M11482 PROBLEM SET 2

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*Problems marked with * are more difficult; we think...*

problem numba Problem statement

1: Open balls are open Given a metric space (X, d), we defined the open ball, $N_r(p)$ about a point $p \in X$ of radius r to be

$$N_r(p) = \{q \in X |\, d(p, q) < r\}$$

We defined a set $S \subset X$ to be open if for every element in S an open ball around that point is contained in S. Show that open balls are open and conclude after that we can equivalently define a set S to be open if it is the union of a collection of open balls.

2: Open/Closed relationship We defined a closed set to be a subset which contains all of it's limit points. Show that a set $S \subset X$ is closed if and only if the complement of it (in X) is open (and vice versa).

3: Topology? Show that an arbitrary union of open sets $\{U_\alpha\}$ is open in X and that finite intersections of open sets are open. Conclude by this and problem 2 that arbitrary intersections of closed sets are closed and finite unions of closed sets are closed.[1]

4: Circle, diamond, and square metric With $X = \mathbb{R}^2$, recall the following three metrics defined in class:

$$d_1(p, q) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$d_2(p, q) := |x_1 - x_2| + |y_1 - y_2|$$
$$d_3(p, q) := \max(|x_1 - x_2|, |y_1 - y_2|)$$

With these three metrics, we get three different metric spaces: (X, d_1), (X, d_2), and (X, d_3). Show that a set is open in one of these spaces if and only if it is open in another. *hint: Show given $N_{r,d_i}(p)$ there exists $r' > 0$ such that $N_{r',d_j}(p) \subset N_{r,d_i}(p)$.*

Date: July 16, 2017.

[1] Good exercise to find an arbitrary intersection of open sets which isn't open and and an arbitrary union of closed sets which isn't closed

5*: Closure Given an arbitrary subset $S \subset X$ of some metric space (X, d), define $\bar{S} = \bigcap_{C \in \mathscr{C}_S} C$ where $\mathscr{C}_S$ is the collection of all closed sets containing S. Show that $\bar{S}$ is the union of S and all of S's limit points.
Can you define a corresponding notion to closure for open sets?

6*: 2 Boxes We define the following two boxes:

$$B_1 = \{(x, y) \in \mathbb{R}^2 |\, 0 \leqslant x, y \leqslant 1\}$$
$$B_2 = \{(x, y) \in \mathbb{R}^2 |\, 2 \leqslant x, y \leqslant 3\}$$

We let $B = B_1 \bigcup B_2$ and consider it as a metric subspace or $\mathbb{R}^2$ (under any of the metrics d_i mentioned in problem 4). What are the subsets of B which are closed and open (sometimes called "clopen")? (no need/attempt for proof on this... didn't introduce enough things to prove the answer for this question).