Lesson 14

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Stochastic volatility

Suppose that under $\mathbb{P}$

$$dS_t = S_t(\mu_t dt + \sigma(W_t^2, t)dW_t^1)$$

where W^1 and W^2 are two independent Brownian motions, we also assume an interest rate r (deterministic) and no dividends. Then (we take $\sigma(W_t^2, t) \geq K > 0$)

$$\begin{aligned} d\tilde{S}_t &= \sigma(W_t^2, t)\tilde{S}_t \left(\frac{\mu_t - r_t}{\sigma(W_t^2, t)} dt + dW_t^1 \right) \\ &= \sigma(W_t^2, t)\tilde{S}_t dW_t^* \end{aligned}$$

(observe that then

$$dS_t = S_t(r_t dt + \sigma(W_t^2, t)dW_t^*)$$

) with

$$dW_t^* = \frac{\mu_t - r_t}{\sigma(W_t^2, t)} dt + dW_t^1$$

and if we consider the probability measure (provided it is well defined)

$$\mathrm{d}\mathbb{P}^* = \exp\left\{ -\int_0^T \frac{\mu_t - r_t}{\sigma(W_t^2, t)} \mathrm{d}W_t^1 - \frac{1}{2}\int_0^T \left(\frac{\mu_t - r_t}{\sigma(W_t^2, t)} \right)^2 \mathrm{d}t \right\} \mathrm{d}\mathbb{P}.$$

W^* will be a BM (independent of W^2) and $\tilde{S}$ will be a $\mathbb{P}^*$-local martingale (w.r.t. the filtration generated by $\left(W^*, W^2\right)$), consequently admissible portfolios will be $\mathbb{P}^*$-local martingales, and the model free of arbitrage.

However the martingales are $\left(W^*, W^2\right)$-martingales with representation

$$M_t = M_0 + \int_0^t Y_s^1 \mathrm{d}W_s^* + \int_0^t Y_s^2 \mathrm{d}W_s^2$$

but we cannot write, in general, $\mathrm{d}W_t^*$ and $\mathrm{d}W_t^2$ in terms of $\mathrm{d}\tilde{S}_t$, so the model is incomplete.

Nevertheless notice that $\sigma(W_\cdot^2, \cdot)$ is adapted to the filtration generated by S :

$$\int_0^\cdot \sigma^2(W_s^2, s)\mathrm{d}s = \mathbb{P}\text{-}\lim_{\pi \to 0} \sum_{i=1}^n \left(\log S_{t_i} - \log S_{t_i - 1}\right)^2$$

and consequently W^* is also S-adapted.

Notice that, when modelling prices by stochastic volatility models, $\mathbb{P}^*$ is part of the pricing model. For instance we can assume that under $\mathbb{P}^*$

$$\mathrm{d}S_t = S_t(r_t \mathrm{d}t + \sigma(W_t^2, t)\mathrm{d}W_t^1)$$

W^1, W^2 independent Brownian motions.
Then the price of a call option with strike K is given by

$$\begin{aligned} C_t &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)_+ \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)_+ \middle| \sigma(W_s^2, s), t \le s \le T, \mathcal{F}_t \right) \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(S_t \Phi(d_+) - K e^{-\int_t^T r_s \mathrm{d}s} \Phi(d_-) \middle| \mathcal{F}_t \right), \end{aligned}$$

with

$$d_\pm = \frac{\log \frac{S_t}{K} + \int_t^T (r_s \pm \frac{1}{2}\sigma^2(W_s^2, s))\mathrm{d}s}{\sqrt{\int_t^T \sigma^2(W_s^2, s)\mathrm{d}s}},$$

Note that if (S, σ) is a Markovian process,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)_+ \middle| \mathcal{F}_t\right) &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)_+ \middle| S_t, \sigma_t\right) \\ &= C(t, S_t, \sigma_t^2) = C(t, S_t, v_t).\end{aligned}$$

where $v_t := \sigma_t^2$. This is the case for the Heston model. It is assumed that v is the solution of the SDE

$$\mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \delta\sqrt{v_t}\mathrm{d}W_t^2$$

with κ, θ and δ positive parameters satisfying $2\kappa\theta \geq \delta^2$. Therefore

$$\mathrm{d}S_t = S_t(r_t\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_t^1).$$

Now if we assume that the function $C(S_t, v_t, t)$ is smooth enough to apply the Itô formula, we have that

$$\begin{aligned} \mathrm{d}C(t, S_t, v_t) &= \partial_t C\mathrm{d}t + \partial_S C\mathrm{d}S_t + \partial_v C\mathrm{d}v_t \\ &\quad + \frac{1}{2} v_t S_t^2 \partial_{SS} C\mathrm{d}t + \frac{1}{2} v\delta^2 \partial_{vv} C\mathrm{d}t, \end{aligned}$$

that is

$$\begin{aligned} \mathrm{d}C(t, S_t, v_t) &= \partial_S C\mathrm{d}S_t + \delta\sqrt{v_t}\partial_v C\mathrm{d}W_t^2 + \kappa(\theta - v_t)\partial_v C\mathrm{d}t \\ &\quad + \frac{1}{2} v_t S_t^2 \partial_{SS} C\mathrm{d}t + \frac{1}{2} v\delta^2 \partial_{vv} C\mathrm{d}t + \partial_t C\mathrm{d}t \end{aligned}$$

if we take $r = 0$, for simplicity, we have that $C(\cdot, S_\cdot, v_\cdot)$ and S are martingales, then identifying the martingale and the absolutely continuous part in the previous equation we obtain that

$$\frac{1}{2}v_t S_t^2 \partial_{SS} C + \frac{1}{2} v \delta^2 \partial_{vv} C + \kappa(\theta - v_t)\partial_v C + \partial_t C = 0$$

If $r_t \neq 0$ we have that $\tilde{C}(\cdot, S_\cdot, v_\cdot)$ and $\tilde{S}$ are martingales we will get that

$$\frac{1}{2}v_t S_t^2 \partial_{SS} C + \frac{1}{2} v_t \delta^2 \partial_{vv} C + \kappa(\theta - v_t)\partial_v C + \partial_t C - r_t C + r_t S_t \partial_S C = 0,$$

with the boundary condition

$$C(T, S_T, v_T) = (S_T - K)_+.$$

If we assume that W_t^1, W_t^2 are Brownian motions w.r.t. the same filtration $(\mathcal{F}_t)$ with quadratic covariation $\int_0^t \rho_s \mathrm{d}s$, we obtain

$$\mathbb{E}\left(S_t \xi_t \Phi(d_+) - K e^{-\int_t^T r_s \mathrm{d}s} \Phi(d_-) \middle| \mathcal{F}_t \right),$$

with

$$d_\pm = \frac{\log \frac{S_t \xi_t}{K} + \int_t^T (r_s \pm \frac{1}{2}(1-\rho_s^2)\sigma^2(W_s^2, s))\mathrm{d}s}{\sqrt{\int_t^T (1-\rho_s^2)\sigma^2(W_s^2, s))\mathrm{d}s}},$$

and

$$\xi_t = \exp\left\{ \int_t^T \rho_s \sigma(W_s^2, s) \mathrm{d}W_s^2 - \frac{1}{2}\int_t^T \rho_s^2 \sigma^2(W_s^2, s)\mathrm{d}s \right\}.$$

In fact, first note that we can write

$$dW_t^1 = \sqrt{1-\rho_t^2}\,d\hat{W}_t + \rho_t dW_t^2,$$

where $\hat{W}$ is a Brownian motion independent of W^2. Therefore we have

$$dS_t = S_t\left(r dt + \sigma(W_t^2, t)\left(\sqrt{1-\rho_t^2}\,d\hat{W}_t + \rho_t dW_t^2\right)\right)$$

and by the Itô formula:

$$\begin{aligned} S_T &= S_t \exp\left\{\int_t^T r_s ds + \int_t^T \rho_s \sigma(W_s^2, s) dW_s^2 - \frac{1}{2}\int_t^T \rho_s^2 \sigma^2(W_s^2, s) ds\right\} \\ &\times \exp\left\{\int_t^T \sqrt{1-\rho_s^2}\,\sigma(W_s^2, s) m d\hat{W}_s - \frac{1}{2}\int_t^T (1-\rho_s^2)\sigma^2(W_s^2, s) ds\right\}. \end{aligned}$$

Now we can proceed as in the previous case.

Local volatility models

In these models, under a risk neutral measure, say $\mathbb{P}^*$,

$$\mathrm{d}S_t = S_t(r_t\mathrm{d}t + \sigma(S_t, t)\mathrm{d}W_t),$$

then under mild conditions on $\sigma(x, t)$ we can have a solution of this SDE and consequently we have that S and W generate the same filtration. Then discounted prices are Brownian (local) martingales and any integrable positive payoff (w.r.t. $\mathbb{P}^*$) is replicable by an admissible portfolio and the model is complete.
Note that in stochastic volatility models

$$\mathrm{d}S_t = S_t(r_t\mathrm{d}t + \sigma(S_s, 0 \leq s \leq t)\mathrm{d}W_t),$$

for certain functional $\sigma(\cdot)$. Then why these models are not complete?

Theorem

Every Itô's martingale X with quadratic variation $\langle X, X\rangle_t = t$, is a Brownian motion.

Proof.

By the Itô formula

$$e^{i\lambda X_t} = e^{i\lambda X_u} + i\lambda \int_u^t e^{i\lambda X_s} \mathrm{d}X_s - \frac{\lambda^2}{2} \int_u^t e^{i\lambda X_s} \mathrm{d}s.$$

Consequently

$$\mathbb{E}(e^{i\lambda(X_t - X_u)}|\mathcal{F}_u) = 1 - \frac{\lambda^2}{2} \int_u^t \mathbb{E}(e^{i\lambda(X_s - X_u)}|\mathcal{F}_u)\mathrm{d}s$$

and

$$\mathbb{E}(e^{i\lambda(X_t - X_u)}|\mathcal{F}_u) = e^{-\frac{1}{2}\lambda^2(t-u)}.$$

Hence X has continuous trajectories, with independent and homogeneous increments with law $\mathrm{N}(0, t)$. In other words, X is a Brownian motion. □

Theorem

Two Brownian motions W^1,W^2, with respect to the same filtration, are independent if and only if $\langle W^1, W^2\rangle_t = 0$.

Proof.

If W^1,W^2 are independent Brownian motions then

$$W := \left(W^1 + W^2\right) / \sqrt{2}$$

is a Brownian motion and

$$\begin{aligned} t &= \langle W, W\rangle_t = \frac{1}{2}\langle W^1, W^1\rangle_t + \frac{1}{2}\langle W^2, W^2\rangle_t + \langle W^1, W^2\rangle_t \\ &= t + \langle W^1, W^2\rangle_t. \end{aligned}$$

□

Proof.

Let $A \in \sigma(W_t^1, 0 \leq t \leq T)$ and $B \in \sigma(W_t^2, 0 \leq t \leq T)$. By the Representation Theorem for Brownian random variables, there exist processes a and b such that

$$\mathbf{1}_A = \mathbb{E}\left(\mathbf{1}_A\right) + \int_0^T a_s \mathrm{d}W_s^1, \quad \mathbf{1}_B = \mathbb{E}\left(\mathbf{1}_B\right) + \int_0^T b_s \mathrm{d}W_s^2.$$

Define

$$X_t := \mathbb{E}\left(\mathbf{1}_A\right) + \int_0^t a_s \mathrm{d}W_s^1, \quad Y_t := \mathbb{E}\left(\mathbf{1}_B\right) + \int_0^t b_s \mathrm{d}W_s^2,$$

□

Proof.

then, by the Itô formula for Itô's processes (w.r.t. the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma(W_s^1, W_s^2, 0 \leq s \leq t,))$

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X_s \mathrm{d} Y_s + \int_0^t Y_s \mathrm{d} X_s + \int_0^T a_s b_s \mathrm{d} \langle W^1, W^2 \rangle_s \\ &= \mathbb{E}\left(\mathbf{1}_A\right) \mathbb{E}\left(\mathbf{1}_B\right) + \int_0^t b_s X_s \mathrm{d} W_s^2 + \int_0^t a_s Y_s \mathrm{d} W_s^1 . \end{aligned}$$

Finally

$$\mathbb{E}\left(\mathbf{1}_A \mathbf{1}_B\right) = \mathbb{E}\left(X_T Y_T\right) = \mathbb{E}\left(\mathbf{1}_A\right) \mathbb{E}\left(\mathbf{1}_B\right).$$

So A and B are independent. □