Birkhoff Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism of class C^1

If it has a saddle point p and a transverse homoclinic point q

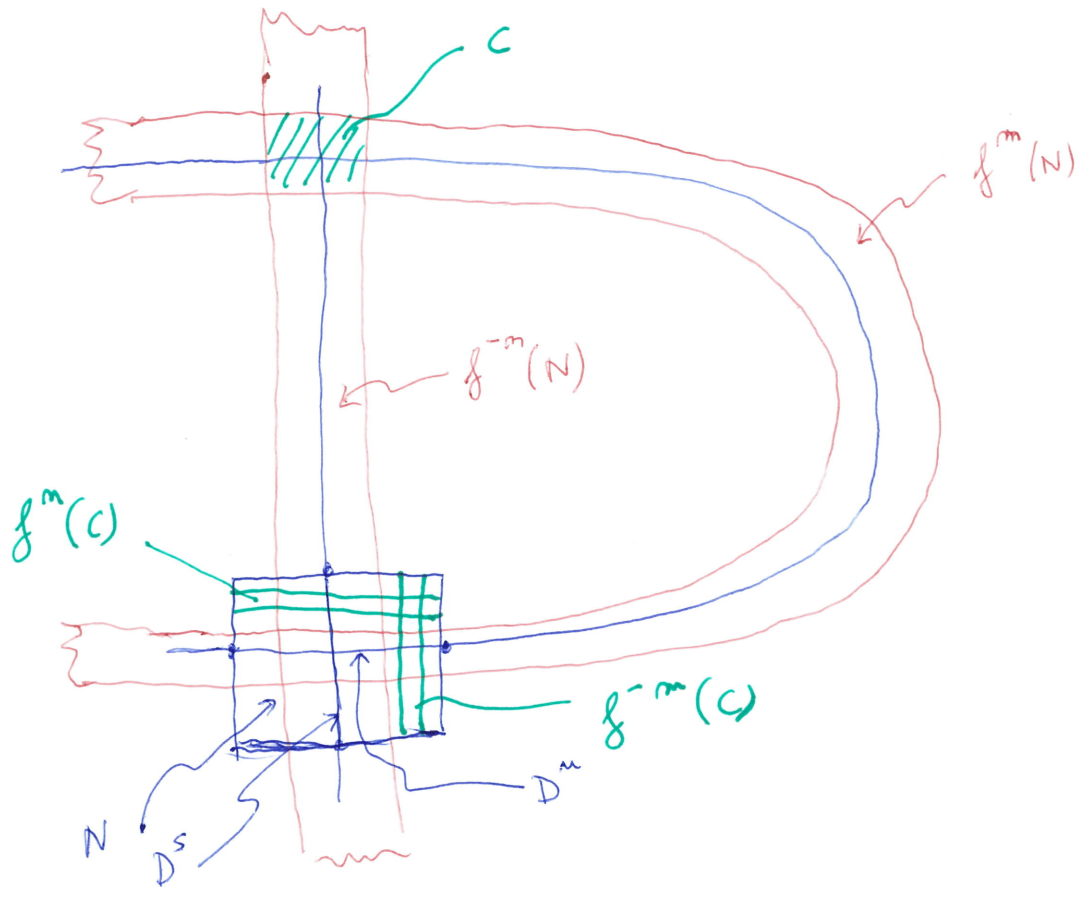
then, in any neighbourhood of q f has a periodic point

Proof We assume $p = 0$.

(1) $\exists$ coordinates such that close to $p = 0$ the stable and unstable manifolds are E^s and E^u respectively.

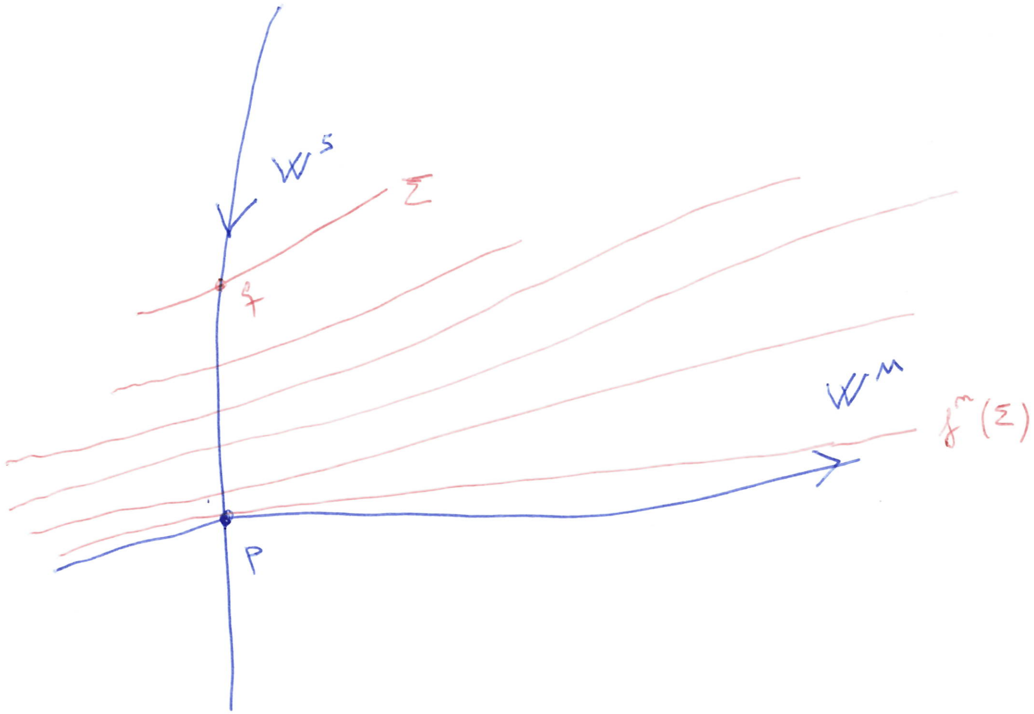
Let D^u disc in E^u and D^s disc in E^s centered at 0.

Let $N = D^u \times D^s$



N small $\Rightarrow$ C small

λ - lemma (J. Palis)



p hyperbolic point

Σ transversal to W^s at $q \neq p$

$f^n(\Sigma)$ converge to W^u in the C^1 - topology

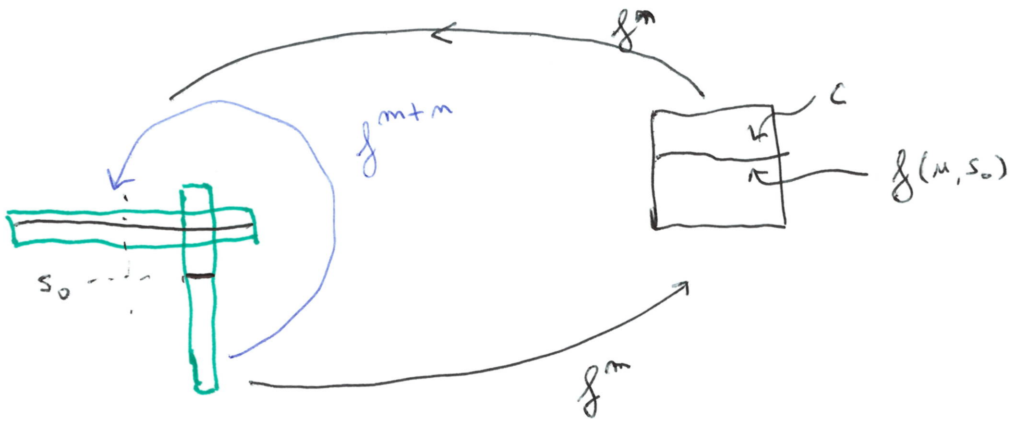
We define $\quad g_1 : f^{-m}(C) \longrightarrow N, \qquad g_1(u,s) = \left(\pi_1 f^{m+n}(u,s), s\right)$

$\qquad g_2 : f^{n}(C) \longrightarrow N \qquad g_2(u,s) = \left(u, \pi_2 f^{-m-n}(u,s)\right)$

<u>Claim</u> g_1, g_2 are bijections and homeomorphisms

<u>Proof</u> Let $(u_0, s_0) \in N$. The condition $g_1(u,s) = (u_0, s_0)$

implies $s = s_0$ and $\pi_1 f^{m+n}(u, s_0) = u_0$



$\exists!\, u$ such that $\pi_1 f^{m+n}(u, s_0) = u_0$. Then g_1 is bijective

Since N is compact, g_1 is continuous and bijective then g_1 is homeomorphism

Analogously for g_2.

We define $h : N \longrightarrow N$ by

$$\boxed{h(u,s) = \big(\underbrace{\pi_1 g_1^{-1}(u,s)}_{\in D^n},\ \underbrace{\pi_2 g_2^{-1}(u,s)}_{\in D^s}\big)}$$

h is continuous

Brouwer thm $\Rightarrow$ h has a fixed point (u_0, s_0)

<u>Claim</u> (u_0, s_0) is a fixed point for g_1 and g_2

<u>Proof</u>

$h(u_0,s_0) = \big(\pi_1 g_1^{-1}(u_0,s_0),\ \pi_2 g_2^{-1}(u_0,s_0)\big) \Rightarrow \pi_1 g_1^{-1}(u_0,s_0) = u_0$

From the definition of g_1, $\pi_2 g_1^{-1}(u_0,s_0) = s_0$

$\Rightarrow g_1^{-1}(u_0,s_0) = (u_0,s_0) \Longrightarrow g_1(u_0,s_0) = (u_0,s_0)$.

Analogously for g_2

<u>Claim</u> $f^{m+n} = g_2^{-1} \circ g_1$

<u>Proof</u> We will prove that $g_2(f^{m+n}(u,s)) = g_1(u,s)$.

$$g_2(f^{m+n}(u,s)) = g_2(\Pi_1 f^{m+n}(u,s), \Pi_2 f^{m+n}(u,s))$$

$$= \Big(\underbrace{\Pi_1 f^{m+n}(u,s)}_{\Pi_1 g_1(u,s)}, \underbrace{\Pi_2 g_2(\Pi_1 f^{m+n}(u,s), \Pi_2 f^{m+n}(u,s))}_{\underbrace{\Pi_2 f^{-m-n}(f^{m+n}(u,s))}_{\underbrace{\Pi_2(u,s)}_{s \,=\, \Pi_2 g_1(u,s)}}}\Big)$$

$$= g_1(u,s) \qquad \blacksquare$$

<u>Final</u>

(u_0, s_0) is fixed point of f^{m+n}. Since $(u_0,s_0) \in N$, $f^{m}(u_0,s_0) \in C$