HW9 2023010747

Problem 9.1 1. $C^TA = \det(A)I$

$\det(C^TA) = \det(\det(A)I) = (\det(A))^3$

$\det(C) = \det(C^T) = (\det(A))^2 = 25$

2. $$\begin{bmatrix} a_1 & 1 & 1 & \cdots & 1 \\ 1 & a_2 & & & \\ \vdots & & \ddots & & \\ 1 & & & & a_n \end{bmatrix} = a_1 \begin{bmatrix} a_2 & & \\ & \ddots & \\ & & a_n \end{bmatrix} - \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & a_3 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} + \cdots + (-1)^{n+1} \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ a_2 & & & \\ & a_3 & & \\ & & \ddots & \\ & & & a_{n-1} \end{bmatrix} = \prod_{i=1}^{n} a_i - \prod_{i=3}^{n} a_i + \cdots + (-1)^{n+1}(-1)^{n} \prod_{i=2}^{n-1} a_i = \prod_{i=2}^{n} a_i \left(a_1 - \sum_{j=2}^{n} a_j\right)$$

So $\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 5 \end{bmatrix}\right) = 120\left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5}\right) = 120 - 60 - 40 - 30 - 24 = -34$

Problem 9.2 1. if we want to get x^4, then each column and row we must pick entry with x

$$\begin{bmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}$$ so the coefficient is 2

similarly, if we want to get x^3 then we must pick entry with x in three columns and rows

$$\begin{bmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}$$ so the coefficient is -1

2. $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & 2a_{12} & 3a_{13} & 4a_{14} \\ a_{21} & 2a_{22} & 3a_{23} & 4a_{24} \\ a_{31} & 2a_{32} & 3a_{33} & 4a_{34} \\ a_{41} & 2a_{42} & 3a_{43} & 4a_{44} \end{bmatrix}$$ when we pick entries in each columns, their mutiple scales up $4!$

so $\det(A)$ change to $24\det(A)$

3. $A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ e & 0 & f & 0 \\ 0 & g & 0 & h \end{bmatrix}$ $\det(A) = \det\left(\left[\begin{array}{cc|cc} a & b & & \\ e & f & \multicolumn{2}{c}{0} \\ \hline & & c & d \\ \multicolumn{2}{c|}{0} & g & h \end{array}\right]\right) = \det\left(\begin{bmatrix} a & b \\ e & f \end{bmatrix}\right)\det\left(\begin{bmatrix} c & d \\ g & h \end{bmatrix}\right) = (af-be)(ch-gd) = acfh + bdeg - adfg - bceh$

Problem 9.3 1. $P_n = L_nU_n = L_nL_n^T$

$$P_n = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 3 & 6 & 10 \cdots & \\ \vdots & 4 & 10 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & n & & & C_{2n-2}^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ \vdots & 2 & 1 & & \\ \vdots & 3 & 3 & \ddots & \\ 1 & n-1 & \cdots & n-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ \vdots & 2 & 1 & & \\ \vdots & 3 & 3 & \ddots & \\ 1 & n-1 & \cdots & n-1 & 1 \end{bmatrix}^T$$

$\det(P_n) = \det(L_n)\det(L_n^T) = 1$

2. $\det(A_n) = \det(P_n) - (-1)^{2n}\det(P_{n-1}) = 0$

Problem 9.4 1. we define $R_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ 0 & \cdots & 1 & 2 \end{bmatrix}$

$\det(H_n) = 2\cdot\det(H_{n-1}) - 1\cdot\det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ 0 & 0 \cdots & 1 & 2 \end{bmatrix}\right) = 2\det(H_{n-1}) - \det(R_{n-1})$

$\det(R_n) = \det(H_{n-1}) - \det(R_{n-1})$

$$\begin{cases} \det(H_n) = 2\det(H_{n-1}) - \det(R_{n-1}) \\ \det(H_{n-1}) = 2\det(H_{n-2}) - \det(R_{n-2}) \end{cases} \Rightarrow \det(H_n) = \det(H_{n-1}) + 2\det(H_{n-2}) - \det(H_{n-2}) = \det(H_{n-1}) + \det(H_{n-2})$$

so $\det(H_n)$ is the Fibonacci sequence

2. we defin $T_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 1 & 3 \end{bmatrix}$

$\det(S_n) = 3\det(S_{n-1}) - \det(T_{n-1})$

$\det(T_n) = \det(S_{n-1}) - 0$

$\Rightarrow \det(S_n) = 3\det(S_{n-1}) - \det(S_{n-2})$

$\det(S_1)=3.\ \det(S_2)=8\ \ \det(S_3)=21\ \ \det(S_4)=55\ \ \det(S_5)=144\ \cdots$

$1\ 1\ 2\ 3\ 5\ 8\ 13\ 21\ 34\ 55\ 89\ 144\ \cdots$

Problem 9.5 1. $\frac{\partial f}{\partial a}=\frac{d}{ad-bc}\quad \frac{\partial f}{\partial b}=\frac{-c}{ad-bc}\quad \frac{\partial f}{\partial c}=\frac{-b}{ad-bc}\quad \frac{\partial f}{\partial d}=\frac{a}{ad-bc}$

2. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix}=\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \\ \frac{\partial f}{\partial c} & \frac{\partial f}{\partial d} \end{bmatrix}^T$

3. $\frac{\partial(\ln(\det(A)))}{\partial a_{ij}}=\frac{1}{\det(A)}\cdot\frac{\partial \det(A)}{\partial a_{ij}}=\frac{1}{\det(A)}C_{ij}$

let $f=\ln(\det(A))$. then since $C^TA=\det(A)I$

$$A^{-1}=\frac{1}{\det(A)}C^T=\begin{bmatrix} \frac{1}{\det(A)}C_{11} & \frac{1}{\det(A)}C_{12} & \cdots & \frac{1}{\det(A)}C_{1n} \\ \vdots & & & \\ \frac{1}{\det(A)}C_{n1} & \cdots & \cdots & \frac{1}{\det(A)}C_{nn} \end{bmatrix}^T=\begin{bmatrix} \frac{\partial f}{\partial a_{11}} & \cdots & \frac{\partial f}{\partial a_{1n}} \\ \vdots & & \vdots \\ \frac{\partial f}{\partial a_{n1}} & \cdots & \frac{\partial f}{\partial a_{nn}} \end{bmatrix}^T$$

Problem 9.6 1. $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A-\lambda I)=(1-\lambda)^3\quad \operatorname{Ker}(A-\lambda I)=\operatorname{Ker}(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$

so the eigenvalue of A is 1. and the eigenvectors of A for the eigenvalue 1 are $\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$ $(t\neq 0)$

2. $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$\det(A-\lambda I)=(1-\lambda)(2-\lambda)(3-\lambda)\quad \operatorname{Ker}(A-1\cdot I)=\operatorname{Ker}(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})\quad \operatorname{Ker}(A-2I)=\operatorname{Ker}(\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix})\quad \operatorname{Ker}(A-3I)=\operatorname{Ker}(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix})$

so the eigenvalues of A are 1.2.3 and the eigenvectors of A for the eigenvalue 1 are $\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$ $(t\neq 0)$

and the eigenvectors of A for the eigenvalue 2 are $\begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$ $(t\neq 0)$

and the eigenvectors of A for the eigenvalue 3 are $\begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix}$ $(t\neq 0)$

3. $A=\begin{bmatrix} 10 & 11 \\ 3 & 18 \end{bmatrix}$

$\det(A-\lambda I)=(10-\lambda)(18-\lambda)-33=\lambda^2-28\lambda+147=(\lambda-21)(\lambda-7)\quad \operatorname{Ker}(A-7I)=\operatorname{Ker}(\begin{bmatrix} 3 & 11 \\ 3 & 11 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 11 \\ -3 \end{bmatrix})\quad \operatorname{Ker}(A-21I)=\operatorname{Ker}(\begin{bmatrix} -11 & 11 \\ 3 & -3 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$

so the eigenvalues of A are 7.21 and the eigenvectors of A for the eigenvalue 7 are $\begin{bmatrix} 11t \\ -3t \end{bmatrix}$ $(t\neq 0)$

and the eigenvectors of A for the eigenvalue 21 are $\begin{bmatrix} t \\ t \end{bmatrix}$ $(t\neq 0)$

4. $A=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\det(A-\lambda I)=\lambda^2(1-\lambda)-(1-\lambda)=-(\lambda+1)(1-\lambda)^2\quad \operatorname{Ker}(A+I)=\operatorname{Ker}(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix})\quad \operatorname{Ker}(A-I)=\operatorname{Ker}(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix})=\operatorname{Ker}(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})$

so the eigenvalues of A are -1, 1 and the eigenvectors of A for the eigenvalue -1 are $\begin{bmatrix} t \\ 0 \\ -t \end{bmatrix}$ $(t\neq 0)$

and the eigenvectors of A for the eigenvalue 1 are $\begin{bmatrix} r \\ t \\ r \end{bmatrix}$ $(t\neq 0 \vee r\neq 0)$

5. $A=\begin{bmatrix} b & -a \\ a & b \end{bmatrix}$

$\det(A-\lambda I)=(b-\lambda)^2+a^2\quad \lambda=b\pm ai\quad \operatorname{Ker}(A-(b-ai)I)=\operatorname{Ker}(\begin{bmatrix} ai & -a \\ a & ai \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 1 \\ i \end{bmatrix})\quad \operatorname{Ker}(A-(b+ai)I)=\operatorname{Ker}(\begin{bmatrix} -ai & -a \\ a & -ai \end{bmatrix})=\operatorname{span}(\begin{bmatrix} -1 \\ i \end{bmatrix})$

so the eigenvalues of A are $b\pm ai$ and the eigenvectors of A for the eigenvalue $b-ai$ are $\begin{bmatrix} t \\ ti \end{bmatrix}$ $(t\in\mathbb{R}\wedge t\neq 0)$

and the eigenvectors of A for the eigenvalue $b+ai$ are $\begin{bmatrix} -t \\ ti \end{bmatrix}$ $(t\in\mathbb{R}\wedge t\neq 0)$

6. $A=\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix}$ $\begin{matrix} -\lambda & 1 & 2 \\ -1 & -\lambda & 2 \\ -2 & -2 & -\lambda \end{matrix}$ $A^T=-A$

$\det(A-\lambda I)=-\lambda^3+4-4-(4\lambda+4\lambda+\lambda)=-\lambda(\lambda^2+9)\quad \operatorname{Ker}(A)=\operatorname{Ker}(\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix})=\operatorname{span}(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix})\quad \operatorname{Ker}(A-3i)=\operatorname{Ker}(\begin{bmatrix} -3i & 1 & 2 \\ -1 & -3i & 2 \\ -2 & -2 & -3i \end{bmatrix})=\operatorname{span}(\begin{bmatrix} -1-3i \\ 1-3i \\ 4 \end{bmatrix})\quad \operatorname{Ker}(A+3i)=\operatorname{span}(\begin{bmatrix} -1+3i \\ 1+3i \\ 4 \end{bmatrix})$

so the eigenvalues of A are $0.\pm 3i$ and the eigenvectors of A for the eigenvalue 0 are $\begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix}$ $(t\neq 0)$

and the eigenvectors of A for the eigenvalue $3i$ are $\begin{bmatrix} (-1-3i)t \\ (1-3i)t \\ 4t \end{bmatrix}$ $(t\in\mathbb{R}\wedge t\neq 0)$

and the eigenvectors of A for the eigenvalue $-3i$ are $\begin{bmatrix} (-1+3i)t \\ (1+3i)t \\ 4t \end{bmatrix}$ $(t\in\mathbb{R}\wedge t\neq 0)$

Problem 9.7 1. $M_p=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\quad P_{M_p}(x)=x^2+1$ the eigenvalues of M_p are $\pm i$

2. $P_{M_p}(x)=x^2-3x+2$ the eigenvalues of M_p are 1, 2

3. the eigenvalues of M_p are $1, 2, 3$

4. $\det(xI - M_p) = \det\left(\begin{bmatrix} x & & & & a_0 \\ -1 & x & & & \vdots \\ & \ddots & \ddots & & \\ & & & x & a_{n-2} \\ & & & -1 & x+a_{n-1} \end{bmatrix}\right) = (-1)^{n+1} a_0 \cdot \det\left(\begin{bmatrix} -1 & x & & \\ & -1 & \ddots & \\ & & \ddots & x \\ & & & -1 \end{bmatrix}\right) + (-1)^{n+2} a_1 \det\left(\begin{bmatrix} x & & & \\ -1 & x & & \\ & \ddots & \ddots & x \\ & & & -1 \end{bmatrix}\right) + \cdots + (-1)^{2n}(x+a_{n-1}) \det\left(\begin{bmatrix} x & & & \\ -1 & x & & \\ & \ddots & \ddots & \\ & & -1 & x \end{bmatrix}\right)$

$= (-1)^{n+1} a_0 \cdot (-1)^{n-1} + (-1)^{n+2} a_1 (-1)^{n-2} x + \cdots + (-1)^{2n-1} a_{n-2} (-1)^1 x^{n-2} + (-1)^{2n} (x + a_{n-1}) x^{n-1}$

$= x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$

$= p(x)$

So the roots of p are just the eigenvalues of M_p

Problem 9.8 1. $4, 7$ are solutions of $p_A(x) = x(x-b) - a$

so $x^2 - bx - a = (x-4)(x-7) \Rightarrow a = -28,\ b = 11 \Rightarrow \begin{cases} a = -28 \\ b = 11 \end{cases}$

2. $2, 3, 4$ are solutions of $p_A(x) = x^2(x-c) - a - bx$

so $x^3 - cx^2 - bx - a = (x-2)(x-3)(x-4) \Rightarrow a = 24,\ b = -26,\ c = 9 \Rightarrow \begin{cases} a = 24 \\ b = -26 \\ c = 9 \end{cases}$

3. $p_A(\lambda) = \lambda(\lambda-2)(\lambda-x) + 8 - 4\lambda - 4(\lambda-2) = (\lambda-2)(\lambda^2 - \lambda x - 8)$

$p_B(\lambda) = (\lambda-2)^2(\lambda-y) = (\lambda-2)(\lambda^2 - (2+y)\lambda + 2y)$

$p_A(\lambda) = p_B(\lambda) \Rightarrow y = -4,\ x = -2 \Rightarrow \begin{cases} x = -2 \\ y = -4 \end{cases}$

4. $Ax = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & a \end{bmatrix} \begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} 3+b \\ 2+2b \\ a+b+1 \end{bmatrix} \qquad \lambda x = \begin{bmatrix} \lambda \\ \lambda b \\ \lambda \end{bmatrix}$

so $\begin{cases} 3+b = \lambda \\ 2+2b = b\lambda \\ a+b+1 = \lambda \end{cases}$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow 2+2b = b(3+b) \Rightarrow b = 1, -2$

① $b = 1,\ \lambda = 4 \Rightarrow a = 2$ $\qquad$ ② $b = -2,\ \lambda = 1\ \ a = 2$

$\begin{cases} a = 2 \\ b = 1 \end{cases} \qquad \begin{cases} a = 2 \\ b = -2 \end{cases}$