Research proposal: Topos theory for a new physics

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April 2021

1 Abstract

This project aims to submit and study a new concept of formal language allowing the creation of a new mathematical framework for the construction of physical theory. The desire for such a project is based on two distinct motivations.

First of all we want to try to find an answer to the current problem encountered in theoretical physics. Particularly in fields such as quantum gravity or problems related to space-time encountered at the quantum scale. To do this we will take the path traced by the theoretical physicist Christofer Isham, and will largely take up this work that we will aim to outline and continue. For this we will use the fact that, "constructing a theory of physics is equivalent to finding a representation in a topos of a certain formal language that is attached to the system"[1]. Our research will be done on the development of this language and on it's application in physics. We will therefore seek to demonstrate the great results of modern physics by using another formal language. And in doing so we will emphasize the difference in interpretation that this framework implies. In particular we will confront it with the so-called Copenhagen interpretation which is today the leader in terms of interpretation of modern physics models.

The second is to make affordable for physicists the structural aspect of the mathematical frameworks in which physics is developed. Indeed, we noticed during our research that the documentation on such abstract subjects was clearly lacking and that it was difficult for a non-mathematician to approach such a concept directly. The rare current works are intended only for "insiders" and requires a very high mathematical banding on this particular topic. We therefore aim to give in a succinct way sufficient intuitive element to be able to do this a first idea of the subject.Therefore, at the same time this project is carried out, we produce a series of reports allowing for non-specialized physicists an easy learning of bases of our research in order to make this subject more accessible to the physics community.

[1]Quote from the article *A Topos Foundation for Theories of Physics: I. Formal Languages for Physics by A. Doering and C. Isham, [11]*

Contents

2 Introduction

Scientific theories and more particularly physics theories have always been confronted at a moment of their existence with a behavior, a value which will drive the theory obsolete. To illustrate this we can take for example the theory of gravitation of Newton defeated by his incapacity to predict the perihelion of Mercury. Fortunately, when such things appears new theories emerge before falling in failure in their turn leading to the emergence of other theories and so on. Furthermore, some current physical interpretations come from hypotheses that are not really justified. An example of that is the Copenhagen's interpretation which considers that a system exists only when there is an observer which appears to be wrong when we come to speak of truly closed systems. Nowadays, the two great theories of general relativity and quantum mechanics both fails as we try to explain the gravitation in a quantum state. Indeed, it seems impossible to get a particle as a conveyor of gravity allowing to obtain the observed and predicted effects by relativity.

From that problem will emerge the idea that physical theories could be expressed with a new algebra that implies a new interpretation of mathematics in physics. Indeed, physics uses tools in order to understand a phenomenon and to be able to shape the phenomenon into a mathematical form. This formulation helps us in understanding nature and in handling the observed phenomenon. Therefore the mathematical framework and its interpretation have a great importance in the development of a theory. As theoretical physics has its problems and as we can establish a mathematical framework to theories of physics we wonder if we could create, find a new algebra, a new framework to solve the difficulties encountered by physics. Iniated by Heisenberg and pick up by scientists such as Von Neumann the idea will however be put aside for a long period before physicists as C.Isham, Butterfield or Döring take another look at it.

Currently, the physical theories are built with a Boolean algebra as they are based on the classical logic. This last is the logic we use on our everyday life with propositions like "a proposition is either true or wrong but not in-between". Even if this logic is very strong (all of our theories in physics are based on it and that's a lot) it encounter some serious problems with modern physics. Indeed the beginning of the XXth has seen one of the strangest theory of our time : the quantum mechanics. This theory counters the classical logic in a very famous problems that kept physicists on a fierce debate : the hidden variables. Some physicists thought that there were hidden variables and others thought the contrary and both sides tried to prove their proposition but no mathematical criterion could be given to decide between both points of view. A solution had been proposed as an equality between the average of phase space and the quantum mechanical average of an observable, this is a classical view of the problem. But as Kochen and Specker wrote it, "this statistical condition does not take into account the algebraic structure of the quantum mechanical observables"[2]. Nowadays the non-existence of hidden variables has been proved but this marks one of the first and not the least failure of classical logic to explain problems of quantum mechanics. Many years later, in the modern days of quantum mechanics, a new and very big problem appeared : the quantification of gravity, or the prediction of the particle that convey gravity, the graviton. Here again classical logic used in general relativity and quantum mechanics cannot explain, predict the existence of such a particle and therefore a new logic is needed which implies that a new algebra must be developed. The Heyting algebra developed in the 1930s could be an elegant and powerful solution to it with a new formalism. In that kind of algebra we have the law of excluded middle (a proposition is either true or false) which does not need to be verified. To do this "we will use the idea that the construction of such a structure amounts to find a representation in a topos of a certain formal language". In order to do that, we will need to introduce the concept of topos and more generally the concept of categories.

The goal is to develop new mathematical tools bases on that new language to formulate the theories of physics, and particularly to try to answer the question of unification of the quantum mechanics and general relativity. Furthermore if the new algebra is able to properly describe nature then the parts of research on these subjects will be stimulated and it will be a bigger opening on new discoveries and innovations.
In order to achieve this goal we desire to create a new research team composed of theoretical physi-

[2] Journal of Mathematics and Mechanics, vol.17 n°1 (1967), §2 p.59, *The Problem of Hidden Variables in Quantum Mechanics [11]*

cists and mathematicians from the Theoretical Physics Centre of Marseille as some researchers already wrote about this subject and therefore could help us to achieve our objectives which will be presented in the following section but which we can began to sum up here :

- Make the inventory of what's wrong with actual theoretical physics to get a bigger picture and therefore to try to find the solution that solves most of these problems, even all but that's a bit of an utopia as science history taught us.

- Make a bibliography about topos theory and its possible applications in physics

- Develop Heyting algebra in a physical context and understand what's modified compare to actual physics (Boolean algebra), how these modifications occurs, ...

- Develop a new formal language in this algebra for theoretical physics

3 Objectives

To be more precise about the objectives, the main goal of our project is the development and the study of a new mathematical formal language for theories in physics which will be interpreted in the topoï theory. This will be achieved by first come up to the problems of contemporary physics that required such changes.

Therefore we shall first look on the side of actual logic, the classical one with which we are very familiar as we use it in our everyday life through the Boolean algebra in mathematics and physics. We will also show how it influenced physical models through the set theory. Then, as we will have understood the philosophy behind the connection between logic and algebra, we will study others logics and algebras such as the Heyting's algebra and the intuitionnistic logic which are what we want to develop. From that point we shall express more deeply the formalism with the theory of categories, the theory of topoï. Finally we will establish a new mathematical framework and formal language for the physics we want to develop.

4 Theoretical approach

4.1 Symbols

Let P be a proposition.

1. $\neg$P is the negation of P
2. $\perp$ designs falsehood
3. T designs truth or provability
4. $\Rightarrow$ and $\rightarrow$ symbolize implication
5. $\wedge$ is the conjunction/and
6. $\vee$ is the disjunction/or

4.2 Logic and algebra

4.2.1 Classical logic

The mathematical framework behind every theories developed in physics can be seen as a matter of logic. Logic is the fruit of philosophers from Ancient Greece as they tried to explain the world in a philosophic way. It gave birth to mathematics that we, physicists, use everyday for our work. It helped us to establish theories and to generalize them to any situation that we can imagine, of course until we find something that can't be described by the theory.

The logic we use in our everyday life is called the classical logic and was developed at the end of the XIXth century as the foundational crisis occurs. As it heritates the thoughts of the Greek philosophers the classical logic was developed on the three laws of thoughts :

1. the law of identity that stipulate that every object is equal to itself : P=P
2. the law of non-contradiction : if P is true then ¬P is necessarily false
3. the law of excluded middle : either P is true or ¬P is true, there is no proposition in-between P and ¬P

As time passed philosophers and mathematicians kept thinking about these laws and therefore they developed mathematics based on them. But after many breakthrough the time for problems came in the XIXth century with the emergence of the Set theory or non-euclidean geometries that brought paradoxes strong enough to shake the basis of mathematics. This is why mathematicians started to build a formalism for the laws of thoughts and for propositions considered true or false by everyone such as "1≠0 is True" and "1=0 is False". This formalism is the propositional calculus and is used to correctly express the laws of thoughts in logic :

1. the law of excluded middle
2. reductio ad absurdum : $\neg\neg A => $ A or in other words "If the negation of the negation of A is true then A is true"
3. contraposition : $(\neg B => \neg A) => (A => B)$ which means "If non-B implies non-A then A implies B"
4. the material conditional : $(A => B) \Leftrightarrow (\neg A \vee B)$ or "The operation 'implies' between two propositions is equivalent to say 'non-A or B' "

I put words on the mathematical expressions just above and this is the results of my intuition but how does $\Leftrightarrow, \vee, \Rightarrow, \neg, ...$ keep this semantic while being expressed with propositional calculus ?

Well the answer of that question lies in the semantic part of mathematics and in truth tables. These lasts will help us to represent with words logical expressions that will give us the possibility to establish if a proposition is true or false. You will find truth table for 'and', 'or', 'exclusive or' and 'implies' in the appendix 1. Now that's fixed there is still the problem of the symbols '∨' and '∧' and their significations because they are very important for our definition of the Boolean algebra.

The first written one is '∨', is called the logical disjunction and corresponds to the 'or'. Its truth table is therefore :

P	Q	P and Q
False	False	False
False	True	True
True	False	True
True	True	True

Table 1: Truth table for ∨ with P and Q two propositions

As it link two propositions to give just one after the operation we call '∨' a binary operator. Another binary operator as you may guess is the operation $'\wedge'$ and is named the logical conjunction and is defined as the 'and' so the truth table of the operator $\wedge$ is :

Another important characteristic point of the classical logic is that the disjunction can be written as a function of the conjunction and vice versa according to the Morgan's laws. Another characteristic of the classical logic is that it applies correctly to finite but problems appears with infinite but we shall discuss of it in the section 4.2.3. Finally the formal framework for this logic is called the Boolean algebra.

P	Q	P and Q
False	False	False
False	True	False
True	False	False
True	True	True

Table 2: Truth table for $\wedge$ with P and Q two propositions

4.2.2 Boolean algebra

To have a correct definition of an algebra we must wait 1939 the year when the group of French speaking mathematicians Bourbaki wrote *Eléments de mathématiques*[4][6]. In that series of books, authors wanted to find common points between all the branches of mathematics that were very compartmentalized. It is an euphemism to say that they succeeded. More than that they revolutionized the learning, the teaching and the research of mathematics. Of course the subject is still evolving and has evolved quite a lot since Bourbaki. Nowadays mathematicians define structures as a "class of mathematical object described by axioms"[3]. To be more exhaustive, a structure is a list of operations, operators, objects, properties, etc that describe a part of mathematics. For example a topological space is an object described by the topological structure in the following way : the departure point is axioms and from them by using a set vocabulary we can define the topological space. For most structures we use the set vocabulary to define every mathematical objects described by the structure.

Therefore, an algebra is characterized by the algebraic structure, more particularly it can be defined by a lattice (even if we could describe an algebra by a ring for example). A lattice is a set (remember that we use the set vocabulary) equiped with two binary laws and a partial order relation. The binary laws respect commutativity, associativity and the absorption law.

Hence, the Boolean algebra is defined as a particular lattice with $\wedge$ and $\vee$ as binary laws. From the truth tables it is easy to see that both respect the three laws of a lattice but another necessarily condition is that they are distributive one to another which also verified. As the lattice is a set we can define the set on which the Boolean algebra is based as[4] :

$$2^E = \{a : a \subset E\} \tag{1}$$

Actually, it is more complicated than that because 2^E refers to a particular set which is the power set. The power set of E is the set of all subsets of E, including the empty set and E itself. The Boolean algebra is therefore define as the power set of E equiped with intersection, union and passage to the complement. Moreover, as '$\cap \Leftrightarrow$ *and*', '$\cup \Leftrightarrow$ *or*', '$\wedge \Leftrightarrow$ *and*' and '$\vee \Leftrightarrow$ *or*' we have '$\cap \Leftrightarrow \wedge$' and '$\cup \Leftrightarrow \vee$'. Because of such a definition and of the absorption law we have $a \leq b \Leftrightarrow a \wedge b = a$ for $a, b \subset E$, that implies that the partial order relation can be written as follows :

$$\begin{aligned} sup(a,b) &= a \vee b \\ inf(a,b) &= a \wedge b \end{aligned} \tag{2}$$

Usually sup(a,b) and inf(a,b) are respectively written 1 and 0.

As we defined the Boolean algebra it is absolutely not obvious that our theories relies on this particular mathematical structure. Hence, the following paragraph is dedicated to help us see where does the Boolean algebra steps in physics.

Let S be a closed system. As we analyze the system and its evolution by taking values of temperature, pressure, ... we are able to establish propositions on how works the system, why does it evolve like that, etc. These propositions are handled by Boolean logic as we, humans, think with this logic.

[3] Cf p.179 *Structure in Mathematics[8]*

[4] What follows is a short summary of the rigorous definition of the Boolean algebra in [10]

4.2.3 Intuitionistic logic and Heyting algebra

As rises the classical logic, several mathematicians, especially L.E.J. Brouwer, contested axioms chosen by the defenders of classical logic. This led to a strong and fierce fight between both schools that ended by the win of the most accepted logic, the classical one. It led also to the depression and the departure from mathematics studies of Brouwer as he was fighting against recognized mathematicians, such as David Hilbert, and a philosophy so obvious for many scientists that they couldn't accept the existence of another manner of organizing thoughts in mathematics.
But let us go back in time in order to understand the philosophies of Hilbert and Brouwer at the beginning of the XXth century. At that time, the set theory was developed and it revolutionized mathematics (as well as every other sciences) by reassuring the foundations of mathematics that were in crisis (the foundational crisis) because of paradoxes at the base itself of mathematics. After that appears the classical logic and David Hilbert was one of its fiercer defender and who developed a program, named the Hilbert's program[5], which point was to list 23 problems of mathematics which once solved could end the crisis that shook mathematics. Let us give a list of the goals of the Hilbert's program :

1. Every mathematical propositions can be written in a formal language such as the classical logic. We call it the formalism.

2. All of the mathematical statement can be proved in the formalism, this is named the completeness.

3. Consistency : There are no combinations of rules of a theory that can prove a proposition P and¬P, its negation.

4. If a proposition P is true or its negation is true then we can decide about the truthfulness of P. Therefore we could establish an algorithm that can decide if a statement is true or false in the formalism.

The main aim of the Hilbert's programm was to prove that all of mathematics are consistent in a theory based on finite objects, or in other words, that mathematics as Hilbert thought them were enough to stabilise the basis of mathematics and did not need other approaches such as the intuitionistic logic.

A consequence of Hilbert's work is a theory called the Proof Theory which represents proofs as objects of the mathematics that can be written with the formalism (i.e. the classical logic, the semantic that comes out this logic,...). This theory is set on the arithmetic of natural numbers. But it happens that the theory was disproved by the Austro-American mathematician Kurt Gödel in his famous incompleteness theorems that are counted as two :

1. 1^{st} theorem : In any theory based on a finite number of axioms, or on infinite number of axioms but which can be described by a finite number of axioms and that can give a formalism, then we have that it exists a statement cannot be either demonstrated or refuted in the theory. This established the non-decidability in oppostion to the decidabilty that tried to prove Hilbert.

2. 2^{nd} theorem : The coherence of a theory cannot be proved inside the theory itself.

The consequence of Gödel's theorems is the establishment of a deep difference (and the formalism of it) between the truth of a mathematical statement and its provability : it can be possible to define the provability in a theory by using some arithmetics relations, but concerning the truthfulness in a theory we principally need to suppose that the theory is consistent (i.e. its axioms are not at odds ones to each others).

The dichotomy between truth and proof is at the very center of the intuitionistic logic which was further developed especially by Brouwer's student, Arend Heyting [6].

Let start with a bit of philosophy in order to clearly catch what were the thoughts of intuitionists and their difference with Hilbert's and classical logician's minds. Hilbert's goal was to prove with

[5] Cf $The Hilbert's programm$ from Kosta Do en [13]

[6] To have a good view on the philosophy of intuitionist, we invite you to see the Disputation part of [14]

philosophy and logic that mathematics exists independently from human thought/mind such as physics exists without any formulation from humans. On the other hand, Brouwer and its students thought that use such "means of demonstration would be to lock oneself into a circle, because to formulate these thesis you need to suppose mathematical concepts as already build".[7] Therefore, mathematics lies onto intuition as a beginning to develop its concepts and ideas. Although the name "Formalism" given to mathematics born from classical logic, the intuitionnistic logic can be written in another formalism. In that formalism mathematicians rejects the *reductio ad absurdum* (RaA) and the excluded middle (EM) because they think that we must build the mathematical statement in order to prove it, a construction which both RaA and EM do not bring as they stipulate as true, i.e. proved in the Formalism, the statement that respects both axioms.

The first writing of the basic ideas of intuitionnistic logic had been made around the 1930s by Heyting based on the work of Brouwer and by Kolmogorov a Soviet mathematician in the Brouwer-Heyting-Kolmogorov interpretation :
Let a and b be respectively the proofs of two statements P and Q.

1. the pair <a,b> is a proof of $P \wedge Q$
2. to prove $P \vee Q$ a proof of P or a proof of Q is needed
3. the construction that transforms a proof of P into a proof of Q is a proof of $P \to Q$
4. ¬P is defined as $P \to \perp$ which proof is a construction that transforms the proof of P into the proof of $\perp$

On this basis Heyting developed a type of algebra that wear his name, the Heyting algebra H. This last is define as a lattice which is bounded by a least element noted 0 and a greatest element noted 1 and the lattice is also furnished with the binary operation $\to$. It is also equiped with a pseudo-complement a so named because its properties are very close from the complement define in the Boolean algebra. Here are the properties of a in H :

$$\begin{gathered}\neg a = a \to 0 \\ a \wedge \neg a = 0 \\ a \vee \neg a \text{ can be different from 1 on the contrary of the complement in a Boolean algebra}\end{gathered} \tag{3}$$

Furthermore the binary operation $\to$ is distributive, $a \to a = 1$ and the following operations must be followed if We want H to be a Heyting algebra :

$$\begin{gathered}a \to a = 1 \\ b \wedge (a \to b) = b \\ a \to (b \wedge c) = (a \to b) \wedge (a \to c), \text{ the distributivity of } \to\end{gathered} \tag{4}$$

The Heyting algebra is a formalism for the intuitionnistic logic and has many properties that can be very useful in every field of mathematics : it is a pseudo-Boolean algebra if $a \to b$ is replaced by $a \vee b$, it can also be described as category which is a mathematical object that will be presented in the next section.

4.3 Category theory

4.3.1 An introduction to the category

The category theory is a very abstract branch of mathematics and in mathematics the more abstract is something, the more it can be used for different things. So it's a big theory and we will try to make a small summary of the main axes. [8]

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[7] *Les fondements des mathématiques. Intuitionnisme. Théorie de la démonstration* by A. Heyting p.13-14 section 5, subsection 1.

[8] For more complete explanations, you can consult the following books [1,2,3] which served as a reference to understand category theory.

First what is a category?
The important thing to understand is that what characterizes a category are not the objects which composes it but the relation between these objects. These relations are named arrow or morphism, and the objects inside a category therefore only serve as starting points and ending points to these morphisms.
So we have a structure made up of objects linked together by morphisms. Furthermore, we need one property to have a category, a binary operation ($\circ$), called composition of morphisms, such as: For any of these three objects α, β, γ, and two morphisms, $h : \alpha \to \beta$ and $g : \beta \to \gamma$,
The composition of $h : \alpha \to \beta$ and $g : \beta \to \gamma$ is written as $g \circ h$ and is governed by two axioms:

-The identity (id), called the identity morphism : Each object has a morphism which sends it on itself such as $(id \circ h) = (h \circ id) = h$
-The associativity: $(f \circ g) \circ h = f \circ (g \circ h)$

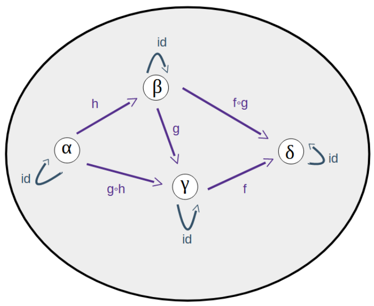

Figure 1: Example of a category made up of four objects $(\alpha, \beta, \gamma, \delta)$ linked together by morphisms (f, g, h, id, $g \circ h$, $f \circ g$)

There is also a transformation between category called functor, which will link all the objects and morphisms of a category to the objects and morphisms of a second category while keeping the structure of the first category. We have to imagine a functor as a mapping from one category to another. We must also report that the mapping being bijective or surjective, information can be lost during the transformation.
To finish, just as there is a transformation between categories named functor, there is a transformation between functors named natural transformation, which is used when two structures of different appearances actually give the same information. We can then go from one structure to another through a natural transformation.

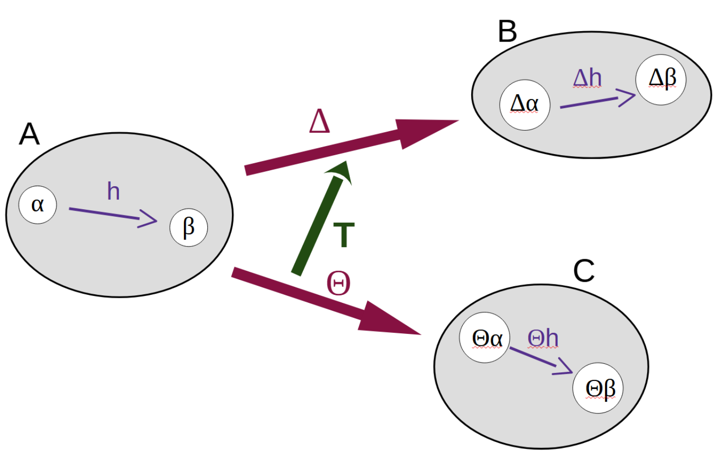

Figure 2: Example of three category (A,B,C), where B and C are mappings of A by functors Δ and Θ, with T the natural transformation between Δ and Θ

With that we have a good idea of what a category is, we will just make a short summary and explain the usual notation.

A category (A) is made up by three things:

-Objects, named ob(A)

-Morphisms or arrows between the objects, named hom(α, β) or $h : \alpha \to \beta$ for two objects α and β

-A binary operation, named $\circ$

We can also have a transformation Δ named functor that preserves the structure between two categories A and B. If two functors are applied to a category A and that the resulting category B holds the same information, then we have a natural transformation going from Δ to Θ

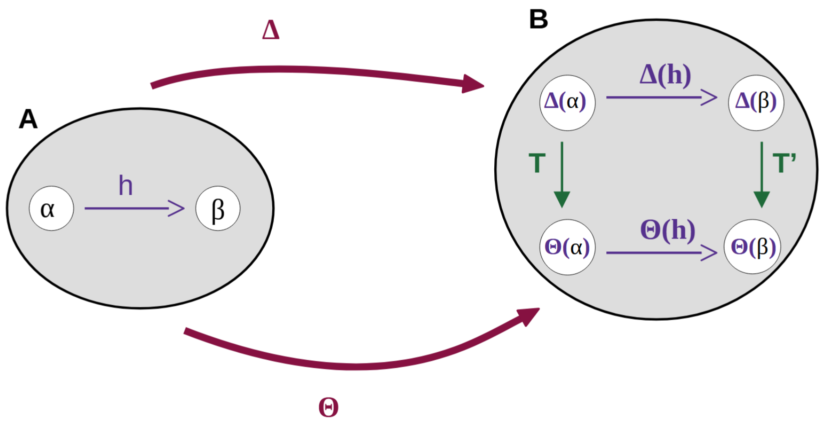

Figure 3: Example of two categories (A,B), the information between the transformation Δ and Θ are the same so they both give the same category B. With T and T' the natural transformation between Δ and Θ

4.3.2 The sets category

Now that we have seen what a category is, we will apply it on a concrete example. This example will make up a link between the different parts of our presentation. We are going to be interested in a generating structure of the most part of the mathematical theory currently used, the theory of sets.

As we know, set theory is ruled by classical logic. Indeed, one of the fundamental property of set theory is that for any set, the set of these parts is a Boolean algebra.

Furthermore we can easily see that it is a category. Its objects are all sets, and its morphisms are all applications between sets. This category is usually noted Ens. It's interesting to know that defining the theory of sets from the notion of category has advantages. First of all, it allows to bypass the problematic Russell paradox encountered when using the axiomatic definition ZF[9] to define the theory of sets. Paradox, which I remind you, tells us that it is impossible to define a set of all the sets.

In addition, this allows to see it as a particular case of a more general structure[10]. As we mentioned in the section on classical logic and Boolean algebra, the current mathematical model is based on set theory. From this it becomes obvious to see how the choice of a boolean algebra as a fundamental structure to describe the sets will directly influence the rest of the mathematics and at the same time define a framework for the physical theories. We can then begin to see why it would be interesting to define another category similar to the category of sets but having a Heyting algebra instead of a Boolean algebra and what the consequences would be on the framework of current physical theories. To better understand it, we will need a concept called topos, which is nothing more than a particular case of category

[9]The axiomatic definition ZF (or ZFC) is the usual way to define set theory. It is this definition that is used when we talk about set and which is given in the 3rd page of the following book [4]

[10]We'll explain this in more detail when we get to the topos part.

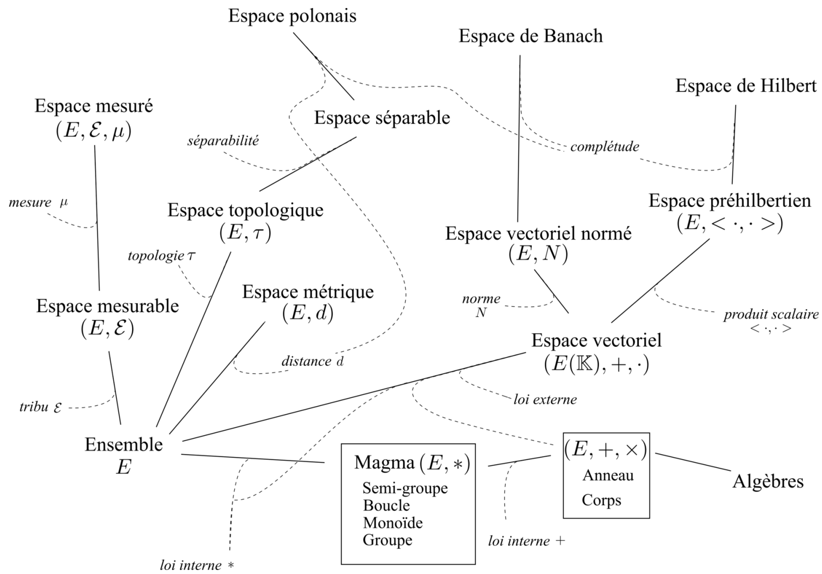

Figure 4: Graphic summary in French of the different mathematical structures and their interweaving. With at the bottom left the sets from which all the other spaces derive. `https://www.entropologie.fr/2018/10/les-structures-algebriques-partie-1-magma-groupe.html`

4.4 About topoï and formal language

Let's start with a little bit of history. In the 1950s, a young mathematician Alexander Grothendieck seized on Weil's [11] conjectures and sought a method to solve them. To make this simple, Weil's conjectures seek to define a relation between the solutions of polynomial equations in finite forms, which are therefore arithmetic and belong to the world of the discrete. Complex solutions (with an imaginary component) which are part of the topology and therefore of the continuous world.

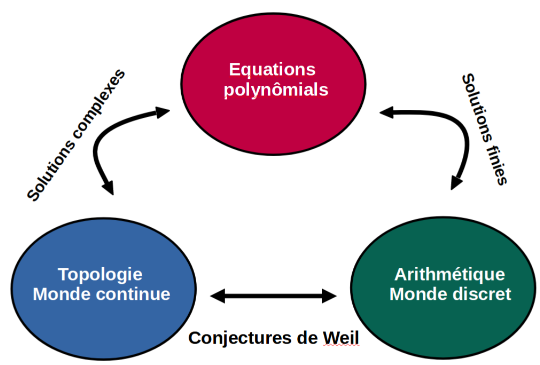

Figure 5: Schematic representation of Weil's conjectures, copied from the document "Hommage à Alexandre Grothendieck" by Bertrand Toen.

To do this, Grothendieck created a new concept which was called topos. The notion of topos is quite complicated to conceive and once again I will try here to give an incomplete vision but sufficiently rich so that we can make an intuitive vision out of it. In order not to get lost, let's keep in mind that a topos is a special category. So a topos has similar properties to the category seen before but with a very important notion of space.

If with my current knowledge on this subject I had to describe what a topos is in one sentence, I would say that it is a mathematical space containing objects and morphisms of this space, but also containing objects and morphisms that do not belong to this space. It is an idea which seems quite

[11] We will not go into details here because that is not the subject. But if you are curious, we suggest that you take a look at the excellent course at the University of Rennes on Weil's conjectures.. [15]

contradictory and in order to illustrate this I will give an example which one finds quite frequently when seeking to introduce this idea.

We invite you to imagine a plane made of the x and y axis. Then, suppose that our topos is the space composed only of the x axis. Therefore, there would be, among the objects of the topos, the real numbers of the x axis and, among the morphisms, the interactions between these numbers. But in addition to that, our topos would also have as objects, the real numbers of the y axis and the morphisms between these numbers. This may seems impossible because our topos is only made of the x axis. Yet, this is the case and to conceive it we have to imagine the y axis as being of an infinitesimal length, like if we were crushing it to such an extent that it becomes like a pseudo thickness of the x axis.

Here is one of the ways to imagine how a space could contain objects which do not belong to this space. But keep in mind that this is just a picture.

Now that we have a vision of what can be a topos as space, we will see the three current ways of approaching topos: -The first approach is the one introduced by Grothendieck where a topos is considered as a generalized space.

-The second approach is to see a topos as a mathematical universe linked to a certain logic

-Finally the last approach is to start from modulo Morita-equivalence theory, which consists in studying the equivalence module on the rings, and to generalize them in the form of topos

Here, the interested approach is the second one, so we will not further talk about the two others. For this, we need to introduce the elementary topos such as a closed Cartesian category with kernels and a classifier of sub-objects. I am not going to define all these concepts here [12] but I will just dwell on what a closed Cartesian category is because it seems to me that it is the most important point, in particular with the notion of exponential.

So we have a closed Cartesian category which is a category endowed with the binary product as defined previously and which is stable by exponentiation. The exponentiation is a morphism of morphism (an arrow which connects two arrows between them), for example, if we have two morphism A and B, the exponential between A and B called A^B is a morphism between A and B. The fact that the category is closed means that the morphisms are stable by the exponential i.e: let A, B be two morphisms and C a category, if A,B $\in$ C, then $A^B \in$ C. This notion of exponential, and therefore of "morphism of morphism" is very important in the notion of topos. For the moment, this is the definition we will use for a topos. It is not yet complete, but it will suffice to support our remarks.

Now we shall speak about the relation between topos and logic. From the point of view of the theory of topos, a space is the place where unfolds a certain universe characterized by a certain logic. Each topos is therefore the representation of a logic, and each logic has a representation in a topos being specific to it. For instance, the category of sets is a topos which represents Boolean logic. But there are a lot of other elementary topos and logic that go with it.

This is where our research work will really begin. Indeed all of our studies will relate onto topos represented by a Heyting algebra and more particularly onto the creation of a formal language attached to these topos. As a reminder, a formal language is a representation of the axiomatic systems of a mathematical structure using a particular syntax. Once the language is created, the mathematics in this structure can be reduced to the syntactic manipulation of this language. Once this language is formed we will seek to create a mathematical structure as complete as the one we had when we started from the category of sets and Boolean logic. We can then use this new framework to re-express the physical theories as they relate propositions about a system to form a physical phenomenon.

[12] All these concepts are very well defined in the course of Alain Prouté, [5]

5 Research Team

In order to complete successfully, or at least to go further in that field of research, the team needs a mathematician that can develop the framework on which we will work. But that must go hand in hand with the work of a theoretical physicist that can manipulate, understand and modify tools created by his/her colleague in order to correctly describe the phenomenons that are taking place in the Universe. With both points of view, the development of a theory that can describe what is yet not understood could be possible.

Furthermore, as it is a new field of research, it would be great to start its development by attracting students in order to prepare future scientists to work on the subject and to continue to make it grow, with, maybe, a great breakthrough into the bargain.

6 Budget

Staff	PhD Mathematician, PhD Theoretical Physicist, PhD Student, 2 Interns M2 student	310 000 €
Communication	Conferences in France and in foreign countries, publications	20000 €
Materials	2 PC and 1 printer	1500 €

Table 3: Table summarizing the costs of the research project

We plan a three year project therefore the numbers presented in the table are calculated as follows :

1. 2800€ per month for a PhD Researcher Director of Research
2. 2000€ per month for PhD Researcher
3. 1700€ per month for a PhD Student
4. 1000€ per month for a M2 Intern

We also consider two seminars in foreign countries, two to three in France and 3/4 publications per year. The part about materials consists in loans to students in internship and in PhD.
Therefore the approximate cost including charges is around 383 600€.

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Appendices

Let P and Q be two propositions and 1 is for true as 0 is for false.

A Appendix A : Truth tables in classical logic

P	Q	P and Q
0	0	0
0	1	0
1	0	0
1	1	1

Table 4: Truth table for 'and'

P	Q	P or Q
0	0	0
0	1	1
1	0	1
1	1	1

Table 5: Truth table for 'or'

P	Q	P $\Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Table 6: Truth table for the implication

P	Q	P $\Leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

Table 7: Truth table for the equivalence

P	Q	P xor Q
0	0	0
0	1	1
1	0	1
1	1	0

Table 8: Truth table for 'xor'