The continuous wavelet transform:

All we have seen so far about wavelets can be seen as a discretisation of a continuous "wavelet transform".

The general idea is similar to what we saw in the Fourier transform, but instead of using a sliding window here the window is translated and streched.

Consider $\psi \in L^2(\mathbb{R})$ (the mother wavelet) such that:

ⓐ $\|\psi\|_2 = 1$

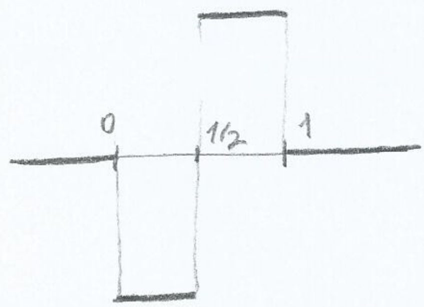
ⓑ ψ compactly supported o rapidly decaying

ⓒ $\int_{\mathbb{R}} \psi(t)\, dt = 0$

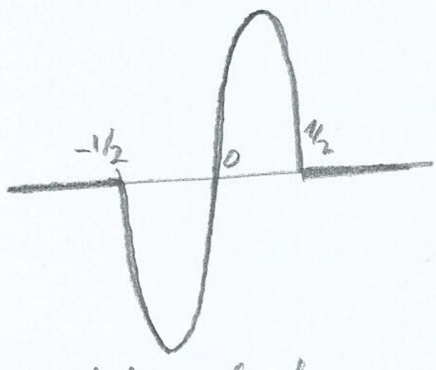
In applications it is usually important that ψ is regular (continuous or continuously differentiable).

<u>Examples:</u> 1. Haar wavelet

$$\psi_H(t) = \begin{cases} -1 & 0 \le t < 1/2 \\ 1 & 1/2 \le t < 1 \\ 0 & t \notin [0,1) \end{cases}$$



2. $\psi_S(t) = \sqrt{2} \sin(2\pi t)\, \chi_{(-1/2, 1/2)}(t)$

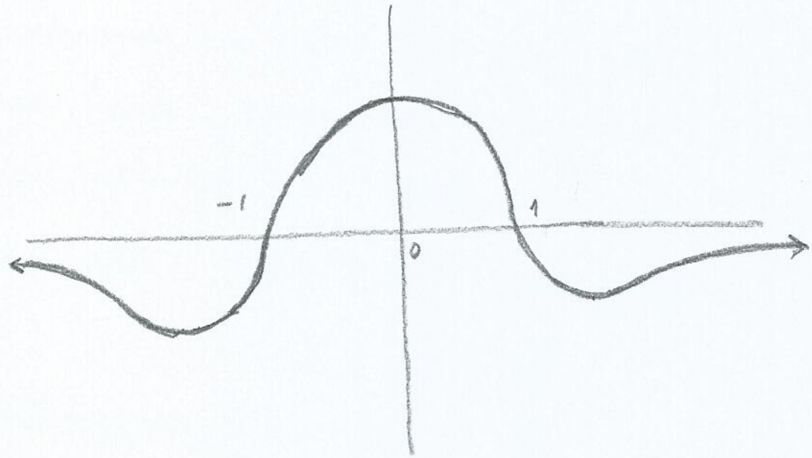


3. Shannon wavelet $\psi \in L^2(\mathbb{R})$ defined by

$$\hat{\psi}(\xi) = e^{\pi i \xi}\, \chi_{[-1,-1/2)\cup(1/2,1]}(\xi)$$

4. Ricker wavelet (Mexican hat wavelet). This is the normalised second derivative of the Gaussian.

$$\psi(t) = \frac{2}{\sqrt{3}\,\pi^{1/4}} (1-t^2) e^{-t^2/2}$$



As we said, the general idea is not only to translate ψ around, as we did in the STFT, but also to scale it to give streched (or squeezed) versions of the original ψ with the same basic shape, but a different scale of frequency. Given $a>0$ and $b \in \mathbb{R}$ define

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$$

This is a single wavelet with support translated by b and streched a times the previous length. If we regard ψ as a single "cycle" of a wave-like function, then $\psi_{a,b}$ has a "frequency" that is $1/a$ times the original. The factor $1/\sqrt{a}$ is just so that $\|\psi_{a,b}\|_2 = 1$

Def: The <u>continuous wavelet transform</u> of $f \in L^2(\mathbb{R})$ corresponding to the wavelet ψ is

$$Wf(a,b) = \langle f, \psi_{a,b} \rangle = \int_{\mathbb{R}} f(t) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{t-b}{a}\right)}\, dt \qquad \begin{matrix} a>0 \\ b\in\mathbb{R} \end{matrix}$$

The key to all this is that one can choose ψ so that $Wf(a,b)$ contains enough information to reconstruct f. We have already seen this with wavelet bases of a MRA, where a system $\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$, $n,k \in \mathbb{Z}$ is an orthonormal basis of $L^2(\mathbb{R})$, and therefore for $f \in L^2(\mathbb{R})$

$$f(t) = \sum_{n,k\in\mathbb{Z}} \langle f, \psi_{n,k}\rangle \psi_{n,k}(t) = \sum_{n,k\in\mathbb{Z}} Wf(2^{-n}, 2^{-n}k)\,\psi_{n,k}(t).$$

As in the Fourier and STF transforms, one can recover $f \in L^2(\mathbb{R})$ from the wavelet transform.

<u>Theorem:</u> (Inverse wavelet transform)

If $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a real-valued function satisfying the admissibility condition $c_\psi := \int_0^\infty \frac{|\hat\psi(\xi)|^2}{|\xi|}\,d\xi < \infty$, then, for $f \in L^2(\mathbb{R})$:

$$f(t) = \frac{1}{c_\psi}\int_{\mathbb{R}}\int_0^\infty Wf(a,b)\,\psi_{a,b}(t)\,\frac{da}{a^2}\,db$$

$$\|f\|_2 = \sqrt{c_\psi}\left(\int_{\mathbb{R}}\int_0^\infty |Wf(a,b)|^2\,\frac{da}{a^2}\,db\right)^{1/2}.$$

<u>Example:</u> For the Haar wavelet:

$$\hat\psi(\xi) = -\int_0^{1/2} e^{-2\pi i\xi t}\,dt + \int_{1/2}^1 e^{-2\pi i\xi t}\,dt = \frac{1}{2\pi i\xi}\left(e^{-\pi i\xi} - 1 + e^{-\pi i\xi} - e^{-2\pi i\xi}\right)$$

$$= -\frac{(1-e^{-\pi i\xi})^2}{2\pi i\xi} = -\frac{\left[e^{-i\frac{\pi}{2}\xi}\left(e^{i\frac{\pi}{2}\xi} - e^{-i\frac{\pi}{2}\xi}\right)\right]^2}{2\pi i\xi} = i e^{-i\pi\xi}\,\frac{\sin^2(\frac{\pi}{2}\xi)}{\frac{\pi}{2}\xi},$$

so ψ is admissible:

$$c_\psi = \int_0^\infty \frac{\sin^4(\frac{\pi}{2}\xi)}{\frac{\pi^2}{4}\xi^3}\,d\xi < +\infty$$