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微积分习题 4.2

3.(1) $\int_L dl = \int_0^1 \sqrt{9+36t^2+36t^4}\,dt = \int_0^1 3(2t^2+1)\,dt = 3(\frac{2}{3}t^3+t)\Big|_0^1 = 5$

(2) $\int_L dl = \int_0^{+\infty}\sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2+(e^{-t}\sin t+e^{-t}\cos t)^2+(-e^{-t})^2}\,dt = \int_0^{+\infty}\sqrt{3e^{-2t}}\,dt = \sqrt{3}\int_0^{+\infty}e^{-t}dt = \sqrt{3}e^{-t}\Big|_0^{+\infty} = \sqrt{3}$

4. 质量为 $\int_L\rho(x,y)dl = \int_L x^2 dl = \int_{\sqrt3}^{\sqrt{15}} x^2\sqrt{\frac{1}{x^2}+1}\,dx = \int_{\sqrt3}^{\sqrt{15}} x\sqrt{x^2+1}\,dx = \frac13(x^2+1)^{\frac32}\Big|_{\sqrt3}^{\sqrt{15}} = \frac{56}{3}$

5. 面积为 $\int_L z\,dl$ 令 $x=a\cos\theta$. $y=a\sin\theta$.

则 $z=a+\frac{a^2\cos^2\theta}{a}=a+a\cos^2\theta$

$$\int_L z\,dl = \int_0^{2\pi} a(1+\cos^2\theta)\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}\,d\theta = a^2\int_0^{2\pi}(1+\cos^2\theta)\,d\theta = 2\pi a^2 + a^2\cdot4\cdot\frac12\cdot\frac\pi2 = 3a^2\pi$$

6. 假设其质量分布均匀，设线密度为1

$$m=\int_L dl = \int_0^\pi\sqrt{a^2(1-\cos t)^2+a^2\sin^2t}\,dt = \int_0^\pi\sqrt{2a^2(1-\cos t)}\,dt = 2a\int_0^\pi \sin\frac t2\,dt = 4a$$

$$x_0=\frac1m\int_L x\,dl = \frac{2a^2}{m}\int_0^\pi(t-\sin t)\sin\frac t2\,dt$$

$$\int_0^\pi (t-\sin t)\sin\frac t2\,dt = -2\int_0^\pi(t-\sin t)\,d\cos\frac t2 = -2\left((t-\sin t)\cos\frac t2\Big|_0^\pi - \int_0^\pi\cos\frac t2(1-\cos t)\,dt\right)$$

$$\int_0^\pi\cos\frac t2(1-\cos t)\,dt = 2\int_0^\pi\cos\frac t2\sin^2\frac t2\,dt = 4\int_0^\pi \sin^2\frac t2\,d\sin\frac t2 = 4\,\frac{\sin^3\frac t2}{3}\Big|_0^\pi = \frac43$$

故 $\int_0^\pi (t-\sin t)\sin\frac t2\,dt = -2(-\frac43) = \frac83$, $x_0 = \frac{2a^2}{4a}\cdot\frac83 = \frac{4a}{3}$.

$$y_0 = \frac1m\int_L y\,dL = \frac{2a^2}{m}\int_0^\pi(1-\cos t)\sin\frac t2\,dt = \frac{4a^2}{m}\int_0^\pi\sin^3\frac t2\,dt = \frac{8a^2}{m}\int_0^{\frac\pi2}\sin^3 t\,dt = \frac{8a^2}{4a}\cdot\frac23 = \frac{4a}{3}$$

习题 4.4

1.(1) $\int_{\frac\pi2}^0 \frac{a^2\cos^6t\, a\cdot3\sin^2t\cdot\cos t + a^2\sin^6 t\, a\cdot 3\cos^2 t\sin t}{a^{\frac53}(\cos^5t+\sin^5t)}\,dt = \int_{\frac\pi2}^0\frac{3a^3}{a^{\frac53}}\cdot\sin^2t\cos^2t\,dt = -3a^{\frac43}\left(\int_0^{\frac\pi2}\sin^2t\,dt - \int_0^{\frac\pi2}\sin^4 t\,dt\right)$

$= -3a^{\frac43}\cdot\left(\frac12\cdot\frac\pi2 - \frac{3}{4\cdot2}\cdot\frac\pi2\right)$

$= -\frac{3a^{\frac43}\pi}{16}$

(3) $\int_0^{2\pi}\left(\frac{-a\sin t\, a(-\sin t)+a\cos t\, a\cos t}{a^2}+b^2\right)dt = \int_0^{2\pi}(1+b^2)\,dt = 2\pi(1+b^2)$

2.(3) 第一象限中 $\oint_{L_1^+}\frac{dx+dy}{|x|+|y|} = \int_1^0\frac{dx+(-1)dx}{|x|+|y|}=0$ 第二象限 $\oint_{L_2^+}\frac{dx+dy}{|x|+|y|} = \int_0^{-1}\frac{dx+dx}{-x+1+x} = -2$

第三象限中 $\oint_{L_3^+}\frac{dx+dy}{|x|+|y|}=0$ 第四象限 $\oint_{L_4^+}\frac{dx+dy}{|x|+|y|} = \int_0^1\frac{dx+dx}{x+1-x}=2$

故 $\oint_{L^+}\frac{dx+dy}{|x|+|y|}=0$

(5) $x=\cos\theta$. $z=\frac{\sqrt2}{2}\sin\theta$

$$\int_{L^+}xyz\,dz = \int_0^{2\pi}\cos\theta\left(\frac{\sqrt2}{2}\sin\theta\right)^2\cdot\frac{\sqrt2}{2}\cos\theta\,d\theta = \frac{\sqrt2}{4}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta = \frac{\sqrt2\pi}{16}$$

4.(1) $x=a\cos\theta$ $y=b\sin\theta$ $\vec F=(-x\vec i - y\vec j)$

$$\int_{L^+}\vec F\,d\vec l = \int_0^{\frac\pi2} -(a\cos\theta\cdot a(-\sin\theta)+b\sin\theta\, b\cos\theta)\,d\theta = (a^2-b^2)\int_0^{\frac\pi2}\sin\theta\cos\theta\,d\theta = \frac{a^2-b^2}{2}$$

(2) 由对称性，F所做的功为0

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微积分作业9

习题4.5 1. (1) $\iint\limits_{\substack{x^2+y^2\le R^2\\ 2R\ge z\ge R}} dxdy - \iint\limits_{\substack{x^2+y^2\le R^2\\ R\ge z\ge 0}} dxdy = 0$

(2) $\iint\limits_{\substack{x^2+y^2\le R^2\\ 2R\ge z\ge R}} zdxdy - \iint\limits_{\substack{x^2+y^2\le R^2\\ R\ge z\ge 0}} zdxdy = \iint\limits_{x^2+y^2\le R^2}(\sqrt{R^2-x^2-y^2}+R)dxdy - \iint\limits_{x^2+y^2\le R^2}(-\sqrt{R^2-x^2-y^2}+R)dxdy$

$= 2\iint\limits_{x^2+y^2\le R^2}\sqrt{R^2-x^2-y^2}\,dxdy$

$= 2\cdot 2\pi\int_0^R \rho\sqrt{R^2-\rho^2}\,d\rho$

$= 4\pi(R^2-\rho^2)^{\frac{3}{2}}\cdot\frac{2}{3}\cdot(-\frac{1}{2})\Big|_R^0$

$= \frac{4\pi}{3}R^3$

(3) $\oiint\limits_{S^+} z^2dxdy = 4\iint\limits_{x^2+y^2\le R^2} R\sqrt{R^2-x^2-y^2}\,dxdy = \frac{8\pi}{3}R^4$

3. (2) $\iint\limits_{S^+}x^2dy\wedge dz + \iint\limits_{S^+}y^2dz\wedge dx + \iint\limits_{S^+}z^2dx\wedge dy$

$= 0+0+0$

$= 0$

5. 由对称性，流量为 $3\iint\limits_{S^+}xydydz = 3\iint\limits_{\substack{y^2+z^2\le 1\\ 0\le y\\ 0\le z}}\sqrt{1-y^2-z^2}\,ydydz = 3\int_0^{\frac{\pi}{2}}d\theta\int_0^1\sqrt{1-\rho^2}\,\rho\cos\theta\cdot\rho d\rho$

$= 3\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{\pi}{2}}^{0}\sin\varphi\cos^2\varphi\cos\theta(-\sin\varphi)d\varphi$

$= 3\int_0^{\frac{\pi}{2}}\cos\theta d\theta(-\int_0^{\frac{\pi}{2}}\sin^4\varphi d\varphi + \int_0^{\frac{\pi}{2}}\sin^2\varphi d\varphi)$

$= \frac{3\pi}{16}$

7. $\frac{D(y,z)}{D(u,v)} = \begin{vmatrix}\sin v & u\cos v\\ 0 & a\end{vmatrix} = a\sin v$

$\frac{D(z,x)}{D(u,v)} = \begin{vmatrix}0 & a\\ \cos v & -u\sin v\end{vmatrix} = -a\cos v$

$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix}\cos v & -u\sin v\\ \sin v & u\cos v\end{vmatrix} = u$

(A,B,C) 始终与法向量同向

故所求即为 $\iint\limits_{\substack{0\le u\le 1\\ 0\le v\le 2\pi}}(u^3+u^2a\sin^2 v - a^3v^2\cos v)dudv$.

$= 2\pi\cdot\frac{u^4}{4}\Big|_0^1 + a\cdot\frac{1}{3}\cdot 4\cdot\frac{2}{3} - a^3\cdot 4\pi$

$= \frac{\pi}{2} + \frac{8a}{9} - 4\pi a^3$