Lesson 17

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Consider an evolution of the form,

$$\mathrm{d}r(t) = \mu(t, r(t))\mathrm{d}t + \sigma(t, r(t))\mathrm{d}W_t \tag{1}$$

under the real probability $\mathbb{P}$, where μ and σ are such that $r(t)$ is a well defined strong solution. We also can write

$$\begin{aligned} \mathrm{d}r(t) &= \mu(t, r(t))\mathrm{d}t + \sigma(t, r(t))\mathrm{d}W_t \\ &= (\mu(t, r(t)) + \sigma(t, r(t))q(t))\,\mathrm{d}t + \sigma(t, r(t))\mathrm{d}W_t^* \\ &= \mu^*(t)\mathrm{d}t + \sigma(t, r(t))\mathrm{d}W_t^* \end{aligned}$$

Where W^* is a $\mathbb{P}^*$ Brownian motion.

Assume that r is Markovian under the risk-neutral probability $\mathbb{P}^*$, this happens if

$$q(t) = \lambda(t, r(t))$$

then

$$P(t,T) = F(t, r(t); T), \tag{2}$$

in fact

$$P(t,T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s} \middle| r(t) \right) = F(t, r(t); T).$$

Obviously the boundary condition $F(T, r(T); T) = 1$, should be fulfilled for all values of $r(T)$.

Theorem

Let $\mathbb{P}^*$ be equivalent to $\mathbb{P}$such that

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \exp\left\{ \int_0^T \lambda(s, r(s))\mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda^2(s, r(s))\mathrm{d}s \right\},$$

assume that

$$F(t, r(t); T) := \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right)$$

is $C^{1,2}$, then $F(t, r(t); T)$ is a solution of

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}(\mu + \lambda\sigma) + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = 0 \text{ (structure equation)} \qquad (3)$$

with the boundary condition $F(T, r(T); T) = 1$.

Proof.

If we apply the Itô formula to $e^{-\int_0^t r(s)\mathrm{d}s}F(t, r(t); T)$ we have:

$$
\begin{aligned}
& e^{-\int_0^t r(s)\mathrm{d}s}F(t, r(t); T) \\
&= F(0, r(0); T) + \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF\right)\mathrm{d}s \\
&+ \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\frac{\partial F}{\partial r}\sigma \mathrm{d}W_s \\
&= F(0, r(0); T) \\
&+ \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF + \lambda\sigma\frac{\partial F}{\partial r}\right)\mathrm{d}s \\
&+ \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\frac{\partial F}{\partial r}\sigma \mathrm{d}W_s^*.
\end{aligned}
$$

$\square$

Proof.

Then, since

$$e^{-\int_0^t r(s)\mathrm{d}s} F(t, r(t); T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T r(u)\mathrm{d}u} \middle| \mathcal{F}_t) \right)$$

it turns out that

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}(\mu + \lambda\sigma) + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = 0.$$

The boundary condition $F(T, r(T); T) = 1$ is obviously satisfied. □

Examples

These are the most popular short-rate models in the literature. The dynamics for $r(t)$ is given under the risk neutral probability $\mathbb{P}^*$

Vasicek

$$\mathrm{d}r(t) = (b - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t^*.$$

Cox-Ingersoll-Ross (CIR)

$$\mathrm{d}r(t) = a(b - r(t))\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W_t^*$$

Dothan

$$\mathrm{d}r(t) = ar(t)\mathrm{d}t + \sigma r(t)\mathrm{d}W_t^*$$

Ho-Lee

$$\mathrm{d}r(t) = \Theta(t)\mathrm{d}t + \sigma \mathrm{d}W_t^*$$

Black–Karasinski

$$\mathrm{d}\log r(t) = (b(t) + a(t)\log r(t))\mathrm{d}t + \sigma(t)\mathrm{d}W_t^*.$$

Examples

Black-Derman-Toy (Dohan generalized)

$$\mathrm{d}r(t) = \Theta(t)r(t)\mathrm{d}t + \sigma(t)r(t)\mathrm{d}W_t^*$$

Hull-White (Vasicek generalized)

$$\mathrm{d}r(t) = (\Theta(t) - a(t)r(t))\mathrm{d}t + \sigma(t)\mathrm{d}W_t^*$$

Hull-White (CIR generalized)

$$\mathrm{d}r(t) = (\Theta(t) - a(t)r(t))\mathrm{d}t + \sigma(t)\sqrt{r(t)}\mathrm{d}W_t^*$$

In the previous models we have several unknown parameters, that we can denote by α. These parameters cannot be estimated from the observed values of r, since they give the dynamics or r under $\mathbb{P}^*$. Where we can note the effect of dynamics under $\mathbb{P}^*$ is in the real prices of the bonds, because if the model is correct

$$P(t,T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s} \,\middle|\, \mathcal{F}_t \right) = F(t, r(t); T, \alpha),$$

this latter equality if the model is Markovian under $\mathbb{P}^*$. Then, if, for instance, the evolution of r under $\mathbb{P}^*$ is given by

$$\mathrm{d}r(t) = \mu^*(t, r(t); \alpha)\mathrm{d}t + \sigma(t, r(t); \alpha)\mathrm{d}W_t^*$$

we can try to solve the partial differential equation

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu^* + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = 0, \tag{4}$$

$$F(T, r(T); T, \alpha) = 1 \tag{5}$$

and then try to adjust the value of α for fitting $P(t,T) = F(t, r(t); T, \alpha)$ to the observed values of the bonds.

Definition

If the term structure $\{P(t, T); 0 \leq t \leq T\}$ has the form

$$P(t, T) = F(t, r(t); T)$$

where F is given by

$$F(t, r; T) = e^{A(t,T) - B(t,T)r}$$

and where $A(t, T)$ and $B(t, T)$ are smooth deterministic functions, then we say that the model has an affine term structure (ATS).

If we have an ATS

$$
\begin{aligned}
0 &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu^* + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF \\
&= F\left(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r - \mu^* B + \frac{1}{2}\sigma^2 B^2 - r\right) \\
&= F\left(\frac{\partial A}{\partial t} - \mu^* B + \frac{1}{2}\sigma^2 B^2 - \left(1 + \frac{\partial B}{\partial t}\right) r\right),
\end{aligned}
$$

therefore

$$
\frac{\partial A}{\partial t} - \mu^* B + \frac{1}{2}\sigma^2 B^2 - \left(1 + \frac{\partial B}{\partial t}\right) r = 0, \text{ a.e.} \tag{6}
$$

and the boundary condition (5) is

$$
\begin{aligned}
A(T, T) &= 0 \\
B(T, T) &= 0.
\end{aligned}
$$

Definition

We will say that the our *model* (1) is affine if $\mu^*(t,r)$ and $\sigma^2(t,r)$ are also affine, that is

$$\mu^*(t,r) = \alpha(t)r + \beta(t)$$
$$\sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}.$$

for some continuous functions $\alpha, \beta, \gamma, \delta$.

Theorem

If the term structure is affine corresponding to (1) then the model is affine.

Proof.

By (6)

$$\mu^*(t,r)B(t,T) - \frac{1}{2}\sigma^2(t,r)B^2(t,T) = \frac{\partial A(t,T)}{\partial t} - \left(1 + \frac{\partial B(t,T)}{\partial t}\right) r,$$

then if we take T_1, T_2 such that $(B(t,T_1), B(t,T_2))$ are linearly independent of $\left(B^2(t,T_1), B^2(t,T_2)\right)$ we can solve

$$\begin{aligned}
\mu^*(t,r)B(t,T_1) - \frac{1}{2}\sigma^2(t,r)B^2(t,T_1) &= \frac{\partial A(t,T_1)}{\partial t} - \left(1 + \frac{\partial B(t,T_1)}{\partial t}\right) r \\
\mu^*(t,r)B(t,T_2) - \frac{1}{2}\sigma^2(t,r)B^2(t,T_2) &= \frac{\partial A(t,T_2)}{\partial t} - \left(1 + \frac{\partial B(t,T_2)}{\partial t}\right) r
\end{aligned}$$

but if we cannot find such T_1, T_2 means that $B(t,T) = c(t)B^2(t,T)$ for all T and that implies that $B(t,T) = 0$, at the same time we have that $1 + \frac{\partial B(t,T)}{\partial t} = 0$ a.e. so this is imposible a.e. Finally the continuity of μ^* and σ guarantees the affinity. $\square$

Then we are going to look for ATS in affine models. If we have an affine model that has an ATS, (6) becomes

$$\frac{\partial A}{\partial t} - (\alpha r + \beta) B + \frac{1}{2}(\gamma r + \delta) B^2 - \left(1 + \frac{\partial B}{\partial t}\right) r = 0,$$

equivalently

$$\frac{\partial A}{\partial t} - \beta B + \frac{1}{2}\delta B^2 - \left(1 + \alpha B + \frac{1}{2}\gamma B^2 + \frac{\partial B}{\partial t}\right) r = 0,$$

since this is true for all ω a.s., this implies that

$$\begin{aligned} \frac{\partial A}{\partial t} - \beta B + \frac{1}{2}\delta B^2 &= 0, \\ 1 + \alpha B + \frac{1}{2}\gamma B^2 + \frac{\partial B}{\partial t} &= 0. \end{aligned}$$

If we have a solution for this equation we have finished.

The Vasicek model

We shall apply the previous technique to the Vasicek model (for simplicity we write W instead of W^*)

$$\mathrm{d}r(t) = (b - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t, \quad a, b, \sigma > 0$$

Note that

$$\begin{aligned} \mathrm{d}r(t) + ar(t)\mathrm{d}t &= b\mathrm{d}t + \sigma \mathrm{d}W_t \\ &= e^{-at}\mathrm{d}\left(e^{at} r(t)\right). \end{aligned}$$

Hence

$$\mathrm{d}\left(e^{at} r(t)\right) = e^{at} b\mathrm{d}t + e^{at}\sigma \mathrm{d}W_t,$$

so

$$e^{at} r(t) - r(0) = \frac{b}{a}\left(e^{at} - 1\right) + \sigma \int_0^t e^{as}\mathrm{d}W_s$$

and finally

$$r(t) = \frac{b}{a} + e^{-at}\left(r(0) - \frac{b}{a}\right) + \sigma \int_0^t e^{-a(t-s)}\mathrm{d}W_s.$$

Then, we have that r is a Gaussian process and when $t \longrightarrow \infty$, the distribution of $r(t)$ tends to a limit distribution $N(b/a, \sigma^2/(2a))$. This process is named the Ornstein-Uhlenbeck process and its main feature is its mean reverting property: if the process $r(t)$ is greater than $\frac{b}{a}$, then the drift is negative and the process tends to go down. If the process $r(t)$ is less than $\frac{b}{a}$ then it tends to go up. So, in the end, it finished oscillating around the mean value $\frac{b}{a}$ with a constant variance. A drawback of this model is that it can give negative values for $r(t)$.

This model is an affine model with $\alpha(t) = -a, \beta(t) = b, \gamma(t) = 0$ y $\delta(t) = \sigma^2$, so if we look for an ATS, it will satisfies

$$\frac{\partial A}{\partial t} - bB + \frac{1}{2}\sigma^2 B^2 = 0, \quad A(T, T) = 0 \tag{7}$$

$$1 + \frac{\partial B}{\partial t} - aB = 0, \quad B(T, T) = 0 \tag{8}$$

From (8) we have that

$$B(t,T)=\frac{1}{a}(1-e^{-a(T-t)}),$$

now, from (7), we get

$$A(t,T)=\frac{\sigma^2}{2}\int_t^T B^2\mathrm{d}s-b\int_t^T B\mathrm{d}s$$

and substituting for B we obtain

$$A(t,T)=\frac{B(t,T)-(T-t)}{a^2}\left(ab-\frac{1}{2}\sigma^2\right)-\frac{\sigma^2}{4a}B^2(t,T).$$

If we consider the continuous spot interest rate for the period $[t, T]$: $R(t, T)$, since

$$P(t, T) = \exp\{-(T - t)R(t, T)\}$$

and since

$$P(t, T) = \exp\{A(t, T) - B(t, T)r(t)\},$$

it turns out that

$$R(t, T) = -\frac{A(t, T) - B(t, T)r(t)}{T - t}.$$

So, in this model

$$\lim_{T\to\infty} R(t, T) = \frac{b}{a} - \frac{\sigma^2}{2a^2}$$

and this is consider as another imperfection of the model by practitioners since it does not depend on $r(t)$.

The Ho-Lee model

In the Ho-Lee model

$$dr(t) = \Theta(t)dt + \sigma dW_t$$

So, $\alpha(t) = \gamma(t) = 0$, $\beta(t) = \Theta(t)$ and $\delta(t) = \sigma^2$. Then, we have the equations

$$\begin{aligned} \frac{\partial A}{\partial t} - \Theta(t)B + \frac{\sigma^2}{2}B^2 &= 0, \quad A(T,T) = 0 \\ 1 + \frac{\partial B}{\partial t} &= 0, \quad B(T,T) = 0, \end{aligned}$$

therefore

$$\begin{aligned} B(t,T) &= T - t \\ A(t,T) &= \int_t^T \Theta(s)(s-T)\mathrm{d}s + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}. \end{aligned}$$

Note that, contrarily to the previous model, we do not have an explicit expression in terms of the parameters. Now, we have an infinite dimensional parameter $\Theta(s)$. One way of estimating it is to try to fit the initially observed term structure $\left\{\hat{P}(0,T), T \geq 0\right\}$ to the theoretical values. That is

$$P(0,T) \approx \hat{P}(0,T), T \geq 0.$$

This gives

$$-\frac{\partial^2 \log P(0,T)}{\partial T^2} \approx -\frac{\partial^2 \log \hat{P}(0,T)}{\partial T^2} = \frac{\partial \hat{f}(0,T)}{\partial T}$$

but

$$\frac{\partial \hat{f}(0,T)}{\partial T} = -\frac{\partial^2 \log P(0,T)}{\partial T^2} = -\frac{\partial^2 A(0,T)}{\partial T^2} = \frac{\partial}{\partial T} \int_t^T \Theta(s)\mathrm{d}s - \sigma^2 T$$

and therefore

$$\Theta(T) = \frac{\partial \hat{f}(0,T)}{\partial T} + \sigma^2 T$$

The CIR model

In this model model

$$\mathrm{d}r(t) = a(b - r(t))\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W_t$$

where $a, b, \sigma > 0$. As in the Vasicek model there is a reversion to the mean, here given by b, but the volatility factor $\sqrt{r(t)}$ keeps the process above zero: when the process is close to zero there is only contribution of a positive drift.

Theorem

Let W_1, W_2 *be two independent Brownian motions and let* $X_i, i = 1, 2$ *be two Ornstein-Uhlenbeck process, solutions of*

$$\mathrm{d}X_i(t) = -\frac{a}{2}X_i(t)\mathrm{d}t + \frac{\sigma}{2}\mathrm{d}W_i(t), i = 1, 2.$$

Then the process

$$r(t) := X_1^2(t) + X_2^2(t),$$

satisfies

$$\mathrm{d}r(t) = \left(\frac{\sigma^2}{2} - ar(t)\right)\mathrm{d}t + \sigma\sqrt{r(t)})\mathrm{d}W(t)$$

where W *is a standard Brownian motion.*

Proof.

By the Itô formula for the bidimensional case

$$
\begin{aligned}
\mathrm{d}r(t) &= 2\sum_{i=1,2} X_i(t)\mathrm{d}X_i(t) + \frac{\sigma^2}{2}\mathrm{d}t \\
&= -ar(t)\mathrm{d}t + \sigma\sum_{i=1,2} X_i(t)\mathrm{d}W_i(t) + \frac{\sigma^2}{2}\mathrm{d}t \\
&= \left(\frac{\sigma^2}{2} - ar(t)\right)\mathrm{d}t + \sigma\sqrt{r(t)}\sum_{i=1,2}\frac{X_i(t)}{\sqrt{r(t)}}\mathrm{d}W_i(t).
\end{aligned}
$$

Write

$$
\mathrm{d}W(t) := \sum_{i=1,2}\frac{X_i(t)}{\sqrt{r(t)}}\mathrm{d}W_i(t),
$$

□

Proof.

then W is an Itô process with quadratic variation t:

$$[W,W]_t = \sum_{i=1,2} \int_0^t \frac{X_i^2(s)}{r(s)} \mathrm{d}s = \int_0^t \frac{\sum_{i=1,2} X_i^2(s)}{r(s)} \mathrm{d}s = t.$$

And by the Itô formula

$$e^{i\lambda W_t} = e^{i\lambda W_u} + i\lambda \int_u^t e^{i\lambda W_s} \mathrm{d}W_s - \frac{\lambda^2}{2} \int_u^t e^{i\lambda W_s} \mathrm{d}s.$$

Consequently

$$\mathbb{E}(e^{i\lambda(W_t - W_u)} | \mathcal{F}_u) = 1 - \frac{\lambda^2}{2} \int_u^t \mathbb{E}(e^{i\lambda(W_s - W_u)} | \mathcal{F}_u) \mathrm{d}s,$$

and

$$\mathbb{E}(e^{i\lambda(W_t - W_u)} | \mathcal{F}_u) = e^{-\frac{1}{2}\lambda^2(t-u)}.$$

We have to solve

$$\frac{\partial A}{\partial t} - \beta(t)B + \frac{1}{2}\delta(t)B^2 = 0,$$
$$1 + \frac{\partial B}{\partial t} + \alpha(t)B - \frac{1}{2}\gamma(t)B^2 = 0.$$

con $\beta = ab, \delta = 0, \alpha = -a$ y $\gamma = \sigma^2$. That is

$$\frac{\partial A}{\partial t} - abB = 0,$$
$$1 + \frac{\partial B}{\partial t} - aB - \frac{1}{2}\sigma^2 B^2 = 0,$$

with the boundary condition $B(T,T)=A(T,T)=0$. It is easy to see that, by taking derivatives, we have

$$B(t,T)=\frac{2(e^{c(T-t)}-1)}{d(t)}$$

with $c=\sqrt{a^2+2\sigma^2}$ and $d(t)=(c+a)(e^{c(T-t)}-1)+2c$. By integrating

$$A(t,T)=\frac{2ab}{\sigma^2}\left(\frac{(a+c)(T-t)}{2}+\log\frac{2c}{d(t)}\right).$$

In the calibration step we try to adjust the real bond prices to the the theoretical ones. If we use the notation $\{\hat{P}(0,T), T \geq 0\}$ for the observed prices, we want to obtain that

$$P(0,T;\alpha) = \hat{P}(0,T), \quad T \geq 0.$$

but this is not possible if our set of parameters, α, is finite dimensional. We have seen that in the Ho-Lee model this was possible due to the fact that the involved parameter $\Theta(t)$ was infinite dimensional. The Hull-White model combines this fact with the mean reverting property we have in the Vasicek model. By this reason it is quite popular. The dynamics we consider is

$$\mathrm{d}r(t) = (\Theta(t) - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t, \quad a, \sigma > 0.$$

Then, we have

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

and

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)\mathrm{d}s - \int_t^T \Theta(s)B(s,T)\mathrm{d}s,$$

so the theoretical forward rates are given by

$$\begin{aligned} f(0,T) &= -\partial_T \log P(0,T) = \partial_T \left(B(0,T)r(0) - A(0,T)\right) \\ &= \partial_T \left(B(0,T)\right) r(0) - \sigma^2 \int_0^T B(s,T)\partial_T B(s,T)\mathrm{d}s \\ &+ \int_0^T \Theta(s)\partial_T B(s,T)\mathrm{d}s, \end{aligned}$$

that is

$$
\begin{aligned}
f(0,T) &= e^{-aT} r(0) - \sigma^2 \int_0^T \frac{1}{a}(1 - e^{-a(T-s)}) e^{-a(T-s)} \mathrm{d}s \\
&\quad + \int_0^T \Theta(s) e^{-a(T-s)} \mathrm{d}s \\
&= e^{-aT} r(0) - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2 + \int_0^T \Theta(s) e^{-a(T-s)} \mathrm{d}s.
\end{aligned}
$$

By differentiating with respect to T and if we call $g(T) := e^{-aT}r(0) - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2$, we have

$$\begin{aligned}\partial_T f(0,T) &= \partial_T g(T) + \Theta(T) - a\int_0^T \Theta(s)e^{-a(T-s)}\mathrm{d}s \\ &= \partial_T g(T) + \Theta(T) - a(f(0,T) - g(T)),\end{aligned}$$

so

$$\Theta(T) = \partial_T f(0,T) - \partial_T g(T) + a(f(0,T) - g(T)).$$

We can then to capture $\hat{f}(0,T)$ by taking

$$\Theta(T) = \partial_T \hat{f}(0,T) - \partial_T g(T) + a(\hat{f}(0,T) - g(T)).$$