Mathematics for Computer Science: Homework 4

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Exercise 11

Answer: (a) We only toss once. Therefore, $\Pr((X = 1) \cap (Y = 0)) = p$ and $\Pr((X = 0) \cap (Y = 1)) = 1 - p$. While $\Pr(X = 1) = \Pr(Y = 0) = p$ and $\Pr(Y = 1) = \Pr(X = 0) = 1 - p$.

If X and Y are independent, then $\Pr((X = 1) \cap (Y = 0)) = \Pr(X = 1)\Pr(Y = 0)$ and $\Pr((X = 0) \cap (Y = 1)) = \Pr(X = 0)\Pr(Y = 1)$ Therefore, $p = p(1 - p)$ and $1 - p = p(1 - p)$. We have $p = 0$ and $p = 1$, which is a contradiction.

Thus, X and Y are dependent.

(b) $N\sim$ Poisson(λ) therefore $\Pr(N = i) = \frac{\lambda^i}{i!}e^{-\lambda}$. λ is the expected number of toss in a given time interval. So λp is the expected number of heads in the same time interval. Similarly, $\lambda(1 - p)$ is the expected number of tails in the same time interval.

$\Pr(X = n_1) = \frac{(\lambda p)^{n_1}}{n_1!}e^{-\lambda p}$
$\Pr(Y = n_2) = \frac{(\lambda(1-p))^{n_2}}{n_2!}e^{-\lambda(1-p)}$

Next, we calculate $\Pr((X = n_1) \cap (Y = n_2))$.

$$\begin{aligned}
\Pr((X = n_1) \cap (Y = n_2)) &= \sum_{i=0}^{+\infty} \Pr((X = n_1) \cap (Y = n_2) \mid N = i)\Pr(N = i) \\
&= \Pr((X = n_1) \cap (Y = n_2) \mid N = n_1 + n_2)\Pr(N = n_1 + n_2) \\
&= \binom{n_1 + n_2}{n_1} p^{n_1}(1 - p)^{n_2} \frac{\lambda^{n_1+n_2}}{(n_1 + n_2)!}e^{-\lambda} \\
&= p^{n_1}(1 - p)^{n_2} \frac{\lambda^{n_1+n_2}}{n_1!n_2!}e^{-\lambda} \\
&= \Pr(X = n_1)\Pr(Y = n_2)
\end{aligned}$$

Therefore, X and Y are independent.

Exercise 14

Answer: (X, Y) uniformly distributed on the unit disk, $R = \sqrt{X^2 + Y^2}$, thus

$$F_R(a) = \Pr(R \le a) = \begin{cases} 0 \text{ if } a \le 0 \\ a^2 \text{ if } 0 < a \le 1. \\ 1 \text{ if } a > 1 \end{cases}$$

Take derivative of $\Pr(R \le a)$, we have

$$f_R(a) = \begin{cases} 0 \text{ if } a \le 0 \\ 2a \text{ if } 0 < a \le 1. \\ 0 \text{ if } a > 1 \end{cases}$$

Exercise 15

Answer: $0 \leq a < 1$, $F_Y(a) = \Pr(Y \leq a) = \Pr(F(X) \leq a) = \Pr(X \leq F^{-1}(a)) = F(F^{-1}(a)) = a$.

For $a < 0$, $F_Y(a) = 0$. For $a \geq 1$, $F_Y(a) = 1$.

We can know that $f_Y(a) = {F_Y}'(a) = \begin{cases} 1 \text{ if } 0 \leq a < 1 \\ 0 \text{ if } a < 0 \text{ or } a \geq 1 \end{cases}$.

Since $U \sim$ Uniform(0, 1), $F_X(a) = \Pr(X \leq a) = \Pr(F^{-1}(U) \leq a) = \Pr(U \leq F(a)) = F(a)$, thus $X \sim F$.

To generate Exponential distribution, we can use the probability integral transform.

Exponential distribution has CDF $F(a) = 1 - e^{-\frac{a}{\beta}}$, so $a = -\beta \ln(1 - F(a))$. We have $F^{-1}(a) = -\beta \ln(1 - a)$.

So for Uniform(0, 1) random variables X, we can use $-\beta \ln(1 - X)$ to generates random variables from an Exponential distribution.

Exercise 16

Answer: We only need to prove that $\Pr(X = x \mid X + Y = n) = \binom{n}{x} \pi^x (1 - \pi)^{n-x}$.

Since $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ), so $X + Y \sim$ Poisson($\lambda + \mu$).

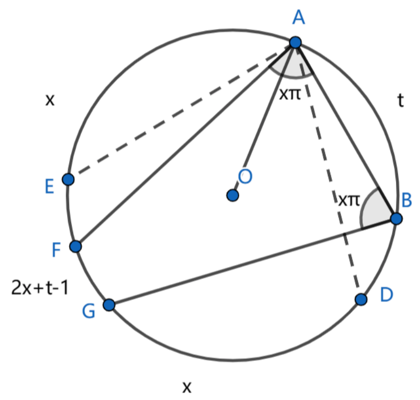
$$\begin{aligned}
\Pr(X = x \mid X + Y = n) &= \frac{\Pr(X = x \cap X + Y = n)}{\Pr(X + Y = n)} \\
&= \frac{\Pr(X = x \cap Y = n - x)}{\Pr(X + Y = n)} \\
&= \frac{\Pr(X = x) \Pr(Y = n - x)}{\Pr(X + Y = n)} \\
&= \frac{\frac{\lambda^x}{x!} e^{-\lambda} \frac{\mu^{n-x}}{(n-x)!} e^{-\mu}}{\frac{(\lambda+\mu)^n}{n!} e^{-\lambda-\mu}} \\
&= \frac{n!}{x!(n-x)!} \frac{\lambda^x \mu^{n-x}}{(\lambda + \mu)^n} \\
&= \binom{n}{x} \left(\frac{\lambda^x}{(\lambda + \mu)^x} \right) \left(\frac{\mu^{n-x}}{(\lambda + \mu)^{n-x}} \right) \\
&= \binom{n}{x} \pi^x (1 - \pi)^{n-x}.
\end{aligned}$$

Problem 1

Answer: Notice that when $x \geq 1$, $a(x) = 0$, when $x \leq \frac{1}{3}$, $a(x) = 1$. We only need to consider the case when $\frac{1}{3} < x < 1$.

First, we consider the case when $\frac{1}{3} < x < \frac{1}{2}$.

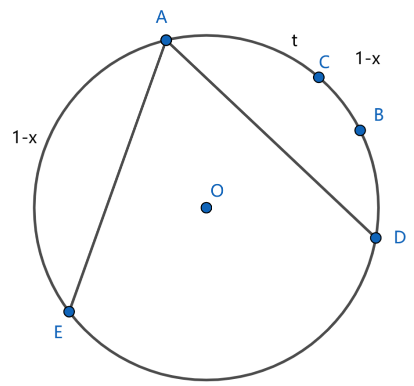
We fix point A and consider the position of B and C. We can consider the situation when all angles are less than $x\pi$.



If all angels ar less than $x\pi$, B must on arc AD or arc AE. Let's first assume it's on arc AD. We then draw AF and BG, thus point C must on arc FG.

$$a(x) = 1 - 2\int_{2x+t-1\geq 0}^{x} (2x+t-1)dt = 1-(3x-1)^2.$$

Then, we consider the case when $\frac{1}{2} < x < 1$.



WLOG, we just need to consider $\angle BCA$ is obtuse, and multiply the result by 3.

First, we fix point A, then B and C must be in the same segment of arc. The relative position of B and C is uncertain and only one can have $\angle BCA$ obtuse.

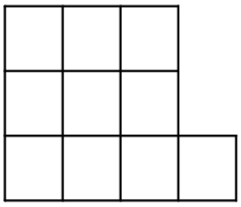
$$a(x) = 3 \cdot 2(1-x)(1-x) \cdot \frac{1}{2} = 3(1-x)^2.$$

Therefore,

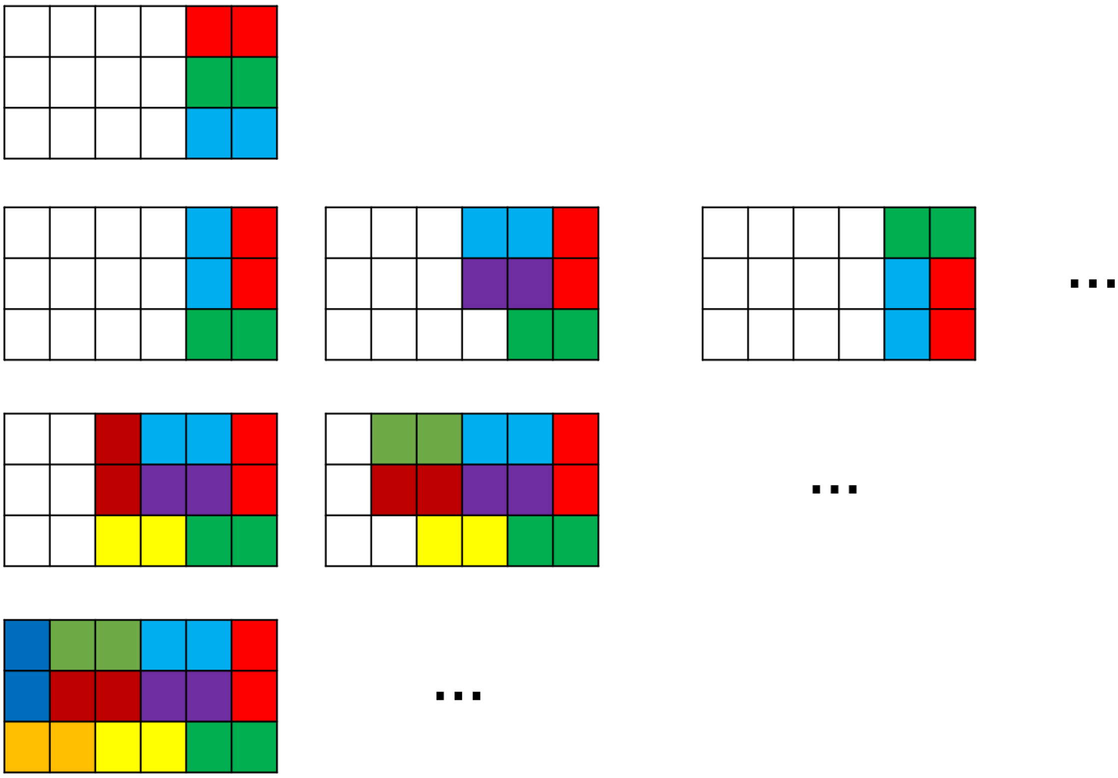
$$a(x) = \begin{cases} 1 \text{ if } x \leq \frac{1}{3} \\ 1-(3x-1)^2 \text{ if } \frac{1}{3} < x \leq \frac{1}{2} \\ 3(1-x)^2 \text{ if } \frac{1}{2} < x < 1 \\ 0 \text{ if } x \geq 1 \end{cases}$$

Problem 2

Answer: (a)First, we donote b_n be the number of ways to tiling a $3 \times n+1$ board like the image below.



Then, we manage to get the recurrence relation of a_n and b_n.



From the image, we can see that

$$\begin{aligned} a_n &= a_{n-2} + 2(a_{n-2} + b_{n-3}) \\ b_n &= a_{n-1} + b_{n-2}. \end{aligned}$$

Therefore, $a_n - a_{n-2} = 3a_{n-2} + 2b_{n-3} - 3a_{n-4} - 2b_{n-5} = 3a_{n-2} - 3a_{n-4} + 2a_{n-4} = 3a_{n-2} - a_{n-4}$. $a_n = 4a_{n-2} - a_{n-4}$.

$$\begin{aligned} A(x) &= \sum_{n=\text{even}} a_n x^n \\ &= \sum_{n=\text{even}} (4a_{n-2} - a_{n-4}) x^n \\ &= 4x^2 \sum_{n=\text{even}} a_n x^n - x^4 \sum_{n=\text{even}} a_n x^n + 1 - x^2 \\ &= (4x^2 - x^4) A(x) + 1 - x^2 \end{aligned}$$

Hence we get $A(x) = \frac{x^2-1}{4x^2-x^4-1}$

(b) For even n, $a_0 = 1$, $a_2 = 3$.

We let $t = x^2$, so $A(x) = \frac{t-1}{-t^2+4t-1}$.

The root of $-t^2 + 4t - 1 = 0$ is $t_1 = 2 + \sqrt{3}$ and $t_2 = 2 - \sqrt{3}$.

$A(x) = -\frac{t}{(t-t_1)(t-t_2)} + \frac{1}{(t-t_1)(t-t_2)} = \frac{1}{t_1-t_2}\left(-\frac{t_1}{t-t_1} + \frac{t_2}{t-t_2}\right) + \frac{1}{t_1-t_2}\left(\frac{1}{t-t_1} - \frac{1}{t-t_2}\right)$

$\frac{t_1}{t-t_1} = -\frac{1}{1-\frac{t}{t_1}} = -\left(1 + \frac{t}{t_1} + \left(\frac{t}{t_1}\right)^2 + \dots\right)$, $\frac{t_2}{t-t_2} = -\frac{1}{1-\frac{t}{t_2}} = -\left(1 + \frac{t}{t_2} + \left(\frac{t}{t_2}\right)^2 + \dots\right)$

$\frac{1}{t-t_1} = -\frac{1}{t_1}\frac{1}{1-\frac{t}{t_1}} = -\frac{1}{t_1}\left(1 + \frac{t}{t_1} + \left(\frac{t}{t_1}\right)^2 + \dots\right)$, $\frac{1}{t-t_2} = -\frac{1}{t_2}\frac{1}{1-\frac{t}{t_2}} = -\frac{1}{t_2}\left(1 + \frac{t}{t_2} + \left(\frac{t}{t_2}\right)^2 + \dots\right)$

Therefore, the coefficient of t^n is $\frac{1}{t_1-t_2}\left(\frac{1}{t_1^n} - \frac{1}{t_2^n}\right) + \frac{1}{t_1-t_2}\left(\frac{1}{t_2}\frac{1}{t_2^n} - \frac{1}{t_1}\frac{1}{t_1^n}\right)$.

We know that $t = x^2$, $t^n = x^{2n}$, thus

$$a_{2n} = \frac{1}{2\sqrt{3}}\left(\frac{1}{\left(2+\sqrt{3}\right)^n} - \frac{1}{\left(2-\sqrt{3}\right)^n}\right) + \frac{1}{2\sqrt{3}}\left(\frac{1}{\left(2-\sqrt{3}\right)^{n+1}} - \frac{1}{\left(2+\sqrt{3}\right)^{n+1}}\right)$$

To sum up,

$$a_n = \frac{3+\sqrt{3}}{6}\left(2+\sqrt{3}\right)^{\frac{n}{2}} + \frac{3-\sqrt{3}}{6}\left(2-\sqrt{3}\right)^{\frac{n}{2}}.$$

Problem 3

Answer: (1) $\tan(z) = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} = \frac{e^{2iz}-1}{i(e^{2iz}+1)} = \frac{1}{i}\left(1 - \frac{2}{e^{2iz}+1}\right)$. Thus the singularities are z_m that satisfy $1 + e^{2iz_m} = 0$. So when $z \to z_m$,

$$1 + e^{2iz} = 1 + \frac{e^{2i(z-z_m)}}{e^{2iz_m}} = 1 - e^{2i(z-z_m)} = 1 - \left(1 + \frac{2i(z-z_m)}{1!} + \frac{(2i(z-z_m))^2}{2!} + \dots\right).$$

We now know

$$\lim_{z\to z_m}(z - z_m)\tan z = \lim_{z\to z_m}\frac{z-z_m}{i}\left(1 + \frac{2}{\sum_{j\geq 1}(2i(z-z_k))^j}\right) = -1.$$

Thus the residues $r_m = -1$.

(2) Since $\tan(z) = \frac{1}{i}\left(1 - \frac{2}{e^{2iz}+1}\right)$. We only need to prove $1 + \frac{2}{|e^{2iz}+1|} \leq 10 \Leftrightarrow \left|e^{2iz}+1\right| \geq \frac{2}{9}$.

For $z = \pm m\pi + iy$, $\left|e^{2iz}+1\right| = \left|e^{-2y}+1\right| > 1$. So $|\tan(z)| < 1 + 2 < 10$,

For $z = x \pm im\pi$, $\left|e^{2iz}+1\right| \geq \left|1 - \left|e^{2iz}\right|\right| \geq \min\{1 - e^{-2m\pi}, e^{2m\pi} - 1\} > \frac{2}{9}$.

Therefore, $\tan(z) \leq 10$ at all z on Γ_m.

(3) When $m < n-1$, the graph can't have spanning tree, so the number is 0.

We need to prove that the determinant of any cofactors of L is 0.

Since $m < n-1$, the graph can't have connectivity. There has to be at least two separate parts. Therefore, we can swap columns of L to get $\begin{pmatrix} L_1 & O \\ O & L_2 \end{pmatrix}$. So for any L', L_1 and L_2 at least one is complete. WLOG, we assume is L_1 is complete and L_2 turns to L_2'. Since L_1 every rows and columns add up to 0, $\det L_1 = 0$. Therefore, $\det L' = \det L_1 \det L_2' = 0$.

(4) This is because L_G every rows add up to 0, so L_G has eigenvalue 0, which means L_G kernel space is at least 1 dimension. Hence rank $< n$.

Problem 4

Answer: (a) $A's$ singularities are z_k that satisfy $\lambda - e^z = 0$, so $z_k = \ln\lambda + 2k\pi i$. They are isolated singularities.

For any z_k,

$$\lim_{z\to z_k}(z-z_k)\frac{1}{\lambda-e^z} = \lim_{z\to z_k}(z-z_k)\frac{1}{\lambda(1-e^{z-z_k})} = \lim_{z\to z_k}(z-z_k)\frac{1}{\lambda\Big(-(z-z_k)-\frac{(z-z_k)^2}{2!}-\dots\Big)} = -\frac{1}{\lambda}.$$

Hence the residues for all singularities are $-\frac{1}{\lambda}$.

(b)

$$\begin{aligned} a_n &= -\sum\left(\text{residues of } \frac{A(z)}{z^{n+1}} \text{ for } z = z_k\right) \\ &= -\sum\left(\text{residues of } \frac{1}{(\lambda-e^z)z^{n+1}} \text{ for } z = z_k\right) \\ &= \frac{1}{\lambda}\sum\frac{1}{z_k^{n+1}} \\ &= \frac{1}{\lambda}\sum\frac{1}{(\ln\lambda+2k\pi i)^{n+1}} \\ &= \frac{1}{\lambda}\frac{1}{(\ln\lambda)^{n+1}}\sum\frac{1}{\left(1+\frac{2k\pi i}{\ln\lambda}\right)^{n+1}}. \end{aligned}$$

Let $g(n) = \frac{1}{\lambda}\frac{1}{(\ln\lambda)^{n+1}}$,

$$\begin{aligned} \lim_{n\to\infty}\frac{a_n}{g(n)} &= \lim_{n\to\infty}\sum\frac{1}{\left(1+\frac{2k\pi i}{\ln\lambda}\right)^{n+1}} \\ &= 1+\lim_{n\to\infty}\sum_{k\neq 0}\frac{1}{\left(1+\frac{2k\pi i}{\ln\lambda}\right)^{n+1}} \end{aligned}$$

When k is large enough, $\left|1+\frac{2k\pi i}{\ln\lambda}\right| = \sqrt{1+4k^2\frac{\pi^2}{\ln\lambda}} > \sqrt{k}$, so when $k > t$(t is a large number),

$$\left|\sum_{k>t}\frac{1}{\left(1+\frac{2k\pi i}{\ln\lambda}\right)^{n+1}}\right| < \left|\sum_{k>t}\frac{1}{k^{\frac{n+1}{2}}}\right| \to 0(\text{when } n\to\infty).$$

So $\lim_{n\to\infty}\frac{a_n}{g(n)} = 1+\lim_{n\to\infty}\left(\sum_{0<k\le t}\frac{1}{\left(1+\frac{2k\pi i}{\ln\lambda}\right)^{n+1}}+\sum_{k>t}\frac{1}{\left(1+\frac{2k\pi i}{\ln\lambda}\right)^{n+1}}\right) = 1+0+0 = 1.$

(c) Let $a_n = \frac{b_n}{(2n)!}$, then

$$\begin{aligned} b_n &= \sum_{k=0}^{n-1} b_k\frac{(2n)!}{(2k)!(2(n-k))!} \\ \Leftrightarrow \frac{b_n}{(2n)!} &= \sum_{k=0}^{n-1}\frac{b_k}{(2k)!}\frac{1}{(2(n-k))!} \\ \Leftrightarrow a_n &= \sum_{k=0}^{n-1} a_k\frac{1}{(2(n-k))!}. \end{aligned}$$

Let $f(x) = \sum_{i\geq 0} a_i x^{2i}$, then $f(x) = 1 + \sum_{i\geq 1}\sum_{k=1}^{i} \frac{a_{i-k}}{(2k)!}x^{2i} = 1 + \sum_{i\geq 1} \frac{x^{2i}}{(2i)!} f(x)$.

$$f(x) = \frac{1}{1 - \sum_{i\geq 1} \frac{x^{2i}}{(2i)!}} = \frac{2}{4 - e^x - e^{-x}} = \frac{2e^x}{-e^{2x} + 4e^x - 1} = -\frac{\frac{2\sqrt{3}+3}{3}}{e^x - \left(2+\sqrt{3}\right)} + \frac{\frac{2\sqrt{3}-3}{3}}{e^x - \left(2-\sqrt{3}\right)}.$$

Therefore from (b), we know that

$$\begin{aligned} h(n) &= (2n)!\left(\frac{\frac{2\sqrt{3}+3}{3}}{\left(2+\sqrt{3}\right)\left(\ln\left(2+\sqrt{3}\right)\right)^{2n+1}} - \frac{\frac{2\sqrt{3}-3}{3}}{\left(2-\sqrt{3}\right)\left(\ln\left(2-\sqrt{3}\right)\right)^{2n+1}}\right) \\ &= \frac{(2n)!}{\sqrt{3}}\left(\frac{1}{\left(\ln\left(2+\sqrt{3}\right)\right)^{2n+1}} - \frac{1}{\left(\ln\left(2-\sqrt{3}\right)\right)^{2n+1}}\right) \\ &= \frac{2(2n)!}{\sqrt{3}\left(\ln\left(2+\sqrt{3}\right)\right)^{2n+1}}. \end{aligned}$$

Problem 5

Answer: We can consider the *Schur decomposition* of A, $A = QUQ^{-1}$, where $U = \begin{pmatrix} \lambda_1 & * & ... & * \\ 0 & \lambda_2 & ... & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$ with A's eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ on it's diagonal.

Then, we let $A'_t = QU_tQ^{-1}$ where $U_t = \begin{pmatrix} \lambda_1+\frac{1}{t} & * & ... & * \\ 0 & \lambda_2+\frac{2}{t} & ... & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \lambda_n+\frac{n}{t} \end{pmatrix}$. So we have $\lim_{t\to\infty} A'_t = A$, now we only need to prove A'_t is invertible starting from t'. Then we let $A_1 = A'_t$ and we'll get a invertible sequence of matrices A_i with $\lim_{i\to\infty} A_i = A$.

Let g be the smallest non-zero gap between eigenvalues of A and between non-zero eigenvalues of A and zero.

Now we can prove a stronger result. We claim that for all $t > \frac{2n}{g}$, A'_t is invertible and has distinct eigenvalues.

We only need to prove that eigenvalues of A_t are distinct and non-zero.

For any $i \neq j$, if $\lambda_i = \lambda_j$, then $\lambda_i + \frac{i}{t} \neq \lambda_j + \frac{j}{t}$. If $\lambda_i \neq \lambda_j$, then

$$\left|\left(\lambda_i + \frac{i}{t}\right) - \left(\lambda_j + \frac{j}{t}\right)\right| \geq |\lambda_i - \lambda_j| - \left|\frac{i}{t}\right| - \left|\frac{j}{t}\right| > g - 2\frac{n}{t} > 0.$$

So we have $\lambda_i + \frac{i}{t} \neq \lambda_j + \frac{j}{t}$ as well.

For any i, if $\lambda_i = 0$, then $\lambda_i + \frac{i}{t} > 0$. If $\lambda_i \neq 0$ we also have

$$\left|\lambda_i + \frac{i}{t} - 0\right| = \left|\lambda_i + \frac{i}{t}\right| > |\lambda_i| - \left|\frac{i}{t}\right| > g - \frac{n}{t} > 0.$$

So we have $\lambda_i + \frac{i}{t} \neq 0$ as well.

Therefore, A'_t is invertible and has distinct eigenvalues when $t > \frac{2n}{g}$.

Problem 6

Answer: (a)Notice that

$$\begin{pmatrix} I_n & B \\ A & I_m \end{pmatrix} = \begin{pmatrix} I_n & B \\ O & I_m \end{pmatrix} \begin{pmatrix} I_n - BA & O \\ O & I_m \end{pmatrix} \begin{pmatrix} I_n & O \\ A & I_m \end{pmatrix}$$

$$\begin{pmatrix} I_n & B \\ A & I_m \end{pmatrix} = \begin{pmatrix} I_n & O \\ A & I_m \end{pmatrix} \begin{pmatrix} I_n & O \\ O & I_m - AB \end{pmatrix} \begin{pmatrix} I_n & B \\ O & I_m \end{pmatrix}$$

Therefore, $\det\begin{pmatrix} I_n & B \\ A & I_m \end{pmatrix} = \det(I_n - BA) = \det(I_m - AB)$.

Replace A by $-A$ and we have *Sylvester's determinant identity.*

(b)

$x^m \det(xI_n + BA) = x^n \det(xI_m + AB)$. WLOG, we assume $m \leq n$.

We compare the coefficient of x^n on both sides, we have

$$\sum_{|S|=m} \det\Big((BA)_{S,S}\Big) = \det(0I_m + AB) = \det(AB).$$

LHS is

$$\sum_{|S|=m} \det\Big((BA)_{S,S}\Big) = \sum_{|S|=m} \det\Big(B_{S,\{1,2,\dots,m\}}\Big) \det\Big(A_{\{1,2,\dots,m\},S}\Big) = \sum_{|S|=m} \det(A_S)\det(B_S).$$

Therefore, we have $\det(AB) = \sum_{|S|=m} \det(A_S)\det(B_S)$, which is *Cauchy-Binet Formula.*

Problem 7

Answer: WLOG, we assume A' is A without the first row.

We only consider the determinant of $A_S^{(1)}$.

If $\{e_k | k \in S\}$ doesn't form a spanning tree, then it must contain a cycle path.

Consider the edges $f_1, f_2, \dots, f_k$ which form a cycle, we prove that their corresponding columns are dependent.

We assign a indicator $t_1, t_2, \dots, t_k$ to $f_1, f_2, \dots, f_k$. If f_i is in the same direction as counterclockwise, then $f_i = 1$, else $f_i = -1$.

Thus we know that $\sum_{i=1}^k t_i f_i = 0$ since all points on the cycle has an edge in and an edge out.

Hence $\det A_S^{(1)} = 0$.

If $\{e_k | k \in S\}$ forms a spanning tree, then we use the property that every trees have at least two leaves.

Even if we had removed the first row, there's must exist a leaf.

Let's assume the leaf is v_r, with edge f_u connecting to it. Then we know that row $r-1$(since we remove the first row) only have one non-zero element(1 or -1) on their intersection.

Therefore, we cancel row $r-1$ and column u to get $A_S^{(1)((r-1)(u))}$ and it's determinant is $\pm \det A_S^{(1)}$.

We can cancel leaf and its related edge multiple times like above, and we can get $\det A_S^{(1)} = \pm 1$, $\left|\det A_S^{(1)}\right| = 1$.

Thus we prove the lemma.

Problem 8

Answer: The Laplacian matrix of K_n is

$$L = \begin{pmatrix} n-1 & -1 & ... & -1 \\ -1 & n-1 & ... & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & ... & n-1 \end{pmatrix}.$$

We only need to calculate determinant of $L^i = \begin{pmatrix} n-1 & -1 & ... & -1 \\ -1 & n-1 & ... & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & ... & n-1 \end{pmatrix}$, which is a $n-1$ by $n-1$ matrix.

Notice that $L_i - nI = \begin{pmatrix} -1 & -1 & ... & -1 \\ -1 & -1 & ... & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & ... & -1 \end{pmatrix}$, has rank $n-2$, thus it has at least $n-2$ eigenvalues 0.

$\text{Trace}(L_i - nI) = 1 - n$, so the remaining eigenvalue is $1 - n$.

Therefore, L_i has $n-1$ eigenvalues, $n-2$ of them are $0 + n = n$ and 1 of them is $1 - n + n = 1$.

Thus $\det L_i = n^{n-2}$.

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