Lesson 13

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Analysis of sensitivity. The Greeks.

Let $V(t, S_t)$ be the value of a portfolio based in a risky asset S and money in a bank account. By practical reasons is often very important to have an idea of the sensitivity of V with respect to changes in the value of S (to measure the risk of our portfolio for instance) and with respect to changes in the parameters of the model (to measure a bad specification of the model). The standard notation is:

- $\Delta = \frac{\partial V}{\partial S_t}$
- $\Gamma = \frac{\partial^2 V}{\partial S_t^2}$
- $\rho = \frac{\partial V}{\partial r}$
- $\Theta = \frac{\partial V}{\partial t}$
- $\mathcal{V} = \frac{\partial V}{\partial \sigma}$

All these indexes of sensitivity are known as the Greeks. These include $\mathcal{V}$ that is pronounced Vega and that is not a Greek letter (κ was previously used). We can add derivatives to our portfolio, then such a portfolio that is not sensitive to small changes with respect to some parameter is said to be neutral: delta neutral, gamma neutral,..

In the BS model the portfolio that replicates a call with strike K and maturity time T has the following Greeks:

- $\Delta = \Phi(d_+) > 0$
- $\Gamma = \frac{\phi(d_+)}{S_t \sigma \sqrt{(T-t)}} > 0$ (where ϕ is the density of a standard normal random variable)
- $\rho = K(T-t)e^{-r(T-t)}\Phi(d_+) > 0$
- $\Theta = -\frac{S_t \sigma}{2\sqrt{(T-t)}}\phi(d_+) - Kre^{-r(T-t)}\Phi(d_-) < 0$
- $\mathcal{V} = S_t \phi(d_+)\sqrt{(T-t)} > 0$

Implied volatility

One important feature in the BS formula

$$C_t^{BS}(S_t, K, T, r, \sigma) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

with

$$d_\pm = \frac{\log\left(\frac{S_t}{K}\right) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

is that only depends of one no directly observable parameter: the volatility σ.

One way of getting a value of σ is to take into account that the r.v. $\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$ with $t_i = i\frac{t}{n}$, $i = 1, ..., n$ are i.i.d. with

$$\begin{aligned}\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) &= \sigma\left(W_{t_i} - W_{t_{i-1}}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)(t_i - t_{i-1}) \\ &\sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)\frac{t}{n}, \sigma^2\frac{t}{n}\right)\end{aligned}$$

then a good estimator of σ^2 is given by

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}\left(\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right)^2}{n-1}\frac{n}{t}.$$

BS prices of calls (and puts) are an increasing function of σ. When σ moves in $(0, \infty)$ call prices move in $\left(S_t - Ke^{-r(T-t)}, S_t\right)$. It can be seen that out of this interval these prices produce arbitrage. Then given the *market prices* of a call, say $C_t^*(K, T)$ one can always find a value of σ, say $\sigma_t^{imp}(K, T)$

$$C_t^*(K, T) = C_t^{BS}(S_t, K, T, r, \sigma_t^{imp}(K, T))$$

by inverting the BS formula. This $\sigma_t^{imp}(K, T)$ is called *the implied volatility.*

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In practice is observed that these curves have the form of a "**smile"** with a minimum close the the value $K = S_t$ (at the money (ATM) call), or a "**smirk**" with higher implied volatilities "in the money" calls $(S_t > K)$ and lower implied volatilities "out of the money" calls $(S_t < K)$.

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Practitioners use σ_t^{imp} to quote prices of calls (and puts) (vanilla options) more than the prices themselves, since it gives more idea of their relative value (taking out the effect of the value of S_t).

The BS formula with continuous dividends

Assume that the risky stock S produce continuous dividends with rate δ, that is if we have a self-financing portfolio $\left(\phi^0, \phi^1\right)$, its value

$$V_t = \phi_t^0 e^{rt} + \phi_t^1 S_t$$

change as

$$\mathrm{d}V_t = r\phi_t^0 e^{rt}\mathrm{d}t + \phi_t^1 \mathrm{d}S_t + \phi_t^1 S_t \delta \mathrm{d}t.$$

This could be the case for a foreign currency (dollar, yen,...) when these currencies are look as risky stocks with price in local currency (euros), δ would be the foreign interest rate and r the domestic interest rates.

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$$
\begin{aligned}
\mathrm{d}\tilde{V}_t &= \mathrm{d}\left(e^{-rt}V_t\right) = -re^{-rt}V_t\mathrm{d}t + e^{-rt}\mathrm{d}V_t \\
&= re^{-rt}\left(\phi_t^0 e^{rt} - V_t\right)\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t + e^{-rt}\phi_t^1 S_t\delta\mathrm{d}t \\
&= -re^{-rt}\phi_t^1 S_t\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t + e^{-rt}\phi_t^1 S_t\delta\mathrm{d}t \\
&= e^{-rt}\phi_t^1 S_t\left(\delta - r\right)\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t \\
&= \phi_t^1 e^{-\delta t}\mathrm{d}\left(S_t e^{-(r-\delta)t}\right).
\end{aligned}
$$

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$$\begin{aligned} \mathrm{d}\tilde{V}_t &= \mathrm{d}\left(e^{-rt}V_t\right) = -re^{-rt}V_t\mathrm{d}t + e^{-rt}\mathrm{d}V_t \\ &= re^{-rt}\left(\phi_t^0 e^{rt} - V_t\right)\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t + e^{-rt}\phi_t^1 S_t\delta\mathrm{d}t \\ &= -re^{-rt}\phi_t^1 S_t\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t + e^{-rt}\phi_t^1 S_t\delta\mathrm{d}t \\ &= e^{-rt}\phi_t^1 S_t\left(\delta - r\right)\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t \\ &= \phi_t^1 e^{-\delta t}\mathrm{d}\left(S_t e^{-(r-\delta)t}\right). \end{aligned}$$

Now we look for a probability $\bar{\mathbb{P}} \sim \mathbb{P}$ such that $\bar{S}$ with

$$\bar{S}_t := S_t e^{-(r-\delta)t}$$

is a martingale! in such a way that the discounted values (w.r.t. the bank account) of the replicating portfolios will be martingales!

We know that this probability is given by

$$\mathrm{d}\bar{\mathbb{P}} = \exp\left\{ \frac{r-\delta-\mu}{\sigma} W_T - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T \right\} \mathrm{d}\mathbb{P}.$$

since under $\bar{\mathbb{P}}$, W^* given by

$$W_t^* := W_t - \frac{r-\delta-\mu}{\sigma} t$$

is a Brownian motion,

In fact we have that

$$\begin{aligned} \mathrm{d}\bar{S}_t &= \mathrm{d}\left(S_t e^{-(r-\delta)t}\right) = e^{-(r-\delta)t}(\mathrm{d}S_t - S_t(r-\delta)\mathrm{d}t) \\ &= \sigma e^{-(r-\delta)t} S_t \mathrm{d}\left(W_t - \frac{r-\delta-\mu}{\sigma}t\right) \\ &= \sigma e^{-(r-\delta)t} S_t \mathrm{d}W^* \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}\tilde{V}_t &= \phi_t^1 e^{-\delta t}\sigma e^{-(r-\delta)t}\mathrm{d}W^* \\ &= \phi_t^1 \sigma \tilde{S}_t \mathrm{d}W^*. \end{aligned}$$

Price of a call option with dividends.

We know that in the BS model we can replicate $X = (S_T - K)_+$ by and admissible portfolio (self-financing and bounded from below), say V, then $\tilde{V}_T = \frac{X}{e^{rT}}$, and since $\tilde{V}$ is a martingale under $\bar{\mathbb{P}}$ then

$$\begin{aligned} \tilde{V}_t &= \mathbb{E}_{\bar{\mathbb{P}}}\left(\tilde{V}_T \middle| \mathcal{F}_t \right) = \mathbb{E}_{\bar{\mathbb{P}}}\left(\frac{X}{e^{rT}} \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{\bar{\mathbb{P}}}\left(\frac{(S_T - K)_+}{e^{rT}} \middle| \mathcal{F}_t \right), \end{aligned}$$

equivalently

$$C_t = V_t = \mathbb{E}_{\bar{\mathbb{P}}}\left(\frac{(S_T - K)_+}{e^{r(T-t)}} \middle| \mathcal{F}_t \right).$$

We have to calculate this expectation under $\bar{\mathbb{P}}$. We have that under $\bar{\mathbb{P}}$

$$\mathrm{d}\bar{S}_t = \sigma e^{-(r-\delta)t} S_t \mathrm{d}W^*,$$

equivalently

$$\mathrm{d}S_t = (r-\delta)S_t\mathrm{d}t + \sigma S_t \mathrm{d}W^*.$$

where W^* is a $\bar{\mathbb{P}}$-Brownian motion so we can do the same calculations as in the case without dividends replacing r by $r-\delta$, that is

$$\mathbb{E}_{\bar{\mathbb{P}}}\left(\left. \frac{(S_T-K)_+}{e^{(r-\delta)(T-t)}} \right| \mathcal{F}_t \right) = S_t\Phi(d_+) - Ke^{-(r-\delta)(T-t)}\Phi(d_-)$$

with

$$d_\pm = \frac{\log\left(\frac{S_t}{K}\right) + (r-\delta\pm\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

Then

$$\begin{aligned} C_t &= \mathbb{E}_{\bar{\mathbb{P}}}\left(\left. \frac{(S_T-K)_+}{e^{r(T-t)}} \right| \mathcal{F}_t \right) = e^{-\delta(T-t)}\mathbb{E}_{\bar{\mathbb{P}}}\left(\left. \frac{(S_T-K)_+}{e^{(r-\delta)(T-t)}} \right| \mathcal{F}_t \right) \\ &= e^{-\delta(T-t)}S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-). \end{aligned}$$

Definition

We call a standard d-dimensional Brownian motion an $\mathbb{R}^d$-value process

$$W = (W^1, ..., W^d)$$

where all the W^i, $i = 1, ..., d$ are independent standard Brownian motions.

Definition

An Itô process with respect to a d-dimensional Brownian motion W is a stochastic process X of the form

$$X_t = X_0 + \int_0^t K_s \mathrm{d}s + \sum_{i=1}^{d} \int_0^t H_s^i \mathrm{d}W_s^i,\ 0 \leq t \leq T.$$

where X_0 is $\mathcal{F}_0$-measurable, K and H are adapted and $\int_0^T \left(|K_s| + (H_s^i)^2\right) \mathrm{d}s < \infty$ a.s.

Theorem

Let $X = (X^1, ..., X^n)$ *be a vector of n Itô processes*

$$X_t^i = X_0^i + \int_0^t K_s^i \mathrm{d}s + \sum_{j=1}^d \int_0^t H_s^{ij} \mathrm{d}W_s^j,\ 0 \leq t \leq T.$$

and $f(t,x)$ *two times differentiable w.r.t.* x *and once w.r.t.* t *with continuous partial derivatives in* (t,x)*,then*

$$\begin{aligned} f(t,X) &= f(t,X_0) + \int_0^t \partial_s f(s,X)\mathrm{d}s + \sum_{i=1}^n \int_0^t \partial_{x_i} f(s,X)\mathrm{d}X_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t \partial_{x_i x_j} f(s,X)\mathrm{d}\langle X^i, X^j\rangle_s, \end{aligned}$$

where $dX_s^i = K_s^i \mathrm{d}s + \sum_{j=1}^d H_s^{ij} \mathrm{d}W_s^j$, $d\langle X^i, X^j\rangle_s = \sum_{k=1}^n H_s^{ik} H_s^{jk} \mathrm{d}s.$

The model of the financial market consists in $(d+1)$ stocks S_t^0, S_t^1, ..., S_t^d in such a way that

$$\mathrm{d}S_t^0 = S_t^0 r(t)\mathrm{d}t, S_0^0 = 1,$$

and

$$\mathrm{d}S_t^i = S_t^i(\mu^i(t)\mathrm{d}t + \sum_{j=1}^{d} \sigma^{ij}(t)\mathrm{d}W_t^j), i = 1, ..., d$$

where $W = (W^1, ..., W^d)$ is a d-dimensional Brownian motion. By simplicity we assume that μ, σ and r are deterministic and càdlàg. We shall consider the natural filtration associated with W.

An investment strategy will be an adapted process $\phi = ((\phi_t^0, \phi_t^1, ..., \phi_t^d))_{0\leq t\leq T}$ in R^{d+1}. The value of the portfolio at time t is given by the scalar product

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{i=0}^{d} \phi_t^i S_t^i,$$

and its discounted value is

$$\tilde{V}_t(\phi) = e^{-\int_0^t r_s ds} V_t(\phi) = \phi_t \cdot \tilde{S}_t.$$

We assume that the stocks can give continuous dividends: $((\delta_t^1, ..., \delta_t^d))_{0\leq t\leq T}$, deterministic and càdlàg, in such a way that, if the strategy is self-financing,

$$\mathrm{d}V_t(\phi) = \sum_{i=0}^{d} \phi_t^i \mathrm{d}S_t^i + \sum_{i=1}^{d} \phi_t^i S_t^i \delta_t^i \mathrm{d}t,$$

Then we can write

$$
\begin{aligned}
\mathrm{d}\tilde{V}_t &= \mathrm{d}\left(e^{-\int_0^t r_s \mathrm{d}s} V_t(\phi)\right) = -r_t e^{-\int_0^t r_s \mathrm{d}s} V_t \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \mathrm{d}V_t \\
&= -r_t e^{-\int_0^t r_s \mathrm{d}s} V_t \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \left(\sum_{i=0}^d \phi_t^i \mathrm{d}S_t^i + \sum_{i=1}^d \phi_t^i S_t^i \delta_t^i \mathrm{d}t\right) \\
&= e^{-\int_0^t r_s \mathrm{d}s} r_t (\phi_t^0 S_t^0 - V_t) \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \sum_{i=1}^d \left(\phi_t^i \mathrm{d}S_t^i + \phi_t^i S_t^i \delta_t^i \mathrm{d}t\right) \\
&= e^{-\int_0^t r_s \mathrm{d}s} \sum_{i=1}^d \phi_t^i S_t^i (\delta_t^i - r_t) \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \sum_{i=1}^d \phi_t^i \mathrm{d}S_t^i \\
&= \sum_{i=1}^d \phi_t^i e^{-\int_0^t \delta_s^i \mathrm{d}s} \left(e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} S_t^i (\delta_t^i - r_t) \mathrm{d}t + e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \mathrm{d}S_t^i\right) \\
&= \sum_{i=1}^d \phi_t^i e^{-\int_0^t \delta_s^i \mathrm{d}s} \mathrm{d}\bar{S}_t^i, \qquad (1)
\end{aligned}
$$

Where

$$\bar{S}_t^i := \frac{S_t^i}{e^{\int_0^t (r_s - \delta_s^i)\mathrm{d}s}} = \frac{S_t^i e^{\int_0^t \delta_s^i \mathrm{d}s}}{e^{\int_0^t r_s \mathrm{d}s}}$$

Now we look for a probability, $\bar{\mathbb{P}}$, under which the *discounted* values of $\left(S_t^i e^{\int_0^t \delta_s^i \mathrm{d}s}\right)_{0\le t\le T}$, $i = 1, ...d$, are martingales.

$$\begin{aligned}
\mathrm{d}\bar{S}_t^i &= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s}\left(S_t^i(\delta_t^i - r_t)\mathrm{d}t + \mathrm{d}S_t^i\right) \\
&= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i \left(\left(\delta_t^i + \mu_t^i - r_t\right)\mathrm{d}t + \sum_{j=1}^d \sigma_t^{ij}\mathrm{d}W_t^j\right) \\
&= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i \sum_{j=1}^d \sigma_t^{ij}\left(\sum_{k=1}^d \left(\sigma_t^{-1}\right)^{jk}(t)(\delta_t^k + \mu_t^k - r_t)\mathrm{d}t + \mathrm{d}W_t^j\right) \\
&= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i \sum_{j=1}^d \sigma^{ij}(t)\mathrm{d}\bar{W}_t^j \qquad (2)
\end{aligned}$$

with

$$\mathrm{d}\bar{W}_t^j = \mathrm{d}W_t^j + \sum_{k=1}^d \left(\sigma^{-1}\right)^{jk}(t)(\delta_t^k + \mu_t^k - r_t)\mathrm{d}t, \quad j = 1, ..., d$$

Now, if X is a $\bar{\mathbb{P}}$-square integrable payoff, by the representation theorem of (mutidimensional) Brownian martingales, we can write

$$\mathbb{E}_{\bar{\mathbb{P}}}\left(\tilde{X}|\mathcal{F}_t\right) = \mathbb{E}_{\bar{\mathbb{P}}}\left(\tilde{X}\right) + \sum_{j=1}^{d}\int_0^t Y_s^j \mathrm{d}\bar{W}_s^j,$$

where Y^j are $\bar{\mathbb{P}}$-square integrable (w.r.t. $\mathrm{d}\bar{\mathbb{P}} \otimes \mathrm{d}t$) adapted processes
Then, by (1) and (2)

$$\begin{aligned}
\tilde{V}_t &= V_0 + \sum_{i=1}^{d}\int_0^t \phi_s^i e^{-\int_0^s \delta_u^i \mathrm{d}u} \mathrm{d}\bar{S}_s^i \\
&= V_0 + \sum_{i=1}^{d}\int_0^t \phi_s^i e^{-\int_0^s \delta_u^i \mathrm{d}u} e^{-\int_0^s (r_u - \delta_u^i)\mathrm{d}u} S_s^i \sum_{j=1}^{d} \sigma_s^{ij} \mathrm{d}\bar{W}_s^j \\
&= V_0 + \sum_{i=1}^{d}\int_0^t \phi_s^i \tilde{S}_s^i \sum_{j=1}^{d} \sigma_s^{ij} \mathrm{d}\bar{W}_s^j \\
&= V_0 + \sum_{j=1}^{d}\int_0^t \sum_{i=1}^{d} \phi_s^i \tilde{S}_s^i \sigma_s^{ij} \mathrm{d}\bar{W}_s^j
\end{aligned}$$

In such a way that if we take $V_0 = \mathbb{E}_{\mathbb{P}^*}(\tilde{X})$ and

$$\phi_t^i = \frac{1}{\tilde{S}_t^i} \sum_{k=1}^{d} \left(\sigma_t^{-1}\right)^{ki} Y_t^k, i = 1, ..., d.$$

we have that

$$\begin{aligned} \sum_{i=1}^{d} \phi_t^i \tilde{S}_t^i \sigma_t^{ij} &= \sum_{i=1}^{d} \left(\sum_{k=1}^{d} \left(\sigma_t^{-1}\right)^{ki} Y_t^k \right) \sigma^{ij}(t) \\ &= \sum_{k=1}^{d} \delta^{kj} Y_t^k = Y_t^j \end{aligned}$$

and consequently

$$\begin{aligned} \tilde{V}_t &= \mathbb{E}_{\mathbb{P}^*}(\tilde{X}) + \sum_{j=1}^{d} \int_0^t Y_s^j \mathrm{d} \bar{W}_s^j \\ &= \mathbb{E}_{\bar{\mathbb{P}}}(\tilde{X} | \mathcal{F}_t), \end{aligned}$$

is the discounted value at time t of a self-financing portfolio that replicates X, and $\tilde{V}$ is a $\bar{\mathbb{P}}$-square integrable martingale.

Price of a call option

First note that under $\bar{\mathbb{P}}$

$$\mathrm{d}S_t^i = S_t^i \left(\left(r_t - \delta_t^i\right) \mathrm{d}t + \sum_{j=1}^d \sigma_t^{ij} \mathrm{d}\bar{W}_t^j \right), i = 1, ..., d,$$

so $\left(S_t^i e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \right)$ are $\bar{\mathbb{P}}$-martingales:

$$\begin{aligned} \mathrm{d}\left(S_t^i e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \right) &= e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \left(-S_t^i \left(r_t - \delta_t^i\right) \mathrm{d}t + \mathrm{d}S_t^i \right) \\ &= e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \sum_{j=1}^d \sigma_t^{ij} S_t^i \mathrm{d}\bar{W}_t^j . \end{aligned}$$

Also we have that, by the multidimensional Itô formula, under $\bar{\mathbb{P}}$

$$S_t^i = S_0^i \exp\left\{ \sum_{j=1}^d \int_0^T \sigma_t^{ij} \mathrm{d}\bar{W}_t^j + \int_0^T \left(r_t - \delta_t^i - \frac{1}{2} \sum_{j=1}^d \left(\sigma_t^{ij}\right)^2 \right) \mathrm{d}t \right\} \quad (3)$$

Then

$$
\begin{aligned}
C_t \quad : \quad &= \mathbb{E}_{\bar{\mathbb{P}}}\left(\left. \frac{(S_T^i - K)_+}{\exp\left\{\int_t^T r_s \mathrm{d}s\right\}} \right| \mathcal{F}_t \right) \\
&= \exp\left\{ -\int_t^T \delta_s^i \mathrm{d}s \right\} \mathbb{E}_{\bar{\mathbb{P}}}\left(\left. \frac{(S_T^i - K)_+}{\exp\left\{\int_t^T (r_s - \delta_s^i) \mathrm{d}s\right\}} \right| \mathcal{F}_t \right),
\end{aligned}
$$

under $\bar{\mathbb{P}}$, and conditional to $\mathcal{F}_t$, by (3)

$$
\begin{aligned}
&\log S_T^i - \log S_t^i \\
\sim \quad & N\left(\int_t^T (r_s - \delta_s^i) \mathrm{d}s - \frac{1}{2} \int_t^T \sum_{j=1}^d \left(\sigma_s^{ij}\right)^2 \mathrm{d}s, \int_t^T \sum_{j=1}^d \left(\sigma_s^{ij}\right)^2 \mathrm{d}s \right).
\end{aligned}
$$

Remember that if, conditional to $\mathcal{F}_t$,

$$\log X_T - \log X_t \sim N\left(r(T-t) - \frac{1}{2}\sigma^2(T-t), \sigma^2(T-t)\right)$$

$$\mathbb{E}\left(\left.\frac{(X_T-K)_+}{\exp\{r(T-t)\}}\right| \mathcal{F}_t\right) = X_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

with

$$d_\pm = \frac{\log(\frac{X_t}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

Therefore

$$C_t = \exp\left\{-\int_t^T \delta_s^i \mathrm{d}s\right\}\left(S_t^i \Phi(d_+^i) - K \exp\left\{-\int_t^T (r_s - \delta_s^i)\mathrm{d}s\right\}\Phi(d_-^i)\right),$$

with

$$d_\pm^i = \frac{\log \frac{S_t^i}{K} + \int_t^T \left(r_s - \delta_s^i \pm \frac{1}{2}\sum_{j=1}^d \left(\sigma_s^{ij}\right)^2\right)\mathrm{d}s}{\sqrt{\int_t^T \sum_{j=1}^d \left(\sigma_s^{ij}\right)^2 \mathrm{d}s}}.$$

We have assumed that $\left(\sigma_t^{ij}\right)_{0\leq t\leq T}$ is invertible, in particular that the number of stocks and independent Brownian motions are the same, and from here we conclude that the model is free of arbitrage and complete:

$$\begin{aligned}
\mathrm{d}\bar{S}_t^i &= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i \left(\left(\delta_t^i + \mu_t^i - r_t\right)\mathrm{d}t + \sum_{j=1}^d \sigma_t^{ij}\mathrm{d}W_t^j \right) \\
&= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i \sum_{j=1}^d \sigma_t^{ij} \left(\sum_{k=1}^d \left(\sigma_t^{-1}\right)^{jk} (\delta_t^k + \mu_t^k - r_t)\mathrm{d}t + \mathrm{d}W_t^j \right) \\
&= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i \sum_{j=1}^d \sigma^{ij}(t)\mathrm{d}\bar{W}_t^j
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{d}\bar{W}_t^j &= \mathrm{d}W_t^j + \sum_{k=1}^d \left(\sigma_t^{-1}\right)^{jk} (\delta_t^k + \mu_t^k - r_t)\mathrm{d}t, \quad j = 1, \dots, d \\
&= \mathrm{d}W_t^j + \theta_j(t)\mathrm{d}t, \quad j = 1, \dots, d,
\end{aligned}$$

where $\theta_j(t) = \sum_{k=1}^d \left(\sigma_t^{-1}\right)^{jk} (r_t - \delta_t^k - \mu_t^k)$.

Then if we take

$$\mathrm{d}\bar{\mathbb{P}} = \Pi_{j=1}^{n} \exp\left\{-\int_0^T \theta_j(t)\mathrm{d}W_t^j - \frac{1}{2}\int_0^T \theta_j^2(t)\mathrm{d}t\right\}\mathrm{d}\mathbb{P}.$$

It turns out that $\bar{W}$ is a d-dimensional Brownian motion with respect to the probability $\bar{\mathbb{P}}$, the admissible portfolios are $\bar{\mathbb{P}}$-local martingales, so the model is free of arbitrage and if X is a $\bar{\mathbb{P}}$-square integrable payoff, by the representation theorem of (mutidimensional) Brownian martingales, we can replicate it.

Assume now that the number of independent BM is l. For the lack of arbitrage it is sufficient to have $\theta(t)$ such that $\sum_{j=1}^{l}\sigma_t^{ij}\theta_j(t)=\delta_t^i+\mu_t^i-r_t$, $i=1,\dots,d$:

$$
\begin{aligned}
\mathrm{d}\bar{S}_t^i &= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i\left(\left(\delta_t^i+\mu_t^i-r_t\right)\mathrm{d}t+\sum_{j=1}^{l}\sigma_t^{ij}\mathrm{d}W_t^j\right)\\
&= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i\sum_{j=1}^{l}\sigma_t^{ij}\left(\theta_j(t)\mathrm{d}t+\mathrm{d}W_t^j\right)\\
&= e^{-\int_0^t (r_s-\delta_s^i)\mathrm{d}s} S_t^i\sum_{j=1}^{l}\sigma_t^{ij}\mathrm{d}\bar{W}_t^j
\end{aligned}
$$

In general we can do this if $l\geq d$.
But for completeness we need to solve $\sum_{i=1}^{d}\phi_t^i\tilde{S}_t^i\sigma_t^{ij}=Y_t^j, j=1,\dots,l$ and this can be done if $d\geq l$.

In this way we can have viable models where the dimension of W is strickly greater than the number of risky stocks but then they are no complete. Then how can we price derivatives in these incomplete, but arbitrage free, models? The answer is quite simple: If we have a risk-neutral probability measure $\bar{\mathbb{P}}$ and we **define** the discounted value of the payoff X by

$$\tilde{C}_t := \mathbb{E}_{\bar{\mathbb{P}}}\left(\tilde{X}\middle|\,\mathcal{F}_t\right)$$

we are introducing new (if X is not replicable) risky stocks, in a way that discounted values of all the assets in the market are $\bar{\mathbb{P}}$-martingales! So we have a model free of arbitrage! Obviously we have different pricing models for different risk-neutral probabilities.