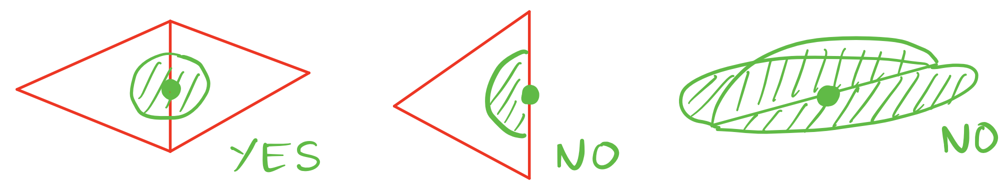
Topological Data Analysis

2022–2023

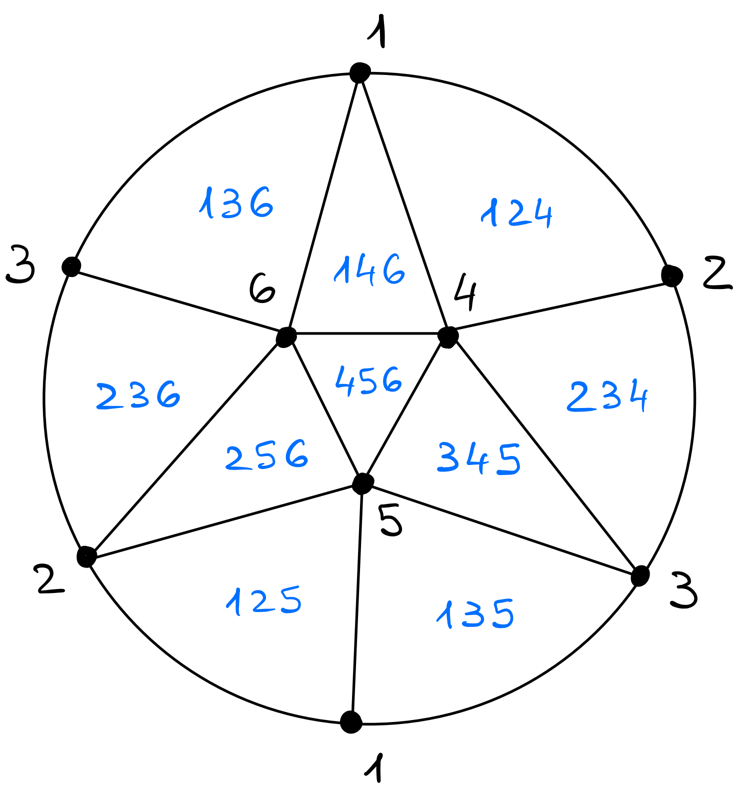
Solutions of Exercises

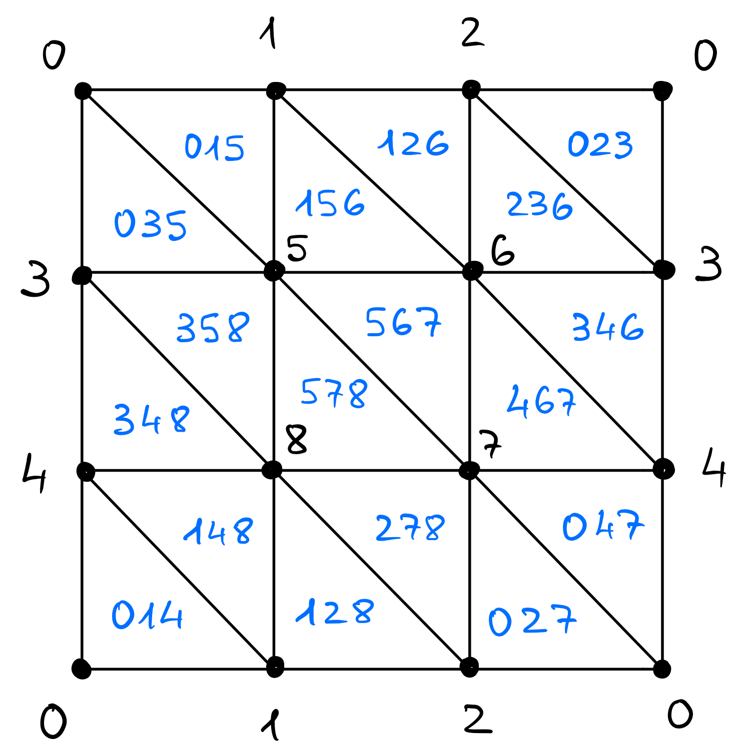
10 November 2022

(1) The underlying topological space of a geometric simplicial complex is a surface if and only if its maximal faces are 2-faces, and every edge is adjacent to two and only two 2-faces. The geometric realizations of K and L have this property. If some edge appears only once, then its points do not have any neighbourhood which is homeomorphic to an open disk, and the same happens if some edge appears more than twice.



Visualization of $|K|$ and $|L|$:





Hence $|K|$ is homeomorphic to a real projective plane $\mathbb{R}P^2$ and $|L|$ is homeomorphic to a torus $S^1 \times S^1$.

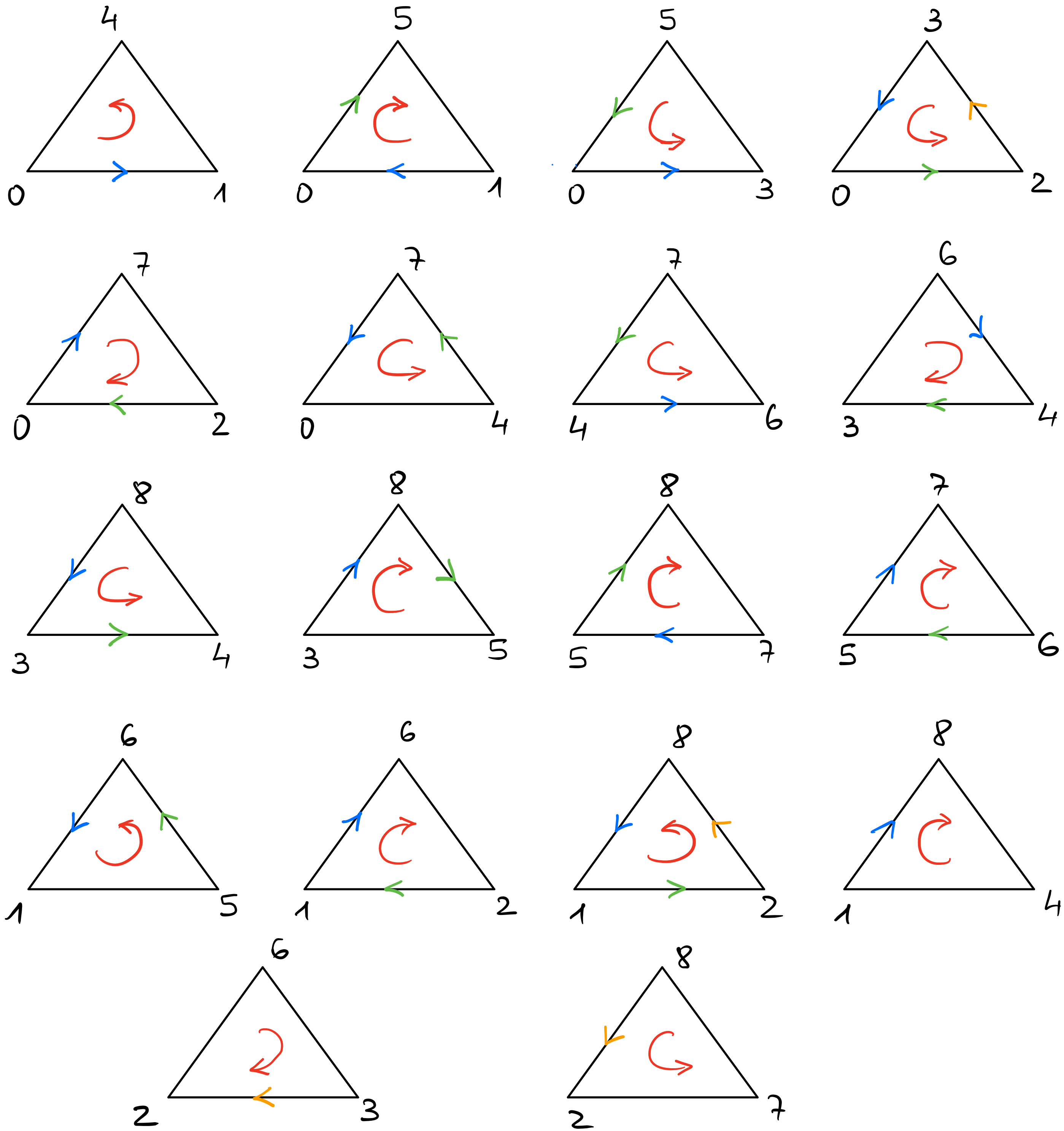
Compactness follows from the fact that $|K|$ and $|L|$ are quotients of finite disjoint unions of compact spaces, namely closed triangles.

One way of guessing which surfaces are $|K|$ and $|L|$ without drawing pictures is to compute their Euler characteristics:

$$\chi(K) = 6 - 15 + 10 = 1,$$

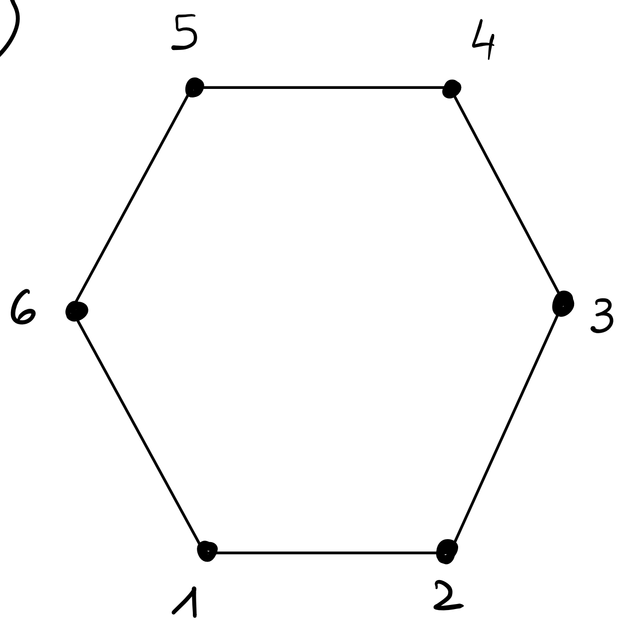
$$\chi(L) = 9 - 27 + 18 = 0.$$

This implies that $|K| \cong \mathbb{R}P^2$ but leaves undecided whether $|L|$ is a torus or a Klein bottle. The fact that $|L|$ is orientable and hence a torus can be deduced by finding compatible local orientations on the 2-faces, as follows:



4 0 1
5 0 1
5 0 3
3 0 2
7 0 2
7 0 4
7 4 6
6 3 4
8 3 4
8 3 5
8 5 7
7 5 6
6 1 5
6 1 2
8 1 2
8 1 4
6 2 3
8 2 7

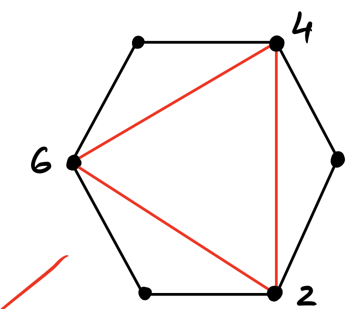
2



The Rips complex is determined by the distance matrix:

	1	2	3	4	5	6
1	0	1	$\sqrt{3}$	2	$\sqrt{3}$	1
2	1	0	1	$\sqrt{3}$	2	$\sqrt{3}$
3	$\sqrt{3}$	1	0	1	$\sqrt{3}$	2
4	2	$\sqrt{3}$	1	0	1	$\sqrt{3}$
5	$\sqrt{3}$	2	$\sqrt{3}$	1	0	1
6	1	$\sqrt{3}$	2	$\sqrt{3}$	1	0

R_ε



$0 \leq \varepsilon < 1$: (1) (2) (3) (4) (5) (6)

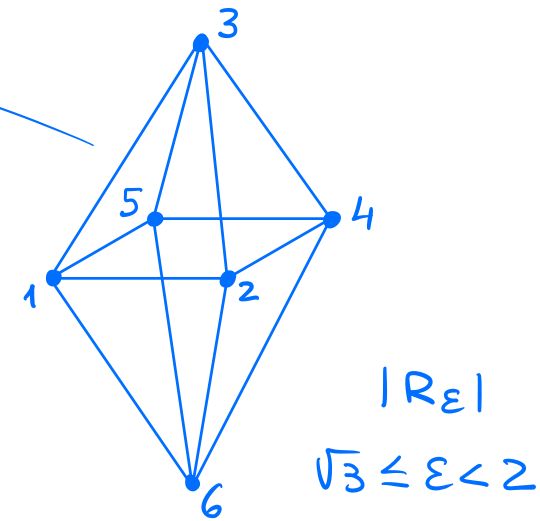
$1 \leq \varepsilon < \sqrt{3}$: (12) (16) (23) (34) (45) (56)

$\sqrt{3} \leq \varepsilon < 2$: (123) (126) (135) (156) (234) (246) (345) (456)

$\varepsilon \geq 2$: (123456)

$\operatorname{diam}\{2,4,6\} = \sqrt{3}$

Note that $|R_\varepsilon| \cong S^1$ for $1 \leq \varepsilon < \sqrt{3}$, while $|R_\varepsilon| \cong S^2$ for $\sqrt{3} \leq \varepsilon < 2$, since $|R_\varepsilon|$ is an octahedron in this range of ε values. Of course $|R_\varepsilon| = \Delta^5$ for $\varepsilon \geq 2$.



Next we compute the Čech complex:

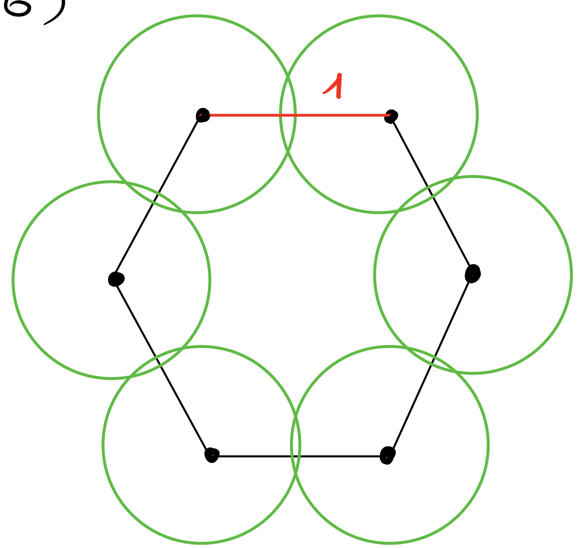
C_ε

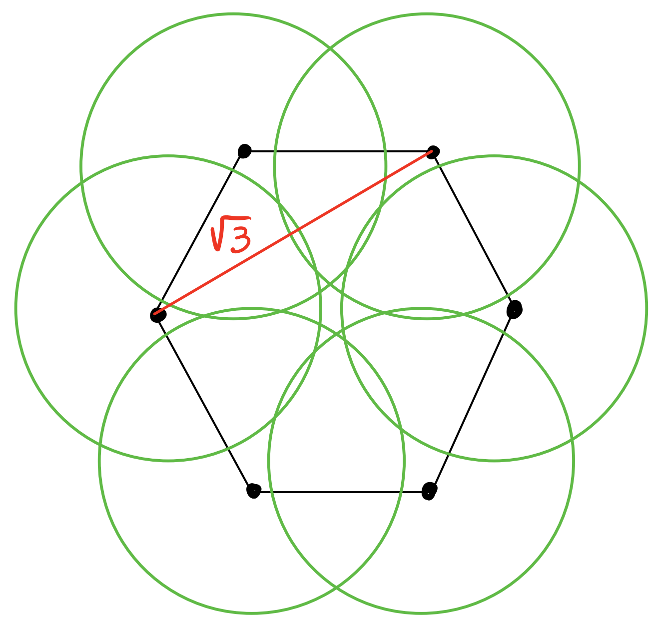
$0 \leq \varepsilon < 1$: (1)(2)(3)(4)(5)(6)

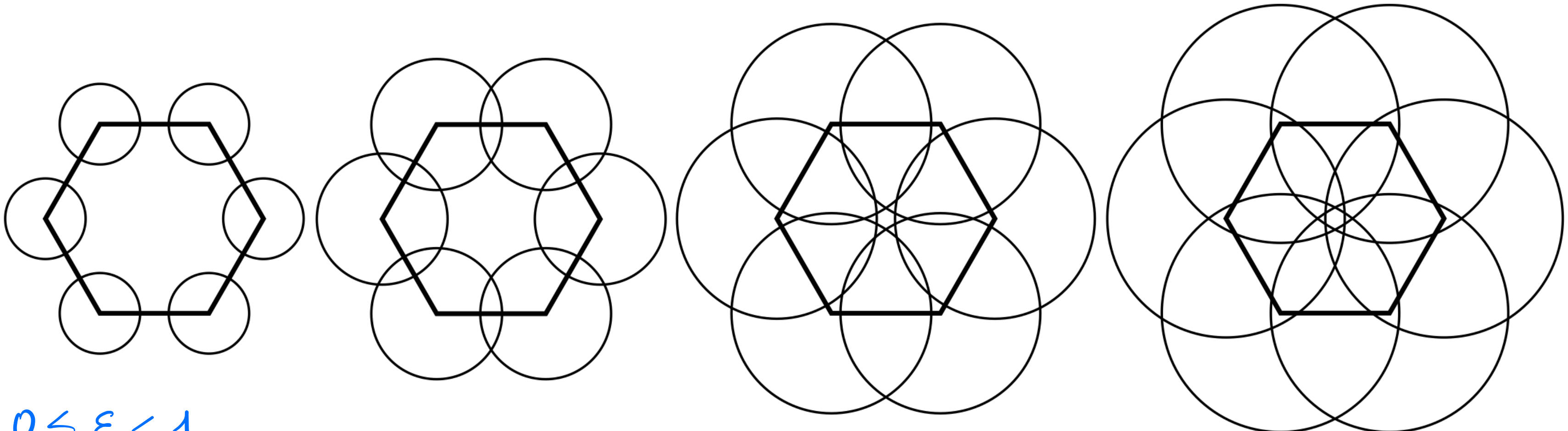
$1 \leq \varepsilon < \sqrt{3}$: (12)(16)(23)(34)(45)(56)

$\sqrt{3} \leq \varepsilon < 2$: (123)(126)(156)(234)(345)(456)

$\varepsilon \geq 2$: (123456)



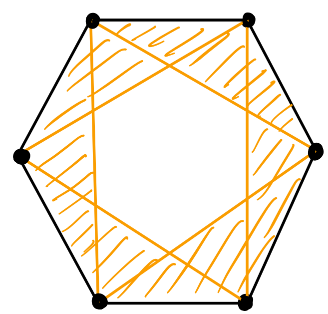




$0 \le \varepsilon < 1$ $\quad$ $1 \le \varepsilon < \sqrt{3}$ $\quad$ $\sqrt{3} \le \varepsilon < 2$ $\quad$ $\varepsilon \geq 2$

Here $|C_\varepsilon| \cong S^1$ for $1 \le \varepsilon < \sqrt{3}$, and $|C_\varepsilon| \simeq S^1$ for $\sqrt{3} \le \varepsilon < 2$, and $|C_\varepsilon| = \Delta^5$ for $\varepsilon \geq 2$.

homotopy equivalence



$|C_\varepsilon|$

$\sqrt{3} \le \varepsilon < 2$