})$

又 $(\vec{\alpha},\vec{\beta})=(a_1\ \ a_2)\begin{pmatrix}1&-1\\-1&2\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}$ 其中 $\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$ 正定. 故$f$为内积.

5. 设 $\vec{\beta}=a\vec{\alpha}_1+b\vec{\alpha}_2=\begin{pmatrix}a-b\\bi\\ai+b\end{pmatrix}$

$\vec{\beta}\perp\vec{\alpha}_1\Rightarrow a-b-i(ai+b)=0\Rightarrow a=\frac{i+1}{2}b$

又 $|\vec{\beta}|=1\Rightarrow(a-b)^2+(bi)^2+(ai+b)^2=1$. 解得 $\begin{cases}a=\frac{\sqrt2(i+1)}{4}\\b=\frac{\sqrt2}{2}\end{cases}$

因此 $\vec{\beta}=\begin{pmatrix}\frac{\sqrt2(i-1)}{4}\\\frac{\sqrt2 i}{2}\\\frac{\sqrt2(i+1)}{4}\end{pmatrix}$

6. $|\vec{\alpha}+\vec{\beta}|^2+|\vec{\alpha}-\vec{\beta}|^2=(\vec{\alpha}+\vec{\beta},\vec{\alpha}+\vec{\beta})+(\vec{\alpha}-\vec{\beta},\vec{\alpha}-\vec{\beta})=(\vec{\alpha},\vec{\alpha})+(\vec{\alpha},\vec{\beta})+(\vec{\beta},\vec{\alpha})+(\vec{\beta},\vec{\beta})+(\vec{\alpha},\vec{\alpha})-(\vec{\beta},\vec{\alpha})-(\vec{\alpha},\vec{\beta})+(\vec{\beta},\vec{\beta})$

$=2|\vec{\alpha}|^2+2|\vec{\beta}|^2$

7. 设 $\vec{\alpha}=x_1\vec{e}_1+\cdots+x_n\vec{e}_n$

$\vec{\beta}=y_1\vec{e}_1+\cdots+y_n\vec{e}_n$

$(\vec{\alpha}-\vec{\beta},\vec{\alpha}-\vec{\beta})=\sum_{i=1}^{n}(x_i-y_i)(\vec{\alpha}-\vec{\beta},\vec{e}_i)=0\Rightarrow\vec{\alpha}-\vec{\beta}=0\Rightarrow\vec{\alpha}=\vec{\beta}$

8. 设 $\vec{\alpha}=x_1\vec{e}_1+\cdots+x_n\vec{e}_n$. 由 $(\vec{\alpha},\vec{v}_i)=c_i$ 知 $\begin{cases}(\vec{v}_1,\vec{v}_1)x_1+\cdots+(\vec{v}_n,\vec{v}_1)x_n=c_1\\(\vec{v}_1,\vec{v}_2)x_1+\cdots+(\vec{v}_n,\vec{v}_2)x_n=c_2\\\vdots\\(\vec{v}_1,\vec{v}_n)x_1+\cdots+(\vec{v}_n,\vec{v}_n)x_n=c_n\end{cases}\Rightarrow\begin{pmatrix}(\vec{v}_1,\vec{v}_1)&\cdots&(\vec{v}_n,\vec{v}_1)\\(\vec{v}_1,\vec{v}_2)&\cdots&(\vec{v}_n,\vec{v}_2)\\&\vdots&\\(\vec{v}_1,\vec{v}_n)&\cdots&(\vec{v}_n,\vec{v}_n)\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}$

令 $A=(\vec{v}_1,\cdots,\vec{v}_n)$, 则 $A^TA\vec{x}=\vec{c}$

$\Rightarrow\vec{x}=(A^TA)^{-1}\vec{c}$.

因此$\vec{\alpha}$存在且唯一