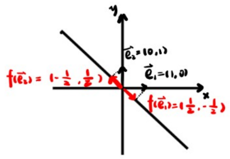
Linear algebra homework by Leonardo (2023010747) 刘一[illegible].

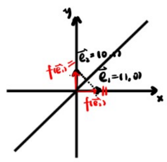
Problem 1.1 1. this projection can be discribed as a matrix $M = [f(e_1)\ f(e_2)]$



$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ neither injective nor surjective

since $f(\vec{e}_1) = f(-\vec{e}_2)$ and f can't reach points that is not on the line $x+y=0$

2. the reflection map send $\vec{e}_1$ to $\vec{e}_2$, $\vec{e}_2$ to $\vec{e}_1$



$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ bijective since it is its own inverse

3. f is not linear since $f(0) \neq 0$

bijective since it has an inverse map $g: v \to v - e_1$

4. $f(e_1) = e_1$ $f(e_2) = e_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ bijective since it is its own inverse

5. it's not linear, if it is linear

consider $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + f(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = f(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = f(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$

however $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 1$. $f(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 1$ $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \sqrt{2}$. $1+1 \neq \sqrt{2}$

hence it's not linear

neither surjective nor injective since f can't reach negative numbers

and $f(e_1) = f(e_2) = 1$

6. similar to question 1. since $\vec{e}_1, \vec{e}_2$ are both above the line $x+y=0$

$f(e_1) = \frac{\sqrt{2}}{2}$. $f(e_2) = \frac{\sqrt{2}}{2}$

$[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ surjective but not injective since $\forall x_0$, we have at least a point $v(\sqrt{2}x_0, 0)$ that $f(v) = x_0$ but $f(e_1) = f(e_2) = \frac{\sqrt{2}}{2}$

7. $f(e_1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ injective but not surjective since if $f(x_1) = f(x_2)$ we have $\begin{bmatrix} 2x_1 \\ 3x_1 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 3x_2 \end{bmatrix} \Rightarrow x_1 = x_2$. but f cannot reach $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Problem 1.2 1.

$M = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + 1\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + 1\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1+x_2+x_3 \\ y_1+y_2+y_3 \\ z_1+z_2+z_3 \end{bmatrix}$

we know that $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in \{1, 2, \cdots, 9\}$

$x_1+x_2+x_3+y_1+y_2+y_3+z_1+z_2+z_3 = 45$

$24 \geq \underbrace{x_1+x_2+x_3}_{t_1} \geq 6$ $24 \geq \underbrace{y_1+y_2+y_3}_{t_2} \geq 6$ $24 \geq \underbrace{z_1+z_2+z_3}_{t_3} \geq 6$

meantime. since we know M is a magic matrix. $t_1 = t_2 = t_3 = 15$

therefore $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$

2. $M = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{19} \\ x_{21} & \cdots & & \vdots \\ \vdots & & & \vdots \\ x_{91} & x_{92} & \cdots & x_{99} \end{bmatrix}$ $M\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x_{11}+\cdots+x_{19} \\ \vdots \\ x_{91}+\cdots+x_{99} \end{bmatrix}$

since M is a Sudoku matrix

hence we have $\sum_{i=1}^{9} x_{1i} = 1+\cdots+9 = 45$

$\vdots$

$\sum_{i=1}^{9} x_{9i} = 1+\cdots+9 = 45$

so $M\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 45 \\ 45 \\ \vdots \\ 45 \end{bmatrix}$

Problem 1.3 1. for one direction. consider its opposite direction

the sum of these two vectors is zero

therefore we can pair those twelve vectors into 6 pairs

the sum of each pair is 0 so the sum of all twelve vectors is 0

2. except the 2 o'clock vector and 8 o'clock vector. the rest 5 pairs sum is 0

so the sum of all vectors except the 2 o'clock vector is 8 o'clock vector

which is $\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}$

3. let's say $\vec{v}_1$ and $\vec{v}_2$ are two unit vectors with opposite directions.

if the center of the clock is $\vec{v}$. then $\vec{v}_1 + \vec{v}_2 = 2\vec{v}$

after moving the starting points. the vector of clock center is $\begin{bmatrix}0\\1\end{bmatrix}$

therefore, 6 pairs of vectors sum is $6\cdot 2\cdot\begin{bmatrix}0\\1\end{bmatrix}$ which is $\begin{bmatrix}0\\12\end{bmatrix}$

Problem 1.4

1. $b=\begin{bmatrix}1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}2\\4\\5\end{bmatrix}$

2. $b=\begin{bmatrix}1&5\\2&4\\3&3\\4&2\\5&1\end{bmatrix}\begin{bmatrix}6\\4\end{bmatrix}$

3. $b=\begin{bmatrix}1&2&1\\0&-1&1\\1&1&1\\1&1&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$

4. $b=f(1e_1+2\cdot e_2+3\cdot e_3+4e_4+5e_5)=f(e_1)+2\cdot f(e_2)+3\cdot f(e_3)+4\cdot f(e_4)+5\cdot f(e_5)=e_5+4e_4+9e_3+16e_2+25e_1=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}25\\16\\9\\4\\1\end{bmatrix}$ or we can directly $b=\begin{bmatrix}0&0&0&0&5\\0&0&0&4&0\\0&0&3&0&0\\0&2&0&0&0\\1&0&0&0&0\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$

5. $b=\begin{bmatrix}0.8&0.3\\0.2&0.7\end{bmatrix}\begin{bmatrix}p'\\1-p'\end{bmatrix}=\begin{bmatrix}0.8&0.3\\0.2&0.7\end{bmatrix}\begin{bmatrix}0.5\\0.5\end{bmatrix}$

Problem 1.5

1.

$a+t(b-a)=(1-t)a+tb$

let $s=1-t$, $t'=t$ then we have $\{sa+t'b : s,t'\in R \text{ and } s+t'=1\}$

2. suppose $sa+tb=s'a+t'b$

then we have $(s-s')a=(t-t')b$

if $s=s'$. then $(t-t')b=0$. since $b\neq 0$ $t-t'$ must be 0, so we can get $s=s'$, $t=t'$

if $t=t'$ same as above, we can get $s=s'$, $t=t'$

if $s\neq s'$ and $t\neq t'$, since a. b is not parallel. the equation cannot hold true.

then we know there is a unique pair s, t such that $p=sa+tb$ and $s+t=1$

3.

the point on the line segment connecting a. b should be $sa+tb$

p is on the segment $\Leftrightarrow 0\le\vec{ap}\cdot\vec{ab}\le|\vec{ab}|^2\Leftrightarrow 0\le((s-1)a+tb)(b-a)\le|b-a|^2\Leftrightarrow 0\le t(b-a)^2\le|b-a|^2\Leftrightarrow 0\le t\le 1\Leftrightarrow 0\le t,s\le 1$

therefore we have the set $\{sa+tb : s,t\in R \text{ and } s+t=1 \text{ and } 0\le s,t\le 1\}$

4. the plane through three points a.b.c is the set $\{sa+tb+vc : s.t.v\in R \text{ and } s+t+v=1\}$

5. $\{sa+tb+vc : s.t.v\in R. \text{ and } s+t+v=1 \text{ and } 0\le s.t.v\le 1\}$

Problem 1.6

1. let $y=z=0$ we have $a=\begin{bmatrix}6\\0\\0\end{bmatrix}$

let $x=z=0$ we have $b=\begin{bmatrix}0\\3\\0\end{bmatrix}$

let $x=y=0$ we have $c=\begin{bmatrix}0\\0\\2\end{bmatrix}$

2. $x_1=x$. $x_2=y$. $x_3=z$

$a_1=1$. $a_2=2$. $a_3=3$ $b=6$

the solution to the equation $a_1x_1+a_2x_2+a_3x_3=6$, which is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, is a hyperplane

$a=\begin{bmatrix}6\\0\\0\end{bmatrix}$. $b=\begin{bmatrix}0\\3\\0\end{bmatrix}$. $c=\begin{bmatrix}0\\0\\2\end{bmatrix}$ are all solutions, so $H=\{\begin{bmatrix}x\\y\\z\end{bmatrix}: x+2y+3z=6\}$ is exact a plane through a.b.c

3. $(a-b)\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}6\\-3\\0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=6\cdot 1+(-3)\cdot 2+0\cdot 3=0$

$(b-c)\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}0\\3\\-2\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=0+6-6=0$

$(c-a)\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}-6\\0\\2\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=-6+0+6=0$

4. $\vec{v}_1=\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$

$\vec{v}_2=\begin{bmatrix}x_1'\\\vdots\\x_n'\end{bmatrix}$

$\vec{v}=\vec{v}_1-\vec{v}_2=\begin{bmatrix}x_1-x_1'\\\vdots\\x_n-x_n'\end{bmatrix}$

Since $\begin{cases}a_1x_1+\cdots+a_nx_n=b\\a_1x_1'+\cdots+a_nx_n'=b\end{cases}\Rightarrow a_1(x_1-x_1')+\cdots+a_n(x_n-x_n')=0$

$\vec{v}\cdot\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}=0\Rightarrow\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}$ is the normal vector