Dynamical System Exercise Set 2.2

Leonardo Bocchi

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1 Exercise Statement

1. Consider the equation

$$x' = x - y - x\left(x^2 + y^2\right),$$
$$y' = x + y - y\left(x^2 + y^2\right).$$

Compute the Poincaré map of it with respect to the section $\Sigma = \left\{(x, y) \in R^2 \mid x > 0, y = 0\right\}$ in explicit form. Hint: use polar coordinates.

2. Let $f(x) = \lambda x + bx^2$ be a map from $\mathbb{R}$ to $\mathbb{R}$ with $|\lambda| \neq 0, 1$. Compute the Taylor expansion of a conjugation h between f and $Ax = \lambda x$, such that $h(0) = 0$ and $h'(0) = 1$, up to order 3. Do you think it is possible to find the Taylor expansion to all orders? If so, are the coefficients uniquely determined?

3. Consider the map

$$f(x, y) = \left(\lambda x, \lambda^2 y + x^2\right), \quad 0 < \lambda < 1.$$

Prove that f cannot be linearized with a C^2 conjugation.

2 Exercise 1

1. Consider the equation

$$x' = x - y - x\left(x^2 + y^2\right),$$
$$y' = x + y - y\left(x^2 + y^2\right).$$

Compute the Poincaré map of it with respect to the section $\Sigma = \left\{(x, y) \in R^2 \mid x > 0, y = 0\right\}$ in explicit form. Hint: use polar coordinates.

Proof

Consider the dynamical system described by the equations:

$$\dot{x} = x - y - x(x^2 + y^2)$$
$$\dot{y} = x + y - y(x^2 + y^2)$$

Using polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), the system becomes:

$$\dot{r} = r - r^3$$
$$\dot{\theta} = 1$$

Consider the following system of differential equations in polar coordinates, $(\theta, r) \in \mathbb{S}^1 \times \mathbb{R}^+$:

$$\begin{cases} \dot{\theta} = 1 \\ \dot{r} = (1 - r^2)r \end{cases}$$

The flow of the system can be obtained by integrating the equation: for the θ component we simply have $\theta(t) = \theta_0 + t$, while for the r component we need to separate the variables and integrate:

$$\int \frac{1}{(1-r^2)r}\,dr = \int dt \Longrightarrow \log\left(\frac{r}{\sqrt{1-r^2}}\right) = t + c$$

Inverting the last expression gives:

$$r(t) = \sqrt{\frac{e^{2(t+c)}}{1 + e^{2(t+c)}}}$$

Since $r(0) = \sqrt{\frac{e^{2c}}{1+e^{2c}}}$, we find:

$$r(t) = \sqrt{\frac{e^{2t}r_0^2}{1 + r_0^2(e^{2t} - 1)}} = \sqrt{\frac{1}{1 + e^{-2t}\left(\frac{1}{r_0^2} - 1\right)}}$$

The flow of the system is therefore:

$$\Phi_t(\theta, r) = \left(\theta + t, \sqrt{\frac{1}{1 + e^{-2t}\left(\frac{1}{r_0^2} - 1\right)}}\right)$$

The behavior of the flow is the following:

- The angle θ increases monotonically and at a constant rate.
- The radius r tends to the equilibrium $\bar{r} = 1$ for every value.

Therefore, the solution with initial data $(\theta_0, r_0 \neq 1)$ draws a spiral that tends towards the radius 1 circle.
The given section $\Sigma = \{(x, y) \in R^2 \mid x > 0, y = 0\}$ in polar coordinates corresponds to the positive horizontal axis $\Sigma = \{(\theta, r) : \theta = 0\}$. So we take this section as the Poincaré section for this flow. Naturally, we can use r as the coordinate on the section. Every point in Σ returns to the section after a time $t = 2\pi$, which can be understood by looking at the evolution of the angle. We

can take the Poincaré map as the restriction of Φ to the section Σ computed at the time 2π, $\Phi_{2\pi}|_\Sigma$. The Poincaré map is therefore:

$$\Psi(r) = \sqrt{\frac{1}{1 + e^{-4\pi}\left(\frac{1}{r^2} - 1\right)}}$$

The behavior of the orbits of the discrete dynamical system $(\Sigma, \mathbb{Z}, \Psi)$ is the following:

- The point $r = 1$ is fixed, so $\Psi^n(1) = 1$ for every n.
- Every other point tends monotonically to the equilibrium, $\Psi^n(z) \to 1$ for $n \to \pm\infty$. ■

3 Exercise 2

2. Let $f(x) = \lambda x + bx^2$ be a map from $\mathbb{R}$ to $\mathbb{R}$ with $|\lambda| \neq 0, 1$. Compute the Taylor expansion of a conjugation h between f and $Ax = \lambda x$, such that $h(0) = 0$ and $h'(0) = 1$, up to order 3. Do you think it is possible to find the Taylor expansion to all orders? If so, are the coefficients uniquely determined?

Proof

Consider a conjugation

$$h(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

for h to be a conjugation between f and $Ax = \lambda x$, we need

$$h \circ f = g \circ h$$

Let us expand the two terms

$$(h \circ f)(x) = c_0 + c_1(\lambda x + bx^2) + c_2(\lambda x + bx^2)^2 + c_3(\lambda x + bx^2)^3 + \dots$$

$$(g \circ h)(x) = \lambda(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots)$$

Now, let us compare the terms for each degree of x

- degree 0: $c_0 = \lambda c_0 \implies c_0 = 0$
- degree 1: $\lambda c_1 = \lambda c_1$ is satisfied $\forall c_1$
- degree 2: $\lambda c_2 = bc_1 + \lambda^2 c_2 \implies c_2 = \frac{bc_1}{\lambda - \lambda^2}$
- degree 3: $c_3 = 2\lambda b c_2 + \lambda^3 c_3 \implies c_3 = \frac{2\lambda b c_2}{1-\lambda^3} = \frac{2b^2 c_1}{(1-\lambda)(1-\lambda^3)}$
- degree 4: ... (each coefficient c_2 is expressed as a function of the previously determined ones, meaning the only free parameter left is c_1)

The Taylor expansion of $h(x)$ is

$$h(x) = h(0) + th'(0) + \frac{1}{2}t^2h''(0) + \frac{1}{3!}t^3h'''(0) + \dots$$

Let us compute each term

- degree 0: $h(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$

$$h(0) = c_0 = 0$$

- degree 1: $h'(x) = c_1 + 2c_2x + 3c_3x^2 + \dots$

$$h'(0) = c_1 = 1$$

  Hence, $c_2 = \frac{b}{\lambda-\lambda^2}$ and, $c_3 = \frac{2b^2}{(1-\lambda)(1-\lambda^3)}$

- degree 2: $h''(x) = 2c_2 + 3!c_3x + \dots$

$$h''(0) = 2c_2$$

- degree 3: $h'''(x) = 3!c_3 + \dots$

$$h'''(0) = 3!c_3$$

- degree 4: $\dots$ (each term $h^{(n)}(0)$ can be computed as a function of the parameter c_n)

Therefore, the Taylor expansion of h can be written as

$$h(x) = t + t^2c_2 + t^3c_3 + \dots$$

The Taylor expansion to higher orders can be computed extending the method used above, obtaining the terms $h^{(n)}(0)$ as a function of the parameters c_n, determined by the values of the given parameters λ, and b.

4 Exercise 3

3. Consider the map

$$f(x,y) = \left(\lambda x, \lambda^2 y + x^2\right), \quad 0 < \lambda < 1.$$

Prove that f cannot be linearized with a C^2 conjugation.

Proof

Consider the map

$$f(x,y) = (\lambda x, \lambda^2 y + x^2), \quad 0 < \lambda < 1$$

To prove that f cannot be linearized with a C^2 conjugation, let's assume, for the sake of contradiction, that there exists a C^2 conjugation h such that $h \circ f \circ h^{-1}$ is a linear map. The linearized map A is given by

$$A = \left.\frac{\partial(h \circ f \circ h^{-1})}{\partial(x, y)}\right|_{(0,0)}$$

Now, let's evaluate f at $(0, 0)$:

$$f(0, 0) = (0, 0)$$

The linearized map A must satisfy the condition

$$A \cdot (0, 0) = (0, 0)$$

However, computing $A \cdot (0, 0)$ involves the derivatives of h at the origin. Since h is a C^2 function, the second-order partial derivatives of h must exist. Let's denote h_x and h_y as the partial derivatives of h with respect to x and y, respectively. The composition $h \circ f \circ h^{-1}$ involves terms like $\lambda^2 h_y$, which will introduce a λ^2 term in the linearized map A. This term cannot be canceled out by any linear transformation, and since λ^2 is nonzero, it implies that A is not linear.
Therefore, the assumption that f can be linearized with a C^2 conjugation leads to a contradiction, and we conclude that f cannot be linearized with a C^2 conjugation. $\blacksquare$