Lesson 19

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Forward rate models. Heath-Jarrow-Morton approach

As we have seen one drawback of the short rate models is their difficulty in capturing the term structure observed at initial time. An alternative is to model the forward rates $f(t, T)$ and to use the relation $r(t) = f(t, t)$. This is the so-called Heath-Jarrow-Morton (HJM) approach. By definition we have that

$$P(t, T) = \exp\left\{-\int_t^T f(t, s)\mathrm{d}s\right\},$$

where $f(t, s)$ represents the (instantaneous) forward rates. Suppose that under a risk neutral probability $\mathbb{P}^*$

$$\mathrm{d}f(t, T) = \alpha(t, T)\mathrm{d}t + \sigma(t, T)\mathrm{d}W_t \quad , T \geq 0, \tag{1}$$

where $\alpha, \sigma > 0$ are adapted continuous (w.r.t T and t) processes and with

$$f(0, T) = \hat{f}(0, T).$$

We shall try to deduce the evolution of $P(t, T)$ from that of $f(t, T)$.

If we write $X_t = -\int_t^T f(t,s)\mathrm{d}s$, we have $P(t,T) = e^{X_t}$ and from the equation (1) we obtain

$$
\begin{aligned}
\mathrm{d}X_t &= f(t,t)\mathrm{d}t - \int_t^T \mathrm{d}f(t,s)\mathrm{d}s = \\
&= f(t,t)\mathrm{d}t - \int_t^T \alpha(t,s)\mathrm{d}t\mathrm{d}s - \left(\int_t^T \sigma(t,s)\mathrm{d}W_t\right)\mathrm{d}s \\
&= \left(f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s\right)\mathrm{d}t - \left(\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}W_t,
\end{aligned}
$$

where we have applied a *stochastic* Fubini theorem (a sufficient condition to allow that is $\int_0^T \int_0^T \mathbb{E}\left(\sigma^2(t,s)\right)\mathrm{d}s\mathrm{d}t < \infty$)

Then

$$
\begin{aligned}
\frac{\mathrm{d}P(t,T)}{P(t,T)} &= \mathrm{d}X_t + \frac{1}{2}\mathrm{d}\langle X\rangle_t \\
&= (f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s)\mathrm{d}t - \left(\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}W_t \\
&+ \frac{1}{2}\left(\int_t^T \sigma(t,s)\mathrm{d}s\right)^2 \mathrm{d}t \\
&= \left(f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s + \frac{1}{2}\left(\int_t^T \sigma(t,s)\mathrm{d}s)^2\right)\right)\mathrm{d}t \\
&- \left(\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}W_t.
\end{aligned}
$$

And if we take into account that $f(t,t) = r(t)$ and that we are modeling under a risk neutral probability, it turns out that

$$-\int_t^T \alpha(t,s)\mathrm{d}s + \frac{1}{2}\left(\int_t^T \sigma(t,s)\mathrm{d}s\right)^2 = 0,$$

therefore, by taking derivatives w.r.t. T,

$$\alpha(t,T) = \left(\int_t^T \sigma(t,s)ds\right)\sigma(t,T)$$

and we can write the evolution equation (1) as

$$\mathrm{d}f(t,T) = \sigma(t,T)\left(\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t.$$

Note that all depends on $\sigma(t,s)$, that is, on the volatility. We have *eliminated* the drift $\alpha(t,T)$.

Then the algorithm to use the HJM approach is

1. Specify the volatilities $\sigma(t,s)$
2. Integrate $\mathrm{d}f(t,T) = \sigma(t,T)(\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t$ with the initial condition $f(0,T) = \hat{f}(0,T)$.
3. Calculate the prices of the bonds from the formula $P(t,T) = \exp\left\{-\int_t^T f(t,s)\mathrm{d}s\right\}$.
4. To use the previous results to calculate contingent claim prices.

Example

Suppose that $\sigma(t, T)$ has a constant value denoted by σ. Then

$$\mathrm{d}f(t, T) = \sigma^2(T - t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

so

$$f(t, T) = \hat{f}(0, T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W_t.$$

In particular

$$r(t) = f(t, t) = \hat{f}(0, t) + \frac{\sigma^2 t^2}{2} + \sigma W_t$$

and therefore

$$\mathrm{d}r(t) = \left(\frac{\partial \hat{f}(0, T)}{\partial T}|_{T=t} + \sigma^2 t\right)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

but this is the Ho-Lee adjusted to the initial structure of the forward rates.

Example

A usual assumption consist of assuming that the forward rates with greater maturity time has a lower fluctuation than that with a lower maturity time. To capture this feature we can take, for instance, $\sigma(t, T) = \sigma e^{-a(T-t)}$, $a > 0$. We have then

$$\int_t^T \sigma(t, s) ds = \int_t^T e^{-a(s-t)} ds = -\frac{\sigma}{a}\left(e^{-a(T-t)} - 1\right),$$

and

$$\mathrm{d}f(t, T) = -\frac{\sigma^2}{a} e^{-a(T-t)}(e^{-a(T-t)} - 1)\mathrm{d}t + \sigma e^{-a(T-t)}\mathrm{d}W_t.$$

Example

Therefore

$$f(t,T) = f(0,T) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-aT}\right)^2 + \sigma e^{-aT}\int_0^t e^{as}\mathrm{d}W_s.$$

In particular

$$r(t) = f(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 + \sigma e^{-at}\int_0^t e^{as}\mathrm{d}W_s$$

that corresponds to the Hull-White model considered in the previous lesson.

Change of numeraire. The forward measure

Theorem

Consider a self-financing portfolio such that its value, say N, is strictly positive and that is a martingale under $\mathbb{P}^*$. Then if $\mathbb{P}^*$ is a risk-neutral probability when the discount factor is the value of the unit of money, the probability in $\mathcal{F}_T$, $\mathbb{P}^{(N)}$ given by

$$\frac{\mathrm{d}\mathbb{P}^{(N)}}{\mathrm{d}\mathbb{P}^*} = \frac{\tilde{N}_T}{N_0}$$

is a risk-neutral probability when the discount factor is the value N. As a consequence the price of a replicable payoff with maturity T, at $t \le T$, is given by

$$C_t = N_t \mathbb{E}_{\mathbb{P}^{(N)}}\left(\frac{Y}{N_T} \middle| \mathcal{F}_t \right),$$

Proof.

Let V the value of a selfinancing portfolio, for all $0 \leq s \leq t \leq T$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^{(N)}}\left(\left.\frac{V_t}{N_t}\right| \mathcal{F}_s\right) &= \frac{\mathbb{E}_{\mathbb{P}^*}\left(\left.\frac{V_t}{N_t}\frac{\tilde{N}_T}{N_0}\right| \mathcal{F}_s\right)}{\frac{\tilde{N}_s}{N_0}} = \frac{\mathbb{E}_{\mathbb{P}^*}\left(\left.\frac{V_t}{N_t}\mathbb{E}_{\mathbb{P}^*}\left(\tilde{N}_T \middle| \mathcal{F}_t\right)\right| \mathcal{F}_s\right)}{\tilde{N}_s} \\
&= \frac{\mathbb{E}_{\mathbb{P}^*}\left(\left.\frac{V_t}{N_t}\tilde{N}_t\right| \mathcal{F}_s\right)}{\tilde{N}_s} = \frac{\mathbb{E}_{\mathbb{P}^*}\left(\tilde{V}_t \middle| \mathcal{F}_s\right)}{\tilde{N}_s} \\
&= \frac{\tilde{V}_s}{\tilde{N}_s} = \frac{V_s}{N_s}.
\end{aligned}$$

For the price of Y just consider a replicating portfolio, V. Then $Y = V_T$ and apply the previous formula. $\square$

Note also that

$$\mathbb{E}_{\mathbb{P}^*}\left(Ye^{-\int_t^T r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right) = N_t \mathbb{E}_{\mathbb{P}^{(N)}}\left(\frac{Y}{N_T} \middle| \mathcal{F}_t \right).$$

In fact

$$e^{\int_0^t r(s)\mathrm{d}s}\mathbb{E}_{\mathbb{P}^*}\left(\tilde{Y} \middle| \mathcal{F}_t \right) = N_t \mathbb{E}_{\mathbb{P}^{(N)}}\left(\frac{Y}{N_T} \middle| \mathcal{F}_t \right)$$

When we take a zero coupon bond with maturity T as the numeraire the corresponding risk-neutral measure is called the *forward measure*, we write it $\mathbb{P}^T$ and is given by

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}^*} = \frac{\tilde{P}(T,T)}{P(0,T)} = \frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T)}. \tag{2}$$

As we shall see this probability is particularly interesting when we want to calculate prices of options on bonds. Notice that if we consider another bond with maturity time $\tilde{T} \geq T$ then

$$U_{T,\tilde{T}}(t) := \frac{P(t,\tilde{T})}{P(t,T)}, \; 0 \leq t \leq T,$$

will be $\mathbb{P}^T$ martingales, and the price of a replicable T-payoff Y is given by

$$P(t,T)\mathbb{E}_{\mathbb{P}^T}(Y|\mathcal{F}_t).$$

Note also

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^*}\left(Ye^{-\int_t^T r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right) &= P(t,T)\mathbb{E}_{\mathbb{P}^T}\left(Y \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right) \mathbb{E}_{\mathbb{P}^T}\left(Y \middle| \mathcal{F}_t \right) \end{aligned}$$

Let $(S_t)_{0\leq t\leq T}$ be a stock (without dividends), we will denote by $\mathbb{P}^{(S)}$ the risk-neutral probability (in $\mathcal{F}_T$) when we take S as numeraire that is

$$\frac{\mathrm{d}\mathbb{P}^{(S)}}{\mathrm{d}\mathbb{P}^{*}} = \frac{\tilde{S}_T}{S_0}.$$

the price of a replicable T-payoff Y is given by

$$S_t \mathbb{E}_{\mathbb{P}^{(S)}}\left(\left.\frac{Y}{S_T}\right| \mathcal{F}_t \right),$$

and

$$S_t \mathbb{E}_{\mathbb{P}^{(S)}}\left(\left.\frac{Y}{S_T}\right| \mathcal{F}_t \right) = \mathbb{E}_{\mathbb{P}^{*}}\left(\left. Ye^{-\int_t^T r(s)\mathrm{d}s} \right| \mathcal{F}_t \right),$$

Now, we have a general formula for the price of a call option.

Theorem

Let $(S_t)_{0\leq t\leq T}$ be an asset strictly positive, then the price of a call option on the asset S, with maturity T and strike K is given by

$$\Pi(t;S) = S_t \mathbb{P}^{(S)}(S_T \geq K|\mathcal{F}_t) - KP(t,T)\mathbb{P}^{T}(S_T \geq K|\mathcal{F}_t).$$

$$
\begin{aligned}
\Pi(t;S) &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T-K)_+ \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T-K)\mathbf{1}_{\{S_T \geq K\}} \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s} S_T \mathbf{1}_{\{S_T \geq K\}} \middle| \mathcal{F}_t \right) - K\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s} \mathbf{1}_{\{S_T \geq K\}} \middle| \mathcal{F}_t \right) \\
&= S_t \mathbb{P}^{(S)}(S_T \geq K | \mathcal{F}_t) - KP(t,T)\mathbb{P}^T(S_T \geq K | \mathcal{F}_t),
\end{aligned}
$$

Suppose that S is another bond with maturity $\bar{T} > T$, then the option (with maturity T) on this bond has a price given by

$$
\begin{aligned}
\Pi(t;S) &= P(t,\bar{T})\mathbb{P}^{\bar{T}}(P(T,\bar{T}) \geq K|\mathcal{F}_t)) - P(t,T)\mathbb{P}^{T}(P(T,\bar{T}) \geq K|\mathcal{F}_t)) \\
&= P(t,\bar{T})\mathbb{P}^{\bar{T}}\left(\frac{P(T,T)}{P(T,\bar{T})} \leq \frac{1}{K}\middle|\mathcal{F}_t\right) \\
&- KP(t,T)\mathbb{P}^{T}\left(\frac{P(T,\bar{T})}{P(T,T)} \geq K\middle|\mathcal{F}_t\right).
\end{aligned}
$$

Define,

$$U(t,T,\bar{T}) := \frac{P(t,T)}{P(t,\bar{T})}.$$

In the context of affine structures

$$
\begin{aligned}
U(t,T,\bar{T}) &= \frac{P(t,T)}{P(t,\bar{T})} \\
&= \exp\left\{-A(t,\bar{T}) + A(t,T) + \left(B(t,\bar{T}) - B(t,T)\right) r_t\right\}
\end{aligned}
$$

and with respect to $\mathbb{P}^*$

$$\mathrm{d}U(t) = U(t)(...\mathrm{d}t + (B(t, \bar{T}) - B(t, T))\sigma_t \mathrm{d}W_t).$$

Then under $\mathbb{P}^{\bar{T}}$ and $\mathbb{P}^{T}$, respectively, we have

$$\mathrm{d}U(t) = U(t)(B(t, \bar{T}) - B(t, T))\sigma_t \mathrm{d}W_t^{\bar{T}},$$
$$\mathrm{d}U^{-1}(t) = -U^{-1}(t)(B(t, \bar{T}) - B(t, T))\sigma_t \mathrm{d}W_t^{T}.$$

in such a way that

$$U(T) = \frac{P(t,T)}{P(t,\bar{T})} \exp\left\{-\int_t^T \sigma_{\bar{T},T}(s)\mathrm{d}W_s^{\bar{T}} - \frac{1}{2}\int_t^T \sigma_{\bar{T},T}^2(s)\mathrm{d}s\right\},$$
$$U^{-1}(T) = \frac{P(t,\bar{T})}{P(t,T)} \exp\left\{\int_t^T \sigma_{\bar{T},T}(s)\mathrm{d}W_s^{T} - \frac{1}{2}\int_t^T \sigma_{\bar{T},T}^2(s)\mathrm{d}s\right\}.$$

with

$$\sigma_{\bar{T},T}(t) = -(B(t,\bar{T}) - B(t,T))\sigma_t.$$

Therefore, if σ_t is **deterministic** the law of $\log U(T)$ conditional to $\mathcal{F}_t$ is Gaussian with respect to $\mathbb{P}^T$ and $\mathbb{P}^{\bar{T}}$, with variance

$$\Sigma^2_{t,T,\bar{T}} := \int_t^T \sigma^2_{\bar{T},T}(s)\mathrm{d}s,$$

$$\mathrm{Law}\left(\left. \frac{\log U(T) - \log \frac{P(t,T)}{P(t,\bar{T})} + \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}} \right| \mathcal{F}_t \right) \sim N(0,1) \text{ under } \mathbb{P}^{\bar{T}}$$

$$\mathrm{Law}\left(\left. \frac{\log U^{-1}(T) - \log \frac{P(t,\bar{T})}{P(t,T)} + \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}} \right| \mathcal{F}_t \right) \sim N(0,1) \text{ under } \mathbb{P}^{T}.$$

Note finally that

$$
\begin{aligned}
&\Pi(t;S) = \\
&= P(t,\bar{T})\mathbb{P}^{\bar{T}}\left(\frac{P(T,T)}{P(T,\bar{T})} \leq \frac{1}{K}\middle|\,\mathcal{F}_t\right) - KP(t,T)\mathbb{P}^{T}\left(\frac{P(T,\bar{T})}{P(T,T)} \geq K\middle|\,\mathcal{F}_t\right) \\
&= P(t,\bar{T})\mathbb{P}^{\bar{T}}\left(U(T) \leq \frac{1}{K}\middle|\,\mathcal{F}_t\right) - KP(t,T)\mathbb{P}^{T}(U^{-1}(T) \geq K|\,\mathcal{F}_t) \\
&= P(t,\bar{T})\mathbb{P}^{\bar{T}}(\log U(T) \leq -\log K|\mathcal{F}_t) \\
&-KP(t,T)\mathbb{P}^{T}(\log U^{-1}(T) \geq \log K|\,\mathcal{F}_t) \\
&= P(t,\bar{T})\Phi(d_+) - KP(t,T)\Phi(d_-),
\end{aligned}
$$

with

$$
d_\pm = \frac{\log\frac{P(t,\bar{T})}{KP(t,T)} \pm \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}}.
$$

Example

In the Ho-Lee model, since $B(t, T) = T - t$

$$\sigma_{\bar{T},T} = -\sigma(\bar{T} - T),$$
$$\Sigma_{t,T,\bar{T}} = \sigma(\bar{T} - T)\sqrt{T - t}.$$

Example

For the Vasicek model,

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

and

$$\sigma_{\bar{T},T} = \frac{\sigma}{a} e^{at}(e^{-a\bar{T}} - e^{-aT}),$$
$$\Sigma^2_{t,T,\bar{T}} = \frac{\sigma^2}{2a^3}(1 - e^{-2a(T-t)})(1 - e^{-a(\bar{T}-T)})^2.$$

and the same for the Hull-White model!.