Linear algebra HW5 2023010747

Problem 5.1 1. $tr(xA+yB) = \sum_{i=1}^{n} e_i^T(xA+yB)e_i = \sum_{i=1}^{n} e_i^T xAe_i + \sum_{i=1}^{n} e_i^T yBe_i = x\sum_{i=1}^{n} e_i^T Ae_i + y\sum_{i=1}^{n} e_i^T Be_i = x\,tr(A) + y\,tr(B)$

2. $tr(I_n) = \sum_{i=1}^{n} e_i^T I_n e_i = \sum_{i=1}^{n} 1 = n$

3. A is a $n\times n$ matrix. its diagnol entries remain the same after the transpose, therefor $tr(A)=tr(A^T)$

4. $u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $u^T = [a_1 \cdots a_n]$ $uu^T = \begin{bmatrix} a_1^2 & a_2a_1 & \cdots & a_na_1 \\ a_1a_2 & a_2^2 & \cdots & a_na_2 \\ \vdots & \vdots & & \vdots \\ a_1a_n & a_2a_n & & a_n^2 \end{bmatrix}$ $tr(uu^T) = \sum_{i=1}^{n} a_i^2$ since we know u is a unit vector. therefore we already have $\sum_{i=1}^{n} a_i^2 = 1$, so $tr(uu^T)=1$

From the first question we know trace is linear, so $tr(I_n - uu^T) = tr(I_n) - tr(uu^T) = n-1$

5. $A^TB = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1b_1+a_3b_3 & \cdots \\ \cdots & a_2b_2+a_4b_4 \end{bmatrix}$ $AB^T = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} = \begin{bmatrix} a_1b_1+a_2b_2 & \cdots \\ \cdots & a_3b_3+a_4b_4 \end{bmatrix}$ so $tr(A^TB) = tr(AB^T) = a_1b_1+a_2b_2+a_3b_3+a_4b_4 = \sum_{i=1}^{4} a_ib_i$

6. assum $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $w^T = [b_1\ b_2 \cdots b_n]$ $vw^T = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & & a_nb_n \end{bmatrix}$ $tr(vw^T) = \sum_{i=1}^{n} a_ib_i$. $w^Tv = [b_1 \cdots b_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [\sum_{i=1}^{n} a_ib_i]$ so $tr(vw^T) = tr(w^Tv)$

7. $A = [a_1\ a_2 \cdots a_n]$ $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix}$

$tr(AB) = \sum_{i=1}^{m} e_i^T ABe_i = \sum_{i=1}^{m} e_i^T A(e_ie_i^T)Be_i = \sum_{i=1}^{m} e_i^T a_ib_i^T e_i = tr(a_ib_i^T)$ From Problem 5.1.6. we know $tr(a_ib_i^T) = tr(b_i^Ta_i) = \sum_{i=1}^{n} b_i^Ta_i$

$tr(BA) = \sum_{i=1}^{n} e_i^T BAe_i = \sum_{i=1}^{n} b_i^Ta_i = tr(b_i^Ta_i)$ so $tr(AB) = tr(BA)$

8. From the first question we know trace is linear, so $tr(AB-BA) = tr(AB) - tr(BA) = 0$

but $tr(I_n) = n \neq 0$. so AB-BA cannot be the identity matrix

Problem 5.2 1. $P = \begin{bmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix}$

2. identity matrix I after permutation is PI, since $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$, the number of fixed elements is just how many 1's remain on the diagonal, which is $tr(PI)$, $PI=P$. so the number of fixed elements is $tr(P)$

3. From Problem 5.1.7 we know $tr(P_1P_2) = tr(P_2P_1)$. according to the last question, we know they have the same number of fixed points

Problem 5.3 1. $f \in V$ then $-f \in V$. $f(t)-f(t) \in V$. $f(t)-f(t)$ is a function: $R \to 0$ which is continuous, thus V is not a vector space

2. this is not a vector space, $\vec{v}_1 = (1,1,\sqrt{2})$. $\vec{v}_2 = (1,1,-\sqrt{2})$ then $\vec{v}_1+\vec{v}_2 = (2,2,0)$ doesn't satisfy $x^2+y^2 \leq z$

3. this is a vector space with dimension 3. a basis can be $\{\{1\},\{1,2\},\{1,2,3\}\}$

4. this is a vector space with dimension 2 a basis can be $\{\begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}\}$

5. this is a vector space with dimension 2 a basis can be $\{\begin{bmatrix} 1&0 \\ 0&0 \end{bmatrix}, \begin{bmatrix} 0&0 \\ 0&1 \end{bmatrix}\}$

6. this is a vector space with dimension 3 a basis can be $\{\begin{bmatrix} 2&9&4 \\ 7&5&3 \\ 6&1&8 \end{bmatrix}, \begin{bmatrix} 2&7&6 \\ 9&5&1 \\ 4&3&8 \end{bmatrix}, \begin{bmatrix} 8&3&4 \\ 1&5&9 \\ 6&7&2 \end{bmatrix}\}$

Problem 5.4 1. $\left[\begin{array}{ccc|ccc} 1&1&1&1&0&0 \\ e&e^2&e^3&0&1&0 \\ e^2&e^4&e^6&0&0&1 \end{array}\right] \xrightarrow[r_2 \to r_2 - er_1]{r_3 \to r_3 - e^2r_1} \left[\begin{array}{ccc|ccc} 1&1&1&1&0&0 \\ 0&e^2-e&e^3-e&-e&1&0 \\ 0&e^4-e^2&e^6-e^2&-e^2&0&1 \end{array}\right] \xrightarrow{r_3 \to r_3 - e(e+1)r_2} \left[\begin{array}{ccc|ccc} 1&1&1&1&0&0 \\ 0&e^2-e&e^3-e&-e&1&0 \\ 0&0&e^3(e-1)^2(e+1)&e^3&-e^2-e&1 \end{array}\right]$

the left side is a upper triangle matrix and the diagonal entries are non-zero. thus A is invertible

2. if e^x, e^{2x}, e^{3x} is dependent. then the rows of A is dependent. so A is not bijective thus A is not invertible which forms a contradiction

so e^x, e^{2x}, e^{3x} is linearly independent

3. suppose the coordinates of f are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $f(x)=x_1e^x+x_2e^{2x}+x_3e^{3x}$ so $\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$

Problem 5.5 1. suppose $L_1=\{p_1+tv_1 : t\in R\}$, $L_2=\{p_2+tv_2 : t\in R\}$

$L_1, L_2 \in V/W \Rightarrow v_1=rv_2$ so $L_1+L_2=\{p_1+p_2+t(r+1)v_2 : t\in R\}$, $(r+1)v_2 \parallel v_2 \Rightarrow L_1+L_2$ is parallel to $L_2 \Rightarrow L_1+L_2$ is parallel to W so $L_1+L_2\in V/W$

$kL_1=\{kp_1+tkv_1 : t\in R\}$. $kv_1\parallel v_1 \Rightarrow kL_1 \parallel L_1 \Rightarrow kL_1 \parallel W \Rightarrow kL_1\in V/W$

2. $k(L_1+L_2)=\{k(p_1+p_2)+tk(r+1)v_2 : t\in R\}$ $\quad kL_1+kL_2=\{kp_1+tkrv_2+kp_2+tkv_2 : t\in R\}$ thus $k(L_1+L_2)=kL_1+kL_2$

3. W is the "zero vector" because $W+L=L$

4. we assume W is in the direction of $v=(x,y,z)$, x,y,z cannot all be zero. Without loss of generality, we assume $z\neq 0$

for $L_1=\{(1,0,0)+tv : t\in R\}$, $L_1\in V/W$, we put L_1 into the basis, we can easily verify that $(0,1,0)$ is not in $\{(1+tx,ty,tz), t\in R\}$

L_1 can't span $L_2=\{(0,1,0)+tv : t\in R\}$, $L_2\in V/W$ so we put L_2 into the basis

then for arbitrary $L=\{(a,b,c)+tv : t\in R\}$, $L\in V/W$, we have $L=\{(a-\frac{cx}{z}, b-\frac{cy}{z}, 0)+t'v : t'\in R\}=(a-\frac{cx}{z})\cdot L_1+(b-\frac{cy}{z})\cdot L_2$

thus $\{L_1, L_2\}$ is a basis and the dimension of V/W is 2

Problem 5.6 1. Yes. we assume for finitely many polynomials, the maximum of the degree of polynomial is n

we choose $f(x)=x^{n+1}$. if $f(x)$ is in the span, then $f(x)=a_1f_1(x)+a_2f_2(x)+\cdots+a_mf_m(x) \Rightarrow |f(x)|\le(|a_1|+\cdots+|a_n|)(|x|^n+|x|^{n-1}+\cdots+|x|+1)$

$|a_1|+\cdots+|a_n|=M$, then $|x|^{n+1}\le M\cdot\frac{|x|^{n+1}-1}{|x|-1}\le M\frac{|x|^{n+1}}{|x|-1}$. if $x\ge M+10$, we get $|x|^{n+1}>M\frac{|x|^{n+1}}{|x|-1}$ which forms a contradiction

therefore $f(x)$ is not in the span

2. Yes. for $f(x), g(x)\in W$ $\quad f(x)=p(x)\cdot h(x)$ $\quad g(x)=p(x)\cdot r(x)$ $\quad kf(x)+lg(x)=(kh(x)+lr(x))\cdot p(x)$. so $k\cdot f(x)+l\cdot g(x)\in W$, thus W is a subspace

3. $[r_1(x)]=\{r_1(x)+p(x)q_1(x) : q_1(x)\in V\}$ $\qquad [r_2(x)]=\{r_2(x)+p(x)q_2(x) : q_2(x)\in V\}$

$[r_1(x)]+[r_2(x)]=\{r_1(x)+r_2(x)+p(x)(q_1(x)+q_2(x)) : q_1(x), q_2(x)\in V\}$

$[r_1(x)+r_2(x)]=\{r_1(x)+r_2(x)+p(x)q_3(x) : q_3(x)\in V\}$. let $q_1(x)+q_2(x)=q_3(x)\in V$. then $[r_1(x)]+[r_2(x)]=[r_1(x)+r_2(x)]$

$[kr_1(x)]=\{kr_1(x)+p(x)\cdot q_4(x) : q_4(x)\in V\}$ $\quad k[r_1(x)]=\{kr_1(x)+p(x)\cdot kq_1(x) : q_1(x)\in V\}$, let $q_4(x)=kq_1(x)\in V$. then $k[r_1(x)]=[kr_1(x)]$

4. if the degree of $r(x)\ge 2$. we can use $p(x)=x^2+3x+2$ and $q(x)$ accordingly to reduce its degree

let's assume the degree of $r(x)$ is n. $r(x)=a_1x^n+a_2x^{n-1}+\cdots$. choose $q(x)=ax^{n-2}$ and we can eliminate x^n and get $r'(x)$ whose degree $<r(x)$

proceed the process similarly and we can get a polynomial whose degree is less than 2

we put $[r(x)=1]=\{1+p(x)q(x) : q(x)\in V\}$ into the basis. but we cannot express x^2+4x+2 because it's $\{x+p(x)\cdot q(x) : q(x)\in R\}$

so we put $[r(x)=x]$ into the basis. then for polynomials whose degree ≤ 1 are in the span. for those degree ≥ 2. similar as above.

we can use $p(x)\cdot q(x)$ to reduce its degree below 2. so it's also in the span

therefore the dimension of V/W is 2 and a basis is $\{[1], [x]\}$.