HW12 2023010747

Problem 12.1 1. if $A \geq B$ and $B \geq C$

the $v^*(A-B)v \geq 0$ and $v^*(B-C)v \geq 0$

therefore $v^*(A-B)v + v^*(B-C)v \geq 0$

$\Leftrightarrow v^*Av - v^*Bv + v^*Bv - v^*Cv \geq 0$

$\Leftrightarrow v^*(A-C)v \geq 0$

$\Leftrightarrow A \geq C$

2. if $A \geq B$ and $B \geq A$

then $v^*(A-B)v \geq 0$ and $v^*(B-A)v \geq 0$

$\Rightarrow v^*Av \geq v^*Bv$ and $v^*Bv \geq v^*Av$

so $v^*Av = v^*Bv$

since $v^*Aw = \frac{1}{2}((v+w)^*A(v+w) - v^*Av - w^*Aw)$ (A is Hermitian matrix)

so $v^*Aw = v^*Bw$

$e_i^*Ae_j = e_i^*Be_j$

so $A = B$

3. by spectral theorem. there exists a unitary matrix U

$A = UDU^*$, $U^*AU = D$

since A is Hermitian matrix. all eigenvalues are real

so let λ_{min} and λ_{max} be the smallest and the largest eigenvalue

$\lambda_{min}I \leq D \leq \lambda_{max}I$

$U(\lambda_{min}I)U^* = \lambda_{min}I$

$U(\lambda_{max}I)U^* = \lambda_{max}I$

we then show that if $A \geq B$. the $C^*AC \geq C^*BC$ for invertible C

this is because $A-B$ is positive definite

$A - B = XX^*$ for invertible X

$C^*AC = C^*BC + \underbrace{(X^*C)^*(X^*C)}_{\text{this is positive semidefinite}}$

so $C^*AC \geq C^*BC$

therefore we have $UDU^* \geq U\lambda_{min}IU^*$, so $A \geq \lambda_{min}I$

Similarly $A \leq \lambda_{max}I$

Problem 12.2 let $S = \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$

1. $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalue ± 1

so this Rayleigh quotient must be between -1 and 1.

2. $S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalue 2, 4

so maximum value is 4

minimum value is 2

3. $S = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ has eigenvalue

so maximum value is $\frac{5+\sqrt{13}}{2}$

minimum value is $\frac{5-\sqrt{13}}{2}$

Problem 12.3

1. $\begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{4}{5} & \frac{3}{5} & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^T & 0 \\ 0 & I \end{bmatrix}$

Problem 12.4

for eigenvalue λ

$\|Ax\| = |\lambda| \|x\|$

$\|Ax\| = \|U\Sigma V^T x\| = \|\Sigma V^T x\|$

let's assume σ_1 is the largest singular value

$\|\Sigma V^T x\| \leq \sigma_1 \|V^T x\| = \sigma_1 \|x\|$

so $\sigma_1 \geq |\lambda|$

Problem 12.5 $\quad 5x + y + 7 = 0$

Problem 12.6

1. singular values are $1, 1, 1, 0$

   eigenvalues are $0, 0, 0, 0$

2. singular values are $1, 1, 1, 10^{-4}$

   eigenvalues are $\pm 0.1, \pm 0.1i$