EXERCISES 1.3

1) The goal of this exercise is to see the limitations of Newton's method, with an example in which Newton's method is divergent while a descent method is convergent to the minimum.
Let us consider the function $g(x) = -e^{-x^2}$
that has a unique minimum at $x=0$. Note that $g'(x)<0$ if $x<0$ and $g'(x)>0$ if $x>0$, which implies that any reasonable descent method should be able to find the minimum, no matter the starting point. Instead, let us use a Newton's method on the function g' (i.e., to solve $g'(x)=0$).

(a) Let $\{x_n\}_n$ be the sequence of points produced by the Newton's method starting at the seed $x_0=1$. Prove that $\lim_{n\to\infty} x_n = \infty$.

(b) Find a value $\alpha>0$ such that, if $x_0\in[0,\alpha)$ the Newton's method converges to 0, and if $x_0>\alpha$ the Newton's method diverges.

(a) Newton's method:

x^k, $x^{k+1} = x^k - [\nabla^2 f(x^k)]^{-1} \nabla f(x^k)$

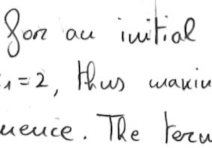
In this case: $g(x) = -e^{-x^2}$, $g:\mathbb{R}\to\mathbb{R}$, $g\in C^2$

$$x^{k+1} = x^k - \frac{g'(x^k)}{g''(x^k)}, \quad g'(x) = 2xe^{-x^2}, \quad g''(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}(1-2x^2)$$

$$\frac{g'(x)}{g''(x)} = \frac{2x\cancel{e^{-x^2}}}{\cancel{2e^{-x^2}}(1-2x^2)} = \frac{x}{1-2x^2}, \quad x^{k+1} = x^k - \frac{x^k}{1-2x^{k2}} \longrightarrow x^{k+1} - x^k = -\frac{x^k}{1-2x^{k2}}$$

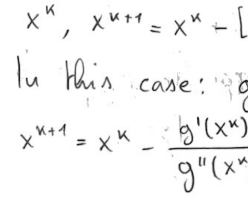
Consider $f(x) = -\dfrac{x}{1-2x^2}$, which represents the increment $f(x^k) = x^{k+1} - x^k$

$f(x)\geq 0$: $\dfrac{x}{1-2x^2}\leq 0$ $\quad$ $x\geq 0$, $\quad -\frac{1}{\sqrt{2}}\leq x\leq \frac{1}{\sqrt{2}}$



and $f(x)=1$ for $x=1$

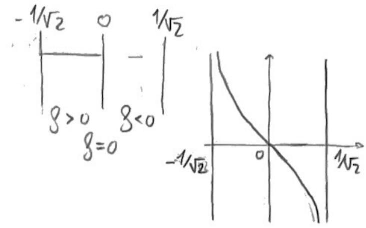
Thus we have that f has a vertical asymptote at $x=1/\sqrt{2}$, it is positive for $x>1/\sqrt{2}$ and $f(1)=1$; for $x>1$ $f(x)\geq 0$ and $\lim_{x\to+\infty} f(x) = 0$



Therefore, for an initial value of $x_0=1$ we have $f(x_0)=1$ leading to $x_1=2$, thus making the sequence $\{f(x_k)\}_k$ a positive power sequence. The term $f(x_k)$ goes to 0 with $k\to+\infty$ but it does so with an order of $-\dfrac{x}{1-2x^2}\sim\dfrac{1}{2x}$, thus making the serie $\sum_k^\infty f(x_k)$ not converge.

Since $x^{k+1} = x_0 + \sum_{k=0}^{\infty} f(x_k)$, x^{k+1} does not converge, and neither does the Newton's method ■

(b) For $x \in (-1/\sqrt{2}, 1/\sqrt{2})$ $f(x)$ has the following behaviour:



with $f(0)=0$ and an asymptotic behaviour towards $x=-1/\sqrt{2}$ and $x=1/\sqrt{2}$, which means that, as long as the initial value doesn't give an increment large enough to make x_{k+1} leave this interval the method will converge to $x=0$.

Consider $x_0 \in [0,\alpha)$, we have in general $x_{k+1} = x_0 + \sum_{k=0}^{N} f(x)$, which can be equivalently studied as $x_{k+1} = x_0 + \sum_{k=0}^{\infty} f(x)$.

Consider α such that $|x_1| = |\alpha + f(\alpha)| < \alpha$, $0<\alpha<1/\sqrt{2}$

$$-\alpha < \alpha + f(\alpha) < \alpha$$

$$-\cancel{\alpha} < \cancel{\alpha} - \frac{\alpha}{1-2\alpha^2} < \cancel{\alpha} \quad \| \alpha > 0$$

$$-1 < \frac{\cancel{1}-2\alpha^2-\cancel{1}}{1-2\alpha^2} < 1 \quad \| 1-2\alpha^2>0$$

$$-1+2\alpha^2 < -2\alpha^2 < 1-2\alpha^2$$

$$2\alpha^2 - 1 < 2\alpha^2 < 1-2\alpha^2$$

$$\begin{cases} 2\alpha^2 > 2\alpha^2 - 1 \quad \forall \alpha \in (0, 1/\sqrt{2}) \\ 4\alpha^2 < 1 \end{cases}$$

$$\alpha^2 < 1/4$$

$$\cancel{-1/2 <} \alpha < 1/2 \qquad \underline{\alpha < 1/2}$$

Thus we have that for $x_0 \in [0, 1/2)$, $|x_1| < x_0$.

If $x_0 = 1/2$, $x_1 = -1/2$, $x_2 = 1/2$, ... and the method does not converge to 0.

If $x_0 > 1/2$, x_{k+1} will increase in module until it will satisfy $|x_{k+1}| > 1/\sqrt{2}$, thus following the behavior of the previous case and not converging.

If $x_0 < 1/2$, we have $|x_1| < x_0$ and if $|x_{k+1}| < |x_k|$, since $\lim_{x\to 0} f(x) = 0$ the method will converge to 0.

$$|x_{k+1}| = |x_k + f(x_k)| < |x_k| \quad \begin{cases} |x_k + f(x_k)| < x_k, \quad x_k > 0 \quad ① \\ |x_k + f(x_k)| < -x_k, \quad x_k < 0 \quad ② \end{cases}$$

① This case is the same as the prior, with $x_k = \alpha$, thus resulting in $\underline{x_k < 1/2}$

② $x_k < x_k + f(x_k) < -x_k$

$$\cancel{x_k} < \cancel{x_k} - \frac{x_k}{1-2x_k^2} < -\cancel{x_k} \quad \| x_k < 0$$

$$-1 < 1 - \frac{1}{1-2x_k^2} < 1$$

$$-2 < -\frac{1}{1-2x_k^2} < 0$$

$$0 < \frac{1}{1-2x_k^2} < 2 \quad \| 1-2x_k^2 > 0$$

$$0 < 1 < 2 - 4x_k^2$$

$$-4x_k^2 > -1$$

$$4x_k^2 < 1$$

$$x_k^2 < 1/4$$

$$-1/2 < x_k \cancel{< 1/2}$$

$$\underline{x_k < -1/2}$$

Therefore, the interval of convergence of the Newton's method is $(-1/2, 1/2)$ and for the specific request $\alpha = 1/2$, resulting in $[0, 1/2)$.

2) Discuss if the following functions are unimodal:

(a) $g(x) = x^3 - x$ on $x \in [-2,0]$, and on $x \in [0,2]$

(b) $g(x) = e^{-x}$ on $x \in [0,1]$

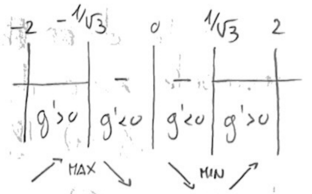
(c) $g(x) = |x| + |x-1|$ on $x \in [-2,2]$

> A function is said to be unimodal on an interval $[a,b]$ if it has a minimum (or a maximum) $\bar{\alpha} \in [a,b]$ and if $\forall \alpha_1 \in [a,b]$, $\forall \alpha_2 \in [a,b]$, with $\alpha_1 < \alpha_2$ the following hold:

- $\alpha_2 \leq \bar{\alpha} \Rightarrow g(\alpha_1) > g(\alpha_2)$ (or $g(\alpha_1) < g(\alpha_2)$)
- $\alpha_1 \geq \bar{\alpha} \Rightarrow g(\alpha_1) < g(\alpha_2)$ (or $g(\alpha_1) > g(\alpha_2)$)

(a) $g(x) = x^3 - x$ on $x \in [-2,0]$, and on $x \in [0,2]$.

In this case $g \in C^1$, $g'(x) = 3x^2 - 1$ can be studied on $\mathbb{R}$



$g'(x) \geq 0$: $3x^2 - 1 \geq 0 \longrightarrow x \leq -1/\sqrt{3} \vee x \geq 1/\sqrt{3}$

For completeness, $g''(x) = 6x$, $g'(-1/\sqrt{3}) < 0$, $g'(1/\sqrt{3}) > 0$

Thus $x = -1/\sqrt{3}$ is a maximum and $x = 1/\sqrt{3}$ is a minimum

The function in the interval $[-2,0]$ increases for $x \in [-2, -1/\sqrt{3}]$ and it decreases for $x \in [-1/\sqrt{3}, 0]$. Therefore, it is unimodal on $[-2,0]$ with a maximum for $\bar{\alpha} = -1/\sqrt{3}$.

Similarly, it decreases for $x \in [0, 1/\sqrt{3}]$ and it increases for $x \in [1/\sqrt{3}, 2]$. Therefore, it is unimodal on $[0,2]$ with a minimum for $\bar{\alpha} = 1/\sqrt{3}$

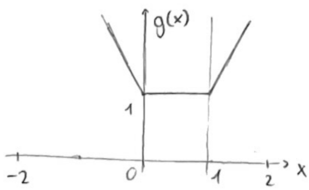
(b) $g(x) = e^{-x}$ on $x \in [0,1]$.

$g \in C^1 \longrightarrow g'(x) = -e^{-x}$, $g'(x) = -\underbrace{\underbrace{e^{-x}}_{>0}}_{<0} < 0 \Rightarrow$ the function monotone strictly decreasing

Therefore, g does not present any maximum or minimum on $\mathbb{R} \Rightarrow$ it is not unimodal on $[0,1]$

(c) $g(x) = |x| + |x-1|$ on $x \in [-2,2]$.

$$g(x) = \begin{cases} -2x+1 & x < 0 \\ 1 & 0 \leq x < 1 \\ 2x - 1 & x \geq 1 \end{cases}$$



In this case $g \notin C^1$, but it follows the above-shown linear behavior.

Since $g(x) = 1$ for $x \in [0,1]$, any $x \in [0,1]$ minimizes $g(x)$.

$[0,1] \subset [-2,2] \Rightarrow$ the definition cannot be applied, thus g is not unimodal on $[-2,2]$. To be precise, given $\bar{\alpha} \in [0,1]$ at least one of the two conditions doesn't hold with a strict inequality. (for either $\alpha_2 = \bar{\alpha} - \varepsilon$, $\alpha_1 = \alpha_2 - \varepsilon$ or $\alpha_1 = \bar{\alpha} + \varepsilon$, $\alpha_2 = \alpha_1 + \varepsilon$, ε small enough)

3) (OPTIONAL) Look for the Golden section search method. Explain it.

> This method is similar to the bisection method but uses a different criteria to restrict the interval in which the minimum (or maximum) can be found. Therefore, it can be applied under the same hypothesis as the bisection method.

It follows a description of the way the method works.

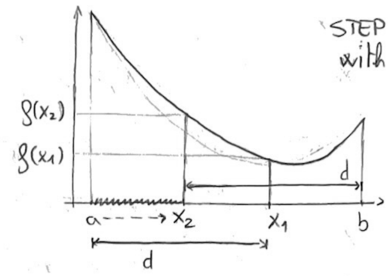
- step 0: initialization

given $f(x)$, $f: \mathbb{R} \to \mathbb{R}$, an interval $[a,b]$ (the method will find the minimum on $[a,b]$)
compute the golden ratio $GR = \frac{\sqrt{5}-1}{2} \approx 0.618$

- step 1:

compute $d = GR \cdot (b-a)$
compute $x_1 = a+d$, $x_2 = b-d$
$f(x_1)$, $f(x_2)$

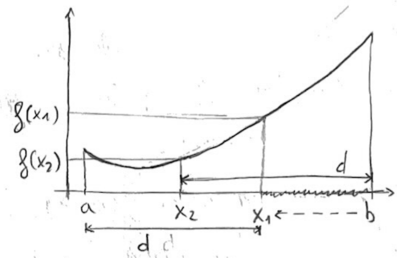
STEP 1
with $f(x_1) < f(x_2)$



if $f(x_1) < f(x_2)$:

$a \leftarrow x_2$
$x_2 \leftarrow x_1$

STEP 1
with $f(x_1) > f(x_2)$



if $f(x_1) > f(x_2)$:

$b \leftarrow x_1$
$x_1 \leftarrow x_2$

- step 2:

if STOP condition is satisfied:
return $(a+b)/2$
else:
repeat from step 1