Advanced Mathematics for Scientific Challenges
Autumn 2022

EXERCISES 1.1

1. Prove the extreme value theorem of Weierstrass: If f is a real continuous function on a compact set $K \subset \mathbb{R}^n$, then the problem of optimization

$$\begin{cases} \text{Minimize } f(x) \\ x \in K \end{cases}$$

has an optimal solution $x^* \in K$.

2. Let f be a real continuous function on $\mathbb{R}^n$ satisfying that $f(x) \to +\infty$ when $\|x\| \to +\infty$. Show that the problem of optimization

$$\begin{cases} \text{Minimize } f(x) \\ x \in \mathbb{R}^n \end{cases}$$

has an optimal solution $x^* \in K$.

3. **(Optional)** Let S be a convex subset of $\mathbb{R}^n$, and let λ_1 and λ_2 be positive scalars.
   (a) Show that $(\lambda_1 + \lambda_2)S = \lambda_1 S + \lambda_2 S$.
   (b) Give an example that shows that this does not need to be true when S is not convex.

4. **(Optional)** Let S be a nonempty closed convex set in $\mathbb{R}^n$, not containing the origin. Show that there exists a hyperplane that strictly separates S and the origin.

5. **(Optional)** Show that a convex function $f : (a, b) \to \mathbb{R}$ is continuous.

6. Consider a function $f : (a, b) \to \mathbb{R}$ of class C^2. Show that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

7. Let f be a real valued function on an open convex set $S \subset \mathbb{R}^n$, of class C^2. Show that f is convex on S if and only if its Hessian matrix,

$$Q(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right),$$

is positive semi-definite for all $x \in S$.

8. Assume that $S \subset \mathbb{R}^n$ is a convex set and that $g : S \to \mathbb{R}$. Show that the set $g(x) \leq 0$ is convex if g is convex. What about the opposite implication?

Deadline to deliver it: October 9, 23:59.