

1. A的特征值为 $\lambda_1, \dots, \lambda_n$. 则 $\lambda_i = 0 \Rightarrow \lambda_i = 0$, 因此A的特征值均为0. $A = O_{n \times n}$

2. (1) $\bar{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix}$ $F_4 \bar{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 4I$ 因此 $\bar{F}_4^{-1} = \frac{1}{4} \bar{F}_4$

(2) $C_4 \bar{F}_4 = \begin{pmatrix} p_{11} & p_{1\omega} & p_{1\omega^2} & p_{1\omega^3} \\ p_{1\omega} & \omega p_{1\omega} & \omega^2 p_{1\omega} & \omega^3 p_{1\omega} \\ p_{1\omega^2} & \omega^2 p_{1\omega} & \omega^4 p_{1\omega} & \omega^6 p_{1\omega} \\ p_{1\omega^3} & \omega^3 p_{1\omega} & \omega^6 p_{1\omega} & \omega^9 p_{1\omega} \end{pmatrix} = \bar{F}_4 \begin{pmatrix} p_{11} & & & \\ & p_{1\omega} & & \\ & & p_{1\omega^2} & \\ & & & p_{1\omega^3} \end{pmatrix}$

(3) $\bar{F}_4^{-1} C_4 \bar{F}_4 = \begin{pmatrix} p_{11} & & & \\ & p_{1\omega} & & \\ & & p_{1\omega^2} & \\ & & & p_{1\omega^3} \end{pmatrix}$ 令 $U = \frac{1}{2} \bar{F}_4$. 则 $U^H C_4 U = \begin{pmatrix} p_{11} & & & \\ & p_{1\omega} & & \\ & & p_{1\omega^2} & \\ & & & p_{1\omega^3} \end{pmatrix} \Rightarrow C_4$ 为复正规矩阵.

3. (1) $(\frac{1}{\sqrt{\pi}}, \frac{1}{\sqrt{\pi}}) = \int_0^{2\pi} \frac{1}{\sqrt{\pi}} dx = 2$ $(\frac{1}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}) = \int_0^{2\pi} \frac{\sin x}{\sqrt{\pi}} dx = 0$
 $(\frac{\sin x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}) = \int_0^{2\pi} \frac{\sin^2 x}{\pi} dx = \frac{2}{\pi} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 1$ $\cos x$ 同理. $(\frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}) = \int_0^{2\pi} \frac{\sin x \cos x}{\pi} dx = \int_0^{2\pi} \frac{\sin 2x}{2\pi} dx = 0$
 $(\frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}) = \int_0^{2\pi} \frac{\sin^2 2x}{\pi} dx = \frac{1}{\pi} \int_0^{2\pi} \sin^2 2x dx = 1$ $\cos 2x$ 同理. 同理可知 $\frac{1}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}$ 这五个内积均为0

又 $V = \{C_0 + C_1(\cos x + i \sin x) + C_2(\cos 2x + i \sin 2x) + C_3(\cos x - i \sin x) + C_4(\cos 2x - i \sin 2x)\}$ 可由 $\frac{1}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}$ 表示

因此 $\frac{1}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}$ 为一组标准正交基.

(2) 验证可得 $1, e^{ix}, e^{-ix}$ 为 ω -组正交基 $\Rightarrow \frac{1}{\sqrt{2\pi}}, \frac{e^{ix}}{\sqrt{2\pi}}, \frac{e^{-ix}}{\sqrt{2\pi}}$ 为一组标准正交基

因此 $f(x)$ 在 ω 上的正交投影为 $(f(x), \frac{1}{\sqrt{2\pi}}) \cdot \frac{1}{\sqrt{2\pi}} + (f(x), \frac{e^{ix}}{\sqrt{2\pi}}) \cdot \frac{e^{ix}}{\sqrt{2\pi}} + (f(x), \frac{e^{-ix}}{\sqrt{2\pi}}) \cdot \frac{e^{-ix}}{\sqrt{2\pi}}$

4. 若 $\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$. 则成立. 否则. 我们可以知道 $\begin{pmatrix} a \\ b \end{pmatrix}$ 和 $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 分别为 $\lambda=0, 2$ 的特征向量.

由于复正规 $(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = 0 \Rightarrow a+b+c=0$

5. A为实正规矩阵 $\Rightarrow AA^T = A^T A$ 又 $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ 故 $AA^T = A^T A$ 可得 $\begin{cases} A_1 A_1^T + A_2 A_2^T = A_1^T A_1 \\ A_2 A_3^T = A_1^T A_2 \\ A_2 A_2^T + A_3 A_3^T = A_3 A_3^T \end{cases}$
因此 A_3 实正规

6. 特征值为 $\pm 1, -2$. 故存在酉阵P使 $f = X^H A X \xrightarrow{\tilde{X} = P^H X} f = Y^H \begin{pmatrix} 1 & & \\ & -1 & \\ & & -2 \end{pmatrix} Y$ 令 $Y = \begin{pmatrix} 1 \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} z$. 则 $f = z_1 \bar{z}_1 - z_2 \bar{z}_2 - 2 z_3 \bar{z}_3$

7. (1) 设 $A = C^H C, B = D^H D$. C,D均可逆 则 $AB = C^H C D^H D \Rightarrow (C^H)^{-1} A B C^H = C D^H D C^H = C D^H (C D^H)^H \Rightarrow AB$ 相似于正定阵 $\Rightarrow AB$ 特征值 > 0

(2) $(AB)^H = B^H A^H = BA = AB \Rightarrow AB$ 为 Hermitian 阵. 又由(1)知 AB 正定. 故 AB 为正定 Hermitian 阵.

8. A特征值最小为 a . 则令 $t = |a| + 1$. 故 $A + I$ 特征值均大于0. 故 $A + I$ 正定. 又 $(A + I)^H = A^H + I^H = A + I$. 故 $A + I$ 为正定 Hermitian 阵

9. $A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $A^T A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 6 \\ 2 & 5 & 12 \\ 6 & 12 & 30 \end{pmatrix}$ 特征值为 $0, 1, 36$. 对应特征向量分别为 $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

归一化为 $\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{36}} \\ \frac{1}{\sqrt{36}} \\ \frac{5}{\sqrt{36}} \end{pmatrix}$. 因此奇异值分解为 $\begin{pmatrix} \frac{5}{\sqrt{36}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{36}} & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{36}} & \frac{1}{\sqrt{5}} & \frac{5}{\sqrt{36}} \end{pmatrix} \begin{pmatrix} 36 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{6}} \\ \frac{5}{\sqrt{5}} & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}^T$

令 $S = U \Sigma U^T = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ 为对称阵. $Q = U V^T = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$ 为正交阵. 故极分解为 $S Q = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$