Wavelets in $\mathbb{R}^2$

As mentioned previously, wavelets are important in image processing. Keeping in mind this applications we introduce here "separable" multiresolutions (or wavelets), that is multiresolutions in $\mathbb{R}^2$ obtained as products of 1-dimensional resolutions.

A first attempt, given a wavelet orthonormal basis $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ would be to consider the products in $L^2(\mathbb{R}^2)$:

$$\{\psi_{n_1,k_1}(x_1)\ \psi_{n_2,k_2}(x_2)\}_{\substack{n_1,n_2\in\mathbb{Z}\\ k_1,k_2\in\mathbb{Z}}}$$

These functions mix information at two different scales $2^{n_1}, 2^{n_2}$, along the axes x_1, x_2. This is not convenient: we would like to have the same scale in all directions. However, this construction can be slightly modified to provide another separable wavelet basis whose elements are products of functions dilated at the same scale. These multiresolution approximations have important applications in computer vision, where they are used to process images at different levels of detail.

Separable multiresolutions:

As in dimension $d=1$, the notion of resolution is formalised with orthogonal projections on spaces of various sizes. The approximation of $f(x_1,x_2)$ at the resolution n is defined as the orthogonal projection of f on a space V_n^2 that is included in $L^2(\mathbb{R}^2)$. The space V_n^2 is the set of all approximations at the resolution n". When the resolution decreases, the size of V_j^2 decreases as well.

The formal definition of a MRA is as in dimension 1: $\{V_n\}_{n\in\mathbb{Z}} \subseteq L^2(\mathbb{R}^2)$ with the properties previously listed.

We consider the particular case of separable multi-resolutions. Given a multiresolution $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, a <u>separable 2-dimensional multiresolution</u> is composed of the tensor product spaces $V_n^2 = V_n \otimes V_n$.

Then $f \in V_n^2 \in L^2(\mathbb{R}^2)$ if it has the form

$$f(x_1,x_2) = \sum_{m\in\mathbb{Z}} c_m f_m(x_1) g_m(x_2) \qquad f_m, g_m \in V_n;\ \sum_m |c_m|^2 < \infty.$$

Then $\{V_n^2\}_{n\in\mathbb{Z}}$ is a multiresolution of $L^2(\mathbb{R}^2)$. Let φ be the scaling function of $\{V_n\}_{n\in\mathbb{Z}}$ (so that $\{\varphi_{n,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_n. Since $V_n^2 = V_n \otimes V_n$, then system

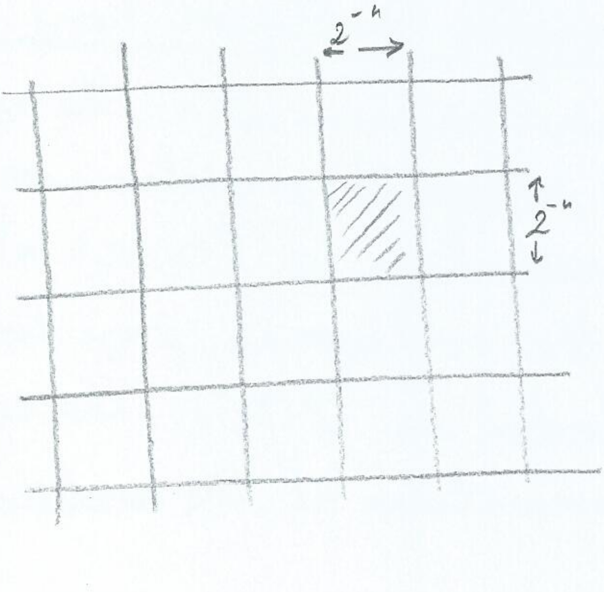
$$k = (k_1,k_2) \in \mathbb{Z}^2, \quad n \in \mathbb{Z}$$

$$\varphi^2_{n,k}(x) := \varphi_{n,k_1}(x_1)\,\varphi_{n,k_2}(x_2)$$
$$= 2^n \varphi(2^n x_1 - k_1)\,\varphi(2^n x_2 - k_2)$$

is an orthonormal basis of V_n^2. Notice that the scaling function here is just $\varphi(x_1,x_2) := \varphi(x_1)\varphi(x_2)$.

<u>Example:</u> Piecewise approximation (Haar). Let V_n be the approximation space of functions that are constant on intervals $[2^{-n}k, 2^{-n}\cdot k]$, $k\in\mathbb{Z}$. The 2-dimensional scaling function is

$$\varphi(x_1,x_2) = \chi_{[0,1)}(x_1)\,\chi_{[0,1)}(x_1)$$
$$= \chi_{[0,1)\times[0,1)}(x_1,x_2)$$



Given a MRA $\{V_n^2\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ as before let W_n^2 denote the corresponding detail space; that is, W_n^2 is the orthogonal complement of V_n^2 in V_{n+1}^2: $V_n^2 \oplus W_n^2 = V_{n+1}^2$

<u>Theorem:</u> Let φ be the scaling function of a MRA $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ and let ψ denote the associated wavelet (which generates the corresponding wavelet basis)

Define
$$\psi^1(x) = \varphi(x_1)\,\psi(x_2) \qquad x=(x_1,x_2)$$
$$\psi^2(x) \doteq \psi(x_1)\,\varphi(x_2)$$
$$\psi^3(x) = \psi(x_1)\,\psi(x_2)$$

and denote, for $k_1,k_2\in\mathbb{Z}$ and $j=1,2,3$, $n\in\mathbb{Z}$

$$\psi^j_{n,k}(x_1,x_2) = 2^n\,\psi^j(2^n x_1 - k_1,\, 2^n x_2 - k_2).$$

The family $\{\psi^1_{n,k},\ \psi^2_{n,k},\ \psi^3_{n,k}\}_{k\in\mathbb{Z}^2}$ is an orthonormal basis of W_n^2, and therefore $\{\psi^1_{n,k},\psi^2_{n,k},\psi^3_{n,k}\}_{\substack{n\in\mathbb{Z}\\ k\in\mathbb{Z}^2}}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$

<u>Proof:</u> Since by definition $V_{n+1} = V_n \oplus W_n$, we have
$$V_{n+1}^2 = (V_n \otimes V_n) \oplus W_n^2$$

Also
$$V_{n+1}^2 = (V_n \oplus W_n) \otimes (V_n \oplus W_n) = V_n^2 \oplus (W_n\otimes V_n) \oplus (V_n \otimes W_n) \oplus (W_n \otimes W_n)$$

and therefore
$$W_n^2 = (W_n \otimes V_n) \oplus (V_n \otimes W_n) \oplus (W_n \otimes V_n)$$

Then: $\{\psi^2_{n,k}\}_k$ is a basis of $W_n \otimes V_n$

$\{\psi^1_{n,k}\}_k$ is a basis of $V_n \otimes W_n$

$\{\psi^3_{n,k}\}_k$ is a basis of $W_n \otimes W_n$ $\boxtimes$

The three wavelets of the theorem extract image details at different scales and orientations. Notice also that

$$\hat{\psi}^1(\xi_1,\xi_2) = \hat{\varphi}(\xi_1)\,\hat{\psi}(\xi_2)$$

$$\hat{\psi}^2(\xi_1,\xi_2) = \hat{\psi}(\xi_1)\,\hat{\varphi}(\xi_2)$$

$$\hat{\psi}^3(\xi_1,\xi_2) = \hat{\psi}(\xi_1)\,\hat{\psi}(\xi_2)$$

Sometimes ψ^1 is called ψ^h (horizontal) because each
ψ^2 is called ψ^v (vertical)
ψ^3 is called ψ^d (diagonal)
of the subspaces favours details in those directions.

<u>Example</u>: (Shannon aproximation). Let $\{V_n\}_{n\in\mathbb{Z}}$ be the MRA of Exercise 2, with φ given by $\hat{\varphi}(\xi) = \chi_{[-1/2,1/2]}(\xi)$.
You will (hopefully) see that $\hat{\psi}(\xi)$ is, up to a constant of modulus 1, the function $\chi_{[-1,1]\setminus[-1/2,1/2]}(\xi)$. The two dimensional basis explained above paves the Fourier plane with dilated rectangles

$\hat{\psi}^1(\xi_1,\xi_2) = \hat{\varphi}(\xi_1)\,\hat{\psi}(\xi_2)$

$\hat{\psi}^2(\xi_1,\xi_2) = \hat{\psi}(\xi_1)\,\hat{\varphi}(\xi_2)$

$\hat{\psi}^3(\xi_1,\xi_2) = \hat{\psi}(\xi_1)\,\hat{\psi}(\xi_2)$