Mathematics for CS & AI
Spring 2024
Due: 23:59, March 25, 2024

Homework Set 4

Reading Assignments: Finish reading Chapter 2 of [Wasserman].

Written Assignments:(1) From [Wasserman] Chapter 2.14, do exercises 11, 14, 15, 16. (For exercise 15, the term "program" only means that you should give a precise description of the algorithm.)
(2) Do the following problems:

Problem 1 Three random points A, B, C are independently and uniformly chosen from the circumference of a circle. Let $a(x)$ be the probability that at least one of the angles of the triangle ABC exceeds $x\pi$. Determine $a(x)$ as an explicit function of x.

Problem 2 (Generalization of dominos-tiling a $2\times n$ board) Consider the tiling of a $3\times n$ rectangle using dominos, where n is an even integer. Let a_n be the number of ways such a tiling can be accomplished. Note that $a_2 = 3$
(a) Determine a closed-form formula for the the generating function

$$A(x) = \sum_{n=\text{even}} a_n x^n.$$

(b) Determine a closed-form formula for a_n.

Problem 3 Complete the following technical results claimed in today's lecture:

(1) On page 13 of slides, prove Theorem 1, namely, $\tan z$ has residues -1 at all its singularities.

(2) On page 17 of slides, prove Fact 2, namely, $|\tan z| \leq 10$ at all z on Γ_m.

(3) On page 25 of slides, at the bottom, it says that we can assume that $m \geq n-1$ as the Matrix Tree theorem is obviously true. Why it that? (This is a linear algebra problem.)

(4) On page 26 of slides, it says that the $n \times n$ matrix L_G has rank $< n$. Give a proof.

Problem 4 Let $A(z) = \frac{1}{\lambda - e^z}$ be a function over the complex plane, where $\lambda > 1$ is a real number.
(a) Where are all the singularities of A on the complex plane? Are they isolated singularities? Determine the residue of A at each of its isolated singularities.
(b) Consider the power series expansion $A(z) = \sum_{n\geq 0} a_n z^n$ in the neighborhood of $z = 0$. Find a closed-form expression $g(n)$ in variable n, such that $\lim_{n\to\infty} \frac{a_n}{g(n)} = 1$. You should give your reasoning rigorously.

(c) Consider the following recurrence relation: $b_0 = 1$, and for $n \geq 1$,

$$b_n = \sum_{0 \leq k \leq n-1} b_k \binom{2n}{2k}.$$

Find a closed-form expression $h(n)$ in variable n, such that $\lim_{n\to\infty} \frac{b_n}{h(n)} = 1$.

Problem 5 Consider any $n \times n$ matrix $A = (a_{i,j})$ as a point in the Euclidean space R^{n^2}. Show that, for any matrix A, there exists an infinite sequence of non-singular matrices $A_1, A_2, \cdots$ (i.e. $\det(A_i) \neq 0$) such that $\lim_{i\to\infty} A_i = A$.

Problem 6 Sylvester's determinant identity: If A and B are matrices of sizes $m \times n$ and $n \times m$ respectively, then $\det(I_m + AB) = \det(I_n + BA)$, where I_m, I_n denote the $m \times m$ and $n \times n$ identity matrices, respectively.
(a) Prove Sylvester's determinant identity.

(Hint: for any real numbers x, y, we have

$$\begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - xy & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}.)$$

(b) Derive the *Cauchy-Binet Formula* from the *Sylvester's Determinant Identity.*

Problem 7 Prove Lemma 3 on page 26 of slides. That is, for each $S \subseteq \{1, 2, \cdots, m\}$ with $|S| = n - 1$, $|\det(A'_S)| = 1$ if $\{e_k | k \in S\}$ forms a spanning tree of G and 0 otherwise.

Problem 8 Let s_n be the number of spanning trees for K_n (the complete graph on n vertices). Use the Matrix Tree Theorem to show $s_n = n^{n-2}$.