Simulation methods. Exercise 2.
Spring 2023

1.- In order to integrate $y' = f(x, y)$ we want to use a Runge-Kutta method of the form

$$y_{n+1} = y_n + h(c_1 k_1 + c_2 k_2)$$

with

$$k_1 = f(x_n + ah, y_n + hak_1), \qquad k_2 = f(x_n + bh, y_n + hbk_1).$$

1. Using what we have seen in the theoretical part, determine which relations have to satisfy the coefficients a, b, c_1, c_2 in order to have global order of convergence equal to 3.
2. Find the regions of stability corresponding to these methods.
3. (optional) Determine which methods are stable.

2.- (optional) Consider the Runge method of order 3:

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1 & 0 & 1 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 2/3 & 0 & 1/6 \end{array}$$

Compute the error when one uses this method to integrate the Cauchy problem $x'' + x = 0$, $x(0) = 1$, $x'(0) = 0$ from $t = 0$ until $t = 1$ and step h small, using the following procedure:

1. Define the complex variable $z = x + ix'$ and write the Cauchy problem and the method of Runge using this variable.
2. Compute n iterates of this method with stepsize $h = 1/n$.
3. If we define $r_k = |z_k|$ and $\theta_k = \mathrm{Arg}(z_k)$, prove that

   $$r_n = a_0 + a_1 h^m + o(h^m), \qquad \mathrm{Arg}(z_n) = b_0 + b_1 h^s + o(h^s)$$

   giving the values of the constants a_0, a_1, b_0, b_1, m and s.
4. If $z(t) = x(t) + ix'(t)$ is the solution of the Cauchy problem and $r(t) = |z(t)|$, $\theta(t) = \mathrm{Arg}(z(t))$, compute $r(1) - r_n$ and $\theta(1) - \theta_n$ as functions of $h = 1/n$.

Hint: In order to compute some Taylor expansions it is useful to take into account that if $\alpha > 0$ we can write $\alpha^\beta = \exp(\beta \log(\alpha))$.

Delivery: "Campus Virtual" before March 27th.