Lesson 12

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Not all the options have a payoff $X = f(S_T)$. For instance we have the Asian options whose payoff is

$$X = \left(\frac{1}{T} \int_0^T S_u \mathrm{d}u - K \right)_+$$

the lookback options,

$$(\text{"lookback call"})\ X = S_T - S_*, \text{ where } S_* = \min_{0 \le t \le T} S_t$$

$$(\text{"lookback put"})\ X = S^* - S_T, \text{ where } S^* = \max_{0 \le t \le T} S_t,$$

or the barrier options

$$(\text{"down-and-out-call"})\ X = (S_T - K)_+ \mathbf{1}_{\{S_* \ge K\}}$$

$$(\text{"down-and-in-call"})\ X = (S_T - K)_+ \mathbf{1}_{\{S_* \le K\}}.$$

Consider an Asian option with payoff

$$X = \left(\frac{1}{T} \int_0^T S_u \mathrm{d}u - K \right)_+ ,$$

by the previous theorem $C_t = \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)} X | \mathcal{F}_t)$. Define

$$\varphi(t, x) = \mathbb{E}_{\mathbb{P}^*} \left(\left(\frac{1}{T} \int_t^T \frac{S_u}{S_t} \mathrm{d}u - x \right)_+ \right).$$

Then

$$\begin{aligned}
C_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*} \left(\left(\frac{1}{T} \int_0^T S_u \mathrm{d}u - K \right)_+ \middle| \mathcal{F}_t \right) \\
&= e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*} \left(\left(\frac{1}{T} \int_t^T S_u \mathrm{d}u - \left(K - \frac{1}{T} \int_0^t S_u \mathrm{d}u \right) \right)_+ \middle| \mathcal{F}_t \right) \\
&= e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*} \left(\left(\frac{1}{T} \int_t^T \frac{S_u}{S_t} \mathrm{d}u - \frac{K - \frac{1}{T} \int_0^t S_u \mathrm{d}u}{S_t} \right)_+ \middle| \mathcal{F}_t \right) \\
&= e^{-r(T-t)} S_t \varphi(t, Z_t)
\end{aligned}$$

where $Z_t = \frac{K - \frac{1}{T} \int_0^t S_u \mathrm{d}u}{S_t}$.

Is easy to see that

$$\mathrm{d}Z_t = \left(\left(\sigma^2 - r\right) Z_t - \frac{1}{T} \right) \mathrm{d}t - \sigma Z_t \mathrm{d}\bar{W}_t.$$

In fact, applying the integration by parts formula and the Itô formula:

$$\begin{aligned}
\mathrm{d}Z_t &= \mathrm{d}\left(\frac{K}{S_t}\right) - \frac{1}{TS_t}\mathrm{d}\left(\int_0^t S_u \mathrm{d}u\right) - \mathrm{d}\left(\frac{1}{S_t}\right)\frac{1}{T}\int_0^t S_u \mathrm{d}u \\
&= -\frac{K}{S_t^2}\mathrm{d}S_t + \frac{K}{S_t^3}\mathrm{d}\langle S\rangle_t - \frac{S_t}{TS_t}\mathrm{d}t + \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t^2}\mathrm{d}S_t - \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t^3}\mathrm{d}\langle S\rangle_t,
\end{aligned}$$

but since $\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}\bar{W}_t$, we have that

$$\begin{aligned}
\mathrm{d}Z_t &= \left(-\frac{K}{S_t} r + \frac{K}{S_t}\sigma^2 + r\frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t} - \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t}\sigma^2 - \frac{1}{T} \right) \mathrm{d}t \\
&+ \left(-\frac{K}{S_t}\sigma + \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t}\sigma \right) \mathrm{d}\bar{W}_t = \left((\sigma^2 - r) Z_t - \frac{1}{T} \right) \mathrm{d}t - \sigma Z_t \mathrm{d}\bar{W}_t.
\end{aligned}$$

Then, we know that $\tilde{C}_t = e^{-r(T-t)}\tilde{S}_t\varphi(t, Z_t)$, $t \leq T$ is a martingale. So if we assume that $\varphi(t,x) \in C^{1,2}$ we will have that

$$\begin{aligned}
\mathrm{d}\varphi &= \frac{\partial\varphi}{\partial t}\mathrm{d}t + \frac{\partial\varphi}{\partial Z_t}\mathrm{d}Z_t + \frac{1}{2}\frac{\partial^2\varphi}{\partial {Z_t}^2}\sigma^2 Z_t^2\mathrm{d}t \\
&= \left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial Z_t}\left(\sigma^2 - r)Z_t - \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2\right)\mathrm{d}t \\
&- \frac{\partial\varphi}{\partial Z_t}\sigma Z_t\mathrm{d}\bar{W}_t.
\end{aligned}$$

Also we have that

$$
\begin{aligned}
\mathrm{d}\tilde{C}_t &= re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t + e^{-r(T-t)}\varphi\mathrm{d}\tilde{S}_t + e^{-r(T-t)}\tilde{S}_t\mathrm{d}\varphi \\
&\quad + e^{-r(T-t)}\mathrm{d}\langle\tilde{S},\varphi\rangle_t \\
&= re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t + e^{-r(T-t)}\varphi\mathrm{d}\tilde{S}_t + e^{-r(T-t)}\tilde{S}_t\mathrm{d}\varphi \\
&\quad - e^{-r(T-t)}\frac{\partial\varphi}{\partial Z_t}\sigma^2\tilde{S}_t Z_t\mathrm{d}t \\
&= e^{-r(T-t)}\left(\varphi - Z_t\frac{\partial\varphi}{\partial Z_t}\right)\mathrm{d}\tilde{S}_t \\
&\quad + re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t - e^{-r(T-t)}\frac{\partial\varphi}{\partial Z_t}\sigma^2\tilde{S}_t Z_t\mathrm{d}t \\
&\quad + e^{-r(T-t)}\tilde{S}_t\left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial Z_t}\left((\sigma^2 - r)Z_t - \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2\right)\mathrm{d}t,
\end{aligned}
$$

by identifying the martingale parts

$$\mathrm{d}\tilde{C}_t = e^{-r(T-t)}\left(\varphi - Z_t \frac{\partial \varphi}{\partial Z_t}\right)\mathrm{d}\tilde{S}_t$$

$$r\varphi + \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial Z_t}\left(rZ_t + \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2 \varphi}{\partial Z_t^2}\sigma^2 Z_t^2 = 0.$$

Therefore the hedging strategy is given by $\left(\phi_t^0, \phi_t^1\right)$ with $\phi_t^0 S_t^0 = C_t - \phi_t^1 S_t$ and

$$\phi_t^1 = e^{-r(T-t)}\left(\varphi - Z_t \frac{\partial \varphi}{\partial Z_t}\right),$$

where φ is the solution of the partial differential equation

$$r\varphi + \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial x}\left(rx + \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2}\sigma^2 x^2 = 0$$

with the boundary condition $\varphi(T, x) = x_-$ (negative part of x). This equation can be solved numerically.

To prove the representation theorem for Brownian martingales it is very useful to consider the below result about the Laplace transform of a measure.

Definition

Let μ be a positive measure in $\mathbb{R}^m$, the so called *real definition subset*, denoted by $E_r(\mu)$, is the subset of $\mathbb{R}^m$ defined as:

$$\lambda \in E_r(\mu) \Longleftrightarrow \int_{\mathbb{R}^m} \exp(\lambda \cdot x)\, \mathrm{d}\mu(x) < \infty.$$

If $E_r(\mu)$ is not empty, the Laplace transform of μ is defined as the complex function

$$\mathcal{L}_\mu(z) = \int_{\mathbb{R}^m} \exp(z \cdot x)\, \mathrm{d}\mu(x),\ z \in \mathbb{C}^m.$$

It can be seen that $\mathcal{L}_\mu(z)$ is defined in the complex set

$$E(\mu) = \{z, \mathrm{Re}(z) \in E_r(\mu)\},$$

Definition

If μ is a real measure in $\mathbb{R}^m$ that is a difference of two positive measures, say $\mu = \mu_+ - \mu_-$, then

$$\mathcal{L}_\mu(z) = \mathcal{L}_{\mu_+}(z) - \mathcal{L}_{\mu_-}(z),$$

and

$$E_r\left(\mu\right) = E_r\left(\mu_+\right) \cap E_r\left(\mu_-\right)$$

Theorem

If $\mathcal{L}_\mu(z) = 0$ **in an open set** *of* $E_r\left(\mu\right)$ *then* $\mu \equiv 0$.

Proof.

See Theorem 4, Chapter XXI in A. Monfort. Cours de Probabilités. 1980. Economica. $\square$

Lemma

Set $\mathcal{F}_T = \sigma\left(B_t, 0 \le t \le T\right)$, *where* B *is a Brownian motion. Consider stepwise functions*

$$f(t) = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{(t_{i-1}, t_i]}(t)$$

with $\lambda_i \in \mathbb{R}$ *and* $0 = t_0 < t_1 \ldots < t_n \le T$. *Denote by* $\mathcal{J}$ *that set of functions. Set* $\mathcal{E}_T^f = \exp\left\{\int_0^T f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^T f^2(s)\mathrm{d}s\right\}, f \in \mathcal{J}$. *If* $Y \in L^2(\mathcal{F}_T, \mathbb{P})$ *is orthogonal to* $\mathcal{E}_T^f, f \in \mathcal{J}$ *then* $Y = 0$, $\mathbb{P}$ *a.s.*

Proof.

Consider $Y \in L^2(\mathcal{F}_T, P)$, orthogonal to $\mathcal{E}_T^f$. Let $\mathcal{G}_n := \sigma\left(B_{t_1}, ..., B_{t_n}\right)$, we have

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}}) - \frac{1}{2}\sum_{i=1}^n \lambda_i^2(t_i - t_{i-1})\right\} Y\right) = 0,$$

then,

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}})\right\} Y\right) = 0$$

and, because $\mathcal{G}_n = \sigma\left(B_{t_1}, ..., B_{t_n}\right) = \sigma\left(B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}\right)$,

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}})\}\right\} \mathbb{E}(Y|\mathcal{G}_n)\right) = 0.$$

$\square$

Proof.

(continuation) Y can be decomposed as $Y = Y_+ - Y_-$, so

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^{n}\lambda_i(B_{t_i} - B_{t_{i-1}})\}\right\}\mathbb{E}(Y_+|\mathcal{G}_n)\right)$$

$$= \mathbb{E}\left(\exp\left\{\sum_{i=1}^{n}\lambda_i(B_{t_i} - B_{t_{i-1}})\}\right\}\mathbb{E}(Y_-|\mathcal{G}_n)\right).$$

Let X be the map

$$X : \Omega \to \mathbb{R}^n$$

$$\omega \longmapsto X(\omega) = (B_{t_1}(\omega), B_{t_2}(\omega) - B_{t_1}(\omega), ..., B_{t_n}(\omega) - B_{t_{n-1}}(\omega)),$$

□

Proof.

(continuation) then

$$\int_{\mathbb{R}^n} \exp\left\{\sum_{i=1}^{n} \lambda_i x_i\right\} \mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n) \mathrm{d}\mathbb{P}^X(x_1, x_2, ..., x_n)$$

$$= \int_{\mathbb{R}^n} \exp\left\{\sum_{i=1}^{n} \lambda_i x_i\right\} \mathbb{E}(Y_-|\mathcal{G}_n)(x_1, x_2, ..., x_n) \mathrm{d}\mathbb{P}^X(x_1, x_2, ..., x_n),$$

in such a way that the Laplace transform of $\mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n)\mathrm{d}\mathbb{P}^X$ is equal to that of $\mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n)\mathrm{d}\mathbb{P}^X$ and therefore, by the uniqueness of the Laplace transform,
$\mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n) = \mathbb{E}(Y_-|\mathcal{G}_n)(x_1, x_2, ..., x_n)$, $\mathbb{P}^X$ a.s. From here
$\mathbb{E}(Y_+|\mathcal{G}_n) = \mathbb{E}(Y_-|\mathcal{G}_n)$ $\mathbb{P}$ a.s., and finally since this is true for any $\mathcal{G}_n$ of the previous type it turns out that Y is zero $\mathbb{P}$ a.s., since
$\mathbb{E}(Y_\pm| \vee_{k=1}^n \mathcal{G}_k) \underset{n\to\infty}{\to} \mathbb{E}(Y_\pm| \vee_{k=1}^\infty \mathcal{G}_k) = Y_\pm.$ □

Theorem

For all random variable $F \in L^2(\mathcal{F}_T, \mathbb{P})$ *there exists and adapted process* Y *with* $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right) < \infty$, *such that*

$$F = \mathbb{E}(F) + \int_0^T Y_t \mathrm{d}B_t$$

Proof.

Suppose that $F - \mathbb{E}(F)$ is orthogonal to $\int_0^T Y_t \mathrm{d}B_t$ for all Y, with $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right) < \infty$, then if we prove that $F - \mathbb{E}(F) = 0$, $\mathbb{P}$ a.s. then we have finished, since the linear space of centered random variables of $L^2(\mathcal{F}_T, \mathbb{P})$ will coincide with its linear **closed** subspace of random variables $\int_0^T Y_t \mathrm{d}B_t$ with $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right) < \infty$. Write $Z = F - \mathbb{E}(F)$, we have

$$\mathbb{E}\left((F - \mathbb{E}(F)) \int_0^T Y_t \mathrm{d}B_t\right) = 0.$$

$\square$

Proof.

(continuation) Take $Y_t = \mathcal{E}_t^f f(t)$, with the $\mathcal{E}_t^f$ define previously, then

$$\mathbb{E}\left((F - \mathbb{E}(F)) \int_0^T \mathcal{E}_t^f f(t) \mathrm{d}B_t\right) = 0$$

and also, because $\mathbb{E}(F - \mathbb{E}(F)) = 0$, we have that

$$\mathbb{E}(\left((F - \mathbb{E}(F)) \left(1 + \int_0^T \mathcal{E}_t^f f(t) \mathrm{d}B_t\right)\right) = 0,$$

but, by the Itô formula,

$$\mathcal{E}_T^f = 1 + \int_0^T \mathcal{E}_t^f f(t) \mathrm{d}B_t.$$

So

$$\mathbb{E}\left((F - \mathbb{E}(F))\mathcal{E}_T^f\right) = 0$$

and by the previous lemma $F - \mathbb{E}(F) = 0$, $\mathbb{P}$ a.s. □

Theorem

Any square integrable martingale M can be written as

$$M_t = M_0 + \int_0^t Y_s \mathrm{d}B_s, 0 \leq t \leq T$$

where Y_s is an adapted process with $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right) < \infty$.

Proof.

We can write

$$M_t = \mathbb{E}(M_T | \mathcal{F}_t)$$

and by the previous proposition

$$M_T = \mathbb{E}(M_T) + \int_0^T Y_s \mathrm{d}B_s$$

then it is enough to take conditional expectations. □

Remark

In the case that F is in $L^1(\mathcal{F}_T, \mathbb{P})$ we can find Y, with $\int_0^T Y_t^2 \mathrm{d}t < \infty$, a.s., such that

$$\mathbb{E}\left(F|\mathcal{F}_t\right) = \mathbb{E}(F) + \int_0^t Y_t \mathrm{d}B_t.$$

In fact in can ve proved that any local Brownian martingale M can be written as

$$M_t = M_0 + \int_0^t Z_t \mathrm{d}B_t$$

with $\int_0^T Z_t^2 \mathrm{d}t < \infty$, a.s.