Chapter 4

The short time Fourier transform

A drawback of both Fourier series and Fourier transform is that they destroy local information. Both allow the reconstruction of L^2-functions, but the whole transform is needed to recover the function. For example, a small shift in a in $f = \frac{1}{2a}\chi_{[-a,a]}$ affects the Fourier transform at each ξ:

$$\widehat{\frac{1}{2a}\chi_{[-a,a]}}(\xi) = \frac{\sin(2\pi a\xi)}{2\pi a\xi} = \mathrm{sinc}(2a\xi).$$

The short time Fourier transform intends to avoid this inconvenience by dividing a long time signal into shorter segments of equal length and then compute separately the Fourier transform of each shorter segment.

Definition 11. Let $f, g \in L^2(\mathbb{R})$. The *short-time Fourier transform* (STFT) of f with window g is

$$V_g f(t,\xi) = \int_{\mathbb{R}} f(s)\overline{g(s-t)}e^{-2\pi i\xi s}ds.$$

Usually g is concentrated around 0, so that $g(s-t)$ gives a "window" to measure f around t. This is why the STFT is sometimes called "windowed" Fourier transform.

Quite often it is convenient to think in terms of time-frequency shifts in the complex plane. Let $z = t + i\xi \in \mathbb{C}$ and define

$$\pi(z)g(s) = g(s-t)e^{2\pi i\xi s} = M_\xi \tau_t g(s).$$

In this language

$$V_g f(z) = \int_{\mathbb{R}} f(s)\overline{\pi(z)g(s)}\,ds = \langle f, \pi(z)g\rangle.$$

Example 1. Let $a > 0$ and let $g = \frac{1}{2a}\chi_{[-a,a]}$. Then

$$V_g f(t) = \int_{\mathbb{R}} f(s)\frac{1}{2a}\chi_{[-a,a]}(s-t)e^{-2\pi i\xi s}ds = \frac{1}{2a}\int_{t-a}^{t+a} f(s)e^{-2\pi i\xi s}ds.$$

This is the Fourier content of f in the frequency ξ for time $t \in [t-a, t+a]$.

The abrupt edges of $\chi_{[-a,a]}$ cause problems in applications. Common windows are the Gaussian $g(s) = e^{-\pi s^2}$ and the Fourier transform of the one we have considered here, that is, $g(s) = \mathrm{sinc}(2as)$.

Exercise. Let $f, g \in L^2(\mathbb{R})$.

(a) Prove the *Fundamental identity of time-frequency analysis*

$$V_g f(t,\xi) = e^{-2\pi i t\xi} V_{\hat{g}} \hat{f}(\xi, -t)\,, \qquad t, \xi \in \mathbb{R}.$$

(b) Prove the *covariance property*: for $t, s, \xi, \omega \in \mathbb{R}$

$$V_g(\tau_s M_\omega f)(t,\xi) = e^{-2\pi i s\xi} V_g f(t-s, \xi-\omega).$$

4.1 Orthogonality theorem and inversion formula

A fundamental result in this theory is the following.

Theorem 10. *(Orthogonality Theorem) Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then $V_{g_j} f_j \in L^2(\mathbb{R}^2)$, $j = 1, 2$, and*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}.$$

In particular, taking $g_1 = g_2 = g$ for a window with $\|g\|_2 = 1$,

$$\langle V_g f_1, V_g f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle$$

and $V_g : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is an isometry.

Proof. Assume that $g_1, g_2 \in (L^1 \cap L^\infty)(R) \subset L^2(\mathbb{R})$. Notice that $f_j \cdot \tau_t \bar{g}_j \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$, $j = 1, 2$, and that

$$V_g f(t,\xi) = \int_{\mathbb{R}} f(s)\overline{g(s-t)}\, e^{-2\pi i s\xi} ds = (f \cdot \tau_t \bar{g})^\wedge(\xi).$$

Then, by Plancherel on the ξ-integral

$$\begin{aligned}
\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} (f_1 \cdot \tau_t \bar{g}_1)^\wedge(\xi) \overline{(f_2 \cdot \tau_t \bar{g}_2)^\wedge(\xi)}\, d\xi\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(s) \overline{g_1(s-t)}\, \overline{f_2(s)} g_2(s-t)\, ds\, dt \\
&= \int_{\mathbb{R}} f_1(s)\, \overline{f_2(s)} \left(\int_{\mathbb{R}} \overline{g_1(s-t)}\, g_2(s-t)\, dt \right) ds \\
&= \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}.
\end{aligned}$$

The general case is obtained by a standard density argument: fixed a window $g_1 \in (L^1 \cap L^\infty)(\mathbb{R})$, the map $g_2 \longmapsto \langle V_{g_1} f_1, V_{g_2} f_2 \rangle$ is a linear functional coinciding with $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on the dense subspace $L^1 \cap L^\infty$, so it extends to all $L^2(\mathbb{R}^2)$. The analogous statement holds for the map $g_1 \longmapsto \langle V_{g_1} f_1, V_{g_2} f_2 \rangle$. □

This result shows that f is completely determined by its STFT $V_g f$, but it doesn't give a way to recover f from $V_g f$; we need an "inversion formula" for the STFT.

Inversion formula for the STFT. *Let $g, \gamma \in L^2(\mathbb{R})$ with $\langle g, \gamma \rangle \neq 0$. Then, for all $f \in L^2(\mathbb{R})$*

$$f(t) = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) M_\xi \tau_s \gamma(t) \, d\xi \, ds \,, \quad t \in \mathbb{R}.$$

In particular if $g = \gamma$ and $\|g\|_2 = 1$,

$$f(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) M_\xi \tau_s g(t) \, d\xi \, ds \,, \quad t \in \mathbb{R}.$$

Remarks 3. 1. This is, in a way, similar to the Fourier inversion formula: f is expressed as a superposition of time-frequency shifts, with $V_g f(t, \xi)$ as a weight. However, in the Fourier case the elementary functions $e_\xi(t) = e^{2\pi i t \xi}$ are not in $L^2(\mathbb{R})$, whereas $M_\xi \tau_t \gamma$ are particularly nice functions in $L^2(\mathbb{R})$.

2. Assume that γ is concentrated on $T \subset \mathbb{R}$ –meaning that the integral of $|g|^2$ outside T is very small– and that $\widehat{\gamma}$ is concentrated on $\Omega \subset \mathbb{R}$. Then $M_\xi \tau_t \gamma$ is concentrated on $t + T$ and its spectrum on $\xi + \Omega$. Thus $M_\xi \tau_t \gamma$ occupies (approximately) a cell

$$(t + T) \times (\xi + \Omega)$$

in the time-frequency plane, and the size of $V_g f(t, \xi)$ measures the contribution of this time-frequency atom in the decomposition of f. The uncertainty principle limits this concentration to sets T, Ω with $|T|\,|\Omega| \gtrsim 1$.

Good time resolution requieres a window with small support; this comes at the price of poor frequency resolution. Similarly, good frequency resolution, by means of a band-limited window, implies poor resolution in time. In practice one choses g such that both g and $\hat{g}$ have fast decay. The choice of g Gaussian is thus canonical.

Proof. Since $V_g f \in L^2(\mathbb{R}^2)$, the function

$$\tilde{f}(t) := \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) M_\xi \tau_s \gamma(t) \, d\xi \, ds \,, \quad t \in \mathbb{R}$$

is well-defined in $L^2(\mathbb{R})$, in the sense that for all $h \in L^2(\mathbb{R})$ the product

$$\langle \tilde{f}, h \rangle = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) \langle M_\xi \tau_s \gamma, h \rangle \, d\xi \, ds \tag{4.1}$$

is finite. This is a consequence of the Cauchy-Schwartz inequality and the orthogonality theorem:

$$\begin{aligned}
|\langle \tilde{f}, h\rangle| &= \left| \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s,\xi)\, \overline{V_\gamma h(s,\xi)}\ d\xi\, ds \right| \\
&\leq \frac{1}{|\langle \gamma, g\rangle|} \|V_g f\|_2^2\, \|V_\gamma h\|_2^2 = \frac{1}{|\langle \gamma, g\rangle|} \|f\|_2^2 \|g\|_2^2 \|h\|_2^2 \|\gamma\|_2^2
\end{aligned}$$

Therefore, by Riesz representation theorem, $\tilde{f}$ is given by a function in $L^2(\mathbb{R})$. We shall be done as soon as we show that this function is f, that is, that

$$\langle \tilde{f}, h\rangle = \langle f, h\rangle \qquad \forall h \in L^2(\mathbb{R}).$$

From the identity above (4.1) and the orthogonality theorem we have

$$\begin{aligned}
\langle \tilde{f}, h\rangle &= \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s,\xi)\, \overline{\langle h, M_\xi \tau_s \gamma\rangle}\, d\xi\, ds \\
&= \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s,\xi)\, \overline{V_\gamma h(s,\xi)}\, d\xi\, ds \\
&= \frac{1}{\langle \gamma, g\rangle} \langle V_g f, V_\gamma h\rangle = \frac{1}{\langle \gamma, g\rangle} \langle f, h\rangle \overline{\langle g, \gamma\rangle} = \langle f, h\rangle,
\end{aligned}$$

as desired. □

Remark 11. In practise, given a signal f and a normalised window g three steps are followed:

1. Analysis: $f \longmapsto V_g f$,
2. Processing. This may involve truncation, separation of signal components, compression, etc. It results in a function $F(t,\xi) \sim V_g f(t,\xi)$.
3. Synthesis:
$$\tilde{f}(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} F(s,\xi)\, M_\xi \tau_s g(t)\, ds\, d\xi.$$

4.2 Discrete time-frequency representations

Instead of the continuous expansion of $f \in L^2(\mathbb{R})$ given by the previous inversion formula we would like to have a series expansion with respect to a countable subset of time-frequency shifts, in the spirit of Shannon's formula for band-limited functions.

Let us do this for the simple window $g = \chi_{[-1/2,1/2)}$. Since $g(t-n) = \chi_{[n-\frac{1}{2}, n+\frac{1}{2})}(t)$, we have

$$f(t) = \sum_{n\in\mathbb{Z}} f(t)\, g(t-n)\,, \quad t \in \mathbb{R}. \tag{4.2}$$

Since $f \in L^2(\mathbb{R})$, also $f \in L^2[n-1/2, n+1/2]$ for all $n \in \mathbb{Z}$, so we can write f as a Fourier series

$$f(t) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikt}, \qquad t \in [n-1/2, n+1/2],$$

where

$$c_k = \int_{n-1/2}^{n+1/2} f(s)e^{-2\pi iks} ds = \int_{\mathbb{R}} f(s)\, g(s-n)e^{-2\pi iks} ds = V_g(n,k).$$

Thus, on $[n-1/2, n+1/2]$ we have the representation

$$f(t) = \sum_{k\in\mathbb{Z}} V_g(n,k)e^{2\pi ikt},$$

and, going back to (4.2),

$$f(t) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} V_g(n,k)e^{2\pi ikt}\, g(t-n) = \sum_{n,k\in\mathbb{Z}} V_g(n,k)M_k\tau_n g(t). \tag{4.3}$$

We can summarise all this in the following statement.

Proposition 6. *Let $g = \chi_{[-1/2,1/2)}$. The system $\{M_k\tau_n g\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. In particular, for all $f \in L^2(\mathbb{R})$,*

$$f = \sum_{n,k\in\mathbb{Z}} V_g(n,k)M_k\tau_n g = \sum_{n,k\in\mathbb{Z}} \langle f, M_k\tau_n g\rangle\, M_k\tau_n g$$

and

$$\|f\|_2^2 = \sum_{n,k\in\mathbb{Z}} |V_g(n,k)|^2.$$

Proof. It is clear that $\{M_k\tau_n g\}_{n,k\in\mathbb{Z}}$ is an orthogonal system. To begin with, $M_k\tau_n g$ is just a translation and a modulation of an L^2 function, so it belongs to $L^2(\mathbb{R})$ and $\|M_k\tau_n g\|_2 = \|g\|_2 = 1$.

The orthogonality is also clear. For $n \neq m$ the supports of $M_k\tau_n g$ and $M_j\tau_m g$ are disjoint, hence

$$\langle M_k\tau_n g, M_k\tau_m g\rangle = 0.$$

Also, for $n = m$

$$\langle M_k\tau_n g, M_j\tau_n g\rangle = \int_{n-1/2}^{n+1/2} e^{2\pi i(k-m)t} dt = \delta_{jk}.$$

On the other hand, we have seen in (4.3) that $\{M_k\tau_n g\}_{n,k\in\mathbb{Z}}$ is a generating system.

Finally, by Parseval on each interval $[n-1/2, n+1/2]$,

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} |f(s)|^2 ds = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |V_g(n,k)|^2.$$

□

At this point we ask ourselves whether there are similar bases of $L^2(\mathbb{R})$ obtained with thanslations and modulations of the Gaussian (instead of the characteristic function $\chi_{[-1/2,1/2]}$). Unfortunately, the answer is no...

The Balian-Low theorem. *Let $g \in L^2(\mathbb{R})$ be a general window. Discretise $M_\xi \tau_t g$ on a latice $a\mathbb{Z} \times b\mathbb{Z}$, where $a, b > 0$ and $ab = 1$ ($a = b = 1$ is the canonical case). If the system $\{M_{bk}\tau_{an}g\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ then*

$$\left(\int_{\mathbb{R}} t^2|g(t)|^2 dt\right)\left(\int_{\mathbb{R}} \xi^2|\hat{g}(\xi)|^2 d\xi\right) = +\infty$$

Remarks 4. 1. This is a particular manifestation of the uncertainty principle: if $\{M_{bk}\tau_{an}g\}_{n,k\in\mathbb{Z}}$ is a basis of $L^2(\mathbb{R})$ the window g cannot be concentrated both in time and frequency.

2. For the case $g = \chi_{[-1/2,1/2]}$ considered previously $\hat{g}(\xi) = \mathrm{sinc}(\xi)$, so

$$\int_{\mathbb{R}} \xi^2|\hat{g}(\xi)|^2 d\xi = \int_{\mathbb{R}} \xi^2 \frac{|\sin(\pi\xi)|^2}{\pi^2\xi^2}\, d\xi = +\infty.$$

3. For the Gaussian $G(t) = 2^{1/4}e^{-\pi t^2}$ (normalised so that $\|G\|_2 = 1$) the two integrals appearing in the statement are the same and they are obviously finite. Therefore, the system $\{M_k\tau_n G\}_{n,k\in\mathbb{Z}}$ is not an orthonormal basis of $L^2(\mathbb{R})$.

Proof. This is a sort of physics proof. We consider only the case $a = b = 1$; the general case can be done similarly. We have thus the system

$$g_{n,k}(t) = M_k\tau_n g(t) = e^{2\pi ikt}g(t-n).$$

Consider the operators "position" X and "momentum" P defined by

$$Xf(t) = tf(t)\,, \qquad Pf(t) = \frac{1}{2\pi i}f'(t).$$

These operators are self-adjoint: on the one hand, for $f, h \in L^2(\mathbb{R})$,

$$\langle Xf, h\rangle = \int_{\mathbb{R}} t\, f(t)\overline{h(t)}\, dt = \langle f, Xh\rangle\,,$$

and on the other hand, by an integration by parts,

$$\langle Pf, h\rangle = \int_{\mathbb{R}} \frac{1}{2\pi i}f'(t)\overline{h(t)}\, dt = \int_{\mathbb{R}} \frac{1}{2\pi i}f(t)\overline{h'(t)}\, dt = \langle f, Ph\rangle.$$

With no loss of generality, assume that $Xg(t) = tg(t) \in L^2(\mathbb{R})$ and that $Pg \in L^2(\mathbb{R})$, which by Theorem 4(b) is equivalent to $\xi\hat{g}(\xi) \in L^2(\mathbb{R})$.

Assuming that $\{M_k\tau_n g\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis we have

$$Xg = \sum_{n,k\in\mathbb{Z}} \langle Xg, M_k\tau_n g\rangle\, M_k\tau_n g$$
$$Pg = \sum_{n,k\in\mathbb{Z}} \langle Pg, M_k\tau_n g\rangle\, M_k\tau_n g,$$

and therefore, by the orthonormality

$$\langle Xg, Pg\rangle = \sum_{n,k\in\mathbb{Z}} \langle Xg, M_k\tau_n g\rangle\, \langle M_k\tau_n g, Pg\rangle.$$

Claim: $\langle Xg, M_k\tau_n g\rangle = \langle M_{-k}\tau_{-n} g, Xg\rangle$ and $\langle M_k\tau_n g, Pg\rangle = \langle Pg, M_{-k}\tau_{-n} g\rangle$.

Assuming this, since $\{M_{-k}\tau_{-n} g\}_{n,k\in\mathbb{Z}}$ is the same basis we started with, we have

$$\langle Xg, Pg\rangle = \sum_{n,k\in\mathbb{Z}} \langle Pg, M_{-k}\tau_{-n} g\rangle\, \langle M_{-k}\tau_{-n} g, Xg\rangle = \langle Pg, Xg\rangle.$$

Since both operators are self-adjoint, this gives a contradiction (provided g is in the domain of both P and X); notice that

$$\begin{aligned}(PX - XP)g(t) &= P(t\,g(t)) - X\big(\frac{1}{2\pi i} g'(t)\big) = \frac{1}{2\pi i}(g(t) + t\,g'(t)) - \frac{t}{2\pi i} g'(t)\\ &= \frac{1}{2\pi i} g(t),\end{aligned}$$

and therefore

$$\langle (PX - XP)g, g\rangle = \frac{1}{2\pi i}\|g\|_2^2. \tag{4.4}$$

But the above identity shows that

$$\langle PXg, g\rangle = \langle XPg, g\rangle,$$

thus yielding

$$\langle (PX - XP)g, g\rangle = 0,$$

which contradicts (4.4).

It remains to prove the Claim. Let us prove the first identity; the second one can be proved similarly. By the self-adjointness

$$\langle Xg, M_k\tau_n g\rangle = \langle g, XM_k\tau_n g\rangle,$$

where

$$\begin{aligned}XM_k\tau_n g(t) &= t\,e^{2\pi ikt} g(t-n) = n\,e^{2\pi ikt} g(t-n) + (t-n)\,e^{2\pi ikt} g(t-n)\\ &= nM_k\tau_n g(t) + M_k\tau_n(Xg)(t).\end{aligned}$$

Hence

$$\langle Xg, M_k\tau_n g\rangle = n\langle g, M_k\tau_n g\rangle + \langle g, M_k\tau_n(Xg)\rangle.$$

Since $g = M_0\tau_0 g$, the orthogonality of the system $\{M_k\tau_n g\}_{n,k\in\mathbb{Z}}$ shows that $n\langle g, M_k\tau_n g\rangle$ is always 0. Then, finally

$$\begin{aligned}\langle Xg, M_k\tau_n g\rangle = \langle g, M_k\tau_n(Xg)\rangle &= \int_{\mathbb{R}} g(t)\, e^{-2\pi ikt}\, \overline{(Xg)(t-n)}\, dt \\ &= \int_{\mathbb{R}} g(s+n)\, e^{-2\pi ik(s+n)}\, \overline{Xg(s)}\, ds = \langle M_{-k}\tau_{-n}g, Xg\rangle.\end{aligned}$$

□

4.3 Gabor frames

So far we have seen that a so-called *Gabor system* $\{M_{bk}\tau_{an}g\}_{n,k\in\mathbb{Z}}$ $(a, b > 0, ab = 1)$ cannot form an orthonormal basis unless either g or $\hat{g}$ are badly localised. Consider for example the Gaussian $G(t) = 2^{1/4}e^{-\pi t^2}$, for which both G and $\hat{G} = G$ are well localised; is there really no way to recover $f \in L^2(\mathbb{R})$ from its action on a Gabor system? The answer is, of course, yes, but not in the same way as it is done with an orthonormal basis.

Observe that ultimately we would like to have an expression of the form (letting $g_{n,k} = M_{bk}\tau_{an}g$)

$$f = \sum_{n,k\in\mathbb{Z}} c_{n,k}(f)\, M_{bk}\tau_{an}g = \sum_{n,k\in\mathbb{Z}} c_{n,k}(f)\, g_{n,k}.$$

Recall that $V_g(an, bk) = \langle f, M_{bk}\tau_{an}g\rangle = \langle f, g_{n,k}\rangle$. If we want the system $\{g_{n,k}\}_{n,k}$ to capture most of the energy of any $f \in L^2(\mathbb{R})$ we need that for some $A > 0$

$$A\|f\|_2^2 \le \sum_{n,k\in\mathbb{Z}} \left|V_g(an, bk)\right|^2 = \sum_{n,k\in\mathbb{Z}} \left|\langle f, g_{n,k}\rangle\right|^2.$$

On the other hand, if we want the system not to "oversample" excessively, in the sense that not too much of $\|f\|_2$ is given by the samples, we ask for an estimate of the form

$$\sum_{n,k\in\mathbb{Z}} \left|\langle f, g_{n,k}\rangle\right|^2 = \sum_{n,k\in\mathbb{Z}} \left|V_g(an, bk)\right|^2 \le B\|f\|_2^2,$$

for some $B > 0$.

These two conditions give a notion that is more relaxed and more robust than being a basis. For example, we could double the number of elements in $\{g_{n,k}\}_{n,k}$ and we would still get such a family. Or we could delete one g_{n_0,k_0} and maybe still preserve the inequalities. This robustness, which orthonormal bases lack, is very helpful in applications.

Definition 12. A sequence $\{e_j\}_{j\in J}$ in a separable Hilbert space $\mathcal{H}$ is a *frame* if there exist $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{j\in J} |\langle f, e_j\rangle|^2 \le B\|f\|_2^2 \,, \qquad \forall f \in \mathcal{H}. \tag{4.5}$$

Notice that, in general, the elements of a frame are neither orthogonal nor independent.

In this language, we are interested in Gabor frames $\{g_{n,k}\}_{n,k}$ in $\mathcal{H} = L^2(\mathbb{R})$. Of course an orthonormal basis is a frame, with $A = B = 1$. Also, the union of two different orthonormal basis is also a frame (with $A = B = 2$).

Let us see next how the "synthesis" (reconstructing the signal from the samples) can be done for frames, that is, let us see how we can express a given $f \in L^2(\mathbb{R})$ in the form

$$f = \sum_{n,k\in\mathbb{Z}} c_{n,k}\, g_{n,k},$$

with a canonical choice of coefficients $c_{n,k}$.

4.3.1 Dual frame. Representation

In general, given a subset $\{e_j\}_j \subset \mathcal{H}$, consider the operators defined by the action on $f \in \mathcal{H}$ and $c = \{c_j\}_j \subset \mathbb{C}$ in the following way:

$$\begin{aligned} C(f) &= \big\{\langle f, e_j\rangle\big\}_j && \text{(analysis)}\\ S(c) &= \sum_j c_j\, e_j && \text{(synthesis)}\\ T(f) &= \sum_j \langle f, e_j\rangle\, e_j && \text{(frame operator).}\end{aligned}$$

Proposition 7. *Suppose that $\{e_j\}_j$ is a frame for $\mathcal{H}$ with constants A, B (as in* (4.5)*). Then,*

(a) $C : \mathcal{H} \longrightarrow \ell^2$ and it has closed range.

(b) $S = C^*$ and consequently S extends to a bounded operator from ℓ^2 into $\mathcal{H}$ with

$$\|S(c)\| = \Big\|\sum_j c_j\, e_j\Big\| \le \sqrt{B}\,\|c\|_2, \qquad c \in \ell^2.$$

(c) $T = C^*C = SS^*$ maps $\mathcal{H}$ into $\mathcal{H}$ and it is a positive invertible operator with

$$A\cdot I \le T \le B\cdot I\,, \qquad B^{-1}\cdot I \le T^{-1} \le A^{-1}\cdot I.$$

Note: It can also be seen that the optimal frame bounds are $B = \|T\|$ and $A = \|T^{-1}\|$.

Proof. (a) It is clear that $C(f) \in \ell^2$ and $\|C\| \leq \sqrt{B}$, by the second inequality in (4.5).

That C has closed range is a consequence of the first frame inequality $\sqrt{A}\|f\| \leq \|C(f)\|_2$ in (4.5): if $\{C(f_n)\}_n \longrightarrow \alpha$ in ℓ^2 then $\{f_n\}_n$ is a Cauchy sequence and there exists $f = \lim_n f_n \in \mathcal{H}$. That $C(f) = \alpha$ is then clear, since

$$\|C(f) - \alpha\|_2 \leq \|C(f) - C(f_n)\|_2 + \|C(f_n) - \alpha\|_2 \leq \sqrt{B}\|f - f_n\| + \|C(f_n) - \alpha\|_2.$$

(b) Let $c = \{c_j\}_j$ be finite. Then

$$\langle C^*(c), f\rangle = \langle c, C(f)\rangle = \sum_j c_j \overline{\langle f, e_j\rangle} = \langle \sum_j c_j\, e_j, f\rangle = \langle S(c), f\rangle.$$

Since C is bounded with $\|C\| \leq \sqrt{B}$ it follows that $S = C^* : \ell^2 \longrightarrow \mathcal{H}$ is also bounded, with $\|S\| \leq \sqrt{B}$.

(c) By definition $T = C^*C = SS^*$ and therefore T is self-adjoint and positive. Since

$$\langle T(f), f\rangle = \sum_j \left|\langle f, e_j\rangle\right|^2,$$

the frame inequality (4.5) is precisely $A \cdot I \leq T \leq B \cdot I$. Also, T is invertible, because $A > 0$. Since the previous inequalities are preserved under multiplication by positive commuting operators we obtain $A \cdot T^{-1} \leq TT^{-1} \leq B \cdot T^{-1}$. □

Note: Let $\{e_j\}_j$ be a frame and let $f = \sum_j c_j e_j$, with $c = \{c_j\}_j \in \ell^2$. Then for all $\epsilon > 0$ there exists F_0 finite such that for all F finite with $F_0 \subset F$

$$\Big\|f - \sum_{j \notin F} c_j e_j\Big\| < \epsilon.$$

This can be proved by simply taking F_0 so that

$$\sum_{j \notin F_0} |c_j|^2 < \frac{\epsilon^2}{B}.$$

It is said that $\sum_j c_j e_j$ *converges unconditionally* to f.

Corollary 2. Let $\{e_j\}_j$ be a frame satisfying the inequalities (4.5). Then $\{T^{-1}e_j\}_j$ is also a frame (called the *dual frame*), with associated frame constants $B^{-1}, A^{-1} > 0$. Also

$$f = \sum_j \langle f, T^{-1}e_j\rangle\, e_j\,, \quad f = T(T^{-1}f) = \sum_j \langle T^{-1}f, e_j\rangle\, e_j,$$

and similarly

$$f = \sum_j \langle f, e_j\rangle\, T^{-1}e_j\,, \quad f = T^{-1}(Tf) = \sum_j \langle Tf, e_j\rangle\, T^{-1}e_j.$$

Thus, we can represent f by means of a frame $\{e_j\}_j$, but the coefficients are given in terms of the dual frame, not the original one. In case $\{e_j\}_j$ is an orthonormal basis then $T = I$, and therefore the dual frame coincides with the original one.

After the Balian Low theorem, a natural question is: for the Gaussian window G what lattices $a\mathbb{Z} \times b\mathbb{Z}$ in the time-frequency plane the system $\{M_{bk}\tau_{an}G\}_{n,k\in\mathbb{Z}}$ is a frame (so that sampling on the lattice allows to codify completely a signal and recover it through the dual frame representation)?

Theorem 11. *(Daubechies-Grossman; Seip) Let $G(t) = 2^{1/4}e^{-\pi t^2}$ be the normalised Gaussian and let $a, b > 0$. The system $\{M_{bk}\tau_{an}G\}_{n,k\in\mathbb{Z}}$ is a frame if and only if $ab < 1$.*

Since the product ab is the area of the lattice cell, condition $ab < 1$ can be viewed as a density condition that guarantees, in a very precise way, that there are enough samples to digitalise $f \in L^2(\mathbb{R})$.

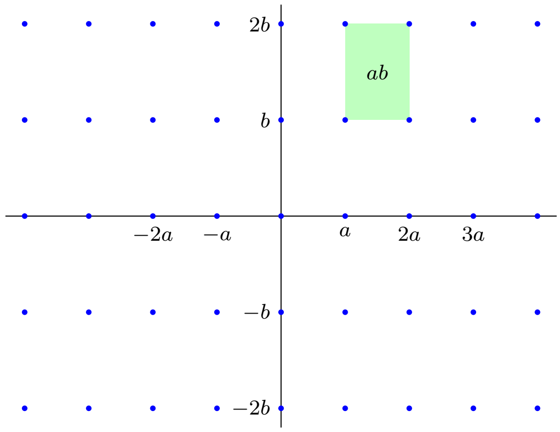

Figure 4.1: Lattice $a\mathbb{Z} \times b\mathbb{Z}$

In a square of side length $N \to \infty$ there are roughly $\frac{N}{a} \times \frac{N}{b}$ lattice nodes. In this sense, the density of the lattice (number of nodes per area unit) is $1/(ab)$.

The previous result has a remarkable generalisation: a Gabor system $\{M_{y_k}\tau_{x_k}G\}_k$ is a frame if the sequence of complex values $\lambda_k = x_k + iy_k$ is separeted (in the sense that $|\lambda_k - \lambda_j| \geq \delta, j \neq k$, for some $\delta > 0$) and

$$D(\{\lambda_k\}_k) := \lim_{r\to\infty} \inf_{z\in C} \frac{\#\{\lambda_k\}_k \cap D(z,r)}{\pi r^2} > 1.$$

Here $D(z, r)$ indicates the disk of center z and radius r.

The quantity $D(\{\lambda_k\}_k)$ is called the *lower density* of $\{\lambda_k\}_k$, and it counts the infimum number of points of the sequence per area unit. In accordance with our intuition, a sequence $\{\lambda_k\}_k$ produces a frame when it is "dense enough".

Remark 12. Frames are useful in many other situations. For example, Shannon's theorem for functions in $L^2(\mathbb{R})$ with $\text{supp}(f) \subset [-\pi, \pi]$ says that any such function can be represented as

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\ \text{sinc}(t-k) \qquad \text{with} \qquad \|f\|_2^2 = \sum_{k\in\mathbb{Z}} |f(k)|^2.$$

We might want to ask whether similar representations exist when replacing the integers for other sequences $\{t_k\}_k \subset \mathbb{R}$. This leads to inequalities of the form

$$A\|f\|_2^2 \leq \sum_k |f(t_k)|^2 \leq B\|f\|_2^2\,, \qquad f \in L^2(\mathbb{R}).$$

Looking at he Fourier side, and letting $g = \hat{f}$, this is

$$A\|g\|_2^2 \leq \sum_k \big\langle g, \frac{1}{\sqrt{2\pi}} e^{-it_k\xi}\big\rangle\big|^2 \leq B\|g\|_2^2\,, \qquad g \in L^2(\mathbb{R}).$$

This is precisely saying that $\left\{\frac{1}{\sqrt{2\pi}} e^{-it_k\xi}\right\}_k$ is a frame. This are called *Fourier frames*.