DYNAMICAL SYSTEMS
MASTER IN ADVANCED MATHEMATICS
Fall semester, 2023

Exercise set # 2.1

$\boxed{\text{Due: Thursday 19/10/23}}$

This exercise contains the basic Liapunov theory for maps.

Let $U \subset \mathbb{R}^n$ be an open set and $x_0 \in U$. Let $f : U \to \mathbb{R}^n$ be continuous with $f(x_0) = x_0$.

Given $V : U \subset \mathbb{R}^n \to \mathbb{R}$ we define $\Delta V(x) = V(f(x)) - V(x)$ in $U \cap f^{-1}(U)$.

Definition 0.1 *$V : U \subset \mathbb{R}^n \to \mathbb{R}$ is a Liapunov function (associated to f and x_0) if*

(1) V is continuous.

(2) $V(x) > 0$ for $x \in U \setminus \{x_0\}$ and $V(x_0) = 0$.

(3) $\Delta V(x) \leq 0$ for $x \in U \cap f^{-1}(U)$.

Prove that

(a) Let $U \subset \mathbb{R}^n$ be an open set and $x_0 \in U$. Let $f : U \to \mathbb{R}^n$ be continuous with $f(x_0) = x_0$. If there exists a Liapunov function associated to f and x_0 then x_0 is stable.

Let $Z = \{x \in U \cap f^{-1}(U) \mid \Delta V(x) = 0\}$. Prove that

(b) If there exists a Liapunov function associated to f and x_0 and if no positive semiorbit $O_+(x) = \{f^k(x) \mid k \geq 0\}$ is contained in Z, except $O_+(x_0) = \{x_0\}$, then x_0 is asymptotically stable.

Remark. Note that if V also satisfies $\Delta V(x) < 0$ for $x \in U \cap f^{-1}(U) \setminus \{x_0\}$ we are under the assumptions of this statement.