Lesson 11

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Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. We say that X is a Brownian motion in $[0, T]$ if it satisfies:

i) $s \longmapsto X_s(\omega)$ is continuous a.s.-$\mathbb{P}$
ii) $X_0 = 0$ a.s.-$\mathbb{P}$
iii) $X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are ind. r.v. for all $0 \leq t_1 \leq ... \leq t_n \leq T$,
iv) $X_t - X_s \sim N(0, t - s)$ for $0 \leq s < t \leq T$.

Lemma

If X is a Brownian motion in $[0, T]$ and $\theta \in \mathbb{R}$ then

$$Z_t := \exp\left\{\theta X_t - \frac{1}{2}\theta^2 t\right\}, 0 \le t \le T$$

is a martingale.

Proof.

Let $t \ge s$

$$\begin{aligned}
& \mathbb{E}\left(Z_t \middle| Z_u, 0 \le u \le s\right) \\
= \quad & Z_s \mathbb{E}\left(\frac{Z_t}{Z_s} \middle| Z_u, 0 \le u \le s\right) \\
= \quad & Z_s \mathbb{E}\left(\exp\left\{\theta\left(X_t - X_s\right) - \frac{1}{2}\theta^2 (t-s)\right\} \middle| X_u, 0 \le u \le s\right) \\
= \quad & Z_s \mathbb{E}\left(\exp\left\{\theta\left(X_t - X_s\right) - \frac{1}{2}\theta^2 (t-s)\right\}\right),
\end{aligned}$$

Proof.

(continuation) and

$$
\begin{aligned}
& \mathbb{E}\left(\exp\left\{\theta\left(X_t - X_s\right)\right\}\right) \\
= & \int_{\mathbb{R}} e^{\theta x} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2(t-s)}x^2} \mathrm{d}x \\
= & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2(t-s)}(x^2 - 2\theta x(t-s))} \mathrm{d}x \\
= & e^{\frac{1}{2}\theta^2(t-s)} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2(t-s)}(x^2 - 2\theta x(t-s) + \theta^2(t-s)^2)} \mathrm{d}x \\
= & e^{\frac{1}{2}\theta^2(t-s)} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2(t-s)}(x - \theta(t-s))^2} \mathrm{d}x \\
= & e^{\frac{1}{2}\theta^2(t-s)}.
\end{aligned}
$$

□

Theorem

(Girsanov's theorem) Let X *be a Brownian motion in* $[0, T]$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$*, set* $\tilde{\mathbb{P}}(A) := \mathbb{E}(Z\mathbf{1}_A)$ *with*

$$Z = \exp\left\{\theta X_T - \frac{1}{2}\theta^2 T\right\}$$

for $A \in \mathcal{F}$*, then* $Y_t := X_t - \theta t$, $0 \leq t \leq T$ *is a Brownian motion in* $[0, T]$ *on* $\left(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\right)$.

Proof.

In the Definition 1 i) and ii) are trivially fulfilled. Set $0 = t_0 \leq t_1 \leq ... \leq t_n \leq t_{n+1} = T$ and let $B_1, B_2, ..., B_n, B_{n+1}$ be Borelian sets in $\mathbb{R}$ with $B_{n+1} = \mathbb{R}$. Set $A = \{X_{t_1} \in B_1, X_{t_2} - X_{t_1} \in B_2, ..., X_{t_n} - X_{t_{n-1}} \in B_n\}$

$$
\begin{aligned}
\tilde{\mathbb{P}}(A) &= \mathbb{E}\left(\exp\left\{\theta X_T - \frac{1}{2}\theta^2 T\right\}\mathbf{1}_A\right) \\
&= \mathbb{E}\left(\exp\left\{\sum_{i=1}^{n+1}\left(\theta\left(X_{t_i} - X_{t_{i-1}}\right) - \frac{1}{2}\theta^2(t_i - t_{i-1})\right)\right\}\mathbf{1}_A\right) \\
&= \int_{B_1}\int_{B_2}\cdots\int_{B_n}\int_{B_{n+1}}\prod_{i=1}^{n+1}\exp\left\{\theta u_i - \frac{1}{2}\theta^2(t_i - t_{i-1})\right\} \\
&\quad \times \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}}\exp\left\{-\frac{u_i^2}{2(t_i - t_{i-1})}\right\}\mathrm{d}u
\end{aligned}
$$

$\square$

Proof.

Therefore

$$\begin{aligned}\tilde{\mathbb{P}}(A) &= \prod_{i=1}^{n+1} \int_{B_i} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left\{ -\frac{(u_i - \theta(t_i - t_{i-1}))^2}{2(t_i - t_{i-1})} \right\} \mathrm{d}u_i \\ &= \prod_{i=1}^{n} \int_{B_i} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left\{ -\frac{(u_i - \theta(t_i - t_{i-1}))^2}{2(t_i - t_{i-1})} \right\} \mathrm{d}u_i,\end{aligned}$$

since $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}} \exp\left\{ -\frac{(u_{n+1}-\theta(t_{n+1}-t_n))^2}{2(t_{n+1}-t_n)} \right\} \mathrm{d}u_{n+1} = 1$ □

Proof.

(continuation) Then we conclude that

$$\begin{aligned}\tilde{\mathbb{P}}(X_{t_1} &\in B_1, X_{t_2} - X_{t_1} \in B_2, ..., X_{t_n} - X_{t_{n-1}} \in B_n) \\ &= \prod_{i=1}^{n} \tilde{\mathbb{P}}(X_{t_i} - X_{t_{i-1}} \in B_i),\end{aligned}$$

with $X_{t_i} - X_{t_{i-1}} \sim N(\theta(t_i - t_{i-1}), t_i - t_{i-1})$ or equivalently $X_{t_i} - X_{t_{i-1}} - \theta(t_i - t_{i-1}) \sim N(0, t_i - t_{i-1})$ and they are independent, so $X_t - \theta t$, satisfies *iii*) and *iv*). $\square$

Example

Consider the discounted price of the stock in the BS-model. We have that

$$\begin{aligned}
\mathrm{d}\tilde{S}_t &= \mathrm{d}\left(e^{-rt}S_t\right) = -re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t \\
&= e^{-rt}S_t\left(-r\mathrm{d}t + \mu\mathrm{d}t + \sigma\mathrm{d}W_t\right) \\
&= \sigma\tilde{S}_t\mathrm{d}\left(-\frac{r-\mu}{\sigma}t + W_t\right) \\
&= \sigma\tilde{S}_t\mathrm{d}\bar{W}_t \qquad (1)
\end{aligned}$$

with

$$\bar{W}_t = W_t - \frac{r-\mu}{\sigma}t.$$

Then by the Girsanov theorem with $\theta = \frac{r-\mu}{\sigma}$ it turns out that $\bar{W}$ is a Brownian motion with respect to the probability $\mathbb{P}^*$

$$\mathrm{d}\mathbb{P}^* = \exp\left\{\frac{r-\mu}{\sigma}W_T - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right\}\mathrm{d}\mathbb{P}.$$

Example

(continuation) From (1) we deduce that

$$\tilde{S}_t = S_0 \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma \bar{W}_t\right\}$$

and that $\tilde{S}$ is a $\mathbb{P}^*$-martingale. We also have that

$$S_t = S_0 \exp\left\{rt - \frac{1}{2}\sigma^2 t + \sigma \bar{W}_t\right\}.$$

Theorem

The Black-Scholes model is free of arbitrage.

Proof.

It is a particular case of modelling the stock process with an Itô process and where we can find a risk-neutral probability. □

Definition

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Theorem

The Black-Scholes model is complete. Any option with payoff $X \geq 0$, $\mathcal{F}_T$*-measurable and square integrable under* $\mathbb{P}^*$ *is replicable and its value, at time* $t \in [0, T]$*, is given by*

$$C_t = \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}X|\mathcal{F}_t).$$

Proof.

Under $\mathbb{P}^*$

$$M_t := \mathbb{E}_{\mathbb{P}^*}(e^{-rT} X | \mathcal{F}_t), 0 \le t \le T$$

is a square integrable martingale, then by the representation theorem of Brownian martingales, (that we shall discuss later) there exists a unique adapted process Y such that

$$M_t = M_0 + \int_0^t Y_s \mathrm{d}\bar{W}_s, 0 \le t \le T$$

with

$$\mathbb{E}_{\mathbb{P}^*}\left(\int_0^T Y_s^2 \mathrm{d}s\right) < \infty.$$

(It is important to note that the filtration generated by $\bar{W}$ and W is the same!) $\square$

Proof.

Then we can define ϕ_t^1 by

$$\phi_t^1 = \frac{Y_t}{\sigma \tilde{S}_t}$$

and we have that

$$\begin{aligned} M_t &= M_0 + \int_0^t Y_s \mathrm{d}\bar{W}_s = \int_0^t \phi_s^1 \sigma \tilde{S}_s \mathrm{d}\bar{W}_t \\ &= M_0 + \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_t = \tilde{V}_t(\phi) \end{aligned}$$

that is

$$\tilde{C}_t = C_0 + \int_0^t \phi_t^1 \mathrm{d}\tilde{S}_t.$$

Therefore the strategy (ϕ_t^0, ϕ_t^1) with $\phi_t^0 e^{rt} = C_t - \phi_t^1 S_t$ is self-financing and replicates X. To see that it is admissible it is enough to take into account that since $X \geq 0$, $C_t \geq 0$. $\square$

Theorem

In the BS model the price of an option with payoff $X = f(S_T) \geq 0$ *and square integrable with respect to* $\mathbb{P}^*$*, is given by* $C(t, S_t) = \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}X|\mathcal{F}_t)$ *and if* $C(t,x)$ *is* $C^{1,2}$*, the strategy that replicates* X *is given by* (ϕ_t^0, ϕ_t^1) *con*

$$\phi_t^1 = \frac{\partial C(t, S_t)}{\partial S_t}$$
$$\phi_t^0 e^{rt} = C(t, S_t) - \phi_t^1 S_t,$$

and $C(t, S_t)$ *is the solution of*

$$\frac{\partial C(t, S_t)}{\partial t} + rS_t \frac{\partial C(t, S_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t, S_t)}{\partial S_t^2} = rC(t, S_t). \qquad (2)$$

with the boundary condition $C(T, S_T) = f(S_T)$.

Proof.

First of all, by the independence of the relative increments

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}(\left(e^{-r(T-t)} f(S_T) \middle| \mathcal{F}_t \right) &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-r(T-t)} f\left(\frac{S_T}{S_t} S_t \right) \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(e^{-r(T-t)} f\left(\frac{S_T}{S_t} x \right) \right)_{x=S_t} \\
&= C(t, S_t),
\end{aligned}$$

so the price at t is a function of on S_t and t. Now define $\bar{C}(t,x) := e^{-rt} C(t, xe^{rt})$. Notice that $\bar{C}(t, \tilde{S}_t)$ is a $\mathbb{P}^*$-martingale:

$$\bar{C}(t, \tilde{S}_t) = e^{-rt} C(t, S_t) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-rT} f(S_T) \middle| \mathcal{F}_t \right),$$

□

Proof.

If If we apply now the Itô formula to $\bar{C}(t, \tilde{S}_t) = e^{-rt}C(t, \tilde{S}_t e^{rt})$, we have

$$\begin{aligned} &\bar{C}(t, \tilde{S}_t) - C(0, S_0) \\ &= \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial s} \mathrm{d}s + \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s} \mathrm{d}\tilde{S}_s + \frac{1}{2}\int_0^t \frac{\partial^2 \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s^2} \mathrm{d}\langle \tilde{S}, \tilde{S}\rangle_s \end{aligned}$$

and since

$$\mathrm{d}\tilde{S}_t = \sigma \tilde{S}_t \mathrm{d}\bar{W}_t$$

we obtain

$$\begin{aligned} &\bar{C}(t, \tilde{S}_t) - C(0, S_0) \\ &= \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s} \sigma \tilde{S}_s \mathrm{d}\bar{W}_s + \int_0^t \left(\frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial s} + \frac{1}{2} \frac{\partial^2 \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s^2} \sigma^2 \tilde{S}_s^2 \right) \mathrm{d}s. \end{aligned}$$

□

Proof.

Therefore, since $\bar{C}(t, \tilde{S}_t)$ is a martingale and the decomposition of an Itô process is unique we have that

$$\tilde{C}(t, S_t) = C(0, S_0) + \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s} \mathrm{d}\tilde{S}_s$$

$$\frac{\partial \bar{C}(t, \tilde{S}_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 \bar{C}(t, \tilde{S}_t)}{\partial \tilde{S}_t^2} \sigma^2 \tilde{S}_t^2 = 0.$$

$\square$

Proof.

Now since

$$\frac{\partial \bar{C}(t,\tilde{S}_t)}{\partial t} = -re^{-rt}C(t,S_t) + e^{-rt}\frac{\partial C(t,S_t)}{\partial t} + re^{-rt}S_t\frac{\partial C(t,S_t)}{\partial S_t}$$

$$\frac{\partial \bar{C}(t,\tilde{S}_t)}{\partial \tilde{S}_s} = e^{-rt}\frac{\partial C(s,S_t)}{\partial S_t}\frac{\partial S_t}{\partial \tilde{S}_t} = \frac{\partial C(t,S_t)}{\partial S_t}$$

and

$$\frac{\partial^2 \bar{C}(t,\tilde{S}_t)}{\partial \tilde{S}_t^2} = \frac{\partial^2 C(t,S_t)}{\partial S_t^2}\frac{\partial S_t}{\partial \tilde{S}_t} = e^{rt}\frac{\partial^2 C(t,S_t)}{\partial S_t^2},$$

□

Proof.

and we obtain that

$$\tilde{C}(t, S_t) = C(0, S_0) + \int_0^t \frac{\partial C(s, S_s)}{\partial S_s} \mathrm{d}\tilde{S}_s$$

that is

$$\phi_t^1 = \frac{\partial C(t, S_t)}{\partial S_t}, \ \phi_t^0 = \tilde{C}(t, S_t) - \frac{\partial C(t, S_t)}{\partial S_t} \tilde{S}_t$$

and

$$\frac{\partial C(t, S_t)}{\partial t} + rS_t \frac{\partial C(t, S_t)}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C(t, S_t)}{\partial S_t^2} = rC(t, S_t).$$

$\square$

Pricing and hedging of a call option. The Black-Scholes formula

For simplicity we write W instead of $\bar{W}$

$$
\begin{aligned}
& C(t, S_t) \\
& = \mathbb{E}_{\mathbb{P}^*}\left(e^{-r(T-t)}(S_T - K)_+ | \mathcal{F}_t\right) \\
& = e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left(S_T \mathbf{1}_{\{S_T > K\}} | \mathcal{F}_t\right) - Ke^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}(\mathbf{1}_{\{S_T > K\}} | \mathcal{F}_t) \\
& = e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left(\frac{S_T}{S_t}\mathbf{1}_{\left\{\frac{S_T}{S_t} > \frac{K}{x}\right\}}\right)_{x=S_t} - Ke^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\left\{\frac{S_T}{S_t} > \frac{K}{x}\right\}}\right)_{x=S_t},
\end{aligned}
$$

but

$$
\begin{aligned}
\frac{S_T}{S_t} & = \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\} \\
& \overset{\text{Law}}{=} \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma W_{T-t}\}
\end{aligned}
$$

then

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\left\{\frac{S_T}{S_t}>\frac{K}{x}\right\}}\right) &= \mathbb{P}^*\left(\frac{S_T}{S_t}>\frac{K}{x}\right) \\
&= \mathbb{P}^*\left(\log\frac{S_T}{S_t}>\log\frac{K}{x}\right) \\
&= \mathbb{P}^*\left(\frac{W_{T-t}}{\sqrt{(T-t)}}>\frac{\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\right) \\
&= \Phi\left(\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\right) \\
&= \Phi(d_-) \text{ (replacing } x \text{ by } S_t)
\end{aligned}
$$

Moreover, if we write Y to indicate a standard normal r.v.

$$e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left(\frac{S_T}{S_t}\mathbf{1}_{\left\{\frac{S_T}{S_t}>\frac{K}{x}\right\}}\right)$$

$$= e^{-r(T-t)}$$

$$\times\, \mathbb{E}_{\mathbb{P}^*}\left(\exp\left\{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}\right\}\mathbf{1}_{\left\{\sigma W_{T-t}>\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)\right\}}\right)$$

$$= \mathbb{E}_{\mathbb{P}^*}\left(\exp\left\{\left(-\frac{1}{2}\sigma^2\right)(T-t)+\sigma W_{T-t}\right\}\mathbf{1}_{\left\{\sigma W_{T-t}>\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)\right\}}\right)$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}}\exp\left\{-\frac{1}{2}\sigma^2(T-t)-\sigma\sqrt{(T-t)}y-\frac{1}{2}y^2\right\}\mathrm{d}y$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}}\exp\left\{-\frac{1}{2}\left(\sigma\sqrt{(T-t)}+y\right)^2\right\}\mathrm{d}y$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}}\exp\left\{-\frac{1}{2}u^2\right\}\mathrm{d}u = \Phi(d_+)\ \text{(replacing } x \text{ by } S_t\text{)}$$

Therefore

$$C(t, S_t) = S_t \Phi(d_+) - Ke^{-r(T-t)} \Phi(d_-)$$

where $\Phi(x)$ is the standard normal distribution function and

$$d_{\pm} = \frac{\log\left(\frac{S_t}{K}\right) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

It is easy to see that

$$\phi_t^1 = \frac{\partial C(t, S_t)}{\partial S_t} = \Phi(d_+) := \Delta.$$

and consequently that

$$\phi_t^0 = -Ke^{-rT}\Phi(d_-)$$