Mathematics for Computer Science
Spring 2024
Due: 23:59, March 18, 2024

Homework Set 3

[Wasserman] "*All of Statistics*" by Larry Wasserman, 2004 edition. Free copy available at this site: https://link.springer.com/book/10.1007/978-0-387-21736-9

Reading Assignments: Read Chapter 1 , and Chapter 2.1-2.4 of [Wasserman].

Written Assignments:
(1) From [Wasserman] Chapter 1.10, do Exercises 12, 18, 20.
(2) Also, do the following Problems:

Problem 1 Consider the rectangle $A = [0, a] \times [0, b]$ on the plane, where $a, b > 2$. Drop a needle of length 1 randomly on the plane as follows: the center of the needle is chosen uniformly distributed over A, and the orientation of the needle is random and uniform over any direction. Compute the probability that the needle intersects any boundary edge of A. Your answer should be an explicit expression of a, b.

Problem 2 Let $\rho_0(x) = e^{-x}$ for $x \geq 0$, and $\rho_0(x) = 0$ for $x < 0$. It is easy to check that ρ_0 is a probability density function defined in R^1. Consider two random x_1, x_2, each independently generated according to ρ_0.

Question: Determine the probability of $x_1 + x_2 > t$ (as a closed-form expression in t).

Problem 3 Give a *true* or *false* answer to each of the following statements, and give concise proofs for your answers. Let X and Y be any random variables.

Questions:

(a) $E(X + Y) = E(X) + E(Y)$.

(b) $(E(X^2Y^2))^{1/2} \geq E(X) \cdot E(Y)$.

(c) $(E(X^2Y^2))^{1/2} \geq E(X) \cdot E(Y)$, if X and Y are independent random variables.

(d) $E(e^X) \geq e^{E(X)}$.

Problem 4 Prove the following inequality, which is needed to complete the proof on the max clique size (presented in the class lecture on March 12):

$$\sum_{2 \leq k \leq m} \binom{m}{k} \binom{n-m}{m-k} \frac{1}{2^{\binom{m}{2} - \binom{k}{2}}} \leq \frac{m^5}{n - m + 1} E(X)$$

Problem 5 [*Randomized Routing*]

The $n-bit$ *hypercube network* consists of $N = 2^n$ nodes, and Nn (directed) edges (for each pair i, j of nodes with Hamming distance $d_H(i,j) = 1$, there are two directed edges (i,j) and (j,i)). In class, we discussed a randomized *bit fixing algorithm* on the $n-bit$ hypercube network. In Phase I, each node j will deliver a message M_j to a random node $\hat{\sigma}(j)$ as follows. For each node j, let $Path_j = e_1 e_2 \cdots e_{\ell_j}$ be the path (i.e. the sequence of edges) to be followed by packet M_j under the bit-fixing algorithm. Now fix i. Let S be the set of $j \neq i$ such that the paths $Path_j$ and $Path_i$ share at least one common edge. The following theorem is important as mentioned in class.

Theorem A The number of steps used in delivering packet M_i is no more than $\ell_i + |S|$. That is, the extra delay D_j for packet M_i is at most $|S|$.

Questions:

(a) Prove Theorem A for the special case $|S| = 2$.

(b) Prove Theorem A for any $|S|$.

Hint: *Please think about the problem for at least 30 minutes before looking at the Hint.* At time 0, packet M_i is at node i (waiting to cross edge e_1) is said to have $lag = 0$. In general any packet M_j at time t waiting (and hoping) to cross edge e_{k+1} is said to have $lag = t - k$. During the routing, M_i increases its lag from 0 to D_j when it arrives at destination.

Problem 6 In class we claimed that $E(Y_e) = 1/2$ for any edge e in the n-dimensional hypercube network. Prove that statement.

Problem 7 Consider the n-bit Hypercube network randomized routing algorithm discussed in class. In Phase I of the algorithm, each message M_i ($i \in \{0,1\}^n$) is sent to a random intermediate node $\eta(i)$, where the $path_i$ has length $|path_i|$ equal to the Hamming distance between i and $\eta(i)$. Let T_i be the number of steps used by M_i in the algorithm. Let Z denote the set of messages M_i such $T_i > |path_i|$. ($|Z|$ counts the number of messages that encounter at least some extra delay due to traffic congestion.)

Question: Prove that $E(|Z|) = \Omega(2^n)$.