FOURIER INTEGRALS

<u>Definition:</u> Given $f \in L^1(\mathbb{R})$, its Fourier transform is the function $\hat{f}: \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i t \xi}\, dt \qquad \xi \in \mathbb{R}.$$

Notice that this is well defined and bounded:

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(t)|\, dt = \|f\|_1$$

Thus $\|\hat{f}\|_\infty \leq \|f\|_1$

This can be roughly interpreted as the density of the frequency ξ in the signal $f(t)$.

<u>Remarks:</u> ① This can be seen as the limit as $T \to \infty$ of the Fourier coefficients for T-periodic functions, in the following sense. Assume $f \in \mathcal{C}^1(\mathbb{R})$, not necessarily periodic. For any $T > 0$ let f_T denote the T-periodic function in $\mathbb{R}$ coinciding with f on $[-T/2, T/2)$.

Then, for $|t| < T/2$,

$$f(t) = f_T(t) = \sum_{n \in \mathbb{Z}} \hat{f}_T(n)\, e^{i \frac{2\pi}{T} n t}$$

Hence

$$f(t) = \lim_{T\to\infty} \sum_{n\in\mathbb{Z}} \left(\frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i\frac{2\pi}{T}nx} dx \right) e^{i\frac{2\pi}{T}nt}$$

Let us try to identify, at least at a formal level, this limit. Let $\xi_n = \frac{n}{T}$, $n \in \mathbb{Z}$, and consider the partition of $\mathbb{R}$ provided by these nodes. The sum in the limit above becomes, written in these terms:

$$\sum_{n\in\mathbb{Z}} \left(\int_{-T/2}^{T/2} f(x) e^{-i2\pi\xi_n x} dx \right) e^{i2\pi\xi_n t} (\xi_{n+1} - \xi_n)$$

Letting $T \to \infty$ in the integral this turns into

$$\sum_{n\in\mathbb{Z}} \hat{f}(\xi_n) e^{2\pi i \xi_n t} (\xi_{n+1} - \xi_n),$$

which is a Riemann sum of the integral

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi t} d\xi .$$

Thus formally, we have the "inversion formula"

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi t} d\xi \qquad f \in \mathcal{C}_c(\mathbb{R}).$$

If we think in terms of signals and frequencies this is not surprising: any reasonable signal can be recovered if we know the densities of all frequencies (just by superposing all these frequencies).

② As in the case of Fourier series, the Fourier transform gives a decomposition of f. Then $\hat{f}(\xi)$ is sometimes called the "analysis" of f. Now, instead of only discrete frequencies we have a continuum. The reconstruction of f from $\hat{f}$ is called the "synthesis".

<u>Basic properties.</u> Assume $f \in L^1(\mathbb{R})$

1. The Fourier transform is <u>linear</u>: if $f, g \in L^1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$, then

$$(\alpha f + \beta g)^\wedge(\xi) = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi) \qquad \xi \in \mathbb{R}$$

2. <u>Conjugation</u>: $\quad (\bar{f})^\wedge(\xi) = \overline{\hat{f}(-\xi)} \qquad \xi \in \mathbb{R}$

3. <u>Translations</u>: Given $a \in \mathbb{R}$ let $\tau_a f(x) = f(x-a)$. Then

$$(\widehat{\tau_a f})(\xi) = \hat{f}(\xi)\, e^{-2\pi i a \xi} \qquad \xi \in \mathbb{R}.$$

④ <u>Modulations</u>: Given $a \in \mathbb{R}$ let $M_a f(x) = f(x) e^{2\pi i a x}$. Then

$$\widehat{M_a f}(\xi) = \tau_a \hat{f}(\xi) \qquad \xi \in \mathbb{R}.$$

⑤ <u>Dilations</u>. Let, for $\lambda > 0$, $D_\lambda f(x) = \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right)$.

Then $$\widehat{D_\lambda f}(\xi) = \hat{f}(\lambda \xi) \qquad \xi \in \mathbb{R}.$$

All these properties are straightforward from the definition.

We gather next some relevant properties of analytic type.

<u>Proposition:</u> Let $f \in L^1(\mathbb{R})$

① $\hat{f}$ is uniformly continuous and $|\hat{f}(\xi)| \le \|f\|_1$.

② If $f, f' \in L^1(\mathbb{R})$, then

$$\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi) \qquad \xi \in \mathbb{R}.$$

③ If $x f(x) \in L^1(\mathbb{R})$ then $\hat{f}$ is differentiable and

$$(-2\pi i x f)^\wedge = (\hat{f})'(\xi) \qquad \xi \in \mathbb{R}$$

④ <u>Riemann - Lebesgue lemma:</u> $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.

⑤ If $f, g \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} f(x)\, \hat{g}(x)\, dx = \int_{\mathbb{R}} \hat{f}(t)\, g(t)\, dt.$$

This is called the <u>multiplication formula</u>.

<u>Proof:</u> ① It only remains to see the uniform continuity. Given $\xi, h \in \mathbb{R}$, let

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(x)|\, |e^{-2\pi i x h} - 1|\, dx$$

Notice that

$$|f(x)|\,|e^{-2\pi i x h} - 1| \leq 2|f(x)|$$

independently of ξ. Then, by the dominated convergence theorem $|\hat{f}(\xi+h) - \hat{f}(\xi)| \xrightarrow{h \to 0} 0$ at a speed that is independent of ξ.

② Since $f \in L^1(\mathbb{R})$ there exist sequences $\{a_n\}_n \to -\infty$ and $\{b_n\}_n \to +\infty$ with $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = 0$. We can thus assume that $\lim_{x\to\pm\infty} f(x) = 0$.

Then, integrating by parts

$$\widehat{f'}(\xi) = \int_{\mathbb{R}} f'(x) e^{-2\pi i x \xi} \, dx =$$

$$= \left[f(x) e^{-2\pi i x \xi} \right]_{-\infty}^{\infty} + \int_{\mathbb{R}} f(x) (2\pi i \xi) e^{-2\pi i x \xi} \, dx$$

$$= 0 + (2\pi i \xi) \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx \quad .$$

③ That $\hat{f}$ is differentiable is a consequence of the theorem about differentiation under the integral. Then

$$(\hat{f})'(\xi) = \int_{\mathbb{R}} f(x)(-2\pi i x) e^{-2\pi i x \xi} \, dx \qquad \xi \in \mathbb{R}$$

④ We could proceed as in the case of Fourier series. Alternatively, multiply by $-1 = e^{i\pi}$ and change the variable

$$\hat{f}(\xi) = -\int_{\mathbb{R}} f(t) e^{-2\pi i \xi (t - \frac{1}{2\xi})} \, dt = \qquad \begin{matrix} t - \frac{1}{2\xi} = x \\ dt = dx \end{matrix}$$

$$= -\int_{\mathbb{R}} f(x + \tfrac{1}{2\xi}) e^{-2\pi i \xi x} \, dx .$$

Then

$$\hat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}} \left[f(x) - f\left(x + \frac{1}{2\xi}\right) \right] e^{-2\pi i x \xi} \, dx$$

and $$|\hat{f}(\xi)| \leq \frac{1}{2} \int_{\mathbb{R}} \left| f(x) - f\left(x + \frac{1}{2\xi}\right) \right| dx .$$

Here we state a Lemma that we will use very often. This will end the proof of ④.

Lemma: Let $f \in L^1(\mathbb{R})$ and let the translations $\tau_a f(t) = f(t-a)$, $a \in \mathbb{R}$. Then

$$\lim_{a \to 0} \| f - \tau_a f \|_1 = \lim_{a \to 0} \int_{\mathbb{R}} | f(t) - \tau_a f(t) | \, dt = 0 .$$

Note: As we shall see in the proof, the statement is also true if we replace the L^1 norm by any L^p, $1 \leq p < \infty$.

Proof: Assume first that $f \in \mathcal{C}_c(\mathbb{R})$ (continuous with compact support. Then there exist, $M, A > 0$ such that $|f| \leq M \chi_{I_A}$, where $I_A = [-A, A]$

For $|a|$ small we have also $|\tau_a f| \leq M \chi_{I_{2A}}$, and therefore $$|f(t) - \tau_a f(t)| \leq 2M \chi_{I_{2A}} \in L^1(\mathbb{R})$$

By the dominated convergence theorem

$$\lim_{a\to 0}\int_{\mathbb{R}} |f(t)-\tau_a f(t)|\,dt = \int_{\mathbb{R}} \lim_{a\to 0} |f(t)-\tau_a f(t)|\,dt = 0.$$

For general $f \in L^1(\mathbb{R})$ take a sequence $\{f_n\} \subseteq \mathcal{C}_c(\mathbb{R})$ such that $\|f_n - f\|_{L^1} \xrightarrow{n\to\infty} 0$. Then

$$\|f-\tau_a f\|_1 \le \|f-f_n\|_1 + \|f_n - \tau_a f_n\|_1 + \|\tau_a f_n - \tau_a f\|_1$$
$$= 2\|f-f_n\|_1 + \|f_n - \tau_a f_n\|_1$$

Given any $\varepsilon > 0$ take f_n such that $\|f-f_n\|_1 < \varepsilon$ and take $\delta > 0$ so that for $|a| < \delta$ then $\|f_n - \tau_a f_n\|_1 < \varepsilon$. Then $\|f-\tau_a f\|_1 < 3\varepsilon$, as desired.

5 By Fubini's theorem

$$\int_{\mathbb{R}} f(x) \int_{\mathbb{R}} g(t) e^{-2\pi i t x}\,dt\,dx = \int_{\mathbb{R}} g(t) \int_{\mathbb{R}} f(x) e^{-2\pi i t x}\,dx\,dt.$$

<u>Remark:</u> Properties 2 and 3 can be applied successively if more derivatives of f are integrable, or if more powers $x^m f(x)$ are integrable. Then, if $f, f', \ldots, f^{(k)} \in L^1(\mathbb{R})$, then

$$(f^{(k)})^\wedge(\xi) = (2\pi i \xi)^k \hat{f}(\xi).$$

In particular, if $P(D) = a_0 + a_1 \frac{\partial}{\partial t} + \cdots + a_n \frac{\partial^n}{\partial t^n}$ is a differential operator, then

$$(P(D)f)^\wedge(\xi) = P(2\pi i\xi)\hat{f}(\xi).$$

This is useful in solving some differential equations of the form $P(D)f = g$.

On the other hand, if f is compactly supported, then $x^m f(x) \in L^1(\mathbb{R})$ for all $m \geq 1$, and therefore $\hat{f} \in \mathcal{E}^\infty(\mathbb{R})$. A bit more can be said.

<u>Theorem:</u> If f has compact support $\hat{f}$ is analytic.

<u>Proof:</u> Assume that $\operatorname{supp} f \subseteq [-A, A]$, $A > 0$.

Then

$$\hat{f}(\xi) = \int_{-A}^{A} f(t) e^{-2\pi i t \xi} dt = \int_{-A}^{A} f(t) \sum_{n=0}^{\infty} \frac{(-2\pi i t \xi)^n}{n!} dt =$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-2\pi i)^n}{n!} \int_{-A}^{A} f(t) t^n dt \right] \xi^n =: \sum_{n=0}^{\infty} c_n \xi^n$$

where

$$|c_n| = \left| \frac{-(2\pi i)^n}{n!} \int_{-A}^{A} f(t) t^n dt \right| \leq \frac{(2\pi)^n}{n!} \int_{-A}^{A} |f(t)| |t|^n dt$$

$$\leq \frac{(2\pi A)^n}{n!} \|f\|_1 .$$

This is a simplified version of the Paley-Wiener theorem, that we shall comment soon when dealing with Shannon's formula.

Exercises: ① Let the Gaussian $G(t) = e^{-\pi t^2}$. Check that $G \in L^1(\mathbb{R})$ and prove that $\hat{G}(\xi) = G(\xi)$, $\xi \in \mathbb{R}$. Hint: prove that both G and $\hat{G}$ solve the differential equation $\begin{cases} f'(x) = -2\pi x f(x) \\ f(0) = 1 \end{cases}$

② Let $f = \chi_{[-1/2, 1/2]}$. Show that $\hat{f}(\xi) = \dfrac{\sin(\pi\xi)}{\pi\xi}$ (cardinal sine).

More generally, compute the Fourier transform of $\chi_{[-a,a]}$, $a > 0$.

③ Let $f(t) = e^{-2\pi|t|}$. Check that $f \in L^1(\mathbb{R})$ and prove that

$$\hat{f}(\xi) = \frac{1}{\pi} \frac{1}{1+\xi^2} \quad \text{(Poisson Kernel)}$$