Lesson 16

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A general framework for short rates

We are going to define the process *bank account* or *riskless* asset. We shall create a random scenario for the instantaneous rates $r(s)$. More concretely we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq T})$, and we assume that $(\mathcal{F}_t)_{0\leq t\leq T}$ is the filtration generated by a Brownian motion $(W_s)_{0\leq t\leq T}$ and that $\mathcal{F}_T = \mathcal{F}$. In this context we introduce the *riskless* asset:

$$S_t^0 = \exp\left\{\int_0^t r(s)\mathrm{d}s\right\}$$

where $(r(t))_{0\leq t\leq T}$ is an adapted process with $\int_0^T |r(s)|\mathrm{d}s < \infty$. In our market we shall assume the existence of risky assets: the bonds! (without coupons) with maturity less or equal than the horizon T. For each time $u \leq T$ we define an adapted process $(P(t,u))_{0\leq t\leq u}$ satisfying $P(u,u) = 1$.

We make the following hypothesis:
(H) There exist a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that for all $0 \leq u \leq T$, $(\tilde{P}(t,u))_{0\leq t\leq u}$ defined by

$$\tilde{P}(t,u) = e^{-\int_0^t r(s)\mathrm{d}s} P(t,u)$$

is a martingale.
This hypothesis has the following interesting consequences:

Theorem

$$P(t,u) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^u r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right)$$

Proof.

$$\begin{aligned}\tilde{P}(t,u) &= \mathbb{E}_{\mathbb{P}^*}(\tilde{P}(u,u)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^u r(s)\mathrm{d}s} P(u,u) \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^u r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right),\end{aligned}$$

so, by eliminating the discount factor

$$P(t,u) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^u r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right)$$

$\square$

The purpose of the following results is to describe the dynamics of $(P(t,u))_{0\le t\le u}$. If we write, as usually, $Z_T = \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}$, we know that $Z_t := \mathbb{E}\left(\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right| \mathcal{F}_t\right)$ is a strictly positive martingale, then, since the filtration is that the generated by the Brownian motion, we have the following representation:

Theorem

There exists an adapted process $(q(t))_{0\le t\le T}$ such that, for all $0 \le t \le T$,

$$Z_t = \exp\left\{\int_0^t q(s)\mathrm{d}W_s - \frac{1}{2}\int_0^t q^2(s)\mathrm{d}s\right\}, \quad a.s.$$

Proof.

Since Z is a Brownian martingale there is a process H satisfying $\int_0^T H_t^2 \mathrm{d}t < \infty$, a.s., such that

$$Z_t = 1 + \int_0^t H_s \mathrm{d}W_s,$$

now since, fixed t, $Z_t > 0$, $\mathbb{P}$-a.s., and continuous, it can be proved that $Z_t > 0, 0 \leq t \leq T$ $\mathbb{P}$-a.s. Now by applying the Itô formula, we have

$$\log Z_t = \int_0^t \frac{H_s}{Z_s} \mathrm{d}W_s - \frac{1}{2} \int_0^t \frac{H_s^2}{Z_s^2} \mathrm{d}s$$

so $q(s) = \frac{H_s}{Z_s}$, $a.s.$ $\square$

It is convenient to know the following Lemma (Abstract Bayes's rule).

Lemma

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T},)$ *be a filtered probability space with* $\mathcal{F}_T = \mathcal{F}$. *Let* $Z_T > 0$ *such that* $\mathbb{E}(Z_T) = 1$ *and* $Z_t := \mathbb{E}(Z_T|\mathcal{F}_t), 0 \leq t \leq T$. *Define* $\mathbb{P}^*(A) := \mathbb{E}(\mathbf{1}_A Z_T), \forall A \in \mathcal{F}$, *then* $\mathbb{P}^* \sim \mathbb{P}$ *and if* Y *is an* $\mathcal{F}_t$*-measurable random variable such that* $\mathbb{E}_{\mathbb{P}^*}(|Y|) < \infty$ *then, for all* $s \leq t \leq T$,

$$\mathbb{E}_{\mathbb{P}^*}(Y|\mathcal{F}_s) = \frac{\mathbb{E}(YZ_t|\mathcal{F}_s).}{Z_s}$$

Proof.

Take $A \in \mathcal{F}_s$ then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}(\mathbf{1}_A Y) &= \mathbb{E}(\mathbf{1}_A YZ_T) = \mathbb{E}(\mathbf{1}_A \mathbb{E}(YZ_t|\mathcal{F}_s)) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_A \frac{1}{Z_s}\mathbb{E}(YZ_t|\mathcal{F}_s)\right).\end{aligned}$$

□

Corollary

The price at time t of a zero-coupon bond with maturity $u \leq T$ is given by

$$P(t,u) = \mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s + \int_t^u q(s)\mathrm{d}W_s - \frac{1}{2}\int_t^u q^2(s)\mathrm{d}s} \middle| \mathcal{F}_t \right)$$

Proof.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^u r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right) &= \frac{\mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s} Z_u \middle| \mathcal{F}_t \right)}{Z_t} \\
&= \mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s} \frac{Z_u}{Z_t} \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s + \int_t^u q(s)\mathrm{d}W_s - \frac{1}{2}\int_t^u q^2(s)\mathrm{d}s} \middle| \mathcal{F}_t \right).
\end{aligned}$$

□

The following theorem gives an economic interpretation of the process q.

Theorem

For each maturity u, there exists an adapted process $(\sigma_t^u)_{0\le t\le u}$ such that, for all $0 \le t \le u$,

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = (r(t) - \sigma_t^u q((t))\mathrm{d}t + \sigma_t^u \mathrm{d}W_t$$

Proof.

Since $\left(\tilde{P}(t,u)\right)$ is a martingale under $\mathbb{P}^*$ it turns out that $\left(\tilde{P}(t,u)Z_t\right)$ is a martingale under $\mathbb{P}$:

$$\tilde{P}(s,u) = \mathbb{E}_{\mathbb{P}^*}\left(\tilde{P}(t,u)\middle|\,\mathcal{F}_s\right) = \frac{\mathbb{E}\left(Z_t\tilde{P}(t,u)\middle|\,\mathcal{F}_s\right)}{Z_s}.$$

$\square$

Proof.

It is strictly positive as well and by reasoning as above we have

$$\tilde{P}(t,u)Z_t = P(0,u)e^{\int_0^t \theta_s^u \mathrm{d}W_s - \frac{1}{2}\int_0^t (\theta_s^u)^2 \mathrm{d}s},$$

for a certain adapted process $(\theta_s^u)_{0\leq t\leq u}$, in such a way that

$$P(t,u) = P(0,u)\exp\left\{\int_0^t r(s)\mathrm{d}s + \int_0^t (\theta_s^u - q(s))\mathrm{d}W_s \right.$$
$$\left. -\frac{1}{2}\int_0^t ((\theta_s^u)^2 - q^2(s))\mathrm{d}s\right\},$$

$\square$

Proof.

consequently, by applying the Itô formula,

$$
\begin{aligned}
\frac{\mathrm{d}P(t,u)}{P(t,u)} &= r(t)\mathrm{d}t + (\theta_t^u - q(t))\mathrm{d}W_t - \frac{1}{2}((\theta_t^u)^2 - q^2(t))\mathrm{d}t \\
&+\frac{1}{2}(\theta_t^u - q(t))^2\mathrm{d}t \\
&= (r(t) + q^2(t) - \theta_t^u q(t))\mathrm{d}t + (\theta_t^u - q(t))\mathrm{d}W_t,
\end{aligned}
$$

and the result follows by taking $\sigma_t^u = \theta_t^u - q(t)$. □

If we compare the formula

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = (r(t) - \sigma_t^u q((t))\mathrm{d}t + \sigma_t^u \mathrm{d}W_t$$

with

$$\frac{\mathrm{d}S_t^0}{S_t^0} = r(t)\mathrm{d}t$$

we find that the bonds are assets with greater risk the *riskless* asset S^0 and $-q(t)$ is the so-called *market price of risk.* Note also that, under $\mathbb{P}^*$

$$W_t^* := W_t - \int_0^t q(s)\mathrm{d}s$$

is a standard $(\mathcal{F}_t)$-Brownian and we can write

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = r(t)\mathrm{d}t + \sigma_t^u \mathrm{d}W_t^*,$$

or equivalently

$$\frac{\mathrm{d}\tilde{P}(t,u)}{\tilde{P}(t,u)} = \sigma_t^u \mathrm{d}W_t^*.$$

Options on bonds

Suppose a European contingent claim with maturity T and payoff

$$(P(T, T^*) - K)_+$$

where $T^* > T$ and $P(T, T^*)$ is the price of a bond with maturity T^*. The purpose is to valuate and hedge this call option on the bond with maturity T^*. It seems sensible to try to hedge this derivative with the riskless stock

$$S_t^0 = e^{\int_0^t r(s)\mathrm{d}s}$$

and the risky one

$$P(t, T^*) = P(0, T^*) \exp\left\{ \int_0^t (r(s) - \frac{1}{2}\left(\sigma_s^{T^*}\right)^2)\mathrm{d}s + \int_0^t \sigma_s^{T^*} \mathrm{d}W_s^* \right\},$$

in such a way that a strategy will be a pair of adapted processes $\left(\phi_t^0, \phi_t^1\right)_{0 \le t \le T^*}$ that represent the number of units of money in the bank account and the bonds with maturity T^* respectively. The value of the self-financing portfolio at time t is given by

$$V_t = \phi_t^0 S_t^0 + \phi_t^1 P(t, T^*).$$

The self-financing condition implies that

$$\begin{aligned}
\mathrm{d}V_t &= \phi_t^0 \mathrm{d}S_t^0 + \phi_t^1 \mathrm{d}P(t, T^*) \\
&= \phi_t^0 r(t) e^{\int_0^t r(s)\mathrm{d}s} \mathrm{d}t + \phi_t^1 P(t, T^*)(r(t)\mathrm{d}t + \sigma_t^{T^*} \mathrm{d}W_t^*) \\
&= (\phi_t^0 r(t) e^{\int_0^t r(s)\mathrm{d}s} + \phi_t^1 r(t) P(t, T^*))\mathrm{d}t + \phi_t^1 \sigma_t^{T^*} P(t, T^*)\mathrm{d}W_t^* \\
&= r(t) V_t \mathrm{d}t + \phi_t^1 \sigma_t^{T^*} P(t, T^*)\mathrm{d}W_t^*,
\end{aligned}$$

we shall impose the conditions $\int_0^T |r(t)V_t|\mathrm{d}t < \infty$ and $\int_0^T |\phi_t^1 \sigma_t^{T^*} P(t, T)|^2 \mathrm{d}t < \infty$, a.s., to get well defined objects.

Definition

A strategy $\phi = (\phi^0, \phi^1)_{0\le t\le T}$ is admissible if it is self-financing and its discounted value, $\tilde{V}$, is bounded from below.

Theorem

Fix $T < T^*$. Suppose that $\sigma_t^{T^*} \neq 0$ a.s. for all $0 \le t \le T$. Let X be a positive random variable $\mathcal{F}_T$-measurable such that $\tilde{X} := e^{-\int_0^T r(s)\mathrm{d}s} X$ is integrable under $\mathbb{P}^*$. Then there exists a unique admissible strategy such that at time T its value is X and at time $t \le T$ is given by

$$V_t = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s} X \middle| \mathcal{F}_t \right).$$

$\tilde{X}$ is a variable $\mathcal{F}_T$-measurable, with $\mathcal{F}_T = \sigma(W_t, 0 \leq t \leq T)$, it is integrable, as well, with respect to $\mathbb{P}^*$, so

$$M_t := \mathbb{E}_{\mathbb{P}^*}(\tilde{X}|\mathcal{F}_t), 0 \leq t \leq T$$

is a $\mathbb{P}^*$-martingale. However we cannot apply the representation theorem with $\mathbb{P}^*$ since we just have $\sigma(W_t^*, 0 \leq t \leq T) \subseteq \sigma(W_t, 0 \leq t \leq T)$ so M is a martingale w.r.t. $(\mathcal{F}_t)$. However by the abstract Bayes rule we know that MZ is a $\mathbb{P}$-martingale. In fact, we know that

$$M_t = \mathbb{E}_{\mathbb{P}^*}\left(\tilde{X}\middle|\,\mathcal{F}_t\right) = \frac{\mathbb{E}\left(\tilde{X}Z_T\middle|\,\mathcal{F}_t\right)}{Z_t}$$

in such a way that

$$M_t Z_t = \mathbb{E}(\tilde{X}Z_T|\mathcal{F}_t)$$

and $\left(\mathbb{E}(\tilde{X}Z_T|\mathcal{F}_t)\right)$ is clearly a $\mathbb{P}$-martingale.

In that way we have

$$M_t Z_t = \mathbb{E}(M_t Z_t) + \int_0^t J_s \mathrm{d}W_s,$$

with J adapted and such that $\int_0^T J_s^2 \mathrm{d}s < \infty$ a.s., so (notice that M is an Itô process)

$$Z_t \mathrm{d}M_t + M_t \mathrm{d}Z_t + \mathrm{d}\langle M, Z\rangle_t = J_s \mathrm{d}W_s,$$

that is

$$\begin{aligned}
\mathrm{d}M_t &= -M_t \frac{\mathrm{d}Z_t}{Z_t} - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t + \frac{J_t}{Z_t}\mathrm{d}W_t \\
&= -M_t q(t)\mathrm{d}W_t - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t + \frac{J_t}{Z_t}\mathrm{d}W_t \\
&= \left(\frac{J_t}{Z_t} - M_t q(t)\right)\mathrm{d}W_t - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t \\
&= \left(\frac{J_t}{Z_t} - M_t q(t)\right)\mathrm{d}W_t - \left(\frac{J_t}{Z_t} - M_t q(t)\right) q(t)\mathrm{d}t \\
&= \left(\frac{J_t}{Z_t} - M_t q(t)\right)\mathrm{d}W_t^* = L_t \mathrm{d}W_t^*,
\end{aligned}$$

with $L_t := \frac{J_t}{Z_t} - M_t q(t), 0 \leq t \leq T$. Where the fourth equality is due to the fact that M is an Itô process and so its covariation with Z, that is another Itô process will be absolutely continuous, then
$\mathrm{d}\langle M, M\rangle_t = \left(\frac{J_t}{Z_t} - M_t q(t)\right)^2 \mathrm{d}t$.

Proof.

Therefore if we take

$$\phi_t^1 = \frac{L_t}{\sigma_t^{T^*}\tilde{P}(t,T^*)}, \phi_t^0 = \mathbb{E}_{\mathbb{P}^*}(\tilde{X}|\mathcal{F}_t) - \frac{L_t}{\sigma_t^{T^*}}$$

we will have a self-financing portfolio with final value $e^{\int_0^T r(s)\mathrm{d}s} M_T = X$. In fact

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= \mathrm{d}\left(e^{-\int_0^t r(s)\mathrm{d}s} V_t\right) = -e^{-\int_0^t r(s)\mathrm{d}s} r(t) V_t \mathrm{d}t + e^{-\int_0^t r(s)\mathrm{d}s} \mathrm{d}V_t \\
&= e^{-\int_0^t r(s)\mathrm{d}s}(-r(t)V_t\mathrm{d}t + r(t)V_t\mathrm{d}t + \phi_t^1\sigma_t^{T^*}P(t,T^*)\mathrm{d}W_t^*) \\
&= \phi_t^1\sigma_t^{T^*}\tilde{P}(t,T^*)\mathrm{d}W_t^* = L_t\mathrm{d}W_t^* = \mathrm{d}M_t
\end{aligned}$$

Since $X \geq 0$ we have that $\tilde{V}_t \geq 0$, so the strategy is admissible. Also note that, since M is $\mathbb{P}^*$- integrable, $\int_0^T L_t^2 \mathrm{d}t < \infty$ and L is unique $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$ a.s. $\square$