Lesson 8

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Continuous-time models for stock markets

The evolutions of the stocks and claims (shares, commodities, options...) will be stochastic processes $(S_t)_{t\geq 0}$ defined in a filter probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Where $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is a filtration and T the horizon of the market.

We shall assume, $\mathcal{F}_0 = \{\emptyset, \Omega, \mathcal{N}\}$, where $\mathcal{N}$ is the collection of the-null sets.

In this sense if we have two versions of a process X and Y, that is $\mathbb{P}(X_t = Y_t) = 1, 0 \leq t \leq T$, and X is $\mathbb{F}$-adapted then Y is also $\mathbb{F}$-adapted.

Let $\phi_t = \left(\phi_t^0, , ...\phi_t^d\right)$ be an adapted processes indicating the number of units invested in the stocks $S = \left(S^0, ..., S^d\right)$ at t, then the portfolio value at t is

$$V_t\left(\phi\right) = \phi_t \cdot S_t$$

and at time $t + \Delta t$, if we keep the investment,

$$V_{t+\Delta t}\left(\phi\right) = \phi_t \cdot S_{t+\Delta t}$$

so,

$$\Delta V_t\left(\phi\right) = \phi_t \cdot \Delta S_t.$$

To freeze ϕ over the period $[t, t + \Delta t)$ is equivalent to the predictability condition. Then we will have, starting at zero, and if the strategy is self-financing $\left(\phi_t \cdot S_{t+\Delta t} = \phi_{t+\Delta t} \cdot S_{t+\Delta t}\right)$, that

$$V_t(\phi) = V_0(\phi) + \sum_{s \in \mathcal{T}} \phi_s \Delta S_s$$

where $\mathcal{T} = \{0, \Delta t, 2\Delta t, .., t\}$. If we trade in a continuous form we will have, passing to the limit when $\Delta t \to 0$, that

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_s \cdot \mathrm{d}S_s.$$

Assume that S^0 is the money account that evolves as:

$$\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t, \quad 0 \leq t \leq T, \quad S_0^0 = 1$$

where r is a non-negative constant, then

$$\left(\int_0^t \phi_t^0 \mathrm{d}S_t^0\right)(\omega) = \int_0^t \phi_t^0(\omega) rS_t^0 \mathrm{d}t$$

so provided that $\int_0^T |\phi_t^0|\mathrm{d}t < \infty$ a.s. $\mathbb{P}$, this integral (as a Lebesgue one) is well defined.

As for the risky assets we need to know how to calculate limits of the form

$$\lim_{n\to\infty}\sum_{i=1}^{n}\phi^j_{t_{i-1,n}}(S^j_{t_{in}}-S^j_{t_{i-1,n}})$$

where $0=t_{0n}<t_{1n}<\ldots<t_{m(n)n}=T$ is a sequence of partitions of $[0,T]$ whose mesh goes to zero. Then

$$V_t=V_0+\int_0^t\phi^0_t\mathrm{d}S^0_t+\sum_{j=1}^{d}\lim_{n\to\infty}\sum_{i=1}^{n}\phi^j_{t_{i-1,n}}(S^j_{t_{in}}-S^t_{t_{i-1,n}}),$$

We shall consider processes S for which, roughly speaking, $\Delta S^j_t\sim h\Delta W^j_t$ where W^j are Brownian motions and h an adapted process, so we have to construct integrals

$$\int_0^t\varphi_s dW_s$$

where $(W_t)_{0\le t\le T}$ is a Brownian motion and $(\varphi_t)_{0\le t\le T}$ is an adapted process. At first glance we can think in a definition ω to ω (path-wise) but though $W_s(\omega)$ is continuous in s, it is not a function with bounded variation and we cannot associate a measure with the increments along the path to see the above limits as Lebesgue-Stieltjes integrals.

Brownian motion

Definition

A (standard) Brownian motion is a stochastic process, say X, that satisfies the following properties:

$s \longmapsto X_s(\omega)$ is continuous $\mathbb{P}$-a.s
$X_0 = 0$ a.s.
$X_t - X_s$ is independent of $\mathcal{F}_s = \sigma(X_u, 0 \leq u \leq s)$ for all $s \leq t$.
$X_t - X_s \sim N(0, t-s)$ for all $0 \leq s < t$.

Proposition

The trajectories of a Brownian motion has not bounded variation with probability one.

Proof.

Given the partition $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{m(n)n} \leq t$ of $[0, t]$ with $\lim_{n\to\infty} \sup |t_{in} - t_{i-1,n}| = 0$, we have:

$$\Delta_n = \sum_{i=1}^{m(n)} (W_{t_{in}} - W_{t_{i-1,n}})^2 \xrightarrow{L^2} t.$$

In fact:

$$\begin{aligned} \mathbb{E}((\Delta_n - t)^2) &= \mathbb{E}(\Delta_n^2 - 2t\Delta_n + t^2) \\ &= \mathbb{E}(\Delta_n^2) - 2t^2 + t^2, \end{aligned}$$

$\square$

Proof.

but

$$
\begin{aligned}
&\mathbb{E}(\Delta_n^2) \\
&= E\left(\sum_{i=1}^{m(n)}\sum_{j=1}^{m(n)}(W_{t_{in}} - W_{t_{i-1,n}})^2(W_{t_{jn}} - W_{t_{j-1,n}})^2\right) \\
&= \sum_{i=1}^{m(n)}\mathbb{E}((W_{t_{in}} - W_{t_{i-1,n}})^4) + 2\sum_{i=1}^{n}\sum_{j<i}\mathbb{E}((W_{t_{in}} - W_{t_{i-1,n}})^2(W_{t_{jn}} - W_{t_{j-1,n}})^2) \\
&= 3\sum_{i=1}^{m(n)}(t_{in} - t_{i-1,n})^2 + 2\sum_{i=1}^{m(n)}\sum_{j<i}(t_{in} - t_{i-1,})(t_{jn} - t_{j-1,n}) \\
&= t^2 + 2\sum_{i=1}^{m(n)}(t_{in} - t_{i-1,n})^2
\end{aligned}
$$

□

Proof.

so

$$\mathbb{E}((\Delta_n - t)^2) = 2\sum_{i=1}^{m(n)} (t_{in} - t_{i-1,n})^2 \leq 2t \sup |t_{in} - t_{i-1,n}| \to 0.$$

Then

$$\mathbb{P}\{|\Delta_n - t| > \varepsilon\} \leq \frac{2t \sup |t_{in} - t_{i-1,n}|}{\varepsilon^2},$$

and if the sequence of partitions is such that $\sum_{n=1}^{\infty} \sup |t_{in} - t_{i-1,n}| < \infty$, by applying the Borel-Cantelli Lemma, we have

$$\Delta_n \overset{\text{a.s.}}{\to} t,$$

and for these partitions

$$\sum_{i=1}^{m(n)} |W_{t_{in}} - W_{t_{i-1n}}| \geq \frac{\sum_{i=1}^{m(n)} |W_{t_{in}} - W_{t_{i-1,n}}|^2}{\sup_i |W_{t_{i,n}} - W_{t_{i-1,n}}|} = \frac{\Delta_n}{\sup_i |W_{t_{in}} - W_{t_{i-1,n}}|} \overset{\text{a.s.}}{\to} \frac{t}{0}.$$

□

Integral with respect to a Brownian motion

Let (W_t) be a Brownian motion, and (τ_n) a sequence of partitions: $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{m(n)n} = t$, with $d_n := \lim_{n\to\infty} \sup |t_{in} - t_{i-1,n}| = 0$, such that for all $0 \leq s \leq t$

$$\lim_{n\to\infty} \sum_{\substack{t_{i,n}\in\tau_n \\ t_{i,n}\leq s}} |W_{t_{in}} - W_{t_{i-1,n}}|^2 \overset{c.s.}{=} s. \tag{1}$$

Let f a C^2 map in $\mathbb{R}$. Then, fixed ω,

$$\begin{aligned} & f(W_{t_{in}}) - f(W_{t_{i-1,n}}) \\ = \; & f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}}) + \frac{1}{2} f''(W_{\tilde{t}_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})^2, \end{aligned}$$

where $\tilde{t}_{i-1,n} \in (t_{i-1,n}, t_{in})$.

Since $f''(W_s(\omega))$ is uniformly continuous in a the compact set $[0, t]$, we have

$$\begin{aligned}
&\sum_{i=1}^{m(n)} |f''(W_{\tilde{t}_{i-1,n}}) - f''(W_{t_{i-1,n}})|(W_{t_{in}} - W_{t_{i-1,n}})^2 \\
\leq \quad & \varepsilon_n \sum_{i=1}^{m(n)} (W_{t_{in}} - W_{t_{i-1,n}})^2 \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

For each n, $\mu_n(A)(\omega) := \sum_{i=1}^{m(n)} |W_{t_{in}}(\omega) - W_{t_{i-1,n}}(\omega)|^2 \mathbf{1}_A(t_{i-1,n})$ defines a measure in $[0, t]$ that converges, by (1), to the Lebesgue measure in $[0, t]$. So

$$\begin{aligned}
\sum_{i=1}^{m(n)} f''(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})^2 &= \int_0^t f''(W_s)\mu_n(\mathrm{d}s) \\
&\underset{n\to\infty}{\longrightarrow} \int_0^t f''(W_s)\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(W_t) - f(0) &= \lim_{n\to\infty} \sum (f(W_{t_{in}}) - f(W_{t_{i-1,n}})) \\
&= \lim_{n\to\infty} \sum f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}}) + \frac{1}{2}\int_0^t f''(W_s)\mathrm{d}s \,.
\end{aligned}$$

Consequently

$$\lim_{n\to\infty}\sum f'(W_{t_{i-1,n}})(W_{t_{in}}-W_{t_{i-1,n}})$$

is well defined since it coincides with $f(W_t)-f(0)-\frac{1}{2}\int_0^t f''(W_s)ds$ and then we can define

$$\int_0^t f'(W_s)\mathrm{d}W_s := \lim_{n\to\infty}\sum f'(W_{t_{i-1,n}})(W_{t_{in}}-W_{t_{i-1,n}}).$$

The drawback of this construction is that this integral depends on the sequences of partitions.

In this way we have established that

$$\int_0^t f'(W_s)\mathrm{d}W_s = f(W_t) - f(0) - \frac{1}{2}\int_0^t f''(W_s)\mathrm{d}s$$

and this result modifies chain rule of the *classical analysis*:

$$\mathrm{d}f(W_t) \neq f'(W_s)\mathrm{d}W_s$$

The knew integral is known as *Itô's integral*.

Example

$$\int_0^t W_s \mathrm{d} W_s = \frac{1}{2} W_t^2 - \frac{1}{2} t,$$

$$\int_0^t \exp\{W_s\} \mathrm{d} W_s = \exp\{W_t\} - 1 - \frac{1}{2} \int_0^t \exp\{W_s\} ds$$

It is straightforward to see that we can extend the previous result to integrands that are $C^{1,2}$-functions $f : [0, t] \times R \to R$ in such a way that

$$
\begin{aligned}
f(t, W_t) &= f(0,0) + \int_0^t f_t(s, W_s)\mathrm{d}s + \int_0^t f_x(s, W_s)\mathrm{d}W_s \\
&\quad + \frac{1}{2}\int_0^t f_{xx}(s, W_s)\mathrm{d}s,
\end{aligned}
$$

where

$$
f_t(s,x) = \left.\frac{\partial}{\partial t} f(t,x)\right|_{t=s}, \quad f_x(s,x) = \left.\frac{\partial}{\partial x} f(t,x)\right|_{t=s},
$$

$$
f_{xx}(s,x) = \left.\frac{\partial^2}{\partial x^2} f(t,x)\right|_{t=s}.
$$

Example

If we take $f(t,x) = \exp(\sigma x - \frac{1}{2}\sigma^2 t)$, $\sigma \in \mathbb{R}_+$, we have

$$
\begin{aligned}
\exp(\sigma W_t - \frac{1}{2}\sigma^2 t) = 1 - \frac{\sigma^2}{2}\int_0^t \exp(\sigma W_s - \frac{1}{2}\sigma^2 s)\mathrm{d}s \\
+ \sigma \int_0^t \exp(\sigma W_s - \frac{1}{2}\sigma^2 s)\mathrm{d}W_s \\
+ \frac{\sigma^2}{2}\int_0^t \exp(\sigma W_s - \frac{1}{2}\sigma^2 s)\mathrm{d}s.
\end{aligned}
$$

That is,

$$\exp(\sigma W_t - \frac{1}{2}\sigma^2 t) = 1 + \sigma \int_0^t \exp(\sigma W_s - \frac{1}{2}\sigma^2 s)\mathrm{d}W_s.$$

so, if we define $S_t := \exp(\sigma W_t - \frac{1}{2}\sigma^2 t)$, we can write

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t.$$