Wavelets via Multi-resolution analysis:

We shall use the notation: $(D_j f)(t) = 2^{j/2} f(2^j t)$, $j \in \mathbb{Z}$.

Definition: A multi-resolution analysis (MRA) is an increasing sequence $\dots V_n \subseteq V_{n+1} \subseteq \dots$ of closed subspaces of $L^2(\mathbb{R})$ such that

1. There exists $\varphi \in V_0$ such that $\varphi_{0,k}(t) = \varphi(t-k)$, $k \in \mathbb{Z}$, forms an orthonormal basis of V_0. φ is the scaling function of the MRA

2. For all $n \in \mathbb{Z}$ $D_1(V_n) = V_{n+1}$. Equivalently $f(t) \in V_n$ iff $f(2t) \in V_{n+1}$.
This implies that $V_n = D_n(V_0)$ and that $\{\varphi_{n,k}\}_{k\in\mathbb{Z}}$ defined by $\varphi_{n,k} = D_n(\varphi_{0,k})$, form an orthonormal basis of V_n

3. $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$

4. $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$.

Remarks: ① As it is clear from the definition, the scaling function determines the MRA

② Condition ④ is actually superfluous; it can be shown that it is implied by the others

Given a MRA as above, let W_n be the orthogonal complement of V_n in V_{n+1}; $V_{n+1} = V_n \oplus W_n$. Here $W_n = D_n(W_0)$:

$$V_{n+1} = D_n(V_1) = D_n(V_0) + D_n(W_0) = V_n + D_n(W_0)$$

and $D_n(W_0)$ is orthogonal to $D_n(V_0) = V_n$ in V_{n+1}.

For $m<n$ we have, iterating,

$$V_{n+1} = V_n \oplus W_n = \cdots = V_m \oplus W_m \oplus \cdots \oplus W_{n-1} \oplus W_n$$

In this sense $V_{n+1} = \bigoplus_{j=-\infty}^{n} W_j$ and

$$L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} W_n$$

Let $P_n : L^2(\mathbb{R}) \longrightarrow V_n$ denote the orthogonal projection.
For $f \in L^2(\mathbb{R})$, $P_n f = \sum_k \langle f, \varphi_{n,k} \rangle \varphi_{n,k}$ is the representation at resolution level n.

Let also $Q_n : L^2(\mathbb{R}) \to W_n$ be the orthogonal projection.
For $f \in L^2(\mathbb{R})$, by the definition of W_n,

$Q_n f = P_{n+1} f - P_n f$: detail between the resolution levels n and $n+1$

<u>Wavelet of the MRA</u>

A (mother) <u>wavelet of the MRA</u> $\{V_n\}_{n\in\mathbb{Z}}$ is $\psi \in L^2(\mathbb{R})$ such that $\psi_{0,k}(t) = \psi(t-k)$, $k\in\mathbb{Z}$, form an orthonormal basis of W_0.

When ψ is a mother wavelet, then the functions $\{\psi_{n,k}\}_{k\in\mathbb{Z}}$
$\psi_{n,k} = D_n(\psi_{0,k})$, $k\in\mathbb{Z}$, form a basis of W_n, and therefore the whole system $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. This is so because $W_n \perp W_m$ for $n\neq m$,
$\{\psi_{m,k}\}_{k\in\mathbb{Z}, m<n}$ form a basis of $V_n = \bigoplus_{m<n} W_m$ and $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$.

A natural question here is whether given a MRA there is always a wavelet. The answer is yes, and it is actually obtained from the scaling function.

We gather all these results in the following statement

<u>Theorem:</u> Let $\{V_n\}_{n\in\mathbb{Z}}$ be a MRA with scaling function φ.

Then ① $\sum_{k\in\mathbb{Z}} |\langle \varphi, \varphi_{1,k}\rangle|^2 = 1$

$$\sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k}\rangle \overline{\langle \varphi, \varphi_{1,k-2\ell}\rangle} = 0 \qquad \forall \ell\in\mathbb{Z},\ \ell\neq 0$$

② Denote $c_k = \langle \varphi, \varphi_{1,k}\rangle$, $k\in\mathbb{Z}$. The function

$\psi = \sum_{k\in\mathbb{Z}} (-1)^k\, \overline{c_{1-k}}\, \varphi_{1,k}$ belongs to V_1 and is orthogonal to V_0. Moreover, its translates $\psi_{0,k}(t) = \psi(t-k)$, $k\in\mathbb{Z}$ form an orthonormal basis of W_0, and the whole collection of translated and scaled functions $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$

<u>Proof:</u> ① This follows from the orthonormality of $\{\varphi_{1,k}\}_{k\in\mathbb{Z}}$ in V_1. On the one hand, since $\varphi\in V_0 \subseteq V_1$,

$\varphi = \sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k}\rangle \varphi_{1,k}$ with $\|\varphi\|^2 = 1 = \sum_{k\in\mathbb{Z}} |\langle \varphi, \varphi_{1,k}\rangle|^2$

On the other hand

$$\varphi_{0,\ell}(t) = \varphi(t-\ell) = \sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k}\rangle \varphi_{1,k}(t-\ell).$$

But $\varphi_{1,k}(t-\ell) = \varphi_{1,k+2\ell}(t)$:

$$\varphi_{1,k}(t-\ell) = \sqrt{2}\,\varphi(2(t-\ell)-k) = \sqrt{2}\,\varphi(2t-(2\ell+k)) = \varphi_{1,k+2\ell}(t),$$

so reindexing the sum above by $m = k+2\ell$,

$$\varphi_{0,\ell}(t) = \sum_{m\in\mathbb{Z}} \langle \varphi, \varphi_{1,m-2\ell}\rangle \varphi_{1,m}$$

Then $$0 = \langle \varphi, \varphi_{0,\ell}\rangle = \sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k}\rangle \overline{\langle \varphi, \varphi_{1,k-2\ell}\rangle} = 0$$

for $\ell\in\mathbb{Z}\setminus\{0\}$.

② Using again that $\varphi_{1,k}(t-\ell) = \varphi_{1,k+2\ell}(t)$, and reindexing the sum with $m = k+2\ell$, we have

$$\psi_{0,\ell}(t) = \psi(t-\ell) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}}\, \varphi_{1,k}(t-\ell) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}}\, \varphi_{1,k+2\ell}(t)$$

$$= \sum_{m\in\mathbb{Z}} (-1)^m \overline{c_{1-m+2\ell}}\, \varphi_{1,m}(t)$$

Then, since $\varphi = \sum_{m\in\mathbb{Z}} \langle \varphi, \varphi_{1,m}\rangle \varphi_{1,m} = \sum_{m\in\mathbb{Z}} c_m \varphi_{1,m}$

$$\langle \varphi, \psi_{0,\ell}\rangle = \sum_{m\in\mathbb{Z}} (-1)^m c_m c_{1-m+2\ell}$$

The indices m in c_m and $c_{1-m+2\ell}$ run in oposite directions and the same values appear twice:

$$(-1)^m c_m c_{1-m+2\ell} + (-1)^{1-m+2\ell} c_{1-m+2\ell}\, c_m$$

Since $(-1)^m$ and $(-1)^{1-m+2\ell}$ have opposite signs, this adds up to 0. Therefore $\langle \varphi, \psi_{0,\ell}\rangle = 0 \quad \forall \ell \in \mathbb{Z}$, and translating $\psi_{0,\ell} \in V_0^{\perp}$.

The orthonormality of the system $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ follows from ①. First, notice that $\langle \psi_{0,m}, \psi_{0,j}\rangle = \langle \psi, \psi_{0,m-j}\rangle$, so it is enough to see that $\langle \psi, \psi_{0,\ell}\rangle = \delta_{0\ell}$. Using the expressions

$\psi = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}}\, \varphi_{1,k}$ and $\psi_{0,\ell} = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k+2\ell}}\, \varphi_{1,k}$ (above)

we have, by ①:

$$\langle \psi, \psi_{0,\ell}\rangle = \sum_{k\in\mathbb{Z}} (-1)^{2k} \overline{c_{1-k}}\, c_{1-k+2\ell} = \qquad (1-k = m)$$

$$= \sum_{m\in\mathbb{Z}} \overline{c_m}\, c_{m+2\ell} = \delta_{0\ell}.$$

It remains to prove the hard part, that $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ span all W_0.

In order to do so assume that $f \in V_1$ is perpendicular to V_0 and that $\langle f, \psi_{0,k} \rangle = 0 \quad \forall k \in \mathbb{Z}$, and let us prove that this forces $f \equiv 0$.

Write $f = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{1,k} \rangle \varphi_{1,k}$ and encode it in $F = (f_k)_{k \in \mathbb{Z}}$, where $f_k = \langle f, \varphi_{1,k} \rangle$.

Similarly, encode $\varphi_{0,\ell}$ in a vector Φ_ℓ containing the coefficents in the basis $(\varphi_{1,k})_{k \in \mathbb{Z}}$. Since

$$\varphi_{0,\ell}(t) = \sum_{k \in \mathbb{Z}} \langle \varphi, \varphi_{1,k} \rangle \varphi_{1,k}(t-\ell) = \sum_{k \in \mathbb{Z}} c_k \, \varphi_{1,k+2\ell}(t) \qquad (k+2\ell = m)$$

$$= \sum_{m \in \mathbb{Z}} c_{m-2\ell} \, \varphi_{1,m}(t)$$

we have $\Phi_\ell = (c_{k-2\ell})_{k \in \mathbb{Z}}$.

Finally, do the same thing for each $\psi_{0,j}$, $j \in \mathbb{Z}$. Since we saw above that $\psi_{0,j} = \sum_{k \in \mathbb{Z}} (-1)^k \, \overline{c_{1-k+2j}} \, \varphi_{1,k}$, we encode this function in the vector $\Psi_j = \left((-1)^k \, \overline{c_{1-k+2j}} \right)_{k \in \mathbb{Z}}$.

In these terms, the orthogonality assumptions on f are that

$$\langle F, \Phi_\ell \rangle_{\ell^2(\mathbb{Z})} = \langle F, \Psi_j \rangle_{\ell^2(\mathbb{Z})} = 0 \qquad \forall \, \ell, j \in \mathbb{Z}$$

Let M the matrix with columns

$$\begin{array}{ccccccc}
\cdots & \Phi_1 & \Psi_1 & \Phi_0 & \Psi_0 & \Phi_{-1} & \Psi_{-1} \quad \cdots \\
 & \downarrow & \downarrow & \downarrow & & & \\
 & c_{-1-2} & (-1)^{-1} c_{1+1+2} & c_{-1} & (-1)^{-1} c_{1+1} & & \\
 & c_{0-2} & (-1)^{0} c_{1-0+2} & c_0 & (-1)^{0} c_{1-0} & & \\
 & c_{1-2} & (-1)^{1} c_{1-1+2} & c_1 & (-1)^{1} c_{1-1} & & \\
 & \vdots & \vdots & \vdots & \vdots & &
\end{array}$$

These columns are orthonormal, by the earlier part of the proof; $MM^* = I$, where M^* indicates the conjugate transpose of M. Also, by assumption $M^*F=0$. Our goal now is to see that $MM^* = I$, so that then

$$F = MM^*F = 0$$

Break M into 2×2 blocks $M_{m,l}$, $m,l \in \mathbb{Z}$ as follows

$$M_{m,l} = \begin{pmatrix} c_{2m-2l} & \overline{c_{1-2m+2l}} \\ c_{2m-2l+1} & -\overline{c_{-2m+2l}} \end{pmatrix}$$

The first column is part of the vector Φ_l and the second part of Ψ_l.

Working out MM^* we get the matrix made up of the blocks (m,l):

$$\sum_j M_{mj}(M_{lj})^* = \sum_j \begin{pmatrix} c_{2m-2j} & \overline{c_{1-2m+2j}} \\ c_{2m-2j+1} & -\overline{c_{-2m+2j}} \end{pmatrix} \begin{pmatrix} \overline{c_{2l-2j}} & \overline{c_{2l-2j+1}} \\ c_{1-2l+2j} & -c_{-2l+2j} \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_j c_{2m-2j}\overline{c_{2l-2j}} + \overline{c_{1-2m+2j}}\, c_{1-2l+2j} & \sum_j c_{2m-2j}\overline{c_{2l-2j+1}} - \overline{c_{1-2m+2j}}\, c_{-2l+2j} \\ \sum_j c_{2m-2j+1}\overline{c_{2l-2j}} - \overline{c_{-2m+2j}}\, c_{1-2l+2j} & \sum_j c_{2m-2j+1}\overline{c_{2l-2j+1}} + \overline{c_{-2m+2j}}\, c_{-2l+2j} \end{pmatrix}$$

A computation, using the orthonormality relations of part ①, shows that this is $\begin{pmatrix} \delta_{ml} & 0 \\ 0 & \delta_{ml} \end{pmatrix}$. For example, for the first entry, reindex the sum by taking $m-j=k$ in the first terms and $k=j-l$ in the second:

$$\sum_j c_{2m-2j}\overline{c_{2l-2j}} + \sum_j \overline{c_{1-2m+2j}}\, c_{1-2l+2j} = \sum_k c_{2k}\overline{c_{2k-2m+2l}} + \sum_k \overline{c_{2k+1-2m+2l}}\, c_{2k+1}$$

$$= \sum_{k\in\mathbb{Z}} c_k \overline{c_{k-2m+2l}} = \delta_{m,l}$$

The other entries are dealt with similarly.

Therefore, as desired $MM^* = I$.

Iterating these ideas one can show that $\{\psi_{n,k}\}_{k\in\mathbb{Z}}$ are all orthonormal. Since $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$, this also shows that any

$f \in V_{n+1}$ which is orthonormal to all $\{\psi_{m,k}\}_{m\le n}$ must be 0 and $\{\psi_{m,k}\}_{\substack{k\in\mathbb{Z}\\ m\le n}}$ is an orthonormal basis for W_n. Since $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$, the whole system $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Remarks. ① It can also be shown that in many cases

- $\sum_{k\in\mathbb{Z}} c_k = \sqrt{2}$
- $\hat{\phi}(\xi) = P(\xi/2)\,\hat{\phi}(\xi/2)$, where $P(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k\xi}$
- $\hat{\phi}(\xi) = \hat{\phi}(0)\prod_{k=1}^{\infty} P(\xi/2^k)$

This will be seen in an exercise.

② This result suggests a four-step scheme to construct wavelet bases $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$:

1. Determine φ scaling function, so that $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ form an orthonormal system
2. Let $V_0 = \langle \varphi_{0,k}\rangle_{k\in\mathbb{Z}}$. Check that $V_n := D_n(V_0)$ is an increasing sequence of subspaces of $L^2(\mathbb{R})$
3. Check that $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$
4. Find, using this last result ψ so that $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $W_0 = V_1 \ominus V_0$

We will use Fourier analysis to carry out, at least partially, this program.

First we state necessary and sufficient conditions for 1. to happen.

<u>Theorem 1</u> Let $\varphi \in L^2(\mathbb{R})$, $\|\varphi\|_2 = 1$. Then $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal system iff $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+k)|^2 = 1$ a.e $\xi \in \mathbb{R}$.

<u>Proof:</u> By Plancherel, since $\hat{\varphi}_{0,k}(\xi) = e^{2\pi i k\xi}\hat{\varphi}(\xi)$

$$\langle \varphi, \varphi_{0,k}\rangle = \langle \hat{\varphi}, \hat{\varphi}_{0,k}\rangle = \int_{\mathbb{R}} \hat{\varphi}(\xi)\,\overline{\hat{\varphi}(\xi)}\, e^{-2\pi i k\xi}\, d\xi =$$

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} |\hat{\varphi}(\xi)|^2 e^{-2\pi i k\xi}\, d\xi = \sum_{n\in\mathbb{Z}} \int_0^1 |\hat{\varphi}(\xi+n)|^2 e^{-2\pi i k\xi}\, d\xi$$

$$= \int_0^1 \Big(\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi+n)|^2\Big) e^{-2\pi i k\xi}\, d\xi$$

This shows that $\langle \varphi, \varphi_{0,k}\rangle = \hat{F}(k)$ (Fourier coefficient), where $F(\xi) = \sum_{n\in\mathbb{Z}} |\varphi(\xi+n)|^2$ 1-periodic.

Writing $F(\xi) = \sum_{k\in\mathbb{Z}} \hat{F}(k)e^{2\pi i k\xi}$ we see that

$\langle \varphi, \varphi_{0,k}\rangle = \delta_{0,k}$ $\forall k\in\mathbb{Z}$ ($\{\varphi_{0,k}\}_k$ orthonormal system)

iff $\hat{F}(k) = \delta_{0,k}$ $\forall k\in\mathbb{Z}$, that is, if

$$F(\xi) = \sum_{k\in\mathbb{Z}} \hat{F}(k)e^{2\pi i k\xi} = \hat{F}(0)e^{2\pi i 0\xi} = 1$$

For the general case $\langle \varphi_{0,m}, \varphi_{0,k}\rangle$ we just translate:

$$\langle \varphi_{0,m}, \varphi_{0,k}\rangle = \langle \tau_m\varphi, \tau_k\varphi\rangle = \langle \varphi, \tau_{k-m}\varphi\rangle = \delta_{km} \quad \boxtimes$$

Next we give conditions for 3. to hold

Theorem 2. Let $\{V_n\}_{n\in\mathbb{Z}}$ be an increasing sequence of closed subspaces of $L^2(\mathbb{R})$ and let $\varphi\in L^2(\mathbb{R})$ be such that $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0 and $D_1(V_n)=V_{n+1}$, $n\in\mathbb{Z}$ (conditions 1 and 2 in the definition of MRA). Assume also that $|\hat{\varphi}|$ is continuous at 0. Then $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$ iff $\hat{\varphi}(0)\neq 0$. In that case

$|\hat{\varphi}(0)|=1$.

With this we obtain the requirements to construct a MRA, according to our program. Once we have that we can construct the associated wavelet, as we already explained. We'll sketch later how to obtain φ using Fourier analysis (that is, defining $\hat{\varphi}$ rather than φ).

Proof. $\boxed{\hat{\varphi}(0)\neq 0 \Rightarrow \overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})}$. Let $W=\overline{\bigcup_{n\in\mathbb{Z}} V_n}$. Our goal is to prove that $W^\perp=\{0\}$.

Let us see first that W is invariant by dyadic translations $\tau_{2^{-l}m}$, $l,m\in\mathbb{Z}$. Given $f\in W$ and fixed $\varepsilon>0$, there exists $n_0\in\mathbb{Z}$ and $h\in V_{n_0}$ such that $\|f-h\|_2<\varepsilon$. Since $\{V_n\}_{n\in\mathbb{Z}}$ is increasing, also $h\in V_n$, $n\geq n_0$.

By the hypothesis (conditions 1. 2. of the MRA), we can write the L^2 series

$$h(x)=\sum_{k\in\mathbb{Z}} c_k^n\, \varphi(2^n x-k)$$

Then

$$\tau_{2^{-l}m}h(x)=h(x-2^{-l}m)=\sum_{k\in\mathbb{Z}} c_k^n\, \varphi(2^n(x-2^{-l}m)-k)$$

For $n\geq l$, $2^{n-l}\in\mathbb{Z}$ and $\varphi(2^n(x-2^{-l}m)-k)=\varphi(2^n x-2^{n-l}m-k)$

belongs to V_n. Since

$$\| \tau_{2^{-\ell}m} f - \tau_{2^{-\ell}m} h \|_2 = \|f-h\|_2 < \varepsilon$$

we deduce that $\tau_{2^{-\ell}m} f \in W$.

Next we observe that W is actually invariant by all translations τ_x, $x \in \mathbb{R}$: take $\ell, m \in \mathbb{Z}$ so that $2^{-\ell}m$ and x are close enough so that for a given $f \in W$ we have $\|\tau_{2^{-\ell}m} f - \tau_x f\|_2 < \varepsilon$. This is possible because, as we saw, translations are continuous in $L^2(\mathbb{R})$.

Let us see finally that $W^\perp = \{0\}$. Since $\hat\varphi(0) \neq 0$ and $|\hat\varphi|$ is continuous at 0, there exists an interval $I = (-\eta, \eta)$ where $\hat\varphi(\xi) \neq 0$. Assume that $g \in W^\perp$. Then, by the invariance by translations

$$\langle \tau_{-x} f, g \rangle = \int_{\mathbb{R}} f(t+x)\, \overline{g(t)}\, dt = 0 \qquad \forall x \in \mathbb{R} \ \forall f \in W$$

By Plancherel, this is

$$\int_{\mathbb{R}} e^{2\pi i x \xi} \hat f(\xi) \overline{\hat g(\xi)}\, d\xi = 0 \qquad \forall x \in \mathbb{R}$$

Since $\hat f \bar{\hat g} \in L^1(\mathbb{R})$, because by Cauchy-Schwarz

$$\int_{\mathbb{R}} |\hat f(\xi) \bar{\hat g}(\xi)|\, d\xi \le \left(\int |\hat f|^2 d\xi\right)^{1/2} \left(\int_{\mathbb{R}} |\hat g|^2 d\xi\right)^{1/2} = \|f\|_2 \|g\|_2 ,$$

we can use the inversion formula to deduce that $\hat f(\xi) \bar{\hat g}(\xi) = 0$ a.e $\xi \in \mathbb{R}$. In particular, for $f(x) = 2^n \varphi(2^n x) \in V_n \subseteq W$, we have $\hat f(\xi) = \hat\varphi(2^{-n}\xi)$. So $\hat\varphi(2^{-n}\xi) \bar{\hat g}(\xi) = 0$ a.e $\xi \in \mathbb{R}$. As soon as $2^{-n}\xi \in I = (-\eta, \eta)$

this forces $\hat{g}(\xi)=0$. This happens for all n such that $|\xi|<2^n\eta$. Letting $n\to\infty$ we obtain $\hat{g}(\xi)=0$ a.e $\xi\in\mathbb{R}$, that is $g=0$ in $L^2(\mathbb{R})$

$\boxed{\overline{\bigcup_{n\in\mathbb{Z}} V_n}=L^2(\mathbb{R}) \Rightarrow \hat{\varphi}(0)\neq 0}$ Let $f\in L^2(\mathbb{R})$ be such that $\hat{f}=\chi_{[-1,1]}$ (as we already know, f is a sinc function).

By Plancherel $\|f\|_2^2=\|\hat{f}\|_2^2=2$.

Let $P_n: L^2(\mathbb{R})\to V_n$ denote the orthogonal projections. Then

$$\big|\|f\|_2-\|P_nf\|_2\big|\leq\|f-P_nf\|_2\xrightarrow{n\to\infty}0$$

Let $\varphi_{n,k}(x)=2^{n/2}\varphi(2^nx-k)$, as usual. Then

$$\|P_nf\|_2^2=\Big\|\sum_{k\in\mathbb{Z}}\langle f,\varphi_{n,k}\rangle\varphi_{n,k}\Big\|_2^2=\sum_{k\in\mathbb{Z}}|\langle f,\varphi_{n,k}\rangle|^2\longrightarrow\|f\|_2^2=2,$$

because $\{\varphi_{n,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_n.

Now, by Plancherel, and using that $\hat{f}=\chi_{[-1,1]}$

$$\|P_nf\|_2^2=\sum_{k\in\mathbb{Z}}|\langle\hat{f},\hat{\varphi}_{n,k}\rangle|^2=\sum_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}}\hat{f}(\xi)\,\overline{e^{2\pi i2^{-n}k\xi}\,2^{-n/2}\,\hat{\varphi}(2^{-n}\xi)}\,d\xi\Big|^2$$

$$=\sum_{k\in\mathbb{Z}}\Big|\int_{-1}^{1}e^{-2\pi i2^{-n}\xi k}\,2^{-n/2}\,\overline{\hat{\varphi}(2^{-n}\xi)}\,d\xi\Big|^2= \qquad (2^{-n}\xi=\omega)$$

$$=\sum_{k\in\mathbb{Z}}\Big|\int_{-2^{-n}}^{2^{-n}}e^{-2\pi i\omega k}\,2^{-n/2}\,\overline{\hat{\varphi}(\omega)}\,2^{n}\,d\omega\Big|^2=$$

$$=2^n\sum_{k\in\mathbb{Z}}\Big|\int_{-2^{-n}}^{2^{-n}}e^{-2\pi i\omega k}\,\overline{\hat{\varphi}(\omega)}\,d\omega\Big|^2$$

For the $n\in\mathbb{Z}$ such that $[-2^{-n},2^{-n}]\subseteq[-1,1]$ (i.e $n\geq0$) this integral is the k-th Fourier coefficient of the function $\chi_{[-2^{-j},2^{-j}]}\bar{\hat{\varphi}}$. Thus, by Plancherel (for Fourier series), and for $n \geq 0$,

$$\|P_n f\|_2^2 = 2^n \|\chi_{[-2^{-j},2^{-j}]}\bar{\hat{\varphi}}\|^2_{L^2[-1,1]} = 2^n \int_{-2^{-n}}^{2^{-n}} |\hat{\varphi}(\omega)|^2 d\omega .$$

Then the condition $\|P_n f\|_2^2 \to 2$ and the continuity of $|\hat{\varphi}|$ at 0 yield respectively

$$2^{n-1}\int_{-2^{-n}}^{2^{-n}} |\hat{\varphi}(\omega)|^2 d\omega \xrightarrow{n\to\infty} 1$$

$$2^{n-1}\int_{-2^{-n}}^{2^{-n}} |\hat{\varphi}(\omega)|^2 d\omega = \fint_{[-2^{-n},2^{-n}]} |\hat{\varphi}(\omega)|^2 d\omega \xrightarrow[n\to\infty]{} |\hat{\varphi}(0)|^2$$

Therefore $|\hat{\varphi}(0)| = 1 \neq 0$ ⊠

Remark. Wavelet of the MRA. So far we have seen conditions on the Fourier side, to construct the scaling function φ of a MRA. It is also possible to construct the associated wavelet ψ using Fourier analysis. The scheme would be as follows.

Developing the scaling function φ of a MRA in the basis of V_1 one sees that there exists a 1-periodic function $H(\xi)$ such that

$$\hat{\varphi}(\xi) = H(\xi/2)\,\hat{\varphi}(\xi/2)$$

The function H is the so-called "refinement mask" of the MRA, and it satisfies the "quadratic mirror filter" property (QMF)

$$|H(\xi)|^2 + |H(\xi+\tfrac{1}{2})|^2 = 1 \qquad (1)$$

(see exercise 2). This is consequence of Theorem 1.

Given $f \in V_1$, and developing $f = \sum_k c_k \varphi_{1,k}$ we see that there exists $m_f \in L^2[0,1]$ 1-periodic such that

$$\hat{f}(\xi) = m_f(\xi/2)\,\hat{\varphi}(\xi/2).$$

Main Lemma: $f \in W_0$ iff there exists $v(\xi)$ 1-periodic such that

$$\hat{f}(\xi) = e^{\pi i \xi}\, v(\xi)\, \overline{H(\xi/2 + 1/2)}\, \hat{\varphi}(\xi/2)$$

The wavelet of the MRA is in V_1, so by this lemma

$$\hat{\psi}(\xi) = m_\psi(\xi/2)\,\hat{\varphi}(\xi/2) \qquad m_\psi(\xi) = e^{2\pi i \xi}\,\sigma(\xi)\,\overline{H(\xi + 1/2)},$$

for some $\sigma(\xi)$ $1/2$-periodic.

Since we want $\{\psi_{0,k}\}_{k \in \mathbb{Z}}$ to form an orthonormal system, we can use Theorem 1 to deduce that m_ψ satisfies a QMF property: $|m_\psi(\xi)|^2 + |m_\psi(\xi + 1/2)|^2 = 1$ a.e $\xi \in \mathbb{R}$

Substituting, this gives, by the periodicity of H, σ, and by (1)

$$1 = |\sigma(\xi)|^2 |H(\xi + 1/2)|^2 + |\sigma(\xi + 1/2)|^2 |H(\xi + 1)|^2 =$$
$$= |\sigma(\xi)|^2 \left(|H(\xi + 1/2)|^2 + |H(\xi)|^2\right) = |\sigma(\xi)|^2$$

Mallat's construction consists of taking the easiest possible $\sigma(\xi)$ with $|\sigma(\xi)| = 1$ a.e, namely $\sigma(\xi) \equiv 1$. Then

$$\hat{\psi}(\xi) = G(\xi/2)\,\hat{\varphi}(\xi/2) \qquad G(\xi) = e^{2\pi i \xi}\,\overline{H(\xi + 1/2)} \quad \text{1-periodic}$$

In this way ψ is defined in terms of φ (actually of $\hat{\varphi}$ and its refinement mask $H(\xi) = \frac{\hat{\varphi}(2\xi)}{\hat{\varphi}(\xi)}$).

By the main lemma $\psi \in W_0$. Also, the family $\{\psi_{0,k}\}_{k \in \mathbb{Z}}$ is orthogonal because $\hat{\psi}_{0,k}(\xi) = e^{-2\pi i k \xi}\, G(\xi/2)\,\hat{\varphi}(\xi/2)$ and the QMF property of $G(\xi)$ implies that Theorem 1 holds.

To see that $W_0 = \text{span}\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ let $f \in W_0$. By the Main Lemma there exists $v(\xi)$ 1-periodic such that

$$\hat{f}(\xi) = v(\xi)e^{\pi i \xi}\, H(\xi/2 + 1/2)\, \hat{\varphi}(\xi/2) = v(\xi)\hat{\psi}(\xi).$$

Here v has the form $v(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k\xi}$, $\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty$,

so

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}} a_k\, e^{2\pi i \xi k}\, \hat{\psi}(\xi) = \sum_{k\in\mathbb{Z}} a_k\, \widehat{\psi(\xi-k)} = \widehat{\sum_{k\in\mathbb{Z}} a_k \psi_{0,k}}\, .$$

Hence $\quad f = \sum_k a_k \psi_{0,k} \in \text{span} \langle \psi_{0,k} \rangle_{k\in\mathbb{Z}}$.

Mallat's theorem provides thus an algorithm for constructing the wavelet from the MRA and the scaling function via the the Fourier coefficients of the 1-periodic function (the refinement mask, or filter). This can be implemented numerically in the so-called "cascade algorithm". Schematically it goes as follows:

$$\underset{\text{scaling MRA}}{\varphi} \longrightarrow \underset{\text{(low pass)}}{H} \longrightarrow \underset{\text{(high pass)}}{G} \longrightarrow \underset{\text{Mallat's wavelet}}{\psi}$$

<u>Example:</u> (Haar) Here $\varphi = \chi_{[0,1)}$ and therefore

$$\hat{\varphi}(\xi) = \frac{1-e^{-2\pi i\xi}}{2\pi i \xi} = \frac{1-e^{-\pi i \xi}}{\pi i \xi}\;\frac{1+e^{-\pi i \xi}}{2} = \hat{\varphi}(\xi/2)\;\frac{1+e^{-\pi i \xi}}{2},$$

so that $H(\xi) = \frac{1}{2}(1+e^{-2\pi i \xi})$. Then $H(\xi + 1/2) = \frac{1}{2}(1-e^{-2\pi i\xi})$ and

$$G(\xi) = e^{2\pi i \xi}\,\overline{H(\xi+1/2)} = e^{2\pi i \xi}\,\frac{1}{2}(1-e^{2\pi i \xi}).$$ Finally

$$\hat{\psi}(\xi) = G(\xi/2)\,\hat{\varphi}(\xi/2) = e^{\pi i \xi}\,\frac{1}{2}(1-e^{\pi i \xi})\;\frac{1-e^{-\pi i \xi}}{\pi i \xi} =$$

$$= e^{\pi i \xi}\,\frac{1}{2}\,e^{\frac{\pi}{2} i \xi}\,(e^{-\frac{\pi}{2} i\xi} - e^{\frac{\pi}{2} i \xi})\;\frac{e^{-\frac{\pi}{2} i \xi}\,(e^{\frac{\pi}{2} i\xi} - e^{-\frac{\pi}{2} i \xi})}{\pi i \xi} =$$

$$= \frac{1}{2}e^{\pi i \xi}\,(-2i\sin(\tfrac{\pi}{2}\xi))\;\frac{2i\sin(\frac{\pi}{2}\xi)}{\pi i \xi} = -i\,e^{\pi i \xi}\;\frac{\sin^2(\frac{\pi}{2}\xi)}{\frac{\pi}{2}\xi}$$