CHAPTER 1

Buffon Needle Problem, Extensions, and Estimation of π

The Buffon needle problem which many of us encountered in our college or even high school days has now been with us for two hundred years. One major aspect of its appeal is that its solution has been tied to the value of π which can then be estimated by physical simulation of the model as was done by a number of investigators in the late 19th and early 20th centuries—and by computer simulation today. Shortly we go into some detail on enlarging the experimental design of the model. Then we discuss a number of ways in which modern statistical procedures can yield estimates of π from these experimental designs with much better precision than the original Buffon procedure employed by many for the estimation of π. In this sense we will be featuring the statistician in geometrical probability and this will continue, where possible, for other topics we discuss in this exposition.

It is interesting that in the original development, Buffon (1777) extols geometry as a companion tool to the calculus in establishing a science of probability and suggests that chance is amenable to the methods of geometry as well as those of the calculus. Buffon indicates that the human mind, because of prior mathematics, preferred numbers to measures of area but that the invention of games revolving around area and ratios of areas could rectify this. To highlight this point he investigated a game already in practice in the 18th century known as "clean tile".

In a room tiled or paved with equal tiles, of any shape, a coin is thrown upwards; one of the players bets that after its fall the coin will rest cleanly, i.e., on one tile only; the second bets that the coin will rest on two tiles, i.e., that it will cover one of the cracks which separate them; a third player bets the coin will rest over 3, 4, or 6 cracks: it is required to find the chances for each of these players.

Buffon investigates this game for square tiles, tiles shaped as equilateral triangles, hexagonal tiles, and diamond shaped tiles; in each case he is interested in the ratios of the diameter of the coin to the equal sides of the particular shaped tile that provides a fair game for each player. In effect, the thrust of his work on geometrical probability is the development of fair games.

As a special case, Buffon in his own words states, "I assume that in a room, the floor of which is merely divided by parallel lines, a stick is thrown upwards and one of the players bets the stick will not interesect any of the parallels on the floor, whereas on the contrary the other one bets the stick will intersect some one of these lines; it is required to find the chances of the two players. It is possible to play this game with a sewing needle or a headless pin."

He then demonstrates that for a fair game between two players, the ratio of the length of the needle, l, to the distance between the parallel lines, d, $(d > l)$ must equal $\pi/4$ for this provides the probability of an intersection equal to $\frac{1}{2}$. This can be seen easily from the Buffon needle result as we now know it, namely the probability of an intersection, $p = 2l/(\pi d)$.

In the Buffon model a needle (line segment) is dropped "at random" on the grid of equidistant parallel lines in the plane. The notion that random elements are geometric objects such as line segments, lines in the plane, circles, rectangles, triangles, etc. requires that a measure be defined for such elements before probabilistic assertions can be made. The Bertrand paradox at the turn of the century suggested a situation wherein future developments could be stymied because of a lack of a natural choice of measure. In that paradox, the probability that a random chord in a circle exceeds the side of an inscribed equilateral triangle can be shown to be $\frac{1}{4}$, $\frac{1}{3}$, or $\frac{1}{2}$ for each of three different models by which the chord is drawn at random. Integral geometry becomes helpful here in establishing appropriate models.

Randomness models will play an important role throughout. As a second example, the question of whether pairs of chromosomes are randomly distributed in the nucleus of a cell during mitosis is translated into a geometrical probability problem regarding the expected number of intersections of n pairs of chords in a circle. Naturally, the expected number depends on the randomization model for obtaining chords in a circle and we develop six models yielding six solutions.

The Bertrand paradox led to proposals by Poincaré and others that probability statements for geometric situations be tied to densities that would be invariant under appropriate transformations. For our purposes, the group of rigid motions, that is, transformations that provide invariance under translation and rotation will serve our interest. For the Bertrand paradox, the appropriate density under the group of rigid motions leads to the solution that the probability is $\frac{1}{2}$. The other two solutions induce densities that are not invariant under the group of rigid motions.

One of the prime developers of integral geometry and its consequences for geometrical probability is L. A. Santaló. We will call on his developments and results for questions of invariance of measure and density where appropriate. Santaló has a prolific and prodigious output that is referenced in his most recent book (1976). That volume is must reading for any student of the subject for its exposition of results to date and the extensive bibliography it contains.

The Buffon needle problem. A needle (line segment) of length l is dropped "at random" on a set of equidistant parallel lines in the plane that are d units apart, $l \leq d$.

Uspensky (1937) provides a proof that the probability of an intersection is $p = 2l/(\pi d)$. He develops this by considering a finite number of possible posi-

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tions for the needle as equally likely outcomes and then treats the limiting case as a representation of the problem. This includes a definition of randomness for the distance x of the needle's midpoint to the nearest line and the acute angle φ formed by the needle and a perpendicular from the midpoint to the line. The solution is obtained by computing the ratio of favorable outcomes to the total set of outcomes and passing to the limit.

A simple way of obtaining the answer is to employ a density that seems intuitively satisfactory and which turns out to be the invariant density under the group of rigid motions. This approach follows. The measure of the set of total outcomes is

$$\int_0^{\pi/2}\int_0^{d/2} dx\, d\varphi = \frac{\pi d}{4}. \tag{1.1}$$

From Fig. 1.1 we evaluate the measure of the set of favorable cases (intersections) as

$$\int_0^{\pi/2}\int_0^{(l/2)\cos\varphi} dx\, d\varphi = \frac{l}{2}; \tag{1.2}$$

therefore

$$p = \frac{l/2}{\pi d/4} = \frac{2l}{\pi d}. \tag{1.3}$$

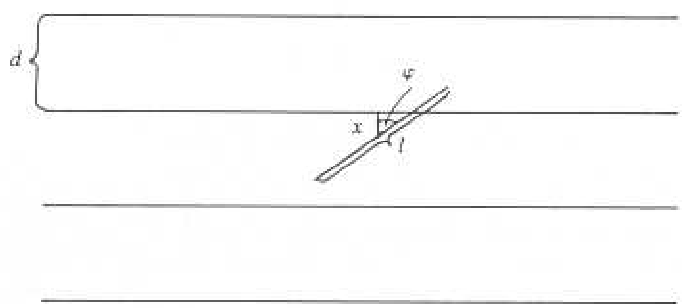

FIG. 1.1

Laplace extension of Buffon problem. Consider two sets of parallel lines over the plane where one set is orthogonal to the other. We now wish to find the probability that the needle dropped at random intersects a line of the grid. Assume the needle is shorter than the smaller sides of the congruent rectangles formed over the plane. This is the Laplace extension. First we find the probability that the needle is contained in one of the rectangles of the set.

Let a, b be the sides of the rectangle that contains the midpoint of the needle whose length is l $(l<a, l<b)$. The position of the needle is determined by the coordinates x, y of its midpoint and, as before, the angle φ formed by the needle with the x-axis. Our randomness model suggests we consider x, y, φ as three

independent variables each with uniform distribution of probability over their ranges $0 \leqq x \leqq a$; $0 \leqq y \leqq b$; and $-\pi/2 \leqq \varphi \leqq \pi/2$. Thus the domain is a parallelepiped for a uniform distribution of the point x, y, φ.

The volume of the domain representing positions of the needle entirely within the rectangle is

$$V^* = \int_{-\pi/2}^{\pi/2} F(\varphi)\, d\varphi = \pi ab - 2b\pi - 2al + l^2 \tag{1.4}$$

where

$$F(\varphi) = ab - bl \cos \varphi - al|\sin \varphi| + \tfrac{1}{2}l^2|\sin 2\varphi|, \tag{1.5}$$

and the volume of the total domain $V = \pi ab$. This is developed in Uspensky (1937, p. 255).

Therefore

$$1 - p = \frac{V^*}{V} = 1 - \frac{2l(a+b) - l^2}{\pi ab} \tag{1.6}$$

and of course, the probability for the needle to intersect the perimeter of one of the rectangles is

$$p = \frac{2l(a+b) - l^2}{\pi ab}. \tag{1.7}$$

If $a = b = 1$,

$$p = \frac{4l - l^2}{\pi}. \tag{1.8}$$

This provides another approach for estimating π where $l < a$, $l < b$.

In an interesting article Schuster (1974) develops the orthogonal lines grid a bit further from the point of view of experimental design. He raises a question about the bright student who drops a needle of length L on a grid of orthogonal lines separated by distance $2L$ and repeats it until 100 observations are made of intersections with, say, lines parallel to the x-axis and these same drops are employed to count intersections with, say, lines parallel to the y-axis. How does the estimate of π from this experiment differ from that obtained by the average student who drops the needle 200 times on a grid of parallel lines separated by distance $2L$. If the x intersections and y intersections are independent, the bright student has accomplished the same purpose with half the drops.

Let A be the event-intersection with the x axis and let

$$x = \begin{cases} 1 & \text{if intersection with the } x \text{ axis,} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for B and the y axis and let

$$y = \begin{cases} 1 & \text{if intersection with the } y \text{ axis,} \\ 0 & \text{otherwise.} \end{cases}$$