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线性代数作业 10

1. $w_1 = v_1$

$(w_1, w_1) = \int_0^1 1\,dx = 1$ $\quad (w_1, v_2) = \int_0^1 x\,dx = \frac{1}{2}$ $\quad (w_1, v_3) = \int_0^1 x^2\,dx = \frac{1}{3}$

$(w_2, v_3) = \int_0^1 x^2 (x-\frac{1}{2})\,dx = \frac{1}{12}$ $\quad (w_2, w_2) = \int_0^1 (x-\frac{1}{2})^2 dx = \frac{1}{12}$

$w_2 = v_2 - \frac{(w_1, v_2)}{(w_1, w_1)} w_1$

$w_3 = v_3 - \frac{(w_1, v_3)}{(w_1, w_1)} w_1 - \frac{(w_2, v_3)}{(w_2, w_2)} w_2$

故 $w_1 = 1$. $w_2 = x - \frac{1}{2}$ $\quad w_3 = x^2 - x + \frac{1}{6}$

将 $w_1 \cdot w_2 \cdot w_3$ 单位化，由于 $(w_1, w_1) = 1$. $(w_2, w_2) = \int_0^1 (x-\frac{1}{2})^2 dx = \frac{1}{12}$ $\quad (w_3, w_3) = \int_0^1 (x^2 - x + \frac{1}{6})^2 dx = \frac{1}{180}$

故 $\alpha_1 = 1$. $\alpha_2 = 2\sqrt{3}(x - \frac{1}{2})$ $\quad \alpha_3 = 6\sqrt{5}(x^2 - 2x + \frac{2}{3})$

2. $w_1 = v_1 = \begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix}$

$w_2 = v_2 - \frac{(w_1, v_2)}{(w_1, w_1)} w_1$ $\quad (w_1, v_2) = 0 \Rightarrow w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$w_3 = v_3 - \frac{(w_1, v_3)}{(w_1, w_1)} w_1 - \frac{(w_2, v_3)}{(w_2, w_2)} w_2$

$(w_1, v_3) = i + 1 - i = 1$ $\quad (w_2, v_3) = 1$

$(w_1, w_1) = 1 + 1 + 1 = 3$, $\quad (w_2, w_2) = 1$

$\Rightarrow w_3 = \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} i - \frac{1}{3} \\ 0 \\ \frac{2-i}{3} \end{pmatrix}$

将 w_1, w_2, w_3 单位化.

故 $\alpha_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix}$ $\quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \alpha_3 = \frac{\sqrt{15}}{5}\begin{pmatrix} i - \frac{1}{3} \\ 0 \\ \frac{2-i}{3} \end{pmatrix}$

3. $(\frac{\cos kx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}}) = \int_{-\pi}^{\pi} \frac{1}{\pi}(\cos kx)^2 dx = \frac{1}{2\pi}\left(\frac{\sin 2kx}{2k} + x\right)\Big|_{-\pi}^{\pi} = 1$ 同理 $(\frac{\sin kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}) = 1$

又 $(\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}) = 1$.

$(\frac{\cos kx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}}) = \int_{-\pi}^{\pi} \frac{1}{\pi}(\cos kx)(\sin mx)\,dx = 0.$ $\quad (\frac{\cos kx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}}) = \int_{-\pi}^{\pi} \frac{1}{\pi}(\cos kx)(\cos mx) = 0$

$(\frac{1}{\sqrt{2\pi}}, \frac{\cos kx}{\sqrt{\pi}}) = (\frac{1}{\sqrt{2\pi}}, \frac{\sin kx}{\sqrt{\pi}}) = 0$ $\quad (\frac{\sin kx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}}) = 0$

故 $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \cdots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}$ 为标准正交基.

4. 若 $\vec{v} = (\vec{v}, \vec{e_1})\vec{e_1} + \cdots + (\vec{v}, \vec{e_n})\vec{e_n}$

则 $|\vec{v}|^2 = (\vec{v}, \vec{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} (\vec{v}, \vec{e_i})\overline{(\vec{v}, \vec{e_j})}(\vec{e_i}, \vec{e_j})$. 又 (e_i, e_j) 当 $i \neq j$ 时为0. 故 $|\vec{v}|^2 = \sum_{i=1}^{n} |(\vec{v}, \vec{e_i})|^2$

若 $|\vec{v}|^2 = \sum_{i=1}^{n} |(\vec{v}, \vec{e_i})|^2$

则正交投影 $\vec{v}' = (\vec{v}, \vec{e_1})\vec{e_1} + \cdots + (\vec{v}, \vec{e_n})\vec{e_n}$. 由上可知 $|\vec{v}'|^2 = |\vec{v}|^2$. 又 $|\vec{v}|^2 = |\vec{v}'|^2 + |(v - v')|^2 \Rightarrow |v - v'|^2 = 0 \Rightarrow v = v' \Rightarrow \vec{v}$ 是 $\vec{e_1} \cdots \vec{e_n}$ 线性组合

v 在 $e_1 \cdots e_n$ 构成空间上

5. (1) 对 $S^{\perp}$ 中的 $\vec{\alpha}$. 有 $(\vec{\alpha}, \vec{\beta_i}) = 0$. 故 $(\vec{\alpha}, c_1\vec{\beta_1} + \cdots + c_k\vec{\beta_k}) = 0 \Rightarrow \vec{\alpha} \in W^{\perp}$. $\Rightarrow S^{\perp} \subseteq W^{\perp}$

$\forall \vec{\alpha} \in W^{\perp}$. 有 $(\vec{\alpha}, \vec{\beta}) = 0$ $\forall \vec{\beta} \in S \Rightarrow \vec{\alpha} \in S^{\perp} \Rightarrow W^{\perp} \subseteq S^{\perp}$

$\Rightarrow S^{\perp} = W^{\perp}$

(2) $V = W^{\perp} \oplus W = (W^{\perp}) \oplus (W^{\perp})^{\perp} \Rightarrow \dim (W^{\perp})^{\perp} = \dim V - \dim W^{\perp} = \dim W$. 又 $W \subseteq (W^{\perp})^{\perp} \Rightarrow W = (W^{\perp})^{\perp}$

6. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 为一组标准正交基. φ 在该基下矩阵为 $\begin{pmatrix} 1 & i \\ -2 & -1 \end{pmatrix}$. φ^* 为 $\begin{pmatrix} 1 & i \\ -2 & -1 \end{pmatrix}^H = \begin{pmatrix} 1 & -2 \\ -i & -1 \end{pmatrix}$. 故 $\varphi^*\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x - 2y \\ -ix - y \end{pmatrix}$

7. $\forall \vec{\alpha} \in \ker\varphi$. $\vec{\beta} \in \mathrm{Im}\,\varphi^*$. $\vec{\beta} = \varphi^*(\vec{\gamma})$

$(\vec{\alpha}, \vec{\beta}) = (\vec{\alpha}, \varphi^*(\vec{\gamma})) = (\varphi(\vec{\alpha}), \vec{\gamma}) = 0 \Rightarrow \vec{\alpha} \in (\mathrm{Im}\,\varphi^*)^{\perp} \Rightarrow \ker\varphi \subseteq (\mathrm{Im}\,\varphi^*)^{\perp}$

$\forall \vec{\alpha} \in (\mathrm{Im}\,\varphi^*)^{\perp}$.

$0 = (\vec{\alpha}, \varphi^*(\varphi(\vec{\alpha}))) = (\varphi(\vec{\alpha}), \varphi(\vec{\alpha})) \Rightarrow \varphi(\vec{\alpha}) = 0 \Rightarrow \vec{\alpha} \in \ker(\varphi) \Rightarrow (\mathrm{Im}\,\varphi^*)^{\perp} \subseteq \ker\varphi$

因此 $\ker\varphi = (\mathrm{Im}\,\varphi^*)^{\perp}$

因此 $\ker\varphi^{\perp} = \mathrm{Im}\,\varphi^*$

8. (1) $\varphi(a\vec{x}+b\vec{y})=(a\vec{x}+b\vec{y},\vec{\alpha})\vec{\beta}=a(\vec{x},\vec{\alpha})\vec{\beta}+b(\vec{y},\vec{\alpha})\vec{\beta}=a\varphi(\vec{x})+b\varphi(\vec{y})$

故φ为线性变换

(2) $(\varphi(\vec{x}),\vec{y})=(\vec{x},\vec{\alpha})(\vec{\beta},\vec{y})=(\vec{x},\overline{(\vec{\beta},\vec{y})}\vec{\alpha})$. 令 $\varphi^*: V\to V$ $\quad \varphi^*(\vec{x})=\overline{(\vec{\beta},\vec{x})}\vec{\alpha}$

(3) $\varphi(\vec{e_1})=(\vec{\alpha},\vec{\alpha})\vec{\beta}=\vec{\beta}=\vec{e_2}$

$\varphi(\vec{e_2})=(\vec{\beta},\vec{\alpha})\vec{\beta}=0.$

$\varphi(\vec{e_3})=0$

$\vdots$

$\varphi(\vec{e_n})=0.$

故 $(\varphi(\vec{e_1}),\cdots,\varphi(\vec{e_n}))=(\vec{e_1},\cdots,\vec{e_n})\begin{pmatrix}0 & \\ 1 & \\ 0 & O \\ \vdots & \\ 0 & \end{pmatrix}$ $\quad (\varphi^*(\vec{e_1}),\cdots,\varphi^*(\vec{e_n}))=(\vec{e_1},\cdots,\vec{e_n})\begin{pmatrix}0 & 1 & \\ 0 & 0 & O \\ \vdots & \vdots & \\ 0 & 0 & \end{pmatrix}$

φ. φ^*在基下矩阵为 $\begin{pmatrix}0 & \\ 1 & \\ 0 & O \\ \vdots & \\ 0 & \end{pmatrix}$ 和 $\begin{pmatrix}0 & 1 & \\ 0 & 0 & O \\ \vdots & \vdots & \\ 0 & 0 & \end{pmatrix}$