Dynamical System Exercise Set Final 1,2

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December 2023

Exercise Statement

1. Given the map
$$f(x,y,z) = A(x,y,z)^\top + \left(yz, x^2, y^2\right)^\top$$
with
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$
1) Compute the approximation of the center manifold of the origin up to order 4 (included), 2) compute the reduction of the map to the center manifold, 3) study the stability of the origin.

2. Let $X(x)$ a vector-field of $\mathbb{R}^2$ such that $X(0) = 0$ and with 0 a non-hyperbolic equilibrium point such that
$$DX(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
Prove that a normal form in a neighborhood of the origin, up to order 2, is
$$\begin{aligned} x' &= y + O_3 \\ y' &= ax^2 + bxy + O_3 \end{aligned}$$

Exercise 1

Given the map
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Proof.
1. The map is already given in a convenient form, with the linear part diagonalized, and the non-linear part separated. Let us consider the origin, which is a fixed point for the map. Hence, the eigenvalues are immediate, $1, \frac{1}{2}, \frac{1}{3}$. We identify the variable associated with the center manifold, for which we have

$$x \mapsto Ax + f(x, y, z)$$

where in this case $A = 1$, and $f(x, y, z) = yz$. Then, we identify the variables associated with the stable manifold

$$\begin{bmatrix} y \\ z \end{bmatrix} \mapsto B \begin{bmatrix} y \\ z \end{bmatrix} + g(x, y, z)$$

where $B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$, and $g(x, y, z) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$. We can now define the manifold h as

$$h : E^c \to E^s \tag{1}$$
$$x \mapsto h(x) \tag{2}$$

and impose that $h(x) - \begin{bmatrix} y \\ z \end{bmatrix} = 0$, which results in imposing

$$h(Ax + f(x, h(x))) - Bh(x) - g(x, h(x)) = 0 \tag{3}$$

Therefore, let us determine the coefficient of

$$h(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} a_1x^2 + b_1x^3 + c_1x^4 + O(5) \\ a_2x^2 + b_2x^3 + c_2x^4 + O(5) \end{bmatrix} = ax^2 + bx^3 + cx^4 + O(5)$$

so that it satisfies equation 3. For the first term, we have a common expression for $h_1(x)$, and $h_2(x)$

$$\begin{aligned}
&h(Ax + f(x, h(x))) = \\
&a\left[x + (a_1x^2 + b_1x^3 + c_1x^4 + O(5))(a_2x^2 + b_2x^3 + c_2x^4 + O(5))\right]^2 + \\
&+ b\left[x + (a_1x^2 + b_1x^3 + c_1x^4 + O(5))(a_2x^2 + b_2x^3 + c_2x^4 + O(5))\right]^3 + \\
&+ x\left[x + (a_1x^2 + b_1x^3 + c_1x^4 + O(5))(a_2x^2 + b_2x^3 + c_2x^4 + O(5))\right]^4 = \\
&= \begin{bmatrix} a_1x^2 + b_1x^3 + c_1x^4 + O(5) \\ a_2x^2 + b_2x^3 + c_2x^4 + O(5) \end{bmatrix}
\end{aligned}$$

while for the negative part of equation 3, we need to separate them to obtain

$$\begin{aligned}
-Bh(x) - g(x, h(x)) &= \\
&= -\begin{bmatrix} \frac{1}{2}(a_1x^2 + b_1x^3 + c_1x^4 + O(5)) + x^2 \\ \frac{1}{3}(a_2x^2 + b_2x^3 + c_2x^4 + O(5)) + (a_1x^2 + \ldots)^2 \end{bmatrix} = \\
&= -\begin{bmatrix} \frac{1}{2}a_1x^2 + \frac{1}{2}b_1x^3 + \frac{1}{2}c_1x^4 + x^2 + O(5)) \\ \frac{1}{3}a_2x^2 + \frac{1}{3}b_2x^3 + \frac{1}{3}c_2x^4 + a_1^2x^4 + O(5) \end{bmatrix}
\end{aligned}$$

adding the positive and negative parts, similar terms are reduced and we obtain the following condition

$$\begin{bmatrix} (\frac{1}{2}a_1 - 1)x^2 + \frac{1}{2}b_1x^3 + \frac{1}{2}c_1x^4 + O(5)) \\ \frac{2}{3}a_2x^2 + \frac{2}{3}b_2x^3 + (\frac{2}{3}c_2x^4 - a_1^2) + O(5) \end{bmatrix} = 0$$

Therefore, from the first we have that

$$a_1 = 2,\ b_1 = 0,\ c_1 = 0$$

and substituting a_1 in the second, we obtain

$$a_2 = 0,\ b_2 = 0,\ c_2 = 6$$

Hence, the approximation of the center manifold of the origin up to order 4 (included) is

$$h(x) = \begin{bmatrix} 2x^2 + O(5) \\ 6x^4 + O(5) \end{bmatrix}$$

2. To compute the reduction of the map to the center manifold, we compute the reduced equation

$$x \mapsto Ax + f(x, h(x))$$

which in this case is

$$x \mapsto x + (2x^2)(6x^4) = x + 12x^6$$

3. To determine the stability of the origin, we can rely on the stability theorem and study the nature of the dynamics on the reduced map. In doing so, we have that, close to the origin, negative values move toward the origin, as

$$(x + 12x^6)|_{x<0} > x$$

while positive values move away from the origin

$$(x + 12x^6)|_{x>0} > x$$

This implies that the origin is unstable.

Exercise 2

Let $X(x)$ a vector-field of $\mathbb{R}^2$ such that $X(0) = 0$ and with 0 a non-hyperbolic equilibrium point such that

$$DX(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Prove that a normal form in a neighborhood of the origin, up to order 2, is

$$\begin{aligned} x' &= y + O_3 \\ y' &= ax^2 + bxy + O_3 \end{aligned}$$

Proof.

We have the following vector-field in $\mathbb{R}^2$

$$x' = y + f(x,y) \tag{4}$$
$$y' = g(x,y) \tag{5}$$

where $f(x,y)$, $g(x,y)$ are the non-linear parts. Let us consider

$$\begin{aligned} H_2 &= \operatorname{span}\left\{ \begin{bmatrix}\tilde{x}^2\\0\end{bmatrix}, \begin{bmatrix}\tilde{x}\tilde{y}\\0\end{bmatrix}, \begin{bmatrix}\tilde{y}^2\\0\end{bmatrix}, \begin{bmatrix}0\\\tilde{x}^2\end{bmatrix}, \begin{bmatrix}0\\\tilde{x}\tilde{y}\end{bmatrix}, \begin{bmatrix}0\\\tilde{y}^2\end{bmatrix} \right\} \\ &= \operatorname{span}\{P_1, P_2, P_3, P_4, P_5, P_6\} \end{aligned}$$

and let us look at the image of $L_J^{(2)}$ by operating on each basis element P_i of H_2, meaning we compute

$$L_J^{(2)}(h_2(\tilde{x},\tilde{y})) = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} h_2(\tilde{x},\tilde{y}) - Dh_2(\tilde{x},\tilde{y}) \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

for each element of the basis P_i. Let us start with P_1

$$L_J^{(2)} P_1 = L_J^{(2)}\left(\begin{bmatrix}\tilde{x}^2\\0\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}^2\\0\end{bmatrix} - \begin{bmatrix}2\tilde{x} & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\\tilde{y}\end{bmatrix} = -2\begin{bmatrix}\tilde{x}\tilde{y}\\0\end{bmatrix}$$

which means

$$L_J^{(2)} P_1 = -2P_2$$

so we can represent $L_J^{(2)}$ in matrix form, with the first column being $\begin{bmatrix}0\\-2\\0\\0\\0\\0\end{bmatrix}$.

Computing the other columns by computing $L_J^{(2)}P_2, L_J^{(2)}P_3, \ldots$ we obtain the following matrix representation for $L_J^{(2)}$

$$L_J^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ -2 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Notice that $L_J^{(2)}$ has a column of zeros, meaning it is not full rank, and thus it does not admit inverse. This implies that we cannot eliminate all second-order terms, but we can select some. There are two columns

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

which are orthogonal to each column. Therefore, these two vectors represent a basis for the orthogonal complement G_2.

$$G_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ x^2 \end{bmatrix}, \begin{bmatrix} x^2 \\ \frac{1}{2}xy \end{bmatrix} \right\}$$

It follows that we cannot eliminate second-order terms that are linear combinations of $\begin{bmatrix} 0 \\ x^2 \end{bmatrix}$, and $\begin{bmatrix} x^2 \\ \frac{1}{2}xy \end{bmatrix}$ Therefore, we can transform $(x, y) \mapsto (\tilde{x}, \tilde{y})$ where

$$\tilde{x}' = \tilde{y} + a_2\tilde{x}^2 + O(3) \tag{6}$$

$$\tilde{y}' = a_1\tilde{x}^2 + \frac{a_2}{2}\tilde{x}\tilde{y} + O(3) \tag{7}$$

with this transformation, we have 3 second-order terms, with two related by a_2. However, there are other choices to write span(G_2), for example

$$G_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ xy \end{bmatrix}, \begin{bmatrix} 0 \\ x^2 \end{bmatrix} \right\}$$

which leads to the transformation

$$\tilde{x}' = y + O(3) \tag{8}$$

$$\tilde{y}' = a_1\tilde{x}^2 + a_2\tilde{x}\tilde{y} + O(3) \tag{9}$$

which is known as Bogdonov normal form, and coincides with the one given in the statement.