Lesson 2

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We put some constraints about the self-financing strategies.

Definition

A strategy ϕ is *admissible* if it is self-financing and $V_n(\phi) \geq 0$, for all $0 \leq n \leq N$.

Definition

An *arbitrage (opportunity)* is an admissible strategy ϕ with zero initial value and with final value different from zero, that is

1. $V_0(\phi) = 0$,
2. $V_N(\phi) \geq 0$,
3. $\mathbb{P}(V_N(\phi) > 0) > 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. With $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega\}) > 0$, for all ω. Consider a filtration $\mathbb{F} = (\mathcal{F}_n)_{0 \leq n \leq N}$, with $\mathcal{F}_0 = \{\phi, \Omega\}$.

Definition

We say that a sequence of random variables $X = (X_n)_{0 \leq n \leq N}$ is adapted (to $\mathbb{F}$) if X_n is $\mathcal{F}_n$-measurable, $0 \leq n \leq N$.

Definition

An adapted sequence $(M_n)_{0\leq n\leq N}$, is said to be a

$$\begin{array}{ll} \text{submartingale if} & \mathbb{E}(M_{n+1}|\mathcal{F}_n) \geq M_n \\ \text{martingale if} & \mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n \\ \text{supermartingale if} & \mathbb{E}(M_{n+1}|\mathcal{F}_n) \leq M_n \end{array}$$

for all $0 \leq n \leq N-1$

Remark

Remember that $\mathbb{E}(X|\mathcal{G})$ *means a* $\mathcal{G}$*-measurable random variable, say* Y*, such that for every* $A \in \mathcal{G}$

$$\mathbb{E}\left(\mathbf{1}_A Y\right) = \mathbb{E}(\mathbf{1}_A X).$$

Let $\mathcal{G}$ *a* σ*-field generated by a partition of* Ω *say* $(A_i)_{i=1}^n$ *, then*

$$\mathbb{E}(X|\mathcal{G}) = \sum_{i=1}^{n} \mathbb{E}(X|A_i)\mathbf{1}_{A_i}$$

where

$$\mathbb{E}(X|A_i) = \sum_j x_j \mathbb{P}\left(X = x_j \middle| A_i\right)$$

Remark

We have the following important properties:

1. *If $\mathcal{H}$ is a sub-σ-field of $\mathcal{G}$, then*
$$\mathbb{E}\left(\mathbb{E}(X|\mathcal{G})|\,\mathcal{H}\right) = \mathbb{E}(X|\mathcal{H})$$
2. *If Z is $\mathcal{G}$-measurable*
$$\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$$

Remark

The above definition can be extended to the multi-dimensional case in a component-wise fashion. If $(M_n)_{0\leq n\leq N}$ *is a martingale is easy to see that* $\mathbb{E}(M_{n+j}|\mathcal{F}_n) = M_n, j \geq 0; \mathbb{E}(M_n) = M_0, n \geq 0$ *and that if* $(N_n)_{0\leq n\leq N}$ *is another martingale,* $(aM_n + bN_n)_{0\leq n\leq N}$ *is also a martingale.*

Proposition

Let $(M_n)_{0\leq n\leq N}$ *be a d-dimensional martingale and* $(H_n)_{1\leq n\leq N}$ *a d-dimensional predictable sequence, let* $\Delta M_n = M_n - M_{n-1}$. *Then, the sequence defined by*

$$X_n = X_0 + \sum_{j=1}^{n} H_j \cdot \Delta M_j, 1 \leq n \leq N \textit{ is a martingale, } X_0 \in \mathbb{R}$$

Proof.

It is enough to see that for all $0 \leq n \leq N$

$$\mathbb{E}(X_{n+1} - X_n | \mathcal{F}_n) = \mathbb{E}(H_{n+1} \cdot \Delta M_{n+1} | \mathcal{F}_n) = H_{n+1} \cdot \mathbb{E}(\Delta M_{n+1} | \mathcal{F}_n) = 0$$

$\square$

Remark

The previous transform is called martingale transform of $(M_n)_{0\leq n\leq N}$ *by* $(H_n)_{1\leq n\leq N}$. *Remind that*

$$\tilde{V}_n(\phi) = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta\tilde{S}_j \ ,$$

with $(\phi_n)_{1\leq n\leq N}$ *predictable. Then if* $(\tilde{S}_n)_{0\leq n\leq N}$ *is a martingale, we will have that* $(\tilde{V}_n)_{0\leq n\leq N}$ *is a martingale and in particular* $\mathbb{E}(\tilde{V}_n(\phi)) = V_0$.

Proposition

An adapted process $(M_n)_{0\leq n\leq N}$ is a d-dimensional martingale if and only if for all d-dimensional predictable process $(H_n)_{1\leq n\leq N}$ we have

$$\mathbb{E}\left(\sum_{j=1}^{N} H_j \cdot \Delta M_j\right) = 0 \tag{1}$$

Proof.

Assume that $(M_n)_{0\leq n\leq N}$ is a d-dimensional martingale, then (1) follows by the previous proposition. Assume that (1) is satisfied, then we can take $H_n^i = 0, 1 \leq n \leq j, H_{j+1}^i = 1_A$ with $A \in \mathcal{F}_j$, $H_n^i = 0, n > j, H_n^k = 0, 1 \leq n \leq N,\ k \neq i$. So

$$\mathbb{E}(\mathbf{1}_A(M_{j+1}^i - M_j^i)) = 0.$$

Since this is true for all $A \in \mathcal{F}_j$, this is equivalent to $\mathbb{E}(M_{j+1}^i|\mathcal{F}_j) = M_j^i$, and this is also true for all $j \geq 0$ and $i = 1, ..., d$. □

Definition

We say that a probability $\mathbb{P}^*$ in $\mathcal{F}$ is equivalent to $\mathbb{P}$, we write $\mathbb{P}^* \sim \mathbb{P}$, if $\mathbb{P}^*(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$.

Remark

In our case $\mathbb{P}^* \sim \mathbb{P}$ *if and only if* $\mathbb{P}^*(\{\omega\}) > 0$ *for all* ω.

Now we have the First Fundamental Theorem of Asset Pricing (FFTAP).

Theorem (FFTAP)

A financial market is viable (free of arbitrage opportunities) if and only if there exists $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$ *such that the discounted prices of the stocks* $((\tilde{S}_n^j)_{0\leq n\leq N}\ , j = 1, ..., d)$ *are* $\mathbb{P}^*$*-martingales.*

Proof.

Assume there exists $\mathbb{P}^* \sim \mathbb{P}$ and let φ be an admissible strategy with zero initial value, then

$$\tilde{V}_n = \sum_{i=1}^{n} \varphi_i \cdot \Delta \tilde{S}_i, n \geq 1, \tilde{V}_0 = 0,$$

is a $\mathbb{P}^*$-martingale and consequently

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N) = \tilde{V}_0 = 0,$$

and since $\tilde{V}_N \geq 0$ we have $\tilde{V}_N = 0$ (because $\mathbb{P}^*(\omega) > 0$ for all ω). So, there is not an arbitrage. $\square$

Proof.

Suppose now that there is not arbitrage. First, let L^0 be the linear space of all random variables. In our case if $|\Omega| = k$ we can identify each random variable X to a vector in $(X(\omega_1), ..., X(\omega_k))$, so $L^0 = \mathbb{R}^k$. Denote the set of positive random variables:

$$L^0_+ := \{X, X \geq 0\}$$

and $\Lambda = \{X, X = \tilde{V}_N(\phi), (\phi_n)_{1\leq n\leq N}$ predictable *and admissible:* $\tilde{V}_n = \sum_{j=1}^n \phi_j \cdot \Delta\tilde{S}_j,\ \tilde{V}_n(\phi) \geq 0$ for all $1 \leq n \leq N\}$. No-arbitrage means that

$$\Lambda \cap L^0_+ = \{0\} \tag{2}$$

□

Proof.

Consider the subset $S \subseteq L^0_+ \setminus \{0\}$ of random variables such that $\sum_{i=1}^k X(\omega_i) = 1$, it is compact and convex. In fact is bounded and closed and if $X, Y \in S$

$$\lambda X + (1-\lambda) Y \in S, \text{ for all } 0 \leq \lambda \leq 1.$$

Set $L = \{X, X = \tilde{V}_N(\phi), (\phi_n)_{1\leq n \leq N} \text{ predictable}, \ \tilde{V}_N = \sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j \}$, then L is a vector subspace of L_0 since

$$\alpha \tilde{V}_N(\phi^{(1)}) + \beta \tilde{V}_N(\phi^{(2)}) = \tilde{V}_N(\alpha\phi^{(1)} + \beta\phi^{(2)})$$

for all $\alpha, \beta \in \mathbb{R}$ and $L \supseteq \Lambda$. Now, assume the, *a priori*, stronger condition

$$L \cap L^0_+ = \{0\}, \tag{3}$$

□

Proof.

Later, in the next lesson, we will see that (2)$\Rightarrow$(3). Then we have that (3) implies that $L \cap S = \phi$. Therefore we have a vector space L and a convex a compact set that are disjoint in $L^0 = \mathbb{R}^k$. By *the separating hyperplane theorem*, there exists a linear map, say A, such that $A(Y) > 0$ for all $Y \in S$ and $A(Y) = 0$ if $Y \in L$. By the linearity we can write $A(Y) = \sum_{i=1}^k \lambda_i Y(\omega_i)$. Then, all $\lambda_i > 0$, since $A(Y) > 0$ for all $Y \in S$, and we can define

$$\mathbb{P}^*(\omega_i) = \frac{\lambda_i}{\sum_{i=1}^k \lambda_i}, i = 1, ..., k.$$

Now for all ϕ predictable

$$\mathbb{E}_{\mathbb{P}^*}\left(\sum_{i=1}^N \phi_i \cdot \Delta \tilde{S}_i\right) = \mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N) = \frac{A(\tilde{V}_N)}{\sum_{i=1}^K \lambda_i} = 0.$$

So, by the previous proposition, $\tilde{S}$ is a $\mathbb{P}^*$-martingale. $\square$