Chapter 3

Distributions and Fourier transform

The distributions are objects that generalize the notion of funtion. They also allow the generalization of related notions such us the derivative and the Fourier transform.

3.1 Distributions

Definition 7. A *distribution* is an element of the dual of $\mathcal{C}_c^\infty(\mathbb{R})$ that is, a linear continuous map

$$T : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}.$$

In order to understand what this continuity means we need to recall the topology on $\mathcal{C}_c^\infty(\mathbb{R})$. A sequence $\{\varphi_n\}_n \subset \mathcal{C}_c^\infty(\mathbb{R})$ converges to $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ if and only if there exist a compact $K \subset \mathbb{R}$ such that suppφ, supp$\varphi_n \subset K$ for all $n \geq 1$, and for all $j \geq 0$ the sequences $\{\varphi_n^{(j)}\}_n$ converge uniformly to $\varphi^{(j)}$ in K. This is equivalent to saying that for all $m \geq 0$,

$$\lim_{n\to\infty} \sup_{j\leq m} \sup_{x\in K} |\varphi_n^{(j)}(x) - \varphi^{(j)}(x)| = 0.$$

In this form the convergence can be given in terms of the seminorms

$$\|\varphi\|_{K,m} := \sup_{j\leq m} \sup_{x\in K} |\varphi^{(j)}(x)|.$$

Then, as usual, $T : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ is continuous if and only if for all compact $K \subset \mathbb{R}$ there exist $m = m(K) \geq 1$ and $C = C(K) > 0$ such than

$$|T(\varphi)| \leq C\|\varphi\|_{K,m} \qquad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}) \text{ such that supp } \varphi \subset K.$$

Notation. Functions φ in $\mathcal{C}_c^\infty(\mathbb{R})$ are usually called *test functions* . Sometimes the notation $D(\mathbb{R})$ is used instead of $\mathcal{C}_c^\infty(\mathbb{R})$; this explains why the set of distributions is usually denoted $D'(\mathbb{R})$.

Examples 3. 1. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a measurable function with some mild regularity (usually $f \in L^1_{\text{loc}}$). This defines the distribution $T_f : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ by means of

$$T_f(\varphi) = \langle f, \varphi \rangle = \int_{\mathbb{R}} \varphi \cdot f \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

In this sense, any (reasonable) function is a distribution.

2. Let μ be a Borel measure. Define $T_\mu : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$T_\mu(\varphi) = \int_{\mathbb{R}} \varphi \, d\mu \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

In particular, if $a \in \mathbb{R}$ and $\mu = \delta_a$ is the associated *Dirac delta*, we have the distribution

$$\langle \delta_a, \varphi \rangle = \int_{\mathbb{R}} \varphi \, d\delta_a = \varphi(a) \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

More generally, given a sequence $\{a_n\}_n \subset \mathbb{R}$ such that $\lim_n |a_n| = +\infty$ and given values $\lambda_n \in \mathbb{C}$, $n \geq 1$, we can consider the *Dirac comb* $T = \sum_n \lambda_n \delta_{a_n}$, defined by

$$\langle T, \varphi \rangle = \sum_n \lambda_n \varphi(a_n) \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

A particular case which appears in applications (see Shannon's theorem in the previous chapter, for example) is $a_n = an$, with $a \in \mathbb{R}$ and $n \in \mathbb{Z}$. The choice $\lambda_n = 1$ for all n yields the Dirac comb with equispaced atoms:

$$S_a = \sum_{n \in \mathbb{Z}} \delta_{an}.$$

Several operations can be performed on distributions. Next we introduce two easy ones.

3.1.1 Product of $\psi \in \mathcal{C}^\infty(\mathbb{R})$ with $T \in D'(\mathbb{R})$

This is simply defined by the identity

$$\langle \psi T, \varphi \rangle = \langle T, \psi \varphi \rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

This is well defined because $\psi\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ and ψT is continuous. To see the continuity let $K \subset \mathbb{R}$ be compact; since T is a distribution there exist $m \geq 1$ and $C > 0$ such that for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp} \varphi \subset K$

$$|\langle \psi T, \varphi \rangle| = |\langle T, \psi \varphi \rangle| \leq C \|\psi \varphi\|_{K,m}.$$

Since K is compact and $\psi \in \mathcal{C}^\infty(\mathbb{R})$ all derivatives $\psi^{(j)}$, $j \leq m$ are uniformly bounded on K. Hence there exists a constant $M = M(\psi, K)$ such that $\|\psi\varphi\|_{K,m} \leq M \|\varphi\|_{K,m}$. Then

$$|\langle \psi T, \varphi \rangle| \leq CM \|\varphi\|_{K,m} \, ,$$

as desired.

As an example let us take $\psi \in \mathcal{C}^\infty(\mathbb{R})$ and $T = \delta_a$, for some $a \in \mathbb{R}$. Then, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\langle \psi\delta_a, \varphi\rangle = \langle \delta_a, \psi\varphi\rangle = \psi(a)\varphi(a).$$

Thus $\psi\delta_a = \psi(a)\delta_a$. In particular $(x-a)\delta \equiv 0$.

3.1.2 Differentiation of $T \in D'(\mathbb{R})$

The derivative $T' \in D'(\mathbb{R})$ is the distribution defined by

$$\langle T', \varphi\rangle = -\langle T, \varphi'\rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

This generalizes the formula of integration by parts for a function $f \in \mathcal{C}^1$:

$$\langle T_f', \varphi\rangle = \int_{\mathbb{R}} f' \cdot \varphi = -\int_{\mathbb{R}} f \cdot \varphi' = -\langle T_f, \varphi'\rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

This formula can be iterated to define the successive derivatives $T^{(j)}$, $j \geq 1$:

$$\langle T^{(j)}, \varphi\rangle = (-1)^j \langle T, \varphi^{(j)}\rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

As an example let $T = \delta_a$ as above and observe that δ_a' is given by

$$\langle \delta_a', \varphi\rangle = -\langle \delta_a, \varphi'\rangle = -\varphi'(a) \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

This is an example of a distribution which is neither a function not a measure.

3.2 Fourier transform and distributions

For $f \in L^1(\mathbb{R})$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ the multiplication formula gives

$$\int_{\mathbb{R}} \hat{f} \cdot \varphi = \int_{\mathbb{R}} f \cdot \hat{\varphi}.$$

Following the previous analogy we could be tempted to define the Fourier transform $\widehat{T}$ of $T \in D'(\mathbb{R})$ by

$$\langle \widehat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

This fails, because $\hat{\varphi} \notin \mathcal{C}_c^\infty(\mathbb{R})$ (actually, the Fourier transform of a $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ is an analytic function in $\mathbb{C}$; Theorem 5).

The way to circumvent this obstacle it to consider only distributions T for which the right hand side of the above identity is well behaved. That leads us to consider only distributions which are well defined for a bigger space that $\mathcal{C}_c^\infty(\mathbb{R})$. In consequence $\widehat{T}$ is not defined for all distributions T but only for some "good" ones.

Definition 8. The *Schwartz class*, denoted by $\mathcal{S}$, consists of the functions $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ such that for all $m, k \in \mathbb{N}$

$$P_{m,k}(\varphi) = \sup_{j\leq k} \sup_{x\in\mathbb{R}} (1+|t|)^m |\varphi^{(j)}(t)| < +\infty.$$

Observe that $\varphi \in \mathcal{S}$ when φ and all its derivatives decay faster at infinity than any polynomial. In particular, if $\varphi \in \mathcal{S}$ and P is a polynomial, then also $P\varphi \in \mathcal{S}$.

Notice also that $\mathcal{C}_c^\infty(\mathbb{R}) \subset \mathcal{S}$ and that there are functions which are in $\mathcal{S}$ but not in $\mathcal{C}_c^\infty(\mathbb{R})$, such as $\varphi(t) = e^{-t^2}$. Functions in $\mathcal{S}$ have very fast decay, but not necessarily compact support.

The topology of $\mathcal{S}$ is given in terms of the seminorms $P_{m,k}$ above. Thus, a sequence $\{\varphi_n\}_n$ tends to 0 in $\mathcal{S}$ if and only if

$$\lim_{n\to\infty} P_{m,k}(\varphi_n) = 0 \qquad \forall\, m, k \in \mathbb{N}.$$

As a first step to define the Fourier transform of certain distributions let us see that the Fourier transform of functions in $\mathcal{S}$ stays in $\mathcal{S}$.

Lemma 9. *If $\varphi \in \mathcal{S}$ then $\hat\varphi \in \mathcal{S}$. In particular, by Plancherel, $\|\hat\varphi\|_2 = \|\varphi\|_2$.*

Proof. As seen in Theorem 5 (b) and (c), for $j, m \geq 0$

$$(\hat\varphi)^{(j)}(\xi) = [(-2\pi i t)^i \varphi]^\wedge(\xi)\,, \qquad \widehat{\varphi^{(m)}}(\xi) = (2\pi i\xi)^m \hat\varphi(\xi),$$

and therefore, by Theorem 5 (a)

$$\begin{aligned}(1+|\xi|)^m \big|(\hat\varphi)^{(j)}(\xi)\big| &= (1+|\xi|)^m \big|[(-2\pi i t)^j\varphi]^\wedge(\xi)\big| \lesssim \left|\big[\frac{\partial^m}{\partial x^m}\big((-2\pi i t)^j\varphi\big)\big]^\wedge(\xi)\right| \\ &\leq \left\|\frac{\partial^m}{\partial x^m}\big[(-2\pi i t)^j\varphi(t)\big]\right\|_1 .\end{aligned}$$

This L^1 norm is finite, because by hypothesis $P_{M,K}(\varphi) < +\infty$ for all $k, l \geq 0$. □

Definition 9. A *tempered distribution* is a linear continuous map $T : \mathcal{S} \longrightarrow \mathbb{C}$. The space of tempered distributions is denoted by $\mathcal{S}'$.

Remarks 2. 1. A tempered distribution T restricted to $\mathcal{C}_c^\infty(\mathbb{R}) \subset \mathcal{S}$ is a distribution.

2. If $\psi \in \mathcal{S}$ and T is a tempered distribution, then $\psi T \in \mathcal{S}'$ as well: $\langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle$ and $\psi\varphi \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$.

3. If $T \in \mathcal{S}'$ then also $T' \in \mathcal{S}'$: $\langle T', \varphi\rangle = -\langle T, \varphi'\rangle$ and $\varphi' \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$.

Examples 4. (a) $L^1(\mathbb{R}), L^2(\mathbb{R}) \subset \mathcal{S}$. Let $f \in L^p(\mathbb{R})$, $p = 1, 2$. By Hölder's inequality, with q such that $1/p + 1/q = 1$ (i.e. $q = 2$ if $p = 2$ and $q = \infty$ if $p = 1$)

$$\big|\langle T, \varphi\rangle\big| = \Big|\int_{\mathbb{R}} f(t)\varphi(t)\,dt\Big| \leq \|f\|_p \|\varphi\|_q,$$

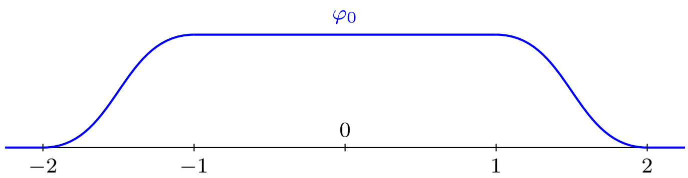
and this is finite because $\|\varphi\|_q < +\infty$ for all $q > 0$ and all $\varphi \in \mathcal{S}$.

(b) Let $f(t) = e^t$ and let us see that $T_f \in D'(\mathbb{R}) \setminus \mathcal{S}'$. It is clear that T_f is a distribution, because $f \in L^1_{\text{loc}}$. In order to prove that T_f is not a tempered distribution let us see that there exists $\varphi \in \mathcal{S}$ such that for some m, k there is no constant $C > 0$ such that

$$|\langle T_f, \varphi\rangle| = \left|\int_{\mathbb{R}} e^t \varphi(t)\, dt\right| \le C P_{m,k}(\varphi) = \sup_{j\le k} \sup_{x\in\mathbb{R}} (1+|t|)^m |\varphi^{(j)}(t)|. \tag{3.1}$$

Let $\varphi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ be a smoothed version of the indicator $\chi_{(-1,1)}$, that is, a smooth function with $0 \le \varphi \le 1$ with

$$\varphi_0(t) = \begin{cases} 1 & t \in [-1,1] \\ 0 & t \notin [-2,2] \end{cases}$$



Consider the translates $\varphi_n(t)\varphi_0(t-n)$, whose support is in $[n-2, n+2]$. Observe also that for all $n, j \ge 0$

$$\sup_{x\in\mathbb{R}} |\varphi_n^{(j)}(t)| = \sup_{x\in\mathbb{R}} |\varphi_0^{(j)}(t)|.$$

On the one hand it is clear that

$$I_n := \int_{\mathbb{R}} e^t \varphi_n(t)\, dt = \int_{n-2}^{n+2} e^t \varphi_n(t)\, dt \simeq e^n,$$

because

$$I_n \le \int_{n-2}^{n+2} e^t\, dt = e^n(e^2 - 1/e^2) \quad \text{and} \quad I_n \le \int_{n-1}^{n+1} e^t\, dt = e^n(e - 1/e).$$

On the other hand, on the support of φ_n we have $|t| \le n+2$; hence

$$\sup_{j\le k} \sup_{t\in\mathbb{R}} (1+|t|)^m |\varphi_n^{(j)}(t)| \le \sup_{j\le k} (3+n)^m \sup_{t\in\mathbb{R}} |\varphi_n^{(j)}(t)| = (3+n)^m\, C$$

where $C = \sup_{j\le k} \sup_{t\in\mathbb{R}} |\varphi_0^{(j)}(t)|$

This shows that (3.1) would be

$$e^n \le C n^m,$$

which cannot hold for all n regardless of how big is C.

Definition 10. The Fourier transform of a tempered distribution $T \in \mathcal{S}'$ is the tempered distribution $\widehat{T}$ defined by the action

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle \qquad \varphi \in \mathcal{S}.$$

Examples 5. 1. $\widehat{\delta_a} = e^{-2\pi ita}$ for $a \in \mathbb{R}$ (by this we mean that the Fourier transform of δ_a is the distribution given by the function $e^{-2\pi ita}$). Let us check this; by definiton, for $\varphi \in \mathcal{S}$:

$$\langle \widehat{\delta_a}, \varphi \rangle = \langle \delta_a, \widehat{\varphi} \rangle = \widehat{\varphi}(a) = \int_{\mathbb{R}} \varphi(t) e^{-2\pi ita} dt = \langle e^{-2\pi ita}, \varphi \rangle.$$

2. $\widehat{e^{2\pi ita}} = \delta_a$. This is proved similarly, using the inversion formula.

3. $\widehat{\delta'_a} = (2\pi it)e^{-2\pi ita}$. Given $\varphi \in \mathcal{S}$

$$\begin{aligned}\langle \widehat{\delta'_a}, \varphi \rangle &= \langle \delta'_a, \widehat{\varphi} \rangle = -(\widehat{\varphi})'(a) = -[(-2\pi it)\varphi(t)]^\wedge(a) = \int_{\mathbb{R}} (2\pi it)\varphi(t) e^{-2\pi ita} dt \\ &= \langle (2\pi it)e^{-2\pi ita}, \varphi \rangle.\end{aligned}$$

Many properties of the Fourier transform for functions have the equivalent version for tempered distributions. These follow directly from the definiton.

Proposition 5. *Let $T \in \mathcal{S}'$. Then,*

(a) $\widehat{T}^{(k)} = [(-2\pi it)^k T]^\wedge$.

(b) $\widehat{T^(k)} = (2\pi i\xi)\widehat{T}$

(c) Given $a \in \mathbb{R}$ let $\tau_a T$ be defined by $\langle \tau_a T, \varphi \rangle = \langle T, \tau_{-a}\varphi \rangle$, $\varphi \in \mathcal{S}$, and let $M_a T = e^{2\pi iat} T$. Then

$$\widehat{\tau_a T} = M_{-a}\widehat{T}\,, \qquad [M_a T]^\wedge = \tau_a \widehat{T}.$$

3.2.1 Poisson summation formula

Let us finish this chapter with a classical result to the effect that the Fourier transform of the tempered distribution $T = \sum_{n\in\mathbb{N}} \delta_n$ is the same T.

The Poisson formula. *For $\varphi \in \mathcal{S}$,*

$$\sum_{n\in\mathbb{N}} \varphi(n) = \sum_{n\in\mathbb{N}} \widehat{\varphi}(n).$$

Remark 9. Taking a modulation by ξ and a dilation by $a \in \mathbb{R}$, the Poisson formula reads as the seemingly more general formula

$$\frac{1}{a}\sum_{n\in\mathbb{N}} \varphi(\frac{n}{a}) e^{-2\pi i \frac{n}{a}\xi} = \sum_{n\in\mathbb{N}} \widehat{\varphi}(an+\xi).$$

Proof. Consider the 1-periodic function

$$F(t) = \sum_{n\in\mathbb{Z}} \varphi(t+n).$$

Since $F \in L^2[0,1]$ we can write it as the Fourier series:

$$F(t) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikt},$$

where

$$\begin{aligned} c_k &= \int_0^1 F(t)e^{-2\pi ikt}dt = \sum_{n\in\mathbb{Z}} \int_0^1 \varphi(t+n)e^{-2\pi ikt}dt = \sum_{n\in\mathbb{Z}} \int_n^{n+1} \varphi(s)e^{-2\pi iks}ds \\ &= \int_{\mathbb{R}} \varphi(s)e^{-2\pi iks}ds = \widehat{\varphi}(k). \end{aligned}$$

Thus we have

$$F(t) = \sum_{n\in\mathbb{Z}} \varphi(t+n) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)e^{2\pi ikt}.$$

Since $F(0) = F(1)$ we can evaluate this at $t = 0$ and get the result. $\square$