Lesson 6

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Optimal stopping times

Now we want to characterize all the optimal stopping times. Let, as above, $(X_n)_{0\le n\le N}$ be the Snell envelope of $(Y_n)_{0\le n\le N}$.

Theorem

ν is an optimal stopping time if and only if

$$\begin{cases} X_\nu = Y_\nu \\ (X_n^\nu)_{0\le n\le N} \quad \textit{is a martingale} \end{cases}$$

Proof.

We know that

$$\sup_{\tau \in \tau_{0,N}} \mathbb{E}(Y_\tau | \mathcal{F}_0) = X_0.$$

Then if $(X_n^\nu)_{0 \le n \le N}$ is a martingale and $X_\nu = Y_\nu$

$$\begin{aligned}\sup_{\tau \in \tau_{0,N}} \mathbb{E}(Y_\tau | \mathcal{F}_0) &= X_0 = X_0^\nu \\ &= \mathbb{E}(X_N^\nu | \mathcal{F}_0) = \mathbb{E}(X_{N \wedge \nu} | \mathcal{F}_0) \\ &= \mathbb{E}(X_\nu | \mathcal{F}_0) = \mathbb{E}(Y_\nu | \mathcal{F}_0).\end{aligned}$$

$\square$

Proof.

Reciprocally, if ν is optimal

$$\sup_{\tau\in\tau_{0,N}} \mathbb{E}(Y_\tau|\mathcal{F}_0) = X_0 = \mathbb{E}(Y_\nu|\mathcal{F}_0) \le \mathbb{E}(X_\nu|\mathcal{F}_0) = \mathbb{E}(X_{N\wedge\nu}|\mathcal{F}_0) \le X_0, \tag{1}$$

where the last inequality is due to the fact that (X_n^ν) is a supermartingale. So, we have

$$\mathbb{E}(X_\nu - Y_\nu|\mathcal{F}_0) = 0$$

and since $X_\nu - Y_\nu \ge 0$, because X is the Snell envelope of Y, we conclude that $X_\nu = Y_\nu$. □

Proof.

Now we can also see that $(X_n^\nu)_{0\le n\le N}$ is a martingale. We know that it is a supermartingale, then

$$X_0 \ge \mathbb{E}(X_n^\nu|\mathcal{F}_0) \ge \mathbb{E}(X_N^\nu|\mathcal{F}_0) = \mathbb{E}(X_\nu|\mathcal{F}_0) = X_0$$

as we saw in (1). Then, for all n

$$\mathbb{E}(X_n^\nu - \mathbb{E}(X_\nu|\mathcal{F}_n)|\mathcal{F}_0) = 0,$$

and since $(X_n^\nu)_{0\le n\le N}$ is supermartingale,

$$X_n^\nu \ge \mathbb{E}(X_N^\nu|\mathcal{F}_n) = \mathbb{E}(X_\nu|\mathcal{F}_n)$$

therefore $X_n^\nu = E(X_\nu|\mathcal{F}_n)$. □

Decomposition of supermartingales (Doob's decomposition)

Proposition

Any supermartingale $(X_n)_{0\leq n\leq N}$ *has a unique decomposition:*

$$X_n = M_n - A_n, 0 \leq n \leq N,$$

where $(M_n)_{0\leq n\leq N}$ *is a martingale and* $(A_n)_{0\leq n\leq N}$ *is non-decreasing predictable with* $A_0 = 0$*.*

Proof.

It is enough to write

$$X_n = \sum_{j=1}^{n}(X_j - \mathbb{E}(X_j|\mathcal{F}_{j-1})) - \sum_{j=1}^{n}(X_{j-1} - \mathbb{E}(X_j|\mathcal{F}_{j-1})) + X_0$$

and to identify

$$M_n = \sum_{j=1}^{n}(X_j - \mathbb{E}(X_j|\mathcal{F}_{j-1})) + X_0,$$

$$A_n = \sum_{j=1}^{n}(X_{j-1} - \mathbb{E}(X_j|\mathcal{F}_{j-1}))$$

where we define $M_0 = X_0$ and $A_0 = 0$. □

Proof.

So $(M_n)_{0\le n\le N}$ is a martingale:

$$M_n - M_{n-1} = X_n - \mathbb{E}(X_n|\mathcal{F}_{n-1}), \quad 1 \le n \le N$$

in such a way that

$$\mathbb{E}(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0, \quad 1 \le n \le N.$$

Finally since $(X_n)_{0\le n\le N}$ is supermartingale

$$A_n - A_{n-1} = X_{n-1} - \mathbb{E}(X_n|\mathcal{F}_{n-1}) \ge 0, \quad 1 \le n \le N.$$

$\square$

Proof.

Now we can see the uniqueness. If

$$M_n - A_n = M'_n - A'_n, \quad 0 \le n \le N$$

we have

$$M_n - M'_n = A_n - A'_n, \quad 0 \le n \le N,$$

but then since $(M_n)_{0\le n\le N}$ y $(M'_n)_{0\le n\le N}$ are martingales and $(A_n)_{0\le n\le N}$ y $(A'_n)_{0\le n\le N}$ predictable, it turns out that

$$\begin{aligned} A_{n-1} - A'_{n-1} &= M_{n-1} - M'_{n-1} = E(M_n - M'_n|\mathcal{F}_{n-1}) \\ &= E(A_n - A'_n|\mathcal{F}_{n-1}) = A_n - A'_n, \quad 1 \le n \le N, \end{aligned}$$

that is

$$A_N - A'_N = A_{N-1} - A'_{N-1} = ... = A_0 - A'_0 = 0,$$

since by hypothesis $A_0 = A'_0 = 0$. □

This decomposition is known as the Doob decomposition.

Proposition

The largest optimal stopping time for $(Y_n)_{0\leq n\leq N}$ *is given by*

$$\nu_{\max} = \begin{cases} N & si\ A_N = 0 \\ \inf\{n, A_{n+1} > 0\} & si\ A_N > 0 \end{cases},$$

where $(X_n)_{0\leq n\leq N}$, *Snell envelope of* $(Y_n)_{0\leq n\leq N}$, *has a Doob decomposition* $X_n = M_n - A_n, 0 \leq n \leq N$.

Proof.

$\{\nu_{\max} = n\} = \{A_1 = 0, A_2 = 0, ..., A_n = 0, A_{n+1} > 0\} \in \mathcal{F}_n$, $0 \leq n \leq N-1$, $\{\nu_{\max} = N\} = \{A_N = 0\} \in \mathcal{F}_{N-1}$. So, it is a stopping time.

$$X_n^{\nu_{\max}} = X_{n\wedge\nu_{\max}} = M_{n\wedge\nu_{\max}} - A_{n\wedge\nu_{\max}} = M_{n\wedge\nu_{\max}}$$

since $A_{n\wedge\nu_{\max}} = 0$. Therefore $(X_n^{\nu_{\max}})_{0\leq n\leq N}$ is a martingale. □

Proof.

So, to see that this stopping time is optimal we have to prove that

$$X_{\nu_{\max}} = Y_{\nu_{\max}}$$

$$\begin{aligned} X_{\nu_{\max}} &= \sum_{j=0}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} X_j + \mathbf{1}_{\{\nu_{\max}=N\}} X_N \\ &= \sum_{j=0}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} \max(Y_j, \mathbb{E}(X_{j+1}|\mathcal{F}_j)) + \mathbf{1}_{\{\nu_{\max}=N\}} Y_N, \end{aligned}$$

but in $\{\nu_{\max} = j\}$, $A_j = 0, A_{j+1} > 0$ so

$$\mathbb{E}(X_{j+1}|\mathcal{F}_j) = \mathbb{E}(M_{j+1}|\mathcal{F}_j) - A_{j+1} < \mathbb{E}(M_{j+1}|\mathcal{F}_j) = M_j = X_j$$

therefore $X_j = Y_j$ en $\{\nu_{\max} = j\}$ and consequently $X_{\nu_{\max}} = Y_{\nu_{\max}}$. □

Proof.

Finally we see that is the largest optimal stopping time. Let $\tau \geq \nu_{\max}$ and $\mathbb{P}\{\tau > \nu_{\max}\} > 0$. Then

$$\begin{aligned}\mathbb{E}(X_{\tau\wedge N}) &= \mathbb{E}(X_\tau) = \mathbb{E}(M_\tau) - \mathbb{E}(A_\tau) = M_0 - \mathbb{E}(A_\tau) \\ &= X_0 - \mathbb{E}(A_\tau) < X_0,\end{aligned}$$

since $\{\tau > \nu_{\max}\} = \{A_\tau > 0\}$, so $(X_n^\tau)_{0\leq n\leq N}$ cannot be a martingale. $\square$

Remark

The Doob decomposition of the Snell envelope, X, of Y is given by

$$X_n = M_n - A_n,$$

with

$$M_n = X_0 + \sum_{i=1}^{n} \left(X_i - \mathbb{E}(X_i | \mathcal{F}_{i-1}) \right),$$
$$A_n = \sum_{i=1}^{n} \left(X_{i-1} - \mathbb{E}(X_i | \mathcal{F}_{i-1}) \right),$$

then,

$$\begin{aligned} \nu_{\max} &= \inf\{n, A_{n+1} > 0\} = \inf\{n, \mathbb{E}(X_{n+1} | \mathcal{F}_n) - X_n < 0\} \\ &= \inf\{n, \mathbb{E}(X_{n+1} | \mathcal{F}_n) - Y_n < 0\}. \end{aligned}$$

Hedge of American options

By the previous results, if U_n denotes the price of an American option and $\tilde{U}_n$ its discount value, we know that we can decompose

$$\tilde{U}_n = M_n - A_n$$

where $(M_n)_{0\leq n\leq N}$ is a positive $\mathbb{P}^*$-martingale and $(A_n)_{0\leq n\leq N}$ is a non-decreasing and predictable process with $A_0 = 0$. If we receive the amount U_0 we can built a self-financing portfolio replicating $(1+r)^N M_N$. In fact, since the market is complete, any positive payoff can be replicated, so there will exist ϕ such that

$$V_N(\phi) = (1+r)^N M_N$$

with $V_0(\phi) = M_0 = U_0$ or what is the same

$$\tilde{V}_N(\phi) = M_N, \; V_0(\phi) = U_0$$

but $(\tilde{V}_n(\phi))_{0\leq n\leq N}$ and $(M_n)_{0\leq n\leq N}$ are $\mathbb{P}^*$-martingales in such a way that $\tilde{V}_n(\phi) = M_n, 0 \leq n \leq N$.

Note that, consequently, we have

$$U_n = (1+r)^n M_n - (1+r)^n A_n = V_n(\phi) - (1+r)^n A_n$$

and therefore

$$V_n(\phi) = U_n + (1+r)^n A_n \geq U_n. \tag{2}$$

In other words with the money we receive we can *super*-hedge the derivative.

Optimal exercise of the American option

Let, as before, $(Z_n)_{0 \leq n \leq N}$ the payoffs of the American option. It seems natural to exercise the option at a stopping time ν that gives the optimal value to a contract with payoff Z_ν in such a way that ν is an optimal stopping time as defined above. That is, a time ν, such that

$$U_0 = \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\nu) = \sup_{\tau \in \tau_{0,N}} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau).$$

In fact if ν is such that $U_{\nu(\omega)}(\omega) > Z_{\nu(\omega)}(\omega)$ it is not worth to exercise the option since the value of the contract $U_{\nu(\omega)}(\omega)$ is greater than what you obtain if you exercise it: $Z_{\nu(\omega)}(\omega)$. Note that it would be better to sell the contract.

So, you will look for ν such that $U_v = Z_v$. On the other hand you will look as well for $A_n = 0$, for all $1 \leq n \leq \nu$, (or equivalently $A_\nu = 0$) otherwise, according with (2) it is better to exercice it before and to built a portfolio with the strategy ϕ:

$$V_\nu(\phi) - U_\nu = (1+r)^\nu A_\nu > 0.$$

So ν the optimal time to exercice the option is *an optimal stopping time* as defined above: $U_v = Z_v$ and $A_\nu = 0$.

Example

The following example is a compute program written in *Mathematica* to calculate the value of a *call* and *put* for a CRR mode with the following data: $S_0 = 100€$, $K = 100€$ $b = 0.2$, $a = -0.2$, $r = 0.02$, $n = 4$ periods.

```
Clear[s, call, pu];
s[0] = Table[100, 1];
a = -0.2; b = 0.2; r = 0.02; n = 4;
p = (r - a)/(b - a);
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
ColumnForm[Table[s[i],{i, 0, n}], Center]
pp[x_]:= Max[x, 0]
call[n] = Map[pp, s[n] - 100]; pu[n] = Map[pp, 100 - s[n]];
call[x_] := call[x] = Drop[p*call[x + 1]/(1 + r) + (1 -
p)*RotateLeft[call[x + 1], 1]/(1 + r), -1]
ColumnForm[Table[call[i],{i, 0, n}], Center]
pu[x_] := pu[x] = Drop[p*pu[x + 1]/(1 + r) + (1 - p)*RotateLeft[pu[x +
1],1]/(1 + r), -1]
ColumnForm[Table[pu[i],{i, 0, n}], Center]
```

Stock Price

100,

120., 80.,

144., 96., 64.,

172.8, 115.2, 76.8, 51.2,

207.36, 138.24, 92.16, 61.44, 40.96

Call price

19.6558,

31.5473, 5.99522,

49.4091, 11.1184, 0.,

74.7608, 20.6196, 0., 0.,

107.36, 38.24, 0, 0, 0

Put price

12.0403,

5.77956, 20.2275,

1.52595, 11.2353, 32.1169,

0., 3.45882, 21.2392, 46.8392,

0, 0, 7.84, 38.56, 59.04

Example

Consider a CRR model with 91 periods $a = -b$. We want to calculate the initial value of a European call where the underlying is a share of Telefónica.

- Maturity: 3 months (91 days$= n$) ($T = 91/365$).
- Current price of the share of Telefónica 15.54€.
- Strike 15.54€.
- Annual interest rate 4.11 %.
- Annual volatility: 23,20%
  $\left(\text{Implied b: } b^2 = (\text{annual volatility})^2 \times T/n\right)$

```
Clear[s, c];n = 91;so = 15.54;
K = 15.54;vol = 0.232;T = 91/365;
r = 0.0411*T/n; b =vol*Sqrt[T/n]; a =-b;
p = (r - a)/(b - a);q = 1 - p;
s[0] = Table[so, 1];
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
pp[x_] := Max[x, 0];
c[n] = Map[pp, s[n] - K];
c[x_] := c[x] = Drop[p*c[x + 1]/(1 + r) + q*RotateLeft[c[x + 1], 1]/(1
+ r), -1];
c[0][[1]]
```

Initial call price 0.796828

Example

Here it is shown how to calculate the premium of an American put option with maturity of 3 months on stocks whose current value is 60€, the strike price is also 60€ (*at the money: ATM*) , the annual interest rate is 10% and the annual volatility 45%. We assume a CRR model with 12 periods. It is also analyzed in which nodes is convenient to exercise the option.

```
Clear[s, pa, vc, vi];
T = 1/4;n =12;so = 60;K = 60;vol = 0.45;ra = 0.10;
r = ra*T/n;b =vol*Sqrt[T/n];a=-b;
p = (r - a)/(b - a); q = 1 - p; pp[x_] := Max[x, 0]
s[0] = Table[so, 1];
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
ColumnForm[Table[s[i], {i, 0, n}], Center]
pa[n] = Map[pp, K - s[n]];
pa[x_] := pa[x] = K - s[ x] + Map[pp, Drop[p*pa[x + 1]/(1 + r) +
q*RotateLeft[pa[x + 1], 1]/(1 + r), -1] - K + s[x]]
ColumnForm[Table[pa[i], {i, 0, n}], Center]
vc[n] = Map[pp, K - s[n]]; vc[x_] := Drop[p*pa[x + 1]/(1 + r) +
q*RotateLeft[pa[x + 1], 1]/(1 + r), -1]
vi[i_] := Map[pp, K - s[i]]
ColumnForm[Table[vc[i] - vi[i],{i, 0, n}], Center]
ColumnForm[Table[pa[i] - vi[i],{i, 0, n}], Center]
```