Mathematics for Computer Science: Homework 7

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Problem 1

1.We let $y_i = x_i - c_1, z_i = x_i' - c_2$, then

$$\Pr(|y_i| \le 1-\varepsilon) = (1-\varepsilon)^d \le e^{-\varepsilon d}.$$

$$\Pr\left(|y_i y_j| \ge \frac{c}{\sqrt{d-1}}\right) \le \frac{2}{c} e^{-\frac{c^2}{2}}.$$

$$\Pr\left(|y_i(c_1-c_2)| \ge \delta \frac{c}{\sqrt{d-1}}\right) \le \frac{2}{c} e^{-\frac{c^2}{2}}.$$

It's similar for z_i.

Let $\varepsilon = \frac{\log n \ln n}{d}, c = \sqrt{\log n \ln n}$, then

$$\Pr(|y_i| \le 1-\varepsilon) \le e^{-\varepsilon d} = n^{-\log n} = n^{-\Omega(\log n)}.$$

$$\Pr\left(|y_i y_j| \ge \frac{c}{\sqrt{d-1}}\right) \le \frac{2}{\sqrt{\log n \ln n}} e^{-\frac{\log n \ln n}{2}} = \frac{2}{\sqrt{\log n \ln n}} n^{-\frac{\log n}{2}} = n^{-\Omega(\log n)}$$

$$\Pr\left(|y_i(c_1-c_2)| \ge \delta \frac{c}{\sqrt{d-1}}\right) \le \frac{2}{\sqrt{\log n \ln n}} e^{-\frac{\log n \ln n}{2}} = \frac{2}{\sqrt{\log n \ln n}} n^{-\frac{\log n}{2}} = n^{-\Omega(\log n)}.$$

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Back to the original problem, let $L^2(d,n) \ge 2 + 2\frac{c}{\sqrt{d-1}}$

$$\begin{aligned}
\Pr(|x_i - x_j| \le L(d,n)) &= \Pr\left(|y_i - y_j|^2 \le L^2(d,n)\right) \\
&\ge \Pr(y_i^2 + y_j^2 - 2y_i y_j \le L^2(d,n)) \\
&\ge \Pr(y_i^2 \le 1) \Pr(y_j^2 \le 1) \Pr\left(|y_i y_j| \le \frac{c}{\sqrt{d-1}}\right) \\
&\ge 1 - n^{-\Omega(\log n)}.
\end{aligned}$$

It's similar for x_i'.

Then, let $L^2(d,n) \le 2(1-\varepsilon)^2 + \delta^2 - 2\frac{c}{\sqrt{d-1}} - 4\delta\frac{c}{\sqrt{d-1}}$

$$\begin{aligned}
\Pr(|x_i - x_j'| \ge L(d,n)) &= \Pr\left(|y_i - c_1 - z_j + c_2|^2 \ge L^2(d,n)\right) \\
&= \Pr\left(y_i^2 + z_j^2 + (c_1-c_2)^2 - 2y_i z_j + 2(c_1-c_2)y_i - 2(c_1-c_2)z_j \ge L^2(d,n)\right) \\
&\ge \Pr\left(y_i^2 + z_j^2 + (c_1-c_2)^2 - 2y_i z_j + 2(c_1-c_2)y_i - 2(c_1-c_2)z_j \ge L^2(d,n)\right) \\
&\ge \Pr\left(y_i^2 \ge (1-\varepsilon)^2\right) \Pr\left(z_j^2 \ge (1-\varepsilon)^2\right) \Pr\left((c_1-c_2)^2 = \delta^2\right) \\
&\Pr\left(|y_i z_j| \le \frac{c}{\sqrt{d-1}}\right) \Pr\left(|(c_1-c_2)y_i| \le \delta\frac{c}{\sqrt{d-1}}\right) \Pr\left(|(c_1-c_2)z_j| \le \delta\frac{c}{\sqrt{d-1}}\right) \\
&\ge \left(1 - n^{-\Omega(\log n)}\right)^5 = 1 - n^{-\Omega(\log n)}.
\end{aligned}$$

so, we only need to have

$$2+2\frac{c}{\sqrt{d-1}} \leq L^2(d,n) \leq 2(1-\varepsilon)^2+\delta^2-2\frac{c}{\sqrt{d-1}}-4\delta\frac{c}{\sqrt{d-1}}.$$

There exists $L(d,n)$ if we have

$$2+4\left(1+\frac{(\log n)^2}{d^{\frac{1}{4}}}\right)\frac{\sqrt{\log n \ln n}}{\sqrt{d-1}} \leq 2\left(1-\frac{\log n \ln n}{d}\right)^2+\frac{(\log n)^4}{\sqrt{d}}$$

$$\Leftarrow 2+4\left(1+\frac{(\log n)^2}{d^{\frac{1}{4}}}\right)\frac{\sqrt{\log n \ln n}}{\sqrt{d-1}} \leq 2+2\left(\frac{\log n \ln n}{d}\right)^2-4\frac{\log n \ln n}{d}+\frac{(\log n)^4}{\sqrt{d}}$$

$$\Leftarrow 4\frac{1+\frac{(\log n)^2}{d^{\frac{1}{4}}}}{\sqrt{d-1}} \leq 2\frac{\log n \ln^2 n}{d^2}-4\frac{\ln n}{d}+\frac{(\log n)^3}{\sqrt{d}}$$

$$\Leftarrow 4\frac{1+\frac{(\log d^3)^2}{d^{\frac{1}{4}}}}{\sqrt{d-1}}+4\frac{\ln d^3}{d} \leq 2\frac{\log d^3 \ln^2 d^3}{d^2}+\frac{(\log d^3)^3}{\sqrt{d}}.$$

It's true for large enough d. Therefore, there exists $L(d,n)$ and we can just let $L(d,n)=\sqrt{2+2\sqrt{\frac{\log n \ln n}{d-1}}}$.

Problem 2.5

Answer: 1.$\Pr(x \geq 3) \leq \frac{E[x]}{3}=\frac{\int_0^4 xf(x)\,\mathrm{dx}}{3}=\frac{\int_0^4 \frac{x}{4}\,\mathrm{dx}}{3}=\frac{2}{3}$.

2.$\Pr(|x| \geq a)=\Pr(x^2 \geq a^2) \leq \frac{E[x^2]}{a^2}$. For $a=3$, we have

$$\Pr(|x| \geq 3) \leq \frac{E[x^2]}{9}=\frac{\int_0^4 \frac{x^2}{4}\,\mathrm{dx}}{9}=\frac{16}{27}.$$

3.For $a=3$,

$$\Pr(|x| \geq 3)=\Pr(x^r \geq 3^r) \leq \frac{E[x^r]}{3^r}=\frac{\int_0^4 \frac{x^r}{4}\,\mathrm{dx}}{3^r}=\frac{4^r}{3^r(r+1)}.$$

Problem 2.12

Answer: We know that $V(d)=\frac{2\pi^{\frac{d}{2}}}{d\Gamma\left(\frac{d}{2}\right)}$, so we have

$$\frac{V(d)}{V(d-1)}=\frac{2\pi^{\frac{d}{2}}}{d\Gamma\left(\frac{d}{2}\right)}\frac{(d-1)\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{d-1}{2}}}.$$

If $d=2k$ and $k \geq 3$, then

$$\Gamma\left(\frac{d}{2}\right)=\Gamma(k)=(k-1)!$$

$$\Gamma\left(\frac{d-1}{2}\right)=\Gamma\left(k-\frac{1}{2}\right)=\frac{(2k-3)!!}{2^{k-1}}\sqrt{\pi}$$

$$\frac{V(d)}{V(d-1)}=\pi\frac{(2k-1)!!}{2^k k!}=\frac{(2k-1)...(5\cdot 3\cdot 1\cdot \pi)}{(2k)...(6\cdot 4\cdot 2)}<1.$$

If $d = 2k+1$ and $k \geq 3$, then

$$\Gamma\left(\frac{d}{2}\right) = \Gamma\left(k+\frac{1}{2}\right) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}$$
$$\Gamma\left(\frac{d-1}{2}\right) = \Gamma(k) = (k-1)!$$
$$\frac{V(d)}{V(d-1)} = \frac{2^{k+1}(k)!}{(2k+1)!!} = \frac{(2k)...(6\cdot 4\cdot 2\cdot 2)}{(2k+1)...(7\cdot 5\cdot 3)} < 1.$$

Therefore, the volume of a sphere decreases as the dimension increases above 5. Since

$$V(2) = \pi < V(3) = \frac{4}{3}\pi < V(4) = \frac{\pi^2}{2} < V(5) = \frac{8}{15}\pi^2.$$

Therefore, the volume of a d-dimensional unit ball take on its maximum when $d = 5$.

Problem 2.19

Answer:

1. We integral x_1. For $x_1 = t$, then volume of the remaining ball is $V(d-1)(1-t^2)^{\frac{d-1}{2}}$.

Therefore, we have $V(d) = \int_{-1}^{1}(1-t^2)^{\frac{d-1}{2}}V(d-1)\mathrm{dt}$.

2. $V(1) = 2$, $V(2) = 2\int_{-1}^{1}\sqrt{1-t^2}\mathrm{dt} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)d(\sin t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos(2t)+1)\mathrm{dt} = \frac{\sin(2t)}{2} + t|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi$ and $V(3) = \int_{-1}^{1}(1-t^2)\pi\mathrm{dt} = \pi\left(t-\frac{t^3}{3}\right)|_{-1}^{1} = \frac{4}{3}\pi$.

Problem 2.22

Answer: We assume the height is h, then the face of the hypercube is a ball of dimension $d-1$ and radius $\sqrt{1-\left(\frac{h}{2}\right)^2}$, so the volume of the cylinder is $h\left(1-\left(\frac{h}{2}\right)^2\right)^{\frac{d-1}{2}}V(d-1)$.

$$f'(h) = \left(1-\frac{h^2}{4}\right)^{\frac{d-1}{2}} + h\frac{d-1}{2}\left(-\frac{h}{2}\right)\left(1-\frac{h^2}{4}\right)^{\frac{d-3}{2}}$$
$$= \left(1-\frac{h^2}{4}\right)^{\frac{d-1}{2}} - h^2\frac{d-1}{4}\left(1-\frac{h^2}{4}\right)^{\frac{d-3}{2}} = (1-h^2)^{\frac{d-3}{2}}\left(1-\frac{dh^2}{4}\right).$$

Since $h \leq 1$, we have $f'(h) \geq 0$ for $h \leq \frac{2}{\sqrt{d}}$ and $f'(h) \leq 0$ for $h \geq \frac{2}{\sqrt{d}}$. Therefore, the maximum volume is achieved when $h = \frac{2}{\sqrt{d}}$ and the volume is $h^d = \left(\frac{2}{\sqrt{d}}\right)^d$.

Problem 2.43

Answer: 1.We assume there are n samples, then $\|\mu - m_s\|_\infty = \max(|\mu_1 - m_{s1}|, |\mu_2 - m_{s,2}|, ..., |\mu_d - m_{sd}|)$. Union bound gives us $\Pr(\|\mu - m_s\|_\infty \geq \varepsilon) \leq d\Pr(|\mu_i - m_{si}| \geq \varepsilon)$. We assume that $m_{si} = \frac{x_1+x_2+...+x_n}{n}$, then $\Pr(|\mu_i - m_{si}| \geq \varepsilon) = \Pr\left(\sum_{i=1}^{n}(x_i-\mu_i) \geq n\varepsilon\right)$. Let $y_i = x_i - \mu_i$, then $\Pr\left(\sum_{i=1}^{n}(x_i-\mu_i) \geq n\varepsilon\right) = \Pr\left(\sum_{i=1}^{n}y_i \geq n\varepsilon\right)$.

y_i has normal distribution with mean 0 and variance 1, so we have $\left|E\left[y_i^{2r+1}\right]\right| = 0$, $\left|E\left[y_i^{2r}\right]\right| = E\left(\left|y_i^{2r}\right|\right) = \frac{(2r)!}{2^r r!} \leq (2r)!$. And we also know that $n\varepsilon \in \left[0, \sqrt{2}n\right]$, thus

$$\Pr\left(\sum_{i=1}^{n} y_i \geq n\varepsilon\right) \leq 3e^{-\frac{n^2\varepsilon^2}{12n}} = 3e^{-n\frac{\varepsilon^2}{12}}.$$

$$\Pr(\|\mu - m_s\|_\infty \geq \varepsilon) \leq 3de^{-n\frac{\varepsilon^2}{12}}$$

Take $n > \frac{1200 \ln d}{\varepsilon^2}$ and we can see that $\Pr(\|\mu - m_s\|_\infty \geq \varepsilon) < 3e^{-100} < 0.01$, thus $\Pr(\|\mu - m_s\|_\infty \geq \varepsilon) > 99\%$.

2.Let $z_i = \mu_i - m_{si}$ and $Z \coloneqq \sum_{i=1}^{d} z_i^2$. Then, since $z_i = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $y_i \sim N(0,1)$, we know that $z_i \sim N\left(0, \frac{1}{n}\right)$. Let $t_i = \sqrt{n} z_i$ thus $t_i \sim N(0,1)$. Then, we have $Z = \sum_{i=1}^{d} z_i^2 = \frac{1}{n}\sum_{i=1}^{d} t_i^2$. so $nZ \sim \chi_d^2$.

$E[\chi_d^2] = d(\mathrm{var}[t_i] + E[t_i]) = d$ therefore $E[Z] = \frac{d}{n}$.

$$\Pr(\|\mu - m_s\|_2 \geq \varepsilon) = \Pr\left(\sqrt{\sum_{i=1}^{d} (\mu_i - m_{si})^2} \geq \varepsilon\right) = \Pr\left(\sqrt{\sum_{i=1}^{d} z_i^2} \geq \varepsilon\right) = \Pr(Z \geq \varepsilon^2) \leq \frac{E[Z]}{\varepsilon^2} = \frac{d}{n\varepsilon^2}.$$

When $n > 100\frac{d}{\varepsilon^2}$, we get $\Pr(\|\mu - m_s\|_2 \geq \varepsilon) < 0.01, \Pr(\|\mu - m_s\|_2 \leq \varepsilon) > 99\%$. So we can have $n = O\left(\frac{d}{\varepsilon^2}\right)$.