Numerical Analysis
Introduction

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Nested Multiplication: Horner's method / synthetic division

- One equation has different calculation methods
  - $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$
    - $a_0 + a_1x + a_2x^2 + a_3x^3$
  - $f(x) = a_0 + x(a_1 + x(a_2 + x(\ldots x(a_n))))$
    - $a_0 + x(a_1 + x(a_2 + x(a_3)))$
- Why we do this?
  - Less computation
    - "+": n, "×: "$1 + 2 + 3 + \ldots + n = n(n+1)/2$
    - "+": n, "×": n
  - Higher computation accuracy
    - Computation of values with different significant digits will cause more error.

Errors: Absolute and Relative

- α, β: two numbers
  - Approximated to each other
- Absolute error of β as an approximation to α :$|\alpha - \beta|$
- Relative error of β as an approximation to α :$|\alpha - \beta|/|\alpha|$
- $\alpha_1 = 1.333$, $\beta_1 = 1.334$
- $\alpha_2 = 0.001$, $\beta_2 = 0.002$
  - Which β_i approximates α_i better?
    - A.E: 0.001 R.E: $\frac{0.001}{1.333}$
    - A.E: 0.001 R.E: 1

Rounding(四捨五入) and Chopping(無條件捨去)

- Rounding to n-th digits:
  - If the digits beyond the n-th digit are greater than 50000...
  - $123.4567 \Rightarrow 123.46$
    - Round up the n-th digit
  - If the digits beyond the n-th digit are less than 50000...
  - $765.4321 \Rightarrow 765.43$
    - Round down the n-th digit
  - If the digits beyond the n-th digit equals to 50000...
    - Round the n-th digit to the nearest even number
    - $124.9750 \Rightarrow 124.98$
    - $124.9650 \Rightarrow 124.96$
- Chopping to n-th digits:
  - Remains the first n digits.
  - Removes the digits beyond the n-th digit.
    - $123.4567 \Rightarrow 123.45$
    - $765.4321 \Rightarrow 765.43$
    - $124.9750 \Rightarrow 124.97$
    - $124.9650 \Rightarrow 124.96$

Significant Digits of Precision

- Consider the following equations:
  - $0.1036x + 0.2122y = 0.7381$
  - $0.2081x + 0.4247y = 0.9327$
- Consider only three significant digits
  - We will have $y = 547$
- If we keep four significant digits
  - We will have $y = 343.9$
- Different significant digits will cause different results
- Usually we use the *double-precision* floating points.
  - 64-bit floating point

Programming Example

- $f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- Find $f^{'}(x)$ for $h = 4^{-1}, 4^{-2}, \ldots 4^{-10}$
- Pseudo code:
  - integer parameters n$\leftarrow$10
  - integer i
  - real error, h, x, y
  - $x \leftarrow 0$
  - $h \leftarrow 1$
  - for $i = 1$ to n do
    - h$\leftarrow$0.25h
    - $y \leftarrow$[sin(x+h)-sin(x)]/h
    - error$\leftarrow |cos(x) - y|$
    - output i, h, y, error
  - endfor

Taylor Series

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \ |x| < \infty$
- $cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \ |x| < \infty$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k, \ |x| < 1$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}, \ (-1 < x < 1)$

Taylor Series Example

- Calculate $\ln(1.1)$
  - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$
  - $\ln(1.1) \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} = 0.095310333$
- Calculate e^8
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
  - $e^8 \approx 1 + 8 + \frac{8^2}{2!} + \frac{8^3}{3!} + \frac{8^4}{4!} + \frac{8^5}{5!} = 570.06666$
  - Real $e^8 = 2980.987987$

Taylor Series of sin(x)

- $sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad |x| < \infty$
- $S_1 = x$
- $S_2 = x - \frac{x^3}{6}$
- $S_3 = x - \frac{x^3}{6} + \frac{x^5}{120}$
- Reference to s1.m, s2.m s3.m and sinplot.m
  - https://web.ma.utexas.edu/CNA/NMC7/nmc7-matlab.html

Taylor's Theorem for $f(x)$

- A function f is continuous and derivative of order 0, 1, 2, ..., $(n+1)$ in close interval $I = [c, d]$, then for any c and x in I
- $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$
- The error $E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{(n+1)}$
  - where ξ is some value between c and x
- $e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^\xi}{(n+1)!}x^{n+1}$, for $-s \leq x \leq s$
- $\lim_{n\to\infty} \left| \frac{e^\xi}{(n+1)!}x^{(n+1)} \right| \leq \lim_{n\to\infty} \frac{e^s}{(n+1)!}s^{(n+1)} = 0$

Mean Value Theorem

- $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$
- $E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{(n+1)}$
- Special case of Taylor's Theorem with $n = 0$
- $f(x) = \sum_{k=0}^{0} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_1$
- $f(x) = \frac{f^{(0)}(c)}{0!}(x-c)^0 + E_1 =$
  $f(c) + E_1 = f(c) + \frac{f^{(1)}(\xi)}{(1)!}(x-c)^{(1)} = f(c) + f'(\xi)(x-c)$
- $f(x) - f(c) = f'(\xi)(x-c)$
- $\frac{f(x)-f(c)}{x-c} = f'(\xi)$, for some $\xi \in [c, x]$

Taylor's Theorem for $f(x+h)$

- A function f is continuous and derivative of order 0, 1, 2, ..., $(n+1)$ in close interval $I=[c,d]$, then for any c and x in I
- $f(x+h)=\sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}(h)^k+E_{n+1}$
- The error $E_{n+1}=\frac{f^{(n+1)}(\xi)}{(n+1)!}(h)^{(n+1)}=O(h^{n+1})$
  - where ξ is some value between c and x

Example of Taylor Series

- $f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(h)^k + E_{n+1}$,
  $E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(h)^{(n+1)} = O(h^{n+1})$
- Consider $\sqrt{(1+h)}$ in power's of h
  - Let $f(x) = x^{1/2}$
  - $f^{'}(x) = \frac{1}{2}x^{-1/2}$, $f^{''}(x) = -\frac{1}{4}x^{-3/2}$, $f^{'''}(x) = \frac{3}{8}x^{-5/2}$,
  - $\sqrt{(1+h)} = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3\xi^{-5/2}$ where $1 < \xi < 1+h$
- Computer $\sqrt{1.00001} = \sqrt{1+0.00001} \approx$
  $1 + 0.5 \times 10^{-5} - 0.125 \times 10^{-10} = 1.000004999987500$
  - Error: $\frac{1}{16}h^3\xi^{-5/2} < \frac{1}{16}10^{-15}$
  - 有效位數有15位

Alternating Series Theorem

Theorem

If $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots \geq 0$ *for all* n *and* $\lim_{n\to\infty} a_n = 0$, *then the alternating series* $a_1 - a_2 + a_3 - a_4 + \ldots$ *converges. That is:* $\sum_{k=1}^{\infty}(-1)^{k-1}a_k = \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}a_k = \lim_{n\to\infty} S_n = S$, *where* S_n *is the sum of the first n-th* a_i *and* $|S - S_n| \leq a_{n+1}$ *for all* n.

- Example: $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \ldots$
  - Choose the value of n such that $\sin(1)$ ˜ $S_n = 1 - \frac{1}{3!} + \frac{1}{5!} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)!}$ with error less than $\frac{1}{2} \times 10^{-6}$
  - Error bound: $|S - S_n| \leq a_{n+1} = \frac{1}{(2n+1)!} \leq \frac{1}{2} \times 10^{-6}$
  - $\log_{10}(2n+1)! \geq \log_{10} 2 + 6 \approx 6.3$
    - $\log_{10} 9! \approx 5.6 \to n > 4$
    - $\log_{10} 10! \approx 6.6 \to n \geq 5$

Programming Exercise

- Implement ibin(m, n) to calculate $\binom{n}{m}$ accoroding to $\binom{n}{m} = \frac{n!}{m!(n-m)!}$
- Implement jbin(m, n) to calculate $\binom{n}{m}$ accoroding to
$$\binom{n}{m} = \prod_{k=1}^{\min(m,n-m)} \frac{n-k+1}{k}$$

Extra Exercise – Calculation of π

- Implementation the following algorithm to calculate π with parameter n
  - integer k
  - real a, b, c, d, e, f, g
  - a $\leftarrow$ 0
  - b $\leftarrow$ 1
  - c $\leftarrow 1/\sqrt{2}$
  - d $\leftarrow$ 0.25
  - e $\leftarrow$ 1
  - for $k = 1$ to n do
    - $a \leftarrow b$
    - $b \leftarrow (b + c)/2$
    - $c \leftarrow \sqrt{ca}$
    - $d \leftarrow d - e(b - a)^2$
    - $e \leftarrow 2e$
    - $f \leftarrow b^2/d$
    - $g \leftarrow (b + c)^2/(4d)$
    - output $k, f, |f - \pi|, g, |g - \pi|$
  - endfor