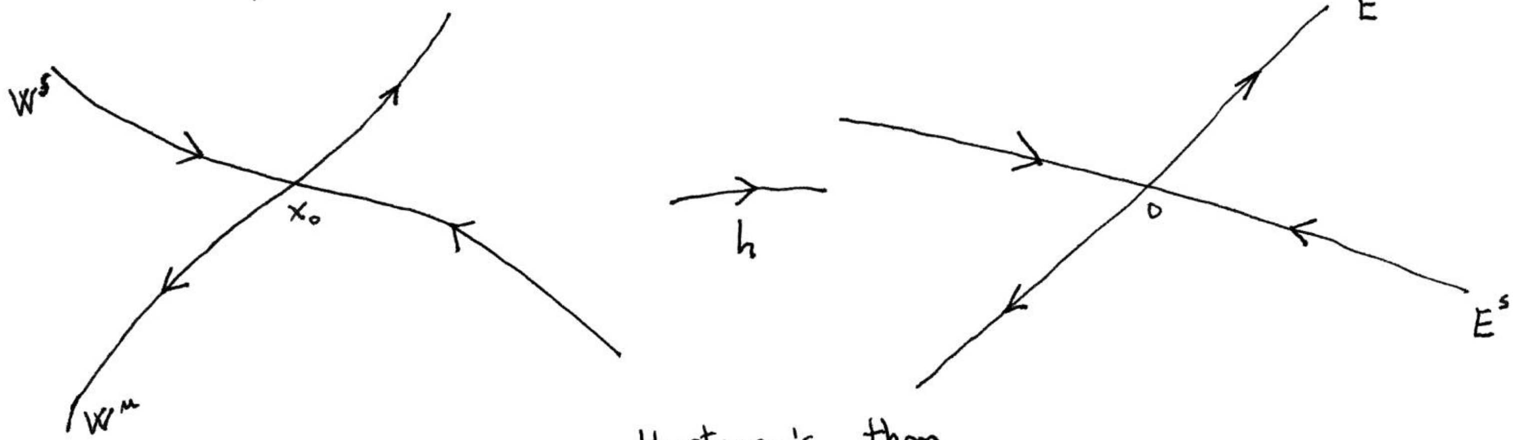
INVARIANT MANIFOLDS (hyperbolic case)

Let $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f \in C^r(U)$, $x_0 \in U$ hyperbolic fixed point

$A = Df(x_0)$, $\mathbb{R}^n = E^s \oplus E^u$



Hartman's thm

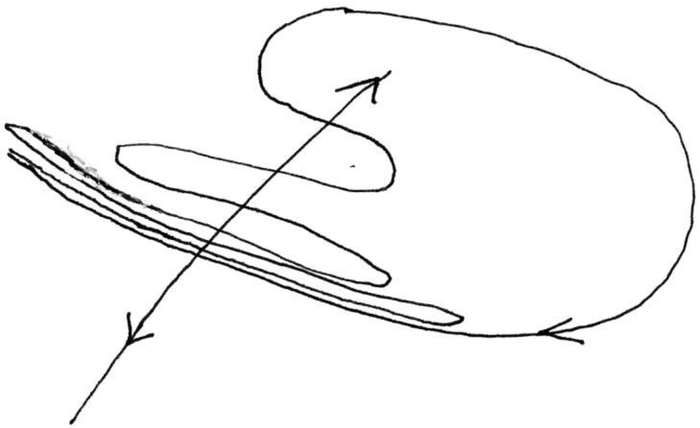
$\overline{W}^s = h^{-1}(E^s)$, $\overline{W}^u = h^{-1}(E^u)$ are continuous manifolds (defined locally)

$W^s = \{ x \in U \mid f^n(x) \in U, \forall n \geq 0, \lim_{n\to\infty} f^n(x) = x_0 \} = W^s(f, x_0)$

$W^u = \{ x \in U \mid f^{-n}(x) \in U, \forall n \geq 0, \lim_{n\to\infty} f^{-n}(x) = x_0 \} = W^u(f, x_0)$

Clearly
- $W^u(f, x_0) = W^s(f^{-1}, x_0)$, $W^s(f, x_0) = W^u(f^{-1}, x_0)$
- $f(W^s) \subset W^s$, $f^{-1}(W^u) \subset W^u$

Local invariant manifolds



$$W^s_{loc,\delta} = \{ x \in M \mid f^n(x) \in B(x_0, \delta),\ \forall n \geq 0,\ \lim_{n \to \infty} f^n(x) = x_0 \}$$

$$W^u_{loc,\delta} = \{ x \in M \mid f^{-n}(x) \in B(x_0, \delta),\ \forall n \geq 0,\ \lim_{n \to \infty} f^{-n}(x) = x_0 \}$$

Given $\delta > 0$

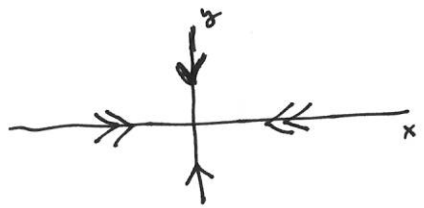
$$W^s = \bigcup_{k \geq 0} f^{-k}(W^s_{loc,\delta})$$

$$W^u = \bigcup_{k \geq 0} f^{k}(W^u_{loc,\delta})$$

Example

$$\boxed{f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad f(x,y) = \left(\frac{1}{2}x, \frac{y}{\sqrt{1+2y^2}}\right)}$$

The axes $\{y=0\}$ and $\{x=0\}$ are invariant



Let $\varphi_1(x) = \frac{1}{2}x$, $\varphi_2(y) = \frac{y}{\sqrt{1+2y^2}}$

if $n \gg 1$ and $y \neq 0$

Induction $\Rightarrow$ $\varphi_1^n(x) = \left(\frac{1}{2}\right)^n x$, $\varphi_2^n(y) = \frac{y}{\sqrt{1+2ny^2}} \approx \frac{1}{\sqrt{2n}} \frac{y}{|y|}$

Then

$$f^n(x,y) = (\varphi_1^n(x), \varphi_2^n(y)) \longrightarrow (0,0)$$

$$Df(0,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{R}^n = E^s \oplus E^c$$

Stable and unstable manifold theorem

Let $f: \mathcal{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathcal{U}$ open set, $f \in C^r(\mathcal{U})$, $r \geq 1$, $x_0 \in \mathcal{U}$ fixed point.

Let $A = Df(x_0)$ and $\mathbb{R}^n = E^s \oplus E^c \oplus E^u$ be the decomposition associated to A.

Then, there exist manifolds $W^s = W^s(f, x_0)$, $W^u = W^u(f, x_0)$ such that

(1) W^s, W^u are invariant by f.

(2) W^s, W^u are C^r

(3) $T_{x_0} W^s = E^s$, $T_{x_0} W^u = E^u$

(4) W^s, W^u are the unique C^1 manifolds satisfying (1) and (3).

(5) $\forall x \in W^s$, $\forall a > \max\{|\lambda| \mid \lambda \in \operatorname{Spec} A,\ |\lambda| < 1\}$ $\exists C_1 = C_1(x, a)$ s.t. $\|f^n(x) - x_0\| \leq C_1 a^n$, $\forall n \geq 0$

$\forall x \in W^u$, $\forall b > \max\{|\lambda|^{-1} \mid \lambda \in \operatorname{Spec} A,\ |\lambda| > 1\}$ $\exists C_2 = C_2(x, b)$ s.t. $\|f^{-n}(x) - x_0\| \leq C_2 b^n$, $\forall n \geq 0$

<u>Remark</u> (3) $\Rightarrow$ $\dim W^s = \dim E^s$ and $\dim W^u = \dim E^u$

The cases $r = \infty$, $r = \omega$ are included

More generally, assume that $\operatorname{Spec} A = \sigma_1 \cup \sigma_2$

with $\sigma_1 \subset \{\lambda \in \mathbb{C} \mid |\lambda| < \nu\}$, $\sigma_2 \subset \{\lambda \in \mathbb{C} \mid |\lambda| > \nu\}$ for some $\nu \leq 1$

Let $\mathbb{R}^n = E_1 \oplus E_2$, $\quad E_1 = \bigoplus_{\lambda \in \sigma_1} \operatorname{Ker}(A - \lambda I)^{m_\lambda}$, $\quad E_2 = \bigoplus_{\lambda \in \sigma_2} \operatorname{Ker}(A - \lambda I)^{m_\lambda}$

<u>Strong stable manifold theorem</u>

Under the previous notation and conditions there exists a manifold W s.t.

(1) W is invariant by f

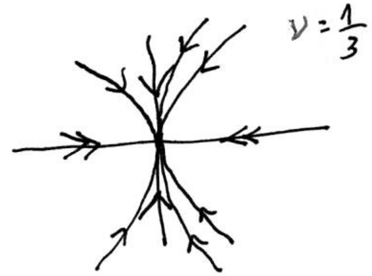
(2) W is C^r

(3) $T_{x_0} W = E_1$

(4) W is the unique C^1 manifold satisfying (1) and (3)

(5) $\forall x \in W$, $\forall a > \max\{|\lambda| \mid \lambda \in \sigma_1\}$ $\exists C = C(x, a)$ s.t. $\|f^m(x) - x_0\| \leq C a^m$, $\forall m \geq 0$

<u>Remark</u> When $\nu = 1$, W is the stable manifold



<u>Example</u> $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(0,0) = (0,0)$ and $Df(0,0) = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

Preliminaries for the proof

We write $\mathbb{R}^n = E_1 \oplus E_2$. With respect to this decomposition, $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$

$$\operatorname{Spec} A_1 \subset \{\lambda \in \mathbb{C} \mid |\lambda| < \nu\}, \qquad \operatorname{Spec} A_2 \subset \{\lambda \in \mathbb{C} \mid |\lambda| > \nu\}$$

$$\mu_1 := \max\{|\lambda| \mid \lambda \in \sigma_1\}, \qquad \mu_2 = \min\{|\lambda| \mid \lambda \in \sigma_2\} \longrightarrow \quad \mu_1 < \nu < \mu_2$$

Given $\varepsilon > 0$, $\exists \|\cdot\|_1$ in E_1, $\exists \|\cdot\|_2$ in E_2 s.t.

$$\|A_1\| \leq \mu_1 + \varepsilon \qquad \|A_2^{-1}\| \leq \mu_2^{-1} + \varepsilon$$

We take the norm $\|z\| = \max\{\|x\|_1, \|y\|_2\}$ with $z = x + y \in E_1 \oplus E_2$

We can assume that $x_0 = 0$ and we write

$$f(x,y) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g(x,y) \\ h(x,y) \end{pmatrix} = \begin{pmatrix} A_1 x + g(x,y) \\ A_2 y + h(x,y) \end{pmatrix}$$

with $g(0,0) = h(0,0) = 0$, $\quad Dg(0,0) = Dh(0,0) = 0$

Sets (manifolds) of points whose iterates converge exponentially to the fixed point

Let $a \in (0,1)$

$$M_a = \{ z \in \mathcal{U} \mid f^m(z) \in \mathcal{U},\ \forall m \geq 0 \ ; \ \sup_{m \geq 0} a^{-m} \| f^m(z) \| < \infty \}$$

$$M_{a,\delta} = \{ z \in \mathcal{U} \mid f^m(z) \in \mathcal{U} \cap B(0,\delta),\ \forall m \geq 0 \ ; \ \sup_{m \geq 0} a^{-m} \| f^m(z) \| < \infty \}$$

$z \in M_a \iff \exists C > 0$ s.t. $\| f^m(z) \| \leq C a^m, \ \forall m \geq 0$

$a < \tilde{a} \Rightarrow M_a \subset M_{\tilde{a}}, \quad M_{a,\delta} \subset M_{\tilde{a},\delta}$

$\delta < \tilde{\delta} \Rightarrow M_{a,\delta} \subset M_{a,\tilde{\delta}}$

Choice of parameters to be used in the proof

Let $\boxed{\mu_1 < a < \nu}$

We take $\varepsilon > 0$, $\eta > 0$ such that

(a) $\boxed{\mu_1 + \varepsilon + \eta < a}$ (b) $\boxed{\mu_1 + \varepsilon + 2\eta < \nu}$

(c) $\boxed{a + \eta < \nu}$ (d) $\boxed{\dfrac{\mu_2}{1 + \varepsilon \mu_2} > \nu} \iff \varepsilon < \dfrac{\mu_2 - \nu}{\mu_2 \nu}$

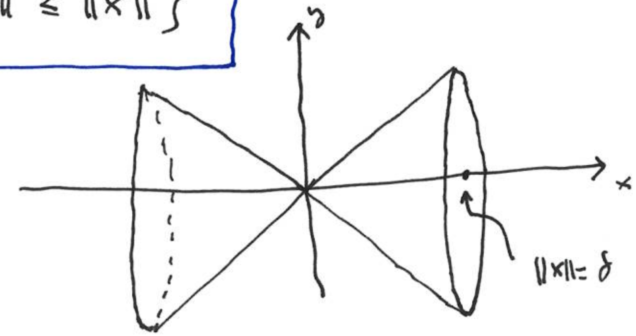
Continuity of Dg, Dh $\Rightarrow$ $\exists \delta > 0$ s.t. $\| Dg(x,y) \|, \| Dh(x,y) \| < \eta \quad \forall (x,y) \in B(0,\delta)$

(b) + (d) $\Rightarrow$ (e) $\boxed{\dfrac{\mu_2}{1 + \varepsilon \mu_2} - \eta > \mu_1 + \varepsilon + \eta}$

Proposition Let $a \in (\mu_2, \nu)$ and $\delta = \delta(a)$ as above.

Then

$$M_{a,\delta} \subset S := \{ z = (x,y) \in \mathbb{R}^n \mid \|y\| \le \|x\| \}$$



Proof Assume $M_{a,\delta} \neq \{0\}$. If not, is obviously true.

Assume that $M_{a,\delta} \not\subset S$ to get a contradiction:

$$\exists\, z_0 = (x_0, y_0) \in M_{a,\delta} \quad \text{s.t.} \quad \|y_0\| > \|x_0\| \quad (\Rightarrow y_0 \neq 0)$$

Notation

$$z_m = (x_m, y_m) = f(x_{m-1}, y_{m-1}) = \begin{pmatrix} A_1 x_{m-1} + g(x_{m-1}, y_{m-1}) \\ A_2 y_{m-1} + h(x_{m-1}, y_{m-1}) \end{pmatrix} = f^m(x_0, y_0)$$

Note that $\|x_m\|, \|y_m\| < \delta$, $\forall m \geq 0$.

Claim $\forall m \geq 0$, $z_m \notin S$ and $\|y_m\| \geq (\nu - \eta)^m \|y_0\|$

$$\boxed{\underline{\text{Claim}} \quad \forall m \geq 0, \quad z_m \notin S \quad \text{and} \quad \|y_m\| \geq (\nu - \eta)^m \|y_0\|}$$

<u>Proof</u> We prove the inductive step. If $z_n \notin S$

$$\|x_{m+1}\| = \|A_1 x_m + g(x_m, y_m)\| \leq (\mu_1 + \varepsilon)\|x_m\| + \eta \|(x_n, y_m)\| \leq (\mu_1 + \varepsilon + \eta) \|y_m\|,$$

$$\|y_{m+1}\| = \|A_2 y_m + h(x_m, y_m)\| \geq \|A_2^{-1}\|^{-1} \|y_m\| - \|h(x_m, y_m)\| \geq \frac{1}{\mu_2^{-1} + \varepsilon} \|y_m\| - \eta \|y_m\|$$

$$\geq \left(\frac{\mu_2}{1 + \varepsilon \mu_2} - \eta\right) \|y_m\|$$

By (e) $\quad \|y_{m+1}\| > \|x_{m+1}\| \quad \Longrightarrow \quad z_{n+1} \notin S$

Also, $\|y_{m+1}\| > (\nu - \eta)\|y_m\|$, $\forall m \quad \Longrightarrow \quad \|y_m\| > (\nu - \eta)^m \|y_0\|$, $\forall m$

<u>End of the proof of the Proposition</u>

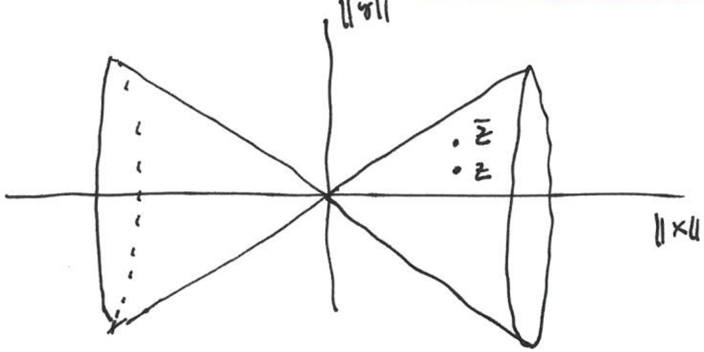
Since $\|y_0\| > \|x_0\|$, by the claim $\|y_n\| > \|x_m\|$, $\forall m \Rightarrow \|z_m\| = \|y_m\| > 0$

Then $\quad a^{-m}\|z_m\| = a^{-m}\|y_m\| > \left(\frac{\nu - \eta}{a}\right)^m \|y_0\|$. By (c) $\frac{\nu - \eta}{a} > 1 \quad \Longrightarrow \quad z_0 \notin M_{a,\delta}$.

(contradiction)

Proposition Let $\mu < a < \gamma$. Using the values of ε, η and $\delta = \delta(a)$ of the preliminaries, if

$$z = (x, y), \quad \bar{z} = (x, \bar{y}) \in M_{a,\delta} \qquad \text{then} \qquad y = \bar{y}$$



Proof Assume $\bar{y} \neq y$. We write $\bar{z}_n = (\bar{x}_n, \bar{y}_n) = f^n(\bar{z})$, $z_n = (x_n, y_n) = f^n(z)$

def. $C = \{ (x, y) \in \mathbb{R}^n \mid \|y\| > \|x\| \}$

Claim We have

$$\bar{z}_n - z_n \in C, \qquad \forall n \geq 0$$

$$\|\bar{y}_n - y_n\| \geq (\gamma - \eta)^n \|\bar{y} - y\|, \qquad \forall n \geq 0$$

Claim $\quad \bar z_n - z_n \in C, \quad \forall n \qquad (\Rightarrow \|\bar z_n - z_n\| = \|\bar y_n - y_n\|)$

$$\|\bar y_n - y_n\| \geq (\nu - \eta)^n \|\bar y - y\|$$

Proof of the claim $\quad$ Induction. The case $n=0$ is immediate

Assume the claim for $n \geq 0$.

$$\|\bar x_{n+1} - x_{n+1}\| = \|A_1 \bar x_n - A_1 x_n + g(\bar x_n, \bar y_n) - g(x_n, y_n)\| \leq \|A_1\| \|\bar x_n - x_n\| + \eta \|(\bar x_n, \bar y_n) - (x_n, y_n)\|$$

$$\leq (\mu_1 + \varepsilon + \eta) \|\bar y_n - y_n\|,$$

$$\|\bar y_{n+1} - y_{n+1}\| = \|A_2 \bar y_n - A_2 y_n + h(\bar x_n, \bar y_n) - h(x_n, y_n)\| \geq \|A_2^{-1}\|^{-1} \|\bar y_n - y_n\| - \eta \|(\bar x_n, \bar y_n) - (x_n, y_n)\|$$

$$\geq \left(\frac{1}{\mu_2^{-1} + \varepsilon} - \eta\right) \|\bar y_n - y_n\| = \left(\frac{\mu_2}{1 + \varepsilon \mu_2} - \eta\right) \|\bar y_n - y_n\|,$$

$$\Rightarrow \quad \|\bar y_{n+1} - y_{n+1}\| > \|\bar x_{n+1} - x_{n+1}\| \quad \Rightarrow \quad \bar z_{n+1} - z_{n+1} \in C$$

Moreover $\quad \|\bar y_{n+1} - y_{n+1}\| \geq (\nu - \eta) \|\bar y_n - y_n\| \geq (\nu - \eta)^{n+1} \|\bar y - y\|$

End of the proof of the Proposition

$$(\nu-\eta)^m \, \|\bar{y}-y\| \overset{\text{Claim}}{\leq} \|\bar{y}_m - y_m\| \leq \|\bar{y}_m\| + \|y_m\| \leq \|\bar{z}_n\| + \|z_n\| \overset{\bar{z}_n, z_n \in M_{a,\delta}}{\leq} \bar{C} a^m + C a^m$$

Using (c) : $\quad a+\eta<\nu \iff \nu-\eta > a$

$$\|\bar{y}-y\| \leq \frac{\bar{C} a^m + C a^m}{(\nu-\eta)^m} = \left(\frac{a}{\nu-\eta}\right)^m (\bar{C}+C), \quad \forall m \geq 0$$

$\Rightarrow \quad \bar{y}-y=0, \qquad$ contradiction!

Consequence

On any set $\{(x,y) \mid x = x^*\}$ with $\|x^*\| < \delta$

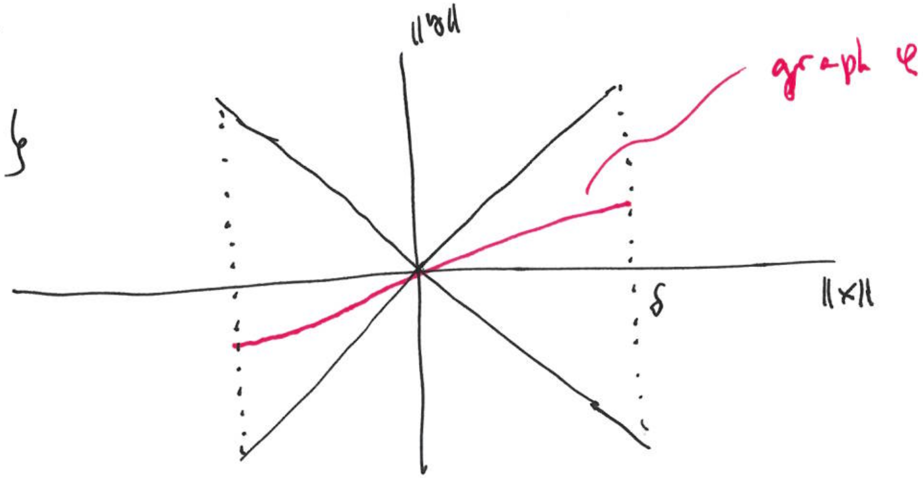
$M_{a,\delta}$ has at most one point.

Then, if $M_{a,\delta} \neq \{0\}$ has to be (or contained in)

a graph of

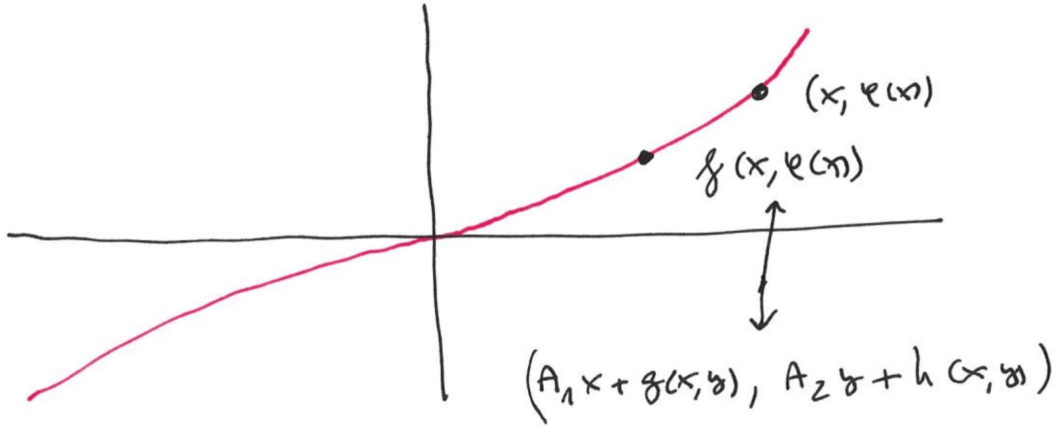
$$\varphi : B(0,\delta) \subset E_1 \longrightarrow E_2$$

contained in $S \cap \{(x,y) \mid \|x\| < \delta\}$



Proof of the Theorem

Equation for φ:



$$f(x, \varphi(x)) \in \text{graph } \varphi$$

$$\Updownarrow$$

$$A_2 \varphi(x) + h(x, \varphi(x)) = \varphi(A_1 x + g(x, \varphi(x)))$$

$$\Updownarrow$$

$$\boxed{\varphi(x) = A_2^{-1}\left[\varphi(A_1 x + g(x, \varphi(x))) - h(x, \varphi(x))\right]} \equiv \Gamma(\varphi)(x)$$

Two difficulties: $\|A_2^{-1}\|$ is not less than 1; the term $\varphi(A_1 x + g(x, \varphi(x)))$

However, $\|A_2^{-1}\| \, \|A_1\| \leq (\mu_2^{-1} + \varepsilon)(\mu_1 + \varepsilon) = \frac{\mu_1}{\mu_2} + \left(\mu_1 + \frac{1}{\mu_2}\right)\varepsilon + \varepsilon^2 < 1$ if ε small

Rescaling It is a change of variables to minimize the effect of the non-linear terms.

It amplifies a neighborhood of size ρ to one of size 1.

Given $\rho > 0$ we define $S_\rho(\xi, \eta) = (\rho\xi, \rho\eta)$; $S_\rho(B(0,1)) = B(0,\rho)$

Let $\tilde{f} = S_\rho^{-1} f S_\rho$

$$\tilde{f}(\xi,\eta) = S_\rho^{-1} \begin{pmatrix} A_1 \rho \xi + g(\rho\xi, \rho\eta) \\ A_2 \rho \eta + h(\rho\xi, \rho\eta) \end{pmatrix} = \begin{pmatrix} A_1 \xi + \rho^{-1} g(\rho\xi, \rho\eta) \\ A_2 \eta + \rho^{-1} h(\rho\xi, \rho\eta) \end{pmatrix}$$

Let $\tilde{g}(\xi,\eta) = \rho^{-1} g(\rho\xi, \rho\eta)$; $\tilde{h}(\xi,\eta) = \rho^{-1} h(\rho\xi, \rho\eta)$

We have

$$D\tilde{g}(\xi,\eta) = \rho^{-1} Dg(\rho\xi,\rho\eta)\cdot\rho = Dg(\rho\xi, \rho\eta)$$

$$D^2\tilde{g}(\xi,\eta) = \rho\, D^2 g(\rho\xi, \rho\eta) \qquad \Rightarrow \quad \| D^2 \tilde{g} \| = \rho \| D^2 g \|$$

We will assume that we have rescaled our map F with $\rho > 0$ as small as necessary.

Space (in the case $r=2$ to prove that $\varphi \in C^1$)

$$\Sigma = \{ \varphi : B(0,\delta) \subset E_1 \longrightarrow E_2 \mid \varphi \in C^1, \ \varphi(0)=0, \ D\varphi(0)=0, \ \mathrm{Lip}\,\varphi \leq 1, \ \mathrm{Lip}\, D\varphi \leq 1 \}$$

Norm

$$\|\varphi\|_\Sigma = \|D\varphi\|_{C^0} = \sup_{x\in B(0,\delta)} \|D\varphi(x)\|$$

Note that $\|\varphi\|_{C^0}$ is controlled by $\|D\varphi\|_{C^0}$ because $\varphi(0)=0$. Indeed,

$$\|\varphi\|_{C^0} = \sup_x \|\varphi(x)\| = \sup_x \|\varphi(x)-\varphi(0)\| \leq \sup_x \sup_\xi \|D\varphi(\xi)\| \, \|x-0\| \leq \|D\varphi\|_{C^0}\,\delta$$

Σ is a complete metric space contained in the Banach space $C^1(B(0,\delta), E_2)$

We recall

$$(\Gamma\varphi)(x) = A_2^{-1}\left[\varphi(A_1x + g(x,\varphi(x))) - h(x,\varphi(x))\right]$$

$$D(\Gamma\varphi)(x) = A_2^{-1}\left[D\varphi(A_1x+g(x,\varphi(x)))\left(A_1 + Dg(x,\varphi(x))\,(Id, D\varphi(x))\right) - Dh(x,\varphi(x))\,(Id, D\varphi(x))\right]$$

If $\delta>0$ and $\rho>0$ are small enough Γ is a contraction and has a unique fixed point

Check that indeed $\text{graph}\,\varphi \subset M_{a,\delta}$

recall that $\text{Lip}\,\varphi \leq 1 \quad \Rightarrow \quad \|\varphi(x)\| \leq \|x\|$

$\|z\| = \max\{\|x\|, \|y\|\}$

If $z = (x, \varphi(x)) \qquad z_1 = f(z) = (x_1, \varphi(x_1))$ and

$$\|x_1\| = \|A_1 x + g(x, \varphi(x))\| \leq \|A_1\|\,\|x\| + \|g(x, \varphi(x))\| \leq (\mu_1 + \varepsilon)\|x\| + \eta\|x\| = (\mu_1 + \varepsilon + \eta)\|x\|$$

$$\|\varphi(x_1)\| \leq \|x_1\|$$

and, in general, if $z_{n+1} = f(z_n) = f(x_n, \varphi(x_n))$ with $\|x_n\| \leq (\mu_1 + \varepsilon + \eta)^n \|x\|$

$$\|x_{n+1}\| = \|A x_n + g(x_n, \varphi(x_n))\| \leq (\mu_1 + \varepsilon + \eta)\|x_n\| \leq (\mu_1 + \varepsilon + \eta)^{n+1}\|x\|$$

Then

$$\|z_n\| = \max\{\|x_n\|, \|\varphi(x_n)\|\} = \|x_n\| \leq (\mu_1 + \varepsilon + \eta)^n \|x\| \leq a^n \|x\| .$$

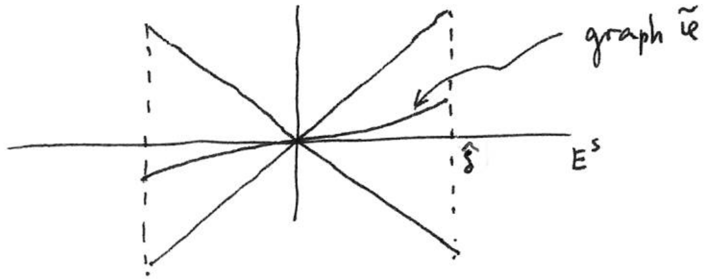
Proposition If $\mu_1 < a < \tilde{a} < \nu$ then $M_a = M_{\tilde{a}}$

Proof We have $M_a \subset M_{\tilde{a}}$.

Let $\tilde{\delta} > 0$ be s.t. $M_{\tilde{a},\tilde{\delta}} = \text{graph } \tilde{\varphi} \subset S$, $\tilde{\varphi} : B(0,\tilde{\delta}) \subset E_1 \longrightarrow E_2$

Let $\delta > 0$ be s.t. $M_{a,\delta} = \text{graph } \varphi \subset S$, $\varphi : B(0,\delta) \subset E_1 \longrightarrow E_2$

Let $\hat{\delta} = \min(\delta, \tilde{\delta})$. We have already seen that $M_{a,\hat{\delta}} \subset M_{\tilde{a},\hat{\delta}}$



$\Longrightarrow \quad M_{a,\hat{\delta}} = M_{\tilde{a},\hat{\delta}}$

Let's prove $M_a \supset M_{\tilde{a}}$

Let $z \in M_{\tilde{a}} \longrightarrow \lim\limits_{k \to \infty} f^k(z) = 0 \longrightarrow \exists\, k_0$ s.t. $f^k(z) \in B(0, \hat{\delta}) \quad \forall k \geq k_0$

$\longrightarrow f^{k_0}(z) \in M_{\tilde{a},\hat{\delta}} = M_{a,\hat{\delta}} \longrightarrow z = f^{-k_0}(f^{k_0}(z)) \in \bigcup\limits_{k \geq 0} f^{-k}(M_{a,\hat{\delta}}) = M_a$

Definition

We write $W = M_a$ for $\mu_1 < a < 1$.

$W_{loc,\delta} = M_{a,\delta}$ for $\mu_1 < a < 1$ and δ small enough

Invariant manifolds and conjugacies

Let $f : \mathcal{U} \subset \mathbb{R}^n \longrightarrow f(\mathcal{U}) \subset \mathbb{R}^n$ and $g : V \subset \mathbb{R}^n \longrightarrow g(V) \subset \mathbb{R}^n$ homeomorphisms

$p \in \mathcal{U}$ with $f(p) = p$, $q \in V$ with $g(q) = q$

$h : \mathcal{U} \to V$ a topological conjugacy from f to g such that $h(p) = q$

Let

$$W^{\pm}(f,p) = \{ x \in \mathcal{U} \mid f^{\pm m}(x) \in \mathcal{U},\ \forall m \geq 0;\quad \lim f^{\pm m}(x) = p \}$$

$$W^{\pm}(g,q) = \{ x \in V \mid g^{\pm m}(x) \in V,\ \forall m \geq 0;\quad \lim g^{\pm m}(x) = q \}$$

Then

$$h(W^{\pm}(f,p)) = W^{\pm}(g,q)$$

Proof $\boxed{h(W^{+}(f,p)) \subset W^{+}(g,q)}$

$$y \in h(W^{+}(f,p)) \Rightarrow y = h(x),\ f^{m}(x) \to p$$

$$\Rightarrow g^{m}(y) = g^{m} h(x) = h f^{m}(x) \to h(p) = q \quad \Rightarrow \quad y \in W^{+}(g,q)$$

$$\boxed{h(W^+(f,p)) \supset W^+(g,q)}$$

h^{-1} is a conjugacy from g to f $\quad$ ($h\circ f = g\circ h \Rightarrow h^{-1}\circ h\circ f\circ h^{-1} = h^{-1}\circ g\circ h\circ h^{-1}$
$\Rightarrow h^{-1}\circ g = f\circ h^{-1}$)

Then, by the previous argument,

$$h^{-1}(W^+(g,q)) \subset W^+(f,p) .$$

The results for W^- follow from the fact $\quad h\circ f^{-1} = g^{-1}\circ h$

and $\quad W^-(f,p) = W^+(f^{-1},p), \quad W^-(g,q) = W^+(g^{-1},q)$

In the hyperbolic case,

$$W^s(f,p) = W^+(f,p), \quad W^u(f,p) = W^-(f,q)$$

Then

Corollary In the previous setting, if p and q are hyperbolic fixed points the conjugacies send the stable manifold of p to the stable manifold of q. Idem for the unstable manifolds

Hyperbolic case

Now $E^c = \{0\}$ $\longrightarrow$ $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^s x + g(x,y) \\ A^u y + h(x,y) \end{pmatrix}$

Let $\boxed{\tilde{W}^s_{loc,\delta} = \{ z \in M \mid f^n(z) \in B(0,\delta), \ \forall n \geq 0 \}}$ δ small

Obviously $W^s_{loc,\delta} \subset \tilde{W}^s_{loc,\delta}$

$\boxed{\text{Proposition} \quad W^s_{loc,\delta} = \tilde{W}^s_{loc,\delta}, \quad \text{if } \delta \text{ is small enough}}$

Proof

Let $h : M_0 \longrightarrow \mathbb{R}^n$ be the conjugation from f to $A = Df(0)$ given by Hartman's thm. Let $\delta > 0$ be s.t. $B(0,\delta) \subset M_0$:

$h \circ f = A \circ h \longrightarrow h \circ f^n = A^n \circ h$

If $z \in \tilde{W}^s_{loc,\delta}$ $A^n h(z) = h(f^n(z))$ bounded for all $n \geq 0$

then $h(z) \in E^s$ and hence $z \in W^s$ and moreover $z \in W^s_{loc,\delta}$

Lemma Let $z = (x,y)$, $\bar{z} = (x,\bar{y}) \in \tilde{W}^s_{loc,\delta}$ with δ small enough

Then $y = \bar{y}$

Proof It is a small variation of the previous ones.

Proposition. Continuing in the hyperbolic case, $\forall a \in (\mu_1, 1)$ $M_{a,\delta} = W^s_{loc,\delta} = \tilde{W}^s_{loc,\delta}$, δ small

We have $\underset{\substack{\uparrow \\ \text{It is a graph}}}{M_{a,\delta}} \subset W^s_{loc,\delta} = \underset{\substack{\uparrow \\ \text{It is at most a graph}}}{\tilde{W}^s_{loc,\delta}}$

Then

$$\boxed{M_{a,\delta} = W^s_{loc,\delta}}$$

for $\mu_1 < a < 1$ and δ sufficiently small

Computation of the stable manifold (locally)

Write the map in the form:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^s x + g(x,y) \\ A^u y + h(x,y) \end{pmatrix}$$

(translation of the fixed point to the origin and transforming the linear part into block diagonal form)

W^s = graph φ

$$\varphi(A^s x + g(x, \varphi(x))) = A^u \varphi(x) + h(x, \varphi(x))$$

Since φ is C^r and $T_0(\text{graph } \varphi) = E^s$ $(\Leftrightarrow D\varphi(0) = 0)$

$$\underset{\substack{\uparrow \\ \text{Taylor}}}{\varphi(x) =} \varphi_2(x) + \varphi_3(x) + \dots + \varphi_k(x) + \dots$$

$\varphi_k(x)$ $\underbrace{\text{homogeneous}}^{\text{polynomial}}$ of degree k

Substitute $\sum_{k \geq 2} \varphi_k(x)$ into the invariance eq and equate terms of the same order. You will obtain linear equations for the coefficients of $\varphi_2, \varphi_3, \dots$

Analogously for W^u

Invariant manifolds for vector fields, hyperbolic case

Let X be a vector field, $X: \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^n$, $x_0 \in \mathcal{U}$, hyperbolic equilibrium point

Let $\dot{x} = X(x)$ and $\varphi(t,x)$ its flow.

Stable and unstable invariant manifolds of x_0.

$$W^s = W^s(X, x_0) = \{ x \in \mathcal{U} \mid \varphi(t,x) \text{ is defined for } t \geq 0 \text{ and } \lim_{t\to\infty} \varphi(t,x) = x_0 \}$$

$$W^u = W^u(X, x_0) = \{ x \in \mathcal{U} \mid \varphi(t,x) \text{ is defined for } t \leq 0 \text{ and } \lim_{t\to-\infty} \varphi(t,x) = x_0 \}$$

Relation between the stable and unstable manifolds of a vector field and the ones of its time τ map, in the hyperbolic case

Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^r vector field and $x_0 \in U$ an equilibrium point

Let $\varphi(t,x)$ be the flow of $x' = X(x)$.

Given $\tau > 0$ we define $f(x) = f_\tau(x) = \varphi(\tau, x)$ (time τ map)

$$f(x_0) = \varphi(\tau, x_0) = x_0$$

$$Df(x_0) = D_x\varphi(\tau, x_0) = e^{DX(x_0)\tau}$$

If $\lambda \in \operatorname{Spec} DX(x_0)$, $e^{\lambda\tau} \in \operatorname{Spec} Df(x_0)$

$\Rightarrow$ if x_0 is a hyperbolic equilibrium point of X, x_0 is a hyperbolic fixed point of f.

Moreover E^s, E^u are the same for $DX(x_0)$ and $Df(x_0)$

$$\boxed{\text{Proposition.} \quad W^s(X, x_0) = W^s(f, x_0) \; ; \qquad W^u(X, x_0) = W^u(f, x_0)}$$

Proof We will prove the stable case.

For the unstable one consider that $W^u(X, x_0) = W^s(-X, x_0) = W^s(f^{-1}, x_0) = W^u(f, x_0)$

$$\boxed{W^s(X, x_0) \subset W^s(f, x_0)}$$

Let $x \in W^s(X, x_0)$. The solution $\varphi(t, x)$ is defined $\forall t \geq 0$.

By induction

$$f^n(x) = \varphi(n\tau, x)$$

$$\varphi(n\tau, x) \longrightarrow x_0 \quad \Rightarrow \quad f^n(x) \longrightarrow x_0 \quad \Rightarrow \quad x \in W^s(f, x_0)$$

$$\boxed{W^s(f, x_0) \subset W^s(X, x_0)}$$

Let $x \in W^s(f, x_0)$. $\quad \underset{\underset{\varphi(m\tau, x)}{\parallel}}{f^m(x)}$ exists and belongs to M $\Rightarrow$ $\varphi(t, x)$ is defined for $t \in [0, m\tau]$, $\forall m \geq 0$

Since $\varphi(t, x_0) = x_0$ $\forall t \in [0, \tau]$ there exists $\rho > 0$ such that

$\varphi(t, x)$ exists and belongs to M $\forall t \in [0, \tau]$, $\forall x \in \overline{B(x_0, \delta)}$. (compactness of $[0,\tau] \times \{x_0\}$)

Since $D_x \varphi(t, x)$ is continuous, $\exists M$ s.t. $\|D_x \varphi(t, x)\| \leq M$ $\forall (t, x) \in [0, \tau] \times \overline{B(x_0, \delta)}$

We can write

$$\varphi(t, x) = \varphi(t, x_0) + \int_0^1 D_x \varphi(t, x_0 + \xi(x - x_0))(x - x_0)\, d\xi$$

$$\Rightarrow \|\varphi(t, x) - x_0\| \leq M \|x - x_0\|, \qquad \forall (t, x) \in [0, \tau] \times \overline{B(x_0, \delta)}$$

$x \in W^s(f, x_0) \Rightarrow f^m(x) \to x_0 \Rightarrow \exists m_0$ s.t. $f^m(x) \in B(x_0, \delta)$, $\forall m \geq m_0$

Let $t \geq m_0 \tau$. There exist $m = m(t)$ s.t.

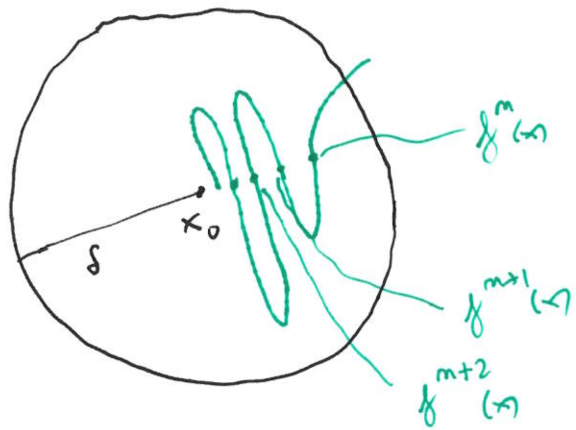
$$t - \tau < m\tau \leq t \iff 0 \leq t - m\tau < \tau$$

we write

$$\varphi(t,x) = \varphi(t - m\tau, \varphi(m\tau, x)) = \varphi(t - m\tau, f^m(x))$$

$$\Rightarrow \|\varphi(t,x) - x_0\| = \|\varphi(t - m\tau, f^m(x)) - x_0\| \leq M \|f^m(x) - x_0\|$$

Making $t \to \infty$ ($\Rightarrow m \to \infty$) we obtain $\varphi(t,x) \to x_0 \Rightarrow x \in W^s(X, x_0)$



This picture is impossible because $X(x_0) = 0$. From $t = m\tau$ to $t = (m+1)\tau$ the length of $\varphi(t,x)$ has to be small.

Computation of the stable manifold for flows (locally)

We write the differential equation in the form

$$\dot{x} = A^s x + g(x,y)$$
$$\dot{y} = A^u y + h(x,y) \qquad \varphi(t,x,y) = (\varphi_1(t,x,y), \varphi_2(t,x,y)) \ ; \ \varphi(0,x,y) = (\varphi_1(0), \varphi_2(0)) = (x,y)$$

$W^s = \text{graph}\, \phi$, with $\phi(0) = 0$, $D\phi(0) = 0$.

Invariance eq. for ϕ: $\quad (x, \overset{\varphi(x)}{\overset{\shortparallel}{y}}) \in W^s \Rightarrow \varphi(t,x,y) \in W^s, \ \forall t$

$$\Rightarrow \varphi_2(t,x,y) = \phi(\varphi_1(t,x,y)), \ \forall t$$
$$\Rightarrow \dot{\varphi}_2(t,x,y) = D\phi(\varphi_1(t,x,y))\, \dot{\varphi}_1(t,x,y), \ \forall t$$
$$\Rightarrow A^u \varphi_2(t) + h(\varphi_1(t), \varphi_2(t)) = D\phi(\varphi_1(t))\left[A^s \varphi_1(t) + g(\varphi_1(t), \varphi_2(t))\right], \ \forall t$$
$$(t=0) \Rightarrow \boxed{A^u y + h(x,y) = D\phi(x)\left[A^s x + g(x,y)\right], \ y = \varphi(x)}$$ It is a 1^{st} order PDE

We look for $\phi(x) = \phi_2(x) + \phi_3(x) + \dots = \sum_{k \geq 2} \phi_k(x)$, $\quad \phi_k(x)$ homogeneous polynomial or degree k

Analogously for W^u