Mathematics for Computer Science: Homework 8

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Exercise 3.7

Answer: Notice that orthonormal rows imply that

$$AA^T = I.$$

Therefore $A^{-1} = A^T$, we have $A^T A = I$, so the columns of A are orthonormal.

Exercise 3.12

Answer: (1) From Singular Value Decomposition, we have $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$, so

$$A^T A = \left(\sum_{i=1}^{r} \sigma_i v_i u_i^T\right)\left(\sum_{i=1}^{r} \sigma_i u_i v_i^T\right) = \sum_{i,j} \sigma_i \sigma_j v_i (u_i^T u_j) v_j^T = \sum_{i=1}^{r} \sigma_i^2 v_i v_i^T.$$

(2) For arbitrary v_j, we have

$$A^T A v_j = \sum_{i=1}^{r} \sigma_i^2 v_i v_i^T v_i = \sigma_i^2 v_j v_j^T v_j = \sigma^2 v_j.$$

Therefore, v_j is an eigenvector of $A^T A$ with eigenvalue σ^2.

(3) If two singular vectors are not unique up to a sign, we assume they are v_i and v_j. Since they are also eigenvectors of $A^T A$, they must be unique up to multiplicative constants, which is impossible because $v_i = \pm v_j$.

Exercise 3.13

Answer: (1)

$$\|A_k\|_F^2 = \sum_{ij} a_{ij}^2 = \sum_{i=1}^{n} |a_i|^2 = \sum_{i=1}^{n}\sum_{j=1}^{k} \left(a_i \cdot v_j\right)^2 = \sum_{j=1}^{k}\sum_{i=1}^{n} \left(a_i \cdot v_j\right)^2 = \sum_{j=1}^{k} \left|A_k v_j\right|^2 = \sum_{i=1}^{k} \sigma_i^2.$$

(2) Let's assume that $x = \sum_{i=1}^{k} c_i v_i$. Then

$$\|A_k\|_2 = \max_{|x|\le 1} |A_k x| = \max_{|x|\le 1} \left| \left(\sum_{i=1}^{k} \sigma_i u_i v_i^T\right)\left(\sum_{i=1}^{k} c_i v_i\right) \right| = \max_{|x|\le 1} \left| \sum_{i=1}^{k} \sigma_i c_i u_i \right| = \max_{|x|\le 1} \sqrt{\sum_{i=1}^{k} \sigma_i^2 c_i^2} = \sigma_1.$$

Thus, $\|A_k\|_2^2 = \sigma_1^2$.

(3) $A - A_k = \sum_{i=k+1}^{r} \sigma_i u_i v_i^T$, so

$$
\begin{aligned}
\|A-A_k\|_F^2 &= \sum_{ij} {a'_{ij}}^2 \\
&= \sum_{i=1}^{n} |a'_i|^2 \\
&= \sum_{i=1}^{n} \sum_{j=k+1}^{r} \left(a'_i \cdot v_j\right)^2 \\
&= \sum_{j=k+1}^{r} \sum_{i=1}^{n} \left(a'_i \cdot v_j\right)^2 \\
&= \sum_{j=k+1}^{r} \left|(A-A_k)v_j\right|^2 \\
&= \sum_{i=k+1}^{r} \sigma_i^2.
\end{aligned}
$$

(4) Let's assume that $x = \sum_{i=1}^{r} c_i v_i$. Then

$$
\begin{aligned}
\|A-A_k\|_2 &= \max_{|x|\le 1} |(A-A_k)x| \\
&= \max_{|x|\le 1} \left| \left(\sum_{i=k+1}^{r} \sigma_i u_i v_i^T \right) \left(\sum_{i=1}^{r} c_i v_i \right) \right| \\
&= \max_{|x|\le 1} \left| \sum_{i=k+1}^{r} \sigma_i c_i u_i \right| \\
&= \max_{|x|\le 1} \sqrt{\sum_{i=k+1}^{r} \sigma_i^2 c_i^2} \\
&= \sigma_{k+1}.
\end{aligned}
$$

Thus, $\|A-A_k\|_2^2 = \sigma_{k+1}^2$.

Exercise 3.21

Answer: We assume that $x = \sum_{i=1}^{r} c_i v_i$. Then

$$
\begin{aligned}
BAx &= \left(\sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T \right) \left(\sum_{i=1}^{r} \sigma_i u_i v_i^T \right) \left(\sum_{i=1}^{r} c_i v_i \right) \\
&= \left(\sum_{i,j} v_i u_i^T u_i v_i^T \right) \left(\sum_{i=1}^{r} c_i v_i \right) \\
&= \left(\sum_{i=1}^{r} v_i v_i^T \right) \left(\sum_{i=1}^{r} c_i v_i \right) \\
&= \sum_{i,j} v_i v_i^T c_j v_j \\
&= \sum_{i=1}^{r} c_i v_i \\
&= x.
\end{aligned}
$$

Exercise 3.22

Answer: (1) $\|A\|_F^2 = \sum_{i=1}^r \sigma_i^2 \geq \sum_{i=1}^k \sigma_i^2 \geq k\sigma_k^2$. So $\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$.

(2) Let $B = \sum_{i=1}^{k-1} \sigma_i u_i v_i^T$, then $\|A - B\|_2 = \sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$.

(3) No. If we can, since $\|A - B\|_F^2 \geq \|A - A_k\|_F^2 = \sum_{i=k+1}^r \sigma_i^2$, we must have

$$\frac{\sum_{i=1}^r \sigma_i^2}{k} = \frac{\|A\|_F^2}{k} \geq \sum_{i=k+1}^r \sigma_i^2.$$

This is not necessarily true. For example, if $\sigma_i = 1, r \geq 5$ and $k = 2$, then this is false. Thus, we cannot replace 2-norm by Frobenius norm.