Chapter 5

Fourier multipliers

A particular class of operators, important in applications, is the so-called *Fourier multipliers*. The defining feature of these operators is that their action on a given function f consists in multiplying its Fourier transform $\hat{f}(\xi)$ by a fixed function $m(\xi)$ (called a "filter" in signal processing). Thus, the general scheme when applying one such operator T is as follows: given f,

1. take its Fourier transform $\hat{f}$,
2. multiply it by m: $\hat{f} \longmapsto m\hat{f}$.
3. take the inverse Fourier transform: $T(f) = (m\hat{f})^\vee$.

To perform the third step some regularity on m is required (usually $m \in L^\infty$), so that T has some boundedness in L^2. The general formalism, seen from the non-Fourier side, would be $T(f) = f * m^\vee$, where $m^\vee$ is a distribution. In general we shall write instead $T = f * \mu$, so that $(Tf)^\vee = \widehat{f} \cdot \widehat{\mu}$.

Examples 6. 1. *Low pass filter*. The function m is 1 for low frequencies and it attenuates (or kills) high frequencies. Given a threshold c, the frequencies above c will be attenuated.

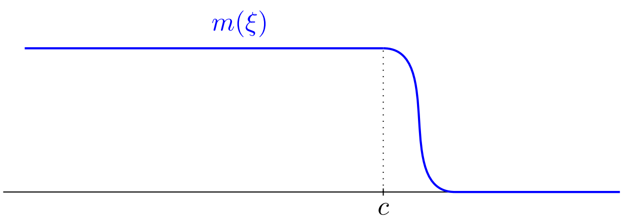

Figure 5.1: Low pass filter m

This is commonly used to clear noise in sound files.

2. *High pass filter*. Now the function m is 1 for high frequencies (above a threshold c) and it attenuates, or kills, low frequencies.

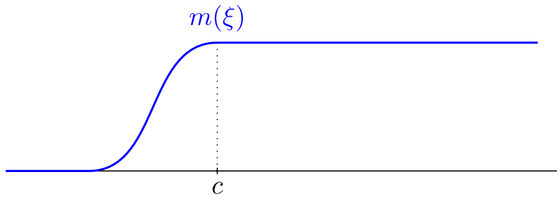

Figure 5.2: High pass filter m

3. *Band pass filter*. Here the extreme frequencies, both high and low, are killed. Given $c_1 < c_2$ one preserves the frequencies in the band $[c_1, c_2]$ and attenuates the rest.

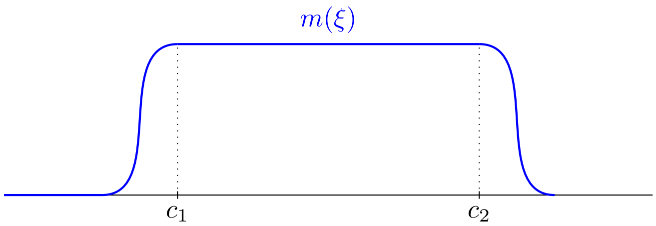

Figure 5.3: Band pass filter m

The human ear perceives frequencies between approximately 20Hz and 20.000Hz. Thus, when recording and reproducing a sound (a piece of music, etc.) nothing is lost if we restrict ourselves to this band (band-pass filter). On the other hand many reproducing devices are restricted to a certain range of frequencies (high or low, usually). This is also the case in other applications, like astronomy. For example, a woofer reproduces frequencies between 100Hz and 500Hz, a subwoofer between 20Hz and 100Hz, and a tweeter between 2.000Hz and 20.000Hz.

These kind of filters were used also, for example, in telephone lines with DSL splitters. By separating low and high frequencies the same wires carried:

- digital data (DSL: digital subscriber line),
- voice (POTS: plain old telephone service).

4. Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$, extended to be 2π-periodic in the whole $\mathbb{R}$. Its Fourier transform is essentially the sequence $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$. The multiplier is just another sequence $\{m_n\}_{n\in\mathbb{Z}}$, and the corresponding Fourier multiplier is

$$(Tf)(t) = \sum_{n\in\mathbb{Z}} m_n \hat{f}(n) e^{int}.$$

5. In general, given $f \in L^1(\mathbb{R})$ and $m \in L^\infty(\mathbb{R})$ one has

$$(Tf)(t) = \int_{\mathbb{R}} m(\xi)\, \hat{f}(\xi) e^{2\pi i \xi t} d\xi.$$

Particular examples of this are 1, 2,3 and

(i) Derivation. As seen in Theorem 4(b) $\widehat{f'}(\xi) = m(\xi)\hat{f}(\xi)$, with $m(\xi) = 2\pi i \xi$. Observe however that $m \notin L^\infty(\mathbb{R})$, so that this cannot be applied to all $f \in L^2(\mathbb{R})$.

This can also be seen through the general formalism of distributions: since $f' = f * \delta_0'$ then $\widehat{f'} = \hat{f} \cdot \widehat{\delta_0'}$. Since the action of δ_0' is given by $\langle \delta_0', \varphi \rangle = -\varphi'(0)$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, we have

$$\langle \widehat{\delta_0'}, \varphi \rangle = \langle \delta_0', \widehat{\varphi} \rangle = -(\widehat{\varphi})'(0) = \int_{\mathbb{R}} (2\pi i t)\, \varphi(t)\, dt = \langle 2\pi i t, \varphi \rangle.$$

So $\widehat{\delta_0'}(\xi) = 2\pi i \xi$ and $\widehat{f'}(\xi) = \hat{f}(\xi)\, (2\pi i \xi)$.

More generally, as seen in Remark 6, given a differential operator

$$P(D)(f) = a_0 + a_1 \frac{\partial f}{\partial t} + \cdots + a_N \frac{\partial^N f}{\partial t^N}$$

we have

$$\widehat{P(D)(f)}(\xi) = P(2\pi i \xi) \hat{f}(\xi).$$

This also works in several variables. For example, the Fourier transform of the Laplace operator in $\mathbb{R}^n$

$$\Delta f = \frac{\partial^2 f}{\partial t_1^2} + \cdots + \frac{\partial^2 f}{\partial t_n^2}$$

is

$$\widehat{\Delta f}(\xi) = \sum_{j=1}^n (2\pi i \xi_j)^2 \hat{f}(\xi) = -4\pi^2 |\xi|^2 \hat{f}(\xi).$$

(See Section 7.5 for the definiton of $\hat{f}(\xi)$ for a function of n variables).

These identities suggest also how to define fractional derivatives: given $\alpha > 0$ define $\frac{\partial^\alpha f}{\partial t^\alpha}$ through the identity

$$\widehat{\frac{\partial^\alpha f}{\partial t^\alpha}}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi).$$

(ii) Translation. Write $\tau_a(f) = f * \delta_a$. Then

$$\widehat{\tau_a(f)}(\xi) = \hat{f}(\xi)\, \widehat{\delta_a}(\xi) = \hat{f}(\xi)\, e^{-2\pi i a \xi}.$$

6. *Linear time-invariant filters*. Time invariant operators appear naturally in many applications. By this we mean operators T such that for all $a \in \mathbb{R}$ (called the "delay")

$$T\big(\tau_a(f)\big) = \tau_a\big(T(f)\big).$$

Writing

$$f(t) = \langle f, \delta_t \rangle = \int_{\mathbb{R}} f(u)\delta_t(u)\,du = \int_{\mathbb{R}} f(u)(\tau_t\delta_0)(u)\,du = \langle f, \tau_t\delta_0 \rangle,$$

the linearity and invariance of T yields

$$Tf(t) = \langle f, T\delta_t \rangle = \langle f, T\big(\tau_t\delta_0\big) \rangle = \langle f, \tau_t\big(T\delta_0\big) \rangle.$$

The distribution $H := T\delta_0$ is called the *impulse response* of T, and therefore

$$Tf(t) = \langle f, \tau_t H \rangle = \int_{\mathbb{R}} f(u)H(u-t)\,du = (f * H)(t).$$

This shows that Tf is a Fourier multiplier operator given by H.

Remark 13. The exponentials $e_\xi(t) = e^{2\pi i\xi t}$ are eigenfunctions of time-invariant operators (with eigenvalue $\widehat{H}(\xi)$):

$$(Te_\xi)(t) = (H * e_\xi)(t) = \int_{\mathbb{R}} H(u)e^{2\pi i\xi(t-u)}du = e_\xi(t)\widehat{H}(\xi).$$

Thus, decomposing a function as a superposition of exponentials can be seen as decomposing it as a superposition of eigenfunctions of a time-invariant-operator. This explains the ubiquity of Fourier analysis in operator theory.