A theorem of unstability

Theorem

Let $U \subset \mathbb{R}^n$ *be open,* $0 \in U$*, and* $g : \mathbb{R} \times U \longrightarrow \mathbb{R}^n$*. Consider the equation*

$$x' = Ax + g(t, x)$$

satisfying

1) $\operatorname{spec} A \cap \{\operatorname{Re} z > 0\} \neq \emptyset$.
2) g *is continuous and* $o(x)$ *uniformly in* t*, for* $t \in [0, \infty)$.
3) *The IVP has unique solution for every initial condition.*

Then 0 *is unstable.*

Lemma

Let B be a matrix in Jordan form with η in the non-zero, non-diagonal terms of the boxes of real eigenvalues (and the non-zero, non-diagonal boxes associated to complex ones). Let

$$\beta = \min\{\mathrm{Re}\,\lambda \mid \lambda \in \mathrm{spec}\,B\}, \qquad \alpha = \max\{\mathrm{Re}\,\lambda \mid \lambda \in \mathrm{spec}\,B\}.$$

Then, if $\|\cdot\|$ denotes the Euclidean norm

$$(\beta - \eta)\|x\|^2 \leq x^\top B\, x \leq (\alpha + \eta)\|x\|^2, \qquad \forall x \in \mathbb{R}^n.$$

Proof.

We write $B = \mathrm{diag}(J_1, \ldots, J_m)$ where J_j is a Jordan box of dimension $n_j \times n_j$. We also write $x = (x_1, \ldots, x_m)$ with $x_j \in \mathbb{R}^{n_j}$ and $x_j = (x_j^1, \ldots, x_j^{n_j})$. We have

$$x^T B\, x = (x_1, \ldots, x_m)^T \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1^T J_1 x_1 + \cdots + x_m^T J_m x_m.$$

The boxes are either of the form $J = \lambda \mathrm{Id} + \varepsilon N_1$, with $\varepsilon = 0$ or $\varepsilon = \eta$ and

$$N_1 = \begin{pmatrix} 0 & 0 & & & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}$$

if λ is real,

or $J = \operatorname{diag}\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \ldots, \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right) + \varepsilon N_2$ with $\varepsilon = 0$ or $\varepsilon = \eta$ and

$$N_2 = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 1 & 0 & & & \\ & & & \ddots & \\ & & 1 & 0 & 0 \end{pmatrix}$$

if $\lambda = a + ib$ is complex. If J is a box assosiated to $\lambda \in \mathbb{R}$ of dimension p

$$y^T J y = y^T \lambda \operatorname{Id} y + y^T \varepsilon N_1 y = \lambda \|y\|^2 + \varepsilon (y_2 y_1 + \cdots + y_p y_{p-1})$$

and using that for $r, s \in \mathbb{R}$, $|rs| \leq \frac{1}{2}(r^2 + s^2)$, [1] we obtain

$$(\lambda - \eta)\|y\|^2 \leq y^T J\, y \leq (\lambda + \eta)\|y\|^2.$$

[1] Also $|y_2 y_1 + \cdots + y_p y_{p-1}| \leq |(y_2, \ldots, y_p) \cdot (y_1, \ldots, y_{p-1})| \leq \|y\|^2$

If J is a box assosiated to $\lambda = a + ib \in \mathbb{C}$ we can write

$$J = a\mathrm{Id} + b\widetilde{N} + \varepsilon N_2,$$

where

$$\widetilde{N} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

We have $y^T \widetilde{N} y = 0$, $|y^T N_2 y| \leq \|y\|^2$ and then

$$(a - \eta)\|y\|^2 \leq y^T J\, y \leq (a + \eta)\|y\|^2.$$

Finally

$$\begin{aligned} x^\top B\, x &= \sum_{i=1}^{m} x_i^\top J_i\, x_i \leq \sum_{i=1}^{m} (\mathrm{Re}\,\lambda_i + \eta)\|x_i\|^2 \leq (\max\{\mathrm{Re}\,\lambda_i\} + \eta) \sum_{i=1}^{m} \|x_i\|^2 \\ &= (\alpha + \eta)\|x\|^2, \end{aligned}$$

since the Euclidean norm $\|x\|^2 = \sum_{i=1}^{m} \|x_i\|^2$, and analogously for the lower bound. $\square$

Proof of the theorem. We make a linear change of coordinates to put the matrix A in Jordan form $B = C^{-1}AC$ with η out of the diagonal in the corresponding boxes, to be fixed later on. We write the matrix decomposed in two boxes B_1, B_2; B_1 associated to the eigenvalues with positive real part and B_2 associated to the eigenvalues with negative or zero real part. Let γ be the minimum of the real parts of the eigenvalues of B_1. We choose $\eta \leq \gamma/10$.

By the lemma we have

$$x^T B_1 x \geq (\gamma - \gamma/10)\|x\|^2, \qquad y^T B_2 y \leq (\gamma/10)\|y\|^2.$$

In the new variables the equation has the form (C is the linear change)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + C^{-1}g(t, Cz), \qquad z = (x, y).$$

From now on we work with the Euclidean norm in these variables.

We denote $h(t, z) = C^{-1}g(t, Cz)$.

Let $r > 0$ be such that if $\|z\| < r$ then $\|h(t, z)\| \leq \frac{\gamma}{10}\|z\|$ for all $t \in [0, \infty)$.

The final part of the proof consists of assuming that $z(t) = 0$ is stable to arrive to a contradiction. We take $\varepsilon = r$ in the definition of stability. Then there exists $\delta > 0$ such that $\|z_0\| < \delta$ implies $\|\varphi(t, 0, z_0)\| < r$ for all $t \geq 0$.

We write $\varphi = (\varphi_1, \varphi_2)$ and $h = (h_1, h_2)$. For $t \geq 0$ we compute

$$\begin{aligned}
\frac{d}{dt}\|\varphi_1(t)\|^2 &= \frac{d}{dt}\langle \varphi_1(t), \varphi_1(t)\rangle = 2\varphi_1(t)^T \left(B_1\varphi_1(t) + h_1(t, \varphi(t))\right) \\
&\geq 2\left(\frac{9}{10}\gamma\|\varphi_1(t)\|^2 - \frac{1}{10}\gamma\|\varphi_1(t)\|\,\|\varphi(t)\|\right)
\end{aligned}$$

and analogously

$$\begin{aligned}
\frac{d}{dt}\|\varphi_2(t)\|^2 &= \frac{d}{dt}\langle \varphi_2(t), \varphi_2(t)\rangle = 2\varphi_2(t)^T \left(B_2\varphi_2(t) + h_2(t, \varphi(t))\right) \\
&\leq 2\left(\frac{1}{10}\gamma\|\varphi_2(t)\|^2 + \frac{1}{10}\gamma\|\varphi_2(t)\|\,\|\varphi(t)\|\right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{dt}[\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2] \geq & \left(\frac{9}{5}\gamma\|\varphi_1(t)\|^2 - \frac{1}{5}\gamma\|\varphi_1(t)\|\,\|\varphi(t)\|\right) \\
& - \left(\frac{1}{5}\gamma\|\varphi_2(t)\|^2 + \frac{1}{5}\gamma\|\varphi_2(t)\|\,\|\varphi(t)\|\right).
\end{aligned}$$

Using that

$$\|\varphi(t)\| \leq \|\varphi_1(t)\| + \|\varphi_2(t)\|$$

and

$$\|\varphi_1(t)\| \, \|\varphi_2(t)\| \leq \frac{1}{2}(\|\varphi_1(t)\|^2 + \|\varphi_2(t)\|^2)$$

we arrive to

$$\frac{d}{dt}[\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2] \geq \gamma(\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2)$$

so that if $\|\varphi_1(0)\|^2 - \|\varphi_2(0)\|^2 > 0$, then for $t \geq 0$

$$\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2 \geq (\|\varphi_1(0)\|^2 - \|\varphi_2(0)\|^2)e^{\gamma t}.$$

[2]

Then if $\|\varphi_1(0)\| > \|\varphi_2(0)\|$ we have $\|\varphi_1(t)\|^2 - \|\varphi_2(t)\|^2 \longrightarrow \infty$. We have arrived to a contradiction with the fact that φ is bounded.

A final simple argument shows that 0 is unstable in the original variables. $\square$

[2] Let $\gamma > 0$. Assume that $\psi : [0, \infty) \to \mathbb{R}$ satisfies $\frac{d}{dt}\psi \geq \gamma\psi$, and $\psi(0) > 0$. Let $[0, \omega_+)$ be the maximal interval where $\psi(t) > 0$. If $\omega_+ < \infty$, $\psi(\omega_+) = 0$ and $\frac{1}{\psi}\frac{d}{dt}\psi \geq \gamma$ and then $\psi(t) \geq \psi(0)e^{\gamma t}$ in $[0, \omega_+)$ and $\lim_{t \to \omega_+} \psi(t) = \psi(0)e^{\gamma\omega_+} > 0$, contradiction. Then $\omega_+ = \infty$ and $\psi(t) \geq \psi(0)e^{\gamma t}$ for all $t \geq 0$.

Theorem (Cetaev theorem)

Let $U \subset \mathbb{R}^n$ be open, $f : U \longrightarrow \mathbb{R}^n$ Lipschitz, $x' = f(x)$ and $x_0 \in U$ an equilibrium point. Let $D \subset U$ be open such that $x_0 \in \partial D$. Assume there exists $V : U \longrightarrow \mathbb{R}$ such that

(a) *V is differentiable.*

(b) *$V > 0$ on D and there exists $r > 0$ such that $V = 0$ on $\partial D \cap B(x_0, r)$.*

(c) *$\dot{V} > 0$ on D.*

Then x_0 is unstable.

Remark D may be $U \setminus \{x_0\}$.

Proof. We argue by contradiction. Assume x_0 is stable.

We take $\varepsilon > 0$ such that $\overline{B(x_0, \varepsilon)} \subset U$ and $\varepsilon < r$.

Let $\delta(\varepsilon) > 0$ be given by the definition of stability.

$B(x_0, \delta) \cap D$ is non-empty since $x_0 \in \partial D$.

We take $x \in B(x_0, \delta) \cap D$.Then

- $\varphi(t, x)$ is defined for all $t \geq 0$.
- $\varphi(t, x) \in B(x_0, \varepsilon)$, $\forall t \geq 0$.

Moreover $a := V(x) > 0$.

We consider the set

$$\Omega_a = \{x \in \overline{B(x_0, \varepsilon) \cap D} \mid \; V(x) \geq a\}.$$

Ω_a is a compact set and $\Omega_a \cap \partial D = \emptyset$:

We decompose $\partial D = \partial_1 D \cup \partial_2 D := \partial D \cap B(x_0, r) \cup \partial D \cap (\mathbb{R}^n \setminus B(x_0, r))$. If $z \in \Omega_a \cap \partial_1 D$, $V(z) \geq a$ and $V(z) = 0$, which is a contradiction. Moreover $\Omega_a \cap \partial_2 D = \emptyset$ because $\Omega_a \subset \overline{B(x_0, \varepsilon)} \subset B(x_0, r)$ and $\partial_2 D := \mathbb{R}^n \setminus B(x_0, r)$.

We claim that $\varphi(t,x) \in \Omega_a$, $\forall t \geq 0$.

Indeed, by the stability condition it is clear that $\varphi(t,x) \in B(x_0,\varepsilon) \subset \overline{B(x_0,\varepsilon)}$, $\forall t \geq 0$.

To leave the set D the solution has to pass through its boundary. It cannot be through $\partial_2 D$ because first it should cross the boundary of the ball of radius r.

It cannot be through $\partial_1 D$ because $\dot{V} > 0$.

Moreover, the condition $\dot{V} > 0$ also implies that $V(\varphi(t,x)) \geq a$, $\forall t \geq 0$.

Then by the compactness of Ω_a and $\Omega_a \subset D$

$$m := \inf\{\dot{V}(z) \mid \ z \in \Omega_a\} > 0.$$

Also

$$V(\varphi(t,x)) = V(\varphi(0,x)) + \int_0^t \dot{V}(\varphi(s,x))\, ds \geq a + mt \to \infty$$

when $t \to \infty$, while V is bounded on $\overline{B(x_0,\varepsilon)}$. Contradiction. $\square$

General Lotka Volterra equations

A general equation for n species is

$$\dot{x}_i = x_i(r_i + \sum_{j=1}^{n} a_{ij}x_j), \qquad 1 \leq i \leq n.$$

$A = (a_{ij})$ is called interaction matrix. The phase space is

$$\{x \in \mathbb{R}^n \mid x_i \geq 0\}.$$

The faces of the phase space are invariant.

The 2-dim case can be completely studied. But for $n \geq 3$ the system may be complicated.

Theorem Let $\mathbb{R}_0^+ = (0, \infty)$. Consider the previous system. Then, there are ω-limit sets in $(\mathbb{R}_0^+)^n$ iff it has an equilibrium point in $(\mathbb{R}_0^+)^n$. The same is true for α-limits.

Proof. The if part is immediate. For the only if part we suppose that there is no a fixed point in $(\mathbb{R}_0^+)^n$. We define $L : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Lx = r + Ax, \qquad \text{where } A = (a_{ij}), \quad r = (r_1, \ldots, r_n).$$

Let $C = L((\mathbb{R}_0^+)^n)$. $0 \notin C$. $(\mathbb{R}_0^+)^n$ is convex and hence C is convex.

There exists a hyperplane H, with $0 \in H$, such that $H \cap C = \emptyset$.

We represent H as $\{x \in \mathbb{R}^n \mid \langle c, x \rangle = 0\}$ for some $c \in \mathbb{R}^n$ and $\langle c, y \rangle > 0, \ \forall\, y \in C$.

We define $V : (\mathbb{R}_0^+)^n \to \mathbb{R}$ by

$$V(x) = \sum_{i=1}^{n} c_i \ln x_i.$$

We have $\dot{V}(x) = \sum_{i=1}^{n} c_i \frac{\dot{x}_i}{x_i} = \sum_{i=1}^{n} c_i (r + Ax)_i > 0$.

Then, the ω-limit sets have to be in $\{\dot{V} = 0\}$. Then $r + Ax \in H$ and also $r + Ax \in C$ but both sets are disjoint. Contradiction.

Hence, there are not ω-limit sets in $(\mathbb{R}_0^+)^n$.