Problem 2.1 by 刘一[illegible] 2023010747

1. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 4 \\ 7 & 8 & 9 & 7 \end{array}\right] \xrightarrow{r_2 \to r_2 - 4r_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 0 \\ 7 & 8 & 9 & 7 \end{array}\right] \xrightarrow{r_3 \to r_3 - 7r_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \xrightarrow{r_3 \to r_3 - 2r_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{REF}} \xrightarrow{r_2 \to -\frac{1}{3}r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \to r_1 - 2r_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RREF}}$

$\begin{cases} x = z+1 \\ y = -2z \end{cases}$

2. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 4 \end{array}\right] \xrightarrow{r_2 \to r_2 - 4r_1} \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 0 \end{array}\right]}_{\text{REF}} \xrightarrow{r_2 \to -\frac{1}{3}r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{r_1 \to r_1 - 2r_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right]}_{\text{RREF}}$

$\begin{cases} x = z+1 \\ y = -2z \end{cases}$

3. $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 7 & 8 & 8 \end{array}\right] \xrightarrow{r_2 \to r_2 - 4r_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 7 & 8 & 8 \end{array}\right] \xrightarrow{r_3 \to r_3 - 7r_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & -6 & 1 \end{array}\right] \xrightarrow{r_3 \to r_3 - 2r_2} \underbrace{\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{\text{REF}} \xrightarrow{r_2 \to -\frac{1}{3}r_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \to r_1 - 2r_2} \underbrace{\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{\text{RREF}}$

we get $0=1$ which is impossible. therefore there is no solution

Problem 2.2

1. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & b & 7 \end{array}\right] \xrightarrow{r_2 \to r_2 - 3r_1} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & b-6 & -2 \end{array}\right]$

if $b-6 \neq 0$ then we can do $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & b-6 & -2 \end{array}\right] \xrightarrow{r_2 \to \frac{r_2}{b-6}} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & \frac{-2}{b-6} \end{array}\right] \xrightarrow{r_1 \to r_1 - 2r_2} \left[\begin{array}{cc|c} 1 & 0 & 3+\frac{4}{b-6} \\ 0 & 1 & \frac{-2}{b-6} \end{array}\right]$ which means that $\begin{cases} x = 3+\frac{4}{b-1} \\ y = \frac{-2}{b-6} \end{cases}$ thus we have a solution

if $b-6=0 \Rightarrow b=6$ $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & -2 \end{array}\right]$ we get $0=-2$ which is impossible. therefore there is no solution

so $b=6$

2. $\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 6 & 4 & b \end{array}\right] \xrightarrow{r_2 \to r_2 - 2r_1} \left[\begin{array}{cc|c} 3 & 2 & 10 \\ 0 & 0 & b-20 \end{array}\right]$

if $b=20$. then $\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \to \frac{1}{3}r_1} \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{10}{3} \\ 0 & 0 & 0 \end{array}\right]$ we have infinite solutions $x = -\frac{2}{3}y + \frac{10}{3}$

if $b \neq 20$ then $\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 0 & 0 & b-20 \end{array}\right]$ we get $0=b-20$ which is impossible. therefore there is no solution

so $b \neq 20$. b can be 1

4 10 2

3. $\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & b & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{r_2 \to r_2 - 2r_1} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & b-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & b-10 & -1 & 2 \end{array}\right]$

if $b=10$. then $\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & b-10 & -1 & 2 \end{array}\right] \xrightarrow[r_3 \to -r_3]{r_1 \to \frac{1}{2}r_1} \left[\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[r_2 \to r_2 + r_3]{r_1 \to r_1 - \frac{1}{2}r_3} \left[\begin{array}{ccc|c} 1 & \frac{5}{2} & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{r_1 \to r_1 - \frac{5}{2}r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$ we get a solution $\begin{cases} x = -\frac{3}{2} \\ y = 1 \\ z = -2 \end{cases}$

if $b \neq 10$ then $\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & b-10 & -1 & 2 \end{array}\right] \xrightarrow{r_3 \to r_3 - (b-10)r_2} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1+b-10 & 2-3(b-10) \end{array}\right]$

if $b=11$. $\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$ we get $0=-1$ which is impossible. therefore there is no solution

if $b \neq 11$ $\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & b-11 & -3b+32 \end{array}\right]$ since the last column isn't pivotal column. there has to be at least one solution.

so $b=11$

4. $\left[\begin{array}{cc|c} b & 3 & 6 \\ 3 & b & -6 \end{array}\right]$ if $b=0$ $\left[\begin{array}{cc|c} 0 & 3 & 6 \\ 3 & 0 & -6 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} 3 & 0 & -6 \\ 0 & 3 & 6 \end{array}\right] \xrightarrow[r_1 \to \frac{1}{3}r_1]{r_2 \to \frac{1}{3}r_2} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 2 \end{array}\right]$ $\begin{cases} x = -2 \\ y = 2 \end{cases}$

if $b \neq 0$ $\left[\begin{array}{cc|c} b & 3 & 6 \\ 3 & b & -6 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} 3 & b & -6 \\ b & 3 & 6 \end{array}\right] \xrightarrow{r_2 \to r_2 - \frac{b}{3}r_1} \left[\begin{array}{cc|c} 3 & b & -6 \\ 0 & 3-\frac{b^2}{3} & 6+2b \end{array}\right]$ if $3 - \frac{b^2}{3} = 0$

(1) $b=3$ $\left[\begin{array}{cc|c} 3 & 3 & -6 \\ 0 & 0 & 12 \end{array}\right]$ we get $0=12$ which is impossible. therefore there is no solution

(2) $b=-3$ $\left[\begin{array}{cc|c} 3 & -3 & -6 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \to \frac{1}{3}r_1} \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 0 & 0 \end{array}\right]$

so $b=-3$

5. $\left[\begin{array}{cc|c} 2 & b & 16 \\ 4 & 8 & c \end{array}\right] \xrightarrow{r_2 \to r_2 - 2r_1} \left[\begin{array}{cc|c} 2 & b & 16 \\ 0 & 8-2b & c-32 \end{array}\right]$

if $8-2b=0 \Rightarrow b=4$ if $c=32$ $\left[\begin{array}{cc|c} 2 & 4 & 16 \\ 0 & 0 & 0 \end{array}\right]$ no pivot in the last column and has less pivots than variables so there are infinit solutions

if $c \neq 32$ we get $0=c-32$ which is impossible. therefore there is no solution

if $8-2b \neq 0$ the last column can't be pivotal column.
and we have the same number of pivots as variables.
then there is a unique solution

if $3-\frac{b^2}{3} \neq 0$

no pivot in the last column
and has less pivots than variables
so there are infinit solutions

the last column can't be pivotal column.
and we have the same number of pivots as variables.
then there is a unique solution

So $b=4$. $c=32$

6. $\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 1 & b & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_2 \to r_2 - r_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & b+2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & b+2 & 1 & 0 \end{array}\right]$

if $b+2=0$ $\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ we have $z=0$, $y+z=0 \Rightarrow y=0$. $x-2y-z=0 \Rightarrow x=0$ X

if $b+2 \neq 0$

$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & b+2 & 1 & 0 \end{array}\right] \xrightarrow{r_3 \to r_3 - (b+2)r_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1-(b+2) & 0 \end{array}\right]$ if $1-(b+2) \neq 0$ we get $1-(b+2)=0$ which is impossible. therefore there is no solution

if $1-(b+2)=0 \Rightarrow b=-1$ $\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ we have a non-zero solution $\begin{cases} x=-1 \\ y=-1 \\ z=1 \end{cases}$

so $b=-1$

7. $\left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ b & b & 4 & 0 \\ b & b & b & 0 \end{array}\right]$ if $b=0$ $\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ we have a non-zero solution $\begin{cases} x=1 \\ y=0 \\ z=0 \end{cases}$

if $b \neq 0$ $\left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ b & b & 4 & 0 \\ b & b & b & 0 \end{array}\right] \xrightarrow[r_3 \to r_3 - r_1]{r_2 \to r_2 - r_1} \left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ 0 & b-2 & 1 & 0 \\ 0 & b-2 & b-3 & 0 \end{array}\right]$

if $b-2=0$ $\left[\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$ we have a non-zero solution $\begin{cases} x=-1 \\ y=1 \\ z=0 \end{cases}$

if $b-2 \neq 0$ $\left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ 0 & b-2 & 1 & 0 \\ 0 & b-2 & b-3 & 0 \end{array}\right] \xrightarrow{r_3 \to r_3 - r_2} \left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ 0 & b-2 & 1 & 0 \\ 0 & 0 & b-4 & 0 \end{array}\right]$

if $b-4 \neq 0$
we get $b-4=0$ which is impossible. therefore there is no solution

if $b-4=0$ $\left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ we have a non-zero solution $\begin{cases} x=1 \\ y=1 \\ z=-2 \end{cases}$

so $b=0$ or $b=2$ or $b=4$

Problem 2.3

$\left[\begin{array}{ccc|c} p & 1 & 1 & 1 \\ 1 & p & 1 & p \\ 1 & 1 & p & p^2 \end{array}\right] \xrightarrow{r_1 \to r_1 + r_2 + r_3} \left[\begin{array}{ccc|c} p+2 & p+2 & p+2 & p^2+p+1 \\ 1 & p & 1 & p \\ 1 & 1 & p & p^2 \end{array}\right]$

if $p+2=0$ $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 3 \\ 1 & -2 & 1 & -2 \\ 1 & 1 & -2 & 4 \end{array}\right]$ we get $0=3$ which is impossible. therefore there is no solution

p

if $p+2 \neq 0$ $\left[\begin{array}{ccc|c} p+2 & p+2 & p+2 & p^2+p+1 \\ 1 & p & 1 & p \\ 1 & 1 & p & p^2 \end{array}\right] \xrightarrow{r_1 \to \frac{r_1}{p+2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{p^2+p+1}{p+2} \\ 1 & p & 1 & p \\ 1 & 1 & p & p^2 \end{array}\right] \xrightarrow[r_2 \to r_2 - r_1]{r_3 \to r_3 - r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{p^2+p+1}{p+2} \\ 0 & p-1 & 0 & \frac{p-1}{p+2} \\ 0 & 0 & p-1 & \frac{p^3+p^2-p-1}{p+2} \end{array}\right]$

if $p-1=0 \Rightarrow p=1$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x+y+z=1$ we get infinite solutions $x=1-y-z$

if $p-1 \neq 0$ $(p-1)y=\frac{p-1}{p+2}$, $(p-1)z=\frac{p^3+p^2-p-1}{p+2} \Rightarrow y=\frac{1}{p+2}$, $z=\frac{(p+1)^2(p-1)}{(p+2)(p-1)}=\frac{(p+1)^2}{p+2}=\frac{p^2+2p+1}{p+2}=p+\frac{1}{p+2}$

$x+y+z=\frac{p^2+p+1}{p+2} \Rightarrow x=\frac{p^2+p+1-1-p^2-2p-1}{p+2}=\frac{-p-1}{p+2}=-1+\frac{1}{p+2}$

Problem 2.4

1. $A\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. A is a 3×3 matrix $(f(\vec{e}_1), f(\vec{e}_2), f(\vec{e}_3))$

$A\begin{bmatrix}2\\0\\0\end{bmatrix} = 2f(\vec{e}_1) \Rightarrow f(\vec{e}_1) = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $A\begin{bmatrix}1\\1\\0\end{bmatrix} = f(\vec{e}_1)+f(\vec{e}_2) \Rightarrow f(\vec{e}_2) = \begin{bmatrix}0\\0\\0\end{bmatrix} - \begin{bmatrix}1\\2\\1\end{bmatrix}$, $A\begin{bmatrix}0\\0\\1\end{bmatrix} = f(\vec{e}_3) \Rightarrow f(\vec{e}_3) = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$A = \begin{bmatrix}1 & -1 & 0\\ 2 & -2 & 0\\ 1 & -1 & 0\end{bmatrix}$$

Problem 2.5

1. two free variables. one dependent variables

2. $\left[\begin{array}{ccc|c}1 & -1 & 0 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$

3. $\left[\begin{array}{ccc|c}1 & -1 & 0 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \xrightarrow{r_2 \to r_2 + 2r_1} \left[\begin{array}{ccc|c}1 & -1 & 0 & 2\\ 2 & -2 & 0 & 4\\ 0 & 0 & 0 & 0\end{array}\right] \xrightarrow{r_3 \to r_3 + r_1} \left[\begin{array}{ccc|c}1 & -1 & 0 & 2\\ 2 & -2 & 0 & 4\\ 1 & -1 & 0 & 2\end{array}\right]$

$$A = \begin{bmatrix}1 & -1 & 0\\ 2 & -2 & 0\\ 1 & -1 & 0\end{bmatrix}$$

Problem 2.6

1. Swapping: swap the first two columns
2. scaling: multiply the first column by $\frac{1}{2}$
3. shearing: add the second column multiplied by -2 to the first column
4. add the first column to the "augmented column"