Linear algebra HW4 刘一[illegible] 2023010727 计32

Problem 4.1

1. $$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$
$$=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}3&2&1\\6&5&4\\9&8&7\end{bmatrix}$$
$$=\begin{bmatrix}9&8&7\\6&5&4\\3&2&1\end{bmatrix}$$

2. $$\begin{bmatrix}1&0&0\\-1&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\1&3&1\\1&4&0\end{bmatrix}=\begin{bmatrix}1&2&3\\0&1&-2\\0&2&-3\end{bmatrix}$$

3. $$\begin{bmatrix}1&0&0&0&0\\-1&1&0&0&0\\-1&0&1&0&0\\-1&0&0&1&0\\-1&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&2&1&0&0\\1&3&3&1&0\\1&4&6&4&1\end{bmatrix}$$
$$=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&2&1&0&0\\0&3&3&1&0\\0&4&6&4&1\end{bmatrix}$$

4. $$\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}^k=\begin{cases}\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix} & k\equiv 2 \pmod 4\\[2ex] \begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix} & k\equiv 3 \pmod 4\\[2ex] \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} & k\equiv 0 \pmod 4\\[2ex] \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix} & k\equiv 1 \pmod 4\end{cases}$$

Problem 4.2

1. $$\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\-1&2&-1&0&0&1&0&0\\0&-1&2&-1&0&0&1&0\\0&0&-1&2&0&0&0&1\end{array}\right]\xrightarrow{r_2\to r_2+\frac{r_1}{2}}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&\frac32&-1&0&\frac12&1&0&0\\0&-1&2&-1&0&0&1&0\\0&0&-1&2&0&0&0&1\end{array}\right]\xrightarrow{r_3\to r_3+\frac23 r_2}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&\frac32&-1&0&\frac12&1&0&0\\0&0&\frac43&-1&\frac13&\frac23&1&0\\0&0&-1&2&0&0&0&1\end{array}\right]\xrightarrow{r_4\to r_4+\frac34 r_3}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&\frac32&-1&0&\frac12&1&0&0\\0&0&\frac43&-1&\frac13&\frac23&1&0\\0&0&0&\frac54&\frac14&\frac24&\frac34&1\end{array}\right]$$

$$\xrightarrow{r_4\to \frac45 r_4}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&\frac32&-1&0&\frac12&1&0&0\\0&0&\frac43&-1&\frac13&\frac23&1&0\\0&0&0&1&\frac15&\frac25&\frac35&\frac45\end{array}\right]\xrightarrow{r_3\to r_3+r_4}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&\frac32&-1&0&\frac12&1&0&0\\0&0&\frac43&0&\frac{8}{15}&\frac{16}{15}&\frac85&\frac45\\0&0&0&1&\frac15&\frac25&\frac35&\frac45\end{array}\right]\xrightarrow{r_3\to \frac34 r_3}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&\frac32&-1&0&\frac12&1&0&0\\0&0&1&0&\frac25&\frac45&\frac65&\frac35\\0&0&0&1&\frac15&\frac25&\frac35&\frac45\end{array}\right]\xrightarrow{r_3\to \frac34 r_3}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&\frac32&0&0&\frac{9}{10}&\frac95&\frac65&\frac35\\0&0&1&0&\frac25&\frac45&\frac65&\frac35\\0&0&0&1&\frac15&\frac25&\frac35&\frac45\end{array}\right]$$

$$\xrightarrow{r_3\to \frac34 r_3}\left[\begin{array}{cccccccc}2&-1&0&0&1&0&0&0\\0&1&0&0&\frac35&\frac65&\frac45&\frac25\\0&0&1&0&\frac25&\frac45&\frac65&\frac35\\0&0&0&1&\frac15&\frac25&\frac35&\frac45\end{array}\right]\xrightarrow{r_3\to \frac34 r_3}\left[\begin{array}{cccccccc}2&0&0&0&\frac85&\frac65&\frac45&\frac25\\0&1&0&0&\frac35&\frac65&\frac45&\frac25\\0&0&1&0&\frac25&\frac45&\frac65&\frac35\\0&0&0&1&\frac15&\frac25&\frac35&\frac45\end{array}\right]\xrightarrow{r_3\to \frac34 r_3}\left[\begin{array}{cccccccc}1&0&0&0&\frac45&\frac35&\frac25&\frac15\\0&1&0&0&\frac35&\frac65&\frac45&\frac25\\0&0&1&0&\frac25&\frac45&\frac65&\frac35\\0&0&0&1&\frac15&\frac25&\frac35&\frac45\end{array}\right]$$

so $$\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}^{-1}=\begin{bmatrix}\frac45&\frac35&\frac25&\frac15\\\frac35&\frac65&\frac45&\frac25\\\frac25&\frac45&\frac65&\frac35\\\frac15&\frac25&\frac35&\frac45\end{bmatrix}$$

2. $$\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

$$\left[\begin{array}{cccccccc}1&0&0&0&1&0&0&0\\-1&1&0&0&0&1&0&0\\0&-1&1&0&0&0&1&0\\0&0&-1&1&0&0&0&1\end{array}\right]\xrightarrow{r_2\to r_2+r_1}\left[\begin{array}{cccccccc}1&0&0&0&1&0&0&0\\0&1&0&0&1&1&0&0\\0&-1&1&0&0&0&1&0\\0&0&-1&1&0&0&0&1\end{array}\right]\xrightarrow{r_3\to r_3+r_2}\left[\begin{array}{cccccccc}1&0&0&0&1&0&0&0\\0&1&0&0&1&1&0&0\\0&0&1&0&1&1&1&0\\0&0&-1&1&0&0&0&1\end{array}\right]\xrightarrow{r_4\to r_4+r_3}\left[\begin{array}{cccccccc}1&0&0&0&1&0&0&0\\0&1&0&0&1&1&0&0\\0&0&1&0&1&1&1&0\\0&0&0&1&1&1&1&1\end{array}\right]$$

so $$\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$$, symmetrically we have $$\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$$

so $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

3.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[r_1 \leftrightarrow r_3]{r_2 \leftrightarrow r_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

so $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

4.

$$\begin{bmatrix} & & 1 & 1 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & 1 \end{bmatrix} \xrightarrow{r_i \leftrightarrow r_{n+1-i}} \begin{bmatrix} 1 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 1 & 1 & & \end{bmatrix}$$

so $\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}^{-1} = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$

5.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \to r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \xrightarrow{r_2 \to -\frac{r_2}{2}} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow{r_1 \to r_1 - 2r_2} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

so $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

6.

$$\begin{bmatrix} 3 & -1 & -1 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} -1 & 3 & -1 & 0 & 1 & 0 \\ 3 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \\ 3 & -1 & -1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[r_2 \to r_2 - r_1]{r_3 \to r_3 + 3r_1} \begin{bmatrix} -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & 0 & -1 & 1 \\ 0 & 8 & -4 & 1 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{r_3 \to r_3 + 2r_2} \begin{bmatrix} -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & 0 & -1 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{r_3 \to \frac{r_3}{4}} \begin{bmatrix} -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & 0 & -1 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \xrightarrow[r_2 \to r_2 - 4r_3]{r_1 \to r_1 + r_3} \begin{bmatrix} -1 & 3 & 0 & \frac{1}{4} & \frac{5}{4} & \frac{1}{2} \\ 0 & -4 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \xrightarrow{r_2 \to \frac{-r_2}{4}} \begin{bmatrix} -1 & 3 & 0 & \frac{1}{4} & \frac{5}{4} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{r_1 \to -r_1 + 3r_2} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

so $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

7.

$$\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \to r_1 + ar_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ so $\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

similarly $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

so $\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Problem 4.3 northwest matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \iddots & \\ a_{n1} & & \end{bmatrix}_{n \times n}$

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ a_{1n} & & \end{bmatrix}_{n\times n}$$ is also northwest matrix

A^2 is an undefined matrix that we can't know its kind (just a normal matrix)

$$A \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} = \begin{bmatrix} a_{n1} & \cdots & a_{11} \\ & \ddots & \vdots \\ & & a_{1n} \end{bmatrix} \text{ so } \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} A^{-1} = \begin{bmatrix} a_{n1}^{-1} & \cdots & \\ & \ddots & \vdots \\ & & a_{1n}^{-1} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \begin{bmatrix} a_{n1}^{-1} & \cdots & \\ & \ddots & \vdots \\ & & a_{1n}^{-1} \end{bmatrix} = \begin{bmatrix} & & a_{1n}^{-1} \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ a_{n1}^{-1} & \cdots & \end{bmatrix}$$ which is a southeast matrix

if we multiply northwest matrix A with southeast matrix B and get $\begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$

$$x_{ij} = a_i^T b_j = (a_{i1} \cdots a_{i(n-i+1)} 0 \cdots 0) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_{(n-j+1)j} \\ \vdots \\ b_{nj} \end{pmatrix} \text{ if } n-i+1 < n-j+1 \Leftrightarrow i > j \text{. then } x_{ij} = 0$$

so AB is a northeast matrix

Problem 4.4 $$(A+uv^T)^{-1} = A^{-1} - \frac{(A^{-1}u)(v^TA^{-1})}{1+v^TA^{-1}u}$$

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v^T = [0\ 0\ 1]$$

so the new inverse is $$\begin{bmatrix} 1 & 9 & 0 \\ 1 & 0 & 8 \\ 0 & 1 & 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} [0\ 1\ 1]}{1 + [0\ 0\ 1] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 9 & 0 \\ 1 & 0 & 8 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 & -1 \\ 1 & -1 & 7 \\ 0 & 1 & 1 \end{bmatrix}$$

Problem 4.5 1.
$$\begin{bmatrix} I_n & 0 \\ A & I_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -A & I_m \end{bmatrix}$$

2. $$\begin{bmatrix} 0 & I_m \\ I_n & A \end{bmatrix} = \begin{bmatrix} 0 & I_m \\ I_n & -A \end{bmatrix}$$

3. $$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}$$

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^{-1} = \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}^{-1} \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix}$$

4. $$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} I & CB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{-1} \begin{bmatrix} I & CB^{-1} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} I & -CB^{-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

Problem 4.6 1. $$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$A^T = \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}$$

$A^T = -A$ thus $x_{11} = x_{22} = x_{33} = 0$

$x_{ij} = -x_{ji}$ ($i \neq j$ and $i, j \in \{1,2,3\}$)

so $$Ax = \begin{bmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_2x_{12} + a_3x_{13} \\ -a_2x_{12} + a_3x_{23} \\ -a_1x_{13} - a_2x_{23} \end{bmatrix}$$ choose $b_1 = x_{23}$. $b_2 = -x_{13}$. $b_3 = x_{12}$ then we find v

2. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ which is skew symmetric but not symmetric

3. let $A = \begin{bmatrix} B & & \\ & B & \\ & & \ddots & \\ & & & B \end{bmatrix}_{2n\times 2n}$ $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ thus $A^2 = \begin{bmatrix} B^2 & & \\ & B^2 & \\ & & \ddots & \\ & & & B^2 \end{bmatrix} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & \ddots & \\ & & & -1 \end{bmatrix} = -I$

4. for $\begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}$ which only has one 1 at ith row jth column

if $i=j$. then itself is a symmetric matrix

while $i \neq j$ we have $$\begin{array}{c} \\ i \end{array}\begin{bmatrix} & \overset{j}{\vdots} & \\ \cdots & 1 & \\ & & \end{bmatrix} = \begin{array}{c} j \\ i \end{array}\begin{bmatrix} & \overset{j}{\vdots} & \overset{i}{\vdots} \\ \cdots & \cdots & \frac{1}{2} \\ \cdots & \frac{1}{2} & \end{bmatrix} + \begin{array}{c} i \\ j \end{array}\begin{bmatrix} & \overset{i}{\vdots} & \overset{j}{\vdots} \\ \cdots & \cdots & -\frac{1}{2} \\ \cdots & \frac{1}{2} & \end{bmatrix}$$ which is the sum of a symmetric matrix and skew symmetric matrix and for an arbitary A. we can decompose A into several matrix like above

$A = \begin{bmatrix} & \\ 1 & \end{bmatrix} + \begin{bmatrix} 1 & \\ & \end{bmatrix} + \begin{bmatrix} & \\ 1 & \end{bmatrix} + \begin{bmatrix} & 1 \\ & \end{bmatrix} + \cdots$ thus A can be written as the sum of a symmetric matrix and skew symmetric matrix

Problem 4.7 1. $BA = R$

$B[A \;\; 2A] = [R \;\; 2R]$ which is RREF

2. $BA = R$

$\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -2I & I \end{bmatrix}\begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A \\ 0 \end{bmatrix} = \begin{bmatrix} AB \\ 0 \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}$ which is RREF

3. $BA = R$

$\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$ which is RREF

4. $BA = R$

$\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}\begin{bmatrix} A & A \\ 0 & A \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ which is RREF

Problem 4.8 1. $\begin{bmatrix} A & b \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax+b \\ 1 \end{bmatrix} = \begin{bmatrix} f(x) \\ 1 \end{bmatrix}$

2. $M_f M_g = \begin{bmatrix} A_1 & b_1 \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} A_2 & b_2 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} A_1A_2 & A_1b_2+b_1 \\ 0^T & 1 \end{bmatrix}$

$f \circ g = A_1(A_2x+b_2)+b_1 = A_1A_2x + A_1b_2+b_1$

$M_{f\circ g} = \begin{bmatrix} A_1A_2 & A_1b_2+b_1 \\ 0^T & 1 \end{bmatrix} = M_f M_g$

3. f is invertible $\Leftrightarrow$ f is bijective $\Leftrightarrow$ A is bijective $\Leftrightarrow$ A is invertible

$M_f M_{f^{-1}} = M_{f\circ f^{-1}} = \begin{bmatrix} I & 0 \\ 0^T & 1 \end{bmatrix} = I \Rightarrow M_f^{-1} = M_{f^{-1}}$ $\quad$ $f(f^{-1}(x)) = x \Rightarrow Af^{-1}(x)+b = x \Rightarrow f^{-1}(x) = A^{-1}(x-b) = A^{-1}x - bA^{-1}$

$M_f^{-1} = M_{f^{-1}} = \begin{bmatrix} A^{-1} & -bA^{-1} \\ 0^T & 1 \end{bmatrix}$

Problem 4.9 1. $x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. $x(A_1 \diamond A_2)(B_1 \diamond B_2)x^{-1} = x(A_1 \diamond A_2)x^{-1}x(B_1 \diamond B_2)x^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} A_1B_1 & 0 \\ 0 & A_2B_2 \end{bmatrix}$

$x(A_1B_1 \diamond A_2B_2)x^{-1} = \begin{bmatrix} A_1B_1 & 0 \\ 0 & A_2B_2 \end{bmatrix}$

$x(A_1 \diamond A_2)(B_1 \diamond B_2)x^{-1} = x(A_1B_1 \diamond A_2B_2)x^{-1} \Rightarrow x^{-1}x(A_1 \diamond A_2)(B_1 \diamond B_2)x^{-1}x = x^{-1}x(A_1B_1 \diamond A_2B_2)x^{-1}x \Rightarrow (A_1 \diamond A_2)(B_1 \diamond B_2) = A_1B_1 \diamond A_2B_2$

3. $(A \diamond B)(A^{-1} \diamond B^{-1}) = (AA^{-1} \diamond BB^{-1}) = (I \diamond I) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore (A \diamond B)^{-1} = A^{-1} \diamond B^{-1}$