TOPOLOGICAL DATA ANALYSIS

EXERCISES 2.3

1) Prove that a morphism f of persistence modules is an isomorphism if and only if f_t is an isomorphism of vector spaces for all t.

Proof

($\Rightarrow$) $f:(V,\pi)\to(V',\pi')$ is an isomorphism of persistence modules

so we have $g:(V',\pi')\to(V,\pi)$ such that $\begin{cases} g\circ f = id_V \\ f\circ g = id_{V'} \end{cases}$

with $(g\circ f)_t = g_t\circ f_t$, $(f\circ g)_t = f_t\circ g_t$.

Thus we have $(g\circ f)_t = g_t\circ f_t = id_{V_t}$, $(f\circ g)_t = f_t\circ g_t = id_{V'_t}$

Meaning $\forall t$ $\exists g$ such that g is the inverse of f, and since they are all linear maps, f and g give a bijection between V and V_t, $\forall t$

($\Leftarrow$) f_t is an isomorphism of vector spaces $\forall t$, thus $\exists g$ such that g_t is the inverse of f_t, meaning $f_t\circ g_t = id_{V_t}$ and $g_t\circ f_t = id_{V_t}$ $\forall t$

Consider now $f_t:(V_t,\pi)\to(V'_t,\pi)$ on two persistence modules;

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s,t}"] \arrow[d, shift right, "f_s"'] & V_t \arrow[d, shift left, "f_t"] \\ V'_s \arrow[u, shift right, "g_s"'] \arrow[r, "\pi'_{s,t}"'] & V'_t \arrow[u, shift left, "g_t"] \end{tikzcd}$$

We have that

$f_t\circ\pi_{s,t} = \pi'_{s,t}\circ f_s$

$f_t\circ\pi_{s,t}\circ g_s = \pi'_{s,t}\circ \underbrace{f_s\circ g_s}_{id_{V'}}$

$\underbrace{g_t\circ f_t}_{id_V}\circ\pi_{s,t}\circ g_s = g_t\circ\pi'_{s,t} \Rightarrow g_t\circ\pi'_{s,t} = \pi_{s,t}\circ g_s$

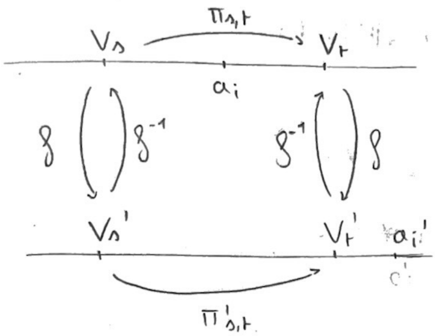
which shows $g_t:(V'_t,\pi)\to(V,\pi)$ is also a morphism and it is such that $\begin{cases} g\circ f = id_V \\ f\circ g = id_{V'} \end{cases}$

Therefore, f is an isomorphism of persistence modules ∎

2) Prove that two isomorphic persistence modules of finite type have the same spectrum

Proof

Suppose two isomorphic persistence modules of finite type have a different spectrum, and consider the following possibility for some $a_i \in A = \{a_0, \dots, a_n\} \subset \mathbb{R}^n$ and with $f : (V, \pi) \to (V', \pi')$ isomorphism.



In this situation $\pi_{s,t}$ is not an isomorphism, while $\pi'_{s,t}$ is.

However, since f is an isomorphism we have

$f : (V, \pi) \to (V', \pi')$
$g : (V', \pi') \to (V, \pi)$
s.t. $\begin{cases} g \circ f = id_V \\ f \circ g = id_{V'} \end{cases}$ with $g = f^{-1}$

and $f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$ if $s \leq t$

thus we have $\pi_{s,t} = f_t^{-1} \circ \pi'_{s,t} \circ f_s$ and $\pi'_{s,t} = f_t \circ \pi_{s,t} \circ f_s^{-1}$

This implies the following:

- $\pi'_{s,t}$ is an isomorphism $\Rightarrow$ $\pi_{s,t}$ is an isomorphism (because it is a combination of isomorphisms)
- $\pi_{s,t}$ is an isomorphism $\Rightarrow$ $\pi'_{s,t}$ is an isomorphism (because it is a combination of isomorphisms)

Therefore, the previous scenario is not admissible ($\pi'_{s,t}$ is an isomorphism while $\pi_{s,t}$ is not) and thus the hypothesis of the two persistence modules having a different spectrum cannot hold. So the two persistence modules must have the same spectrum. ∎

3) Prove that there is a nonzero morphism $\mathbb{F}[a,b) \longrightarrow \mathbb{F}[c,d)$ if and only if $c \leq a$ and $a < d \leq b$

Proof

We have $\mathbb{F}[a,b)_t = \begin{cases} \mathbb{F} & \text{if } t \in [a,b) \\ 0 & \text{otherwise} \end{cases}$ and $\pi_{s,t} = id$ if $s \leq t$, $s,t \in [a,b)$
or $\pi_{s,t} = 0$ otherwise

($\Rightarrow$) There is a nonzero morphism $\mathbb{F}[a,b) \to \mathbb{F}[c,d)$

meaning for $t \in [a,b)$: $\forall s \in [a,b), s \leq t$

$$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s \neq 0$$

since $\pi_{s,t} = \begin{cases} id & \text{if } s,t \in [a,b) \\ 0 & \text{otherwise} \end{cases}$ and $f_t \circ \pi_{s,t} \neq 0$,

we have $f_t = \pi'_{s,t} \circ f_s$

where $\pi'_{s,t}$ is an $\mathbb{F}$-linear map to $\mathbb{F}[c,d)$ and $\pi'_{s,t} = \begin{cases} id & \text{if } s,t \in [c,d) \\ 0 & \text{otherwise} \end{cases}$

Thus, $\forall s,t \in [c,d)$, $s \leq t$ we have $c \leq a$, $\cancel{c<d}$ a (otherwise the ~~interval~~ would be empty and give a null morphism) and $d \leq b$

($\Leftarrow$) We have $c \leq a$, $c < d$ and $d \leq b$ (i.e. $[c,d) \subset [a,b)$)

We define $\mathbb{F}[a,b)$ as $\mathbb{F}[a,b)_t = \begin{cases} \mathbb{F} & \text{if } t \in [a,b) \\ 0 & \text{otherwise} \end{cases}$

and $\pi_{s,t} = \begin{cases} id & \text{if } s,t \in I \\ 0 & \text{otherwise} \end{cases}$

Since $\forall k,l \in [c,d)$ it follows that $k,l \in [a,b)$ we have

$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s \neq 0$ $\forall s,t \in [a,b)$, $s \leq t$ with $\pi'_{s,t} \neq 0$ since $s,t \in [c,d)$

Therefore, it is a nonzero morphism. ■