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微积分 习题2.1

4. (1) $|x^5e^{-x}| \le x^b e^{-x}$

又 $\int_1^{+\infty} x^b e^{-x}dx$ 收敛

故 $\int_1^{+\infty} x^5 e^{-x}dx$ 一致收敛.

(2) $\left|\frac{\cos yx}{1+x^2}\right| \le \frac{1}{1+x^2}$

又 $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx = \arctan x\Big|_{-\infty}^{\infty} = \pi$ 收敛.

故 $\int_{-\infty}^{+\infty} \frac{\cos yx}{1+x^2}dx$ 一致收敛

(4) $|e^{-tx}\sin x| \le e^{-tx}$

又 $\int_0^{+\infty} e^{-tx}dx$ 收敛

故 $\int_1^{+\infty} e^{-tx}\sin x dx$ 一致收敛.

(10) $\frac{1}{x^{p+1}}$ 关于 x 单调且一致收敛至0

又 $\left|\int_1^A x\sin x^2 dx\right| \le 2$

故由 Dirichlet 判别法知 $\int_1^{+\infty}\frac{\sin x^2}{x^p}dx$ 一致收敛.

又 $\int_0^{\delta}\frac{\sin x^2}{x^p}dx \ge \frac{2}{\pi}\int_0^{\delta}x^{2-p}dx = \frac{2}{\pi(3-p)}\delta^{3-p} > \frac{1}{2\pi}$ 对 $\forall\, 0<\delta<\frac{1}{2}$. $\exists p$ 使 $\delta^{3-p}=\frac{1}{2}$

故 $\int_0^{+\infty}\frac{\sin x^2}{x^p}dx$ 非一致收敛

5. $\frac{e^{-tx}}{x+t}$ 在 $[0,+\infty)$ 上单调且一致收敛至0

又 $\left|\int_0^A \sin 3x dx\right| \le 1$

由 Dirichlet 判别法知 $\int_0^{+\infty} e^{-tx}\frac{\sin 3x}{x+t}dx$ 一致收敛

8. 设 $0 \le t \le b$. 对 $\forall A>0$. 当 $t\to 0^+$ 时

$\int_A^{+\infty}\frac{\sin tx}{x}dx = \int_{tA}^{+\infty}\frac{\sin y}{y}dy \to \int_0^{+\infty}\frac{\sin y}{y}dy = \frac{\pi}{2}$

因此 t 充分小时有 $\int_A^{+\infty}\frac{\sin tx}{x}dx > \frac{\pi}{4}$. 因此 $\int_0^{+\infty}\frac{\sin tx}{x}dx$ 不一致收敛

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微积分. 习题 2.2

1. (1) $\int_{-1}^{1}\sqrt{x^2+a^2}\,dx = 2\int_0^1\sqrt{x^2+a^2}\,dx = 2\left(\frac{1}{2}\left(x\sqrt{x^2+a^2} + a^2\ln\left|x+\sqrt{x^2+a^2}\right|\right)\right)\Big|_0^1 = \sqrt{1+a^2} + a^2\ln\left(1+\sqrt{1+a^2}\right) - a^2\ln|a| = \sqrt{1+a^2} + a^2\ln\left|\frac{1}{a}+\sqrt{1+\frac{1}{a^2}}\right|$

$\lim_{a\to 0}\int_{-1}^{1}\sqrt{x^2+a^2}\,dx = 1+0=1$

2. (4) $F'(t) = \int_0^t\left(\frac{\partial}{\partial t}f(x+t,x-t)\right)dx + f(2t,0)\cdot 1$

$= \int_0^t\left(\frac{\partial f(x+t,x-t)}{\partial(x+t)} - \frac{\partial f(x+t,x-t)}{\partial(x-t)}\right)dx + f(2t,0)$

$= \int_0^t\left(f_1'(x+t,x-t) - f_2'(x+t,x-t)\right)dx + f(2t,0)$

4. $\frac{\partial u}{\partial x} = \frac{1}{2}\left(\varphi'(x+at) + \varphi'(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at}\frac{\partial}{\partial x}\psi(s)\,ds + \frac{1}{2a}\left(\psi(x+at) - \psi(x-at)\right)$

$\frac{\partial u}{\partial t} = \frac{a}{2}\left(\varphi'(x+at) - \varphi'(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at}\frac{\partial}{\partial t}\psi(s)\,ds + \frac{1}{2}\left(\psi(x+at) + \psi(x-at)\right)$

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(\varphi''(x+at) + \varphi''(x-at)\right) + \frac{1}{2a}\left(\psi'(x+at) - \psi'(x-at)\right)$

$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{2}\left(\varphi''(x+at) + \varphi''(x-at)\right) + \frac{a}{2}\left(\psi'(x+at) - \psi'(x-at)\right)$

因此 $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$

习题 2.3

1. (2) $|xe^{-ax^2}\sin yx| \le xe^{-ax^2}$

又 $\int_0^{+\infty}xe^{-ax^2}dx = e^{-ax^2}\left(-\frac{1}{a}\right)\Big|_0^{+\infty} = \frac{1}{a}$ 收敛. 故原式一致收敛.

故 $\underbrace{\int_0^y\left(\int_0^{+\infty}xe^{-ax^2}\sin tx\,dx\right)dt}_{J(y)} = \int_0^{+\infty}\left(\int_0^y xe^{-ax^2}\sin tx\,dt\right)dx$

$= \int_0^{+\infty}e^{-ax^2}(1-\cos xy)\,dx$

$= \frac{\sqrt{\pi}}{2\sqrt{a}} - \frac{1}{y}e^{-ax^2}\sin yx\Big|_0^{+\infty} - \frac{1}{y}\int_0^{+\infty}2axe^{-ax^2}\sin yx\,dx$

$= \frac{\sqrt{\pi}}{2\sqrt{a}} - \frac{2a}{y}\cdot J'(y)$

又 $J(0)=0$. 故 $J(y) = \frac{\sqrt{\pi}}{2\sqrt{a}} - \frac{\sqrt{\pi}}{2\sqrt{a}}\cdot e^{-\frac{y^2}{4a}}$ $\quad I(y) = J'(y) = \frac{1}{4a}\sqrt{\frac{\pi}{a}}\cdot ye^{-\frac{y^2}{4a}}$.

2. (1) $|e^{-tx^2}x^{2n}| \le e^{-ax^2}x^{2n}$.

故 $\int_0^{+\infty}e^{-tx^2}x^{2n}dx$ 一致收敛 (关于 $t\in[a,b]$)

又 $\int_0^{+\infty}e^{-tx^2}dx = \frac{\sqrt{\pi}}{2\sqrt{t}}$. 两边对 t 求 n 次导有

$\int_0^{+\infty}x^{2n}e^{-tx^2}dx = \frac{\sqrt{\pi}}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)t^{-\frac{2n+1}{2}}$

故 $\int_0^{+\infty}x^{2n}e^{-tx^2}dx = \frac{\sqrt{\pi}(2n-1)!!}{2^{n+1}}\cdot t^{-\frac{2n+1}{2}}$