Lesson 4

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Definition

A derivative defined by h is said to be replicable if there exists an admissible strategy ϕ such that replicates h that is $V_N(\phi) = h$.

Proposition

If ϕ is a self-financing strategy that replicates h and the market is viable then it is admissible.

Proof.

$\tilde{V}_N(\phi) = \tilde{h}$ and since there exists $\mathbb{P}^*$ such that $\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N(\phi)|\mathcal{F}_n) = \tilde{V}_n(\phi)$, we have $\tilde{V}_n(\phi) \geq 0$. $\square$

Definition

A market is said to be complete if any derivative is replicable.

We have the second fundamental theorem of asset pricing (SFTAP)

Theorem

(SFTAP) A viable market is complete if and only if there is a unique probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted prices of the stocks $((\tilde{S}_n^j)_{0\leq n\leq N}, j=1,...,d)$ are $\mathbb{P}^*$-martingales.

Proof.

Assume that the market is viable and complete, then, given h $\mathcal{F}_N$-measurable there exists ϕ admissible, such that $V_N(\phi) = hS_N^0$ that is:

$$\tilde{V}_N(\phi) = V_0(\phi) + \sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j = \frac{hS_N^0}{S_N^0} = h.$$

Assume there exist $\mathbb{P}_1$ and $\mathbb{P}_2$ martingale measures, then

$$\mathbb{E}_{\mathbb{P}_1}\left(\frac{hS_N^0}{S_N^0}\right) = V_0(\phi)$$
$$\mathbb{E}_{\mathbb{P}_2}\left(\frac{hS_N^0}{S_N^0}\right) = V_0(\phi),$$

so $\mathbb{E}_{\mathbb{P}_1}(h) = \mathbb{E}_{\mathbb{P}_2}(h)$ and since this is true for all h, $\mathcal{F}_N$-measurable, both probabilities are the same in $\mathcal{F}_N = \mathcal{F}$. $\square$

Proof.

Assume now that the market is viable but incomplete, we shall see that we can construct more than one risk-neutral probability. Let H be the subset of random variables of the form

$$V_0 + \sum_{j=1}^{N} \phi_j \cdot \Delta \tilde{S}_j$$

with $V_0 \in \mathbb{R}$ and $\phi = ((\phi_n^1, \ldots, \phi_n^d))_{1 \leq n \leq N}$ *predictable*. H is a vector subspace of the vectorial space, L^0, formed by all random variables. If the market is incomplete there will exist $h \geq 0$ in L^0 such that h is not an element of H with ϕ *admissible* but, by Proposition 2, these are the only elements of H that could replicate h, so $h \notin H$. Therefore H it is not a trivial subspace. Let $\mathbb{P}^*$ be a risk-neutral probability, we can define the scalar product in L^0, $\langle X, Y \rangle := \mathbb{E}_{\mathbb{P}^*}(XY)$. □

Proof.

Let X be a non trivial random variable in L^0 orthogonal to H and set

$$\mathbb{P}^{**}(\omega) = \left(1 + \frac{X(\omega)}{2||X||_\infty}\right) \mathbb{P}^*(\omega).$$

Then we have an equivalent probability to $\mathbb{P}^*$:

$$\mathbb{P}^{**}(\omega) = \left(1 + \frac{X(\omega)}{2||X||_\infty}\right) \mathbb{P}^*(\omega) > 0,$$

$\square$

Proof.

$$\begin{aligned}\sum \mathbb{P}^{**}(\omega) &= \sum \mathbb{P}^{*}(\omega) + \frac{\sum X(\omega)\mathbb{P}^{*}(\omega)}{2\|X\|_\infty} \\ &= \sum \mathbb{P}^{*}(\omega) + \frac{\mathbb{E}_{\mathbb{P}^*}(X)}{2\|X\|_\infty} = \sum \mathbb{P}^{*}(\omega) = 1,\end{aligned}$$

since $1 \in H$ and X is orthogonal to H. Also, by this orthogonality, and for any predictable process ϕ, we have that

$$\mathbb{E}_{\mathbb{P}^{**}}\left(\sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j\right) = \mathbb{E}_{\mathbb{P}^{*}}\left(\sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j\right) + \frac{\mathbb{E}_{\mathbb{P}^*}\left(X \sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j\right)}{2\|X\|_\infty} = 0$$

in such a way that $\tilde{S}$ is a $\mathbb{P}^{**}$-martingale by previous proposition. □

Assume we have a derivative with payoff $h \geq 0$ and that the market is viable and complete. We know that there exists ϕ admissible, such that $V_N(\phi) = h$ and if $\mathbb{P}^*$ is the risk-neutral probability neutral we have that

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j$$

is a $\mathbb{P}^*$-martingale, in particular

$$\mathbb{E}_{\mathbb{P}^*}\left(\left. \frac{h}{S_N^0} \right| \mathcal{F}_n \right) = \mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N(\phi) | \mathcal{F}_n) = \tilde{V}_n(\phi)$$

that is

$$V_n(\phi) = S_n^0 \mathbb{E}_{\mathbb{P}^*}\left(\left. \frac{h}{S_N^0} \right| \mathcal{F}_n \right) = \mathbb{E}_{\mathbb{P}^*}\left(\left. \frac{h}{(1+r)^{N-n}} \right| \mathcal{F}_n \right)$$

so, the value of the replicating portfolio of h is given by the previous formula and this gives us the price of the derivative at time n that we shall denote by C_n, that is $C_n = V_n(\phi)$. Notice that if we have that $\tilde{C}_n = \tilde{C}(\tilde{S}_n)$ a single risky stock $(d = 1)$ then

$$\frac{\tilde{C}_n - \tilde{C}_{n-1}}{\Delta \tilde{S}_n} = \phi_n$$

and we can calculate the hedging portfolio if we have an expression of C as a function of S.

The binomial model of Cox-Ross-Rubinstein (CRR)

Assume a model with one risky stock that evolves as:

$$S_n(\omega) = S_0(1+b)^{U_n(\omega)}(1+a)^{n-U_n(\omega)}$$

where

$$U_n(\omega) = \xi_1(\omega) + \xi_2(\omega) + \dots + \xi_n(\omega)$$

and where ξ_i are random variables wit values 0 or 1, that is Bernoulli random variables, and $-1 < a < r < b$:

$$\begin{array}{ccc}
n=0 & n=1 & n=2\dots \\
 & & S_0(1+b)^2 \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} \\
 & S_0(1+b) \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} & \\
S_0 \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} & & S_0(1+b)(1+a) \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} \\
 & S_0(1+a) \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix} & \\
 & & S_0(1+a)^2 \begin{smallmatrix}\nearrow \\ \searrow\end{smallmatrix}.
\end{array}$$

We can also write

$$S_n = S_{n-1}(1+b)^{\xi_n}(1+a)^{1-\xi_n(\omega)},$$

then

$$\tilde{S}_n = S_0 \left(\frac{1+b}{1+r}\right)^{U_n} \left(\frac{1+a}{1+r}\right)^{n-U_n} = \tilde{S}_{n-1} \left(\frac{1+b}{1+r}\right)^{\xi_n} \left(\frac{1+a}{1+r}\right)^{1-\xi_n}.$$

For $\tilde{S}_n$ to be a martingale with respect to $\mathbb{P}^*$ we need

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{S}_n|\mathcal{F}_{n-1}) = \tilde{S}_{n-1}$$

and if we take $\mathcal{F}_n = \sigma(S_0, S_1, ..., S_n)$ we have that the previous condition is equivalent to

$$\mathbb{E}_{\mathbb{P}^*}\left(\left(\frac{1+b}{1+r}\right)^{\xi_n}\left(\frac{1+a}{1+r}\right)^{1-\xi_n}\middle|\mathcal{F}_{n-1}\right) = 1$$

that is

$$\left(\frac{1+b}{1+r}\right)\mathbb{P}^*(\xi_n = 1|\mathcal{F}_{n-1}) + \left(\frac{1+a}{1+r}\right)\mathbb{P}^*(\xi_n = 0|\mathcal{F}_{n-1}) = 1$$

and consequently

$$\mathbb{P}^*(\xi_n = 1|\mathcal{F}_{n-1}) = \frac{r-a}{b-a},$$
$$\mathbb{P}^*(\xi_n = 0|\mathcal{F}_{n-1}) = 1 - \mathbb{P}^*(\xi_n = 1|\mathcal{F}_{n-1}) = \frac{b-r}{b-a}$$

Note that this conditional probability is deterministic and does not depend on n, so *under it* $\xi_i, i = 1, ..., N$ are independent, identically distributed random variables with common distribution Bernoulli(p), for $p = \frac{r-a}{b-a}$. $\mathbb{P}^*$ is unique as well, so the market is viable and complete. Therefore, under the neutral probability $\mathbb{P}^*$

$$
\begin{aligned}
S_N &= S_n(1+b)^{\xi_{n+1}+\ldots+\xi_N}(1+a)^{N-n-(\xi_{n+1}+\ldots+\xi_N)} \\
&= S_n(1+b)^{W_{n,N}}(1+a)^{N-n-W_{n,N}}
\end{aligned}
$$

with $W_{n,N} \sim$Bin$(N-n, p)$ independent of $S_n, S_{n-1}, ...S_1$.

Since we have the risk neutral probability we can, for instance, calculate the price of a *call option.* At time n, its price C_n is given by

$$
\begin{aligned}
C_n &= \mathbb{E}_{\mathbb{P}^*}\left(\left. \frac{(S_N - K)_+}{(1+r)^{N-n}} \right| \mathcal{F}_n \right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\left. \frac{(S_n(1+b)^{W_{n,N}}(1+a)^{N-n-W_{n,N}} - K)_+}{(1+r)^{N-n}} \right| \mathcal{F}_n \right) \\
&= \sum_{k=0}^{N-n} \frac{(S_n(1+b)^k(1+a)^{N-n-k} - K)_+}{(1+r)^{N-n}} \binom{N-n}{k} p^k (1-p)^{N-n-k} \\
&= S_n \sum_{k=k^*}^{N-n} \binom{N-n}{k} \frac{(p(1+b))^k((1-p)(1+a))^{N-n-k}}{(1+r)^{N-n}} \\
&\quad - K(1+r)^{n-N} \sum_{k=k^*}^{N-n} \binom{N-n}{k} p^k (1-p)^{N-n-k}
\end{aligned}
$$

where

$$k^* = \inf\{k, S_n(1+b)^k(1+a)^{N-n-k} > K\}$$
$$= \inf\left\{k, k > \frac{\log \frac{K}{S_n} - (N-n)\log(1+a)}{\log(\frac{1+b}{1+a})}\right\}$$

Note that

$$\frac{p(1+b)}{1+r} + \frac{(1-p)(1+a)}{1+r} = 1,$$

so, if we define

$$\bar{p} = \frac{p(1+b)}{1+r}$$

we can write

$$C_n = S_n \sum_{k=k^*}^{N-n} \binom{N-n}{k} \bar{p}^k (1-\bar{p})^{N-n-k}$$
$$- K(1+r)^{n-N} \sum_{k=k^*}^{N-n} \binom{N-n}{k} p^k (1-p)^{N-n-k}$$
$$= S_n \Pr\{\mathrm{Bin}(N-n, \bar{p}) \geq k^*\} - K(1+r)^{n-N} \Pr\{\mathrm{Bin}(N-n, p) \geq k^*\}$$

It can be seen that if we consider a sequence of CRR binomial models where the number of periods depends of N in a way that

$$1 + r_N = e^{\frac{rT}{N}},$$

$$1 + b_N = e^{\sigma\sqrt{\frac{T}{N}}},$$

$$1 + a_N = e^{-\sigma\sqrt{\frac{T}{N}}},$$

σ, $T > 0$ constants, we have, for $0 \le t < T$ that

$$C_t = \lim_{N\to\infty} C_{\left[\frac{t}{\frac{T}{N}}\right]} = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

where Φ is the c.d.f. of a standard normal distribution:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u$$

and

$$d_{\pm} = \frac{\log \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}},$$

this is the celebrated Black-Scholes formula.

Hedging portfolio in the CRR model

We have that

$$V_n = \phi_n^0 (1+r)^n + \phi_n^1 S_n.$$

Fixed S_{n-1}, S_n can take two value $S_n^u = S_{n-1}(1+b)$ ó $S_n^d = S_{n-1}(1+a)$ and analogously V_n. Then

$$\phi_n^1 = \frac{V_n^u - V_n^d}{S_{n-1}(b-a)}. \tag{1}$$

and

$$\phi_n^0 = \frac{V_n^u - \phi_n^1 S_n^u}{(1+r)^n}$$

In the case of a call, if we take $n = N$ we have:

$$\phi_N^1 = \frac{V_N^u - V_N^d}{S_{N-1}(b-a)} = \frac{(S_{N-1}(1+b) - K)_+ - (S_{N-1}(1+a) - K)_+}{S_{N-1}(b-a)}.$$

Now we can calculate by the self-financing condition the value of the portfolio at $N-1$: $V_{N-1} = \phi_N^0 (1+r)^{N-1} + \phi_N^1 S_{N-1}$ and from here ϕ_{N-1}^1 using (1) again.