Week 4

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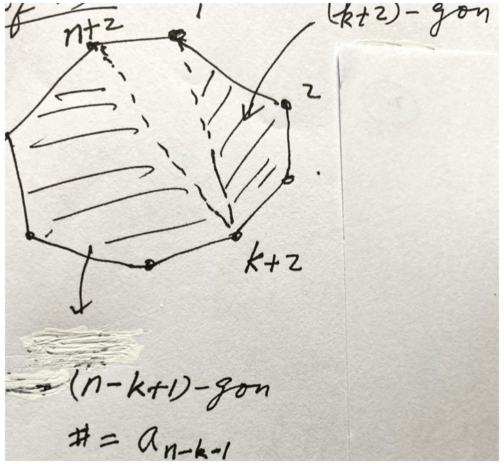
Example 2. Number of triangulations for a convex n-gon (Catalan Number)



- Let a_n = # triangulations of a convex (n+2)-gon

  a_1 = # triangulations of a 3-gon $=1$,

  $a_2 = 2$

  $a_3 = 5$

  Define $a_0 = 1$ (to simplify the recurrence relation; see below)

- For $n \geq 1$, $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0$ (*)

  Proof of (*). A triangulation of a (n+2)-gon has a triangle $\Delta = (1, n+2, k+2)$ where $2 \leq k \leq n+1$. Thus for $n \geq 12$,

$$a_n = \sum_{1 \leq k \leq n-2} a_k\, a_{n-k-1} + 1 \cdot a_{n-1} + a_{n-1} \cdot 1$$
$$= \sum_{0 \leq k \leq n-1} a_k\, a_{n-k-1}$$

It is easy to check that the recurrence is also true for n=1, and this proves (*).

We now solve (*). Let $A(x) = \sum_{n \geq 0} a_n x^n$

Example 2. Number of triangulations for a convex n-gon (continued)

- From (1) $a_n = a_0a_{n-1} + a_1a_{n-2} + \cdots + a_{n-1}a_0$

we obtain $A(x) = 1 + \sum_{n\geq 1} a_n x^n$

$$= 1 + \sum_{n\geq 1} \sum_{0\leq k\leq n-1} (a_k\, a_{n-k-1})\, x^n$$

$$= 1 + x(A(x))^2.$$

Hence $x(A(x))^2 - A(x) + 1 = 0$

$$A(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

To ensure that A(0) is finite, we eliminate the "+" sign possibility. Thus, we have determined our generating function

$$A(x) = \frac{1 - \sqrt{1-4x}}{2x} \qquad (2)$$

We now show how to obtain an explicit formula for a_n .

Example 2. Number of triangulations for a convex n-gon (continued)

Recall the Binomial Theorem $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

- There is a generalization of this called Newton's Generalized Binomial Theorem.

First generalize binomial coefficients to any real z by $\binom{z}{0} \equiv 1$ and $\binom{z}{k} \equiv \frac{z(z-1)\cdots(z-k+1)}{k!}$ for integer k>0. Then $(1+x)^z = \sum_{k\geq 0} \binom{z}{k} x^k$, and this power series is convergent in a neighborhood around x=0.

We apply this theorem to obtain $\sqrt{1-4x} = \sum_{k\geq 0} \binom{1/2}{k} (-4)^k x^k$, and by (2)

$$A(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{-\sum_{k\geq 1} \binom{1/2}{k} (-4)^k x^k}{2x}$$

$$= -\frac{1}{2} \sum_{k\geq 1} \binom{1/2}{k} (-4)^k x^{k-1}$$

Thus $a_n = \frac{1}{2} (-1)^n \binom{1/2}{n+1} 4^{n+1}$.

Example 2. Number of triangulations for a convex n-gon (continued)

<u>Fact</u> $\binom{1/2}{n+1} = \frac{(-1)^n}{2n+1} \frac{1}{2^{2n+1}} \binom{2n+1}{n}$

pf. Homework

This implies $a_n = \frac{1}{2}(-1)^n \frac{(-1)^n}{2n+1} \frac{1}{2^{2n+1}} \binom{2n+1}{n} 4^{n+1}$

$$= \frac{1}{2n+1}\binom{2n+1}{n}$$

The a_n's are known as the <u>Catalan numbers</u>. They appear in many applications.

Example 3. Up-down permutations – *counting via complex analysis*

- Let n=odd, a permutation σ of {1,2,…, n} is an <u>up-down permutation</u>

  if $\sigma[1] < \sigma[2] > \sigma[3] < \sigma[4] > \cdot\ \cdot < \sigma[n-1] > \sigma[n]$'

  e.g. $\sigma = (2\ 5\ 4\ 7\ 1\ 6\ 3)$

- Let a_n = # of up-down permutations of {1,2,…, n}.

  Then $a_1=1$, $a_3=2$ (1 3 2), (2, 3, 1)

What's the recurrence for a_n ?

Example 3. Up-down permutations (continued)

Recurrence for a_n :

- In any up-down permutation σ, let $\sigma(k+1)=n$, then $\sigma[1: k]$ and $\sigma[k+2:n]$ form two up-down permutations. Thus $a_1=1$ and

$$a_n = \sum_{k=odd} \binom{n-1}{k} a_k\, a_{n-1-k} \quad \text{for odd } n\geq 3. \qquad (3)$$

Recurrence relation (3) is similar to the Catalan number recurrence. But there is an extra factor $\binom{n-1}{k}$ in the equation. Let us try to remove this complication, by writing it as

$$a_n = \sum_{k=odd} \frac{(n-1)!}{k!(n-1-k)!} a_k\, a_{n-1-k} \quad , \text{ or}$$

$$n\,\frac{a_n}{n!} = \sum_{k=odd} \frac{a_k}{k!} \cdot \frac{a_{n-1-k}}{(n-1-k)!}$$

- Define a new sequence <b_n> by $b_n \equiv \frac{a_n}{n!}$, then we have $b_1=1$ and

$$n\, b_n = \sum_{k=odd} b_k\, b_{n-1-k} \quad \text{for odd } n\geq 3. \qquad (4)$$

Consider the generating function $B(x)=\sum_{odd\ n} b_n\, x^n$

Example 3. Up-down permutations (continued)

For the generating function $B(x)=\sum_{odd\ n} b_n x^n$ we have $B'(x)=\sum_{odd\ n} n b_n x^{n-1}$,

and (4) leads to $B'(x) = b_1 + \sum_{odd\ n\geq 3} n b_n x^{n-1}$

$$= 1 + \sum_{odd\ n\geq 3} \sum_{k=odd} b_k\, b_{n-1-k}\ x^{n-1}$$

$$= 1 + B(x)^2$$

Thus, $B(x)$ is a solution to the differential equation $y'=1+y^2$. This gives

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + C$$

$$B(x) = y = \tan(x + C)$$

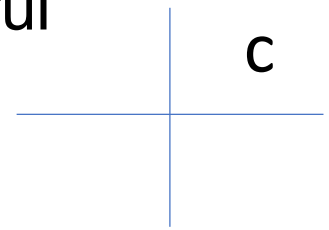
Note $B(0)=0 \Rightarrow C=0$ (or any $k\pi$, as tan is periodic with period π).

- We have reduced the problem of evaluating a_n to the problem of b_n $(\equiv \frac{a_n}{n!})$ which are the coefficients of the power expansion

$$\tan x = \sum_{odd\ n} b_n x^n$$

Example 3. Up-down permutations (continued)

- We are interested in the power-series behavior of $f(x)=\tan x$ near the point $x=0$ on the real line. It turns out that it can be understood by investigating the behavior of $f(z)$ on the much larger complex plane. (Complex analysis is useful in quantum computing, signal processing etc.)



- Let us restrict our attention to two classes of complex functions C → C.

Class A Functions include (1) all polynomials with complex coefficients $p(z)$,

(2) $e^z = \sum_{n\geq 0} \frac{1}{n!} z^n$, (3) inductively, if f, g are included, then $f * g$, $f+g$, $f(g(z))$ are included, for example $\exp(iz^2 - 10z+1)$.

*These functions are special cases of entire functions in the following sense: $f(z)$ is analytic at each point $z_0 \in C$, i.e., $f(z)$ has a power-series expansion $f(z) = \sum_{n\geq 0} a_n (z - z_0)^n$, convergent in a neighborhood of z_0 (i.e. $|z - z_0| < \epsilon$).

Class B Functions of the form $f(z)/g(z)$ where f, g are from Class A above.

Example 3. Up-down permutations (continued)

Class B Functions of the form f(z)/g(z) where f, g are from Class A above, e.g. $\frac{1}{e^{iz}-e^{-iz}}$

These functions are special cases of meromorphic functions in the following sense: F(z) is analytic at all points in C, except possibly at a set of isolated pole singularities $z_i \in C$. Meaning, at a neighborhood of z_i, f(z) has a convergent Laurent series

Laurent series $f(z) = \frac{\lambda_{i,m}}{(z-z_i)^m} + \cdots + \frac{\lambda_{i,1}}{z-z_i} + \sum_{n\geq 0} a_{i,n}(z-z_i)^n$

where $\lambda_{i,k}$, $a_{i,n}$ are complex numbers. We say f(z) has a pole of order m if $\lambda_{i,m} \neq 0$; $\lambda_{i,1}$ is called the residue of f(z) at $z= z_0$.

Example. $f(z) = \frac{1}{z(z-3)^2}$

f(z) is a Class B function, with pole singularities at z_1=0 and z_2=3. At z_1, its Laurent series is as follows:

$$f(z) = \frac{1}{9z\left(1-\frac{z}{3}\right)^2} = \frac{1}{9z}\left(1-\frac{z}{3}\right)^{-2}$$

$$= \frac{1}{9z}\sum_{n\geq 0}\binom{-2}{n}\left(-\frac{z}{3}\right)^n$$

$$= \frac{1}{9z}\left(1+\binom{-2}{1}\left(-\frac{z}{3}\right)+\binom{-2}{2}\left(-\frac{z}{3}\right)^2+\cdots\right)$$

$$= \frac{1}{9z}\left(1+\frac{2}{3}z+\frac{1}{3}z^2+\cdots\right) = \frac{1}{9z}+\frac{2}{27}+\frac{1}{27}z+\cdots$$

Thus at z_1, f(z) has a pole of order 1 (also called a simple pole) and residue 1/9.

$$f(z) = \frac{1}{(3+\Delta)\Delta^2} = \frac{1}{\Delta^2}\,\frac{1}{3}\,\frac{1}{1+\frac{\Delta}{3}}$$

$$= \frac{1}{3\Delta^2}\left(1-\frac{\Delta}{3}+\left(\frac{\Delta}{3}\right)^2-\left(\frac{\Delta}{3}\right)^3+\cdots\right)$$

$$= \frac{1}{3\Delta^2}-\frac{1}{9\Delta}+\frac{1}{27}-\frac{\Delta}{81}+\cdots.$$

Thus, at z_2, $f(z)$ has a pole of order 2 and has residue $-\frac{1}{9}$.

Example 3. Up-down permutations (continued)

Complex integration

- Consider a path Γ from a ∈ C to b ∈ C on the complex plane defined by a parametrization t: [0,1]→ C, t(0)=a, t(1)=b. Let f be a function from Class B (or any meromorphic function) such that Γ doesn't contain any singularities of f.
- Divide the curve Γ evenly by m points $a=z_0, z_1, z_2, \ldots, z_m=b$, and let $D_m = \sum_{0\le k\le m-1} f(z_k)(z_{k+1} - z_k)$. If $\lim_{n\to\infty} D_m$ exists, then we say the complex integral $\int_\Gamma f(z)dz$ exists and equals the value $\lim_{n\to\infty} D_m$. We will need two results.

  The first one is elementary and obvious:

  Length Theorem $|\int_\Gamma f(z)dz| \le \max_{z\in\Gamma} f(z) \cdot$ *length of* Γ.

  The second result, Cauchy's Residue Theorem, is a fundamental and most useful result for applications.

Example 3. Up-down permutations (continued)

Cauchy's Residue Theorem Let Γ be a closed simple curve (not crossing itself) inside an open set W on the complex plane. Let f be a function of Class B (or just any meromorphic function over W) that has no singularities on Γ. Assume that $S=\{z_k\}$ is the set of (pole) singularities of f inside the curve Γ. Then

$$\frac{1}{2\pi i}\oint_\Gamma f(z)dz = \sum_k (\text{residue of f at } z_k)$$

*The direction of integration is counter-clockwise on Γ.

Main Theorem For $b_n \equiv \frac{a_n}{n!}$ where a_n is the number of up-down permutations, we have $b_n = 0$ for even $n \geq 0$ and $b_n = 2(\frac{2}{\pi})^{n+1}(\frac{1}{1^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \cdots)$ for odd $n>0$.

This implies, whenever b_n is known for some n, it leads to a non-obvious equation:

$$\frac{1}{1^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \cdots = \frac{1}{2}\left(\frac{2}{\pi}\right)^{n+1} b_n$$

e.g., $\tan x = x + \cdots$, so $b_1 = \frac{1}{1!} = 1$, this give the value of the infinite sum

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

[This implies $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$, a well-known result.]

Example 3. Up-down permutations (continued)

We have developed important tools in complex analysis, and can now apply them to study the power series coefficients in tan x = $\sum_{odd\ n} b_n x^n$.

We first extend tan x to be a function over the complex plane.

Definition For any z ∈ C, define

$$\tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz}+e^{-iz})}$$

*Note that for real x, this definition gives $\frac{2\, sinx}{2\, cosx}$ = tan x, consistent with the standard definition for real x.

Theorem 1 *tan* z is a function of Class B, having only simple pole singularities at $z_m = (m - ½)\pi$ for integer m, with residues $r_m = -1$ at z_m.

Theorem 2 For all odd n>0, $b_n = -\sum_{\text{integer } m}$ [residue of $\frac{tanz}{z^{n+1}}$ at z_m].

We'll prove Theorem 1, 2 later. For the present, we show how to prove Main Theorem Using Theorem 1, 2.

Let Δ = z − zm

$$\frac{\cancel{\tan z}\,1}{z^{n+1}} = \frac{1}{((z-z_m)+z_m)^{n+1}} \cancel{= \frac{1}{z_m^{n+1}}\left(1-\frac{z}{z_m}\right)}$$

$$= \frac{1}{z_m^{n+1}} \frac{1}{\left(1+\frac{\Delta}{z_m}\right)^{n+1}}$$

$$\tan z = \frac{r_m}{\Delta} + c_0 + c_1\Delta + c_2\Delta^2 + \cdots$$

$$\frac{\tan z}{z^{n+1}} = \frac{r_m}{z_m^{n+1}}\frac{1}{\Delta} + \left(\sum_{k\geq 0} c_k\Delta^k\right)\left(1+\frac{\Delta}{z_m}\right)^{-(n+1)}$$

$$= \frac{r_m}{z_m^{n+1}}\frac{1}{\Delta} + \sum_{k\geq 0} d_k\Delta^k$$

By Thm 1

Thus, $\frac{\tan z}{z^{n+1}}$ has residue $\frac{r_m}{z_m^{n+1}} = -\frac{1}{z_m^{n+1}}$

By Thm 2

$$b_n = \cancel{-}\sum_m \frac{1}{z_m^{n+1}} = \sum_m \frac{1}{((m-\frac{1}{2})\pi)^{n+1}}$$

$$= 2\sum_{m\geq 1}\left(\frac{2}{(2m-1)\pi}\right)^{n+1}.$$

☐ This gives the Main Theorem.

We leave the proof of Theorem 1 as homework. It remains to prove Theorem 2.

Clearly, for each odd n, the function $\dfrac{\tan z}{z^{n+1}}$ has pole singularities at

~~$z = (m+2-\frac{1}{2})\pi$~~

$z_m = (m-\frac{1}{2})\pi$ (singularities of $\tan z$) plus at $z=0$.

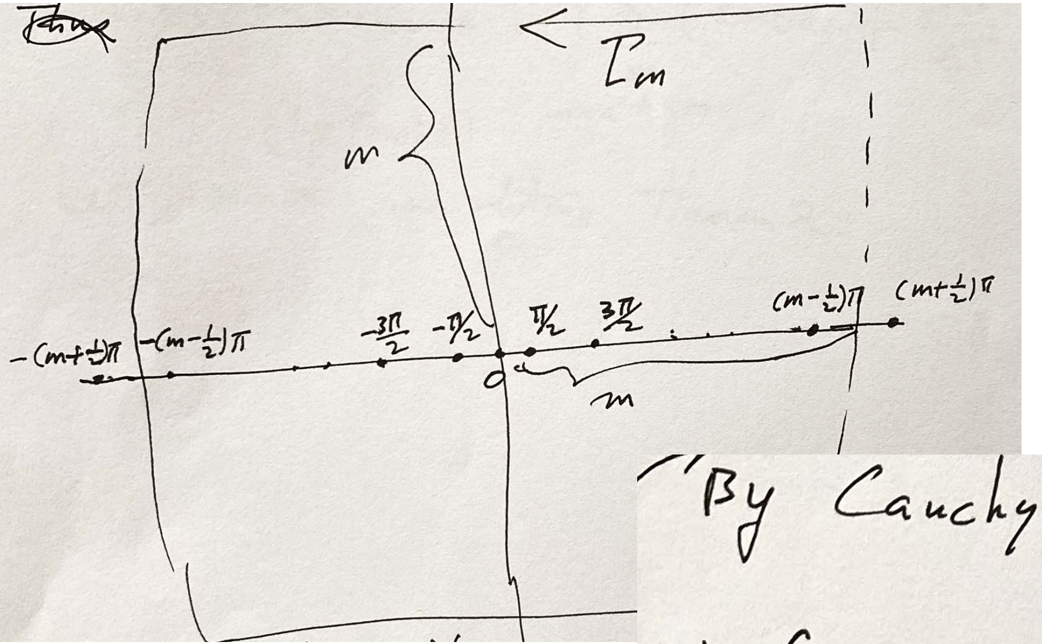
Let β_n be the residue of $\dfrac{\tan z}{z^{n+1}}$ at $z=0$.

<u>Fact</u> $\beta_n = b_n$

<u>proof</u>
$$\frac{\tan z}{z^{n+1}} = \frac{1}{z^{n+1}}\sum_{k\geq 0} b_k z^k = \frac{b_0}{z^{n+1}} + \frac{b_1}{z^n} + \cdots + \frac{b_n}{z} + b_{n+1} + b_{n+2}z + \cdots$$

Let $m > 0$ be any integer.

Construct a closed curve Γ_m traversing the boundary of the $2m \times 2m$ symmetric square.



Fact 2 $|\tan z| \leq 10$ for all $z \in \Gamma_m$

pf. homework. $\blacksquare$

By Cauchy's Residue Theorem,

$$\frac{1}{2\pi i}\oint_{\Gamma_m} \frac{\tan z}{z^{n+1}}\,dz = \left(\text{Residue of } \frac{\tan z}{z^{n+1}} \text{ at } z=0\right)$$

$$+ \sum_{\text{integers } m} \left(\text{Residue of } \frac{\tan z}{z^{n+1}} \text{ at } z_m = (m-\tfrac{1}{2})\pi\right)$$

Example 3. Up-down permutations (continued)

By the length Theorem, in the equation above,

LHS has absolute value $\leq$ 8m $\cdot$ max value of $\left|\frac{\tan z}{z^{n+1}}\right|$

$$\leq 8m \frac{1}{m^{n+1}} \cdot 10 \qquad \text{(by Fact 2)}$$

$$= 80 \frac{1}{m^n}$$

$$\text{RHS} = b_n + \sum_{-m \leq k \leq m} [\text{residue of } \frac{\tan z}{z^{n+1}} \text{ at } z_k].$$

Let m $\rightarrow\infty$, we obtain Theorem 2.

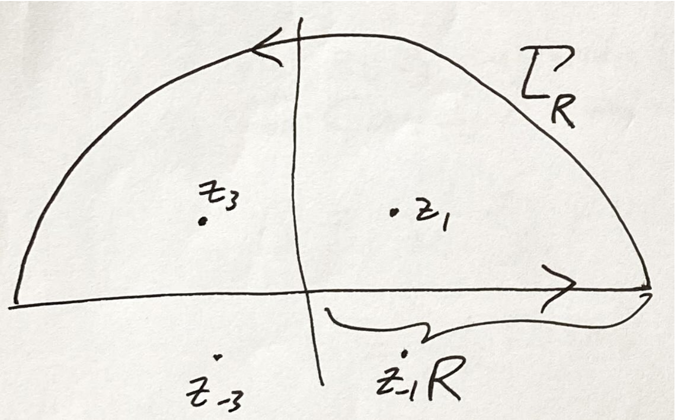
This finishes the problem of counting up-down permutations.

QED

More on Cauchy's Residue Theorem :

A typical application of Cauchy's residue theorem:

Evaluation of the integral $\propto = \int_{-\infty}^{\infty} \frac{dx}{1+x^4}$



- Let $f(z) = \frac{1}{1+z^4}$ and the complex integral

$$\beta_R = \frac{1}{2\pi i} \oint_{\Gamma_R} f(z)dz$$

where Γ_R is the closed semi-circle of radius R>2 on the complex plane.

Lemma 1 The singularities of f(z) are $z_j = \exp(i\frac{2j+1}{4}\pi)$ (j $= \pm 1, \pm 3$), with residues $r_j = -z_j/4$.

pf. The equation has exactly 4 distinct roots $z_j = \exp(i\frac{2j+1}{4}\pi)$

$j = \pm 1, \pm 3$. Thus $f(z) = \frac{1}{(z-z_1)(z-z_3)(z-z_{-1})(z-z_{-3})}$ has simple pole singularities at z_j, with residue

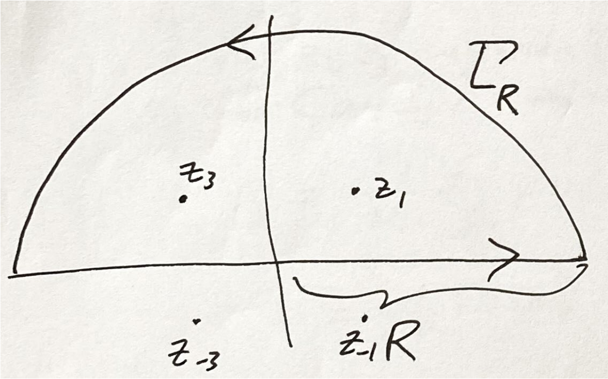
$$r_j = \lim_{z \to z_j} \frac{z - z_j}{1+z^4} = \lim_{z \to z_j} \frac{1}{4z^3} = \frac{1}{4z_j^3} = -\frac{z_j}{4} \quad \square$$

*We have used L'Hopital's calculus rule for real variables here. See homework for justifying its use here

Note that z_1, z_3 are inside Γ_R

We now use Lemma 1 to evaluate β_R :

By Cauchy's residue theorem, we have



$$\beta_R = \frac{1}{2\pi i}\oint_{\Gamma_R} f(z)\,dz = r_1 + r_3 = -\frac{1}{4}(z_1 + z_3)$$

$$= -\frac{1}{4}\cdot i2\sin\frac{\pi}{4} = -\frac{i}{2\sqrt{2}} \qquad (1)$$

On the other hand, using the Length Bound for complex integral,

$$\beta_R = \frac{1}{2\pi i}\left[\int_{-R}^{R} f(x)\,dx + \int_{\text{semi-circle}} \frac{1}{1+z^4}\,dz\right]$$

$$= \frac{1}{2\pi i}\left[\int_{-R}^{R} f(x)\,dx + O\left(\frac{R}{R^4}\right)\right] \qquad (2)$$

(1)+(2) implies

$$\int_{-R}^{R} f(x)\,dx = (2\pi i)\left(-\frac{i}{2\sqrt{2}}\right) + O\left(\frac{1}{R^3}\right)$$

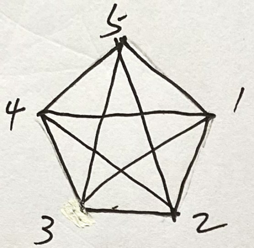
Taking $R \to \infty$, we have $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$

- Counting problems in general are pretty hard to solve. We will see later why this is so. But there are exceptions, and we will present two famous classical counting functions, in which Linear Algebra Theory plays a prominent role. The first one is as follows.

Matrix tree Theorem

- To fix the terminology, a graph G is a pair (V, E), where the vertex set V is a finite set, and $E \subseteq V^{(2)}$ is the edge set where $V^{(2)}$ is the set of all size-2 subsets of V.
  -- For example, let V={1,2,3,4} E={{1,3}, {4,2}, {3,4}, {1,4}}
  -- A complete graph is defined as $K_n=(V,E)$, $|V|=n$ and $E= V^{(2)}$.

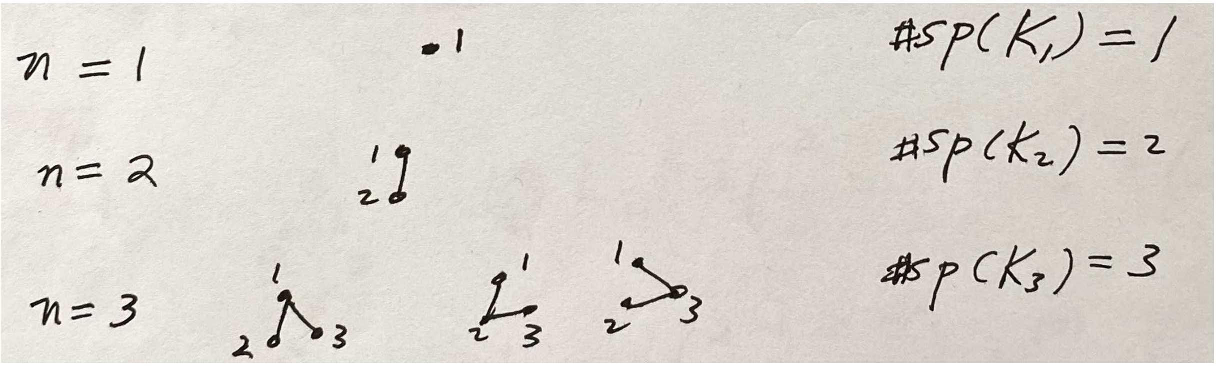
  Here is K_5 :

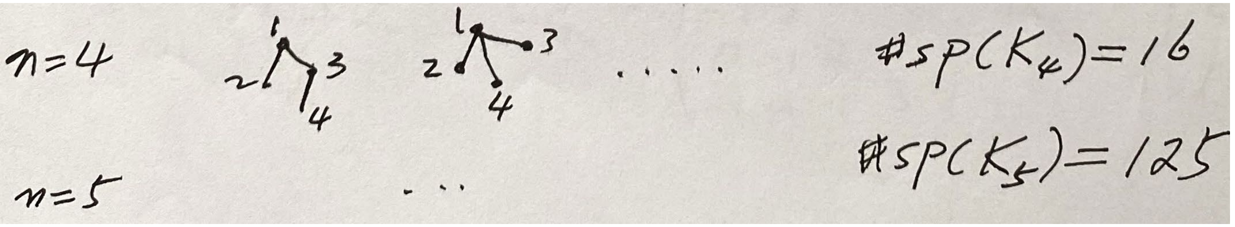


- For a vertex v, degree(v) is the number of edges of the form {v,u} in E. A graph is connected if for every pair of vertices $u \neq v$, there exists a path from u to v. A tree T=(V, E) is a connected graph with $|E| = |V| - 1 \geq 0$. A graph G'=(V', E') is a subgraph of G=(V, E) if $V' \subseteq V$ and $E' \subseteq E$.

Matrix tree Theorem

- A spanning tree of a graph G=(V, E) is a subgraph G'=(V', E') such that (1) G' is a tree and (2) V'=V. Thus any spanning tree of G=(V, E) contains |V| – 1 edges and can connect any two vertices of G.
- How many spanning trees does K_n contain? Call this number #sp(K_n). This is the same as asking how many trees are there on n (labeled) vertices. Let us enumerate them for small n.





- *Is there a general formula in n?*

Carley’s Formula

$$\#sp(K_n) = n^{n-2}$$

There are many ways to prove this formula. (Lovasz’s book contains one such proof.) Instead of giving a proof, we will present a more general result, known as the Matrix Tree Theorem. It gives an explicit elegant formula of #sp(G) for any graph G!

- Let $G=(V, E)$ where $V=\{v_1, v_2,\ldots, v_n\}$. The Laplacian of G is an nxn matrix $L_G=(\ell_{ij})$

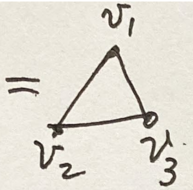
  where ℓ_{ii} = degree(v_i)

  $\ell_{ij} = -1$ if $\{i,j\} \in E$

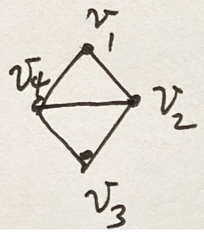
  $\ell_{ij} = 0$ otherwise

For any square matrix A and i, let $A^{(i)}$ be the matrix A with i-th row & i-th column deleted.

Example 1.

$G =$ 

$L_G = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ $\quad L_G^{(2)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$H =$ 

$L_H = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$ $\quad L_H^{(2)} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$

Matrix Tree Theorem (Kirchhoff 1847):

Let $G=(V, E)$, then $\#sp(G)=\det(L_G^{(i)})$ for any $1 \le i \le |V|$.

- In the examples above, clearly $\#sp(G)=3$, and $\det(L_G^{(2)})=\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}=3$, agreeing with the theorem. For H, we have $\det(L_H^{(2)})=\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}= 2\cdot 2\cdot 3-2(-1)(-1)-2(-1)(-1) = 8.$

It is easy to enumerate and verify that $\#sp(H)=8$.

Before proving the theorem, we need a standard linear algebra result:

- Let $n \le m$ and $A=(a_{ij})$, $B=(b_{ij})$ be nxm real matrices. For any $S \subseteq \{1,2,\ldots, m\}$ with $|S|=n$, let A_S denote the A's nxn submatrix $\begin{pmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{ni_1} & a_{ni_2} & \cdots & a_{ni_n} \end{pmatrix}$ where $S=\{i_1 < i_2 < \ldots < i_n\}$. Similarly for B_S.

Cauchy-Binet Formula (1812):

$$\det(A\cdot B^T) = \sum_{|S|=n} \det(A_S) \cdot \det(A_S)$$

[Proof left as homework.]

Cauchy-Binet Formula (1812):

$$\det(A\cdot B^T) = \sum_{|S|=n} \det(A_S) \cdot \det(A_S)$$

For example, let n=1, $A=\{a_1, a_2,\ldots, a_m\}$, $B=\{a_1, a_2,\ldots, a_m\}$. Then for the LHS of formula, $A\cdot B^T$ is just the inner product of A, B as vectors in R^m, and the RHS is $\sum_{1\le i\le m} a_i b_i \cdot$

- Return now to the proof of the Matrix Tree Theorem.

Let $n=|V|$, $m=|E|$, and $V=\{v_1, v_2,\ldots, v_n\}$, $E=\{e_1, e_2,\ldots, e_m\}$. Let A be the nxm matrix $(a_{i\,j})$ where for each $e_j =\{v_{j_1}, v_{j_2}\}$ $(j_1 < j_2)$, $a_{j_1\,j} =1$, $a_{j_2\,j} = -1$, and other $a_{i\,j} =0$. Clearly, each row i of A has exactly degree(v_j) non-zero entries, with each entry being ± 1.

Fact 1 $\quad \text{row}_j \cdot (\text{row}_j)^T = \text{degree}(v_j)$

$$\text{row}_j \cdot (\text{row}_k)^T = \begin{cases} 0 & \text{if } \{vj, vk\} \notin E \\ -1 & \text{if } \{vj, vk\} \in E \end{cases}$$

Proof. Obvious.

Lemma 1 $\;A \cdot A^T = L_G$ (Follows from Fact 1)

- Fix any $1\le i \le n$. Let A' be the (n-1)xm matrix obtained from A by deleting its i-th row.

Corollary $\;A'\cdot (A')^T = L_G^{(i)}$

<u>Lemma 1</u> $A \cdot A^T = L_G$

- Fix any $1 \le i \le n$. Let A' be the (n-1)xm matrix obtained from A by deleting its i-th row.

<u>Corollary</u> $A' \cdot (A')^T = L_G^{(i)}$

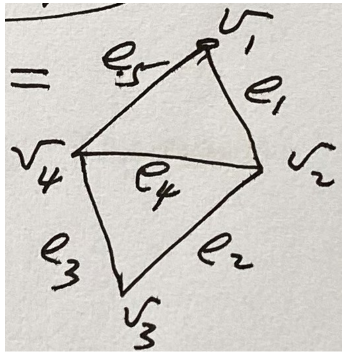
(Imagine setting the i-th row of A to be all-zero. The effect on the product $A \cdot A^T$ is to set all entries on the i-th row and i-th column to 0. This effectively turns L_G into $L_G^{(i)}$).

QED

- For example, consider the earlier example:

then

$$A = \begin{array}{c|c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix} \end{array}$$

$H =$ 

$$A \cdot A^T = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix},$$

which is exactly the same as L_H computed before.

Deleting the i=1 row in A will have the effect of deleting 1st row & column in the matrix product $A \cdot A^T$. QED

Before taking next step, we remark that we can assume $m \ge n-1$. (why?)

Lemma 2 $\det(L_G^{(i)}) = \sum_{\substack{|S|=n-1 \\ S \subseteq \{1,2,\ldots,m\}}} (\det(A'_S))^2$

pf. By Lemma 1 and Cauchy-Binet formula. QED

Lemma 3 Let $S \subseteq \{1,2,\ldots, m\}$ with $|S|=n-1$, then

$|\det(A'_S)|=1$ if $\{e_k \mid k \in S\}$ forms a spanning tree of G,

and $\det(A'_S)=0$ otherwise.

pf. Homework.

QED

It follows from Lemma 2 and Lemma 3 that $\det(L_G^{(i)}) = \#sp(G)$.

We have proven the Matrix Tree Theorem.

- Recall from linear algebra, for an k x k matrix M, its characteristic polynomial is defined as $\det(M-\lambda I)$ where I is the k x k identity matrix. (Some authors define it as $\det(\lambda I - M)$.) Let $f_G(\lambda) = \det(L_G - \lambda I)$ be the characteristic polynomial of L_G. As $L_G = A \cdot A^T$, the roots of $f_G(\lambda)$ are all real and non-negative. Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be its roots. In fact $\lambda_1=0$ as the matrix L_G has rank < n (homework).

- $\lambda_1 \leq \cdots \leq \lambda_n$ are the roots of $f_G(\lambda)$ where $\lambda_1=0$.

Corollary to the Matrix Theorem

$$\#sp(G) = \frac{1}{n}\, \lambda_2 \lambda_3 \cdots \lambda_n$$

Proof of Corollary. Since $f_G(\lambda) = \det(L_G - \lambda I)$, using the Matrix Tree Theorem we have

$$f'_G(0) = \sum_{1\le i\le n} \left(\frac{d}{d\lambda}(dii-\lambda)\right)_{\lambda=0} \cdot \det(L_G^{(i)})$$

$$= -\sum_{1\le i\le n} \det(L_G^{(i)}) = -n \cdot \#sp(G) \qquad \text{(A)}$$

On the other hand,

$$f_G(\lambda) = (-1)^n (\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

and hence

$$f_G'(0) = (-1)^n \left[(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)\right]\Big|_{\lambda=0}$$

$$+ \left(\lambda \sum_{i=2}^{n} \frac{d}{d\lambda}(\lambda-\lambda_i)\right)_{\lambda=0} \cdot \prod_{j\neq\lambda} (-\lambda_j)$$

$$= (-1)^n (-1)^{n-1} \lambda_2\lambda_3\cdots\lambda_n \qquad (B)$$

From (A)+(B), we obtain

$$\# sp(G) = \frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n$$

QED

End