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习题4.6

2.(2) $X=\frac{x+y}{x^2+y^2}$，$Y=\frac{y-x}{x^2+y^2}$，故 $\frac{\partial X}{\partial y}=\frac{x^2+y^2-2y(x+y)}{(x^2+y^2)^2}=\frac{x^2-y^2-2xy}{(x^2+y^2)^2}$，$\frac{\partial Y}{\partial x}=\frac{-(x^2+y^2)-2x(y-x)}{(x^2+y^2)^2}=\frac{x^2-y^2-2xy}{(x^2+y^2)^2}$

$\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}=0$

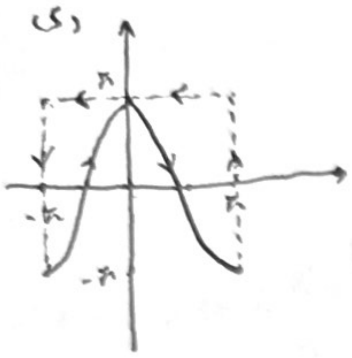
又D包含原点，故作出半径为r的圆周 L_0^+（顺时针）．则 $\int_{L^+}(Xdx+Ydy)+\int_{L_0^+}(Xdx+Ydy)=0$

$\int_{L_0^+}\frac{(x+y)dx+(y-x)dy}{x^2+y^2}=\int_{2\pi}^{0}\frac{r^2(\sin\theta+\cos\theta)(-\sin\theta)+r^2(\sin\theta-\cos\theta)\cdot\cos\theta}{r^2}d\theta=-\int_{2\pi}^{0}d\theta=2\pi$

故所求为 -2π

(3) 同(2)知为 -2π

(5)



增添边界 L_0^+ 则 $\int_{L^+}(Xdx+Ydy)+\int_{L_0^+}(Xdx+Ydy)=0$

$\int_{L_0^+}(Xdx+Ydy)=\int_{\pi}^{-\pi}\frac{x+\pi}{x^2+\pi^2}dx=\left(\frac{1}{2}\ln(x^2+\pi^2)+\arctan\frac{x}{\pi}\right)\Big|_{\pi}^{-\pi}=-\frac{\pi}{2}$

3.(2) $\int_{L^+}(2xy+3x\sin x)dx+(x^2-ye^y)dy$

$=\iint_D(-2x+2x)dxdy-\int_{\pi a}^{0}3x\sin x\,dx=-3\int_0^{\pi a}x\,d\cos x=-3\left(x\cos x\Big|_0^{\pi a}-\int_0^{\pi a}\cos x\,dx\right)=-3\pi a\cos(\pi a)+3\sin\pi a$

4.(2) 设 $x=r\cos\theta$，$y=r\sin\theta$

则 $r^4=a^2r^2(\cos^2\theta-\sin^2\theta)\Rightarrow r=a\sqrt{\cos^2\theta-\sin^2\theta}=a\sqrt{\cos2\theta}$

又 $a>0\Rightarrow\theta\in[-\frac{\pi}{4},\frac{\pi}{4}]\cup[\frac{3\pi}{4},\frac{5\pi}{4}]$

面积为 $\frac{1}{2}\oint_{\partial D}a\cos\theta\sqrt{\cos2\theta}\left(a\cos\theta\sqrt{\cos2\theta}-a\sin\theta\frac{\sin2\theta}{\sqrt{\cos2\theta}}\right)-a\sin\theta\sqrt{\cos2\theta}\left(-a\sin\theta\sqrt{\cos2\theta}-a\cos\theta\frac{\sin2\theta}{\sqrt{\cos2\theta}}\right)d\theta$

$=\frac{1}{2}\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^2\cos2\theta\,d\theta+\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}a^2\cos2\theta\,d\theta\right)=a^2$

8.(2) $LHS=\oint_{\partial D}\left(v\frac{\partial u}{\partial x}dy-v\frac{\partial u}{\partial y}dx\right)=\iint_D\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+v\frac{\partial^2u}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2u}{\partial y^2}\right)dxdy=\iint_D\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)dxdy+\iint_D v\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)dxdy$

$=\iint_D\nabla u\cdot\nabla v\,dxdy+\iint_D v\Delta u\,dxdy=RHS$

(3) $LHS=\oint_{\partial D}v\frac{\partial u}{\partial n}dl-\oint_{\partial D}u\frac{\partial v}{\partial n}dl=\iint_D(v\Delta u-u\Delta v)dxdy=\iint_D\begin{vmatrix}\Delta u&\Delta v\\u&v\end{vmatrix}dxdy=RHS$

9. $\cos\langle n,i\rangle=\frac{dy}{dl}$，$\cos\langle n,j\rangle=\frac{-dx}{dl}$

故所求即为 $\oint_L xdy-ydx=2S$，S为L围的面积

11.(5) $\left(1-\frac{\sin^2y}{x^2}\right)dx+\frac{\sin2y}{x}dy=0$

$\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x}=-\frac{\sin2y}{x^2}$，故$\exists u(x,y)$ $du=Xdx+Ydy\Rightarrow\frac{\partial u}{\partial x}=1-\frac{\sin^2y}{x^2}\Rightarrow u=x+\frac{\sin^2y}{x}+C(y)$

又 $\frac{\partial u}{\partial y}=\frac{\sin2y}{x}+C'(y)=\frac{\sin2y}{x}\Rightarrow C(y)$为常数，故解为 $x+\frac{\sin^2y}{x}=C$

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习题4.7

3. (5) 考虑 S_0^+ 为 $x^2+y^2+z^2=\varepsilon^2$

则由 Gauss 公式. 我们知道 $\iint\limits_{S^+} A\cdot dS - \iint\limits_{S_0^+} A\cdot dS = 0$

又 $\iint\limits_{S_0^+} A\cdot dS = \iint\limits_{S_0^+} \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{\varepsilon^3} = 3\cdot 2\iint\limits_{\partial D} \frac{\sqrt{\varepsilon^2-y^2-z^2}}{\varepsilon^3}\,dy\,dz = 6\int_0^{2\pi} d\theta \int_0^{\varepsilon} \frac{\sqrt{\varepsilon^2-r^2}}{\varepsilon^3}\cdot r\,dr = 4\pi$

因此原式 $=4\pi$

5. (1) 由 Stokes 公式

$$\oint\limits_{L^+} y\,dx+z\,dy+x\,dz = \iint\limits_{S^+}(-1)\,dy\wedge dz - dz\wedge dx - dx\wedge dy = -3\iint\limits_{S^+} dy\wedge dz = -3\cdot\pi\cdot\left(\frac{\sqrt{6}}{6}\cdot\sqrt{2}\right)\cdot\left(\frac{\sqrt{2}}{2}\cdot\sqrt{2}\right)R^2 = -\sqrt{3}\pi R^2$$

6. (2) $X_z = Z_x = -\frac{2y(x+z)}{((x+z)^2+y^2)^2}$ $\quad X_y = Y_x = \frac{(x+z)^2+y^2-y\cdot 2y}{((x+z)^2+y^2)^2} = Y_z = Z_y$

故存在 u $\quad du = X\,dx + Y\,dy + Z\,dz$

解得 $u = \arctan\left(\frac{x+z}{y}\right) + C$

7. (1) $X_y = Y_x = Z_x = X_z = Y_z = Z_y = 1$. 故 $\operatorname{rot} V = 0$. 积分与路径无关

故存在 u. $du = X\,dx + Y\,dy + Z\,dz \Rightarrow u = xy+yz+zx+C$

所求为 $u(1,2,1)-u(0,0,0)=5$