微积分A(2) 2023010747 刘一鹞 计32

习题1.3

1.(4) 当(x,y)沿直线$y=-kx$ $(k>0)$ 从$x>0$趋于$(0,0)$时

$$\frac{x+y}{|x|+|y|}=\frac{x-kx}{x+kx}=\frac{1-k}{1+k}$$

不同的k值导致不同的极限. 因此$\frac{x+y}{|x|+|y|}$的极限不存在.

(7) 当(x,y)沿曲线$y=x^5-x$趋于$(0,0)$时

$$\frac{x^3-y^3}{x+y}=\frac{x^3-(x^5-x)^3}{x^5}=\frac{1-(x^4-1)^3}{x^2}=\frac{2+o(x^4)}{x^2}$$

在$x\to 0$时极限不存在

故$\lim\limits_{(x,y)\to(0,0)}\frac{x^3-y^3}{x+y}$不存在

(10) 令$xy=t$

则$(x,y)\to(0,0)\Rightarrow t\to 0$

$$\lim_{t\to 0}\frac{t-\sin t}{t-t\cos t}=\lim_{t\to 0}\frac{\frac{t^3}{6}+o(t^3)}{t\cdot(\frac{t^2}{2}+o(t^3))}=\frac{1}{3}$$ 由复合函数极限性质知

$$\lim_{(x,y)\to(0,0)}\frac{xy-\sin xy}{xy-xy\cos xy}=\frac{1}{3}$$

2.(1) $$\lim_{\substack{x\to 3\\ y\to 0}}\frac{\ln(x+\sin y)}{\sqrt{x^2+y^2}}=\frac{\ln(3+0)}{\sqrt{9+0}}=\frac{\ln 3}{3}$$

(2) 我们知道

$$|x^2+xy+y^2|\geqslant\big||x^2+y^2|-|xy|\big|\geqslant\big|2|xy|-|xy|\big|=|xy|$$

故$$\left|\frac{x+y}{x^2+xy+y^2}\right|\leqslant\frac{|x|+|y|}{|xy|}=\frac{1}{|x|}+\frac{1}{|y|}$$

又$\lim\limits_{\substack{x\to\infty\\ y\to\infty}}(\frac{1}{|x|}+\frac{1}{|y|})=0$ 且$\left|\frac{x+y}{x^2+xy+y^2}\right|\geqslant 0$

由夹逼定理知$\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\frac{x+y}{x^2+xy+y^2}=0$

6.(1)
$$0\leqslant\left|\frac{\sin(x^3+y^3)}{x^2+y^2}\right|\leqslant\left|\frac{x^3+y^3}{x^2+y^2}\right|\leqslant\frac{|x^3|}{|x^2+y^2|}+\frac{|y^3|}{|x^2+y^2|}\leqslant|x|+|y|$$

$$\lim_{(x,y)\to(0,0)}(|x|+|y|)=0$$

由夹逼定理$\lim\limits_{(x,y)\to(0,0)}\frac{\sin(x^3+y^3)}{x^2+y^2}=0=f(0,0)$

故函数在$(0,0)$点连续.

7.(1) 考虑分母趋于0的情况

$x^3+y^3\to 0$

此时分子$\to x-x^2$.

若$x\neq 0$且$x\neq 1$. 则$\lim\limits_{(x,y)\to(x,-x)}f(x,y)$不存在. 故不连续.

若$x\to 0$. 则$y\to 0$.

当(x,y)沿$y=x$趋于$(0,0)$时 $\lim\limits_{(x,y)\to(0,0)}\frac{x-y^2}{x^3+y^3}=\lim\limits_{(x,y)\to(0,0)}\frac{1-x}{2x^2}$ 不存在. 故不连续

若$x\to 1$. 则$y\to -1$

当(x,y)沿$y=-x^2$趋于$(1,-1)$时 $\lim\limits_{(x,y)\to(1,-1)}f(x,y)=\lim\limits_{(x,y)\to(1,-1)}\frac{1}{x^2}=1\neq 0$. 故不连续.

因此除$y=-x$上的点不连续. 其余点连续.

8. 任取点 (x_0,y_0).

由于 $\lim\limits_{x^2+y^2\to\infty} f(x,y)=+\infty$. 故 $\exists M$. 使 $x^2+y^2>M$ 时有 $f(x,y)>f(x_0,y_0)$

又在有界闭集 $\{(x,y)\mid x^2+y^2\le M\}$ 上. $f(x,y)$ 存在最小值 $f(x_1,y_1)$.

因为 f 有最小值 $\min\{f(x_0,y_0),f(x_1,y_1)\}$

10. (1) $\lim\limits_{(x,y)\to(0,0)}\frac{\sin(x^2+y^2)}{x^2+y^2}$ 令 $x^2+y^2=t$ $(x,y)\to(0,0)\Rightarrow t\to 0$

有 $\lim\limits_{(x,y)\to(0,0)}\frac{\sin(x^2+y^2)}{x^2+y^2}=\lim\limits_{t\to 0}\frac{\sin t}{t}=1$

故 $\sin(x^2+y^2)$ 的阶为2

(3) 令 $x^2+y^2=t$ $(x,y)\to(0,0)\Rightarrow t\to 0$

$\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}\Big/(\sqrt{x^2+y^2})^{\alpha}=\lim\limits_{t\to 0}t^{1-\frac{\alpha}{2}}\sin\frac{1}{\sqrt{t}}$

$\alpha\geq 2$. 极限不存在.

$0<\alpha<2$. 极限为0

故无法为不为0的常数. 因此无阶

$\sqrt{\text{极限}}$

(5) 令 $x^2+y^2=t$ $(x,y)\to(0,0)\Rightarrow t\to 0$

$\lim\limits_{(x,y)\to(0,0)}\frac{ax^2+2bxy+cy^2}{(\sqrt{x^2+y^2})^{\alpha}}$ 若 $b=0$. 则 $\alpha>2$. 极限不存在

$\alpha<2$ 极限为0.

故 $\alpha=2$. $\lim\limits_{x\to 0}\lim\limits_{y\to 0}\frac{ax^2+cy^2}{x^2+y^2}=a$

$\lim\limits_{y\to 0}\lim\limits_{x\to 0}\frac{ax^2+cy^2}{x^2+y^2}=c$.

因此需 $a=c$ 否则极限不存在.

又极限不为0. 故 $a=c\neq 0$.

若 $b\neq 0$. 令 (x,y) 沿 $y=kx$ 趋近于 $(0,0)$.

则极限为 $\lim\limits_{(x,y)\to(0,0)}\frac{(a+2bk+ck^2)x^2}{\sqrt{1+k^2}\cdot x^{\alpha}}$

$\alpha=2$. 则随 k 变化极限值不同. 故极限不存在(或 $a=b=c=0$ 极限为0)

$\alpha\neq 0$. 则极限为0或不存在.

综上. $b=0$. $a=c\neq 0$ 时阶为2.

其余情况阶不存在.

2023010747 刘一鸣 计32.

习题1.4. 2. (1) 对 $f(x,y)$ 求偏导

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt{|\Delta x|}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\sqrt{|\Delta x|}}$$ 不存在.

因此 $f(x,y)$ 在坐标原点不可微.

(3) 对 $f(x,y)$ 求偏导

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

$$f'_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0}{\Delta y}=0$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f(\Delta x,\Delta y)-f(0,0)-(f'_x(0,0)\Delta x+f'_y(0,0)\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta x)^2(\Delta y)^2}{((\Delta x)^2+(\Delta y)^2)^2}$$

当 $\Delta y=\Delta x$ 趋于0时极限为 $\frac{1}{4}\neq 0$

因此 $f(x,y)$ 在 $(0,0)$ 处不可微

4. (5) $\frac{\partial z}{\partial x}=\frac{x+y-(x-y)}{(x+y)^2}=\frac{2y}{(x+y)^2}$

$\frac{\partial z}{\partial y}=\frac{-x-y-(x-y)}{(x+y)^2}=-\frac{2x}{(x+y)^2}$

$dz=\frac{2y\,dx-2x\,dy}{(x+y)^2}$

(8) 记 $X=(x_1,\cdots,x_n)^T$

$\Delta X=(\Delta x_1,\cdots,\Delta x_n)^T$

$A=\begin{pmatrix}1&\cdots&1\\ \vdots&&\vdots\\ 1&\cdots&1\end{pmatrix}_{n\times n}$

$z=X^TAX$

$\Delta z=(X+\Delta X)^TA(X+\Delta X)-X^TAX=2X^TA\Delta X+\Delta X^TA\Delta X$

$\therefore dz=2(x_1,\cdots,x_n)\begin{pmatrix}1&\cdots&1\\ \vdots&&\vdots\\ 1&\cdots&1\end{pmatrix}\begin{pmatrix}dx_1\\ \vdots\\ dx_n\end{pmatrix}=2\sum_{i=1}^{n}\sum_{j=1}^{n}x_i\,dx_j$

8. $\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\sqrt[3]{xy}=0$

$f(0,0)=0$. 故 f 在原点处连续

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0$$

由对称性 $f'_y(0,0)=0$

$$\frac{\partial f}{\partial l}(0,0)=\lim_{\rho\to 0^+}\frac{f(\rho\cos\alpha,\rho\sin\alpha)-f(0,0)}{\rho}=\lim_{\rho\to 0^+}\frac{\rho^{\frac{2}{3}}\cos^{\frac{1}{3}}\alpha\sin^{\frac{1}{3}}\alpha}{\rho}=\lim_{\rho\to 0^+}\frac{2^{-\frac{1}{3}}\sin^{\frac{1}{3}}2\alpha}{\rho^{\frac{1}{3}}}$$

若存在. 则 $\sin 2\alpha=0\Rightarrow\alpha=\frac{k\pi}{2}$.

其余情况不存在.

故沿方向 $l=(a,b)$ 的方向导数不存在.

9. $f(0,0)=0$

$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^3}{y}$ 当以 $y=x^4$ 趋于 $(0,0)$ 时有 $\lim_{(x,y)\to(0,0)}\frac{1}{x}$ 不存在

故 $f(x,y)$ 在 $(0,0)$ 处不连续

$\alpha\neq k\pi$ 时 $\frac{\partial f}{\partial l}(0,0)=\lim_{\rho\to 0^+}\frac{f(\rho\cos\alpha,\rho\sin\alpha)-f(0,0)}{\rho}=\lim_{\rho\to 0^+}\frac{\rho^2\cdot\frac{\cos^3\alpha}{\sin\alpha}}{\rho}=\lim_{\rho\to 0^+}\frac{\rho\cos^3\alpha}{\sin\alpha}=0$

$\alpha=k\pi$ 时 $\frac{\partial f}{\partial l}(0,0)=\lim_{\rho\to 0^+}\frac{f(\rho,0)-f(0,0)}{\rho}=0$

因此沿任何方向的方向导数均存在

13. $\frac{\partial u}{\partial x}=2x-y-z$ $\quad\frac{\partial u}{\partial y}=2y+z-x$ $\quad\frac{\partial u}{\partial z}=2z-x+y$

$\mathrm{grad}\,u(P)=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)_P$

方向导数最大为 $2\sqrt{2}$. 此时方向为 $(0,2,2)$

与该方向垂直的方向导数为0

即所有形如 $(a,b,-b)$ 的方向上导数均为0

$(a^2+b^2\neq 0)$

2023010747 刘一铭 计32

习题1.4. 15.(2) $\frac{\partial u}{\partial x}=-\frac{1}{x^2+y^2+z^2}\cdot 2x=-\frac{2x}{x^2+y^2+z^2}$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{-2\cdot(x^2+y^2+z^2)+2x\cdot 2x}{(x^2+y^2+z^2)^2}=\frac{2x^2-2y^2-2z^2}{(x^2+y^2+z^2)^2}$$

由对称性可得 $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial z^2}$.

故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2x^2-2y^2-2z^2+2y^2-2x^2-2z^2+2z^2-2x^2-2y^2}{(x^2+y^2+z^2)^2}=0$

(3) $\frac{\partial u}{\partial x}=e^x\cos y$ $\quad\frac{\partial u}{\partial y}=e^x(-\sin y)$

$\frac{\partial v}{\partial y}=e^x\cdot\cos y$ $\quad\frac{\partial v}{\partial x}=e^x\sin y$

$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=e^x\cos y$ $\quad\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=-e^x\cos y$ $\quad$故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)=e^x\sin y$ $\quad\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=-e^x\sin y$ $\quad$故 $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}=0$