Mathematics for CS & AI
Spring 2024
Due: 23:59, March 11, 2024

Homework Set 2

Notation Let f, g be non-negative real-valued functions. We are often interested in comparing the asymptotic behavior of $f(n), g(n)$ as $n \to \infty$.

Big-O notation: We write $f(n) = O(g(n))$, if there exists a constant $c > 0$ such that $f(n) \leq cg(n)$ for all $n \geq 0$.

little-O notation: We write $f(n) = o(g(n))$, if $f(n)/g(n) \to 0$ a $n \to \infty$.

Big-Omega notation: We write $f(n) = \Omega(g(n))$, if there exists constant $c > 0$ such that $f(n) \geq cg(n)$ for all $n \geq 0$.

Theta notation: We write $f(n) = \Theta(g(n))$, if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

For example, $\log_2 n = O(n^{1/3})$, the Harmonic number $H_n = \Theta(\ln n)$, $H_n = \ln n + O(1)$, $1/(log_2 n)^2 = o(1)$, $n^{20} = e^{\Omega(\log n)}$ etc.

A *closed-form formula (or expression)* has only a bounded number of terms involving familiar functions. For example, $\log_2 x + x^5$ is a closed-form formula in x, $n^3 + \binom{n^2}{n}/(n+6) - \cos(1/n)$ is a closed-form formula in n, while $\sum_{1 \leq i \leq n} i^2$ is *not* a closed-form formula in n. You may consider the quantities $H_n = \sum_{1 \leq i \leq n} \frac{1}{i}$ and $H_n^{(2)} = \sum_{1 \leq i \leq n} \frac{1}{i^2}$ as familiar functions.

Problem 1 In class, we show that by each using the cycle-following strategy, Person 1 and Person 2 have a probability close to 3/8 for both finding their pets.

Question: Prove that, in the decision-tree model as discussed in today's class, no strategy can achieve a strictly better probability than the cycle-following strategy.

Problem 2 Alice and Bob each independently tosses an unbiased coin n times. Let X and Y be the random variables corresponding to the number of HEADs in Alice' and Bob's results. Your solutions must be *closed-form* formulas in n.

(a) Determine the expected value of the random variable $X - Y$.

(b) Determine the variance of the random variable $X - Y$.

(c) Let S denote the event that $X = Y$, and let $s(n) = \Pr\{S\}$. Determine $s(n)$.

(d) Let T denote the event that $X = Y + 1$, and let $t(n) = \Pr\{T\}$. Determine $t(n)$.

Problem 3 Consider the probability space (U, p), where U is the set of all $n!$ permutations of $\{1, 2, \cdots, n\}$, and $p(u) = 1/|U|$ is the uniform probability function. Define random variables X as follows. For each $u \in U$, let $X(u)$ be the number of cycles in the cycle representation of u. In class it was shown that $E(X) = H_n = \ln n + O(1)$.

(a) Derive an explicit exact formula for $Var(X)$. Your formula may involve summations and products of terms.

(b) Determine two functions $g(n)$ and $h(n)$ such that $Var(X) = g(n) + O(h(n))$ and $h(n) = o(g(n))$ for large n.

Remarks The purpose of (b) is to figure out the "asymptotic" behavior of $Var(X)$ when n is large. By comparing $E(X)$ and $(Var(X))^{1/2}$, we get a sense whether $E(X)$ gives a good estimate of typical values for the random variable X, when n is large.

Problem 4 Let $f(t) = \mu(e^t - 1) - t(1+\delta)\mu$, where $\delta > 0, \mu > 0$. Prove that $f(t)$ achieves minimum value at $t = \ln(1+\delta)$.

Problem 5 We proved in today's class one of the Chernoff's bounds. In this problem, prove the other inequalities. Let $0 < \delta < 1$.
(a) Prove:

$$\Pr\{X \leq (1-\delta)\mu\} \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

(b) Prove: $\Pr\{X \geq (1+\delta)\mu\} \leq e^{-\mu\delta^2/3}$,
(c) Prove: $\Pr\{X \leq (1-\delta)\mu\} \leq e^{-\mu\delta^2/2}$.

Problem 6 Let A and B be two events in some probability space, with $\Pr\{A\} \neq 0$ and $\Pr\{B\} \neq 0$. We will say A *attracts* B, if $\Pr\{B|A\} > \Pr\{B\}$. We say A *repels* B, if $\Pr\{B|A\} < \Pr\{B\}$.

(a) Prove that A *attracts* B if and only if B *attracts* A.

(b) Assume that (1)A attracts B, (2) A attracts C, and (3) A repels $B \cap C$. Prove that A attracts $B \cup C$.

(c) In a poker game of two players, where from a random deck of 52 cards, each player receives a hand of 5 cards. Let A be the event that player 1 receives a royal flush, and let B be the event that player 2 receives a royal flush. Prove that A attract B. (That is, if you get a royal flush, then it increases the probability that your opponent also has received a royal flush!)

Remark A royal flush means you get a hand of $Ace, K, Q, J, 10$ all in one suit (e.g. Spade). There are only 4 possible royal flushes.