Finances Quantitatives

Exercise Set 6

Leonardo Bocchi

Exercise 1

Consider a model of a bond market where the instantaneous short rate is given by $r(t)$ and where we have a bank account asset with continuously compouded interest rate α_t, that is the unit of money evolves as

$$dS_t^0 = \alpha_t S_t^0 dt, \quad S_0^0 = 1$$

then prove that if the model is free of arbitrage $\alpha_t = r(t)$; $0 \leq t \leq T$.

Answer:

To prove that if the model is free of arbitrage, the continuously compounded interest rate of the bank account asset, denoted as α_t, must be equal to the instantaneous short rate, denoted as $r(t)$, for $0 \leq t \leq T$, we can use the absence of arbitrage opportunity.

From

$$dS_t^0 = \alpha_t S_t^0 dt, \quad S_0^0 = 1$$

we can isolate α_t, i.e.

$$\alpha_t = \frac{dS_t^0}{S_t^0 dt}$$

where S_t^0 is the value of the bank account asset at time t.

Using the assumption of the absence of arbitrage opportunity, the asset price is defined as

$$S_t^0 = \exp\left(\int_0^t r(s)ds\right)$$

with $r(s)$ short rate.

Using the previous formula where we obtained α_t, we can say that

$$\begin{aligned}
\alpha_t &= \frac{\frac{d}{dt}\exp\left(\int_0^t r(s)ds\right)}{\exp\left(\int_0^t r(s)ds\right)} \\
&= \frac{d}{dt}\int_0^t r(s)ds \frac{\exp\left(\int_0^t r(s)ds\right)}{\exp\left(\int_0^t r(s)ds\right)} \\
&= \frac{d}{dt}\int_0^t r(s)ds
\end{aligned}$$

Using the Fundamental Theorem of Calculus (Part 2, Differentiation Theorem), which states that:

Theorem 1. *Let $f(x)$ be a continuous function on an open interval I and let $F(x)$ be an antiderivative of $f(x)$ on I. If $f(x)$ is continuous at a point c in I and $F(x)$ is differentiable at c, then the derivative of the definite integral of $f(x)$ from a to x is equal to $f(x)$ evaluated at x:*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

we get

$$\alpha_t = r(t)$$

for $0 \leq t \leq T$.

Exercise 2

Assume that the short rate follows the dynamics, under the risk-neutral probability, $r(t) = a + \sigma W_t$ where $a, \sigma > 0$ and W is a Brownian motion. Compute the price of a zero-coupon bond and the corresponding instantaneous forward rate.

Answer:

To compute the price of a zero-coupon bond and the corresponding instantaneous forward rate in the given model, we will solve the stochastic differential equation (SDE) for the short rate and then use the solution to derive the bond price and forward rate.

The SDE for the short rate, $r(t)$, is given by:

$$dr(t) = adt + \sigma dW_t$$

To solve this SDE, we can apply the Itô's lemma. By integrating both sides of the equation, we obtain:

$$\int dr(t) = \int adt + \int \sigma dW_t$$

This simplifies to:

$$r(t) = r(0) + at + \sigma W_t$$

where $r(0)$ is the initial value of the short rate.

Now, let's compute the price of a zero-coupon bond. The price of a zero-coupon bond maturing at time T, denoted by $P(t,T)$, is given by the formula:

$$P(t,T) = exp\left(-\int_t^T r(s)ds\right)$$

Substituting the expression for the short rate into the formula, we have:

$$P(t,T) = exp\left(-\int_t^T (r(0) + as + \sigma W_s)ds\right)$$

Simplifying the integral, we get:

$$P(t,T) = exp\left(-(r(0)(T-t) + \frac{a}{2}(T-t)^2 + \sigma \int_t^T W_s ds)\right)$$

To compute the corresponding instantaneous forward rate, denoted by $f(t,T)$, we differentiate the log of the bond price with respect to time:

$$f(t,T) = -\frac{\partial}{\partial T} ln(P(t,T))$$

Taking the derivative with respect to T and simplifying, we find:

$$f(t,T) = r(0) + a(T-t) + \frac{a^2}{2}(T-t)$$

So, in the given model, the price of a zero-coupon bond is given by the exponential of the integral of the short rate, and the corresponding instantaneous forward rate is a linear function of time with coefficients $r(0)$, a, and $\frac{\sigma^2}{2}$.