Simulation Methods

Strange attractors

Joan Carles Tatjer

Departament de Matemàtiques i Informàtica

Universitat de Barcelona

Introduction

When we talked about bifurcations of periodic points, we introduce what is called a dissipative family of diffeomorphism: a family $\{f_a\}_{a \in \mathbb{R}^m}$ such that for all $x \in \mathbb{R}^n$ $|\det Df_a(x)| < 1$. Then the most important objects of this kind of families are the attracting set. If $n = 3$ we can have:

1. Attracting periodic orbits (Three negative Lyapunov exponents)
2. Attracting invariant closed curves (one zero Lyapunov exponent and two negative) and
3. Strange attractors of two types. If $L_1 \geq L_2 \geq L_3$ are the Lyapunov exponents: a) One dimensional: $L_1 > 0$ and $L_1 + L_2 < 0$, two dimensional: $L_1 > 0$ and $L_1 + L_2 > 0$. These invariant sets are fractal (non-integer dimension) and can be hyperbolic (persistent in front of perturbation of the parameter) or not (but persistent in the measure sense).

We will see som examples of strange attractors and their invariant measures.

Comment

- *Attracting periodic orbits and attracting invariant closed curves appear when the family f_a undergoes a saddle-node, flip or Neimark-Sacker bifurcation.*
- *Typically, strange attractors appear when there are homoclinic bifurcations, that is bifurcations associated to the invariant manifolds of some hyperbolic fixed or periodic point. But they can be appear also due to a codimension-two bifurcation of a periodic point.*

Strange attractors

The first proof of the existence of an attractor of a 2d family of dissipative diffeomorphisms which is not a periodic orbit was obtained for the Hénon family. Precisely, if $f_{a,b}(x,y) = (1 + y - ax^2, bx)$ is the Hénon family

Theorem (Benedicks and Carleson 1991)

There exists a subset $E \subset \mathbb{R}^2$ with positive Lebesgue measure such that, for every $(a,b) \in E$ the map $f_{a,b}$ admits a compact invariant set Λ s. t.

1. *The basin $B(\Lambda) = \{z \in \mathbb{R}^2 \, : \, f^n_{a,b}(z) \to \Lambda \text{ as } n \to +\infty\}$ has nonempty interior.*
2. *$\exists \; z_1 \in \Lambda$ whose forward orbit $\{f^n_{a,b}(z_1) \, : \, n \geq 0\}$ is dense in Λ and $\exists$ $c > 0$ and a tangent vector v to $M = \mathbb{R}^2$ at z_1, s.t.*

$$\|D_{a,b} f^n(z_1) v\| > e^{cn} \|v\| \;\; \forall \, n \geq 1.$$

In particular, for $(a,b) \in E$, $f_{a,b}$ has an attracting set Λ which does not contain attracting periodic orbits, and with a positive Lyapunov exponent for at least one dense orbit.

Physical or SRB measures and strange attractors

Let the time evolution of some natural process be described by $f : M \to M$ on a manifold M. Physicallly observable quantities correspond to functions $\varphi : M \to \mathbb{R}$. Experimental data, comes in the form of measurements $\varphi(f^j(x))$. This time-series can behave in a complicated way as j varies and may depend very sensitively on the initial state of the system.
A first basic question concerns the existence of asymptotic time-averages:

$$E_x(\varphi) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

for $x \in M$.

Comment

1. *By the Birkhoff's Ergodic Theorem* $E_x(\varphi)$ *exists for almost all points* $x \in M$, *with respect to an invariant measure* μ ($\mu(A) = \mu(f^{-1}(A))$, $\forall A \subset M$ *borelian).*
2. *If for every continous function* $\varphi : M \to \mathbb{R}$ *the average exists and it is independent of* $x \in B$, *with* $B \subset M$, *with positive Lebesgue measure, then*

$$\varphi \mapsto E(\varphi) = E_x(\varphi), \qquad \forall x \in B,$$

defines a non-negative linear operator on $C(M, \mathbb{R})$. *By the Riesz representation theorem, there exists a Borel measure* μ *on* M *s.t.*

$$\int_M \varphi \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)), \qquad x \in B.$$

The previous comments motivates the following definition:

Definition (Physical or SRB (Sinai-Ruelle-Bowen) measure)

An f invariant Borel measure μ is a physical or SRB measure for f if $\exists$ a positive Lebesgue measure set of points $x \in M$ s.t.

$$\int_M \varphi \, d\mu = E(\varphi) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)), \qquad \forall \, \varphi \in C(M, \mathbb{R}).$$

The set of points $x \in M$ satisfying this property is called the ergodic basin of μ and it is denoted by $B(\mu)$.

Comment

Let μ *be a SRB measure and* $K \subset M$ *a compact set. We define* $\mathcal{F}_K \subset C(M, \mathbb{R})$ *s.t.* $\varphi \in \mathcal{F}_K$ *iff* $0 \leq \varphi \leq 1$ *and* $\varphi(x) = 1, \forall x \in K$*. From the Riesz Representation Theorem*

$$\mu(K) = \inf\{E(\varphi) \ : \ \varphi \in \mathcal{F}_K\}.$$

Suppose that $\varphi_n \in \mathcal{F}_K$ *s.t.* $\varphi_n \geq \varphi_{n+1}$ *and* $(\varphi_n)_n \to \chi_K$ *pointwise. Then*

$$\mu(K) \leq E(\chi_K) = \lim_{n\to\infty} \frac{1}{n} \#\{0 \leq j \leq n-1 \ : \ f^j(x) \in K\}.$$

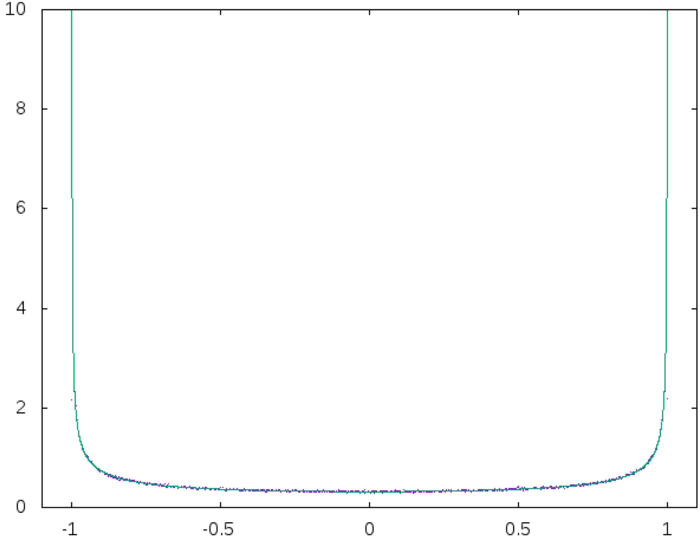
Moreover, if μ *is ergodic (the invariant sets of* f *have measure 0 or 1), and* Λ *is an invariant set* ($f(\Lambda) = \Lambda$) *s.t.* $\mu(\Lambda) = 1$

$$\mu(F) = \lim_{n\to\infty} \frac{1}{n} \#\{0 \leq j \leq n-1 \ : \ f^j(x) \in F\},$$

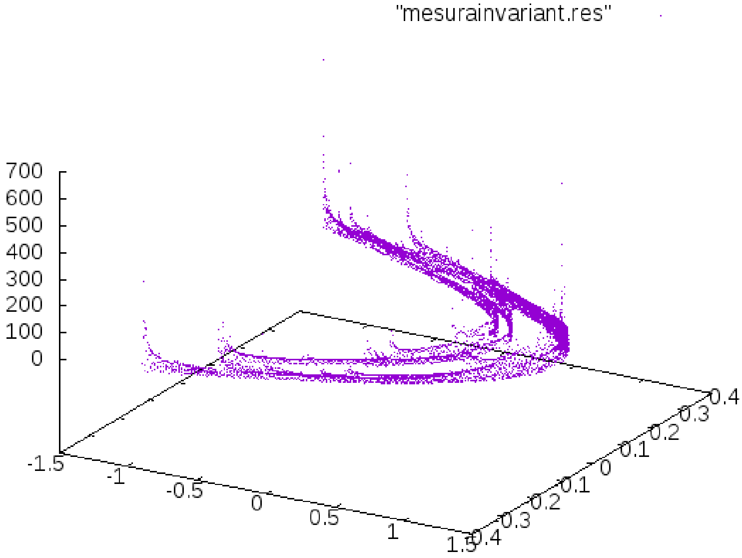
for all borelian set $F \subset \Lambda$.

This means that this measure is related with the frequency that a point in the attractor visits some region, and we can approximate it if it is ergodic.

Approximation of the density of the invariant measure for the logistic map: $x \mapsto 1 - 2x^2$. The exact value is $g(x) = \frac{1}{\pi\sqrt{1-x^2}}$, $x \in (-1, 1)$.



Approximation of the density of the invariant measure for the Hénon map: $a = 1.4,\ b = 0.3$.



Comment

In order to compute an approximation of the density of the invariant measure, we do the following:

1. *We take a point* x_0 *in the basin of attraction of the attractor and iterate it* n_1 *times, obtaining a point* x_1 *which we can consider that it is in the attractor.*
2. *Consider an open set* U *s.t. it contains the attractor and take a partition* $U = \cup_{i=1}^{p} C_i$.
3. *We compute* n *iterates of* x_1. *Then, we choose a point* $y_i \in C_i$, $i = 1, \ldots, p$ *and define* $\varphi_n(y_i) = \frac{\#\{k\, :\, 0 \le k \le n-1 \text{ and } f^k(x_1) \in C_i\}}{n|C_i|}$. *Then* φ_n *is an approximation of the density.*

Theorem (Benedicks-Young 1993, Benedicks-Viana 2001)

The Hénon maps of the previous theorem and similar families ('Hénon-like maps') support a unique invariant SBR measure. Moreover, it is ergodic and has a positive Lyapunov exponent

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n\| > 0, \quad \mu \textit{ almost everywhere.}$$

Moreover, for every continous function φ and for Lebesgue almost every $x \in B(\Lambda)$

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \to \int \varphi \, d\mu, \textit{ as } n \to \infty.$$

On the definition of strange attractor

The definition of strange attractor is motivated by the theorems we have seen:

We say that an invariant set $\Lambda \subset M$ is a (chaotic)strange attractor if

1. The basin $B(\Lambda)$ has non-empty interior.
2. Λ contains a dense orbit along which the derivative of f grows exponentially fast in norm, as $n \to \infty$.
3. Contains a dense subset consisting of periodic saddles and coincides with the closure of the unstable manifold of some of those saddles.
4. Supports an ergodic SRB-measure μ having some positive Lyapunov exponent and whose basin $B(\mu)$ has full Lebesgue measure in $B(\Lambda)$. Moreover, if f is invertible, the sum of all the Lyapunov exponents is negative.
5. Is persistent under perturbations of the system: generic parametrized families of diffeomorphism f_s, $s \in \mathbb{R}^p$, with $f_0 = f$, exhibit a similar attractor for a positive Lebesgue measure set of parameters s close to zero.

A two-parametric family of non invertible quadratic maps

We want to look into the dynamics of the two-parametric family of planar quadratic non-invertible maps

$$f_{a,b}(x,y) = (a + y^2, x + by).$$

We can ask ourselves why to study this family. It will be good if

1. It describes a very general behaviour of certain types of dynamical systems.
2. It is simple but it has a complicated dynamics.

We know [T. 2001]:

1. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is quadratic map s.t. $f \circ f$ is quadratic and it has not any invariant linear foliation then there exist $a, b \in \mathbb{R}$ such that f is linearly conjugate to $f_{a,b}$.
2. The family $f_{a,b}$ is a family of limit return map of some type of codimension-two homoclinic bifurcation called generalized homoclinic tangency for families of two-parameter diffeomorphisms with a saddle fixed points with real eigenvalues λ_i, $i = 1, 2, 3$, s.t. $|\lambda_1| > |\lambda_2| > 1 > |\lambda_3|$ and $|\lambda_1 \lambda_2 \lambda_3| < 1$.

Comment

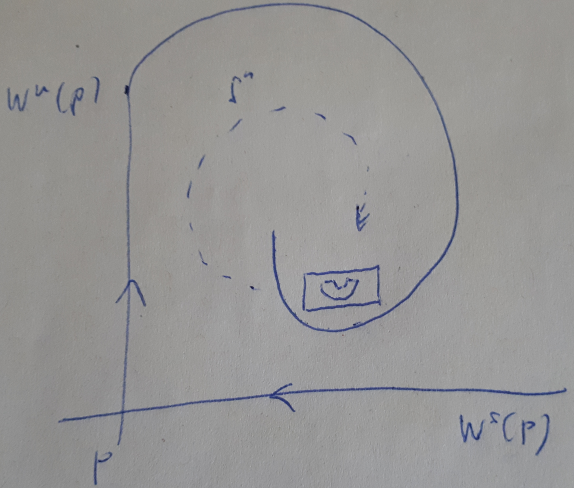
We say that a family of maps f_b, $b \in \mathbb{R}^k$, is a family of limit return maps associated to some homoclinic bifurcation, if for any family of maps g_a, $a \in \mathbb{R}^k$, having this type of bifurcation for $a = a_0$, and taking a point q in the orbit of the corresponding homoclinic orbit, $\exists$ a natural number N, s.t. $\forall\, n \geq N$, $\exists$ reparametrizations $a = M_n(b)$, and b-dependent coordinate transformations $x = \Psi_{n,b}(\tilde{x})$, s.t.

1. *$\forall K$ compact in the $(b, \tilde{x})$ space, the images of K under the maps*

$$(b, \tilde{x}) \mapsto (M_n(b), \Psi_{n,b}(\tilde{x}))$$

   converge for $n \to \infty$ to (a_0, q).

2. *The maps $(b, \tilde{x}) \mapsto (b, \Psi^{-1}_{n,b} \circ g^n_{M_n(b)} \circ \Psi_{n,b})$, converge to $(b, \tilde{x}) \mapsto (b, f_b(\tilde{x}))$, as $n \to \infty$.*

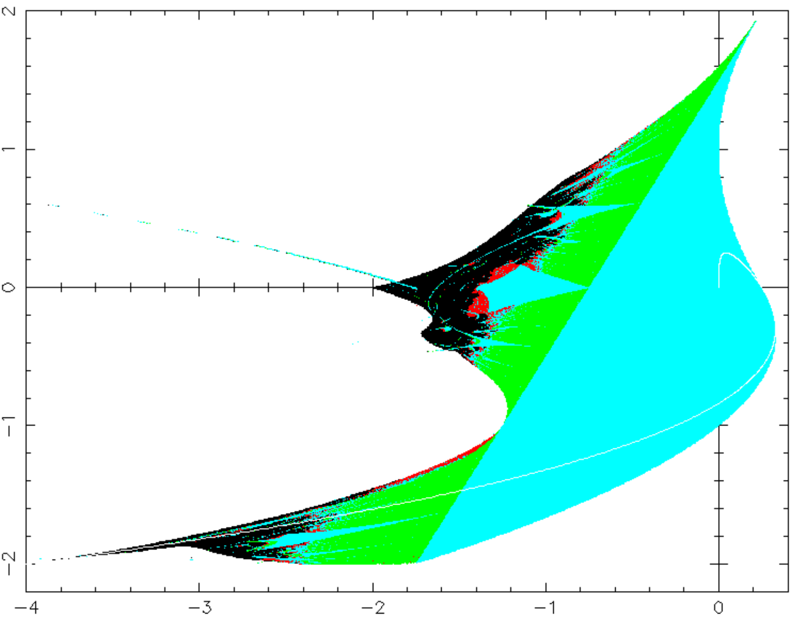


$W^u(p)$
S^n
$W^s(p)$
p

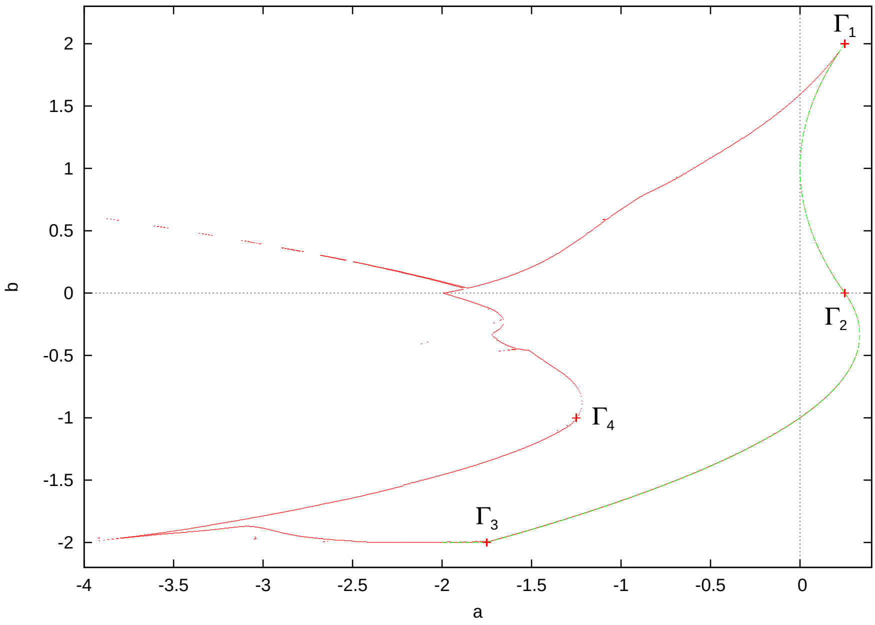
Comment

The logistic map, $f_a(x) = 1 - ax^2$, is a limit return map of one-parameter families of two dimensional dissipative diffeomorphisms having a quadratic homoclinic tangency.

In the next figure, we have drawn the different types of attractors one can find in the parameter plane (a, b) : blue corresponds to a periodic orbit, green to a closed invariant curve, red to a strange attractor with 1 positive Lyapunov exponent, and black to a strange attractor for which the sum of the two largest Lyapunov exponents is positive. The white part corresponds to values of the parameters for which there are not attractors.



2
1
0
−1
−2
−4
−3
−2
−1
0



Γ_1
Γ_2
Γ_3
Γ_4
b
a
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0

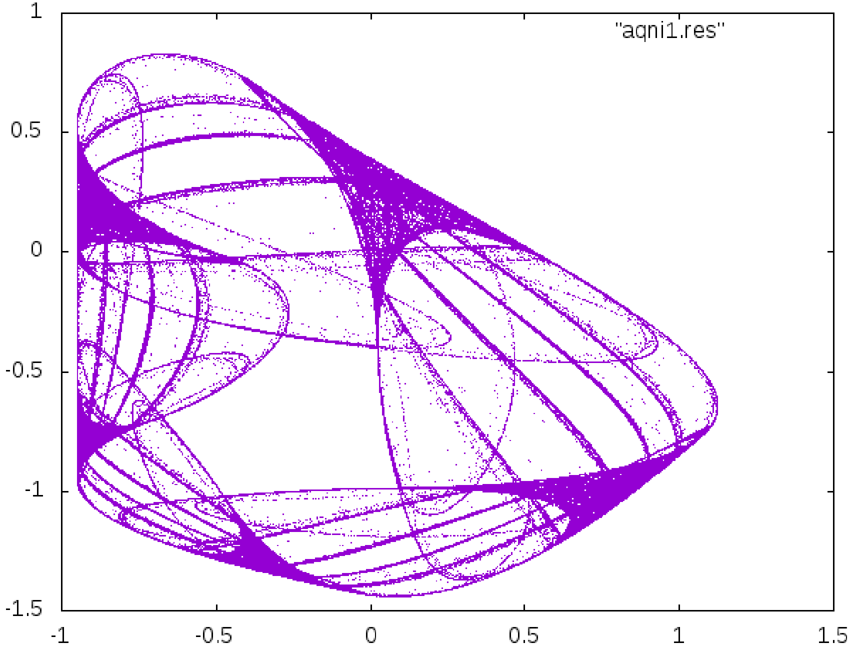
Visible bifurcations:

1. Γ_1: Bogdanov-Takens bifurcation, eigenvalues 1,1.
2. Γ_2 Eigenvalues -1, 1.
3. Γ_3 eigenvalues $-1, -1$.
4. Γ_4 eigenvalues $\exp(\pm 2\pi i/3)$.
5. Saddle-node bifurcation curve joins Γ_1 and Γ_2.
6. Flip bifurcation curve joins Γ_2 and Γ_3.
7. The straight line from Γ_1 to Γ_3 passing through Γ_4 is a curve of Neimark-Sacker bifurcation.

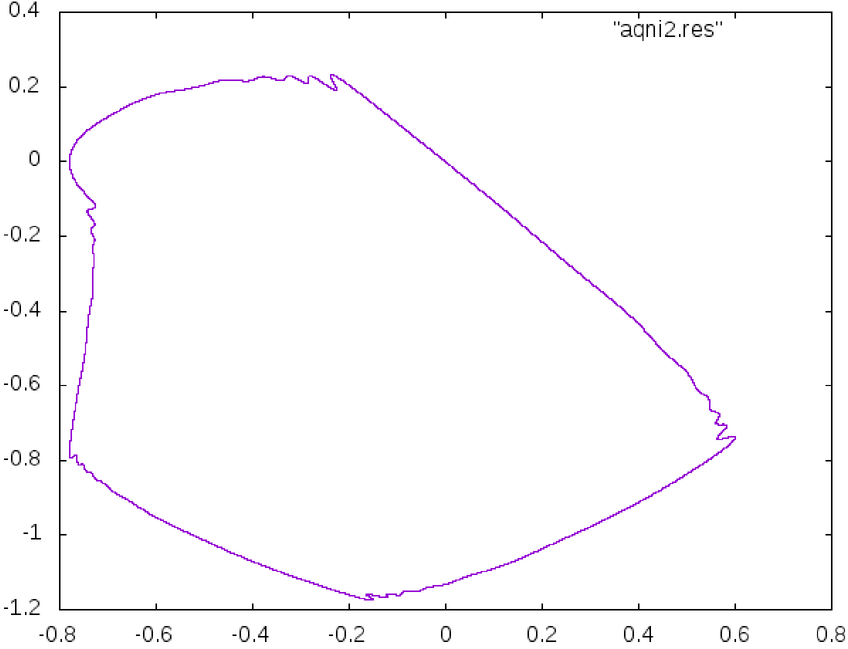
Comment

There are only two values of the parameter for which one knows that there exists an strange attractor with two positive Lyapunov exponents and an SRB ergodic invariant measure: $a = -2$, $b = 0$ *and* $a = -4$ *and* $b = -2$ *(Pumariño-T. 2006).*

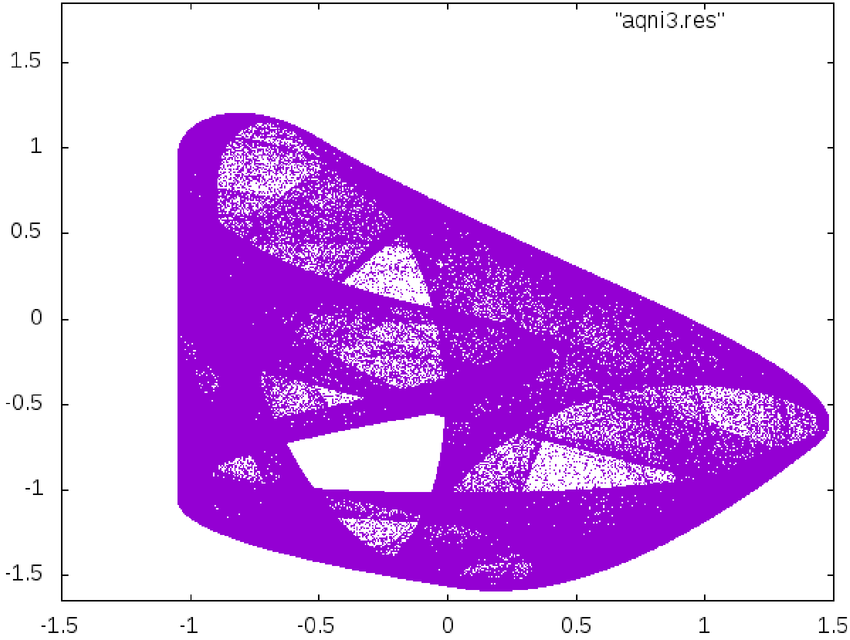
Strange attractor with one positive Lyapunov exponent:



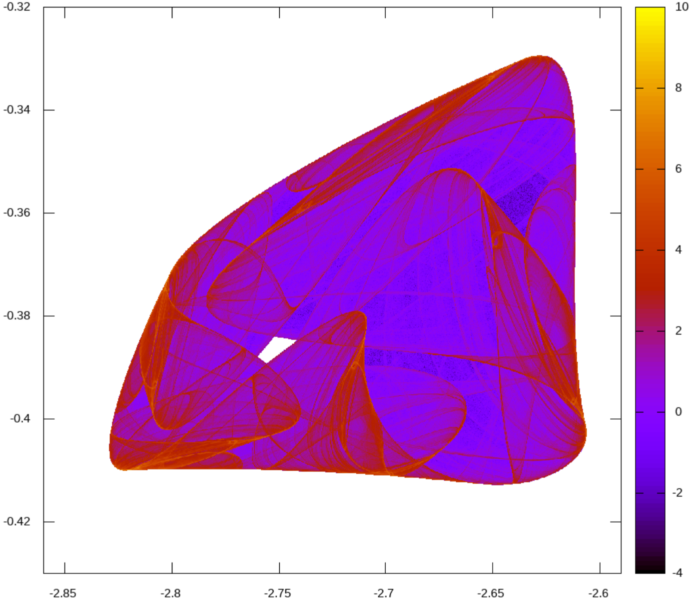
Attracting invariant curve:

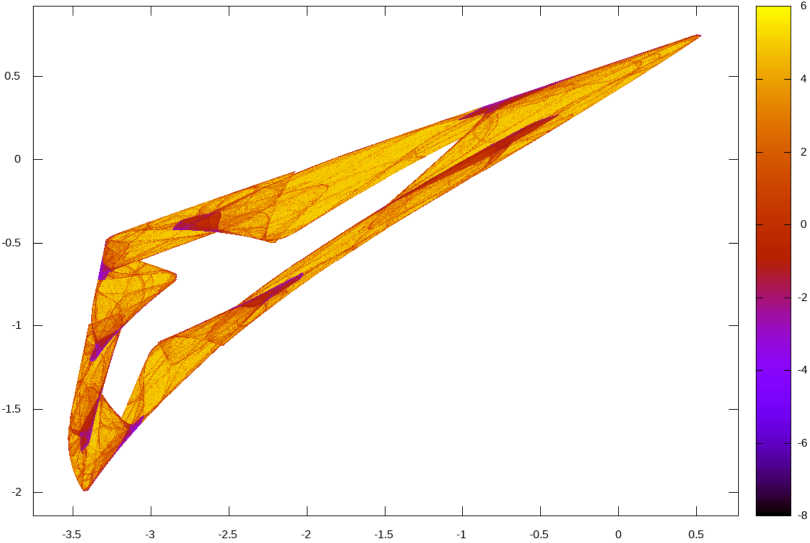


Strange attractor with positive sum of the two largest Lyapunov exponents:



Strange attractors with invariant measures:




0.5
0
-0.5
-1
-1.5
-2
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
6
4
2
0
-2
-4
-6
-8

For more information:

1. M. Viana: *Stochastic Dynamics of Deterministic Systems*. 21 Colóquio Brasileiro de Matemática. Instituto de Matematica Pura e Aplicada (1997).
2. J. C. Tatjer: Three-dimensional dissipative diffeomorphisms with homoclinic tangencies. *Ergodic Th. & Dynam. Sys.* (2001),**21**, 249-302.
3. A. Pumariño, J. C. Tatjer: Dynamics near homoclinic bifurcations of three-dimensional dissipative diffeomorphisms. *Nonlinearity* **19** (2006) 2833-2852.
4. A. Pumariño, J. C. Tatjer: Attractors for return maps near homoclinic tangencies of three-dimensional dissipative diffeomorphisms. *Discrete and Continuous Dynamical Systems Series B* (2007), **8**, 971-1005.