Exercise 1 *Show that any random variable measurable w.r.t. the σ-algebra $\mathcal{F} = \{\phi, \Omega\}$ is constant. What if $\mathcal{F} = \{\phi, A, A^c, \Omega\}$? and if $\mathcal{F} = \sigma(A_1, A_2, ..., A_n)$ with $(A_i)_{1 \leq i \leq n}$ a partition of Ω?*

Exercise 2 *Prove that if $\mathbb{F} = (\mathcal{F}_n)_{0 \leq n \leq N}$ is a filtration and X a random variable $(\mathbb{E}(X|\mathcal{F}_n))_{0 \leq n \leq N}$ is an $\mathbb{F}$-martingale.*

Exercise 3 *Prove from the definition of conditional expectation that if $X = \mathbf{1}_B$ with is $B \in \mathcal{F}$ and Y another random variable then*

$$\mathbb{E}(XY|\mathcal{F}) = X\mathbb{E}(Y|\mathcal{F}),$$

and prove that if Y is independent of $\mathcal{F}$, $\mathbb{E}(Y|\mathcal{F}) = \mathbb{E}(Y)$.

Exercise 4 *Let $\{Y_n\}_{n \geq 1}$ be a sequence of independent, identically distributed random variables*

$$\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}.$$

Set $S_0 = 0$ and $S_n = Y_1 + \cdots + Y_n$ if $n \geq 1$. Check if the following sequences are martingales:

$$M_n^{(1)} = \frac{e^{\theta S_n}}{(\cosh\theta)^n},\ n \geq 0$$
$$M_n^{(2)} = \sum_{k=1}^{n} \operatorname{sign}(S_{k-1})Y_k,\ n \geq 1,\ M_0^{(2)} = 0$$
$$M_n^{(3)} = S_n^2 - n$$

Exercise 5 *Consider a market model with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, interest rate $r = 0$ and three risky assets S^1, S^2, S^3 and only one period. Assume that $S_0^1 = S_0^2 = S_0^3 = 1$ and that $S_1^i = x_i \mathbf{1}_{\{\omega_i\}}$, $i = 1, 2, 3$. Find a condition on the x_i to have a market free of arbitrage.*