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线性代数 HW3

习题8. 12. 设 $f(x) = g(x)\cdot(x-1)^2 + 2x$

$f(x) = h(x)(x-2)^3 + 3x$

$g(x)(x-1)^2 = h(x)(x-2)^3 + x$

$(x-2)^3 = (x-1-1)^3 = (x-1)^3 - 3(x-1)^2 + 3(x-1) - 1$

故 $(x-1)^2 \mid (3x-4)h(x) + x$

若想使 $h(x)$ 次数尽量低，则令 $h(x) = ax+b$

$x^2-2x+1 \mid 3ax^2 + (3b-4a+1)x - 4b$

$$\begin{cases} -6a = 3b-4a+1 \\ 3a = -4b \end{cases} \Rightarrow \begin{cases} a = 4 \\ b = -3 \end{cases}$$

此时 $f(x) = (4x-3)(x-2)^3 + 3x$

14. $f(f(x)) = f^k(x)$

设 $\deg(f(x)) = t \geq 0$

则 $\deg(f(f(x))) = t^2$，$\deg(f^k(x)) = kt$

故 $t^2 = kt$

若 $t=0$. 则 $f(x) \equiv c$. $\Rightarrow c = c^k$ $\begin{cases} 若 k=1. 则 f(x)\equiv c, c \neq a \\ 若 k \geq 2. 则 f(x) \equiv 1 \end{cases}$

若 $t \neq 0$. 则 $t = k$. 设 $f(x) = ax^k + g(x)$ $\deg(g(x)) < k$

$a(f(x))^k + g(f(x)) = (f(x))^k$. 故 $a=1$. $g(f(x)) \equiv 0$.

又 $f(x)$ 不为常数. $\deg(f(x)) \geq 1$. 故 $g(x) \equiv 0$.

$\therefore f(x) = x^k$ 此时成立.

18.

$q_2(x) = -x-2$

$$\begin{array}{l|l}
x^4+2x^3-x^2-4x-2 & x^4+x^3-x^2-2x-2 \\
x^4 \quad -2x^2 & x^4+2x^3-x^2-4x-2 \\
\hline
2x^3+x^2-4x-2 & r_1(x) = -x^3 \quad +2x \\
2x^3 \quad -4x & -x^3 \quad +2x \\
\hline
r_2(x) = x^2 \quad -2 & r_3(x) = 0
\end{array}$$

$q_1(x) = 1$

$q_3(x) = -x$

$g(x) = q_1(x)f(x) + r_1(x)$

$f(x) = q_2(x)r_1(x) + r_2(x)$

$r_1(x) = r_2(x)\cdot q_3(x) + r_3(x)$.

故 $r_2(x) = f(x) - q_2(x)r_1(x)$

$= f(x) + (x+2)(g(x) - f(x))$

即 $u(x) = -x-1$

$v(x) = x+2$

有 $u(x)f(x) + v(x)g(x) = (f(x), g(x)) = x^2-2$.

19.

$$\begin{array}{l|l}
x^3+(1+t)x^2+2x+2u & x^3+tx^2 \quad +u \\
x^3+tx^2 \quad +u & x^3+2x^2+ux \\
\hline
x^2+2x+u & (t-2)x^2-ux+u \\
 & (t-2)x^2+2(t-2)x+u(t-2) \\
\hline
 & (-u-2(t-2))x+u(3-t)
\end{array}$$

由于 $f(x)$. $g(x)$ 最大公因式是二次多项式. 故 $(-u-2(t-2)) = 0$. $u(3-t) = 0$.

解得 $\begin{cases} u=0 \\ t=2 \end{cases}$ 或 $\begin{cases} u=-2 \\ t=3 \end{cases}$

20. 若 $a=0$.

$(f(x), g(x)) = (cf(x), g(x))$

$= (cf(x) + dg(x), g(x))$

$= (cf(x) + dg(x), bg(x))$ 成立.

若 $c=0$ 同理成立. 下设 $a, c \neq 0$.

$(f(x), g(x)) = (af(x), g(x))$

$= (af(x) + bg(x), g(x))$

$= (af(x) + bg(x), \frac{ad-bc}{a}g(x))$

$= (af(x) + bg(x), \frac{c}{a}(af(x) + bg(x)) + \frac{ad-bc}{a}g(x))$

$= (af(x) + bg(x), cf(x) + dg(x))$ 成立.

22. 若 $(f(x), g(x)h(x)) = 1$

若 $(f(x), g(x)) \neq 1$. 设 $(f(x), g(x)) = t(x)$.

则 $t(x) \mid f(x)$. $t(x) \mid g(x)h(x)$ $\Rightarrow (f(x), g(x)h(x)) \geq t(x)$ 矛盾.

故 $(f(x), g(x)) = 1$. 同理 $(f(x), h(x)) = 1$.

若 $(f(x), g(x)) = (f(x), h(x)) = 1$.

$\exists u_1(x), u_2(x), v_1(x), v_2(x)$

$u_1(x)f(x) + v_1(x)g(x) = 1$

$u_2(x)f(x) + v_2(x)h(x) = 1$

$\Rightarrow (u_1(x)u_2(x)f(x) + u_2(x)v_1(x)g(x) + u_1(x)v_2(x)h(x))f(x)$

$+ (v_1(x)v_2(x))g(x)h(x) = 1$

故 $(f(x), g(x)h(x)) = 1$.

25. $x^2+bx+c=0$ 的根为 x_1, x_2.

则 $x_1+x_2 = -b$，$x_1x_2 = c$

$x_1^2 + x_2^2 = b^2-2c$，$x_1^2x_2^2 = c^2$

故二次方程 $x^2 - (b^2-2c)x + c^2 = 0$ 以 x_1^2, x_2^2 为根

26.(1) 有理根只可能为 $\pm1, \pm2, \pm7, \pm14$

$x<0$ 则 $x^3<0$. $15x<0$. 故式子 <0.

肯定不为根. 只需验证 1, 2, 7, 14.

经检验 $x=2$ 是唯一的有理零点

(2) 有理根只可能为 $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}$

经检验. $x = -\frac{1}{2}$ 是唯一的有理根.

(3) 有理根只可能为 $\pm1, \pm3$

经检验、$x=-1, 3$ 为有理根.