FOURIER MULTIPLIERS

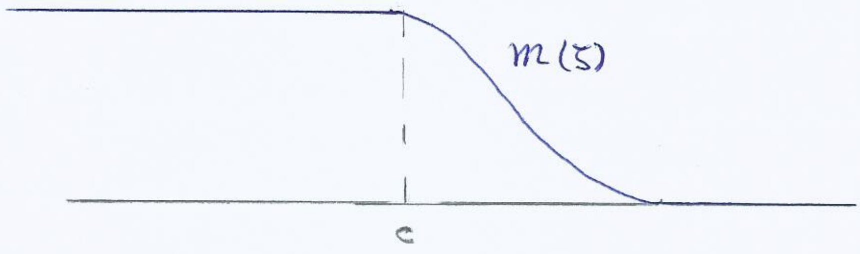
A particular class of operators, important in applications, is the so-called "Fourier multipliers". The main feature is that the Fourier transform is multiplied by a function $m(\xi)$, called a "filter" in signal processing. Thus, the general scheme when applying one of such operators T is as follows: given f

1. Take its Fourier transform $f \longmapsto \hat{f}$
2. Multiply it by m: $\hat{f} \longmapsto m\hat{f}$
3. Undo the Fourier transform: $T(f) = (m\hat{f})^{\vee}$.

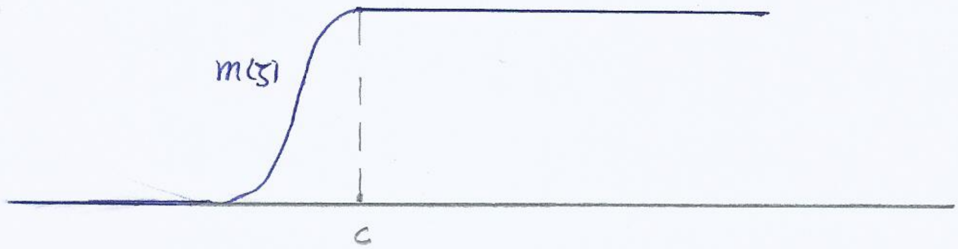
To perform the third step some regularity on m is required (usually $m \in L^{\infty}$, so that T has some boundedness in L^2, or L^p). The general formalism would be $T(f) = f * \check{m}$, where $\check{m}$ is a distribution. In general, we shall write instead $T(f) = f * \mu$, so that $\widehat{Tf} = \hat{f} \cdot \hat{\mu}$.

Examples:

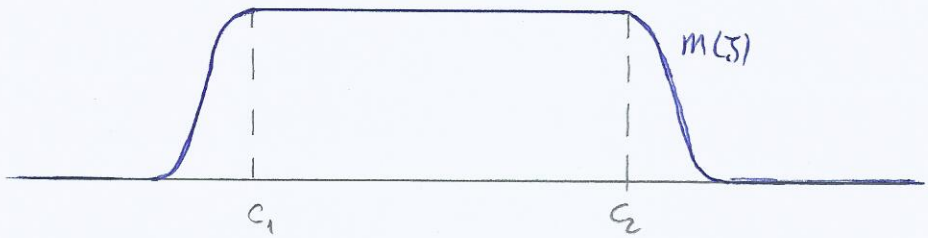
① Low pass filter. The function m is 1 for low frequencies and it attenuates (or kills) high frequencies. Given a threshold c, we will have:



② <u>High-pass filter</u>. Here we preserve high frequencies and attenuate low ones



③ <u>Band-pass filter</u>. Here we kill extreme frequencies, both low and high. If $c_1 < c_2$



The human ear perceives frequencies between 20 Hz and 20.000 Hz. Thus, when recording and reproducing a sound (music, etc.) nothing is lost if we restrict ourselves to this band (band-pass filter). Sometimes reproducing devices are restricted to either low or high frequencies (this is also the case for other applications, like astronomy). For example, a woofer reproduces frequencies between 100Hz and 500Hz, a subwoofer between 20 Hz and 100Hz, and a tweeter between 2000Hz and 20.000Hz.

These kind of filters were used also, for example, in telephone lines with DSL splitters. By separating low and high frequencies, the same wires carried:

- digital data (DSL: digital subscriber line)
- voice (POTS: plain old telephone service)

There are other Fourier multipliers:

④ Let $f: [0, 2\pi] \to \mathbb{R}$, extended to be 2π-periodic in the whole $\mathbb{R}$. The Fourier transform of is a sequence $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$. The multiplier is just another sequence $\{m_n\}_{n\in\mathbb{Z}}$, and the corresponding Fourier multiplier is

$$(Tf)(t) = \sum_{n\in\mathbb{Z}} m_n \hat{f}(n)\, e^{int}$$

⑤ In general, given $f \in L^1(\mathbb{R})$ and $m \in L^\infty(\mathbb{R})$ we have

$$(Tf)(t) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)\, e^{2\pi i x\xi}\, dt$$

Particular examples of this are ①, ②, ③ and

(a) <u>Derivation</u>. Since $\hat{f'}(\xi) = 2\pi i\xi\, \hat{f}(\xi)$, i.e $\hat{f'}(\xi) = m(\xi)\hat{f}(\xi)$, for $m(\xi) = 2\pi i\xi$, we have a multiplier of this kind. Observe though that $m \notin L^\infty(\mathbb{R})$, so extra conditions on f are necessary.

This can also be seen through the general formalism with distributions, since $f' = f * \delta_0'$: for a test function ψ we have $\langle \delta_0', \psi\rangle = -\psi'(0)$, so

$$\hat{\delta}_0'(\xi) := \langle \delta_0', e^{2\pi i t\xi}\rangle = 2\pi i\xi .$$

More generally, given a differential operator

$$P(D)(f) = a_0 + a_1 \frac{\partial f}{\partial x} + \dots + a_N \frac{\partial^N f}{\partial x^N}$$

we have

$$(\widehat{P(D)f})(\xi) = \sum_{j=0}^{N} a_j (2\pi i\xi)^j \hat{f}(\xi) = P(2\pi i\xi)\hat{f}(\xi).$$

This also works in several variables. For example, if

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \qquad \text{in } \mathbb{R}^n,$$

we get

$$\widehat{\Delta f}(\xi) = \sum_{j=1}^{n} (2\pi i\xi_j)^2 \hat{f}(\xi) = -4\pi^2 |\xi|^2 \hat{f}(\xi).$$

These identities suggest how to define fractional derivatives: given $\alpha > 0$ we define $\frac{\partial^\alpha f}{\partial x^\alpha}$ by establishing

$$\widehat{\frac{\partial^\alpha f}{\partial x^\alpha}}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi).$$

⑤ Translation Write $\tau_{x_0}(f) = f * \delta_{x_0}$. Then

$$\widehat{\tau_{x_0}(f)}(\xi) = \hat{f}(\xi) \cdot \hat{\delta}_{x_0}(\xi) = \hat{f}(\xi)\, e^{-2\pi i \xi x_0}.$$

⑥ Linear time-invariant filters

Time-invariant operators T appear in many applications. By this we mean T such that $T(\tau_a(f)) = \tau_a(T(f))$; a is called the "delay".

Writing
$$f(t) = \int_{\mathbb{R}} f(u)\, \delta_u(t)\, du,$$

by the linearity and continuity of T we have

$$Tf(t) = \int_{\mathbb{R}} f(u)\, T\delta_u(t)\, du, \quad T\delta_u = T\tau_u\delta_0 = \tau_u T\delta_0.$$

Thus, letting $H(t) := T\delta_0(t)$ be the "impulse response of T", and by the time invariance of T:

$$(Tf)(t) = \int_{\mathbb{R}} f(u)\, H(t-u)\, dt = \int_{\mathbb{R}} H(u) f(t-u)\, du = (H * f)(t).$$

So, Tf is the Fourier multiplier operator given by H.

Observe that exponentials $e_\omega(t) = e^{2\pi i \omega t}$ are eigenfunctions of these time-invariant operators:

$$T(e_\omega)(t) = (H * e_\omega)(t) = \int_{\mathbb{R}} H(x)\, e_\omega(t-x)\, dx$$

$$= \int_{\mathbb{R}} H(x)\, e^{2\pi i \omega (t-x)}\, dx = e^{2\pi i t \omega}\, \hat{H}(\omega)$$

$$= \hat{H}(\omega)\, e_\omega(t)$$

This is one of the reasons that explain the ubiquity of Fourier analysis in operator theory.