

Exercise 1 Consider a model of a bond market where the instantaneous short rate is given by $r(t)$ and where we have a bank account asset with continuously compounded interest rate α_t , that is the unit of money evolves as

$$dS_t^0 = \alpha_t S_t^0 dt, \quad S_0^0 = 1.$$

then prove that if the model is free of arbitrage $\alpha_t = r(t), 0 \leq t \leq T$.

Exercise 2 Assume that the short rate follows the dynamics, under the risk-neutral probability,

$$r(t) = a + \sigma W_t$$

where $a, \sigma > 0$ and W is a Brownian motion. Compute the price of a zero-coupon bond and the corresponding instantaneous forward rate.