Lesson 1

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Discrete time models

The values of the stocks and claims (shares, commodities, options...) will be random variables defined in a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will consider an increasing sequence of σ-fields (filtration) : $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_N \subseteq \mathcal{F}$. $\mathcal{F}_n$ represents the collection of all events that are observable up to time n. The horizon N, will correspond with the maturity of the claims. We shall assume that Ω is *finite*, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F} = \mathcal{P}(\Omega)$ and that $\mathbb{P}(\{\omega\}) > 0$, for all $\omega \in \Omega$.

The financial market will consist on $(d+1)$ stocks whose prices at time n will be given by *positive* random variables S_n^0, $S_n^1, \ldots, S_n^d$ measurable with respect to $\mathcal{F}_n$ (that is, the prices at n are part of that observed until n). In some cases we shall assume that $\mathcal{F}_n = \sigma(S_k^1, \ldots, S_k^d, 0 \leq k \leq n)$, in such that prices are the only thing we observe.

Strategies of investment, portfolios

The super-index zero corresponds to a *riskless* stock (a bank account) and by convention we take $S_0^0 = 1$. If the relative profit (return) of the riskless stock is constant:

$$\frac{S_{n+1}^0 - S_n^0}{S_n^0} = r \geq 0$$

we will have

$$S_{n+1}^0 = S_n^0(1+r) = S_0^0(1+r)^{n+1} = (1+r)^{n+1}.$$

A *trading strategy* is a stochastic process (a sequence or random variables in the discrete time setting) $\phi = ((\phi_n^0, \phi_n^1, ..., \phi_n^d))_{1 \leq n \leq N}$ in R^{d+1}. ϕ_n^i indicates the number of stocks of kind i in the portfolio at time n and ϕ is *predictable* that is ϕ_n^i is $\mathcal{F}_{n-1}$-measurable, for all $1 \leq n \leq N$. This means that the positions in the portfolio at n is decided at $n-1$ and held till time n. In other words, ϕ_n^i is the quantity of stocks of i type during the period $(n-1, n]$.

The value of the portfolio associated with a trading strategy ϕ is given by

$$V_n(\phi) = \phi_n \cdot S_n := \sum_{i=0}^{d} \phi_n^i S_n^i, \quad n \geq 1, \quad V_0(\phi) = \phi_1 \cdot S_0.$$

and its *discounted* value

$$\tilde{V}_n(\phi) = \frac{V_n(\phi)}{(1+r)^n} = \phi_n \cdot \tilde{S}_n$$

with

$$\tilde{S}_n = \left(1, \frac{S_n^1}{(1+r)^n}, ..., \frac{S_n^d}{(1+r)^n}\right) = (1, \tilde{S}_n^1, ..., \tilde{S}_n^d)$$

Definition

An investment strategy is said to be *self-financing* if

$$V_n = \phi_{n+1} \cdot S_n, \quad 0 \leq n \leq N-1$$

Remark

The meaning is that at n, once the new prices S_n *are announced, investors change their portfolio without adding or taking out wealth: if at time n there is an increment* $\phi_{n+1} - \phi_n$ *of the risky stocks the cost of this trade is* $\sum_{i=1}^{d}(\phi_{n+1}^i - \phi_n^i)S_n^i$, *the change in the bank account will be*

$$(\phi_{n+1}^0 - \phi_n^0)S_n^0 = -\sum_{i=1}^{d}(\phi_{n+1}^i - \phi_n^i)S_n^i$$

so

$$\sum_{i=0}^{d}(\phi_{n+1}^i - \phi_n^i)S_n^i = (\phi_{n+1} - \phi_n) \cdot S_n = 0$$

and $V_n = \phi_{n+1} \cdot S_n$, $0 \leq n \leq N-1$.

Proposition

A trading strategy is self-financing if and only if

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot (S_{n+1} - S_n), 0 \leq n \leq N-1.$$

Proof.

Assume that the strategy is self-financing then

$$\begin{aligned} V_{n+1}(\phi) - V_n(\phi) &= \phi_{n+1} \cdot S_{n+1} - \phi_{n+1} \cdot S_n \\ &= \phi_{n+1} \cdot (S_{n+1} - S_n). \end{aligned}$$

If $V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot (S_{n+1} - S_n)$ then

$$\phi_{n+1} \cdot S_{n+1} - V_n = \phi_{n+1} \cdot (S_{n+1} - S_n),$$

and consequently $V_n = \phi_{n+1} \cdot S_n$. □

Proposition

The following statements are equivalent:

(i) The strategy ϕ is self-financing,

$$\textit{(ii)}\ V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot (S_j - S_{j-1})$$

$$= V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j = V_0(\phi) + \sum_{j=1}^{n} \sum_{i=0}^{d} \phi_j^i \Delta S_j^i, \quad 1 \le n \le N$$

$$\textit{(iii)}\ \tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1})$$

This last formula needs to consider only the risky assets (because we assumed S_0^0=1)

$$= V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j = V_0(\phi) + \sum_{j=1}^{n} \sum_{i=1}^{d} \phi_j^i \Delta \tilde{S}_j^i, \ 1 \le n \le N$$

Proposition

For any predictable process $\hat{\phi} = ((\phi_n^1, \ldots, \phi_n^d))_{1 \leq n \leq N}$ *and any value* V_0, *there exists a unique predictable process* $\left(\phi_n^0\right)_{1 \leq n \leq N}$ *such that the strategy* $\phi = ((\phi_n^0, \phi_n^1, \ldots, \phi_n^d))_{1 \leq n \leq N}$ *is self-financing with initial value* V_0.

Proof.

For $1 \leq n \leq N$

$$\begin{aligned}\tilde{V}_n(\phi) &= V_0 + \sum_{j=1}^{n} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) \\ &= \phi_n \cdot \tilde{S}_n = \phi_n^0 + \sum_{i=1}^{d} \phi_n^i \tilde{S}_n^i.\end{aligned}$$

$\square$

Proof.

Therefore

$$
\begin{aligned}
\phi_n^0 &= V_0 + \sum_{j=1}^{n} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) - \sum_{i=1}^{d} \phi_n^i \tilde{S}_n^i \\
&= V_0 + \sum_{j=1}^{n} \sum_{i=1}^{d} \phi_j^i \cdot (\tilde{S}_j^i - \tilde{S}_{j-1}^i) - \sum_{i=1}^{d} \phi_n^i \tilde{S}_n^i \\
&= V_0 + \sum_{j=1}^{n-1} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) - \sum_{i=1}^{d} \phi_n^i \tilde{S}_{n-1}^i \in \mathcal{F}_{n-1}.
\end{aligned}
$$

□

The arbitrage condition

First of all note that we are not doing any assumption about the sign of the quantities . $\phi_n^i < 0$ amounts to borrowing this number of stocks and converting them into cash (*short-selling*) or, if $i = 0$, borrowing this number of monetary units and converting them into stocks (a loan to buy stocks). In fact, we do not put any restriction on ϕ_n^i, it can be any real number, so divisibility and total liquidity conditions of the market are assumed, including no transaction costs. For simplicity we suppose that any unit of cash at 0 becomes $(1 + r)^n$ at n and this happens independently of it is borrowed or invested in the bank account.

We put some constraints about the self-financing strategies.

Definition

A strategy ϕ is *admissible* if it is self-financing and $V_n(\phi) \geq 0$, for all $0 \leq n \leq N$.

Definition

An *arbitrage (opportunity)* is an admissible strategy ϕ with zero initial value and with final value different from zero, that is

1. $V_0(\phi) = 0$,
2. $V_N(\phi) \geq 0$,
3. $\mathbb{P}(V_N(\phi) > 0) > 0$.