Lab 1: Basic Digital Signal Processing

1. Sketch of concepts

The functions that we will consider represent digital signals. Thus the natural model to present them are sequences of real numbers.

$$x = \{x_n\}_{n=-\infty}^{\infty} = x[n].$$

In practice we only have a finite sequence of terms $x_1, \dots, x_k$ but to analyze it we may consider it infinite by padding it with infinite zeros before and after it. Sometimes, it is also convenient to consider complex valued sequences although they may have not a physical interpretation.

We are interested in transforming this signal, to eliminate noise from it for instance. The type of transformations that we are going to consider are linear operators $y = T(x)$, $y = \{y_n\}_{n=-\infty}^{\infty}$, i.e

$$T(\lambda x + \mu z) = \lambda T(x) + \mu T(z).$$

This operators are usually denoted in the engineering literature as Linear Systems.

Another important property that allows us to use Fourier Analysis techniques is that we will only consider time invariant linear systems (LTI for short).

DEFINITION 1. A system T is invariant if it commutes with the translations, i.e. $T \circ \tau_m = \tau_m \circ T$, where τ_m is a delay of the sequence: $\tau_m(x[n]) = x[n-m]$.

That is, in a LTI "if we delay the input, we delay the output". If $Tx = y$ then $T(\tau_m(x)) = \tau_m(y)$.

In the following, we will use infinite series but we will not worry about convergence because most of our signals will be compactly supported or very rapidly decaying.

DEFINITION 2. A basic example of discrete signal if the impulse signal δ, this signal is defined as

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

This is a basic piece because we can express any sequence as a linear combination of delays of the delta function, that is $x = \sum_k x[k]\tau_k\delta$. If we have a linear system, we give a special name as how it responds to the delta function.

DEFINITION 3. Given any linear time invariant system T we define its impulse response function h as

$$h = T(\delta).$$

The impulse response function $h[n]$ of an LTI system is very important because it completely characterizes the system.

PROPOSITION 1. *Given any LTI system T, with impulse response function h, then the system $y = T(x)$ acts as*

$$y[n] = \sum_{k\in\mathbb{Z}} x[k]h[n-k].$$

PROOF. We know that for all x, $x = \sum_k x[k]\tau_k\delta$. Then by linearity

$$T(x) = \sum_k T(x[k]\tau_k\delta) = \sum_k x[k]T(\tau_k\delta).$$

By the time invariance

$$T(x) = \sum_k x[k]T(\tau_k\delta) = \sum_k x[k]\tau_k(T(\delta)) = \sum_k x[k]\tau_k h.$$

□

We recognize immediately the convolution:

DEFINITION 4. Given two signals x, z. We define its convolution $y = x \star z = z \star x$ as

$$y[n] = \sum_{k\in\mathbb{Z}} x[k]z[n-k].$$

This is the complete analogous definition that we had in the real setting. We have just proved that any LTI system is a convolution operator with the impulse response function. It is a very easy exercise to check that given any impulse response function h (with compact support to avoid problems of convergence) defines an operator by convolution that is LTI. Thus we have a complete characterization.

1.1. Fourier analysis and LTI. We are going to use Fourier analysis to describe and understand LTI systems.

DEFINITION 5. We define the Fourier transform of a signal $x[n]$ as a complex valued function X defined on $[-\pi, \pi]$.

$$X(\omega) = \sum_{k\in\mathbb{Z}} x[k]e^{-ik\omega}.$$

Observe that we may recover the signal from its Fourier transform as:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{in\omega}\,d\omega.$$

Observe that the role of the Fourier coefficients and the function is reversed with the Fourier series that you know.

A signal $x[n]$ is characterized by its Fourier transform X. As the physical signals that we will consider are real valued signals, observe that $X(-\omega) = \overline{(X(\omega))}$.

If a signal x has Fourier transform X with support in $(-\delta, \delta) \subset [-\pi, \pi]$ near the origin we say that x is a low frequency signal. If the support intersects the complement $\operatorname{supp} X \cap ([-\pi, \pi] \setminus (-\delta, \delta))$, we say that x has high frequency content.

Let us see that the complex valued signals $x[n] = e^{i\omega n}$, for some fixed $\omega \in [-\pi, \pi]$ are the eigensequences of any LTI system T with response function h.

We know that $y = T(x) = x \star h$. That is

$$y[n] = \sum_k h[k]x[n-k] = \sum_{k\in\mathbb{Z}} h[k]e^{i\omega(n-k)} = e^{i\omega n}\left(\sum_{k\in\mathbb{Z}} h[k]e^{-i\omega k}\right).$$

If we denote by $H(\omega)$ the Fourier transform of the impulse response function $h[n]$, i.e., $H(\omega) = \sum_{k\in\mathbb{Z}} h[k]e^{-i\omega k}$, we obtain

$$y[n] = T(x[n]) = H(\omega)x[n].$$

That means that $x[n]$ is an eigenfunction of the TLI system with eigenvalue $H(w)$. Therefore all TLI system "diagonalize" through the Fourier transform. If we know how the system acts against the exponentials, we know the eigenvalues $H(\omega)$ and therefore we know $h[n]$ which describes the system.

This can also be seen by the convolution theorem.

THEOREM 1. *If we have signals y, x, h such that*

$$y = x \star h$$

and $Y(\omega)$, $X(\omega)$, $H(\omega)$ are its corresponding Fourier transforms, then

$$Y(\omega) = H(\omega)X(\omega)$$

.

PROOF. We compute the Fourier transform of y:

$$\begin{aligned} Y(\omega) &= \sum_{n\in\mathbb{Z}} y[n]e^{-in\omega} = \sum_{n\in Z}\left(\sum_{k\in\mathbb{Z}} x[k]h[n-k]\right)e^{-in\omega} = \\ &= \sum_{k\in Z} x[k]\left(\sum_{n\in\mathbb{Z}} h[n-k]e^{-in\omega}\right). \end{aligned}$$

We make the change of variables $m = n - k$ and we get

$$Y(w) = \sum_{k\in Z} x[k] \left(\sum_{m\in\mathbb{Z}} h[m] e^{-i(m+k)\omega} \right) =$$
$$= \left(\sum_{k\in Z} x[k] e^{-ik\omega} \right) \left(\sum_{m\in\mathbb{Z}} h[m] e^{-im\omega} \right).$$

That is $Y(\omega) = X(\omega)H(\omega)$. $\square$

Thus one can interpret $H(\omega)$, the Fourier transform of the impulse response function $h[n]$, as the spectral multiplier of the LTI system. It is usually called the frequency response of the LTI system. If $|H(\omega)|$ is very small for some values of $\omega \in [-\pi, \pi]$ then the system T attenuates the frequencies of a signal x at that frequencies. A low pass system is a system such that $|H(\omega)|$ is small for big w and a high pass system has frequency response small for small values of ω.

For many applications the relevant signal is low-frequency and the high frequency components are due to noise. Thus it is interesting to construct low pass filters to eliminate the noise from the signal.

Exercise 1. *Take a look at the data compiled by el Departament de Salut of new infections by COVID-19 in Catalunya since March 1, 2020. I have extracted the total data (without segregation by sex or age) in the file "newcases.txt" Observe that the daily data is very noisy. In some graphic representations widely used they have opted to reduce the noise by drawing a curve using points obtained by taking a five day moving average of the data. That is they replace* $x[n]$ *by* $y[n] = (x[n-2] + x[n-1] + x[n] + x[n+1] + x[n+2])/5$ *and draw a curve passing through the new points* $y[n]$ *to observe the trend.*

- *Express the new data* $y[n]$ *as a convolution against some impulse response function* h, $y = x \star h$.
- *Compute and draw the absolute value of the Fourier transform of the filter* h.
- *Try to explain why the noise is reduced. You can either plot the Fourier transform of use the Octave/Matlab command freqz to analyze it.*
- *Propose alternative filters* h.

Exercise 2. *Eliminate the noise from a GPS, signal. Take the heights obtained from a GPS from one of the stages of the circuit "Carros de Foc". Concretely from the stage from Refugi Ventura Calvell to Estany Llong. You can get the original file here. I have parsed the heights and they are in the file heights.txt.*

- *Compute the total height climbed and descended, that is compute*

$$Climbing = \sum_{n:x[n+1]>x[n]} x[n+1] - x[n].$$

and

$$Descending = \sum_{n:x[n+1]<x[n]} x[n] - x[n+1].$$

where $x[n]$ *is a vector that contains the heights that you should have read previously.*
- *Apply a low pass filter, i.e. remove part of the noise produced by errors in the GPS measurement. For this you can try to find the coefficients* $h[n]$ *of a low pass filter such that its Fourier transform mimics a characteristic function. Alternatively you can use a filter as in the previous exercise.*
- *Recompute Climbing and Descending once we have applied a low pass filter.*
- *Explain the difference.*

Bibliography

[1] Ronald W. Schafer. Alan V. Oppenheim, *Discrete-time signal processing.*, Prentice Hall, 2010.