Lesson 9

José M. Corcuera.
University of Barcelona.

Example

Suppose a financial market with a single risky stock, $S_t = S_0 + \sigma W_t$, $t \in [0, T]$ and a bank account with interest rate $r = 0$. Suppose that the value of a call option is a smooth function

$$C_t := f(t, S_t),$$

and consider a portfolio with β_t calls and α_t stocks, the value of this portfolio is

$$\beta_t C_t + \alpha_t S_t =: V_t,$$

and when it evolves its value changes as

$$\mathrm{d}V_t = \beta_t \mathrm{d}C_t + \alpha_t \mathrm{d}S_t,$$

since it is self-financed.

Example

Then by applying the Itô formula

$$\mathrm{d}V_t = \beta_t \left(\partial_t f \mathrm{d}t + \partial_x f \sigma \mathrm{d}W_t + \frac{1}{2} \partial_{xx} f \sigma^2 \mathrm{d}t \right) + \alpha_t \sigma \mathrm{d}W_t.$$

Now if we take $\alpha_t = - \beta_t \partial_x f$ we have that the cost of this portfolio is

$$\mathrm{d}V_t = \beta_t \left(\partial_t f + \frac{1}{2} \partial_{xx} f \sigma^2 \right) \mathrm{d}t.$$

The profit of this portfolio behaves like a bank account with continuosly compounded interest rate, say r_t, such that

$$\beta_t \left(\partial_t f + \frac{1}{2} \partial_{xx} f \sigma^2 \right) = r_t V_t,$$

but if we have an equilibrium (non arbitrage) $r_t = r = 0$.

Example

Therefore we look for a function $f(t,x)$ such that

$$\partial_t f(t,x) + \frac{\sigma^2}{2}\partial^2_{xx} f(t,x) = 0, \qquad (1)$$

$$\text{with} \quad f(T,x) = (x-K)_+$$

also we have that

$$\mathrm{d}C_t = -\frac{\alpha_t}{\beta_t}\mathrm{d}S_t = \partial_x f \mathrm{d}S_t. \qquad (2)$$

Example

It is easy to see that

$$p(t,x) := \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left\{-\frac{x^2}{2\sigma^2(T-t)}\right\},$$

is a solution of (1) with $p(T,x) = \delta(x)$, where δ is the Dirac's delta. That is $p(t,x)$ is *the fundamental solution*.

Example

Then if the boundary is $f(T,x)$, we will have that

$$f(t,x) = \int_{\mathbb{R}} f(T,y) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left\{-\frac{(y-x)^2}{2\sigma^2(T-t)}\right\} \mathrm{d}y,$$

and

$$\begin{aligned} f(t,S_t) &= \int_{\mathbb{R}} f(T,y) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left\{-\frac{(y-S_t)^2}{2\sigma^2(T-t)}\right\} \mathrm{d}y \\ &= \mathbb{E}(f(T,S_T)|S_t). \end{aligned}$$

Example

If $f(T, S_T) = (S_T - K)_+$ we have that

$$
\begin{aligned}
C_t &= f(t, S_t) = \mathbb{E}((S_T - K)_+ | S_t) = (S_t - K)\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \\
&\quad + \sigma\sqrt{T-t}\,\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

where Φ and ϕ are, respectively, the cumulative distribution function and the density of a standard normal distribution. The equation (2) solves the hedging problem.

Martingales in continuous time

Definition

Let $(M_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted process, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}(|M_t|) < \infty$, then it is:

- a martingale if $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, for all $s \leq t$
- a submartingale if $\mathbb{E}(M_t|\mathcal{F}_s) \geq M_s$, for all $s \leq t$
- a supermartingale if $\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s$, for all $s \leq t$.

In the previous definition equalities and inequalities are $\mathbb{P}$ *almost surely*.

Proposition

If (X_t) is a Brownian motion then:

- (X_t) is a martingale.
- $(X_t^2 - t)$ is a martingale.
- $(\exp(\sigma X_t - \frac{\sigma^2}{2} t))$ is an martingale.

Proof.

We take $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$

$$\begin{aligned}
\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(X_t - X_s + X_s|\mathcal{F}_s) \\
&= \mathbb{E}(X_t - X_s|\mathcal{F}_s) + X_s \\
&= \mathbb{E}(X_t - X_s) + X_s = X_s,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X_t^2 - t|\mathcal{F}_s) &= \mathbb{E}((X_t - X_s + X_s)^2|\mathcal{F}_s) - t \\
&= \mathbb{E}((X_t - X_s)^2 + X_s^2 + 2(X_t - X_s)|\mathcal{F}_s) - t \\
&= t - s + X_s^2 - t \\
&= X_s^2 - s,
\end{aligned}$$

$\square$

Proof.

$$
\begin{aligned}
&\mathbb{E}(\exp(\sigma X_t - \frac{\sigma^2}{2}t)|\mathcal{F}_s) \\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}t)\mathbb{E}(\exp(\sigma(X_t - X_s))|\mathcal{F}_s) \\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}t)\mathbb{E}(\exp(\sigma(X_t - X_s))) \\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}t)\exp(\frac{\sigma^2}{2}(t-s)) \text{ (since } X_t - X_s \sim N(0, t-s)) \\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}s)
\end{aligned}
$$

$\square$

Definition

A *stopping time* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a random variable

$$\tau : \Omega \longrightarrow [0, \infty]$$

such that for all $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$.

Theorem

If τ is a bounded stopping time and $(M_t)_{t\geq 0}$ is a martingale, then $\mathbb{E}(M_\tau) = M_0$.

Proof.

This is a corollary of the Optional Sampling Theorem. $\square$

Definition

Let $(M_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted process, we say that $(M_t)_{t\geq 0}$ is a *local* martingale, if it exists an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ with $\tau_n \uparrow \infty$, such that, fixed n, $(M_{t\wedge\tau_n})_{t\geq 0}$ is a martingale, for all $n \geq 0$.

Proposition

Let $(M_t)_{t\geq 0}$ *be an* $(\mathcal{F}_t)$*-adapted process, set* $s \leq t$ *and* $A \in \mathcal{F}_s$. *Then:* *(i)* $\tau_{ts} = t\mathbf{1}_{A^c} + s\mathbf{1}_A$ *is an* $(\mathcal{F}_t)_{t\geq 0}$ *stopping time; (ii) If* $\mathbb{E}(M_{\tau_{ts}}) = M_0$ *for all* $0 \leq s \leq t$ *then* $(M_t)_{t\geq 0}$ *is a martingale.*

Proof.

(i) Let $s \leq u < t$, then $\{\tau_{ts} \leq u\} = A \in \mathcal{F}_s \subseteq \mathcal{F}_u$. Otherwise $\{\tau_{ts} \leq u\}$ is ϕ or Ω. (ii) $\mathbb{E}(M_{\tau_{ts}}) =$
$\mathbb{E}(M_t \mathbf{1}_{A^c}) + \mathbb{E}(M_s \mathbf{1}_A) = \mathbb{E}(M_{\tau_{tt}}) = \mathbb{E}(M_t) = \mathbb{E}(M_t \mathbf{1}_{A^c}) + \mathbb{E}(M_t \mathbf{1}_A)$.
Therefore $\mathbb{E}(M_t \mathbf{1}_A) = \mathbb{E}(M_s \mathbf{1}_A)$ for all $A \in \mathcal{F}_s$. $\square$

Corollary

If τ is a stopping time and and $(M_t)_{t\geq 0}$ is a martingale, then $(M_{t\wedge\tau})_{t\geq 0}$ is a martingale.

Proof.

By using the same notation as in the previous theorem $\tau_{ts} \wedge \tau$ is a bounded stopping time. So by the previous theorem $\mathbb{E}(M_{\tau_{ts}\wedge\tau}) = M_0 = \mathbb{E}(\tilde{M}_{\tau_{ts}})$, with $\tilde{M}_t := M_{t\wedge\tau}$and we can apply the previous proposition. $\square$

Proposition

A local martingale $(M_t)_{0\leq t\leq T}$ *such that* $\mathbb{E}(\sup_{0\leq t\leq T}|M_t|) < \infty$ *is in fact a martingale in* $[0, T]$.

Proof.

Let $(\tau_n)_{n\geq 0}$ with $\tau_n \uparrow \infty$ be the sequence of stopping times such that $(M_{t\wedge\tau_n})_{0\leq t\leq T}$ is, for all fixed n, a martingale. Then, for all $s \leq t$

$$\mathbb{E}(\, M_{t\wedge\tau_n} |\, \mathcal{F}_s) = M_{s\wedge\tau_n}$$

and

$$\lim_{n\to\infty} \mathbb{E}(\, M_{t\wedge\tau_n} |\, \mathcal{F}_s) = \lim_{n\to\infty} M_{s\wedge\tau_n} = M_s.$$

Now , since $\mathbb{E}(\sup_{0\leq t\leq T}|M_t|) < \infty$ we can apply the dominated convergence theorem. □

Proposition

A local martingale $(M_t)_{0\leq t\leq T}$ that is bounded from below is a supermartingale

Proof.

Let $(\tau_n)_{n\geq 0}$ with $\tau_n \uparrow \infty$ be the sequence of stopping times such that $(M_{t\wedge\tau_n})_{0\leq t\leq T}$ is, for all fixed n, a martingale. Then, for all $s \leq t$

$$\mathbb{E}(M_{t\wedge\tau_n} | \mathcal{F}_s) = M_{s\wedge\tau_n}$$

and since M is bounded from below we can apply the Fatou lemma and

$$M_s = \lim\inf_{n\to\infty} \mathbb{E}(M_{t\wedge\tau_n} | \mathcal{F}_s) \geq \mathbb{E}\left(\lim\inf_{n\to\infty} M_{t\wedge\tau_n} | \mathcal{F}_s\right) = \mathbb{E}\left(M_t | \mathcal{F}_s\right).$$

$\square$