2023010747 刘一鸣 计32.

线性代数.

1. 若A不幂零，则由凯莱-哈密尔顿定理知，0的代数重数 $< n$

因此有 $N(A^{n-1}) = N(A^n) \Rightarrow C(A^{n-1}) = C(A^n)$

若 $v \in N(A^n) \cap C(A^n)$，则 $\exists w$ 使 $v = A^n w$. $A^n v = 0 \Rightarrow A^{2n} w = 0 \Rightarrow w \in N(A^{2n}) = N(A^n)$

$\therefore v = \vec{0} \Rightarrow N(A^n) \cap C(A^n) = \{\vec{0}\} \Rightarrow Ker(T^{n-1}) \cap Im(T^{n-1}) = \{\vec{0}\}$.

又 $\dim C^n = \dim(Ker(T^{n-1})) + \dim(Im(T^{n-1}))$

因此 $C^n = Ker(T^{n-1}) \oplus Im(T^{n-1})$

2. 由于 $Im\,T^4 \neq Im\,T^5$ 且 $C(A^5) \subseteq C(A^4)$，故 $r(A^5) < r(A^4)$

因此 $N(A^4) \subsetneq N(A^5)$，0的代数重数 ≥ 5，故0的代数重数为5.

$\therefore A^5 = 0$，T是幂零变换.

3. $C(A^{m+1}) \subseteq C(A^m)$

因此 $r(A^{m+1}) \leq r(A^m)$，取等时 $C(A^{m+1}) = C(A^m)$，

$\therefore r(A^{m+1}) = r(A^m) \Leftrightarrow C(A^{m+1}) = C(A^m)$.

又 $r(A^{m+1}) = r(A^m) \Leftrightarrow N(A^{m+1}) = N(A^m)$

故 $Ker(T^m) = Ker(T^{m+1}) \Leftrightarrow Im(T^m) = Im(T^{m+1})$

4. 令 $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$，$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

A, A^2 均不可对角化.

5. $A^5 + 2A^4 - 7A^3 - 6A^2 + 5A + 4I = 0$

$I + 2A^{-1} - 7A^{-2} - 6A^{-3} + 5A^{-4} + 4A^{-5} = 0$

故 A^{-1} 的化零多项式为 $4x^5 + 5x^4 - 6x^3 - 7x^2 + 2x + 1$

6. $A = PJP^{-1}$

(1) $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(2) $P = \begin{bmatrix} 3 & 3 & 4 \\ 1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(3) $P = \begin{bmatrix} 1 & 0 & -1 & \frac{1}{3} \\ 1 & -1 & -1 & \frac{4}{3} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. $r(A) = 1$，则 $A = uv^T$，$tr(A) = v^T u$

若 $tr(A) = 0$，则 $|\lambda I - A| = \lambda^n \Rightarrow$ A幂零. 反之，若A幂零，则特征值均为0，$tr(A) = 0$

$r(A) = 1 \Rightarrow \dim(N(A)) = n-1$，有 $n-1$ 个循环子空间，$A^2 = 0$

$\begin{array}{|c|c|c|c|}\hline T(\vec{v}_1) & & & \\ \hline \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{n-1} \\ \hline \end{array}$

$A^2 = 0 \Rightarrow$ A的非零列 $\in N(A) \cap C(A)$.

取出其中非零列，设为 $\vec{\alpha}_1$. $Ax = \vec{\alpha}_1 \Rightarrow x = \vec{v}_1$

再取出N(A)基中另外 $n-2$ 个向量 $\vec{\alpha}_2, \cdots, \vec{\alpha}_{n-1}$，使 $\vec{\alpha}_1, \cdots, \vec{\alpha}_{n-1}$ 为 N(A) 一组基.