# Mathematics for Computer Science: Homework 1

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## Problem 1

Answer: (a) From Pascal's Formula we know that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. So

$$\begin{aligned}
\binom{n+1}{k+1} &= \binom{n}{k} + \binom{n}{k+1} \\
&= \binom{n}{k} + \left(\binom{n-1}{k} + \binom{n-1}{k+1}\right) \\
&= \binom{n}{k} + \binom{n-1}{k} + \left(\binom{n-2}{k} + \binom{n-2}{k+1}\right) \\
&= \ldots \\
&= \binom{n}{k} + \binom{n-1}{k} + \ldots + \binom{k}{k}
\end{aligned}$$

(b) From induction: We can first easily verify that the equation is true for n = 1.

If it's true for n = k, then for n = k + 1, we have:

$$\begin{aligned}
1^2 + 2^2 + \ldots + (k+1)^2 &= (1^2 + 2^2 + \ldots + k^2) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6}
\end{aligned}$$

Therefore we can continue the induction and get the results.

From (a): Let k = 2 in (a) and we will get:

$$\binom{n+1}{3} = \binom{n}{2} + \binom{n-1}{2} + \ldots + \binom{k}{2}$$

So

$$\begin{aligned}
\frac{(n+1)n(n-1)}{6} &= \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} + \ldots + \frac{2(2-1)}{2} \\
&= \frac{n^2 + (n-1)^2 + \ldots + 2^2 + 1^2 - (n + (n-1) + \ldots + 2 + 1)}{2}
\end{aligned}$$

$$\begin{aligned}
n^2 + (n-1)^2 + \ldots + 2^2 + 1^2 &= \frac{(n+1)n(n-1)}{3} + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

# Problem 2

Answer: (a) For n < 365

$$\begin{aligned}
&\overline{q}(n) \le \overline{d}(n) \\
\Leftrightarrow &\prod_{1\le i\le n-1}\left(1-\frac{i}{365}\right) \le e^{-\frac{n(n-1)}{730}} \\
\Leftarrow &\prod_{1\le i\le n-1} e^{-\frac{i}{365}} \le e^{-\frac{n(n-1)}{730}} \\
\Leftrightarrow &e^{-\frac{\sum_{1\le i\le n-1} i}{365}} \le e^{-\frac{n(n-1)}{730}} \\
\Leftrightarrow &e^{-\frac{n(n-1)}{2\cdot 365}} \le e^{-\frac{n(n-1)}{730}}
\end{aligned}$$

(b) We first prove that $e^{-x-x^2} \le 1-x \le e^{-x-\frac{x^2}{2}}$ for $0 \le x \le \frac{1}{2}$

LHS: Let $f(x) = 1-x-e^{-x-x^2}$, $f'(x) = -1+(1+2x)e^{-x-x^2}$

Let $g(x) = (1+2x)e^{-x-x^2}$, $g'(x) = \left(-(1+2x)^2+2\right)e^{-x-x^2}$

$f'(0) = 0, f'\left(\frac{1}{2}\right) = -1+2e^{-\frac{3}{4}} > 0, f'\left(\frac{\sqrt{2}-1}{2}\right) = -1+\sqrt{2}e^{-\frac{1}{4}} > 0$. So $f(x) \ge f(0) = 0$

RHS: Let $f(x) = 1-x-e^{-x-\frac{x^2}{2}}$, $f'(x) = -1+(1+x)e^{-x-\frac{x^2}{2}}$

Let $g(x) = (1+x)e^{-x-\frac{x^2}{2}}$, $g'(x) = \left(-(1+x)^2+1\right)e^{-x-\frac{x^2}{2}} \le 0$.

So $f'(x) \le f'(0) = 0, f(x) \le f(0) = 0$

For $1 \le n \le \frac{365}{2}$

$$\begin{aligned}
\exp\left(\frac{n(n-1)(2n-1)}{12\cdot(365)^2}\right)\overline{q}(n) &= \exp\left(\frac{1^2+2^2+...+(n-1)^2}{2\cdot(365)^2}\right)\overline{q}(n) \\
&\le e^{-\frac{\sum_{1\le i\le n-1} i}{365}-\frac{\sum_{1\le i\le n-1} i^2}{2\cdot(365)^2}+\frac{1^2+2^2+...+(n-1)^2}{2\cdot(365)^2}} \\
&= \overline{d}(n)
\end{aligned}$$

$$\begin{aligned}
\exp\left(\frac{n(n-1)(2n-1)}{6\cdot(365)^2}\right)\overline{q}(n) &= \exp\left(\frac{1^2+2^2+...+(n-1)^2}{365^2}\right)\overline{q}(n) \\
&\ge e^{-\frac{\sum_{1\le i\le n-1} i}{365}-\frac{\sum_{1\le i\le n-1} i^2}{365^2}+\frac{1^2+2^2+...+(n-1)^2}{365^2}} \\
&= \overline{d}(n)
\end{aligned}$$

# Problem 3

Answer: Using the inequality above, we know that

$$\exp\left(-\frac{n(n-1)(2n-1)}{6\cdot(2^m)^2}\right)\overline{d}(n) \le 1-\epsilon_m \le \exp\left(-\frac{n(n-1)(2n-1)}{12\cdot(2^m)^2}\right)\overline{d}(n)$$

Therefore, $\epsilon_m \approx 1-e^{-\frac{n(n-1)}{2^{m+1}}} < 10^{-10}$. We get $m > \log_2\left(10^{10}n(n-1)\right)-1 \approx 62.9$ so m = 63.

# Problem 4

Answer: (a)P = (U, p), where U =$\{a, b, c, d\}$ ("a" represents the situation when gift behind door 1 while host open door 2; "b" represents the situation when gift behind door 1 while host open door 3; "c" represents the situation when gift behind door 2 while host open door 3; "d" represents the situation when gift behind door 3 while host open door 2).

P(a) = P(b) = $\frac{r_1}{2}$, P(c) = $r_2$, P(c) = $r_3$

If g is "$2 \to$ Switch, $3 \to$ Switch", $T_g$ = {d, c}, Pr$\{T_g\}$ = P(c) + P(d) = $r_2 + r_3$.

If g is "$2 \to$ Switch, $3 \to$ No-Switch", $T_g$ = {c, b}, Pr$\{T_g\}$ = P(b) + P(d) = $\frac{r_1}{2} + r_2$.

If g is "$2 \to$ No-Switch, $3 \to$ Switch", $T_g$ = {a, d}, Pr$\{T_g\}$ = P(a) + P(d) = $\frac{r_1}{2} + r_3$.

If g is "$2 \to$ No-Switch, $3 \to$ No-Switch", $T_g$ = {a, b}, Pr$\{T_g\}$ = P(a) + P(b) = $r_1$.

(b) $f\big(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\big) = \max\big\{r_2 + r_3, \frac{r_1}{2} + r_2, \frac{r_1}{2} + r_3, r_1\big\} = \max\big\{\frac{1}{2}, \frac{7}{12}, \frac{5}{12}, \frac{1}{2}\big\} = \frac{7}{12}$

(c) $f(r_1, r_2, r_3) = \max\big\{r_2 + r_3, \frac{r_1}{2} + r_2, \frac{r_1}{2} + r_3, r_1\big\} \geq \frac{r_2+r_3+\frac{r_1}{2}+r_2+\frac{r_1}{2}+r_3+r_1}{4} = \frac{1}{2}$

When $r_2 + r_3 = \frac{r_1}{2} + r_2 = \frac{r_1}{2} + r_3 = r_1$, which means $r_1 = \frac{1}{2}, r_2 = \frac{1}{4}, r_3 = \frac{1}{4}$, we have $f(r_1, r_2, r_3) = \frac{1}{2}$

(d) $f(r_1, r_2, r_3) = \max\big\{r_2 + r_3, \frac{r_1}{2} + r_2, \frac{r_1}{2} + r_3, r_1\big\}$

When host open door 2, guest doesn't know about door 1 or door 3 has the gift, so he should choose the door with higher probability, which is max{P(a), P(d)} = $\max\{\frac{r_1}{2}, r_3\}$. It's the same when host open door 3.

# Problem 5

Answer: It is because the y in the question is different when we are calculating yuans we gain or lose. For example, If y = x, then the probability to lose is 0 but not $\frac{1}{2}$.

# Problem 6

Answer: (a) Any G with only one point has a clique of size 1 and has an independent set of size 1. Therefore R(1, s) = R(r, 1) = 1.

(b) We colored the graph's edges to red and if vertical $v$ and $w$ don't have an edge between them, we draw an edge and color it to blue.

We assume $R(r-1, s) = n_1$, $R(r, s-1) = n_2$.

Construct a graph with $R(r-1, s) + R(r, s-1) = n_1 + n_2$ verticles.

We random choose a vertical $v$, let $v_r$ denote the red edges and $v_b$ denote the blue edges. So $v_r + v_b \geq n_1 + n_2 - 1$. Then we know that $v_r \geq n_1$ and $v_b \geq n_2$ at least one must be right.

Without lose of generosity, we assume $v_r \geq n_1$, then for all verticles with a red edge connected to $v$, this new subgraph must has a red $K_{r-1}$ or blue $K_s$.

If it has a red $K_{r-1}$, since $v$ connects to all of $K_{r-1}$'s verticles, so we get a $K_r$.

If it has a blue $K_s$, then we are done.

To sum up, a graph with $R(r-1, s) + R(r, s-1)$ verticles must contain a clique of size r or an independent set of size s. So $R(r, s) \leq R(r-1, s) + R(r, s-1)$.

(c) We use induction.

For r = 1, $R(1,s) = 1 \le \binom{s-1}{0}$; s = 1, $R(r,1) = 1 \le \binom{r-1}{0}$.

If $r \le k$, $s \le t$ is true, then for r = k + 1:

$$\begin{aligned} R(k+1,t) &\le R(k,t) + R(k+1,t-1) \\ &\le \binom{k+t-2}{k-1} + \binom{k+t-2}{k} \\ &= \binom{k+t-1}{k} \end{aligned}$$

For s = t + 1:

$$\begin{aligned} R(k,t+1) &\le R(k-1,t+1) + R(k,t) \\ &\le \binom{k+t-2}{k-2} + \binom{k+t-2}{k-1} \\ &= \binom{k+t-1}{k-1} \end{aligned}$$

Therefore $R(r,s) \le \binom{r+s-2}{r-1}$ for all integers $r, s \ge 2$

Let r = s = k, we can get $R(k) \le \binom{2k-2}{k-1}$.

We use induction once again, for k = 2, $\binom{2}{1} < 4^2$.

If k = p is true, then for k = p + 1:

$$\binom{2p}{p} = \frac{(2p)!}{(p!)^2} = \frac{2p(2p-1)}{p^2}\binom{2p-2}{p-1} < 4\binom{2p-2}{p-1} < 4^{p+1}$$

Therefore $R(k) \le \binom{2k-2}{k-1} < 4^k$.

# Problem 7

Answer: Note that move empty cell left or right will not change the number of inversion.

When move empty cell up or down, empty cell jump though 3 cells, therefore the change of the number of inversion could be +3, +1, −1, −3.

We can also note that if empty cell remains in the same position, then the number of move must be an even number.

The first configuration has 0 inversion, so when we change the first configuration, the number of inversion must remain even. While the second configuration has 1 inversion, which is impossible.