Chapter 2

Fourier integrals

2.1 Fourier transform and main properties

Consider the space of integrable functions in $\mathbb{R}$:

$$L^1(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow \mathbb{C} : \|f\|_1 = \int_{\mathbb{R}} |f(t)|\, dt < +\infty\}.$$

Definition 4. Given $f \in L^1(\mathbb{R})$, its *Fourier transform* is the function $\hat{f} : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t} dt, \quad \xi \in \mathbb{R}.$$

Notice that the Fourier transform is a well-defined and bounded:

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}} |f(t)|\, dt = \|f\|_1.$$

As we shall see, this can be roughly interpreted as the content of frequency ξ in the original signal $f(t)$.

Remark 5. 1. The Fourier transform can be seen as the limit of the value of the Fourier coefficient of a T-periodic function as T tends to ∞, in the following sense. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ (not necessarily periodic). For any T let f_T denote the T-periodic function that coincides with f on $(-T/2, T/2)$. Then, for $|t| < T/2$

$$f(t) = f_T(t) = \sum_{n \in \mathbb{Z}} \hat{f}_T(n) e^{i \frac{2\pi}{T} n t}.$$

Hence

$$f(t) = \lim_{T \to \infty} \sum_{n \in \mathbb{Z}} \frac{1}{T} \Big(\int_{-T/2}^{T/2} f(s)\, e^{-i \frac{2\pi}{T} n s} ds \Big)\, e^{i \frac{2\pi}{T} n t}.$$

Let us try to identify this limit, at least at a formal level. Let $\xi_n = n/T$, $n \in \mathbb{Z}$, and consider the partition of $\mathbb{R}$ given by these nodes. In this terms the sum above is

$$\sum_{n \in \mathbb{Z}} \Big(\int_{-T/2}^{T/2} f(s)\, e^{-i 2\pi \xi_n s} ds \Big)\, e^{i 2\pi \xi_n t} (\xi_{n+1} - \xi_n).$$

Letting $T \to \infty$ in the integral this turns into

$$\sum_{n \in \mathbb{Z}} \hat{f}(\xi_n)\, e^{2\pi i \xi_n t}(\xi_{n+1} - \xi_n),$$

which is a Riemann sum of the integral

$$\int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi t} d\xi.$$

Thus, formally, the "inversion formula"

$$f(t) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi t} d\xi$$

works with the definition of $\hat{f}$ just given.

2. As in the case of Fourier series, the Fourier transform gives a decomposition of f. The operator $f \mapsto \hat{f}$ is sometimes called the *analysis*. Now, instead of only a discrete set of frequencies, as in the Fourier series, we have a continuum. The reconstruction operator $\hat{f} \mapsto f$ is usually called *synthesis*.

Proposition 3. *Assume that* $f, g \in L^1(\mathbb{R})$, $\alpha, \beta \in \mathbb{C}$ *and* $a \in \mathbb{R}$.

1. *The Fourier transform is linear:* $(\alpha f + \beta g)^\wedge(\xi) = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi)$, $\xi \in \mathbb{R}$.
2. *Conjugation:* $(\bar{f})^\wedge(\xi) = \overline{\hat{f}(-\xi)}$, $\xi \in \mathbb{R}$.
3. *Translations. Let* $\tau_a f(t) = f(t-a)$. *Then*
$$(\widehat{\tau_a f})(\xi) = \hat{f}(\xi)\, e^{-2\pi i a \xi} \qquad \xi \in \mathbb{R}.$$
4. *Modulations: Let* $M_a f(t) = f(t)\, e^{2\pi i a t}$. *Then*
$$(\widehat{M_a f})(\xi) = \tau_a \hat{f}(\xi) = \hat{f}(\xi - a) \qquad \xi \in \mathbb{R}.$$
5. *Dilations. Given* $\lambda > 0$ *let* $D_\lambda f(t) = \frac{1}{\lambda} f(\frac{1}{\lambda})$. *Then*
$$\widehat{D_\lambda f}(\xi) = \hat{f}(\lambda \xi) \qquad \xi \in \mathbb{R}.$$

Remark 6. Observe that for any $a \in \mathbb{R}$ and $\lambda > 0$,

$$\|f\|_1 = \|\tau_a f\|_1 = \|M_a f\|_1 = \|D_\lambda f\|_1.$$

Proof. All these properties are straightforward from the definition. For example, translating the variable,

$$(\widehat{\tau_a f})(\xi) = \int_{\mathbb{R}} f(t-a) e^{-2\pi i \xi t} dt = \int_{\mathbb{R}} f(s) e^{-2\pi i \xi (s+a)} ds = e^{-2\pi i \xi a} \hat{f}(\xi).$$

□

We gather next some relevant properties of a more analytic nature. For that we need the following application of the Dominated Convergence Theorem (see Annex 2.5).

Lemma 5. *Let $f \in L^1(\mathbb{R})$. The translations $\tau_h f$ are continuous in the L^1 norm; that is*

$$\lim_{h \to 0} \|f - \tau_h f\|_1 = 0.$$

Proof. Assume first that $f \in \mathcal{C}_c(\mathbb{R})$. Then there exist $A, M > 0$ such that $|f| \leq M\chi_{[-A,A]}$. Then, for h small enough, $|\tau_h f| \leq M\chi_{[-2A,2A]} \in L^1(\mathbb{R})$, and since obviously $f(t) = \lim_{h \to 0} \tau_h f(t)$ pointwise, by the Dominated Convergence theorem we deduce the result.

For the general case $f \in L^1(\mathbb{R})$ consider a sequence $\{f_n\}_{n \geq 1} \subset \mathcal{C}_c(\mathbb{R})$ such that $\lim_{n \to \infty} \|f - f_n\|_1 = 0$. Given $\epsilon > 0$ take $n \geq 1$ big enough so that $\|f - f_n\|_1 < \epsilon/3$; then

$$\begin{aligned} \|f - \tau_h f\|_1 &\leq \|f - f_n\|_1 + \|f_n - \tau_h f_n\|_1 + \|\tau_h f_n - f_n\|_1 \\ &= 2\|f - f_n\|_1 + \|f_n - \tau_h f_n\|_1 < \frac{2\epsilon}{3} + \|f_n - \tau_h f_n\|_1. \end{aligned}$$

Once this n is fixed, take $\delta > 0$ so that $\|f_n - \tau_h f_n\|_1 < \epsilon/3$ if $|h| < \delta$ and finally obtain

$$\|f - \tau_h f\|_1 \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

Now we are ready to state and prove the following properties.

Theorem 4. *Let $f \in L^1(\mathbb{R})$. Then*

(a) $\hat{f}$ is uniformly continuous and $|\hat{f}(\xi)| \leq \|f\|_1$.

(b) If $f, f' \in L^1(\mathbb{R})$, then

$$\widehat{f'}(\xi) = 2\pi i\xi\, \hat{f}(\xi) \qquad \xi \in \mathbb{R}.$$

(c) If $tf(t) \in L^1(\mathbb{R})$ then $\hat{f}$ is differentiable and

$$(\hat{f})'(\xi) = (-2\pi i t f)^{\wedge}(\xi) \qquad \xi \in \mathbb{R}.$$

(d) Riemann-Lebesgue lemma: $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.

(e) Multiplication formula: if $f, g \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} f(t)\, \hat{g}(t)\, dt = \int_{\mathbb{R}} \hat{f}(t)\, g(t)\, dt.$$

Remark 7. Property (b) can be applied iteratively; if $f, f', \dots, f^{(k)} \in L^1(\mathbb{R})$ we have

$$\widehat{f^{(k)}}(\xi) = (2\pi i\xi)^k \hat{f}(\xi).$$

In particular, if

$$P(D) = a_0 + a_1 \frac{\partial}{\partial t} + \cdots a_n + \frac{\partial^n}{\partial t^n}$$

is a differential operator associated to a polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, then

$$\widehat{P(D)(f)}(\xi) = P(2\pi i\xi)\, \hat{f}(\xi).$$

This is very useful in solving equations of the form $P(D)f = g$, but we shall not discuss this here.

Proof. (a) It is immediate that $|\hat{f}(\xi)| \leq \|f\|_1$. To prove the uniform continuity observe that for a given $\xi \in \mathbb{R}$

$$|\hat{f}(\xi + h) - \hat{f}(\xi)| = \Big| \int_{\mathbb{R}} f(t)\, e^{2\pi i(\xi+h)t} - f(t)\, e^{2\pi i\xi t} dt \Big| \leq \int_{\mathbb{R}} |f(t)| \left| e^{2\pi i h t} - 1 \right| dt.$$

Observe that for all $h \in \mathbb{R}$

$$|f(t)| \left| e^{2\pi i h t} - 1 \right| \leq 2|f(t)| \in L^1(\mathbb{R}),$$

so by the Dominated Convergence theorem the right had side of the above estimate tends to 0 as $h \to 0$, and it does so at a speed that does not depend on ξ.

(b) By hypothesis there exist sequences $\{a_n\}_{n\geq 1} \to -\infty$ and $\{b_n\}_{n\geq 1} \to +\infty$ such that

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = 0.$$

Since obviously $|\chi_{(a_n,b_n)} f| \leq |f|$, the Dominated Convergence theorem ensures that

$$\lim_{n\to\infty} \int_{a_n}^{b_n} f(t) e^{-2\pi i\xi t}\, dt = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{(a_n,b_n)}(t)\, f(t)\, e^{-2\pi i\xi t} dt = \hat{f}(\xi)$$

and similarly

$$\lim_{n\to\infty} \int_{a_n}^{b_n} f'(t) e^{-2\pi i\xi t}\, dt = \widehat{f'}(\xi).$$

Integrating by parts

$$\begin{aligned} \int_{a_n}^{b_n} f'(t) e^{-2\pi i\xi t}\, dt &= \left[f(t) e^{-2\pi i\xi t} \right]_{a_n}^{b_n} - \int_{a_n}^{b_n} f(t)(-2\pi i\xi) e^{-2\pi i\xi t}\, dt \\ &= (2\pi i\xi) \int_{a_n}^{b_n} f(t) e^{-2\pi i\xi t}\, dt, \end{aligned}$$

and taking the limit as $n \to \infty$ we get the result.

(c) That $\hat{f}$ is differentiable is an immediate consequence of the theorem of differentiation (Theorem 9 in Annen 2.5). Fix ξ_0 and take $F(\xi, t) = f(t)e^{-2\pi i \xi t}$, where ξ is in a fixed interval I centered in ξ_0. Observe that

$$\left|\frac{\partial F}{\partial \xi}(\xi, t)\right| = \big|(-2\pi i t) f(t)\big| \leq 2\pi \big|t f(t)\big| \in L^1(\mathbb{R}),$$

hence $\hat{f}(\xi) = \int_{\mathbb{R}} F(\xi, t)\, dt$ is differentiable at ξ_0 and

$$(\hat{f})'(\xi_0) = \int_{\mathbb{R}} \frac{\partial F}{\partial \xi}(\xi_0, t)\, dt = \int_{\mathbb{R}} (-2\pi i t) f(t) e^{-2\pi i \xi_0 t} dt = (-2\pi i t f)^\wedge(\xi_0).$$

(d) We could proceed as in the analogue for Fourier series, assuming first that f is $\mathcal{C}^1$ with compact support and then proving the general case by approximation. We take instead a different path.

Multiplying the identity that defines $\hat{f}(\xi)$ by $-1 = e^{\pi i}$ we get

$$\hat{f}(\xi) = -\int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} e^{i\pi}\, dt = -\int_{\mathbb{R}} f(t) e^{-2\pi i t(\xi - \frac{1}{2\xi})t}\, dt = -\int_{\mathbb{R}} f(s + \frac{1}{2\xi}) e^{-2\pi i \xi s} ds.$$

Adding to this the usual expression of the Fourier transform we obtain

$$2\hat{f}(\xi) = \int_{\mathbb{R}} \big(f(s) - f(s + \frac{1}{2\xi})\big) e^{-2\pi i \xi s} ds,$$

hence from Lemma 5

$$2|\hat{f}(\xi)| \leq \int_{\mathbb{R}} \big|f(s) - f(s + \frac{1}{2\xi})\big| ds = \|f - \tau_{-\frac{1}{2\xi}} f\|_1 \overset{|\xi|\to\infty}{\longrightarrow} = 0.$$

(e) By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}} f(t)\, \hat{g}(t)\, dt &= \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} g(s) e^{-2\pi i s t} ds\, dt = \int_{\mathbb{R}} g(s) \int_{\mathbb{R}} f(t)\, e^{-2\pi i s t} dt\, ds \\ &= \int_{\mathbb{R}} g(s)\, \hat{f}(s)\, ds.\end{aligned}$$

□

Examples 1. 1. Let $f = \chi_{[-1/2, 1/2]}$. Obviously $f \in L^1(\mathbb{R})$ and $\|f\|_1 = 1$. Its Fourier transform is

$$\hat{f}(\xi) = \int_{-1/2}^{1/2} e^{-2\pi i \xi t} dt = \left[\frac{e^{-2\pi i \xi t}}{-2\pi i \xi}\right]_{t=-1/2}^{t=1/2} = \frac{e^{\pi i \xi} - e^{-\pi i \xi}}{2\pi i \xi} = \frac{\sin(\pi \xi)}{\pi \xi}.$$

We shall use the following definition of the *cardinal sine*

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \tag{2.1}$$

so that $\widehat{\chi_{[-1/2,1/2]}}(\xi) = \mathrm{sinc}(\xi)$.

2. Consider the Gaussian $G(t) = e^{-\pi t^2}$, which is normalised so that $\|G\|_1 = 1$. A direct computation of $\hat{G}(\xi)$ leads to complicated integral, so we take a different path, using the relationship between Fourier transform and differentiation. Notice that G satisfies the differential equation

$$\begin{cases} G'(t) = -2\pi i G(t) \\ G(0) = 1. \end{cases}$$

Taking Fourier transform on both sides of this equation we get $\widehat{G'}(\xi) = (-2\pi i G(t))^\wedge(\xi)$, which by Theorem 4 (b) and (c), is $(2\pi i\xi)\hat{G}(\xi) = -(\hat{G})'(\xi)$. Since

$$\hat{G}(0) = \int_{\mathbb{R}} G(t)\,dt = 1,$$

we see that $\hat{G}$ is also the solution to the above system. Therefore $\hat{G}(\xi) = e^{-\pi\xi^2}$.

We finish this section by seeing a new instance in which a function which is well concentrated has a Fourier transform that is really spread out.

Theorem 5. *Let $f \in L^1(\mathbb{R})$ have compact support. Then $\hat{f}(\xi)$ defines an entire function of exponential type, that is, there exist $A, B > 0$ such that $|\hat{f}(\xi)| \le Ae^{B|\operatorname{Im}\xi|}$, $\xi \in \mathbb{C}$.*

Proof. Assume $\mathrm{supp}(f) \subset [-C, C]$, for some $C > 0$. Then, for $\xi \in \operatorname{Re}\xi + i\operatorname{Im}\xi$ we have

$$|\hat{f}(\xi)| \le \int_{-C}^{C} |f(t)| e^{2\pi t \operatorname{Im}\xi} dt \le e^{2\pi C|\operatorname{Im}\xi|} \int_{-C}^{C} |f(t)|\,dt = e^{2\pi C|\operatorname{Im}\xi|}\|f\|_1\,,$$

so $\hat{f}(\xi)$ is well defined for all $\xi \in \mathbb{C}$ and has exponential type (take $A = \|f\|_1$ and $B = 2\pi C$).

That $\hat{f}$ is holomorphic is an immediate application of Morera's theorem: for any close simple, piecewise $\mathcal{C}^1$ curve γ we have

$$\int_\gamma \hat{f}(\xi)\,d\xi = \int_{\mathbb{R}} f(t)\Big(\int_\gamma e^{-2\pi i\xi t} d\xi\Big)\,dt = \int_{\mathbb{R}} f(t) = 0,$$

since, by Cauchy's theorem, $\int_\gamma e^{-2\pi i\xi t} d\xi = 0$ for all $t \in \mathbb{R}$. $\square$

2.2 The inversion formula

We want to see that $f \in L^1(\mathbb{R})$ can be recovered from the set of values $\hat{f}(\xi)$, at least when $\hat{f} \in L^1(\mathbb{R})$. From the point of view of sound processing this seems natural: knowing the frequency density of any possible frequency allows to recover the signal.

An important tool in proving this will be the convolution of functions.

2.2.1 Convolution of L^1 functions and approximate identities

Definition 5. Given $f, g \in L^1(\mathbb{R})$, the *convolution of f and g* is the function $f * g$ defined by

$$(f * g)(t) = \int_{\mathbb{R}} f(s)\, g(t-s)\, ds\,, \qquad t \in \mathbb{R}.$$

Lemma 6. *If $f, g \in L^1(\mathbb{R})$ then $f * g = g * f$, $f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.*

Proof. That $f * g = g * f$ is readily checked by the substitution $t - s = u$ in the definition above.

On the other hand, by Fubini's theorem,

$$\begin{aligned} \|f * g\|_1 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(s)\, g(t-s)\, ds \right| dt \leq \int_{\mathbb{R}} |f(s)| \left(\int_{\mathbb{R}} |g(t-s)|\, dt \right) ds \\ &= \int_{\mathbb{R}} |f(s)|\, \|g\|_1\, ds = \|f\|_1\, \|g\|_1. \end{aligned}$$

□

Another property that will be used systematically is the analogue of Theorem 2.

Theorem 6. *Let $f, g \in L^1(\mathbb{R})$. Then, for $\xi \in \mathbb{R}$,*

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi).$$

Proof. By definition

$$\widehat{(f * g)}(\xi) = \int_{\mathbb{R}} \big(\int_{\mathbb{R}} f(t-s)\, g(s)\, ds \big)\, e^{-2\pi i t\xi} dt = \int_{\mathbb{R}} g(s) \big(\int_{\mathbb{R}} f(t-s)\, e^{-2\pi i t\xi} dt \big)\, ds.$$

Substituting $t - s = u$ we finally obtain

$$\widehat{(f * g)}(\xi) = \int_{\mathbb{R}} g(s) \big(\int_{\mathbb{R}} f(u)\, e^{-2\pi i u\xi} dt \big)\, e^{-2\pi i s\xi} ds = \int_{\mathbb{R}} g(s)\, \hat{f}(\xi)\, e^{-2\pi i s\xi} ds = \hat{f}(\xi)\, \hat{g}(\xi).$$

□

We shall use the convolution mostly with regular functions $g \geq 0$ such that $\|g\|_1 = 1$. Such a function can be seen as the density function of a random variable Y. From this point of view the convolution of f ang g is a sort of weighted average of the values of f weighted by g; more specifically,

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s)\, g(s)\, ds = \mathbb{E}(f(t-Y)).$$

For example, if we take a uniform density on the interval $[-\delta/2, \delta/2]$, that is $g_\delta(t) = \frac{1}{\delta}\chi_{[-\delta/2,\delta/2]}(t)$, we have

$$(f * g)(t) = \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} f(t-s)\,ds = \frac{1}{\delta}\int_{t-\delta/2}^{t+\delta/2} f(s)\,ds,$$

which is the (ordinary) average of f around t.

The process of convolving f with a regular function g concentrated around 0, as in the example above, produces in general a regular function that is "similar" to f.

Definition 6. Let $g \in L^1(\mathbb{R})$ be non-negative and with $\|g\|_1 = 1$. For $\delta > 0$ consider the dilations

$$g_\delta(t) = D_\delta f(t) = \frac{1}{\delta} g\big(\frac{t}{\delta}\big).$$

The family $\{g_\delta\}_{\delta>0}$ is called an *approximate identity* if for any $\eta > 0$

$$\lim_{\delta\to 0}\int_{|t|>\eta} g_\delta(t)\,dt = 0. \tag{2.2}$$

There is a more general notion of approximate identity, but we shall only use this specific kind.

Remark 8. Observe that the condition above forces g_δ to be increasingly concentrated around 0 as δ tends to 0.

Examples 2. 1. Let $g_\delta = \frac{1}{\delta}\chi_{[-\delta/2,\delta/2]}$, as considered above. It is clear that for $\eta > \delta/2$

$$\int_{|t|>\eta} g_\delta(t)\,dt = \frac{1}{\delta}\int_{|t|>\eta} \chi_{[-\delta/2,\delta/2]}(t)\,dt = 0,$$

hence $\{g_\delta\}_{\delta>0}$ satisfies (2.2)

2. The family $\{G_\delta\}_{\delta>0}$ obtained from the Gaussian $G(t) = e^{-\pi t^2}$ is also an approximate identity, since

$$\int_{|t|>\eta} g_\delta(t)\,dt = \frac{1}{\delta}\int_{|t|>\eta} e^{-\pi\frac{t^2}{\delta^2}}\,dt = \int_{|s|>\eta/\delta} e^{-\pi s^2}\,ds = \int_{\mathbb{R}} e^{-\pi s^2}\chi_{[-\eta/\delta,\eta/\delta]}(s)\,ds$$

tends to 0 as $\delta \to 0$, by the Dominated Convergence theorem.

Proposition 4. *Let $f \in L^1(\mathbb{R})$ and let $\{g_\delta\}_{\delta>0}$ be an approximate identity. Then*

$$\lim_{\delta\to 0}\|f * g_\delta - f\|_1 = 0.$$

In particular, $\lim_{\delta\to 0}(f * g_\delta)(t) = f(t)$ a.e. $t \in \mathbb{R}$.

Proof. By definition, and since $\int_{\mathbb{R}} g_\delta = 1$,

$$\|f * g_\delta - f\|_1 = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t-s)\, g_\delta(s)\, ds - f(t) \right| dt = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(t-s) - f(t))\, g_\delta(s)\, ds \right| dt$$
$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t-s) - f(s)|\, g_\delta(s)\, ds\, dt.$$

Substituting $s/\delta = u$ we get

$$\|f * g_\delta - f\|_1 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t - \delta u) - f(t)|\, g(u)\, du\, dt = \int_{\mathbb{R}} g(u) \int_{\mathbb{R}} |f(t-\delta u) - f(t)|\, dt\, du$$
$$= \int_{\mathbb{R}} g(u)\, \|\tau_{\delta u} f - f\|_1\, du.$$

It is clear, by Lemma 5, that for all $u \in \mathbb{R}$

$$\lim_{\delta \to 0} g(u)\, \|\tau_{\delta u} f - f\|_1 = 0.$$

Since obviously $g(u)\, \|\tau_{\delta u} f - f\|_1 \leq 2\|f\|_1 g(u) \in L^1(\mathbb{R})$, we can apply the Dominated Convergence theorem to the integral above and deduce the result. □

2.2.2 The inversion formula

The goal in this section is to prove the following theorem.

Theorem 7 (Inversion formula for L^1 functions)**.** *Let $f \in L^1(\mathbb{R})$ be such that $\hat{f} \in L^1(\mathbb{R})$. Then*

$$f(t) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i t \xi} d\xi \qquad a.e. \quad t \in \mathbb{R}.$$

Moreover, the identity holds for the points t where f is continuous.

Recall that the right-hand side of this identity defines a continuous function of t (see Theorem 4 (a)), so in general there is no hope to have the equality for all $t \in \mathbb{R}$.

Observe also that a consequence of this statement is that for functions $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$ the estimate $|f(t)| \leq \|\hat{f}\|_1$ holds a.e. $t \in \mathbb{R}$.

Remark 9. The Gaussian and its Fourier transform will be instrumental in the proof of this result. Recall that $G(t) = e^{-\pi t^2}$ is normalised so that $\|G\|_1 = 1$ and that $\hat{G} = G$ (Example 1, 2). Also, by Theorem 3, 4,

$$\hat{G}_\delta(\xi) = \hat{G}(\delta \xi) = e^{-\pi \delta^2 \xi^2}.$$

Reciprocally, leting $F_\delta(t) = e^{-\pi \delta^2 t^2}$ we see that $\hat{F}_\delta = G_\delta$:

$$\hat{F}_\delta(\xi) = \int_{\mathbb{R}} e^{-\pi \delta^2 t^2} e^{-2\pi i t \xi} dt = \int_{\mathbb{R}} e^{-\pi s^2} e^{-2\pi i s \frac{\xi}{\delta}} \frac{ds}{\delta} = \frac{1}{\delta} \hat{G}(\frac{\xi}{\delta}) = \frac{1}{\delta} G(\frac{\xi}{\delta}) = G_\delta(\xi).$$

Proof of the inversion formula. Consider G_δ as in the previous remark. By the multiplication formula (Theorem 5 (e)),

$$\begin{aligned}(f * G_\delta)(t) &= \int_{\mathbb{R}} f(t-s)\, G_\delta(s)\, ds = \int_{\mathbb{R}} f(t+u)\, G_\delta(u)\, du = \int_{\mathbb{R}} f(t+u)\, \hat{F}_\delta(u)\, du \\ &= \int_{\mathbb{R}} \widehat{\tau_{-t} f}(u)\, F_\delta(u)\, du.\end{aligned}$$

By Proposition 3, 3, the Fourier transform of a translation is a modulation, hence

$$(f * G_\delta)(t) = \int_{\mathbb{R}} \hat{f}(u)\, e^{2\pi i t u} e^{-\pi \delta^2 u^2} du \tag{2.3}$$

and it only remains to see that this identity can be taken to the limit as $\delta \to 0$.

It is clear by Proposition 4 that the left hand side tends to $f(t)$ a.e. $t \in \mathbb{R}$. On the other hand

$$|\hat{f}(u)\, e^{2\pi i t u} e^{-\pi \delta^2 u^2}| \leq |\hat{f}(u)| \in L^1(\mathbb{R}),$$

so the Dominated Convergence theorem ensures that the right hand side tends to the stated integral.

It only remains to see that that the identity holds for the points t where f is continuous at t. By translating if necessary, we can assume that f is continuous at t. By (2.3) it is enough to see that

$$\lim_{\delta \to 0} |(f * G_\delta)(0) - f(0)| = 0.$$

Here, since $\int G_\delta = 1$, for any $\eta > 0$,

$$\begin{aligned}|(f * G_\delta)(0) - f(0)| &= \Big|\int_{\mathbb{R}} (f(0-s) - f(0))\, G_\delta(s)\, ds\Big| \leq \int_{\mathbb{R}} |f(u) - f(0)|\, G_\delta(u)\, du \\ &= \int_{|u| \leq \eta} |f(u) - f(0)|\, G_\delta(u)\, du + \int_{|u| > \eta} |f(u) - f(0)|\, G_\delta(u)\, du.\end{aligned}$$

The first integral here is small because f is continuous at 0, and the second one because G_δ is an approximate identity.

Given $\epsilon > 0$ take $\eta > 0$ so that $|f(u) - f(0)| < \epsilon/2$ for $|u| \leq \eta$. Then

$$\int_{|u| \leq \eta} |f(u) - f(0)|\, G_\delta(u)\, du \leq \frac{\epsilon}{2} \int_{|u| \leq \eta} G_\delta(u)\, du < \frac{\epsilon}{2}.$$

As shown in Example2 2, $\{G_\delta\}_\delta$ is an approximate identity, hence it satisfies (2.2). Since also $|f(t)| \leq \|\hat{f}\|_1$ a.e. $t \in \mathbb{R}$, given η there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$

$$\int_{|u| > \eta} |f(u) - f(0)|\, G_\delta(u)\, du \leq \|\hat{f}\|_1 \int_{|u| > \eta} G_\delta(u)\, du < \epsilon/2.$$

This finishes the proof. □

Corollary 1 (Uniqueness theorem)**.** If $f \in L^1(\mathbb{R})$ is such that $\hat{f} = 0$ a.e. $\xi \in \mathbb{R}$ then $f = 0$ a. e. $t \in \mathbb{R}$.

Final remark. Given $f \in L^1(\mathbb{R})$ the operator

$$\check{f}(\xi) = \int_{\mathbb{R}} f(t) e^{2\pi i t\xi} dt = \hat{f}(-\xi)$$

is sometimes called the *Fourier co-transform*. For $f \in L^1(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$ we have just proved that $\check{\hat{f}}(t) = f(t)$ a.e $t \in \mathbb{R}$.

2.3 Fourier transform in L^2

We would like to take advantage of the Hilbert structure of $L^2(\mathbb{R})$ in the Fourier analysis. An initial obstacle is that L^2 functions are not necessarily in L^1, so we have to be careful.

As usual, in L^2 we have the Hermitian product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\, \overline{g(t)}\, dt\,, \qquad f, g \in L^2(\mathbb{R}),$$

which gives the norm

$$\|f\|_2 = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2}.$$

In L^2 the rôles of f and $\hat{f}$ are equivalent, and this symmetry is often quite useful. This is clear in the following result.

Plancherel theorem. *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\hat{f} \in L^2(\mathbb{R})$ and $\|f\|_2 = \|\hat{f}\|_2$. In particular, if$\in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*

$$\int_{\mathbb{R}} f(t)\, \overline{g(t)}\, dt = \int_{\mathbb{R}} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi.$$

The requirement $f \in L^1(\mathbb{R})$ can be removed as long as we interpret $\hat{f}$ for $f \in L^2(\mathbb{R})$ in the appropriate way. We shall see this later.

Observe that the Fourier transform is thus an isometry on $L^2(\mathbb{R})$.

The first step in the proof is the good behaviour of convolution in L^2.

Lemma 7. *Let $f, g \in L^2(\mathbb{R})$. Then $f * g$ is a continuous bounded function such that*

$$\|f * g\|_\infty = \sup_{t \in \mathbb{R}} |(f * g)(t)| \le \|f\|_2 \|g\|_2.$$

Proof. Notice first that the convolution is well defined, since

$$|f(t-s)\,g(s)| \le \frac{1}{2}\big(|f(t-s)|^2 + |g(s)|^2\big)$$

and therefore $\int_{\mathbb{R}} f(t-s)\,g(s)\,ds$ is finite.

That $f * g$ is bounded is just a consequence of the Cauchy-Schwartz inequality:

$$\begin{aligned}
|(f*g)(t)| &= \left|\int_{\mathbb{R}} f(t-s)\,g(s)\,ds\right| \le \left(\int_{\mathbb{R}} |f(t-s)|\,ds\right)^{1/2} \left(\int_{\mathbb{R}} |g(s)|\,ds\right)^{1/2} \\
&= \left(\int_{\mathbb{R}} |f(u)|\,du\right)^{1/2} \left(\int_{\mathbb{R}} |g(s)|\,ds\right)^{1/2} = \|f\|_2 \|g\|_2.
\end{aligned}$$

In order to prove the continuity use again the Cauchy-Schwartz inequality:

$$\begin{aligned}
|(f*g)(t+h) - (f*g)(t)| &\le \int_{\mathbb{R}} |f(t+h-s) - f(t-s)|\,|g(s)|\,ds \\
&\le \left(\int_{\mathbb{R}} |f(t+h-s) - f(t-s)|^2\,ds\right)^{1/2} \|g\|_2 \\
&= \left(\int_{\mathbb{R}} |f(+h) - f(t)|^2\,ds\right)^{1/2} \|g\|_2 = \|\tau_h f - f\|_2 \|g\|_2.
\end{aligned}$$

By the same arguments as in the proof of Lemma 5 the factor $\|\tau_h f - f\|_2$ tends to 0 as $h \to 0$. □

Proof of Plancherel theorem. Define $\tilde{f}(t) = \overline{f(-t)}$, so that $\widehat{\tilde{f}}(\xi) = \overline{\hat{f}(\xi)}$. Then, by Theorem 6

$$|\hat{f}(\xi)|^2 = \hat{f}(\xi)\overline{\hat{f}(\xi)} = (f * \tilde{f})^\wedge(\xi).$$

Defining $g = f * \tilde{f}$, which by Lemma 7 is a continuous function, we have thus

$$\|\hat{f}\|_2^2 = \int_{\mathbb{R}} \hat{g}(\xi)\,d\xi.$$

Also

$$g(0) = (f * \tilde{f})(0) = \int_{\mathbb{R}} f(0-s)\overline{f(-s)}\,ds = \int_{\mathbb{R}} |f(u)|^2 du = \|f\|_2^2,$$

hence we shall be done as soon as we prove that $\hat{f} \in L^2(\mathbb{R})$ and

$$g(0) = \int_{\mathbb{R}} \hat{g}(\xi)\,d\xi. \tag{2.4}$$

As in the proof of the inversion formula for $L^1(\mathbb{R})$ (Theorem 7), we prove this by convolution with G_δ, being $G(t) = e^{-\pi t^2}$. Letting $F_\delta(t) = e^{-\pi\delta^2 t^2}$, by Remark 9 and the multiplication formula

$$
\begin{aligned}
(g * G_\delta)(0) &= \int_{\mathbb{R}} g(s)\, G_\delta(0-s)\, ds = \int_{\mathbb{R}} g(s)\, \hat{F}_\delta(s)\, ds \int_{\mathbb{R}} \hat{g}(\xi)\, F_\delta(\xi)\, d\xi \\
&= \int_{\mathbb{R}} \hat{g}(\xi) e^{-\pi\delta^2\xi^2} d\xi
\end{aligned}
$$

Since g is continuous and $\{G_\delta\}_\delta$ is an approximate identity we obtain that $\lim_{\delta\to 0}(g*G_\delta)(0) = g(0)$. This gives the left hand side of (2.4).

On the other hand for any $R > 0$ there exists $\delta > 0$ small enough so that $e^{-\pi\delta^2 R^2} \geq 1/2$ and therefore

$$
\int_{-R}^{R} |\hat{f}(\xi)|^2 d\xi \leq \frac{1}{2}\int_{-R}^{R} \hat{g}(\xi) e^{-\pi\delta^2\xi^2} d\xi \leq \frac{1}{2}\int_{\mathbb{R}} \hat{g}(\xi) e^{-\pi\delta^2\xi^2} d\xi \leq g(0).
$$

This shows that $\hat{f} \in L^2(\mathbb{R})$ and finally, by the Dominated Convergence theorem,

$$
\lim_{\delta\to 0} \int_{\mathbb{R}} \hat{g}(\xi) e^{-\pi\delta^2\xi^2} d\xi = \int_{\mathbb{R}} \hat{g}(\xi) d\xi.
$$

This completes the proof of (2.4). □

2.3.1 Getting rid of the assumption $f \in L^1(\mathbb{R})$

Given $f \in L^2(\mathbb{R})$ it is easy to find $f_n \in (L^1 \cap L^2)(\mathbb{R})$ such that $\lim_n \|f_n - f\|_2 = 0$; for instance $f_n = f\chi_{[-n,n]}$. Then, by Plancherel, $\{\hat{f}_n\}_n$ is a Cauchy sequence in $L^2(\mathbb{R})$:

$$
\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2 \overset{n\to\infty}{\longrightarrow} 0.
$$

Thus one can define the Fourier transform of f as

$$
\hat{f}(\xi) = \lim_{n\to\infty} \hat{f}_n(\xi),
$$

with convergence in the L^2-sense.

This definition does not depend on the particular sequence $\{f_n\}_n$ that converges to f: if $\{g_n\}_n \subset L^1 \cap L^2$ is any other such sequence then, again by Plancherel

$$
\|\hat{f}_n - \hat{g}_n\|_2 = \|f_n - g_n\|_2 \leq \|f_n - f\|_2 + \|f - g_n\|_2 \overset{n\to\infty}{\longrightarrow} 0
$$

We can summarise all this in the following statement.

Theorem 8. *For $f \in L^2(\mathbb{R})$*

(a) $\hat{f}(\xi) = \lim_{n\to\infty} \int_{-n}^{n} f(t)\, e^{-2\pi i t\xi} dt,$

(b) *Plancherel identity:* $\|f\|_2 = \|\hat{f}\|_2$.

(c) *If also* $g \in L^2(\mathbb{R})$, *then*

$$\int_{\mathbb{R}} f(t)\,\hat{g}(t)\,dt = \int_{\mathbb{R}} \hat{f}(t)\,g(t)\,dt$$

and

$$\int_{\mathbb{R}} f(t)\,\overline{g(t)}\,dt = \int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,dt.$$

With this interpretation the inversion formula also holds in L^2. To see this we need the following property of the convolution.

Lemma 8. *Let* $f \in L^2(\mathbb{R})$ *and* $g \in L^1(\mathbb{R})$. *Then* $f * g \in L^2(\mathbb{R})$, $\|f * g\|_2 \leq \|f\|_2\,\|g\|_1$ *and*

$$(f * g)^\wedge(\xi) = \hat{f}(\xi)\,\hat{g}(\xi).$$

Proof. It is clear that $f * g \in L^2(\mathbb{R})$; by the Cauchy-Schwartz inequality

$$\begin{aligned}
\int_{\mathbb{R}} |(f * g)(t)|^2\,dt &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t-s)\,g(s)\,ds \right|^2 dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t-s)|\,|g(s)|^{1/2}\,|g(s)|^{1/2}\,ds \right)^2 dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t-s)|^2\,|g(s)|\,ds \right) \|g\|_1\,dt \\
&= \|g\|_1 \int_{\mathbb{R}} |g(s)| \left(\int_{\mathbb{R}} |f(t-s)|^2 dt \right) ds = \|g\|_1^2 \|f\|_2^2.
\end{aligned}$$

Let now $f_n \in L^1 \cap L^2$ be such that $\|f_n - f\|_2 \overset{n\to\infty}{\longrightarrow} 0$. Then also $f_n * g \in L^1 \cap L^2$ (by Lemma 6) and

$$\|(f_n * g) - (f * g)\|_2 = \|(f_n - f) * g\|_2 \leq \|f_n - f\|_2 \|g\|_1 \overset{n\to\infty}{\longrightarrow} 0.$$

Since $(f_n * g)^\wedge(\xi) = \hat{f}_n(\xi)\,\hat{g}(\xi)$, by Theorem 6, we only need to see that $\hat{f}_n(\xi)\,\hat{g}(\xi)$ converges to $\hat{f}(\xi)\,\hat{g}(\xi)$ in L^2. But this is clear because $\hat{g}$ is bounded (Theorem 5 (a)):

$$\|\hat{f}_n\,\hat{g} - \hat{f}\,\hat{g}\|_2 \leq \|\hat{f}_n - \hat{f}\|_2 \|\hat{g}\|_\infty = \|\hat{f}_n - \hat{f}\|_2 \|g\|_1 \overset{n\to\infty}{\longrightarrow} 0.$$

□

Theorem. *Let* $f \in L^2(\mathbb{R})$. *Then* $\check{\hat{f}}(t) = f(t)$ *in* $L^2(\mathbb{R})$, *and therefore a.e.* $t \in L^2(\mathbb{R})$. *In particular*

$$f(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi t} d\xi = \lim_{n\to\infty} \int_{-n}^{n} \hat{f}(\xi) e^{2\pi i \xi t} d\xi,$$

as a limit in L^2.

Proof. Consider the dilations $\{G_\delta\}_\delta$ of the Gaussian and observe first that, as in Proposition 4 for the L^1 case, $\lim\limits_{\delta\to 0} f*G_\delta = f$ in $L^2(\mathbb{R})$; by the Cauchy-Schwartz inequality and the Dominated Convergence theorem

$$\begin{aligned}
\|f * G_\delta - f\|_2^2 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(t-s) - f(t))\, G_\delta(s)\, ds \right|^2 dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t-s) - f(t)|^2\, G_\delta(s)\, ds \right) \|G_\delta\|_1 dt \\
&= \int_{\mathbb{R}} G(u) \left(\int_{\mathbb{R}} |f(t-\delta u) - f(t)|^2 dt \right) du \\
&= \int_{\mathbb{R}} G(u)\, \|\tau_{\delta u} f - f\|_2^2\, du \overset{\delta\to 0}{\longrightarrow} 0.
\end{aligned}$$

Also, $\lim\limits_{\delta\to 0}(f * G_\delta)^\wedge = \hat{f}$ in $L^2(\mathbb{R})$; since

$$(f * G_\delta)^\wedge(\xi) = \hat{f}(\xi)\, \hat{G}_\delta(\xi) = \hat{f}(\xi)\, F_\delta(\xi),$$

where $F_\delta(\xi) = e^{-\pi\delta^2\xi^2}$, we have:

$$\|(f * G_\delta)^\wedge - \hat{f}\|_2^2 = \int_{\mathbb{R}} \left|\hat{f}(\xi)e^{-\pi\delta^2\xi^2} - \hat{f}(\xi)\right|^2 d\xi = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left|e^{-\pi\delta^2\xi^2} - 1\right|^2 d\xi.$$

This tends to 0, again by the dominated convergence theorem.

In particular, $(f * G_\delta)^\wedge = \hat{f}\, F_\delta$ is a family in L^2 tending to $\hat{f}$, and therefore, by Plancherel,

$$\check{\hat{f}} = \lim_{\delta\to 0}[(f * G_\delta)^\delta]^\vee. \tag{2.5}$$

If $f \in L^1 \cap L^2$, both $f * G_\delta$ and $(f * G_\delta)^\wedge$ are in L^1, because, by the previous Lemma 8

$$\|(f * G_\delta)^\wedge\|_1 = \int_{\mathbb{R}} |\hat{f}(\xi)|\, |F_\delta(\xi)|\, d\xi \leq \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2\, d\xi \right)^{1/2} \left(\int_{\mathbb{R}} |F_\delta(\xi)|^2\, d\xi \right)^{1/2}.$$

Then the inversion formula in L^1 yields $[(f * G_\delta)^\wedge]^\vee = f * G_\delta$ and by (2.5) we have

$$\check{\hat{f}}(t) = \lim_{\delta\to 0}[(f * G_\delta)^\delta]^\vee(t) = \lim_{\delta\to 0}(f * G_\delta)(t) = f(t).$$

If we assume only $f \in L^2(\mathbb{R})$ we take $f * G_\delta \in L^1 \cap L^2$ and take the identity just proved

$$[(f * G_\delta)^\wedge]^\vee = f * G_\delta$$

to the limit as $\delta \to 0$ in (2.5). □

2.4 Two applications of Fourier analysis

Fourier analysis was born in the estudy of the heat equation, so one could say, at least from a historical perspective, that differential equations are its more important applications. Here we illustrate the power of Fourier analysis in two different famous results.

2.4.1 Heisenberg's uncertainty principle

We have already noticed that time and frequency cannot be localised simultaneously. Here we have a precise statement that formalised this impossibility.

Heisenberg uncertainty. *Let $f \in L^2(R)$ and let $a, b \in \mathbb{R}$. Then*

$$\left(\int_{\mathbb{R}} (t-a)|f(t)|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} (\xi-b)|\hat{f}(\xi)|^2 d\xi\right)^{1/2} \geq \frac{\|f\|_2^2}{4\pi}.$$

Moreover, the equality if and only if f is a Gaussian of the form $f(t) = ce^{ibt}e^{-\gamma(t-a)^2}$, for some $c \in \mathbb{C}$ and $\gamma > 0$.

In Quantum Mechanics $f(t)$ is the wave function of a particle and the condition $f \in L^2(R)$ expresses that it has finite energy. The *position operator*

$$Pf(t) = t\,f(t)$$

indicates the density of probability of finding the particle at position t.

The *momentum operator* is

$$Qf = \frac{1}{2\pi i} f'\,.$$

In this language, by Plancherel,

$$\int_{\mathbb{R}} |Pf(t)|^2 dt = \int_{\mathbb{R}} t^2\,|f(t)|^2 dt$$

and

$$\int_{\mathbb{R}} |Qf(\xi)|^2 d\xi = \int_{\mathbb{R}} \left|\frac{1}{2\pi i} f'(t)\right|^2 dt = \int_{\mathbb{R}} \left|\frac{1}{2\pi i} \widehat{f'}(\xi)\right|^2 d\xi = \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi.$$

In these terms the statement above shows that there is a limit to localising both position and momentum, and that the best compromise is obteined with the eigenvalues of the *annihilation operator* $P+iQ$ (the so-called coherent states).

Proof. By the basic identities on translations and modulations for the Fourier transform, there is no restriction in assuming that $a = b = 0$. Assume also that $tf(t), \xi\hat{f}(\xi) \in L^2(\mathbb{R})$, otherwise the inequality has no content. Notice that this implies that $f, \hat{f} \in L^1(\mathbb{R})$, since by the Cauch-Schwartz inequality

$$\int_{\mathbb{R}} |f(t)|\, dt = \int_{\mathbb{R}} (1+|t|)|f(t)| \frac{dt}{1+|t|} \le \left(\int_{\mathbb{R}} (1+|t|)^2 |f(t)|^2\, dt\right)^{1/2} \left(\int_{\mathbb{R}} \frac{dt}{(1+|t|)^2}\right)^{1/2}.$$

In particular, by the Riemann-Lebesgue applied to $\hat{f}$, we deduce that f is continuous and

$$\lim_{|t|\to 0} f(t) = 0.$$

Since $\widehat{f'}(\xi) = (2\pi i\xi)\, \hat{f}(\xi) \in L^2(\mathbb{R})$ we can apply Plancherel to deduce that $f' \in L^2(\mathbb{R})$. Since

$$\big(|f|^2\big)' = (f \cdot \bar{f})' = 2\,\mathrm{Re}(f \cdot f'),$$

given any $c < d$ we have

$$2\,\mathrm{Re}\big(\int_c^d t\, f(t)\overline{f'(t)}\, dt\big) = \big(\int_c^d t\, 2\,\mathrm{Re}\, f(t)\overline{f'(t)}\, dt\big) = \big[t|f(t)|^2\big]_c^d - \int_c^d |f(t)|^2 dt.$$

Since $f, tf, f' \in L^2(\mathbb{R})$, there exist sequences $\{c_n\}_n \searrow -\infty$ and $\{d_n\}_n \nearrow +\infty$ such that

$$\lim_{n\to\infty} d_n |f(d_n)|^2 = \lim_{n\to\infty} d_n |f(d_n)|^2 = 0.$$

Thus, using that $f'(t) = [(2\pi i\xi)\hat{f}]^\wedge(t)$ we get

$$\int_{\mathbb{R}} |f(t)|^2\, dt = -2\,\mathrm{Re} \int_{\mathbb{R}} t f(t) \overline{[(2\pi i\xi)\hat{f}]^\wedge(t)} dt = 4\pi\, \mathrm{Im} \int_{\mathbb{R}} t f(t) \overline{(\xi\hat{f})^\wedge(t)} dt.$$

Squaring and applying consecutively the Cauchy-Schwartz inequality and Plancherel's identity we finally get

$$\begin{aligned} \|f\|_2^4 &\le 16\pi^2 \left(\int_{\mathbb{R}} t^2 |f(t)|^2 dt\right) \left(\int_{\mathbb{R}} |(\xi\hat{f})^\wedge(t)|^2 dt\right) \\ &= 16\pi^2 \left(\int_{\mathbb{R}} t^2 |f(t)|^2 dt\right) \left(\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi\right). \end{aligned}$$

The identity holds only when $tf(t) = \gamma f'(t)$ for some $\gamma \in \mathbb{R}$, that is, when $f(t) = Ce^{\gamma t^2}$ for some C. The condition $\gamma < 0$ is necessary so that $f \in L^2(\mathbb{R})$. □

2.4.2 The Kotelnikov-Shannon sampling theorem

This is a fundamental result in digital signal processing, establishing a sufficient condition for a sample rate to recover completely a continuous time signal of finite band-width.

Assume that $f(t)$ is a continuous signal (a sound, for example) of finite energy, that is, with $f \in L^2(\mathbb{R})$. Assume that f has a finite band-width, that is, that the signal has a finite range of frequencies: there exists $\tau > 0$ so that $\mathrm{supp}\, \hat{f} \subset [-\tau, \tau]$. This is a natural assumption for at least two reasons: 1) the range of frequencies perceived by the human ear is limited (between 20 Hz and 20.000 Hz); 2) transporting media attenuate extreme frequencies.

The Kotelnikov-Shannon-Whittaker theorem. *Let $f \in L^2(\mathbb{R})$ with supp $\hat{f} \subset [-\tau, \tau]$. Then f can be completely recovered from its samples $\{f(k/2\tau)\}_{k\in\mathbb{Z}}$ through the cardinal series*

$$f(t) = \sum_{k\in Z} f(\frac{k}{2\tau}) \ \mathrm{sinc}[2\tau(t - \frac{k}{2\tau})] \tag{2.6}$$

Moreover

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(t)|^2 dt = \frac{1}{2\tau} \sum_{k\in Z} |f(\frac{k}{2\tau})|^2. \tag{2.7}$$

In this statement $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

Remarks 1. (a) This statement is sometimes referred to as the "Fundamental theorem in information theory". It allows to encode a signal through a sequence of numbers (*digitalisation*) from which it can be completely recovered.

(b) The sampling rate $1/(2\tau)$ is called the *Nyquist rate*. Harry Nyquist was a communications engineer working first for AT&T and later for Bell Telephone Laboratories.

(c) The result was first proved in 1933 by Vladimir A. Kotelnikov, a pioneer in information theory and radar astronomy working at the Moscow Energy Institute. Independently it was proved also by Claude Shannon, an electrical engineer, and by Edmund Whittaker, (just) a mathematician.

Proof. By Plancherel's identity

$$\int_{\mathbb{R}} |f(\xi)|^2 d\xi = \int_{-\tau}^{\tau} |f(\xi)|^2 d\xi = \|f\|_2^2 < +\infty.$$

Since $\left\{e^{i\frac{\pi}{\tau}kt}\right\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-\tau, \tau]$ (see Remark 4) we can write

$$\hat{f}(\xi) = \sum_{k\in Z} c_k e^{i\frac{\pi}{\tau}k\xi},$$

where, by the inversion formula

$$c_k = \langle \hat{f}, e^{i\frac{\pi}{\tau}k\xi}\rangle = \frac{1}{2\tau}\int_{-\tau}^{\tau} \hat{f}(\xi)\, e^{i\frac{\pi}{\tau}k\xi} d\xi = \frac{1}{2\tau}\int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i\frac{k}{2\tau}\xi} d\xi = \frac{1}{2\tau} f\big(-\frac{k}{2\tau}\big).$$

Then, by the inversion formula,

$$\begin{aligned} f(t) &= \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i\xi t} dt = \int_{-\tau}^{\tau} \hat{f}(\xi) e^{2\pi i\xi t} dt = \sum_{k\in\mathbb{Z}} c_k \int_{-\tau}^{\tau} e^{2\pi i\xi(\frac{k}{2\tau}+t)} d\xi \\ &= \sum_{k\in\mathbb{Z}} \frac{1}{2\tau} f\big(-\frac{k}{2\tau}\big) \int_{-\tau}^{\tau} e^{2\pi i\xi(\frac{k}{2\tau}+t)} d\xi. \end{aligned}$$

Since

$$\int_{-\tau}^{\tau} e^{2\pi i\xi(\frac{k}{2\tau}+t)}d\xi = \left[\frac{e^{2\pi i\xi(\frac{k}{2\tau}+t)}}{2\pi i(\frac{k}{2\tau}+t)}\right]_{\xi=-\tau}^{\xi=\tau} = \frac{e^{2\pi i\tau(\frac{k}{2\tau}+t)} - e^{-2\pi i\tau(\frac{k}{2\tau}+t)}}{2\pi i(\frac{k}{2\tau}+t)}$$
$$= \frac{\sin\big(2\pi\tau(\frac{k}{2\tau}+t)\big)}{\pi(\frac{k}{2\tau}+t)} = 2\tau \ \mathrm{sinc}\big(2\tau(\frac{k}{2\tau}+t)\big)$$

we get

$$f(t) = \sum_{k\in\mathbb{Z}} f\big(-\frac{k}{2\tau}\big)\,\mathrm{sinc}\big(2\tau(\frac{k}{2\tau}+t)\big),$$

which after replacing k by $-k$ gives (2.6).

In order to prove (2.7) observe that, by Plancherel's identity for Fourier series

$$\sum_{k\in\mathbb{Z}}|c_k|^2 = \frac{1}{(2\tau)^2}\sum_{k\in\mathbb{Z}}\big|f(\frac{k}{2\tau})\big|^2 = \|\hat{f}\|^2_{L^2[-\tau,\tau]} = \frac{1}{2\tau}\int_{-\tau}^{\tau}|\hat{f}(\xi)|^2 d\xi = \frac{1}{2\tau}\|\hat{f}\|_2^2.$$

Thus, by Plancherel (for $L^2(\mathbb{R})$)

$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \frac{1}{2\tau}\sum_{k\in\mathbb{Z}}\big|f(\frac{k}{2\tau})\big|^2,$$

as stated. □

Remark 10. The family

$$\big\{\sqrt{2\tau}\,\mathrm{sinc}\big(2\tau(t-\frac{k}{2\tau})\big)\big\}_{k\in\mathbb{Z}}$$

is an orthonormal system. To see this just notice that, by Proposition 3 and Example 1,

$$\big[\mathrm{sinc}\big(2\tau(t-\frac{k}{2\tau})\big)\big]^{\wedge}(\xi) = e^{\pi i\frac{k}{\tau}\xi}\big[\mathrm{sinc}(2\tau t)\big]^{\wedge}(\xi) = e^{\pi i\frac{k}{\tau}\xi}\frac{1}{2\tau}\chi_{[-\tau,\tau]}(\xi).$$

Let $g_k(t) = \sqrt{2\tau}\,\mathrm{sinc}\big(2\tau(t-\frac{k}{2\tau})\big)$; then, by Plancherel,

$$\langle g_k, g_m\rangle = (2\tau)\int_{\mathbb{R}} e^{\pi i\frac{k}{\tau}\xi}e^{-\pi i\frac{m}{\tau}\xi}\frac{1}{(2\tau)^2}\chi_{[-\tau,\tau]}(\xi)\,d\xi = \frac{1}{2\tau}\int_{-\tau}^{\tau} e^{\pi i(k-m)\frac{\xi}{\tau}}d\xi = \delta_{km}.$$

This shows, in particular, that for any sequence $\{a_k\}_{k\in\mathbb{Z}} \in \ell^2$ the series

$$f(t) := \sum_{k\in\mathbb{Z}} a_k\,\mathrm{sinc}\big(2\tau(t-\frac{k}{2\tau})\big)$$

defines a function $f \in L^2(\mathbb{R})$ with $\mathrm{supp}\hat{f} \subset [-\tau,\tau]$ such that $f\big(\frac{k}{2\tau}\big) = a_k$, $k\in\mathbb{Z}$.

Digression. Fourier transform and analytic functions.

For the sake of simplicity let us momentarily reverse the rôles of f and $\hat{f}$ (which, by Plancherel, are equivalent). Let $f \in L^2(\mathbb{R})$ be supported in $[-\tau, \tau]$ and consider

$$F(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t\xi} dt = \int_{-\tau}^{\tau} f(t) e^{-2\pi i t\xi} dt.$$

As we saw in Theorem 5 this defines an entire function of exponential type.

The reciprocal is also true: if $F(\xi) = \hat{f}(\xi)$ belongs to $L^2(\mathbb{R})$ and extends holomorphically to an entire function of exponential type then supp$\hat{f} \subset [-\tau, \tau]$. The proof goes along the same lines as the proof of Theorem 5: defining

$$f(t) \text{``} = \text{``} \int_{\mathbb{R}} F(\xi) e^{-2\pi i \xi t} d\xi,$$

applying the residue theorem to a rectangle with vertices $\pm R + i\epsilon$ and $\pm R + iR$ and letting $R \to \infty$, $\epsilon \to 0$.

Similarly one can prove that for $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp} \phi \subset [-A, A]$, $A > 0$, the Fourier transform $\hat{\phi}$ can be extended to an entire function such that for for all $m \in \mathbb{N}$ there exists $c_m > 0$ such that

$$|\hat{\phi}(\xi)| \leq c_m (1 + |\xi|)^{-m} e^{2\pi A |\operatorname{Im} \xi|}, \qquad \xi \in \mathbb{C}.$$

Note. Going back to the original situation (reversing the rôles of f and $\hat{f}$we see that when $f \in L^2(\mathbb{R})$ is band-limited, it can be extended to an entire function $f(z)$, $z \in \mathbb{C}$. In particular, f can only vanish on a discrete set with no accumulation points in $\mathbb{C}$. Thus the signal $f(t)$ has to be non-zero everywhere (except for maybe this sequence). This seems to contradict our intuition. Here we just copy Joseph Slepian's reflections: "it makes no sense to discuss whether real life functions are band-limited or time-limited, since this would mean to measure the signal in remote and future times with arbitrarily high precision". The *Paley-Wiener space*

$$PW_\tau = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\tau, \tau] \right\}$$

is just a mathematical model.

2.5 Annex. The dominated convergence theorem

Dominated convergence theorem. *Let $E \subset \mathbb{R}$ be measurable and let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions $f_n : E \longrightarrow \mathbb{C}$ for which the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists a.e. $x \in E$. If there exists $g \in L^1(E)$ such that for all n big enough*

$$|f_n(t)| \leq g(t) \qquad t \in E,$$

then

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(E)} = \lim_{n\to\infty} \int_E |f_n(t) - f(t)| \, dt = 0.$$

In particular,

$$\lim_{n\to\infty}\int_E f_n(t)\,dt = \int_E f(t)\,dt.$$

A consequence of this is the following differentiation theorem.

Theorem 9. *Let $I = (x_0 - r, x_0 + r) \subset \mathbb{R}$ be an interval and let $E \subset \mathbb{R}$ be measurable. Assume that $f : I \times E \longrightarrow \mathbb{C}$ is a function such that:*

(i) each $f(x,\cdot)$ is integrable in E,

(ii) $f(\cdot,t) \in \mathcal{C}^1(I)$ for all $t \in E$.

(ii) there exists $g \in L^1(E)$ such that

$$\left|\frac{\partial f}{\partial x} f(x,t)\right| \leq g(t), \qquad (x,t) \in I \times E.$$

Then the function on $F : I \longrightarrow \mathbb{C}$ defined by

$$F(x) = \int_E f(x,t)\,dt$$

is differentiable and

$$F'(x) = \int_E \frac{\partial f}{\partial x} f(x,t)\,dt.$$