DYNAMICAL SYSTEMS
MÀSTER EN MATEMÀTICA AVANÇADA
Year 2023-2024, Fall semester

Problem set #2. Due December 8th, 2023

1 (3p) Let $P(z) = (z-\alpha)(z-\beta)$ where $\alpha, \beta \in \mathbb{C}$, with $\alpha \neq \beta$. Let $N_P(z) = z - \frac{P(z)}{P'(z)}$ be the Newton's method of P. Describe precisely (with proofs) the basins of attraction of α and β, the Fatou set and the Julia set. What can you say about the dynamics on the Julia set? *(Hint: Conjugate N_P (on the whole Riemann sphere) by the Möbius transformation $M(z) = \frac{z-\alpha}{z-\beta}$ and see what the resulting map is.)*

OPTIONAL (1p): Make a program that draws the basins of attraction of Newton's method of the cubic polynomial $P(z) = z(z-1)(z-i)$. (Please include the code and the image in the same pdf file where the rest of the problems are).

2 **The quadratic family $Q_c(z) = z^2 + c$.** Let $A_c(\infty)$ denote the basin of attraction of ∞ for Q_c, and $K_c := \mathbb{C} \setminus A_c(\infty)$ denote the filled Julia set.

(a) (1p) Prove that $K_c \subset \overline{D(0,R)}$ where $R = \max\{|c|, 2\}$.

(b) (0,5p) Deduce that if $|c| > 2$ then the orbit of the critical point $z = 0$ escapes to infinity.

(c) (0,5p) Show that for every value of $c \in \mathbb{C}$, Q_c has at most one attracting cycle.

(d) (1p) Calculate and draw the sets

$$\Omega_1 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting fixed point}\}$$

and

$$\Omega_2 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting 2-cycle}\}.$$

3 (4p) Let R be a rational function and suppose that C is a round circle such that $R^{-1}(C) \subset C$. Prove that $J(R) = C$ or $J(R)$ is a totally disconnected subset of C. *Hint: There are several ways of solving this problem. Some key words that* **might** *be related to possible solutions are: conjugacy, unit circle, Schwarz reflection, invariance, Denjoy-Wolff, normality...*