Resonances and linearization: case of maps

Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be a linear map, and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues (they may be repeated). We will use multiindex notation, and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n, \quad |m| = |m|_1 = m_1 + \cdots + m_n.$$

Definition A resonance is a relation

$$\lambda_j = \lambda_1^{m_1} \cdot \lambda_2^{m_2} \cdots \lambda_n^{m_n} = \lambda^m, \qquad m \in \mathbb{Z}_+^n, \quad |m| \geq 2.$$

In such case we say that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is resonant.

- $|m|$ is called the order of the resonance,
- If $\lambda_j = \lambda^m$ is a resonance, we say that $x^m e_j = x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n} e_j$ is a resonant monomial.

Some examples in low dimensions:

Example $\boxed{n = 1}$ The resonance condition is written $\lambda = \lambda^m \;\; \Leftrightarrow \;\; \lambda(\lambda^{m-1} - 1) = 0$. The condition is satisfied if and only if

$$\begin{cases} \lambda = 0, & \text{or} \\ \lambda \;\; \text{is a } m-1 \text{ root of } 1. \end{cases}$$

Example $\boxed{n=2}$ There are a lot of possibilities. Let for instance

$$\lambda_1 = 2, \qquad \lambda_2 = \frac{1}{4}.$$

First resonance relation. The equation is

$$2 = 2^{m_1}\left(\frac{1}{4}\right)^{m_2} = 2^{m_1 - 2m_2}.$$

This implies $m_1 - 2m_2 = 1 \Leftrightarrow m_1 = 1 + 2m_2$, and then the resonances are given by the pairs

$$\left\{(m_1, m_2) \in \mathbb{Z}_+^2 \mid \ m_1 = 1 + 2m_2, \ |m| \geq 2\right\} = \{(3,1),(5,2),(7,3),\dots\}.$$

Second resonance relation. The equation is

$$\frac{1}{4} = 2^{m_1}\left(\frac{1}{4}\right)^{m_2} = 2^{m_1 - 2m_2}.$$

Then $m_1 - 2m_2 = -2 \Leftrightarrow m_1 = -2 + 2m_2$ and the resonant pairs are

$$\left\{(m_1, m_2) \in \mathbb{Z}_+^2 \mid \ m_1 = -2 + 2m_2, \ |m| \geq 2\right\} = \{(2,2),(4,3),(6,4),\dots\}.$$

Definition

(a) We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to the Poincaré domain if either $|\lambda_j| < 1$ for all j or $|\lambda_j| > 1$ for all j.

(b) We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to the Siegel Domain otherwise.

In the case $n = 2$, we can easily split the plane into the union of Siegel's and Poincaré's domains as shown in Figure 1.

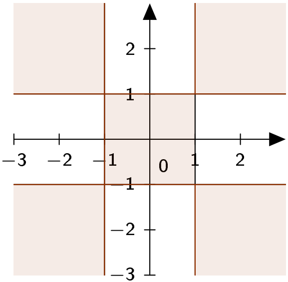

Figure: Poincaré and Siegel domains in the (λ_1, λ_2) plane, when $\lambda_1, \lambda_2 \in \mathbb{R}$.

Observation If λ is in the Poincaré domain, it has at most finitely many resonances.
Indeed, assume we are in the contractive case. Let

$$s = \min |\lambda_i|, \qquad r = \max |\lambda_i| < 1.$$

We have

$$s \leq |\lambda_j| \qquad \text{and} \qquad |\lambda^m| \leq r^{m_1} r^{m_2} \cdots r^{m_n} = r^{|m|}.$$

If $\lambda_j = \lambda^m$ then

$$s \leq |\lambda_j| = |\lambda^m| \leq r^{|m|}.$$

Therefore resonances can only occur for values of m such that $|m|$ satisfies

$$|m| \leq \frac{\log s}{\log r}.$$

This condition is only satisfied by a finite number of m's.

Geometry of resonances

Assume for the moment that $n = 2$ and both eigenvalues are real. In the first quadrant of the (λ_1, λ_2) plane, we shall look for couples of eigenvalues which are resonant. The two possible resonance conditions are

$$\lambda_1 = \lambda_1^{m_1}\lambda_2^{m_2}, \tag{1}$$

$$\lambda_2 = \lambda_1^{m_1}\lambda_2^{m_2}. \tag{2}$$

Study of (1). This can be easily done by considering 4 cases,

- ⋆ If $m_1 = 0$ and $m_2 \geq 2$

$$\lambda_1 = \lambda_2^{m_2} \qquad \Leftrightarrow \qquad \boxed{\lambda_2 = (\lambda_1)^{1/m_2}} \tag{3}$$

- ⋆ If $m_1 = 1$ and $m_2 \geq 1$
  (1) $\Leftrightarrow$ $\lambda_1 = \lambda_1\lambda_2^{m_2}$ $\Leftrightarrow$ $\boxed{\lambda_1 = 0 \text{ or } \lambda_2 = 1}$.
- ⋆ If $m_1 \geq 2$ and $m_2 = 0$
  (1) $\Leftrightarrow$ $\lambda_1 = \lambda_1^{m_1}$ $\Leftrightarrow$ $\boxed{\lambda_1 = 0 \text{ or } \lambda_1 = 1}$.
- ⋆ If $m_1 \geq 2$ and $m_2 \geq 1$ we have (see Figure 2)

$$\lambda_1 = \lambda_1^{m_1}\lambda_2^{m_2} \quad \Leftrightarrow \quad \boxed{\lambda_2 = \left(\frac{1}{\lambda_1^{m_1-1}}\right)^{1/m_2} = \lambda_1^{-p/q}} \quad \frac{p}{q} \in \mathbb{Q}_+ \setminus \{0\} \tag{4}$$

The study of (2) leads to similar results.

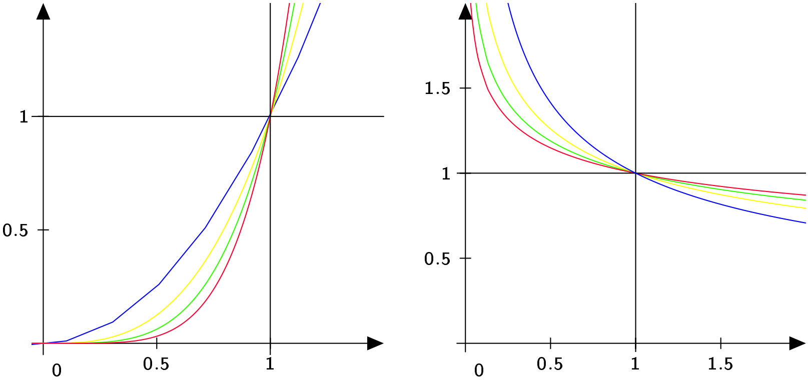

Figure: Left: Resonant pairs of real eigenvalues (3). They live always in the Poincaré domain. Here the first axis is λ_2 and the secons axis is λ_1. Right: Resonant pairs of real eigenvalues (4). They live always in the Siegel domain. Here the first axis is λ_1 and the secons axis is λ_2

Observation Resonances are dense in the Siegel domain. Indeed, let $(x, y) \in \mathcal{S}$, with $y > 1$ and $0 < x < 1$. Then there exists $\alpha > 0$ such that

$$y = x^{-\alpha}, \qquad \left(\alpha = \frac{\log y}{\log x}\right).$$

Next we can approximate α by a rational number p/q. Then (x, y) would be near a resonant couple (λ_1, λ_2) with $\lambda_2 = \lambda_1^{-p/q}$. For instance
$\lambda_1 = x,\ \lambda_2 = x^{-p/q}$.

Now we can state the two main theorems on linearization, in the analytic case.

Theorem [Poincaré] Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be analytic with $f(0) = 0$. Let $A = Df(0)$. If the set of eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$ of A is in the Poincaré domain and it is non-resonant, then f is locally analytically conjugate to A.

Definition We say that $(\lambda_1, \ldots, \lambda_n)$ satisfies a Diophantine condition $DC(c, \tau)$ with $\tau > 0$ and $c > 0$ if

$$|\lambda_j - \lambda^m| \geq \frac{c}{|m|^\tau}, \quad m \in \mathbb{Z}_+^n, \quad |m| \geq 2, \quad j = 1, \ldots, n. \tag{5}$$

Theorem [Siegel] If $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is analytic, $0 \in U$, $f(0) = 0$, $A = Df(0)$. If the set of eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ of A is in the Siegel domain and satisfies a Diophantine condition $DC(c, \tau)$. then f is locally analytically conjugate to A.

Observation If $\tau > (n-1)$, then

$$\left\{ \lambda \in \mathbb{R}^n \mid \text{ exists } c \text{ such that } |\lambda_j - \lambda^m| \geq \frac{c}{|m|^\tau}, \, \forall m \in \mathbb{Z}_+^n, \, |m| \geq 2, \, j = 1, \ldots, n \right\}$$

is a full measure set.

Normal Forms for maps

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0$. Assume $f \in \mathcal{C}^r$. We can write its Taylor expansion as

$$f(x) = Ax + f_2(x) + f_3(x) + \cdots + f_r(x) + o(\|x\|^r),$$

where $f_k(x)$ is a homogeneous polynomial of n variables of degree k. Write also

$$E_k = \{\text{ homogeneous polynomials of degree } k \text{ in } n \text{ variables and } n \text{ components }\},$$

which is a vector space. A basis of such space is given by

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} e_j = x^m e_j, \quad |m| = k, \quad 1 \le j \le n.$$

Example

If $n = 2$, E_2 is given by

$$E_2 = \left\{ \begin{pmatrix} a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 \\ b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2 \end{pmatrix} \;\middle|\; a_{i,j}, b_{i,j} \in \mathbb{R} \right\}$$

Theorem Under the previous conditions, there exists a polynomial change of variables h with $h(0) = 0$ and $Dh(0) = \mathrm{Id}$ such that $\tilde{f} = h^{-1} \circ f \circ h$ is of the form

$$\tilde{f}(y) = Ay + f_2^{(r)}(y) + f_3^{(r)}(y) + \cdots + f_r^{(r)}(y) + o(\|y\|^r),$$

where $f_k^{(r)}(y) = \sum_{\substack{|m|=k \\ \lambda_j=\lambda^m}} c_{m,j} y^m e_j$. In particular, if $\lambda_j \neq \lambda^m$ for all $m \in \mathbb{Z}_+^n$, with $2 \leq |m| \leq r$, $j = 1, \ldots, m$, then

$$\tilde{f}(y) = Ay + o(\|y\|^r).$$

Before proceeding with the proof, we shall give a preliminary result on inverses of near the identity polynomial changes of variables.

Lemma Let $\tilde{h}_k(y) = y + h_k(y)$ with $h_k \in E_k$. Then $\tilde{h}_k$ is invertible and its inverse has the form

$$\tilde{h}_k^{-1}(y) = y - h_k(y) + Dh_k(y)h_k(y) + O(\|y\|^{2k}).$$

Proof of the lemma Since $D\tilde{h}_k(0) = \mathrm{Id}$, the inverse function theorem ensures the existence of $\tilde{h}_k^{-1}$, which is such that

$$\tilde{h}_k^{-1}(0) = 0, \qquad D\tilde{h}_k^{-1}(0) = \mathrm{Id}, \qquad \tilde{h}_k^{-1} \text{ is analytic.}$$

We use the following notation for its Taylor expansion around the origin

$$\tilde{h}_k^{-1}(y) = y + g_2(y) + g_3(y) + \cdots .$$

Technical remark

$$h_k(y + g_2(y) + \dots) = h_k(y) + Dh_k(y)g_2(y) + \cdots = h_k(y) + O(y^{k+1}).$$

Truncating it to order k and imposing $y = \tilde{h}_k \circ \tilde{h}_k^{-1}(y)$ we obtain

$$\begin{aligned}
y &= \tilde{h}_k^{-1}(y) + h_k(\tilde{h}_k^{-1}(y)) + \dots \\
&= y + g_2(y) + g_3(y) + \cdots + O(\|y\|^{k+1}) + h_k(y + g_2(y) + \cdots) + \dots \\
&= y + g_2(y) + g_3(y) + \cdots + g_k(y) + O(\|y\|^{k+1}) + h_k(y) + O(\|y\|^{k+1}).
\end{aligned}$$

Equating terms of the same order yields

$$g_2(y) = g_3(y) = \cdots = g_{k-1}(y) = 0, \quad \text{and} \quad g_k(y) + h_k(y) = 0. \qquad (6)$$

Let us return to $y = \tilde{h}_k \circ \tilde{h}_k^{-1}(y)$. Consider the expansion of $\tilde{h}_k^{-1}$ up to order $2k-1$,

$$\begin{aligned}
y &= y + g_k(y) + \cdots + g_{2k-1}(y) + O(\|y\|^{2k}) + h_k(y + g_k(y) + \cdots) \\
&= y + g_k(y) + \cdots + g_{2k-1}(y) + O(\|y\|^{2k}) + h_k(y) \\
&\quad + Dh_k(y)\,[g_k(y) + g_{k+1}(y) + \cdots] + \frac{1}{2} D^2 h_k(y)\,[g_k(y) + g_{k+1}(y) + \cdots]^2 \\
&= y + g_k(y) + \cdots + g_{2k-1}(y) + O(\|y\|^{2k}) + h_k(y) + Dh_k(y) g_k(y) \\
&\quad + O(\|y\|^{2k}) + O(\|y\|^{3k-2})
\end{aligned}$$

and again collecting terms of the same order in both sides we obtain the relations

$$g_{k+1}(y) = \cdots = g_{2k-2}(y) = 0, \quad \text{and} \quad g_{2k-1}(y) + Dh_k(y) g_k(y) = 0. \quad (7)$$

Finally, from (6) and (7) we obtain

$$\tilde{h}_k^{-1}(y) = y - h_k(y) + Dh_k(y) h_k(y) + \cdots$$

$\square$

Proof of the main Theorem

Let us restrict ourselves to the case that A diagonalizes, and assume that the linear part is already in diagonal form.
We make a polynomial change of variables $\tilde{h}_k(y) = y + h_k(y)$, as the one in the lemma, to f, expressed as a Taylor polynomial of order r plus a remainder.
We get

$$
\begin{aligned}
& \tilde{h}_k^{-1} \circ f \circ \tilde{h}_k = \\
= \quad & \tilde{h}_k^{-1} \left(A(y + h_k(y)) + \overbrace{f_2(y + h_k(y))}^{f_2(y)+O(\|y\|^{k+1})} + \cdots + \overbrace{f_k(y + h_k(y))}^{f_k(y)+O(\|y\|^{2k-1})} + o(\|y\|^k) \right) \\
= \quad & \tilde{h}_k^{-1} \left(Ay + f_2(y) + \cdots + f_{k-1}(y) + f_k(y) + Ah_k(y) + o(\|y\|^k) \right) \\
= \quad & Ay + f_2(y) + \cdots + f_{k-1}(y) + \boxed{f_k(y) + Ah_k(y) - h_k(Ay)} + o(\|y\|^k)
\end{aligned}
$$

so the first terms in the expansion modified by the change of variables $\tilde{h}_k$ are the ones of order k, i.e. it does not change terms of order less than k.
Then, we want to design a sequence of changes such that they modify (and reduce to only resonant monomials) terms of order $2, 3, 4 \ldots$

We define the operator

$$
\begin{array}{rccl}
L_k : & E_k & \to & E_k \\
 & h & \mapsto & (L_k h)(x) = h(Ax) - Ah(x)
\end{array}
$$

which is linear between finite dimensional vector spaces. Note that when making the change of variables $\tilde{h}_k$, the first modified term is changed to

$$f_k(y) \to f_k(y) - (L_k h)(y).$$

We are assuming that A is diagonal: $A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then, the vectors of the canonical basis are eigenvectors of A.

Lemma

The set

$$\{x^m e_j \mid |m| = k,\ 1 \leq j \leq m\}$$

is a basis of eigenvectors of L_k, and the eigenvalue of the eigenvector $x^m e_j$ is $\lambda^m - \lambda_j$.

Proof. Let $x^m e_j$ with $|m| = k$.

$$
\begin{aligned}
L_k(x^m e_j) &= (\lambda_1 x_1)^{m_1} \cdots (\lambda_n x_n)^{m_n} e_j - \mathrm{diag}(\lambda_1, \ldots, \lambda_n) x^m e_j \\
&= \lambda^m x^m e_j - \lambda_j x^m e_j = (\lambda^m - \lambda_j) x^m e_j.
\end{aligned}
$$

$\square$

With all this, we can write

$$E_k = \underbrace{\mathrm{Ker}\, L_k}_{\substack{\text{generated by}\\ \text{resonant}\\ \text{monomials}}} \oplus \mathrm{Im} L_k$$

If we decompose $f_k = f_k^{(r)} + f_k^{(nr)} \in \mathrm{Ker} L_k \oplus \mathrm{Im} L_k$ and choose h_k such that $L_k h_k = f_k^{(nr)}$ only the resonant part of the f_k would be left.
Hence the composition of such changes of orders $2, 3, 4, \ldots$ leads to the desired change, which ends the proof of the theorem.

How do resonances appear as denominators when performing the change $\tilde{h}_k(x) = x + h_k(x)$ in the case $k = 2$.

We write the m-th compoment of $\tilde{h}_2$ as $\tilde{h}_2^{(m)} = x_m + \sum_{i,j} Q_{i,j}^{(m)} x_i x_j$, and assume that the linear part of f is already reduced to a diagonal matrix, so that $f^{(m)}$ has the form

$$f^{(m)}(x) = \lambda_m x_m + \sum_{i,j} P_{i,j}^{(m)} x_i x_j + \cdots .$$

We perform the change to f. Then, $\tilde{h}_2 \circ f \circ h_2$ has the form

$$x \overset{\tilde{h}_2}{\mapsto} \left(x_m + \sum_{i,j} Q_{i,j}^{(m)} x_i x_j \right)_m \overset{f}{\mapsto} \left(\lambda_m \left(x_m + \sum_{i,j} Q_{i,j}^{(m)} x_i x_j \right) + \sum_{i,j} P_{i,j}^{(m)} x_i x_j + \cdots \right)_m$$

$$\overset{\tilde{h}_2^{-1}}{\mapsto} \left(\lambda_m x_m + \lambda_m \sum_{i,j} Q_{i,j}^{(m)} x_i x_j + \sum_{i,j} P_{i,j}^{(m)} x_i x_j - \sum_{i,j} Q_{i,j}^{(m)} \lambda_i x_i \lambda_j x_j + \cdots \right)_m .$$

If we want to kill the terms of order 2, we have to choose $Q_{i,j}^{(m)}$ as

$$\lambda_m Q_{i,j}^{(m)} + P_{i,j}^{(m)} - \lambda_i \lambda_j Q_{i,j}^{(m)} = 0 \quad \Leftrightarrow \quad \boxed{Q_{i,j}^{(m)} = \frac{P_{i,j}^{(m)}}{\lambda_i \lambda_j - \lambda_m}}, \quad 1 \le i, j \le n.$$

Sternberg's theorem for maps

Next result is the differentiable version of Siegel's theorem.

Theorem Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, U open set, $0 \in U$. Write

$$f(x) = Ax + u(x),$$

with $u(0) = 0$, $Du(0) = 0$, and $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Assume that

$$\lambda_i \neq \lambda^m \qquad \text{for all } m \in \mathbb{Z}_+^n, \quad |m| \geq 2, \;\; 1 \leq i \leq n.$$

Then, there exists $\tau = \tau(\lambda_1, \dots, \lambda_n, r)$ such that if $f \in \mathcal{C}^r$, f is $\mathcal{C}^\tau$ (locally) conjugate to its linear part A.
Furthermore, $\lim_{r\to\infty} \tau(\lambda_1, \dots, \lambda_n, r) = \infty$.