Simulation Methods

Bifurcations of periodic points

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Outline

1. Computation of bifurcations of fixed and periodic points

2. An example: the Lorenz discrete map

Introduction

Suppose that we have a family $\{f_a\}_{a\in\mathbb{R}^m}$ such that for all a, $f_a : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Moreover, we suppose that f_a is dissipative, that is $|\det Df_a(x)| < 1$. In order to understand the dynamical behaviour of this family we need to know its invariant objects and how they evolve when we change the parameter. We want to find invariant objects: periodic orbits, invariant manifolds (compact or not) and attracting sets. The attracting sets are specially important, because they show us which is the behavior of an orbit $(f_a^n(x))_n$ when $n \to \infty$. Moreover we want know:

- Has the system sensitive dependence on initial conditions? Indicator: positive Lyapunov exponent
- Are there changes in the dynamical behaviour when we change the parameter? Indicator: bifurcations: periodic orbits, invariant circles, invariant tori,..

Comment

If we have a family of vector fields, we have to add the equilibrium points and we can use the Poincaré map (if possible) in order to obtain a family of diffeomorphisms associated to the family of vector fields.

We recall:

Definition

The appearance of a topologically nonconjugate (nonequivalent for vector fields) phase portrait under variation of parameters is called a bifurcation.

Definition

- The codimension of a bifurcation is the smallest dimension of the parameter space which contains the bifurcation in a persistent way.
- An unfolding of a bifurcation is a family which contains the bifurcation in a persistent way.

Fact

If $x_0 \in \mathbb{R}^n$ is a fixed (resp. p-periodic) point of f_{a_0} and then f_a is locally topologically conjugate to f_{a_0} if x_0 is hyperbolic, that is, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $Df_{a_0}(p)$ (resp. $Df_{a_0}^p(x_0)$) then $|\lambda_i| \neq 1$, $\forall\, i \in \{1, \ldots, n\}$. $\lambda_1, \ldots, \lambda_n$ are called the eigenvalues of the periodic point x_0.

Therefore, in order to have bifurcations of periodic orbits, we need that for a certain $a_0 \in \mathbb{R}^m$ and a p-periodic point $x_0 \in \mathbb{R}^n$ for f_{a_0} s.t. $\exists\, i$ with $|\lambda_i| = 1$.

Proposition

The codimension 1 bifurcations of fixed (or periodic points) are:

- *Saddle-node:* $\lambda_1 = 1$, $|\lambda_i| \neq 1$, $i \neq 1$.
- *Flip:* $\lambda_1 = -1$, $|\lambda_i| \neq 1$, $i \neq 1$.
- *Neimark-Sacker:* $|\lambda_1| = |\lambda_2| = 1$, $\overline{\lambda}_2 = \lambda_1$, $\lambda_1 \notin \mathbb{R}$, $|\lambda_i| \neq 1$, $i = 2, \dots, n$.

Comment

To have generic bifurcations, one has to add non-degeneracy conditions.

Comment

In order to get the behaviour of a bifurcation we use the restriction of the maps to a centre manifold (an invariant manifold associated to the eigenvalues of modulus one) and normal forms. We obtain also a model family of the bifurcation

Model families: After reducing the family to the center manifolds and considering suitable nondegeneracy conditions, we obtain the following model families (unfoldings)

- Saddle-node (fold) $f_a(x) = x + a - x^2$.
- Flip: $f_a(x) = -(1+a)x \pm x^3$.
- Neimark-Sacker: $w \mapsto e^{i\theta(a)}(1 + a + d(a)|w|^2)w$, where $d(a) = b(a) + ic(a)$, with $b(0) \neq 0$, $0 < \theta(0) < \pi$ and $w = x_1 + ix_2 \in \mathbb{C}$.

In all the cases, the bifurcation takes place for $a = 0$ with fixed point $x = 0$. Using these models we can understand the behaviour of these bifurcations.

fold Fixed points: $x_\pm = \pm\sqrt{a}$ if $a \geq 0$, $f'(x_\pm) = 1 \mp 2\sqrt{a}$. Then a_+ attracting and x_- repelling if $a > 0$.

flip
- Fixed point $x = 0$ for all a. Attracting if $a < 0$ and repelling if $a > 0$.
- Two-periodic points:

$$f_a^2(x) = (1+a)^2 x \mp [(1+a) + (1+a)^3]x^3 + O(x^4).$$

If $f_{a(x)}^2(x) = x$ then $a(x) = \pm x^2 + O(x^3)$. If $a > 0$ is attracting, and if $a < 0$ is repelling.

NS If we write $w = \rho e^{i\varphi}$, we obtain

$$\begin{cases} \rho & \mapsto & \rho(1 + a + b(a)\rho^2) + \rho^4 R_a(\rho), \\ \varphi & \mapsto & \varphi + \theta(a) + \rho^2 Q_a(\rho), \end{cases}$$

for functions R and Q, which are smooth functions of (ρ, a).

- $(x_1, x_2) = (0, 0)$ is a fixed point, repelling if $a > 0$ and attracting if $a < 0$.
- If $b(0) > 0$ (supercritical), the circle of radius $\rho = \sqrt{-a/b(a)} + O(a)$ is invariant and repelling if $a < 0$, and if $b(0) < 0$ (subcritical) and $a > 0$ is invariant and attracting.

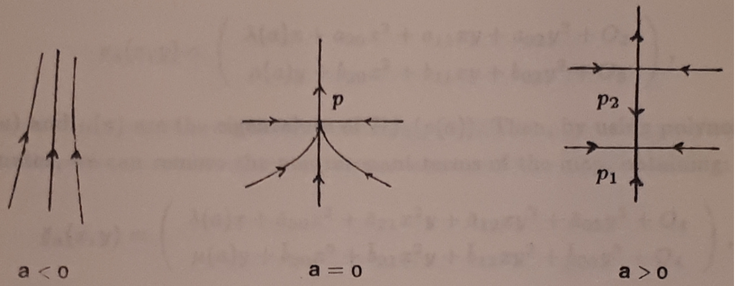

Figure – 4.1: Qualitative behaviour of a saddle-node $g_a(x,y) = (\lambda x, y + a + y^2)$, $|\lambda| < 1$.

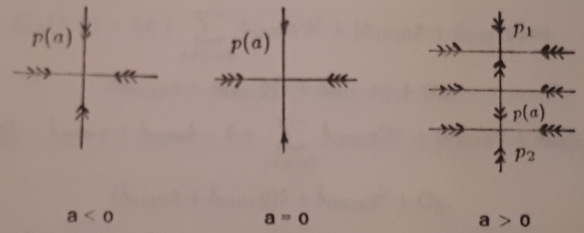

Figure 4.2: Qualitative behaviour of a flip $g_a(x,y) = (\lambda x, (-1+a)y + y^3)$, $|\lambda| < 1$.

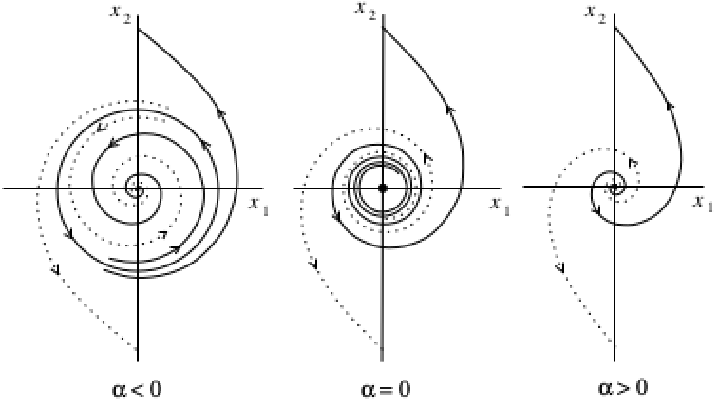

FIGURE 4.9. Subcritical Neimark-Sacker bifurcation. [1]

[1] From Kuznetsov's book.

Proposition

Normal form near a fixed point:

Let $f = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ *be a sufficiently smooth map such that*

$$f_j(x) = \lambda_j x_j + \sum_{k=2}^{s} \sum_{|\alpha|=k} a_{j\alpha} x^\alpha + O(|x|^{s+1}), \quad j = 1, \dots, n,$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ *and* $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. *Then, there exists a polynomial change of variables*

$$\bar{x}_j = x_j + \sum_{k=2}^{s} \sum_{|\alpha|=k} c_{j,\alpha} x^\alpha, \quad j = 1, \dots, n$$

such that f *is transformed into another map with the same linear part and with all coefficients of index* (j, α) $1 \le j \le n$, $2 \le |\alpha| \le s$, *such that* $\lambda_j \neq \lambda^\alpha$ (*nonresonant terms*) *equal to zero.*

Computation of a saddle-node bifurcation:

Theorem

Let $f_a : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$ such that for $a = a_0$, $\exists\ x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$, $f'_{a_0}(x_0) = 1$, $f''_{a_0}(x_0) \neq 0$ and $D_a f(x_0, a_0) \neq 0$, where $f(x, a) = f_a(x)$. Then f_a is topologically conjugate to the model family for the saddle-node.

Now we want to find computable conditions for the existence of a saddle-node bifurcation for familiy of maps $f_a : \mathbb{R}^n \to \mathbb{R}^n$, where $a \in \mathbb{R}$. Suppose:

- $f_{a_0}(x_0) = x_0$ and $\lambda_1 = 1$ is a simple eigenvalue of $Df_a(x_0)$.
- If $\{\lambda_1, \dots, \lambda_n\} = \mathrm{Spec}(Df_{a_0}(x_0))$ then $\lambda_1 = 1$ and $|\lambda_j| \neq 1$ if $j \neq 1$.

$$f_a(x) = f_a(x_0) + Df_a(x_0)(x - x_0) + \frac{1}{2}D^2 f_a(x_0)(x - x_0)^2 + O(|x - x_0|^3).$$

As $f_{a_0}(x_0) = x_0$, then

$$f_a(x) = x_0 + D_a f(z_0)(a - a_0) +$$

$$\frac{1}{2}D_a^2 f(z_0)(a - a_0)^2 + Df_a(x_0)(x - x_0) + \frac{1}{2}D^2 f_a(x_0)(x - x_0)^2 + O(|z - z_0|^3),$$

where $z = (a, x)$ and $f(a, x) = f_a(x)$.
If we perform a translation of x_0 to the origin, and the change of parameter $a \mapsto a - a_0$ (that we call again a) we obtain a conjugate map g_a :

$$g_a(y) = aD_a f(z_0) + \frac{1}{2}a^2 D_{aa} f(z_0) + Df_a(x_0)y + \frac{1}{2}D^2 f_a(x_0)y^2 + O(|w|^3),$$

where $y = x - x_0$ and $w = (a, y)$.

As $\lambda_1 = 1$ is a simple eigenvalue of $Df_a(x_0)$, there exists $\lambda_1(a)$, $|a|$ small, such that it is an eigenvalue of $Df_a(x_0)$ and it is different from the rest of the eigenvalues. Then, $\exists$ an $n \times n$ matrix $M = M_a$ s. t.

$$M_a^{-1} Df_a(x_0) M_a = \begin{pmatrix} \lambda_1(a) & 0^T \\ 0 & \Lambda \end{pmatrix}.$$

Performing the change of variables $y = M_a t$ we get a conjugate map h such that

$$h(t) = aM_0^{-1} D_a f(z_0) + \begin{pmatrix} \lambda_1(a) & 0^T \\ 0 & \Lambda \end{pmatrix} t + \frac{1}{2} M_0^{-1} D^2 f_{a_0}(x_0)(M_0 t)^2 + O(a^2) + O_3.$$

Then, by the IFT, we can perform an a-dependent translation $(t_1, \tilde{t}) \mapsto (t_1, \tilde{t} - \varphi(a))$, where $t = (t_1, \tilde{t}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and $\varphi(0) = 0$, s.t. we can write (if we maintain the variable t)

$$h(t) = a \begin{pmatrix} a_{100} \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_1(a) t_1 \\ \Lambda \tilde{t} \end{pmatrix} + \frac{1}{2} M_0^{-1} D^2 f_{a_0}(x_0)(M_0 t)^2 + O(|at|) + O(a^2) + O_3,$$

where $O(a^2) = (O(a^2), 0)^T$ and $a_{100} = e_1^T M_0^{-1} D_a f(z_0)$, for $e_1^T = (1, 0, \ldots, 0)$.

Finally, if $e_1^T M_0^{-1} D^2 f_{a_0}(x_0)(M_0 e_1)^2 \neq 0$ and $e_1^T M_0^{-1} D_a f(z_0) \neq 0$, we can perform a translation $\begin{pmatrix} t_1 - \psi(a) \\ \tilde{t} \end{pmatrix}$, a change of scale of t_1 and a change of parameter, such that the initial family is conjugate to

$$h_b(t) = \begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} t_1 \\ \Lambda(0)\tilde{t} \end{pmatrix} + \begin{pmatrix} t_1^2 + O(|a\tilde{t}|) + O(|\tilde{t}|^2) \\ O(|at|) + O(|t|^2) \end{pmatrix} + O_3.$$

Comment

It is possible to prove that this map is topologically conjugate to the map

$$\begin{pmatrix} t_1 \\ \tilde{t} \end{pmatrix} \mapsto \begin{pmatrix} b + t_1 + t_1^2 \\ \Lambda(b)\tilde{t} \end{pmatrix},$$

such that $1 \notin Spec(\Lambda(0))$.

Summary

Suppose that $f_a : \mathbb{R}^n \to \mathbb{R}^n$, $a \in \mathbb{R}$ *has the following properties:*

- $f_{a_0}(x_0) = x_0$,
- *1 is a simple eigenvalue of* $Df_{a_0}(x_0)$, *and the other eigenvalues have modulus different from 1.*
- $w^T D_a f(a_0, x_0) \neq 0$,
- $w^T D^2 f_{a_0}(x_0) v^2 \neq 0$,

where w *is a left eigenvector of eigenvalue 1 and* v *is a right eigenvector of eigenvalue 1. Then we can conjugate the family to a model family having a saddle-node bifurcation unfolding generically.*

Comment

We have used that $w = e_1^T M(0)$ *is the left eigenvector with eigenvalue 1 and* $M(0)e_1$ *is the right eigenvector of eigenvalue 1.*

Comment

Suppose that we have a two-parameter family $f_{a,b}$, such that for f_{a_0,b_0} has a saddle node bifurcation at $x = x_0$. If we want to get a curve $(a, b) = (a(t), b(t))$ such that $f_{a(t),b(t)}$ has a saddle-node bifurcation, we have to solve the system:

$$\begin{aligned} f_{a,b}(x) - x &= 0 \\ \det(Df_{a,b}(x) - I) &= 0. \end{aligned}$$

We know that this system has a solution $(a, b, x) = (a_0, b_0, x_0)$. We note that we cannot apply the IFT to the first equation to find $x = x(a, b)$, because of the second equation. In general, this means that we have a return point.
If we consider two parameters in the previous construction, we can prove that there exists the curve $(a(t), b(t))$ if $w^T D_a f(a_0, b_0, x_0) \neq 0$ or $w^T D_b f(a_0, b_0, x_0) \neq 0$.

Another way to obtain the normal form I

We can obtain the same result if we consider the following auxiliar map: $F(a,x) = (a, f(a,x))$. Then we take

$$f(a,x) = f(z_0) + Df(z_0)(z - z_0) + \frac{1}{2}D^2 f(z_0)(z - z_0)^2 + O(|z - z_0|^3),$$

where $z = (a,x)$ and $z_0 = (a_0, x_0)$. Then

$$DF(z_0) = \begin{pmatrix} 1 & 0 \\ D_a f(z_0) & Df_{a_0}(x_0) \end{pmatrix}.$$

We can prove that there exists a matrix

$$A = \begin{pmatrix} 1 & 0 \\ v & M_0 \end{pmatrix}$$

such that

$$A^{-1} DF(z_0) A = \begin{pmatrix} 1 & 0 & 0 \\ w^T D_a f(z_0) & 1 & 0 \\ 0 & 0 & \Lambda \end{pmatrix},$$

Another way to obtain the normal form II

where $e_1^T = (1, 0, \ldots, 0) \in \mathbb{R}^n$.
Indeed, if we suppose that such v exists then

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -M_0^{-1}v & M_0^{-1} \end{pmatrix}$$

and if $\mathbb{R}^n = \langle v_1 \rangle \oplus E_2$ where v_1 is an eigenvector of eigenvalue 1 and E_2 is the invariant subspace corresponding to the eigenvalues different from 1, then

$$(Df_{a_0}(x_0) - I)v = [w^T D_a f(z_0)]v_1 - D_a f(z_0).$$

This equation has a unique solution v such that $w^T v = 0$. Then we have that the original map is conjugate to

$$h(t) = \begin{pmatrix} a_{100}a + t_1 + O_2 \\ \Lambda \tilde{t} + O_2 \end{pmatrix},$$

where O_2 means terms of order 2 in the $a, t, \tilde{t}$ variables.

Center manifolds theorems I

Consider a discrete system $x \mapsto f(x)$, $x \in \mathbb{R}^n$, f smooth and $f(0) = 0$. Using an eigenbasis of $Df(0)$, we can rewrite the system as:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} Bu + g(u,v) \\ Cv + h(u,v) \end{pmatrix}, \tag{1}$$

where $u \in \mathbb{R}^{n_0}$, $\mathrm{Spec(B)} \subset S(0;1)$ and $\mathrm{Spec(C)} \cap S(0;1) = \emptyset$, $g = O(|(u,v)|^2)$, $h = O(|(u,v)|^2)$.

Theorem

There is a locally defined smooth n_0-dimensional invariant manifold $W^c_{loc}(0)$ of (1) that posseses the local representation $v = V(u)$, near $u = 0$, s.t. $V = O(|u|^2)$.

Center manifolds theorems II

Theorem (Reduction principle)

System (1) is locally topologically conjugate near the origin to the system

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} Bu + g(u, V(u)) \\ Cv \end{pmatrix}.$$

Consider now a smooth parameter dependent system $x \mapsto f(x, \alpha)$, $\alpha \in \mathbb{R}$ such that $f(0, 0) = 0$. We consider the auxiliar system $y = (\alpha, x) \mapsto F(y) = (\alpha, f(x, \alpha))$. Then there exists a linear transformation such that F is linearly conjugate to

$$\begin{pmatrix} \alpha \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ \alpha w_0 + Bu + g(u, v, \alpha) \\ Cv + h(u, v, \alpha) \end{pmatrix},$$

where $x \in \mathbb{R}^{n_0}$, $w_0 \in \mathbb{R}^{n_0}$, $\mathrm{Spec(B)} \subset S(0; 1)$, $\mathrm{Spec(C)} \cap S(0; 1) = \emptyset$, $g = O(|(u, v, \alpha)|^2)$, $h = O(|(u, v, \alpha)|^2)$, and $w_0 = 0$ if $1 \notin \mathrm{Spec(B)}$.

Center manifolds theorems III

Applying the center manifold theorem we have

Theorem

There is a locally defined smooth n_0-dimensional invariant manifold W^c_α of the map

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \alpha w_0 + Bu + g(u,v,\alpha) \\ Cv + h(u,v,\alpha) \end{pmatrix}, \tag{2}$$

that posseses the local representation $v = V(u,\alpha)$, near the origin, s. t. $V = O(|(u,\alpha)|^2)$.

Theorem

The sytem (2) is locally topollogically conjugate near the origin to the system

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \alpha w_0 + Bu + g(u,V(u,\alpha),\alpha) \\ Cv \end{pmatrix}.$$

Center manifolds theorems IV

Comment (Application to the saddle-node case)

If we have a family $f_a : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

- $f_{a_0}(x_0) = x_0$,
- *1 is a simple eigenvalue of* $Df_{a_0}(x_0)$, *and the other eigenvalues have modulus different from 1.*
- $w^T D_a f(a_0, x_0) \neq 0$,
- $w^T D^2 f_{a_0}(x_0) v^2 \neq 0$,

where w *is a left eigenvector of eigenvalue 1 and* v *is a right eigenvector of eigenvalue 1, we know that* f_a *is conjugate to*

$$h(t) = \begin{pmatrix} a_{100}a + t_1 + a_{020}t_1^2 + O(a^2) + O(a|\tilde{t}|) + O(|\tilde{t}|^2) \\ \Lambda\tilde{t} + O_2 \end{pmatrix}.$$

Then, using the center manifold theorem, we see that it is conjugate to

$$\begin{pmatrix} t_1 \\ \tilde{t} \end{pmatrix} \mapsto \begin{pmatrix} a_{100}a + t_1 + a_{020}t_1^2 + O(|(a, t_1)|^3) \\ \Lambda\tilde{t} \end{pmatrix}$$

Computation of a flip bifurcation The idea is similar to the case the saddle-node bifurcation. As it is more complex, we restrict ourselves to the case $f_a : \mathbb{R}^2 \to \mathbb{R}^2$.

Theorem

Let $f_a : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$ such that for $a = a_0$ $\exists\, x_0 \in \mathbb{R}$ such that $f_{a_0}(x_0) = x_0$, $f'_{a_0}(x_0) = -1$, $\frac{1}{2} f''_{a_0}(x_0)^2 + \frac{1}{3} f'''_{a_0}(x_0) \neq 0$ and $f_{xa}(a_0, x_0) \neq 0$. Then there are smooth invertible coordinate and parameter changes transforming the system into

$$y \mapsto -(1+b)y \pm y^3 + O(y^4).$$

Let $f_a : \mathbb{R}^2 \to \mathbb{R}^2$ and suppose:

- $f_{a_0}(x_0, y_0) = x_0$ and $\lambda_1 = -1$ is a simple eigenvalue of $Df_{a_0}(x_0)$.
- The other eigenvalue satisfies $|\lambda_2| \neq 1$.

As $\lambda_i \neq 1$, $i = 1, 2$, there exists $p = p(a)$ such that $p(a_0) = (x_0, y_0)$ and $f_a(p) = p$, if $|a - a_0|$ small enough. Then

$$f_a(x,y) = f_a(p(a)) + Df_a(p(a)) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} +$$

$$\frac{1}{2} D^2 f_a(p(a)) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^2 + \frac{1}{6} D^3 f_a(p(a)) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^3 + \cdots$$

Then we move the fixed point $p(a)$ to the origin and diagonalize the linear part by the change $(z, w)^T = M(x, y)^T$, where M is a 2×2 matrix. Then we have (giving the same name to the variables)

$$g_a(x,y) = \begin{pmatrix} \lambda(a)x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \sum_{j=0}^{3} a_{j,3-j}x^j y^{3-j} + O_4 \\ \mu(a)y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \sum_{j=0}^{3} b_{j,3-j}x^j y^{3-j} + O_4 \end{pmatrix}.$$

By the normal form theory, we can transform this system to

$$\bar{g}_a(x,y) = \begin{pmatrix} \lambda(a)x + \bar{a}_{12}xy^2 + O_4 \\ \mu(a)y + \bar{b}_{30}y^3 + O_4 \end{pmatrix},$$

which is conjugate to the model of the flip bifurcation if $\mu'(a_0) \neq 0$ and $\bar{b}_{30} \neq 0$.
To compute $\bar{b}_{30}$, we consider the change of coordinates:

$$h(x,y) = \begin{pmatrix} x + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 \\ y + \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 \end{pmatrix}.$$

We get

$$\bar{b}_{03}(a_0) = \frac{b_{11}(a_0)a_{02}(a_0)}{1-\lambda(a_0)} + (b_{02}(a_0))^2 + b_{03}(a_0).$$

As

$$b_{11} = w^T D^2 f_a(p(a))(v, v_1), \quad b_{02} = \frac{1}{2} w^T D^2 f_a(p(a))(v,v),$$

$$b_{03} = \frac{1}{6} w^T D^3 f_a(p(a))(v,v,v),$$

where w is the left eigenvector of eigenvalue -1, v is the eigenvector of eigenvalue -1 and v_1 is the eigenvector of eigenvalue $\lambda(a_0)$.

Summary

Suppose that $f_a : \mathbb{R}^2 \to \mathbb{R}^2$ *has the following properties:*

- $f_{a_0}(z_0) = z_0$,
- $\mu(a_0) = -1$ *is a simple eigenvalue of* $Df_{a_0}(z_0)$ *and the other eigenvalue* $\lambda(a_0)$ *satisfies* $|\lambda(a_0)| \neq 1$,
- $\frac{1}{2(1-\lambda(a_0))}\left[w^T D^2 f_{a_0}(z_0)(v, v_1)\right] \cdot \left[w_1^T D^2 f_{a_0}(z_0)(v, v)\right] + \frac{1}{4}(w^T D^2 f_{a_0}(z_0)(v, v))^2 + \frac{1}{6} w^T D^3 f_{a_0}(z_0)(v, v, v) \neq 0$,
- *If* $p(a)$ *satisfies* $f_a(p(a)) = p(a)$ *and* $p(a_0) = z_0$, *and* $\mu(a)$ *is the eigenvalue of* $p(a)$ *such that* $\mu(a_0) = -1$, *then* $\mu'(a_0) \neq 0$.

Then we can conjugate the family to the model family of a flip.

Comment

We have to add the following conditions on the eigenvectors: $w_1^T v_1 = w^T v = 1$, *because we have taken as the right eigenvectors the columns of* M *and as the left eigenvectors the rows of* M^{-1}.

Computation of Neimark-Sacker bifurcations In Neimark-Sacker bifurcations, $r(a)e^{\pm i\varphi(a)}$, where a is the parameter, are eigenvalues of the fixed point of f_a. Moreover, $r(a_0) = 1$, $r'(0) \neq 0$, $\varphi(a_0) = \theta_0$ and $e^{ik\theta_0} \neq 1$, $k = 1, 2, 3, 4$.
If we have a map $f_a : \mathbb{R}^n \to \mathbb{R}^n$, $(n \geq 2)$ in order to have a Neimark Sacker bifurcation the following system has to be satisfied:

$$f_{a_0}(x_0) = x_0, \qquad e^{i\theta} \in \mathrm{Spec}(Df_{a_0}(x_0)),$$

and the other eigenvalues with modulus different from 1.

- $n = 3$: The second condition implies that $\det Df_{a_0}(x_0) \neq \pm 1$ and it is an eigenvalue of $Df_{a_0}(x_0)$. Indeed, if $\lambda_1 = e^{i\theta_0}$, $\lambda_2 = e^{-i\theta_0}$ anf λ_3 are the eigenvalues of $Df(x_0)$ then $\det Df_{a_0}(x_0) = \lambda_3$, so we have:

$$\begin{aligned} f_a(x) &= x, \\ \det(Df_a(x) - \det Df_a(x) I) &= 0. \end{aligned}$$

  If $-\det(Df_a(x) - \lambda I) = \lambda^3 - c_2\lambda^2 + c_1\lambda - c_0$ then $c_0 + \lambda_1 + \lambda_2 = c_2$ and $1 + c_0(\lambda_1 + \lambda_2) = c_1$. This means that $c_1 = 1 + c_0(c_2 - c_0)$ and $(c_2 - c_0)^2 < 4$.

- $n = 4$: Let $\lambda_1 = e^{i\theta_0}$, $\lambda_2 = e^{-i\theta_0}$, λ_3 and λ_4 be the eigenvalues of $Df_{a_0}(x_0)$. If

$$\det(Df_{a_0}(x_0) - \lambda I) = \lambda^4 - c_3\lambda^3 + c_2\lambda^2 - c_1\lambda + c_0,$$

then

$$\begin{aligned} c_0 &= \lambda_3\lambda_4, \\ c_1 &= \lambda_3 + \lambda_4 + (\lambda_1 + \lambda_2)\lambda_3\lambda_4, \\ c_2 &= 1 + (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) + \lambda_3\lambda_4 \end{aligned}$$

If we call $S = \lambda_1 + \lambda_2$, $s = \lambda_3 + \lambda_4$ then

$$\begin{aligned} c_1 &= s + c_0 S, \\ c_2 &= 1 + sS + c_0, \\ c_3 &= S + s \end{aligned}$$

Solving this system we see that there exists a polynomial $P(c_0, c_1, c_2, c_3)$ such that if

$$\begin{aligned} f_a(x) &= x, \\ P(c_0, c_1, c_2, c_3) &= 0 \end{aligned}$$

Then x is a fixed point of f_a with two eigenvalues such that $\lambda_1\lambda_2 = 1$. We note that we can also recover the value of S:

$$S = \frac{c_3 - c_1}{1 - c_0}, \text{ if } c_0 \neq 1.$$

.

Comment

In the general case, one can use the *bialternate product* *of matrices. If A and B are $n \times n$ matrices, then the bialternate product of A and B, denoted $A \odot B$, is an $m \times m$ matrix ($m = \frac{1}{2}n(n-1)$) such that*

$$(A \odot B)_{(p,q),(r,s)} = \frac{1}{2}\left\{ \begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{vmatrix} \right\},$$

where $(p,q),(r,s) \in Q_{2,n} = \{(i,j) \mid i,j \in \{1,\ldots,n\} \text{ and } i > j\}$. We have the following property: if $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A then $A \odot A$ has eigenvalues $\lambda_i\lambda_j$ and $2A \odot I_n$ has eigenvalues $\lambda_i + \lambda_j$, where $i = 2,3,\ldots,n$; $j = 1,2,\ldots,i-1$.
Therefore, if A has two complex conjugate eigenvalues of modulus one then $\det(A \odot A - I_n) = 0$.

Example

If A is an 3×3 matrix then

$$A \odot A = \begin{pmatrix} b_{(2,1),(2,1)} & b_{(2,1),(3,1)} & b_{(2,1),(3,2)} \\ b_{(3,1),(2,1)} & b_{(3,1),(3,1)} & b_{(3,1),(3,2)} \\ b_{(3,2),(2,1)} & b_{(3,2),(3,1)} & b_{(3,2),(3,2)} \end{pmatrix}.$$

Example

We consider the Lorenz discrete map $(\bar{x}, \bar{y}) = f(x, y)$, such that:

$$\left\{ \begin{array}{lcl} \bar{x} & = & (1+ah)x - hxy, \\ \bar{y} & = & (1-h)y + hx^2 \end{array} \right. ,$$

where $h > 0$, $a \in \mathbb{R}$.

The fixed points of f are $E_0 = (0,0)$, $E_1 = (\sqrt{a}, a)$ and $E_2 = (-\sqrt{a}, a)$. We want to analize the bifurcations of the fixed points of this map.

- Differential at the fixed points:

$$Df(x,y) = \begin{pmatrix} 1+ah-hy & -hx \\ 2hx & 1-h \end{pmatrix}, \; Df(E_0) = \begin{pmatrix} 1+ah & 0 \\ 0 & 1-h \end{pmatrix},$$

$$Df(E_1) = \begin{pmatrix} 1 & -h\sqrt{a} \\ 2h\sqrt{a} & 1-h \end{pmatrix}, \; Df(E_2) = \begin{pmatrix} 1 & h\sqrt{a} \\ -2h\sqrt{a} & 1-h \end{pmatrix}.$$

- One eigenvalue equal to 1 (possible saddle-node): $E_0 : a = 0$ (we discard this case because it is degenerate) and $E_1, E_2 : a = 0$ (same case as before).

- One egenvalue equal to -1 (possible flip): $E_0 : a = -2/h$ or $h = 2$ and $E_1, E_2 : a = (h-2)/h^2$.
- Two non-real eigenvalues of modulus 1 (possible Neimark-Saker): E_1 and $E_2 : h = 1/(2a)$ and $a > 1/8$.

Flip bifurcation for E_0:

- -1 is a simple eigenvalue: This implies that $(a, h) \neq (-1, 2)$ If not, there is a possible codimension-two bifurcation with two eigenvalues equal to -1.
- 1 is not an eigenvalue: $a \neq 0$ and $h \neq 0$.
- Non-degeneracy condition:
  1. Case $a = -2/h$: We have $\lambda = 1 - h$, $v = e_1$, $w = e_1$, $v_1 = e_2$ $w_1 = e_2$: Then
  $$\frac{1}{2h}(-h)2h = -h < 0.$$
  and $\frac{\partial \mu}{\partial a} = h \neq 0$.
  2. Case $h = 2$. We have: $\lambda = 2a + 1$ $v = e_2$, $w = e_2$, $v_1 = e_1$, $w_1 = e_1$. Then the non-degeneracy condition is equal to 0 always.

Comment

As we said, we can also use the technique of the center manifold: We fix h*, consider the new parameter* μ *such that* $a = \mu - 2/h$*, and define* $M : U \subset \mathbb{R}^2 \to \mathbb{R}^2$*, where* U *is an open set such that* $0 \in U$ *and* $M(0,\mu) = 0$*, for all* μ*, since* $(0,0)$ *is always a fixed point. Then*

$$M(x,\mu) = m_1(\mu)x + m_2(\mu)x^2 + m_3(\mu)x^3 + O(x^4),$$

and $(x, M(x,\mu))$ *is the center manifold. As it is invariant, if* $g = (g_1, g_2)$*, where*

$$g_1(x,y,\mu) = (\mu h - 1)x - hxy, \quad g_2(x,y,\mu) = (1-h)y + hx^2,$$

then

$$M(g_1(x, M(x,\mu),\mu),\mu) = g_2(x, M(x,\mu),\mu).$$

Moreover, $M_x(0,\mu) = 0$*, which implies that* $m_1(\mu) = 0$*. By the invariance* $m_2(\mu) = \frac{1}{h\mu^2 - 2\mu + 1}$ *and* $m_3(\mu) = 0$*, that is*

$$M(x,\mu) = (h\mu^2 - 2\mu + 1)^{-1}x^2 + O(x^4).$$

Comment

Once we have te center manifold, the map restricted to it is $g(x,\mu) = g_1(x, M(x,\mu), \mu)$, *that is*

$$g(x,\mu) = (\mu h - 1)x - \frac{h}{h\mu^2 - 2\mu + 1}x^3 + O(x^5).$$

Then we have: $g(0,0) = 0$, $g_x(0,0) = -1$, $\frac{1}{2}g_{xx}(0,0)^2 + \frac{1}{3}g_{xxx}(0,0) = -2h \neq 0$ *and* $g_{x\mu}(0,0) = h \neq 0$, *which implies that it is a generic flip bifurcation (subcritical).*

Flip bifurcation for E_1 : We take $a=(h-2)/h^2$. Then

$$f(x,y)=\left(\left(1+\frac{h-2}{h}\right)x-hxy,(1-h)y+hx^2\right),$$

$$Df(E_1)=\left(\begin{array}{cc} 1 & -\sqrt{h-2} \\ 2\sqrt{h-2} & 1-h \end{array}\right).$$

As before, we can use the theorem:

- -1 is a simple eigenvalue if $h\neq 4$.
- 1 is not an eigenvalue if $h\neq 2$ (the other eigenvalue is $3-h$).
- $v=\frac{1}{4-h}(\sqrt{h-2},2)$, $w=(-\sqrt{h-2},1)$, $v_1=\frac{1}{4-h}(1,\sqrt{h-2})$, $w_1=(2,-\sqrt{h-2})$.
- If $v=(\alpha_1,\alpha_2)$ and $w=(\beta_1,\beta_2)$ then

$$D^2f_{a(h),h}(E_1)(v,w)=\left(\begin{array}{c} -(\alpha_1\beta_2+\alpha_2\beta_1)h \\ 2\alpha_1\beta_1 h \end{array}\right),$$

where $a(h)=\frac{h-2}{h^2}$.

- $D^2 f_{a(h)}(E_1)(v, v_1) = \frac{1}{(4-h)^2} \begin{pmatrix} -h^2 \\ 2h\sqrt{h-2} \end{pmatrix}$
- $w^T D^2 f_{a(h)}(E_1)(v, v_1) = \frac{1}{(4-h)^2}(h^2 + 2h)\sqrt{h-2}$.
- $D^2 f_{a(h)}(E_1)(v, v) = \frac{1}{(4-h)^2} \begin{pmatrix} -4h\sqrt{h-2} \\ 2(h-2)h \end{pmatrix}$.
- $w_1^T D^2 f_{a(h)}(E_1)(v, v) = -\frac{1}{(4-h)^2}(4h + 2h^2)\sqrt{h-2}$.
- $w^T D^2 f_{a(h)}(E_1)(v, v) = \frac{6h(h-2)}{(4-h)^2}$.
- $1/(2(1-\lambda)) = 1/(2(h-2))$.
- Non-degeneracy condition:

$$\frac{1}{2(1-\lambda(a_0))} \left[w^T D^2 f_{a_0}(z_0)(v, v_1) \right] \cdot \left[w_1^T D^2 f_{a_0}(z_0)(v, v) \right] +$$

$$\frac{1}{4}(w^T D^2 f_{a_0}(z_0)(v, v))^2 + \frac{1}{6} w^T D^3 f_{a_0}(z_0)(v, v, v) = \frac{8(h-1)h^2}{(h-4)^3}.$$

- Non-degeneracy with respect to parameter (check!).

Note that for $h = 1$ there is not fixed point.

Neimark-Sacker bifurcation for E_1 : In this case $h = 1/(2a)$ and $h > 1/8$. Then

$$f(x,y) = \left(\frac{3}{2}x - \frac{1}{2a}xy, \left(1 - \frac{1}{2a}\right)y + \frac{1}{2a}x^2\right),$$

$$Df_{a,h}(E_1) = \begin{pmatrix} 1 & -\frac{1}{2\sqrt{a}} \\ \frac{1}{\sqrt{a}} & 1 - \frac{1}{2a} \end{pmatrix}.$$

Eigenvalues: $\mu_{1,2} = \frac{4a-1 \pm i\sqrt{8a-1}}{4a}$. Then

1. If $a = 1/8$ then $\mu_{1,2}^2 = 1$. (two multipliers equal to -1).
2. If $a = 1/2$ then $\mu_{1,2}^3 = 1$. (multipliers $\frac{1}{2}(-1 \pm i\sqrt{3})$).
3. If $a = 1/4$ then $\mu_{1,2}^4 = 1$. (multipliers $\pm i$).

Eigenvectors of μ_1 :

1. Right eigenvector: $(1 + i\sqrt{8a-1}, 4\sqrt{a})$.
2. Left eigenvector: $\left(\frac{1}{2\sqrt{8a-1}}i, \frac{1}{8\sqrt{a}} - \frac{1}{8\sqrt{a}\sqrt{8a-1}}i\right)$.

In order to study this bifurcation we need more information about it:

The Neimark-Sacker bifurcation in $\mathbb{R}^2$ I

Suppose that we have a two-dimensional discrete-time system $x \mapsto f(x, \alpha)$, $x \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, with smooth f , has, for all sufficiently small $|\alpha|$, the fixed point $x = 0$ with eigenvalues

$$\mu_{1,2}(\alpha) = r(\alpha)e^{\pm i\varphi(\alpha)},$$

where $r(0) = 1$, $\varphi(0) = \theta_0$. Let the following conditions be satisfied:

1. $r'(0) \neq 0$;
2. $e^{ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$.

We write $r(\alpha) = 1 + \beta(\alpha)$, $\beta'(0) \neq 0$, and we perform a change of parameter. Then $\mu(\beta) = (1 + \beta)e^{i\theta(\beta)}$.
If $\mathbf{x} \in \mathbb{R}^2$ then we can write

$$\mathbf{x} = zv(\beta) + \bar{z}\bar{v}(\beta),$$

The Neimark-Sacker bifurcation in $\mathbb{R}^2$ II

where $v(\beta)$ is the eigenvector of eigenvalue $\mu(\beta)$. Moreover, if $w(\beta)$ is the left eigenvector of eigenvalue $\mu(\beta)$, that is

$$w(\beta)^H Df_\beta(0) = \mu(\beta) w(\beta)^H,$$

such that $w(\beta)^H v(\beta) = 1$, it is easy to prove that $w(\beta)^H \overline{v(\beta)} = 0$, since $\mu(\beta)$ is non-real. Then

$$z = w(\beta)^H \mathbf{x}$$

and

$$z \mapsto \mu(\beta) z + w(\beta)^H F(zv(\beta) + \bar{z}\bar{v}(\beta), \beta),$$

where

$$f(x,\beta) = D_x f(0,\beta)x + F(x,\beta), \quad F(x,\beta) = O(|x|^2).$$

The Neimark-Sacker bifurcation in $\mathbb{R}^2$ III

Proposition

Define

$$a(0) = Re\ \left(\frac{e^{-i\theta_0} g_{21}}{2}\right) - Re\ \left(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})} g_{20} g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2,$$

where

$$g_{jk} = \frac{\partial^{j+k}}{\partial z^j \partial \bar{z}^k}\ w(\beta)^H F(zv(\beta) + \bar{z}\bar{v}(\beta))\Big|_{z=0}.$$

If $a(0) \neq 0$, and items 1 and 2 of the previous comment hold, then $f(x, \alpha)$ undergoes a generic Neimark-Sacker bifurcation.

Comment

If some of the conditions does not hold, we will have a codimension-two bifurcation.

The Neimark-Sacker bifurcation in $\mathbb{R}^2$ IV

Comment

In order to compute g_{jk}, we write

$$f(x) = Df(0)x + \frac{1}{2}D^2 f(0)x^2 + \frac{1}{3!}D^3 f(0)x^3 + \cdots .$$

As $x = zv + \bar{z}\bar{v}$, where v and $\bar{v}$ are eigenvectors, we have

$$w^H F(zv + \bar{z}\bar{v}) = \frac{1}{2}w^H D^2 f(0)(zv + \bar{z}\bar{v})^2 + \frac{1}{3!}D^3 f(0)(zv + \bar{z}\bar{v})^3 + \cdots .$$

For example

$$\begin{aligned} g_{20} &= w^H D^2 f(0)(v, v), \\ g_{11} &= w^H D^2 f(0)(v, \bar{v}), \\ g_{02} &= w^H D^2 f(0)(\bar{v}, \bar{v}), \\ g_{21} &= w^H D^3 f(0)(v, v, \bar{v}). \end{aligned}$$

Comment

We can obtain normal forms of maps
$z \mapsto \mu z + \sum_{j=2}^{m} \sum_{k=0}^{j} g_{k,j-k} z^k \bar{z}^{j-k} + O(|z|^{m+1})$, *where* $|\mu| = 1$. *Indeed, there is a polynomial change of variables*

$$\tilde{z} = z + \sum_{j=2}^{m} \sum_{k=0}^{j} \alpha_{k,j-k} z^k \bar{z}^{j-k},$$

such that the map is transformed into another map with the same linear part and with all coefficients of index (j,k), $j,k \geq 0$, $j+k \leq m$, *such that* $\mu \neq \mu^j \bar{\mu}^k$ *(nonresonant terms) equal to zero.*

Note that, if $|\mu| = 1$, $\mu \neq 1$, $\mu^2 \neq 1$, $\mu^3 \neq 1$ and $\mu^4 \neq 1$ then the terms with index $(2,0)$, $(1,1)$, $(0,2)$ $(3,0)$, $(1,2)$, and $(0,3)$ are nonresonant and $(2,1)$ is resonant. This means that we can conjugate our map to

$$z \mapsto \mu z + a z^2 \bar{z} + O(|z|^4).$$

In this case, only $\operatorname{Re} a$ is relevant for the dynamics.

The Neimark-Sacker bifurcation in dimension 3 I

Let $f_a : U \subset \mathbb{R} \to \mathbb{R}^3$, $a \in \mathbb{R}$, such that

- $\exists\ x^{(0)} \in U$ and $a_0 \in \mathbb{R}$, s.t. $f_{a_0}(x^{(0)}) = x^{(0)}$,
- $\mathrm{Spec}\left(Df_{a_0}(x^{(0)})\right) = \{\lambda, \mu_1, \mu_2\}$, such that
  - $|\lambda| \neq 1$,
  - $\mu_{1,2} = e^{\pm i\theta_0}$, with $e^{ik\theta_0} \neq 1$, for $k = 1, 2, 3, 4$.

By using the IFT and a translation in the parameter, we can suppose that $a_0 = 0$ and $f_a(0) = 0$, for $|a|$ small enough. Moreover

$$\mathrm{Spec}\left(Df_a(0)\right) = \{\lambda(a), \mu_1(a), \mu_2(a)\},$$

such that $|\lambda(a)| \neq 1$, $\mu_{1,2}(a) = r(a)e^{i\varphi(a)}$, where $r(0) = 1$ and $\varphi(0) = \theta_0$. We know that $c_1(0) = 1 + c_0(0)(c_2(0) - c_0(0))$. Moreover, if $|a|$ small enough and $r(a) = |\mu_{1,2}(a)|$ then

$$r^6 - c_1(a)r^4 + c_0(a)c_2(a)r^2 - c_0(a)^2 = 0.$$

The Neimark-Sacker bifurcation in dimension 3 II

Using the IFT, we have that there exists $r = r(a)$ for $|a|$ small enough, s.t. $r(0) = 1$. Moreover, if $c_1'(0) \neq (c_0(0)c_2(0))' - 2c_0(0)c_0'(0)$ then $r'(0) \neq 0$. Then, by means of a linear change of coordinates, one can write

$$f_a(x, y) = \begin{pmatrix} C(a)x + O(|(x, y)|^2) \\ \lambda(a)y + O(|(x, y)|^2) \end{pmatrix},$$

where $x \in \mathbb{R}^2$ and $C(a)$ is a 2×2 matrix with eigenvalues $\mu_{1,2}(a) = r(a)e^{\pm i\varphi(a)}$, and the same conditions as before.
Now, we define the new parameter β such that $r(a) = 1 + \beta(a)$. Then

$$\mu_{1,2}(\beta) = (1 + \beta)e^{\pm i\theta(\beta)}.$$

Let $v = v(\beta) \in \mathbb{C}^2$ the eigenvector of eigenvalue $\mu = \mu_1(\beta)$ of $C(\beta)$ and $w \in \mathbb{C}^2$ the left eigenvector of eigenvalue μ (that is $w^H C = \mu w^H$), such that $w^H v = 1$. If $x \in \mathbb{R}^2$ we can write

$$x = zv(\beta) + \bar{z}\bar{v}(\beta).$$

The Neimark-Sacker bifurcation in dimension 3 III

As $\mu \notin \mathbb{R}$, we have $w^H \bar{v} = 0$. Then

$$z = w^H x$$

and we can transform our map to

$$\begin{pmatrix} z \\ y \end{pmatrix} \mapsto \begin{pmatrix} \mu(\beta) z + w(\beta)^H F(zv(\beta) + \bar{z}\bar{v}(\beta), y, \beta) \\ \lambda(a) y + G(zv(\beta) + \bar{z}\bar{v}(\beta), y, \beta) \end{pmatrix},$$

where $F(x, y, \beta) = O(|(x, y)|^2)$, $G(x, y, \beta) = O(|(x, y)|^2)$. If we write

$$F((zv(\beta) + \bar{z}\bar{v}(\beta), y, \beta) = \sum_{k=2}^{3} \sum_{|\alpha|=k} a_\alpha t^\alpha + O(|t|^4),$$

$$G((zv(\beta) + \bar{z}\bar{v}(\beta), y, \beta) = \sum_{k=2}^{3} \sum_{|\alpha|=k} b_\alpha t^\alpha + O(|t|^4),$$

The Neimark-Sacker bifurcation in dimension 3 IV

where $t = (z, \bar{z}, y)$, $\alpha \in \mathbb{N}_0^3$ and $b_{\alpha_1,\alpha_2,\alpha_3} = \bar{b}_{\alpha_2,\alpha_1,\alpha_3}$, we have that the terms of order two are non-resonant, and the only resonant terms of order three have indices $(1,1,1)$ for G and $(2,1,0)$ for F. This means that we can transform the map to

$$\begin{pmatrix} z \\ y \end{pmatrix} \mapsto \begin{pmatrix} \mu(\beta)z + c_1 z^2\bar{z} + O_4 \\ \lambda(a)y + c_2|z|^2 y + O_4 \end{pmatrix},$$

where $c_1 = c_1(\beta) \in \mathbb{C}$ and $c_2 = c_2(\beta) \in \mathbb{R}$. The transformation has the form

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} w + \frac{h^{(1)}_{200}}{2}w^2 + h^{(1)}_{110}w\bar{w} + \frac{h^{(1)}_{020}}{2}\bar{w}^2 + \frac{h^{(1)}_{002}}{2}s^2 + h^{(1)}_{101}sw + h^{(1)}_{011}s\bar{w} \\ s + \frac{h^{(2)}_{200}}{2}w^2 + h^{(2)}_{110}w\bar{w} + \frac{h^{(2)}_{020}}{2}\bar{w}^2 + \frac{h^{(2)}_{002}}{2}s^2 + h^{(2)}_{101}sw + h^{(2)}_{011}s\bar{w} \end{pmatrix}.$$

As the center manifold $y = H(x_1, x_2) = O_2$, we have that the map restricted to the center manifold is

$$w \mapsto \mu(\beta)w + c_1 w^2\bar{w} + O_4.$$

For more information:

1. Y. A. Kuznetsov: *Elements of Applied Bifurcation Theory*. Springer (1995)
2. J. Guckenheimer, P. Holmes: *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer (1983).