DYNAMICAL SYSTEMS
Fall semester 2023/24

LAB 1
The real quadratic family: Conjugacies and symbolic dynamics.

Consider the one-dimensional real dynamical system

$$Q_c(x) = x^2 + c,\ \ c \in \mathbb{R}.$$

1. General facts about Q_c

(a) Prove that every quadratic polynomial of the form $P(x) = Ax^2+Bx+C$ is globally conjugate to a member of the quadratic family, by a linear map $h(x) = \alpha x+\beta$ ($\alpha \neq 0$). In other words, show that given $A, B, C \in \mathbb{R}$, $A \neq 0$, there exist $\alpha, \beta, c \in R$, $\alpha \neq 0$, such that $h \circ P = Q_c \circ h$.

(b) Compute the fixed points of Q_c and determine their existeince and character in terms of the parameter c.

(c) Let $c < 0$ and let β_c denote the fixed point that is always repelling. Draw the graph of Q_c and the diagonal $y = x$ and prove that the following conditions hold.

  (b.1) If $x \notin [-\beta_c, \beta_c]$ then $|Q_c^n(x)| \to \infty$ as $n \to \infty$.

  (b.2) If $c \in [-2, 0)$ then $-\beta_c \leq c$ and $|Q_c^n(x)| \leq \beta_c$ for all $x \in [-\beta_c, \beta_c]$, and for all $n \geq 0$.

(d) Set $c = -2$ ($\beta_{-2} = 2$). Prove that $\#\mathrm{Per}_n(Q_c) = 2^n$ (Draw the graph of Q_c and of its iterates, and and use the intervals of monotonicity to count the intersections with the diagonal).

2. The non-escaping set Λ

From now one we fix $\boxed{c < -\frac{1}{4}\left(5 + 2\sqrt{5}\right) < -2}$. Set $\mathcal{I} := [-\beta_c, \beta_c]$, and define

$$\begin{aligned} A_n &:= \{x \in \mathcal{I} \mid Q_c^{n-1}(x) \in \mathcal{I},\ Q_c^n(x) \notin \mathcal{I},\ n > 0\} \\ \Lambda := \Lambda_c &:= \{x \in \mathcal{I} \mid Q_c^n(x) \in \mathcal{I}\ \forall\ n \geq 1\} = \mathcal{I} \setminus \bigcup_{n\geq 1} A_n \end{aligned} \tag{1}$$

Notice that A_n is the set of points that escape I at the n^{th} iteration, while Λ is the set of points that remain in I forever. Prove the following statements.

(a) $|Q_c'(x)| > 1$ for all $x \in \mathcal{I} \setminus A_1$.

(b) Λ is closed.

(c) Λ is nowhere dense (i.e. it contains no intervals). *(Hint: use the expansivity of Q_c outside of A_0 and the Mean Value Theorem).*

(d) Λ is perfect (it has no isolated points).

In other words Λ is a *Cantor set.*

Note: Λ is a Cantor set as long as $c < -2$. However, the proof of (c) is harder in the case when $Q_c'(x)$ is not larger than one in modulus.

We shall consider the dynamical system $Q|_\Lambda : \Lambda \to \Lambda$ (we ignore the dependence on c for the sake of exposition).

3. The space of sequences and the shift map

We define the abstract space of infinite sequences

$$\Sigma_2 = \{s = s_0 s_1 s_2 \dots \text{ where } s_j \in \{0,1\}\},$$

with the distance

$$\mathrm{d}\,(s,t) = \sum_{j\geq 0} \frac{|s_j - t_j|}{2^j}.$$

On this space, let σ denote **the shift map**, which is defined as

$$\begin{aligned} \sigma: \quad (\Sigma_2, d) \quad &\longrightarrow \quad (\Sigma_2, d) \\ s_0 s_1 s_2 s_3 \dots &\mapsto s_1 s_2 s_3 \dots \end{aligned}$$

Prove that:

(a) (Σ, d) is a metric space.

(b) σ is a continuous map.

(c) $\mathrm{Per}(\sigma)$ is dense in Σ_2.

(d) There exists $s \in \Sigma_2$ whose σ-orbit is dense.

(e) σ has sensitive dependence with respect to initial conditions.

In other words, $\sigma : \Sigma_2 \to \Sigma_2$ is **chaotic**.

4. Symbolic dynamics: the itinerary map

Let I_0 (resp. I_1) be the left (resp. right) interval of $\mathcal{I} \setminus A_0$. Given $x \in \Lambda$, we define the **itinerary of** x as follows

$$s(x) = s_0 s_1 s_2 \dots, \text{ where } s_j = k \iff Q_c^j(x) \in I_k, k \in \{0,1\},$$

and the **itinerary map** S as

$$\begin{aligned} S: \quad \Lambda \quad &\longrightarrow \quad \Sigma_2 \\ x \quad &\to \quad s(x). \end{aligned}$$

Prove that:

(a) S is a homeomorphism.

(b) S is a topological conjugacy between $Q|_\Lambda : \Lambda \to \Lambda$ and $\sigma : \Sigma_2 \to \Sigma_2$.

(c) Deduce that $Q|_\Lambda : \Lambda \to \Lambda$ is chaotic.