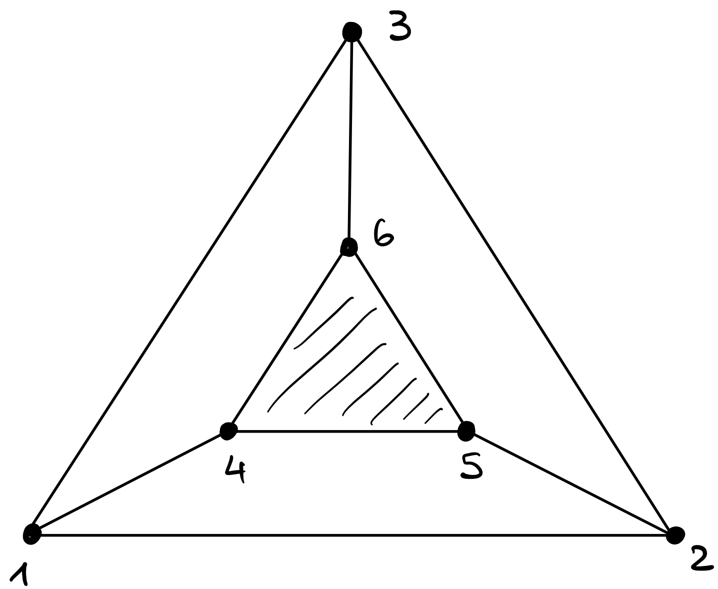
Topological Data Analysis

2022–2023

Solutions of Exercises

17 November 2022

$K: \ (12)\ (13)\ (14)\ (23)\ (25)\ (36)\ (456)$



$C_0 = \mathbb{Z}(1) \oplus \dots \oplus \mathbb{Z}(6) \cong \mathbb{Z}^6$

$C_1 = \mathbb{Z}(12) \oplus \dots \oplus \mathbb{Z}(56) \cong \mathbb{Z}^9$

$C_2 = \mathbb{Z}(456) \cong \mathbb{Z}$

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

∂_1:

	(12)	(13)	(14)	(23)	(25)	(36)	(45)	(46)	(56)
(1)	−1	−1	−1	0	0	0	0	0	0
(2)	1	0	0	−1	−1	0	0	0	0
(3)	0	1	0	1	0	−1	0	0	0
(4)	0	0	1	0	0	0	−1	−1	0
(5)	0	0	0	0	1	0	1	0	−1
(6)	0	0	0	0	0	1	0	1	1

$$\begin{matrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

$$\begin{matrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{matrix}$$

rank $\partial_1 = 5$

rank Ker $\partial_1 = 4$

∂_2:

$$\begin{array}{c|c} & (456) \\ \hline (12) & 0 \\ (13) & 0 \\ (14) & 0 \\ (23) & 0 \\ (25) & 0 \\ (36) & 0 \\ (45) & 1 \\ (46) & -1 \\ (56) & 1 \end{array}$$

rank $\partial_2 = 1$

Ker $\partial_2 = 0$

$$H_0(K) = C_0 / \operatorname{Im} \partial_1 \cong \mathbb{Z}$$ generated by any vertex

$$H_2(K) = \operatorname{Ker} \partial_2 = 0$$

$$H_1(K) = \operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2 \cong \mathbb{Z}^3$$

For $H_1(K)$, we know that $\operatorname{Ker} \partial_1$ has a basis of 4 vectors. One of these is $v = (56) - (46) + (45)$, which was obtained from the column reduction of the last three columns of ∂_1. This vector v is a generator of $\operatorname{Im} \partial_2$. Therefore $H_1(K)$ is a free abelian group of rank 3.