1. FOURIER SERIES.

Consider the space $L^2[0,T]$, $T>0$, which is a model for the T-periodic functions $f: \mathbb{R} \to \mathbb{C}$ with finite energy (in a period). To simplify start with $T = 2\pi$; then we will just rescale if we need to consider $T \neq 2\pi$.

<u>Lemma</u>: $\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\}_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2[0,2\pi]$.

<u>Proof:</u> Let us check first the orthogonality:

$$\langle e^{int}, e^{imt} \rangle = \int_0^{2\pi} e^{i(n-m)t}\, dt = 2\pi\, \delta_{nm}.$$

Thus $\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\}_{n\in\mathbb{Z}}$ is an orthonormal system.

In order to prove the completeness we use Weierstrass approximation theorem. Identify $[0,2\pi]$ with $\partial\mathbb{D} = \{ e^{it} : t \in [0,2\pi] \}$. Any $f \in \mathcal{C}_c([0,2\pi])$ is of the form $f(t) = g(e^{it})$; with $g \in \mathcal{C}_c(\partial\mathbb{D})$. Then, by Weierstrass approximation theorem, there exist polynomials p_k in $\mathbb{C}$ $p_k(z,\bar{z}) = p_k(e^{it}, e^{-it})$ such that

$$|f(t) - p_K(e^{it}, e^{-it})| = |g(e^{it}) - p_K(e^{it}, e^{-it})| < \varepsilon$$

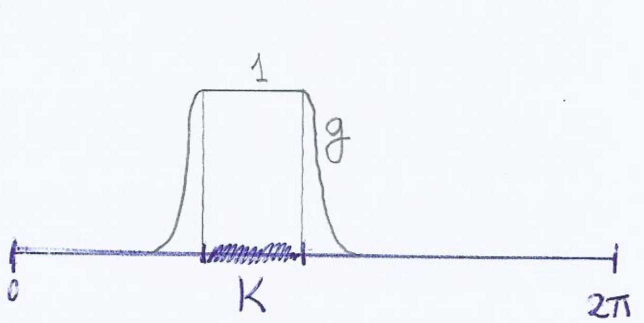
$$\forall t \in [0, 2\pi].$$

Since $p_K(e^{it}, e^{-it})$ is a linear combination of the elements $\{e^{int}\}_{n\in\mathbb{Z}}$ we obtain the approximation of any $f \in \mathcal{C}_c([0,2\pi])$ by our system.

The approximation of $f \in L^2[0,2\pi]$ by continuous functions with compact support $\varphi \in \mathcal{C}_c[0,2\pi]$ is a well-known result in measure theory. First, since f is limit of simple functions, it is enough to approximate an indicator χ_E, with $E \subset (0,2\pi)$ measurable. Since for every E and every $\varepsilon > 0$ there exists $K \subseteq E$ compact such that $|E \setminus K| < \varepsilon$, it is enough to approximate χ_K. By Urysohn's lemma, there exists $g \in \mathcal{C}[0,2\pi]$ with

$$\chi_K \leq g \leq 1$$

and $|\operatorname{supp} g \setminus K| < \varepsilon$.



Then

$$\int_0^{2\pi} |\chi_K(t) - g(t)|^2 \, dt = \int_{\operatorname{supp} g \setminus K} |g(t)|^2 \, dt$$

$$\leq |\operatorname{supp} g \setminus K| < \varepsilon.$$

Remark: By the Corollary seen in the preliminaries, for any $f \in L^2[0,2\pi]$,

$$f(t) = \sum_{n\in\mathbb{Z}} \langle f, \tfrac{1}{\sqrt{2\pi}} e^{int} \rangle \frac{1}{\sqrt{2\pi}} e^{int}$$

$$= \sum_{n\in\mathbb{Z}} \left(\int_0^{2\pi} f(t) e^{-int} \frac{dt}{2\pi} \right) e^{int}$$

Definition: For $f \in L^2[0,2\pi]$ and $n \in \mathbb{Z}$, the n-th Fourier coefficient of f is

$$\hat{f}(n) = \int_0^{2\pi} f(t) e^{-int} \frac{dt}{2\pi} = \sqrt{2\pi} \langle f, e_n \rangle,$$

where $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{-int}$.

Then, the identity above is

$$f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int} \qquad t \in [0,2\pi]$$

Note: This identity is in $L^2[0,2\pi]$, so strictly speaking $\lim_{N\to\infty} \int_0^{2\pi} |f(t) - \sum_{|n|\le N} \hat{f}(n) e^{int}|^2 dt = 0.$

The right hand side of the identity is called the Fourier series of f.

Also, as a consequence of the general theory of Hilbert spaces, we have the Plancherel identity: for $f \in L^2[0,2\pi]$

$$\|f\|_2^2 = \int_0^{2\pi} |f(t)|^2 \, dt = 2\pi \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 .$$

Exercise: Reorganise the Fourier series above, using $e^{it} = \cos t + i \sin t$ and show that

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + i \sum_{n=1}^{\infty} b_n \sin(nt),$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt)\, dt \qquad n \geq 1$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt)\, dt \qquad n \geq 1$$

$$a_0 = \hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt .$$

Extension and pointwise convergence

As mentioned before the identity $f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{int}$ $f \in L^2[0,2\pi]$ is to be understood in the L^2-sense.

At this point at least two fundamental questions arise:

1) Does the Fourier series make sense for other functions? If so, does it coincide with $f(t)$?

2) For what $t \in [0, 2\pi]$ the series above converges pointwise to $f(t)$?

These are delicate matters that we shall not discuss. We shall just mention a couple of results.

1) By the Cauchy-Schwarz inequality $L^2[0,2\pi] \subseteq L^1[0,2\pi]$ and $\|f\|_1 \leq \sqrt{2\pi}\,\|f\|_2$:

$$\|f\|_1 = \int_0^{2\pi} |f(t)|\,dt \leq \left(\int_0^{2\pi} 1\,dt\right)^{1/2} \left(\int_0^{2\pi} |f(t)|^2\,dt\right)^{1/2} = \sqrt{2\pi}\,\|f\|_2 .$$

Then, for $f \in L^1[0,2\pi]$ the <u>n-th Fourier coefficient</u>

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt \qquad n \in \mathbb{Z}$$

is well defined. Actually:

<u>Lemma:</u> Let $f \in L^1[0,2\pi]$. Then

ⓐ $|\hat{f}(n)| \leq \|f\|_1$

ⓑ $\lim\limits_{|n|\to\infty} \hat{f}(n) = 0$ (Riemann-Lebesgue Lemma)

Proof: ⓐ Straightforward estimate.

ⓑ Assume first that $f \in \mathcal{C}^1(0,2\pi)$ and that $f(0)=f(2\pi)$. Integration by parts yields

$$\hat{f}(n) = \int_0^{2\pi} f(t)e^{-int}\frac{dt}{2\pi} = \left[-f(t)\frac{e^{-int}}{in\,2\pi}\right]_0^{2\pi} + \int_0^{2\pi} f'(t)\frac{e^{-int}}{in}\frac{dt}{2\pi} =$$

$$= 0 + \frac{1}{in}\widehat{f'}(n).$$

By ⓐ $|\hat{f}(n)| = \frac{1}{|n|}|\widehat{f'}(n)| \leq \frac{\|f'\|_1}{|n|} \xrightarrow{n\to\infty} 0.$

For a general $f \in L^1[0,2\pi]$, take approximations by functions $g \in \mathcal{C}^1(0,2\pi)$ with $g(0)=g(2\pi)$ (in a similar way as recently done for functions in $L^2[0,2\pi]$). Given $\varepsilon>0$ let $g_\varepsilon \in \mathcal{C}(0,2\pi)$ with $g_\varepsilon(0)=g_\varepsilon(2\pi)$ and $\|f-g_\varepsilon\|_1 < \varepsilon$. Then:

$$|\hat{f}(n)| \leq |\hat{f}(n)-\hat{g}_\varepsilon(n)| + |\hat{g}_\varepsilon(n)| \leq \|f-g_\varepsilon\|_1 + |\hat{g}_\varepsilon(n)|.$$

There exists $n_0 \in \mathbb{N}$ such that $|\hat{g}_\varepsilon(n)| < \varepsilon \;\forall n \geq n_0$, and therefore $|\hat{f}(n)| < 2\varepsilon$, as desired. □

Exercise: Let $f \in \mathcal{C}^1[0,2\pi]$ be 2π-periodic (i.e. $f(0)=f(2\pi)$). Prove that $\widehat{f'}(n) = in\hat{f}(n)$, $n\in\mathbb{Z}$.
Deduce a similar formula for $f \in \mathcal{C}^{(k)}(0,2\pi)$ 2π-periodic ($f^{(j)}(0)=f^{(j)}(2\pi)$, $j<k$). Observe that the decay improves with the regularity of f.

2) Given $f \in L^1[0,2\pi]$ (or in another space), when does its Fourier series converge? When it converges, does it do it to $f(t)$?

<u>Theorem</u> (Dirichlet). Let $f \in L^1[0,2\pi]$ and let $t \in [0,2\pi)$ be such that $f(t^-)$, $f(t^+)$ exist, as well as $f'(t^-)$, $f'(t^+)$. Then

$$\sum_{n \in \mathbb{Z}} \hat{f}(n)\, e^{int} = \frac{f(t^-)+f(t^+)}{2} .$$

In particular, for $f \in \mathcal{C}^1[0,2\pi]$ everything works well (the series converges to $f(t)$). However, Du Bois-Reymond (a German mathematician) gave an example of $f \in \mathcal{C}[0,2\pi]$ whose Fourier series diverges at a point. Later, in 1923, Kolmogorov produced a L^1-function f whose Fourier series diverges a.e. On the other hand, Carleson proved that for L^2-functions there is pointwise convergence. This is a very delicate matter, that we will avoid completely (the functions we shall consider in applications do not involve this subtleties).

In order to understand better the convergence of the partial sums

$$S_N f(t) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{int} \qquad \text{as } N \to \infty$$

use the definition of $\hat{f}(n)$ and group exponentials:

$$S_N f(t) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sum_{n=-N}^{N} e^{in(t-x)}\, dx = (f * D_N)(t),$$

where

$$2\pi D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}} = \frac{\sin[(N+\frac{1}{2})x]}{\sin(\frac{x}{2})}.$$

This function $$D_N(x) = \frac{1}{2\pi} \frac{\sin[(N+\frac{1}{2})x]}{\sin(\frac{x}{2})}$$

is called the Dirichlet Kernel.

Note: Maybe we should recall here that the convolution of $f, g \in L^1[0,2\pi]$ is defined as

$$(f*g)(t) = \int_0^{2\pi} f(x)\, g(t-x)\, dx.$$

Thus, the behaviour of $S_N f$ as $N \to \infty$ is somehow dependent on the behaviour of D_N, which is not very good:

<u>Proposition</u>: (a) $\frac{1}{2\pi}\int_0^{2\pi} D_N(x)\,dx = 1$

(b) $\frac{1}{2\pi}\int_0^{2\pi} |D_N(x)|\,dx \gtrsim \log N$ as $N\to\infty$

In more technical terms $\{D_N(x)\}_N$ is not an approximation of the identity.

A way to improve the convergence is to take, instead of $S_N f$, the averages

$$\sigma_N(f) = \frac{S_0 f + \dots + S_N f}{N+1}.$$

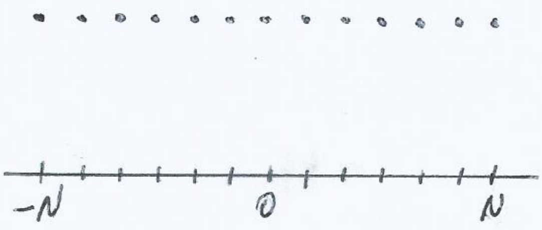
It is easy to see, as before, that $\sigma_N(f) = f * K_N$, where K_N is the Fejér kernel

$$K_N(t) = \frac{D_0(t)+\dots+D_N(t)}{N+1} = \sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)e^{inx}$$

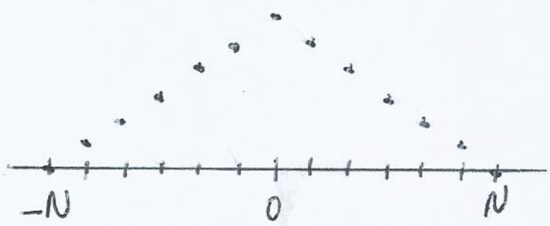
$$= \frac{1}{N+1}\,\frac{\sin^2\left(\frac{N+1}{2}x\right)}{\sin^2\left(\frac{x}{2}\right)} = \frac{1}{N+1}\,\frac{1-\cos[(N+1)x]}{1-\cos x}.$$

Lipót Fejér, born Leopold Weisz in 1880, was the advisor of John Von Neumann, Paul Erdös, György Pólya, Tibor Radó, Mihály Fekete, Marcel Riesz...

Notice that Fejér's Kernel charges the same exponentials (from $-N$ to N) but with different weight:



Dirichlet



Fejér

The Fejér Kernel gives a good approximation of the identity. In consequence:

ⓐ For $f \in L^1[0,2\pi]$ $\quad \sigma_N(f) \xrightarrow[N \to \infty]{} f$ in L^1

ⓑ If $f \in \mathcal{C}(0,2\pi)$ is 2π-periodic, then

$$\lim_{N \to \infty} \sigma_n(f)(t) = f(t) \text{ uniformly in } t.$$

So, we have just seen that the behaviour of the summation formula depends on the properties of the kernel we convolve with. The action of convolution is transparent at the Fourier side.

<u>Theorem:</u> Let $f, g \in L^1[0,2\pi]$. Then

$$(f * g)^\wedge(n) = \hat{f}(n)\,\hat{g}(n) \qquad n \in \mathbb{Z}$$

<u>Proof:</u> Use the definition and apply Fubini's theorem:

$$(f*g)^\wedge(n) = \int_0^{2\pi} \left(\int_0^{2\pi} f(t)\, g(x-t)\, dx \right) e^{-inx}\, dt =$$

$$= \int_0^{2\pi} f(t) \left(\int_0^{2\pi} g(x-t) e^{-in(x-t)} dx \right) e^{-int} \, dt$$

$$= \int_0^{2\pi} f(t) \, \hat{g}(n) \, e^{-int} \, dt = \hat{g}(n) \, \hat{f}(n) . \qquad \square$$

Final remark: In case we work with an arbitrary interval $[0,T]$ (i.e, with T-periodic functions) we follow the same arguments, starting with the orthonormal basis $\{e_n(t)\}_{n\in\mathbb{Z}}$, where now

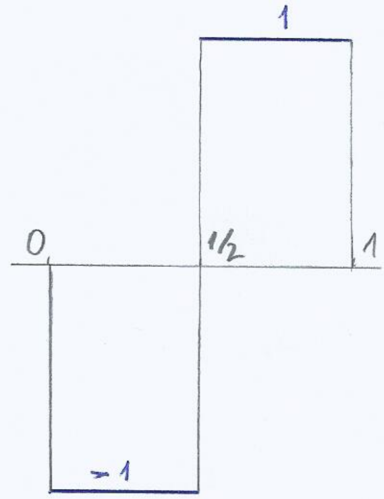
$$e_n(t) = \frac{1}{\sqrt{T}} e^{in\frac{2\pi}{T}t} .$$

ANNEX. THE HAAR SYSTEM

Let us see a different orthonormal basis of $L^2[0,1]$, which will be relevant in applications.

Consider $\chi(t) \equiv 1$ in $L^2[0,1]$ (called "scaling function" in the theory of wavelets, as we shall see in the future) and let

$$\psi_0(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & 1/2 \le t < 1 \end{cases}$$



Now rescale and translate this function:

$$\psi_{n,k}(t) = 2^{n/2}\,\psi_0(2^n t - k) \qquad n \geq 0;\ k = 0, \dots, 2^n - 1.$$

Explicitly:

$$\psi_{n,k}(t) = \begin{cases} -2^{n/2} & \text{if} \quad t \in \left[\frac{k}{2^n}, \frac{k+1/2}{2^n}\right) \\ 2^{n/2} & \text{if} \quad t \in \left[\frac{k+1/2}{2^n}, \frac{k+1}{2^n}\right) \end{cases}$$

<u>Properties</u> ① $\psi_{n,k}$ is supported in the dyadic interval of the n-th generation $I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$.

② $\|\psi_{n,k}\|_{L^2[0,1]} = 1 \quad \forall n \geq 0;\ k = 0, \dots, 2^n - 1.$

③ $\int_0^1 \psi_{n,k}(t)\, dt = 0$

<u>Theorem:</u> χ, $\psi_{n,k}$, $n \geq 0$; $k = 0, \dots, 2^n - 1$ form an orthonormal basis of $L^2[0,1]$.

<u>Proof:</u> By ② and ③ we just need to prove that the system $\{\psi_{n,k}\}_{n,k}$ is orthogonal and that χ, $\psi_{n,k}$, $n \geq 0$, $k = 0, \dots, 2^n - 1$ is complete in $L^2[0,1]$.

In order to see the orthogonality assume first that $(n,k) \neq (m,j)$, and let us see that $\langle \psi_{n,k}, \psi_{m,j} \rangle = 0$. If $I_{n,k} \cap I_{m,j} = \emptyset$ then $\psi_{n,k}$, $\psi_{m,j}$ have disjoint supports and the identity is obvious. If $I_{n,k} \cap I_{m,j} \neq 0$ then, by construction of the dyadic intervals, either $I_{n,k} \subseteq I_{m,j}$ or $I_{m,j} \subseteq I_{n,k}$ (since $I_{n,k} \cap I_{n,j} = \emptyset$ if $j \neq k$). Assume $I_{n,k} \subseteq I_{m,j}$, that is $n > m$. Then $I_{n,k}$ is contained in one of the two halves of $I_{m,j}$, and therefore $\psi_{m,j}$ is constant on $I_{n,k}$.

Hence, by (3)

$$\langle \psi_{n,k}, \psi_{m,j} \rangle = \pm 2^{-m/2} \int_{I_{n,k}} \psi_{n,k}(t)\, dt = 0$$

In order to see that the system is complete let us observe first that it is enough to approximate continuous functions f, which are dense in $L^2[0,1]$.

Let us observe next that it is enough to approximate functions which are constant on dyadic intervals of the same n-th generation:

<u>Claim:</u> $\forall \varepsilon > 0 \quad \exists n \geq 1 \quad \exists g_n = \sum_{k=0}^{2^n-1} \alpha_k \chi_{I_{n,k}}, \alpha_k \in \mathbb{C},$

such that $\quad \|f - g_n\|_\infty < \varepsilon.$

In particular, $\|f - g_n\|_{L^2[0,1]} < \varepsilon$ as well.

We prove this using the uniform continuity of f on $[0,1]$: $\forall \varepsilon > 0 \quad \exists \delta > 0$:

$$|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$$

Take $n \geq 1$ with $2^{-n} < \delta$ and let $\alpha_k = f(t_k)$, where t_k is the centre (or any point) of $I_{n,k}$. Let

$$g = \sum_{k=0}^{2^n-1} \alpha_k \chi_{I_{n,k}}.$$

Then, given $t \in [0,1]$, take (n,k) such that $t \in I_{n,k}$

and check that

$$|f(t)-g_n(t)| = |f(t)-f(t_{n,k})| < \varepsilon .$$

It remains to see that the system $\chi, \psi_{n,k}$ can generate all the functions that are constant on dyadic intervals of the same generation.

This is just a matter of linear algebra: given g_n as above, set the system

$$g_n = \sum_{k=0}^{2^n-1} \alpha_k \chi_{I_{n,k}} = \langle g_n, \chi \rangle + \sum_{m<n} \sum_{j=0}^{2^m-1} \langle g_n, \psi_{mj} \rangle \psi_{m,j} .$$

On the left hand side we have 2^n given values α_k, and on the right hand side we have

$$1 + \sum_{m<n} 2^m = 1 + (1+\dots+2^{n-1}) = 1 + \frac{2^n-1}{2-1} = 2^n$$

unknowns.

Actually notice that if $m<n$ and $I_{n,k} \subseteq I_{m,j}$, then

$$\langle \alpha_k \chi_{I_{n,k}}, \psi_{m,j} \rangle = \alpha_k \int_{I_{n,k}} \psi_{m,j}(t)\,dt = \pm \alpha_k 2^{m/2} 2^{-n}$$

and therefore, on $I_{n,k}$,

$$\langle \alpha_k \chi_{I_{n,k}}, \psi_{m,j} \rangle \psi_{m,j} = \alpha_k 2^{m/2} 2^{-n} 2^{m/2} = \alpha_k 2^{m-n} .$$

Summing up all the ancestors of $I_{n,k}$ we have

$$\sum_{m<n} \sum_{j=0}^{2^m-1} \langle \alpha_k \chi_{I_{n,k}}, \psi_{m,j} \rangle \psi_{m,j} = \sum_{m<n} \alpha_k 2^{m-n} = \alpha_k 2^{-n}(2^n-1) = \alpha_k - \alpha_k 2^{-n}$$

Adding the term $\langle \alpha_k \chi_{I_{n,k}}, \chi \rangle \chi = \alpha_k 2^{-n}$ we get the identity directly for each part $\alpha_k \chi_{I_{n,k}}$.