DYNAMICAL SYSTEMS
MÀSTER EN MATEMÀTICA AVANÇADA
Year 2023-2024, Fall semester

Problem set #1. Due October 31st, 2023

1 (1p) Consider the following homeomorphism of the circle

$$f(x) = \begin{cases} \frac{1}{4} + 2x \pmod 1 & \text{if } x \in [0, \frac{1}{4}] \\ \frac{5}{8} + \frac{x}{2} \pmod 1 & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ x + \frac{1}{4} \pmod 1 & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

Draw a lift of f and compute its rotation number.

2 (1p) Consider $F_1(x) := x + \frac{1}{2}\sin(2\pi x)$ and $F_2(x) := x + \frac{1}{4\pi}\sin(2\pi x)$. Decide whether F_1 and F_2 are lifts of circle homeomorphisms. If so, decide whether that homeomorphism is orientation preserving. If it is, determine the rotation number.

3 (2p) Let $f(\theta) = \theta + \frac{\varepsilon}{2\pi}\sin(2\pi n\theta) \pmod 1$ for $0 < \varepsilon < 1/n$ and $n \in \mathbb{N}$. Find an expression for the lifts F. Calculate the periodic points of f and determine their character. Draw the phase portrait of f and calculate its rotation number.

4 (2p) Let f be an orientation preserving homeomorphism of the circle. Show that all periodic orbits of f must have the same period. Is this also true for orientation reversing homeomorphisms? Prove it or give a counterexample.

5 (**The Arnold family of circle maps**) Given $\alpha, \epsilon \in [0, 1)$ and $\theta \in [0, 1)$, consider the circle map

$$f_{\varepsilon,\alpha}(\theta) = \theta + \alpha + \frac{\varepsilon}{2\pi}\sin(2\pi\theta) \pmod 1.$$

with one of its lifts

$$F_{\varepsilon,\alpha}(x) = x + \alpha + [\varepsilon/(2\pi)]\sin(2\pi x), \quad x \in \mathbb{R}.$$

Let $\rho(f_{\varepsilon,\alpha})$ denote the rotation number of the map $f_{\varepsilon,\alpha}$.

Fixed $\epsilon \in (0, 1)$, and writing $f_\alpha = f_{\varepsilon,\alpha}$, the graph of $\alpha \mapsto \rho(f_\alpha)$ is a *devil's staircase* since it increases from 0 to 1 continuously, while having a derivative equal to 0 almost everywhere.

(a) (2p) Show that the map $\alpha \mapsto \rho(f_\alpha)$ is not absolutely continuous.

(b) (2p) Make a computer program (in whatever language you choose) that draws the graph of this function for different values of ϵ.

(c) (Extra credit 2 p) Let T_λ denote the level set of the rotation number λ (known as the $\lambda-$Arnold Tongue). In other words,

$$T_\lambda = \{(\alpha, \epsilon) \in [0, 1] \times [0, 1] \mid \rho(f_{\epsilon,\alpha}) = \lambda\}.$$

Make a computer program (in whatever program you choose) to draw the tongues T_λ for $\lambda = 0, 1/2, 1/4$ and the tongue (actually curve) T_λ for $\lambda = \frac{1+\sqrt{5}}{2}$, the golden mean. For the latter, use the bisection method to locate the point in the curve for every ϵ.