Local Dynamics

Conjugations

- Discrete systems: Let $f: M \to M$ and $g: N \to N$. They are topologically conjugate if $\exists\, h: M \to N$ homeomorphism s.t.

$$h \circ f = g \circ h$$

If h is a C^r-diffeo we say that f, g are C^r conjugate

Note: $h \circ f = g \circ h \iff h \circ f^m = g^m \circ h,\ \forall m \geq 0$. If f and g are invertible we also have $h \circ f^m = g^m \circ h$ $\forall m \in \mathbb{Z}$

- Continuous systems: Let $x' = X(x)$, $x' = Y(x)$ defined on open sets U, V of $\mathbb{R}^n$ resp.

Let $\varphi(t,x)$, $\psi(t,x)$ be the corresponding flows

They are top conjugate if $\exists\, h: U \to V$ homeomorphism s.t.

$$h(\varphi(t,x)) = \psi(t, h(x)), \quad x \in U, \quad t \in (\omega_-(x), \omega_+(x))$$

If h is a C^r diffeo we say the systems are C^r-conjugate

Classification: being conjugate is an equivalence relation

Changes of variable in o.d.e

Let $x' = X(x)$ and $y = h(x)$ be a change of variable (differentiable)

The transformed equation is

$$y' = Dh(x)\, x' = Dh(x)\, X(x) = Dh(h^{-1}(y))\, X(h^{-1}(y)) =: Y(y)$$

and if $x(t)$ is a solution of $x' = X(x)$, then $y(t) = h(x(t))$ is a solution of $y' = Y(y)$.

On the other hand if h is a conjugation from $x' = X(x)$ to $y' = Y(y)$

$$h(\varphi(t,x)) = \psi(t, h(x))$$

If h is differentiable, $Dh(\varphi(t,x))\, \varphi'(t,x) = \psi'(t, h(x))$

$$\rightarrow \quad Dh(\varphi(t,x))\, X(\varphi(t,x)) = Y(\psi(t, h(x)))$$

$$(t=0) \quad \rightarrow \quad Dh(x)\, X(x) = Y(h(x))$$

$$\rightarrow \quad Y(y) = Dh(h^{-1}(y))\, X(h^{-1}(y))$$

Dynamics of linear maps

Let $A : \mathbb{R}^m \longrightarrow \mathbb{R}^m$. We decompose the space as $\mathbb{R}^m = E^s \oplus E^c \oplus E^u$

where E^s = sum of eigenspaces associated to eigenvalues of modulus less than 1

E^c = idem ... equal to 1

E^u = idem ... bigger than 1

With respect to this decomposition

$$A = \begin{pmatrix} A^s & 0 & 0 \\ 0 & A^c & 0 \\ 0 & 0 & A^u \end{pmatrix}$$

$$\mathrm{Spec}\, A^s \subset \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$$
$$\mathrm{Spec}\, A^c \subset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$$
$$\mathrm{Spec}\, A^u \subset \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}$$

$\exists$ norms in E^s, E^c, E^u s.t.

$$\|A^s\| \leq \max_{|\lambda|<1,\ \text{eigenvalues}} \{|\lambda|\} + \varepsilon =: \gamma < 1$$

$$\|A^c\| \leq 1 + \varepsilon$$

$$\|(A^u)^{-1}\| \leq \max_{\substack{|\lambda|>1 \\ \text{eigenvalues}}} \left\{\frac{1}{|\lambda|}\right\} + \varepsilon =: \beta < 1$$

A useful norm in $\mathbb{R}^m$ is $\|x\| = \max\{\|x_s\|, \|x_c\|, \|x_u\|\}$ if $x = x_s + x_c + x_u$

- If $x \in E^s$, $A^m x = \begin{pmatrix} (A^s)^m x \\ 0 \\ 0 \end{pmatrix}$, $\|A^m x\| = \|(A^s)^m x\| \leq \gamma^m \|x\| \xrightarrow[n \to \infty]{} 0$

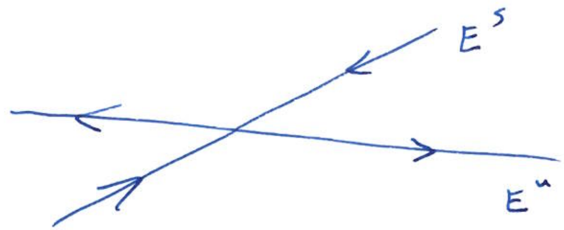
- If $x \in E^u$, $A^{-m} x = \begin{pmatrix} 0 \\ 0 \\ (A^u)^{-m} x \end{pmatrix}$, $\|A^{-m} x\| = \|((A^u)^{-1})^m x\| \leq \beta^m \|x\| \xrightarrow[n \to \infty]{} 0$

and also $\|A^m x\| = \|(A^u)^m x\| \geq \|(A^u)^{-m}\|^{-1} \|x\| \geq \beta^{-m} \|x\| \xrightarrow[m \to \infty]{} \infty$ if $x \neq 0$

<u>Def</u> $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ is hyperbolic if $\operatorname{Spec} A \cap \{\lambda \mid |\lambda| = 1\} = \emptyset$.

Then $\mathbb{R}^n = E^s \oplus E^u$ and

$$A = \begin{pmatrix} A^s & 0 \\ 0 & A^u \end{pmatrix}$$



In this case we have

- $x \in E^s \iff A^m x \to 0$, $m \to \infty$
- $x \in E^u \iff A^{-m} x \to 0$, $m \to \infty$
- $x \notin E^s \cup E^u \Rightarrow \|A^m x\| \to \infty$, $m \to \pm\infty$

Prop Let f, g be C^1 diffeos. Let $p = f(p)$, $q = g(q)$, and h a C^1 conjugation from f to g such that $h(p) = q$.

Then

$$Dh(p)\, Df(p) = Dg(q)\, Dh(p)$$

Consequence: $\quad \operatorname{Spec} Df(p) = \operatorname{Spec} Dg(q)$

Even more, Jordan form of $Df(p)$ = Jordan form of $Dg(q)$

Proof Taking derivatives in $h(f(x)) = g(h(x))$

$$Dh(f(x))\, Df(x) = Dg(h(x))\, Dh(x)$$

<u>Example</u> $f(x) = 2x$, $g(x) = 3x$ are not C^1 - conjugate

Are they top. conjugate ? We look for $h(x) = x^\alpha$

$$h(2x) = 3h(x) \longrightarrow (2x)^\alpha = 3x^\alpha \longrightarrow 2^\alpha = 3 \longrightarrow \alpha = \frac{\log 3}{\log 2} > 1$$

We take
$$h(x) = \begin{cases} x^{\log 3/\log 2}, & x \geq 0 \\ -|-x|^{\log 3/\log 2}, & x < 0 \end{cases}$$

It is C^1 and $h'(0) = 0 \Longrightarrow$ It is not diffeo in a nbh of 0

$$h^{-1}(x) = \begin{cases} x^{\log 2/\log 3} \\ -|-x|^{\log 2/\log 3} \end{cases}$$

Prop Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be hyperbolic and invertible. Then it is topologically conjugate to one of the following eight linear maps

$$\begin{pmatrix} \pm 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{2} & & & & & & & \\ & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & & & & & \\ & & & \ddots & & & & \\ & & & & 1/2 & & & \\ & & & & & \pm 2 & & \\ & & & & & & 2 & \\ & & & & & & & \ddots \\ & & & & & & & & 2 \end{pmatrix}, \quad \begin{pmatrix} \pm\frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2} \end{pmatrix}$$

Prop Two linear maps $A, B \in L(\mathbb{R}^n, \mathbb{R}^n)$ are C^1 conjugate $\iff$ A, B are conjugate matrices

Proof If there exists $h \in C^1$ s.t. $h(Ax) = Bh(x)$

$Dh(Ax)A = BDh(x) \rightarrow Dh(0)A = BDh(0)$

Conversely, if there exists $C \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\det C \neq 0$, s.t. $CA = BC$

then $h(x) = Cx$ is the desired (linear, and hence C^1) conjugation

Results on conjugations of vector fields. Let X, Y be vector fields of class C^1

Prop Let h be a C^1 diffeomorphism.
h is a C^1-conjugation from X to Y $\iff$ $Dh(x)\,X(x) = Y(h(x))$

Proof $\boxed{\Rightarrow}$ Already done

$\boxed{\Leftarrow}$ Given $x \in M$ let $\alpha(t) = h(\varphi(t,x))$, $\beta(t) = \psi(t, h(x))$

$$\begin{cases} \alpha'(t) = Dh(\varphi(t,x))\,\varphi'(t,x) = Dh(\varphi(t,x))\,X(\varphi(t,x)) = Y(h(\varphi(t,x))) = Y(\alpha(t)) \\ \alpha(0) = h(\varphi(0,x)) = h(x) \end{cases}$$

$$\begin{cases} \beta'(t) = \psi'(t, h(x)) = Y(\psi(t, h(x))) = Y(\beta(t)) \\ \beta(0) = \psi(0, h(x)) = h(x) \end{cases}$$

Proposition Let X, Y vector fields of class C^1, and $x' = X(x)$, $x' = Y(x)$.

Let p, q be such that $X(p) = 0$, $Y(q) = 0$

Then, if h is a C^1-conjugation from X to Y such that $h(p) = q$,

$$Dh(p)\, DX(p) = DY(q)\, Dh(p)$$

Proof Let $\varphi(t,x)$, $\psi(t,x)$ be the solutions of $x' = X(x)$, $x' = Y(x)$ resp.

By def. of conjugation $h(\varphi(t,x)) = \psi(t, h(x))$

Taking derivatives with respect to x:

$$Dh(\varphi(t,x))\, D_x\varphi(t,x) = D_x\psi(t, h(x))\, Dh(x) \xrightarrow{x=p} Dh(p)\, D_x\varphi(t,p) = D_x\psi(t,q)\, Dh(p)$$

$D_x\varphi(t,p)$ satisfies the V.E. $(D_x\varphi(t,p))' = DX(\varphi(t,p))\, D_x\varphi(t,p)$, $D_x\varphi(0,p) = Id$.

$\Rightarrow D_x\varphi(t,p) = \exp(DX(p)\, t)$. In the same way $D_x\psi(t,q) = \exp(DY(q)\, t)$

Then $Dh(p)\, e^{DX(p)t} = e^{DY(q)t}\, Dh(p)$.

Taking derivative w.r.t. t, $Dh(p)\, DX(p)\, e^{DX(p)t} = DY(q)\, e^{DY(q)t}\, Dh(p)$. Put $t = 0$.

Example $\quad x' = -x, \quad x' = -2x \quad$ are top. conjugate but not C^1-conjugate

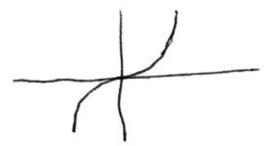
$\varphi(t,x) = e^{-t}x, \quad \psi(t,x) = e^{-2t}x$

If $\exists h \in C^1 \quad h(e^{-t}x) = e^{-2t}h(x) \quad \to \quad Dh(e^{-t}x)\,e^{-t} = e^{-2t}Dh(x)$

$(x=0) \quad \to \quad Dh(0)\,e^{-t} = e^{-2t}Dh(0)$

$\frac{\partial}{\partial t},\ t=0 \to \quad -Dh(0) = -2Dh(0) \quad \to \quad Dh(0) = 0$

$$\exists\, h(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$



$$h^{-1}(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$

If $\quad x \geq 0 \quad \begin{cases} h(e^{-t}x) = e^{-2t}x^2 \\ e^{-2t}h(x) = e^{-2t}x^2 \end{cases} \qquad$ If $\quad x < 0 \quad \ldots$

<u>Def</u> Linear hyperbolic vector field : $x' = Ax$ s.t. $\operatorname{spec} A \cap \{\lambda \mid \operatorname{Re} \lambda = 0\} = \emptyset$

s = index of stability = number of eigenvalues s.t. $\operatorname{Re} \lambda < 0$ (counting multiplicities)

<u>Theorem</u> Two hyperbolic linear vector fields are topologically conjugate

$\iff$ they have the same index of stability

<u>Theorem</u> $x' = Ax$, $x' = Bx$ are C^1 conjugate $\iff$ A, B are conjugate matrices

<u>Proof</u> $\boxed{\Rightarrow}$ Let h be a conjugation, of class C^1: $h(e^{At}x) = e^{Bt} h(x)$

Taking derivative w.r.t. x $\rightarrow$ $Dh(e^{At}x)\, e^{At} = e^{Bt} Dh(x)$

Putting $x = 0$ $\rightarrow$ $Dh(0)\, e^{At} = e^{Bt} Dh(0)$

Taking derivative w.r.t. t $\rightarrow$ $Dh(0) A e^{At} = B e^{Bt} Dh(0) \xrightarrow{t=0} Dh(0) A = B\, Dh(0)$

$\boxed{\Leftarrow}$ If $\exists C$ s.t. $CA = BC$, $h(x) = Cx$ is a conjugation (linear and therefore C^1)