HW8 2023010747

Problem 8.1 1. P is an orthogonal projection

$$\begin{cases} P^2 = P \\ P^T = P \end{cases} \qquad P_{ii} = \| e_i^T P e_i \| = \| e_i^T P P e_i \| = \| e_i^T P^T P e_i \| = \| P e_i \|^2$$

2. $AB = BA \Leftrightarrow AB = B^T A^T \Leftrightarrow AB = (AB)^T \Leftrightarrow AB$ is an orthogonal projection

(since we already have $A^2B^2 = AB$)

$Ran(AB) = Ran(A) \cap Ran(B)$

3. $(A+B)^2 = A^2 + AB + BA + B^2 = A + B + AB + BA$

$(A+B)^T = A^T + B^T = A + B.$

so $A+B$ is an orthogonal projection $\Leftrightarrow AB + BA = 0 \Leftrightarrow Ran(AB) + Ran(BA) = 0 \Leftrightarrow Ran(A) \cap Ran(B) + Ran(B) \cap Ran(A) = 0 \Leftrightarrow Ran(A) \perp Ran(B)$

$Ran(A+B) = Ran(A) + Ran(B)$

Problem 8.2 1. $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ solve $A^T A b = A^T y$ we can get $b = 9$

2. $A = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ $y = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ solve $A^T A k = A^T y$ we can get $k = \frac{56}{13}$

3. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix}$ $y = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ solve $A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T y$ we can get $a = \frac{2}{3}, b = \frac{4}{3}, c = 2$

Problem 8.3 1. $\det\left(\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right) = 1 \cdot (-1) \cdot 2 \cdot 3 = -6$

2. $\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & -1 & 1 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 7 \\ 1 & 2 & 8 & 26 \\ 1 & -2 & 0 & -2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 3 & 7 \\ 2 & 8 & 26 \\ -2 & 0 & -2 \end{bmatrix}\right) = -2\det\left(\begin{bmatrix} 3 & 7 \\ 8 & 26 \end{bmatrix}\right) - 2\det\left(\begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}\right) = -2(78 - 56 + 8 - 6) = -48$

3. $\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}\right) = 160$

4. $\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 0 & 0 \\ 13 & 14 & 0 & 0 & 0 \\ 15 & 16 & 0 & 0 & 0 \end{bmatrix}\right) = 15 \cdot 14 \det\left(\begin{bmatrix} 3 & 4 & 5 \\ 8 & 9 & 10 \\ 0 & 0 & 0 \end{bmatrix}\right) - 16 \cdot 13 \cdot \det\left(\begin{bmatrix} 3 & 4 & 5 \\ 8 & 9 & 10 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$

5. $\det\left(\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}\right) = (-1)^{n-1}(-1)^{n-2} \cdots (-1)^2 \cdot \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = (-1)^{\frac{(n-2)(n+1)}{2} + 1} = (-1)^{\frac{n(n-1)}{2}}$

Problem 8.4 1. $\det(2A) = 2^3 \det(A) = 40$

2. $\det(-A) = (-1)^3 \det(A) = -5$

3. $\det(A^2) = \det(A)\det(A) = 25$

4. $\det(A^{-1})\det(A) = \det(I) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{5}$

5. $\det(A^T) = \det(A) = 5$

6. $\det\left(\begin{bmatrix} a_1^T - a_3^T \\ a_2^T - a_1^T \\ a_3^T - a_2^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1^T - a_2^T \\ a_2^T - a_1^T \\ a_3^T - a_2^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 \\ a_2^T - a_1^T \\ a_3^T - a_2^T \end{bmatrix}\right) = 0$

7. $\begin{bmatrix} a_1^T + a_3^T \\ a_2^T + a_1^T \\ a_3^T + a_2^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$

$\det\left(\begin{bmatrix} a_1^T + a_3^T \\ a_2^T + a_1^T \\ a_3^T + a_2^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) \det\left(\begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}\right) = 2 \times 5 = 10$

Problem 8.5 1. $A = QR \Rightarrow a_i = Q r_i \Rightarrow \|a_i\| = \|Q r_i\|$ $\quad r_i = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ $\quad Q = [q_1 \cdots q_m]$ $\quad Q r_i = [q_1 x_1 \cdots q_m x_m]$

$\|Q r_i\| = \sqrt{q_1^2 x_1^2 + \cdots + q_m^2 x_m^2} = \sqrt{x_1^2 + \cdots + x_m^2} = \|r_i\| \Rightarrow \|a_i\| = \|r_i\|$

2. $\det(R) = R_{11} \cdot R_{22} \cdot \cdots \cdot R_{nn} \le \sqrt{R_{11}^2} \cdot \cdots \cdot \sqrt{R_{nn}^2} \le \|r_1\| \cdot \cdots \|r_n\|$

3. $|\det(A)| = |\det(Q) \cdot \det(R)|$

$\det(Q) = \pm 1$, so $|\det(A)| = |\det(R)| \le \|r_1\| \cdot \cdots \cdot \|r_n\| = \|a_1\| \cdots \|a_n\|$

Problem 8.6 1. False $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $B=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ $AB=\begin{bmatrix}1&1\\1&2\end{bmatrix}$ $BA=\begin{bmatrix}2&1\\1&1\end{bmatrix}$ $AB-BA=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ $\det(AB-BA)=-1\neq 0$

2. False.

$$\left.\begin{array}{l} A=\begin{bmatrix}0&1\\1&0\end{bmatrix} \quad \det(A)=-1 \\ -A=\begin{bmatrix}0&-1\\-1&0\end{bmatrix} \quad \det(-A)=-1 \end{array}\right\} \det(A)=\det(-A)$$

3. $A^T=-A$, $\det(-I)=(-1)^n$, since n is odd. so $\det(-I)=-1$

$\det(A)=\det(A^T)=\det(-A)=\det(-IA)=\det(-I)\det(A)=-\det(A)$

so $\det(A)=0$. A is not invertible

4. $A=\begin{bmatrix}A_i & \vdots\\ \cdots & \end{bmatrix}=LDU=\begin{bmatrix}L_i & 0\\ \cdots\cdots & \end{bmatrix}\begin{bmatrix}d_1 & & & \vdots\\ & \ddots & & \\ & & d_i & \vdots\\ \cdots & \cdots & \cdots & \end{bmatrix}\begin{bmatrix}U_i & \vdots\\ 0 & \end{bmatrix} \Rightarrow A_i=L_i\begin{bmatrix}d_1 & & \\ & \ddots & \\ & & d_i\end{bmatrix}U_i$, $\det(L_i)=\det(U_i)=1 \Rightarrow \det(A_i)=\det\left(\begin{bmatrix}d_1 & & \\ & \ddots & \\ & & d_i\end{bmatrix}\right)=d_1\cdots d_i$

$\therefore \det(A_i)=d_i\cdot\det(A_{i-1}) \Rightarrow d_i=\dfrac{\det(A_i)}{\det(A_{i-1})}$

Problem 8.7 1. $\begin{bmatrix}A&B\\C&D\end{bmatrix}=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$

$AD-BC=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}-\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}-\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&-1\\0&0\end{bmatrix}$

$\det\left(\begin{bmatrix}A&B\\C&D\end{bmatrix}\right)=(-1)^2\det(I)=1\neq AD-BC$

2. $\det\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}\right)=\det\left(\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}\right)=\left(\frac{1}{ad-bc}\right)^2\cdot(ab-bc)=\frac{1}{ad-bc}$

3. $\det(-BA)\overset{?}{=}-\det(B)\det(A)$

for some A.B. like $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\det(-AB)=\det\left(\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\right)=1$. $-\det(A)\det(B)=-1\neq 1$