Dynamical System Exercise Set 1.2

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1 Exercise Statement

1. (3p) Let $P(z) = (z-\alpha)(z-\beta)$ where $\alpha, \beta \in \mathbb{C}$, with $\alpha \neq \beta$. Let $N_P(z) = z - \frac{P(z)}{P'(z)}$ be the Newton's method of P. Describe precisely (with proofs) the basin of attraction of α and β, the Fatou set and the Julia set. What can you say about the dynamics on the Julia set? (*Hint: Conjugate N_P (on the whole Riemann sphere) by the Mobius transformation $M(z) = \frac{z-\alpha}{z-\beta}$ and see what the resulting map is.*)

   OPTIONAL (1p). Make a program that draws the basins of attraction of Newton's method of the cubic polynomial $P(z) = z(z-1)(z-i)$. (Please include the code and the image in the same pdf file where the rest of the problems are).

2. **The quadratic family** $Q_c(z) = z^2 + c$. Let $A_c(\infty)$ denote the basin of attraction of ∞ for Q_c, and $K_c := \mathbb{C} \setminus A_c(\infty)$ denote the filled Julia set.

   (a) (1p) Prove that $K_c \subset \overline{D(0,R}$ where $R = \max\{|c|, 2\}$.

   (b) (0,5p) Deduce that if $|c| > 2$ then the orbit of the critical point $z = 0$ escapes to infinity.

   (c) (0,5p) Show that for every value of $c \in \mathbb{C}$, Q_c has at most one attracting cycle.

   (d) (1p) Calculate and draw the sets

   $$\Omega_1 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting fixed point}\}$$

   and

   $$\Omega_2 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting 2-cycle}\}$$

3. (4p) Let R be a rational function and suppose that C is a round circle such that $R^{-1}(C) \subset C$. Prove that $J(R) = C$ or $J(R)$ is a totally disconnected subset of C. *Hint: There are several ways of solving this problem. Some key words that* **might** *be related to possible solutions are: conjugacy, unit circle, Schwarz reflection, invariance Denjoy-Wolff, normality . . .*

2 Exercise 1

(3p) Let $P(z) = (z-\alpha)(z-\beta)$ where $\alpha, \beta \in \mathbb{C}$, with $\alpha \neq \beta$. Let $N_P(z) = z - \frac{P(z)}{P'(z)}$ be the Newton's method of P. Describe precisely (with proofs) the basin of attraction of α and β, the Fatou set and the Julia set. What can you say about the dynamics on the Julia set? (*Hint: Conjugate N_P (on the whole Riemann sphere) by the Möbius transformation $M(z) = \frac{z-\alpha}{z-\beta}$ and see what the resulting map is.*)

Proof.

We now that the Newton method is a local root finding method, meaning iterating $N_P(z)$ will lead to one of the two roots of P, either α, or β, depending on the choice of the starting point. This means that the two roots define a basin of attraction for the Newton method, which is what we are interested in determining.

Let us begin by following the problem statement's suggestion. We seek to find a conjugation for N_P using the Möbius transformation M. Let us compute $N_P(z)$ and simplify its expression

$$\begin{aligned} N_P(z) &= z - \frac{P(z)}{P'(z)} = z - \frac{(z-\alpha)(z-\beta)}{2z-\alpha-\beta} = \\ &= \frac{2z^2 - \alpha z - \beta z - (z^2 - \alpha z - \beta z + \alpha\beta)}{2z-\alpha-\beta} = \frac{z^2-\alpha\beta}{2z-\alpha-\beta} \end{aligned}$$

Now we seek a conjugate function h, such that $h \circ M = M \circ N_P$. We could compute it by simplifying $h = M \circ N_P \circ M^{-1}$. However, in this case, it computing $M \circ N_P$ first leads to better computations.

$$\begin{aligned} M \circ N_P = M(N_P(z)) &= \frac{\frac{z^2-\alpha\beta}{2z-\alpha-\beta} - \alpha}{\frac{z^2-\alpha\beta}{2z-\alpha-\beta} - \beta} = \\ &= \frac{z^2 - \alpha\beta - 2\alpha z + \alpha^2 + \alpha\beta}{z^2 - \alpha\beta - 2\beta z + \alpha\beta + \beta^2} = \frac{z^2 - 2\alpha z + \alpha^2}{z^2 - 2\beta z + \beta^2} = \\ &= \frac{(z-\alpha)^2}{z-\beta^2} = \left(\frac{z-\alpha}{z-\beta}\right)^2 = h(M(z)) = h \circ M \end{aligned}$$

with $h(z) = z^2$. Hence, we have a conjugation of N_P and h through the Möbius transformation M. By looking at the expression of M we see that α and β are mapped, respectively, to 0 and ∞.

$$\begin{cases} \alpha \mapsto 0 \\ \beta \mapsto \infty \end{cases}$$

Hence, to study the basins of attraction of α and β for N_p, we can study the basins of attraction of 0 and ∞ for $h(z) = z^2$. As we have already seen, we can consider the unit circle S^1. We have that, considering a point z in the disk $\mathbb{D}$, and a neighborhood of such point U, $h(U) \to 0$. On the other hand, considering a point in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and its neighborhood U, then $h(U) \to \infty$. Hence, those are the two basins of attraction for 0, and ∞, which correspond to the basins of attraction of α and β, through an inverse Möbius transformation M^{-1}, since M sends orbits of N_P to orbits of h. As the Möbius transformation is a bijective conformal map, it comprises of simpler transformations, preserving the angles. For this reason, the boundary between the basins of attraction 0 and ∞ for h, S^1, is transformed to the boundary between the basins of α and β, which divides the space into the two basins. Hence, this boundary for N_P corresponds to the line for which $z - \alpha = z - \beta$. We show in figure 1 an example with $\alpha = 1$, $\beta = i$

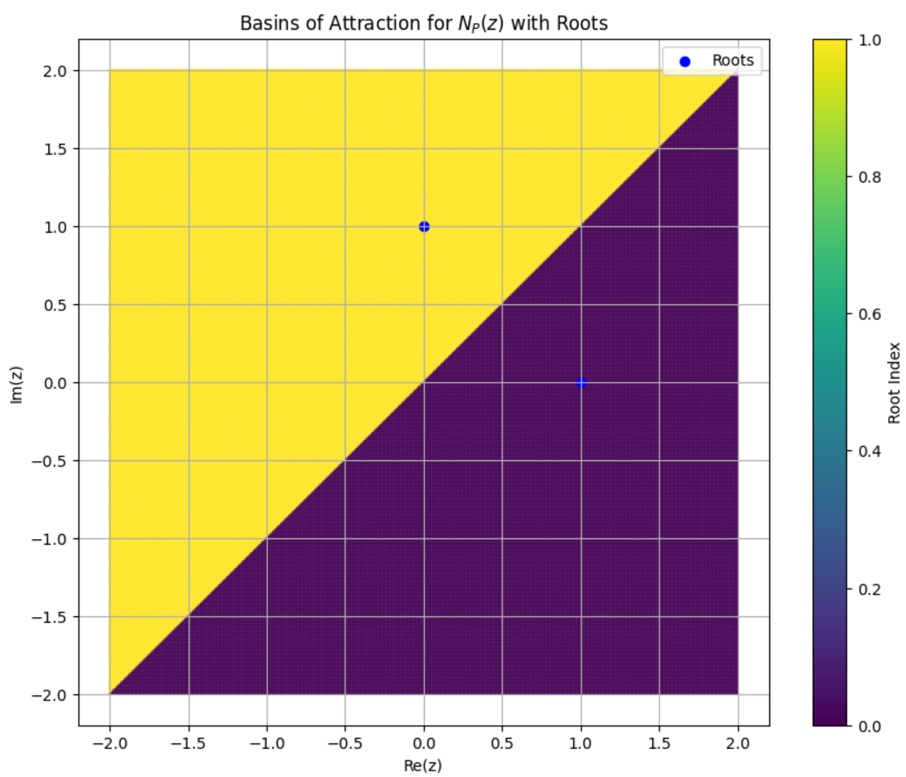

Figure 1: Basins of attraction of roots 1, i

In the sets $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and $\mathbb{D}$, h is normal, and hence the Fatou set is $\hat{C} \setminus S^1$, while the Julia set is S^1. The inverse Möbius transformation M^{-1} of these sets, corresponds to the Fatou set and the Julia set of N_P. As we have seen, taking a point z in the Julia set, and consider a neighborhood of z, shows the 'blow-up' property, meaning we have points in the neighborhood displaying 'diverging' dynamics, with some going towards 0, and some towards ∞. Analogously, taking points in the Julia set of N_P, $M^{-1}(S^1)$, will have the orbits of N_P moving towards different roots of the method, either α or β in this case. Therefore, the Julia set of N_P, corresponds to the boundary of the regions of seed values of the method that converge to each one of the two roots, meaning the boundary between the two basins of attraction. ■

OPTIONAL (1p). Make a program that draws the basins of attraction of Newton's method of the cubic polynomial $P(z) = z(z-1)(z-i)$. (Please include the code and the image in the same pdf file where the rest of the problems are).

Proof.

In the following we include the code and the obtained image. The code is written in Python. It is very simple and, in particular, the implementation of the Newton method is not sophisticated but it is just the most basic version possible. However, since optimizing the method is not exactly the scope of the exercise, it should be sufficient. (The computational time to produce the resulting image is of few seconds even with this basic implementation) As we would expect, in the case of a cubic polynomial, the boundary between the basins of attraction becomes a fractal and it is not so easy and intuitive as in the previous case.

```python
import numpy as np
import matplotlib.pyplot as plt

def P(z):
    return z * (z - 1) * (z - 1j)

def P_prime(z):
    return 3 * z**2 - 2 * (1 + 1j) * z + 1j

def newton_method(initial_guess, max_iter=100, tolerance=1e-6):
    z = initial_guess
    for i in range(max_iter):
        dz = P(z) / P_prime(z)
        z = z - dz
        if np.abs(dz) < tolerance:
            break
    return z

# Generate a grid of complex numbers
x = np.linspace(-2, 2, 400)
y = np.linspace(-2, 2, 400)
X, Y = np.meshgrid(x, y)
Z = X + 1j * Y

# Initialize an array to store the indices of the roots
root_indices = np.zeros_like(Z, dtype=int)
```

```python
# Apply Newton's method to each element in the grid
for i in range(Z.shape[0]):
    for j in range(Z.shape[1]):
        root = newton_method(Z[i, j])
        # Find the index of the root
        root_index = np.argmin(np.abs([root - r for r in [0, 1, 1j]]))
        root_indices[i, j] = root_index

# Define colors for each root
colors = ['red', 'green', 'blue']

# Plot the results
plt.figure(figsize=(10, 8))
plt.scatter(X, Y, c=root_indices, cmap='viridis', marker='.', s=1)
plt.title('Basins of Attraction for $N_P(z)$')
plt.xlabel('Re(z)')
plt.ylabel('Im(z)')
plt.colorbar(label='Root Index')
plt.grid(True)
plt.show()
```

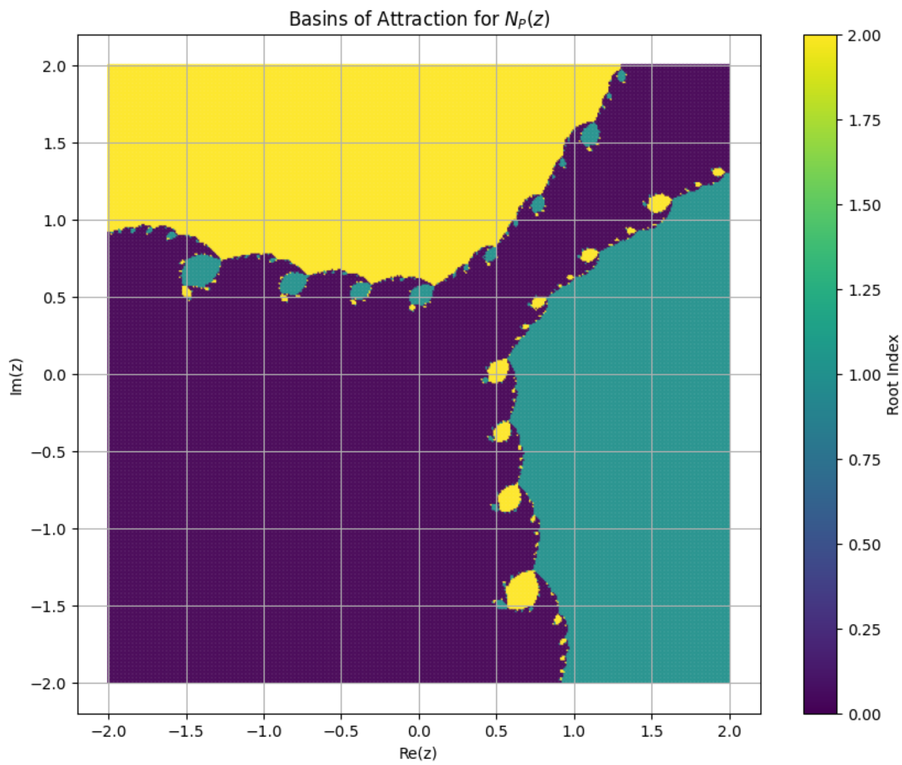

Figure 2: Basins of attraction of roots of P(z)

3 Exercise 2

The quadratic family $Q_c(z) = z^2 + c$. Let $A_c(\infty)$ denote the basin of attraction of ∞ for Q_c, and $K_c := \mathbb{C} \setminus A_c(\infty)$ denote the filled Julia set.

1. (1p) Prove that $K_c \subset \overline{D(0, R}$ where $R = \max\{|c|, 2\}$.
2. (0,5p) Deduce that if $|c| > 2$ then the orbit of the critical point $z = 0$ escapes to infinity.
3. (0,5p) Show that for every value of $c \in \mathbb{C}$, Q_c has at most one attracting cycle.
4. (1p) Calculate and draw the sets

$$\Omega_1 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting fixed point}\}$$

and

$$\Omega_2 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting 2-cycle}\}$$

Proof.

1. Let us see that $K_c^c = A_c(\infty)$, and so, to show $K_c \subset \overline{D(0,R)}$, we can show that
$$\overline{D(0,R)}^c \subset K_c^c \Longleftrightarrow \overline{D(0,R)}^c \subset A_c(\infty)$$
Hence, let us show that, given a point $z_0 \in \overline{D(0,R)}^c$, its orbit escapes to infinity. In particular, let us show that $|Q_c(z_0)| > |z_0|$, with $z_0 = re^{i\theta}$, $r > R = \max(|c|, 2)$.
$$|Q_c(z_0)| = |z_0^2 + c| \geq ||z_0^2| - |c|| \geq |z_0^2| - |c|$$
Now suppose $|c| > 2$, then
$$|Q_c(z_0)| \geq |z_0^2| - |c| = |z_0||z_0| - |c| > |z_0|$$
where the last inequality holds as, if $|c| > 2$
$$|z_0||z_0| > |z_0| + |c|, \quad |z_0| > |c| > 2$$
$$|z_0| > 2 > 1 + \frac{|c|}{|z_0|}, \quad |z_0| > |c| > 2$$
and if $|c| < 2$
$$|z_0||z_0| > |z_0| + |c|, \quad |z_0| > 2 > |c|$$
$$|z_0| > 2 > 1 + \frac{|c|}{|z_0|}, \quad |z_0| > 2 > |c|$$

2. This is a direct consequence of the previous point. Suppose $z = 0$, then we have that
$$Q_c(z) = Q_c(0) = c$$
and thus $|Q_c(z)| = |c|$, meaning we have $Q(0) \in \partial\overline{D(0,R)}$, which is not exactly included in the previously shown case, but it is very close. To see that its orbit escapes to infinity, we can use an analogous approach, seeing that $|Q_c(c)| > |c|$
$$|Q_c(c)| \geq |c^2| - |c| > |c|$$
where the last inequality holds as
$$|c^2| > 2|c|, \quad |c| > 2$$
$$|c| > 2, \quad |c| > 2$$

3. We have already seen the following
Theorem. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. If $< z_0 >$ is an attracting cycle of period k, then there exists a critical point of f^k in every component of $\mathcal{A}^*(< z_0 >)$. In particular, there exists a critical point of f in at least one component of $\mathcal{A}^*(< z_0 >)$.

With this result, since the map $Q_c(z) = z^2 + c$ has only one critical point, in $z = 0$, since

$$Q_c'(z) = 2z = 0 \Longleftrightarrow z = 0$$

it follows directly that it can have at most one attracting cycle.

4. Let

$$\Omega_1 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting fixed point}\}$$

we can compute the region explicitly by solving

$$Q_c(z) = z^2 + c = z$$

to find the fixed points, and then impose $|Q_c'(z)| = |2z| < 1$ to find the attractive points for the values of $c \in \mathbb{C}$. We can compute and represent the boundary of this region explicitly, by solving $|Q_c'(z)| = |2z| = 1$, which yields $|z| = 1/2$, and thus $z = 1/2e^{i\theta}$. Now, substituting and solving for c we find the values for which we have this boundary,

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$$

On the other hand, for

$$\Omega_2 := \{c \in \mathbb{C} \mid Q_c \text{ has an attracting 2-cycle}\}$$

we need to solve

$$Q_c(Q_c(z)) = (z^2 + c)^2 + c = z$$

imposing that the multiplier is less than one. We can solve the degree 4 equation by factorizing the fixed points and reducing it to $z^2+z+c+1 = 0$, which yields $z = (-1 \pm \sqrt{-3-4c})/2$, and its multiplier is $2z_+2z_- = 1 - (-3-4c) = 4(c+1)$. Therefore, we have an attracting cycle for the values

$$4|c+1| < 1$$

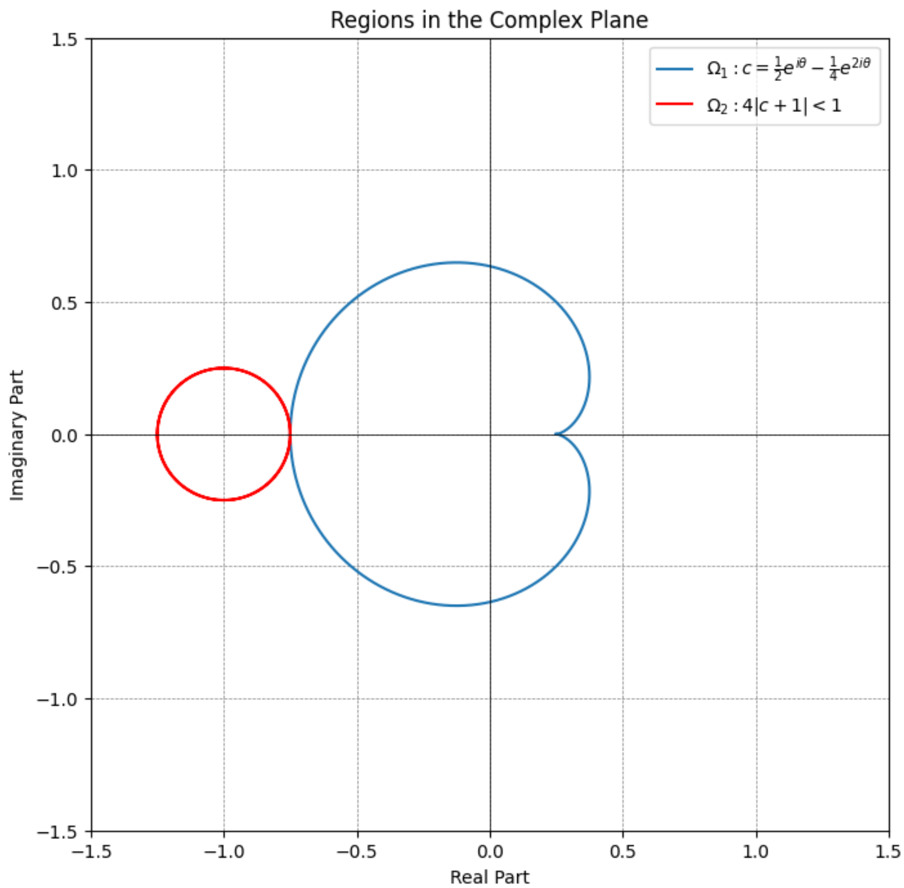

Figure 3: Ω_1 and Ω_2 regions

4 Exercise 3

(4p) Let R be a rational function and suppose that C is a round circle such that $R^{-1}(C) \subset C$. Prove that $J(R) = C$ or $J(R)$ is a totally disconnected subset of C. *Hint: There are several ways of solving this problem. Some key words that* **might** *be related to possible solutions are: conjugacy, unit circle, Schwarz reflection, invariance Denjoy-Wolff, normality ...*

Proof.

We begin by seeing that $J(R) \subseteq C$. To see this we do the following considerations

- Let $z \in C$, then its pre-image with respect to R is in C, meaning $R^{-1}(z) \in$

C. This is true for all the points in C, granting that all the orbits reaching a point in C, are coming from C.

- Consider a neighborhood U of $z \in C$. We don't know anything about the dynamics of the points in U, but we know that, given $z^* \in U \setminus C$, the orbit $< z^* >$ will not end up in U, otherwise it would contradict the hypothesis $R^{-1}(C) \subset C$.

- We don't know anything about the dynamics of the points in C with forward iterations of the rational map R. Hence, we cannot affirm that R is invariant on C.