Linear algebra HW5 2023010747

Problem 5.1

1. $tr(xA+yB) = \sum_{i=1}^{n} e_i^T(xA+yB)e_i = \sum_{i=1}^{n} e_i^T xAe_i + \sum_{i=1}^{n} e_i^T yBe_i = x\sum_{i=1}^{n} e_i^T Ae_i + y\sum_{i=1}^{n} e_i^T Be_i = x\,tr(A) + y\,tr(B)$

2. $tr(I_n) = \sum_{i=1}^{n} e_i^T I_n e_i = \sum_{i=1}^{n} 1 = n$

3. A is a $n\times n$ matrix. its diagnol entries remain the same after the transpose, therefor $tr(A) = tr(A^T)$

4. $u = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $u^T = [a_1 \cdots a_n]$ $uu^T = \begin{bmatrix} a_1^2 & a_2a_1 & \cdots & a_na_1 \\ a_1a_2 & a_2^2 & \cdots & a_na_2 \\ \vdots & \vdots & & \vdots \\ a_1a_n & a_2a_n & & a_n^2 \end{bmatrix}$ $tr(uu^T) = \sum_{i=1}^{n} a_i^2$ since we know u is a unit vector. therefore we already have $\sum_{i=1}^{n} a_i^2 = 1$, so $tr(uu^T) = 1$

From the first question we know trace is linear, so $tr(I_n - uu^T) = tr(I_n) - tr(uu^T) = n-1$

5. $A^TB = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1b_1+a_3b_3 & \cdots \\ \cdots & a_2b_2+a_4b_4 \end{bmatrix}$ $AB^T = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} = \begin{bmatrix} a_1b_1+a_2b_2 & \cdots \\ \cdots & a_3b_3+a_4b_4 \end{bmatrix}$ So $tr(A^TB) = tr(AB^T) = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = \sum_{i=1}^{4} a_ib_i$

6. assum $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $w^T = [b_1\ b_2 \cdots b_n]$ $vw^T = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & & a_nb_n \end{bmatrix}$ $tr(vw^T) = \sum_{i=1}^{n} a_ib_i$. $w^Tv = [b_1 \cdots b_n]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [\sum_{i=1}^{n} a_ib_i]$ so $tr(vw^T) = tr(w^Tv)$

7. $A = [a_1\ a_2 \cdots a_n]$ $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix}$

$tr(AB) = \sum_{i=1}^{m} e_i^T ABe_i = \sum_{i=1}^{m} e_i^T A(e_ie_i^T)Be_i = \sum_{i=1}^{m} e_i^T a_ib_i^T e_i = tr(a_ib_i^T)$ From Problem 5.1.6. we know $tr(a_ib_i^T) = tr(b_i^Ta_i) = \sum_{i=1}^{n} b_i^Ta_i$

$tr(BA) = \sum_{i=1}^{n} e_i^T BAe_i = \sum_{i=1}^{n} b_i^T a_i = tr(b_i^Ta_i)$ So $tr(AB) = tr(BA)$

8. From the first question we know trace is linear, so $tr(AB-BA) = tr(AB) - tr(BA) = 0$

but $tr(I_n) = n \neq 0$. so $AB-BA$ cannot be the identity matrix

Problem 5.2

1. $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

2. identity matrix I after permutation is PI, since $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$, the number of fixed elements is just how many 1's remain on the diagonal, which is $tr(PI)$, $PI = P$. so the number of fixed elements is $tr(P)$

3. From Problem 5.1.7 we know $tr(P_1P_2) = tr(P_2P_1)$. according to the last question, we know they have the same number of fixed points

Problem 5.4

1. $f \in V$ then $-f \in V$, $f + (-f) \in V$. $f + (-f)$ is a function: $R \to 0$ which is continuous, thus V is not a vector space

2. this is a vector space with dimension 3. a basis can be $\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{pmatrix}\}$

3. this is a vector space with dimension 3. a basis can be $\{\{1\}, \{1,2\}, \{1,2,3\}\}$

4. this is a vector space with dimension 2 a basis can be $\{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\}$

5. this is a vector space with dimension 2 a basis can be $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$

6. this is a vector space with dimension 3 a basis can be $\{\begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}\}$

Problem 5.4

1. $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ e & e^2 & e^3 & 0 & 1 & 0 \\ e^2 & e^4 & e^6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_2 \to r_2 - er_1]{r_3 \to r_3 - e^2r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & e^2-e & e^3-e & -e & 1 & 0 \\ 0 & e^4-e^2 & e^6-e^2 & -e^2 & 0 & 1 \end{array}\right] \xrightarrow{r_3 \to r_3 - e(e+1)r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & e^2-e & e^3-e & -e & 1 & 0 \\ 0 & 0 & e^3(e-1)^2(e+1) & e^3 & -e^2-e & 1 \end{array}\right]$

the left side is a upper triangle matrix and the diagonal entries are non-zero. thus A is invertible

2. if e^x, e^{2x}, e^{3x} is dependent. then the rows of A is dependent. so A is not bijective thus A is not invertible which forms a contradiction

so e^x, e^{2x}, e^{3x} is linearly independent

3. suppose the coordinates of f are $[x_1\ x_2\ x_3]$, we have $f = x_1 f(0) + x_2 f(1) + x_3 f(2) = A[x_1\ x_2\ x_3] \Rightarrow$ coordinates of f are $A^{-1}f$

Problem 5.5 1. suppose $L_1 = \{p_1 + tv_1 : t \in R\}$, $L_2 = \{p_2 + tv_2 : t \in R\}$

$L_1, L_2 \in V/W \Rightarrow v_1 = rv_2$ so $L_1 + L_2 = \{p_1 + p_2 + t(r+1)v_2 : t \in R\}$, $(r+1)v_2 \parallel v_2 \Rightarrow L_1 + L_2$ is parallel to $L_2 \Rightarrow L_1 + L_2$ is parallel to W so $L_1 + L_2 \in V/W$

$kL_1 = \{kp_1 + tkv_1 : t \in R\}$. $kv_1 \parallel v_1 \Rightarrow kL_1 \parallel L_1 \Rightarrow kL_1 \parallel W \Rightarrow kL_1 \in V/W$

2. $k(L_1 + L_2) = \{k(p_1 + p_2) + tk(r+1)v_2 : t \in R\}$ $\quad kL_1 + kL_2 = \{kp_1 + tkrv_2 + kp_2 + tkv_2 : t \in R\}$ thus $k(L_1 + L_2) = kL_1 + kL_2$

3. W is the "zero vector" because $W + L = L$

4. for $L_1 = \{(1,0,0) + tv : t \in R\}$, $L_1 \in V/W$, we put L_1 into the basis

L_1 can't span $L_2 = \{(0,1,0) + tv : t \in R\}$, $L_2 \in V/W$ so we put L_2 into the basis

L_1 and L_2 can't span $L_3 = \{(0,0,1) + tv : t \in R\}$ $L_3 \in V/W$. so we put L_3 into the basis

then for arbitrary $L = \{(a,b,c) + tv : t \in R\}$, $L \in V/W$, we have $L = aL_1 + bL_2 + cL_3$ thus $\{L_1, L_2, L_3\}$ is a basis and the dimension of V/W is 3

Problem 5.6 1. Yes. we assume for finitely many polynomials, the maximum of the degree of polynomial is n

we choose $f(x) = x^{n+1}$. if $f(x)$ is in the span, then $f(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_m f_m(x) \Rightarrow |f(x)| \le (|a_1| + \cdots + |a_n|)(|x|^n + |x|^{n-1} + \cdots + |x| + 1)$

$|a_1| + \cdots + |a_n| = M$, then $|x|^{n+1} \le M \cdot \frac{|x|^{n+1} - 1}{|x| - 1} \le M \frac{|x|^{n+1}}{|x| - 1}$. if $x \ge M + 10$, we get $|x|^{n+1} > M \frac{|x|^{n+1}}{|x| - 1}$ which forms a contradiction

therefore $f(x)$ is not in the span

2. Yes. for $f(x), g(x) \in W$ $\quad f(x) = p(x) \cdot h(x)$ $\quad g(x) = p(x) \cdot r(x)$ $\quad kf(x) + lg(x) = (kh(x) + lr(x)) \cdot p(x)$. so $k \cdot f(x) + l \cdot g(x) \in W$, thus W is a subspace

3. $[r_1(x)] = \{r_1(x) + p(x) q_1(x) : q_1(x) \in V\}$ $\quad [r_2(x)] = \{r_2(x) + p(x) q_2(x) : q_2(x) \in V\}$

$[r_1(x)] + [r_2(x)] = \{r_1(x) + r_2(x) + p(x)(q_1(x) + q_2(x)) : q_1(x), q_2(x) \in V\}$

$[r_1(x) + r_2(x)] = \{r_1(x) + r_2(x) + p(x) q_3(x) : q_3(x) \in V\}$. let $q_1(x) + q_2(x) = q_3(x) \in V$. then $[r_1(x)] + [r_2(x)] = [r_1(x) + r_2(x)]$

$[kr_1(x)] = \{kr_1(x) + p(x) \cdot q_4(x) : q_4(x) \in V\}$ $\quad k[r_1(x)] = \{kr_1(x) + p(x) \cdot kq_1(x) : q_1(x) \in V\}$. let $q_4(x) = kq_1(x) \in V$. then $k[r_1(x)] = [kr_1(x)]$

4. if the degree of $r(x) \ge 2$. we can use $p(x) = x^2 + 3x + 2$ and $q(x)$ accordingly to reduce its degree

let's assume the degree of $r(x)$ is n. $r(x) = ax^n + a_1 x^{n-1} + \cdots$. choose $q(x) = ax^{n-2}$ and we can eliminate x^n and get $r'(x)$ whose degree $<$ $r(x)$

proceed the process similarly and we can get a polynomial whose degree is less than 2

we put $[r(x) = 1] = \{1 + p(x) q(x) : q(x) \in V\}$ into the basis. but we cannot express $x^2 + 4x + 2$ because it's $\{x + p(x) \cdot q(x) : q(x) \in R\}$

so we put $[r(x) = x]$ into the basis. then for polynomials whose degree ≤ 1 are in the span. for those degree ≥ 2. similar as above.

we can use $p(x) \cdot q(x)$ to reduce its degree below 2. so it's also in the span

therefore the dimension of V/W is 2 and a basis is $\{[1], [x]\}$.