M11482 PROBLEM SET 2

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*Problems marked with * are more difficult; we think...*

problem numba Problem statement

1: Open balls are open Given a metric space (X, d), we defined the open ball, $N_r(p)$ about a point $p \in X$ of radius r to be

$$N_r(p) = \{q \in X |\, d(p, q) < r\}$$

We defined a set $S \subset X$ to be open if for every element in S an open ball around that point is contained in S. Show that open balls are open and conclude after that we can equivalently define a set S to be open if it is the union of a collection of open balls.

2: Open/Closed relationship We defined a closed set to be a subset which contains all of it's limit points. Show that a set $S \subset X$ is closed if and only if the complement of it (in X) is open (and vice versa).

3: Topology? Show that an arbitrary union of open sets $\{U_\alpha\}$ is open in X and that finite intersections of open sets are open. Conclude by this and problem 2 that arbitrary intersections of closed sets are closed and finite unions of closed sets are closed.[1]

4: Circle, diamond, and square metric With $X = \mathbb{R}^2$, recall the following three metrics defined in class:

$$\begin{aligned} d_1(p, q) &:= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ d_2(p, q) &:= |x_1 - x_2| + |y_1 - y_2| \\ d_3(p, q) &:= \max(|x_1 - x_2|, |y_1 - y_2|) \end{aligned}$$

With these three metrics, we get three different metric spaces: (X, d_1), (X, d_2), and (X, d_3). Show that a set is open in one of these spaces if and only if it is open in another. *hint: Show given $N_{r,d_i}(p)$ there exists $r' > 0$ such that $N_{r',d_j}(p) \subset N_{r,d_i}(p)$.*

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[1] Good exercise to find an arbitrary intersection of open sets which isn't open and and an arbitrary union of closed sets which isn't closed

5*: Closure Given an arbitrary subset $S \subset X$ of some metric space (X, d), define $\bar{S} = \bigcap_{C \in \mathscr{C}_S} C$ where $\mathscr{C}_S$ is the collection of all closed sets containing S. Show that $\bar{S}$ is the union of S and all of S's limit points.
Can you define a corresponding notion to closure for open sets?

6*: 2 Boxes We define the following two boxes:

$$B_1 = \{(x, y) \in \mathbb{R}^2 |\ 0 \leqslant x, y \leqslant 1\}$$
$$B_2 = \{(x, y) \in \mathbb{R}^2 |\ 2 \leqslant x, y \leqslant 3\}$$

We let $B = B_1 \bigcup B_2$ and consider it as a metric subspace or $\mathbb{R}^2$ (under any of the metrics d_i mentioned in problem 4). What are the subsets of B which are closed and open (sometimes called "clopen")? (no need/attempt for proof on this... didn't introduce enough things to prove the answer for this question).

7: Analytic functions Show if the following functions are analytic on C or not:

$$f_1(z) = \frac{z}{\overline{z}}$$
$$f_2(z) = e^z$$

8: Polar form and z-bar Prove that if z has magnitude 1, $arg(\overline{z}) = -arg(z)$

9: Analytic function with z-bar Prove that if f(z) is analytic on A, $\overline{f(\overline{z})}$ is also analytic on A

10*: Constant functions Prove that if f(z) is constant in a square inside an area A and f(z) is analytic on A, f(z) is a constant function of A