Quantitative Finance Problem Set 4

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1 Exercise 1

To prove that X_t is a martingale with respect to the filtration $\mathcal{F}_t$, we need to verify that:
1) X_t is $\mathcal{F}_t$ adapted
2) $\mathbb{E}(|X_t|) < \infty \ \ \forall t$
3) $\forall s \leq t, \ \mathbb{E}(X_t|\mathcal{F}_t) = X_s$

Points 1 and 2 are trivial in all cases. We want to prove point 3
$X_t = t^2 B_t - 2\int_0^t sB_s ds$
$\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}(t^2 B_t|\mathcal{F}_s) - 2\mathbb{E}(\int_0^t uB_u du|\mathcal{F}_s) = t^2 B_s - 2\mathbb{E}(\int_0^s uB_u du|\mathcal{F}_s) - 2\mathbb{E}(\int_s^t uB_u du|\mathcal{F}_s) =$
$= t^2 B_s - 2\int_0^s uB_u du - 2\int_s^t u\mathbb{E}(B_u|\mathcal{F}_s)du = t^2 B_s - 2\int_0^s uB_u du - 2B_s \int_s^t udu =$
$= t^2 B_s - 2\int_0^s uB_u du - 2B_s \frac{t^2-s^2}{2} = t^2 B_s - 2\int_0^s uB_u du - t^2 B_s + s^2 B_s = s^2 B_s - s\int_0^s uB_u du \quad \blacksquare$
$X_t = e^{\frac{t}{2}} cos(B_t)$
$\mathbb{E}\left(e^{t/2}\cos\left(B_t\right) \mid \mathcal{F}_s\right) = \mathbb{E}\left(e^{t/2}\left(\frac{e^{iB_t}+e^{-iB_t}}{2}\right) \mid \mathcal{F}_s\right) = \frac{1}{2}\mathbb{E}\left(e^{t/2+iB_t} + e^{t/2-iB_t} \mid \mathcal{F}_s\right) =$
$\{\{\ i = \sqrt{-1} = \sigma_1, \quad \sigma_1^2 = -1, \quad -i = -\sqrt{-1} = \sigma_2, \quad \sigma_2^2 = -1\ \}\}$
$= \frac{1}{2}\left(\mathbb{E}\left(\exp\left(\sigma_1 B_t - \frac{\sigma_1^2}{2}t\right) \mid \mathcal{F}_s\right) + \mathbb{E}\left(\exp\left(\sigma_2 B_t - \frac{\sigma_2^2}{2}t\right) \mid \mathcal{F}_s\right)\right) =$
$\frac{1}{2}\left(\exp\left(\sigma_1 B_s - \frac{\sigma_1^2}{2}s\right) + \exp\left(\sigma_2 B_s - \frac{\sigma_2^2}{2}s\right)\right) = \frac{1}{2}\left(\exp\left(\sigma_1 B_s - \frac{\sigma_1^2}{2}s\right) + \exp\left(\sigma_2 B_s - \frac{\sigma_2^2}{2}s\right)\right) =$
$= \frac{1}{2}\left(\exp\left(\frac{s}{2} + iBs\right) + \exp\left(\frac{s}{2} - iBs\right)\right) = \frac{1}{2}\left(e^{\frac{s}{2}}\left(e^{iB_s} + e^{-is}\right)\right) = e^{s/2}\cos\left(B_s\right) \quad \blacksquare$
Here we used the fact that $e^{\sigma B_t - \frac{\sigma^2}{2}t}$ is a martingale
$X_t = e^{\frac{t}{2}} sin(B_t)$
$\mathbb{E}\left(e^{t/2}\sin\left(B_t\right) \mid \mathcal{F}_s\right) = \mathbb{E}\left(e^{t/2}\left(\frac{e^{iB_t}-e^{-iB_t}}{2i}\right) \mid \mathcal{F}_s\right) = \frac{1}{2i}\mathbb{E}\left(e^{t2+iB_t} - e^{t/2-iB_t} \mid \mathcal{F}_s\right) =$
$\{\{\ i = \sqrt{-1} = \sigma_1, \quad \sigma_1^2 = -1, \quad -i = -\sqrt{-1} = \sigma_2, \quad \sigma_2^2 = -1\ \}\}$
$= \frac{1}{2i}\left(\mathbb{E}\left(\exp\left(\sigma_1 B_t - \frac{\sigma_1^2}{2}t\right) \mid \mathcal{F}_s\right) - \mathbb{E}\left(\exp\left(\sigma_2 B_t - \frac{\sigma_2^2}{2}t\right) \mid \mathcal{F}_s\right)\right) =$
$= \frac{1}{2i}\left(\exp\left(\sigma_1 B_s - \frac{\sigma_1^2}{2}s\right) - \exp\left(\sigma_2 B_s - \frac{\sigma_2^2}{2}s\right)\right) = \frac{1}{2i}\left(\exp\left(\frac{s}{2} + iB_s\right) - \exp\left(\frac{s}{2} - iB_s\right)\right) =$
$= \frac{1}{2i}\left(e^{\frac{s}{2}}\left(e^{iB_s} - e^{-iB_s}\right)\right) = e^{s/2} sin(B_s) \quad \blacksquare$
$X_t = (B_t + t)e^{-B_t - \frac{1}{2}t}$
$\mathbb{E}\left[\left(B_t + t\right)e^{-B_t - \frac{1}{2}t} \mid \mathcal{F}_s\right] = \mathbb{E}\left[B_t e^{-B_t - \frac{1}{2}t} \mid \mathcal{F}_s\right] + \mathbb{E}\left[te^{-B_t - \frac{1}{2}t} \mid \mathcal{F}_s\right] =$
$\{\{\ \sigma = -1, \quad \sigma^2 = 1\ \}\}$
This part can be written as: $\mathbb{E}\left[B_t \exp\left(\sigma B_t - \frac{\sigma^2}{2}t\right) \mid \mathcal{F}_s\right] =$
$= \mathbb{E}\left[\left(B_t - B_s + B_s\right)\exp\left(\sigma B_t - \frac{\sigma^2}{2}t\right) \mid \mathcal{F}_s\right] =$
$\mathbb{E}\left[\left(B_t - B_s\right)\exp\left(\sigma B_t - \frac{\sigma^2}{2} \mid \mathcal{F}_s\right]\right) + \mathbb{E}\left[B_s \exp\left(\sigma B_t - \frac{\sigma^2}{2}t\right) \mid \mathcal{F}_s\right] = B_s \exp\left(\sigma B_s - \frac{\sigma^2}{2}s\right)$
Then $\mathbb{E}\left[\left(B_t + t\right)e^{-B_t - \frac{1}{2}t} \mid \mathcal{F}_s\right] = s\exp\left(\sigma B_s - \frac{\sigma^2}{2}s\right) + B_s\left(\sigma B_s - \frac{\sigma^2}{2}s\right) = \left(s + B_s\right)\mathrm{e}^{-B_s - \frac{1}{2}s} \quad \blacksquare$
$X_t = B_t^1 B_t^2$
We want to prove that $\mathbb{E}\left[{B_t}^2 1 {B_t}^2 \mid F_s\right] = {B_s}^1 {B_s}^2$ We can write X_t in the following form: $\left({B_t}^1 - {B_s}^1\right)\left({B_t}^2 - {B_s}^2\right) - {B_t}^1 {B_t}^2 - {B_t}^1 {B_s}^2 - {B_s}^1 {B_t}^2 + {B_s}^1 {B_s}^2$
Then we can rearrange:

$B_t{}^1 B_t^2 - B_s{}^1 B_t^2 + B_s{}^1 B_s{}^2 - B_t{}^1 B_s{}^2 = B_t{}^1 B_t{}^2 - B_s{}^1 B_t{}^2 + B_s{}^1 B_s{}^2 - B_t{}^1 B_s{}^2 + B_s{}^1 B_s{}^2 - B_s{}^1 B_s{}^2 =$
$= B_t{}^1 B_t{}^2 - B_s{}^1 \left(B_t{}^2 - B_s{}^2\right) - B_s{}^2 \left(B_t{}^1 + B_s{}^1\right) - B_s{}^1 B_s{}^2$
$\Rightarrow \left(B_t{}^1 - B_s{}^1\right)\left(B_t{}^2 - B_s{}^2\right) = B_t{}^1 B_t{}^2 - B_s{}^1 \left(B_t{}^2 - B_s{}^2\right) - B_s{}^2 \left(B_t{}^1 + B_s{}^1\right) - B_s{}^1 B_s{}^2$
We can then rewrite it as:
$B_t{}^1 B_t^2 = \left(B_t{}^1 - B_s^1\right)\left(B_t^2 - B_s{}^2\right) + B_s^1 \left(B_t{}^2 - B_s{}^2\right) + B_s{}^2 \left(B_t{}^1 + B_s{}^1\right) + B_s^1 B_s{}^2$
So we obtain:
$\mathbb{E}\left[B_t^1 B_t^2 \mid \mathcal{F}_s\right] = \mathbb{E}\left[\left(B_t{}^1 - B_s{}^1\right)\left(B_t{}^2 - B_s{}^2\right) \mid \mathcal{F}_s\right] + \mathbb{E}\left[B_s{}^1 \left(B_t{}^2 - B_s{}^2\right) \mid \mathcal{F}_s\right] +$
$+ \mathbb{E}\left[B_s{}^2 \left(B_t{}^2 + B_s{}^2\right) \mid \mathcal{F}_s\right] + \mathbb{E}\left[B_s{}^1 B_s{}^2 \mid \mathcal{F}_s\right]$
$\Rightarrow \mathbb{E}\left[\left(B_t{}^1 - B_s{}^1\right)\left(B_t{}^2 - B_s{}^2\right) \mid \mathcal{F}_s\right] + B_s{}^1 B_s{}^2 = \mathbb{E}\left[B_t{}^1 - B_s{}^1 \mid \mathcal{F}_s\right] \mathbb{E}\left[B_t{}^2 - B_s{}^2 \mid \mathcal{F}_s\right] + B_s{}^1 B_s{}^2 = B_s{}^1 B_s{}^2$
Where to separate the conditional expected value we used the independence of the two B.H. ■

2 Exercise 2

The Bachelier model assumes that the T-forward price of an asset at lime t, S_t, follows a standard Brownian Motion with volatility σ,

$$S_t = S_0 + \sigma \omega_t$$

where s_s is the initial forward price. S_t is a martingale.
The payoff of a European call with strike price K and maturity time T is given by

$$C_T = (S_T - K)_+$$

so at time $t = 0$ one obtains the call option price

$$C_0 = \mathbb{E}\left[(S_T - K)_+\right]$$

Since S_t folds a standard Brownian Motion, $S_T = S_0 + \sigma\omega_T$. $\mathbb{E}[S_T] = S_0, \operatorname{Var}[S_T] = \sigma^2 T$, so we can rewrite the option price at $t = 0$ as

$$C_0 = \mathbb{E}\left[(S_T - K)_+\right] = \mathbb{E}\left[\left(S_0 + \sqrt{\sigma^2 T} Z - K\right)_+\right]$$

since we know that $S_T = S_0 + \sigma B_t$ we can convert it into

$$Z = \frac{S_T - S_0}{\sqrt{\sigma^2 T}} \Leftrightarrow \sqrt{\sigma^2 T} \cdot Z + S_0 = S_T$$

where Z is a standard normal random variable.
We may now apply this result to $t \in [0, T]$ and take a closer look at C_t:

$$C_t = \mathbb{E}\left[(S_t - K)_+ \mid S_t\right]$$

since S_t is known, we may write

$$Z = \frac{S_T - S_t}{\sqrt{\sigma^2 (T - t)}} \Leftrightarrow Z \cdot \sigma\sqrt{T \cdot t} + S_t = S_T$$

$$\Rightarrow \mathbb{E}\left[(S_T - K)_+ \mid S_t\right] = \mathbb{E}\left[\left(S_t + Z\sigma\sqrt{T \cdot t} - K\right)_+ \mid S_t\right].$$

We can substitute the max function using indicators:

$$\mathbb{E}\left[\left(S_t + Z\sigma\sqrt{T - t} - K\right) \mathbb{1}_{\left\{Z \leq \frac{S_t - K}{\sigma\sqrt{T - t}}\right\}}\right]$$

since Z is a standard normal random variable We can express this expectation as follows

$$= (S_t - K)\,\Phi\left(\frac{S_t - K}{\sigma\sqrt{T - t}}\right) + \mathbb{E}\left[Z\sigma\sqrt{T - t}\mathbb{1}_{\left\{Z \leq \frac{S_t - K}{\sigma\sqrt{T - t}}\right\}}\right\}]$$

with Φ the cumulative distribution function of a standard normal random variable this is the probability of the indicator

$$= (S_t - K)\,\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right)$$

where ϕ is the density of a standard normal distribution. ■