TOPOLOGICAL DATA ANALYSIS

EXERCISES 2.5

1) Consider the functions $f, g : [-1,1] \to \mathbb{R}$ given by

$$f(x) = x^5 - x \qquad g(x) = \frac{1}{5}(x^9 + 7x^5 - 10x)$$

(a) Find the persistence modules $V(f)$ and $V(g)$ and the spectrum of each

(b) Compute the interleaving distance $d_{int}(V(f), V(g))$

(c) Check that $d_{int}(V(f), V(g)) < \|f-g\|_\infty$ on $[-1,1]$

(a) Note that both f and g are differentiable on $\mathbb{R}$

$f'(x) = 5x^4 - 1 = 0$

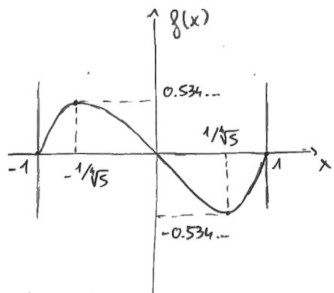
$\quad \hookrightarrow x = \pm \frac{1}{\sqrt[4]{5}}$

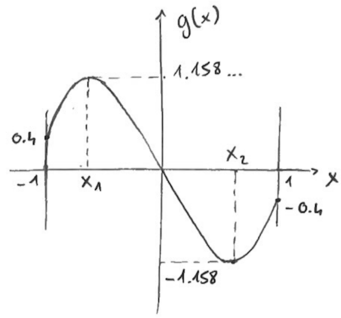
$g'(x) = \frac{1}{5}(9x^8 + 35x^4 - 10) = 0$

$\quad \hookrightarrow$ 2 real roots:

$\quad x_1 = -0.719...$

$\quad x_2 = 0.719...$





critical points of f: $\{-1, -\frac{1}{\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}}, 1\}$

critical points of g: $\{-1, -0.719..., 0.719..., 1\}$

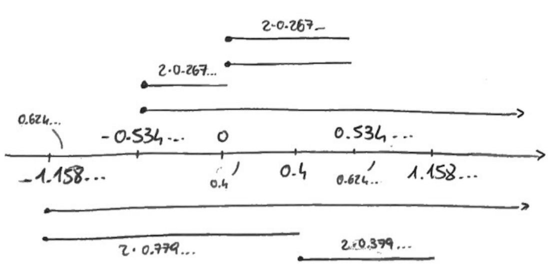
$$V_t(f) = H_0(L_t(f)) = \begin{cases} 0 & t < -0.534... \\ 1 & -0.534... \leq t < 0 \text{ and } t \geq 0.534... \\ 2 & 0 \leq t < 0.534... \end{cases} \qquad V_t(g) = H_0(L_t(g)) = \begin{cases} 0 & t < -1.158... \\ 1 & -1.158... \leq t < 0.4 \text{ and } t \geq 1.158... \\ 2 & 0.4 \leq t < 1.158 \end{cases}$$

Spectrum of $V(f) = \{-1, -1/\sqrt[4]{5}, 1/\sqrt[4]{5}\}$

Spectrum of $V(g) = \{-1, -0.719..., 0.719...\}$

(b) $V(f) = \mathbb{F}[-0.534..., 0) \oplus \mathbb{F}[0, 0.534...)^2 \oplus \mathbb{F}[-0.534..., \infty)$

$\quad V(g) = \mathbb{F}[-1.158..., 0.4) \oplus \mathbb{F}[0.4, 1.158...)^2 \oplus \mathbb{F}[-1.158..., \infty)$



$d_{int}(V(f), V(g)) = \inf\{\delta > 0 : V(f) \text{ and } V(g) \text{ are } \delta\text{-interleaved}\}$

$d_{int}(V(f), V(g)) = \min\{0.624, ..., 0.379...\} = 0.379...$

(c) $\|f-g\|_\infty = \sup\{|f(x) - g(x)| : -1 \leq x \leq 1\}$

$\quad \|f-g\|_\infty = 0.624...$

Stability theorem: $d_{int}(V(f), V(g)) \leq \|f-g\|_\infty$ on $[-1,1]$

In this case $d_{int}(V(f), V(g)) = 0.379... < 0.624... = \|f-g\|_\infty$

2) Consider the following point clouds in $\mathbb{R}^2$

$X = \{(0.81, 2.87), (2.15, 1.18), (3.19, 3.62), (4.17, 2.01), (5.32, 4.88), (6.21, 3.13)\}$

$Y = \{(0.75, 2.80), (2.33, 1.25), (3.28, 3.66), (4.15, 2.15), (5.24, 4.78), (6.34, 3.12)\}$

(a) Compute the Hausdorff distance $d_H(X,Y)$ and the Gromov-Hausdorff distance $d_{GH}(X,Y)$

(b) Compute the bottleneck distance $W_\infty(D(X), D(Y))$ between the Vietoris-Rips persistence diagrams of X and Y

(c) Check that $W_\infty(D(X), D(Y)) < 2 d_{GH}(X,Y)$

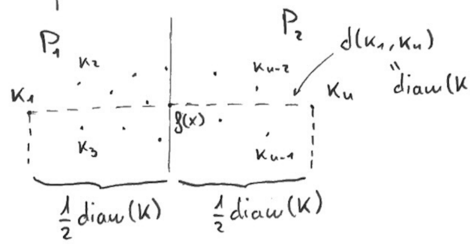
(a), (b) and (c) developed in the notebook submitted with this document.

3) Prove that the Gromov-Hausdorff distance between a single point and a non-empty compact subset K of a metric space is equal to half the diameter of K.

Proof

Suppose, without loss of generality, that $K = \{\underbrace{k_1}_{\text{"extreme"}}, \underbrace{k_2, \dots, k_{n-1}}_{\text{"interior points"}}, \underbrace{k_n}_{\text{"extreme"}}\}$

meaning that k_1 and k_n are the two points such that $d(k_1, k_n) \geq d(k_i, k_j) \ \forall i,j$.



For such a set we have

$$\operatorname{diam}(K) = \sup\{d(p,q) : p,q \in K\} = d(k_1, k_n)$$

Introduce now a single point x, with $X = \{x\}$

Now, in order to compute $d_{GH}(X,K)$, where

$$d_{GH}(X,K) = \inf\{d_H^M(f(X), g(K)) : f : X \hookrightarrow M,\ g : K \hookrightarrow M \text{ isometrically}\}$$

we consider $g : K \to \mathbb{R}^2$, $g(x,y) = (x,y)$

and $f : X \to \mathbb{R}^2$, $f(x,y)$ such that it minimizes $d_H^M(f(X), g(K))$

The set K can be seen as partitioned, by the axis orthogonal to the distance between k_1 and k_n through the center point, into two halfs P_1 and P_2

Suppose $f(x) \in P_1$, then $d(x, k_n) > \frac{1}{2}\operatorname{diam}(K) \longrightarrow d_H(f(X), g(K)) > \frac{1}{2}\operatorname{diam}(K)$

Suppose $f(x) \in P_2$, then $d(x, k_1) > \frac{1}{2}\operatorname{diam}(K) \longrightarrow d_H(f(X), g(K)) > \frac{1}{2}\operatorname{diam}(K)$

However, if $f(x)$ is coincides with the middle point of the distance $d(k_1, k_n)$, then we have $d_H(f(X), g(K)) = \frac{1}{2} d(k_1, k_n) = \frac{1}{2}\operatorname{diam}(K)$

Since this last case minimizes $d_H(f(X), g(K))$, then we have

$d_{GH}(X,K) = d_H(f(X), g(K)) = \frac{1}{2}\operatorname{diam}(K)$ with f such that $f(x)$ coincides with the middle point of the distance $d(k_1, k_n)$ ■