Advanced Mathematics for Scientific Challenges
Autumn 2022

EXERCISES 1.2

1. Let us consider the convex set (polyhedron),
$$C = \{(x,\, y) \in \mathbb{R}^2 \text{ such that } 0 \le x \le 1,\ 0 \le y \le 1\}.$$
Write this set in the form $\{z \in \mathbb{R}^n \text{ such that } Az = b,\, z \ge 0\}$, compute the basic feasible solutions and, from them, the vertices.

2. Assume that $a_1, \dots, a_m$ are given vectors in $\mathbb{R}^3$ (all different from 0). Let $b_1, \dots, b_m$ strictly positive numbers and let us define the set
$$M = \{x \in \mathbb{R}^3 \text{ such that } a_i^t x \le b_i \text{ for } i = 1, \dots, m\}.$$
   (a) Show that the interior of this set is not empty.
   (b) We want to determine the centre and the radius of the biggest sphere contained in M. Write this problem as a linear programme.

3. Use a software package to solve
$$(a)\left\{\begin{array}{lcl} \text{Minimize } -8x_1 - 9x_2 - 5x_3 & & \\ \text{subject to} & & \\ \quad x_1 + x_2 + 2x_3 & \le & 2 \\ 2x_1 + 3x_2 + 4x_3 & \le & 3 \\ 6x_1 + 6x_2 + 2x_3 & \le & 8 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 & & \end{array}\right. \qquad (b)\left\{\begin{array}{lcl} \text{Minimize } 5x_1 - 3x_2 & & \\ \text{subject to} & & \\ \quad x_1 - x_2 & \ge & 2 \\ 2x_1 + 3x_2 & \le & 4 \\ -x_1 + 6x_2 & = & 10 \\ x_1 \ge 0,\ x_2 \ge 0 & & \end{array}\right.$$
$$(c)\left\{\begin{array}{lcl} \text{Maximize } 3x_1 + 2x_2 - 5x_3 & & \\ \text{subject to} & & \\ 4x_1 - 2x_2 + 2x_3 & \le & 4 \\ -2x_1 + x_2 - x_3 & \le & -1 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 & & \end{array}\right. \qquad (d)\left\{\begin{array}{lcl} \text{Maximize } 4x_1 + 6x_2 + 3x_3 + x_4 & & \\ \text{subject to} & & \\ 1.5x_1 + 2x_2 + 4x_3 + 3x_4 & \le & 550 \\ 4x_1 + x_2 + 2x_3 + x_4 & \le & 700 \\ 2x_1 + 3x_2 + x_3 + 2x_4 & \le & 200 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0 & & \end{array}\right.$$

4. **(Optional)** Consider a linear programme (P) in standard form and its dual programme (D),
$$(P)\left\{\begin{array}{l} \text{Min } z = c \cdot x \\ A \cdot x = b \\ x \ge 0 \end{array}\right. \qquad (D)\left\{\begin{array}{l} \text{Max } w = u \cdot b \\ u \cdot A \le c \end{array}\right.$$
Let us denote by A_j the jth column of A. Prove that two solutions $(\bar{x}, \bar{u})$ of, respectively, (P) and (D) are optimal if and only if
$$(\bar{u} \cdot A_j - c_j)\bar{x}_j = 0, \qquad \forall j = 1, \dots, n$$

Deadline: October 23, 23:50.