Liapunov Functions

We consider $x' = X(x)$, $X : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^n$ of class C^1

Given $V : \mathcal{U} \to \mathbb{R}$ differenciable we define

$$\dot{V}(x) = DV(x)\, X(x) = D_{x_1} V(x)\, X_1(x) + \dots + D_{x_n} V(x)\, X_n(x)$$

Interpretation: if φ is the solution s.t. $\varphi(0) = x$

$$\frac{d}{dt} V(\varphi(t))\Big|_{t=0} = DV(\varphi(t))\,\varphi'(t)\Big|_{t=0} = DV(\varphi(t))\, X(\varphi(t))\Big|_{t=0} = DV(x)\, X(x) = \dot{V}(x)$$

Definition let $x_0 \in \mathcal{U}$ be an equilibrium point of $x' = X(x)$

A Liapunov function for x_0 is a differentiable function $V : \mathcal{U} \to \mathbb{R}$ s.t.

(1) $V(x_0) = 0$, $V(x) > 0$ if $x \in \mathcal{U} - \{x_0\}$

(2) $\dot{V}(x) \leq 0$, $\forall x \in \mathcal{U}$

A strict Liapunov function for x_0 is a Liapunov function such that

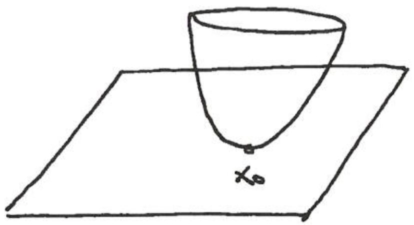
(3) $\dot{V}(x) < 0$, $\forall x \in \mathcal{U} - \{x_0\}$

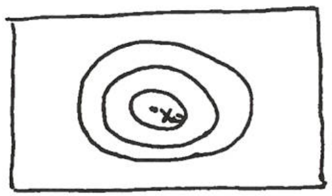
Geometric interpretation of the conditions which define Liapunov functions

(1) $\Rightarrow$ V has a global strict minimum

Then, locally, the graph of V behaves like a paraboloid

Consider the level sets of V near the fixed point x_0

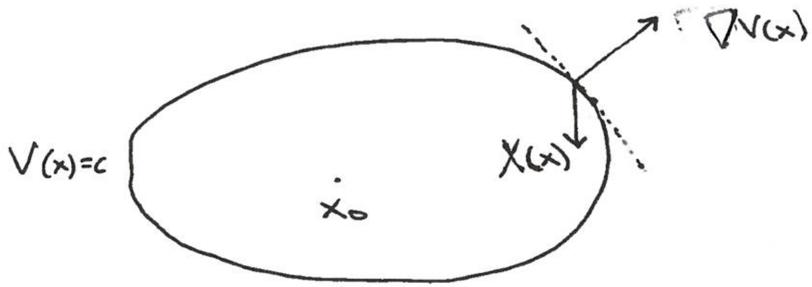




$$\dot V(x) = DV(x)\cdot X(x) = \langle \nabla V(x), X(x)\rangle$$

$\dot V(x) < 0 \Rightarrow$ Angle between $\nabla V(x)$ and $X(x)$ is bigger than $\frac{\pi}{2}$

$$|\text{angle}(\nabla V(x), X(x))| > \frac{\pi}{2}$$

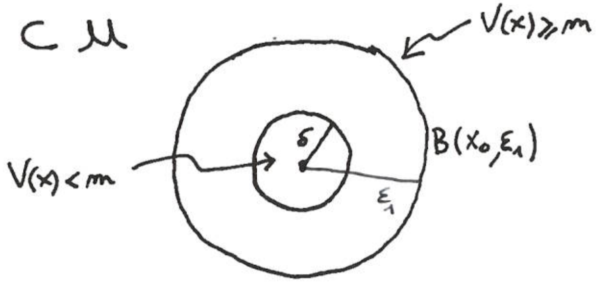


Thm x_0 equilibrium point of $x' = X(x)$

If $\exists$ a Liapunov function for x_0 then x_0 is stable

If $\exists$ a strict Liapunov function for x_0 then x_0 is asymptotically stable

Proof Let $\varepsilon > 0$. Let $0 < \varepsilon_1 < \varepsilon$ s.t. $\overline{B(x_0, \varepsilon_1)} \subset \mathcal{U}$



Let $m = \min\limits_{x \in \partial B(x_0, \varepsilon_1)} V(x) = V(x^*) > 0$

V continuous and $V(x_0) = 0 \Rightarrow \exists \delta > 0$ s.t. $x \in B(x_0, \delta)$, $|V(x) - \underbrace{V(x_0)}_{=0}| < m$

Let $x \in B(x_0, \delta)$ and $\varphi(t,x)$ the solution defined in the maximal interval (ω_-, ω_+)

(2) $\Rightarrow V(\varphi(t,x))$ is decreasing $\Rightarrow V(\varphi(t,x)) < m$, $\forall t \in [0, \omega_+)$

$\Rightarrow \varphi(t,x) \in B(x_0, \varepsilon_1)$, $\forall t \in [0, \omega_+)$ since φ can not have passed through the boundary

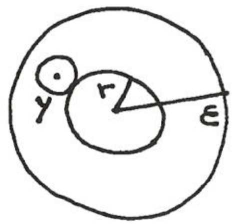
If $\omega_+ < \infty$ we have a contradiction. Then $\omega_+ = \infty$

Therefore x_0 is stable since $\varphi(t,x) \in B(x_0, \varepsilon_1) \subset B(x_0, \varepsilon)$ $\forall t \geq 0$

Proof of the asymptotic stability

Let V be a strict Liapunov function. We already know that x_0 is stable.

Let $x \in B(x_0, \delta)$ and assume that $\lim_{t\to\infty} \varphi(t,x) \neq x_0$



Then $\exists r > 0$ and $t_n \to \infty$ s.t. $d(\varphi(t_n, x), x_0) \geq r > 0$

By compactness there is a subsequence, which we denote again $\varphi(t_n, x_0)$ which has a limit $y \in \bar{B}(x_0, \varepsilon_1) \setminus B(x_0, r)$

Since $V(\varphi(t,x))$ is decreasing $\Rightarrow V(\varphi(t,x)) > \lim_{n\to\infty} V(\varphi(t_n, x)) = V(y)$

Since y is not a fixed point $\dot{V}(y) < 0 \qquad \left(\Rightarrow \frac{d}{dt} V(\varphi(t,y))_{|t=0} < 0\right)$

$$V(\varphi(1,y)) < V(\varphi(0,y)) = V(y)$$

By continuity of $V(\varphi(1,y))$ with respect to y

$\exists \rho > 0$ s.t. $\forall z \in B(y, \rho) \qquad V(\varphi(1,z)) < V(y)$

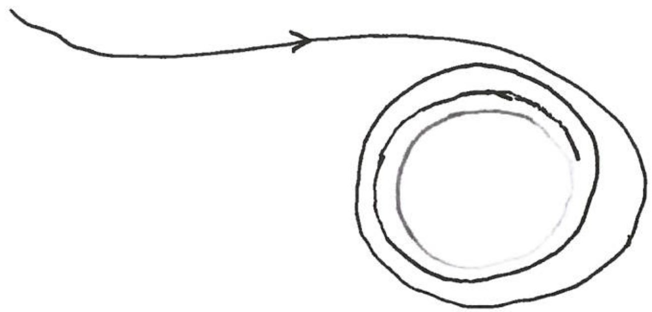
$\exists m$ s.t. $\varphi(t_m, x) \in B(y, \rho) \Rightarrow V(\varphi(1, \varphi(t_m, x))) = V(\varphi(t_m + 1, x)) > V(y)$

Contradiction

Given $x' = X(x)$ let $\varphi(t,x)$ be its flow.

We recall the def. of ω-limit set of a point:

$$\omega(x) = \{ y \in M \mid \exists t_k \longrightarrow \infty \ \text{ s.t. } \lim_{k\to\infty} \varphi(t_k, x) = y \}$$



Properties:

- $\omega(x)$ is invariant
- If $\{\varphi(t,x) \mid t \geq 0\} \subset K$ compact set, then $\omega(x) \neq \emptyset$.

Lemma Let M be an open set, $X : M \subset \mathbb{R}^n \to \mathbb{R}^n$ and $\dot{x} = X(x)$.

Let $V : M \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function such that $\dot{V}(x) = DV(x)\, X(x)$ satisfies either $\dot{V} \leq 0$ or $\dot{V} \geq 0$ on M. Then given $x \in M$

$$\omega(x) \subset \{ x \in M \mid \dot{V}(x) = 0 \}$$

Proof Assume $\omega(x) \neq \phi$. If $y \in \omega(x)$, $y = \lim_{k \to \infty} \varphi(t_k, x)$.

Then, for t small

$$V(\varphi(t, y)) = V\left(\varphi\left(t, \lim_{k\to\infty} \varphi(t_k, x)\right)\right) = V\left(\lim_{k\to\infty} \varphi(t + t_k, x)\right) = \lim_{k\to\infty} V(\varphi(t + t_k, x))$$

$$= \lim_{s\to\infty} V(\varphi(s, x)) = \lim_{k\to\infty} V(\varphi(t_k, x)) = V\left(\lim_{k\to\infty} \varphi(t_k, x)\right) = V(y).$$

This means that $V(\varphi(t, y))$ is constant. Then

$$0 = \frac{d}{dt}\left[V(\varphi(t, y))\right] = (DV \cdot X)(\varphi(t, y)) \quad \text{for } t \text{ small}$$

$\Rightarrow$ $\varphi(t, y) \in \{x \mid \dot{V}(x) = 0\}$ and in particular $y = \varphi(0, y) \in \{x \mid \dot{V}(x) = 0\}$

Theorem (LaSalle principle)

Consider the eq. $x' = X(x)$, $X \in C^1(\mathcal{U})$, $x_0 \in \mathcal{U}$ an equilibrium pt., and $V : \mathcal{U} \to \mathbb{R}$ a Liapunov function for x_0.

Let $Z = \{x \in \mathcal{U} \mid \dot{V}(x) = 0\}$

If for every solution φ, $\{\varphi(t) \mid t \in [0, \infty)\} \not\subset Z$ except $\varphi(t) = x_0$, then x_0 is asymptotically stable

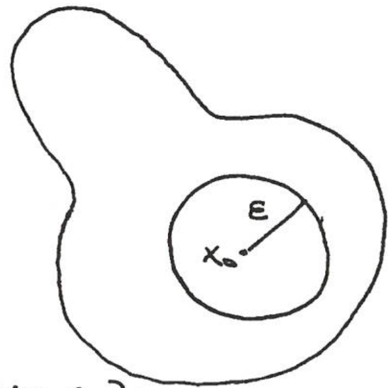
Remark If V is a strict Liapunov function it satisfies the property of the Theorem

Proof.

Since V is a Liapunov function for x_0, x_0 is stable.

Let $\varepsilon > 0$ be such that $\overline{B(x_0, \varepsilon)} \subset \mathcal{U}$ and $\delta > 0$ be given by the definition of stability, depending on ε.

Let $x \in B(x_0, \delta)$ and consider $\omega(x)$ which is



- non-empty
- contained in $\overline{B(x_0, \varepsilon)}$

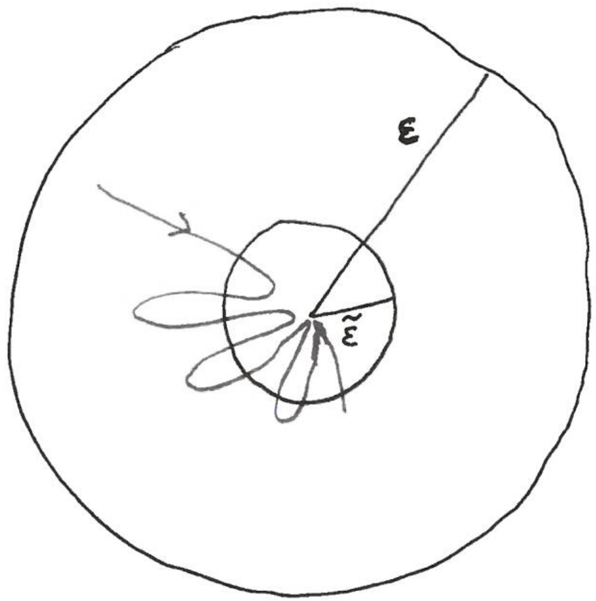
By the lemma $\omega(x) \subset Z$ and if $y \in \omega(x)$, $\{\varphi(t, y) \mid t \geq 0\} \subset \omega(x) \subset Z$

Then $\varphi(t, y) = x_0 \ \forall t \Rightarrow y = x_0 \Rightarrow \lim\limits_{t \to \infty} \varphi(t, x) \overset{(*)}{=} x_0$

(*) If $\lim \varphi(t,x) \neq x_0$ $\quad \exists \tilde{\varepsilon} > 0$ s.t. $\forall T$ $\quad \exists t > T$ s.t. $\|\varphi(t,x) - x_0\| > \tilde{\varepsilon}$

$$\Rightarrow \quad \exists \, t_n \to \infty \text{ s.t. } \|\varphi(t_n,x) - x_0\| > \tilde{\varepsilon}$$

Since $\varphi(t_n,x) \in \overline{B(x_0,\varepsilon)} \setminus B(x_0,\tilde{\varepsilon})$ $\exists$ subsequence of $\varphi(t_n,x)$ converging to a point in $\overline{B(x_0,\varepsilon)} \setminus B(x,\tilde{\varepsilon})$. This point, by def, belongs to $\omega(x)$. Contradiction.



Example

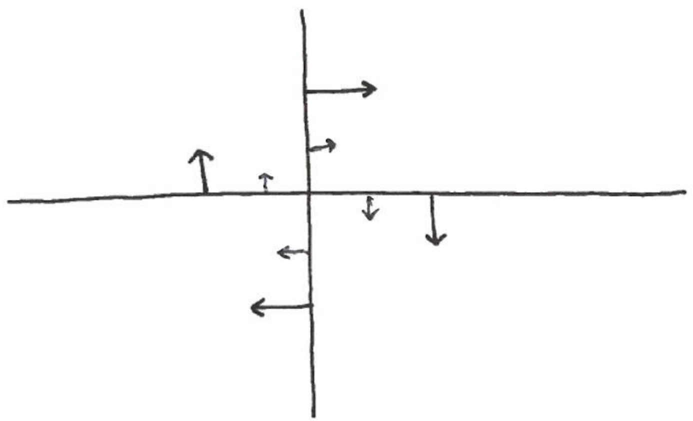
$$\dot{x} = y - xy^2$$
$$\dot{y} = -x^3$$

$(0,0)$ is equilibrium pt., $DF(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $(0,0)$ is linearly unstable.

Consider $V(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2$

$$\dot{V} = x^3(y - xy^2) + y(-x^3) = -x^4y^2 \le 0 \quad \Rightarrow (0,0) \text{ is stable}$$

$Z = \{x=0\} \cup \{y=0\}$



at $(x,0)$ the v.f. is $\begin{pmatrix} 0 \\ -x^3 \end{pmatrix}$

at $(0,y)$ " " is $\begin{pmatrix} y \\ 0 \end{pmatrix}$

Example Consider the mechanical system

$$H(x,y) = \frac{y^2}{2} + W(x), \qquad W \in C^2$$

(then the vector field is C^1 and there is existence and uniqueness of solutions)

The equations are

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -W'(x) \end{aligned}\right\} \qquad \ddot{x} = -W'(x)$$

Claim:

If x_0 is a minimum of W then $(x_0, 0)$ is a stable fixed point

Indeed, let us take the function $V = H$

$$\left.\begin{aligned} \dot{V} &= \frac{\partial H}{\partial x} \cdot y + \frac{\partial H}{\partial y}(-W'(x)) = W'(x)\, y + y\,(-W'(x)) = 0 \\ V(x,y) &= H(x,y) = \frac{y^2}{2} + W(x) \geq \frac{y^2}{2} + W(x_0) \geq V(x_0, 0) \end{aligned}\right| \Rightarrow \tilde{V} = V - V(x_0,0) \text{ is a Liapunov function}$$

Ex The pendulum $\ddot{x} = -\sin x$; $W(x) = 1 - \cos x$ $x=0$ is a minum of W

$$H(x,y) = \frac{y^2}{2} + 1 - \cos x \qquad (0,0) \text{ is stable}$$

If we add friction, i.e. a force depending on the velocity, oposite to the motion

$$\ddot{x} = -W'(x) - K\dot{x}$$

the system becomes

$$\dot{x} = y$$
$$\dot{y} = -W'(x) - Ky$$

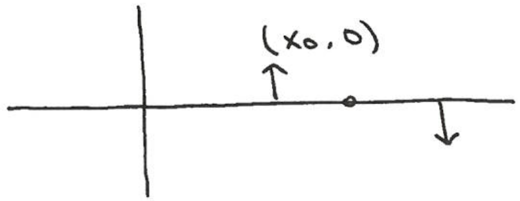
We assume that W' is strictly increasing (A sufficient condition is $W'' > 0$).
Taking the same Liapunov function as before

$$\dot{V} = W'(x)y + y\left[-W'(x) - Ky\right] = -Ky^2 \leq 0 \longrightarrow Z = \{(x,y) \mid y = 0\}$$

If $(x,0) \in Z$, the vector field is $\begin{pmatrix} 0 \\ -W'(x) \end{pmatrix}$

If $x > x_0 \quad W'(x) > W'(x_0) = 0$

$\quad x < x_0 \quad W'(x) < W'(x_0) = 0$



Then no solution can be contained in Z, apart from the equilibrium pt.

Dirichlet's theorem

Let $H : M \longrightarrow \mathbb{R}$ be a Hamiltonian function of class C^2, M a phase space, $2n$-dimensional

Let (x,y) the variables in M (for instance $M = \mathbb{R}^n \times \mathbb{R}^n$ or $\mathbb{T}^n \times \mathbb{R}^n$)

The equations of motion are

$$\dot{x} = \frac{\partial H}{\partial y}(x,y)$$

$$\dot{y} = -\frac{\partial H}{\partial x}(x,y)$$

Theorem If $(x_0, y_0) \in M$ is an isolated equilibrium point and H has a minimum or maximum at (x_0,y_0), then (x_0,y_0) is stable

Proof Take $V = H - H(x_0,y_0)$ if (x_0,y_0) is minimum or

$$V = -(H - H(x_0,y_0)) \quad \text{if } (x_0,y_0) \text{ is maximum.}$$

Consider the minimum case. $V(x,y) \geq 0$ in a nbh $\mathcal{U}$ of (x_0,y_0). I claim that $\exists\, \mathcal{U}_1 \subset \mathcal{U}$ nbh of (x_0,y_0) such that $V(x,y) > 0$ for $(x,y) \in \mathcal{U}_1 \setminus \{(x_0,y_0)\}$. If not $\exists$ a sequence $(x_k,y_k) \longrightarrow (x_0,y_0)$ such that $V(x_k,y_k) = 0$. They are minimums of H and therefore $\nabla H(x_k,y_k) = 0 \Rightarrow (x_k,y_k)$ are equilibrium pts.

On the other hand $\dot{V}(x,y) = \frac{\partial H}{\partial x}(x,y)\,\dot{x} + \frac{\partial H}{\partial y}(x,y)\,\dot{y} = 0$

<u>Remark</u> If (x_0, y_0) is not an isolated equilibrium point it may not be stable

Consider $H(x,y) = \frac{x^2}{2}$

$$\dot{x} = \frac{\partial H}{\partial y} = 0$$

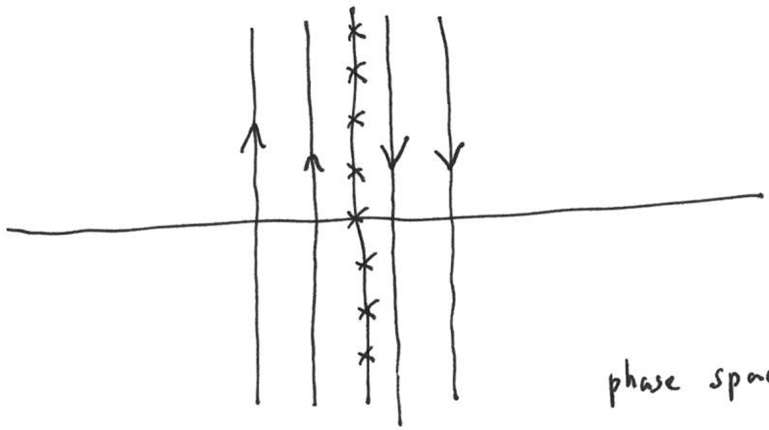
$$\dot{y} = -\frac{\partial H}{\partial x} = -x$$

The line $\{(0,y) \mid y \in \mathbb{R}\}$ is a line of equilibrium points

All points $(0,y)$ are minimums of H.

The system of eq. is linear with matrix $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

All equilibrium points are unstable



phase space

Basin of attraction let x_0 be an asymptotically stable equilibrium point

we denote basin of attraction of x_0 the set

$$\Omega(x_0) = \{ x \in M ; \lim_{t \to \infty} \varphi(t,x) = x_0 \}$$

Def. We say that $P \subset M$ is positively invariant if $\forall x \in P$, $\varphi(t,x)$ is defined for all $t \geq 0$ and $\varphi(t,x) \in P$ $\forall t \geq 0$.

We recall the def of ω-limit set of a pt:

$$\omega(x) = \{ y \in M ; \exists t_k \to \infty \text{ and } \lim_{k \to \infty} \varphi(t_k, x) = y \}$$

If $\{ \varphi(t,x) ; t \geq 0 \} \subset K$ compact set then $\omega(x) \neq \emptyset$.

Prop

Let x_0 be an equilibrium point of $\dot{x} = X(x)$, $X : M \subset \mathbb{R}^n \to \mathbb{R}^n$

Let $P \subset M$, $x_0 \in P$, positively invariant and compact.

Let $V : M \to \mathbb{R}$ be a diff. function such that $\dot{V} \leq 0$. Let $Z = \{x \in M \mid \dot{V}(x) = 0\}$.

Assume that for all solutions φ, $\{\varphi(t) \mid t \in [0,\infty)\} \not\subset Z$ except the ctant. solution $\varphi(t) = x_0$

Then x_0 is asymptotically stable and $P \subset \Omega(x_0)$

Proof Let $x \in P$. $\omega(x) \neq \emptyset$ and $\omega(x) \subset P$.

By the lemma $\omega(x) \subset Z$.

For $y \in \omega(x)$, $\{\varphi(t,y) \mid t \in [0,\infty)\} \subset Z \Rightarrow \varphi(t,y) = x_0 \Rightarrow y = x_0$

Then for all $x \in P$, $\lim_{t \to \infty} \varphi(t,x) = x_0$.

It remains to be proved that x_0 is stable.

Let $x \in P$.

$V(\varphi(t,x))$ is decreasing $\Longrightarrow$ $V(x) = V(\varphi(0,x)) \geqslant V(\varphi(t,x)) \geqslant V(x_0)$, $t \geqslant 0$.

We define $\tilde{V}(x) = V(x) - V(x_0)$. We have

- $\tilde{V}(x) > 0$ if $x \in P \setminus \{x_0\}$

  Indeed, if $x \neq x_0$, $\{\varphi(t,x) \mid t > 0\} \not\subset Z$.

  $$\Rightarrow \exists t_1 \text{ s.t. } \dot{V}(\varphi(t_1,x)) < 0$$

  $$\Rightarrow V(x) > V(x_0)$$

- $\dot{\tilde{V}}(x) = \dot{V}(x) \leq 0$

Then $\tilde{V}$ is a Liapunov function for x_0 and therefore x_0 is stable.

$\square$

Ex Lotka – Volterra

$$\begin{cases} \dot{x} = x(a-by) \\ \dot{y} = y(-c+dx) \end{cases} \qquad a,b,c,d > 0$$

x prey population

y predator population

Fixed pts $(0,0)$, $\left(\frac{c}{d}, \frac{a}{b}\right)$

$$DF(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \qquad DF\left(\frac{c}{d}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -bc/d \\ ad/b & 0 \end{pmatrix}$$

One looks for a Liapunov function as a separate variables function

$$V(x,y) = H(x) + G(y)$$

we impose $\dot{V} = 0$

$$H'(x)\, x(a-by) + G'(y)\, y(-c+dx) = 0$$

$$\Leftrightarrow \quad \frac{xH'(x)}{-c+dx} = -\frac{y\, G'(y)}{a-by}$$

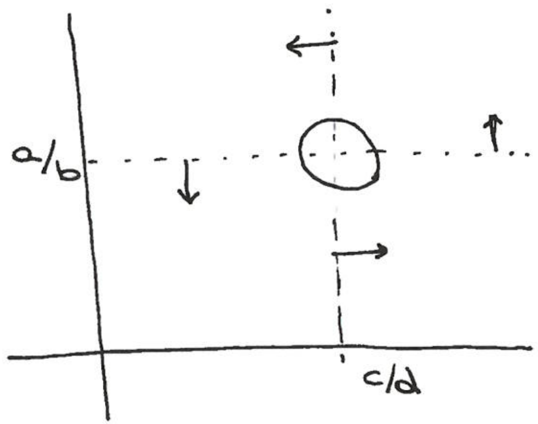
$$\Longleftrightarrow \quad \frac{xH'(x)}{-c+dx} = -\frac{yG'(y)}{a-by}$$

$$\begin{cases} \Rightarrow \dfrac{xH'(x)}{-c+dx} = A \text{ constant} & \Rightarrow H'(x) = A\left(\dfrac{-c}{x}+d\right) \Rightarrow H(x) = A(dx - c\ln x) \\[2ex] \Rightarrow \dfrac{yG'(y)}{a-by} = -A & \Rightarrow G'(y) = -A\left(\dfrac{a}{y} - b\right) \Rightarrow G(y) = A(by - a\ln y) \end{cases}$$

Take

$V(x,y) = dx - c\ln x + by - a\ln y$. It is a first integral, that has a strict min. in $(c/d, a/b)$

Sketch of phase portrait (null-clines)



Ex Lotka-Volterra with intraspecific competition

$$\begin{cases} \dot{x} = x(a - ex - by) \\ \dot{y} = y(-c + dx - fy) \end{cases}$$

we are interested in the dynamics on Q_1

$a, b, c, d, e, f > 0$

Fixed points $(0,0)$, $\begin{cases} x = 0 \\ -c - fy = 0 \end{cases}$ $y = \frac{-c}{f}$, $\begin{cases} y = 0 \\ x = \frac{a}{e} \end{cases}$

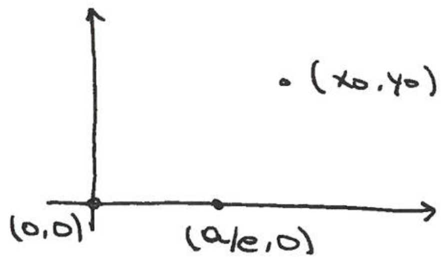
(x_0, y_0) s.t. $a - ex_0 - by_0 = 0$

$-c + dx_0 - fy_0 = 0$

$$\begin{pmatrix} e & b \\ d & -f \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \longrightarrow \begin{cases} x_0 = \dfrac{\begin{vmatrix} a & b \\ c & -f \end{vmatrix}}{\begin{vmatrix} e & b \\ d & -f \end{vmatrix}} = + \dfrac{af + cb}{ef + bd} \\[2em] y_0 = \dfrac{\begin{vmatrix} e & a \\ d & c \end{vmatrix}}{\begin{vmatrix} e & b \\ d & -f \end{vmatrix}} = \dfrac{ec - ad}{-(ef + bd)} \end{cases}$$

We consider the case $ad - ce > 0$

$(\Rightarrow (x_0, y_0) \in Q_1)$



$$DX = \begin{pmatrix} a - 2ex - by & -bx \\ dy & -c + dx - 2Fy \end{pmatrix}$$

$$DX(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \qquad \text{Saddle}$$

$$DX(a/e, 0) = \begin{pmatrix} -a & -ab/e \\ 0 & -c + 2da/e = \frac{ad - ce}{e} > 0 \end{pmatrix}$$

Saddle

$$DX(x_0, y_0) = \begin{pmatrix} -ex_0 & -bx_0 \\ dy_0 & -Fy_0 \end{pmatrix}$$

$$\operatorname{tr} DX = -(ex_0 + Fy_0) < 0$$

$$\det DX = x_0 y_0 (eF + db) > 0$$

$$\lambda^2 - \operatorname{tr}\lambda + \det = 0, \qquad \lambda_{1,2} = \frac{\operatorname{tr} \pm \sqrt{\operatorname{tr}^2 - 4\det}}{2}$$

If $\lambda_{1,2} \in \mathbb{R} \Rightarrow \lambda_{1,2} < 0$

$\lambda_{1,2} \in \mathbb{C} \Rightarrow \operatorname{Re} \lambda_{1,2} < 0$

Claim The basin of attraction of (x_0, y_0) is $\{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$

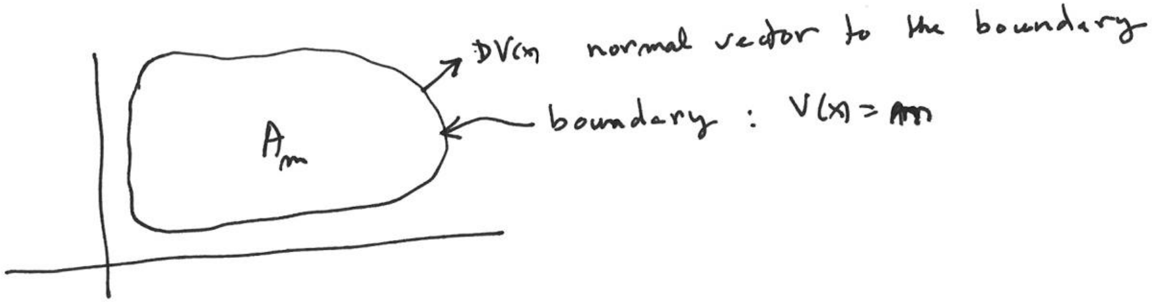
We consider the Liapunov function $V(x,y) = d(x - x_0 \log x) + b(y - y_0 \log y)$

A calculation gives $\dot{V}(x,y) = -de(x-x_0)^2 - bf(y-y_0)^2 < 0$ if $(x,y) \neq (x_0,y_0)$

Given $m > V(x_0,y_0)$ we have that $A_m = \{(x,y) \mid x>0, y>0, V(x,y) \leq m\}$ is compact.

Indeed, assume A_m is not bounded. Then $\exists$ a sequence (x_k, y_k) of points of A_m such that $\|(x_k,y_k)\| \to \infty$. This means that either $\{x_k\}$ or $\{y_k\}$ are unbounded. Then $V(x_k,y_k) \to \infty$ but also $V(x_k,y_k) \leq m$ (contradiction)

Also A_m is positively invariant:



The condition $\dot{V}(x) = DV(x) \cdot X(x) < 0$ implies the solutions enter into A_m

By a proposition $A_m \subset \Omega(x_0,y_0)$

Let $(\xi,\eta) \in \{(x,y) \mid x>0, y>0\}$. $(\xi,\eta) \in A_{m=V(\xi,\eta)} \subset \Omega(x_0,y_0)$ $\Rightarrow$ $\{(x,y) \mid x>0, y>0\} \subset \Omega(x_0,y_0)$

Stability of a solution

Let $x' = F(t,x)$ and $\gamma(t)$ be a solution defined on $[t_0, \infty)$

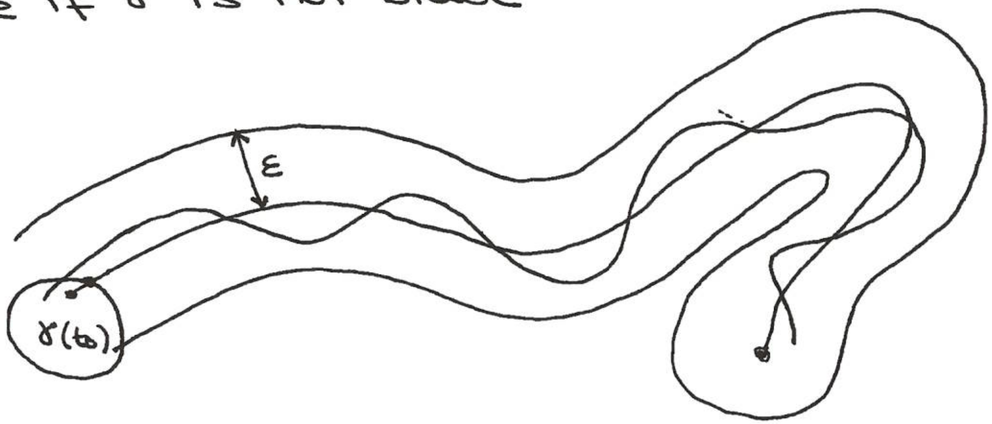
γ is stable if $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. if $\|x - \gamma(t_0)\| < \delta$

- $\varphi(\cdot, t_0, x)$ is defined for all $t \geq t_0$
- $\|\varphi(t, t_0, x) - \gamma(t)\| < \varepsilon \quad \forall t \geq t_0$

γ is asymptotically stable if

- γ is stable
- $\exists \eta > 0$ s.t. if $\|x - \gamma(t_0)\| < \eta$, $\|\varphi(t, t_0, x) - \gamma(t)\| \underset{t \to \infty}{\longrightarrow} 0$

γ is unstable if γ is not stable



The study of the stability of an orbit can be reduced to the study of the stability of a fixed point.

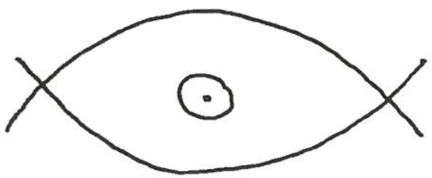
Indeed, we do the change $y = x - \gamma(t)$ and

$$y' = x' - \gamma'(t) = F(t,x) - \gamma'(t) = F(t, y+\gamma(t)) - \gamma'(t) = g(t,y)$$

Note that $g(t,0) = F(t,\gamma(t)) - \gamma'(t) = 0$ because γ is a solution.

If $x' = F(x)$ and γ is a P.O. then $y' = g(t,y)$ is a periodic system.

Example Pendulum



The P.O. are not stable in the above sense since the period depends on the P.O.

orbital stability

Theorem Consider the equation $\dot{x} = A(t)x + h(t,x)$ such that

1) A is continuous and T-periodic

2) h is continuous and $o(x)$ uniformly in t.

3) the I.V.P. has unique solution for every initial condition

Then

(a) If the real parts of the characteristic exponents of $\dot{x} = A(t)x$ are negative,

0 is asymptotically stable.

(b) If there is a characteristic exponent of $\dot{x} = A(t)x$ positive,

0 is unstable

Stability of fixed points of maps

Let $M \subset \mathbb{R}^n$ be open and $f : M \longrightarrow \mathbb{R}^n$

$x_0 \in M$ is a fixed point of f if $f(x_0) = x_0$

- x_0 is stable if

  $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that if $\|x - x_0\| < \delta$, $\quad \|f^n(x) - x_0\| < \varepsilon, \quad \forall n \geq 0.$

- x_0 is asimptotically stable if
  - x_0 is stable
  - $\exists \eta > 0$ such that if $\|x - x_0\| < \eta$, $\quad \lim_{n \to \infty} f^n(x) = x_0$

- x_0 is unstable if it is not stable

Lemma

Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $m = \max\{|\lambda| ; \lambda \in \operatorname{Spec} A\}$

Then, $\forall \varepsilon > 0 \quad \exists \|\cdot\|$ in $\mathbb{R}^n$ s.t.

$$\|A\| \leq m + \varepsilon$$

Stability Criterium for fixed points of maps

Prop Let $\mathcal{U}$ be an open set of $\mathbb{R}^n$, $0 \in \mathcal{U}$, $f : \mathcal{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Assume

- $f(0) = 0$
- f is differentiable at 0
- $\operatorname{Spec} Df(0) \subset \{ \lambda \in \mathbb{C} \mid |\lambda| < 1 \}$

Then 0 is asymptotically stable.

Proof Let $A = Df(0)$. We have

- $\exists \, \|\cdot\|$ in $\mathbb{R}^n$ s.t. $a \equiv \|A\| < 1$
- $f(x) = Ax + \mu(x)$, $\displaystyle\lim_{x \to 0} \frac{\mu(x)}{\|x\|} = 0$

$\forall \eta > 0 \quad \exists r > 0$ s.t. if $\|x\| < r$, $\|\mu(x)\| < \eta \|x\|$

Let η be such that $\boxed{a + \eta < 1}$ and r be the corresponding radius.

<u>Claim</u> $\forall m \geq 0$, $\forall x \in B(0,r)$

$f^m(x) \in B(0,r)$ and $\| f^m(x) \| \leq (a+\eta)^m \|x\|$.

- $m=0$ true
- Assume true for m

$$\| f^{m+1}(x) \| = \| f(f^m(x)) \| \leq \| A f^m(x) \| + \| \mu (f^m(x)) \|$$

$$\leq a \| f^m(x) \| + \eta \| f^m(x) \| = (a+\eta) \| f^m(x) \|$$

$$\leq (a+\eta)^{m+1} \|x\|$$

Stability: Given $\varepsilon > 0$ let $\delta = \min(\varepsilon, r)$

If $\|x\| < \delta$, $\| f^m(x) \| \leq (a+\eta)^m \|x\| \leq \|x\| < \delta \leq \varepsilon$

$\forall m \geq 0$

Asymptotic stability:

If $\|x\| < \delta$, $f^m(x) \longrightarrow 0$

Remarks

(1) If the fixed point is $p \neq 0$ then we would have

$$\| f^m(x) - p \| \leq (a+\eta)^m \| x - p \|$$

(2) The proofs works if $\det A = 0$