Technical University of Crete
School of Electrical and Computer Engineering
Course: **Convex Optimization**
Exercise 3
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100/1000 points

In this exercise, we shall solve simple (but important) optimization problems by explicitly solving the KKT conditions. In all cases, start with a drawing of the problem. We shall also use the projected gradient method.

1. (10) Compute the projection of $\mathbf{x}_0 \in \mathbb{R}^n$ onto the set $\mathbf{B}(\mathbf{0}, r) := \{\mathbf{x} \in \mathbb{R}^n \,|\, \|\mathbf{x}\|_2 \leq r\}$.
   (a) Draw a scheme of the problem.
   (b) Write down the optimization problem you must solve, in terms of **differentiable** functions.
   (c) Write down the KKT conditions, in terms of the optimal parameters $\mathbf{x}_*$ and λ_*.
   (d) Consider the case $\lambda_* > 0$. What is the conclusion?
   (e) Consider the case $\lambda_* = 0$. What is the conclusion?
2. (10) Repeat the steps of the previous question and compute the projection of $\mathbf{x}_0 \in \mathbb{R}^n$ onto the set $\mathbf{B}(\mathbf{y}, r) := \{\mathbf{x} \in \mathbb{R}^n \,|\, \|\mathbf{x} - \mathbf{y}\|_2 \leq r\}$ (for given $\mathbf{y} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{++}$).
3. (10) Let $\mathbf{a} \in \mathbb{R}^n$. Compute the projection of $\mathbf{x}_0 \in \mathbb{R}^n$ onto set $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{a} \leq \mathbf{x}\}$.
4. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$ and consider the problem

$$\text{(P)} \qquad \min_{\mathbf{x}} f_0(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}\|_2^2, \text{ subject to } \mathbf{x} \in \mathbb{H} := \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{a}^T\mathbf{x} = b\}. \qquad (1)$$

   (a) (10) Write and solve the KKT for problem (P).
   (b) (10) Compute the solution of problem (P) using the projected gradient descent method

$$\mathbf{x}_{k+1} = \mathbf{P}_{\mathbb{H}}\left(\mathbf{x}_k - \frac{1}{L}\nabla f_0(\mathbf{x}_k)\right), \qquad (2)$$

   where $L := \max(\text{eig}(\nabla^2 f_0(\mathbf{x})))$. What do you observe?

5. Let $\mathbf{A} \in \mathbb{R}^{p \times n}$, with $\operatorname{rank}(\mathbf{A}) = p$, and $\mathbf{b} \in \mathbb{R}^p$.

   (a) (10) Find the distance of a point $\mathbf{x}_0 \in \mathbb{R}^n$ from the set $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ (you must compute the projection of $\mathbf{x}_0$ onto $\mathbb{S}$).

   (b) Let the $(n \times n)$ positive definite matrix $\mathbf{P} = \mathbf{P}^T \succ \mathbf{0}$, $q \in \mathbb{R}^n$ and

$$f_0(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{q}^T\mathbf{x}.$$

   Consider the problem

$$(\mathrm{Q}) \qquad \min_{\mathbf{x} \in \mathbb{S}} f_0(\mathbf{x}). \tag{3}$$

   i. Solve problem (Q) using `cvx`.

   ii. (10) Write the KKT for problem (Q) and compute the optimal solution by solving them using `matlab` (no `cvx`).

   iii. (30) Compute the optimal solution via the projected gradient method

$$\mathbf{x}_{k+1} = \mathbf{P}_{\mathbb{S}}\left(\mathbf{x}_k - \frac{1}{L}\nabla f_0(\mathbf{x}_k)\right), \tag{4}$$

   where $L := \max(\operatorname{eig}(\nabla^2 f_0(\mathbf{x})))$.