Chapter 6

Wavelets

6.1 Introduction. An example.

Many signals –sound, for example– exhibit in general slowly changing trends. On the other hand images have usually smooth regions interrupted by edges, or abrupt changes in contrast, which quite often provide the most relevant information (contours, etc.). The Fourier transform does not represent abrupt changes efficiently, because exponentials are not well localised.

Continuing the ideas introduced in the study of the Short Time Fourier Transform we want to represent functions in $L^2(\mathbb{R})$ by means of a new class of well localised functions called wavelets. Informally, wavelets are zero mean rapidly decaying wave-like functions which by translation and dilation can represent any $f \in L^2(\mathbb{R})$. Unlike exponentials, they either exist for finite time or they decay very rapidly.

Instead of starting with the definitions and main results let's examine a known example, which hopefully will help to understand the main ideas.

The Haar wavelet and its multi-resolution analysis. Let us recall the basis described in Section 1.3. Consider $\varphi(t) = \chi_{[0,1]}(t)$ and its translates $\varphi_{0,k}(t) = \varphi(t-k)$, $k \in \mathbb{Z}$, which form an orthonormal basis of the closed subspace of L^2 consisting of the functions which are constant between integers:

$$V_0 = \left\{ f \in L^2(\mathbb{R}) : f_{|[k,k+1)} = c_k \text{ constant for all } k \in \mathbb{Z} \right\}.$$

Observe that a function f is in V_0 if and only if is of the form

$$f = \sum_{k\in\mathbb{Z}} c_k \varphi_{0,k} \quad \text{with} \quad \|f\|^2 = \sum_{k\in\mathbb{Z}} |c_k|^2 < +\infty.$$

Observe also that, given an arbitrary $f \in L^2(\mathbb{R})$, its projection on V_0 is $P_0 f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{0,k} \rangle \varphi_{0,k}$, where

$$\langle f, \varphi_{0,k} \rangle = \int_k^{k+1} f(t)\, dt = \fint_{I_{0,k}} f$$

is the average of f on the interval $[k, k+1)$. From this point of view $P_0 f$ can be seen as a low resolution approximation of f.

Consider now the space V_1 of L^2 functions which are constant on all intervals $I_{1,k} = \left[\frac{k}{2}, \frac{k+1}{2}\right)$, that is

$$V_1 = \left\{ f \in L^2(\mathbb{R}) : f_{|I_{1,k}} = c_k \text{ constant for all } k \in \mathbb{Z} \right\}.$$

We can obtain an orthonormal basis of V_1 just by rescaling the basis of V_0: for $k \in \mathbb{Z}$ let

$$\varphi_{1,k}(t) = \sqrt{2}\,\varphi(2t-k) = \sqrt{2}\,\chi_{I_{1,k}}(t).$$

It is clear that $\|\varphi_{1,k}\|_2 = 1$ and that

$$\langle \varphi_{1,k}, \varphi_{1,j} \rangle = 0 \qquad \text{for } j \neq k.$$

As before, a function of the form $f = \sum_{k\in\mathbb{Z}} c_k \varphi_{1,k}$ is in V_1 if and only if $\|f\|^2 = \sum_{k\in\mathbb{Z}} |c_k|^2 < +\infty$.

Now an arbitrary $f \in L^2(\mathbb{R})$ projected on V_1 gives

$$P_1 f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{1,k} \rangle\, \varphi_{1,k} = \sum_{k\in\mathbb{Z}} c_k\, \chi_{I_{1,k}},$$

where

$$c_k = \sqrt{2}\langle f, \varphi_{1,k} \rangle = 2 \int_{k/2}^{(k+1)/2} f(t)\,dt = \fint_{I_{1,k}} f.$$

Thus

$$P_1 f = \sum_{k\in\mathbb{Z}} \Big(\fint_{I_{1,k}} f \Big) \chi_{I_{1,k}}$$

is an approximation of f with "double resolution" that $P_0 f$.

Let us examine the detail we add to $P_0 f$ when increasing the resolution to produce $P_1 f$. For the sake of simplicity we just look at the interval $I_{0,0} = [0,1)$. The detail in this interval is

$$\begin{aligned}(P_1 f - P_0 f)\chi_{[0,1)} &= \Big(\fint_{I_{1,0}} f\Big)\chi_{I_{1,0}} + \Big(\fint_{I_{1,1}} f\Big)\chi_{I_{1,1}} - \Big(\fint_{I_{0,0}} f\Big)\chi_{I_{0,0}} \\ &= \Big(\fint_{I_{1,0}} f - \fint_{I_{0,0}} f\Big)\chi_{I_{1,0}} + \Big(\fint_{I_{1,1}} f - \fint_{I_{0,0}} f\Big)\chi_{I_{1,1}}.\end{aligned}$$

Observe that this is a multiple of the function $\psi := -\chi_{I_{1,0}} + \chi_{I_{1,1}}$, since

$$\begin{aligned}\fint_{I_{1,0}} f - \fint_{I_{0,0}} f + \fint_{I_{1,1}} f - \fint_{I_{0,0}} f &= 2\int_{I_{1,0}} f - \int_{I_{0,0}} f + 2\int_{I_{1,1}} f - \int_{I_{0,0}} f \\ &= 2\Big(\int_{I_{1,0}} f + \int_{I_{1,1}} f\Big) - 2\int_{I_{0,0}} = 0.\end{aligned}$$

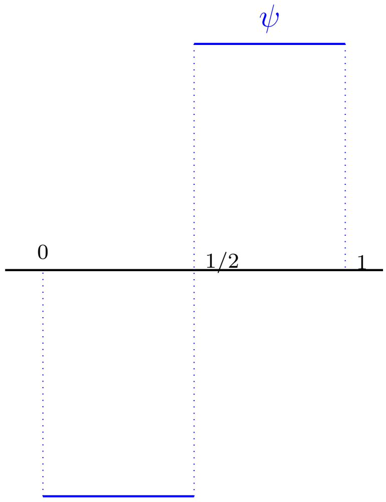


Figure 6.1: The detail added when passing from V_0 to V_1, in the part corresponding to the interval $[0,1)$, is a multiple of ψ

The same arguments are valid for all intervals $[k, k+1)$, where the detail added when passing from V_0 to V_1 is a multiple of $\psi_{0,k}(t) := \psi(t-k)$.

Let W_0 denote the orthogonal complement of V_0 in V_1 (the space of details added to $P_0 f$ to get $P_1 f$, for $f \in L^2(\mathbb{R})$), so that $V_1 = V_0 \oplus W_0$ and $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_0.

This scheme can be reproduced at all scales: consider the dyadic intervals of the n^{th} generation $I_{n,k} = \big[k/2^n, (k+1)/2^n\big)$, $k \in \mathbb{Z}$, together with the closed subspace

$$V_n = \big\{f \in L^2(\mathbb{R}) : f_{|I_{n,k}} = c_k \text{ constant for all } k \in \mathbb{Z}\big\}.$$

The system $\{\varphi_{n,k}\}_{k\in\mathbb{Z}}$, with $\varphi_{n,k}(t) = 2^{n/2}\varphi(2^n t - k)$, is an orthonormal basis of V_n, and a function of the form $f = \sum_{k\in\mathbb{Z}} \alpha_k \chi_{I_{n,k}}$ is in V_n if and only if $\|f\|_2^2 = \sum_{k\in\mathbb{Z}} |\alpha_k|^2 < +\infty$.

The orthogonal projection $P_n : L^2(\mathbb{R}) \longrightarrow V_n$ produces the best approximation (in terms of the L^2-norm) of a given $f \in L^2(\mathbb{R})$ by functions which are constant on each dyadic interval $I_{n,k}$, $k \in \mathbb{Z}$.

The detail added when passing from resolution V_n to resolution V_{n+1} forms a close space, denoted by W_n; hence $V_{n+1} = V_n \oplus W_n$. The rescaled functions $\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$, $k \in \mathbb{Z}$, form an orthonormal basis of W_n.

In the way they are defined it is clear that

$$\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R}),$$

as we have seen in Appendix 1.3. It is also clear that

$$\bigcap_{n\in\mathbb{Z}} V_n = \{0\}.$$

Then, from the iteration ($m < n$)

$$V_n = V_{n-1} \oplus W_{n-1} = V_{n-2} \oplus W_{n-2} \oplus W_{n-1} = \cdots = V_m \oplus W_m \oplus \cdots \oplus W_{n-1}$$

we deduce that

$$V_n = \bigoplus_{j=-\infty}^{n-1} W_j$$

and therefore

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

That is, any function can be viewed as the superposition of the details at all possible resolutions. Also $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$.

This is an example of what is called in general a *multi-resolution analysis* (MRA); this particular one is called Haar MRA. The initial function φ is called the *scaling function* and ψ is the (*mother*) *wavelet* of the MRA. The system $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is the wavelet basis of $L^2(\mathbb{R})$.

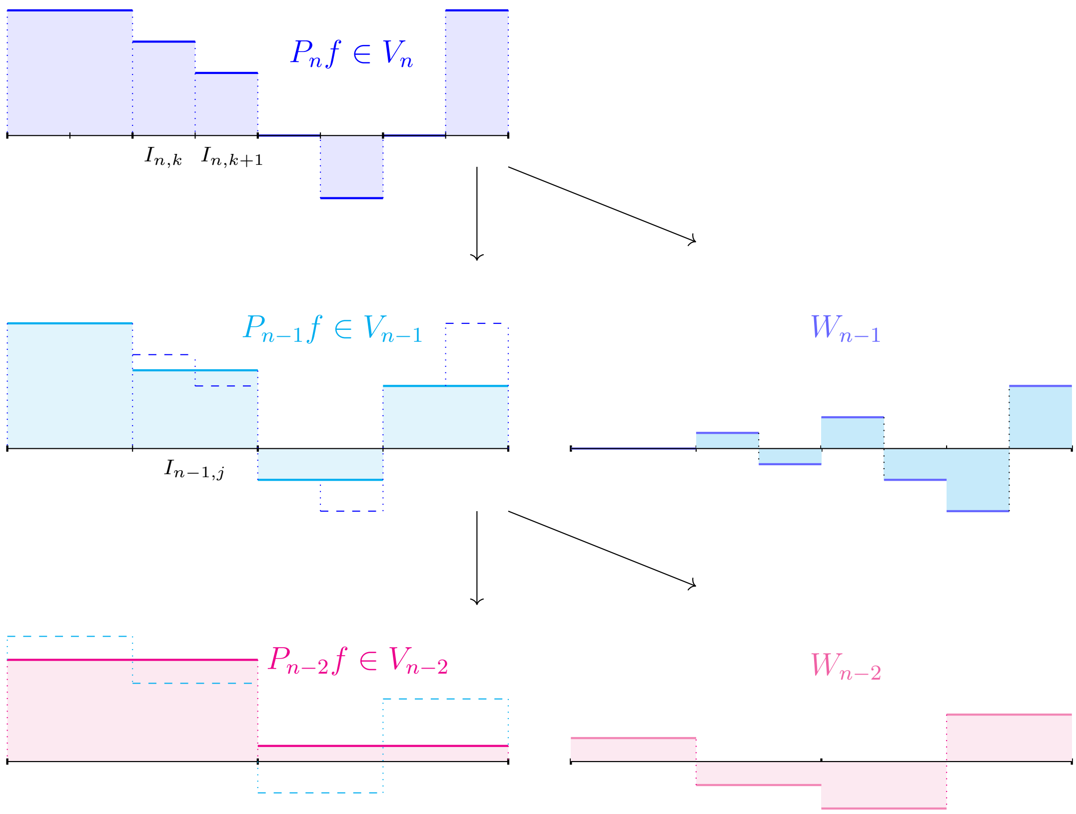


Figure 6.2: Multi-resolution of a signal f. Two consecutive dyadic intervals $I_{n,k}$, $I_{n,k+1}$ of length 2^{-n} correspond to a dyadic interval $I_{n-1,j}$ of length 2^{n-1}. The value of $P_{n-1}f$ on $I_{n-1,j}$ is the average of the values of $P_n f$ on $I_{n,k}$ and $I_{n,k+1}$. The difference $Q_{n-1}f = P_n f - P_{n-1}f \in W_{n-1}$ can be viewed as the detail that must be added to $P_{n-1}f$ to obtain the representation of f at resolution V_n.

6.2 Wavelets via multi-resolution analysis

For a function $f \in L^2(\mathbb{R})$ and an integer $j \in \mathbb{Z}$ let us denote $(D_j f)(t) = 2^{j/2} f(2^j t)$, $t \in \mathbb{R}$.

Definition 13. A *multi-resolution analysis* (MRA) is an increasing sequence $\cdots V_n \subset V_{n+1} \subset \cdots$ of closed subspaces of $L^2(\mathbb{R})$ such that:

1. There exists $\varphi \in V_0$ such that the translates $\varphi_{0,k}(t) = \varphi(t-k)$, $k \in \mathbb{Z}$, form an orthonormal base of V_0. The function φ is the *scaling function* of the MRA.
2. $V_{n+1} = D_1(V_n)$ for all $n \in \mathbb{Z}$. Equivalently $f(t) \in V_n$ if and only if $f(2t) \in V_{n+1}$.
   This implies that $V_n = D_n(V_0)$ and $\varphi_{n,k} = D_n(\varphi_{0,k})$, $k \in \mathbb{Z}$, form an orthonormal base of V_n.
3. $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = L^2(\mathbb{R})$.
4. $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$.

Remarks 5. 1. It is clear from the definition that the scaling function determines the MRA.

2. Condition 4 is actually redundant, it is implied by the others. (We will not prove this).

Given a MRA $\{V_n\}_{n \in \mathbb{Z}}$ as above consider the orthogonal complement of V_n in V_{n+1} and call it W_n; hence $V_{n+1} = V_n \oplus W_n$. Observe that $W_n = D_n(W_0)$, because

$$V_{n+1} = D_n(V_1) = D_n(V_0 \oplus W_0) = D_n(V_0) \oplus D_n(W_0) = V_n \oplus D_n(W_0).$$

As in the Haar example, one has $V_{n+1} = \bigoplus_{j \leq n} W_j$ and

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j. \tag{6.1}$$

Associated to the MRA there are the orthogonal projections $P_n : L^2(\mathbb{R}) \longrightarrow V_n$, given by

$$P_n f = \sum_{k \in \mathbb{Z}} \langle f, \varphi_{n,k} \rangle \, \varphi_{n,k},$$

which may be thought of as the representations at resolution n of a given f.

Similarly, one has the orthogonal projections $Q_n : L^2(\mathbb{R}) \longrightarrow W_n$, given by

$$Q_n f = P_{n+1} f - P_n f,$$

which may be interpreted as the detail to add to $P_n f$ to obtain $P_{n+1} f$.

6.2.1 Wavelet of the MRA. Mallat's theorem

A (mother) *wavelet of the MRA* $\{V_n\}_{n\in\mathbb{Z}}$ is a function $\psi \in W_0$ such that the translates $\psi_{0,k}(t) = \psi(t-k)$, $k \in \mathbb{Z}$, form an orthonormal basis of W_0.

Proposition 8. *If ψ is a wavelet of the MRA $\{V_n\}_{n\in\mathbb{Z}}$, then the system $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$, where $\psi_{n,k} = D_n(\psi_{0,k})$, is an orthonormal basis of $L^2(\mathbb{R})$ (called the wavelet basis of the MRA).*

Proof. Since $W_n = D_n(W_0)$, $n \in \mathbb{Z}$, when ψ is a wavelet the family $\psi_{n,k} = D_n(\psi_{0,k})$, $k \in \mathbb{Z}$, is an orthonormal basis of W_n. By (6.1), the system $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ spans then the whole $L^2(\mathbb{R})$.

It remains to see that $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal system. By definition of wavelet it is clear that $\langle \psi_{n,k}, \psi_{n,j}\rangle = \delta_{jk}$, so it will enough to prove that $W_n \perp W_k$ for $k \neq n$. But this is clear, since (assuming without loss of generality that $k < n$) one has $W_k \subset V_{k+1} \subset V_n$ and $V_n \perp W_n$. $\square$

At this juncture the natural question is whether every MRA has a wavelet. The answer is affirmative, and the wavelet can be obtained from the scaling function.

Theorem 12 (Mallat's theorem)**.** *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a MRA with scaling function φ. Then*

(a) $\sum\limits_{k\in\mathbb{Z}} |\langle \varphi, \varphi_{1,k}\rangle|^2 = 1$ *and* $\sum\limits_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k}\rangle \overline{\langle \varphi, \varphi_{1,k-2l}\rangle} = 0$ *for all $l \in \mathbb{Z}\setminus\{0\}$.*

(b) The function

$$\psi = \sum_{k\in\mathbb{Z}} (-1)^k\, \overline{\langle \varphi, \varphi_{1,1-k}\rangle}\, \varphi_{1,k}$$

is a wavelet for the MRA.

Example 2. (Haar) Let $\varphi = \chi_{[0,1)}$ be the scaling function of the Haar wavelet. Then

$$\varphi = \chi_{[0,1/2)} + \chi_{[1/2,1)} = \frac{1}{\sqrt{2}}\varphi_{1,0} + \frac{1}{\sqrt{2}}\varphi_{1,1},$$

so that

$$c_k = \langle \varphi, \varphi_{1,k}\rangle = \begin{cases} 1/\sqrt{2} & \text{if } k = 0,1 \\ 0 & \text{if } k \neq 0,1. \end{cases}$$

Then, by (b)

$$\begin{aligned} \psi &= (-1)^0 \bar{c}_1 \varphi_{1,0} + (-1)^1 \bar{c}_0 \varphi_{1,1} = \frac{1}{\sqrt{2}}\sqrt{2}\chi_{[0,1/2)} - \frac{1}{\sqrt{2}}\sqrt{2}\chi_{[1/2,1)} \\ &= \chi_{[0,1/2)} - \chi_{[1/2,1)} \end{aligned}$$

Proof. (a) The first identity follows immediately from the decomposition $\varphi = \sum\limits_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k}\rangle\, \varphi_{1,k}$ of φ in V_1.

(b) For the second one, let $l \neq 0$ and try to express the identity $\langle \varphi, \varphi_{0,l} \rangle = 0$ in terms of the coefficients of φ and $\varphi_{0,l}$ in the basis $\{\varphi_{1,k}\}_{k\in\mathbb{Z}}$. Since

$$\varphi_{0,l}(t) = \varphi(t-l) = \sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k} \rangle \, \varphi_{1,k}(t-l)$$

and

$$\varphi_{1,k}(t-l) = \sqrt{2}\varphi\big(2t-l)-k\big) = \sqrt{2}\,\varphi\big(2t-(2l+k)\big) = \varphi_{1,k+2l}(t) \tag{6.2}$$

if follows that

$$\varphi_{0,l}(t) = \sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k} \rangle \, \varphi_{1,k+2l}(t) = \sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k-2l} \rangle \, \varphi_{1,k}(t).$$

This and the decomposition of φ in terms of $\{\varphi_{1,k}\}_{k\in\mathbb{Z}}$ yield,

$$0 = \langle \varphi, \varphi_{0,l} \rangle = \sum_{k\in\mathbb{Z}} \langle \varphi, \varphi_{1,k} \rangle \, \overline{\langle \varphi, \varphi_{1,k-2l} \rangle},$$

as stated.

(b) We need to see that $\psi \in V_1 \ominus V_0$ and that the translates $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ form an orthonormal basis of W_0.

It is clear by the definition that $\psi \in V_1$.

In order to see that $\psi \in V_0^\perp$ it is enough to see that for all $l \in \mathbb{Z}$

$$\langle \varphi_{0,-l}, \psi \rangle = \langle \varphi, \psi_{0,l} \rangle = 0.$$

Using again that $\varphi_{1,k}(t-l) = \varphi_{1,k+2l}(t)$ (see (6.2)), denoting $c_k = \langle \varphi, \varphi_{1,k} \rangle$ and re-indexing the sum defining ψ,

$$\begin{aligned}\psi_{0,l}(t) = \psi(t-l) &= \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} \, \varphi_{1,k}(t-l) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} \, \varphi_{1,k+2l}(t) \\ &= \sum_{m\in\mathbb{Z}} (-1)^m \overline{c_{1-m+2l}} \, \varphi_{1,m}(t).\end{aligned}$$

Since

$$\varphi = \sum_{m\in\mathbb{Z}} \langle \varphi, \varphi_{1,m} \rangle \, \varphi_{1,m} = \sum_{m\in\mathbb{Z}} c_m \, \varphi_{1,m}$$

this yields

$$\langle \varphi, \psi_{0,l} \rangle = \sum_{m\in\mathbb{Z}} (-1)^m c_m c_{1-m+2l}.$$

Since $(-1)^m$ and $(-1)^{1-m+2l}$ have opposite signs this adds up to 0 (each term appers twice, and with opposite signs).

By translation, this also shows that $\psi_{0,l} \in W_0$ for all $l \in \mathbb{Z}$.

Let us see next that $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal system. Since $\langle\psi_{0,m},\psi_{0,j}\rangle = \langle\psi,\psi_{0,j-m}\rangle$, it is enough to see that $\langle\psi,\psi_{0,l}\rangle = \delta_{0l}$, $l\in\mathbb{Z}$.

Using the definition of ψ and the expression $\psi_{0,l} = \sum_{m\in\mathbb{Z}}(-1)^m\overline{c_{1-m+2l}}\,\varphi_{1,m}$ seen above we obtain, by the first identity in (a),

$$\langle\psi,\psi_{0,l}\rangle = \sum_{k\in\mathbb{Z}}(-1)^{2k}\overline{c_{1-k}}\,c_{1-m+2l} = \sum_{m\in\mathbb{Z}}\overline{c_m}\,c_{m+2l} = \delta_{0,l}.$$

It remains to prove the hard part, that $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ generates the whole W_0. To do so we want to see that any $f\in V_1\ominus V_0$ such that $\langle f,\psi_{0,k}\rangle = 0$ for all $k\in\mathbb{Z}$ is necessarily $f=0$. Since all functions appearing here are in V_1, they are completely determined by their coordinates in the basis $\varphi_{1,k}$, $k\in\mathbb{Z}$. Thus

$$f \longleftrightarrow F = (f_k)_{k\in\mathbb{Z}}\,, \qquad f_k = \langle f,\varphi_{1,k}\rangle.$$

Similarly, since $\varphi_{0,l} = \sum_{k\in\mathbb{Z}}\langle\varphi,\varphi_{1,k-2l}\rangle\varphi_{1,k}$ (see the proof of the first identity in (a) above), these functions can be identified with the sequence of coefficients

$$\varphi_{0,l} \longleftrightarrow \Phi_l = (c_{k-2l})_{k\in\mathbb{Z}}.$$

In the same way, from the expression above

$$\psi_{0,l} \longleftrightarrow \Psi_l = \big((-1)^k\overline{c_{1-k+2l}}\big)_{k\in\mathbb{Z}}.$$

The orthogonality assumptions are then expressed as

$$\langle F,\Phi_j\rangle_{\ell^2(\mathbb{Z})} = \langle F,\Psi_j\rangle_{\ell^2(\mathbb{Z})} = 0 \qquad \forall j\in\mathbb{Z}. \tag{6.3}$$

Consideren now the infinite matrix M whose columns are the vectors Φ_l and Ψ_l writen alternately:

$$\begin{array}{cccccccc}
\cdots & \Phi_1 & \Psi_1 & \Phi_0 & \Psi_0 & \Phi_{-1} & \Psi_{-1} & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & c_{-1-2} & (-1)^{-1}\bar{c}_{1-1+2} & c_{-1} & (-1)^{-1}\bar{c}_{1+1} & c_{-1+2} & (-1)^{1}\bar{c}_{1+1-2} & \cdots \\
\cdots & c_{0-2} & (-1)^{0}\bar{c}_{1-0+2} & c_{0} & (-1)^{0}\bar{c}_{1-0} & c_{0+2} & (-1)^{0}\bar{c}_{1-0-2} & \cdots \\
\cdots & c_{1-2} & (-1)^{1}\bar{c}_{1-1+2} & c_{1} & (-1)^{1}\bar{c}_{1-1} & c_{1+2} & (-1)^{0}\bar{c}_{1-1-2} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Observe that the columns of this matrix are orthogonal.

Claim. $MM^* = I$, where M^* indicates the conjugate transpose of M.

Accepting this the proof is finished: by assumption (see (6.3)) $M^*F = 0$, and therefore

$$F = MM^*F = 0.$$

The proof of the claim is essentialy a computation. Break M into 2×2 blocks of the form

$$M_{m,l} = \begin{pmatrix} c_{2m-2l} & \bar{c}_{1-2m+2l} \\ c_{2m-2l+1} & -\bar{c}_{-2m+2l} \end{pmatrix} \qquad m, l \in \mathbb{Z}.$$

The first column is part of the vector Φ_l and the second part of Ψ_l.

Then the matrix MM^* is made up of the blocks

$$\sum_{j\in\mathbb{Z}} M_{m,j}(M_{l,j})^* = \begin{pmatrix} c_{2m-2j} & \bar{c}_{1-2m+2j} \\ c_{2m-2j+1} & -\bar{c}_{-2m+2j} \end{pmatrix} \begin{pmatrix} \bar{c}_{2l-2j} & \bar{c}_{2l-2j+1} \\ c_{1-2l+2j} & -c_{-2l+2j} \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_j c_{2m-2j}\bar{c}_{2l-2j} + \bar{c}_{1-2m+2j}c_{1-2l+2j} & \sum\limits_j c_{2m-2j}\bar{c}_{2l-2j+1} - \bar{c}_{1-2m+2j}c_{-2l+2j} \\ \sum\limits_j c_{2m-2j+1}\bar{c}_{2l-2j} - \bar{c}_{-2m+2j}c_{1-2l+2j} & \sum\limits_j c_{2m-2j+1}\bar{c}_{2l-2j+1} + \bar{c}_{-2m+2j}c_{-2l+2j} \end{pmatrix}$$

A computation, using the orthogonality relations seen previously shows that this is

$$\begin{pmatrix} \delta_{ml} & 0 \\ 0 & \delta_{ml} \end{pmatrix}.$$

For example, for the firts entry re-index the sum by $m - j = k$ in the first term and by $k = j - l$ in the second one:

$$\sum_j c_{2m-2j}\bar{c}_{2l-2j} + \bar{c}_{1-2m+2j}c_{1-2l+2j} = \sum_k c_{2k}\bar{c}_{2k-2m+2l} + \sum_k \bar{c}_{2k+1-2m+2l}\, c_{2k+1}$$
$$= \sum_k c_k\, \bar{c}_{k-2m+2l} = \delta_{ml}.$$

The other entries are dealt with similarly.

The same procedure shows that $\{\psi_{n,k}\}_{k\in\mathbb{Z}}$ is an orthonormal system and that any $f \in V_{n+1} \ominus V_n$ which is orthogonal to that system must be 0, so $\{\psi_{n,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_n. Since $\cap_n V_n = \{0\}$ and $\overline{\cup_n V_n} = L^2(\mathbb{R})$ we deduce that $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. $\square$

6.2.2 Construction of a MRA

The results in the previous section suggest a scheme to construct wavelet basis:

1. Determine a $\varphi \in L^2(\mathbb{R})$ (scaling function) such that $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal system and define $V_0 = \overline{\langle\varphi_{0,k}\rangle}_{k\in\mathbb{Z}}$.

2. Check that $V_n := D_n(V_0)$ is an increasing sequence of close subspaces in $L^2(\mathbb{R})$ and that $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$.

3. Find, using Mallat's theorem, the associated wavelet ψ, so that $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $W_0 = V_1 \ominus V_0$.

In order to (partially) carry out this program we shall use Fourier analysis. We begin with a necessary and sufficient condition for φ to satisfy condition 1.

Theorem 13. *Let $\varphi \in L^2(\mathbb{R})$. Then $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal system if and only if*

$$\sum_{n\in\mathbb{Z}} \left|\widehat{\varphi}(\xi+n)\right|^2 = 1 \qquad \text{a.e. } \xi \in \mathbb{R}.$$

Proof. By Plancherel, since $\widehat{\varphi_{0,k}}(\xi) = e^{2\pi i k\xi}\widehat{\varphi}(\xi)$,

$$\begin{aligned}
\langle \varphi, \varphi_{0,k}\rangle = \langle \widehat{\varphi}, \widehat{\varphi}_{0,k}\rangle &= \int_{\mathbb{R}} \widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\xi)}\, e^{-2\pi i k\xi} d\xi = \sum_{n\in\mathbb{Z}} \int_n^{n+1} |\widehat{\varphi}(\xi)|^2 e^{-2\pi i k\xi} d\xi \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 |\widehat{\varphi}(\xi+n)|^2 e^{-2\pi i k\xi} d\xi = \int_0^1 \left(\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\xi+n)|^2 \right) e^{-2\pi i k\xi} d\xi.
\end{aligned}$$

Letting $F(\xi) = \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\xi+n)|^2$, which is 1-periodic and in $L^1[0,1]$ (since $\widehat{\varphi} \in L^2(\mathbb{R})$), we have thus

$$\langle \varphi, \varphi_{0,k}\rangle = \widehat{F}(k).$$

By the uniqueness theorem for L^1 functions (see Remark 2) the conclusion is immediate: if the system is orthonormal $\widehat{F}(k) = 0$ for all $k \neq 0$ and $\widehat{F}(0) = 1$, so $F(\xi) = 1$. On the other hand, if $F(\xi) = 1$ a.e. $\xi \in \mathbb{R}$ then $\widehat{F}(0) = 1$ and $\widehat{F}(k) = 0$ for all $k \neq 0$. □

Let us give next a condition for 2 to hold.

Theorem 14. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be an increasing sequence of closed subspaces of $L^2(\mathbb{R})$ such that $D_1(V_n) = V_{n+1}$ for all $n \in \mathbb{Z}$. Let φ be such that $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0 and assume also that $|\widehat{\varphi}|$ is continuous at 0. Then $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$ if and only if $\widehat{\varphi}(0) \neq 0$. (In that case $|\widehat{\varphi}(0)| = 1$).*

Once we have this kind of scaling function φ we can construct the associated wavelet ψ and wavelet basis $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$, as explained before. We will sketch later how to obtain ψ using Fourier analysis (that is, obtaining $\hat{\psi}$ rather than ψ).

Proof. Let's prove first that $\widehat{\varphi}(0) \neq 0$ implies $\mathcal{V} := \overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$.

The first step in proving that $\mathcal{V}^\perp = \{0\}$ is proving that $\mathcal{V}$ is invariant by translations. Assume first that $f \in \mathcal{V}$ is translated by a dyadic number $m/2^l$, $m, l \in \mathbb{Z}$. Given $\epsilon > 0$ there exists $n_0 \in \mathbb{Z}$ and $h \in V_{n_0}$ such that $\|f - h\|_2 < \epsilon$. Since $\{V_n\}_{n\in\mathbb{Z}}$ is an increasing sequence we have $h \in V_n$ for all $n \geq n_0$, and therefore it has an expression of the form

$$h(t) = \sum_{k\in\mathbb{Z}} c_k^n \varphi(2^n t - k).$$

Then

$$\tau_{\frac{m}{2^l}} h(t) = h\big(t - \frac{m}{2^l}\big) = \sum_{k\in\mathbb{Z}} c_k^n\, \varphi\big(2^n(t - \frac{m}{2^l}) - k\big) = \sum_{k\in\mathbb{Z}} c_k^n\, \varphi\big(2^n t - (2^{n-l}m + k)\big),$$

and for $n \geq l$ this belongs to V_n as well. Then $\tau_{\frac{m}{2^l}} f \in \mathcal{V}$, since

$$\big\|\tau_{\frac{m}{2^l}} f - \tau_{\frac{m}{2^l}} h\big\|_2 = \|f - h\|_2 < \epsilon.$$

For a general translation $\tau_x f$, $x \in \mathbb{R}$, just take a dyadic number $m/2^l$ close enough to x so that

$$\big\|\tau_{\frac{m}{2^l}} f - \tau_x f\big\|_2 < \epsilon.$$

This is possible because, as we have seen in Lemma 5, translations are continuous in $L^2(\mathbb{R})$.

Let us see finally that $\mathcal{V}^\perp = \{0\}$. Assume that $g \in \mathcal{V}^\perp$. By the invariance by translations

$$\langle \tau_{-x} f, g\rangle = \int_{\mathbb{R}} f(t+x)\,\overline{g(t)}\,dt = 0 \qquad \text{for all } x \in \mathbb{R} \text{ and all } f \in \mathcal{V}.$$

By Plancherel this is

$$\int_{\mathbb{R}} e^{2\pi i x\xi}\,\widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\,d\xi = 0 \qquad \text{for all } x \in \mathbb{R} \text{ and all } f \in \mathcal{V}.$$

By the Cauchy-Schwartz inequality $\hat{f}\overline{\hat{g}} \in L^1(\mathbb{R})$, so we can use the inversion formula (Theorem 7) to deduce that

$$\widehat{f}(\xi)\,\widehat{g}(\xi) = 0 \quad \text{a.e } \xi \in \mathbb{R}.$$

In particular, for $f(t) = D_{1/2^n}(t) = 2^n\varphi(2^n t)$, which is in $V_n \subset \mathcal{V}$ and has $\widehat{f}(\xi) = \widehat{\varphi}(\xi/2^n)$ (see Proposition 3, 5) one has

$$\widehat{\varphi}\big(\frac{\xi}{2^n}\big)\,\widehat{g}(\xi) = 0 \quad \text{a.e } \xi \in \mathbb{R}.$$

Since $\widehat{\varphi}(0) \neq 0$ and $|\widehat{\varphi}|$ is continuous at 0 there exists and interval $I = (-\eta, \eta)$ where $\widehat{\varphi}(\xi) \neq 0$. As soon as $\xi/2^n \in I$, that is, $|\xi| < 2^n$ this implies $\widehat{g}(\xi) = 0$. Since this is valid for all $n \in \mathbb{Z}$ we deduce that $\hat{g} \equiv 0$, as desired.

Let us see now that $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$ implies $\widehat{\varphi}(0) \neq 0$. Let $f \in L^2(\mathbb{R})$ be such that $\hat{f} = \chi_{[-1,1]}$ (i.e. $f(t) = \mathrm{sinc}(2t)$). Let $P_n : L^2(\mathbb{R}) \longrightarrow V_n$ denote the orthogonal projection; then

$$\big|\|f\|_2 - \|P_n f\|_2\big| \leq \|f - P_n f\|_2 \overset{n\to\infty}{\longrightarrow} 0.$$

Since $\{\varphi_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of V_n, Parseval's identity and Plancherel's theorem yield

$$\|P_n f\|_2^2 = \sum_{k\in\mathbb{Z}} \big|\langle f, \varphi_{n,k}\rangle\big|^2 \overset{n\to\infty}{\longrightarrow} \|f\|_2^2 = \|\hat{f}\|_2^2 = 2. \tag{6.4}$$

Since

$$\varphi_{n,k}(t) = 2^{n/2}\varphi\big(2^n(t-k/2^n)\big) = 2^{-n/2}\tau_{\frac{k}{2^n}} D_{\frac{1}{2^n}}\varphi(t),$$

and therefore

$$\widehat{\varphi}_{n,k}(\xi) = 2^{-n/2}e^{2\pi i\frac{k}{2^n}\xi}\,\widehat{\varphi}\big(\frac{\xi}{2^n}\big),$$

Plancherel yields

$$\begin{aligned}
\|P_n f\|_2^2 &= \sum_{k\in\mathbb{Z}}\big|\langle \hat{f}, \widehat{\varphi}_{n,k}\rangle\big|^2 = \sum_{k\in\mathbb{Z}}\Big|\int_{\mathbb{R}} \hat{f}(\xi)\,\overline{e^{2\pi i\frac{k}{2^n}\xi}\,2^{-n/2}\widehat{\varphi}(2^{-n}\xi)}\,d\xi\Big|^2 \\
&= \sum_{k\in\mathbb{Z}}\Big|\int_{-1}^{1} e^{-2\pi i\frac{k}{2^n}\xi}\,2^{-n/2}\overline{\widehat{\varphi}(2^{-n}\xi)}\,d\xi\Big|^2 \qquad (2^{-n}\xi=\omega) \\
&= \sum_{k\in\mathbb{Z}}\Big|\int_{-2^{-n}}^{2^{-n}} e^{-2\pi i k\omega}\,2^{-n/2}\,\overline{\widehat{\varphi}(\omega)}\,2^n d\omega\Big|^2 \\
&= 2^n\sum_{k\in\mathbb{Z}}\Big|\int_{-2^{-n}}^{2^{-n}} e^{-2\pi i k\omega}\,\overline{\widehat{\varphi}(\omega)}\,d\omega\Big|^2.
\end{aligned}$$

For $n \geq 0$ the integral appearing in this last sum is twice the k^{th} Fourier coefficient of the function $\chi_{[-2^{-n},2^{-n}]}\overline{\widehat{\varphi}}$, which is clearly in $L^2[-1,1]$. Thus, again by Plancherel (for Fourier series)

$$\|P_n f\|_2^2 = 2^n\big\|\chi_{[-2^{-n},2^{-n}]}\overline{\widehat{\varphi}}\big\|_{L^2[-1,1]}^2 = 2^n\int_{-2^{-n}}^{2^{-n}}|\widehat{\varphi}(\omega)|^2 d\omega = 2\fint_{-2^{-n}}^{2^{-n}}|\widehat{\varphi}(\omega)|^2 d\omega.$$

By the continuity of $|\widehat{\varphi}|$ at 0

$$\fint_{-2^{-n}}^{2^{-n}}|\widehat{\varphi}(\omega)|^2 d\omega \overset{n\to\infty}{\longrightarrow} |\widehat{\varphi}(0)|^2.$$

This together with (6.4) shows that $\widehat{\varphi}(0)\neq 0$ (and $|\widehat{\varphi}(0)|=1$). □

6.2.3 The wavelet of the MRA from the Fourier side

So far we have seen conditions to construct the scaling function φ of a MRA in terms of its Fourier transform $\widehat{\varphi}$. It is also possible to construct the associated wavelet ψ using Fourier analysis. The scheme would be as follows.

Developing the scaling function in terms of the basis $\{\varphi_{1,k}\}_k$ of V_1 one sees that there exists a 1-periodic function $H(\xi)$ such that

$$\widehat{\varphi}(\xi) = H(\xi/2)\,\widehat{\varphi}(\xi/2).$$

The function H is called the *refinement mask* or *low pass filter* of the MRA, and it satisfies the *quadratic mirror filter* (QMF) identity

$$\big|H(\xi)\big|^2 + \big|H\big(\xi+\frac{1}{2}\big)\big|^2 = 1. \tag{6.5}$$

This is consequence of Theorem 13.

Main Lemma. *A function $f \in L^2(\mathbb{R})$ is in the detail space W_0 if and only if there exists a 1-periodic function v such that*

$$\hat{f}(\xi) = e^{i\pi\xi} v(\xi)\, \overline{H\big(\xi + \frac{1}{2}\big)}\, \widehat{\varphi}(\xi/2).$$

Since the wavelet ψ of the MRA is in W_0, it must have this form. We write this as

$$\widehat{\psi}(\xi) = m_\psi(\xi/2)\, \widehat{\varphi}(\xi/2),$$

where

$$m_\psi(\xi) = e^{2\pi i\xi}\sigma(\xi)\overline{H\big(\xi + \frac{1}{2}\big)}$$

and σ is $1/2$-periodic.

Since we want $\{\psi_{0,k}\}_k$ to form an orthonormal system we can use again Theorem 13 to deduce that m_ψ also satisfies the QMF identity above. Developing this, one gets

$$\begin{aligned} 1 &= |\sigma(\xi)|^2 \big|H\big(\xi + \frac{1}{2}\big)\big|^2 + |\sigma(\xi + \frac{1}{2})|^2 \big|H\big(\xi + 1\big)\big|^2 = |\sigma(\xi)|^2 \Big(\big|H\big(\xi + \frac{1}{2}\big)\big|^2 + |H(\xi)|^2\Big) \\ &= |\sigma(\xi)|^2. \end{aligned}$$

Mallat's construction consists of taking the easiest possible $\sigma(\xi)$ with $|\sigma(\xi)| = 1$, that is $\sigma(\xi) \equiv 1$. Then

$$\widehat{\psi}(\xi) = G(\xi/2)\widehat{\varphi}(\xi/2)\,, \qquad G(\xi) = e^{2\pi i\xi}\, \overline{H\big(\xi + \frac{1}{2}\big)} \quad \text{1-periodic.}$$

In this way ψ is defined in terms of φ (actually in terms of $\widehat{\varphi}$ and its refinement mask $H(\xi) = \frac{\widehat{\varphi}(2\xi)}{\widehat{\varphi}(\xi)}$).

By the Main Lemma $\psi \in W_0$. Also, the family $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ is orthogonal, because

$$\widehat{\psi_{0,k}}(\xi) = e^{-2\pi i k\xi} G(\xi/2)\, \widehat{\varphi}(\xi/2)$$

and the QMF property of $G(\xi)$ implies that Theorem 13 holds.

To see that $W_0 = \overline{\langle\psi_{0,k}\rangle}_{k\in\mathbb{Z}}$ let $f \in W_0$ and use the Main Lemma to see that

$$\hat{f}(\xi) = v(\xi) e^{\pi i\xi} H\big(\xi + \frac{1}{2}\big)\widehat{\varphi}(\xi/2) = v(\xi)\, \widehat{\psi}(\xi)$$

for some 1-periodic function v. Here v has the form $v(\xi) = \sum_k a_k e^{2\pi i k\xi}$, with $\{c_k\}_k \in \ell^2(\mathbb{Z})$, so

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k\xi} \widehat{\psi}(\xi) = \sum_{k\in\mathbb{Z}} \widehat{a_k \psi_{0,k}}.$$

Hence, $f = \sum_{k\in\mathbb{Z}} a_k \psi_{0,k}$, as desired.

Mallat's theorem provides thus an algorithm for constructing the wavelet from the MRA and the scaling function via the Fourier coefficients of the 1-periodic refinement mask. This can be implementes numerically in the so-called *cascade algorithm*, which schematically goes as follows:

$$\begin{array}{ccccccc} \varphi & \longrightarrow & H & \longrightarrow & G & \longrightarrow & \psi \\ \text{scaling function} & & \text{low pass} & & \text{high pass} & & \textit{Mallat's wavelet} \end{array}$$

Example 3. (Haar). Start with $\varphi = \chi_{[0,1)}$, so that

$$\widehat{\varphi}(\xi) = \frac{1-e^{-2\pi i\xi}}{2\pi i\xi} = \frac{1-e^{-\pi i\xi}}{\pi i\xi}\,\frac{1+e^{-\pi i\xi}}{2} = \widehat{\varphi}\big(\frac{\xi}{2}\big)\,\frac{1+e^{-\pi i\xi}}{2}$$

and therefore $H(\xi) = \frac{1}{2}(1+e^{-2\pi i\xi})$. Then

$$H\big(\xi+\frac{1}{2}\big) = \frac{1+e^{-2\pi i\xi}e^{-\pi i}}{2} = \frac{1-e^{-2\pi i\xi}}{2}$$

and

$$G(\xi) = e^{2\pi i\xi}\overline{H\big(\xi+\frac{1}{2}\big)} = e^{2\pi i\xi}\frac{1-e^{2\pi i\xi}}{2}.$$

Finally

$$\begin{aligned}\widehat{\psi}(\xi) &= G(\xi/2)\widehat{\varphi}(\xi/2) = e^{\pi i\xi}\frac{1}{2}(1-e^{2\pi i\xi})\,\frac{1-e^{-\pi i\xi}}{\pi i\xi}\\ &= e^{\pi i\xi}\frac{e^{\pi i\frac{\xi}{2}}}{2}\,\big(e^{-\pi i\frac{\xi}{2}}-e^{\pi i\frac{\xi}{2}}\big)\frac{e^{-\pi i\frac{\xi}{2}}(e^{\pi i\frac{\xi}{2}}-e^{-\pi i\frac{\xi}{2}})}{\pi i\xi} = \frac{1}{2}e^{\pi i\xi}\big(-2i\,\sin(\frac{\pi}{2}\xi)\big)\,\frac{2i\sin(\frac{\pi}{2}\xi)}{\pi i\xi}\\ &= -ie^{\pi i\xi}\frac{\sin^2(\frac{\pi}{2}\xi)}{\frac{\pi}{2}\xi} = -ie^{\pi i\xi}\,\operatorname{sinc}(\frac{\pi}{2}\xi)\,\sin(\frac{\pi}{2}\xi).\end{aligned}$$

Of course this is the same we obtain by computing the Fourier transform of $\psi = \chi_{[0,1/2)} - \chi_{[1/2,0)}$.

6.3 Wavelets in $\mathbb{R}^2$

As mentioned previously, wavelets are important in image processing. Keeping in mind these applications we introduce here "separable" multi-resolutions, that is, multi-resolutions in $\mathbb{R}^2$ obtained as products of one-dimensional multi-resolutions.

A first attempt, given $\{\psi_{n,k}\}_{n,k\in\mathbb{Z}}$ wavelet orthonormal basis of $L^2(\mathbb{R})$, would be to consider the products $L^2(\mathbb{R}^2)$:

$$\{\psi_{n_1,k_1}(t_1)\,\psi_{n_2,k_2}(t_2)\}_{\substack{n_1,n_2\in\mathbb{Z}\\ k_1,k_2\in\mathbb{Z}}}.$$

These functions mix information at two different scales, 2^{n_1} and 2^{n_2}, along the axes t_1 and t_2. This is not convenient; it is desirable to have the same scale in all directions.

This construction can be slightly modified to provide another separable wavelet basis whose elements are products of one variable functions dilated by the same factor in all coordinate directions. These multi-resolution approximations have important applications in computer vision, where they are used to process images at different levels of detail.

6.3.1 Separable multi-resoltions

The formal definition of a MRA in $\mathbb{R}^2$ is as in one dimension: it is an increasing collection of closed subspaces $\{V_n^{(2)}\}_{n\in\mathbb{Z}}$ with the properties previously listed (see Definiton 13). As in dimension one, the notion of resolution is formalised with orthogonal projections on the spaces V_n; hence the approximation of $f(t_1, t_2)$ at resolution n is the orthogonal projection of f on $V_n^{(2)}$ and the space $V_n^{(2)}$ is the set of all approximations at resolution n.

We consider only the particular case of separable multi-resoltions. Given a multi-resolution $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, the associated *separable 2-dimensional multi-resolution* is $\{V_n^{(2)}\}_{n\in\mathbb{Z}}$, where $V_n^{(2)} = V_n \otimes V_n$. Thus $f \in V_n^{(2)}$ if it has the form

$$f(t_1, t_2) = \sum_{m\in\mathbb{Z}} c_m f_m(t_1)\, g_m(t_2)\,,$$

where $f_m, g_m \in V_n$, $\|f_m\| = \|g_m\| = 1$ and $\sum_m |c_m|^2 = \|f\|^2_{L^2(\mathbb{R}^2)} < \infty$.

It is immediate to ckeck that $\{V_n^{(2)}\}_{n\in\mathbb{Z}}$ is a multi-resolution of $L^2(\mathbb{R}^2)$.

Let φ be the scaling function of $\{V_n\}_{n\in\mathbb{Z}}$, so that $\{\varphi_{n,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_n. Then the system

$$\begin{gathered}\varphi^{(2)}_{n,k}(t_1, t_2) = \varphi_{n,k_1}(t_1)\,\varphi_{n,k_2}(t_2) = 2^n\varphi(2^n t_1 - k_1)\,\varphi(2^n t_2 - k_2) \\ n \in \mathbb{Z}, \quad k = (k_1, k_2) \in \mathbb{Z}^2\end{gathered}$$

is an orthonormal basis of $V_n^{(2)}$. Notice that the scaling function of the two-dimensional multi-resolution is just

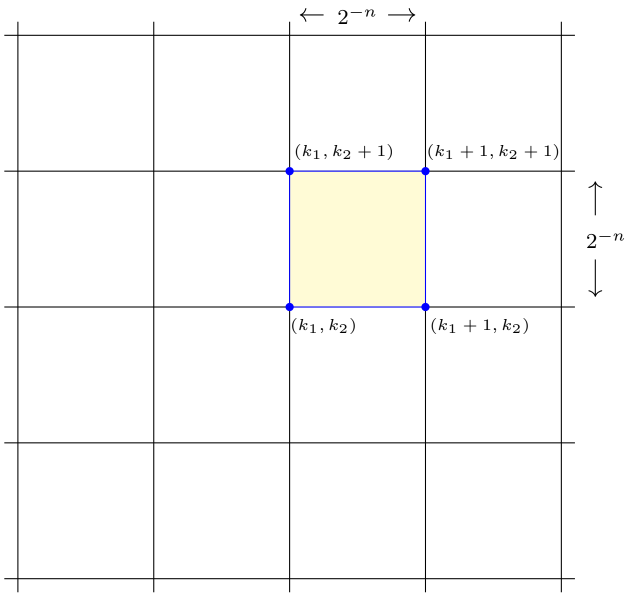
$$\varphi(t_1, t_2) = \varphi(t_1)\,\varphi(t_2).$$

Example 4. (Haar) Let $\{V_n\}_{n\in\mathbb{Z}}$ be the Haar MRA given in the introduction of Chapter 6.3. Then $V_n^{(2)}$ is the approximation space consisting on the functions in $L^2(\mathbb{R}^2)$ which are constant on dyadic squares

$$Q_{n,k} = [2^{-n}k_1, 2^{-n}(k_1+1)) \times [2^{-n}k_2, 2^{-n}(k_2+1))\,, \quad k_1, k_2 \in \mathbb{Z}.$$

The two-dimensional scaling function is therefore

$$\varphi(t_1, t_2) = \chi_{[0,1)}(t_1)\,\chi_{[0,1)}(t_2) = \chi_{[0,1)\times[0,1)}(t_1, t_2).$$



Given a MRA $\{V_n^{(2)}\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ let $W_n^{(2)} = V_{n+1}^{(2)} \ominus V_n^{(2)}$ denote the corresponding detail space at level n.

Theorem 15. *Let φ be the scaling function of a MRA $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ and let ψ denote the associated wavele. Define*

$$\begin{aligned}\psi^1(t_1,t_2) &= \varphi(t_1)\,\psi(t_2),\\ \psi^2(t_1,t_2) &= \psi(t_1)\,\varphi(t_2),\\ \psi^3(t_1,t_2) &= \psi(t_1)\,\psi(t_2)\end{aligned}$$

and denote, for $n, k_1, k_2 \in \mathbb{Z}$ and $j = 1, 2, 3$,

$$\psi^j_{n,k}(t_1,t_2) = 2^n\,\psi^j\big(2^n t_1 - k_1, 2^n t_2 - k_2\big).$$

The collection $\{\psi^1_{n,k}, \psi^2_{n,k}, \psi^3_{n,k}\}_{k\in\mathbb{Z}^2}$ is an orthonormal basis of $W_n^{(2)}$, $n \in \mathbb{Z}$, and therefore the system $\{\psi^1_{n,k}, \psi^2_{n,k}, \psi^3_{n,k}\}_{\substack{n\in\mathbb{Z}\\ k\in\mathbb{Z}^2}}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$.

Proof. By definition $V_{n+1}^{(2)} = V_n^{(2)} \oplus W_n^{(2)}$. Since by definition $V_{n+1} = V_n \oplus W_n$ we also have

$$V_{n+1}^{(2)} = \big(V_n \oplus W_n\big) \otimes \big(V_n \oplus W_n\big) = V_n^{(2)} \oplus \big(W_n \otimes V_n\big) \oplus \big(V_n \otimes W_n\big) \oplus \big(W_n \otimes W_n\big)$$

we deduce that

$$W_n^{(2)} = \big(W_n \otimes V_n\big) \oplus \big(V_n \otimes W_n\big) \oplus \big(W_n \otimes W_n\big).$$

It is clear that $\{\psi^1_{n,k}\}_{k\in\mathbb{Z}^2}$ is an orthonormal basis of $V_n \otimes W_n$, $\{\psi^2_{n,k}\}_{k\in\mathbb{Z}^2}$ is an orthonormal basis of $W_n \otimes V_n$, and $\{\psi^3_{n,k}\}_{k\in\mathbb{Z}^2}$ is an orthonormal basis of $W_n \otimes W_n$, so the statement follows. $\square$

The three wavelets of the previous theorem extract image details at different scales and orientations. Notice also that

$$\begin{aligned}\widehat{\psi}^1(\xi_1,\xi_2) &= \widehat{\varphi}(\xi_1)\,\widehat{\psi}(\xi_2),\\ \widehat{\psi}^2(\xi_1,\xi_2) &= \widehat{\psi}(\xi_1)\,\widehat{\varphi}(\xi_2),\\ \widehat{\psi}^3(\xi_1,\xi_2) &= \widehat{\psi}(\xi_1)\,\widehat{\psi}(\xi_2).\end{aligned}$$

Sometimes ψ^1 is denoted ψ^h and called *horizontal wavelet*, because the corresponding subspaces favour details in the horizontal direction. Similarly $\psi^2 = \psi^v$ is called *vertical wavelet* and $\psi^3 = \psi^d$ is called *diagonal wavelet*.

Example 5. (Shannon wavelet) Let $\{V_n\}_{n\in\mathbb{Z}}$ be the MRA given by the scaling function $\widehat{\varphi}(\xi) = \chi_{[-1/2,1/2]}(\xi)$. Observe that, by Theorem 13, such a MRA exists. Actually, by Shannon's theorem, V_n coincides with the subspace of $L^2(\mathbb{R})$ consisting of the functions f such that $\mathrm{supp}(\hat{f}) \subset [-2^{n-1}, 2^{n-1}]$. It can be proved that the corresponding wavelet ψ is, up to a unimodular constant, given by the identity $\widehat{\psi} = \chi_{[-1,1]\setminus[-1/2,1/2]}$.

In this setting the two-dimensional basis explained above paves the Fourier plane with the dyadic dilations of the rectangles shown below. Here

$$\begin{aligned}\widehat{\psi}^1(\xi_1,\xi_2) &= \chi_{[-1/2,1/2]}(\xi_1)\,\chi_{[-1,1]\setminus[-1/2,1/2]}(\xi_2)\\ \widehat{\psi}^2(\xi_1,\xi_2) &= \chi_{[-1,1]\setminus[-1/2,1/2]}(\xi_1)\,\chi_{[-1/2,1/2]}(\xi_2)\\ \widehat{\psi}^3(\xi_1,\xi_2) &= \chi_{[-1,1]\setminus[-1/2,1/2]}(\xi_1)\,\chi_{[-1,1]\setminus[-1/2,1/2]}(\xi_1).\end{aligned}$$

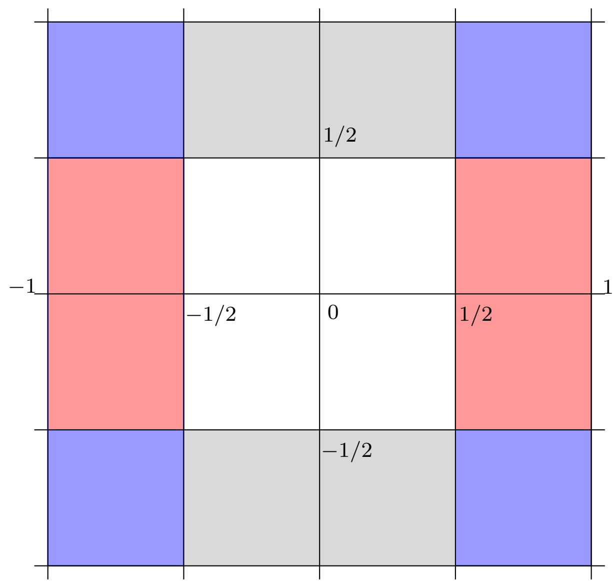


Figure 6.3: Supports of the horizontal (grey), vertical (red) and diagonal (blue) Shannon wavelets.

6.4 The continuous wavelet transform

What we have seen so far in the previous sections of this chapter can be thought of as a discretisation of the continuous *wavelet transform*, which we shall quickly review next.

The general idea is similar to that of the Short-Time Fourier Transform, but now instead of sliding a window we allow translations and dilations.

Consider $\psi \in L^2(\mathbb{R})$ (*mother wavelet*) such that,

(a) $\|\psi\|_2 = 1$,

(b) ψ is compactly supported, or very rapidly decaying,

(c) $\displaystyle\int \psi(t)\, dt = 0$.

In applications it is also important that ψ has some regularity (continuous or continuously differentiable).

Examples 7. 1. *Haar wavelet.* Here $\psi_H = -\chi_{[0,1/2)} + \chi_{[1/2,1)}$, as seen in the Annex 1.3, or in the first section of Chapter 6.3.

2. Define $\psi_S(t) = \sqrt{2}\sin(2\pi t)\chi_{(-1/2,1/2)}(t)$. Observe that ψ_S is continuous on $\mathbb{R}$, supported on the interval $(-1/2, 1/2)$ and smooth everywhere except for the two points $\pm 1/2$.

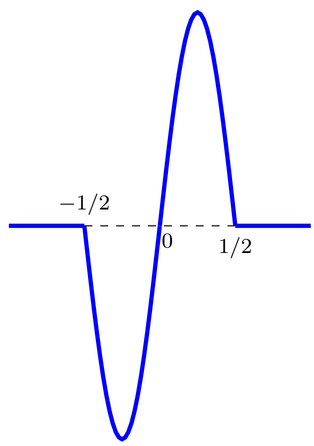


Figure 6.4: Wavelet $\psi_S(t) = \sqrt{2}\sin(2\pi t)\chi_{(-1/2,1/2)}(t)$.

3. *Shannon wavelet.* This is the wavelet ψ associated to the MRA with scaling function (See Examples 1 (1)).

$$\varphi(t) = \widehat{\chi}_{[-1/2,1/2]}(t) = \operatorname{sinc}(t).$$

It has Fourier transform

$$\widehat{\psi}(\xi) = e^{\pi i \xi} \chi_{[-1,-1/2)\cup(1/2,1]}(\xi).$$

4. *Ricker wavelet (mexican hat).* This is the normalised second derivative of the Gaussian:

$$\psi_R(t) = \frac{2}{\pi^{1/4}\sqrt{3}}\,(1-t^2)\,e^{-t^2/2}.$$

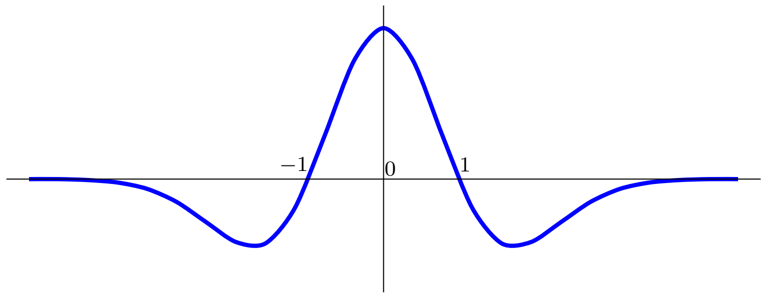


Figure 6.5: Mexican hat.

As previously mentioned, the general idea is not only to translate ψ, as in the STFT, but also to rescale it to give streched (or squeezed) versions with the same shape, but with different scale of frequency.

Given $a > 0$ and $b \in \mathbb{R}$ define

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\Big(\frac{t-b}{a}\Big).$$

This is a single wavelet with support translated by b and streched by a. If we regard ψ as a single "cycle" of a wave-like function, then $\psi_{a,b}$ has a "frequency" that is $1/a$ times the original. The factor $1/\sqrt{a}$ is just so that $\|\psi_{a,b}\|_2 = 1$.

Definition 14. The *continuous wavelet transform* of $f \in L^2(\mathbb{R})$ associated to a wavelet ψ is

$$Wf(a,b) = \langle f, \psi_{a,b}\rangle = \frac{1}{\sqrt{a}}\int_{\mathbb{R}} f(t)\,\overline{\psi\Big(\frac{t-b}{a}\Big)}\,dt.$$

A key feature is that, with mild conditions on ψ, the wavelet transform $Wf(a,b)$ contains enough information to reconstruct f. We have already seen this for the wavelets associated to a MRA. In that case the discretisation

$$\psi_{n,k}(t) = \psi_{2^{-n},2^{-n}k}(t) = 2^{n/2}\psi(2^n t - k), \quad n,k \in \mathbb{Z},$$

forms an orthonormal basis of $L^2(\mathbb{R})$, so for any $f \in L^2(\mathbb{R})$,

$$f(t) = \sum_{n,k\in\mathbb{Z}} \langle f, \psi_{n,k}\rangle\,\psi_{n,k}(t) = \sum_{n,k\in\mathbb{Z}} Wf(2^{-n}, 2^{-n}k)\,\psi_{n,k}(t).$$

Theorem 16. *(Inverse wavelet transform) If $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a real valued function satisfying the admissibility condition*

$$c_\psi := \int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\, d\xi < \infty$$

Then, for $f \in L^2(\mathbb{R})$,

$$f(t) = \frac{1}{c_\psi} \int_{\mathbb{R}} \int_0^\infty Wf(a,b)\, \psi_{a,b}(t)\, \frac{da}{a^2}\, db$$

and

$$\|f\|_2 = \sqrt{c_\psi} \left(\int_{\mathbb{R}} \int_0^\infty |Wf(a,b)|^2\, \frac{da}{a^2}\, db \right)^{1/2}.$$

Examples 8. 1. For the Haar wavelet

$$\begin{aligned}\widehat{\psi}(\xi) &= -\int_0^{1/2} e^{-2\pi i t\xi} dt + \int_{1/2}^1 e^{-2\pi i t\xi} dt = \frac{1}{2\pi i\xi}\big(e^{-\pi i\xi} - 1 + e^{-\pi i\xi} - e^{-2\pi i\xi}\big) \\ &= -\frac{(1-e^{-\pi i\xi})^2}{2\pi i\xi} = -\frac{(e^{-i\frac{\pi}{2}\xi}(e^{i\frac{\pi}{2}\xi} - e^{-i\frac{\pi}{2}\xi})^2}{2\pi i\xi} = ie^{-\pi i\xi} \frac{\sin^2(\frac{\pi}{2}\xi)}{\frac{\pi}{2}\xi},\end{aligned}$$

so ψ is admissible:

$$c_\psi = \int_0^\infty \frac{\sin^4(\frac{\pi}{2}\xi)}{\frac{\pi^2}{4}|\xi|^3} d\xi < \infty.$$

2. For the Shannon wavelet ψ, defined by $\widehat{\psi} = e^{\pi i\xi}\chi_{[-1,-1/2)\cup(1/2,1]}$ it is also clear that the admissibility condition holds:

$$c_\psi = \int_{[-1,-1/2)\cup(1/2,1]} \frac{d\xi}{|\xi|} = 2\int_{1/2}^1 \frac{d\xi}{\xi} = 2\log 2.$$