

# Mathematics for Computer Science:

## Homework 1

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### Problem 1

Answer: (a) From Pascal's Formula we know that  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ . So

$$\begin{aligned}
 \binom{n+1}{k+1} &= \binom{n}{k} + \binom{n}{k+1} \\
 &= \binom{n}{k} + \left( \binom{n-1}{k} + \binom{n-1}{k+1} \right) \\
 &= \binom{n}{k} + \binom{n-1}{k} + \left( \binom{n-2}{k} + \binom{n-2}{k+1} \right) \\
 &= \dots \\
 &= \binom{n}{k} + \binom{n-1}{k} + \dots + \binom{k}{k}
 \end{aligned}$$

(b) From induction: We can first easily verify that the equation is true for  $n = 1$ .

If it's true for  $n = k$ , then for  $n = k + 1$ , we have:

$$\begin{aligned}
 1^2 + 2^2 + \dots + (k+1)^2 &= (1^2 + 2^2 + \dots + k^2) + (k+1)^2 \\
 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
 &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
 &= \frac{(k+1)(k+2)(2(k+1) + 1)}{6}
 \end{aligned}$$

Therefore we can continue the induction and get the results.

From (a): Let  $k = 2$  in (a) and we will get:

$$\binom{n+1}{3} = \binom{n}{2} + \binom{n-1}{2} + \dots + \binom{k}{2}$$

So

$$\begin{aligned}
 \frac{(n+1)n(n-1)}{6} &= \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} + \dots + \frac{2(2-1)}{2} \\
 &= \frac{n^2 + (n-1)^2 + \dots + 2^2 + 1^2 - (n + (n-1) + \dots + 2 + 1)}{2} \\
 n^2 + (n-1)^2 + \dots + 2^2 + 1^2 &= \frac{(n+1)n(n-1)}{3} + \frac{n(n+1)}{2} \\
 &= \frac{n(n+1)(2n+1)}{6}
 \end{aligned}$$

## Problem 2

Answer: (a) For  $n < 365$

$$\begin{aligned}
 \bar{q}(n) &\leq \bar{d}(n) \\
 \Leftrightarrow \prod_{1 \leq i \leq n-1} \left(1 - \frac{i}{365}\right) &\leq e^{-\frac{n(n-1)}{730}} \\
 \Leftrightarrow \prod_{1 \leq i \leq n-1} e^{-\frac{i}{365}} &\leq e^{-\frac{n(n-1)}{730}} \\
 \Leftrightarrow e^{-\frac{\sum_{1 \leq i \leq n-1} i}{365}} &\leq e^{-\frac{n(n-1)}{730}} \\
 \Leftrightarrow e^{-\frac{n(n-1)}{2 \cdot 365}} &\leq e^{-\frac{n(n-1)}{730}}
 \end{aligned}$$

(b) We first prove that  $e^{-x-x^2} \leq 1-x \leq e^{-x-\frac{x^2}{2}}$  for  $0 \leq x \leq \frac{1}{2}$

LHS: Let  $f(x) = 1-x-e^{-x-x^2}$ ,  $f'(x) = -1+(1+2x)e^{-x-x^2}$

Let  $g(x) = (1+2x)e^{-x-x^2}$ ,  $g'(x) = -(1+2x)^2+2)e^{-x-x^2}$

$f'(0) = 0$ ,  $f'(\frac{1}{2}) = -1+2e^{-\frac{3}{4}} > 0$ ,  $f'(\frac{\sqrt{2}-1}{2}) = -1+\sqrt{2}e^{-\frac{1}{4}} > 0$ . So  $f(x) \geq f(0) = 0$

RHS: Let  $f(x) = 1-x-e^{-x-\frac{x^2}{2}}$ ,  $f'(x) = -1+(1+x)e^{-x-\frac{x^2}{2}}$

Let  $g(x) = (1+x)e^{-x-\frac{x^2}{2}}$ ,  $g'(x) = -(1+x)^2+1)e^{-x-\frac{x^2}{2}} \leq 0$ .

So  $f'(x) \leq f'(0) = 0$ ,  $f(x) \leq f(0) = 0$

For  $1 \leq n \leq \frac{365}{2}$

$$\begin{aligned}
 \exp\left(\frac{n(n-1)(2n-1)}{12 \cdot (365)^2}\right) \bar{q}(n) &= \exp\left(\frac{1^2+2^2+\dots+(n-1)^2}{2 \cdot (365)^2}\right) \bar{q}(n) \\
 &\leq e^{-\frac{\sum_{1 \leq i \leq n-1} i}{365} - \frac{\sum_{1 \leq i \leq n-1} i^2}{2 \cdot (365)^2} + \frac{1^2+2^2+\dots+(n-1)^2}{2 \cdot (365)^2}} \\
 &= \bar{d}(n) \\
 \exp\left(\frac{n(n-1)(2n-1)}{6 \cdot (365)^2}\right) \bar{q}(n) &= \exp\left(\frac{1^2+2^2+\dots+(n-1)^2}{365^2}\right) \bar{q}(n) \\
 &\geq e^{-\frac{\sum_{1 \leq i \leq n-1} i}{365} - \frac{\sum_{1 \leq i \leq n-1} i^2}{365^2} + \frac{1^2+2^2+\dots+(n-1)^2}{365^2}} \\
 &= \bar{d}(n)
 \end{aligned}$$

## Problem 3

Answer: Using the inequality above, we know that

$$\exp\left(-\frac{n(n-1)(2n-1)}{6 \cdot (2^m)^2}\right) \bar{d}(n) \leq 1 - \epsilon_m \leq \exp\left(-\frac{n(n-1)(2n-1)}{12 \cdot (2^m)^2}\right) \bar{d}(n)$$

Therefore,  $\epsilon_m \approx 1 - e^{-\frac{n(n-1)}{2^{m+1}}} < 10^{-10}$ . We get  $m > \log_2(10^{10}n(n-1)) - 1 \approx 61.9$  so  $m = 62$ .

## Problem 4

Answer: (a)  $P = (U, p)$ , where  $U = \{a, b, c, d\}$  (“a” represents the situation when gift behind door 1 while host open door 2; “b” represents the situation when gift behind door 1 while host open door 3; “c” represents the situation when gift behind door 2 while host open door 3; “d” represents the situation when gift behind door 3 while host open door 2).

$$P(a) = P(b) = \frac{r_1}{2}, P(c) = r_2, P(d) = r_3$$

If  $g$  is “2  $\rightarrow$  Switch, 3  $\rightarrow$  Switch”,  $T_g = \{d, c\}$ ,  $\Pr\{T_g\} = P(c) + P(d) = r_2 + r_3$ .

If  $g$  is “2  $\rightarrow$  Switch, 3  $\rightarrow$  No-Switch”,  $T_g = \{c, b\}$ ,  $\Pr\{T_g\} = P(b) + P(c) = \frac{r_1}{2} + r_2$ .

If  $g$  is “2  $\rightarrow$  No-Switch, 3  $\rightarrow$  Switch”,  $T_g = \{a, d\}$ ,  $\Pr\{T_g\} = P(a) + P(d) = \frac{r_1}{2} + r_3$ .

If  $g$  is “2  $\rightarrow$  No-Switch, 3  $\rightarrow$  No-Switch”,  $T_g = \{a, b\}$ ,  $\Pr\{T_g\} = P(a) + P(b) = r_1$ .

$$(b) f\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right) = \max\left\{r_2 + r_3, \frac{r_1}{2} + r_2, \frac{r_1}{2} + r_3, r_1\right\} = \max\left\{\frac{1}{2}, \frac{7}{12}, \frac{5}{12}, \frac{1}{2}\right\} = \frac{7}{12}$$

$$(c) f(r_1, r_2, r_3) = \max\left\{r_2 + r_3, \frac{r_1}{2} + r_2, \frac{r_1}{2} + r_3, r_1\right\} \geq \frac{r_2 + r_3 + \frac{r_1}{2} + r_2 + \frac{r_1}{2} + r_3 + r_1}{4} = \frac{1}{2}$$

When  $r_2 + r_3 = \frac{r_1}{2} + r_2 = \frac{r_1}{2} + r_3 = r_1$ , which means  $r_1 = \frac{1}{2}, r_2 = \frac{1}{4}, r_3 = \frac{1}{4}$ , we have  $f(r_1, r_2, r_3) = \frac{1}{2}$

$$(d) f(r_1, r_2, r_3) = \max\left\{r_2 + r_3, \frac{r_1}{2} + r_2, \frac{r_1}{2} + r_3, r_1\right\}$$

When host open door 2, guest doesn’t know about door 1 or door 3 has the gift, so he should choose the door with higher probability, which is  $\max\{P(a), P(d)\} = \max\{\frac{r_1}{2}, r_3\}$ . It’s the same when host open door 3.

## Problem 5

Answer: It is because the  $y$  in the question is different when we are calculating yuans we gain or lose. For example, If  $y = x$ , then the probability to lose is 0 but not  $\frac{1}{2}$ .

## Problem 6

Answer: (a) Any  $G$  with only one point has a clique of size 1 and has an independent set of size 1. Therefore  $R(1, s) = R(r, 1) = 1$ .

(b) We colored the graph’s edges to red and if vertical  $v$  and  $w$  don’t have an edge between them, we draw an edge and color it to blue.

We assume  $R(r-1, s) = n_1$ ,  $R(r, s-1) = n_2$ .

Construct a graph with  $R(r-1, s) + R(r, s-1) = n_1 + n_2$  vertices.

We random choose a vertical  $v$ , let  $v_r$  denote the red edges and  $v_b$  denote the blue edges. So  $v_r + v_b \geq n_1 + n_2 - 1$ . Then we know that  $v_r \geq n_1$  and  $v_b \geq n_2$  at least one must be right.

Without lose of generosity, we assume  $v_r \geq n_1$ , then for all vertices with a red edge connected to  $v$ , this new subgraph must has a red  $K_{r-1}$  or blue  $K_s$ .

If it has a red  $K_{r-1}$ , since  $v$  connects to all of  $K_{r-1}$ ’s vertices, so we get a  $K_r$ .

If it has a blue  $K_s$ , then we are done.

To sum up, a graph with  $R(r-1, s) + R(r, s-1)$  vertices must contain a clique of size  $r$  or an independent set of size  $s$ . So  $R(r, s) \leq R(r-1, s) + R(r, s-1)$ .

(c) We use induction.

For  $r = 1$ ,  $R(1, s) = 1 \leq \binom{s-1}{0}$ ;  $s = 1$ ,  $R(r, 1) = 1 \leq \binom{r-1}{0}$ .

If  $r \leq k$ ,  $s \leq t$  is true, then for  $r = k + 1$ :

$$\begin{aligned} R(k+1, t) &\leq R(k, t) + R(k+1, t-1) \\ &\leq \binom{k+t-2}{k-1} + \binom{k+t-2}{k} \\ &= \binom{k+t-1}{k} \end{aligned}$$

For  $s = t + 1$ :

$$\begin{aligned} R(k, t+1) &\leq R(k-1, t+1) + R(k, t) \\ &\leq \binom{k+t-2}{k-2} + \binom{k+t-2}{k-1} \\ &= \binom{k+t-1}{k-1} \end{aligned}$$

Therefore  $R(r, s) \leq \binom{r+s-2}{r-1}$  for all integers  $r, s \geq 2$

Let  $r = s = k$ , we can get  $R(k) \leq \binom{2k-2}{k-1}$ .

We use induction once again, for  $k = 2$ ,  $\binom{2}{1} < 4^2$ .

If  $k = p$  is true, then for  $k = p + 1$ :

$$\binom{2p}{p} = \frac{(2p)!}{(p!)^2} = \frac{2p(2p-1)}{p^2} \binom{2p-2}{p-1} < 4 \binom{2p-2}{p-1} < 4^{p+1}$$

Therefore  $R(k) \leq \binom{2k-2}{k-1} < 4^k$ .

## Problem 7

Answer: Note that move empty cell left or right will not change the number of inversion.

When move empty cell up or down, empty cell jump through 3 cells, therefore the change of the number of inversion could be +3, +1, -1, -3.

We can also note that if empty cell remains in the same position, then the number of move must be an even number.

The first configuration has 0 inversion, so when we change the first configuration, the number of inversion must remain even. While the second configuration has 1 inversion, which is impossible.