Local study

The aim of the local study is to understand and describe the dynamics in a nbd of a fixed point. One of the goals is to represent a system in the simplest way by means of a conjugation or a change of variables. Also, we want to classify (locally) systems using conjugacy classes.

Regular points:
We will study the neighbourhood of a regular point (only for continuous systems).
The main result is the flow-box theorem or straightening theorem.

Theorem Let $x' = f(x)$, $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, $x_0 \in U$, $f(x_0) \neq 0$, $f \in C^r$, $r \geq 1$. Then, there exist a neighbourhood V of x_0 and $h : V \to \mathbb{R}^n$ a diffeomorphism of class C^r, such that h conjugates $x' = f(x)$ to $x' = g_1(x)$, where $g_1(x) = (1, 0, 0, \dots, 0)^\top$.

Fixed points:
In order to study the dynamics near a fixed point, we begin with discrete systems.

Definition Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map.
A is a hyperbolic linear map if $\mathrm{Spec}\,(A) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \neq 1\}$.

Definition Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$ and $x_0 \in U$ such that $f(x_0) = x_0$.
We say that x_0 is hyperbolic if $Df(x_0)$ is a hyperbolic linear map.

Hartman-Grobman Theorem (local version for maps)

Theorem

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$, $x_0 \in U$, $f(x_0) = x_0$. Suppose x_0 is a hyperbolic point and $A = Df(x_0)$ is invertible.

Then, there exist V open set such that $x_0 \in V$ and $h : V \to \mathbb{R}^n$ homeomorphism onto its image, such that $h(x_0) = 0$ and

$$h \circ f = A \circ h, \qquad \text{on} \quad V \cap f^{-1}(V)$$

$$\begin{array}{ccc} f^{-1}(V) \cap V & \xrightarrow{f} & V \\ h \downarrow & & \downarrow h \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \end{array}$$

Note that $f^{-1}(V) \cap V \subset U$.

(Hartman 1959, Grobman 1958)

Remarks

1. A clear proof can be found in Palis-de Melo's book.
2. A has to be invertible. **Example**

   $$f(x) = x^3$$

   The linear part is zero, the corresponding linear map is zero. Obviously, $A = 0$ can not be conjugated to $f(x) = x^3$:

   $$h(x^3) = 0 \cdot h(x) \implies h = 0 \text{ constant on } (-\delta, \delta)$$

   for some $\delta > 0$.
3. Even if f is analytic, in general h is only a homeomorphism. **Example** $f : \mathbb{R}^3 \to \mathbb{R}^3$, given by

   $$f(x, y, z) = (ax, ac(y + \varepsilon xz), cz),$$

   with $0 < c < 1 < a$, $1 < ac$ and $\varepsilon \neq 0$.
4. The function h is not unique.

Hartman-Grobman Theorem (global version for maps)

First, we consider a special case, for which the proof of the existence of a conjugation is easier. Then, the local version follows as a corollary.

Theorem (Global version for maps)
Let A be a hyperbolic linear invertible map. Then $\exists \varepsilon > 0$ such that if

- $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz,
- $f - A, g - A$ are bounded,
- $\mathrm{Lip}\,(f - A), \mathrm{Lip}\,(g - A) < \varepsilon$,

we have that, $\exists h : \mathbb{R}^n \to \mathbb{R}^n$ global homeomorphism such that

$$h \circ f = g \circ h.$$

Scheme of the proof I

Proof of the theorem: A hyperbolic $\implies$ $\exists$ decomposition $\mathbb{R}^n = E^s \oplus E^u$ such that with respect to it $A = \begin{pmatrix} A^s & 0 \\ 0 & A^u \end{pmatrix}$.
$\exists$ a norm in E^s s.t. $\|A^s\| \leq a < 1$ and a norm in E^u s.t. $\|(A^u)^{-1}\| \leq a < 1$.

In $\mathbb{R}^n$ we take the norm $\|x\| = \max\{\|x_s\|, \|x_u\|\}$ where $x = x_s + x_u$.

Let $C_b^0(\mathbb{R}^n, \mathbb{R}^n) = \{\varphi : \mathbb{R}^n \to \mathbb{R}^n \mid \varphi \in C^0, \sup_{x \in \mathbb{R}^n} \|\varphi(x)\| < \infty\}$ with the norm $\|\varphi\| = \sup_{x \in \mathbb{R}^n} \|\varphi(x)\|$. It is a Banach space.

We look for h in the form $h = I + u$, $u \in C_b^0$.
We write $\varphi_1 = f - A$, $\varphi_2 = g - A$. By the hypotheses $\varphi_1, \varphi_2 \in C_b^0$.

The conjugation condition is

$$(I + u) \circ (A + \varphi_1) = (A + \varphi_2) \circ (I + u). \tag{1}$$

Scheme of the proof II

Lemma $\exists \eta > 0$ such that, if $\varphi_1, \varphi_2 \in C_b^0$, $\mathrm{Lip}\,\varphi_1, \mathrm{Lip}\,\varphi_2 < \eta$, then $\exists! u \in C_b^0$ satisfying (1). Actually, we can take $\eta < \min\{1 - a, \|A^{-1}\|^{-1}\}$.

Proof of the Lemma: The condition

$$(I + u) \circ (A + \varphi_1) = (A + \varphi_2) \circ (I + u)$$

is equivalent to

$$A + \varphi_1 + u \circ (A + \varphi_1) = A + Au + \varphi_2 \circ (I + u)$$

and to

$$Au - u \circ (A + \varphi_1) = \varphi_1 - \varphi_2 \circ (I + u)$$

Consider $\varphi_1 \in C_b^0$ fixed with $\mathrm{Lip}\,\varphi_1 < \eta$ (η to be determined) and let $L : C_b^0 \to C_b^0$,

$$Lu = Au - u \circ (A + \varphi_1).$$

Scheme of the proof III

Lemma If $\mathrm{Lip}\,\varphi_1 < \|A^{-1}\|^{-1}$ then $\Phi = A + \varphi_1$ is (globally) invertible. Moreover, Φ^{-1} is Lipschitz and $\mathrm{Lip}\,\Phi^{-1} \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\mathrm{Lip}\,\varphi_1}$.

Proof:
one to one: assume $Ax + \varphi_1(x) = Ay + \varphi_1(y)$ or equivalently $A(x-y) = \varphi_1(y) - \varphi_1(x)$.

$$\|A^{-1}\|^{-1}\|x-y\| \leq \|A(x-y)\| = \|\varphi_1(y) - \varphi_1(x)\| \leq \mathrm{Lip}\,\varphi_1\|x-y\|.$$

onto: let $y \in \mathbb{R}^n$; we look for x s.t. $Ax + \varphi_1(x) = y$ or $x = -A^{-1}\varphi_1(x) + A^{-1}y$.
Let $H_y(x) = -A^{-1}\varphi_1(x) + A^{-1}y$. We compute its Lipschitz constant:

$$\|H_y(x_2) - H_y(x_1)\| = \|A^{-1}\varphi_1(x_2) - A^{-1}\varphi_1(x_1)\| \leq \|A^{-1}\|\,\mathrm{Lip}\,\varphi_1\|x_2 - x_1\|$$

There exists a unique x s.t. $\Phi(x) = y$. $\Phi^{-1}(y)$ is the fixed point of H_y.
Φ^{-1} is Lipschitz:

$$\Phi^{-1}(y_2) - \Phi^{-1}(y_1) = -A^{-1}\varphi_1(\Phi^{-1}(y_2)) + A^{-1}y_2 + A^{-1}\varphi_1(\Phi^{-1}(y_1)) - A^{-1}y_1,$$

$$\|\Phi^{-1}(y_2) - \Phi^{-1}(y_1)\| \leq \|A^{-1}\|\mathrm{Lip}\,\varphi_1\|\Phi^{-1}(y_2)) - \Phi^{-1}(y_1)\| + \|A^{-1}\|\,\|y_2 - y_1\|$$

Scheme of the proof IV

Lemma L is linear. Moreover, if $\mathrm{Lip}\,\varphi_1 < \|A^{-1}\|^{-1}$, L is invertible and $\|L^{-1}\| \le \frac{1}{1-a}$.

Proof *(sketch)* Given (v_1, v_2) we write

$$L\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A^s & 0 \\ 0 & A^u \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} u_1 \circ (A+\varphi_1) \\ u_2 \circ (A+\varphi_1) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$\begin{array}{lcl} A^s u_1 - u_1 \circ (A+\varphi_1) = v_1 & \to & u_1 = A^s u_1 \circ (A+\varphi_1)^{-1} - v_1 \circ (A+\varphi_1)^{-1}, \\ A^u u_2 - u_2 \circ (A+\varphi_1) = v_2 & \to & u_2 = (A^u)^{-1}[v_2 + u_2 \circ (A+\varphi_1)]. \end{array}$$

Let $\mathcal{L}^s : C_b^0 \to C_b^0$ be defined by

$$\mathcal{L}^s(u_1) = A^s u_1 \circ (A+\varphi_1)^{-1} - v_1 \circ (A+\varphi_1)^{-1}.$$

It is a contraction. Moreover, $\|u_1\| \le \|A^s\|\|u_1\| + \|v_1\| \le a\|u_1\| + \|v_1\|$. Then $\|u_1\| \le \frac{1}{1-a}\|v_1\|$.
Analogously for $\mathcal{L}^u(u_2) = (A^u)^{-1}[v_2 + u_2 \circ (A+\varphi_1)]$. Now $\|u_2\| \le \frac{a}{1-a}\|v_2\|$.

Scheme of the proof V

We write the equation

$$Lu = \varphi_1 - \varphi_2 \circ (I+u) \qquad \Leftrightarrow \qquad u = L^{-1}(\varphi_1 - \varphi_2 \circ (I+u)).$$

This defines an operator

$$\begin{array}{rccc} \Gamma : & C_b^0 & \to & C_b^0 \\ & u & \mapsto & L^{-1}(\varphi_1 - \varphi_2 \circ (I+u)) \end{array}$$

- It is well defined.
-
$$\begin{aligned} \|\Gamma(u_1) - \Gamma(u_2)\| &= \|L^{-1}(\varphi_1 - \varphi_2 \circ (I+u_1)) - L^{-1}(\varphi_1 - \varphi_2 \circ (I+u_2))\| \\ &\leq \|L^{-1}\| \|\varphi_2 \circ (I+u_1) - \varphi_2 \circ (I+u_2)\|, \end{aligned}$$

  where

$$\begin{aligned} \|\varphi_2 \circ (I+u_1) - \varphi_2 \circ (I+u_2)\| &= \sup_x \|\varphi_2(x+u_1(x)) - \varphi_2(x+u_2(x))\| \\ &\leq \sup_x \operatorname{Lip} \varphi_2 \|u_1(x) - u_2(x)\| = \operatorname{Lip} \varphi_2 \|u_1 - u_2\|. \end{aligned}$$

  Then $\operatorname{Lip} \Gamma \leq \|L^{-1}\| \operatorname{Lip} \varphi_2 < 1$, if η is small enough.

$\square$

Scheme of the proof VI

Until now, we have $\exists h = I + u$, $u \in C_b^0$ such that $(I+u) \circ f = g \circ (I+u)$.
Now exchange f and g

$$\exists v \in C_b^0 \qquad \text{such that} \qquad (I+v) \circ g = f \circ (I+v).$$

We have

$$(I+v)(I+u)f = (I+v)g(I+u) = f(I+v)(I+u),$$

$$(I+u)(I+v)g = (I+u)f(I+v) = g(I+u)(I+v).$$

$$\left.\begin{array}{l} (I+v)(I+u) = I + \underbrace{u + v(I+u)}_{C_b^0} \text{ conjugates } f \text{ to } f \\ I = I + 0 \text{ also conjugates } f \text{ to } f. \end{array}\right\} \Longrightarrow \begin{array}{l} (I+v)(I+u) = I \\ \text{(by uniqueness)} \end{array}$$

In the same way $(I+u)(I+v) = I$. Then $I + u$ is a homeomorphism.

$\square$

Lemma for the local version of Hartman's theorem

Lemma Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of class C^r, $r \geq 1$, such that $f(0) = 0$. Let $A = Df(0)$. Given $\varepsilon > 0$ there exist $\rho > 0$ and $\overline{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that
(1) $\overline{f}$ is of class C^r,
(2) $\overline{f}_{|B(0,\rho/2)} = f_{|B(0,\rho/2)}$ and $\overline{f}_{|\mathbb{R}^n \setminus B(0,\rho)} = A$,
(3) $\overline{f} = A + \varphi, \quad \varphi \in C_b^0, \quad \text{Lip}\, \varphi < \varepsilon$.

Proof Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy

- $\alpha \in C^\infty$,
- $\alpha(t) = 1$, if $t \leq 1/2$,
- $\alpha(t) = 0$, if $t \geq 1$,
- $\alpha(t) \in [0, 1]$, $\forall t \in \mathbb{R}$.

Let $M = \sup|\alpha'(t)|$. Given $\rho > 0$ let $\beta : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $\beta(x) = \alpha\Big(\frac{\|x\|}{\rho}\Big)$, where $\|\cdot\|$ is the euclidean norm, which is C^∞ except at 0. β depends on ρ and satisfies

- $\beta \in C^\infty$, because $\|\cdot\|$ is differentiable outside the origin and in a neighbourhood of the origin β is constant,
- $\beta(x) = 1$ if $x \in B(0, \rho/2)$,
- $\beta(x) = 0$ if $x \notin B(0, \rho)$,
- $\beta(x) \in [0, 1]$, $\forall x \in \mathbb{R}^n$,
- $\| D\beta(x) \| \leq M/\rho$, because $D\beta(x) = \alpha'\Big(\frac{\|x\|}{\rho}\Big)\frac{x}{\rho\|x\|}$.

Take

$$\varphi(x) = \begin{cases} \beta(x)[f(x) - Ax] & \text{if} \quad x \in U, \\ 0 & \text{if} \quad x \notin U. \end{cases}$$

and $\overline{f}(x) = Ax + \varphi(x)$,

$$\overline{f}(x) = \begin{cases} Ax + \beta(x)[f(x) - Ax] & \text{if} \quad x \in U, \\ Ax & \text{if} \quad x \notin U. \end{cases}$$

Let us determine ρ. It must satisfy $\overline{B}(0, \rho) \subset U$. Clearly $\overline{f}_{|B(0,\rho/2)} = f_{|B(0,\rho/2)}$.

φ is a C^r function: if $x \in \partial U$ there exists a neighbourhood where it is constant 0.

Moreover $\varphi \in C^0_b$ because it has compact support.

If $x \in B(0,\rho)$

$$D\varphi(x) = D\beta(x)[f(x) - Ax] + \beta(x)[Df(x) - A].$$

Since Df is continuous,

$$\exists \rho_1 > 0 \text{ s.t. if } x \in B(0,\rho_1),\ \| Df(x) - Df(0) \| < \varepsilon/(M+1).$$

Then if $x \in B(0,\rho)$, with ρ satisfying both conditions above

$$\begin{aligned}
\| D\varphi(x) \| &\leq \| D\beta(x) \| \| f(x) - Ax \| + \| \beta(x) \| \| Df(x) - A \| \\
&\leq M/\rho \sup_{\xi \in B(0,\rho)} \| Df(\xi) - A \| \| x \| + \| Df(x) - A \| \\
&< M\varepsilon/(M+1) + \varepsilon/(M+1) = \frac{M+1}{M+1}\varepsilon = \varepsilon.
\end{aligned}$$

If $x \notin B(0,\rho)$, $D\varphi(x) = 0$.

Therefore Lip $\varphi < \varepsilon$.

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Proof of the local version of Hartman's theorem

We may suppose that $x_0 = 0$.
Given $\varepsilon(A)$ given by the Global Hartman thm we extend f from a nbd of 0 to $\mathbb{R}^n$. We have

$$\overline{f}_{|B(0,\frac{\rho}{2})} = f_{|B(0,\frac{\rho}{2})} \qquad \text{for some } \rho$$

and Lip $(\overline{f} - A) < \varepsilon(A)$.
Then $\exists h : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$h\overline{f} = Ah.$$

We take

$$V = B(0, \frac{\rho}{2}).$$

Then $h \circ f = Ah$, in $V \cap f^{-1}(V)$.

Example of non-uniqueness of the conjugation in Hartman's theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

- $f(0) = 0$
- f is derivable, $f'(0) = \lambda$ and $0 < f'(x) \leqslant m < 1$.

We claim there exist ∞ many conjugations between f and f.
If h is a conjugation from f to f: $h \circ f = f \circ h$ and
k is a conjugation from f to the linear map $\lambda : x \mapsto \lambda x$:
$k \circ f = \lambda k$, then

$$(k \circ h) \circ f = k \circ (h \circ f) = k \circ (f \circ h) = (k \circ f) \circ h = \lambda k \circ h.$$

We do the construction for the right-hand side (the left-hand side being analogous).

Let $a_0 = 1,\ a_1 = f(1)$ and $a_n = f^n(1)$.
Clearly $a_n \to 0$ monotonically.
Let $I_n = [a_n, a_{n-1}],\ n \geqslant 1$.
Clearly

$$(0,1] = \bigcup_{n\geqslant 1} I_n, \qquad I_{n+1} \cap I_n = \{a_n\}, \qquad f(I_n) = I_{n+1}.$$

We look for $h : [0,1] \to [0,1]$ such that $h \circ f = f \circ h \ \Leftrightarrow\ h = f \circ h \circ f^{-1}$.
Very important: to know h on I_n we only need h on I_{n-1}.
Then we take any $h : I_1 \to I_1$ homeomorphism increasing:

$$h(a_1) = a_1, \qquad h(1) = 1$$

and we extend h to $[0,1]$ recursively by the formulas

$$\begin{aligned} &h(x) = f^n \circ h \circ f^{-n}(x), \qquad x \in I_{n+1},\ n \geqslant 1 \\ &h(0) = 0. \end{aligned}$$

Note that (by induction)

$$x \in I_{n+1} \Rightarrow h(x) = f^n \circ h \circ f^{-n}(x) \qquad (f^{-n}(x) \in I_1,\ h(f^{-n}(x)) \in I_1)$$

$\Rightarrow h_{|I_{n+1}}$ is homeo onto I_{n+1}, is increasing, $h(a_n) = a_n$.

$$\lim_{x \to 0} h(x) = 0$$

It satisfies the conjugation relation:

$$\text{if } x \in I, \quad \exists n \quad s.t. \quad x \in I_n \quad \text{and} \quad f(x) \in I_{n+1}$$

$$h \circ f(x) = f^n \circ h \circ f^{-n} \circ f(x) = f \circ f^{n-1} \circ h \circ f^{-n+1}(x) = f \circ h(x).$$