Topological Data Analysis

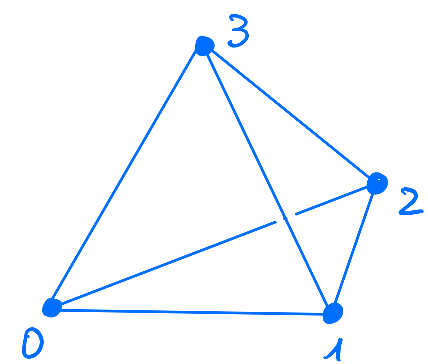
2022–2023

Lecture 3

Simplicial Homology

10 November 2022

Example: Find the homology groups with $\mathbb{Z}$ coefficients of (0123) and those of each of its skeleta $K^{(i)}$.



0-skeleton: $(0)\ (1)\ (2)\ (3)$

$$C_0(K^{(0)}) = \mathbb{Z}(0) \oplus \mathbb{Z}(1) \oplus \mathbb{Z}(2) \oplus \mathbb{Z}(3)$$

$$0 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$H_0(K^{(0)}) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = C_0 \cong \mathbb{Z}^4, \quad \text{and} \quad H_n(K^{(0)}) = 0 \quad \forall n \neq 0.$$

1-skeleton: $(01)\ (02)\ (03)\ (12)\ (13)\ (23)$

$$C_0(K^{(1)}) = C_0(K^{(0)})$$

$$C_1(K^{(1)}) = \mathbb{Z}(01) \oplus \mathbb{Z}(02) \oplus \mathbb{Z}(03) \oplus \mathbb{Z}(12) \oplus \mathbb{Z}(13) \oplus \mathbb{Z}(23)$$

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Matrix of ∂_1:

$$
\begin{array}{c|cccccc}
 & (01) & (02) & (03) & (12) & (13) & (23) \\
\hline
(0) & -1 & -1 & -1 & 0 & 0 & 0 \\
(1) & 1 & 0 & 0 & -1 & -1 & 0 \\
(2) & 0 & 1 & 0 & 1 & 0 & -1 \\
(3) & 0 & 0 & 1 & 0 & 1 & 1
\end{array}
$$

Column reduction:

rank Ker $\partial_1 = 3$

$$
\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}
\to
\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}
\to
\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)
$$

$$\operatorname{Ker} \partial_1 = \langle (12)-(02)+(01),\ (13)-(03)+(01),\ (23)-(03)+(02) \rangle$$

This basis of Ker ∂_1 is obtained from the column reduction process

$$H_0(K^{(1)}) = C_0 / \operatorname{Im} \partial_1 = \langle (0),(1),(2),(3) \rangle / \langle (1)-(0),\ (2)-(1),\ (3)-(2) \rangle \cong \mathbb{Z}$$

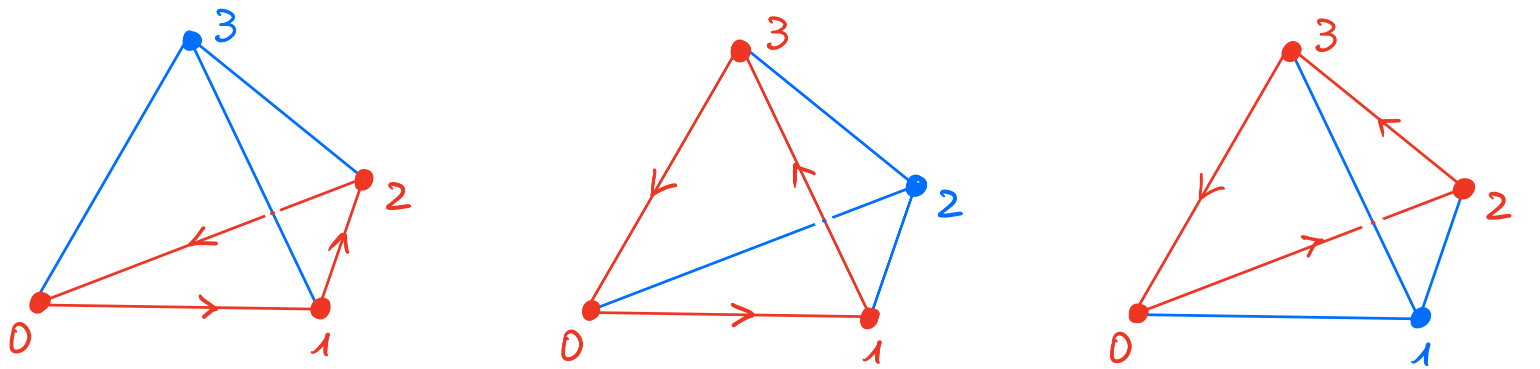
generated by the class $[0]$ of (0).

$[0] = [1] = [2] = [3]$

$$H_1(K^{(1)}) = \operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2 = \operatorname{Ker} \partial_1 \cong \mathbb{Z}^3$$

it is free abelian since $\operatorname{Ker} \partial_1 \leq C_1 = \mathbb{Z}^6$

We can visualize the generators of $H_1(K^{(1)})$ as linearly independent 1-cycles:



Note that the fourth 1-cycle $(23)-(13)+(12)$ is a linear combination of the three chosen 1-cycles.

<u>2-skeleton:</u> $(012)\ (013)\ (023)\ (123)$

$$C_0(K^{(2)}) = C_0(K^{(1)})$$

$$C_1(K^{(2)}) = C_1(K^{(1)})$$

$$C_2(K^{(2)}) = \mathbb{Z}(012) \oplus \mathbb{Z}(013) \oplus \mathbb{Z}(023) \oplus \mathbb{Z}(123)$$

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Matrix of ∂_2:

$$
\begin{array}{c|cccc}
 & (012) & (013) & (023) & (123) \\
\hline
(01) & 1 & 1 & 0 & 0 \\
(02) & -1 & 0 & 1 & 0 \\
(03) & 0 & -1 & -1 & 0 \\
(12) & 1 & 0 & 0 & 1 \\
(13) & 0 & 1 & 0 & -1 \\
(23) & 0 & 0 & 1 & 1
\end{array}
$$

Column reduction:

rank Ker $\partial_2 = 1$

$$
\begin{matrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{matrix}
\;\longrightarrow\;
\begin{matrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{matrix}
\;\longrightarrow\;
\begin{matrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{matrix}
\;\longrightarrow\;
\underbrace{\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}}_{\text{a basis of } \mathrm{Im}\,\partial_2}
\;\boxed{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}}
$$

$$\mathrm{Ker}\,\partial_2 = \langle (123) - (023) + (013) - (012) \rangle$$

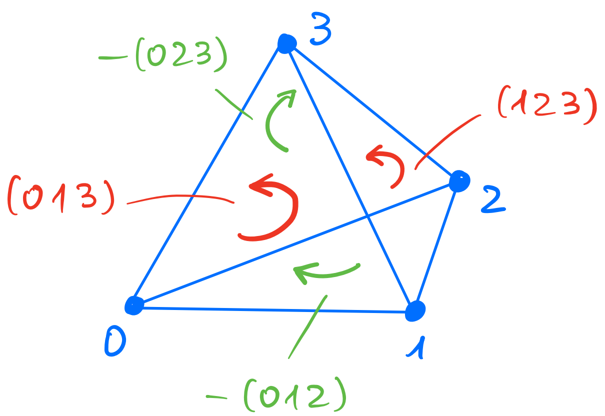
$H_0(K^{(2)}) \cong \mathbb{Z}$ generated by $[0]$ since ∂_0 and ∂_1 have not changed.

$$H_1(K^{(2)}) = \frac{\operatorname{Ker}\partial_1}{\operatorname{Im}\partial_2} = 0 \quad \text{since } \operatorname{Im}\partial_2 \overset{(*)}{=} \operatorname{Ker}\partial_1.$$

$$H_2(K^{(2)}) = \frac{\operatorname{Ker}\partial_2}{\operatorname{Im}\partial_3} = \operatorname{Ker}\partial_2 \cong \mathbb{Z},$$

generated by the class of the 2-cycle $(123)-(023)+(013)-(012)$.

We can interpret it as representing the <u>cavity</u> inside $|K^{(2)}| \cong S^2$.



Signs correspond to a choice of compatible orientations on the 2-faces. The orientations induced on every edge by the adjacent 2-faces are opposite.

(*) Column reduction provides a basis of $\operatorname{Im}\partial_2$:
$$\begin{cases} v_1 = (01)-(02)+(12) \\ v_2 = (02)-(03)-(12)+(13) \\ v_3 = (12)-(13)+(23). \end{cases}$$

We need to prove that v_1, v_2, v_3 span the same subgroup as

$$\begin{cases} w_1 = (12)-(02)+(01) \\ w_2 = (13)-(03)+(01) \\ w_3 = (23)-(03)+(02). \end{cases}$$

Indeed, $\begin{cases} v_1 = w_1 \\ v_2 = w_2 - w_1 \\ v_3 = w_3 - w_2 + w_1 \end{cases}$ $\begin{cases} w_1 = v_1 \\ w_2 = v_2 + v_1 \\ w_3 = v_3 + v_2 \end{cases}$ ✓

3-skeleton: $(0\,1\,2\,3)$, $K^{(3)} = K$.

$C_3(K) = \mathbb{Z}(0\,1\,2\,3)$ and $C_i(K) = C_i(K^{(2)})$ for $i \neq 3$.

$$0 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Matrix of ∂_3:

$$\begin{array}{c|c} & (0\,1\,2\,3) \\ \hline (012) & -1 \\ (013) & 1 \\ (023) & -1 \\ (123) & 1 \end{array}$$

Hence $\operatorname{Ker}\partial_3 = 0$.

$H_0(K) \cong \mathbb{Z}$, generated by $[0]$.

$H_1(K) = 0$.

both generated by $(123)-(023)+(013)-(012)$

$H_2(K) = \operatorname{Ker}\partial_2 / \operatorname{Im}\partial_3 = 0$ since $\operatorname{Im}\partial_3 = \operatorname{Ker}\partial_2$.

$H_3(K) = \operatorname{Ker}\partial_3 / \operatorname{Im}\partial_4 = \operatorname{Ker}\partial_3 = 0$.

This corresponds to the fact that $|K| = \Delta^3$, which is a contractible space (no nontrivial n-cycles for any $n \geq 1$).

Betti numbers

Enrico Betti (1823–1892)

For a finite ordered abstract simplicial complex K, the **Betti numbers** $\beta_n(K)$ are defined for $n \geq 0$ as

$$\beta_n(K) = \operatorname{rank} H_n(K) = \dim_{\mathbb{Q}} H_n(K;\mathbb{Q}).$$

More generally, for any field $\mathbb{F}$, one defines

$$\beta_n(K;\mathbb{F}) = \dim_{\mathbb{F}} H_n(K;\mathbb{F}).$$

We next prove that the number

$$\boxed{\chi(K) = \sum_{n=0}^{\infty} (-1)^n \beta_n(K;\mathbb{F})}$$

does not depend on $\mathbb{F}$. It is called the Euler characteristic of K.

Let D_n denote the matrix of the boundary operator $\partial_n : C_n(K;\mathbb{F}) \to C_{n-1}(K;\mathbb{F})$ in any chosen bases.

If we denote by f_n the number of n-faces of K, then

$$\beta_n(K;\mathbb{F}) = \dim_{\mathbb{F}} \operatorname{Ker} \partial_n - \dim_{\mathbb{F}} \operatorname{Im} \partial_{n+1} =$$

$$= \left[\dim_{\mathbb{F}} C_n(K;\mathbb{F}) - \dim_{\mathbb{F}} \operatorname{Im} \partial_n\right] - \dim_{\mathbb{F}} \operatorname{Im} \partial_{n+1} =$$

$$= f_n - \operatorname{rank} D_n - \operatorname{rank} D_{n+1}.$$

Therefore, if $d = \dim K$,

$$\sum_{n=0}^{\infty} (-1)^n \beta_n(K;\mathbb{F}) = \sum_{n=0}^{d} (-1)^n \beta_n(K;\mathbb{F}) =$$

$$= \sum_{n=0}^{d} (-1)^n \left(f_n - \operatorname{rank} D_n - \operatorname{rank} D_{n+1}\right) =$$

$$= f_0 - \operatorname{rank} D_0 - \cancel{\operatorname{rank} D_1} - f_1 + \cancel{\operatorname{rank} D_1} + \cancel{\operatorname{rank} D_2} + f_2 - \cancel{\operatorname{rank} D_2} - \dots$$

$$\dots + (-1)^d \left(f_d - \cancel{\operatorname{rank} D_d} - \operatorname{rank} D_{d+1}\right).$$

Here $\operatorname{rank} D_0 = 0$ since $\partial_0 : C_0 \to 0$ and $\operatorname{rank} D_{d+1} = 0$ since $\partial_{d+1} : 0 \to C_d$. Therefore, $\displaystyle\sum_{n=0}^{\infty} (-1)^n \beta_n(K;\mathbb{F}) = \sum_{n=0}^{\infty} (-1)^n f_n$

which is independent from $\mathbb{F}$. ✓

Example: For $K: (0123)$ we found

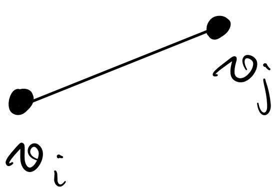
$$\chi(K^{(0)}) = 4 = f_0$$

$$\chi(K^{(1)}) = 1 - 3 = -2 = f_0 - f_1 = 4 - 6$$

$$\chi(K^{(2)}) = 1 - 0 + 1 = 2 = f_0 - f_1 + f_2 = 4 - 6 + 4$$

$$\chi(K^{(3)}) = 1 - 0 + 0 - 0 = 1 = f_0 - f_1 + f_2 - f_3 = 4 - 6 + 4 - 1$$

H_0 counts connected components



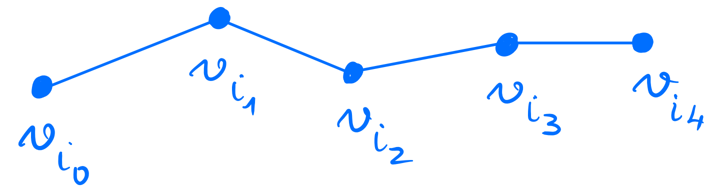
$$\partial_1(\{v_i, v_j\}) = \begin{cases} \{v_j\} - \{v_i\} & \text{if } v_i < v_j \\ \{v_i\} - \{v_j\} & \text{if } v_j < v_i \end{cases}$$

Hence $\{v_i\} - \{v_j\} \in \operatorname{Im} \partial_1 \ \forall i \ \forall j$.

Therefore $[v_i] = [v_j]$ in $H_0(K) = C_0(K) / \operatorname{Im} \partial_1$

if $\{v_i, v_j\}$ is an edge of K.

Consequently, $[v_i]=[v_j]$ if there is an edge path from v_i to v_j,



and $\boxed{H_0(K)\cong\mathbb{Z}}$ if the 1-skeleton of K is a connected graph, since $H_0(K)$ is the abelian group generated by the classes $[v_i]$ for $v_i\in V$ with the relations $[v_i]=[v_j]$ $\forall i\neq j$.

Moreover, if $K=A\cup B$ where A and B are disjoint subcomplexes, then $H_n(K)\cong H_n(A)\oplus H_n(B)$ for all n, since the boundary operators ∂_n restrict to A and to B.

i.e., maximal connected subcomplexes

As a special case of this fact, if $|K|$ has N connected components $K_1,\dots,K_N$, then

$$H_0(K)\cong H_0(K_1)\oplus\dots\oplus H_0(K_N)\cong\mathbb{Z}^N.$$

More generally, if $K=A\cup B$ with $A\cap B\neq\emptyset$, then $H_*(K)$ is determined by $H_*(A)$, $H_*(B)$ and $H_*(A\cap B)$ through the Mayer-Vietoris long exact sequence.

Walther Mayer (1887–1948)