Introduction to wavelets. An example

Sound signals, and many others, exhibit in general slowly changing trends. On the other hand, images have usually smooth regions interrupted by edges, or abrupt changes in contrast, which quite often provide the most relevant information (contours, etc.). The Fourier transform does not represent abrupt changes efficiently, because exponentials (waves) are not well localised. Continuing the ideas introduced in the study of the Short Time Fourier Transform, we want to represent functions with a new class of L^2 functions that are well localised, called "wavelets". Informally, wavelets are rapidly decaying wave-like functions with zero mean that, by translation and dilation, can represent any function in $L^2(\mathbb{R})$. Unlike exponentials, they exist for finite time (or have a very fast decay).

A relevant example: Instead of starting with the definitions and main results, let's start with a (known) example that might help to understand the main ideas.

The Haar wavelet and its multi-resolution analysis (MRA)

Let $\varphi(t) = \chi_{[0,1]}(t)$. Its translates $\varphi_{0,k}(t) = \varphi(t-k)$, $k \in \mathbb{Z}$ form an orthonormal basis of the space

$$V_0 = \{ f \in L^2(\mathbb{R}) : f_{|[k,k+1)} = c_k \text{ constant for all } k \in \mathbb{Z} \}$$

Observe that a function that is constant in the unit intervals $[k, k+1)$, $k \in \mathbb{Z}$, is in V_0 if it is of the form

$$f = \sum_{k \in \mathbb{Z}} \alpha_k \varphi_{0,k} \quad , \text{ with } \sum_{k \in \mathbb{Z}} |\alpha_k|^2 = \|f\|_2^2 < +\infty .$$

Now we can consider the space V_1 of L^2-functions which are constant on every half interval of the form $I_{1,k} = \left[\frac{k}{2}, \frac{k+1}{2}\right)$, $k \in \mathbb{Z}$, that is

$$V_1 = \{ f \in L^2(\mathbb{R}) : f|_{I_{1,k}} = c_k \text{ constant } \forall k \in \mathbb{Z} \}.$$

We can obtain an orthonormal basis of V_1 just by rescaling the bases of V_0: define

$$\varphi_{1,k}(t) = \sqrt{2}\,\varphi(2t-k) = \sqrt{2}\,\chi_{I_{1,k}}(t)$$

It is clear that $\|\varphi_{1,k}\|^2 = \int_{I_{1,k}} 2\, dt = 1$ and that

$\langle \varphi_{1,k}, \varphi_{1,j} \rangle = 0$ for $k \neq j$.

Then, a function of the form $f = \sum_{k\in\mathbb{Z}} \alpha_k \varphi_{1,k}$ (constant on the intervals $I_{1,k}$) is in V_1 iff $\sum_{k\in\mathbb{Z}} |\alpha_k|^2 = \|f\|_2^2 < +\infty$.

Given a general $f \in L^2(\mathbb{R})$, its projection onto V_0, which is a closed subspace is $P_0 f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{0,k}\rangle \varphi_{0,k}$, where

$$\langle f, \varphi_{0,k} \rangle = \int_k^{k+1} f(t)\, dt \quad (\text{average of } f \text{ over } I_{0,k} = [k, k+1))$$

This $P_0 f$ can be viewed as an approximation of f at resolution 0 (the best approximation of f by functions which are constant in the unit intervals $[k, k+1)$, $k \in \mathbb{Z}$)

Similarly, the projection of on V_1 is $P_1 f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{1,k}\rangle \varphi_{1,k}$, where $\langle f, \varphi_{1,k}\rangle = \sqrt{2} \int_{I_{1,k}} f(t)\, dt$. With this

$$P_1 f = \sum_{k\in\mathbb{Z}} 2 \left(\int_{I_{1,k}} f(t)\, dt \right) \chi_{I_{1,k}}(t) = \sum_{k\in\mathbb{Z}} \left(⨍_{I_{1,k}} f \right) \chi_{I_{1,k}}.$$

Again, the coefficient for each interval $I_{1,k}$ is just the average of f on that interval, and $P_1 f$ is the approximation of at resolution 1 (constant on all intervals $\left[\frac{k}{2}, \frac{k+1}{2}\right)$, $k \in \mathbb{Z}$).

Let us now examine the "detail" of f we add to $P_0 f$ when increasing the resolution and passing to $P_1 f$. Let us see what happens in the unit interval $I_{0,0} = [0,1)$. The part of the detail $P_1 f - P_0 f$ in $I_{0,0}$ is

$$(P_1 f - P_0 f)\chi_{[0,1)} =$$

$$= \left(\fint_{I_{1,0}} f\right)\chi_{I_{1,0}} + \left(\fint_{I_{1,1}} f\right)\chi_{I_{1,1}} - \left(\fint_{I_{0,0}} f\right)\chi_{I_{0,0}} =$$

$$= \left(\fint_{I_{1,0}} f - \fint_{I_{0,0}} f\right)\chi_{I_{1,0}} + \left(\fint_{I_{1,1}} f - \fint_{I_{0,0}} f\right)\chi_{I_{1,1}}$$

Observe that this is a function of the form $-\alpha\chi_{I_{1,0}} + \alpha\chi_{I_{1,1}}$:

$$\fint_{I_{1,0}} f - \fint_{I_{0,0}} f + \fint_{I_{1,1}} f - \fint_{I_{0,0}} f = 2\int_{I_{1,0}} f + 2\int_{I_{1,1}} f - 2\int_{I_{0,0}} f = 0$$

Thus, the detail added when going from V_0 to V_1 is in the part corresponding to $[0,1)$, a multiple of the function

$$\psi(t) = \begin{cases} -1 & 0 \le t < 1/2 \\ 1 & 1/2 \le t < 1 \end{cases}$$

The same argument holds for all intervals $[k, k+1)$. Denote by W_0 the orthogonal complement of V_0 in V_1 (the detail added to $P_0 f$ to get $P_1 f$); $V_1 = V_0 \oplus W_0$. Then

$$\psi_{0,k}(t) = \psi(t-k) \qquad k \in \mathbb{Z}$$

is an orthonormal basis of W_0.

This scheme can be reproduced at all scales (resolutions) $n \in \mathbb{Z}$. Let

$$V_n = \{ f \in L^2(\mathbb{R}) : f_{|I_{n,k}} = c_k \text{ constant } \forall k \in \mathbb{Z} \} \quad I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

This closed subspace of $L^2(\mathbb{R})$ has an orthonormal basis $\varphi_{n,k}(t) = 2^{n/2}\varphi(2^n t - k)$, $k \in \mathbb{Z}$, and $f = \sum_{k\in\mathbb{Z}} \alpha_k \varphi_{n,k}$ is in V_n iff $\sum_{k\in\mathbb{Z}} |\alpha_k|^2 = \|f\|_2^2 < +\infty$

The orthogonal projection $P_n f : L^2(\mathbb{R}) \to V_n$ indicates the best approximation of f by functions which are constant on dyadic intervals $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, $k \in \mathbb{Z}$.

The detail space added when passing from resolution n (V_n) to resolution $n+1$ (V_{n+1}) is denoted by W_n. Thus $V_{n+1} = V_n \oplus W_n$

The functions $\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$, $k \in \mathbb{Z}$, form an orthonormal basis of W_n.

In the way V_n are defined we have $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$ and $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$. Then, from the iteration $(m<n)$

$$V_n = V_{n-1} \oplus W_{n-1} = V_{n-2} \oplus W_{n-2} \oplus W_{n-1} = \cdots = V_m \oplus W_m \oplus \cdots \oplus W_{n-1}$$

we deduce that $V_n = \bigoplus_{j=-\infty}^{n-1} W_j$ and $\boxed{L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j}$

That is, any function can be viewed as the superposition of the details at all possible resolutions. Also, in particular, $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$.

This is an example of what is called a "multi-resolution analysis" (this one is called Haar MRA). The initial function φ is called the "scaling function" of the MRA and ψ is the wavelet of the MRA (sometimes also called "mother wavelet"). The system $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is the wavelet basis. In this particular case, this is the Haar orthonormal basis that we saw at the beginning of the course.

Mini-example