Radon Inversion in the Computed Tomography Problem

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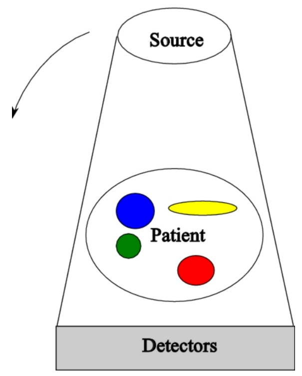
Acknowledgements

- Much of the theoretical background and intuition on the CT scan problem is drawn from a series of brilliant lectures delivered by Gunther Uhlmann, Peter Kuchment, and Leonid Kunyansky at the IPDE Summer School 2010 at the University of Washington.
- A comprehensive introduction to the mathematics of CT scans can be found in Charles Epstein's book, *Introduction to the Mathematics of Medical Imaging* [1].
- A more advanced treatment can be found in the classic book of Frank Natter, *The Mathematics of Computed Tomography*, [3].

The Computed Tomography Problem

- The Computed Tomography problem: Want to view the internal structure of something without cutting it open. Physical setup:
  1. Send radiation through the object and look at how it attenuates.
  2. Determine the object's density by looking at the attenuation patterns.
  3. Recover detailed image of internal structure from the density
- Example: (Medical CT) Tumors and other abnormalities have specific densities, distinct from healthy tissues.

A typical fan beam scan setup. Patient lies between a fixed detector panel and a rotating source. The source emits x-ray radiation in straight beams and the attenuated signals are collected by the detector.



Direct Problem: A model for X-ray Attenuation

Goal: Image a 2D slice of the patient's head.

- Let $\mathbf{I}(\mathbf{x})$ be the flux of radiation at the point $\mathbf{x} \in \mathbb{R}^2$. and let $I(\mathbf{x}) = \|\mathbf{I}(\mathbf{x})\|$ be the intensity of the radiation at $\mathbf{x}$.
- Let $\mu(x)$ be the attenuation coefficient for X-ray radiation. This coefficient describes the energy loss for radiation passing through $\mathbf{x}$ and is determined by the density and materials properties of the body. $\mu(\mathbf{x})$ is a reasonable proxy for the patient's density function.

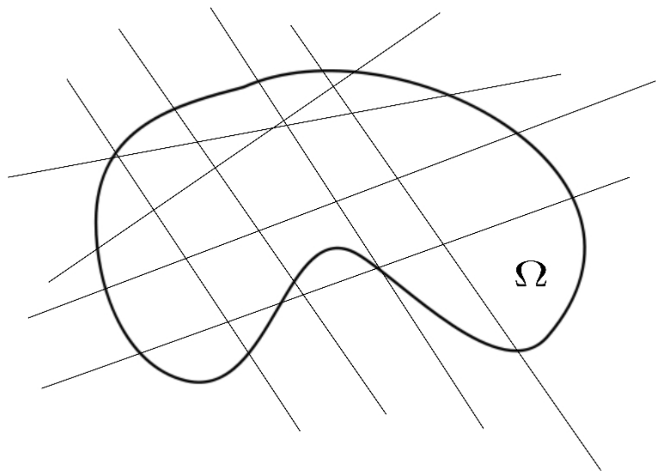
A model for X-ray Attenuation

- For X-ray radiation traveling along a straight line we have Beer's Law:
$$\frac{d\mathbf{I}}{dx} = -\mu(x)\mathbf{I}$$
- So if l is the line segment on the source and detector and the source emits initial intensity I_0, then the detector on the opposite side feels a radiation intensity
$$I = I_0 e^{-\int_l \mu(x)\, dx}$$

Inverse Problem

- We want to find $\mu(\mathbf{x})$.
- We know all the source intensities and we can compute all the detector intensities by computing all the integrals over all the lines on the sources and detectors.
- So solving the inverse problem is equivalent to resolving the following question: If we know all the line integrals of a function $f(\mathbf{x})$ in a domain Ω, is this enough information to recover the function in Ω?

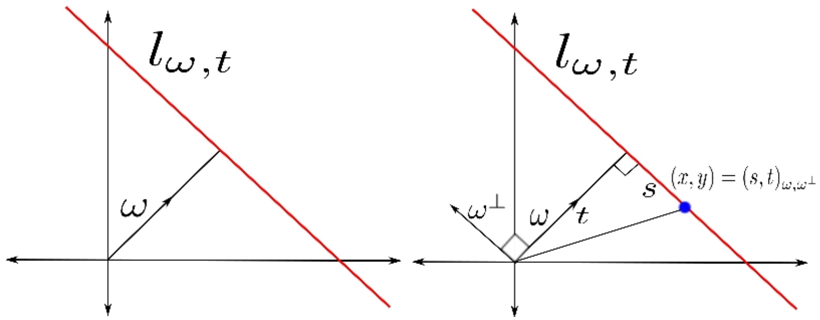
The Inverse Problem Sketch



The Radon Transform

Let l be a line in $\mathbb{R}^2$, t (affine parameter) denote the perpendicular distance from l to the origin, ω a unit vector perpendicular to l, and $\omega^{\perp}$ a unit vector perpendicular to ω. Then the points $\mathbf{x} \in l$ are those which satisfy

$$< x, \omega >= t$$



The Radon Transform

The **Radon Transform** of $f \in S(\mathbb{R})$ is the map given by:

$$\mathcal{R}f(t, \omega) = \int_{L_{t,\omega}} f(x)\, dx = \int_{-\infty}^{\infty} f(t\omega + s\omega^{\perp})\, ds.$$

The major questions:

- Domain
- Range
- Inversion

Easy Properties of the Radon Transform

- $\mathcal{R}$ maps a function on $\mathbb{R}^2$ into the set of its line integrals.
- Even: $\mathcal{R}f(-t, -\omega) = \mathcal{R}f(t, \omega)$.
- An easy computation shows that translations T_a for $a \in \mathbb{R}^2$ and rotations $O(\theta) \in SO(\mathbb{R}^2)$ commute with the Radon transform in the sense that

$$R(O(\theta)f)(t, \omega) = O(\theta)Rf(t, \omega)$$

$$R(T_a f)(t, \omega) = T_{a \cdot \omega} Rf(t, \omega)$$

The Radon and Fourier Transform

The translational and rotational invariance of the Radon transform will suggest to the harmonic analyst that the Radon transform might be related to the Fourier transform.

Theorem (Projection-Slice)

Let $f \in L^1(\mathbb{R})$ and the natural domain of $\mathcal{R}$

$$\int_{-\infty}^{\infty} \mathcal{R}f(t, \omega) e^{-itr}\, dt = \hat{f}(r\omega)$$

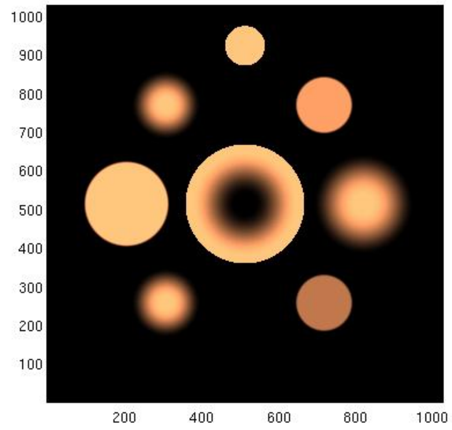
▸ Proof

- The 1D Fourier transform of $\mathcal{R}f$ in the affine parameter t is the 2D Fourier transform of f expressed in polar coordinates.

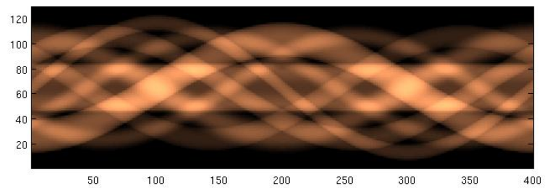
Radon Tranform Data

- Obviously in CT scanning we sample only a finite number of values of $\mathcal{R}(\mu)$.
- The raw output of a CT scan procedure is called a **sinogram**.
- Typically, we visualize this data by a "heat" plot of the values of the Radon transform against t and θ.
- Darker and lighter areas of the sinogram plot correspond to different values of the Radon transform.
- We won't usually have real X-ray data to use, but image files are a great substitute. Pixel locations correspond to points in space and gray scale colors correspond to densities.

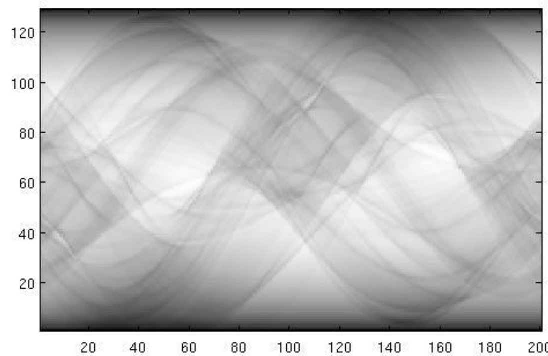
A Phantom:



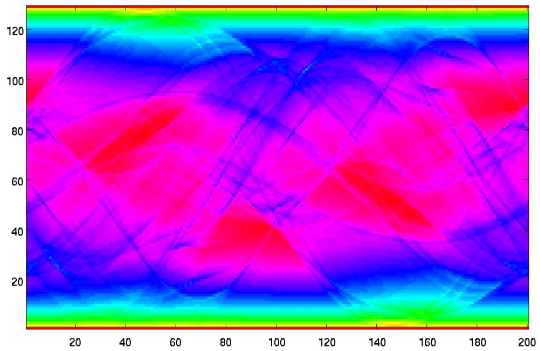
Sinogram



Whose sinogram is this?



Whose sinogram is this?



Whose sinogram is this?

Image Source http://images.hollywood.com/site/homer-simpson.jpg

Sine Waves in Sinograms

- The appearance of sine waves in sinograms is due to the fact that the Radon transform of a delta function has support on a trigonometric curve.
- Why? Regard the point $\mathbf{x}$ as a vector with angle θ_0. By trigonometry the only lines containing $\mathbf{x}$ will be those with angle θ and affine parameter t satisfying

$$t = |x| \cos(\theta - \theta_0).$$

- Since $\mathcal{R}\delta_x(t,\theta) = 0$ whenever x is not in $L_{t,\theta}$, the support of δ_x is a cosine curve in the θ-t plane.
- We can show that the Radon transform is continuous and thus an object made up of many small,sharp-edged features will have a sinogram that is a combination of blurred sine curves.

The Domain

- On first pass, we notice that for the Radon transform of f to be defined on $\mathbb{R} \times S^1$, we must have that the restriction of f to lines makes sense and that f decays fast enough along every line so that the line integrals converge. This suggest the *natural domain* of the Radon transform should be f satisfying
  1. f restricted to any line in $\mathbb{R}^2$ is locally integrable.
  2. f decays rapidly enough for the improper integrals in the definition to converge.
- For CT, this is probably much more general than necessary. Tissue structures should be piecewise continuous and people compactly supported!

More on Domains

- If f is compactly supported then $\mathcal{R}f(t,\omega)$ is compactly supported in the t parameter. (Why?)
- Note that continuous functions of compact support belong to the natural domain of $\mathcal{R}$, so by density we may extend the definition of the Radon transform to $f \in L^1(\mathbb{R})$. For $f \in L^1(\mathbb{R}^2)$, define $\mathcal{R}f$ to be the limit in L^1 norm of the sequence of approximates Rf_n, where $\{f_n\}$ is a sequence of smooth, compactly supported functions converging to f in the L^1 norm.

Continuity

- Continuity: For $f \in L^1(\mathbb{R}^2)$ and the natural domain of $\mathcal{R}$, define

$$\|\mathcal{R}f(t,\omega)\|_{1,\infty} = \sup_{\omega \in S^1} \int_{-\infty}^{\infty} |\mathcal{R}f(t,\omega)|\, dt.$$

Then

$$\|\mathcal{R}f\|_{1,\infty} \leq \|f\|_1.$$

Proof

- Continuity implies the scanning procedure is stable even if the patient can't keep completely still.

Schwartz Functions

- It will be useful to occasionally consider $\mathcal{R}$ defined over the set of Schwartz class functions:

$$\mathcal{S}\left(\mathbb{R}^2\right) = \{f \in C^\infty(\mathbb{R}) \mid \sup_{x \in \mathbf{R}^n} |x^\alpha D^\beta f(x) \leq \infty \quad \forall \alpha, \beta\}.$$

- We can define an analogous space over $T = \mathbb{R} \times S^1$, where the Schwartz decay happens in the affine parameter.

A Reconstruction

Let $f \in S(\mathbb{R}^2)$. Since $\mathcal{R}f$ is even, we can prove:

$$\mathcal{F}_t\mathcal{R}f(-r,-\omega) = \mathcal{F}_t\mathcal{R}f(r,\omega).$$

Then by the Fourier inversion theorem:

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i<\mathbf{x},\xi>}\hat{f}(\xi)\, d\xi \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} e^{ir<\mathbf{x},\omega>}\hat{f}(r\omega) r\, dr d\omega \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} e^{ir<\mathbf{x},\omega>}\mathcal{F}_t\mathcal{R}f(r,\omega) r\, dr d\omega \\ &= \frac{1}{(2\pi)^2}\int_0^{\pi}\int_{-\infty}^{\infty} e^{ir<\mathbf{x},\omega>}\mathcal{F}_t\mathcal{R}f(r,\omega)|r|\, dr d\omega. \end{aligned}$$

The Range Question

- We have the reconstruction formula

$$f(x) = \frac{1}{(2\pi)^2} \int_0^{\pi} \int_{-\infty}^{\infty} e^{irx\cdot\omega} \mathcal{F}_t \mathcal{R} f(t,\omega)|r| \, dr d\omega$$

- By the projection slice theorem, $\mathcal{R}$ has trivial kernel over most reasonable domains.
- Though we recover f uniquely by the above formula, without knowledge of the range there is no guarantee that this is the only reconstruction formula.

The CT Problem
Domain
Ranges
Reconstruction
References
Details for the Curious

Range

- Geometry: Suppose g is in the range of the radon transform. Then g is a function defined on $T = \mathbb{R} \times S^1$, but by evenness, we must identify $g(t,\omega)$ and $g(-t,-\omega)$. So T is in fact an infinitely wide Möbius strip.
- Equip T with the inner product

$$(f,g)_T = \int_T g(t,\omega)\overline{f(t,\omega)} \, d\omega ds$$

and define the Hilbert space $L_2(T)$ as the set of functions on T with $\|f\|_L^2(T) = \sqrt{(f,f)_T} < \infty$.
- The natural domain of f is also dense in $L^2(\mathbb{R}^2)$. Thus we might hope that there is a continuous extension of $\mathcal{R}$ to a map from $L^2(\mathbb{R}^2)$ to $L^2(T)$. *This does not happen.*

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A continuous L^2 extension of $\mathcal{R}$?

- $\mathcal{R}$ does not extend to a continuous operator between L^2 spaces. More importantly, $\mathcal{R}^{-1}$ is not continuous as an operator from $L^2(T) \to L^2(\mathbb{R}^2)$.
- Parseval relation:

$$\int_{\mathbb{R}^2} |f(x)|^2 \, dx = \int_0^{2\pi} \int_0^{\infty} |\mathcal{F}_r(\mathcal{R}(r,\omega))|^2 |r| \, dr d\omega$$

- Consequences:
  1. $\mathcal{R}$ is unbounded on $L^2(\mathbb{R}^2)$.
  2. $\mathcal{R}$ is a smoothing operator

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$\mathcal{R}$ as a smoothing operator

- Recall the Fourier transform identity

$$\mathcal{F}(\frac{d^n}{dx^n} f)(\xi) = i^n \xi^n \mathcal{F}(f)(\xi).$$

- This suggests a meaningful way to think about fractional order derivatives. In particular, define the 1/2 derivative operator by

$$D_{1/2} f = \mathcal{F}^{-1}(|\xi|^{1/2} \mathcal{F} f)$$

- Parseval:

$$\int_{\mathbb{R}^2} |f(x)|^2 \, dx = \frac{1}{(2\pi)} \int_0^{\pi} \int_{-\infty}^{\infty} |D_{1/2} \mathcal{R} f(t,\omega)|^2 \, dt d\omega$$

- Thus, a function which is a Radon transform has some additional smoothness beyond that of an arbitrary L^2 function.

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Range Conditions

- Let us attempt to come up with a description of the range for the Radon transform.
- For suitable choice of domain, the range of $\mathcal{R}$ turns out to be highly structured.
- Evenness.
- For f in the natural domain of $\mathcal{R}$, define

$$M_n(f)(\omega) = \int_{\mathbb{R}} t^n \mathcal{R}f(t,\omega)\, dt.$$

Claim: $M_n(f)(\omega)$ is a homogeneous polynomial of degree n in $\omega = (\omega_1, \omega_2)$ for any n.

Moments

Use the coarea formula to obtain:

$$\begin{aligned} M_n(f)(\omega) &= \int_{\mathbb{R}} t^n \mathcal{R}f(t,\omega)\, dt \\ &= \int_{\mathbb{R}} \int_{x\cdot\omega=t} t^n f(x)\, dsdt \\ &= \int_{\mathbb{R}^2} (x\cdot\omega)^n f(x)\, dx \end{aligned}$$

Thus the moments of the Radon transform of any smooth, compactly supported function are homogeneous polynomials with respect to components of ω.

The Range Result

Theorem

Let $g(t,\omega)$ satisfy

1. $g \in S(T)$
2. g *is even*
3. $M_n g(\omega) = \int_{\mathbb{R}} t^n g(t,\omega)\, dt$ *is a homogeneous polynomial in the components of* ω.

Then there is an f in $S(\mathbb{R}^2)$ so that $\mathcal{R}f = g$. In particular, $\mathcal{R}$ is a bijection from $S(\mathbb{R}^2)$ to $S(T)$.

Sketch of Proof.

Given g satisfying the conditions, the candidate inverse is specified by

$$\hat{f}(r\omega) = \mathcal{F}_t g(s,\omega).$$

If $\hat{f}$ is in the $S(T)$ then by projection-slice

$$\mathcal{F}_t g(s,\omega) = F_t \mathcal{R}f(s,\omega).$$

Thus the proof amounts to obtaining $\hat{f} \in S(T)$.

First Reconstruction Attempt: Back-Projection

- Reconstruction:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_0^{\pi} \int_{-\infty}^{\infty} e^{ir<\mathbf{x},\omega>} \mathcal{F}_t \mathcal{R} f(r,\omega)|r|\, dr d\omega.$$

- For $g \in S(T)$, define the *back-projection* operator by

$$\mathcal{R}^{\sharp} g(\mathbf{x}) = \int_{S^1} g(x \cdot \omega, \theta)\, d\omega$$

- Intuitively, back projection averages g over every line through x.

Back-projection

- Write the reconstruction formula as:

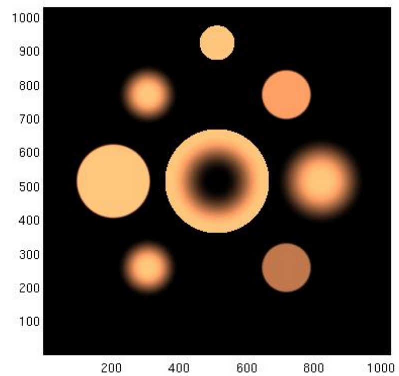
$$f(\mathbf{x}) = \frac{1}{2} \frac{1}{(2\pi)^2} \mathcal{R}^{\sharp} \left(\int_{-\infty}^{\infty} e^{ir<\mathbf{x},\omega>} \mathcal{F}_t \mathcal{R} f(r,\omega)|r|\, dr \right).$$

- This reconstruction can be viewed as a two stage process: inner step is filtration and outer step is back-projection.
  - Radial integral is a filter ($|r|$) applied to the Radon transform. Back-projection is the angular integral.
  - Formula is often called filtered back-projection formula.
  - Factor $|r|$ suppresses low-frequency components and amplifies high frequency components.
  - Numerically useful but not in this form.

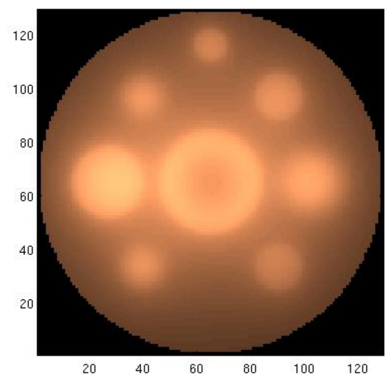
Back-projection

- Does back-projecting alone recover f from $\mathcal{R}f$?
- This might seem like a good guess since we know the value of every $\int_L f\, dx$ where L is a line through x.
- Answer: no.

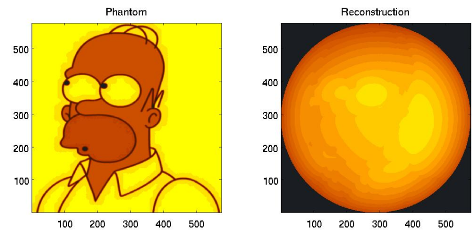
A Phantom:



Back-projection: Blurring



Back-projection: Fail



Back-Projection

For functions $f(x)$ from a "good domain" space, recalling that $t = x \cdot \omega$, compute:

$$\mathcal{R}^{\#}\mathcal{R}f(x) = \int_{S^1} \mathcal{R}f(x \cdot \omega, \theta)\, d\omega$$
$$= \int_{S^1} \int_{-\infty}^{\infty} f((x \cdot \omega)\omega + s\omega^{\perp})\, ds d\omega$$

Basically, a polar integral. Rewrite:

$$\mathcal{R}^{\#}\mathcal{R}f(x) = \int \frac{2f(y)}{|y-x|} dy = \frac{2}{|x|} * f(x).$$

Thus, we should recover not $f(x)$ but a blurred version of $f(x)$ with back-projection.

A Little More on Back-Projection

- Formally, we can show that $\mathcal{R}^{\sharp} : L^2(T) \to L^2(\mathbb{R}^2)$ is up to a constant the (non-continuous) dual to $\mathcal{R} : L^2(\mathbb{R}^2) \to L^2(T)$.
- That $\mathcal{R}^{\sharp}$ does not recover f from $\mathcal{R}f$ reflects the fact the $\mathcal{R}$ is not a unitary transformation on L^2 spaces.

A second look at filtration

It is very interesting to note that if we replace $|r|$ with just r in the formula

$$f(\mathbf{x}) = \frac{1}{2}\frac{1}{(2\pi)^2}\mathcal{R}^\sharp\left(\int_{-\infty}^{\infty} e^{ir<\mathbf{x},\omega>}\mathcal{F}_t\mathcal{R}f(r,\omega)|r|\,dr\right)$$

we could write

$$\begin{aligned} f(x) &= \frac{1}{4\pi i}\mathcal{R}^\sharp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ir<\mathbf{x},\omega>}\mathcal{F}_t(\partial_t\mathcal{R}f)(r,\omega)\,dr\right) \\ &= \frac{1}{4\pi i}\mathcal{R}^\sharp\left(\partial_t\mathcal{R}f(t,\omega)\right) \end{aligned}$$

The Hilbert Transform

This motivates the introduction of the Hilbert transform: for $g \in L^2(\mathbb{R})$ with $\hat{g} \in L^1(\mathbb{R})$: We see that the filtration step in our reconstruction is

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ir<\mathbf{x},\omega>}\mathcal{F}_t\mathcal{R}f(r,\omega)|r|\,dr &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{sgn} r r\mathcal{F}_t\mathcal{R}f(r,\omega)e^{irt}\,dr \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{sgn} r\mathcal{F}_t(\partial_t\mathcal{R}f)(r,\omega)e^{itr}\,dr \end{aligned}$$

Hilbert Transform Formulation

We have the filtered back-projection formula:

$$f(\mathbf{x}) = \frac{1}{4\pi i}\mathcal{R}^\sharp H\frac{d}{dt}(\mathcal{R}f)$$

- Filtration is itself two steps: derivative and Hilbert transform.
- Computationally, this is a superior formulation.

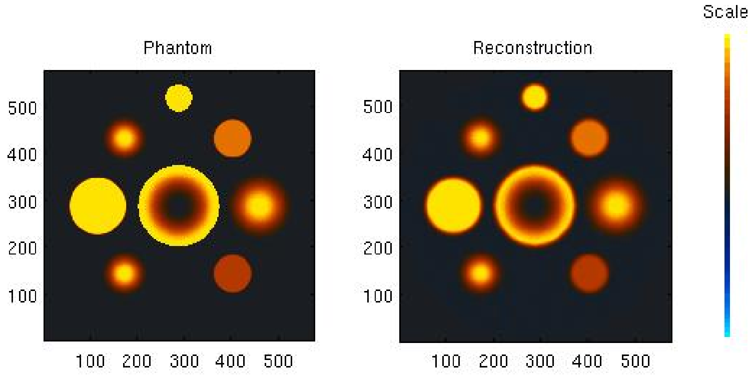
Numerical Methods

- Given: real or simulated Radon transform data over a discrete range of affine parameter values and angles.
- Our samples are approximately 500 pixels x 500 pixels. We sample 200 evenly spaced angles in $[0, 2\pi]$. For each angle sample we approximate the line integral through the angle at 130 different values of t. This gives about 26,000 sample data points.
- Use the Hilbert transform back-projection formula:

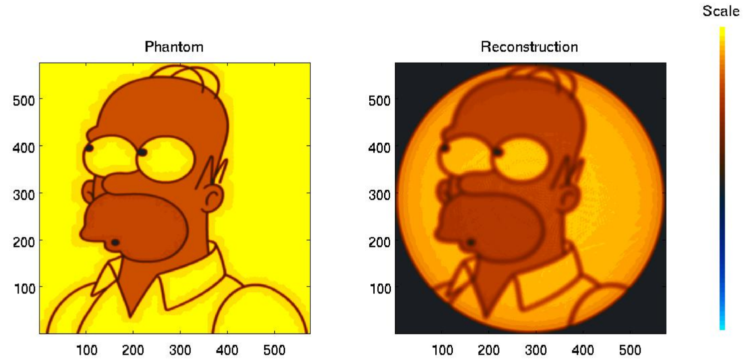
$$f = \frac{1}{4\pi}\mathcal{R}^\#\mathcal{H}\frac{d}{dt}(\mathcal{R}f).$$

- Use differencing to approximate $\frac{d}{dt}Rf$.
- Possible to approximate $\mathcal{H}\frac{d}{dt}(\mathcal{R}f)$ as a convolution
- Integrate approximately to back-project.

Reconstruction of "bump" phantom



Reconstruction of Homer



To-Do

- Prove the range conditions.
- More sophisticated description and analysis for Hilbert transform.
- Incomplete data problem: "hole" theorem
- Edge detection viewpoint.
- Relation to microlocal analysis

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Proof of Continuity Estimate

Fix ω and make the orthogonal change of variables $(x,y)^T = [\omega, \omega^\perp](t,s)^T$ to see

$$\int_{-\infty}^{\infty} |\mathcal{R}f(t,\omega)|\ dt = \int_{\mathbb{R}^2} |f(t\omega + s\omega^\perp)|\ dt = \int_{R^2} |f(x,y)|\ dxdy.$$

Back

Proof of the Projection-Slice Theorem

Proof. By definition

$$\int_{-\infty}^{\infty} \mathcal{R}f(t,\omega)e^{-itr}\ dt = \int_{\mathbb{R}^2} f(t\omega + s\omega^\perp)e^{-itr}\ dsdt.$$

Change variables: $\mathbf{x} = t\omega + s\omega^\perp$ and use the fact that $t =< \mathbf{x}, \omega >$ to obtain

$$\int_{\mathbb{R}^2} f(t\omega + s\omega^\perp)e^{itr}\ dsdt = \int_{\mathbb{R}^2} f(\mathbf{x})e^{-i<\mathbf{x},\omega>r}\ d\mathbf{x} = \hat{f}(r\omega).$$

$\square$ Back