# Resonances and linearization: vector field case

Let $x' = X(x) = Ax + u(x)$, such that $u(0) = 0$ and $Du(0) = 0$.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ we write

$$m \cdot \lambda \qquad \text{for} \qquad m_1\lambda_1 + \ldots + m_n\lambda_n,$$

being $m \in \mathbb{Z}_+^n$ and $|m| = m_1 + \ldots + m_n$.

In this setting we call resonance to

$$\lambda_j = m \cdot \lambda, \qquad m \in \mathbb{Z}_+^n, \quad |m\,| \geq 2, \quad j \in \{1, \ldots, n\}.$$

To each resonance $\lambda_j = m \cdot \lambda$ we associate the monomial $x^m e_j$ and we call it resonant monomial

**Definition (Poincaré's and Siegel's domains for vector fields)**

Given $\lambda = (\lambda_1, \ldots, \lambda_n)$, we say that $\lambda$ is in the Poincaré domain if

$$Re(\lambda_i) > 0, \quad \forall i \qquad \text{or} \qquad Re(\lambda_i) < 0, \quad \forall i$$

Otherwise (the real part of different eigenvalues have different sign or some eigenvalues have real part equal to zero), we say that $\lambda$ is in the Siegel domain.

### Theorem (Poincaré's Theorem for vector fields)

*Let $X$ be analytic in some open $U$, $0 \in U$ and $X(0) = 0$. Assume the eigenvalues of $A$ are in the Poincaré domain and they are non resonant.Then, $X$ is locally analytically conjugate to its lineal part $A$ in a neighbourhood of 0.*

### Theorem (Siegel's Theorem for vector fields)

*Let $X$ be analytic in some open $U$, $0 \in U$ and $X(0) = 0$. Assume the eigenvalues of $A$ are in the Siegel domain and satisfy the Diophantine condition: $\exists C, \tau > 0$ such that*

$$|\lambda_j - m \cdot \lambda| \geq \frac{C}{|m|^\tau}, \qquad \forall m \in \mathbb{Z}_+^n,\ |m| \geq 2,\ 1 \leq j \leq n:$$

*Then, $X$ is locally analytically conjugate to its linear part $A$ in a neighbourhood of 0.*

# Normal forms for vector fields

## Theorem (Normal forms)

*Let* $X(x) = Ax + X_2(x) + \ldots + X_r(x) + o(\|x\|^r)$. *There exists a polynomial change of variables that transforms* $X$ *into*

$$Y(y) = Ay + X_2^{(r)}(y) + \ldots + X_r^{(r)}(y) + o(\|y\|^r), \text{ where}$$

$$X_k^{(r)}(y) = \sum_{\substack{|m|=k \\ \lambda_j = m\cdot\lambda}} c_{m,j} y^m e_j$$

## Theorem (Sternberg's theorem for attractors)

*Given* $X(x)$ *as before,* $Spec(A) = \{\lambda_1, \ldots, \lambda_n\}$ *such that:*
1) $Re(\lambda_i) < 0$
2) $\lambda_i \neq m \cdot \lambda$

*Let* $k > \dfrac{max\{|Re(\lambda_i)|\}}{min\{|Re(\lambda_i)|\}}$. *Then, if* $X \in \mathcal{C}^k$, *we have that* $X$ *is* $\mathcal{C}^k$*-locally conjugate to its lineal part* $A$ *in a neighbourhood of the origin.*

## Sketch of the proof of the normal form theorem for vector fields

Assume that $A$ is in diagonal form.

Suppose we have made consecutive changes of variables $x = y + h_\ell(y)$, $2 \le \ell \le k-1$ and we have reached

$$x' = Ax + X_2^{(r)}(x) + \ldots + X_{k-1}^{(r)}(x) + X_k(x) + o(\|x\|^k).$$

Now we perform the change $x = y + h_k(y)$, $h_k \in E_k$, to obtain

$$\begin{aligned} y' + Dh_k(y)y' &= A(y + h_k(y)) + X_2^{(r)}(y + h_k(y)) + \ldots + X_{k-1}^{(r)}(y + h_k(y)) \\ &\quad + X_k(y + h_k(y)) + o(\|y\|^k), \end{aligned}$$

$$\begin{aligned} y' &= [Id + Dh_k(y)]^{-1}[Ay + X_2^{(r)}(y) + \ldots + X_{k-1}^{(r)}(y) + Ah_k(y) \\ &\quad + X_k(y) + o(\|y\|^k)] \\ &= Ay + X_2^{(r)}(y) + \ldots + X_{k-1}^{(r)}(y) \\ &\quad + Ah_k(y) + X_k(y) - Dh_k(y)Ay \\ &\quad + o(\|y\|^k) \end{aligned}$$

In order to study the terms of order $k$ in the previous equation:

$$Ah_k(y) + X_k(y) - Dh_k(y)Ay$$

we introduce the operator:

$$L_k : E_k \longrightarrow E_k, \qquad (L_k h)(x) = Dh(x)Ax - Ah(x)$$

**Lemma**

Assuming $A$ is a diagonal matrix,

$$\{x^m e_j \mid \ |m| = k,\ \ j \in \{1, \dots, n\}\}$$

is a basis of eigenvectors of $L_k$.
The corresponding eigenvalues are

$$m \cdot \lambda - \lambda_j$$

Proof: The operator $L_k$ is defined by $(L_k h)(x) = Dh(x)Ax - Ah(x)$

$$
\begin{aligned}
L_k(x^m e_j) &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_1 x_1^{m_1-1} x_2^{m_2} \ldots x_n^{m_n} & m_2 \dfrac{x^m}{x_2} & \cdots & m_n \dfrac{x^m}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \vdots \\ \vdots \\ \lambda_n x_n \end{pmatrix} \\
&\quad - \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ x^m \\ \vdots \\ 0 \end{pmatrix} \\
&= m \cdot \lambda x^m e_j - \lambda_j x^m e_j
\end{aligned}
$$

Hence,

$$
L_k(x^m e_j) = (m \cdot \lambda - \lambda_j) x^m e_j .
$$

In particular, in the case there are no resonances of order $k$, $L_k$ becomes exhaustive.

We write $E_k = \mathrm{Ker}\, L_k \oplus \mathrm{Im}\, L_k$

and

$$X_k(y) = X_k^{(r)}(y) + X_k^{(nr)}(y) \in \mathrm{Ker}\, L_k \oplus \mathrm{Im}\, L_k.$$

Therefore,

$$X_k^{(r)}(y)+X_k^{(nr)}(y)-Dh_k(y)+Ah_k(y) = X_k^{(r)}(y)+X_k^{(nr)}(y)-(L_k h_k)(y) = X_k^{(r)}(y),$$

where we have taken

$$L_k h_k = X_k^{(nr)}$$

# Center manifolds for maps

Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $U$ open set, $0 \in U$ and $f(0) = 0$.

Let $A = Df(0)$.

Let $Spec(A) = \sigma^s \cup \sigma^c \cup \sigma^u \subset \mathbb{C}$ and consider the decomposition of $\mathbb{R}^n$ corresponding to this splitting of the spectrum:

$$\mathbb{R}^n = E^s \oplus E^c \oplus E^u.$$

# Center manifold theorem for maps

### Theorem

*Let $f$ be as before such that $f \in \mathcal{C}^r$, $r \geq 1$. Then, there exists a manifold $W^c$ such that:*
*1) $W^c$ is invariant by $f$,*
*2) $T_0 W^c = E^c$,*
*3) $W^c$ is $\mathcal{C}^r$.*

This manifold is called center manifold.

### Remark

1) There is not uniqueness for $W^c$ by imposing $T_0 W^c = E^c$ and $W^c$ invariant. But, if $f \in \mathcal{C}^r$ and $W^c = graph(\varphi)$, the Taylor expansion up to order $r$ of $\varphi$ is unique.
2) As we will see by means of an example, $f \in \mathcal{C}^\omega$ does not imply $W^c \in \mathcal{C}^\omega$.
3) Even more, $f \in \mathcal{C}^\omega$ does not imply $W^c \in \mathcal{C}^\infty$.

## Center manifold computations

Given a map $f$ such that $f(0) = 0$ we write it as

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + g(x, y) \\ By + h(x, y) \end{pmatrix}$$

with

$$Spec(A) \subset \{\lambda | \, |\lambda| = 1\}, \qquad Spec(B) \subset \{\lambda | \, |\lambda| \neq 1\}$$

and

$$g(0, 0) = h(0, 0) = 0, \qquad Dg(0, 0) = Dh(0, 0) = 0.$$

We look for $W^c$ as an invariant graph of some function $\varphi$. We write

$$f\begin{pmatrix} x \\ \varphi(x) \end{pmatrix} = \begin{pmatrix} Ax + g(x,\varphi(x)) \\ B\varphi(x) + h(x,\varphi(x)) \end{pmatrix}.$$

The invariance condition by $f$ of the graph of $\varphi$ is

$$B\varphi(x) + h(x,\varphi(x)) = \varphi(Ax + g(x,\varphi(x))). \tag{1}$$

Assuming $\varphi \in \mathcal{C}^r$, we can expand:

$$\varphi(x) = \varphi_2(x) + \varphi_3(x) + \dots \quad \text{with} \quad \varphi_k \in E_k$$

and substitute this expansion into (1).

Comparing terms of the same order we can obtain $\varphi_j$ recursively.

## Reduction to the center manifold

One can reduce the study of the dynamics on the center manifold on the whole $n$-dimensional space to the one of the lower-dimensional map:

$$x \longmapsto Ax + g(x, \varphi(x))$$

This map is called the reduced map to the center manifold.

# Application to stability

**Remark**

If at least one of the eigenvalues of $B$ has modulus greater than one, then $W^u \neq \{0\}$ and 0 is unstable.

**Theorem (Stability Theorem)**

*If $Spec(B) \subset \{\lambda | \, |\lambda| < 1\}$, then the character of 0 (stable, asymptotically stable or unstable) as a fixed point of $f$ is the same as the character of 0 corresponding to $f$ reduced to $W^c$*

**Remark**

In the case that $W^c$ is one-dimensional, it is very easy to study the stability of 0. Indeed, in this case, if the eigenvalue is 1, the reduced map is

$$x \mapsto x + a_k x^k + \dots$$

The origin is asymptotically stable when $k$ is odd and $a_k < 0$.

**Theorem**

*Using the previous notation, if $f \in \mathcal{C}^1$ there exists a topological conjugation from $f$ to $\tilde{f}$, being*

$$\tilde{f}(x,y) = \begin{pmatrix} Ax + g(x,\varphi(x)) \\ By \end{pmatrix}.$$

# Center manifolds for vector fields

We can develop an analogous theory for vector fields $X$. Assume $X(0) = 0$.

We write $\mathbb{R}^n = E^s \oplus E^c \oplus E^u$, where now $E^c$ is the spectral subspace generated by the eigenvalues that have real part equal to zero.

If $X$ is $C^r$, $r \geq 1$, there exists an invariant manifold (by the flow of $X$), tangent to $E^c$ at 0 of class $C^r$. We denote it by $W^c$.

We can write the system in the form

$$\begin{cases} x' = Ax + g(x,y) \\ y' = By + h(x,y) \end{cases}$$

with

$$\lambda \in Spec(A) \Leftrightarrow Re(\lambda) = 0, \qquad \lambda \in Spec(B) \Leftrightarrow Re(\lambda) \neq 0$$

and

$$g(0,0) = h(0,0) = 0, \qquad Dg(0,0) = Dh(0,0) = 0.$$

In these variables there exists $\varphi$ such that $W^c_{loc} = \{(x,y) |\ y = \varphi(x)\}$ with $\varphi(0) = 0$, $D\varphi(0) = 0$.

## Invariance equation for $\varphi$

Here we find an equation for $\varphi$.

Let $(x(t), y(t))$ be the solution such that $(x(0), y(0)) = (x_0, y_0)$.

The invariance by $X$ of $W^c$ means that if $(x_0, y_0) \in W^c$, $(x(t), y(t)) \in W^c$ and hence $y(t) = \varphi(x(t))$ for all $t$.

Differentiating both sides of $y(t) = \varphi(x(t))$ we have

$$y'(t) = D\varphi(x(t))x'(t).$$

Substituting $y = \varphi(x)$ into the equation:

$$\begin{cases} x'(t) = Ax(t) + g(x(t), \varphi(x(t))) \\ y'(t) = B\varphi(x(t)) + h(x(t), \varphi(x(t))). \end{cases}$$

Then

$$B\varphi(x(t)) + h(x(t), \varphi(x(t))) = D\varphi(x(t))\left[Ax(t) + g(x(t), \varphi(x(t)))\right] .$$

Evaluating at $t = 0$ and writing $(x, y)$ instead of $(x_0, y_0)$ :

$$B\varphi(x) + h(x, \varphi(x)) = D\varphi(x)\left[Ax + g(x, \varphi(x))\right] .$$

The condition

$$B\varphi(x) + h(x, \varphi(x)) = D\varphi(x)[Ax + g(x, \varphi(x))]$$

is a partial differential equation that let us to find, together with $\varphi(0) = 0$ and $D\varphi(0) = 0$ the Taylor polynomial of $\varphi$ at 0. If the equation is $C^\infty$ we can obtain its (formal) power series.

**Reduced equation**

We call

$$x' = Ax + g(x, \varphi(x))$$

the reduced equation to the center manifold.

**Stability**

## Theorem

*Suppose that all eigenvalues of $B$ have negative real part. Then the character (stable, asymptotically stable, unstable) of the origin as an equilibrium point of the reduced equation $x' = Ax + g(x, \varphi(x))$ is the same as the one of the original equation.*

When $E^c$ is one dimensional the reduced equation is a one dimensional equation. The only center eigenvalue is 0. The reduced equation is of the form

$$x' = a_n x^n + a_{n+1} x^{n+1} + \ldots, \qquad n \geq 2.$$

If $a_n \neq 0$, the stability is easy to study.

## Several examples:

**Example 1:**

We look for the center manifold of $(0,0)$ for the system

$$\begin{cases} x' = -x^3 \\ y' = -y \end{cases}, \qquad DX(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

The explicit solutions are obtained as

$$x' = -x^3 \quad \Rightarrow \quad \frac{1}{2x^2} - \frac{1}{2x_0^2} = t \quad \text{if } x_0 \neq 0, \qquad x(t) = \frac{x_0}{(1+2x_0^2 t)^{1/2}}.$$

$$y' = -y \quad \Rightarrow \quad \ln\left(\frac{y_0}{y}\right) = t \quad \text{if } y_0 \neq 0, \qquad y(t) = e^{-t} y_0.$$

Equating $t$ from the above equations we obtain the solutions are contained in $y = y(x)$ where

$$\frac{y_0}{y} = \exp\left(\frac{1}{2x^2} - \frac{1}{2x_0^2}\right).$$

Therefore,

$$y(x) = c\exp\left(\frac{-1}{2x^2}\right), \qquad c = y_0 \exp\left(\frac{1}{2x_0^2}\right).$$

Then, maps $h$ that satisfy the invariance condition are:

$$\varphi(x, c_1, c_2) = \begin{cases} c_1 \exp\left(\frac{-1}{2x^2}\right) & x < 0 \\ 0 & x = 0 \\ c_2 \exp\left(\frac{-1}{2x^2}\right) & x > 0 \end{cases}$$

which is a biparametric family of center manifolds of class $\mathcal{C}^\infty$. This example shows the lack of uniqueness of $W^c$.

## Example 2:

Example that shows we cannot expect analyticity for a center manifold even if $f \in \mathcal{C}^\omega$, as has been asserted in previous pages. The system is

$$\begin{cases} x' = -x^3 \\ y' = -y + x^2 \end{cases}, \qquad DX(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose that the equation has an analytic center manifold. We can then look it as the graph of $\varphi$ having a power series: $y = \varphi(x) = a_2 x^2 + a_3 x^3 + \dots$.

We derive the invariance condition:

$$\begin{aligned} y' &= D\varphi(x)x', \\ -y + x^2 &= D\varphi(x)(-x^3), \\ -\varphi(x) + x^2 &= D\varphi(x)(-x^3), \\ x^3 D\varphi(x) - \varphi(x) + x^2 &= 0. \end{aligned}$$

Substituting the expansion we get

$$x^3 \left( \sum_{n \geq 2} n a_n x^{n-1} \right) - \left( \sum_{n \geq 2} a_n x^n \right) + x^2 = 0.$$

$$\left(\sum_{n\geq 2} na_n x^{n+2}\right) - \left(\sum_{n\geq 0} a_{n+2} x^{n+2}\right) + x^2 = 0$$

Comparing terms of the same degree:

$$\begin{aligned} -a_2 + 1 &= 0\,, \\ -a_3 &= 0\,, \\ na_n - a_{n+2} &= 0\,, \qquad \forall n \geq 2. \end{aligned}$$

Then, it is immediate to deduce that the odd coefficients are zero and the even ones satisfy the recursive relation $a_2 = 1$, $a_{n+2} = na_n$. Therefore, for all $n \geq 0$,

$$a_{2n+1} = 0, \qquad a_{2n} = (2n-2)(2n-4)\dots 2 = 2^{n-1}(n-1)!$$

Then, coefficients of the power expansion of $h$ grow as a factorial as we increase $n$, so the power series is not convergent and $h$ is not analytic.

# Example 3:

This example shows that we cannot expect $\varphi$ to be $\mathcal{C}^\infty$ even if $f \in \mathcal{C}^\omega$. The system of equations is:

$$\left\{ \begin{array}{l} x' = -\varepsilon x - x^3 \\ y' = -y + x^2 \\ \varepsilon' = 0 \end{array} \right. , \qquad DX(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose there exists $\varphi(x,\varepsilon) \in \mathcal{C}^\infty$, defined on $(-\delta,\delta) \times (-\delta,\delta)$ whose graph is $W^c_{loc}$.

Choose $n > \dfrac{1}{2\delta}$, $n \geq 2$, and consider $\varphi(x, \frac{1}{2n})$, which is $\mathcal{C}^\infty$ on $(-\delta,\delta)$. Then, by Taylor's Theorem, there exist coefficients $a_0, \ldots, a_{2n}$ such that

$$\varphi(x, \frac{1}{2n}) = \sum_{i=0}^{2n} a_i x^i + O(x^{2n+1})$$

The invariance condition for graph of $\varphi$ gives

$$
\begin{aligned}
y' = D_t(\varphi(x,\varepsilon)) &= D_x\varphi(x,\varepsilon)x' + D_\varepsilon\varphi(x,\varepsilon)\varepsilon' \ , \\
-y + x^2 &= D_x\varphi(x,\varepsilon)(-\varepsilon x - x^3) \ , \\
-\varphi(x,\varepsilon) + x^2 &= D_x\varphi(x,\varepsilon)(-\varepsilon x - x^3) \ .
\end{aligned}
$$

If $x = 0$ we get $\varphi(0,\varepsilon) = 0$. Taking derivatives:

$$
\begin{aligned}
-D_x\varphi(x,\varepsilon) + 2x &= D_{xx}\varphi(x,\varepsilon)(-\varepsilon x - x^3) + D_x\varphi(x,\varepsilon)(-\varepsilon - 3x^2) \ , \\
-D_x\varphi(0,\varepsilon) &= D_{xx}\varphi(0,\varepsilon)0 + D_x\varphi(0,\varepsilon)(-\varepsilon) \\
(1-\varepsilon)D_x\varphi(0,\varepsilon) &= 0 \ .
\end{aligned}
$$

So, taking $\varepsilon$ small enough (smaller than 1), $D_x\varphi(0,\varepsilon)$ has to be 0.

Therefore, the coefficients $a_0$, $a_1$ of the Taylor polynomial of $\varphi(., \frac{1}{2n})$ are 0. Now we can replace $\varphi(., \frac{1}{2n})$ by its Taylor polynomial of order $2n$ on the invariance condition for $\varphi$:

$$
\begin{aligned}
-\varphi(x,\varepsilon) + x^2 &= D_x\varphi(x,\varepsilon)(-\varepsilon x - x^3)\ , \\
-\sum_{i=2}^{2n} a_i x^i + O(x^{2n+1}) + x^2 &= \left[\sum_{i=2}^{2n} i a_i x^{i-1} + \tilde{O}(x^{2n})\right]\left(\frac{-1}{2n}x - x^3\right)\ , \\
-\sum_{i=2}^{2n} a_i x^i + x^2 + \frac{1}{2n}\sum_{i=2}^{2n} i a_i x^i + \sum_{i=2}^{2n} i a_i x^{i+2} &= O(x^{2n+1}) + \tilde{O}(x^{2n})\left(\frac{-1}{2n}x - x^3\right)\ , \\
-\sum_{i=2}^{2n} a_i x^i + x^2 + \frac{1}{2n}\sum_{i=2}^{2n} i a_i x^i + \sum_{i=4}^{2n}(i-2)a_{i-2}x^i &= O(x^{2n+1})\ .
\end{aligned}
$$

Therefore, proceeding as in the previous example, solving the equations for the coefficients at each degree:

$$-a_2 + 1 + \frac{1}{2n} 2a_2 = 0 \quad \Rightarrow \quad a_2 = \frac{1}{1 - \frac{1}{n}} \neq 0$$
$$-a_3 + \frac{3}{2n} a_3 = 0 \quad \Rightarrow \quad a_3 = 0$$
$$-a_i + \frac{i}{2n} a_i + (i-2) a_{i-2} = 0 \quad \Rightarrow \quad (1 - \frac{i}{2n}) a_i = (i-2) a_{i-2} \,, \quad 4 \leq i \leq 2n.$$

Then for $4 \leq i \leq 2n - 2$, $i$ even, we can write (since $1 - \frac{i}{2n} \neq 0$)

$$a_i = \frac{i-2}{1 - \frac{i}{2n}} a_{i-2} = \ldots = \frac{(i-2)(i-4)\ldots 2}{(1 - \frac{i}{2n})(1 - \frac{i-2}{2n})\ldots(1 - \frac{4}{2n})} a_2, \quad 4 \leq i \leq 2n - 2.$$

for even values of $i$. In particular: $a_{2n-2} \neq 0$. However, when $i = 2n$,

$$0 \cdot a_{2n} = (2n-2) a_{2n-2} \neq 0$$

reaching the wished contradiction.