2.1 5, 9

2.2 3, 5, 6

2.3 4 (a, b, c, d), 6 (a, b, c,,d)

2.4 8 (a, b, c), 9

2.5 3, 4, 9

Solutions to Exercises 2.1

5. a.▷ Prove formula (2.1) for the number of bits in the binary representation of a positive integer.

   b.▷ Prove the alternative formula for the number of bits in the binary representation of a positive integer n:

$$b = \lceil \log_2(n+1) \rceil.$$

Sol:

a. The smallest positive integer that has b binary digits in its binary expansion is $1\underbrace{0...0}_{b-1}$, which is 2^{b-1}; the largest positive integer that has b binary digits in its binary expansion is $1\underbrace{1...1}_{b-1}$, which is $2^{b-1}+2^{b-2}+...+1 = 2^b - 1$. Thus,

$$2^{b-1} \leq n < 2^b.$$

Hence

$$\log_2 2^{b-1} \leq \log_2 n < \log_2 2^b$$

or

$$b - 1 \leq \log_2 n < b.$$

These inequalities imply that $b-1$ is the largest integer not exceeding $\log_2 n$. In other words, using the definition of the floor function, we conclude that

$$b - 1 = \lfloor \log_2 n \rfloor \text{ or } b = \lfloor \log_2 n \rfloor + 1.$$

b. If $n > 0$ has b bits in its binary representation, then, as shown in part a,

$$2^{b-1} \le n < 2^b.$$

Hence

$$2^{b-1} < n+1 \le 2^b$$

and therefore

$$\log_2 2^{b-1} < \log_2(n+1) \le \log_2 2^b$$

or

$$b-1 < \log_2(n+1) \le b.$$

These inequalities imply that b is the smallest integer not smaller than $\log_2(n+1)$. In other words, using the definition of the ceiling function, we conclude that

$$b = \lceil \log_2(n+1) \rceil.$$

Solutions to Exercises 2.2

3. a. $(n^3+1)^6 \approx (n^3)^6 = n^{18} \in \Theta(n^{18})$.

   b. $\sqrt{10n^4 + 7n^2 + 3n} \approx \sqrt{10n^4} = \sqrt{10n^2} \in \Theta(n)$

   c. $2n \lg(2n+2)^3 + (n^2+2)^2 \lg n \in \Theta(n^4 \lg n)$.

   d. $3^{n+1} + 3^{n-1} \in \Theta(3^n) + \Theta(3^n) = \Theta(3^n)$

   e. $2\log_2 n \approx \log_2 n \in \Theta(\log n)$.

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   b. $\sqrt{10n^4 + 7n^2 + 3n} \approx \sqrt{10n^4} = \sqrt{10}n^2 \in \Theta(n^2)$

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   e. $2\log_2 n \approx \log_2 n \in \Theta(\log n)$.

5. $2\lg(n+50)^5,\quad \ln^3 n,\quad \sqrt{n},\quad 0.05n^{10} + 3^{n3} + 1,\quad 3^{2n},\quad 3^{3n},\quad (n^2+3)!$

(*請將題目和解答中的$0.05n^{10} + 3^{n3} + 1$改成$0.05n^{10} + 3n^3 + 1$)

6. a. $\lim_{n\to\infty} \frac{p(n)}{n^k} = \lim_{n\to\infty} \frac{a_k n^k + a_{k-1}n^{k-1} + \ldots + a_0}{n^k} = \lim_{n\to\infty} (a_k + \frac{a_{k-1}}{n} + \ldots + \frac{a_0}{n^k})$

   $= a_k > 0$.

   Hence $p(n) \in \Theta(n^k)$.

Solutions to Exercises 2.3

4. a. Compute $s(n)_{i=1}^{n} = \sum i/i!$

b. Multiplication (for factorial) and division.

c. $\frac{n(n+1)}{2}$ times

說明: $i/i!$ 執行 $i-1$ 次乘法運算和 1 次除法運算共 i 次基本運算(basic operation)

⇨ $C(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

d: $C(n) = \frac{n(n+1)}{2} \in \Theta(n^2)$

6. a. The algorithm returns "true" if its input matrix is symmetric and "false" if it is not.

b. Comparison of two matrix elements.

c. $C_{worst}(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1)$

$= \sum_{i=0}^{n-2} (n-1-i) = (n-1) + (n-2) + ... + 1 = \frac{(n-1)n}{2}.$

d. Quadratic: $C_{worst}(n) \in \Theta(n^2)$ (or $C(n) \in O(n^2)$).

e. The algorithm is optimal because any algorithm that solves this problem must, in the worst case, compare $(n-1)n/2$ elements in the upper-triangular part of the matrix with their symmetric counterparts in the lower-triangular part, which is all this algorithm does.