DYNAMICAL SYSTEMS
MASTER IN ADVANCED MATHEMATICS
Fall semester, 2023

Exercises to be exposed: The solution of each exercise should be exposed in the blackboard by the mentioned participant in the course, on the given order, on 14 or 21 December. The exposition should be between 5 and 10 minutes.

1. (David Ache-Vogel) Prove that if $g : \mathbb{R} \to \mathbb{R}$ is C^1 and commutes with $f(x) = \lambda x$, with $\lambda \in (0, 1)$ then g is linear.

2. (David Arribas) Let $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$. Assume that A is hyperbolic and invertible. Prove that there exists $\varepsilon > 0$ such that if $\|B - A\| < \varepsilon$ then B is hyperbolic and is locally topologically conjugate to A.

3. (Tobias) Let $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$ be invertible hyperbolic maps. Assume that A is a contraction and A, B are locally topologically conjugated. Prove that B is a contraction.

4. (Leonardo) Prove that if $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$ are invertible hyperbolic maps locally topologically conjugated and A is a contraction then the local conjugacy can be extended to a global one.

5. (Salim) Let $0 < \lambda < 1$. Prove that

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

   are topologically conjugate.

6. (Oscar) Prove that

$$\begin{pmatrix} \sqrt{2}/4 & -\sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

   are topologically conjugate.

7. (Guillermo) Prove that $f(x) = \frac{1}{2}x$ and $g(x) = -\frac{1}{2}x$ are not topologically conjugate.

8. (Guillem Carbó) Prove that

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

   are topologically conjugate.

9. (Juan Carlos) Provide an explicit conjugation from

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{to} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

   Do the calculations in terms of the eigenvalues of A.

10. (Anna) Let $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$ be two invertible hyperbolic maps that are conjugate. Let h be the conjugation. Let E^s_A and E^s_B be the stable subspaces of A and B, respectively. Prove that h sends E^s_A to E^s_B and that $h_{|E^s_A}$ conjugates $A_{|E^s_A}$ to $B_{|E^s_B}$.

11. (Junhan) Let $\lambda \in (0, 1)$. Find all resonaces of the vector (λ, λ^{-1}).

12. (Joan) Let p be a hyperbolic periodic point of a C^1 diffeomorphisms f (all eigenvalues of $Df^k(p)$ have modulus different from 1, where k is the period). Prove that given $n \geq 1$ there is a neighbourhood V of p such that any periodic point of f in $V \setminus \{p\}$ has period bigger than n.

13. (Antoni) Assume that $(\lambda, \mu) \in \mathbb{R}^2$ with $0 < \lambda < 1$ and $\mu > 1$. Prove that there exist two sequences of integers $\{n_k\}$, $\{m_k\}$ such that

$$\lambda - \lambda^{n_k} \mu^{m_k} \to 0$$

Use the following fact: from the method of continued fractions one can obtain two sequences of integers $\{p_k\}$, $\{q_k\}$ such that $|x - p_k/q_k| < 1/q_k^2$.

14. (Elena) Let $F(x, y) = (x+y+\varepsilon \sin x, y+\varepsilon \sin x)$, $\varepsilon > 0$, be the Chirikov standard map. Find a linear change of variables that transforms it to $G(x, y) = (y, -x+2y+\varepsilon \sin x)$. Prove that G can be written as the composition of two involutions, one of them being $I_1(x, y) = (y, x)$. Prove that G is conjugate to its inverse. Let $p_0 = (0, 0)$ and $p_1 = (-2\pi, -2\pi)$. Prove that if $W^s(G, p_0)$ intersects the line $y = -x + 2\pi$ at some point q, then q belongs to $W^u(G, p_1)$.

15. (Alba) Consider the pendulum equations

$$\dot{x} = y, \qquad \dot{y} = -\sin x.$$

Let $(x(t), y(t))$ be the solution such that $(x(t), y(t)) = (2\pi, y_0)$ with $y_0 \neq 0$. Prove that

$$x(-t) = 2\pi - x(t), \qquad y(-t) = y(t), \qquad \text{for all} \quad t \in \mathbb{R}.$$

16. (David Rosado) Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^r map such that has a hyperbolic fixed point at $0 \in U$. Prove that there exists a local change of variables C of class C^r such that $C(0) = 0$, $DC(0) = \mathrm{Id}$ and $\tilde{f} = C^{-1} \circ f \circ C$ satisfies $W^s_{\mathrm{loc},\delta}(\tilde{f}, 0) \subset E^s$ and $W^u_{\mathrm{loc},\delta}(\tilde{f}, 0) \subset E^u$.

17. (Eduard) In the setting of the previous exercise, assuming $f : \mathbb{R}^n \to \mathbb{R}^n$, prove that if δ is small enough, the local change C can be extended to a global diffeomorphism $\tilde{C} : \mathbb{R}^n \to \mathbb{R}^n$.

18. (Yating) Consider the equation (Duffing equation)

$$\dot{x} = y, \qquad \dot{y} = x - x^3.$$

Prove that all solutions are defined for all values of $t \in \mathbb{R}$.

19. (Yang) Consider the equation

$$\dot{x} = y, \qquad \dot{y} = x - x^2.$$

Prove that there are solutions that are not defined for all values of $t \in \mathbb{R}$.

NOTES: in exercises 5, 6 and 8 one has to find a path in the spaces of matrices and apply Hartman-Grobman's theorem several times to close enough matrices in the path.