Topological Data Analysis

2022–2023

Solutions of Exercises

24 November 2022

(1) Suppose that $f:(V,\pi)\to(V',\pi')$ is an isomorphism. Then there is a morphism $g:(V',\pi')\to(V,\pi)$ such that $g\circ f=id_V$ and $f\circ g=id_{V'}$. Therefore, for all $t\in\mathbb{R}$,

$$\begin{cases} g_t\circ f_t=(g\circ f)_t=(id_V)_t=id_{V_t}\\ f_t\circ g_t=(f\circ g)_t=(id_{V'})_t=id_{V'_t}\end{cases}$$

This implies that $f_t:V_t\to V'_t$ and $g_t:V'_t\to V_t$ are mutually inverse isomorphisms.

Conversely, suppose that $f_t:V_t\to V'_t$ is an isomorphism for all t. Pick an inverse $g_t:V'_t\to V_t$ for each t. We need to prove that $\{g_t\}_{t\in\mathbb{R}}$ is a morphism of persistence modules. Indeed, since f is a morphism, for all $s\leq t$ we have $f_t\circ\pi_{s,t}=\pi'_{s,t}\circ f_s$ and hence

$$g_t\circ f_t\circ\pi_{s,t}=g_t\circ\pi'_{s,t}\circ f_s;\quad \pi_{s,t}=g_t\circ\pi'_{s,t}\circ f_s;$$

$$\pi_{s,t}\circ g_s=g_t\circ\pi'_{s,t}\circ f_s\circ g_s;\quad \pi_{s,t}\circ g_s=g_t\circ\pi'_{s,t}$$ so g is a morphism.

Moreover, $g\circ f=id_V$ since $g_t\circ f_t=id_{V_t}$ $\forall t$, and likewise $f\circ g=id_{V'}$.

(2) Let (V, π) and (V', π') be persistence modules of finite type and suppose that $f: V \to V'$ is an isomorphism.

Let A and A' be the respective spectra.

Let $a \in A$. We need to prove that $a \in A'$. Suppose the contrary.

If $a \notin A'$, then there is a $\delta > 0$ such that $\pi'_{s,t}$ is an isomorphism for $a - \delta < s \leq t < a + \delta$. Since $a \in A$, there is an $\varepsilon > 0$ such that $\pi_{s,a}$ is <u>not</u> an isomorphism for $a - \varepsilon < s < a$.

Now we have, for $\lambda = \min\{\delta, \varepsilon\}$ and $a - \lambda < s < a$, that

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s,a}"] \arrow[d, "f_s"', "\cong"] & V_a \arrow[d, "f_a", "\cong"'] \\ V'_s \arrow[r, "\pi'_{s,a}"', "\cong"] & V'_a \end{tikzcd}$$

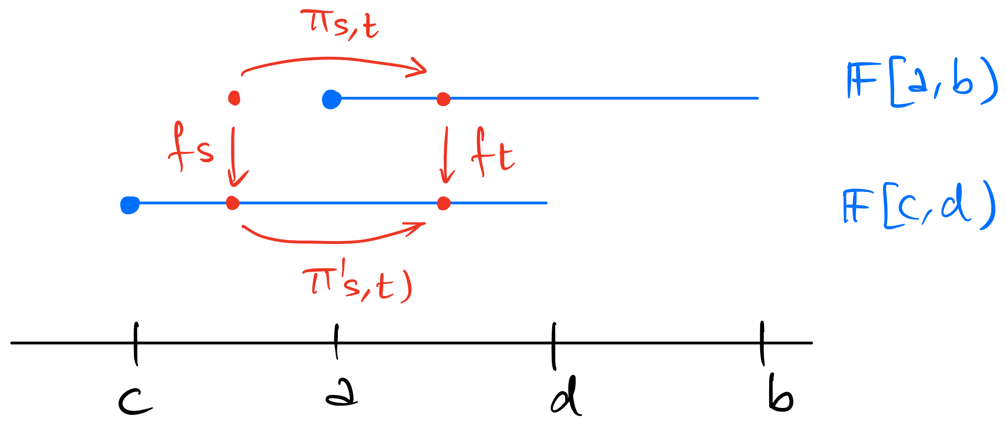
$f_a \circ \pi_{s,a} = \pi'_{s,a} \circ f_s$

where f_a, f_s and $\pi'_{s,a}$ are isomorphisms while $\pi_{s,a}$ is not.

This is a contradiction.

This argument shows that $A \subseteq A'$, and $A' \subseteq A$ by symmetry.

3

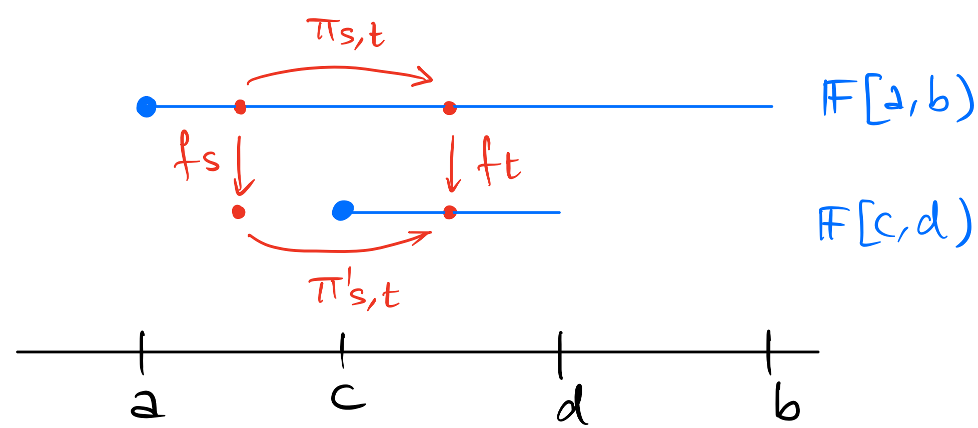


- If $c \le a < d \le b$, we define $f : \mathbb{F}[a,b) \to \mathbb{F}[c,d)$ as follows:

$$f_t = \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } a \le t < d \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s \quad \forall s \le t$.

- If $a < c$, then for $a \le s < c$ and $c \le t < d$ we have either

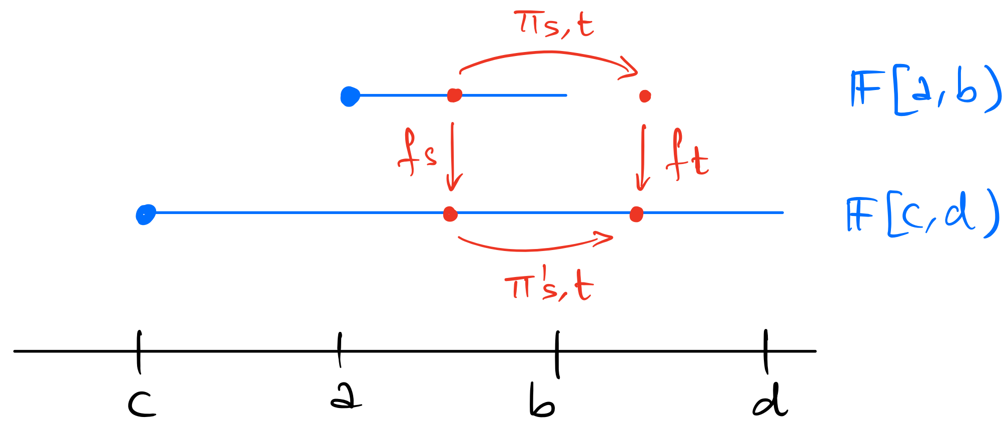


$$\begin{tikzcd} \mathbb{F} \arrow[r, "\cong"] \arrow[d, "f_s"'] & \mathbb{F} \arrow[d, "f_t"] \\ 0 \arrow[r] & \mathbb{F} \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathbb{F} \arrow[r] \arrow[d, "f_s"'] & 0 \arrow[d, "f_t"] \\ 0 \arrow[r] & \mathbb{F} \end{tikzcd}$$

if $t < b$ if $t \ge b$,

which forces $f_t = 0$ for $c \le t < d$ and hence $f = 0$.

- If $c \le a$ and $b < d$, then for $a \le s < b$ and $b \le t < d$ we have



$$\begin{tikzcd}
\mathbb{F} \arrow[r] \arrow[d, "f_s"'] & 0 \arrow[d, "f_t"] \\
\mathbb{F} \arrow[r, "\cong"] & \mathbb{F}
\end{tikzcd}$$

which forces $f_s = 0$ for $a \le s < b$ and therefore $f = 0$.

- If $d \le a$, then necessarily $f_t = 0$ for all t. Hence $f = 0$ is the only possibility.