1. Compute the Fourier transform of the functions

   (a) $f_1(t) = te^{-|t|}$, (b) $f_2(t) = |t|e^{-|t|}$,

   (c) $f_3(t) = te^{-t^2}$, (d) $f_4(t) = (1 - |t|)\chi_{(-1,1)}(t)$.

2. let $a, b \in \mathbb{R}$, $a < b$. Prove that the Fourier transform of $\chi_{(a,b)}$ is $(b-a)e^{-\pi i(a+b)\xi}\,\mathrm{sinc}[\pi(b-a)\xi]$.

3. Prove that if $f \in L^1(\mathbb{R})$ has compact support then $\hat{f}(\xi)$ is actually analytic in $\xi \in \mathbb{C}$. (Hint: use Morera's theorem).

4. Let $f = \chi_{[0,1]}$.
   (a) Compute $f * f$ and $f * f * f$.
   (b) Examine the regularity of these new functions.
   (c) Conjecture, and prove if possible, what is the regularity of $f * \overset{n}{\cdots} * f$.

5. Let $f \in L^1(\mathbb{R})$ be even. Prove:
   (a) The Fourier transform $\hat{f}(\xi)$ and the *cosine transform of* f coincide, that is
   $$\hat{f}(\xi) = \int_{\mathbb{R}} f(t)\, \cos(2\pi\xi t)\, dt.$$
   (b) The following inversion formula holds
   $$f(t) = \int_{\mathbb{R}} \hat{f}(\xi)\, \cos(2\pi\xi t)\, d\xi.$$

   Prove similar formulas for f odd.

6. (a) Prove that the Fourier transform of the function $f(t) = e^{-2\pi|t|}$ is
   $$\widehat{f}(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}.$$
   (Hint: $\int_{\mathbb{R}} e^{-2\pi|t|}e^{-2\pi i\xi t}dt = 2\int_0^\infty e^{-2\pi t}\cos(2\pi\xi t)\, dt$.)
   (b) Let $g \in \mathcal{C}(\mathbb{R}) \cap L^1(\mathbb{R})$. Find $u \in \mathcal{C}^2(\mathbb{R})$ such that $u, u', u'' \in L^1(\mathbb{R})$ and solving the differential equation
   $$u'' - u = g.$$
   Prove also that $u(\infty) = 0$.

7. Use the Fourier transform to find f such that

$$\int_{\mathbb{R}} f(x-y)e^{-|y|}dy = 2e^{-|x|} - e^{-2|x|}.$$

8. Let $0 < a < b$. Use the Fourier transform to compute the integrals

$$\text{(a)} \quad \int_{-\infty}^{+\infty} \frac{\sin(at)\sin(b(u-t))}{t(u-t)}\, dt, \qquad \text{(b)} \quad \int_{-\infty}^{+\infty} \frac{\sin(at)\sin(bt)}{t^2}\, dt$$

9. Let f have Fourier transform $\dfrac{1-i\xi}{1+i\xi}\dfrac{\sin(\pi\xi)}{\pi\xi}$. Compute $\displaystyle\int_{\mathbb{R}} |f(x)|^2\, dx$.

10. Use the Fourier transform to compute

$$\int_{\mathbb{R}} \frac{\sin x}{x(x^2+1)}\, dx.$$

(Hint: use exercise 6 (a)).

11. If $f(t)$ has Fourier transform $\hat{f}(\xi) = \dfrac{1}{1+|\xi|^3}$, compute $\|f * f'\|_{L^2(\mathbb{R})}$.

12. (a) Show that the functions $\varphi_n(x) = \dfrac{\sin(x/2)}{\pi x}e^{inx}$, $n \in \mathbb{Z}$, are pairwise orthogonal in $L^2(\mathbb{R})$. (Hint: use exercise 1 (d)).

    (b) Determine the constants $c_n \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \left| \frac{1}{1+x^2} - \sum_{n=-N}^{N} c_n \varphi_n(x) \right|^2 dx$$

    is minimal. (Hint: use exercise 6 (a)).

    (c) Is the system $(\varphi_n)_{n\in\mathbb{Z}}$ complete?