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Problem 3.1 Yes all columns parallel. since all rows parallel

let's assume $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ k_1a_1 & k_1a_2 & \cdots & k_1a_n \\ \vdots & \vdots & & \\ k_{n-1}a_1 & \cdots & \cdots & k_{n-1}a_n \end{bmatrix}$ for arbitary two columns $b_i = \begin{pmatrix} a_i \\ k_1a_i \\ \vdots \\ k_{n-1}a_i \end{pmatrix}$ $b_j = \begin{pmatrix} a_j \\ k_1a_j \\ \vdots \\ k_{n-1}a_j \end{pmatrix}$ if a_i or $a_j = 0$, since 0 parallel to everything, we see $b_i // b_j$

if $a_i . a_j \neq 0$. we have $b_j = \frac{a_j}{a_i} \cdot b_i$. so $b_i // b_j$

Problem 3.2 1. $a^T = (a_{11}\ a_{12} \cdots a_{1n})$ since the columns are all arithmetic sequences.

$b^T = (a_{21}\ a_{22} \cdots a_{2n})$

we create $d^T = (d_1\ d_2 \cdots d_n)$ $d_i = a_{2i} - a_{1i}$ so $d^T = b^T - a^T$

for the nth row. $(n>2)$. we have $r_n^T = a^T + (n-1)d^T = (n-1)b^T - (n-2)a^T$

2. if rank ≥ 3. we have at least 3 effective equations

let's assume they are r_i^T, r_j^T, r_k^T $(k>j>i)$

$r_j^T - r_i^T = (j-i)b^T - (j-i)a^T = (j-i)(b^T - a^T)$

$r_k^T - r_j^T = (k-j)(b^T - a^T)$

therefore $r_k^T - r_j^T = \frac{k-j}{j-i}(r_j^T - r_i^T) \Rightarrow r_k^T = \frac{k-j}{j-i}(r_j^T - r_i^T) + r_j^T$ the kth row is redundant

so rank $< 3 \Rightarrow$ rank at most 2

Problem 3.3 $A\ {}_3[\underbrace{\cdots\cdots}_{5}]$ $B\ {}_5\left[\overset{3}{\vdots}\right]$ $C\ {}_3\left[\overset{1}{\vdots}\right]$

$\left.\begin{array}{l} \text{BA is } 3\times3 \text{ matrix} \\ \text{AB is } 5\times5 \text{ matrix} \\ \text{ABAB is } 5\times5 \text{ matrix} \\ \text{BABC is } 5\times1 \text{ matrix} \end{array}\right\}$ well-defined

BAC is not well-defined

Problem 3.4 1.

$$\begin{bmatrix} a & c \\ b & -a \end{bmatrix}\begin{bmatrix} a & c \\ b & -a \end{bmatrix} = \begin{bmatrix} a^2+bc & 0 \\ 0 & cb-a^2 \end{bmatrix}$$

therefore $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^2 = \begin{bmatrix} 121-120 & 0 \\ 0 & 121-120 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^3 = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}$

3. $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^n$

$\begin{cases} n=1 \text{ we have } \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} \\ n=2 \text{ we have } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$

if $\begin{cases} n=2k-1 \text{ we have } \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} \\ n=2k \text{ we have } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$

then when $\begin{cases} n=2k+1 & \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^{2k+1} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} \\ n=2k+2 & \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^{2k+1} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$

so $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^{2020} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

we already know that $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}$ is inversible

thus $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^{-1} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}$

5. $\begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}^2 = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}\begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix} = \begin{bmatrix} 4&4&4&4 \\ 4&4&4&4 \\ 4&4&4&4 \\ 4&4&4&4 \end{bmatrix}$

6. $\begin{bmatrix} a&a&a&a \\ a&a&a&a \\ a&a&a&a \\ a&a&a&a \end{bmatrix}\begin{bmatrix} b&b&b&b \\ b&b&b&b \\ b&b&b&b \\ b&b&b&b \end{bmatrix} = \begin{bmatrix} 4ab&4ab&4ab&4ab \\ 4ab&4ab&4ab&4ab \\ 4ab&4ab&4ab&4ab \\ 4ab&4ab&4ab&4ab \end{bmatrix}$

so we use mathematical induction $n=1$ $\begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix} = \begin{bmatrix} 4^0&4^0&4^0&4^0 \\ 4^0&4^0&4^0&4^0 \\ 4^0&4^0&4^0&4^0 \\ 4^0&4^0&4^0&4^0 \end{bmatrix}$

if $n=k$ $\begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}^k = \begin{bmatrix} 4^{k-1} & \cdots & 4^{k-1} \\ \vdots & & \\ 4^{k-1} & & 4^{k-1} \end{bmatrix}$

when $n=k+1$ $\begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}\begin{bmatrix} 4^{k-1} & \cdots & 4^{k-1} \\ \vdots & & \\ 4^{k-1} & & 4^{k-1} \end{bmatrix} = \begin{bmatrix} 4^k & \cdots & 4^k \\ \vdots & & \\ 4^k & & 4^k \end{bmatrix}$

so $\begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}^3 = \begin{bmatrix} 4^2&4^2&4^2&4^2 \\ 4^2&4^2&4^2&4^2 \\ 4^2&4^2&4^2&4^2 \\ 4^2&4^2&4^2&4^2 \end{bmatrix}$

7. $\begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}^{2020} = \begin{bmatrix} 4^{2019}&4^{2019}&4^{2019}&4^{2019} \\ 4^{2019}&4^{2019}&4^{2019}&4^{2019} \\ 4^{2019}&4^{2019}&4^{2019}&4^{2019} \\ 4^{2019}&4^{2019}&4^{2019}&4^{2019} \end{bmatrix}$

8. $$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

9. $$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -4 & -4 & -4 \\ -4 & 4 & -4 & -4 \\ -4 & -4 & 4 & -4 \\ -4 & -4 & -4 & 4 \end{bmatrix}$$

10. $$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^2 = 4I$$

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{2020} = \left(\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^2\right)^{1010} = 4^{1010} I^{1010} = \begin{bmatrix} 4^{1010} & 0 & 0 & 0 \\ 0 & 4^{1010} & 0 & 0 \\ 0 & 0 & 4^{1010} & 0 \\ 0 & 0 & 0 & 4^{1010} \end{bmatrix}$$

11. $$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4I$$

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \frac14 & -\frac14 & -\frac14 & -\frac14 \\ -\frac14 & \frac14 & -\frac14 & -\frac14 \\ -\frac14 & -\frac14 & \frac14 & -\frac14 \\ -\frac14 & -\frac14 & -\frac14 & \frac14 \end{bmatrix} = I$$

we also know $\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{-1} = I$ and $\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ is inversible

so $$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac14 & -\frac14 & -\frac14 & -\frac14 \\ -\frac14 & \frac14 & -\frac14 & -\frac14 \\ -\frac14 & -\frac14 & \frac14 & -\frac14 \\ -\frac14 & -\frac14 & -\frac14 & \frac14 \end{bmatrix}$$

Problem 3.5 1. they are not the same

$$(A+B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix} \neq \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = (A+B)^2$$

2. they are not the same

$$(AB)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$A^2B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \neq \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

Problem 3.6 1. $X_{13} = e_1 e_3^T$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_{32} = e_3 e_2^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_{12} = e_1 e_2^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. $$X_{13}X_{32} = e_1 e_3^T e_3 e_2^T = e_1(e_3^T e_3)e_2^T = e_1 e_2^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_{32}X_{13} = e_3 e_2^T e_1 e_3^T = e_3(e_2^T e_1)e_3^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. $X_{ij}^2 = (e_i e_j^T)(e_i e_j^T) = e_i(e_j^T e_i)e_j^T = 0$ because $i \neq j$

4. $(A-I)(B-I) = AB - BI - AI + I^2$

$(B-I)(A-I) = BA - AI - BI + I^2$

therefore $(A-I)(B-I) = (B-I)(A-I) \Leftrightarrow AB = BA$

Problem 3.7 1. shifting things up

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \cdots & & & \vdots \\ a_{31} & \cdots & & & \vdots \\ a_{41} & \cdots & & & a_{4n} \end{bmatrix}$$

$$JA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \vdots \\ a_{31} & \cdots & & \vdots \\ a_{41} & \cdots & & a_{4n} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & \cdots & & \vdots \\ a_{41} & \cdots & & a_{4n} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (or simply notice that J is a row operation to I)

2. shifting things right

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \vdots & & \cdots & \vdots \\ a_{n1} & & \cdots & a_{n4} \end{bmatrix}$$

$$AJ = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & \cdots & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{n4} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{11} & \cdots & a_{13} \\ 0 & a_{21} & \cdots & a_{23} \\ 0 & \vdots & \cdots & \vdots \\ 0 & a_{n1} & \cdots & a_{n3} \end{bmatrix}$$ (or simply notice that J is a column operation to I)

3. $$PJ = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{bmatrix}$$

$$J^T P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \end{bmatrix}$$

$PJ + J^T P = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = P$ the "coincidence" is because for Pascal's matrix, $a_{ij} + a_{(i+1)(j-1)} = a_{(i+1)j}$

4. $J = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$ $J^2 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$ $J^3 = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$ $J^4 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$

5. $(I-J)(I+J+J^2+J^3)$

$= I^2+IJ+IJ^2+IJ^3-JI-J^2-J^3-J^4$

we already know $J^4=0$, $AI=IA=A$, $I^2=I$

so $(I-J)(I+J+J^2+J^3) = I+J+J^2+J^3-J-J^2-J^3 = I$

therefore $I-J$ has inverse $I+J+J^2+J^3$

6. $(J+I)^2 = J^2+JI+IJ+I^2 = J^2+2J+I = \begin{bmatrix} 1&2&1&0 \\ 0&1&2&1 \\ 0&0&1&2 \\ 0&0&0&1 \end{bmatrix}$

$(J+I)^3 = J^3+2J^2+JI+IJ^2+2IJ+I^2 = J^3+3J^2+3J+I = \begin{bmatrix} 1&3&3&1 \\ 0&1&3&3 \\ 0&0&1&3 \\ 0&0&0&1 \end{bmatrix}$

$(J+I)^4 = J^4+4J^3+6J^2+4J+I = \begin{bmatrix} 1&4&6&4 \\ 0&1&4&6 \\ 0&0&1&4 \\ 0&0&0&1 \end{bmatrix}$

$(J+I)^x = J^x+C_x^1 J^{x-1}+C_x^2 J^{x-2}+\cdots+C_x^{x-1} J+I^x = \begin{bmatrix} 1&x&C_x^2&C_x^3 \\ 0&1&x&C_x^2 \\ 0&0&1&x \\ 0&0&0&1 \end{bmatrix}$

7. $AJ = JA$

$\Leftrightarrow \begin{bmatrix} a_{21}&a_{22}&\cdots&a_{2m} \\ a_{31}&\cdots&&\vdots \\ a_{41}&\cdots&&a_{4m} \\ 0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&a_{11}&\cdots&a_{13} \\ 0&a_{21}&\cdots&a_{23} \\ 0&\vdots&\ddots&\vdots \\ 0&a_{n1}&\cdots&a_{n3} \end{bmatrix}$

$n=m=4$

$a_{21}=a_{31}=a_{41}=0$, $a_{41}=a_{42}=a_{43}=0$

$\Leftrightarrow \begin{bmatrix} 0&a_{22}&a_{23}&a_{24} \\ 0&a_{32}&a_{33}&a_{34} \\ 0&a_{42}&a_{43}&a_{44} \\ 0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&a_{11}&a_{12}&a_{13} \\ 0&a_{21}&a_{22}&a_{23} \\ 0&a_{31}&a_{32}&a_{33} \\ 0&0&0&0 \end{bmatrix}$

$a_{11}=a_{22}=a_{33}=a_{44}$, $a_{42}=a_{31}=0$, $a_{43}=a_{32}=a_{21}=0$

$a_{24}=a_{13}$ $a_{34}=a_{23}=a_{12}$

so A have to be the form $\begin{bmatrix} a&b&c&d \\ 0&a&b&c \\ 0&0&a&b \\ 0&0&0&a \end{bmatrix}$ $\forall a.b.c.d \in R$

Problem 3.8 1. $AI = A$

if $B = 4I = \begin{bmatrix} 4&0&0 \\ 0&4&0 \\ 0&0&4 \end{bmatrix}$ then $AB = 4AI = 4A$

2.

if $B = 4I = \begin{bmatrix} 4&0&0 \\ 0&4&0 \\ 0&0&4 \end{bmatrix}$ then $BA = 4IA = 4A$

3. $B = \begin{bmatrix} 1&0&0 \\ 1&0&0 \\ \vdots&\vdots&\vdots \\ 1&0&0 \end{bmatrix}$ the number of rows are arbitary

4. $B = \begin{bmatrix} \frac{1}{3}&\frac{1}{3}&\cdots&\frac{1}{3} \\ \frac{1}{3}&\frac{1}{3}&\cdots&\frac{1}{3} \\ \frac{1}{3}&\frac{1}{3}&\cdots&\frac{1}{3} \end{bmatrix}$ the number of columns are arbitary

5. $B = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$

6. B is a 2×2 matrix

$B = \begin{bmatrix} a&c \\ b&d \end{bmatrix}$

$B\begin{bmatrix} 1&1 \\ 1&1 \end{bmatrix} = \begin{bmatrix} a+c&a+c \\ b+d&b+d \end{bmatrix}$ $\begin{bmatrix} 1&1 \\ 1&1 \end{bmatrix}B = \begin{bmatrix} a+b&c+d \\ a+b&c+d \end{bmatrix}$

$a+c=a+b$, $b+d=a+b$, $a+c=c+d$, $b+d=c+d$

$\Rightarrow b=c$, $a=d$

$B = \begin{bmatrix} a&b \\ b&a \end{bmatrix}$ $\forall a,b \in R$

Problem 3.9 1. $A^2 = A$

$x \geq 3$ $A^x = A^{x-2}A^2 = A^{x-1}$

so $A^x = A^{x-1} = \cdots = A$

$A^3+2A^2-A-I = A+2A-A-I = 2A-I$

$A^2+3A+4I = A+3A+4I = 4A+4I$

2. $(I+2A)(I+2A)^{-1} = I$

let's compute $(I+2A)(sA+tI) = sIA+2sA^2+tI^2+2tAI = tI+sA+2sA+2tA$

if $t=1$. $3s+2t=0 \Rightarrow \begin{cases} t=1 \\ s=-\frac{2}{3} \end{cases}$ then $(I+2A)(sA+tI) = I$

we also know that $I+2A$ is inversible

thus $(I+2A)^{-1} = sA+tI = -\frac{2}{3}A+I$