2023010747 刘一鸣 计32

微积分

习题1.9 1.(3) $\begin{cases} u_x' = \cos x - \cos(x+y+z) = 0 \\ u_y' = \cos y - \cos(x+y+z) = 0 \\ u_z' = \cos z - \cos(x+y+z) = 0 \end{cases}$

得u的驻点为 $P_1(0,0,0)$. $P_2(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$. $P_3(\pi,\pi,\pi)$

又函数在任意点的Hesse矩阵为 $H(P) = \begin{pmatrix} -\sin x+\sin(x+y+z) & \sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & -\sin y+\sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & \sin(x+y+z) & -\sin z+\sin(x+y+z) \end{pmatrix}$

$H(P_1) = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$ $H(P_2) = \begin{pmatrix} -2&-1&-1 \\ -1&-2&-1 \\ -1&-1&-2 \end{pmatrix}$ $H(P_3) = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$

其中 $H(P_2)$ 负定. 故函数有极大值, 为4

2. $\begin{cases} z_x' = \frac{4x+8z}{1-2z-8x} = 0 \\ z_y' = \frac{4y}{1-2z-8x} = 0 \end{cases}$

驻点为 $P_1(-2,0)$ 和 $P_2(\frac{16}{7},0)$

又 $\begin{cases} 4x+2zz_x'+8z+8xz_x'-z_x'=0 \\ 4y+2zz_y'+8xz_y'-z_y'=0 \end{cases} \Rightarrow \begin{cases} 4+2z_x'^2+2zz_{xx}''+8z_x'+8z_x'+8xz_{xx}''-z_{xx}''=0 \\ 2z_y'z_x'+2zz_{xy}''+8z_y'+8xz_{xy}''-z_{xy}''=0 \\ 4+2z_y'^2+2zz_{yy}''+8xz_{yy}''-z_{yy}''=0 \end{cases}$

$(x,y,z)=(-2,0,1)$ 时 $H = \begin{pmatrix} \frac{4}{15} & 0 \\ 0 & \frac{4}{15} \end{pmatrix}$ 正定. 故z极小值为1

$(x,y,z)=(\frac{16}{7},0,-\frac{8}{7})$ 时 $H = \begin{pmatrix} -\frac{4}{15} & 0 \\ 0 & -\frac{4}{15} \end{pmatrix}$ 负定. 故z极大值为 $-\frac{8}{7}$

7.(3) $L = x^2+y^2+z^2+\lambda(\frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{4}-1)$

$\begin{cases} L_x' = 2x+\frac{\lambda x}{8} = 0 \\ L_y' = 2y+\frac{2\lambda y}{9} = 0 \\ L_z' = 2z+\frac{\lambda z}{2} = 0 \\ \frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{4} = 1 \end{cases}$

解得驻点为 $P_1(4,0,0)$. $P_2(0,3,0)$ $P_3(0,0,2)$. u分别为16, 9, 4.

又u存在极大值和极小值. 因此极大值为16. 极小值为4.

8. 若在内部. $\begin{cases} u_x' = 2x-2y=0 \\ u_y' = 4y-2x-2z=0 \\ u_z' = 2z-2y=0 \end{cases}$

驻点为 (a,a,a). $u=0$.

若在边界上. $u = y^2-2xy-2yz+4$

$L = y^2-2xy-2yz+4+\lambda(x^2+y^2+z^2-4)$

$\begin{cases} L_x' = -2y+2\lambda x=0 \\ L_y' = 2y-2x-2z+2\lambda y=0 \\ L_z' = -2y+2\lambda z=0 \\ x^2+y^2+z^2=4 \end{cases}$

驻点为 $(\sqrt{2},0,-\sqrt{2})$. $(-\sqrt{2},0,\sqrt{2})$. $(\sqrt{\frac{4}{3}},\sqrt{\frac{4}{3}},\sqrt{\frac{4}{3}})$, $(\sqrt{\frac{2}{3}},-2\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}})$. u分别为4, 4, 0, 12

u存在极大.极小值. 故u极大值为12. 极小值为0

9.(3) $L = 4xyz+\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1)$ $\begin{cases} L_x' = 4yz+\frac{2\lambda}{a^2}x=0 \\ L_y' = 4xz+\frac{2\lambda}{b^2}y=0 \\ L_z' = 4xy+\frac{2\lambda}{c^2}z=0 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \end{cases}$ 驻点为 $(a,0,0)$. $(0,b,0)$. $(0,0,c)$, $(\pm\frac{\sqrt{3}a}{3},\pm\frac{\sqrt{3}b}{3},\pm\frac{\sqrt{3}c}{3})$.

u分别为0. 0, 0, $\pm\frac{4\sqrt{3}abc}{9}$.

因此体积最大值为 $|\frac{4\sqrt{3}abc}{9}|$, 最小值为0

10.(1) $L = 2\sqrt{h^2+(\frac{b-a}{2})^2}+a+\lambda(h\cdot\frac{a+b}{2}-S)$.

$f(a,b,h)$ 驻点为 $(t,t,0)$, $(2t,t,\frac{\sqrt{3}}{2}t)$

又a, b, h均大于0. 因此最小值取到时 $a:b:h=2:1:\frac{\sqrt{3}}{2}$.

因此上.下底及腰的比例为2:1:1时用的水泥最省

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微积分. 习题 2.2

1. (1) $\int_{-1}^{1}\sqrt{x^2+a^2}\,dx = 2\int_0^1\sqrt{x^2+a^2}\,dx = 2\left(\frac{1}{2}\left(x\sqrt{x^2+a^2}+a^2\ln|x+\sqrt{x^2+a^2}|\right)\right)\Big|_0^1 = \sqrt{1+a^2}+a^2\ln(1+\sqrt{1+a^2})-a^2\ln|a| = \sqrt{1+a^2}+a^2\ln\left|\frac{1}{a}+\sqrt{1+\frac{1}{a^2}}\right|$

$\lim\limits_{a\to 0}\int_{-1}^{1}\sqrt{x^2+a^2}\,dx = 1+0=1$

2. (4) $F'(t) = \int_0^t\left(\frac{\partial}{\partial t}f(x+t,x-t)\right)dx + f(2t,0)\cdot 1$

$= \int_0^t\left(\frac{\partial f(x+t,x-t)}{\partial(x+t)} - \frac{\partial f(x+t,x-t)}{\partial(x-t)}\right)dx + f(2t,0)$

$= \int_0^t\left(f_1'(x+t,x-t) - f_2'(x+t,x-t)\right)dx + f(2t,0)$

4. $\frac{\partial u}{\partial x} = \frac{1}{2}\left(\varphi'(x+at)+\varphi'(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at}\frac{\partial}{\partial x}\psi(s)\,ds + \frac{1}{2a}\left(\psi(x+at)-\psi(x-at)\right)$

$\frac{\partial u}{\partial t} = \frac{a}{2}\left(\varphi'(x+at)-\varphi'(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at}\frac{\partial}{\partial t}\psi(s)\,ds + \frac{1}{2}\left(\psi(x+at)+\psi(x-at)\right)$

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(\varphi''(x+at)+\varphi''(x-at)\right) + \frac{1}{2a}\left(\psi'(x+at)-\psi'(x-at)\right)$

$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{2}\left(\varphi''(x+at)+\varphi''(x-at)\right) + \frac{a}{2}\left(\psi'(x+at)-\psi'(x-at)\right)$

因此 $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$