$A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\quad f_A : \mathbb{R}^n \to \mathbb{R}^n$, $f_A(x) = Ax$ $\quad f_B : \mathbb{R}^n \to \mathbb{R}^n$, $f_B(x) = Bx$ $\}$ linear, invertible, hyperbolic maps

$\hookrightarrow$ $Spec(A) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\}$

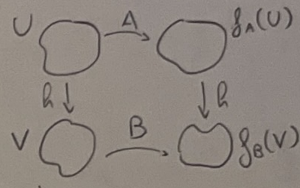
$Spec(B) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\}$

A contraction : $\|Ax_1 - Ax_2\| \leq k_A \|x_1 - x_2\|$

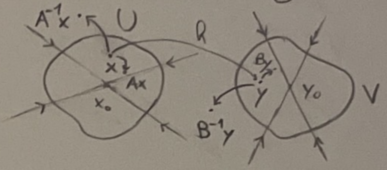
B contraction : $\|Bx_1 - Bx_2\| \leq k_B \|x_1 - x_2\|$ (from previous exercise in the list)

$\hookrightarrow$ $f_A, f_B$ admit one and only one fixed point

A and B are locally topologically conjugated: $\exists h$ homeomorphism s.t. $h : U \to V$, $h \circ A = B \circ h$ where $U$ and $V$ are open neighborhoods of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ respectively.




Since both A and B are contractions, the fixed-point theorem holds in $\mathbb{R}^n$. Let us consider the fixed points $x_0, y_0$ of A, B respectively, and their neighborhoods U, V.




Now, since both maps are hyperbolic and contractions, we can decompose the space into $\mathbb{R}^n = E^s$, since all eigenspaces will have the respective eigenvalues with modulus less than 1.

Hence, we have that for $x \in \mathbb{R}^n$, $\|A^n x\| \leq \gamma_A^n \|x\| \xrightarrow[n \to \infty]{} 0$

$y \in \mathbb{R}^n$, $\|B^n x\| \leq \gamma_B^n \|x\| \xrightarrow[n \to \infty]{} 0$

Now we can use this to construct an homeomorphism H as an extension of the local homeomorphism h.

For a point $x \in \mathbb{R}^n \setminus U$ we iterate A until $A^k x \in U$, which will happen for the above-mentioned decomposition. Also, we know there can be no accumulation in other parts of the space due to the uniqueness of the fixed point.

Therefore, we can write H as $H(x) = A^{k(x)} h B^{-k(x)}$ were $k(x)$ is the minimal integer for which $A^{k(x)} x \in U$. This way H is defined $\forall x \in \mathbb{R}^n$ and it is an homeomorphism directly from the properties of A, B, h which still stand through the composition. ∎

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