0. Quick review of Hilbert spaces

A Hilbert space is a vector space H on $\mathbb{C}$ with a Hermitian product $\langle x,y\rangle$, $x,y\in H$, i.e.:

- $\langle x,y\rangle$ is linear
- $\overline{\langle x,y\rangle} = \langle y,x\rangle$
- $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0 \Leftrightarrow x=0$.

H is a Banach space with the norm $\|x\|=\langle x,x\rangle^{1/2}$

We shall always assume that H is separable, that is, there exists a countable orthonormal basis.

Basic properties

1. Cauchy-Schwarz inequality:

$$|\langle x,y\rangle| \leq \|x\|\,\|y\| \qquad x,y\in H$$

2. Triangle inequality

$$\|x+y\| \leq \|x\|+\|y\| \qquad x,y\in H.$$

Examples: ① $\ell^2=\{\{a_n\}_{n=1}^{\infty} : \|a\|_2^2=\sum_{n=1}^{\infty}|a_n|^2<+\infty\}$ with the scalar product

$$\langle a,b\rangle=\sum_{n=1}^{\infty} a_n\bar{b}_n \qquad a=\{a_n\}_n\ ;\ b=\{b_n\}_n$$

② Let (X,μ) be a measure space and let $A \subseteq X$ measurable. Let

$$H = L^2(A,\mu) = \{ f: A \to \mathbb{C} \text{ measurable} : \int_A |f|^2 \, d\mu < +\infty \}$$

This is a Hilbert space, with the scalar product

$$\langle f, g \rangle = \int_A f \cdot \overline{g} \, d\mu .$$

Two particular cases will appear often:

- For $T > 0$
$$L^2[0,T] = \{ f: [0,T] \to \mathbb{C} : \|f\|_2^2 = \int_0^T |f(t)|^2 dt < +\infty \}$$
- $$L^2(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{C} : \|f\|_2^2 = \int_{\mathbb{R}} |f(t)|^2 dt < +\infty \}$$

<u>Closed subspaces and projection:</u>

Given a closed subspace $M \subseteq H$ let

$$M^\perp = \{ x \in H : \langle x, y \rangle = 0 \quad \forall y \in M \}$$

<u>Projection theorem</u> Let $M \subseteq H$ be closed. Then $H = M \oplus M^\perp$, that is, $\forall x \in H$ $\exists!\ y \in M, z \in M^\perp : x = y + z$.

Moreover:

- $d(x,M) = \|x-y\| = \|z\| = \sup_{\substack{w\in M^\perp \\ \|w\|=1}} |\langle x, w\rangle|$

- The projections $P: H \to M$, $Q: H \to M^\perp$ are linear, continuous and
$$\|x\|^2 = \|Px\|^2 + \|Qx\|^2, \qquad x \in H.$$

<u>Hilbert bases.</u> A Hilbert basis is a complete orthonormal system $\{e_i\}_{i\in I}$ such that $H = \overline{\text{span}\langle e_i\rangle}_{i\in I}$.
That $V = \text{span}\langle e_i\rangle_{i\in I}$ is dense in H is equivalent to $\overline{V}^\perp = \{0\}$, i.e. $\langle x, e_i\rangle = 0 \ \forall i \in I \Rightarrow x = 0$.

<u>Example:</u> Let $H = \ell^2$ and let $e_n = (0, \dots, \overset{n}{1}, \dots 0 \dots)$. It is clear that $\langle e_n, e_m\rangle = \delta_{nm} = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$ and that

for any given $a = (a_n)_{n=1}^\infty$,
$$\langle a, e_n\rangle = a_n = 0 \quad \forall n \geq 1 \Rightarrow a = 0 .$$

<u>Theorem:</u> Let $\{e_n\}_{n=1}^\infty$ be a countable orthonormal system and let $V = \text{span}\langle e_n\rangle_{n=1}^\infty$. Let $x \in H$.

ⓐ $P_{\overline{V}}(x) = \sum_{n=1}^\infty \langle x, e_n\rangle e_n$ ($P_{\overline{V}}$ projection)

ⓑ $\sum_{n=1}^\infty |\langle x, e_n\rangle|^2 \leq \|x\|^2$ (Bessel inequality)

Corollary: Let $\{e_n\}_{n=1}^{\infty}$ be a countable orthonormal system. The following are equivalent:

ⓐ $\{e_n\}_{n=1}^{\infty}$ is a Hilbert basis

ⓑ $\langle x, e_n \rangle = 0 \quad \forall n \geq 1 \Rightarrow x = 0$

ⓒ $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \quad \forall x \in H$ (Parseval identity)

With this we see that, given $\{e_n\}_{n=1}^{\infty}$ Hilbert basis of H, the map

$$\begin{aligned} \ell^2 &\longrightarrow H \\ a &\longmapsto \sum_{n=1}^{\infty} a_n e_n \end{aligned}$$

is an isometric isomorphism with inverse

$$\begin{aligned} H &\longrightarrow \ell^2 \\ x &\longmapsto (\langle x, e_n \rangle)_{n \in 1}^{\infty} \end{aligned}$$