THE INVERSION FORMULA

We want to prove that $f \in L^1(\mathbb{R})$ can be recovered from its Fourier transform. Before we need to introduce a technical section about convolution and its behaviour with the Fourier transform.

Convolution and Fourier transform:

Recall that, given $f, g \in L^1(\mathbb{R})$, the convolution $f * g$ is a function defined by

$$(f*g)(x) = \int_{\mathbb{R}} f(t)\, g(x-t)\, dt .$$

Lemma: If $f, g \in L^1(\mathbb{R})$ then $f*g = g*f$ and $f*g \in L^1(\mathbb{R})$, with $\|f*g\|_1 \leq \|f\|_1 \cdot \|g\|_1$.

Proof: That $f*g = g*f$ is immediate after changing the variable $x-t=s$ in the integral above.

On the other hand, by Fubini's theorem

$$\|f*g\|_1 = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t)\, g(x-t)\, dt \right| dx \leq$$

$$\leq \int_{\mathbb{R}} |f(t)| \left(\int_{\mathbb{R}} |g(x-t)|\, dx \right) dt = \int_{\mathbb{R}} |f(t)|\, \|g\|_1\, dt$$

$$= \|g\|_1 \, \|f\|_1 \qquad \square$$

In this course the convolution will be applied mostly to a function $g \geq 0$ with $\int_{\mathbb{R}} g(t)\,dt = \|g\|_1 = 1$, and usually with a good decay at infinity. If we think of g as the probability density of a random variable Y, and we further assume that Y is centred at 0 (as e.g. a Gaussian or a t-Student) then

$$(f * g)(x) = \mathbb{E}(f(x-Y)).$$

Thus $(f*g)(x)$ is an "average", weighted by g, of the values of f near x. For examble, if we take the "uniform" density

$$g(t) = \chi_\delta(t) := \frac{1}{2\delta}\chi_{(-\delta,\delta)}(t) \qquad \delta > 0,$$

we get

$$(f*g)(x) = \int_{\mathbb{R}} f(x-t)\,\chi_\delta(t) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x-t)\,dt$$

The process of convolving f with a function g as above (centred and concentrated around 0) produces in general a function that is "similar" to f and has more regularity, especially when the function g is regular. We shall see all this in more detail soon.

<u>Def:</u> Let $g \in L^1(\mathbb{R})$, $g \geq 0$, and for $\delta > 0$ let

$$g_\delta(x) = \frac{1}{\delta}\, g\left(\frac{x}{\delta}\right).$$

Notice that $g_\delta \in L^1$ and $\|g_\delta\|_1 = \|g\|_1$.

The system $\{g_\delta\}_{\delta>0}$ is an <u>approximate identity</u> if

ⓐ $\displaystyle\int_{\mathbb{R}} |g_\delta(x)|\,dx = \|g_\delta\|_1 = 1$

ⓑ $\forall \eta > 0 \quad \displaystyle\int_{|x|>\eta} g_\delta(x)\,dx \xrightarrow{\delta \to 0} 0.$

For example, the system $\{\mathcal{X}_\delta\}_{\delta>0}$ defined previously is an approximate identity. The approximate identity we shall use more often is based on the Gaussian $G(t) = e^{-\pi t^2}$. Then $G_\delta(t) = \frac{1}{\delta} e^{-\pi\left(\frac{t}{\delta}\right)^2}$

<u>Exercise:</u> Prove that $\hat{G}_\delta(\xi) = e^{-\pi\delta^2\xi^2}$ (Hint: use that $\hat{G}(\xi) = G(\xi)$ together with the Fourier transform with respect to dilations).

The name "approximate identity" is justified by the following lemma.

<u>Lemma:</u> Let $\{g_\delta\}_{\delta>0}$ be an approximate identity. Then $\displaystyle\lim_{\delta\to 0} \|f * g_\delta - f\|_1 = 0.$

<u>Proof:</u> Since $\int_{\mathbb{R}} G_\delta(t)\,dt = 1$,

$$\|f - f * G_\delta\|_1 = \int_{\mathbb{R}} |f(x) - (f * G_\delta)(x)|\,dx =$$

$$= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(x) - f(x-t))\, G_\delta(t)\,dt \right| dx$$

By the change of variable $\frac{t}{\delta} = s$ and Fubini

$$\|f - (f * G_\delta)\|_1 = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(x) - f(x - \delta s))\, G(s)\,ds \right| dx$$

$$\leq \int_{\mathbb{R}} \|f - \tau_{\delta s} f\|_1\, G(s)\,ds$$

Here we apply the dominated convergence theorem: since

$$\|f - \tau_{\delta s} f\|_1\, G(s) \leq 2\|f\|_1\, G(s) \in L^1(\mathbb{R}),$$

$$\lim_{\delta \to 0} \|f - f * G_\delta\|_1 = \int_{\mathbb{R}} \lim_{\delta \to 0} \|f - \tau_{\delta s} f\|_1\, G(s)\,ds = 0 \quad \boxtimes$$

We are ready to state the first version of the inversion theorem.

<u>Theorem</u> Let $f, \hat{f} \in L^1(\mathbb{R})$. Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi \quad \text{a.e.} \quad x \in \mathbb{R}$$

Moreover, the right hand side is a continuous function which coincides with $f(x)$ when f is continuous at x.

<u>Proof:</u> Take the Gaussian $G(x) = e^{-\pi x^2}$, so that $G_\delta(x) = \frac{1}{\delta} e^{-\pi\left(\frac{x}{\delta}\right)^2}$. By the dilation property of the Fourier transform

$$\hat{G}_\delta(\xi) = G(\delta\xi) = e^{-\pi\delta^2\xi^2}$$

We can also go the opposite way: if $F_\delta(x) = e^{-\pi\delta^2 x^2}$ then

$$\hat{F}_\delta(\xi) = \int_{\mathbb{R}} e^{-\pi\delta^2 x^2} e^{-2\pi i x \xi}\, dx \overset{\delta x = t}{=} \int_{\mathbb{R}} e^{-\pi t^2} e^{-2\pi i t \frac{\xi}{\delta}} \frac{dt}{\delta}$$

$$= \frac{1}{\delta} \hat{G}\left(\frac{\xi}{\delta}\right) = \frac{1}{\delta} G\left(\frac{\xi}{\delta}\right) = G_\delta(\xi).$$

since for the Gaussian $\hat{G} = G$.

Then, by the multiplication formula, and changing $t = -y$

$$(f * G_\delta)(x) = \int_{\mathbb{R}} f(x-t)\, G_\delta(t)\, dt = \int_{\mathbb{R}} f(x+y)\, G_\delta(y)\, dy$$

$$= \int_{\mathbb{R}} (\tau_{-x} f)(y)\, \hat{F}_\delta(y)\, dy = \int_{\mathbb{R}} \widehat{(\tau_{-x} f)}(\xi)\, F_\delta(\xi)\, d\xi$$

$$= \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\; e^{-\pi \delta^2 \xi^2} \cdot d\xi$$

Now, taking limits as $\delta \to 0$ we will get the desired identity.

On the one hand, by the dominated convergence theorem

$$\lim_{\delta \to 0} \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} e^{-\pi \delta^2 \xi^2} d\xi = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi .$$

Let us finally prove that $(f * G_\delta)(x) \xrightarrow{\delta \to 0} f(x)$ a.e.

Changing the variable $t = \delta s$ we have

$$(f * G_\delta)(x) = \int_{\mathbb{R}} f(x-t)\, G_\delta(t)\, dt = \int_{\mathbb{R}} f(x - \delta s)\, e^{-\pi s^2} ds .$$

We shall see that this converges to f in $L^1(\mathbb{R})$:

$$\int_{\mathbb{R}} |(f * G_\delta)(x) - f(x)|\, dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x - \delta s) - f(x))\, e^{-\pi s^2} ds \right| dx \leq$$

$$\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x - \delta s) - f(x)|\, dx \right) e^{-\pi s^2} ds = \int_{\mathbb{R}} \|(\tau_{\delta s} f) - f\|_1\, e^{-\pi s^2} ds$$

By the dominated convergence theorem, as seen previously

$$\lim_{\delta \to 0} \int_{\mathbb{R}} \| (\tau_{\delta s} f - f) \|_1 \, e^{-\pi s^2} \, ds = 0$$

That the expression $\int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi$ defines a continuous function of x follows again from the dominated convergence theorem. In case f is continuous at x:

$$\left| \int_{\mathbb{R}} f(x - \delta s) e^{-\pi s^2} \, ds - f(x) \right| \leq \int_{\mathbb{R}} |f(x - \delta s) - f(x)| \, e^{-\pi s^2} \, ds$$

and, once more by the dominated convergence theorem this goes to 0

Corollary: (Uniqueness theorem)
Let $f \in L^1(\mathbb{R})$. If $\hat{f} = 0$ a.e, then $f = 0$ a.e.

Note: Given g, the operation

$$\check{g}(x) = \int_{\mathbb{R}} g(\xi) e^{2\pi i \xi x} \, dx$$

is called the Fourier co-transform. For $g \in L^1(\mathbb{R})$ with $\hat{g} \in L^1(\mathbb{R})$ we have just proved that

$$g(x) = \check{\hat{g}}(x) \qquad \text{a. e} \quad x \in \mathbb{R}.$$