Hartman's example

It is an analytic example in $\mathbb{R}^3$ that can not be locally conjugated to its linear part by a differentiable diffeo.
Let a, c be such that $0 < c < 1 < a,\ 1 < ac$ and let $\varepsilon \neq 0$.

We define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$f(x, y, z) = (ax, ac(y + \varepsilon xz), cz).$$

$(0, 0, 0)$ is a hyperbolic fixed point:

$$A = Df(0, 0, 0) = \begin{pmatrix} a & & \\ & ac & \\ & & c \end{pmatrix}$$

Claim: $f^k(x, y, z) = (a^k x, (ac)^k(y + k\varepsilon xz), c^k z)$.

Induction: if $k = 1$ it is immediate; assuming it true for k,

$$\begin{aligned} f^{k+1}(x, y, z) &= f(a^k x, (ac)^k(y + k\varepsilon xz), c^k z) \\ &= (a^{k+1}x, ac[(ac)^k(y + k\varepsilon xz) + \varepsilon a^k x c^k z], c^{k+1}z). \end{aligned}$$

Hartman's example (II)

Let $g(x) = Ax$.

Assume h is a differentiable (local) diffeo s.t. $h \circ f = g \circ h$.

We have that $h(0) = 0$ because $h(f(0)) = A(h(0)) \Rightarrow A(h(0)) = h(0) \Rightarrow h(0)$ is an eigenvector of eigenvalue 1, but since 1 is not an eigenvalue of A then $h(0) = 0$.

Let $H = h^{-1}$, $H(0) = 0$, $f \circ H = H \circ g$, and therefore

$$f^k \circ H = H \circ g^k.$$

Let us write this equality in components

$$a^k H_1(x, y, z) = H_1(a^k x, (ac)^k y, c^k z) \tag{1}$$
$$(ac)^k [H_2(x, y, z) + \varepsilon k H_1(x, y, z) H_3(x, y, z)] = H_2(a^k x, (ac)^k y, c^k z) \tag{2}$$
$$c^k H_3(x, y, z) = H_3(a^k x, (ac)^k y, c^k z). \tag{3}$$

Hartman's example (III)

From (1), if $x = y = 0 \rightsquigarrow H_1(0,0,z) = \frac{1}{a^k} H_1(0,0,c^k z) \xrightarrow[k\to\infty]{} 0.$

From (2), if $x = y = 0 \rightsquigarrow H_2(0,0,z) = \frac{1}{(ac)^k} H_2(0,0,c^k z) \xrightarrow[k\to\infty]{} 0.$

From (3), if $y = z = 0,\ x = \frac{t}{a^k} \rightsquigarrow c^k H_3(\frac{t}{a^k},0,0) = H_3(t,0,0)$
$\Rightarrow H_3(t,0,0) = 0.$

From (2), if $y = z = 0,\ x = \frac{t}{a^k} \rightsquigarrow (ac)^k H_2(\frac{t}{a^k},0,0) = H_2(t,0,0)$

$$
\begin{aligned}
H_2(t,0,0) &= (ac)^k [H_2(\tfrac{t}{a^k},0,0) - H_2(0,0,0)] \\
&= c^k t \frac{H_2(\frac{t}{a^k},0,0) - H_2(0,0,0)}{\frac{t}{a^k}} \longrightarrow 0 \cdot D_1 H_2(0,0,0).
\end{aligned}
$$

From (2), if $x = \frac{t}{a^k},\ y = 0,$

$$(ac)^k [H_2(\frac{t}{a^k},0,z) + \varepsilon k H_1(\frac{t}{a^k},0,z) H_3(\frac{t}{a^k},0,z)] = H_2(t,0,c^k z)$$

$$a^k H_2(\frac{t}{a^k},0,z) + \varepsilon k a^k H_1(\frac{t}{a^k},0,z) H_3(\frac{t}{a^k},0,z) = c^{-k} H_2(t,0,c^k z)$$

(note that from (1), $a^k H_1(\frac{t}{a^k},0,z) = H_1(t,0,c^k z)$).

Hartman's example (IV)

$$\lim_{k\to+\infty} a^k H_2\left(\frac{t}{a^k},0,z\right) = \lim_{k\to+\infty} t\,\frac{H_2(\frac{t}{a^k},0,z) - H_2(0,0,z)}{\frac{t}{a^k}} = tD_1H_2(0,0,z)$$

$$\lim_{k\to+\infty} c^{-k} H_2(t,0,c^k z) = \lim_{k\to+\infty} z\,\frac{H_2(t,0,c^k z) - H_2(t,0,0)}{c^k z} = zD_3H_2(t,0,0)$$

Hence,

$$\begin{aligned}\varepsilon k H_1(t,0,c^k z)H_3(\tfrac{t}{a^k},0,z) &= c^{-k}H_2(t,0,c^k z) - a^k H_2(\tfrac{t}{a^k},0,z)\\ &\xrightarrow[k\to\infty]{} zD_3H_2(t,0,0) - tD_1H_2(0,0,z)\end{aligned}$$

implies that

$$\underbrace{\lim_{k\to\infty} H_1(t,0,c^k z)H_3\left(\frac{t}{a^k},0,z\right)}_{H_1(t,0,0)H_3(0,0,z)} = 0$$

because $\underbrace{k}_{\to\infty} H_1(\ )H_3(\ ) \to$ constant.

Then, either $H_1(t,0,0) = 0 \implies H(t,0,0) = 0,\ \forall t \implies H$ is not injective
or $H_3(0,0,z) = 0 \implies H(0,0,z) = 0,\ \forall z \implies H$ is not injective.

Lemma

Let U be an open set of $\mathbb{R}^n$, $0 \in U$, $X : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a vector field of class C^r, $r \geq 1$, such that $X(0) = 0$. Let $L = DX(0)$. Given $\varepsilon > 0$, $\exists \rho > 0$ and $\exists Y : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ verifying

(1) $Y(0) = 0$, Y of class C^r and Y is globally Lipschitz, (and as a result complete),

(2) $Y_{|B(\mathbf{0},\rho/2)} = X_{|B(\mathbf{0},\rho/2)}$ and $Y_{|\mathbb{R}^n \setminus B(\mathbf{0},\rho)} = L_{|\mathbb{R}^n \setminus B(\mathbf{0},\rho)}$.

Let $\phi(t,x)$ and $\psi(t,x)$ the flows of X and Y respectively (the first one is a local flow),

(3) $\exists \rho_1 < \rho/2$ such that $\phi(t,x) = \psi(t,x)$, $\forall (t,x) \in [-1,1] \times B(0,\rho_1)$.

(4) $\psi(t,x)$ can be written as $\psi(t,x) = e^{Lt}x + \tilde{\psi}(t,x)$ and

(a) $\tilde{\psi}(t,0) = 0$ and $D_x\tilde{\psi}(t,0) = 0$.

(b) $\| \tilde{\psi}(t,x) \| \leq M$, $\forall t \in [-1,1]$.

(c) $\operatorname{Lip} \tilde{\psi}(1,x) < \varepsilon$.

Proof: Given $\eta > 0$, by the previous lemma $\exists \rho > 0$ and $Y : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ verifying (1) and (2). $Y = L + \tilde{Y}$ where

$$\tilde{Y}(x) = \begin{cases} \beta(x)[X(x) - Lx] & \text{if} \quad x \in U, \\ 0 & \text{if} \quad x \notin U, \end{cases}$$

supp $\beta = B(0, \rho) \subset U$ and Lip $\tilde{Y} < \eta$.

(3) We can proceed in two different ways. Firstly: Since $\phi(t, 0) = 0$ for all t and ϕ is continuous, $\forall s \in [-1, 1]$ exists a neighbourhood $I_s \times B(0, r_s)$ such that $\forall (t, x) \in I_s \times B(0, r_s)$, $\|\phi(t, x) - 0\| \leq \rho/2$. We take a finite subcover of $[-1, 1] \times \{0\}$ and $\rho_1 = \min r_{s_j}$. Since X, Y coincide in $B(0, \rho/2)$, we have $\phi(t, x) = \psi(t, x)$ in $[-1, 1] \times B(0, \rho_1)$.

Secondly: Since X is continuous and $X(0) = 0$ exists $\rho_1 < \rho/2$ such that $\|X(x)\| < \rho/4$. Then, if $x \in B(0, \rho_1/2)$, by the existence of solutions theorem ϕ is such that $\phi(t, x) \in B(x, \rho/4)$, $\forall t \in [-1, 1]$ and $B(x, \rho/4) \subset B(0, \rho_1/2 + \rho/4) \subset B(0, \rho/2)$.

(4) (a) $\psi(t,0)=0$, $\forall t$, because $Y(0)=0$. Then $\tilde{\psi}(t,0)=\psi(t,0)-e^{Lt}0=0$. $D_x\psi(t,0)$ satisfies the variational equation

$$(D_x\psi(t,0))' = DY(\psi(t,0))D_x\psi(t,0) = LD_x\psi(t,0),$$

with $D_x\psi(0,0)=I$. Then $D_x\psi(t,0)=e^{Lt}$. As a result $D_x\tilde{\psi}(t,0)=D_x\psi(t,0)-e^{Lt}=0$.

(b) Let $\rho_2>\rho$ such that, if $x\in\mathbb{R}^n\backslash B(0,\rho_2)$ and $t\in[-1,1]$ the flow does not enter inside de ball $B(0,\rho)$. A sufficient condition by which the flow does not get into the ball $B(0,\rho)$ can be obtained from

$$\|e^{tL}x\| \geq \|e^{-tL}\|^{-1}\|x\| \geq e^{-\|L\|}\rho_2$$

which implies that $\rho_2 \geq e^{\|L\|}\rho$.

In $\mathbb{R}^n\backslash B(0,\rho)$, $Y=L$ so that, if $x\in\mathbb{R}^n\setminus B(0,\rho_2)$ and $t\in[-1,1]$ then $\psi(t,x)=e^{Lt}x$ and as a result $\tilde{\psi}(t,x)=0$.

Moreover $[-1,1]\times\overline{B}(0,\rho_2)$ is compact and $\tilde{\psi}$ is bounded on this set.

(c) Firstly, we have

$$\| \psi(t,x) - \psi(t,y) \| \leq e^{K|t|} \| x - y \|,$$

where $K = \|L\| + \eta$ is a bound of the global constant of Y. It is a consequence of Gronwall's lemma applied to

$$\psi(t,x) - \psi(t,y) = x + \int_0^t Y(\psi(s,x))\, ds - y - \int_0^t Y(\psi(s,y))\, ds.$$

Now we calculate a bound of the Lipschitz constant of $\tilde{\psi}$

$$\begin{aligned}
&\tilde{\psi}(t,x) - \tilde{\psi}(t,y) \\
&\quad = \quad \psi(t,x) - \psi(t,y) - e^{Lt}x + e^{Lt}y \\
&\quad = \quad x + \int_0^t Y(\psi(s,x))\, ds - y - \int_0^t Y(\psi(s,y))\, ds - x - \int_0^t Le^{Ls}x\, ds \\
&\qquad\qquad + y + \int_0^t Le^{Ls}y\, ds.
\end{aligned}$$

Note that
$Y(\psi(s,x)) = L\psi(s,x) + \tilde{Y}(\psi(s,x)) = Le^{Ls}x + L\tilde{\psi}(s,x) + \tilde{Y}(\psi(s,x))$.
Replacing it in the previous expression, for $0 \leq t \leq 1$ we have

$$
\begin{aligned}
&\| \tilde{\psi}(t,x) - \tilde{\psi}(t,y) \| \\
&\quad \leq \ \| \int_0^t [L(\tilde{\psi}(s,x)) - L(\tilde{\psi}(s,y))]\, ds \| + \| \int_0^t [\tilde{Y}(\psi(s,x)) - \tilde{Y}(\psi(s,y))]\, ds \| \\
&\quad \leq \ \int_0^t \|L\| \, \|\tilde{\psi}(s,x) - \tilde{\psi}(s,y)\| \, ds + \int_0^t \text{ Lip } \tilde{Y} \|\psi(s,x) - \psi(s,y)\| \, ds \\
&\quad \leq \ \int_0^t \|L\| \, \|\tilde{\psi}(s,x) - \tilde{\psi}(s,y)\| \, ds + \eta \int_0^t e^{Ks} \|x - y\| \, ds
\end{aligned}
$$

and applying Gronwall's Lemma, we obtain

$$\| \tilde{\psi}(t,x) - \tilde{\psi}(t,y) \| \leq \eta e^{K} \|x - y\| e^{\|L\| t}.$$

(We have bound e^{Ks} by e^{K} inside the integral. It is possible to integrate the function, but the improvement on the bound is irrelevant).

Finally, taking $t = 1$, we get

$$\| \tilde{\psi}(1,x) - \tilde{\psi}(1,y) \| \leq \eta e^{K} e^{\|L\|} \|x - y\|,$$

so that if η is small enough Lip $\tilde{\psi}(1,x) < \varepsilon$.

Another computation of Lip $\tilde{\psi}(1,x)$ based on the computation of the derivative,

$$[D_x\psi(t,x)]' = DY(\psi(t,x))D_x\psi(t,x).$$

$$\psi(t,x) = e^{Lt}x + \tilde{\psi}(t,x), \qquad D_x\tilde{\psi}(t,0) = 0.$$

$$\begin{aligned}[D_x\tilde{\psi}(t,x)]' &= DY(\psi(t,x))[e^{Lt} + D_x\tilde{\psi}(t,x)] - Le^{Lt} \\ &= [DY(\psi(t,x)) - L]e^{Lt} + DY(\psi(t,x))D_x\tilde{\psi}(t,x).\end{aligned}$$

$$D_x\tilde{\psi}(0,x) = D_x\psi(0,x) - e^{A0} = \mathrm{Id} - \mathrm{Id} = 0$$

$$D_x\tilde{\psi}(t,x) = \int_0^t \underbrace{(DY(\psi(s,x)) - L)}_{D\tilde{Y}(\psi(s,x))} e^{Ls}ds + \int_0^t DY(\psi(s,x))D_x\tilde{\psi}(s,x)ds.$$

Using Gronwall's lemma

$$\|D_x\tilde{\psi}(t,x)\| \le \eta e^{\|L\|}\exp\int_0^1 \|DY(\psi(s,x))\|ds \le \eta e^{\|L\|}e^{(\|L\|+\eta)}, \; 0 \le t \le 1.$$

Hartman's theorem for vector fields

Theorem

Let $X : U \to \mathbb{R}^n$ be a vector field of class C^r, $r \geqslant 1$, $X(0) = 0$, 0 hyperbolic fixed point. Let $L = DX(0)$. Then X is locally topologically conjugated to L in a neighbourhood of 0.

Proof.

Let $\varepsilon = \varepsilon(e^L) > 0$ given by Hartman's global theorem for diffeomorphisms.
Let $Y : \mathbb{R}^n \to \mathbb{R}^n$ given by the previous lemma and let $\psi(t, x)$ be its flow.

$$\psi(1, x) = e^L x + \tilde{\psi}(1, x), \; \tilde{\psi}(1, \cdot) \in C_b^0, \; \operatorname{Lip} \tilde{\psi}(1, \cdot) < \varepsilon,$$

$$\tilde{\psi}(1, 0) = 0, \; D_x \tilde{\psi}(1, 0) = 0$$

0 is an hyperbolic fixed point of $\psi(1, \cdot)$.
Then $\exists! h : \mathbb{R}^n \to \mathbb{R}^n$ homeo. of the form $h = I + u$ with $u \in C_b^0$, such that

$$h(\psi(1, x)) = e^L h(x).$$

Hartman's theorem for vector fields (II)

Define

$$H(x) = \int_0^1 e^{-Lt} h(\psi(t,x))dt.$$

Then $H \in C^0$ and

$$\begin{aligned} H(x) - x &= \int_0^1 [e^{-Lt} h(\psi(t,x)) - x]dt = \int_0^1 [e^{-Lt}[\psi(t,x) + u(\psi(t,x))] - x]dt \\ &= \int_0^1 [e^{-Lt}[e^{Lt}x + \tilde{\psi}(t,x) + u(\psi(t,x))] - x]dt \\ &= \int_0^1 e^{-Lt}[\tilde{\psi}(t,x) + u(\psi(t,x))]dt \qquad \Rightarrow \quad H(x) - x \in C_b^0. \end{aligned}$$

Hartman's theorem for vector fields (III)

Let us see that

$$H(\psi(t,x)) = e^{Lt}H(x), \quad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^n.$$

$$e^{-Lt}H(\psi(t,x)) = e^{-Lt}\int_0^1 e^{-Ls}h(\psi(s,\psi(t,x)))ds = \int_0^1 e^{-L(t+s)}h(\psi(t+s,x))ds =$$

$(u := t + s - 1)$

$$= \int_{-1+t}^{t} e^{-L(u+1)}h(\psi(u+1,x))du = \int_{-1+t}^{0} + \int_0^t =$$

$(v := u + 1)$

$$= \int_t^1 e^{-Lv}h(\psi(v,x))dv + \int_0^t e^{-Lu}e^{-L}h(\psi(1,\psi(u,x)))du =$$

$$= \int_0^1 e^{-Lv}h(\psi(v,x))dv = H(x).$$

Moreover, H conjugates $\psi(1,x)$ to e^L. By uniqueness $H = h$ and H is a homeo. Restricting $\psi(t,x)$ to $[-1,1] \times B(0,\rho_1)$ with ρ_1 given in the previous lemma we obtain

$$h(\varphi(t,x)) = e^{Lt}h(x).$$

Hartman's theorem for vector fields (II')

Given $s \in \mathbb{R}$ we define

$$H^s(x) = e^{-Ls} h(\psi(s,x)).$$

Then $H^s \in C^0$ and

$$\begin{aligned} H^s(x) - x &= e^{-Ls} h(\psi(s,x)) - x = e^{-Ls}[\psi(s,x) + u(\psi(s,x))] - x \\ &= e^{-Ls}[e^{Ls}x + \tilde{\psi}(s,x) + u(\psi(s,x))] - x \\ &= e^{-Ls}[\tilde{\psi}(s,x) + u(\psi(s,x)) \qquad \Rightarrow \quad H^s(x) - x \in C^0_b. \end{aligned}$$

Hartman's theorem for vector fields (III')

We claim that

$$h(\psi(t,x)) = e^{Lt}h(x), \qquad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^n.$$

Indeed,

$$\begin{aligned} H^s(\psi(1,x)) &= e^{-Ls}h(\psi(s,\psi(1,x))) = e^L e^{-Ls} e^{-L} h(\psi(1+s,x)) \\ &= e^L e^{-Ls} h(\psi(s,x)) = e^L H^s(x). \end{aligned}$$

Since H^s conjugates $\psi(1,x)$ to e^L, by uniqueness $H^s = h$, and this proves the claim.
Restricting $\psi(t,x)$ to $[-1,1] \times B(0,\rho_1)$ with ρ_1 given in the previous lemma we obtain

$$h(\varphi(t,x)) = e^{Lt}h(x).$$

■