Quantitative Finance Problem Set 2

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1 Exercise 1

We consider a market model as in the previous lessons. A numèraire is an adapted sequence $Z = (Z_n)_{0 \leq n \leq N}$ s.t. $Z_0 = 1$; $Z_n > 0$ for $n = 1, \dots, N$; and $Z_n = V_n(\varphi)$ for some admissible strategy φ $(n = 1, \dots, N)$. Denote by S^Z the Z-discounted vector price process: $S_n^Z = \frac{S_n}{Z_n}$, $n = 1, \dots, N$

1. Prove that a predictable sequence $\phi = (\phi_n)_{1 \leq n \leq N}$, with values in $\mathbb{R}^{d+1}$, is self-financing if

$$V_n^Z(\phi) := \frac{V_n(\phi)}{Z_n} = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

Proof
To prove that the predictable sequence $\phi = (\phi_n)_{1 \leq n \leq N}$ is self-financing, we need to show that the portfolio value $V_n(\phi)$ satisfies the self-financing property:

$$V_n(\phi) = V_{n-1}(\phi) + \phi_n \cdot \Delta S_n, \quad \text{for} \quad n = 1, \dots, N,$$

where $\Delta S_n = S_n - S_{n-1}$ is the price change of the underlying asset from time $n-1$ to time n.
First, we consider the definition of $V_n^Z(\phi)$:

$$V_n^Z(\phi) = \frac{V_n(\phi)}{Z_n}$$

Multiplying both sides by Z_n, we obtain:

$$V_n(\phi) = Z_n V_n^Z(\phi)$$

Substituting this expression into the self-financing property, we obtain:

$$\begin{aligned} V_n(\phi) &= V_{n-1}(\phi) + \phi_n \cdot \Delta S_n \\ Z_n V_n^Z(\phi) &= Z_{n-1} V_{n-1}^Z(\phi) + \phi_n \cdot \Delta S_n \\ V_n^Z(\phi) &= \frac{Z_{n-1}}{Z_n} V_{n-1}^Z(\phi) + \frac{\phi_n \cdot \Delta S_n}{Z_n} \end{aligned}$$

We recognize the second term on the right-hand side of the equation as the n-th component of the Z-discounted portfolio value $\phi \cdot \Delta S^Z$. Therefore, we can write:

$$V_n^Z(\phi) = V_{n-1}^Z(\phi) + \phi_n \cdot \Delta S_n^Z$$

This equation grants that the Z-discounted portfolio value $V^Z(\phi)$ satisfies the self-financing property, which means that the original portfolio value $V(\phi)$ is self-financing. Hence, the predictable sequence ϕ is self-financing.
Now, if we reiterate the same reasoning for $V_{n-1}^Z(\phi)$ and so on until we reach $V_1^Z(\phi) = V_0^Z(\phi) + \phi_1 \cdot \Delta S_1^Z$, the expression we obtain for the self-financing property is the one proposed in the statement:

$$V_n^Z(\phi) := \frac{V_n(\phi)}{Z_n} = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N \quad \blacksquare$$

2. Prove that

$$\sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z = 0,\ n = 1, \dots, N$$

Proof
In exercise 1 we have proven that given a predictable sequence $\phi = (\phi_n)_{1 \leq n \leq N}$, with values in $\mathbb{R}^{d+1}$, is self-financing if

$$V_n^Z(\phi) := \frac{V_n(\phi)}{Z_n} = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

Using this result we have that, the strategy φ defining the numèraire Z_n is an admissible strategy and hence it is self-financing. Therefore, it satisfies the identity

$$V_n^Z(\varphi) = \frac{V_n(\varphi)}{Z_n} = V_0 + \sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

Rearranging the terms we obtain

$$V_n^Z(\varphi) - V_0 = \sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

$$\frac{V_n(\varphi)}{Z_n} - V_0 = \sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

$$\frac{Z_n}{Z_n} - Z_0 = \sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

$$1 - 1 = \sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

Therefore, we have

$$\sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z = 0,\ n = 1, \dots, N \quad \blacksquare$$

3. Prove that for any predictable sequence $\phi = (\phi_n)_{1 \leq n \leq N}$, there exists a self-financing strategy $\hat{\phi}$ such that

$$\hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^N,\ n = 1, \dots, N$$

Proof
Using the previous results, we have that

$$\sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z = 0,\ n = 1, \dots, N$$

So we can add this term to the identity given in the statement to obtain equivalently

$$\hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^N + \sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

$$\hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^N + \varphi_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

$$\hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} (\phi_j + \varphi_j) \cdot \Delta S_j^N,\ n = 1, \dots, N$$

Now, we want to construct a self-financing strategy $\hat{\phi}$ that satisfies the identity above. Let $\hat{\phi}$ be a self-financing strategy, then the following result holds

$$V_n^Z(\hat{\phi}) := \frac{V_n(\hat{\phi})}{Z_n} = \frac{\hat{\phi}_n S_n}{Z_n} = \hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} \hat{\phi}_j \cdot \Delta S_j^Z,\ n = 1, \dots, N$$

Substituting in the previously obtained identity we obtain

$$\hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} (\phi_j + \varphi_j) \cdot \Delta S_j^N,\ n = 1, \dots, N$$

$$V_0 + \sum_{j=1}^{n} \hat{\phi}_j \cdot \Delta S_j^Z = V_0 + \sum_{j=1}^{n} (\phi_j + \varphi_j) \cdot \Delta S_j^N,\ n = 1, \dots, N$$

$$\sum_{j=1}^{n} \hat{\phi}_j \cdot \Delta S_j^Z = \sum_{j=1}^{n} (\phi_j + \varphi_j) \cdot \Delta S_j^N,\ n = 1, \dots, N$$

Thus we have that the strategy is not necessarily unique, but we have that if $\hat{\phi}$ is constructed such that $\hat{\phi}_j = \phi_j + \varphi_j,\ j = 1, \dots, n$ then the identity we wanted to prove holds. ■

4. Assume that the market is viable (free of arbitrage) and let $\mathbb{P}^*$ be the risk-neutral probability. Define $\mathbb{P}^Z$ by

$$\frac{d\mathbb{P}^Z}{d\mathbb{P}^*} = \frac{Z_n}{S_N^0}$$

that is

$$\mathbb{P}^Z(A) := \mathbb{E}_{\mathbb{P}^*}(\frac{Z_N}{S_N^0} \mathbb{1}_A) \text{ for all } A \in \mathcal{F}$$

Prove that $\mathbb{P}^Z$ is a probability equivalent to $\mathbb{P}^*$ and that for all $n = 0, \dots, N$

$$\mathbb{E}_{\mathbb{P}^Z}(X|\mathcal{F}_n) = \frac{\mathbb{E}_{\mathbb{P}^*}(X \frac{Z_N}{S_N^0} |\mathcal{F}_n)}{\mathbb{E}_{\mathbb{P}^*}(\frac{Z_N}{S_N^0} |\mathcal{F}_n)}$$

Proof
To prove that $\mathbb{P}^Z$ is a probability equivalent to to $\mathbb{P}^*$ we need to show that

$$\mathbb{P}^*(A) = 0 \Longleftrightarrow \mathbb{P}^Z(A) = 0\ \ \forall A \in \mathcal{F}$$

We have that

$$\mathbb{P}^Z(A) = \mathbb{E}_{\mathbb{P}^*}(\frac{Z_n}{S_n^0} \mathbb{1}_A) = \mathbb{E}(\frac{Z_n}{S_n^0})\mathbb{P}^*(A)$$

Since $Z_n > 0$ for $n = 0, \dots, N$ we have

$$\mathbb{E}(\frac{Z_N}{S_N^0})\mathbb{P}^*(A) = 0 \Longleftrightarrow \mathbb{P}^*(A) = 0$$

$$\mathbb{P}^Z = 0 \Longleftrightarrow \mathbb{P}^*(A) = 0$$

Which shows that $\mathbb{P}^Z$ is equivalent to $\mathbb{P}^*$.
Now to show the given identity we can do the following

$$\frac{\mathbb{E}_{\mathbb{P}^*}(X\frac{Z_N}{S_N^0}|\mathcal{F}_n)}{\mathbb{E}_{\mathbb{P}^*}(\frac{Z_N}{S_N^0}|\mathcal{F}_n)} = \frac{\int_{\mathcal{F}_n} X\frac{Z_N}{S_N^0}d\mathbb{P}^*}{\int_{\mathcal{F}_n}\frac{Z_N}{S_N^0}d\mathbb{P}^*}$$

Using the previous result $\frac{Z_N}{S_N^0}d\mathbb{P}^* = d\mathbb{P}^Z$ we have

$$\frac{\mathbb{E}_{\mathbb{P}^*}(X\frac{Z_N}{S_N^0}|\mathcal{F}_n)}{\mathbb{E}_{\mathbb{P}^*}(\frac{Z_N}{S_N^0}|\mathcal{F}_n)} = \frac{\int_{\mathcal{F}_n} X\frac{Z_N}{S_N^0}d\mathbb{P}^*}{\int_{\mathcal{F}_n}\frac{Z_N}{S_N^0}d\mathbb{P}^*} = \frac{\int_{\mathcal{F}_n} X d\mathbb{P}^Z}{\int_{\mathcal{F}_n} d\mathbb{P}^Z} = \frac{\mathbb{E}_{\mathbb{P}^Z}(\mathbb{1}_{\mathcal{F}_n}X)}{\mathbb{P}^Z(\mathcal{F}_n)} = \mathbb{E}_{\mathbb{P}^Z}(X|\mathcal{F}_n)$$

Which shows the given identity holds. ■

5. Prove that the market is viable (free of arbitrage) if there exists a probability $\mathbb{P}^Z \sim \mathbb{P}$ s.t. S^Z is a $\mathbb{P}^Z$-martingale and that in that case there is at most one deterministic numèraire.

Proof
$(\Longrightarrow)$
We have proven that if the market is viable, then there exists a unique risk-neutral probability measure $\mathbb{P}^Z \sim \mathbb{P}$ s.t. S^Z is a $\mathbb{P}^Z$-martingale, meaning we have

$$\mathbb{E}_{\mathbb{P}^Z}(\frac{S_n}{Z_n}|\mathcal{F}_{n-1}) = \frac{S_{n-1}}{Z_{n-1}}$$

To show there is at most one deterministic numèraire, suppose Z_1, Z_2 are deterministic numèrarires such that S^{Z_1} and S^{Z_2} are both $\mathbb{P}^Z$-martingales. We can define a new probability measure $\mathbb{Q}$ as a linear combination of $\mathbb{P}^{Z_1}$ and $\mathbb{P}^{Z_2}$

$$\mathbb{Q} = \lambda\mathbb{P}^{Z_1} + (1-\lambda)\mathbb{P}^{Z_2}, \quad \lambda \in [0,1]$$

Then we have that S^{Z_1} and S^{Z_2} are both $\mathbb{Q}$-martingales, and S^Z is also a $\mathbb{Q}$-martingale.
This contradicts the uniqueness of the risk-neutral probability measure under which S^Z is a $\mathbb{P}^Z$-martingale. Therefore we have that $Z_1 = Z_2$ almost surely.
$(\Longleftarrow)$
Assume there exists a probability measure $\mathbb{P}^Z \sim \mathbb{P}$ s.t. S^Z is a $\mathbb{P}^Z$-martingale and that in that case there is at most one deterministic numèraire.
Assume there exists an arbitrage opportunity, meaning there exists $\hat{\phi}$ self-financing such that

$$V_0(\hat{\phi}) = 0$$
$$V_N(\hat{\phi}) \geq 0$$
$$\mathbb{P}(V_n(\hat{\phi}) > 0) > 0$$

Since $\hat{\phi}$ is self-financing, we have

$$V_n - \hat{\phi}_n S_n^Z = V_0 - \hat{\phi}_0 S_n^Z = 0, \quad \forall n$$
$$V_n = \hat{\phi}_n S_n^Z, \quad \forall n$$

Since $\mathbb{P}(V_n > 0) > 0$, we have

$$\mathbb{P}(\hat{\phi}_n S_n^Z > 0) > 0$$

By assumption S^Z is a $\mathbb{P}^Z$-martingale, therefore we have

$$\mathbb{E}_{\mathbb{P}^Z}(\hat{\phi}_n S_n^Z) = \hat{\phi}_0 S_0^Z = 0$$

This contradicts the fact that

$$\mathbb{P}(\hat{\phi}_n S_n^Z > 0) > 0$$

, and therefore there can be no arbitrage opportunity. ■

6. Assume a market is viable and complete, prove that the price of a payoff X at time n is given by

$$Z_n \mathbb{E}_{\mathbb{P}^Z}(\frac{X}{Z_N}|\mathcal{F}_n)$$

Proof
From the previous results, we have that, since the market is viable (free of arbitrage) there exists a probability $\mathbb{P}^Z \sim \mathbb{P}$ s.t. S^Z is a $\mathbb{P}^Z$-martingale, meaning

$$\mathbb{E}_{\mathbb{P}^Z}(S^Z_{n+1}|\mathcal{F}_n) = S^Z_n$$

And in particular

$$\mathbb{E}_{\mathbb{P}^Z}(S^Z_N|\mathcal{F}_n) = S^Z_n$$

Since the market is complete we have that any derivative can be replicated with a self-financing strategy, meaning that for any derivative with payoff X, there exists ϕ self-financing such that $V_n(\phi) = X$. We thus have

$$\mathbb{E}_{\mathbb{P}^Z}(S^Z_N|\mathcal{F}_n) = S^Z_n$$

$$\mathbb{E}_{\mathbb{P}^Z}(\frac{S_N}{Z_N}|\mathcal{F}_n) = \frac{S_n}{Z_n}$$

$$Z_n \mathbb{E}_{\mathbb{P}^Z}(\frac{S_N}{Z_N}|\mathcal{F}_n) = S_n$$

Furthermore, $\phi_n S_n$ is also a $\mathbb{P}^Z$-martingale, so the same identity holds for $\phi_n S_n$, and we thus obtain the desired identity for pricing a derivative with payoff X at time n

$$Z_n \mathbb{E}_{\mathbb{P}^Z}(\frac{\phi_N S_N}{Z_N}|\mathcal{F}_n) = \phi_n S_n$$

$$Z_n \mathbb{E}_{\mathbb{P}^Z}(\frac{X}{Z_N}|\mathcal{F}_n) = V_n(\phi) \quad ■$$