Mathematics for Computer Science: Homework 10

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Problem 1

Answer:

a.

We use the fact that $\left\langle x_i, x_j \right\rangle_\pi = \delta_{ij}$, which means that in the inner product space $\left(R^\Omega, \langle \cdot, \cdot \rangle_\pi\right)$, $\{f_i\}_{i=1}^{|\Omega|}$ is an orthonormal basis. Then we assume $f = \sum_{i=1}^n a_i f_i$. Since $\langle f, f_1 \rangle_\pi = 0$, whence $a_1 = 0$, $f = \sum_{i=2}^n a_i f_i$. Since $\langle f, f \rangle_\pi = 1$, we have $\sum_{i=2}^n a_i^2 = 1$.

Thus,

$$
\begin{aligned}
\langle Lf, f \rangle_\pi &= \sum_{x \in \Omega} \pi(x)(Lf)(x)f(x) \\
&= \sum_{x \in \Omega} \pi(x)((I-P)f)(x)f(x) \\
&= \sum_{x \in \Omega} \pi(x)f(x)f(x) - \sum_{x \in \Omega} \pi(x)(Pf)(x)f(x) \\
&= \langle f, f \rangle_\pi - \sum_{x \in \Omega} \pi(x) \left(\sum_{i=2}^n a_i \lambda_i f_i(x) \right) \left(\sum_{j=2}^n a_j f_j(x) \right) \\
&= \langle f, f \rangle_\pi - \sum_{i=2}^n a_i^2 \lambda_i \langle f_i, f_i \rangle_\pi \\
&= 1 - \sum_{i=2}^n a_i^2 \lambda_i \\
&\geqslant 1 - \lambda_2 \sum_{i=2}^n a_i^2 \\
&= 1 - \lambda_2.
\end{aligned}
$$

Because $\lambda_1 = 1$, so $\langle Lf, f \rangle_\pi \geqslant 1 - \lambda_2 = \lambda_1 - \lambda_2 = \gamma$.

b.

Similar to a, except we don't have $\langle f, f \rangle_\pi = 1$. We still assume $f = \sum_{i=1}^n a_i f_i$. Since $\langle f, f_1 \rangle_\pi = 0$, whence $a_1 = 0$, $f = \sum_{i=2}^n a_i f_i$. The difference is that now we have $\sum_{i=2}^n a_i^2 = \langle f, f \rangle_\pi$. Thus,

$$
\begin{aligned}
\langle Lf, f\rangle_\pi &= \sum_{x\in\Omega} \pi(x)(Lf)(x)f(x) \\
&= \sum_{x\in\Omega} \pi(x)((I-P)f)(x)f(x) \\
&= \sum_{x\in\Omega} \pi(x)f(x)f(x) - \sum_{x\in\Omega} \pi(x)(Pf)(x)f(x) \\
&= \langle f, f\rangle_\pi - \sum_{x\in\Omega} \pi(x)\left(\sum_{i=2}^{n} a_i\lambda_i f_i(x)\right)\left(\sum_{j=2}^{n} a_j f_j(x)\right) \\
&= \langle f, f\rangle_\pi - \sum_{i=2}^{n} a_i^2\lambda_i\langle f_i, f_i\rangle_\pi \\
&= \langle f, f\rangle_\pi - \sum_{i=2}^{n} a_i^2\lambda_i \\
&\geqslant \langle f, f\rangle_\pi - \lambda_2\sum_{i=2}^{n} a_i^2 \\
&= \langle f, f\rangle_\pi(1-\lambda_2).
\end{aligned}
$$

And if we let $a_1 = a_3 = a_4 = ... = a_n = 0$ and $a_2 = \sqrt{\langle f, f\rangle_\pi}$, the equality is reached and we have $\frac{\langle Lf, f\rangle_\pi}{\langle f, f\rangle_\pi} = 1 - \lambda_2 = \gamma$.

Thus, $\gamma = \min_{f\in R^n, \langle f, f_1\rangle_\pi = 0, \langle f, f\rangle_\pi > 0} \frac{\langle Lf, f\rangle_\pi}{\langle f, f\rangle_\pi}$.

c.

Use $\pi(x)P(x,y) = \pi(y)P(y,x)$ we can know that

$$
\begin{aligned}
\sum_{x\in\Omega} \pi(x)f(x)^2 &= \sum_{x\in\Omega} \pi(x)\sum_{y\in\Omega} P(x,y)f(x)^2 \\
&= \sum_{x,y\in\Omega} \pi(x)P(x,y)f(x)^2 \\
&= \frac{1}{2}\sum_{x,y\in\Omega} \left(\pi(x)P(x,y)f(x)^2 + \pi(y)P(y,x)f(y)^2\right) \\
&= \frac{1}{2}\sum_{x,y\in\Omega} \left(\pi(x)P(x,y)\left(f(x)^2 + f(y)^2\right)\right)
\end{aligned}
$$

Also notice that $(Pf)(x) = \sum_{y\in\Omega} P(x,y)f(y)$. Thus,

$$
\begin{aligned}
\langle Lf, f\rangle_\pi &= \sum_{x\in\Omega} \pi(x)(Lf)(x)f(x) \\
&= \sum_{x\in\Omega} \pi(x)((I-P)f)(x)f(x) \\
&= \sum_{x\in\Omega} \pi(x)f(x)f(x) - \sum_{x\in\Omega} \pi(x)(Pf)(x)f(x) \\
&= \frac{1}{2}\sum_{x,y\in\Omega} \left(\pi(x)P(x,y)\left(f(x)^2 + f(y)^2\right)\right) - \sum_{x\in\Omega} \pi(x)\sum_{y\in\Omega} P(x,y)f(y)f(x) \\
&= \frac{1}{2}\sum_{x,y\in\Omega} \left(\pi(x)P(x,y)(f(x)-f(y))^2\right).
\end{aligned}
$$

For the second equality, we can calculate the right side,

$$\begin{aligned}
\text{RHS} &= \sum_{x,y\in\Omega} \pi(x)\pi(y)f(x)f(y) + \frac{1}{2}\sum_{x,y\in\Omega} \pi(x)\pi(y)(f(x)f(x) + f(y)f(y) - 2f(x)f(y)) \\
&= \frac{1}{2}\sum_{x,y\in\Omega} \pi(x)\pi(y)(f(x)f(x) + f(y)f(y)) \\
&= \sum_{x,y\in\Omega} \pi(x)\pi(y)f(x)f(x) \\
&= \sum_{x\in\Omega} \pi(x)f(x)^2 \sum_{y\in\Omega} \pi(y) \\
&= \sum_{x\in\Omega} \pi(x)f(x)^2 \\
&= \text{LHS}.
\end{aligned}$$

d.

$$\langle f_S, f_S\rangle_\pi = \sum_{x\in\Omega} \pi(x)f_S(x)f_S(x) = \sum_{x\in S} \pi(x)\pi\big(S^C\big)^2 + \sum_{x\in S^C} \pi(x)\pi(S)^2 = \pi(S)\pi\big(S^C\big).$$

$$\langle Lf_S, f_S\rangle_\pi = \sum_{x\in\Omega} \pi(x)(Lf_S)(x)f_S(x) = \langle f_S, f_S\rangle_\pi - \sum_{x\in\Omega} \pi(x)\sum_{y\in\Omega} P(x,y)f_S(y)f_S(x).$$

$$\begin{aligned}
\sum_{x\in\Omega} \pi(x)\sum_{y\in\Omega} P(x,y)f_S(y)f_S(x) &= \sum_{x\in S,y\in S} \pi(x)P(x,y)\big(-\pi\big(S^C\big)\big)\big(-\pi\big(S^C\big)\big) + \sum_{x\in S^C,y\in S^C} \pi(x)P(x,y)\pi(S)\pi(S) \\
&\quad + \sum_{x\in S,y\in S^C} \pi(x)P(x,y)\big(-\pi\big(S^C\big)\big)\pi(S) + \sum_{x\in S^C,y\in S} \pi(x)P(x,y)\pi(S)\big(-\pi\big(S^C\big)\big) \\
&= \pi\big(S^C\big)^2 \sum_{x\in S,y\in S} \pi(x)P(x,y) + \pi(S)^2 \sum_{x\in S^C,y\in S^C} \pi(x)P(x,y) \\
&\quad -2\pi(S)\pi\big(S^C\big) \sum_{x\in S,y\in S^C} \pi(x)P(x,y) \\
&= \pi\big(S^C\big)^2 \sum_{x\in S} \pi(x)\left(1 - \sum_{y\in S^C} P(x,y)\right) + \pi(S)^2 \sum_{x\in S^C} \pi(x)\left(1 - \sum_{x\in S^C} P(x,y)\right) \\
&\quad -2\pi(S)\pi\big(S^C\big) \sum_{x\in S,y\in S^C} \pi(x)P(x,y) \\
&= \pi\big(S^C\big)^2\pi(S) + \pi(S)^2\pi\big(S^C\big) - \big(\pi(S) + \pi\big(S^C\big)\big)^2 \sum_{x\in S,y\in S^C} \pi(x)P(x,y) \\
&= \pi(S)\pi\big(S^C\big) - \sum_{x\in S,y\in S^C} \pi(x)P(x,y).
\end{aligned}$$

Therefore, $\langle Lf_S, f_S\rangle_\pi = \sum_{x\in S,y\in S^C} \pi(x)P(x,y)$. And we can know that

$$\frac{\langle Lf_S, f_S\rangle_\pi}{\langle f_S, f_S\rangle_\pi} = \frac{\sum_{x\in S,y\in S^C} \pi(x)P(x,y)}{\pi(S)\pi(S^C)} = \frac{Q\big(S, S^C\big)}{\pi(S)\pi(S^C)}.$$

e.

We can verify that

$$\langle f_S, f_1\rangle_\pi = \sum_{x\in\Omega}\pi(x)f_S(x)f_1(x) = \sum_{x\in S}\pi(x)\big(-\pi\big(S^C\big)\big) + \sum_{x\in S_C}\pi(x)\pi(S) = 0,$$

and

$$\langle f_S, f_S\rangle_\pi = \pi(S)\pi\big(S^C\big)\big(\pi\big(S^C\big)+\pi(S)\big) > 0.$$

Thus from b and d we can deduce that $\gamma \leqslant \frac{\langle Lf_S, f_S\rangle_\pi}{\langle f_S, f_S\rangle_\pi} = \frac{Q(S,S^C)}{\pi(S)\pi(S^C)} \leqslant \frac{2Q(S,S^C)}{\min(\pi(S),\pi(S^C))} = 2\Phi_*$.

Exercise 4.25

Answer:

(1) We denote the probability of reaching vertex 1 before vertex 5 when starting at vertex i is p_i. Then we know that: $p_1 = 1, p_5 = 0$. And we can also get a recursive formula: $p_i = \frac{1}{2}p_{i-1} + \frac{1}{2}p_{i+1}$ for $i = 2, 3, 4$.

Thus $p_2 = \frac{1}{2} + \frac{p_3}{2}, p_3 = \frac{p_2}{2} + \frac{p_4}{2}, p_4 = \frac{p_3}{2}$. Solve these equations we can get $p_2 = \frac{3}{4}, p_3 = \frac{1}{2}, p_4 = \frac{1}{4}$.

Thus, the probability of reaching vertex 1 before vertex 5 when starting at vertex 4 is $\frac{1}{4}$.

(2) We denote the probability of reaching vertex 1 before vertex 5 when starting at vertex i is p_i. Then we know that: $p_1 = 1, p_5 = 0$. And we can also get some equations:

$$\begin{aligned}
p_2 &= \frac{1}{4}p_1 + \frac{1}{4}p_3 + \frac{1}{4}p_4 + \frac{1}{4}p_6,\\
p_3 &= \frac{1}{4}p_1 + \frac{1}{4}p_2 + \frac{1}{4}p_4 + \frac{1}{4}p_6,\\
p_4 &= \frac{1}{4}p_2 + \frac{1}{4}p_3 + \frac{1}{4}p_5 + \frac{1}{4}p_6,\\
p_6 &= \frac{1}{4}p_2 + \frac{1}{4}p_3 + \frac{1}{4}p_4 + \frac{1}{4}p_5.
\end{aligned}$$

Solve these equations and we can get $p_2 = p_3 = \frac{3}{5}, p_4 = p_6 = \frac{2}{5}$.

Thus, the probability of reaching vertex 1 before vertex 5 when starting at vertex 4 is $\frac{2}{5}$.