MAS 6002 Assignment 1

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1. (a) Set A = the Patient has disease

set B = the test is positive

therefore $P(A|B)$ = the probability that the patient has the disease if the test is positive

according to the Bayes' theorem.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|\bar{A})P(\bar{A}) + P(B|A)P(A)}$$

$P(B|A) = 1$ (known) $\quad P(A) = 0.1$ (known) $\quad P(B|\bar{A}) = 0.01$ (known)

Thus $$P(A|B) = \frac{1 \times 0.1}{0.01 \times (1-0.1) + 1 \times 0.1} = \frac{0.1}{0.109} \approx 0.917$$

the probability that the patient has the disease if the test is positive is 0.917.

(b) $P(A|\bar{B})$ is the probability that the patient has disease if test is not positive.

according to the Bayes' theorem,

$$P(A|\bar{B}) = \frac{P(\bar{B}|A)P(A)}{P(\bar{B}|\bar{A})P(\bar{A}) + P(\bar{B}|A)P(A)}$$

$P(\bar{B}|A) = 1 - P(B|A) = 0$

therefore $P(A|\bar{B}) = 0$

the probability that the patient has disease if test is not positive is 0

2. (a) $P(X>\frac{1}{2}) = \int_{\frac{1}{2}}^{+\infty} f_X(x)\,dx = \int_{\frac{1}{2}}^{1} \frac{3}{4}(1-x^2)dx + \int_{1}^{+\infty} 0\,dx$

$$= \frac{3}{4}\left[x-\frac{1}{3}x^3\right]_{\frac{1}{2}}^{1} = \frac{5}{32}$$

(b) $F_X(x) = \int_{-1}^{x} f_X(x) = \frac{3}{4}\left[x-\frac{1}{3}x^3\right]_{-1}^{x} = \frac{3}{4}x - \frac{1}{4}x^3 + \frac{1}{2}$

(c) $E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} x f_X(x)dx + \int_{1}^{+\infty} 0\,dx$

$$= \int_{-1}^{1} x f_X(x)\,dx = \int_{-1}^{1} \frac{3}{4}(x-x^3) = \left[\frac{3}{8}x^2 - \frac{3}{16}x^4\right]_{-1}^{1} = 0$$

$Var(X) = \int_{-\infty}^{+\infty} (x)^2 f_X(x)\,dx = \int_{-1}^{1} x^2 f_X(x)dx = \int_{-1}^{1} (\frac{3}{4}x^2 - \frac{3}{4}x^4)\,dx$

$$= \left[\frac{1}{4}x^3 - \frac{3}{20}x^5\right]_{-1}^{1} = \frac{1}{5}$$

3.

(a) the maximum is nearly 80. the minimum is about 22. And the medium value is 56. the lower quartile value is 47. the upper quartile is about 65.
The outlier is in 10 ~ 20.

(b) We can see from the box plots that the A's medium is higher than B's. The A's minimum is also higher than B. But the B's maximum is higher than A's. The A's grade is more stable and concentrated. A's grade might be better than B.

(c)

(d) We can't find the difference in ~~scores~~ grades in every range such as 40~50, 50~60, etc.
I think if we use bar chart, we can know the distribution of grades in different ranges.

4. Set H_0:

H_1

5、(a) $\because$ $X = \begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector with a bivariate normal distribution

so $X \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$\because X \sim N(\mu_1, \sigma_1^2) \qquad Y \sim N(\mu_2, \sigma_2^2)$

$\therefore P_X(x) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp\left\{ -\frac{(x-\mu_1)^2}{2\sigma_1^2} \right\}$

according to $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\therefore \sigma_1 = \sigma_2 = \sqrt{3} \qquad \rho = \frac{1}{3} \qquad \mu_1 = 0 \qquad \mu_2 = 2$

$\therefore P_X(x) = \frac{1}{\sqrt{3}\sqrt{2\pi}} \exp\left(-\frac{(x-0)^2}{2(\sqrt{3})^2} \right) = \frac{1}{\sqrt{6\pi}} \exp\left\{ -\frac{x^2}{6} \right)$

(b) $\because \begin{cases} u = X - Y \\ V = X + 2Y + 3 \end{cases}$

$\therefore U = \begin{pmatrix} u \\ V \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$\because \vec{Y} = \vec{a} + \vec{B}\vec{X} \sim N(\vec{a} + \vec{B}\vec{\mu}, \vec{B}\vec{\Sigma}\vec{B}^T)$

$\vec{\mu} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 0 & +4 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$

$\vec{\Sigma} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{vmatrix} 4 & -2 \\ -2 & 19 \end{vmatrix}$

6. (a). Show: for $0<x<1$ and $0<y<1$, $f_{X,Y}(x,y) = k(x^2+y)$,

other wise, $f_{X,Y}(x,y) = 0$

therefore: $\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \int_0^1 dx \int_0^1 f_{X,Y}(x,y)\, dy$

$\because \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = 1$

$\therefore \int_0^1 dx \int_0^1 f_{X,Y}(x,y)\, dy = 1$

therefore $\int_0^1 dx \int_0^1 k(x^2+y)\, dy = 1$

$\Rightarrow \int_0^1 [kx^2 + \frac{k}{2})\ \ dx = 1$

$\Rightarrow \frac{k}{3} + \frac{1}{2}k = 1 \quad \Rightarrow k = \frac{6}{5}$

(b) $P[X \leq Y] = \int_0^1 dy \int_0^y f_{X,Y}(x,y)\, dx = \int_0^1 k(\frac{y^3}{3} + y^2)\, dy$

$= \frac{6}{5}(\frac{y^4}{12} + \frac{y^3}{3})\Big|_0^1 = \frac{1}{2}$

(c) $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\, dx = \int_0^1 k(x^2+y)\, dx$, $0<x<1$ and $0<y<1$

$= (\frac{kx^3}{3} + kyx)\Big|_0^1 = \frac{2}{5} + \frac{6}{5}y$, $0<x<1$ and $0<y<1$

$\therefore f_Y(y) = \begin{cases} \frac{2}{5} + \frac{6}{5}y, & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

7. (a) $B(\alpha,1) = \int_0^1 x^{\alpha-1}(1-x)^{1-1}\, dx = \int_0^1 x^{\alpha-1}\, dx = \frac{1}{\alpha}x^\alpha\Big|_0^1 = \frac{1}{\alpha}$

(b) $X \sim Be(8,1)$ therefore $p(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$a=8, b=1 \Rightarrow p(x) = \frac{\Gamma(8+1)}{\Gamma(8)\Gamma(1)} x^{8-1}(1-x)^{1-1}$

$\because \Gamma(8+1) = 8\Gamma(8) \qquad \Gamma(8) = (8-1)! = \qquad \Gamma(1) = 1$

$\therefore p(x) = \frac{8\times(7!)}{(7!)\times 1} x^7(1-x)^0 = 8 \cdot x^7, \quad 0 \leq x \leq 1$

$\therefore F_X(x) = \int_0^1 P(x)dx = \int_0^1 8x^7 dx = x^8$

$\because Y = \sqrt[3]{X}$ $\quad \therefore F_Y(y) = P(Y \le y) = P(\sqrt[3]{X} \le y)$

$\therefore F_Y(y) = P(X \le y^3) = F_X(y^3) = (y^3)^8$

therefore $\quad F_Y(y) = y^{24}$

$$f_Y(y) = \begin{cases} 24y^{23} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

8. (a) $Cov(U,V) = E[(U-E[U])(V-E[V])]$

$= E[UV] - 2E[V]E[U] + E[U]E[V]$

$\therefore = E[UV] - E[U]E[V]$

$\because U = X+Y \qquad V = X-Y$

X and Y are a pair of independent and identically variables

$\therefore Cov(U,V) = E[(X+Y)(X-Y)] - E[X+Y]E[X-Y]$

$= E(X+Y)E(X-Y) - E(X+Y)E(X-Y)$

$= 0$

(b) $\because U = X+Y \quad , \quad V = X-Y$

$\therefore X = \frac{1}{2}u + \frac{1}{2}v \qquad y = \frac{1}{2}u - \frac{1}{2}v$

jacobian $= |(dx/du)(dy/dv) - (dx/dv)(dy/du)| = |\frac{1}{2} \times (\frac{-1}{2}) - (\frac{1}{2})(\frac{1}{2})|$

$= |-\frac{1}{2}| = \frac{1}{2}$

$f_{U,V}(u,v) = f_{X,Y}(x,y) \times \text{jacobian} = f_{X,Y}(\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u - \frac{1}{2}v) \times \frac{1}{2}$

$\because$ $X \sim exp(\lambda)$ $\quad Y \sim exp(\lambda)$

$\therefore$ $f(x) = \lambda e^{(-\lambda x)}$ $\quad f(y) = \lambda e^{(-\lambda y)}$

$\cancel{f_{X,Y}(x,y) = \int_{\infty}^{x}}$

$f_{X,Y}(x,y) =$

(c) Yes, U and V independent.

$\because$ according to (a) $\quad Cov(U,V) = 0$

and with X, Y as in (b)

So U and V independent.

9. (a) $L(\lambda; X) = \prod_{i=1}^{n} \lambda exp\{-\lambda X_i\} = \lambda^n exp(-\lambda \sum_{i=1}^{n} X_i)$

$= \lambda^n exp(-\lambda n \bar{X}) = \lambda^n e^{-\lambda n \bar{x}}$

$l(\lambda; X) = \ln L(\lambda; X) = n \ln \lambda - \lambda n \bar{x}$

(b) $\because$ $\frac{d \ln L(\lambda, x)}{d\lambda} = \frac{n}{\lambda} - n\bar{x} = 0$

$\Rightarrow \frac{n}{\lambda} - n\bar{x} = 0 \quad \Rightarrow \hat{\lambda} = \frac{1}{\bar{x}}$

(c)