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线性代数作业14

1. 定义内积 $(f(x), g(x)) = \int_{-1}^{1} f(x)g(x)\,dx$. 则根据 Riesz 表示定理. 有 $q(x) = \varphi(e_1)e_1 + \varphi(e_2)e_2 + \varphi(e_3)e_3$ 其中 e_1, e_2, e_3 为一组标准正交基.

因 $\varphi: p(x) \mapsto p(\frac{1}{2})$. 又 $p_2(R)$ 的一组正交基可以是 $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{3}{2}}x$, $\sqrt{\frac{45}{8}}(x^2-\frac{1}{3})$. 故 $q(x) = -\frac{15}{32}x^2 + \frac{3}{4}x + \frac{21}{32}$

2. 设其对偶基为 w_1^*, w_2^*, w_3^*, 则 $(w_1^*, w_2^*, w_3^*) = (v_1^*, v_2^*, v_3^*)(P^T)^{-1}$ $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$(w_1, w_2, w_3) = (v_1, v_2, v_3)P$

解得 $w_1^* = v_1^* - v_2^*$, $w_2^* = v_2^* - v_3^*$, $w_3^* = v_3^*$

3. 若线性相关. 设有 $c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 = 0 \Rightarrow c_1(x+2y+z) + c_2(2x+3y+3z) + c_3(3x+7y+z) = 0 \Rightarrow (c_1\ c_2\ c_3)\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 7 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

又 $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 7 & 1 \end{pmatrix}$ 可逆. 故 $c_1 = c_2 = c_3 = 0$. 因此 $\varphi_1, \varphi_2, \varphi_3$ 线性无关. 又 $\dim V^* = 3$. 故 $\varphi_1, \varphi_2, \varphi_3$ 为一组基.

将 $\varphi(x, y, z) = x+y+z$ 用 $\varphi_1, \varphi_2, \varphi_3$ 表示即需解 $(c_1, c_2, c_3)\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 7 & 1 \end{pmatrix} = (1, 1, 1)$ 解得 $\varphi(x+y+z) = (1-7a)\varphi_1 + 2a\varphi_2 + a\varphi_3$

4. $\forall f, g \in V_1^\perp$. $\forall \vec{w} \in V_1$. $\langle f+g, \vec{w}\rangle = \langle f, \vec{w}\rangle + \langle g, \vec{w}\rangle = 0 \Rightarrow f+g \in V_1^\perp$ $\forall c \in F$. $\langle cf, \vec{w}\rangle = cf(\vec{w}) = c\langle f, \vec{w}\rangle = 0 \Rightarrow cf \in V_1^\perp$ 故 $V_1^\perp \subseteq V^*$

$\forall f \in (V_1+V_2)^\perp$, 则有 $\langle f, \vec{w}\rangle = 0$. $\forall \vec{w} \in V_1+V_2 \Rightarrow \begin{cases} \langle f, \vec{w_1}\rangle = 0 \quad w_1 \in V_1 \Rightarrow f \in V_1^\perp \\ \langle f, \vec{w_2}\rangle = 0 \quad w_2 \in V_2 \Rightarrow f \in V_2^\perp \end{cases} \Rightarrow f \in V_1^\perp \cap V_2^\perp$

$\forall g \in V_1^\perp \cap V_2^\perp$ 则 $\begin{cases} g \in V_1^\perp \Rightarrow \langle g, \vec{w_1}\rangle = 0 \quad \forall \vec{w_1} \in V_1 \\ g \in V_2^\perp \Rightarrow \langle g, \vec{w_2}\rangle = 0 \quad \forall \vec{w_2} \in V_2 \end{cases} \Rightarrow \langle g, \vec{w_1} + \vec{w_2}\rangle = 0 \Rightarrow g \in (V_1+V_2)^\perp$

$\Rightarrow (V_1+V_2)^\perp = V_1^\perp \cap V_2^\perp$

5. 设 $\vec{e_1}, \vec{e_2}, \cdots, \vec{e_r}$ 为 V_1 一组基. 由于 $V_1 \subseteq V$. 可将基扩充为 $\vec{e_1}, \cdots, \vec{e_r}, \vec{e_{r+1}}, \cdots, \vec{e_n}$ 作为 V 的一组基.

其对偶基为 $\vec{e_1}^*, \cdots, \vec{e_n}^*$ 对 $j = r+1, \cdots, n$. $i = 1, 2, \cdots, r$ 有 $\vec{e_j}^*(\vec{e_i}) = 0$. 故 $\vec{e_j}^* \in V_1^\perp$

$\forall f \in V_1^\perp$. 有 $f = a_1\vec{e_1}^* + \cdots + a_r\vec{e_r}^* + \cdots + a_n\vec{e_n}^*$, 对 $i = 1, 2, \cdots, r$. 有 $f(\vec{e_i}) = 0 \Rightarrow a_i = 0 \Rightarrow f = a_{r+1}\vec{e_{r+1}}^* + \cdots + a_n\vec{e_n}^*$

故 $\vec{e_{r+1}}^*, \cdots, \vec{e_n}^*$ 是 $V_1^\perp$ 的一组基. 故 $\dim V_1^\perp = n - \dim V$

6. "⇒" 设 V_1 是不变子空间. $\forall f \in V_1^\perp \subseteq V^*$ $T^*(f) = f \circ T$

$\forall \vec{v} \in V_1$ $T^*(f)(\vec{v}) = f(T(\vec{v})) = 0 \Rightarrow T^*(f) \in V_1^\perp$. 故 $V_1^\perp$ 是 T^* 不变子空间

"⇐" 设 $V_1^\perp$ 是 V^* 的 T^* 不变子空间.

$\forall f \in V_1^\perp$ $f(T(\vec{v})) = 0 \Rightarrow T(\vec{v}) \in (V_1^\perp)^\perp \subseteq (V^*)^* = V$ $V_1 \subseteq (V_1^\perp)^\perp$. 又由上一题知 $\dim V_1 = \dim(V_1^\perp)^\perp \Rightarrow V_1 = (V_1^\perp)^\perp$

故 $T(\vec{v}) \in V_1 \Rightarrow V_1$ 是 T 不变子空间