Quantitative Finance

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Chapter 1

Stock Markets. Financial Derivatives

1.1 Introduction

Mathematical Finance or Quantitative Finance could be defined as the area of mathematics concerned with the quantitative behavior of the Financial markets. The mathematics involved is quite rich: stochastic analysis, stochastic control, stochastic processes, game theory, risk theory, statistics, numerical methods, partial differential equations, equilibrium theory, enlargement of filtrations, filtering, transport theory,... and the goals are also diverse: valuation of derivatives, hedging, optimization of portfolios, formation of prices, and risk control... This monograph is devoted to valuation and hedging of derivatives. A nice and quick introduction to this area is given in ? (?). There are excellent monographs devoted to the issue treated here, see for instance ? (?), ? (?) or ? (?), among others.

A central problem in Mathematical Finance is the following, assume that the price of a stock in a financial market is evolving randomly in such a way that at time t is, say, S_t. That is, we assume a random evolution for the value of such a stock caused by the forces of supply and demand. We do not discuss this price $(S_t)_{t\geq 0}$, but we want to study financial contracts *based* in this asset, more specifically we are interested in *options* or *derivatives* . *An option* is a contract issued by a company that gives the right (but not the obligation) to buy, then is named a CALL, or sell, then named PUT, the stock at price K (strike) at time T (maturity of the contract).

The profit or *payoff* of this contract is

$$(S_T - K)_+ := \max(0, S_T - K)$$

in the case of a CALL. In fact if at time T the value of the stock, S_T, is greater than K, the owner of the CALL can buy the asset by K and to sell it at the market by S_T obtaining a profit of $S_T - K$. On the contrary if $S_T < K$ the owner will not exercise her right to buy it, since she could do it, if she is interested, by a lower price at the market. In such a case the profit of the CALL would be zero. In the case of a PUT the payoff is

$$(K - S_T)_+.$$

Then , we have two typical problems. The first one is how much the buyer should pay for the option. This is called *the pricing problem*. The second problem is how the seller of the contract can guarantee the corresponding payoff, $(S_T - K)_+$ in the case of a CALL, from the price charged. This is the *hedging problem*.

A crucial assumption to price any contract in the market is the assumption that nothing is given for free. If you want to get some profit you have to pay for it and sometimes you lose. In other words in the financial market you cannot do profits without risk. It is said that there are not *arbitrage* opportunities in the market.

As a first consequence of this assumption is that prices of contracts based in the same underlying are related. In particular we have the so-called put-call parity condition. Before showing this parity condition think about the following: which is the price, **at time** t, for receiving the value S_T at time $T > t$? In other words which is the price we have to pay at t if we want to have the asset at a future time T? Obviously S_t! (we assume that there are not dividends!, otherwise we would receive more than S_T by paying S_t). A similar question can be raised for a fixed amount K. How much you should pay **at time** t to get K at time T? As in the case of the stock we have to assume a time evolution for the price of the money! But it is clear that it is not the same to have K at t or K at T. Assume for the moment that to have 1 unit of money at T we have to pay, at t, $P(t, T)$. Then to have K we have to pay $KP(t, T)$. Now the price, at t, of a contract with payoff $S_T - K$ and maturity time T will be $S_t - KP(t, T)$, following the rule that the price of a linear combination of payoffs have to be a linear combination of prices to avoid arbitrage opportunities. Moreover

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K,$$

so if we write C_t for the price of a CALL, at time t, with strike K and maturity T and P_t the put price, with the same strike and maturity, we have the put-call parity condition:

$$C_t - P_t = S_t - KP(t, T) \text{ for all } 0 \leq t \leq T.$$

But still which is the price C_t or P_t? We will see that they depend on the probabilistic model of S.

? in 1973 assume that a unit of money in a bank account evolves as

$$e^{rt}$$

and the stock as

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t},$$

where (W_t) is a Brownian motion.

They obtain, using stochastic analysis and an equilibrium argument, that

$$C_t = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

where $\Phi(x)$ is the standard normal distribution function and

$$d_\pm = \frac{\log(\frac{S_t}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

This is the celebrated Black-Scholes option price formula.

The Brownian motion describes the random movement that is possible to observe in some microscopic particles in a fluid mean (for instance pollen in a water drop). This name is due to the botanist Robert Brown who first observed this phenomenon en 1828.

The zigzagging of these particles is due to the fact that they collide with the molecules of the fluid in an intense way depending of the temperature of the fluid.

The mathematical description of this phenomenon was elaborated by Albert Einstein in 1905, finding that the distribution of position of a Brownian particle at time t, starting at x at time 0, was of the form $N(x, \sigma^2 t)$ for a constant $\sigma^2 > 0$ depending on parameters of the particle and the liquid but independent of events before time 0.

It is also remarkable that ?, in 1900, in his thesis "theorie de la spéculation", proposed a limit of a random walk, an antecedent of the Brownian motion, to describe the market price of a stock and to obtain a formula for a CALL.

Another issue related with the pricing problem is how the seller of the contract can guarantee the quantity $(S_T - K)_+$ (in the case of a CALL) from the price charged. This is the *hedging problem.* From the Black-Scholes formula

$$C_t = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \tag{1.1}$$

we deduce that a portfolio with $\Phi(d_+)$ stocks and $-\Phi(d_-)$ bonds have the same value as the CALL! But it is not clear that we can have this portfolio without injecting or taking money from it.

The way that Black and Scholes obtained the formula is the following. Suppose that the value of the CALL is a smooth function

$$C_t := f(t, S_t),$$

and consider a portfolio with β_t CALL and α_t stocks, the cost of this portfolio is

$$\beta_t C_t + \alpha_t S_t =: V_t,$$

and when it evolves its value changes as

$$\mathrm{d}V_t = \beta_t \mathrm{d}C_t + \alpha_t \mathrm{d}S_t,$$

where we assume that the portfolio is self-financed (its value change because the value of the stocks change but not because we take or inject money in the portfolio). Then

$$\mathrm{d}V_t = \beta_t \left(\partial_t f \mathrm{d}t + \partial_x f \mathrm{d}S_t + \frac{1}{2}\partial_{xx} f S_t^2 \sigma^2 \mathrm{d}t \right) + \alpha_t \mathrm{d}S_t,$$

where we apply the Itô formula for stochastic differentials.

Now if we take $\alpha_t = -\ \beta_t \partial_x f$ we have that the cost of this portfolio is

$$\mathrm{d}V_t = \beta_t \left(\partial_t f + \frac{1}{2}\partial_{xx} f S_t^2 \sigma^2 \right) \mathrm{d}t.$$

It behaves like a bank account! remember that if we put V_t in the bank account then $\mathrm{d}V_t = V_t r \mathrm{d}t$, then if we want an equilibrium situation in such a way that it is not possible to do profit without risk, we must have

$$\beta_t \left(\partial_t f + \frac{1}{2}\partial_{xx} f S_t^2 \sigma^2 \right) = rV_t = r(\beta_t C_t - \beta_t \partial_x f S_t) = r\beta_t (f - \partial_x f S_t).$$

So, the price of a CALL is the solution of the partial differential equation

$$\partial_t f + rx\partial_x f + \sigma^2 x^2 \frac{1}{2} \partial_{xx} f = rf, \tag{1.2}$$

with the boundary condition $f(T, x) = (x - K)_+$.

To solve this, we do the change

$$f(t, x) = e^{-r(T-t)} u(\tau, y), \tag{1.3}$$

with

$$\tau := \frac{2}{\sigma^2} \left(r - \frac{1}{2}\sigma^2 \right)^2 (T - t) \tag{1.4}$$

$$y := \frac{2}{\sigma^2} \left(r - \frac{1}{2}\sigma^2 \right) \log \frac{x}{K} + \tau,$$

then (1.2) becomes into

$$\partial_\tau u = \partial_{yy} u \tag{1.5}$$

with the boundary condition $u(0, y) = K \left(e^{\frac{\sigma^2}{2r - \sigma^2} y} - 1 \right)_+$.

Now, since

$$g(\tau, y) := \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{1}{4}\frac{y^2}{\tau}}$$

is the solution of (1.5) with boundary condition $g(0, y) = \delta(y)$, where $\delta(y)$ is the Dirac delta, we will have that

$$u(\tau, y) = \int_{\mathbb{R}} K \left(e^{\frac{\sigma^2}{2r - \sigma^2} x} - 1 \right)_+ \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{1}{4}\frac{(x-y)^2}{\tau}} \mathrm{d}x.$$

Finally plugging this into (1.3) and taking into account (1.4) we obtain the Black-Scholes formula (1.1).

1.2 Discrete time models

The values of the stocks (shares, commodities or other stocks) will be random variables defined in a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will consider an increasing sequence of σ-fields (filtration) : $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_N \subseteq \mathcal{F}$. $\mathcal{F}_n$ represents the collection of all events that are observable up to time n. The horizon N, will correspond with the maturity of the options. We shall assume that Ω is *finite*, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F} = \mathcal{P}(\Omega)$ and that $\mathbb{P}(\{\omega\}) > 0$, for all $\omega \in \Omega$.

The financial market will consist on $(d+1)$ stocks whose prices at time n will be given by *positive* random variables S_n^0, $S_n^1, ...,$ S_n^d measurable with respect to $\mathcal{F}_n$ (that is, the prices depend on what has been observed so far, there is not privilege information). In many cases we shall assume that $\mathcal{F}_n = \sigma(S_k^1, ...,$ $S_k^d, 0 \leq k \leq n)$, in such a way that the whole information will be in the prices observed until this moment.

The super-index zero corresponds to the riskless stock (a bank account) and by convention we take $S_0^0 = 1$. If the relative profit (return) of the riskless stock is constant:

$$\frac{S_{n+1}^0 - S_n^0}{S_n^0} = r \geq 0$$

we will have

$$S_{n+1}^0 = S_n^0(1+r) = S_0^0(1+r)^{n+1} = (1+r)^{n+1}.$$

1.2.1 Strategies of investment, portfolios.

A *trading strategy* is a stochastic process (a sequence or random variables in the discrete time setting) $\phi = ((\phi_n^0, \phi_n^1, ..., \phi_n^d))_{1 \leq n \leq N}$ in R^{d+1}. ϕ_n^i indicates the number of stocks of kind i in the portfolio at time n and ϕ is *predictable* that is ϕ_n^i is $\mathcal{F}_{n-1}$-measurable, for all $1 \leq n \leq N$. This means that the positions in the portfolio at n is decided at $n-1$ and hold till time n. In other words, ϕ_n^i is the quantity of stocks of i type during the period $(n-1, n]$.

The value of the portfolio associated with a trading strategy ϕ is given by

$$V_n(\phi) = \phi_n \cdot S_n := \sum_{i=0}^{d} \phi_n^i S_n^i, \quad n \geq 1, \quad V_0(\phi) = \phi_1 \cdot S_0.$$

and its *discounted* value

$$\tilde{V}_n(\phi) = \frac{V_n(\phi)}{(1+r)^n} = \phi_n \cdot \tilde{S}_n$$

with

$$\tilde{S}_n = \left(1, \frac{S_n^1}{(1+r)^n}, ..., \frac{S_n^d}{(1+r)^n}\right) = (1, \tilde{S}_n^1, ..., \tilde{S}_n^d)$$

Definition 1.2.1 *An investment strategy is said to be self-financing if*

$$V_n = \phi_{n+1} \cdot S_n, \quad 0 \leq n \leq N-1$$

Remark 1.2.1 *The meaning is that at n, once the new prices S_n are announced, investors change their portfolio without adding or taking out wealth: if at time n there is an increment $\phi_{n+1} - \phi_n$ of the risky stocks the cost of this trade is $\sum_{i=1}^d (\phi^i_{n+1} - \phi^i_n) S^i_n$, the change in the bank account will be*

$$(\phi^0_{n+1} - \phi^0_n) S^0_n = -\sum_{i=1}^d (\phi^i_{n+1} - \phi^i_n) S^i_n$$

so

$$\sum_{i=0}^d (\phi^i_{n+1} - \phi^i_n) S^i_n = (\phi_{n+1} - \phi_n) \cdot S_n = 0$$

and $V_n = \phi_{n+1} \cdot S_n$, $0 \le n \le N-1$.

Proposition 1.2.1 *A trading strategy is self-financing if and only if*

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot (S_{n+1} - S_n), 0 \le n \le N-1.$$

Proof. Assume that the strategy is self-financing then

$$\begin{aligned} V_{n+1}(\phi) - V_n(\phi) &= \phi_{n+1} \cdot S_{n+1} - \phi_{n+1} \cdot S_n \\ &= \phi_{n+1} \cdot (S_{n+1} - S_n). \end{aligned}$$

If $V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot (S_{n+1} - S_n)$ then

$$\phi_{n+1} \cdot S_{n+1} - V_n = \phi_{n+1} \cdot (S_{n+1} - S_n),$$

and consequently $V_n = \phi_{n+1} \cdot S_n$. ■

Proposition 1.2.2 *The following statements are equivalent:*

(i) The strategy ϕ is self-financing,

(ii) $V_n(\phi) = V_0(\phi) + \sum_{j=1}^n \phi_j \cdot (S_j - S_{j-1})$

$$= V_0(\phi) + \sum_{j=1}^n \phi_j \cdot \Delta S_j = V_0(\phi) + \sum_{j=1}^n \sum_{i=0}^d \phi^i_j \Delta S^i_j, \quad 1 \le n \le N$$

(iii) $\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^n \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1})$

$$= V_0(\phi) + \sum_{j=1}^n \phi_j \cdot \Delta \tilde{S}_j = V_0(\phi) + \sum_{j=1}^n \sum_{i=1}^d \phi^i_j \Delta \tilde{S}^i_j, \ 1 \le n \le N$$

Proof. (i) $\Rightarrow$ (ii): By the previous proposition

$$\begin{aligned} V_n(\phi) &= V_0(\phi) + \sum_{j=1}^n (V_j(\phi) - V_{j-1}(\phi)) \\ &= V_0(\phi) + \sum_{j=1}^n \phi_j \cdot (S_j - S_{j-1}), \end{aligned}$$

(i) $\Rightarrow$ (iii): the self-financing condition can be written as $\tilde{V}_n = \phi_{n+1} \cdot \tilde{S}_n, 0 \leq n \leq N-1$, so

$$\tilde{V}_{n+1}(\phi) - \tilde{V}_n(\phi) = \phi_{n+1} \cdot (\tilde{S}_{n+1} - \tilde{S}_n),\ 0 \leq n \leq N-1$$

and

$$\begin{aligned}\tilde{V}_n(\phi) &= \tilde{V}_0(\phi) + \sum_{j=1}^{n} (\tilde{V}_j(\phi) - \tilde{V}_{j-1}(\phi)) \\ &= V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1})\end{aligned}$$

■

Exercise 1.2.1 *In the previous proposition prove that (ii)⇒(i) and that (iii)⇒(i).*

The previous proposition tell us that any self-financing strategy ϕ is defined by its initial value V_0 and for the positions in the risky stocks. More precisely:

Proposition 1.2.3 *For any predictable process* $\hat{\phi} = ((\phi_n^1, ..., \phi_n^d))_{1 \leq n \leq N}$ *and any value* V_0*, there exists a unique predictable process* $\left(\phi_n^0\right)_{1 \leq n \leq N}$ *such that the strategy* $\phi = ((\phi_n^0, \phi_n^1, ..., \phi_n^d))_{1 \leq n \leq N}$ *is self-financing with initial value* V_0.

Proof. For $1 \leq n \leq N$

$$\begin{aligned}\tilde{V}_n(\phi) &= V_0 + \sum_{j=1}^{n} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) \\ &= \ \phi_n \cdot \tilde{S}_n = \phi_n^0 + \sum_{i=1}^{d} \phi_n^i \tilde{S}_n^i.\end{aligned}$$

Therefore

$$\begin{aligned}\phi_n^0 &= V_0 + \sum_{j=1}^{n} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) - \sum_{i=1}^{d} \phi_n^i \tilde{S}_n^i \\ &= V_0 + \sum_{j=1}^{n} \sum_{i=1}^{d} \phi_j^i \cdot (\tilde{S}_j^i - \tilde{S}_{j-1}^i) - \sum_{i=1}^{d} \phi_n^i \tilde{S}_n^i \\ &= V_0 + \sum_{j=1}^{n-1} \phi_j \cdot (\tilde{S}_j - \tilde{S}_{j-1}) - \sum_{i=1}^{d} \phi_n^i \tilde{S}_{n-1}^i \in \mathcal{F}_{n-1}.\end{aligned}$$

■

1.2.2 The arbitrage condition

First of all note that we are not doing any assumption about the sign of the quantities . $\phi_n^i < 0$ amounts to borrowing this number of stocks and converting them into cash (*short-selling*) or, if $i = 0$, borrowing this number of monetary units and converting them into stocks (a loan to buy stocks). In fact, we do not put any restriction on ϕ_n^i, it can be any real number, so divisibility and total liquidity conditions of the market are assumed, including no transaction costs. For simplicity we suppose that any unit of cash at 0 becomes $(1+r)^n$ at n and this happens independently of it is borrowed or invested in the bank account.

We put some constraints about the self-financing strategies.

Definition 1.2.2 *A strategy ϕ is admissible if it is self-financing and $V_n(\phi) \geq 0$, for all $0 \leq n \leq N$.*

Definition 1.2.3 *An* arbitrage (opportunity) *is an admissible strategy ϕ with zero initial value and with final value different from zero, that is*

1. $V_0(\phi) = 0$,
2. $V_N(\phi) \geq 0$,
3. $\mathbb{P}(V_N(\phi) > 0) > 0$.

Remark 1.2.2 *Note that if there is an arbitrage we can get a strictly positive wealth with a null initial investment. Most of the models of prices exclude arbitrage opportunities. A market without arbitrage opportunities is said to be* viable. *The next purpose will be to characterize viable markets with the aid of the notion of martingale.*

Exercise 1.2.2 *Consider a portfolio with initial value $V_0 = 1000$ and formed by the following quantities of risky stocks:*

Trading period	*Stock 1*	*Stock 2*
$(0,1]$	200	100
$(1,2]$	150	120
$(2,3]$	500	60

The prices of he stocks are

	Stock 1	*Stock 2*
$n = 0$	3.4	2.3
$n = 1$	3.5	2.1
$n = 2$	3.7	1.8.

To find out, at any time, the amount invested in the riskless stock in the portfolio assuming that $r = 0.05$ and that the portfolio is self-financing.

Solution. Assuming that the value at time $t = 0$ is $V_0 = 1000$, we can calculate the initial composition of the portfolio according with the positions in

Stock	N^o shares	Price $t=0$	Value $t=0$	Price $t=1$	Value $t=1$
0	90	1	90	1,05	94,5
1	200	3,4	680	3,5	700
2	100	2,3	230	2,1	210
Total			1000		1004,5

the risky stocks $\phi_1 = (200, 100)$ and leaving the remainder of the 1000 euros in the bank account. Later we calculate how the value of the portfolio change in terms of change of prices between instants 0 and 1. We rebuilt our portfolio according with the positions $\phi_2 = (150, 120)$, in the bank account we leave the remainder after buying the indicated quantities of stocks 1 and 2. Later we calculate again how the value of the portfolio evolves.

Stock	N^o assets	Price $t=1$	Value $t=1$	Price $t=2$	Value $t=2$
0	216,67	1,05	2227,5	1,103	238,88
1	150	3,5	525	3,7	555
2	120	2,1	252	1,8	216
Total			1004,5		1009,88

Exercise 1.2.3 *Consider a financial market with one single period, with interest rate r and one stock S. Suppose that $S_0 = 1$ and, for $n = 1$, S_1 can take two different values:* 3, 2. *For which values of r the market is viable viable (free of arbitrage opportunities)? What if S_1 can also take the value* 2.5*?*

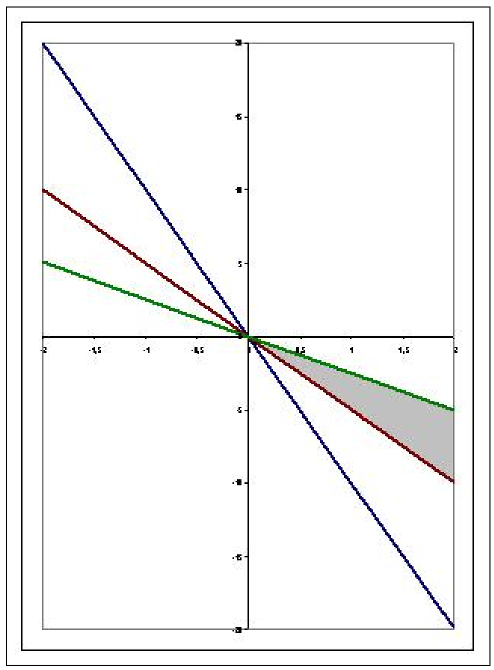
Solution. We want to calculate the values of r such that there is an arbitrage opportunity. We take a portfolio with zero initial value $V_0 = 0$. Then we invest the amount q in the stock without risk, we have to invest $-q$ in the risky stock (q can be negative or positive). We calculate the value of this portfolio at time 2.
$V_1(\omega_1) = q(1 + r) - 3q = q(r - 2)$
$V_1(\omega_2) = q(1 + r) - 2q = q(r - 1)$
So, if $r \geq 2$ there is an arbitrage oppportunity taking q positive (money in the bank account and short position in the risky stock) and if $r \leq 1$ we have an arbitrage opportunity with q negative (borrowing money and investing in the risky stock). The situation does not change if S_1 can take the value 2.5.

Exercise 1.2.4 *Consider a financial market with two risky stocks ($d = 2$) and such that the values at $t = 0$ are $S_0^1 = 9.52$€ and $S_0^2 = 4.76$€. The simple interest rate is 5% during the period [0, 1]. We also assume that at time 1, S_1^1 and S_1^2 can take three different values, depending of the market state: $\omega_1, \omega_2, \omega_3$: $S_1^1(\omega_1) = 20$€ ., $S_1^1(\omega_2) = 15$€ and $S_1^1(\omega_3) = 7.5$€, and $S_1^2(\omega_1) = 6$€, $S_1^2(\omega_2) = 6$€ and $S_1^2(\omega_3) = 4$€. ¿Is that a viable market?*

Solution. To know if he market is viable we have to check if there are arbitrage opportunities. We take a portfolio with initial value equal to zero and

we see if the yield can be non-negative in all states of time 1 with some of them strictly positive yield. Let q_1 and q_2 be the amounts invested in the stocks 1 and 2 respectively. Since the initial value of the portfolio es zero, we should have $-9.52q_1 - 4.76q_2$ in the bank account. Then we calculate the value of our portfolio at time 1 for all possible states.
$V_1(\omega_1) = 10.004q_1 + 1.002q_2$
$V_1(\omega_2) = 5.004q_1 + 1.002q_2$
$V_1(\omega_3) = -2.4964q_1 - 0.998q_2$.

It is easy to see that there is a region of the plane where the three expressions are positive at the same time (see Figure 1), therefore there are arbitrage opportunities.



Exercise 1.2.5 *Consider a market with two periods, a risk-free asset such that* $S_0^0 = 1$, $S_1^0 = 1.1, S_2^0 = 1.21$, *and a risky asset, the price of which can follow three possible evolutions*

	S_0^1	S_1^1	S_2^1
ω_1	1	1.2	1.44
ω_2	1	1.2	0.96
ω_3	1	0.9	0.96

*Is this market free of arbitrage? What if short-**sell**ing is not allow?. Assume that short-**sell**ing is allowed but there is a transaction cost of* 5% *each time a stock is bought or sold.*

1.2.3 Characterization of arbitrage and martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. With $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega\}) > 0$, for all ω. Consider a filtration $(\mathcal{F}_n)_{0\le n\le N}$, with $\mathcal{F}_0 = \{\phi, \Omega\}$.

Definition 1.2.4 *We say that a sequence of random variables $X = (X_n)_{0\le n\le N}$ is adapted if X_n is $\mathcal{F}_n$-measurable, $0 \le n \le N$.*

Definition 1.2.5 *An adapted sequence $(M_n)_{0\le n\le N}$, is said to be a*

submartingale if	$\mathbb{E}(M_{n+1}\vert\mathcal{F}_n) \ge M_n$
martingale if	$\mathbb{E}(M_{n+1}\vert\mathcal{F}_n) = M_n$
supermartingale if	$\mathbb{E}(M_{n+1}\vert\mathcal{F}_n) \le M_n$

for all $0 \le n \le N-1$

Remark 1.2.3 *This definition can be extended to the multi-dimensional case in a component-wise fashion. If $(M_n)_{0\le n\le N}$ is a martingale is easy to see that $\mathbb{E}(M_{n+j}|\mathcal{F}_n) = M_n, j \ge 0; \mathbb{E}(M_n) = M_0, n \ge 0$ and that if $(N_n)_{0\le n\le N}$ is another martingale, $(aM_n + bN_n)_{0\le n\le N}$ is also a martingale. We shall omit the sub-indexes.*

Proposition 1.2.4 *Let $(M_n)_{0\le n\le N}$ be a martingale and $(H_n)_{1\le n\le N}$ a predictable sequence, let $\Delta M_n = M_n - M_{n-1}$. Then, the sequence defined by*

$$X_0 = M_0$$

$$X_n = M_0 + \sum_{j=1}^{n} H_j \Delta M_j, 1 \le n \le N \text{ is a martingale}$$

Proof. It is enough to see that for all $0 \le n \le N$

$$\mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) = \mathbb{E}(H_{n+1}\Delta M_{n+1}|\mathcal{F}_n) = H_{n+1}\mathbb{E}(\Delta M_{n+1}|\mathcal{F}_n) = 0$$

■

Remark 1.2.4 *The previous transform is called martingale transform of $(M_n)_{0\le n\le N}$ by $(H_n)_{1\le n\le N}$ Remind that*

$$\tilde{V}_n(\phi) = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j$$

with $(\phi_n)_{1\le n\le N}$ predictable. Then if $(\tilde{S}_n)_{0\le n\le N}$ is a martingale, we will have that $(\tilde{V}_n)_{0\le n\le N}$ is a martingale and in particular $\mathbb{E}(\tilde{V}_n(\phi)) = V_0$.

Proposition 1.2.5 *An adapted process $(M_n)_{0\le n\le N}$ is a martingale if and only if for all predictable process $(H_n)_{1\le n\le N}$ we have*

$$\mathbb{E}\left(\sum_{j=1}^{N} H_j \Delta M_j\right) = 0 \tag{1.6}$$

Proof. Assume that $(M_n)_{0\leq n\leq N}$ is a martingale, then (1.6) follows by the previous proposition. Assume that (1.6) is satisfied, then we can take $H_n = 0, 1 \leq n \leq j, H_{j+1} = 1_A$ with $A \in \mathcal{F}_j$, $H_n = 0, n > j$. So

$$\mathbb{E}(\mathbf{1}_A(M_{j+1} - M_j)) = 0.$$

Since this is true for all $A \in \mathcal{F}_j$, this is equivalent to $\mathbb{E}(M_{j+1}|\mathcal{F}_j) = M_j$, and this is also true for all $j \geq 0$. ■

Theorem 1.2.1 *A financial market is viable (free of arbitrage opportunities) if and only if there exists $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted prices of the stocks $((\tilde{S}_n^j)_{0\leq n\leq N}\ , j = 1, ..., d)$ are $\mathbb{P}^*$-martingales.*

Proof. Assume there exists $\mathbb{P}^*$ and let φ and admissible strategy with zero initial value, then

$$\tilde{V}_n = \sum_{i=1}^{n} \varphi_i \cdot \Delta\tilde{S}_i, n \geq 1, \tilde{V}_0 = 0,$$

is a $\mathbb{P}^*$-martingale and consequently

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N) = \tilde{V}_0 = 0,$$

and since $\tilde{V}_N \geq 0$ we have $\tilde{V}_N = 0$ (because $\mathbb{P}^*(\omega) > 0$ for all ω). So, there is not an arbitrage.

Suppose now that there is not arbitrage. First, we can identify each random variable X to a vector in $\mathbb{R}^K$ $(X(\omega_1), ..., X(\omega_K))$, where $K = Card(\Omega)$. Denote Γ the set of random variables strictly positive:

$$\Gamma := \{X, X \geq 0, X \neq 0\}$$

and $\Lambda = \{X = \tilde{V}_N(\phi), (\phi_n)_{1\leq n\leq N}$ *predictable,* $\tilde{V}_n = \sum_{j=1}^{n} \phi_j \cdot \Delta\tilde{S}_j$, $\tilde{V}_n(\phi) \geq 0$ for all $1 \leq n \leq N\}$. No-arbitrage means that

$$\Lambda \cap \Gamma = \phi \tag{1.7}$$

Consider the subset $S \subseteq \Gamma$ of the random variables such that $\sum_{i=1}^{K} X(\omega_i) = 1$, it is compact and convex:

$$S \subseteq B(0, 1)$$

where $B(0, 1)$ is the ball with center at the origin and radius one, and if $X, Y \in S$

$$\lambda X + (1 - \lambda)Y \in S, \text{ for all } 0 \leq \lambda \leq 1.$$

Let $L = \{X = \tilde{V}_N(\phi), (\phi_n)_{1\leq n\leq N}$ *predictable,* $\tilde{V}_N = \sum_{j=1}^{N} \phi_j \cdot \Delta\tilde{S}_j$ $\}$, then L is a vector subspace of $\mathbb{R}^K$ since

$$\alpha\tilde{V}_N(\phi^{(1)}) + \beta\tilde{V}_N(\phi^{(2)}) = \tilde{V}_N(\alpha\phi^{(1)} + \beta\phi^{(2)})$$

for all $\alpha, \beta \in \mathbb{R}$ and $L \supseteq \Lambda$.

We shall see later that $\Lambda \cap \Gamma = \phi$ implies that $L \cap \Gamma = \phi$ and, a fortiori, $L \cap S = \phi$. As a result of the separating hyperplane theorem there exists a linear map A such that $A(Y) > 0$ for all $Y \in S$ and $A(Y) = 0$ if $Y \in L$. By the linearity we can write $A(Y) = \sum_{i=1}^{K} \lambda_i Y(\omega_i)$. Then, all $\lambda_i > 0$, since $A(Y) > 0$ for all $Y \in S$, and we can define

$$\mathbb{P}^*(\omega_i) = \frac{\lambda_i}{\sum_{i=1}^{K} \lambda_i}, i = 1, ..., K,$$

and for all ϕ predictable

$$\mathbb{E}_{\mathbb{P}^*}\left(\sum_{i=1}^{N} \phi_i \cdot \Delta \tilde{S}_i\right) = \mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N) = \frac{A(\tilde{V}_N)}{\sum_{i=1}^{K} \lambda_i} = 0.$$

So, by the previous proposition, $\tilde{S}$ is a $\mathbb{P}^*$-martingale.

See now that $L \cap \Gamma = \phi$. Assume that $L \cap \Gamma \neq \phi$ then there exists φ predictable $V_0 = 0$ and $\tilde{V}_N(\varphi) \in \Gamma$. Then from φ, we can see that $\Lambda \cap \Gamma \neq \phi$. Define

$$n = \inf\{j, \tilde{V}_k(\varphi)(\omega) \geq 0 \text{ for all } k > j \text{ and } \omega \in \Omega\},$$

note that $n \leq N - 1$ since $\tilde{V}_N(\varphi) \geq 0$. Let $A = \left\{\tilde{V}_n(\varphi) < 0\right\}$, define the predictable vector process such that for all $i = 1, ..., d$

$$\theta_j^i = \begin{cases} 0 & j \leq n \\ \mathbf{1}_A \varphi_j^i & j > n \end{cases}$$

Then, $\tilde{V}_k(\theta) = 0$, for all $0 \leq k \leq n$ and for all $k > n$

$$\begin{aligned} \tilde{V}_k(\theta) &= \sum_{j=n+1}^{k} \mathbf{1}_A \varphi_j \cdot \Delta \tilde{S}_j = \mathbf{1}_A \left(\sum_{j=1}^{k} \varphi_j \cdot \Delta \tilde{S}_j - \sum_{j=1}^{n} \varphi_j \cdot \Delta \tilde{S}_j\right) \\ &= \mathbf{1}_A \left(\tilde{V}_k(\varphi) - \tilde{V}_n(\varphi)\right) \geq 0, \end{aligned}$$

so θ is admissible and we have that $\tilde{V}_N(\theta) = \mathbf{1}_A \left(\tilde{V}_N(\varphi) - \tilde{V}_n(\varphi)\right) > 0$ in A. So $\tilde{V}_N(\theta) \in \Lambda \cap \Gamma$ and consequently $\Lambda \cap \Gamma \neq \phi$ contradicting the no-arbitrage condition (1.7). ■

Remark 1.2.5 $\mathbb{P}^*$ *is named martingale measure or risk-neutral probability. Notice that the discounted values of self-financing portfolios are* $\mathbb{P}^*$*-martingales.*

Exercise 1.2.6 *Consider a sequence* $\{X_n\}_{n\geq 1}$ *of independent random variables with law* $N(0, \sigma^2)$*. Define the sequence* $Y_n = \exp\left(a \sum_{i=1}^{n} X_i - n\sigma^2\right)$, $n \geq 1$, *for* a *a real parameter, and* $Y_0 = 1$*. Find the values of* a *such that the sequence* $\{Y_n\}_{n\geq 0}$ *is a martingale (supermartingale) (submartingale).*

Exercise 1.2.7 *Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent, identically distributed random variables*

$$\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}.$$

Set $S_0 = 0$ and $S_n = Y_1 + \cdots + Y_n$ if $n \geq 1$.

Check if the following sequences are martingales:

$$\begin{aligned} M_n^{(1)} &= \frac{e^{\theta S_n}}{(\cosh\theta)^n},\ n \geq 0 \\ M_n^{(2)} &= \sum_{k=1}^{n} \operatorname{sign}(S_{k-1}) Y_k,\ n \geq 1,\ M_0^{(2)} = 0 \\ M_n^{(3)} &= S_n^2 - n \end{aligned}$$

Exercise 1.2.8 *Consider a discrete-time financial market, with two periods, interest rate $r \geq 0$, and a single risky stock, S. Suppose that S evolves as:*

$$\begin{array}{ccc} n=0 & n=1 & n=2 \\ & & 9 \\ & 8 \begin{smallmatrix} \nearrow p_2 \\ \searrow 1-p_2 \end{smallmatrix} & \\ 5 \begin{smallmatrix} \nearrow p_1 \\ \searrow 1-p_1 \end{smallmatrix} & & 6 \\ & 4 \begin{smallmatrix} \nearrow p_3 \\ \searrow 1-p_3 \end{smallmatrix} & \\ & & 3 \end{array}$$

a) Find p_1, p_2 i p_3, in terms of r such that the probability is neutral. b) Assuming that $r = 0.1$, give the initial value of a derivative with maturity $N = 2$ and payoff $\frac{S_1+S_2}{2}$. Construct first the portfolio that covers the risk of the derivative (that replicates the derivative) and see its initial value. Check that this value coincides with the expectation, with respect to the risk-neutral probability, of the discounted payoff.

Solution 1.2.1 *a) Our sample space Ω is given by the set of possible trajectories of the process S. That is the different values of $S := (S_0, S_1, S_2)$. We can write*

$$\begin{aligned} S(\omega_1) &= (5, 8, 9) \\ S(\omega_2) &= (5, 8, 6) \\ S(\omega_3) &= (5, 4, 6) \\ S(\omega_4) &= (5, 4, 3) \end{aligned}$$

with $\Omega = \{\omega_1.\omega_2, \omega_3, \omega_4\}$. Then a probability measure $\mathbb{P}$ in $\mathcal{P}(\Omega)$ (the family of all subsets of Ω) (equivalently we look for $\mathbb{P}(\omega_i)$) such that the discounted-value process of S, that we write $\tilde{S}$, is a martingale with respect to the filtration $\mathbb{F}$ generated by the process S itself. $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$, where

$$\begin{aligned} \mathcal{F}_0 &= \sigma(S_0) = \{\phi, \Omega\}, \\ \mathcal{F}_1 &= \sigma(S_0, S_1) = \{\phi, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}, \\ \mathcal{F}_2 &= \mathcal{P}(\Omega). \end{aligned}$$

According to the picture above, we have that

$$\mathbb{P}(\omega_1) = p_1p_2,\ \mathbb{P}(\omega_2) = p_1(1-p_2),\ \mathbb{P}(\omega_3) = (1-p_1)p_3,\ \mathbb{P}(\omega_4) = (1-p_1)(1-p_3).$$

where

$$p_1 = \mathbb{P}(S_1 = 8),\ 1 - p_1 = \mathbb{P}(S_1 = 4) \tag{1.8}$$
$$p_2 = \mathbb{P}(S_2 = 9|S_1 = 8),\ 1 - p_2 = \mathbb{P}(S_2 = 6|S_1 = 8) \tag{1.9}$$
$$p_3 = \mathbb{P}(S_2 = 6|S_1 = 4),\ 1 - p_3 = \mathbb{P}(S_2 = 3|S_1 = 4). \tag{1.10}$$

Then $\tilde{S}$ *is a martingale if* $\mathbb{P}$ *satisfies*

$$\mathbb{E}_{\mathbb{P}}\left(\tilde{S}_2 \middle| \mathcal{F}_1\right) = \tilde{S}_1$$
$$\mathbb{E}_{\mathbb{P}}\left(\tilde{S}_1 \middle| \mathcal{F}_0\right) = \tilde{S}_0.$$

That is

$$\mathbb{E}_{\mathbb{P}}\left(\frac{S_2}{(1+r)^2} \middle| \mathcal{F}_1\right) = \mathbb{E}_{\mathbb{P}}\left(\frac{S_2}{(1+r)^2} \middle| S_1\right) = \frac{S_1}{1+r}, \tag{1.11}$$
$$\mathbb{E}_{\mathbb{P}}\left(\frac{S_1}{1+r} \middle| \mathcal{F}_0\right) = \mathbb{E}_{\mathbb{P}}\left(\frac{S_1}{1+r}\right) = S_0. \tag{1.12}$$

By (1.11), (1.9) and (1.10)

$$\frac{9}{(1+r)^2}p_2 + \frac{6}{(1+r)^2}(1-p_2) = \frac{8}{1+r}$$
$$\frac{6}{(1+r)^2}p_3 + \frac{3}{(1+r)^2}(1-p_3) = \frac{4}{1+r}.$$

By (1.12) y (1.8)

$$\frac{8}{1+r}p_1 + \frac{4}{1+r}(1-p_1) = 5.$$

Therefore

$$p_1 = \frac{1+5r}{4}, p_2 = \frac{2+8r}{3}, p_3 = \frac{1+4r}{3}.$$

We need that $0 < p_i < 1$ *for* $\mathbb{P}$ *to be a probability measure and* $r \geq 0$. *These constraints are satisfied if and only if* $0 \leq r < 1/8$. *b) If we assume* $r = 0.1$ *we have a neutral probability* $\mathbb{P}$, *with* $p_1 = 0.375$, $p_2 = 0.9\bar{3}$ *and* $p_3 = 0.4\bar{6}$. *Now if we want to get a self-financing portfolio* $\phi_i = (\phi_i^0, \phi_i^1)$, $i = 0, 1, 2$, *that replicates the payoff* $X = \frac{S_1+S_2}{2}$, *we will need that*

$$V_2 = \phi_2^0(1+r)^2 + \phi_2^1 S_2 = X = \frac{S_1+S_2}{2}, \text{(replication at 2)} \tag{1.13}$$
$$V_1 = \phi_1^0(1+r) + \phi_1^1 S_1 = \phi_2^0(1+r) + \phi_2^1 S_1 \text{ (self-financing at 1)} \tag{1.14}$$
$$V_0 = \phi_0^0 + \phi_0^1 S_0 = \phi_1^0 + \phi_1^1 S_0 \text{ (self-financing at 0)}. \tag{1.15}$$

The strategy, or portfolio, ϕ *has to be a predictable process, so* ϕ_i *will be a measurable function of* $S_{i-1},..,S_0$*. In that case* $\phi_2^0 = \phi_2^0(S_1), \phi_2^1 = \phi_2^1(S_1)$ *and* ϕ_1 *and* ϕ_0 *will be constants. We have, by (1.13)*

$$\phi_2^0(8)(1+r)^2 + \phi_2^1(8)9 = \frac{8+9}{2},$$
$$\phi_2^0(8)(1+r)^2 + \phi_2^1(8)6 = \frac{8+6}{2},$$

so

$$\phi_2^1(8) = 1/2, \phi_2^0(8) = \frac{8}{2(1+r)^2}. \tag{1.16}$$

And

$$\phi_2^0(4)(1+r)^2 + \phi_2^1(4)6 = \frac{4+6}{2},$$
$$\phi_2^0(8)(1+r)^2 + \phi_2^1(8)3 = \frac{4+3}{2},$$

therefore

$$\phi_2^1(4) = 1/2, \phi_2^0(4) = \frac{4}{2(1+r)^2}. \tag{1.17}$$

Now we use (1.14), (1.16) and (1.17),

$$\phi_1^0(1+r) + \phi_1^1 8 = \phi_2^0(8)(1+r) + \phi_2^1(8)8 = \frac{4}{1+r} + 4$$
$$\phi_1^0(1+r) + \phi_1^1 4 = \phi_2^0(4)(1+r) + \phi_2^1(4)4 = \frac{2}{1+r} + 2,$$

and we get $\phi_1^1 = \frac{1}{2(1+r)} + \frac{1}{2}$ *and* $\phi_1^0 = 0$*. Moreover by (1.15)*

$$V_0 = \frac{S_0}{2(1+r)} + \frac{S_0}{2} = \frac{5}{2.2} + \frac{5}{2} = \frac{10.5}{2.2} = 4.7\overline{72},$$

and $\left(\phi_0^0, \phi_0^1\right)$ *are any values satisfying* $\phi_0^0 + \phi_0^1 S_0 = V_0$*. All these calculations, in this case, can be avoided if we think that to replicate* $\frac{S_1+S_2}{2}$ *at* 2 *it is enough to buy at* 1*,* $\frac{1}{2}$ *of* S *and to put* $\frac{S_1}{2(1+r)}$ *in the bank account. The cost of this portfolio at* 1 *is* $\frac{S_1}{2(1+r)} + \frac{S_1}{2} = S_1(\frac{1}{2(1+r)} + \frac{1}{2})$*. Now this value at* 1 *can be reproduced if we start at* 0 *with* $\frac{1}{2(1+r)} + \frac{1}{2}$ *of* S *and zero units in the bank account. The value of this latter portfolio is* $\frac{S_0}{2(1+r)} + \frac{S_0}{2}$*! However, in general, to calculate the replicating portfolio we need to do all the above calculations. Finally we can see that*

$$V_0 = \mathbb{E}_{\mathbb{P}}(\tilde{X}).$$

In fact, using (1.12) y (1.11), that is, the fact that $\tilde{S}$ *is a* $\mathbb{P}$*-martingale,*

$$\mathbb{E}_{\mathbb{P}}(\tilde{X}) = \mathbb{E}_{\mathbb{P}}\left(\frac{S_1+S_2}{2(1+r)^2}\right) = \frac{1}{2(1+r)}\mathbb{E}_{\mathbb{P}}\left(\frac{S_1}{1+r}\right) + \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left(\frac{S_2}{(1+r)^2}\right)$$
$$= \frac{1}{2(1+r)}\mathbb{E}_{\mathbb{P}}\left(\tilde{S}_1\right) + \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left(\tilde{S}_2\right) = \frac{\tilde{S}_0}{2(1+r)} + \frac{\tilde{S}_0}{2} = \frac{S_0}{2(1+r)} + \frac{S_0}{2}.$$

1.2.4 Complete markets and derivative pricing

We define a *European option, derivative or contingent claim* as a contract with *maturity* N and with a *payoff* $h \geq 0$, where h is $\mathcal{F}_N$-measurable.

For instance a CALL is a European option with payoff $h = (S_N^1 - K)_+$, for a PUT $h = (K - S_N^1)_+$, and an *Asian option* is a European one! with $h = \left(\frac{1}{N}\sum_{j=0}^N S_j^1 - K\right)_+$

Definition 1.2.6 *A derivative defined by h is said to replicable if there exists an admissible strategy ϕ such that replicates h that is $V_N(\phi) = h$.*

Proposition 1.2.6 *If ϕ is a self-financing strategy that replicates h and the market is viable then it is admissible.*

Proof. $\tilde{V}_N(\phi) = \tilde{h}$ and since there exists $\mathbb{P}^*$ such that $\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N(\phi)|\mathcal{F}_n) = \tilde{V}_n(\phi)$, we have $\tilde{V}_n(\phi) \geq 0$. ■

Definition 1.2.7 *A market is said to be complete if any derivative is replicable.*

Theorem 1.2.2 *A viable market is complete if and only if there is a unique probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted prices of the stocks $((\tilde{S}_n^j)_{0\leq n\leq N}, j = 1, ..., d)$ are $\mathbb{P}^*$-martingales.*

Proof. Assume that the market is viable and complete, then, given h $\mathcal{F}_N$-measurable there exists ϕ admissible, such that $V_N(\phi) = h$ that is:

$$\tilde{V}_N(\phi) = V_0(\phi) + \sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j = \frac{h}{S_N^0}.$$

Assume there exist $\mathbb{P}_1$ and $\mathbb{P}_2$ martingale measures, then

$$\mathbb{E}_{\mathbb{P}_1}\left(\frac{h}{S_N^0}\right) = V_0(\phi)$$
$$\mathbb{E}_{\mathbb{P}_2}\left(\frac{h}{S_N^0}\right) = V_0(\phi),$$

so $\mathbb{E}_{\mathbb{P}_1}(h) = \mathbb{E}_{\mathbb{P}_2}(h)$ and since this is true for all h, $\mathcal{F}_N$-measurable, both probabilities are the same in $\mathcal{F}_N = \mathcal{F}$.

Assume now that the market is viable but incomplete, we shall see that we can construct more than one risk-neutral probability. Let H be the subset of random variables of the form

$$V_0 + \sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j$$

with $V_0 \in \mathbb{R}$ and $\phi = ((\phi_n^1, ..., \phi_n^d))_{1\leq n\leq N}$ predictable. H is a vector subspace of the vectorial space, E, formed by all random variables. Moreover it is not a trivial subspace, in fact since the market is incomplete there will exist h such that $\frac{h}{S_n^0} \notin H$ (note that if $h \geq 0$ can be replicated by a non -admissible strategy then the market cannot be viable by Proposition 1.2.6). Let $\mathbb{P}^*$ be a risk-neutral probability in E, we can define the scalar product $\langle X, Y\rangle = \mathbb{E}_{\mathbb{P}^*}(XY)$. Let X be an random variable orthogonal to H and set

$$\mathbb{P}^{**}(\omega) = \left(1 + \frac{X(\omega)}{2||X||_\infty}\right)\mathbb{P}^*(\omega).$$

Then we have an equivalent probability to $\mathbb{P}^*$:

$$\mathbb{P}^{**}(\omega) = \left(1 + \frac{X(\omega)}{2||X||_\infty}\right)\mathbb{P}^*(\omega) > 0,$$

$$\sum \mathbb{P}^{**}(\omega) = \sum \mathbb{P}^*(\omega) + \frac{\mathbb{E}_{\mathbb{P}^*}(X)}{2||X||_\infty} = \sum \mathbb{P}^*(\omega) = 1,$$

since $1 \in H$ and X is orthogonal to H. Also, by this orthogonality, and for any predictable process ϕ, we have that

$$\mathbb{E}_{\mathbb{P}^{**}}\left(\sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j\right) = \mathbb{E}_{\mathbb{P}^*}\left(\sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j\right) + \frac{\mathbb{E}_{\mathbb{P}^*}\left(X\sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j\right)}{2||X||_\infty} = 0$$

in such a way that $\tilde{S}$ is a $\mathbb{P}^{**}$-martingale by Proposition 1.2.5. ■

Pricing and hedging in complete markets

Assume we have a derivative with payoff $h \geq 0$ and that the market is viable and complete. We know that there exists ϕ admissible, such that $V_N(\phi) = h$ and if $\mathbb{P}^*$ is the risk-neutral probability neutral we have that

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^n \phi_j \cdot \Delta\tilde{S}_j$$

is a $\mathbb{P}^*$-martingale, in particular

$$\mathbb{E}_{\mathbb{P}^*}\left(\frac{h}{S_N^0}\middle|\mathcal{F}_n\right) = \mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N(\phi)|\mathcal{F}_n) = \tilde{V}_n(\phi)$$

that is

$$V_n(\phi) = S_n^0\mathbb{E}_{\mathbb{P}^*}\left(\frac{h}{S_N^0}\middle|\mathcal{F}_n\right) = \mathbb{E}_{\mathbb{P}^*}\left(\frac{h}{(1+r)^{N-n}}\middle|\mathcal{F}_n\right)$$

so, the value of the replicating portfolio of h is given by the previous formula and this gives us the price of the derivative at time n that we shall denote by C_n, that is $C_n = V_n(\phi)$. Notice that if we have a single risky stock ($d = 1$) then

$$\frac{\tilde{C}_n - \tilde{C}_{n-1}}{\Delta\tilde{S}_n} = \phi_n$$

and we can calculate the hedging portfolio if we have an expression of C as a function of S.

The binomial model of Cox-Ross-Rubinstein (CRR)

Assume a model with one risky stock that evolves as:

$$S_n(\omega) = S_0(1+b)^{U_n(\omega)}(1+a)^{n-U_n(\omega)}$$

where

$$U_n(\omega) = \xi_1(\omega) + \xi_2(\omega) + ... + \xi_n(\omega)$$

and where ξ_i are random variables wit values 0 or 1, that is Bernoulli random variables, and $-1 < a < r < b$:

$$\begin{array}{ccc}
n=0 & n=1 & n=2... \\
 & & S_0(1+b)^2 \;\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} \\
 & S_0(1+b) \;\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & \\
S_0 \;\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & & S_0(1+b)(1+a) \;\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} \\
 & S_0(1+a)\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & \\
 & & S_0(1+a)\ ^2 \;\begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix}.
\end{array}$$

We can also write

$$S_n = S_{n-1}(1+b)^{\xi_n}(1+a)^{1-\xi_n(\omega)},$$

then

$$\tilde{S}_n = S_0\left(\frac{1+b}{1+r}\right)^{U_n}\left(\frac{1+a}{1+r}\right)^{n-U_n} = \tilde{S}_{n-1}\left(\frac{1+b}{1+r}\right)^{\xi_n}\left(\frac{1+a}{1+r}\right)^{1-\xi_n}.$$

For $\tilde{S}_n$ to be a martingale with respect to $\mathbb{P}^*$ we need

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{S}_n|\mathcal{F}_{n-1}) = \tilde{S}_{n-1}$$

and if we take $\mathcal{F}_n = \sigma(S_0, S_1, ..., S_n)$ we have that the previous condition is equivalent to

$$\mathbb{E}_{\mathbb{P}^*}\left(\left(\frac{1+b}{1+r}\right)^{\xi_n}\left(\frac{1+a}{1+r}\right)^{1-\xi_n}\middle|\mathcal{F}_{n-1}\right) = 1$$

that is

$$\left(\frac{1+b}{1+r}\right)\mathbb{P}^*(\xi_n = 1|\mathcal{F}_{n-1}) + \left(\frac{1+a}{1+r}\right)\mathbb{P}^*(\xi_n = 0|\mathcal{F}_{n-1}) = 1$$

and consequently

$$\mathbb{P}^*(\xi_n = 1|\mathcal{F}_{n-1}) = \frac{r-a}{b-a},$$

$$\mathbb{P}^*(\xi_n = 0|\mathcal{F}_{n-1}) = 1 - \mathbb{P}^*(\xi_n = 1|\mathcal{F}_{n-1}) = \frac{b-r}{b-a}$$

Note that this conditional probability is deterministic and does not depend on n, so *under it* $\xi_i, i = 1, ..., N$ are independent, identically distributed random variables with common distribution Bernoulli(p), for $p = \frac{r-a}{b-a}$. $\mathbb{P}^*$ is unique as well, so the market is viable and complete. Therefore, under the neutral probability $\mathbb{P}^*$

$$\begin{aligned} S_N &= S_n(1+b)^{\xi_{n+1}+\ldots+\xi_N}(1+a)^{N-n-(\xi_{n+1}+\ldots+\xi_N)} \\ &= S_n(1+b)^{W_{n,N}}(1+a)^{N-n-W_{n,N}} \end{aligned}$$

with $W_{n,N}$ $\sim$Bin$(N-n,p)$ independent of $S_n, S_{n-1}, ...S_1$. Since we have the risk neutral probability we can calculate the price of a *call* at time n

$$\begin{aligned} C_n &= \mathbb{E}_{\mathbb{P}^*}\left(\frac{(S_N-K)_+}{(1+r)^{N-n}}\middle|\mathcal{F}_n\right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\frac{(S_n(1+b)^{W_{n,N}}(1+a)^{N-n-W_{n,N}}-K)_+}{(1+r)^{N-n}}\middle|\mathcal{F}_n\right) \\ &= \sum_{k=0}^{N-n}\frac{(S_n(1+b)^k(1+a)^{N-n-k}-K)_+}{(1+r)^{N-n}}\binom{N-n}{k}p^k\,(1-p)^{N-n-k} \\ &= S_n\sum_{k=k^*}^{N-n}\binom{N-n}{k}\frac{(p(1+b))^k((1-p)(1+a))^{N-n-k}}{(1+r)^{N-n}} \\ &-K(1+r)^{n-N}\sum_{k=k^*}^{N-n}\binom{N-n}{k}p^k\,(1-p)^{N-n-k} \end{aligned}$$

where

$$\begin{aligned} k^* &= \inf\{k, S_n(1+b)^k(1+a)^{N-n-k} > K\} \\ &= \inf\left\{k, k > \frac{\log\frac{K}{S_n}-(N-n)\log(1+a)}{\log(\frac{1+b}{1+a})}\right\} \end{aligned}$$

Note that

$$\frac{p(1+b)}{1+r}+\frac{(1-p)(1+a)}{1+r} = 1,$$

so, if we define

$$\bar{p} = \frac{p(1+b)}{1+r}$$

we can write

$$\begin{aligned} C_n &= S_n\sum_{k=k^*}^{N-n}\binom{N-n}{k}\bar{p}^k(1-\bar{p})^{N-n-k} \\ &-K(1+r)^{n-N}\sum_{k=k^*}^{N-n}\binom{N-n}{k}p^k\,(1-p)^{N-n-k} \\ &= S_n\Pr\{\text{Bin}(N-n,\bar{p})\geq k^*\}-K(1+r)^{n-N}\Pr\{\text{Bin}(N-n,p)\geq k^*\} \end{aligned}$$

Hedging portfolio in the CRR model

We have that

$$V_n = \phi_n^0 (1+r)^n + \phi_n^1 S_n.$$

Fixed S_{n-1}, S_n can take two value $S_n^u = S_{n-1}(1+b)$ ó $S_n^d = S_{n-1}(1+a)$ and analogously V_n. Then

$$\phi_n^1 = \frac{V_n^u - V_n^d}{S_{n-1}(b-a)}. \tag{1.18}$$

and

$$\phi_n^0 = \frac{V_n^u - \phi_n^1 S_n^u}{(1+r)^n}$$

In the case of a call, if we take $n = N$ we have:

$$\phi_N^1 = \frac{V_N^u - V_N^d}{S_{N-1}(b-a)} = \frac{(S_{N-1}(1+b) - K)_+ - (S_{N-1}(1+a) - K)_+}{S_{N-1}(b-a)}.$$

Now we can calculate by the self-financing condition the value of the portfolio at $N-1$:

$$V_{N-1} = \phi_N^0 (1+r)^{N-1} + \phi_N^1 S_{N-1}$$

and from here ϕ_{N-1}^1 using (1.18) again.

Example 1.2.1 *The following example is a compute program written in* Mathematica *to calculate the value of a* call *and* put *for a CRR mode with the following data: $S_0 = 100$€ , $K = 100$€ $b = 0.2$, $a = -0.2$, $r = 0.02$, $n = 4$ periods.*

```
Clear[s, call, pu];
s[0] = Table[100, {1}];
a = -0.2; b = 0.2; r = 0.02; n = 4;
p = (r - a)/(b - a);
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
ColumnForm[Table[s[i], {i, 0, n}], Center]
pp[x_] := Max[x, 0]
call[n] = Map[pp, s[n] - 100]; pu[n] = Map[pp, 100 - s[n]];
call[x_] := call[x] =
          Drop[p*call[x + 1]/(1 + r) + (1 - p)*RotateLeft[call[x +
  1], 1]/(1 + r), -1]
ColumnForm[Table[call[i], {i, 0, n}], Center]
pu[x_] := pu[x] = Drop[p*pu[x + 1]/(1 +
r) + (1 - p)*RotateLeft[pu[x + 1], 1]/(1 + r), -1]
ColumnForm[Table[pu[i], {i, 0, n}], Center]
```

Example 1.2.2 *Consider a CRR model with 91 periods $a = -b$. We want to calculate the initial value of a European call where the underlying is a share of Telefónica.*

- *Maturity: 3 months (91 days= n)* ($T = 91/365$).
- *Current price of the share of Telefónica 15.54€.*
- *Strike 15.54€.*
- *Annual interest rate 4.11 %.*
- *Annual volatility: 23,20%* $\left(b^2 = (\textit{annual volatility})^2 \times T/n\right)$

```
Clear[s, c];
n = 91;
so = 15.54;
K = 15.54;
vol = 0.232;
T = 91/365;
r = 0.0411*T/n;
b =vol*Sqrt[T/n];
a =-b;
p = (r - a)/(b - a);
q = 1 - p;
s[0] = Table[so, {1}];
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
pp[x_] := Max[x, 0];
c[n] = Map[pp, s[n] - K];
c[x_] := c[x] = Drop[p*c[x + 1]/(1 + r) +
     q*RotateLeft[c[x + 1], 1]/(1 + r), -1];
c[0][[1]]
```

Exercise 1.2.9 *Consider a financial market with two periods, interest rate* $r = 0$, *and a single risky asset* S^1. *Suppose that* $S_0^1 = 1$ *and for* $n = 1, 2$, $S_n^1 = S_{n-1}^1 \xi_n$, *where the random variables* ξ_1, ξ_2 *are independent, and take two different values:* $2, \frac{3}{4}$, *with the same a probability. a) Is that a viable market? Is it complete? Find the price of a European option with maturity* $N = 2$ *and payoff* $\max_{0 \le n \le 2} S_n^1$. *Find the hedging portfolio of this option.*

Assume now that we have a second risky asset in this market with S_n^2 *such that* $S_0^2 = 1$ *and for* $n = 1, 2$

$$S_n^2 = S_{n-1}^2 \eta_n,$$

where the random variables η_n *take three different values* $2, 1, \frac{1}{2}$, η_1 *and* η_2 *are independent and*

$$\begin{aligned}
\mathbb{P}(\eta_n = 2 | \xi_n = 2) &= 1, \\
\mathbb{P}(\eta_n = \ 1 | \xi_n = \frac{3}{4}) &= \frac{1}{3}, \\
\mathbb{P}(\eta_n = \frac{1}{2} | \xi_n = \frac{3}{4}) &= \frac{2}{3},
\end{aligned}$$

in such a way that the vector (ξ_n, η_n) *takes only the values* $(2,2)$, $(\frac{3}{4},1)$, $(\frac{3}{4},\frac{1}{2})$ *with probabilities* $\frac{1}{2},\frac{1}{6},\frac{1}{3}$. *b) Prove that these two assets* S_n^1, S_n^2 *form a viable and complete market and calculate the neutral probability. Is it possible to know the value of the European option mentioned in a) without doing any calculation? Why?*

Exercise 1.2.10 *Prove that if* $X_n \xrightarrow{\mathcal{L}} X$, X *absolutely continuous, and* $a_n \to a \in \bar{R}$, *then* $\mathbb{P}\{X_n \le a_n\} \to \mathbb{P}\{X \le a\}$.

Exercise 1.2.11 *Let* $\{X_{nj},\ j = 1,...,k_n,\ n \geqq 1\}$, *where* $k_n \xrightarrow{n} \infty$, *a triangular system of centered and independent random variables, fixed* n, *with* $X_{nj} = O(k_n^{-1/2})$, *and such that* $\sum_{j=1}^{k_n} \mathbb{E}(X_{nj}^2) \to \sigma^2 > 0$, *prove that* $S_n = \sum_{j=1}^{k_n} X_{nj} \xrightarrow{\mathcal{L}} N(0,\sigma^2)$.

Exercise 1.2.12 *Assume now a sequence of CRR binomial models where the number of periods depends of* n *and such that*

$$\begin{aligned} 1+r(n) &= e^{\frac{rT}{n}}, \\ 1+b(n) &= e^{\sigma\sqrt{\frac{T}{n}}}, \\ 1+a(n) &= e^{-\sigma\sqrt{\frac{T}{n}}}, \end{aligned}$$

Prove that for n *big enough the markets are viable. Calculate the limit of the price of a call at the initial time when* $n \to \infty$.

Exercise 1.2.13 *Consider the analogous situation as in the previous exercise but with*

$$\begin{aligned} 1+b(n) &= e^{\tau}, \\ 1+a(n) &= e^{\lambda\frac{T}{n}}, \end{aligned}$$

where $\tau > 0$ *y* $0 < \lambda < r$.

Exercise 1.2.14 *We consider a market model as in the above sections. A* numéraire *is an adapted sequence* $Z = (Z_n)_{0\le n\le N}$ *s.t.* $Z_0 = 1, Z_n > 0$ *for* $n = 1,...,N$ *and* $Z_n = V_n(\varphi)$ *for some admissible strategy* φ $(n = 1,...,N)$. *Denote by* S^Z *the* Z*-discounted vector price process:* $S_n^Z = \frac{S_n}{Z_n}$, $n = 0,...,N$.
(1) *Prove that a predictable sequence* $\phi = (\phi_n)_{1\le n\le N}$, *with values in* $\mathbb{R}^{d+1}$, *is self-financing iff*

$$V_n^Z(\phi) := \frac{V_n(\phi)}{Z_n} = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^Z,\ \ n = 1,...,N.$$

(2) *Prove that*

$$\sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z = 0,\ \ n = 1,...,N.$$

(3) *Prove that for any predictable sequence* $\phi = (\phi_n)_{1\leq n\leq N}$, *there exists a self-financing strategy* $\hat{\phi}$ *such that*

$$\hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^Z, \; n = 1, ..., N.$$

(4) *Prove that the market is viable (free of arbitrage) iff there exists a probability* $\mathbb{P}^Z \sim \mathbb{P}$ *s.t.* S^Z *is a* $\mathbb{P}^Z$*-martingale and that in that case there is at most one deterministic numéraire.*
(5) *Assume a market is viable and complete and let* $\mathbb{P}^*$ *be the risk neutral probability, prove that*

$$\frac{\mathrm{d}\mathbb{P}^Z}{\mathrm{d}\mathbb{P}^*} = \frac{Z_N}{S_N^0}$$

and that the price of a payoff X *at time* n *is given by*

$$Z_n \mathbb{E}_{\mathbb{P}^Z}\left(\frac{X}{Z_N} \middle| \mathcal{F}_n \right).$$

1.2.5 American options

An American option is a derivative that can be exercised at *any* time between 0 and N, and consequently it has an associated payoff sequence defined by an $(\mathcal{F}_n)$-adapted positive process $(Z_n)_{0\leq n\leq N}$ to indicate the immediate payoff when it is exercised at time n. In case of an American Call $Z_n := (S_n - K)_+$ and in the case of an American Put $Z_n := (K - S_n)_+$. To obtain the price, U_n, at time n, we assume that the market is complete. If the payoff is Z_τ at the random time τ and this random time is the time when the option is exercised then its price at time $n \leq \tau$ is

$$U_n = (1+r)^n \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau | \mathcal{F}_n),$$

in fact we can always assume that the payoff is paid at time N by taking into account the interest rates, that is

$$Z_\tau (1+r)^{N-\tau},$$

then we can use the pricing formula for the European options

$$U_n = (1+r)^n \mathbb{E}_{\mathbb{P}^*}\left(\left. \frac{Z_\tau (1+r)^{N-\tau}}{(1+r)^N} \right| \mathcal{F}_n \right) = (1+r)^n \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau | \mathcal{F}_n).$$

If the option can be exercised at any time we will have that, to cover any potential payoff, the discounted price should be

$$\tilde{U}_n = \sup_{\tau \geq n} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau | \mathcal{F}_n).$$

In particular

$$U_0 = \sup_{\tau} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau).$$

However the random time τ will be an observed one when it takes a specific value, in other words $\{\tau = n\} \in \mathcal{F}_n$. By this reason we introduce the following definition.

Definition 1.2.8 *A random variable ν taking values in $\{0, 1, ..., N\}$ is a* stopping, *or Markov, time if*

$$\{\nu = n\} \in \mathcal{F}_n, \quad 0 \leq n \leq N$$

Remark 1.2.6 *Equivalently ν is a stopping time if $\{\nu \leq n\} \in \mathcal{F}_n, \quad 0 \leq n \leq N$, definition that can be extended to the continuous time setting. We write $\tau_{0,N}$ for the set of stopping times.*

Then the initial price of an American option with payoff $(Z_n)_{0\leq n\leq N}$ will be given by

$$U_0 = \sup_{\tau \in \tau_{0,N}} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau).$$

Then one goal will be to find $\nu \in \tau_{0,N}$ such that

$$\sup_{\tau \in \tau_{0,N}} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau) = \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\nu).$$

In order to characterize the *optimal stopping time* we need to introduce some definitions and to obtain some results.

Let $(X_n)_{0 \le n \le N}$ be an adapted stochastic process and ν a stopping time, then we define

$$X_n^\nu := X_{n \wedge \nu}, \; 0 \le n \le N.$$

Note that

$$X_n^\nu(\omega) = \begin{cases} X_n(\omega) & \text{si } n \le \nu(\omega) \\ X_{\nu(\omega)}(\omega) & \text{si } n > \nu(\omega) \end{cases}$$

Proposition 1.2.7 *Let $(X_n)_{0 \le n \le N}$ be adapted, then $(X_n^\nu)_{0 \le n \le N}$ is adapted and if $(X_n)_{0 \le n \le N}$ is a (sup, super) martingale, then $(X_n^\nu)_{0 \le n \le N}$ is a (sub, super) martingale.*

Proof.

$$\begin{aligned} X_n^\nu = X_{n \wedge \nu} &= X_0 + \sum_{j=1}^{n \wedge \nu} (X_j - X_{j-1}) \\ &= X_0 + \sum_{j=1}^{n} \mathbf{1}_{\{j \le \nu\}} (X_j - X_{j-1}), \end{aligned}$$

but $\{j \le \nu\} = \overline{\{\nu \le j-1\}} \in \mathcal{F}_{j-1}$, since ν is a stopping time, and consequently $\mathbf{1}_{\{j \le \nu\}}$ is $\mathcal{F}_{j-1}$-measurable and the sequence $(\phi_j)_{1 \le j \le N}$ con $\phi_j = \mathbf{1}_{\{j \le \nu\}}$ is predictable. Obviously X_n^ν es $\mathcal{F}_n$-measurable and

$$\begin{aligned} \mathbb{E}(X_{n+1}^\nu - X_n^\nu | \mathcal{F}_n) &= \mathbb{E}(\mathbf{1}_{\{n+1 \le \nu\}}(X_{n+1} - X_n) | \mathcal{F}_n) \\ &= \mathbf{1}_{\{n+1 \le \nu\}} \mathbb{E}(X_{n+1} - X_n | \mathcal{F}_n) \lesseqgtr 0 \text{ if } (X_n) \text{ is } \begin{array}{l} \text{super} \\ \text{martingala} \\ \text{sub} \end{array} \end{aligned}$$

■

The Snell envelope

Let $(Y_n)_{0 \le n \le N}$ be an adapted process (to $(\mathcal{F}_n)_{0 \le n \le N}$), define

$$\begin{aligned} X_N &= Y_N \\ X_n &= \max(Y_n, E(X_{n+1} | \mathcal{F}_n)), \quad 0 \le n \le N-1, \end{aligned}$$

we say that $(X_n)_{0 \le n \le N}$ is the *Snell envelope* of $(Y_n)_{0 \le n \le N}$.

Proposition 1.2.8 *The process $(X_n)_{0 \le n \le N}$ is the smallest supermartingale that dominates the process $(Y_n)_{0 \le n \le N}$.*

Proof. $(X_n)_{0\leq n\leq N}$ is adapted and by construction

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n.$$

Let $(T_n)_{0\leq n\leq N}$ be another supermartingale that dominates $(Y_n)_{0\leq n\leq N}$, then $T_N \geq Y_N = X_N$. Assume that $T_{n+1} \geq X_{n+1}$. Then, by the monotonicity of the expectation and since $(T_n)_{0\leq n\leq N}$ is a supermartingale:

$$T_n \geq \mathbb{E}(T_{n+1}|\mathcal{F}_n) \geq \mathbb{E}(X_{n+1}|\mathcal{F}_n),$$

moreover $(T_n)_{0\leq n\leq N}$ dominates $(Y_n)_{0\leq n\leq N}$, so $T_n \geq Y_n$

$$T_n \geq \max(Y_n, \mathbb{E}(X_{n+1}|\mathcal{F}_n)) = X_n$$

■

Remark 1.2.7 *Fixed ω if X_m is strictly greater than Y_m for all $m \leq n$, $X_m(\omega) = \mathbb{E}(X_{m+1}|\mathcal{F}_m)(\omega)$ so X_m behaves, until this n, as a martingale, this indicates that if we "stop" the supermartingale X properly we can obtain a martingale.*

Proposition 1.2.9 *The random variable*

$$\nu = \inf\{n \geq 0, X_n = Y_n\}$$

is a stopping time and $(X_n^\nu)_{0\leq n\leq N}$ is a martingale.

Proof.

$$\{\nu = n\} = \{X_0 > Y_0\} \cap ... \cap \{X_{n-1} > Y_{n-1}\} \cap \{X_n = Y_n\} \in \mathcal{F}_n.$$

And

$$X_n^\nu = X_0 + \sum_{j=1}^{n} \mathbf{1}_{\{j\leq\nu\}}(X_j - X_{j-1}).$$

Therefore

$$X_{n+1}^\nu - X_n^\nu = \mathbf{1}_{\{n+1\leq\nu\}}(X_{n+1} - X_n),$$

but on the set $\{n+1 \leq \nu\}$ we have that $X_n > Y_n$ so $X_n = \mathbb{E}(X_{n+1}|\mathcal{F}_n)$ on this set, and we can write

$$X_{n+1}^\nu - X_n^\nu = \mathbf{1}_{\{n+1\leq\nu\}}(X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n)).$$

Therefore

$$\begin{aligned}
\mathbb{E}(X_{n+1}^\nu - X_n^\nu|\mathcal{F}_n) &= \mathbb{E}(\,\mathbf{1}_{\{n+1\leq\nu\}}(X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n)\big|\,\mathcal{F}_n) \\
&= \mathbb{E}(\mathbf{1}_{\{n+1\leq\nu\}}(\mathbb{E}(X_{n+1}|\mathcal{F}_n) - \mathbb{E}(X_{n+1}|\mathcal{F}_n))) \\
&= 0.
\end{aligned}$$

■

The following corollary establishes that ν is an optimal stopping time.

Corollary 1.2.1

$$X_0 = \mathbb{E}(Y_\nu|\mathcal{F}_0) = \sup_{\tau\in\tau_{0,N}} \mathbb{E}(Y_\tau|\mathcal{F}_0)$$

Proof. $(X_n^\nu)_{0\le n\le N}$ is a martingale and consequently

$$\begin{aligned} X_0 &= \mathbb{E}(X_N^\nu|\mathcal{F}_0) = \mathbb{E}(X_{N\wedge\nu}|\mathcal{F}_0) \\ &= \mathbb{E}(X_\nu|\mathcal{F}_0) = \mathbb{E}(Y_\nu|\mathcal{F}_0). \end{aligned}$$

On the other hand $(X_n)_{0\le n\le N}$ is supermartingale and then $(X_n^\tau)_{0\le n\le N}$ as well for all $\tau\in\tau_{0,N}$, so

$$X_0 \ge \mathbb{E}(X_N^\tau|\mathcal{F}_0) = \mathbb{E}(X_\tau|\mathcal{F}_0) \ge \mathbb{E}(Y_\tau|\mathcal{F}_0),$$

therefore

$$\mathbb{E}(Y_\nu|\mathcal{F}_0) \ge \mathbb{E}(Y_\tau|\mathcal{F}_0), \quad \forall\tau\in\tau_{0,N}.$$

■

Remark 1.2.8 *Analogously we could prove*

$$X_n = \mathbb{E}(Y_{\nu_n}|\mathcal{F}_n) = \sup_{\tau\in\tau_{n,N}} \mathbb{E}(Y_\tau|\mathcal{F}_n),$$

where

$$\nu_n = \inf\{j\ge n, X_j = Y_j\}$$

and we denote $\tau_{n,N}$ the set of stopping times with values in $\{n, n+1, ..., N\}$. We say that ν_n are optimal stopping times*.*

Price of American options

Now we can apply these results to give a more explicit expression for the price of American options. In fact if we take as probability, the risk-neutral one, $\mathbb{P}^*$, the process $(Y_n)_{0\le n\le N}$ as $\left(\tilde{Z}_n\right)_{0\le n\le N}$, and

$$\nu_n := \inf\{j\ge n, X_j = \tilde{Z}_j\},$$

we will have that

$$X_n = \mathbb{E}_{\mathbb{P}^*}\left(\tilde{Z}_{\nu_n}|\mathcal{F}_n\right) = \sup_{\tau\in\tau_{n,N}} \mathbb{E}_{\mathbb{P}^*}\left(\tilde{Z}_\tau|\mathcal{F}_n\right),$$

and the discounted prices of the American options, that we write $\left(\tilde{U}_n\right)_{0\le n\le N}$, will be given by $(X_n)_{0\le n\le N}$, in other words, $\left(\tilde{U}_n\right)_{0\le n\le N}$ is the Snell envelope of the discounted payoffs $\left(\tilde{Z}_n\right)_{0\le n\le N}$, and we can write

$$\begin{cases} U_N = Z_N \\ U_n = \max(Z_n, S_n^0 \mathbb{E}_{\mathbb{P}^*}(\tilde{U}_{n+1}|\mathcal{F}_n)) \quad \text{if } 0\le n\le N-1. \end{cases}$$

This formula for the price can be obtained, as well, by doing a backward induction. Define $U_N = Z_N$. At time $N-1$, owners of the option can choose between receiving Z_{N-1} or the *equivalent* amount to Z_N at time $N-1$ that is the amount that replicates Z_N at $N-1$ given by $S^0_{N-1}\mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_N|\mathcal{F}_{N-1})$. Obviously they will choose the maximum of the two values, so we have

$$U_{N-1} = \max(Z_{N-1}, S^0_{N-1}\mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_N|\mathcal{F}_{N-1}))$$

and, by backward induction,

$$U_n = \max(Z_n, S^0_n\mathbb{E}_{\mathbb{P}^*}(\tilde{U}_{n+1}|\mathcal{F}_n)),\ 0 \leq n \leq N-1$$

Optimal stopping times

Now we want to characterize all the optimal stopping times. Let, as above, $(X_n)_{0\leq n\leq N}$ be the Snell envelope of $(Y_n)_{0\leq n\leq N}$.

Theorem 1.2.3 *ν is an optimal stopping time if and only if*

$$\begin{cases} X_\nu = Y_\nu \\ (X^\nu_n)_{0\leq n\leq N} \quad \textit{is a martingale} \end{cases}$$

Proof. We know that

$$\sup_{\tau\in\tau_{0,N}} \mathbb{E}(Y_\tau|\mathcal{F}_0) = X_0.$$

Then if $(X^\nu_n)_{0\leq n\leq N}$ is a martingale and $X_\nu = Y_\nu$

$$\begin{aligned}\sup_{\tau\in\tau_{0,N}} \mathbb{E}(Y_\tau|\mathcal{F}_0) = X_0 = X^\nu_0& \\ &= \mathbb{E}(X^\nu_N|\mathcal{F}_0) = \mathbb{E}(X_{N\wedge\nu}|\mathcal{F}_0) \\ &= \mathbb{E}(X_\nu|\mathcal{F}_0) = \mathbb{E}(Y_\nu|\mathcal{F}_0).\end{aligned}$$

Reciprocally, if ν is optimal

$$\sup_{\tau\in\tau_{0,N}} \mathbb{E}(Y_\tau|\mathcal{F}_0) = X_0 = \mathbb{E}(Y_\nu|\mathcal{F}_0) \leq \mathbb{E}(X_\nu|\mathcal{F}_0) = \mathbb{E}(X_{N\wedge\nu}|\mathcal{F}_0) \leq X_0, \quad (1.19)$$

where the last inequality is due to the fact that (X^ν_n) is a supermartingale. So, we have

$$\mathbb{E}(X_\nu - Y_\nu|\mathcal{F}_0) = 0$$

and since $X_\nu - Y_\nu \geq 0$, since X is the Snell envelope of Y, we conclude that $X_\nu = Y_\nu$.

Now we can also see that $(X^\nu_n)_{0\leq n\leq N}$ is a martingale. We know that it is a supermartingale, then

$$X_0 \geq \mathbb{E}(X^\nu_n|\mathcal{F}_0) \geq \mathbb{E}(X^\nu_N|\mathcal{F}_0) = \mathbb{E}(X_\nu|\mathcal{F}_0) = X_0$$

as we saw in (1.19). Then, for all n

$$\mathbb{E}(X_n^\nu - \mathbb{E}(X_\nu|\mathcal{F}_n)|\mathcal{F}_0) = 0,$$

and since $(X_n^\nu)_{0\leq n\leq N}$ is supermartingale,

$$X_n^\nu \geq \mathbb{E}(X_N^\nu|\mathcal{F}_n) = \mathbb{E}(X_\nu|\mathcal{F}_n)$$

therefore $X_n^\nu = E(X_\nu|\mathcal{F}_n)$. ∎

Decomposition of supermartingales (Doob's decomposition)

Proposition 1.2.10 *Any supermartingale $(X_n)_{0\leq n\leq N}$ has a unique decomposition:*

$$X_n = M_n - A_n, 0 \leq n \leq N,$$

where $(M_n)_{0\leq n\leq N}$ is a martingale and $(A_n)_{0\leq n\leq N}$ is non-decreasing predictable with $A_0 = 0$.

Proof. It is enough to write

$$X_n = \sum_{j=1}^{n}(X_j - \mathbb{E}(X_j|\mathcal{F}_{j-1})) - \sum_{j=1}^{n}(X_{j-1} - \mathbb{E}(X_j|\mathcal{F}_{j-1})) + X_0$$

and to identify

$$M_n = \sum_{j=1}^{n}(X_j - \mathbb{E}(X_j|\mathcal{F}_{j-1})) + X_0,$$
$$A_n = \sum_{j=1}^{n}(X_{j-1} - \mathbb{E}(X_j|\mathcal{F}_{j-1}))$$

where we define $M_0 = X_0$ and $A_0 = 0$. So $(M_n)_{0\leq n\leq N}$ is a martingale:

$$M_n - M_{n-1} = X_n - \mathbb{E}(X_n|\mathcal{F}_{n-1}), \quad 1 \leq n \leq N$$

in such a way that

$$\mathbb{E}(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0, \quad 1 \leq n \leq N.$$

Finally since $(X_n)_{0\leq n\leq N}$ is supermartingale

$$A_n - A_{n-1} = X_{n-1} - \mathbb{E}(X_n|\mathcal{F}_{n-1}) \geq 0, \quad 1 \leq n \leq N.$$

Now we can see the uniqueness. If

$$M_n - A_n = M_n' - A_n', \quad 0 \leq n \leq N$$

we have

$$M_n - M_n' = A_n - A_n', \quad 0 \leq n \leq N,$$

but then since $(M_n)_{0\leq n\leq N}$ y $(M'_n)_{0\leq n\leq N}$ are martingales and $(A_n)_{0\leq n\leq N}$ y $(A'_n)_{0\leq n\leq N}$ predictable, it turns out that

$$\begin{aligned} A_{n-1}-A'_{n-1} &= M_{n-1}-M'_{n-1} = E(M_n - M'_n|\mathcal{F}_{n-1}) \\ &= E(A_n - A'_n|\mathcal{F}_{n-1}) = A_n - A'_n, \quad 1\leq n\leq N, \end{aligned}$$

that is

$$A_N - A'_N = A_{N-1}-A'_{N-1} = ... = A_0 - A'_0 = 0,$$

since by hypothesis $A_0 = A'_0 = 0$. ■

This decomposition is known as the Doob decomposition.

Proposition 1.2.11 *The largest optimal stopping time for* $(Y_n)_{0\leq n\leq N}$ *is given by*

$$\nu_{\max} = \begin{cases} N & si\ A_N = 0 \\ \inf\{n, A_{n+1} > 0\} & si\ A_N > 0 \end{cases},$$

where $(X_n)_{0\leq n\leq N}$*, Snell envelope of* $(Y_n)_{0\leq n\leq N}$*, has a Doob decomposition* $X_n = M_n - A_n, 0\leq n\leq N$.

Proof. $\{\nu_{\max} = n\} = \{A_1 = 0, A_2 = 0, ..., A_n = 0, A_{n+1} > 0\} \in \mathcal{F}_n$, $0\leq n\leq N-1, \{\nu_{\max} = N\} = \{A_N = 0\}\in\mathcal{F}_{N-1}$. So, it is a stopping time.

$$X_n^{\nu_{\max}} = X_{n\wedge\nu_{\max}} = M_{n\wedge\nu_{\max}} - A_{n\wedge\nu_{\max}} = M_{n\wedge\nu_{\max}}$$

since $A_{n\wedge\nu_{\max}} = 0$. Therefore $(X_n^{\nu_{\max}})_{0\leq n\leq N}$ is a martingale. So, to see that this stopping time is optimal we have to prove that

$$X_{\nu_{\max}} = Y_{\nu_{\max}}$$

$$\begin{aligned} X_{\nu_{\max}} &= \sum_{j=0}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} X_j + \mathbf{1}_{\{\nu_{\max}=N\}} X_N \\ &= \sum_{j=0}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} \max(Y_j, \mathbb{E}(X_{j+1}|\mathcal{F}_j)) + \mathbf{1}_{\{\nu_{\max}=N\}} Y_N, \end{aligned}$$

but in $\{\nu_{\max} = j\}$, $A_j = 0, A_{j+1} > 0$ so

$$\mathbb{E}(X_{j+1}|\mathcal{F}_j) = \mathbb{E}(M_{j+1}|\mathcal{F}_j) - A_{j+1} < \mathbb{E}(M_{j+1}|\mathcal{F}_j) = M_j = X_j$$

therefore $X_j = Y_j$ en $\{\nu_{\max} = j\}$ and consequently $X_{\nu_{\max}} = Y_{\nu_{\max}}$. Finally we see that is the largest optimal stopping time. Let $\tau \geq \nu_{\max}$ and $\mathbb{P}\{\tau > \nu_{\max}\} > 0$. Then

$$\begin{aligned} \mathbb{E}(X_{\tau\wedge N}) &= \mathbb{E}(X_\tau) = \mathbb{E}(M_\tau) - \mathbb{E}(A_\tau) = M_0 - \mathbb{E}(A_\tau) \\ &= X_0 - \mathbb{E}(A_\tau) < X_0, \end{aligned}$$

since $\{\tau > \nu_{\max}\} = \{A_\tau > 0\}$, so $(X_n^\tau)_{0\leq n\leq N}$ cannot be a martingale. ■

Remark 1.2.9 *The Doob decomposition of the Snell envelope, X, of Y is given by*

$$X_n = M_n - A_n,$$

with

$$M_n = X_0 + \sum_{i=1}^{n} \left(X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1}) \right),$$
$$A_n = \sum_{i=1}^{n} \left(X_{i-1} - \mathbb{E}(X_i|\mathcal{F}_{i-1}) \right),$$

then,

$$\begin{aligned} \nu_{\max} &= \inf\{n, A_{n+1} > 0\} = \inf\{n, \mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n < 0\} \\ &= \inf\{n, \mathbb{E}(X_{n+1}|\mathcal{F}_n) - Y_n < 0\}. \end{aligned}$$

Hedge of American options

By the previous results, if U_n denotes the price of an American option and $\tilde{U}_n$ its discount value, we know that we can decompose

$$\tilde{U}_n = M_n - A_n$$

where $(M_n)_{0\le n\le N}$ is a positive $\mathbb{P}^*$-martingale and $(A_n)_{0\le n\le N}$ is a non-decreasing and predictable process with $A_0 = 0$. If we receive the amount U_0 we can built a self-financing portfolio replicating $(1+r)^N M_N$. In fact, since the market is complete, any positive payoff can be replicated, so there will exist ϕ such that

$$V_N(\phi) = (1+r)^N M_N$$

with $V_0(\phi) = M_0 = U_0$ or what is the same

$$\tilde{V}_N(\phi) = M_N, \; V_0(\phi) = U_0$$

but $(\tilde{V}_n(\phi))_{0\le n\le N}$ and $(M_n)_{0\le n\le N}$ are $\mathbb{P}^*$-martingales in such a way that $\tilde{V}_n(\phi) = M_n, 0 \le n \le N$. Note that, consequently, we have

$$U_n = (1+r)^n M_n - (1+r)^n A_n = V_n(\phi) - (1+r)^n A_n$$

and therefore

$$V_n(\phi) = U_n + (1+r)^n A_n \ge U_n. \tag{1.20}$$

In other words with the money we receive we can *super*-hedge the derivative.

Optimal exercise of the American option

Let, as before, $(Z_n)_{0\leq n\leq N}$ the payoffs of the American option. It seems natural to exercise the option at a stopping time ν that gives the optimal value to a contract with payoff Z_ν in such a way that ν is an optimal stopping time as defined above. That is, a time ν, such that

$$U_0 = \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\nu) = \sup_{\tau\in\tau_{0,N}} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau).$$

In fact if ν is such that $U_{v(\omega)}(\omega) > Z_{v(\omega)}(\omega)$ it is not worth to exercise the option since the value of the contract $U_{v(\omega)}(\omega)$ is greater than what you obtain if you exercise it: $Z_{v(\omega)}(\omega)$. Note that it would be better to sell the contract. So, you will look for ν such that $U_v = Z_v$. On the other hand you will look as well for $A_n = 0$, for all $1 \leq n \leq \nu$, (or equivalently $A_\nu = 0$) otherwise, according with (1.20) it is better to exercice it before and to built a portfolio with the strategy ϕ:

$$V_\nu(\phi) - Z_\nu = (1+r)^\nu A_\nu > 0.$$

So ν the optimal time to exercice the option is *an optimal stopping time* as defined above.

Note finally that if $U_\nu > Z_\nu$ or $A_\tau > 0$ and the seller has invested the prime U_0 to construct a portfolio with the strategy ϕ, to hedge the option, seller's profit will be

$$V_\nu(\phi) - Z_\nu = U_\nu - Z_\nu + (1+r)^\nu A_\nu > 0.$$

Example 1.2.3 *Here it is shown how to calculate the premium of an American put option with maturity of 3 months on stocks whose current value is 60€, the strike price is also 60€ (*at the money: ATM*) , the annual interest rate is 10% and the annual volatility 45%. We assume a CRR model with 12 periods. It is also analyzed in which nodes is convenient to exercise the option.*

```
Clear[s, pa, vc, vi];
T = 1/4;n =12;so = 60;K = 60;vol = 0.45;ra = 0.10;
r = ra*T/n;b =vol*Sqrt[T/n];a=-b;
p = (r - a)/(b - a);
q = 1 - p;
pp[x_] := Max[x, 0]
s[0] = Table[so, {1}];
s[x_] := s[x] = Prepend[(1 + a)*s[x - 1], (1 + b)*s[x - 1][[1]]];
ColumnForm[Table[s[i], {i, 0, n}], Center]
pa[n] = Map[pp, K - s[n]];
pa[x_] := pa[x] = K - s[
x] + Map[pp, Drop[p*pa[x + 1]/(1 + r) + q*RotateLeft[pa[x + 1],
1]/(1 + r), -1] - K + s[x]]
ColumnForm[Table[pa[i], {i, 0, n}], Center]
vc[n] = Map[pp, K - s[n]];
vc[x_] := Drop[p*pa[x + 1]/(1 + r) + q*RotateLeft[pa[x +
```

```
1], 1]/(1 + r), -1]
vi[i_] := Map[pp, K - s[i]]
ColumnForm[Table[vc[i] - vi[i], {i, 0, n}], Center]
ColumnForm[Table[pa[i] - vi[i], {i, 0, n}], Center]
```

Exercise 1.2.15 *Obtain the following bounds for the call prices (C) and for the put ones (P) European (E) and American (A):*

$$\max(S_n - K, 0) \leq C_n(E) \leq C_n(A);$$
$$\max(0, (1+r)^{-(N-n)}K - S_n) \leq P_n(E) \leq (1+r)^{-(N-n)}K$$

Exercise 1.2.16 *Consider a viable and complete market with N periods of trading. Show that, with the usual notations,*

$$\sup_{\tau,\ stopping\ time} \mathbb{E}_{\mathbb{Q}}\left(\frac{(S_\tau - K)_+}{(1+r)^\tau}\right) = \mathbb{E}_{\mathbb{Q}}\left(\frac{(S_N - K)_+}{(1+r)^N}\right)$$

where $\mathbb{Q}$ is the risk neutral probability.

Exercise 1.2.17 *Let $\{C_n^E\}_{n=0}^N$ be the price of a European option with payoff Z_N and let $\{Z_n\}_{n=0}^N$ be the payoffs of an American option. Demonstrate that if $C_n^E \geq Z_n, n = 0, 1, ..., N-1$, then $\{C_n^A\}_{n=0}^N$ (the prices of the American option) coincide with $\{C_n^E\}_{n=0}^N$.*

Exercise 1.2.18 *Consider a time discrete market with N periods, a bank account with zero interest rate and a risky asset S. In such a market we want to price an American option with payoffs equal to a constant amount $d > 0$ if the trading time $n \leq N-1$ and $X = S_N$ if $n = N$. Prove that its price is equal to that of a European call option with strike d and maturity time $N-1$ plus the fixed amount d.*

Exercise 1.2.19 *Let $X_n = \xi_1 + \xi_2 + ... + \xi_n, n \geq 1$, where the ξ_i are i.i.d. such that $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = 1/2$. Find the Doob decomposition of $|X|$.*

1.2.6 Annex

Theorem 1.2.4 *(Separating Hyperplane Theorem) Let L a subspace of $\mathbb{R}^n$ and K a convex and compact subset of $\mathbb{R}^n$ without intersection with L. Then there exists a linear functional $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi(x) = 0$ for all $x \in L$ and $\phi(x) > 0$ for all $x \in K$.*

The proof is based in the following lemma:

Lemma 1.2.1 *Let C be a closed convex set of $\mathbb{R}^n$ not containing the origin, then there exists $\phi : \mathbb{R}^n \to \mathbb{R}$, linear, such that $\phi(x) > 0$ for all $x \in C$.*

Proof. *Let $B(0,r)$ a ball of radius r and centered at the origin, take r sufficiently big in such a way that $B(0,r)\cap C\neq\phi$. The map*

$$B(0,r)\cap C\to\mathbb{R}_+$$
$$x\longmapsto ||x||=\left(\sum_{i=1}^n x_i^2\right)^{1/2}$$

is continuous and since it is defined in a compact set there will exist $z\in B(0,r)\cap C$ such that $||z||=\inf_{x\in B(0,r)\cap C}||x||$ and it satisfies $||z||>0$ since C does not contain the origin. Let $x\in C$, since C is convex $\lambda x+(1-\lambda)z\in C$ for all $0\leq\lambda\leq 1$. It is obvious that

$$\|\lambda x+(1-\lambda)z\|\geq ||z||>0,$$

then

$$\lambda^2 x\cdot x+2\lambda(1-\lambda)x\cdot z+(1-\lambda)^2 z\cdot z\geq z\cdot z,$$

equivalently

$$\lambda^2(x\cdot x+z\cdot z)+2\lambda(1-\lambda)x\cdot z-2\lambda z\cdot z\geq 0.$$

Take $\lambda>0$, then

$$\lambda(x\cdot x+z\cdot z)+2(1-\lambda)x\cdot z\geq 2z\cdot z$$

and taking the limit when $\lambda\to 0$ we have

$$x\cdot z\geq z\cdot z>0.$$

Then it is enough to take $\phi\ (x)=x\cdot z$. ■

Proof. *(of the Theorem) $K-L=\{u\in\mathbb{R}^n,u=k-l,k\in K,l\in L\}$ is closed and convex. In fact, let $0\leq\lambda\leq 1$ and $u,\tilde{u}\in K-L$*

$$\begin{aligned}\lambda u+(1-\lambda)\tilde{u}&=\lambda k+(1-\lambda)\tilde{k}-(\lambda l+(1-\lambda)\tilde{l})\\&=\bar{k}-\bar{l}\end{aligned}$$

where $\bar{k}\in K$ (by convexity of K) and $\bar{l}\in L$ (since it is a vectorial space), then it is convex. Furthermore, if we take a sequence $(u_n)\in K-L$ converging to u, we have that $u_n=k_n-l_n$ with $k_n\in K,l_n\in L$, that is $l_n=k_n-u_n$. But since K is compact, there exists a subsequence k_{n_r} that converges to a certain $k\in K$, so l_{n_r} will converge to $k-u$, and since l_{n_r} is a convergent sequence in a closed vectorial space ($\mathbb{R}^d$ is closed for all d) we will have $k-u=l\in L$, in such a way that $u=k-l\in K-L$. Now $K-L$ does not contain the origin and by the previous proposition there exists ϕ linear such that

$$\phi(k)-\phi(l)>0,\ \textit{para todo } k\in K\ \textit{y todo } l\in L.$$

Moreover, since L is a vectorial space $\phi(l)$ has to be zero. In fact if we assume for instance that $\phi(l)>0$, then $\lambda l\in L$ for all $\lambda>0$ arbitrary big and we will have that

$$\phi(k)>\lambda\phi(l),$$

but this is impossible if $\phi(k)$ es finite. Finally, since $\phi(l)=0$ we have that $\phi(k)>0$ for all $k\in K$. ■

Chapter 2

Continuous-time models for stock markets

2.1 Introduction

Now we are going to consider continuous-time models and even though the basic ideas of arbitrage, completeness, etc, are the same, the technical aspects are more delicate.

The main reason to consider such models is that it is not necessary to fix the time between trades, the models are more realistic and we can get close formulas for pricing derivatives as well. It was Louis Bachelier in 1900 with his "Théorie de la spéculation" the first in considering the Brownian motion (as an "infinitesimal" random walk) to describe stock prices and in obtaining formulas to price options. However his work was not understood at that time and consequently undervalued. Here we shall consider models based on Brownian motions. Contrarily to the first chapter we shall not study general results for continuous-time models but we will concentrate on some specific models and we will see if these models are free of arbitrage and complete and, if so, we shall calculate the price and the hedging portfolios.The first problem is to know in which extent we can use the definitions of arbitrage, self-financing portfolio, completeness and so on for continuous time models. To see all these stuff we need some new definitions and mathematical tools.

2.2 Continuous-time stochastic processes

Definition 2.2.1 *A stochastic process is a family of real random variables* $(X_t)_{t\in\mathbb{R}_+}$ *defined in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 2.2.1 *Usually, index* t *indicates time and it takes values between* 0 *and* T.

Remark 2.2.2 *A stochastic process can be also seen as a random map: for all $\omega \in \Omega$ we can associate the map from $\mathbb{R}_+$ to $\mathbb{R}$: $t \longmapsto X_t(\omega)$ named* trajectory *of the process. If the trajectories are continuous then the process is said to be* continuous.

Remark 2.2.3 *We shall assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is* complete *in the sense that $\mathcal{F}$ contains all* negligible sets*, it means that any subset, say C, of an element $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ is also in $\mathcal{F}$.*

Definition 2.2.2 *A filtration $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of sub-σ-fields of $\mathcal{F}$ and we say that $(X_t)_{t\geq 0}$ is an* adapted *process if for all t, X_t es $\mathcal{F}_t$-measurable.*

Definition 2.2.3 *We say that a process Y is a version of X if $\mathbb{P}(X_t = Y_t) = 1$ for all t.*

Remark 2.2.4 *We shall work, as well, with filtrations satisfying the property*

$$\text{If } A \in \mathcal{F} \text{ with } \mathbb{P}(A) = 0 \text{ then } A \in \mathcal{F}_t \text{ for all } t.$$

That is $\mathcal{F}_0$ contents all the $\mathbb{P}$-null sets of $\mathcal{F}$. Then if a process $(X_t)_{t\geq 0}$ is adapted and $(Y_t)_{t\geq 0}$ is a version of it then $(Y_t)_{t\geq 0}$ is adapted.

Remark 2.2.5 *We can built the filtration generated by a process $(X_t)_{t\geq 0}$ and write $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$. In general this filtration does not satisfy the previous condition and we shall substitute for $\mathcal{F}_t$ by $\bar{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{N}$ where $\mathcal{N}$ is the collection of null sets of $\mathcal{F}$. We call it the natural filtration generated by $(X_t)_{t\geq 0}$.*

Remark 2.2.6 *Moreover a stochastic process, say X, can be also described as a map from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}$. We shall assume that in $\mathbb{R}_+ \times \Omega$ we have the σ-field $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ and that the map is measurable (*measurable process*). This condition is a bit stronger that the condition of being simply a process. Nevertheless if the process X has continuous trajectories on the left or right sides, then it is measurable. Moreover if X is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and has continuous trajectories on the left or right sides then it is* progressively measurable*, it means that the map*

$$\begin{aligned} [0,t] \times \Omega &\to \mathbb{R} \\ (s, \omega) &\longmapsto X_s(\omega), \end{aligned}$$

is measurable for all $t > 0$, where in $[0,t]\times\Omega$ we consider the σ-field $\mathcal{B}([0,t])\otimes\mathcal{F}_t$.

2.2.1 Brownian motion

The Brownian motion describes the random movement that is possible to observe in some microscopic particles in a fluid mean (for instance pollen in a water drop). This name is due to the botanist Robert Brown who first observed this phenomenon en 1828. The zigzagging of these particles is due to the fact that they are buffeted by the molecules of the fluid in an intense way depending of the temperature of the fluid. The mathematical description of this phenomenon was deduced from Albert Einstein's work on kinetic theory in 1905, finding that the distribution of position of a Brownian particle at time t, starting at x at time 0, was of the form $N(x, \sigma^2 t)$ for a constant $\sigma^2 > 0$ depending on parameters of the particle and the liquid but independent of events before time 0. Previously, in 1900, as we mentioned at the Introduction, Louis Bachelier used a process that approximates a Brownian motion to model the movement of stock prices in a financial market.

Some years later Norbert Wiener, in 1923, gave a characterization of the Brownian motion as an stochastic process, (X_s), proving that $|X_t - X_s|(\omega) < M(\omega)|t-s|^\alpha$ for all $\alpha < \frac{1}{2}$. The name of Wiener process is also used to name the Brownian motion. We consider the one-dimensional case.

Definition 2.2.4 *We say that if for all $0 \le t_1 \le ... \le t_n$, $X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent random variables.*

Remark 2.2.7 *Equivalenty we can say that $(X_t)_{t\ge 0}$ is a process with independent increments if for all $0 \le s \le t$, $X_t - X_s$ is independent of $\sigma(X_u, 0 \le u \le s)$.*

Proposition 2.2.1 *If $(X_t)_{t\ge 0}$ is a continuous process with independent increments and $0 = t_{0n} \le t_{1n} \le ... \le t_{nn} \le t$ is a sequnce of partitions of $[0,t]$ with $\lim_{n\to\infty} \sup |t_{in} - t_{i-1,n}| = 0$, then for all $\varepsilon > 0$*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}\{|X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\} = 0.$$

Proof. For all $\varepsilon > 0$, by a.s. uniform continuity in $[0,t]$,

$$\lim_{n\to\infty} \mathbb{P}\left\{\sup_i |X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\right\} = 0,$$

and

$$\begin{aligned}
\mathbb{P}\left\{\sup_i |X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\right\} &= 1 - \prod_{i=1}^{n} \mathbb{P}\left\{|X_{t_{in}} - X_{t_{i-1,n}}| \le \varepsilon\right\} \\
&= 1 - \prod_{i=1}^{n} \left(1 - \mathbb{P}\{|X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\}\right) \\
&\ge 1 - \exp\left\{-\sum_{i=1}^{n} \mathbb{P}\{|X_{t_{in}} - X_{t_{i-1,n}}| > \varepsilon\}\right\} \ge 0.
\end{aligned}$$

■

Definition 2.2.5 *A Brownian motion is a continuous process with independent and stationary increments. That is:*

$\mathbb{P}$-c.s $s \longmapsto X_s(\omega)$ is continuous.
$s \le t$, $X_t - X_s$ is independent of $\mathcal{F}_s = \sigma(X_u, 0 \le u \le s)$.
$s \le t$, $X_t - X_s \sim X_{t-s} - X_0$.

We deduce that the law of $X_t - X_0$ is Gaussian:

Theorem 2.2.1 *If* $(X_t)_{t\geq 0}$ *is a Brownian motion then*

$$X_t - X_0 \sim N(rt, \sigma^2 t).$$

The proof is based in the following proposition.

Proposition 2.2.2 *Let* $\{Y_{kn}, k = 1, ..., n\}$ *independent random variables such that* $|Y_{kn}| \leq \varepsilon_n$ *with* $\varepsilon_n \downarrow 0$. *Then if* $\liminf \mathrm{Var}\left(\sum_{k=1}^n Y_{kn}\right) > 0$

$$\frac{\sum_{k=1}^n Y_{kn} - \mathbb{E}\left(\sum_{k=1}^n Y_{kn}\right)}{\sqrt{\mathrm{Var}\left(\sum_{k=1}^n Y_{kn}\right)}} \xrightarrow{\mathcal{L}} N(0,1)$$

Proof. Denote $X_{kn} := Y_{kn} - \mathbb{E}(Y_{kn})$ and $v_n^2 := \mathrm{Var}\left(\sum_{k=1}^n Y_{kn}\right)$

$$\begin{aligned}
&\log \mathbb{E}\left(\exp\left\{it\frac{1}{v_n}\sum_{k=1}^n X_{kn}\right\}\right) \\
&= \log\left(\prod_{i=1}^n \mathbb{E}\left(\exp\left\{it\frac{X_{kn}}{v_n}\right\}\right)\right) = \sum_{i=1}^n \log \mathbb{E}\left(\exp\left\{it\frac{X_{kn}}{v_n}\right\}\right) \\
&= -\frac{1}{2}t^2\frac{\sum_{k=1}^n \mathbb{E}(X_{kn}^2)}{v_n^2} - \frac{i}{3!}t^3\frac{\sum_{k=1}^n \mathbb{E}(X_{kn}^3)}{v_n^3} + ... \\
&= -\frac{1}{2}t^2 + O\left(\frac{t\varepsilon_n}{v_n}\right),
\end{aligned}$$

ya que

$$\left|\frac{\sum_{k=1}^n \mathbb{E}\left(X_{kn}^3\right)}{v_n^3}\right| \leq \left|\frac{\varepsilon_n \sum_{k=1}^n \mathbb{E}(X_{kn}^2)}{v_n^3}\right|.$$

■

Proof. (of the Theorem 2.2.1) Given the partition $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{nn} \leq t$ define

$$Y_{nk} = (X_{t_{kn}} - X_{t_{k-1,n}})\mathbf{1}_{\{|X_{t_{kn}} - X_{t_{k-1,n}}| \leq \varepsilon_n\}},$$

then, by the Proposition 2.2.1, we can find a sequence $\varepsilon_n \downarrow 0$ such that

$$\mathbb{P}\left(X_t - X_0 \neq \sum_{k=1}^n Y_{nk}\right) \leq \sum_{k=1}^n \mathbb{P}\left(|X_{t_{kn}} - X_{t_{k-1,n}}| > \varepsilon_n\right) \overset{n\to\infty}{\to} 0.$$

Then $\sum_{k=1}^n Y_{nk} \xrightarrow{\mathbb{P}} X_t - X_0$. Also we have, by Proposition 2.2.2, that if $\liminf \mathrm{Var}\left(\sum_{k=1}^n Y_{kn}\right) > 0$,

$$\frac{\sum_{k=1}^n Y_{kn} - \mathbb{E}\left(\sum_{k=1}^n Y_{kn}\right)}{\sqrt{\mathrm{Var}\left(\sum_{k=1}^n Y_{kn}\right)}} \xrightarrow{\mathcal{L}} N(0,1),$$

so $X_t - X_0$ is normally distributed or a constant (notice that if $\liminf \mathrm{Var}\left(\sum_{k=1}^{n} Y_{kn}\right) = 0$ then $\sum_{k=1}^{n_r} Y_{kn} - \mathbb{E}\left(\sum_{k=1}^{n_r} Y_{kn}\right) \xrightarrow{\mathbb{P}} 0$ for a certain subsequence). Consequently, the law of any increment is normal, and if we take r and σ^2 such that $X_1 - X_0 \sim N(r, \sigma^2)$, from the independence, homogeneity and continuity we have that $X_t - X_0 \sim N(rt, \sigma^2 t)$:

$$X_1 - X_0 = \sum_{i=1}^{p} (X_{i/p} - X_{(i-1)/p})$$

then $X_{1/p} - X_0 \sim N(r/p, \sigma^2/p)$, analogously $X_{q/p} - X_0 \sim N(qr/p, q\sigma^2/p)$. Now we can approximate any value of t by rational numbers and to apply the continuity of X. ■

Definition 2.2.6 *We say that a Brownian motion is standard if $X_0 = 0$ $\mathbb{P}$ a.s. $r = 0$ and $\sigma^2 = 1$. We shall always assume that it is standard.*

Proposition 2.2.3 *If $(X_t)_{t\geq 0}$ is a Brownian motion then* $\mathrm{Cov}(X_t, X_s) = s \wedge t$.

Proof. Fix $t > s$. $\mathrm{Var}(X_t - X_s) = \mathrm{Var}(X_t) + \mathrm{Var}(X_s) - 2\mathrm{Cov}(X_t, X_s)$. That is $t - s = t + s - 2\mathrm{Cov}(X_t, X_s)$. ■

2.2.2 Continuous-time Martingales

Definition 2.2.7 *Let $(M_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted process, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}(|M_t|) < \infty$, then it is:*

- *a martingale if $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, for all $s \leq t$*
- *a submartingale if $\mathbb{E}(M_t|\mathcal{F}_s) \geq M_s$, for all $s \leq t$*
- *a supermartingale if $\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s$, for all $s \leq t$.*

In the previous definition equalities and inequalities are *almost surely* $\mathbb{P}$. We assume that our martingales (sub, super) are càdlàg.

Definition 2.2.8 *A* stopping time *with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a random variable*

$$\tau : \Omega \to [0, \infty]$$

such that for all $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$.

Theorem 2.2.2 *If τ is a bounded stopping time and $(M_t)_{t\geq 0}$ is a martingale, then $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.*

Proof. *This is a corollary of the Optional Sampling Theorem. See ? (?) page 19.* ■

Definition 2.2.9 *Let $(M_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted process, we say that $(M_t)_{t\geq 0}$ is a* local *martingale, if it exists an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ with $\tau_n \uparrow \infty$, such that, fixed n, $(M_{t\wedge\tau_n})_{t\geq 0}$ is a martingale, for all $n \geq 0$.*

Proposition 2.2.4 *Let $(M_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-adapted process, set $s \leq t$ and $A \in \mathcal{F}_s$. Then: (i) $\tau_{ts} = t\mathbf{1}_{A^c} + s\mathbf{1}_A$ is an $(\mathcal{F}_t)_{t\geq 0}$ stopping time; (ii) If $\mathbb{E}(M_{\tau_{ts}}) = \mathbb{E}(M_0)$ for all $0 \leq s \leq t$ then $(M_t)_{t\geq 0}$ is a martingale.*

Proof. (i) Let $s \leq u < t$, then $\{\tau_{ts} \leq u\} = A \in \mathcal{F}_s \subseteq \mathcal{F}_u$. Otherwise $\{\tau_{ts} \leq u\}$ is ϕ or Ω. (ii) $\mathbb{E}(M_{\tau_{ts}}) = \mathbb{E}(M_t\mathbf{1}_{A^c}) + \mathbb{E}(M_s\mathbf{1}_A) = \mathbb{E}(M_{\tau_{tt}}) = \mathbb{E}(M_t) = \mathbb{E}(M_t\mathbf{1}_{A^c}) + \mathbb{E}(M_t\mathbf{1}_A)$. Therefore $\mathbb{E}(M_t\mathbf{1}_A) = \mathbb{E}(M_s\mathbf{1}_A)$ for all $A \in \mathcal{F}_s$. ■

Corollary 2.2.1 *If τ is a stopping time and and $(M_t)_{t\geq 0}$ is a martingale, then $(M_{t\wedge\tau})_{t\geq 0}$ is a martingale.*

Proof. By using the same notation as in the previous theorem $\tau_{ts} \wedge \tau$ is a bounded stopping time. So by Theorem 2.2.2 $\mathbb{E}(M_{\tau_{ts}\wedge\tau}) = \mathbb{E}(M_0) = \mathbb{E}(\tilde{M}_{\tau_{ts}})$, with $\tilde{M}_t := M_{t\wedge\tau}$and we can apply the previous proposition. ■

Exercise 2.2.1 *Let $(M_t)_{t\geq 0}$ be a local martingale and τ a stopping time then prove that $(M_{t\wedge\tau})_{t\geq 0}$ is a local martingale.*

Exercise 2.2.2 *Show that a local martingale $(M_t)_{0\leq t\leq T}$ such that $\mathbb{E}(\sup_{0\leq t\leq T}|M_t|) < \infty$ is in fact a martingale in $[0, T]$.*

Proof. Let $(\tau_n)_{n\geq 0}$ with $\tau_n \uparrow \infty$ be the sequence of stopping times such that $(M_{t\wedge\tau_n})_{0\leq t\leq T}$ is, for all fixed n, a martingale. Then, for all $s \leq t$

$$\mathbb{E}(\, M_{t\wedge\tau_n} | \, \mathcal{F}_s) = M_{s\wedge\tau_n}$$

and

$$\lim_{n\to\infty} \mathbb{E}(\, M_{t\wedge\tau_n} | \, \mathcal{F}_s) = \lim_{n\to\infty} M_{s\wedge\tau_n} = M_s.$$

Now , since $\mathbb{E}(\sup_{0\leq t\leq T}|M_t|) < \infty$ we can apply the dominated convergence theorem. ■

Exercise 2.2.3 *Let $(M_t)_{t\geq 0}$ be a positive continuous local martingale demonstrate that*

$$\mathbb{E}\,(M_0) - \mathbb{E}(M_T) = \lim_{n\to\infty} n\mathbb{P}(\sup_{0\leq t\leq T} M_t > n).$$

Solution 2.2.1 *Set $\tau_n = \inf\{0 \leq t \leq T, M_t = n\}$. By the previous exercise $(M_{t\wedge\tau_n})_{0\leq t\leq T}$ is a martingale, then*

$$\mathbb{E}\,(M_0) = \mathbb{E}(M_{T\wedge\tau_n}) = \mathbb{E}(M_T\mathbf{1}_{\{\tau_n\geq T\}}) + \mathbb{E}(M_{\tau_n}\mathbf{1}_{\{\tau_n<T\}}),$$

so

$$\mathbb{E}\,(M_0) - \mathbb{E}(M_T\mathbf{1}_{\{\tau_n\geq T\}} = \mathbb{E}(M_{\tau_n}\mathbf{1}_{\{\tau_n<T\}}) = n\mathbb{P}(\tau_n < T),$$

and

$$\mathbb{E}\,(M_0) - \lim_{n\to\infty} \mathbb{E}(M_T\mathbf{1}_{\{\tau_n\geq T\}} = \lim_{n\to\infty} n\mathbb{P}(\tau_n < T).$$

The result follows by the dominated convergence theorem and the definition of τ_n.

Exercise 2.2.4 *Let $(M_k)_{0\leq k\leq N}$ be a local martingale in a discrete time setting. Show that it is in fact a martingale.*

Proposition 2.2.5 *If (X_t) is a Brownian motion then:*

- (X_t) is a martingale.
- $(X_t^2 - t)$ is a martingale.
- $(\exp(\sigma X_t - \frac{\sigma^2}{2}t))$ is an martingale.

Proof. We take $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$

$$\begin{aligned}
\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(X_t - X_s + X_s|\mathcal{F}_s) \\
&= \mathbb{E}(X_t - X_s|\mathcal{F}_s) + X_s \\
&= \mathbb{E}(X_t - X_s) + X_s = X_s,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X_t^2 - t|\mathcal{F}_s) &= \mathbb{E}((X_t - X_s + X_s)^2|\mathcal{F}_s) - t \\
&= \mathbb{E}((X_t - X_s)^2 + X_s^2 + 2(X_t - X_s)|\mathcal{F}_s) - t \\
&= t - s + X_s^2 - t \\
&= X_s^2 - s,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(\exp(\sigma X_t - \frac{\sigma^2}{2}t)|\mathcal{F}_s) &= \exp(\sigma X_s - \frac{\sigma^2}{2}t)\mathbb{E}(\exp(\sigma(X_t - X_s))|\mathcal{F}_s) \\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}t)\mathbb{E}(\exp(\sigma(X_t - X_s)) \\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}t)\exp(\frac{\sigma^2}{2}(t-s)) \text{ (since } X_t - X_s \sim N(0, t-s)) \\
&= \exp(\sigma X_s - \frac{\sigma^2}{2}s)
\end{aligned}$$

■

Exercise 2.2.5 *Prove that the following stochastic processes, defined from a Brownian motion B, are martingales, respect to $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$,*

$$\begin{aligned}
X_t &= t^2 B_t - 2\int_0^t sB_s ds \\
X_t &= e^{t/2}\cos B_t \\
X_t &= e^{t/2}\sin B_t \\
X_t &= (B_t + t)\exp(-B_t - \frac{1}{2}t) \\
X_t &= B_t^1 B_t^2.
\end{aligned}$$

In the last case B_t^1 y B_t^2 are two independent Brownian motions and $\mathcal{F}_t = \sigma(B_s^1, B_s^2, 0 \leq s \leq t)$.

2.3 Self-financing strategies in continuous time

Consider a financial market with horizon T and with two assets, one without risk, S^0 (or bank account), that evolves as:

$$dS_t^0 = rS_t^0 dt, \quad 0 \leq t \leq T, \quad S_0^0 = 1$$

where r is a non-negative constant, that is

$$S_t^0 = e^{rt}, \quad t \geq 0,$$

and a risky stock $(S_t)_{0\leq t\leq T}$ that is a continuous-time stochastic process defined in a probability space $(\Omega, \mathcal{F}, P)$ adapted to a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ representing the flow of public information.

Definition 2.3.1 *A process $(\phi_t)_{0\leq t\leq T}$ is said to be a simple predictable process, w.r.t. $(\mathcal{F}_t)_{0\leq t\leq T}$, if it can be written as*

$$\phi(t) = \phi_0 \mathbf{1}_{\{t=0\}}(t) + \sum_{i=1}^{m(n)} h_{t_{i-1,n}}^n \mathbf{1}_{(t_{i-1,n}, t_{in}]}(t),$$

where $0 = t_{0n} < t_{1n} < ... < t_{m(n)n} = T$ and $h_{t_{i-1,n}}^n$ is $\mathcal{F}_{t_{i-1,n}}$-measurable and bounded and $m(n) \uparrow \infty$ when $n \to \infty$. A process is said to be predictable if it belongs to set of processes generated by simple predictable processes. In other words, if it is measurable with respect to the σ-field generated by limits of simple predictable processes.

Definition 2.3.2 *A strategy $\phi = (\phi_t)_{0\leq t\leq T} = \left(\left(\phi_t^0, \phi_t^1\right)\right)_{0\leq t\leq T}$ will be a predictable process w.r.t. $(\mathcal{F}_t)_{0\leq t\leq T}$. $\left(\phi_t^0\right)_{0\leq t\leq T}$ indicates the number of units in the bank account and $\left(\phi_t^1\right)_{0\leq t\leq T}$ the number of risky assets. In such a way that the value of the portfolio is given by*

$$V_t(\phi) = \phi_t^0 S_t^0 + \phi_t^1 S_t$$

Let $\phi = (\phi_t)_{0\leq t\leq T} = \left(\left(\phi_t^0, \phi_t^1\right)\right)_{0\leq t\leq T}$ be now an $(\mathcal{F}_t)_{0\leq t\leq T}$ *adapted* process with values in $\mathbb{R}^2$ and càdlàg. Fix $n \in \mathbb{N}$ and define the discrete approximation to ϕ predictive elementary strategy

$$\phi_t^{i(n)} := \sum_{j=1}^{m(n)} \phi_{t_{j-1,n}}^i \mathbf{1}_{(t_{j-1,n}, t_{j,n}]}(t), \; i = 0, 1, \; \phi_0^{i(n)} = \phi_0^i$$

where $0 = t_{0n} < t_{1n} < ... < t_{m(n)n} = T$. Note that $\lim_{n\to\infty} \phi_t^{i(n)} = \phi_{t-}^i$. Then we have

$$V_t\left(\phi^{(n)}\right) = \phi_{t_{j-1,n}}^0 S_t^0 + \phi_{t_{j-1,n}}^1 S_t, \quad t \in (t_{j-1,n}, t_{j,n}]$$

and to use the self-financing condition in the discrete-time setting:

$$V_{t_{j,n}}\left(\phi^{(n)}\right) = \phi_{t_{j-1,n}}^0 S_{t_{j,n}}^0 + \phi_{t_{j-1,n}}^1 S_{t_{j,n}} = \phi_{t_{j,n}}^0 S_{t_{j,n}}^0 + \phi_{t_{j,n}}^1 S_{t_{j,n}},$$

so

$$\begin{aligned}V_{t_{j+1,n}}\left(\phi^{(n)}\right)-V_{t_{j,n}}\left(\phi^{(n)}\right)&=\phi^0_{t_{j,n}}S^0_{t_{j+1,n}}+\phi^1_{t_{j,n}}S_{t_{j+1,n}}-\left(\phi^0_{t_{j,n}}S^0_{t_{j,n}}+\phi^1_{t_{j,n}}S_{t_{j,n}}\right)\\&=\phi^0_{t_{j,n}}(S^0_{t_{j+1,n}}-S^0_{t_{j,n}})+\phi^1_{t_{j,n}}(S_{t_{j+1,n}}-S_{t_{j,n}}).\end{aligned}$$

Therefore the corresponding version in the continuous time case will be:

$$\mathrm{d}V_t=\phi^0_{t-}\mathrm{d}S^0_t+\phi^1_{t-}\mathrm{d}S_t,$$

We can also use the self-financing condition with the discounted values, and the corresponding version in the continuous time case will be

$$\mathrm{d}\tilde{V}_t=\phi^1_{t-}\mathrm{d}\tilde{S}_t.$$

In integral form, if $\phi=(\phi_t)_{0\le t\le T}=\left((\phi^0_t,\phi^1_t)\right)_{0\le t\le T}$ is self-financing

$$V_t=V_0+\int_0^t\phi^0_{t-}\mathrm{d}S^0_t+\int_0^t\phi^1_{t-}\mathrm{d}S_t.$$

We know how to calculate the first integral

$$\left(\int_0^t\phi^0_{t-}\mathrm{d}S^0_t\right)(\omega)=\int_0^t\phi^0_t(\omega)rS^0_t\mathrm{d}t,$$

so provided that $\int_0^T|\phi^0_t|\mathrm{d}t<\infty$ a.s. P , this integral is well defined. As for the second integral we need to know how to calculate limits of the form

$$\lim_{n\to\infty}\sum_{i=1}^n\phi^1_{t_{i-1,n}}(S_{t_{in}}-S_{t_{i-1,n}})$$

where $0=t_{0n}<t_{1n}<...<t_{m(n)n}=T$ is a sequence of partitions of $[0,T]$ whose mesh goes to zero. Roughly speaking we shall say that, given an adapted càdlàg process $(\phi_t)_{0\le t\le T}=\left((\phi^0_t,\phi^1_t)\right)_{0\le t\le T}$, the corresponding strategy $\varphi:=(\phi_{t-})_{0\le t\le T}=\left((\phi^0_{t-},\phi^1_{t-})\right)_{0\le t\le T}$ is self-financing if

$$V_t=V_0+\int_0^t\phi^0_{t-}\mathrm{d}S^0_t+\lim_{n\to\infty}\sum_{i=1}^n\phi^1_{t_{i-1,n}}(S_{t_{in}}-S_{t_{i-1,n}}),$$

whatever is the kind of limit, in L^2, almost sure,.. etc, we have in the previous expression. We shall consider processes S for which, roughly speaking, $\Delta S_t\propto\Delta W_t$ where W is a Brownian motion, so we have to construct integrals

$$\int_0^t\varphi^1_s dW_s$$

where $(W_s)_{0\le t\le T}$ is a Brownian motion and $\left(\varphi^1_s\right)_{0\le t\le T}$ is a predictable process. In a first glance we can think in a definition ω to ω (path-wise) but though $W_s(\omega)$ is continuous in s, it is not a function with bounded variation and we cannot associate a measure with the increments along the path to build a Lebesgue-Stieltjes integral.

Proposition 2.3.1 *The trajectories of a Brownian motion has not bounded variation with probability one.*

Proof. Given the partition $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{m(n)n} \leq t$ of $[0,t]$ with $\lim_{n\to\infty} \sup |t_{in} - t_{i-1,n}| = 0$, we have:

$$\Delta_n = \sum_{i=1}^{m(n)} (W_{t_{in}} - W_{t_{i-1,n}})^2 \xrightarrow{L^2} t.$$

In fact:

$$\begin{aligned}\mathbb{E}((\Delta_n - t)^2) &= E(\Delta_n^2 - 2t\Delta_n + t^2) \\ &= E(\Delta_n^2) - 2t^2 + t^2,\end{aligned}$$

but

$$\begin{aligned}\mathbb{E}(\Delta_n^2) &= E\left(\sum_{i=1}^{m(n)}\sum_{j=1}^{m(n)} (W_{t_{in}} - W_{t_{i-1,n}})^2 (W_{t_{jn}} - W_{t_{j-1,n}})^2\right) \\ &= \sum_{i=1}^{m(n)} E((W_{t_{in}} - W_{t_{i-1,n}})^4) + 2\sum_{i=1}^{n}\sum_{j<i} E((W_{t_{in}} - W_{t_{i-1,n}})^2(W_{t_{jn}} - W_{t_{j-1,n}})^2) \\ &= 3\sum_{i=1}^{m(n)} (t_{in} - t_{i-1,n})^2 + 2\sum_{i=1}^{m(n)}\sum_{j<i} (t_{in} - t_{i-1,})((t_{jn} - t_{j-1,n}) \\ &= t^2 + 2\sum_{i=1}^{m(n)} (t_{in} - t_{i-1,n})^2\end{aligned}$$

so

$$\mathbb{E}((\Delta_n - t)^2) = 2\sum_{i=1}^{m(n)} (t_{in} - t_{i-1,n})^2 \leq 2t \sup |t_{in} - t_{i-1,n}| \to 0.$$

Then

$$\mathbb{P}\{|\Delta_n - t| > \varepsilon\} \leq \frac{2t \sup |t_{in} - t_{i-1,n}|}{\varepsilon^2},$$

and if the sequence of partitions is such that $\sum_{n=1}^{\infty} \sup |t_{in} - t_{i-1,n}| < \infty$, by applying the Borel-Cantelli Lemma, we have

$$\Delta_n \xrightarrow{a.s.} t, \tag{2.1}$$

and for these partitions

$$\sum_{i=1}^{m(n)} |W_{t_{in}} - W_{t_{i-1n}}| \geq \frac{\sum_{i=1}^{m(n)} |W_{t_{in}} - W_{t_{i-1,n}}|^2}{\sup_i |W_{t_{i,n}} - W_{t_{i-1,n}}|} = \frac{\Delta_n}{\sup_i |W_{t_{in}} - W_{t_{i-1,n}}|} \xrightarrow{a.s.} \frac{t}{0}.$$

■

2.4 Stochastic integral with respect to a Brownian motion

2.4.1 Integrands depending smoothly on the Brownian motion

Let (W_t) be a Brownian motion, and (τ_n) a sequence of partitions: $0 = t_{0n} \leq t_{1n} \leq ... \leq t_{m(n)n} = t$, with $d_n := \lim_{n\to\infty} \sup |t_{in} - t_{i-1,n}| = 0$, such that for all $0 \leq s \leq t$

$$\lim_{n\to\infty} \sum_{\substack{t_{i,n}\in\tau_n \\ t_{i,n}\leq s}} |W_{t_{in}} - W_{t_{i-1,n}}|^2 \overset{c.s.}{=} s. \tag{2.2}$$

Let f a C^2 map in $\mathbb{R}$. Then, fixed ω,

$$f(W_{t_{in}})-f(W_{t_{i-1,n}}) = f'(W_{t_{i-1,n}})(W_{t_{in}}-W_{t_{i-1,n}})+\frac{1}{2}f^{''}(W_{\tilde{t}_{i-1,n}})(W_{t_{in}}-W_{t_{i-1,n}})^2,$$

where $\tilde{t}_{i-1,n} \in (t_{i-1,n}, t_{in})$. Since $f^{''}$ is uniformly continuous in a the compact set $(W_s(\omega))_{0\leq s\leq t}$, we have

$$\sum_{i=1}^{m(n)} |f^{''}(W_{\tilde{t}_{i-1,n}})-f^{''}(W_{t_{i-1,n}})|(W_{t_{in}}-W_{t_{i-1,n}})^2 \leq \varepsilon_n \sum_{i=1}^{m(n)} (W_{t_{in}}-W_{t_{i-1,n}})^2 \underset{n\to\infty}{\to} 0,$$

For each n, $\mu_n(A)(\omega) := \sum_{i=1}^{m(n)} |W_{t_{in}}(\omega) - W_{t_{i-1,n}}(\omega)|^2 \mathbf{1}_A(t_{i-1,n})$ defines a measure in $[0,t]$ that converges, by (2.1), to the Lebesgue measure in $[0,t]$. So

$$\begin{aligned}\sum_{i=1}^{m(n)} f^{''}(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})^2 &= \int_0^t f^{''}(W_s)\mu_n(ds) \\ &\underset{n\to\infty}{\to} \int_0^t f^{''}(W_s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}f(W_t) - f(0) &= \lim_{n\to\infty} \sum (f(W_{t_{in}}) - f(W_{t_{i-1,n}})) = \lim_{n\to\infty} \sum f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}}) \\ &+ \frac{1}{2}\int_0^t f^{''}(W_s)ds.\end{aligned}$$

Consequently

$$\lim_{n\to\infty} \sum f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}})$$

is well defined since it coincides with $f(W_t) - f(0) - \frac{1}{2}\int_0^t f^{''}(W_s)ds$ and then we can define

$$\int_0^t f'(W_s)\mathrm{d}W_s = \lim_{n\to\infty} \sum f'(W_{t_{i-1,n}})(W_{t_{in}} - W_{t_{i-1,n}}).$$

The drawback of this construction is that this integral depends on the sequences of partitions. Nevertheless if we get that our Riemann sums converge in probability or in L^2, independently of the partitions we choose, the limit will be the same by the uniqueness of the limit in probability. In this way we have established that

$$\int_0^t f'(W_s)\mathrm{d}W_s = f(W_t) - f(0) - \frac{1}{2}\int_0^t f''(W_s)\mathrm{d}s$$

and this result modifies the chain rule of the *classical analysis*. This integral is known as *Itô's integral*.

Remark 2.4.1 *It is easy to see that if we take the midpoint in the above Riemann sums:*

$$\sum f'(W_{\frac{t_{i-1,n}+t_{in}}{2}})(W_{t_{in}} - W_{t_{i-1,n}})$$

then

$$\int_0^t f'(W_s) \circ \mathrm{d}W_s := \lim_{n\to\infty} \sum f'(W_{\frac{t_{i-1,n}+t_{in}}{2}})(W_{t_{in}} - W_{t_{i-1,n}}) = f(W_t) - f(0).$$

This integral is known as Stratonovich *integral.*

Example 2.4.1

$$\int_0^t W_s\mathrm{d}W_s = \frac{1}{2}W_t^2 - \frac{1}{2}t,$$

$$\int_0^t \exp\{W_s\}\mathrm{d}W_s = \exp\{W_t\} - 1 - \frac{1}{2}\int_0^t \exp\{W_s\}ds$$

It is straightforward to see that we can extend the previous result to integrands that are $C^{1,2}$-functions $f : [0,t] \times R \to R$ in such a way that

$$f(t, W_t) = f(0,0) + \int_0^t f_t(s, W_s)\mathrm{d}s + \int_0^t f_x(s, W_s)\mathrm{d}W_s + \frac{1}{2}\int_0^t f_{xx}(s, W_s)\mathrm{d}s,$$

where

$$f_t(s,x) = \frac{\partial}{\partial t}f(t,x)\Big|_{t=s}, \quad f_x(s,x) = \frac{\partial}{\partial x}f(t,x)\Big|_{t=s},$$
$$f_{xx}(s,x) = \frac{\partial^2}{\partial x^2}f(t,x)\Big|_{t=s}.$$

Example 2.4.2 *If we take* $f(t,x) = \exp(ax - \frac{1}{2}a^2 t)$, $a \in \mathbb{R}$, *we have*

$$\begin{aligned}\exp(aW_t - \frac{1}{2}a^2 t) = 1 &- \frac{a^2}{2}\int_0^t \exp(aW_s - \frac{1}{2}a^2 s)\mathrm{d}s \\ &+ a\int_0^t \exp(aW_s - \frac{1}{2}a^2 s)\mathrm{d}W_s \\ &+ \frac{a^2}{2}\int_0^t \exp(aW_s - \frac{1}{2}a^2 s)\mathrm{d}s.\end{aligned}$$

That is,

$$\exp(aW_t - \frac{1}{2}a^2t) = 1 + a\int_0^t \exp(aW_s - \frac{1}{2}a^2s)\mathrm{d}W_s.$$

so, if we define $S_t := \exp(aW_t - \frac{1}{2}a^2t)$*, we can write*

$$\mathrm{d}S_t = aS_t\mathrm{d}W_t.$$

Example 2.4.3 *Suppose a financial market with a single risky stock,* $S_t = S_0 + \sigma W_t$*,* $t \in [0,T]$ *and a bank account with interest rate* $r = 0$*. Given a strategy* $\phi_t = (\phi_t^0, \phi_t^1)$ *the value of a portfolio at time* t*, is*

$$V_t = \phi_t^0 + \phi_t^1 S_t,$$

If the strategy is self-financing we will have

$$\mathrm{d}V_t = \phi_t^1\mathrm{d}S_t = \phi_t^1\sigma\mathrm{d}W_t$$

Assume now that $V_t = V(t, S_t)$*, then, by applying the previous stochastic calculus*

$$\mathrm{d}V_t = \mathrm{d}V(t,S_t) = \partial_t V(t,S_t)\mathrm{d}t + \sigma\partial_x V(t,S_t)\mathrm{d}W_t + \frac{\sigma^2}{2}\partial_{xx}^2 V(t,S_t)dt,$$

therefore

$$\partial_t V(t,S_t) + \frac{\sigma^2}{2}\partial_{xx}^2 V(t,S_t) = 0 \tag{2.3}$$

$$\partial_x V(t,S_t) = \phi_t^1 \tag{2.4}$$

if we want to replicate $X = F(S_T)$*, we have to find a solution of (2.3) with the boundary condition* $V(T,S_T) = F(S_T)$*. It is easy to see that*

$$p(t,x) := \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left\{-\frac{x^2}{2\sigma^2(T-t)}\right\},$$

is a solution of (2.3) with $p(T,x) = \delta(x)$*, where* δ *is the Dirac's delta. That is* $p(t,x)$ *is* the fundamental solution*. Then if the boundary condition is* $F(x)$*, we will have that*

$$V(t,x) = \int_{\mathbb{R}} F(y)\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left\{-\frac{(y-x)^2}{2\sigma^2(T-t)}\right\}\mathrm{d}y,$$

and

$$V(t,S_t) = \int_{\mathbb{R}} F(y)\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left\{-\frac{(y-S_t)^2}{2\sigma^2(T-t)}\right\}\mathrm{d}y = \mathbb{E}(F(S_T)|S_t).$$

The equation (2.4) solves the hedging problem. In the particular case that $F(S_T) = (S_T - K)_+$ *we have that*

$$V(t,S_t) = \mathbb{E}((S_T-K)_+|S_t) = (S_t-K)\Phi\left(\frac{S_t-K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{S_t-K}{\sigma\sqrt{T-t}}\right),$$

where Φ *and* ϕ *are, respectively, the cumulative distribution function and the density of a standard normal distribution.*

2.4.2 The Itô integral

The definite integral

We are going to build a stochastic integral in the sense of limit of Riemann sums that converge in L^2 sense.

Definition 2.4.1 *If $(H_t)_{0\leq t\leq T}$ is an elementary process,*

$$H_t = \sum_{i=1}^{n} h_i \mathbf{1}_{(t_{i-1},t_i]}(t),$$

$0 = t_0 \leq t_1 \leq ... \leq t_n = T$, *we define*

$$\int_0^T H_s \mathrm{d}W_s = \sum_{i=1}^{n} h_i (W_{t_i} - W_{t_{i-1}})$$

Proposition 2.4.1 *If $(H_t)_{0\leq t\leq T}$ and $(L_t)_{0\leq t\leq T}$ are two elementary process,* $\mathbb{E}(\int_0^T H_s \mathrm{d}W_s \int_0^T L_s \mathrm{d}W_s) = \int_0^T \mathbb{E}(H_s L_s)\mathrm{d}s$ *(isometry property)*

Proof.

$$\begin{aligned}
\mathbb{E}(\int_0^T H_s \mathrm{d}W_s \int_0^T L_s \mathrm{d}W_s) &= \mathbb{E}(\sum_{i=1}^{n} h_i (W_{t_i} - W_{t_{i-1}}) \sum_{j=1}^{n} l_j (W_{t_j} - W_{t_{i-1}})) \\
&= \mathbb{E}(\sum_{i=1}^{n} h_i l_i (W_{t_i} - W_{t_{i-1}})^2) \\
&+ 2\sum_{i=1}^{n-1}\sum_{j>i} \mathbb{E}(h_i (W_{t_i} - W_{t_{i-1}}) l_j \mathbb{E}(W_{t_j} - W_{t_{i-1}} | \mathcal{F}_{t_{j-1}})) \\
&= \sum_{i=1}^{n} \mathbb{E}(h_i l_i \mathbb{E}(W_{t_i} - W_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}})) \\
&= \sum_{i=1}^{n} \mathbb{E}(h_i l_i)(t_i - t_{i-1}) = \mathbb{E}\int_0^T H_s L_s \mathrm{d}s = \int_0^T \mathbb{E}(H_s L_s)\mathrm{d}s
\end{aligned}$$

■

Now we extend the class of simple integrands, $\mathcal{S}$ to the class $\mathcal{H}$:

$$\mathcal{H} = \{(H_t)_{0\leq t\leq T}, (\mathcal{F}_t)\text{-adapted}, \int_0^T \mathbb{E}(H_s^2)\mathrm{d}s < \infty\}.$$

It can be seen that the class $\mathcal{H}$ with the scalar product $\langle (H_t), (F_t)\rangle = \int_0^T \mathbb{E}(H_s F_s)\mathrm{d}s$ is a Hilbert space. Note that, by the previous proposition, we have defined a linear map $I : \mathcal{S} \to \mathcal{M} = \{$ square integrable $\mathcal{F}_T$-measurable random variables$\}$, $I(H) = \int_0^T H_s \mathrm{d}W_s$. In $\mathcal{M}$ we can also define a scalar product $\langle M, L\rangle := \mathbb{E}(ML)$. We have then that I is an isometry.

Proposition 2.4.2 *The class $\mathcal{S}$ is dense in $\mathcal{H}$ (with respect to the norm $||H||^2 := \int_0^T \mathbb{E}(H_s^2)ds$).*

Definition 2.4.2 *If H is a process in the class $\mathcal{H}$, the integral is defined as the L^2 limit*

$$\int_0^T H_s \mathrm{d}W_s = \lim_{n\to\infty} \int_0^T H_s^n \mathrm{d}W_s, \tag{2.5}$$

where H_s^n is a sequence of simple processes such that

$$\lim_{n\to\infty} ||H^n - H||^2 = \lim_{n\to\infty} \int_0^T \mathbb{E}(H_s^n - H_s)^2 \mathrm{d}s = 0.$$

The existence of the limit (2.5) is due to the fact that the sequence of random variables $\int_0^T H_s^n \mathrm{d}W_s$ is a Cauchy sequence in $L^2(\Omega)$ and $L^2(\Omega)$ is complete, in fact due to the isometry property

$$\begin{aligned}\mathbb{E}\left(\int_0^T H_s^n \mathrm{d}W_s - \int_0^T H_s^m \mathrm{d}W_s\right)^2 &= \int_0^T \mathbb{E}(H_s^n - H_s^m)^2 \mathrm{d}s \\ &\le 2\int_0^T \mathbb{E}(H_s^n - H_s)^2 \mathrm{d}s \\ &+ 2\int_0^T \mathbb{E}(H_s^m - H_s)^2 \mathrm{d}s.\end{aligned}$$

Analogously it can be seen that the limit does not depend on the sequence H^n. It is easy to show that for all H in the class $\mathcal{H}$.

- The isometry property is satisfied,

$$\mathbb{E}\left(\int_0^T H_s \mathrm{d}W_s \int_0^T L_s \mathrm{d}W_s\right) = \int_0^T \mathbb{E}(H_s L_s) \mathrm{d}\mathrm{d}s,$$

- The integral has zero expectation,

$$\mathbb{E}\left(\int_0^T H_s \mathrm{d}W_s\right) = 0,$$

- The integral is linear,

$$\int_0^T (aH_s + bL_s)\mathrm{d}W_s = a\int_0^T H_s \mathrm{d}W_s + b\int_0^T L_s \mathrm{d}W_s,\ a, b \in \mathbb{R}.$$

The indefinite integral

If H is in the class $\mathcal{H}$ then $H\mathbf{1}_{[0,t]}$ it is as well and we can define

$$\int_0^t H_s dW_s := \int_0^T H_s \mathbf{1}_{[0,t]}(s) dW_s,$$

then we have the process

$$\left\{ I(H)_t := \int_0^t H_s dW_s, 0 \leq t \leq T \right\}$$

Proposition 2.4.3 *$I(H)$ is an $(\mathcal{F}_t)$-martingale.*

Proof. The result is obvious if H is elementary: $\int_0^t H_s dW_s$ is $\mathcal{F}_t$-measurable and with finite expectation, then it is sufficient to see that $\forall t > s$

$$\mathbb{E}\left(\int_0^t H_u \mathrm{d}W_u \middle| \mathcal{F}_s \right) = \int_0^s H_u \mathrm{d}W_u.$$

We can asume that s and t are some of the points in the partition $0 = t_0 < t_1 < ... < t_n = T$. So it is enough to see that $(M_n) := \left(\int_0^{t_n} H_u \mathrm{d}W_u \right)$ is a $(\mathcal{G}_n)$-martingale with $G_n = \mathcal{F}_{t_n}$. But (M_n) is the martingale transform of the $(\mathcal{G}_n)$-martingale (W_{t_n}) by the process $(\mathcal{G}_n)$-predictable (h_n) and consequently it is a martingale.

If H is not a simple process the integral is an L^2 -limit of martingales and this limit preserves the martingale property. ■

Remark 2.4.2 *If can be shown, by using the Doob inequality for continuous martingales:*

$$\mathbb{E}(\sup_{0 \leq t \leq T} M_t^2) \leq 4\mathbb{E}(M_T^2)$$

that we have a continuous version of $I(H)$.

Remark 2.4.3 *We shall denote $\forall t > s$, $\int_s^t H_u dW_u := \int_0^t H_u dW_u - \int_0^s H_u dW_u$.*

To do a further extension of the integral the following results are convenient

Proposition 2.4.4 *Let A $\mathcal{F}_t$-measurable, then for all $H \in \mathcal{H}$*

$$\int_0^T \mathbf{1}_A H_s \mathbf{1}_{\{s>t\}} \mathrm{d}W_s = \mathbf{1}_A \int_t^T H_s \mathrm{d}W_s$$

Proof. If H^n is an approximate sequence of H then $1_A H^n 1_{\{\cdot>t\}}$ approximates $1_A H 1_{\{\cdot>t\}}$ and since the result is true for simple processes then the proposition follows. ■

Proposition 2.4.5 *Let τ be an $(\mathcal{F}_t)$-stopping time then*

$$\int_0^{\tau\wedge T} H_s \mathrm{d}W_s = \int_0^T \mathbf{1}_{\{s\leq\tau\}} H_s \mathrm{d}W_s$$

Proof. If τ is of the form $\tau = \sum_{i=1}^n t_i 1_{A_i} + \infty 1_{(\cup A_i)^c}$ where $0 < t_1 < t_2 < ... < t_n = T$ and A_i $\mathcal{F}_{t_i}$-measurable and disjoints, then it is straightforward:

$$\begin{aligned}\int_0^T \mathbf{1}_{\{s>\tau\}} H_s \mathrm{d}W_s &= \int_0^T \sum_{i=1}^n \mathbf{1}_{\{s>t_i\}} \mathbf{1}_{A_i} H_s \mathrm{d}W_s = \sum_{i=1}^n \mathbf{1}_{A_i} \int_{t_i}^T H_s \mathrm{d}W_s \\ &= \int_{\tau\wedge T}^T H_s \mathrm{d}W_s,\end{aligned}$$

Moreover $\int_0^{\tau\wedge T} H_s \mathrm{d}W_s = \int_0^T H_s \mathrm{d}W_s - \int_{\tau\wedge T}^T H_s \mathrm{d}W_s$. In general, it is enough to approximate τ by $\tau_n = \sum_{k=0}^{2^n-1} \frac{(k+1)T}{2^n} 1_{\{\frac{kT}{2^n} < \tau \leq \frac{(k+1)T}{2^n}\}} + \infty 1_{(\tau > T)}$ and to see that $\int_0^T 1_{\{s\leq\tau_n\}} H_s dW_s \overset{L^2}{\to} \int_0^T 1_{\{s\leq\tau\}} H_s dW_s$:

$$\mathbb{E}\left(\left|\int_0^T \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s - \int_0^T \mathbf{1}_{\{s\leq\tau\}} H_s \mathrm{d}W_s\right|^2\right) = \mathbb{E}\left(\int_0^T \mathbf{1}_{\{\tau<s\leq\tau_n\}} H_s^2 \mathrm{d}s\right),$$

and the we apply the dominated convergence theorem. Finally we take a subsequence of $\int_0^T 1_{\{s\leq\tau_n\}} H_s dW_s$ converging almost surely. ■

Extension of the integral

We are going to do a further extension of the integrands, consider the class

$$\tilde{\mathcal{H}} = \{(H_t)_{0\leq t\leq T}, (\mathcal{F}_t)\text{-adapted}, \int_0^T H_s^2 \mathrm{d}s < \infty \ P\text{-c.s.}\}.$$

Given $H \in \tilde{\mathcal{H}}$ sea $\tau_n = \inf\{t \leq T, \int_0^t (H_s)^2 \, ds \geq n\}$ ($+\infty$ if the previous set is empty). That $\int_0^t (H_s)^2 \, ds$ is $\mathcal{F}_t$-measurable can be deduced from the fact that it is an a.s. limit of $\mathcal{F}_t$-measurable random variables, from here τ_n is a stopping time. Set $A_n = \{\int_0^T (H_s)^2 \, ds < n\}$ we can define

$$\tilde{J}(H)_t^n := \left(\int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s\right) \mathbf{1}_{A_n}, \text{ para todo } n \geq 1.$$

Note that this is consistently defined: if $m \geq n$ and $\omega \in A_n$ then

$$\tilde{J}(H)_t^m(\omega) = \tilde{J}(H)_t^n(\omega),$$

in fact:

$$\tilde{J}(H)_t^m(\omega) = \int_0^{t\wedge\tau_n(\omega)} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s,$$

but

$$\int_0^{t\wedge\tau_n} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s = \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s$$
$$= \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s,$$

in such a way that

$$\int_0^{t\wedge\tau_n(\omega)} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s = \tilde{J}(H)_t^n(\omega)$$

Now we can define

$$\tilde{J}(H)_t = \lim_{n\to\infty} \left(\left(\int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s \right) \mathbf{1}_{A_n} \right) = \lim_{n\to\infty} \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s.$$

Note that if $H \in \mathcal{H}$

$$\tilde{J}(H)_t = \lim_{n\to\infty} \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} H_s \mathrm{d}W_s = \lim_{n\to\infty} \int_0^{t\wedge\tau_n} H_s \mathrm{d}W_s$$
$$= \int_0^t H_s \mathrm{d}W_s = J(H)_t,$$

so it is really an *extension* of the integral.

Exercise 2.4.1 *Prove that the previous extension of the integral does not depend on the sequence of localizing stopping times of (H_s). In other words, that if we take $\tilde{\tau}_n \uparrow \infty$ and $\left(\mathbf{1}_{\{\cdot<\tilde{\tau}_n\}} H_\cdot\right)$ is in $\mathcal{H}$) then the limit is the same.*

It can be shown that the previous extension is a limit in probability of integrals for simple processes H^n which converge to H in the sense that

$$\mathbb{P}(\int_0^t |H_s^n - H_s|^2 ds > \varepsilon) \to 0.$$

Note that by construction the extension of the integral is an a.s. limit of another limit in quadratic norm.

We lose then the martingale property. In general we have that if (τ_m) is a localizing sequence

$$\tilde{J}(H)_{t\wedge\tau_m} = \lim_{n\to\infty} \int_0^{t\wedge\tau_m} \mathbf{1}_{\{s\leq\tilde{\tau}_n\}} H_s \mathrm{d}W_s$$
$$= \lim_{n\to\infty} \int_0^t \mathbf{1}_{\{s\leq\tilde{\tau}_n\wedge\tau_m\}} H_s \mathrm{d}W_s$$
$$= \lim_{n\to\infty} \int_0^{t\wedge\tilde{\tau}_n} \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s$$
$$= \int_0^t \mathbf{1}_{\{s\leq\tau_m\}} H_s \mathrm{d}W_s$$

in such a way that $\tilde{J}(H)_{t\wedge\tau_m}$ is a martingale. Then $\tilde{J}(H)$ is a *local martingale.*

2.4.3 The Itô calculus

We are going to develop a differential calculus based in the previous integral. We have seen that

$$\int_0^t f'(W_s)\mathrm{d}W_s = f(W_t) - f(W_0) - \frac{1}{2}\int_0^t f''(W_s)\mathrm{d}s$$

for $f \in C^2$, or in differential form

$$\mathrm{d}f(W_t) = f'(W_s)\mathrm{d}W_s + \frac{1}{2}f''(W_t)\mathrm{dt} \tag{2.6}$$

Then we want to extend this result.

Definition 2.4.3 *A process $(X_t)_{0\le t\le T}$ is said to be an Itô process if it can be written as*

$$X_t = X_0 + \int_0^t K_s\mathrm{d}s + \int_0^t H_s\mathrm{d}W_s$$

where

- *X_0 is $\mathcal{F}_0$-measurable.*
- *(K_t) and (H_t) are $(\mathcal{F}_t)$-adapted.*
- *$\int_0^T(|K_s| + |H_s|^2)\mathrm{d}s < \infty$ $\mathbb{P}$-a.s..*

Proposition 2.4.6 *If $(M_t)_{0\le t\le T}$ is a continuous $(\mathcal{F}_t)$-martingale such that $M_t = \int_0^t K_s\mathrm{d}s$, where (K_s) is a measurable and $(\mathcal{F}_t)$-adapted process with $\int_0^T |K_s|\mathrm{d}s < \infty$ $\mathbb{P}$-a.s., then*

$$M_t = 0, \ \textit{a.s for all } t \le T$$

Proof. Without loss of generality we can assume that $\int_0^t |K_s|\mathrm{d}s \le C < \infty$. Otherwise we can define the stopping time

$$\tau_n = \inf\{t, \int_0^t |K_s|\mathrm{d}s \ge n\} \wedge T,$$

and to apply the result to the martingale $(M_{t\wedge\tau_n})$. This would make $M_{t\wedge\tau_n} \equiv 0$ and we can let n go to infinity to conclude that $M_t \equiv 0$.

Now if $\int_0^t |K_s|\mathrm{d}s$ is bounded by C and we take $t_i^n = T\frac{i}{n}, 0 \le i \le n$, we have

$$\begin{aligned}\sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2 &\le \sup_{1\le i\le n}\left|M_{t_i^n} - M_{t_{i-1}^n}\right| \sum_{i=1}^n \left|M_{t_i^n} - M_{t_{i-1}^n}\right| \\ &\le \sup_{1\le i\le n}\left|M_{t_i^n} - M_{t_{i-1}^n}\right| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} |K_s|\mathrm{d}s \\ &\le C \sup_{1\le i\le n}\left|M_{t_i^n} - M_{t_{i-1}^n}\right| \end{aligned} \tag{2.7}$$

and $(M_t)_{0\le t\le T}$ is continuous, so

$$\lim_{n\to\infty}\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2=0,\ \text{a.s.},$$

Moreover $\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2\le C^2$ by (2.7), so by the dominated convergence theorem

$$\lim_{n\to\infty}\mathbb{E}\left(\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2\right)=0.$$

On the other hand, since $(M_t)_{0\le t\le T}$ is a martingale and simultaneously

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n}(M_{t_i^n}-M_{t_{i-1}^n})^2\right) &= \mathbb{E}\left(\sum_{i=1}^{n}(M_{t_i^n}^2+M_{t_{i-1}^n}^2-2M_{t_i^n}M_{t_{i-1}^n})\right)\\
&= \mathbb{E}\left(\sum_{i=1}^{n}\left(M_{t_i^n}^2+M_{t_{i-1}^n}^2-2M_{t_{i-1}^n}E(M_{t_i^n}|\mathcal{F}_{t_{i-1}^n})\right)\right)\\
&= \mathbb{E}\left(\sum_{i=1}^{n}\left(M_{t_i^n}^2+M_{t_{i-1}^n}^2-2M_{t_{i-1}^n}^2\right)\right)\\
&= \mathbb{E}\left(\sum_{i=1}^{n}\left(M_{t_i^n}^2-M_{t_{i-1}^n}^2\right)\right)\\
&= \mathbb{E}(M_T^2-M_0^2)
\end{aligned}$$

and that consequently that $M_T\equiv 0$ a.s., and so $M_t\equiv\mathbb{E}(M_T|\mathcal{F}_t)=0$ a.s, for all $t\le T$. ■

Corollary 2.4.1 *The expression of an Itô process is unique.*

Theorem 2.4.1 *Let $(X_t)_{0\le t\le T}$ be an Itô process and $f(t,x)\in C^{1,2}$ then:*

$$f(t,X_t)=f(0,X_0)+\int_0^t f_t(s,X_s)\mathrm{d}s+\int_0^t f_x(s,X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^t f_{xx}(s,X_s)\mathrm{d}\langle X,X\rangle_s,$$

where

$$\begin{aligned}
\int_0^t f_x(s,X_s)\mathrm{d}X_s &= \int_0^t f_x(s,X_s)K_s\mathrm{d}s+\int_0^t f_x(s,X_s)H_s\mathrm{d}W_s\\
\langle X,X\rangle_s &= \int_0^t H_s^2\mathrm{d}s.
\end{aligned}$$

Example 2.4.4 *Suppose we want to find a solution $(S_t)_{0\le t\le T}$ for the equation*

$$S_t=x_0+\int_0^t S_s(\mu\mathrm{d}s+\sigma\mathrm{d}W_s)$$

or in differential form

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t), \quad S_0 = x_0.$$

By the previous theorem

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t = \mathrm{d}(\log S_t) + \frac{1}{2S_t^2}\sigma^2 S_t^2 \mathrm{d}t,$$

that is

$$\mathrm{d}(\log S_t) = (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W_t$$

in such a way that

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$$

Proposition 2.4.7 *(Integration by parts formula) Let X_t and Y_t two Itô processes, $X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s$ e $Y_t = Y_0 + \int_0^t K'_s \mathrm{d}s + \int_0^t H'_s \mathrm{d}W_s$. Then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \mathrm{d}Y_s + \int_0^t Y_s \mathrm{d}X_s + \langle X, Y \rangle_t$$

where

$$\langle X, Y \rangle_t = \int_0^t H_s H'_s \mathrm{d}s.$$

Proof. By the Itô formula

$$(X_t + Y_t)^2 = (X_0 + Y_0)^2 + 2\int_0^t (X_s + Y_s)\mathrm{d}(X_s + Y_s) + \frac{1}{2}\int_0^t 2(H_s + H'_s)^2 \mathrm{d}s$$

and

$${X_t}^2 = X_0^2 + 2\int_0^t X_s \mathrm{d}X_s + \frac{1}{2}\int_0^t 2H_s^2 \mathrm{d}s,$$

$${Y_t}^2 = Y_0^2 + 2\int_0^t Y_s \mathrm{d}Y_s + \frac{1}{2}\int_0^t 2H_s'^2 \mathrm{d}s$$

so, by subtracting the sum of these latter expressions from the first one we obtain:

$$2X_t Y_t = 2X_0 Y_0 + 2\int_0^t X_s \mathrm{d}Y_s + 2\int_0^t Y_s \mathrm{d}X_s + \int_0^t 2H_s H'_s \mathrm{d}s.$$

■

Example 2.4.5 *Consider the differential equation*

$$\mathrm{d}X_t = -cX_t \mathrm{d}t + \sigma \mathrm{d}W_t, X_0 = x$$

then if we apply the previous formula to

$$X_t e^{ct}$$

we have

$$\mathrm{d}\left(X_t e^{ct}\right) = e^{ct}\mathrm{d}X_t + cX_t e^{ct}\mathrm{d}t$$

and therefore

$$e^{-ct}\mathrm{d}\left(X_t e^{ct}\right) = \sigma \mathrm{d}W_t$$

in such a way that

$$X_t = xe^{-ct} + \sigma e^{-ct}\int_0^t e^{cs}\mathrm{d}W_s.$$

A new integration by parts lead us to

$$X_t = xe^{-ct} + \sigma e^{-ct}(e^{ct}W_t - \int_0^t ce^{cs}W_s\mathrm{d}s).$$

So, it is a Gaussian process with expectation xe^{-ct} and variance

$$\begin{aligned}\mathrm{Var}(X_t) &= \sigma^2 e^{-2ct}\int_0^t e^{2cs}\mathrm{d}s \\ &= \sigma^2\frac{1-e^{-2ct}}{2c}.\end{aligned}$$

Exercise 2.4.2 *Solve the stochastic differential equation*

$$\mathrm{d}X_t = tX_t\mathrm{d}t + e^{t^2/2}\mathrm{d}B_t,\ X_0 = x_0,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion.

Exercise 2.4.3 *Solve the stochastic differential equation*

$$\mathrm{d}X_t = \frac{1}{2}X_t\mathrm{d}t + \sqrt{1+X_t^2}\mathrm{d}B_t,\ X_0 = 0,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion. (Hint: Consider the function $y = \sinh(x)$)

2.5 The Black-Scholes model

The Samuelson model, more known as the Black-Scholes model, is a model of a financial market with two stocks. One without risk, S^0, (or bank account) that evolves as:

$$S_t^0 = e^{rt}, \quad t \geq 0$$

and a risky stock S that evolves as

$$\mathrm{d}S_t = S_t\left(\mu\mathrm{d}t + \sigma\mathrm{d}B_t\right) \quad t \geq 0$$

where (B_t) is a Brownian motion. As we haven seen this implies that

$$S_t = S_0 \exp\{\mu t - \frac{\sigma^2}{2} t + \sigma B_t\}.$$

Then $\log(S_t)$ is a Brownian motion, no necessarily standard, and by the properties of the Brownian motion we have that S_t :

- has continuous trajectories
- the relative increments $\frac{S_t - S_u}{S_u}$ are independent of $\sigma(S_s, 0 \leq s \leq u)$:

$$\frac{S_t - S_u}{S_u} = \frac{S_t}{S_u} - 1$$

  and

$$\frac{S_t}{S_u} = \exp\{\mu(t-u) - \frac{\sigma^2}{2}(t-u) + \sigma(B_t - B_u)\}$$

  that is independent of $\sigma(B_s, 0 \leq s \leq u) = \sigma(S_s, 0 \leq s \leq u)$.

- the relative increments are homogeneous:

$$\frac{S_t - S_u}{S_u} \sim \frac{S_{t-u} - S_0}{S_0}.$$

  In fact we could formulate the model in terms of these three hypothesis.

2.5.1 Admissible strategies and arbitrage

According with the discussion in subsection 2.3, we give the following *definition.*

Definition 2.5.1 *A self-financing strategy ϕ, is a pair of adapted processes $\left(\phi_t^0\right)_{0 \leq t \leq T}, \left(\phi_t^1\right)_{0 \leq t \leq T}$ that satisfy*

- $\int_0^T \left(|\phi_t^0| + \left(\phi_t^1\right)^2\right) \mathrm{d}s < \infty$ P *a.s.*
- $\phi_t^0 S_t^0 + \phi_t^1 S_t = \phi_0^0 S_0^0 + \phi_0^1 S_0 + \int_0^t \phi_s^0 r e^{rs} \mathrm{d}s + \int_0^t \phi_s^1 \mathrm{d}S_s, \quad 0 \leq t \leq T.$

Denote $\tilde{S}_t = e^{-rt} S_t$, in such a way that we use the tilde as in the discrete-time setting: to indicate any discounted value.

Proposition 2.5.1 *ϕ is self-financing strategy if and only if:*

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_s$$

Proof. Suppose that ϕ is self-financing, then since $\tilde{V}_t = e^{-rt}V_t$, we will have that

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= -re^{-rt}V_t\mathrm{d}t + e^{-rt}\mathrm{d}V_t \\
&= -re^{-rt}V_t\mathrm{d}t + e^{-rt}(\phi_t^0\mathrm{d}S_t^0 + \phi_t^1\mathrm{d}S_t) \\
&= -re^{-rt}(V_t - \phi_t^0 S_t^0)\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t \\
&= -re^{-rt}\phi_t^1 S_t\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t \\
&= \phi_t^1(-re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t) \\
&= \phi_t^1\mathrm{d}\tilde{S}_t.
\end{aligned}$$

Analogously if

$$\mathrm{d}\tilde{V}_t = \phi_t^1\mathrm{d}\tilde{S}_t$$

we have that

$$\mathrm{d}V_t = \phi_t^0\mathrm{d}S_t^0 + \phi_t^1\mathrm{d}S_t.$$

■

Existence of a risk neutral measure in the Black-Scholes model

We have to find a probability under which discounted prices are martingale. We know that

$$\begin{aligned}
\mathrm{d}\tilde{S}_t &= \mathrm{d}\left(e^{-rt}S_t\right) = -re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t \\
&= e^{-rt}S_t\left(-r\mathrm{d}t + \mu\mathrm{d}t + \sigma\mathrm{d}B_t\right) \\
&= \sigma\tilde{S}_t\mathrm{d}\left(-\frac{r-\mu}{\sigma}t + B_t\right) \\
&= \sigma\tilde{S}_t\mathrm{d}W_t
\end{aligned} \tag{2.8}$$

with

$$W_t = B_t - \frac{r-\mu}{\sigma}t.$$

Then by the Girsanov theorem (see subsection 2.6.1 in the Annex) with $\theta_t = \frac{r-\mu}{\sigma}$ it turns out that $(W_t)_{0\leq t\leq T}$ is a Brownian motion with respect to the probability $\mathbb{P}^*$

$$\mathrm{d}\mathbb{P}^* = \exp\left\{\frac{r-\mu}{\sigma}B_T - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right\}\mathrm{d}\mathbb{P}. \tag{2.9}$$

From (2.8) we deduce that

$$\tilde{S}_t = S_0\exp\{-\frac{1}{2}\sigma^2 t + \sigma W_t\}$$

and that $\left(\tilde{S}_t\right)_{0\leq t\leq T}$ is a $\mathbb{P}^*$-martingale. We also have that

$$S_t = S_0\exp\{rt - \frac{1}{2}\sigma^2 t + \sigma W_t\}.$$

Now we put a restriction in the set of self-financing strategies, similar to that we put in the discrete-time setting.

Definition 2.5.2 *A strategy ϕ is admissible if it is self-financing and there exists $K > 0$ such that its discounted value $\tilde{V}_t = \phi_t^0 + \phi_t^1 \tilde{S}_t \geq -K, \forall t$.*

Absence of arbitrage

Definition 2.5.3 *An* arbitrage (opportunity) *is an admissible strategy ϕ with zero initial value and with strictly positive final value, that is*

1. $V_0(\phi) = 0$,
2. $\mathbb{P}\left(V_T(\phi) \geq 0\right) = 1$,
3. $\mathbb{P}(V_T(\phi) > 0) > 0$.

We have changed the definition of arbitrage, slightly, with respect to that we have in the discrete-time setting and also the definition of self-financing strategy is new, so we have to check if the existence of a risk neutral measure is a sufficient condition for absence of arbitrage.

Proposition 2.5.2 *The Black-Scholes model is free of arbitrage.*

Proof. We have seen that there is a probability $\mathbb{P}^*$ such that $\tilde{S}_t$ is a martingale. Then if we consider an admissible strategy ϕ with zero initial value we have

$$\tilde{V}_t(\phi) = \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_s,$$

so $\tilde{V}_t(\phi)$ is a $\mathbb{P}^*$-*local* martingale that is bounded from below, so it is a supermartingale. Consequently

$$\mathbb{E}_{\mathbb{P}^*}\left(\tilde{V}_T(\phi)\right) \leq 0,$$

now, since $\mathbb{P}\left(V_T(\phi) \geq 0\right) = 1$ and $\mathbb{P} \sim \mathbb{P}^*$ we have that $V_T(\phi) = 0$, $\mathbb{P}^*$-a.s. and $\mathbb{P}$-a.s. ■

Completeness

Definition 2.5.4 *We say that an option is replicable if its payoff is equal to the final value of an admissible strategy.*

Definition 2.5.5 *We shall say that the model is complete if any option with not negative payoff X, square integrable with respect to $\mathbb{P}^*$, is replicable.*

Theorem 2.5.1 *In the Black-Scholes model any option with payoff $h \geq 0$, $\mathcal{F}_T$-measurable and square integrable under $\mathbb{P}^*$ is replicable and its value is given by*

$$C_t = \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}h|\mathcal{F}_t)$$

Proof. Under $\mathbb{P}^*$

$$M_t := \mathbb{E}_{\mathbb{P}^*}(e^{-rT}h|\mathcal{F}_t), 0 \leq t \leq T$$

is a square integrable martingale, then by the representation theorem of Brownian martingales, (see Subsection 2.6.2 in the Annex) there exists a unique adapted process $(Y_t)_{0\leq t\leq T}$ such that

$$M_t = M_0 + \int_0^t Y_s \mathrm{d}W_s$$

with

$$\mathbb{E}_{\mathbb{P}^*}\left(\int_0^T Y_s^2 \mathrm{d}s\right) < \infty,$$

then we can define ϕ_t^1 by

$$\phi_t^1 = \frac{Y_t}{\sigma \tilde{S}_t}$$

and we have that

$$M_t = M_0 + \int_0^t \phi_t^1 \mathrm{d}\tilde{S}_t = \tilde{V}_t(\phi)$$

that is

$$\tilde{C}_t = C_0 + \int_0^t \phi_t^1 \mathrm{d}\tilde{S}_t.$$

Therefore the strategy $\left(\phi_t^0, \phi_t^1\right)$ with $\phi_t^0 e^{rt} = C_t - \phi_t^1 S_t$ is self-financing and replicates h. To see that it is admissible it is enough to take into account that since $h \geq 0$, $C_t \geq 0$. ■

Remark 2.5.1 *From Remark 2.6.1 we can see that, given $h \in L^2(\mathcal{F}_T, \mathbb{P}^*)$, we can find $\left(\varphi_t^1\right)_{0\leq t\leq T}$ such that*

$$\tilde{h} = V_0 + \int_0^T \varphi_t^1 \mathrm{d}\tilde{S}_t,$$

with $\int_0^T \varphi_t^1 \mathrm{d}\tilde{S}_t < \infty$ a.s. $\mathbb{P}^*$ different from $\left(\phi_t^1\right)_{0\leq t\leq T}$. However $\left(\int_0^t \varphi_t^1 \mathrm{d}\tilde{S}_t\right)_{0\leq t\leq T}$ will be a local martingale and since it is bounded by zero will be a supermartingale so $\mathbb{E}_{\mathbb{P}^*}\left(\tilde{h}|\mathcal{F}_t\right) \leq V_0 + \int_0^t \varphi_t^1 \mathrm{d}\tilde{S}_t$ and these strategies are more expensive than the above ones and nobody will use them.

In some cases we can give more explicit expressions for the replicating portfolio.

Proposition 2.5.3 *In the Black-Scholes model any option with payoff (non negative) of the form $h = f(S_T)$, square integrable with respect to $\mathbb{P}^*$, with $\mathbb{E}_{\mathbb{P}^*}(h|\mathcal{F}_t)$ a $C^{1,2}$ function of the time and of S_t, is replicable, its value is given by $C(t, S_t) = \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}h|\mathcal{F}_t)$ and the strategy that replicates h is given by (ϕ_t^0, ϕ_t^1) con*

$$\phi_t^1 = \frac{\partial C(t, S_t)}{\partial S_t}$$
$$\phi_t^0 e^{rt} = C(t, S_t) - \phi_t^1 S_t$$

Proof. First of all, by the independence of the relative increments

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}f(S_T)|\mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}f(\frac{S_T}{S_t}S_t)|\mathcal{F}_t) \\ &= \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}f(\frac{S_T}{S_t}x))_{x=S_t} \\ &= C(t, S_t),\end{aligned}$$

so what we shall call price of the contingent claim at t depends only on S_t and t.

If we apply now the Itô formula to $\bar{C}(t, \tilde{S}_t) = e^{-rt}C(t, \tilde{S}_t e^{rt})$, we have

$$\begin{aligned}&\bar{C}(t, \tilde{S}_t) \\ &= C(0, S_0) + \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial s}\mathrm{d}s + \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s}\mathrm{d}\tilde{S}_s + \frac{1}{2}\int_0^t \frac{\partial^2 \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s^2}\mathrm{d}\langle \tilde{S}, \tilde{S}\rangle_s\end{aligned}$$

and since

$$\mathrm{d}\tilde{S}_t = \sigma \tilde{S}_t \mathrm{d}W_t$$

we obtain

$$\begin{aligned}&\bar{C}(t, \tilde{S}_t) \\ &= C(0, S_0) + \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s}\sigma \tilde{S}_s \mathrm{d}W_s + \int_0^t \left(\frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial s} + \frac{1}{2}\frac{\partial^2 \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s^2}\sigma^2 \tilde{S}_s^2\right)\mathrm{d}s\end{aligned}$$

but $\bar{C}(t, \tilde{S}_t)$ is a square integrable martingale:

$$\bar{C}(t, \tilde{S}_t) = \tilde{C}(t, S_t) = \mathbb{E}_{\mathbb{P}^*}(e^{-rT}f(S_T)|\mathcal{F}_t)$$

and therefore, since the decomposition of an Itô process is unique we have:

$$\tilde{C}(t, S_t) = C(0, S_0) + \int_0^t \frac{\partial \bar{C}(s, \tilde{S}_s)}{\partial \tilde{S}_s}\mathrm{d}\tilde{S}_s$$

$$\frac{\partial \bar{C}(t, \tilde{S}_t)}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{C}(t, \tilde{S}_t)}{\partial \tilde{S}_t^2}\sigma^2 \tilde{S}_t^2 = 0.$$

Now since

$$\frac{\partial \bar{C}(t,\tilde{S}_t)}{\partial t} = -re^{-rt}C(t,S_t) + e^{-rt}\frac{\partial C(t,S_t)}{\partial t} + re^{-rt}S_t\frac{\partial C(t,S_t)}{\partial S_t}$$

$$\frac{\partial \bar{C}(t,\tilde{S}_t)}{\partial \tilde{S}_s} = e^{-rt}\frac{\partial C(s,S_t)}{\partial S_t}\frac{\partial S_t}{\partial \tilde{S}_t} = \frac{\partial C(t,S_t)}{\partial S_t}$$

and

$$\frac{\partial^2 \bar{C}(t,\tilde{S}_t)}{\partial \tilde{S}_t^2} = \frac{\partial^2 C(t,S_t)}{\partial S_t^2}\frac{\partial S_t}{\partial \tilde{S}_t} = e^{rt}\frac{\partial^2 C(t,S_t)}{\partial S_t^2},$$

we can write

$$\tilde{C}(t,S_t) = C(0,S_0) + \int_0^t \frac{\partial C(s,S_s)}{\partial S_s}\mathrm{d}\tilde{S}_s \tag{2.10}$$

$$\frac{\partial C(t,S_t)}{\partial t} + rS_t\frac{\partial C(t,S_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C(t,S_t)}{\partial S_t^2} = rC(t,S_t). \tag{2.11}$$

From (2.10) we have a self-financing strategy whose final value is $f(S_T)$ and such that $\left(\phi_t^0, \phi_t^1\right)$ are given by

$$\phi_t^1 = \frac{\partial C(t,S_t)}{\partial S_t}$$

and

$$e^{rt}\phi_t^0 = C(t,S_t) - \frac{\partial C(t,S_t)}{\partial S_t}S_t.$$

■

Pricing and hedging of a call option. The Black-Scholes formula.

If we take $h = (S_T - K)_+$, we have

$$C(t,S_t) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-) \text{ (Black-Scholes' formula)}$$

where $\Phi(x)$ is the standard normal distribution function

$$d_\pm = \frac{\log(\frac{S_t}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

In fact

$$\begin{aligned}
&C(t,S_t)\\
&= \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)}(S_T-K)_+|\mathcal{F}_t)\\
&= e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}(S_T\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t) - Ke^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}(\mathbf{1}_{\{S_T>K\}}|\mathcal{F}_t)\\
&= e^{-r(T-t)}S_t\mathbb{E}_{\mathbb{P}^*}\left(\frac{S_T}{S_t}\mathbf{1}_{\{\frac{S_T}{S_t}>\frac{K}{x}\}}\right)_{x=S_t} - Ke^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\frac{S_T}{S_t}>\frac{K}{x}\}}\right)_{x=S_t},
\end{aligned}$$

but

$$\begin{aligned}\frac{S_T}{S_t} &= \exp\{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\,(W_T-W_t)\}\\ &\overset{\text{Law}}{=} \exp\{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}\}\end{aligned}$$

then

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\left\{\frac{S_T}{S_t}>\frac{K}{x}\right\}}\right) &= \mathbb{P}^*\left(\frac{S_T}{S_t}>\frac{K}{x}\right)\\ &= \mathbb{P}^*\left(\log\frac{S_T}{S_t}>\log\frac{K}{x}\right)\\ &= \mathbb{P}^*\left(\frac{W_{T-t}}{\sqrt{(T-t)}}>\frac{\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\right)\\ &= \Phi\left(\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\right)\\ &= \Phi(d_-)\text{ (after substituting for } x \text{ by } S_t)\end{aligned}$$

On the other hand, if we write Y to indicate a standard normal random variable

$$\begin{aligned}&e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left(\frac{S_T}{S_t}\mathbf{1}_{\left\{\frac{S_T}{S_t}>\frac{K}{x}\right\}}\right)\\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left(\exp\left\{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}\right\}\mathbf{1}_{\left\{\sigma W_{T-t}>\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)\right\}}\right)\\ &= \mathbb{E}_{\mathbb{P}^*}\left(\exp\left\{\left(-\frac{1}{2}\sigma^2\right)(T-t)+\sigma W_{T-t}\right\}\mathbf{1}_{\left\{\sigma W_{T-t}>\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)(T-t)\right\}}\right)\\ &= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}}\exp\left\{-\frac{1}{2}\sigma^2(T-t)-\sigma\sqrt{(T-t)}y-\frac{1}{2}y^2\right\}\mathrm{d}y\\ &= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}}\exp\left\{-\frac{1}{2}\left(\sigma\sqrt{(T-t)}+y\right)^2\right\}\mathrm{d}y\\ &= \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\frac{\log\frac{x}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}}\exp\left\{-\frac{1}{2}u^2\right\}\mathrm{d}u\\ &= \Phi(d_+)\text{ (after substituting } S_t \text{ for } x)\end{aligned}$$

From here

$$\frac{\partial C(t,S_t)}{\partial S_t}=\Phi(d_+):=\Delta.$$

In fact,

$$\begin{aligned}\frac{\partial C(t,S_t)}{\partial S_t} &= \Phi(d_+) + S_t\frac{\partial \Phi(d_+)}{\partial S_t} - Ke^{-r(T-t)}\frac{\partial \Phi(d_-)}{\partial S_t} \\ &= \Phi(d_+) + S_t\frac{1}{\sqrt(2\pi)}e^{-\frac{d_+^2}{2}}\frac{\partial d_+}{\partial S_t} \\ &\quad - Ke^{-r(T-t)}\frac{1}{\sqrt(2\pi)}e^{-\frac{d_-^2}{2}}\frac{\partial d_-}{\partial S_t}.\end{aligned}$$

But

$$\frac{\partial d_\pm}{\partial S_t} = \frac{1}{S_t\sigma\sqrt(T-t)},$$

therefore

$$\begin{aligned}\frac{\partial C(t,S_t)}{\partial S_t} &= \Phi(d_+) + \frac{1}{\sqrt(2\pi)}\frac{\partial d_+}{\partial S_t}\left(S_te^{-\frac{d_+^2}{2}} - Ke^{-r(T-t)}e^{-\frac{d_-^2}{2}}\right) \\ &= \Phi(d_+) + \frac{1}{\sqrt(2\pi)}\frac{\partial d_+}{\partial S_t}S_te^{-\frac{d_+^2}{2}}\left(1 - \frac{K}{S_t}e^{-r(T-t)}e^{\frac{d_+^2}{2}-\frac{d_-^2}{2}}\right).\end{aligned}$$

Moreover

$$d_+ = d_- + \sigma\sqrt(T-t)$$

so

$$\begin{aligned}d_+^2 - d_-^2 &= (d_- + \sigma\sqrt(T-t))^2 - d_-^2 \\ &= 2d_-\sigma\sqrt(T-t) + \sigma^2(T-t) \\ &= 2\log\frac{S_t}{K} + 2r(T-t)\end{aligned}$$

and therefore

$$1 - \frac{K}{S_t}e^{-r(T-t)}e^{\frac{d_+^2}{2}-\frac{d_-^2}{2}} = 0.$$

Exercise 2.5.1 *In the Black-Scholes model compute the price and the self-financing hedging portfolios of contingent claims with payoffs:*

(1) $X = S_T^2$,

(2) $X = S_T/S_{T_0}$, $0 \le T_0 \le T$.

Analysis of sensitivity. The Greeks.

One of the most important things besides pricing and hedging is the calculation of sensitivities of the prices. These sensitivities are given Greek letters and this is why they are called Greeks. Let $C(t,S_t)$ the value of a portfolio based in a risky asset (S_t) (and bonds). By practical reasons is often very important to have an idea of the sensitivity of C with respect to changes in the value of S_t (to measure the risk of our portfolio for instance) and with respect to changes in the parameters of the model (to measure a bad specification of the model). The standard notation is:

- $\Delta = \frac{\partial C}{\partial S_t}$
- $\Gamma = \frac{\partial^2 C}{\partial S_t^2}$
- $\rho = \frac{\partial C}{\partial r}$
- $\Theta = \frac{\partial C}{\partial t}$
- $\mathcal{V} = \frac{\partial C}{\partial \sigma}$

All these indexes of sensitivity are known as the Greeks. These include $\mathcal{V}$ that is pronounced Vega and that is not a Greek letter (κ was previously used). A portfolio that is not sensitive to small changes with respect to some parameter is said to be neutral: : delta neutral, gamma neutral,..

Proposition 2.5.4 *In the Black-Scholes model the portfolio that replicates a call with strike K and maturity time T has the following Greeks:*

- $\Delta = \Phi(d_+) > 0$
- $\Gamma = \frac{\phi(d_+)}{S_t\sigma\sqrt{(T-t)}} > 0$ *(where ϕ is the density of a standard normal random variable)*
- $\rho = K(T-t)e^{-r(T-t)}\Phi(d_+) > 0$
- $\Theta = -\frac{S_t\sigma}{2\sqrt{(T-t)}}\phi(d_+) - Kre^{-r(T-t)}\Phi(d_-) < 0$
- $\mathcal{V} = S_t\phi(d_+)\sqrt{(T-t)} > 0$

Exercise 2.5.2 *Prove that* $\Theta = -\frac{S_t\sigma}{2\sqrt{(T-t)}}\phi(d_+) - Kre^{-r(T-t)}\Phi(d_-)$.

Note that equation (2.11), can be written

$$\Theta + rS_s\Delta + \frac{1}{2}\sigma^2 S_s^2\Gamma = rC(s, S_s).$$

Exotic Options

Not all the options have a payoff $h = f(S_T)$. For instance we have the Asian options whose payoff is

$$h = \left(\frac{1}{T}\int_0^T S_u \mathrm{d}u - K\right)_+$$

the lookback options,

$$\text{("lookback call")}\ h = S_T - S_*, \text{ where } S_* = \min_{0\le t\le T} S_t$$

$$\text{("lookback put")}\ h = S^* - S_T, \text{ where } S^* = \max_{0\le t\le T} S_t,$$

or the barrier options

$$(\text{"down-and-out-call"})\ h = (S_T - K)_+ \mathbf{1}_{\{S_* \geq K\}}$$

$$(\text{"down-and-in-call"})\ h = (S_T - K)_+ \mathbf{1}_{\{S_* \leq K\}}.$$

Example 2.5.1 *Consider an Asian option with payoff*

$$h = \left(\frac{1}{T} \int_0^T S_u \mathrm{d}u - K \right)_+ ,$$

by the previous theorem $C_t = \mathbb{E}_{\mathbb{P}^*}(e^{-r(T-t)} h | \mathcal{F}_t)$. *Define*

$$\varphi(t,x) = \mathbb{E}_{\mathbb{P}^*}\left(\left(\frac{1}{T} \int_t^T \frac{S_u}{S_t} \mathrm{d}u - x \right)_+ \right).$$

Then

$$\begin{aligned}
C_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*}\left(\left(\frac{1}{T} \int_0^T S_u \mathrm{d}u - K \right)_+ \middle| \mathcal{F}_t \right) \\
&= e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*}\left(\left(\frac{1}{T} \int_t^T S_u \mathrm{d}u - \left(K - \frac{1}{T} \int_0^t S_u \mathrm{d}u \right) \right)_+ \middle| \mathcal{F}_t \right) \\
&= e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left(\left(\frac{1}{T} \int_t^T \frac{S_u}{S_t} \mathrm{d}u - \frac{K - \frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t} \right)_+ \middle| \mathcal{F}_t \right) \\
&= e^{-r(T-t)} S_t \varphi(t, Z_t)
\end{aligned}$$

where $Z_t = \frac{K - \frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t}$. *Is easy to see that*

$$\mathrm{d}Z_t = \left(\left(\sigma^2 - r\right) Z_t - \frac{1}{T} \right) \mathrm{d}t - \sigma Z_t \mathrm{d}W_t.$$

In fact, applying the integration by parts formula and the Itô formula:

$$\begin{aligned}
\mathrm{d}Z_t &= \mathrm{d}\left(\frac{K}{S_t} \right) - \frac{1}{T S_t} \mathrm{d}\left(\int_0^t S_u \mathrm{d}u \right) - \mathrm{d}\left(\frac{1}{S_t} \right) \frac{1}{T} \int_0^t S_u \mathrm{d}u \\
&= -\frac{K}{S_t^2} \mathrm{d}S_t + \frac{K}{S_t^3} \mathrm{d}\langle S_t \rangle - \frac{S_t}{T S_t} \mathrm{d}t + \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t^2} \mathrm{d}S_t - \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t^3} \mathrm{d}\langle S_t \rangle,
\end{aligned}$$

but since $\mathrm{d}S_t = rS_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$, *we have that*

$$\begin{aligned}
\mathrm{d}Z_t &= \left(-\frac{K}{S_t} r + \frac{K}{S_t} \sigma^2 + r \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t} - \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t} \sigma^2 - \frac{1}{T} \right) \mathrm{d}t \\
&\quad + \left(-\frac{K}{S_t} \sigma + \frac{\frac{1}{T}\int_0^t S_u \mathrm{d}u}{S_t} \sigma \right) \mathrm{d}W_t \\
&= \left((\sigma^2 - r) Z_t - \frac{1}{T} \right) \mathrm{d}t - \sigma Z_t \mathrm{d}W_t.
\end{aligned}$$

Then, we know that $\tilde{C}_t = e^{-r(T-t)}\tilde{S}_t\varphi(t, Z_t), t \le T$ *is a martingale. So if we assume that* $\varphi(t, x) \in C^{1,2}$ *we will have that*

$$\begin{aligned}
\mathrm{d}\varphi &= \frac{\partial\varphi}{\partial t}\mathrm{d}t + \frac{\partial\varphi}{\partial Z_t}\mathrm{d}Z_t + \frac{1}{2}\frac{\partial^2\varphi}{\partial {Z_t}^2}\sigma^2 Z_t^2\mathrm{d}t \\
&= \left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial Z_t}\left(\sigma^2 - r)Z_t - \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2\right)\mathrm{d}t \\
&- \frac{\partial\varphi}{\partial Z_t}\sigma Z_t\mathrm{d}W_t.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathrm{d}\tilde{C}_t &= re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t + e^{-r(T-t)}\varphi\mathrm{d}\tilde{S}_t + e^{-r(T-t)}\tilde{S}_t\mathrm{d}\varphi \\
&+ e^{-r(T-t)}\mathrm{d}\langle\tilde{S}, \varphi\rangle_t \\
&= re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t + e^{-r(T-t)}\varphi\mathrm{d}\tilde{S}_t + e^{-r(T-t)}\tilde{S}_t\mathrm{d}\varphi \\
&- e^{-r(T-t)}\frac{\partial\varphi}{\partial Z_t}\sigma^2\tilde{S}_t Z_t\mathrm{d}t \\
&= e^{-r(T-t)}\left(\varphi - Z_t\frac{\partial\varphi}{\partial Z_t}\right)\mathrm{d}\tilde{S}_t \\
&+ re^{-r(T-t)}\tilde{S}_t\varphi\mathrm{d}t - e^{-r(T-t)}\frac{\partial\varphi}{\partial Z_t}\sigma^2\tilde{S}_t Z_t\mathrm{d}t \\
&+ e^{-r(T-t)}\tilde{S}_t\left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial Z_t}\left((\sigma^2 - r)Z_t - \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2\right)\mathrm{d}t,
\end{aligned}$$

by identifying the martingale parts

$$\mathrm{d}\tilde{C}_t = e^{-r(T-t)}\left(\varphi - Z_t\frac{\partial\varphi}{\partial Z_t}\right)\mathrm{d}\tilde{S}_t$$

$$r\varphi + \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial Z_t}\left(rZ_t + \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2 = 0.$$

Therefore the hedging strategy is given by $\left(\phi_t^0, \phi_t^1\right)$ *with* $\phi_t^0 S_t^0 = C_t - \phi_t^1 S_t$ *and*

$$\phi_t^1 = e^{-r(T-t)}\left(\varphi - Z_t\frac{\partial\varphi}{\partial Z_t}\right),$$

where φ *is the solution of the partial differential equation*

$$r\varphi + \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial x}\left(rx + \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}\sigma^2 x^2 = 0 \tag{2.12}$$

with the boundary condition $\varphi(T, x) = x_-$ *(negative part of* x*). These equation can be solved numerically.*

Exercise 2.5.3 *Show that the price of an Asian option with floating strike (payoff=*$\left(\frac{1}{T}\int_0^T S_u \mathrm{d}u - S_T\right)_+$*) is given, at the initial time, by*

$$C = e^{-rT} S_0 \varphi(0,0)$$

where φ *is the solution of the equation (2.12) with boundary condition* $\varphi(T,x) = (1+x)_-$.

2.5.2 Multidimensional Black-Scholes model with continuous dividends

The model of the financial market consists in $(d+1)$ stocks S_t^0, $S_t^1, ..., S_t^d$ in such a way that

$$\mathrm{d}S_t^0 = S_t^0 r(t)\mathrm{d}t, S_0^0 = 1,$$

and

$$\mathrm{d}S_t^i = S_t^i(\mu^i(t)\mathrm{d}t + \sum_{j=1}^d \sigma^{ij}(t)\mathrm{d}W_t^j), i = 1, ..., d$$

where $W = (W^1, ..., W^d)$ is a d-dimensional Brownian motion. By simplicity we assume that μ, σ and r are deterministic and cadlag. We shall consider the natural filtration associated with W.

An investment strategy will be an adapted process $\phi = ((\phi_t^0, \phi_t^1, ..., \phi_t^d))_{0\le t\le T}$ in R^{d+1}. The value of the portfolio at time t is given by the scalar product

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{i=0}^d \phi_t^i S_t^i,$$

and its discounted value is

$$\tilde{V}_t(\phi) = e^{-\int_0^t r_s ds} V_t(\phi) = \phi_t \cdot \tilde{S}_t.$$

We assume that the stocks can give continuous and deterministic dividends: $((\delta_t^1, ..., \delta_t^d))_{0\le t\le T}$, in such a way that, if the strategy is self-financing,

$$\mathrm{d}V_t(\phi) = \sum_{i=0}^d \phi_t^i \mathrm{d}S_t^i + \sum_{i=1}^d \phi_t^i S_t^i \delta_t^i \mathrm{d}t,$$

Then we can write

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= \mathrm{d}\left(e^{-\int_0^t r_s \mathrm{d}s} V_t(\phi)\right) = -r_t e^{-\int_0^t r_s \mathrm{d}s} V_t \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \mathrm{d}V_t \\
&= -r_t e^{-\int_0^t r_s \mathrm{d}s} V_t \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \left(\sum_{i=0}^d \phi_t^i \mathrm{d}S_t^i + \sum_{i=1}^d \phi_t^i S_t^i \delta_t^i \mathrm{d}t \right) \\
&= e^{-\int_0^t r_s \mathrm{d}s} r_t (\phi_t^0 S_t^0 - V_t) \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \sum_{i=1}^d \left(\phi_t^i \mathrm{d}S_t^i + \phi_t^i S_t^i \delta_t^i \mathrm{d}t \right) \\
&= e^{-\int_0^t r_s \mathrm{d}s} \sum_{i=1}^d \phi_t^i S_t^i (\delta_t^i - r_t) \mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \sum_{i=1}^d \phi_t^i \mathrm{d}S_t^i \\
&= \sum_{i=1}^d \phi_t^i e^{-\int_0^t \delta_s^i \mathrm{d}s} \left(e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} S_t^i (\delta_t^i - r_t) \mathrm{d}t + e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \mathrm{d}S_t^i \right) \\
&= \sum_{i=1}^d \phi_t^i e^{-\int_0^t \delta_s^i \mathrm{d}s} \mathrm{d}\bar{S}_t^i, \qquad (2.13)
\end{aligned}$$

Where

$$\bar{S}_t^i := \frac{S_t^i}{e^{\int_0^t (r_s - \delta_s^i) \mathrm{d}s}} = \frac{S_t^i e^{\int_0^t \delta_s^i \mathrm{d}s}}{e^{\int_0^t r_s \mathrm{d}s}}$$

Now we look for a probability under which the *discounted* values of $\left(S_t^i e^{\int_0^t \delta_s^i \mathrm{d}s}\right)_{0 \le t \le T}, i = 1, ...d$, are martingales.

$$\begin{aligned}
\mathrm{d}\bar{S}_t^i &= e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} \left(S_t^i (\delta_t^i - r_t) \mathrm{d}t + \mathrm{d}S_t^i \right) \\
&= e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} S_t^i \left(\left(\delta_t^i + \mu_t^i - r_t \right) \mathrm{d}t + \sum_{j=1}^d \sigma^{ij}(t) \mathrm{d}W_t^j \right) \\
&= e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} S_t^i \sum_{j=1}^d \sigma^{ij}(t) \left(\sum_{k=1}^d \left(\sigma^{-1}\right)^{jk}(t) (\delta_t^k + \mu_t^k - r_t) \mathrm{d}t + \mathrm{d}W_t^j \right) \\
&= e^{-\int_0^t (r_s - \delta_s^i) \mathrm{d}s} S_t^i \sum_{j=1}^d \sigma^{ij}(t) \mathrm{d}\tilde{W}_t^j \qquad (2.14)
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{d}\tilde{W}_t^j &= \mathrm{d}W_t^j + \sum_{k=1}^d \left(\sigma^{-1}\right)^{jk}(t) (\delta_t^k + \mu_t^k - r_t) \mathrm{d}t, \quad j = 1, ..., d \\
&= \mathrm{d}W_t^j + \theta_j(t) \mathrm{d}t, \quad j = 1, ..., d,
\end{aligned}$$

with $\theta_j(t) = \sum_{k=1}^d \left(\sigma^{-1}\right)^{jk}(t)(r_t - \delta_t^k - \mu_t^k)$.Then if we take

$$\mathrm{d}\mathbb{P}^* = \Pi_{j=1}^n \exp\left\{ -\int_0^T \theta_j(t) \mathrm{d}W_t^j - \frac{1}{2} \int_0^T \theta_j^2(t) \mathrm{d}t \right\} \mathrm{d}\mathbb{P}.$$

it turns out that $\left(\tilde{W}_t\right)_{0\leq t\leq T}$ is a d-dimensional Brownian motion with respect to the probability $\mathbb{P}^*$ (a multidimensional version of the Girsanov theorem, see subsection 2.6.1). Therefore, if X is a $\mathbb{P}^*$-square integrable payoff, by the representation theorem of (mutidimensional) Brownian martingales (see subsection 2.6.2), we can write

$$\mathbb{E}_{\mathbb{P}^*}\left(\tilde{X}|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(\tilde{X}\right) + \sum_{j=1}^{d}\int_0^t h_s^j \mathrm{d}\tilde{W}_s^j,$$

where $\left(h_t^j\right)$ are $\mathbb{P}^*$-square integrable adapted processes. Then, by (2.13) and (2.14)

$$\begin{aligned}
\tilde{V}_t &= V_0 + \sum_{i=1}^{d}\int_0^t \phi_s^i e^{-\int_0^s \delta_u^i \mathrm{d}u}\mathrm{d}\bar{S}_s^i \\
&= V_0 + \sum_{i=1}^{d}\int_0^t \phi_s^i e^{-\int_0^s \delta_u^i \mathrm{d}u} e^{-\int_0^s (r_u-\delta_u^i)\mathrm{d}u} S_s^i \sum_{j=1}^{d}\sigma^{ij}(s)\mathrm{d}\tilde{W}_s^j \\
&= V_0 + \sum_{i=1}^{d}\int_0^t \phi_s^i \tilde{S}_s^i \sum_{j=1}^{d}\sigma^{ij}(s)\mathrm{d}\tilde{W}_s^j \\
&= V_0 + \sum_{j=1}^{d}\int_0^t \sum_{i=1}^{d}\phi_s^i \tilde{S}_s^i \sigma^{ij}(s)\mathrm{d}\tilde{W}_s^j
\end{aligned}$$

In such a way that if we take $V_0 = \mathbb{E}_{\mathbb{P}^*}(\tilde{X})$ and

$$\phi_t^i = \frac{1}{\tilde{S}_t^i}\sum_{k=1}^{d}\left(\sigma^{-1}\right)^{ki}(t)h_t^k, i = 1, ..., d.$$

we have that

$$\tilde{V}_t = \mathbb{E}_{\mathbb{P}^*}(\tilde{X}|\mathcal{F}_t),$$

is the discounted value at time t of a self-financing portfolio that replicates X, and $\left(\tilde{V}_t\right)$ is a $\mathbb{P}^*$-square integrable martingale.

Remark 2.5.2 *We have assumed that* $\left(\sigma_t^{ij}\right)_{0\leq t\leq T}$ *is invertible and from here we conclude that the model is free of arbitrage and complete. For the lack of arbitrage it is sufficient to have* $\theta(t)$ *such that* $\sum_{j=1}^{d}\sigma^{ij}(t)\theta_j(t) = \delta_t^i + \mu_t^i - r_t$. *But for completeness we need that* $\left(\sigma_t^{ij}\right)_{0\leq t\leq T}$ *is invertible. In this way we can have viable models where the dimension of* W *is greater than the number of stocks but then they are no complete.*

Remark 2.5.3 *Note that a portfolio with a constant number of assets is NOT a self-financing portfolio, except for the trivial case where you have only riskless assets. This is due to the fact that risky assets generate dividends and then your bank account change if you maintain the number of risky assets in your portfolio.*

Price of a call option

First note that under $\mathbb{P}^*$

$$\mathrm{d}S_t^i = S_t^i \left(\left(r_t - \delta_t^i\right) \mathrm{d}t + \sum_{j=1}^{d} \sigma_t^{ij} \mathrm{d}\tilde{W}_t^j \right), i = 1, ..., d,$$

so $\left(S_t^i e^{-\int_0^t (r_s - \delta_s^i)\mathrm{d}s}\right)$ are $\mathbb{P}^*$-martingales:

$$\begin{aligned} \mathrm{d}\left(S_t^i e^{-\int_0^t (r_s - \delta_s^i)\mathrm{d}s}\right) &= e^{-\int_0^t (r_s - \delta_s^i)\mathrm{d}s} \left(-S_t^i \left(r_t - \delta_t^i\right) \mathrm{d}t + \mathrm{d}S_t^i\right) \\ &= e^{-\int_0^t (r_s - \delta_s^i)\mathrm{d}s} \sum_{j=1}^{d} \sigma_t^{ij} S_t^i \mathrm{d}\tilde{W}_t^j. \end{aligned}$$

Also we have that, by the multidimensional Itô formula, under $\mathbb{P}^*$

$$S_t^i = S_0^i \exp\left\{ \sum_{j=1}^{d} \int_0^T \sigma_t^{ij} \mathrm{d}\tilde{W}_t^j + \int_0^T \left(r_t - \delta_t^i - \frac{1}{2} \sum_{j=1}^{d} \left(\sigma_t^{ij}\right)^2 \right) \mathrm{d}t \right\} \qquad (2.15)$$

Then

$$C_t := \mathbb{E}_{\mathbb{P}^*} \left(\left. \frac{(S_T^i - K)_+}{\exp\left\{\int_t^T r_s \mathrm{d}s\right\}} \right| \mathcal{F}_t \right) = \exp\left\{ -\int_t^T \delta_s^i \mathrm{d}s \right\} \mathbb{E}_{\mathbb{P}^*} \left(\left. \frac{(S_T^i - K)_+}{\exp\left\{\int_t^T (r_s - \delta_s^i) \mathrm{d}s\right\}} \right| \mathcal{F}_t \right),$$

under $\mathbb{P}^*$, and conditional to $\mathcal{F}_t$, by 2.15

$$\log S_T^i - \log S_t^i \sim N\left(\int_t^T (r_s - \delta_s^i)\mathrm{d}s - \frac{1}{2}\int_t^T \sum_{j=1}^{d} \left(\sigma_s^{ij}\right)^2 \mathrm{d}s, \int_t^T \sum_{j=1}^{d} \left(\sigma_s^{ij}\right)^2 \mathrm{d}s \right).$$

Therefore

$$C_t = \exp\left\{ -\int_t^T \delta_s^i \mathrm{d}s \right\} \left(S_t^i \Phi(d_+^i) - K \exp\left\{ -\int_t^T (r_s - \delta_s^i)\mathrm{d}s \right\} \Phi(d_-^i) \right),$$

with

$$d_{\pm}^i = \frac{\log \frac{S_t^i}{K} + \int_t^T \left(r_s - \delta_s^i \pm \frac{1}{2} \sum_{j=1}^{d} \left(\sigma_s^{ij}\right)^2 \right) \mathrm{d}s}{\sqrt{\int_t^T \sum_{j=1}^{d} \left(\sigma_s^{ij}\right)^2 \mathrm{d}s}}.$$

If we take $d = 1$ a constant interest rate r and constant dividend rate δ, we have the following formula for a call option, with strike K:

$$C_t = S_t^i e^{-\delta(T-t)}\Phi(d_+^i) - Ke^{-r(T-t)}\Phi(d_-^i),$$

with

$$d_\pm^i = \frac{\log\frac{S_t^i}{K} + (r - \delta \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

2.5.3 Currency options

A foreign currency can be thought as a kind of risky stock whose value at t, say X_t, changes in a random way at that generates some interests (or dividends) at the foreign rate, say r_f. In this way, if we assume a Black-Scholes for X and with domestic interest rates r_d, the price of a call option with strike K can be obtained by using the previous formula with $\delta = r_f$ y $r = r_d$.

Remark 2.5.4 *The previous arguments can be extended to the cases where μ, r and δ are adapted processes, cadlag and such that*

$$\Pi_{j=1}^n \exp\left\{-\int_0^t \theta_j(s)\mathrm{d}W_s^j - \frac{1}{2}\int_0^t \theta_j^2(s)\mathrm{d}s\right\}, 0 \leq t \leq T,$$

is a martingale. Also to the cases where σ is adapted and invertible for all ω and t, but in these cases we will not have formulas of Black-Scholes type since the discounted values of the stocks will not be log-normal distributed.

2.5.4 Stochastic volatility

Suppose that under $\mathbb{P}^*$

$$\mathrm{d}S_t = S_t(r_t\mathrm{d}t + \sigma(W_t^2, t)\mathrm{d}W_t^1)$$

where W^1 and W^2 are two independent Brownian motions (w.r.t. the same filtration $(\mathcal{F}_t)$). Then the price of a call option with strike K is given by

$$\begin{aligned} C_t &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s\mathrm{d}s}(S_T - K)_+\Big|\,\mathcal{F}_t\right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s\mathrm{d}s}(S_T - K)_+\Big|\,\sigma(W_s^2, s), t \leq s \leq T, \mathcal{F}_t\right)\Big|\,\mathcal{F}_t\right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(S_t\Phi(d_+) - Ke^{-\int_t^T r_s\mathrm{d}s}\Phi(d_-)\Big|\,\mathcal{F}_t\right), \end{aligned}$$

with

$$d_\pm = \frac{\log\frac{S_t}{K} + \int_t^T(r_s \pm \frac{1}{2}\sigma^2(W_s^2, s))\mathrm{d}s}{\sqrt{\int_t^T \sigma^2(W_s^2, s)\mathrm{d}s}},$$

If we assume that W_t^1, W_t^2 are Brownian motions w.r.t. the same filtration $(\mathcal{F}_t)$ with quadratic covariation $\int_0^t \rho_s \mathrm{d}s$, we obtain

$$\mathbb{E}\left(S_t \xi_t \Phi(d_+) - K e^{-\int_t^T r_s \mathrm{d}s} \Phi(d_-) \Big| \mathcal{F}_t \right),$$

with

$$d_\pm = \frac{\log \frac{S_t \xi_t}{K} + \int_t^T (r_s \pm \frac{1}{2}(1-\rho_s^2)\sigma^2(W_s^2, s))\mathrm{d}s}{\sqrt{\int_t^T (1-\rho_s^2)\sigma^2(W_s^2, s))\mathrm{d}s}},$$

and

$$\xi_t = \exp\left\{ \int_t^T \rho_s \sigma(W_s^2, s) \mathrm{d}W_s^2 - \frac{1}{2}\int_t^T \rho_s^2 \sigma^2(W_s^2, s)\mathrm{d}s \right\}.$$

In fact, first note that we can write

$$\mathrm{d}W_t^1 = \sqrt{1-\rho_t^2}\mathrm{d}\hat{W}_t + \rho_t \mathrm{d}W_t^2,$$

where $\hat{W}$ is a Brownian motion independent of W^2. Therefore we have

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma(W_t^2, t)\left(\sqrt{1-\rho_t^2}\mathrm{d}\hat{W}_t + \rho_t \mathrm{d}W_t^2 \right) \right)$$

and by the Itô formula:

$$\begin{aligned} S_T = S_t \exp &\left\{ \int_t^T r_s \mathrm{d}s + \int_t^T \rho_s \sigma(W_s^2, s)\mathrm{d}W_s^2 - \frac{1}{2}\int_t^T \rho_s \sigma^2(W_s^2, s)\mathrm{d}s \right\} \\ \times \exp &\left\{ \int_t^T \sqrt{1-\rho_s^2}\sigma(W_s^2, s)\mathrm{d}\hat{W}_s - \frac{1}{2}\int_t^T (1-\rho_s^2)\sigma^2(W_s^2, s)\mathrm{d}s \right\}. \end{aligned}$$

Proposition 2.5.5 *Every Itô's martingale X with quadratic variation $\langle X, X\rangle_t = t$, is a Brownian motion.*

Proof. By the Itô formula

$$e^{i\lambda X_t} = e^{i\lambda X_u} + i\lambda \int_u^t e^{i\lambda X_s}\mathrm{d}X_s - \frac{\lambda^2}{2}\int_u^t e^{i\lambda X_s}\mathrm{d}s.$$

Consequently

$$\mathbb{E}(e^{i\lambda(X_t - X_u)}|\mathcal{F}_u) = 1 - \frac{\lambda^2}{2}\int_u^t \mathbb{E}(e^{i\lambda(X_s - X_u)}|\mathcal{F}_u)\mathrm{d}s$$

and

$$\mathbb{E}(e^{i\lambda(X_t - X_u)}|\mathcal{F}_u) = e^{-\frac{1}{2}\lambda^2(t-u)}.$$

Hence X has continuous trajectories, with independent and homogeneous increments with law $\mathrm{N}(0, t)$. In other words, X is a Brownian motion. ∎

Proposition 2.5.6 *Two Brownian motions W^1,W^2 with $\langle W^1, W^2\rangle_t = 0$ are independent.*

Proof. Let $A \in \sigma(W_t^1, 0 \le t \le T)$ and $B \in \sigma(W_t^2, 0 \le t \le T)$. By the Representation Theorem for Brownian random variables, there exist processes a and b such that

$$\mathbf{1}_A = \mathbb{E}\left(\mathbf{1}_A\right) + \int_0^T a_s \mathrm{d}W_s^1, \quad \mathbf{1}_B = \mathbb{E}\left(\mathbf{1}_B\right) + \int_0^T b_s \mathrm{d}W_s^2.$$

Define

$$X_t := \mathbb{E}\left(\mathbf{1}_A\right) + \int_0^t a_s \mathrm{d}W_s^1, \quad Y_t := \mathbb{E}\left(\mathbf{1}_B\right) + \int_0^t b_s \mathrm{d}W_s^2$$

then, by the Itô formula for Itô's processes (w.r.t. the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma(W_s^1, W_s^2, 0 \le s \le t,))$

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X_t \mathrm{d}Y_t + \int_0^t Y_t \mathrm{d}X_t + \int_0^T a_s b_s \mathrm{d}\langle W^1, W^2\rangle_s \\ &= \mathbb{E}\left(\mathbf{1}_A\right)\mathbb{E}\left(\mathbf{1}_B\right) + \int_0^t b_s X_t \mathrm{d}W_t^2 + \int_0^t a_s Y_t \mathrm{d}W_t^1. \end{aligned}$$

Finally

$$\mathbb{E}\left(\mathbf{1}_A \mathbf{1}_B\right) = \mathbb{E}\left(X_T Y_T\right) = \mathbb{E}\left(\mathbf{1}_A\right)\mathbb{E}\left(\mathbf{1}_B\right).$$

So A and B are independent. ■

2.6 Annex

2.6.1 The Girsanov theorem

Lemma 2.6.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{0\le t\le T}$, $\mathcal{F}_T = \mathcal{F}$. Let $Z_T > 0$ such that $\mathbb{E}(Z_T) = 1$ and $Z_t := \mathbb{E}(Z_T|\mathcal{F}_t), 0 \le t \le T$. Then if we define $\tilde{\mathbb{P}}(A) := \mathbb{E}(\mathbf{1}_A Z_T), \forall A \in \mathcal{F}$, and Y is an $\mathcal{F}_t$-measurable r.v. such that $\tilde{\mathbb{E}}(|Y|) < \infty$ then, for all $s \le t$,*

$$\tilde{\mathbb{E}}(Y|\mathcal{F}_s) = \frac{1}{Z_s}\mathbb{E}(Y Z_t|\mathcal{F}_s). \tag{2.16}$$

Proof. Take $A \in \mathcal{F}_s$ then

$$\begin{aligned} \tilde{\mathbb{E}}(\mathbf{1}_A Y) &= \mathbb{E}(\mathbf{1}_A Y Z_T) = \mathbb{E}(\mathbf{1}_A \mathbb{E}(Y Z_t|\mathcal{F}_s)) \\ &= \tilde{\mathbb{E}}\left(\mathbf{1}_A \frac{1}{Z_s}\mathbb{E}(Y Z_t|\mathcal{F}_s)\right). \end{aligned}$$

■

Theorem 2.6.1 *(Girsanov) Consider a probability space as before and $(\theta_t)_{0\leq t\leq T}$ an adapted process such that $\int_0^T \theta_t^2 \mathrm{d}t < \infty$ a.s. where*

$$Z_t := \exp\left\{\int_0^t \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s\right\},$$

and $\mathbb{E}(Z_T) = 1$ and W is an $(\mathcal{F}_t)$-Brownian motion. Then under the probability $\tilde{\mathbb{P}}(\cdot) := \mathbb{E}(\mathbf{1}.Z_T)$, $X_t = W_t - \int_0^t \theta_s \mathrm{d}s, 0 \leq t \leq T$, is an $(\mathcal{F}_t)$-Brownian motion.

Proof. $(Z_t)_{0\leq t\leq T}$ is, by construction, a positive local martingale. Then by Fatou's lemma it is a supermartingale and since $\mathbb{E}(Z_0) = \mathbb{E}(Z_T) = 1$, it has constant expectation and therefore it is a martingale. $(X_t)_{0\leq t\leq T}$ is adapted and continuous. We can see that the increments are independent and homogeneous.

$$\begin{aligned}
&\tilde{\mathbb{E}}(\exp\{iu(X_t - X_s)\}|\mathcal{F}_s)\\
&= \frac{1}{Z_s}\mathbb{E}(\exp\{iu(X_t - X_s)\}Z_t|\mathcal{F}_s)\\
&= \mathbb{E}(\exp\{\int_s^t (iu + \theta_u)\mathrm{d}W_u - \frac{1}{2}\int_s^t (2iu\theta_u + \theta_u^2)\mathrm{d}u\}|\mathcal{F}_s).
\end{aligned}$$

But, if we write

$$N_t := \exp\{iuX_t\}$$

and we apply the Itô formula to

$$Z_t N_t = \exp\left\{\int_0^t (iu + \theta_s)\mathrm{d}W_s - \frac{1}{2}\int_0^t (2iu\theta_s + \theta_s^2)\mathrm{d}s\right\}$$

we obtain

$$\begin{aligned}
&Z_t N_t\\
&= 1 + \int_0^t Z_s N_s \left((iu + \theta_s)\mathrm{d}W_s - \frac{1}{2}(2iu\theta_s + \theta_s^2)\mathrm{d}s\right) + \frac{1}{2}\int_0^t Z_s N_s (iu + \theta_s)^2 \mathrm{d}s\\
&= 1 + \int_0^t Z_s N_s (iu + \theta_s)\mathrm{d}W_s - \frac{u^2}{2}\int_0^t Z_s N_s \mathrm{d}s.
\end{aligned}$$

Then (by localizing with $\tau_n = \inf\{t \leq T, \int_0^t |(Z_s N_s (iu + \theta_s))|^2 \mathrm{d}s \geq n\}$)

$$\mathbb{E}(Z_{t\wedge\tau_n} N_{t\wedge\tau_n}|\mathcal{F}_s) = Z_{s\wedge\tau_n} N_{s\wedge\tau_n} - \frac{u^2}{2}\mathbb{E}\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} Z_v N_v \mathrm{d}v \middle| \mathcal{F}_s\right),$$

so, on $\{\tau_n \geq s\}$, $\mathbb{E}(Z_{t\wedge\tau_n} N_{t\wedge\tau_n}|\mathcal{F}_s) = \mathbb{E}(Z_T N_{t\wedge\tau_n}|\mathcal{F}_s)$ and we can write

$$\mathbf{1}_{\{\tau_n\geq s\}}\mathbb{E}(Z_T N_{t\wedge\tau_n}|\mathcal{F}_s) = \mathbf{1}_{\{\tau_n\geq s\}}\left(Z_{s\wedge\tau_n} N_{s\wedge\tau_n} - \frac{u^2}{2}\mathbb{E}\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} Z_T N_v \mathrm{d}v \middle| \mathcal{F}_s\right)\right)$$

taking now the limit when $n \to \infty$ and by the dominated convergence theorem

$$\mathbb{E}(Z_T N_t|\mathcal{F}_s) = Z_s N_s - \frac{u^2}{2}\mathbb{E}\left(\int_s^t Z_T N_v \mathrm{d}v \middle| \mathcal{F}_s\right),$$

consequently

$$\mathbb{E}(Z_t N_t|\mathcal{F}_s) = Z_s N_s - \frac{u^2}{2}\mathbb{E}\left(\int_s^t Z_v N_v \mathrm{d}v \middle| \mathcal{F}_s\right).$$

That is

$$\tilde{\mathbb{E}}(N_t|\mathcal{F}_s) = N_s - \frac{u^2}{2}\tilde{\mathbb{E}}\left(\int_s^t N_v \mathrm{d}v \middle| \mathcal{F}_s\right),$$

and we obtain

$$\tilde{\mathbb{E}}\left(\frac{N_t}{N_s}|\mathcal{F}_s\right) = 1 - \frac{u^2}{2}\int_s^t \tilde{\mathbb{E}}\left(\frac{N_v}{N_s}|\mathcal{F}_s\right)\mathrm{d}v.$$

This gives an equation for $g_s(t) := \tilde{\mathbb{E}}\left(\frac{N_t}{N_s}|\mathcal{F}_s\right)(\omega)$, such that

$$\begin{aligned} g_s'(t) &= -\frac{u^2}{2}g_s(t) \\ g_s(s) &= 1 \end{aligned}$$

In such a way that

$$g_s(t) = \exp\left\{-\frac{u^2}{2}(t-s)\right\},$$

that is

$$\tilde{\mathbb{E}}(\exp\{iu(X_t - X_s)\}|\mathcal{F}_s) = \exp\left\{-\frac{u^2}{2}(t-s)\right\}$$

so the increments are independent and homogeneous with law $\mathrm{N}(0, t-s)$. ■

Exercise 2.6.1 *Consider the process* $(S_t)_{0\le t\le T}$

$$\mathrm{d}S_t = S_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}B_t\right),\ 0 \le t \le T,$$

$(B_t)_{0\le t\le T}$ *a standard Brownian motion. Using Girsanov's theorem compute a probability* $\mathbb{Q}$ *under which* $\tilde{S}_t := S_t e^{-rt}$, $0 \le t \le T$ *is a martingale.*

2.6.2 The representation theorem of Brownian martingales

Lemma 2.6.2 *Set* $\mathcal{F}_T = \sigma\left(B_t, 0 \le t \le T\right)$*, where* B *is a Brownian motion. Consider stepwise functions*

$$f(t) = \sum_{i=1}^n \lambda_i \mathbf{1}_{(t_{i-1},t_i]}(t)$$

with $\lambda_i \in \mathbb{R}$ *and* $0 = t_0 < t_1 ... < t_n \leq T$. *Denote by* $\mathcal{J}$ *that set of functions. Set* $\mathcal{E}_T^f = \exp\left\{\int_0^T f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^T f^2(s)\mathrm{d}s\right\}, f \in \mathcal{J}$. *If* $Y \in L^2(\mathcal{F}_T, \mathbb{P})$ *is orthogonal to* $\mathcal{E}_T^f, f \in \mathcal{J}$ *then* $Y = 0$, $\mathbb{P}$ *c.s.*

Proof. Consider $Y \in L^2(\mathcal{F}_T, P)$, orthogonal to $\mathcal{E}_T^f$. Let $\mathcal{G}_n := \sigma\left(B_{t_1}, ..., B_{t_n}\right)$, we have

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}}) - \frac{1}{2}\sum_{i=1}^n \lambda_i^2(t_i - t_{i-1})\right\} Y\right) = 0,$$

then,

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}})\right\} Y\right) = 0$$

and, because $\mathcal{G}_n = \sigma\ (B_{t_1}, ..., B_{t_n}) = \sigma\ (B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})$,

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}})\}\right\} \mathbb{E}(Y|\mathcal{G}_n)\right) = 0.$$

Y can be decomposed as $Y = Y_+ - Y_-$, so

$$\mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}})\}\right\} \mathbb{E}(Y_+|\mathcal{G}_n)\right) = \mathbb{E}\left(\exp\left\{\sum_{i=1}^n \lambda_i(B_{t_i} - B_{t_{i-1}})\}\right\} \mathbb{E}(Y_-|\mathcal{G}_n)\right).$$

Let X be the map

$$\begin{aligned} &X : \Omega \to \mathbb{R}^n \\ &\omega \longmapsto X(\omega) = (B_{t_1}(\omega), B_{t_2}(\omega) - B_{t_1}(\omega), ..., B_{t_n}(\omega) - B_{t_{n-1}}(\omega)), \end{aligned}$$

then

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\left\{\sum_{i=1}^n \lambda_i x_i\right\} \mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n)\mathrm{d}\mathbb{P}^X(x_1, x_2, ..., x_n) \\ &= \int_{\mathbb{R}^n} \exp\left\{\sum_{i=1}^n \lambda_i x_i\right\} \mathbb{E}(Y_-|\mathcal{G}_n)(x_1, x_2, ..., x_n)\mathrm{d}\mathbb{P}^X(x_1, x_2, ..., x_n), \end{aligned}$$

in such a way that the Laplace transform of $\mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n)\mathrm{d}\mathbb{P}^X$ is equal to that of $\mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n)\mathrm{d}\mathbb{P}^X$ and therefore, by the uniqueness of the Laplace transform, $\mathbb{E}(Y_+|\mathcal{G}_n)(x_1, x_2, ..., x_n) = \mathbb{E}(Y_-|\mathcal{G}_n)(x_1, x_2, ..., x_n)$, $\mathbb{P}^X$ a.s. From here $\mathbb{E}(Y_+|\mathcal{G}_n) = \mathbb{E}(Y_-|\mathcal{G}_n)$ $\mathbb{P}$a.s., and finally since this is true for any $\mathcal{G}_n$ of the previous type it turns out that Y is zero $\mathbb{P}$ a.s., since $\mathbb{E}(Y_\pm| \vee_{k=1}^n \mathcal{G}_k) \underset{k\to\infty}{\to} \mathbb{E}(Y_\pm| \vee_{k=1}^\infty \mathcal{G}_k) = Y_\pm$. ■

Proposition 2.6.1 *For all random variable* $F \in L^2(\mathcal{F}_T, \mathbb{P})$ *there exists and adapted process* $(Y_t)_{0\leq t\leq T}$, *with* $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right) < \infty$, *such that*

$$F = \mathbb{E}(F) + \int_0^T Y_t \mathrm{d}B_t$$

Proof. Suppose that $F-\mathbb{E}(F)$ is orthogonal to $\int_0^T Y_t \mathrm{d}B_t$ for all $(Y_t)_{0\le t\le T}$, with $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right)<\infty$, then if we prove that $F-\mathbb{E}(F)=0$, $\mathbb{P}$ a.s. then we have finished, since the Hilbert space of centered random variables of $L^2(\mathcal{F}_T,\mathbb{P})$ will coincide with its Hilbert subspace of random variables $\int_0^T Y_t \mathrm{d}B_t$ with $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right)<\infty$. Write $Z=F-\mathbb{E}(F)$, we have

$$\mathbb{E}((F-\mathbb{E}(F))\int_0^T Y_t \mathrm{d}B_t)=0.$$

Take $Y_t=\mathcal{E}_t^f f(t)$, with the $\mathcal{E}_t^f$ define previously, then

$$\mathbb{E}((F-\mathbb{E}(F))\int_0^T \mathcal{E}_t^f f(t)\mathrm{d}B_t)=0$$

and also, because $\mathbb{E}(F-\mathbb{E}(F))=0$, we have that

$$\mathbb{E}((F-\mathbb{E}(F))(1+\int_0^T \mathcal{E}_t^f f(t)\mathrm{d}B_t))=0,$$

but, by the Itô formula,

$$\mathcal{E}_T^f=1+\int_0^T \mathcal{E}_t^f f(t)\mathrm{d}B_t.$$

So

$$\mathbb{E}((F-\mathbb{E}(F))\mathcal{E}_T^f)=0$$

and by the previous lemma $F-\mathbb{E}(F)=0$, $\mathbb{P}$ a.s.. ■

Remark 2.6.1 *It can be shown that if we remove the condition* $\mathbb{E}\left(\int_0^T Y_t^2 \mathrm{d}t\right)<\infty$ *there are infinitely many adapted processes* Y *such that* $\int_0^T Y_t^2\mathrm{d}t<\infty$ *and*

$$F=\mathbb{E}(F)+\int_0^T Y_t \mathrm{d}B_t. \tag{2.17}$$

In fact consider any real number $0<a<T$ *and*

$$Z_t=\mathbf{1}_{[0,a)}(t)+\frac{1}{T-t}\mathbf{1}_{[a,\tau)}(t),$$

where $\tau=\inf\left\{t\ge a, \int_a^t \frac{1}{T-u}\mathrm{d}B_u=-B_a\right\}$*. It can be seen that* $\mathbb{P}(\tau<T)=1$*, then*

$$\int_0^T Z_t^2\mathrm{d}t=a+\frac{1}{T-\tau}-\frac{1}{T-a}<\infty \ a.s.$$

and that $\mathbb{E}\left(\int_0^T Z_t^2\mathrm{d}t\right)=\infty$*. Therefore* $\int_0^T Z_t\mathrm{d}B_t=0$*! and* $Y+Z$ *is another solution for (2.17).*

Remark 2.6.2 *It can be also shown (Dudley's Representation Theorem) that if $F \in \mathcal{F}_T$ there exist Y with $\int_0^T Y_t^2 \mathrm{dt} < \infty$ such that*

$$F = \int_0^T Y_t \mathrm{d}B_t.$$

Theorem 2.6.2 *Any square integrable martingale $(M_t)_{0 \leq t \leq T}$ can be written as*

$$M_t = M_0 + \int_0^t Y_s \mathrm{d}B_s, 0 \leq t \leq T$$

where Y_s is an adapted process with $\mathbb{E}(\int_0^T Y_t^2 \mathrm{dt}) < \infty$.

Proof. We can write

$$M_t = \mathbb{E}(M_T | \mathcal{F}_t)$$

and by the previous proposition

$$M_T = \mathbb{E}(M_T) + \int_0^T Y_s \mathrm{d}B_s$$

then it is enough to take conditional expectations. ■

Remark 2.6.3 *In the case that F is in $L^1(\mathcal{F}_T, \mathbb{P})$ we can find Y, with $\int_0^T Y_t^2 \mathrm{dt} < \infty$, such that*

$$\mathbb{E}\left(F | \mathcal{F}_t\right) = \mathbb{E}(F) + \int_0^t Y_t \mathrm{d}B_t.$$

Chapter 3

Interest rates models in continuous time

Interest rates models are used mainly for valuing and hedging bonds and options on bonds. It is remarkable that in bond markets there is not a benchmark model as the Black-Scholes model in stock markets .

3.1 Basic facts

3.1.1 The yield curve

In the models we studied so far we assumed a constant interest rate. In practice the interest rate depends on the emission data of a loan, or deposit, and the final or maturity time.

If someone borrows one euro at time t, till maturity T, he will have to pay an amount $F(t,T)$ at time T, this is equivalent to a mean rate of continuously compounded interest $R(t,T)$ given by the equality:

$$F(t,T) = e^{(T-t)R(t,T)}.$$

If we assume that interest rates are known: $(R(t,T))_{0\leq t\leq T}$, and there is not arbitrage then

$$F(t,s) = F(t,u)F(u,s), \forall t \leq u \leq s,$$

and from here together with the condition $F(t,t) = 1$, it follows, if $F(t,s)$ is differentiable as a function of s, that there exist a function $r(t)$ such that

$$F(t,T) = \exp\left(\int_t^T r(s)\mathrm{d}s\right).$$

In fact, let $s \geq t$

$$\begin{aligned} F(t, s+h) - F(t, s) &= F(t, s)F(s, s+h) - F(t, s) \\ &= F(t, s)(F(s, s+h) - 1), \end{aligned}$$

$$\frac{F(t, s+h) - F(t, s)}{F(t, s)h} = \frac{F(s, s+h) - F(s, s)}{h},$$

taking $h \to 0$ we have

$$\frac{\partial_2 F(t, s)/\partial s}{F(t, s)} = \partial_2 F(s, s)/\partial s := r(s)$$

and from here

$$F(t, T) = \exp\left(\int_t^T r(s)\mathrm{d}s\right).$$

Note that

$$R(t, T) = \frac{1}{T - t}\int_t^T r(s)\mathrm{d}s.$$

The function $r(s)$ is interpreted as an instantaneous interest rate, and it is also called *short rate*.

But look the other way round. Suppose that I want a contract to guarantee one euro at time T. We have the so called *bonds*. What is the price of a bond at time t? To receive $F(t, T)$ at time T we have to pay (put in the bank account) one euro, then, to receive one euro, we have to pay $1/F(t, T)$, so the price of the bond is

$$P(t, T) = \frac{1}{F(t, T)}.$$

In practice we do not know the prices of the bonds in different times, these prices are changing randomly, but intuitively it seems that there must exist a relation among all these prices for different initial and maturity times. The interest rate models try to explain these prices.

The main object of our study is what is called the *zero coupon bond*

Definition 3.1.1 *A zero coupon bond with maturity T is a contract that guarantees one euro at time T. Its price at time t shall be denote by $P(t, T)$.*

The bonds with coupons are those that are giving certain amounts (coupons) until the maturity of the bond.

Definition 3.1.2 *The yield curve of a zero coupon bond is the graph corresponding to the map*

$$T \longmapsto R(t, T)$$

We saw above that if we can anticipate the future or we would like to build a bond market with deterministic prices for the different trading and maturity times, the lack of arbitrage lead us to

$$P(t,T) = e^{-\int_t^T r(s)\mathrm{d}s}.$$

and

$$R(t,T) = \frac{1}{T-t}\int_t^T r(s)\mathrm{d}s$$

3.1.2 Yield curve for a random future

For a fixed t, $P(t,T)$ is a function of T whose graph gives us the prices of the bonds at t or the *term structure* at t. It is expected a smooth function. If we fix T, $P(t,T)$ will be a stochastic process. In this context, our bond market will be a market with infinitely many assets: for each T we have an asset and we ask ourselves questions like:

- which models are sensible to valuate bonds?
- what relation must the prices of the bond have to avoid arbitrage opportunities?
- can we obtain the prices of the bonds if we have a model for short rates?
- given a model of bond market how can we calculate prices of derivatives, such as call options of bonds?

3.1.3 Interest rates

Consider the following situation: At time t we sell a bond with maturity S and with the money we receive, $P(t,S)$, we buy $P(t,S)/P(t,T)$ bonds with maturity T , with $T > S$. By this operation we have a contract such that we pay 1 at time S and receive $P(t,S)/P(t,T)$ at time T. This change, from 1 at S to $P(t,S)/P(t,T)$ at T, can be quoted by simple or continuously compounded interest rates in the period $[S,T]$:

- The simple *forward* interest rate (LIBOR), $L = L(t;S,T)$, which is the solution of the equation:

  $$1 + (T-S)L = \frac{P(t,S)}{P(t,T)}$$

  that is the simple interest rate guaranteed for the period $[S,T]$ *at time* t.

- The continuously compounded forward interest rate $R = R(t;S,T)$, solution of the equation:

  $$e^{R(T-S)} = \frac{P(t,S)}{P(t,T)}.$$

analogously to the previous case, is continuously compounded interest guaranteed at time t, for the period $[S,T]$. The quotation using simple interest rates is the usual at financial markets whereas continuously compounded rates are used in theoretical frameworks.

So, in the bond market we can define different interest rates. That is the prices of the bonds can be quoted in different ways.

Definition 3.1.3 *1. The simple forward rate for the interval $[S,T]$ contracted at t, (LIBOR acronym of "London Interbank Offer Rate") is defined as*

$$L(t;S,T) = -\frac{P(t,T) - P(t,S)}{(T-S)P(t,T)}. \tag{3.1}$$

2. The simple spot rate for $[t,T]$, spot LIBOR, is defined as

$$L(t,T) = -\frac{P(t,T) - 1}{(T-t)P(t,T)},$$

it is the previous one with $S = t$. That is to buy at time t a bond with maturity T.

3. The continuously compounded forward rate contracted at t for $[S,T]$ as

$$R(t;S,T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S}$$

4. The continuously compounded spot rate for $[t,T]$ as

$$R(t,T) = -\frac{\log P(t,T)}{T-t}$$

5. The instantaneous forward rate with maturity T contracted at t as (we assume smoothness in T)

$$f(t,S) = -\left.\frac{\partial \log P(t,T)}{\partial T}\right|_{T=S} = \lim_{T\to S} R(t;S,T)$$

6. The instantaneous (spot) short rate at t

$$r(t) = f(t,t) = \lim_{S\to t} f(t,S)$$

Note that the instantaneous forward rate with maturity T contracted at t can be seen as the rate contracted a t for the infinitesimal period $[T, T + \mathrm{d}T]$.

Fixed t, any of the rates defined previously, from 1 to 5, allow us to recover the prices of the bonds. Then, modelling these rates is equivalent to modelling the bond prices.

3.1.4 Bonds with coupons, swaps, caps and floors

Fixed coupons bonds

The simplest of the bonds with coupons is the bond with fixed coupons. It is a bond that at some times in between 0 and T gives predetermined profits (coupons) to the owner of the bond. Its formal description is:

- Let $T_0, T_1, ..., T_n$, fixed times. T_0 is the emission time of the bond, whereas $T_1, ..., T_n$ are the payment times.
- At time T_i the owner receives the amount c_i.
- At time T_n there is an extra payment: K.

It is obvious that this bond can be replicated with a portfolio with c_i zero-coupon bonds with maturities T_i, $i = 1, .., n$ and K zero-coupon bonds with maturity T_n. So, the price at time $t \leq T_1$ will be given by

$$p(t) = KP(t, T_n) + \sum_{i=1}^{n} c_i P(t, T_i).$$

Usually the coupons are expressed in terms of certain rates r_i instead of quantities, in such a way that for instance

$$c_i = r_i(T_i - T_{i-1})K.$$

For a standard coupon the intervals of time are equal:

$$T_i = T_0 + i\delta,$$

y $r_i = r$, de manera que

$$p(t) = K\left(P(t, T_n) + r\delta\sum_{i=1}^{n} P(t, T_i)\right).$$

Floating rate coupon

Quite often the coupons are not fixed in advance, but rather they are updated for every coupon period. One possibility is to take $r_i = L(T_{i-1}, T_i)$ where L is the spot LIBOR. Since

$$L(T_{i-1}, T_i)(T_i - T_{i-1}) = \frac{1}{P(T_{i-1}, T_i)} - 1$$

we have (taking $K = 1$)

$$c_i = L(T_{i-1}, T_i)(T_i - T_{i-1}) = \frac{1}{P(T_{i-1}, T_i)} - 1.$$

We can replicate this amount, c_i, by selling a bond (without coupons) with maturity T_i and buying one with maturity T_{i-1} :

- With the bond sold we will have at T_i a payoff -1.
- With the bond bought, we will have 1 at T_{i-1} and we can buy bonds with maturity T_i giving a payoff $\frac{1}{P(T_{i-1},T_i)}$ at T_i.
- The total cost is $P(t,T_{i-1}) - P(t,T_i)$.

The for any time $t \leq T_0$ the price of this bond with random coupons is

$$p(t) = P(t,T_n) + \sum_{i=1}^{n} \left(P(t,T_{i-1}) - P(t,T_i)\right) = P(t,T_0)!.$$

This means that, with a unit of money at T_0, we can reproduce the cash flow of this coupon. In fact at time T_0 we can buy $\frac{1}{P(T_0,T_1)}$ bonds with maturity T_1 and then we get $\frac{1}{P(T_0,T_1)}$ units of money. Again at time T_1 we can invest one unit buying bonds with maturitity T_2 and so on. The cash flow is $\frac{1}{P(T_0,T_1)} - 1$ at T_1, $\frac{1}{P(T_1,T_2)} - 1$ at T_2,...and $\frac{1}{P(T_{n-1},T_n)} - 1 + 1$ at T_n.

Exercise 3.1.1 *If the instantaneous short rate at t is given by $r(t)$ a bank account is defined as an asset such that the unit of money evolves as*

$$\mathrm{d}S_t^0 = r(t)S_t^0\mathrm{d}t, \quad S_0^0 = 1.$$

Prove that you can obtain the same if you reinvest the money continuously in bonds that mature an infinitesimal time later.

Forward-rate agreement

A forward-rate agreement, or FRA, is the simplest form of interest rate derivative. It is an agreement to pay a fixed simple interest rate on a fixed sum of money between two fixed dates in the future. Asume that the two times are T_1 and T_2 and that at T_1 you receive 1 and you pay a simple interest rate r between T_1 and T_2, that is you pay $1 + r(T_2 - T_1)$ at T_2. Then the price of this contract at $t \leq T_1$ is

$$p(t) := P(t,T_1) - (1 + r(T_2 - T_1))\, P(t,T_2),$$

then if we want the price of the contract be zero we have the *fair* rate

$$r = -\frac{P(t,T_2) - P(t,T_1)}{(T_2 - T_1)P(t,T_2)} = L(t;T_1,T_2)!$$

that is the forward-rate defined in (3.1).

Interest rate Swaps

There are many types of rate swaps but all of them are basically exchanges of payments with fixed rates with random payments. We shall consider the so called *forward swaps settled in arrears*. Denote the principal by K and the swap rate (fixed rate) by R . Suppose equally spaced dates T_i, at time T_i, $i \geq 1$ we receive

$$K\delta L(T_{i-1}, T_i)$$

by paying $K\delta R$, so the cash flow at T_i is $K\delta[L(T_{i-1}, T_i) - R]$,. The value at $t \leq T_0$ of this cash flow is

$$\begin{aligned}
&K(P(t, T_{i-1}) - P(t, T_i)) - K\delta R P(t, T_i) \\
&= K P(t, T_{i-1}) - K(1 + R\delta) P(t, T_i),
\end{aligned}$$

so in total

$$\begin{aligned}
p(t) &= \sum_{i=1}^{n} (K P(t, T_{i-1}) - K(1 + R\delta) P(t, T_i)) \\
&= K P(t, T_0) - K \left(P(t, T_n) + R\delta \sum_{i=1}^{n} P(t, T_i) \right). \qquad (3.2)
\end{aligned}$$

Notice that $P(t, T_0)$ is the price of a coupon bond with floating rates whereas $P(t, T_n) + R\delta \sum_{i=1}^{n} P(t, T_i)$ is the price of a coupon bond with fixed rate R. This fixed rate R is usually taken in such a way that the value of the contract is zero when it is issued, this rate is named the *swap rate*. If $t \leq T_0$ is the time when it is issued, the swap rate is

$$R = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}.$$

Exercise 3.1.2 *Consider the following* discount curve. *(The discount curve is* *just the price of a zero-coupon bond as a function of maturity).*

0.0	1
0.5	0.975609756
1	0.949991019
1.5	0.923971427
2	0.898032347
2.5	0.872449235
3	0.847375747
3.5	0.822893781
4	0.799043132
4.5	0.775839012
5	0.753282381
5.5	0.710078204
6	0.698404609
6.5	0.669835212
7	0.648467232
7.5	0.630911969
8	0.612536891

Find the swap rate for a swap starting in year with six-month payments over 3 years. Find also the forward rates in these periodes.

Caps and Floors

A *cap* is a contract that protects you from paying more than a fixed rate in a loan, (the *cap rate*) R, even though the loan has floating rate. We can also define a *floor* that is a contract that guarantees that the rate is always above the so called *floor rate* R even for an investment with random rate.

A cap is a sum of *captlets*, they consist on these basic contracts.

- The interval $[0, T]$ is divided by equidistant points: $0 = T_0, T_1, ..., T_n = T$, with distance δ. Typically 1/4 of the year or half year.

- The cap works on a principal, say K, and the cap rate is R.

- The floating rate is for instance the LIBOR $L(T_{i-1}, T_i)$.

- The caplet i is defined as a contract with payoff *at* T_i given by

$$K\delta(L(T_{i-1}, T_i) - R)_+.$$

Proposition 3.1.1 *The value of a cap with principal K and* cap rate *R is that of one portfolio with $K(1 + R\delta)$ put options with maturities T_{i-1}, $i = 1, ..., n$ on bonds with maturities T_i and with strike $\frac{1}{1+R\delta}$.*

Proof. This is what you receive at time T_i

$$\begin{aligned} K\delta(L(T_{i-1},T_i)-R)_+ &= K\left(\frac{1}{P(T_{i-1},T_i)}-1-\delta R\right)_+ \\ &= \frac{K(1+R\delta)}{P(T_{i-1},T_i)}\left(\frac{1}{(1+R\delta)}-P(T_{i-1},T_i)\right)_+, \end{aligned}$$

but a payoff $\frac{1}{P(T_{i-1},T_i)}$ in T_i is equivalent to 1 at T_{i-1}. In other words, with the cash amount

$$K(1+R\delta)\left(\frac{1}{(1+R\delta)}-P(T_{i-1},T_i)\right)_+$$

at T_{i-1} I can buy

$$\frac{K(1+R\delta)}{P(T_{i-1},T_i)}\left(\frac{1}{(1+R\delta)}-P(T_{i-1},T_i)\right)_+$$

bonds with maturity T_i and I get this amount at T_i. ■

Note that, since

$$K\delta(L(T_{i-1},T_i)-R)_+ - K\delta(R-L(T_{i-1},T_i))_+ = K\delta(L(T_{i-1},T_i)-R),$$

we have that

$$\mathrm{Cap(t)} - \mathrm{Floor(t)} = \mathrm{Swap(t)}.$$

Swaptions

It is a contract s that gives the right to enter in a swap *at the maturity time* of the *swaption.* A *payer swaption* gives the right to enter in a swap as payer of the fixed rate. A *receiver swaption* gives the right to enter as the receiver of the fixed rates.

A payer swaption has similarities with the cap contract. In the cap the owner has the right to receive a random rate and to pay a constant rate and he will exercise in each period when the random rate is greater than the fixed one. Similarly the owner of payer swaption has the right to receive a floating rate and to pay a constant rate, however in the cap you chose if paying or not at each period, in the case of a swaption te decision is taken once for ever at the maturity time of the swaption. The value of the swap, according to (3.2) with principal 1, at the maturity time of the swaption, say T, is

$$P(T,T_0)-P(T,T_n)-R\delta\sum_{i=1}^{n}P(T,T_i),$$

so the payoff of a swaption is

$$\left(P(T,T_0)-P(T,T_n)-R\delta\sum_{i=1}^{n}P(T,T_i)\right)_+.$$

It is interesting the decomposition of the payer swaption payoff as

$$\left(P(T,T_0) - (P(T,T_n) + R\delta \sum_{i=1}^{n} P(T,T_i))\right)_+$$

where $P(T,T_0)$ is the value of a coupon bond (at T) with floating payments and $P(T,T_n) + R\delta \sum_{i=1}^{n} P(T,T_i)$ of a coupon bond with fixed payments. Then a swaption can be seen as an option to exchange a bond with fixed coupons by another with floating rate coupons. If $T = T_0$ a swaption becomes a put option with strike 1 on a bond with fixed coupons.

3.2 A general framework for short rates

We are going to define the process *bank account* or *riskless* asset. We shall create a random scenario for the instantaneous rates $r(s)$. More concretely we consider a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le T})$, and we assume that $(\mathcal{F}_t)_{0\le t\le T}$ is the filtration generated by a Brownian motion $(W_s)_{0\le t\le T}$ and that $\mathcal{F}_T = \mathcal{F}$. In this context we introduce the *riskless* asset:

$$S_t^0 = \exp\left\{\int_0^t r(s)\mathrm{d}s\right\}$$

where $(r(t))_{0\le t\le T}$ is an adapted process with $\int_0^T |r(s)|\mathrm{d}s < \infty$. In our market we shall assume the existence of risky assets: the bonds! (without coupons) with maturity less or equal than the horizon T. For each time $u \le T$ we define an adapted process $(P(t,u))_{0\le t\le u}$ satisfying $P(u,u) = 1$.

We make the following hypothesis:

(H) There exist a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that for all $0 \le u \le T$, $(\tilde{P}(t,u))_{0\le t\le u}$ defined by

$$\tilde{P}(t,u) = e^{-\int_0^t r(s)\mathrm{d}s} P(t,u)$$

is a martingale.

This hypothesis has the following interesting consequences:

Proposition 3.2.1

$$P(t,u) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^u r(s)\mathrm{d}s}\middle|\, \mathcal{F}_t\right)$$

Proof.

$$\begin{aligned}\tilde{P}(t,u) &= \mathbb{E}_{\mathbb{P}^*}(\tilde{P}(u,u)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^u r(s)\mathrm{d}s} P(u,u)\middle|\, \mathcal{F}_t\right)\\ &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^u r(s)\mathrm{d}s}\middle|\, \mathcal{F}_t\right),\end{aligned}$$

so, by eliminating the discount factor

$$= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^u r(s)\mathrm{d}s}\middle|\, \mathcal{F}_t\right)$$

■

Exercise 3.2.1 *Assume that*

$$r(t) = a + bW_t,$$

where $a > 0$, $b \neq 0$ and W is a $\mathbb{P}^*$-Brownian motion. Give an expression for the (instantaneous) forward rates.

The purpose of the following propositions is to describe the dynamics of $(P(t,u))_{0 \le t \le u}$. If we write, as usually, $Z_T = \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}$, we know that $Z_t := \mathbb{E}\left(\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} \middle| \mathcal{F}_t\right)$ is a strictly positive martingale, then, since the filtration is that the generated by the Brownian motion, we have the following representation:

Proposition 3.2.2 *There exists an adapted process $(q(t))_{0 \le t \le T}$ such that, for all $0 \le t \le T$,*

$$Z_t = \exp\left\{\int_0^t q(s)\mathrm{d}W_s - \frac{1}{2}\int_0^t q^2(s)\mathrm{d}s\right\}, \quad a.s.$$

Proof. Since Z_t is a Brownian martingale it can be seen, by an approximation argument, that it has a continuous version, then by a localization argument (since we do not know if it is square integrable) allows us to extend the Theorem (2.6.2) and to conclude that there is a process (H_t) satisfying $\int_0^T H_t^2 \mathrm{d}t < \infty$, a.s., such that

$$Z_t = 1 + \int_0^t H_s \mathrm{d}W_s,$$

now since $Z_t > 0$, P a.s., by applying the Itô formula, we have

$$\log Z_t = \int_0^t \frac{H_s}{Z_s}\mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{H_s^2}{Z_s^2}\mathrm{d}s$$

so $q(s) = \frac{H_s}{Z_s}$, *c.s.* ∎

Corollary 3.2.1 *The price at time t of a bond (without coupons) with maturity $u \le T$ is given by*

$$P(t,u) = \mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s + \int_t^u q(s)\mathrm{d}W_s - \frac{1}{2}\int_t^u q^2(s)\mathrm{d}s} \middle| \mathcal{F}_t\right)$$

Proof.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^u r(s)\mathrm{d}s} \middle| \mathcal{F}_t\right) &= \frac{\mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s} Z_u \middle| \mathcal{F}_t\right)}{Z_t} \\
&= \mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s} \frac{Z_u}{Z_t} \middle| \mathcal{F}_t\right) \\
&= \mathbb{E}\left(e^{-\int_t^u r(s)\mathrm{d}s + \int_t^u q(s)\mathrm{d}W_s - \frac{1}{2}\int_t^u q^2(s)\mathrm{d}s} \middle| \mathcal{F}_t\right).
\end{aligned}$$

∎

The following proposition gives an economic interpretation of the process q.

Proposition 3.2.3 *For each maturity u, there exists an adapted process $(\sigma_t^u)_{0\leq t\leq u}$ such that, for all $0 \leq t \leq u$,*

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = (r(t) - \sigma_t^u q((t))\mathrm{d}t + \sigma_t^u \mathrm{d}W_t$$

Proof. Since $\left(\tilde{P}(t,u)\right)$ is a martingale under $\mathbb{P}^*$ it turns out that $\left(\tilde{P}(t,u)Z_t\right)$ is a martingale under P, it is strictly positive as well and by reasoning as above we have

$$\tilde{P}(t,u)Z_t = P(0,u)e^{\int_0^t \theta_s^u \mathrm{d}W_s - \frac{1}{2}\int_0^t (\theta_s^u)^2 \mathrm{d}s}$$

for a certain adapted process $(\theta_s^u)_{0\leq t\leq u}$, in such a way that

$$\begin{aligned} P(t,u) = P(0,u)\exp\Bigg\{&\int_0^t r(s)\mathrm{d}s + \int_0^t (\theta_s^u - q(s))\mathrm{d}W_s \\ &-\frac{1}{2}\int_0^t ((\theta_s^u)^2 - q^2(s))\mathrm{d}s\Bigg\}, \end{aligned}$$

consequently, by applying the Itô formula,

$$\begin{aligned} \frac{\mathrm{d}P(t,u)}{P(t,u)} &= r(t)\mathrm{d}t + (\theta_t^u - q(t))\mathrm{d}W_t \\ &- \frac{1}{2}((\theta_t^u)^2 - q^2(t))\mathrm{d}t \\ &+ \frac{1}{2}(\theta_t^u - q(t))^2\mathrm{d}t \\ &= (r(t) + q^2(t) - \theta_t^u q(t))\mathrm{d}t \\ &+ (\theta_t^u - q(t))\mathrm{d}W_t, \end{aligned}$$

and the result follows by taking $\sigma_t^u = \theta_t^u - q(t)$. ■

Remark 3.2.1 *If we compare the formula*

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = (r(t) - \sigma_t^u q((t))\mathrm{d}t + \sigma_t^u \mathrm{d}W_t$$

with

$$\frac{\mathrm{d}S_t^0}{S_t^0} = r(t)\mathrm{d}t$$

we find that the bonds are assets with greater risk the riskless *asset S^0 and $-q(t)$ is the so-called* market price of risk. *Note also that, under $\mathbb{P}^*$*

$$W_t^* := W_t - \int_0^t q(s)\mathrm{d}s$$

is a standard $(\mathcal{F}_t)$- Brownian (by the Girsanov (2.6.1 theorem)) and we can write

$$\frac{\mathrm{d}P(t,u)}{P(t,u)} = r(t)\mathrm{d}t + \sigma_t^u \mathrm{d}W_t^*,$$

or equivalently

$$\frac{\mathrm{d}\tilde{P}(t,u)}{\tilde{P}(t,u)} = \sigma_t^u \mathrm{d}W_t^*,$$

justifying the name of risk neutral probability that we use for $\mathbb{P}^*$.

3.3 Options on bonds

Suppose a European contingent claim with maturity T and payoff

$$(P(T,T^*) - K)_+$$

where $T^* > T$ and $P(T,T^*)$ is the price of a bond with maturity T^*. The purpose is to valuate and hedge this call option of the bond with maturity T^*. It seems sensible to try to hedge this derivative with the riskless stock

$$S_t^0 = e^{\int_0^t r(s)\mathrm{d}s}$$

and the risky one

$$P(t,T^*) = P(0,T^*)\exp\left\{\int_0^t (r(s) - \frac{1}{2}\left(\sigma_s^{T^*}\right)^2)\mathrm{d}s + \int_0^t \sigma_s^{T^*}\mathrm{d}W_s^*\right\},$$

in such a way that a strategy will be a pair of adapted processes $\left(\phi_t^0, \phi_t^1\right)_{0\le t\le T^*}$ that represent the amount od assets without risk and the bonds with maturity T^* respectively. The value of the self-financing portfolio at time t is given by

$$V_t = \phi_t^0 S_t^0 + \phi_t^1 P(t,T^*)$$

and the self-financing condition implies that

$$\begin{aligned}
\mathrm{d}V_t &= \phi_t^0 \mathrm{d}S_t^0 + \phi_t^1 \mathrm{d}P(t,T^*) \\
&= \phi_t^0 r(t) e^{\int_0^t r(s)\mathrm{d}s}\mathrm{d}t + \phi_t^1 P(t,T^*)(r(t)\mathrm{d}t + \sigma_t^{T^*}\mathrm{d}W_t^*) \\
&= (\phi_t^0 r(t) e^{\int_0^t r(s)\mathrm{d}s} + \phi_t^1 r(t) P(t,T^*))\mathrm{d}t + \phi_t^1 \sigma_t^{T^*} P(t,T^*)\mathrm{d}W_t^* \\
&= r(t)V_t\mathrm{d}t + \phi_t^1 \sigma_t^{T^*} P(t,T^*)\mathrm{d}W_t^*,
\end{aligned}$$

we shall impose the conditions $\int_0^T |r(t)V_t|\mathrm{d}t < \infty$ and $\int_0^T |\phi_t^1\sigma_t^{T^*}P(t,T)|^2\mathrm{d}t < \infty$, to get well defined objects.

Definition 3.3.1 *A strategy* $\phi = (\phi_t^0, \phi_t^1)_{0\le t\le T}$ *is admissible if it is self-financing and its discounted value,* $\tilde{V}_t$*, is bounded below.*

Proposition 3.3.1 *Let* $T < T^*$*. Suppose that* $\sigma_t^{T^*} \neq 0$ *a.s. for all* $0 \le t \le T$ *. Let* h *be a positive random variable* $\mathcal{F}_T$*-measurable such that* $\tilde{h} = e^{-\int_0^T r(s)\mathrm{d}s}h$ *is square integrable under* $\mathbb{P}^*$*. Then there exists an admissible strategy such that at time* T *its value is* h *and at time* $t \le T$ *it is given by*

$$V_t = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s}h \,\middle|\, \mathcal{F}_t\right).$$

Proof. $\tilde{h}$ is a variable $\mathcal{F}_T$-measurable, with $\mathcal{F}_T = \sigma(W_t, 0 \leq t \leq T)$, it is square integrable, as well, with respect to $\mathbb{P}^*$, so

$$M_t := \mathbb{E}_{\mathbb{P}^*}(\tilde{h}|\mathcal{F}_t), 0 \leq t \leq T$$

is a, square integrable, $\mathbb{P}^*$-martingale. However we cannot apply the representation theorem with $\mathbb{P}^*$ since we do not know if $(M_t)_{0\leq t\leq T}$ is adapted to the $\mathbb{P}^*$-Brownian motion $\left(\tilde{W}_t\right)_{0\leq t\leq T}$ defined above, since we just have $\sigma(\tilde{W}_t, 0 \leq t \leq T) \subseteq \sigma(W_t, 0 \leq t \leq T)$. However by the abstract Bayes' rule we know that $(M_t Z_t)_{0\leq t\leq T}$ is a $\mathbb{P}$-martingale, thought not necessarily square integrable. In fact, we know that

$$\mathbb{E}_{\mathbb{P}^*}\left(\tilde{h}\middle|\,\mathcal{F}_t\right) = \frac{\mathbb{E}\left(\tilde{h}Z_T\middle|\,\mathcal{F}_t\right)}{Z_t}$$

in such a way that

$$M_t Z_t = \mathbb{E}(\tilde{h}Z_T|\mathcal{F}_t)$$

and $\left(\mathbb{E}(\tilde{h}Z_T|\mathcal{F}_t)\right)$ is clearly a $\mathbb{P}$-martingale. In that way we have, by the mentioned extension in the proof of Proposition 3.2.2 of the Theorem (2.6.2),

$$M_t Z_t = \mathbb{E}(M_t Z_t) + \int_0^t J_s \mathrm{d}W_s,$$

with (J_s) adapted and such that $\int_0^T J_s^2 \mathrm{d}s < \infty$ a.s., so (notice that M is an Itô process)

$$Z_t \mathrm{d}M_t + M_t \mathrm{d}Z_t + \mathrm{d}\langle M, Z\rangle_t = J_s \mathrm{d}W_s,$$

that is

$$\begin{aligned}
\mathrm{d}M_t &= -M_t \frac{\mathrm{d}Z_t}{Z_t} - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t + \frac{J_t}{Z_t}\mathrm{d}W_t \\
&= -M_t q(t)\mathrm{d}W_t - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t + \frac{J_t}{Z_t}\mathrm{d}W_t \\
&= (\frac{J_t}{Z_t} - M_t q(t))\mathrm{d}W_t - \frac{1}{Z_t}\mathrm{d}\langle M, Z\rangle_t \\
&= (\frac{J_t}{Z_t} - M_t q(t))\mathrm{d}W_t - (\frac{J_t}{Z_t} - M_t q(t))q(t)\mathrm{d}t \\
&= (\frac{J_t}{Z_t} - M_t q(t))\mathrm{d}W_t^* = H_t \mathrm{d}W_t^*
\end{aligned}$$

with $H_t := \frac{J_t}{Z_t} - M_t q(t), 0 \leq t \leq T$. Where the fourth equality is due to the fact that $\langle M, Z\rangle$ is a bounded variation process:

$$\langle M, Z\rangle = \frac{1}{2}\left(\langle M+Z, M+Z\rangle - \langle M, M\rangle - \langle Z, Z\rangle\right)$$

and the quadratic variation is an increasing process.

Therefore if we take

$$\phi_t^1 = \frac{H_t}{\sigma_t^{T^*}\tilde{P}(t,T^*)}, \phi_t^0 = \mathbb{E}_{\mathbb{P}^*}(\tilde{h}|\mathcal{F}_t) - \frac{H_t}{\sigma_t^{T^*}}$$

we will have a self-financing portfolio with final value $e^{\int_0^T r(s)\mathrm{d}s}M_T = h$. In fact

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= \mathrm{d}(e^{-\int_0^t r(s)\mathrm{d}s}V_t) = -e^{-\int_0^t r(s)\mathrm{d}s}r(t)V_t\mathrm{d}t + e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}V_t \\
&= e^{-\int_0^t r(s)\mathrm{d}s}(-r(t)V_t\mathrm{d}t + r(t)V_t\mathrm{d}t + \phi_t^1\sigma_t^{T^*}P(t,T^*)\mathrm{d}W_t^*) \\
&= \phi_t^1\sigma_t^{T^*}\tilde{P}(t,T^*)\mathrm{d}W_t^* = H_t\mathrm{d}W_t^* = \mathrm{d}M_t
\end{aligned}$$

Since $h \geq 0$ we have that $\tilde{V}_t \geq 0$, so the strategy is admissible. Also note that, since M is $\mathbb{P}^*$-square integrable, $\mathbb{E}_{\mathbb{P}^*}\left(\int_0^T H_t^2\mathrm{d}t\right) < \infty$ and H is unique $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$ a.s. ∎

3.4 Short rate models

Consider an evolution of the form,

$$\mathrm{d}r(t) = \mu(t,r(t))\mathrm{d}t + \sigma(t,r(t))\mathrm{d}W_t \tag{3.3}$$

under the real probability $\mathbb{P}$, where μ and σ are such that $r(t)$ is well defined. Assume that r is Markovian under the risk-neutral probability $\mathbb{P}^*$, then

$$P(t,T) = F(t,r(t);T), \tag{3.4}$$

in fact

$$\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s}\middle|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s}\middle|r(t)\right) = F(t,r(t);T).$$

Obviously the boundary condition $F(T,r(T);T) = 1$, should be fulfilled for all values of $r(T)$.

As we haven seen, in Proposition 3.3.1 that it is possible to replicate a bond with maturity T_1 by a self-financing portfolio (ϕ_t^0, ϕ_t^1), based on the bank account and a bond with maturity T_2, $T_2 > T_1$, then we have

$$P(t,T_1) = \phi_t^0 S_t^0 + \phi_t^1 P(t,T_2)$$

and, by the self-financing condition,

$$\mathrm{d}P(t,T_1) = r(t)\phi_t^0 e^{\int_0^t r(s)\mathrm{d}s}\mathrm{d}t + \phi_t^1\mathrm{d}P(t,T_2)$$

for all $t \leq T_1$. Assume that $F^{(i)} := F(t,r(t);T_i)$ are $C^{1,2}$ as functions of $(t,r(t))$, then applying the Itô formula to (3.4) we have

$$\begin{aligned}
&\frac{\partial F^{(1)}}{\partial t}\mathrm{d}t + \frac{\partial F^{(1)}}{\partial r}\mathrm{d}r(t) + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2(t,r(t))\mathrm{d}t \\
&= r(t)\phi_t^0 S_t^0\mathrm{d}t + \phi_t^1\frac{\partial F^{(2)}}{\partial t}\mathrm{d}t + \phi_t^1\frac{\partial F^{(2)}}{\partial r}\mathrm{d}r(t) + \phi_t^1\frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2(t,r(t))\mathrm{d}t.
\end{aligned}$$

So, by equating the $\mathrm{d}W_t$ and $\mathrm{d}t$ terms,

$$\begin{aligned}
&\frac{\partial F^{(1)}}{\partial t} + \frac{\partial F^{(1)}}{\partial r}\mu(t,r(t)) + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2(t,r(t)) \qquad (3.5)\\
&= r(t)\phi_t^0 S_t^0 + \phi_t^1 \frac{\partial F^{(2)}}{\partial t} + \phi_t^1 \frac{\partial F^{(2)}}{\partial r}\mu(t,r(t)) + \phi_t^1 \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2(t,r(t))
\end{aligned}$$

$$\frac{\partial F^{(1)}}{\partial r} = \phi_t^1 \frac{\partial F^{(2)}}{\partial r},$$

hence

$$\phi_t^1 = \frac{\frac{\partial F^{(1)}}{\partial r}}{\frac{\partial F^{(2)}}{\partial r}}$$

and

$$r(t)\phi_t^0 S_t^0 = r(t)\left(F^{(1)} - \frac{\frac{\partial F^{(1)}}{\partial r}}{\frac{\partial F^{(2)}}{\partial r}} F^{(2)}\right).$$

Then, by substituting in (3.5) we have

$$\begin{aligned}
&\frac{1}{\frac{\partial F^{(1)}}{\partial r}}\left(\frac{\partial F^{(1)}}{\partial t} + \frac{\partial F^{(1)}}{\partial r}\mu(t,r(t)) + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2(t,r(t)) - r(t)F^{(1)}\right)\\
&= \frac{1}{\frac{\partial F^{(2)}}{\partial r}}\left(\frac{\partial F^{(2)}}{\partial t} + \frac{\partial F^{(2)}}{\partial r}\mu(t,r(t)) + \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2(t,r(t)) - r(t)F^{(2)}\right).
\end{aligned}$$

Since this is true for all, $T_1, T_2 < T$, it turns out that there exists a $\lambda(t, r(t))$ such that

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = \lambda\sigma\frac{\partial F}{\partial r} \text{ (structure equation)} \qquad (3.6)$$

As we see there is an indetermination in λ and this has to do with the fact that the dynamics of $r(t)$ under $\mathbb{P}$ does **not** determine the prices of the bonds.

We have the following proposition

Proposition 3.4.1 *Let $\mathbb{P}^*$ be equivalent to $\mathbb{P}$such that*

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \exp\left\{-\int_0^T \lambda(s,r(s))\mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda^2(s,r(s))\mathrm{d}s\right\},$$

assume that

$$F(t,r(t);T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)ds}\Big| \mathcal{F}_t\right)$$

is $C^{1,2}$, then $F(t,r(t);T)$ is a solution of (3.6) with the boundary condition $F(T,r(T);T) = 1$. Also, under $\mathbb{P}^*$

$$\mathrm{d}r(t) = (\mu(t,r(t)) - \lambda(t,r(t))\sigma(t,r(t)))\mathrm{d}t + (t,r(t))\mathrm{d}\tilde{W}_t$$

and

$$\mathrm{d}P(t,T) = P(t,T)\left(r(t)\mathrm{d}t + \sigma_t^T \mathrm{d}\tilde{W}_t\right)$$

with $\tilde{W}$ *being a* $\mathbb{P}^*$*-Brownian motion and*

$$\sigma_t^T = \frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial r}\sigma.$$

Proof. Let $\mathbb{P}^*$ be equivalent to $\mathbb{P}$ such that

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \exp\left\{-\int_0^T \lambda(s,r)\mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda^2(s,r)\mathrm{d}s\right\} \tag{3.7}$$

(a sufficient condition to guarantee that (3.7) is a change to an equivalent probability is the *Novikov condition*: $\mathbb{E}\left(\exp\left\{\frac{1}{2}\int_0^T \lambda^2(s,r(s))\mathrm{d}s\right\}\right) < \infty$) then we know, by the Girsanov theorem, that

$$\tilde{W}_\cdot = W_\cdot + \int_0^\cdot \lambda(s,r(s))\mathrm{d}s$$

is an $(\mathcal{F}_t)$-Brownian motion with respect to $\mathbb{P}^*$. If we apply the Itô formula to $e^{-\int_0^t r(s)\mathrm{d}s}F(t,r(t);T)$ we have:

$$\begin{aligned}
&e^{-\int_0^t r(s)\mathrm{d}s}F(t,r(t);T)\\
&= F(0,r(0);T) + \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF\right)\mathrm{d}s\\
&+ \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\frac{\partial F}{\partial r}\sigma \mathrm{d}W_s\\
&= F(0,r(0);T) + \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF - \lambda\sigma\frac{\partial F}{\partial r}\right)\mathrm{d}s\\
&+ \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\frac{\partial F}{\partial r}\sigma \mathrm{d}\tilde{W}_s.
\end{aligned}$$

Then, since

$$e^{-\int_0^t r(s)\mathrm{d}s}F(t,r(t);T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T r(u)\mathrm{d}u}\Big|\ \mathcal{F}_t)\right)$$

it turns out that

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF - \lambda\sigma\frac{\partial F}{\partial r} = 0.$$

The boundary condition $F(T,r(T);T) = 1$ is obviously satisfied. Also we have that

$$\begin{aligned}
\mathrm{d}e^{-\int_0^t r(s)\mathrm{d}s}F(t,r(t);T) &= e^{-\int_0^t r(s)\mathrm{d}s}\frac{\partial F}{\partial r}\sigma \mathrm{d}\tilde{W}_t\\
&= e^{-\int_0^t r(s)\mathrm{d}s}\mathrm{d}F(t,r(t);T) + r(t)e^{-\int_0^t r(s)\mathrm{d}s}F(t,r(t);T)\mathrm{d}t
\end{aligned}$$

in such a way that

$$\mathrm{d}F(t, r(t); T) = F(t, r(t); T)\left(r(t)\mathrm{d}t + \frac{1}{F(t, r(t); T)}\frac{\partial F}{\partial r}\sigma \mathrm{d}\tilde{W}_s\right)$$

■

3.4.1 Examples

These are the most popular short-rate models in the literature. The dynamics for $r(t)$ is given under the risk neutral probability $\mathbb{P}^*$ and for simplicity we write W to denote a Brownian motion under $\mathbb{P}^*$.

1. Vasicek
$$\mathrm{d}r(t) = (b - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t.$$

2. Cox-Ingersoll-Ross (CIR)
$$\mathrm{d}r(t) = a(b - r(t))\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W_t$$

3. Dothan
$$\mathrm{d}r(t) = ar(t)\mathrm{d}t + \sigma r(t)\mathrm{d}W_t$$

4. Black-Derman-Toy
$$\mathrm{d}r(t) = \Theta(t)r(t)\mathrm{d}t + \sigma(t)r(t)\mathrm{d}W_t$$

5. Ho-Lee
$$\mathrm{d}r(t) = \Theta(t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

6. Hull-White (Vasicek generalized)
$$\mathrm{d}r(t) = (\Theta(t) - a(t)r(t))\mathrm{d}t + \sigma(t)\mathrm{d}W_t$$

7. Hull-White (CIR generalized)
$$\mathrm{d}r(t) = (\Theta(t) - a(t)r(t))\mathrm{d}t + \sigma(t)\sqrt{r(t)}\mathrm{d}W_t$$

3.4.2 Inversion of the yield curve

In the previous models we have several unknown parameters, that we shall denote by α. These parameters cannot be estimated from the observed values of $r(s)$, since that evolve not $\mathbb{P}^*$ but under the real probability $\mathbb{P}$. Where we can note the effect of $\mathbb{P}^*$ is in the real prices of the bonds, because if the model is correct

$$P(t, T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r(s)\mathrm{d}s}\,\middle|\, \mathcal{F}_t\right) = F(t, r(t); T, \alpha),$$

this latter equality if the model is Markovian under $\mathbb{P}^*$. Then, if, for instance, the evolution of r under $\mathbb{P}^*$ is given by

$$\mathrm{d}r(t) = \mu(t, r(t); \alpha)\mathrm{d}t + \sigma(t, r(t); \alpha)\mathrm{d}W_t$$

we can try to solve the partial differential equation

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = 0, \tag{3.8}$$

$$F(T, r(T); T, \alpha) = 1 \tag{3.9}$$

and then try to adjust the value of α for fitting $P(t,T) = F(t, r(t); T, \alpha)$ to the observed values of the bonds. Evidently some models will be more tractable than others.

3.4.3 Affine term structures

Definition 3.4.1 *If the term structure $\{P(t,T);\ 0 \leq t \leq T\}$ has the form*

$$P(t,T) = F(t, r(t); T)$$

where F is given by

$$F(t, r(t); T) = e^{A(t,T) - B(t,T)r}$$

and where $A(t,T)$ and $B(t,T)$ are deterministic, then we say that the model has an affine term structure (Affine Term Structure: ATS).

The structure equation (3.8) lead us to

$$\frac{\partial A}{\partial t} - \left\{1 + \frac{\partial B}{\partial t}\right\} r - \mu B + \frac{1}{2}\sigma^2 B^2 = 0$$

and the boundary condition (3.9) to

$$A(T,T) = 0$$
$$B(T,T) = 0.$$

We will say that the *model* (3.3) is affine if $\mu(t, r(t))$ and $\sigma^2(t, r(t))$ are also affine, that is

$$\mu(t, r(t)) = \alpha(t)r + \beta(t)$$
$$\sigma(t, r(t)) = \sqrt{(\gamma(t)r + \delta(t))}.$$

In such a case we have

$$\frac{\partial A}{\partial t} - \beta(t)B + \frac{1}{2}\delta(t)B^2 - \left\{1 + \frac{\partial B}{\partial t} + \alpha(t)B - \frac{1}{2}\gamma(t)B^2\right\} r = 0$$

and since this is satisfied for all values of $r(t)(\omega)$ we conclude

$$\frac{\partial A}{\partial t} - \beta(t)B + \frac{1}{2}\delta(t)B^2 = 0$$
$$1 + \frac{\partial B}{\partial t} + \alpha(t)B - \frac{1}{2}\gamma(t)B^2 = 0.$$

Remark 3.4.1 *It can be seen that under mild conditions on the coefficients $\mu(t,x)$ and $\sigma(t,x)$ the term structure is affine if and only if the model is affine. See for instance Section 5.2 in ? (?).*

Exercise 3.4.1 *Consider all the above mentioned models except for the Dothan and Black-Derman-Toy models, and show that they are affine.*

3.4.4 The Vasicek model

We shall apply the previous technique to the Vasicek model

$$\mathrm{d}r(t) = (b - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t, \quad a, b, \sigma > 0$$

Note that

$$\begin{aligned} \mathrm{d}r(t) + ar(t)\mathrm{d}t &= b\mathrm{d}t + \sigma \mathrm{d}W_t \\ &= e^{-at}\mathrm{d}\left(e^{at}r(t)\right). \end{aligned}$$

Hence

$$\mathrm{d}\left(e^{at}r(t)\right) = e^{at}b\mathrm{d}t + e^{at}\sigma\mathrm{d}W_t,$$

and finally

$$r(t) = \frac{b}{a} + e^{-at}\left(r(0) - \frac{b}{a}\right) + \sigma\int_0^t e^{-a(t-s)}\mathrm{d}W_s.$$

Then, we have that r is a Gaussian process and when $t \to \infty$, the distribution of $r(t)$ tends to a limit distribution $\mathrm{N}(b/a, \sigma^2/(2a))$. This process is named the Ornstein-Uhlenbeck process and its main feature is its mean reverting property: if the process $r(t)$ is greater than $\frac{b}{a}$, then the drift is negative and the process tends to go down. If the process $r(t)$ is less than $\frac{b}{a}$ then it tends to go up. So, in the end, it finished oscillating around the mean value $\frac{b}{a}$ with a constant variance. A drawback of this model is that it can give negative values for $r(t)$, producing arbitrage opportunities. This model is an ATS model with $\alpha(t) = -a, \beta(t) = b, \gamma(t) = 0$ y $\delta(t) = \sigma^2$, so

$$\begin{aligned} \frac{\partial A}{\partial t} - bB + \frac{1}{2}\sigma^2 B^2 = 0, &\quad A(T,T) = 0 \\ 1 + \frac{\partial B}{\partial t} - aB = 0, &\quad B(T,T) = 0 \end{aligned} \tag{3.10}$$

It is easy to see that

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

then , from (3.10), we have

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2\mathrm{d}s - b\int_t^T B\mathrm{d}s$$

and substituting for B we obtain

$$A(t,T) = \frac{B(t,T)-(T-t)}{a^2}(ab - \frac{1}{2}\sigma^2) - \frac{\sigma^2}{4a}B^2(t,T).$$

If we consider the continuous spot interest rate for the period $[t,T]$: $R(t,T)$, since

$$P(t,T) = \exp\{-(T-t)R(t,T)\}$$

and since

$$P(t,T) = \exp\{A(t,T) - B(t,T)r(t)\},$$

it turns out that

$$R(t,T) = -\frac{A(t,T) - B(t,T)r(t)}{T-t}.$$

So, in this model

$$\lim_{T\to\infty} R(t,T) = \frac{b}{a} - \frac{\sigma^2}{2a^2}$$

and this is consider as another imperfection of the model by practitioners since it does not depend on $r(t)$.

3.4.5 The Ho-Lee model

In the Ho-Lee model

$$dr(t) = \Theta(t)dt + \sigma dW_t$$

So, $\alpha(t) = \gamma(t) = 0$, $\beta(t) = \Theta(t)$ and $\delta(t) = \sigma^2$. Then, we have the equations

$$\begin{aligned} \frac{\partial A}{\partial t} - \Theta(t)B + \frac{\sigma^2}{2}B^2 = 0, &\quad A(T,T) = 0 \\ 1 + \frac{\partial B}{\partial t} = 0, &\quad B(T,T) = 0, \end{aligned}$$

therefore

$$\begin{aligned} B(t,T) &= T - t \\ A(t,T) &= \int_t^T \Theta(s)(s-T)ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}. \end{aligned}$$

Note that, contrarily to the previous model, we do not have an explicit expression in terms of the parameters. Now, we have an infinite-dimension parameter $\Theta(s)$. One way of estimating it is to try to fit the initially observed term structure $\{\hat{P}(0,T), T \geq 0\}$ to the theoretical values. That is

$$P(0,T) \approx \hat{P}(0,T), T \geq 0.$$

This gives

$$-\frac{\partial^2 \log P(0,T)}{\partial T^2} \approx -\frac{\partial^2 \log \hat{P}(0,T)}{\partial T^2} = \frac{\partial \hat{f}(0,T)}{\partial T}$$

and therefore

$$\Theta(T) = \frac{\partial \hat{f}(0,T)}{\partial T} + \sigma^2 T$$

3.4.6 The CIR model

In this model model

$$dr(t) = a(b - r(t))\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W_t$$

where $a, b, \sigma > 0$. As in the Vasicek model there is a reversion to the mean, here given by b, but the volatility factor $\sqrt{r(t)}$ keeps the process above zero: when the process is close to zero there is only contribution of a positive drift.

Proposition 3.4.2 *Let W_1, W_2 be two independent Brownian motions and let $X_i, i = 1, 2$ be two Ornstein-Uhlenbeck process, solutions of*

$$\mathrm{d}X_i(t) = -\frac{a}{2}X_i(t)\mathrm{d}t + \frac{\sigma}{2}\mathrm{d}W_i(t), i = 1, 2.$$

Then the process

$$r(t) := X_1^2(t) + X_2^2(t),$$

satisfies

$$\mathrm{d}r(t) = \left(\frac{\sigma^2}{2} - ar(t)\right)\mathrm{d}t + \sigma\sqrt{r(t)})\mathrm{d}W(t)$$

where W is a standard Brownian motion.

Proof. By the Itô formula for the bidimensional case

$$\begin{aligned}
\mathrm{d}r(t) &= 2\sum_{i=1,2} X_i(t)\mathrm{d}X_i(t) + \frac{\sigma^2}{2}\mathrm{d}t \\
&= -ar(t)\mathrm{d}t + \sigma\sum_{i=1,2} X_i(t)\mathrm{d}W_i(t) + \frac{\sigma^2}{2}\mathrm{d}t \\
&= \left(\frac{\sigma^2}{2} - ar(t)\right)\mathrm{d}t + \sigma\sqrt{r(t)}\sum_{i=1,2}\frac{X_i(t)}{\sqrt{r(t)}}\mathrm{d}W_i(t).
\end{aligned}$$

Write

$$\mathrm{d}W(t) := \sum_{i=1,2}\frac{X_i(t)}{\sqrt{r(t)}}\mathrm{d}W_i(t),$$

then W is an Itô process with quadratic variation t:

$$\begin{aligned}
[W, W]_t &= \sum_{i=1,2}\int_0^t \frac{X_i^2(s)}{r(s)}\mathrm{d}s \\
&= t.
\end{aligned}$$

And by the Itô formula

$$e^{i\lambda W_t} = e^{i\lambda W_u} + i\lambda\int_u^t e^{i\lambda W_s}\mathrm{d}W_s - \frac{\lambda^2}{2}\int_u^t e^{i\lambda W_s}\mathrm{d}s.$$

Consequently

$$E(e^{i\lambda(W_t-W_u)}|\mathcal{F}_u) = 1 - \frac{\lambda^2}{2}\int_u^t E(e^{i\lambda(W_s-W_u)}|\mathcal{F}_u)\mathrm{d}s,$$

and

$$E(e^{i\lambda(W_t-W_u)}|\mathcal{F}_u) = e^{-\frac{1}{2}\lambda^2(t-u)}.$$

Hence W has continuous trajectories, with independent and homogeneous increments (and $\mathrm{N}(0,t)$). In other words, W is a Brownian motion. ■

Remark 3.4.2 *It can be seen that if $ab \geq \frac{\sigma^2}{2}$, the values of $r(t)$ hold strictly positive.*

Bond prices for the CIR model

We have to solve

$$\frac{\partial A}{\partial t} - \beta(t)B + \frac{1}{2}\delta(t)B^2 = 0,$$
$$1 + \frac{\partial B}{\partial t} + \alpha(t)B - \frac{1}{2}\gamma(t)B^2 = 0.$$

con $\beta = ab, \delta = 0, \alpha = -a$ y $\gamma = \sigma^2$. That is

$$\frac{\partial A}{\partial t} - abB = 0,$$
$$1 + \frac{\partial B}{\partial t} - aB - \frac{1}{2}\sigma^2 B^2 = 0,$$

with the boundary condition $B(T,T) = A(T,T) = 0$. It is easy to see that, by taking derivatives, we have

$$B(t,T) = \frac{2(e^{c(T-t)} - 1)}{d(t)}$$

with $c = \sqrt{a^2 + 2\sigma^2}$ and $d(t) = (c+a)(e^{c(T-t)} - 1) + 2c$. By integrating

$$A(t,T) = \frac{2ab}{\sigma^2}\left(\frac{(a+c)(T-t)}{2} + \log\frac{2c}{d(t)}\right).$$

3.4.7 The Hull-White model

In the calibration step we try to adjust the real bond prices to the the theoretical ones. If we use the notation $\{\hat{P}(0,T), T \geq 0\}$ for the observed prices, we want to obtain that

$$P(0,T;\alpha) = \hat{P}(0,T), \quad T \geq 0.$$

but this is not possible if our set of parameters, α, is finite dimensional. We have seen that in the Ho-Lee model this was possible due to the fact that the involved parameter $\Theta(t)$ was infinite dimensional. The Hull-White model combines this fact with the mean reverting property we have in the Vasicek model. By this reason it is quite popular. The dynamics we consider is

$$\mathrm{d}r(t) = (\Theta(t) - ar(t))\mathrm{d}t + \sigma \mathrm{d}W_t, \quad a, \sigma > 0.$$

Then, we have

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

and

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2 \mathrm{d}s - \int_t^T \Theta(s) B \mathrm{d}s,$$

so the theoretical forward rates are given by

$$\begin{aligned}
f(0,T) &= -\partial_T \log P(0,T) = \partial_T \left(B(0,T)r(0) - A(0,T)\right) \\
&= \partial_T \left(B(0,T)\right) r(0) - \sigma^2 \int_0^T B(s,T)\partial_T B(s,T)\mathrm{d}s + \int_0^T \Theta(s)\partial_T B(s,T)\mathrm{d}s \\
&= e^{-aT} r(0) - \sigma^2 \int_0^T \frac{1}{a}(1 - e^{-a(T-s)})e^{-a(T-s)}\mathrm{d}s + \int_0^T \Theta(s) e^{-a(T-s)}\mathrm{d}s \\
&= e^{-aT} r(0) - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2 + \int_0^T \Theta(s) e^{-a(T-s)}\mathrm{d}s.
\end{aligned}$$

We have to solve $f(0,T) = \hat{f}(0,T)$. By differentiating with respect to T and we call $g(T) := e^{-aT}r(0) - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2$, we have

$$\begin{aligned}
\partial_T f(0,T) &= \partial_T g(T) + \Theta(T) - a\int_0^T \Theta(s)e^{-a(T-s)}\mathrm{d}s \\
&= \partial_T g(T) + \Theta(T) - a(f(0,T) - g(T)),
\end{aligned}$$

so

$$\Theta(T) = \partial_T f(0,T) - \partial_T g(T) + a(f(0,T) - g(T)).$$

We can then to capture $\hat{f}(0,T)$ by taking

$$\Theta(T) = \partial_T \hat{f}(0,T) - \partial_T g(T) + a(\hat{f}(0,T) - g(T)).$$

Exercise 3.4.2 *Let $(W_1, W_2, ..., W_n)$ be n independent standard Brownian motions and let $X_i, i = 1, ..., n$, be Ornstein-Uhlenbeck processes solving*

$$dX_i(t) = -aX_i(t)dt + \sigma dW_i(t), i = 1, ..., n.$$

Consider the process

$$r(t) := X_1^2(t) + ... + X_n^2(t).$$

Show that

$$dr(t) = (n\sigma^2 - 2ar(t))dt + 2\sigma\sqrt{r(t)})dW(t)$$

where W is a standard Brownian motion.

3.5 Forward rate models. Heath-Jarrow-Morton approach

As we have seen one drawback of the short rate models is their difficulty in capturing the term structure observed at initial time. An alternative is to model the forward rates $f(t,T)$ and to use the relation $r(t) = f(t,t)$, this is the so-called Heath-Jarrow-Morton (HJM) approach. By definition we have that

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)\mathrm{d}s\right\},$$

so $f(t,s)$ represents the instantaneous rates (at s) *anticipated* by the market at t, that here we call for brevity *forward rates*. Suppose that under a risk neutral probability $\mathbb{P}^*$

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t \quad ,T \geq 0 \tag{3.11}$$

with

$$f(0,T) = \hat{f}(0,T).$$

We shall try to deduce the evolution of $P(t,T)$ from that of $f(t,T)$. If we write $X_t = -\int_t^T f(t,s)\mathrm{d}s$, we have $P(t,T) = e^{X_t}$ and from the equation (3.11) we obtain

$$\begin{aligned}
\mathrm{d}X_t &= f(t,t)\mathrm{d}t - \int_t^T \mathrm{d}f(t,s)\mathrm{d}s = \\
&= f(t,t)\mathrm{d}t - \int_t^T \alpha(t,s)\mathrm{d}t\mathrm{d}s - \int_t^T \sigma(t,s)\mathrm{d}W_t\mathrm{d}s \\
&= \left(f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s\right)\mathrm{d}t - \left(\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}W_t,
\end{aligned}$$

where we have applied a *stochastic* Fubini theorem. Then

$$\begin{aligned}
\frac{\mathrm{d}P(t,T)}{P(t,T)} &= \mathrm{d}X_t + \frac{1}{2}\mathrm{d}\langle X\rangle_t \\
&= (f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s)\mathrm{d}t - \left(\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}W_t \\
&+ \frac{1}{2}\left(\int_t^T \sigma(t,s)\mathrm{d}s\right)^2\mathrm{d}t \\
&= \left(f(t,t) - \int_t^T \alpha(t,s)\mathrm{d}s + \frac{1}{2}(\int_t^T \sigma(t,s)\mathrm{d}s)^2\right)\mathrm{d}t \\
&- \left(\int_t^T \sigma(t,s)\mathrm{d}s\right)\mathrm{d}W_t.
\end{aligned}$$

And if we compare with that obtained in (3.2.1) and we have into account that $f(t,t) = r(t)$ it turns out that

$$-\int_t^T \alpha(t,s)\mathrm{d}s + \frac{1}{2}(\int_t^T \sigma(t,s)\mathrm{d}s)^2 = 0,$$

therefore

$$\alpha(t,T) = (\int_t^T \sigma(t,s)ds)\sigma(t,T)$$

and we can write the evolution equation (3.11) as

$$\mathrm{d}f(t,T) = \sigma(t,T)(\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t.$$

Note that all depends on $\sigma(t,s)$, that is, on the volatility. We have *eliminated* the drift $\alpha(t,T)$, as it happened for the call prices in the Black-Scholes model.

Then the algorithm to use the HJM approach is

1. Specify the volatilities $\sigma(t,s)$
2. Integrate $\mathrm{d}f(t,T) = \sigma(t,T)(\int_t^T \sigma(t,s)\mathrm{d}s)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t$ with the initial condition $f(0,T) = \hat{f}(0,T)$.
3. Calculate the prices of the bonds from the formula $P(t,T) = \exp\{-\int_t^T f(t,s)\mathrm{d}s\}$.
4. To use the previous results to calculate contingent claim prices.

Example 3.5.1 *Suppose that $\sigma(t,T)$ has a constant value denoted by σ. Then*

$$\mathrm{d}f(t,T) = \sigma^2(T-t)\mathrm{d}t + \sigma\mathrm{d}W_t,$$

so

$$f(t,T) = \hat{f}(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma W_t.$$

In particular

$$r(t) = f(t,t) = \hat{f}(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W_t$$

and therefore

$$\mathrm{d}r(t) = \left(\frac{\partial \hat{f}(0,T)}{\partial T}|_{T=t} + \sigma^2 t\right)\mathrm{d}t + \sigma\mathrm{d}W_t,$$

but this is the Ho-Lee adjusted to the initial structure of the forward rates.

Example 3.5.2 *A usual assumption consist of assuming that the forward rates with greater maturity time has a lower fluctuation than that with a lower maturity time. To capture this feature we can take, for instance, $\sigma(t,T) = \sigma e^{-b(T-t)}$, $b > 0$. We have then*

$$\int_t^T \sigma(t,s)ds = \int_t^T e^{-b(s-t)}ds = -\frac{\sigma}{b}\left(e^{-b(T-t)} - 1\right),$$

and

$$\mathrm{d}f(t,T) = -\frac{\sigma^2}{b} e^{-b(T-t)}(e^{-b(T-t)} - 1)\mathrm{d}t + \sigma e^{-b(T-t)}\mathrm{d}W_t.$$

Therefore

$$\begin{aligned} f(t,T) = f(0,T) &+ \frac{\sigma^2 e^{-2bT}}{2b^2}\left(1 - e^{2bt}\right) - \frac{\sigma^2 e^{-bT}}{b^2}(1 - e^{bt}) \\ &+ \sigma e^{-bT}\int_0^t e^{bs}\mathrm{d}W_s. \end{aligned}$$

In particular

$$\begin{aligned} r(t) = f(0,t) &+ \frac{\sigma^2}{2b^2}\left(e^{-2bt} - 1\right) - \frac{\sigma^2}{b^2}(e^{-bt} - 1) \\ &+ \sigma e^{-bt}\int_0^t e^{bs}\mathrm{d}W_s, \end{aligned}$$

that corresponds to the Hull-White model considered above.

Remark 3.5.1 *A sufficient condition to guarantee the equality* $\int_0^T \sigma(t,s)\mathrm{d}W_t\mathrm{d}s = \int_0^T \sigma(t,s)\mathrm{d}s)\mathrm{d}W_t$ *is* $\int_0^T\int_0^T E(\sigma^2(t,s))\mathrm{d}s\mathrm{d}t < \infty$, *see Lamberton and Lapeyre (2008), page 171 ? (?).*

3.6 Change of numeraire. The forward measure

We are going to study a procedure that is specially useful when we want to calculate prices of options in a bond market. It has to do with the use of the so-called *forward measure*. Let $\mathbb{P}^*$ be the neutral probability. By definition $\mathbb{P}^*$ is a probability such that

$$\left(\tilde{P}(t,T)\right)_{0\le t\le T}$$

are martingales, for all values of T. Fix a maturity time T and consider the values of bonds with another maturity time $\tilde{T} \ge T$ in terms of the bond with maturity T:

$$U_{T,\tilde{T}}(t) := \frac{P(t,\tilde{T})}{P(t,T)}.$$

That is, instead of taking as reference (*numeraire*) the value of a unit of money in the bank account, we take the value of a bond with maturity T. Let $\mathbb{P}^T$ be a probability with respect to which $\left(U_{T,\tilde{T}}(t)\right)_{0\le t\le T}$ are martingales for all $\tilde{T} \ge T$. We call $\mathbb{P}^T$ the *forward measure*. Define a probability on $\mathcal{F}_T$, say $\mathbb{P}^T$, such that

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}^*} = \frac{e^{-\int_0^T r_s\mathrm{d}s}}{P(0,T)}. \tag{3.12}$$

We can see that it is a forward measure.

Proposition 3.6.1 *Let $(V_t)_{0\leq t\leq T}$ be the value of a self-financing portfolio, such that its discounted value $\left(\tilde{V}_t\right)_{0\leq t\leq T}$ is a $\mathbb{P}^*$ martingale, then its discounted value using as reference (*numeraire*) the bond value $P(t,T)$, is a $\mathbb{P}^T$-martingale. That is*

$$\frac{V_t}{P(t,T)}, \quad 0\leq t\leq T,$$

is a $\mathbb{P}^T$-martingale.

Proof. Define

$$Z_t := \mathbb{E}_{\mathbb{P}^*}\left(\left.\frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T)}\right|\mathcal{F}_t\right),$$

then

$$Z_t = \frac{\tilde{P}(t,T)}{P(0,T)}.$$

By the Bayes (2.16) rule

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^T}\left(\left.\frac{V_T}{P(T,T)}\right|\mathcal{F}_t\right) &= \mathbb{E}_{\mathbb{P}^T}(V_T|\mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}^*}(V_T Z_T|\mathcal{F}_t)}{Z_t} \\
&= \frac{\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_T|\mathcal{F}_t)}{P(0,T)Z_t} = \frac{\tilde{V}_t}{\tilde{P}(t,T)} \\
&= \frac{V_t}{P(t,T)}.
\end{aligned}$$

■

Corollary 3.6.1 *The price of a replicable T-payoff Y is given by*

$$P(t,T)\mathbb{E}_{\mathbb{P}^T}(Y|\mathcal{F}_t).$$

Proof. Let $(V_t)_{0\leq t\leq T}$ be the self-financing portfolio that replicates Y, then $V_T = Y$ and therefore

$$\mathbb{E}_{\mathbb{P}^T}(Y|\mathcal{F}_t) = \frac{V_t}{P(t,T)}.$$

■

Proposition 3.6.2 *Suppose that*

$$\frac{\partial}{\partial T}\mathbb{E}_{\mathbb{P}^*}\left(\left. e^{-\int_t^T r_s \mathrm{d}s}\right|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(\left.\frac{\partial}{\partial T}\left(e^{-\int_t^T r_s \mathrm{d}s}\right)\right|\mathcal{F}_t\right),$$

then

$$\mathbb{E}_{\mathbb{P}^T}(r_T|\mathcal{F}_t) = f(t,T).$$

Proof.

$$\begin{aligned} f(t,T) &= -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} = -\frac{1}{P(t,T)}\frac{\partial}{\partial T}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}\middle|\mathcal{F}_t\right) \\ &= -\frac{1}{P(t,T)}\mathbb{E}_{\mathbb{P}^*}\left(\frac{\partial}{\partial T}\left(e^{-\int_t^T r_s \mathrm{d}s}\right)\middle|\mathcal{F}_t\right) = \frac{1}{P(t,T)}\mathbb{E}_{\mathbb{P}^*}\left(r_T e^{-\int_t^T r_s \mathrm{d}s}\middle|\mathcal{F}_t\right) \\ &= \mathbb{E}_{\mathbb{P}^T}\left(r_T|\mathcal{F}_t\right). \end{aligned}$$

■

Let $(S_t)_{0\le t\le T}$ be an asset strictly positive and denote by $\mathbb{P}^{(S)}$ the probability (in $\mathcal{F}_T$) that makes

$$\left(\frac{V_t}{S_t}\right)_{0\le t\le T}$$

a martingale, where $(V_t)_{0\le t\le T}$ is a self-financing portfolio.

Proposition 3.6.3 *The price of a replicable T-payoff Y is given by*

$$S_t\mathbb{E}_{\mathbb{P}^{(S)}}\left(\frac{Y}{S_T}\middle|\mathcal{F}_t\right).$$

Proof. Let $(V_t)_{0\le t\le T}$ be the self-financing portfolio that replicates Y, then $V_T = Y$ and therefore

$$\mathbb{E}_{\mathbb{P}^{(S)}}\left(\frac{V_T}{S_T}\middle|\mathcal{F}_t\right) = \frac{V_t}{S_t}.$$

■

Now, we have a general formula for the price of a call option.

Proposition 3.6.4 *Let $(S_t)_{0\le t\le T}$ be an asset strictly positive, then the price of a call option on the asset S, with maturity T and strike K is given by*

$$\Pi(t;S) = S_t\mathbb{P}^{(S)}(S_T \ge K|\mathcal{F}_t) - KP(t,T)\mathbb{P}^T(S_T \ge K|\mathcal{F}_t).$$

Proof.

$$\begin{aligned} \Pi(t;S) &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T-K)_+\middle|\mathcal{F}_t\right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}(S_T-K)\mathbf{1}_{\{S_T\ge K\}}\middle|\mathcal{F}_t\right) \\ &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}S_T\mathbf{1}_{\{S_T\ge K\}}\middle|\mathcal{F}_t\right) - K\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s}\mathbf{1}_{\{S_T\ge K\}}\middle|\mathcal{F}_t\right) \\ &= S_t\mathbb{P}^{(S)}(S_T \ge K|\mathcal{F}_t) - KP(t,T)\mathbb{P}^T(S_T \ge K|\mathcal{F}_t), \end{aligned}$$

with

$$\frac{\mathrm{d}\mathbb{P}^{(S)}}{\mathrm{d}\mathbb{P}^*} = \frac{e^{-\int_0^T r_s \mathrm{d}s}S_T}{S_0}.$$

■

Suppose that S is another bond with maturity $\bar{T} > T$, then the option (with maturity T) on this bond has a price given by

$$\begin{aligned}\Pi(t;S) &= P(t,\bar{T})\mathbb{P}^{\bar{T}}(P(T,\bar{T}) \geq K|\mathcal{F}_t)) - P(t,T)\mathbb{P}^{T}(P(T,\bar{T}) \geq K|\mathcal{F}_t)) \\ &= P(t,\bar{T})\mathbb{P}^{\bar{T}}\left(\frac{P(T,T)}{P(T,\bar{T})} \leq \frac{1}{K}\middle| \mathcal{F}_t\right) - KP(t,T)\mathbb{P}^{T}\left(\frac{P(T,\bar{T})}{P(T,T)} \geq K\middle| \mathcal{F}_t\right).\end{aligned}$$

Define,

$$U(t,T,\bar{T}) := \frac{P(t,T)}{P(t,\bar{T})}.$$

In the context of affine structures

$$U(t,T,\bar{T}) = \frac{P(t,T)}{P(t,\bar{T})} = \exp\left\{-A(t,\bar{T}) + A(t,T) + \left(B(t,\bar{T}) - B(t,T)\right) r_t\right\}$$

and with respect to $\mathbb{P}^*$

$$\mathrm{d}U(t) = U(t)(...\mathrm{d}t + (B(t,\bar{T}) - B(t,T))\sigma_t \mathrm{d}W_t).$$

Then under $\mathbb{P}^{\bar{T}}$ and $\mathbb{P}^{T}$, respectively, we have

$$\begin{aligned}\mathrm{d}U(t) &= U(t)(B(t,\bar{T}) - B(t,T))\sigma_t \mathrm{d}W_t^{\bar{T}}, \\ \mathrm{d}U^{-1}(t) &= -U^{-1}(t)(B(t,\bar{T}) - B(t,T))\sigma_t \mathrm{d}W_t^{T}.\end{aligned}$$

in such a way that

$$\begin{aligned}U(T) &= \frac{P(t,T)}{P(t,\bar{T})}\exp\left\{-\int_t^T \sigma_{\bar{T},T}(s)\mathrm{d}W_s^{\bar{T}} - \frac{1}{2}\int_t^T \sigma_{\bar{T},T}^2(s)\mathrm{d}s\right\}, \\ U^{-1}(T) &= \frac{P(t,\bar{T})}{P(t,T)}\exp\left\{\int_t^T \sigma_{\bar{T},T}(s)\mathrm{d}W_s^{T} - \frac{1}{2}\int_t^T \sigma_{\bar{T},T}^2(s)\mathrm{d}s\right\}.\end{aligned}$$

with

$$\sigma_{\bar{T},T}(t) = -(B(t,\bar{T}) - B(t,T))\sigma_t.$$

Therefore, if σ_t is **deterministic** the law of $\log U(T)$ conditional to $\mathcal{F}_t$ is Gaussian with respect to $\mathbb{P}^{T}$ and $\mathbb{P}^{\bar{T}}$, with variance

$$\Sigma_{t,T,\bar{T}}^2 := \int_t^T \sigma_{\bar{T},T}^2(s)\mathrm{d}s,$$

$$\begin{aligned}&\text{Law}\left(\frac{\log U(T) - \log\frac{P(t,T)}{P(t,\bar{T})} + \frac{1}{2}\Sigma_{t,T,\bar{T}}^2}{\Sigma_{t,T,\bar{T}}}\middle| \mathcal{F}_t\right) \sim N(0,1) \text{ under } \mathbb{P}^{\bar{T}} \\ &\text{Law}\left(\frac{\log U^{-1}(T) - \log\frac{P(t,\bar{T})}{P(t,T)} + \frac{1}{2}\Sigma_{t,T,\bar{T}}^2}{\Sigma_{t,T,\bar{T}}}\middle| \mathcal{F}_t\right) \sim N(0,1) \text{ under } \mathbb{P}^{T}.\end{aligned}$$

Note finally that

$$\Pi(t;S) = P(t,\bar{T})\mathbb{P}^{\bar{T}}\left(\frac{P(T,T)}{P(T,\bar{T})} \le \frac{1}{K}\bigg|\,\mathcal{F}_t\right) - KP(t,T)\mathbb{P}^{T}\left(\frac{P(T,\bar{T})}{P(T,T)} \ge K\bigg|\,\mathcal{F}_t\right) \tag{3.13}$$

$$= P(t,\bar{T})\mathbb{P}^{\bar{T}}\left(U(T) \le \frac{1}{K}\bigg|\,\mathcal{F}_t\right) - KP(t,T)\mathbb{P}^{T}(U^{-1}(T) \ge K\big|\,\mathcal{F}_t)$$
$$= P(t,\bar{T})\mathbb{P}^{\bar{T}}(\log U(T) \le -\log K|\mathcal{F}_t) - KP(t,T)\mathbb{P}^{T}(\log U^{-1}(T) \ge \log K|\mathcal{F}_t)$$
$$= P(t,\bar{T})\Phi(d_+) - KP(t,T)\Phi(d_-),$$

with

$$d_\pm = \frac{\log\frac{P(t,\bar{T})}{KP(t,T)} \pm \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}}.$$

Example 3.6.1 *In the Ho-Lee model*

$$\sigma_{\bar{T},T} = -\sigma(\bar{T}-T),$$
$$\Sigma_{t,T,\bar{T}} = \sigma(\bar{T}-T)\sqrt{T-t}.$$

Example 3.6.2 *For the Vasicek model*

$$\sigma_{\bar{T},T} = \frac{\sigma}{a}e^{at}(e^{-a\bar{T}} - e^{-aT}),$$
$$\Sigma^2_{t,T,\bar{T}} = \frac{\sigma^2}{2a^3}(1-e^{-2(T-t)})(1-e^{-(\bar{T}-T)})^2.$$

and the same for the Hull-White model!.

3.7 Defaultable bonds

Definition 3.7.1 *A zero-coupon bond with* default *is a contract with maturity time T and payoff*

$$X = \mathbf{1}_{\{\tau > T\}},$$

where τ is the (random) default time*.*

Then the arbitrage price of this bond at time t will be given by

$$D(t,T) := \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s\mathrm{d}s}\mathbf{1}_{\{\tau>T\}}\Big|\,\mathcal{G}_t\right),$$

where $(\mathcal{G}_t)_{0\le t\le T}$ represents the flow of the *total* information we have in the market. So far we use the letter $\mathcal{F}$ to indicate the available information in the market but now we will consider different kind of information and we will use $(\mathcal{F}_t)_{0\le t\le T}$ for *the default free market information* that includes the short rate process. Obviously $D(t,T)$ will depend on the model for τ. There are different approaches.

3.7.1 Merton's approach to pricing defaultable bonds

In the Merton approach there is a firm's value process $(V_t)_{t\geq 0}$, that evolves as a geometric Brownian motion under the risk-neutral martingale measure $\mathbb{P}^*$, specifically

$$\mathrm{d}V_t = V_t((r-\kappa)\mathrm{d}t + \sigma_V \mathrm{d}W_t^*). \tag{3.14}$$

where κ is the dividend rate. Solving the SDE for every $t \in [0,T]$ we obtain

$$V_t = V_0 \exp\left(\left(r-\kappa-\frac{1}{2}\sigma_V^2\right)t + \sigma_V W_t^*\right). \tag{3.15}$$

Therefore, we have that

$$V_T = V_t \exp\left(\left(r-\kappa-\frac{1}{2}\sigma_V^2\right)(T-t) + \sigma_V (W_T^* - W_t^*)\right), \tag{3.16}$$

Then if at the maturity T the total value V_T of the firm's assets is less than the total notional value L of the firm's debt, the firm defaults. Otherwise, the firm does not default, and the debt, that is the L zero-coupon bonds, is paid. Consequently, the default time τ is defined as

$$\tau := T\mathbf{1}_{\{V_T<L\}} + \infty\mathbf{1}_{\{V_T\geq L\}}$$

and

$$\mathbf{1}_{\{\tau>T\}} = \mathbf{1}_{\{V_T\geq L\}}.$$

It correspond to a digital option in a Black-Scholes model, so

$$D(t,T) = \Phi(d_-)e^{-r(T-t)},$$

where

$$d_- = \frac{\log(\frac{V_t}{L}) + (r-\kappa-\frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{(T-t)}}.$$

In his approach Merton considers a *partial recovery* in case of default in such a way that set of bond holders receive V_T if $V_T < L$. So, the price of this bond with this recovery rule will be

$$D(t,T) + \frac{1}{L}\mathbb{E}_{\mathbb{P}^*}\left(\left.\frac{V_T\mathbf{1}_{\{V_T<L\}}}{e^{r(T-t)}}\right| \mathcal{G}_t\right) = e^{-r(T-t)} - \frac{1}{L}P_t,$$

where P_t is the price of a put option with strike L sold by the bond holders to the owners of the firm. To see that note that in that case the payoff for a holder of a zero-coupon bond is

$$\begin{aligned}
\mathbf{1}_{\{\tau>T\}} + \frac{V_T}{L}\mathbf{1}_{\{\tau\le T\}} &= \mathbf{1}_{\{V_T\ge L\}} + \frac{V_T}{L}\mathbf{1}_{\{V_T<L\}} \\
&= 1 - \mathbf{1}_{\{V_T<L\}} + \frac{V_T}{L}\mathbf{1}_{\{V_T<L\}} \\
&= 1 - \left(1 - \frac{V_T}{L}\right)\mathbf{1}_{\{V_T<L\}} \\
&= 1 - \left(1 - \frac{V_T}{L}\right)_+ \\
&= 1 - \frac{1}{L}\left(L - V_T\right)_+ .
\end{aligned}$$

3.7.2 Intensity approach

In this approach the total information available for the investors is given by a filtration $(\mathcal{G}_t)_{0\le t\le T}$ and $\mathcal{G}_t = \sigma(\mathbf{1}_{\{\tau\le s\}}, 0 \le s \le t) \vee \mathcal{F}_t$, where $\mathcal{F}_t$ is the information of the *default free market* that includes the short rate process. On the other hand τ is not necessarily an $(\mathcal{F}_t)$-stopping time, but it is assumed that there exists a non negative $(\mathcal{F}_t)$-adapted process such that

$$\mathbb{P}^*(\tau > t|\mathcal{F}_t) = e^{-\int_0^t \lambda_s \mathrm{d}s} > 0 \text{ for all } t \ge 0.$$

where $\mathbb{P}^*$ is the risk neutral probability. Under this framework we have the following proposition.

Proposition 3.7.1

$$D(t,T) = \mathbf{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T(\lambda_s+r_s)\mathrm{d}s}\middle|\,\mathcal{F}_t\right).$$

The proof of this proposition is based in the following lemma.

Lemma 3.7.1 *For any non-negative random variable X*

$$\mathbf{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{P}^*}\left(X\middle|\,\mathcal{G}_t\right) = \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}.$$

Proof. We have to prove that

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}X\mathbf{1}_A\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}\mathbf{1}_A\right), \tag{3.17}$$

for all $A \in \mathcal{G}_t$. Then it is enough to consider sets of the form $A = \{\tau \le s\} \cap B$, $B \in \mathcal{F}_t$ and $0 \le s \le t$ or $A \in \mathcal{F}_t$. If $A \in \mathcal{F}_t$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}\mathbf{1}_A\right) &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)\frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)}\mathbf{1}_A\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\mathbf{1}_A\middle|\,\mathcal{F}_t\right)\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\mathbf{1}_A\right).
\end{aligned}$$

If $A = \{\tau \leq s\} \cap B, B \in \mathcal{F}_t$

$$\mathbf{1}_{\{t<\tau\}}\mathbf{1}_A = 0,$$

so both sides of (3.17) are zero. ■

Proof. (of the proposition)

$$\begin{aligned}
D(t,T) &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}}\ e^{-\int_t^T r_s \mathrm{d}s}\middle|\,\mathcal{G}_t\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\mathbf{1}_{\{T<\tau\}}\ e^{-\int_t^T r_s \mathrm{d}s}\middle|\,\mathcal{G}_t\right) \\
&= \mathbf{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}}\ e^{-\int_t^T r_s \mathrm{d}s}\middle|\,\mathcal{G}_t\right) \\
&= \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\mathbf{1}_{\{T<\tau\}}\ e^{-\int_t^T r_s \mathrm{d}s}\middle|\,\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}}\middle|\,\mathcal{F}_T\right)\ e^{-\int_t^T r_s \mathrm{d}s}\middle|\,\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\,\mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T \lambda_s \mathrm{d}s}e^{-\int_t^T r_s \mathrm{d}s}\middle|\,\mathcal{F}_t\right)}{e^{-\int_0^t \lambda_s \mathrm{d}s}}.
\end{aligned}$$

■

3.7.3 Credit default swaps (CDS)

CDS is a credit derivative that offers protection against default of a bond. Assume that the nominal of the bond is N and the recovery rate is $R < 1$, in such a way that the owner of the bond receives only NR in case of default. Then the buyer of the CDS, in order to ensure the total nominal N, pays at time T_i

$$sN(T_i - T_{i-1})$$

provided the default time $\tau > T_i$, $i = 1, ..., n$ and receives

$$N(1-R)$$

at time τ if $\tau \leq T_n$. In this way the discounted price of the CDS at time zero will be

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*}\left(N(1-R)\mathbf{1}_{\{\tau\leq T_n\}}e^{-\int_0^\tau r_s \mathrm{d}s} - \sum_{i=1}^n sN(T_i - T_{i-1})\mathbf{1}_{\{\tau>T_i\}}e^{-\int_0^{T_i} r_s \mathrm{d}s}\right) \\
&= N(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau\leq T_n\}}e^{-\int_0^\tau r_s \mathrm{d}s}\right) - \sum_{i=1}^n sN(T_i - T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>T_i\}}e^{-\int_0^{T_i} r_s \mathrm{d}s}\right) \\
&= N(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau\leq T_n\}}e^{-\int_0^\tau r_s \mathrm{d}s}\right) - \sum_{i=1}^n sN(T_i - T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}(r_s+\lambda_s) \mathrm{d}s}\right).
\end{aligned}$$

We can write, for $t_i = i\frac{T_n}{k}$,

$$\mathbf{1}_{\{\tau \le T_n\}} e^{-\int_0^\tau r_s \mathrm{d}s} = \lim_{k\to\infty} \sum_{i=1}^k \mathbf{1}_{\{t_{i-1} < \tau \le t_i\}} e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s},$$

and

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau \le T_n\}} e^{-\int_0^\tau r_s \mathrm{d}s}\right) &= \lim_{k\to\infty} \sum_{i=1}^k \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t_{i-1} < \tau \le t_i\}} e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s}\right) \\
&= \lim_{k\to\infty} \sum_{i=1}^k \mathbb{E}_{\mathbb{P}^*}\left(\left(\mathbf{1}_{\{\tau > t_{i-1}\}} - \mathbf{1}_{\{\tau > t_i\}}\right) e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s}\right) \\
&= \lim_{k\to\infty} \sum_{i=1}^k \mathbb{E}_{\mathbb{P}^*}\left(\left(e^{-\int_0^{t_{i-1}} \lambda_s \mathrm{d}s} - e^{-\int_0^{t_i} \lambda_s \mathrm{d}s}\right) e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s}\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\int_0^{T_n} e^{-\int_0^{t_s}(r_u+\lambda_u)\mathrm{d}u} \lambda_s \mathrm{d}s\right).
\end{aligned}$$

Therefore the price of the CDS is

$$N(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\int_0^{T_n} e^{-\int_0^{t_s}(r_u+\lambda_u)\mathrm{d}u} \lambda_s \mathrm{d}s\right) - \sum_{i=1}^n sN(T_i - T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}(r_s+\lambda_s)\mathrm{d}s}\right).$$

s is chosen in such a way that the price of this contract is zero:

$$s = \frac{(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\int_0^{T_n} e^{-\int_0^{t_s}(r_u+\lambda_u)\mathrm{d}u} \lambda_s \mathrm{d}s\right)}{\sum_{i=1}^n (T_i - T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}(r_s+\lambda_s)\mathrm{d}s}\right)}.$$

3.8 Market models

3.8.1 A market model for Swaptions

Consider a *payer swaption* with maturity $T < T_0$, *tenor structure* $T_1, T_2, ..., T_n$, and *swap rate* R. Its payoff is

$$(S(T) - Z(T))_+$$

con

$$S(T) = P(T, T_0) - P(T, T_n)$$

that is the value of the floating payments minus the last fixed payment and

$$Z(T) = R\delta \sum_{i=1}^n P(T, T_i)$$

the value of payments with fixed rate minus the last payment. We can take $Z(t)$ as *numeraire* and the price will be

$$Z(t)\mathbb{E}_{\mathbb{P}^{(Z)}}\left(\frac{(S(T)-Z(T))_+}{Z(T)}\middle|\mathcal{F}_t\right)=Z(t)\mathbb{E}_{\mathbb{P}^{(Z)}}\left(\left(\frac{S(T)}{Z(T)}-1\right)_+\middle|\mathcal{F}_t\right).$$

Then, if we assume that under $\mathbb{P}$, or $\mathbb{P}^*$ we have an evolution

$$\mathrm{d}\left(\frac{S(t)}{Z(t)}\right)=\frac{S(t)}{Z(t)}\left(\mu\mathrm{d}t+\sigma\mathrm{d}W_t\right),$$

with σ constant, it turns out that, under $\mathbb{P}^{(Z)}$

$$\mathrm{d}\left(\frac{S(t)}{Z(t)}\right)=\frac{S(t)}{Z(t)}\sigma\mathrm{d}W_t^Z,$$

so

$$\frac{S(T)}{Z(T)}=\frac{S(t)}{Z(t)}\exp\left\{\int_t^T\sigma\mathrm{d}W_s^Z-\frac{1}{2}\int_t^T\sigma^2\mathrm{d}s\right\},$$

and we obtain the Black-Scholes formula of a call with strike 1 and $r=0$, multiplied by $Z(t)$:

$$Z(t)\left(\frac{S(t)}{Z(t)}\Phi(d_+)-\Phi(d_-)\right)=S(t)\Phi(d_+)-Z(t)\Phi(d_-),$$

with

$$\Phi(d_\pm)=\frac{\log\frac{S(t)}{Z(t)}\pm\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

This formula is known as *the Margrabe formula* . Remember that the *swap rate* is given by

$$R(t)=\frac{P(t,T_0)-P(t,T_n)}{\delta\sum_{i=1}^n P(t,T_i)},$$

so

$$\frac{S(t)}{Z(t)}=\frac{P(t,T_0)-P(t,T_n)}{R\delta\sum_{i=1}^n P(t,T_i)}=\frac{R(t)}{R}.$$

Therefore the volatility σ corresponds to the volatility of $R(t)$. The previous formula can be written more explicitly as

$$\mathrm{Swaption}_t=\left(P(t,T_0)-P(t,T_n)\right)\Phi(d_+)-\left(R\delta\sum_{i=1}^n P(t,T_i)\right)\Phi(d_-),$$

where

$$\Phi(d_\pm)=\frac{\log\left(P(t,T_0)-P(t,T_n)\right)-\log\left(R\delta\sum_{i=1}^n P(t,T_i)\right)\pm\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

3.8.2 A LIBOR market model

First of all note that

$$L(t;T_{i-1},T_i) = -\frac{P(t,T_i) - P(t,T_{i-1})}{\delta P(t,T_i)},$$

so

$$U(t,T_{i-1},T_i) = \frac{P(t,T_{i-1})}{P(t,T_i)} = 1 + \delta L(t;T_{i-1},T_i)$$

and therefore

$$\mathrm{d}U(t,T_{i-1},T_i) = \delta \mathrm{d}L(t;T_{i-1},T_i),$$

then, respect to $\mathbb{P}^{T_i}$, and if the term structure is, for instance, affine

$$\begin{aligned}\mathrm{d}L(t;T_{i-1},T_i) &= \frac{1}{\delta}U(t,T_{i-1},T_i)(B(t,T_i) - B(t,T_{i-1}))\sigma_t \mathrm{d}W_t^{T_i} \\ &= \frac{1}{\delta}(1 + \delta L(t;T_{i-1},T_i))(B(t,T_i) - B(t,T_{i-1}))\sigma_t \mathrm{d}W_t^{T_i}.\end{aligned}$$

Consequently the structure of LIBORs is established, it is determined by the term structure for bonds. Another approach is to fix directly a model for the LIBORs, but then we have to check the consistency and if the whole model is free of arbitrage. One way is that the whole model implies a model for forward rates free of arbitrage. It can be seen, by a backward induction, that it is possible to build a LIBOR model such that

$$\mathrm{d}L(t;T_{i-1},T_i) = L(t;T_{i-1},T_i)\lambda(t,T_{i-1},T_i)\mathrm{d}W_t^{T_i}, i = 1,...,n$$

with initial conditions

$$L(0;T_{i-1},T_i) = -\frac{P(0,T_i) - P(0,T_{i-1})}{\delta P(0,T_i)}, i = 1,...,n.$$

In particular, if we take $\lambda(t,T_{i-1},T_i)$ deterministic we have that $L(t;T_{i-1},T_i)$ is lognormal (LLM). This model is very popular.

Let $P(t,T_n)$ fixed as numeràire, then

$$U(t,T_i,T_n) = \frac{P(t,T_i)}{P(t,T_n)},$$

are $\mathbb{P}^{T_n}$-martingales for $i = 0,...,n-1$ and since

$$\mathrm{d}U(t,T_i,T_n) = \delta \mathrm{d}L(t;T_i,T_n),$$

in turns out, assuming an affine structure for bond prices, that

$$\mathrm{d}L(t;T_i,T_n) = L(t;T_i,T_n)\lambda_i^n(t)\mathrm{d}W^{T_n}.$$

To fix the model we have arbitrariness choosing $\lambda_i^n(t)$. Fix $\lambda_{n-1}^n(t) = \lambda(t, T_{n-1}, T_n)$. Note that then

$$\begin{aligned} \mathrm{d}U(t, T_{n-1}, T_n) &= \delta \mathrm{d}L(t; T_{n-1}, T_n) \\ &= \delta L(t; T_{n-1}, T_n)\lambda(t, T_{n-1}, T_n)\mathrm{d}W^{T_n}, \end{aligned}$$

and the dynamics of $U(t, T_{n-1}, T_n)$ is fixed. Consider now the market when t moves between 0 and T_{n-1}, take now $P(t, T_{n-1})$ as numeraire, we have that

$$U(t, T_i, T_{n-1}) = \frac{P(t, T_i)}{P(t, T_{n-1})}, i = 0, ..., n-2,$$

are $\mathbb{P}^{T_{n-1}}$-martingales, but

$$\begin{aligned} U(t, T_i, T_{n-1}) &= \frac{P(t, T_i)}{P(t, T_{n-1})} = \frac{\frac{P(t,T_i)}{P(t,T_n)}}{\frac{P(t,T_{n-1})}{P(t,T_n)}} \\ &= \frac{U(t, T_i, T_n)}{U(t, T_{n-1}, T_n)}, \end{aligned}$$

therefore we can calculate the dynamics in terms of W^{T_n}. For simplicity in the notation write

$$\mathrm{d}U(t, T_i, T_n) = \alpha \mathrm{d}W_t^{T_n}, \quad \mathrm{d}U(t, T_{n-1}, T_n) = \beta \mathrm{d}W_t^{T_n}$$

$$\begin{aligned} \mathrm{d}U(t, T_i, T_{n-1}) &= \frac{1}{U(t, T_{n-1}, T_n)}\mathrm{d}U(t, T_i, T_n) + U(t, T_i, T_n)\mathrm{d}\frac{1}{U(t, T_{n-1}, T_n)} \\ &+ \mathrm{d}\langle U(\cdot, T_i, T_n), \frac{1}{U(\cdot, T_{n-1}, T_n)}\rangle_t \\ &= \frac{\alpha}{U(t, T_{n-1}, T_n)}\mathrm{d}W_t^{T_n} - \frac{U(t, T_i, T_n)\beta}{U(t, T_{n-1}, T_n)^2}\mathrm{d}W_t^{T_n} \\ &+ \frac{U(t, T_i, T_n)\beta^2}{U(t, T_{n-1}, T_n)^3}\mathrm{d}t \\ &- \frac{\alpha\beta}{U(t, T_{n-1}, T_n)^2}\mathrm{d}t \\ &= \frac{\alpha U(t, T_{n-1}, T_n) - \beta U(t, T_i, T_n)}{U^2(t, T_{n-1}, T_n)}\left(\mathrm{d}W_t^{T_n} - \frac{\beta}{U(t, T_{n-1}, T_n)}\mathrm{d}t\right) \\ &= \gamma_i^n(t)\left(\mathrm{d}W_t^{T_n} - \frac{\delta L(t; T_{n-1}, T_n)\lambda(t; T_{n-1}, T_n)}{1 + \delta L(t; T_{n-1}, T_n)}\mathrm{d}t\right), \end{aligned}$$

for certain process γ_i^n, that is not yet fixed. Then, we can find a forward measure $\mathbb{P}^{T_{n-1}}$ respect to which $U(t, T_i, T_{n-1})$, $i = 1, ..., n-2$ are martingales, and we will have

$$\mathrm{d}L(t; T_i, T_{n-1}) = L(t; T_i, T_{n-1})\lambda_i^{n-1}(t)\mathrm{d}W^{T_{n-1}}.$$

Now fix $\lambda_{n-2}^{n-1}(t) := \lambda(t, T_{n-2}, T_{n-1})$ and so on. Finally we have that

$$\mathrm{d}L(t; T_{i-1}, T_i) = L(t; T_{i-1}, T_i)\lambda_{i-1}^{i}(t)\mathrm{d}W^{T_i}, i = 1, ..., n.$$

and we have fixed the evolution of all LIBORs in such a way that the market model formed by the bonds

$$(P(t, T_i))_{0 \leq t \leq T_n}, i = 1, ..., n,$$

is free of arbitrage.

3.8.3 A market model for caps

Proposition 3.8.1 *In an LLM model the price of a cap ("in arrears") with swap rate $R = K$, principal 1 and tenor-structure $T_i = T_0 + i\delta$, $i = 1, ..., n$ is given by*

$$\Pi(t) = \sum_{i=1}^{n} \delta P(t, T_i)(L(t; T_{i-1}, T_i)\Phi(d_{i+}) - K\Phi(d_{i-})),$$

where

$$d_{i\pm} = \frac{\log \frac{L(t; T_{i-1}, T_i)}{K} \pm \frac{1}{2}\upsilon_i^2(t)}{\upsilon_i(t)},$$

with

$$\upsilon_i^2(t) = \int_t^{T_{i-1}} \lambda^2(s, T_{i-1}, T_i)\mathrm{d}s.$$

Proof.

$$\Pi(t) = \sum_{i=1}^{n} \delta P(t, T_i)\mathbb{E}_{\mathbb{P}^{T_i}}\left((L(T_{i-1}, T_i) - K)_+ | \mathcal{F}_t\right),$$

and under $\mathbb{P}^{T_i}$,

$$\begin{aligned}\log L(T_{i-1}, T_i) &= \log L(T_{i-1}, T_{i-1}, T_i) \\ &= \log L(t, T_{i-1}, T_i) + \int_t^{T_{i-1}} \lambda(s, T_{i-1}, T_i)\mathrm{d}W_s^{T_i} \\ &- \frac{1}{2}\int_t^{T_{i-1}} \lambda^2(s, T_{i-1}, T_i)\mathrm{d}s.\end{aligned}$$

■

Remark 3.8.1 *If $\lambda^2(s, T_{i-1}, T_i) = \sigma_i^2$, $i = 1, .., n$ for certain constants, then we have the so-called* Black's formula for caps . *This market model is incompatible with a model for swaps with constant volatility for the forward swap rate.*

3.9 Miscelanea

3.9.1 Forwards and Futures

Definition 3.9.1 *Let X be a payoff at T. A forward contract on X with delivering time T is a contract established at $t < T$ that specifies a forward price $f(t;T)$* ***that will be paid at*** *T for receiving X. The price $f(t;T)$ is fixed in such a way that the contract price at t is zero.*

Proposition 3.9.1

$$\begin{aligned} f(t;T) &= \frac{1}{P(t,T)} \mathbb{E}_{\mathbb{P}^*} \left(X \exp\left\{ -\int_t^T r_s ds \right\} \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{P}^T}(X|\mathcal{F}_t). \end{aligned}$$

Proof. The total payoff of this contract at T is

$$X - f(t;T),$$

so if $\mathbb{P}^*$ is the risk neutral probability the price of this contract will be zero at t if and only if

$$\mathbb{E}_{\mathbb{P}^*} \left((X - f(t;T)) \exp\left\{ -\int_0^T r_s ds \right\} \middle| \mathcal{F}_t \right) = 0,$$

therefore

$$\mathbb{E}_{\mathbb{P}^*} \left(X \exp\left\{ -\int_0^T r_s ds \right\} \middle| \mathcal{F}_t \right) = f(t;T) \mathbb{E}_{\mathbb{P}^*} \left(\exp\left\{ -\int_0^T r_s ds \right\} \middle| \mathcal{F}_t \right).$$

If we use $\mathbb{P}^T$

$$\mathbb{E}_{\mathbb{P}^T} \left(\frac{X - f(t;T)}{P(T,T)} \middle| \mathcal{F}_t \right) = 0,$$

Therefore

$$f(t;T) = \mathbb{E}_{\mathbb{P}^T} \left(X \middle| \mathcal{F}_t \right).$$

■

Definition 3.9.2 *Let X a payoff at T. A contract of futures on X and delivering time T is a financial asset with the following properties*

- There exist a *future price* $F(t;T)$ on X at each time t.
- At T the owner of the contract pays $F(T;T)$ and receives X.
- For any arbitrary interval $(s,t]$ the owner receives $F(t;T) - F(s;T)$.
- At each time the price of the contract is zero.

Proposition 3.9.2 *Let $\mathbb{P}^*$ be a risk neutral probability measure such that the discounted values of admissible self-financing portfolios based on futures are $\mathbb{P}^*$-martingales, then*

$$F(t;T) = \mathbb{E}_{\mathbb{P}^*}(X|\mathcal{F}_t).$$

Proof. Let V_t be the value of a self-financing portfolio formed by a bank account and a contract of futures

$$\begin{aligned} V_t &= \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 \cdot 0 \\ &= \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} \end{aligned}$$

but

$$\begin{aligned} \mathrm{d}V_t &= r_t \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} \mathrm{d}t + \phi_t^1 \mathrm{d}F(t;T) \\ &= r_t V_t \mathrm{d}t + \phi_t^1 \mathrm{d}F(t;T), \end{aligned}$$

so

$$\mathrm{d}\tilde{V}_t = e^{\int_0^t r_s \mathrm{d}s} \phi_t^1 \mathrm{d}F(t;T),$$

with $F(T;T) = X$ and since $\tilde{V}$ is a martingale (that we assume a Brownian one) with respect to $\mathbb{P}^*$ it turns out that $F(\cdot;T)$ is a martingale and therefore

$$F(t;T) = \mathbb{E}_{\mathbb{P}^*}(F(T;T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(X|\mathcal{F}_t)$$

∎

Corollary 3.9.1 *Future prices and forward prices coincide if and only if interest rates are deterministic.*

Proof. $f(t,T) = F(t;T)$ if and only if

$$\mathbb{P}^* = \mathbb{P}^T.$$

By (3.12)

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}^*} = \frac{\frac{P(T,T)}{P(0,T)}}{e^{\int_0^T r_s \mathrm{d}s}} = \frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0,T)},$$

then $\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}^*} = 1$ if and only if

$$P(0,T) = e^{-\int_0^T r_s \mathrm{d}s}.$$

∎

3.9.2 Stock options

Suppose that bonds have a volatility $\sigma_B(t,T)$, d-dimensional, **deterministic** and cadlag, that is, that under the risk neutral probability $\mathbb{P}^*$

$$\mathrm{d}P(t,T) = P(t,T)(r_t \mathrm{d}t + \sigma_B(t,T)\cdot dW_t)$$

and that there is a stock S such that under $\mathbb{P}^*$

$$\mathrm{d}S_t = S_t(r_t \mathrm{d}t + \sigma_S(t)\cdot dW_t),$$

where $\|\sigma_S(t) - \sigma_B(t,T)\| > 0$, $\sigma_S(t)$ **deterministic** and càdlàg. Then the price of a call option with strike K is given by

$$C_t = S_t\Phi(d_+) - KP(t,T)\Phi(d_-), \tag{3.18}$$

with

$$d_\pm = \frac{\log\frac{S_t}{KP(t,T)} \pm \frac{1}{2}\Sigma_t^2}{\Sigma_t},$$

where

$$\Sigma_t^2 = \int_t^T \|\sigma_S(u) - \sigma_B(u,T)\|^2\,du.$$

In fact, by the general formula we have seen above

$$\Pi(t;S) = S_t\mathbb{P}^{(S)}(S_T \geq K|\mathcal{F}_t) - KP(t,T)\mathbb{P}^T(S_T \geq K|\mathcal{F}_t),$$

under $\mathbb{P}^*$

$$F_S(t) := \frac{P(t,T)}{S_t} = \frac{P(0,T)}{S_0}\exp\left\{\int_0^t ..\mathrm{d}u + \int_0^t (\sigma_S(u) - \sigma_B(u,T))\cdot dW_u\right\},$$

and under $\mathbb{P}^{(S)}$

$$\mathrm{d}F_S(t) = F_S||\sigma_S(u) - \sigma_B(u,T))||\mathrm{d}W_u^{(S)},$$

where $W^{(S)}$ is a $\mathbb{P}^{(S)}$-Brownian motion. Analogously under $\mathbb{P}^T$

$$F_B(t) := \frac{S_t}{P(t,T)}$$

$$\mathrm{d}F_B(t) = F_B||\sigma_S(u) - \sigma_B(u,T))||\mathrm{d}W_u^T,$$

with W^T Brownian motion under $\mathbb{P}^T$. And doing similar calculations to that in (3.13) we obtain (3.18).

3.10 Portfolio optimization

3.10.1 Optimization in the Black-Sholes model

In the Black-Scholes model we have two assets. One asset without risk, S^0 that evolves as:

$$\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t, \quad S_0^0 = 1,\ 0 \le t \le T,$$

where $r > 0$, and a risky asset S that evolves as

$$\mathrm{d}S_t = S_t\left(\mu\mathrm{d}t + \sigma\mathrm{d}B_t\right), \quad 0 \le t \le T$$

with $\mu \in \mathbb{R}, \sigma > 0$ and $(B_t)_{0\le t\le T}$ is a Brownian motion, that is:

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}.$$

If we assume an investment-consumption process $(c,\phi)_{0\le t\le T}$ and an initial wealth x, we have that the value of a self-financing portfolio $\left(\phi_t^0,\phi_t^1\right)_{0\le t\le T}$ is given by

$$V_t = x + \int_0^t \phi_s^0 \mathrm{d}S_t^0 + \int_0^t \phi_s^1 \mathrm{d}S_t - \int_0^t c_s \mathrm{d}s.$$

In terms of the parameters of the model:

$$\begin{aligned}
\mathrm{d}V_t &= \phi_t^0 r e^{rt}\mathrm{d}t + \phi_t^1 S_t\left(\mu\mathrm{d}t + \sigma\mathrm{d}B_t\right) - c_t\mathrm{d}t \\
&= r(V_t - \phi_t^1 S_t)\mathrm{d}t + \phi_t^1 S_t\left(\mu\mathrm{d}t + \sigma\mathrm{d}B_t\right) - c_t\mathrm{d}t \\
&= (rV_t - c_t)\mathrm{d}t + \phi_t^1 S_t(\mu - r)\mathrm{d}t + \phi_t^1 S_t\mathrm{d}B_t \\
&= (rV_t - c_t)\mathrm{d}t + \pi_t(\mu - r)\mathrm{d}t + \pi_t\sigma\mathrm{d}B_t
\end{aligned}$$

where $\pi_t := \phi_t^1 S_t$ is the amount invested in the risky asset.

A particular case

We shall consider the case where we want to maximize the logaritm of the terminal wealth. In that case $c_t = 0, 0 \le t \le T$, and

$$\begin{aligned}
\mathrm{d}V_t &= rV_t\mathrm{d}t + \pi_t(\mu - r)\mathrm{d}t + \pi_t\sigma\mathrm{d}B_t \\
&= V_t\{(r + \theta_t(\mu - r))\mathrm{d}t + \theta_t\sigma\mathrm{d}B_t\}
\end{aligned}$$

with $\theta_t := \frac{\pi_t}{V_t}$, the fraction of wealth invested in S.

We want to find

$$\max_\theta \mathbb{E}(\log(V_T)).$$

We have, by the Itô formula,

$$\mathrm{d}\log(V_t) = (r + \theta_t(\mu - r) - \frac{1}{2}\theta_t^2\sigma^2)\mathrm{d}t + \theta_t\sigma\mathrm{d}B_t.$$

Therefore

$$\mathbb{E}(\log(V_T)) = \log(V_0) + \mathbb{E}\left(\int_0^T (r + \theta_s(\mu - r) - \frac{1}{2}\theta_s^2\sigma^2)\mathrm{d}s\right) + \mathbb{E}\left(\int_0^T \theta_t \sigma \mathrm{d}B_t\right).$$

Then if we assume that $\int_0^T \mathbb{E}(\theta_t^2)\mathrm{d}t < \infty$, we have that

$$\mathbb{E}(\log(V_T)) = \log(V_0) + \mathbb{E}\left(\int_0^T (r + \theta_s(\mu - r) - \frac{1}{2}\theta_s^2\sigma^2)\mathrm{d}s\right)$$

and we obtain its maximun value if and only if

$$\theta_t = \frac{\mu - r}{\sigma^2}.$$

3.10.2 Dynamic programming method. The Hamilton-Jacobi-Bellman (HJB) equation.

First of all we introduce the notion of utility function

Definition 3.10.1 *A utility function* $U : [0, \infty) \times (0, \infty) \to \mathbb{R}$ *is a* $C^{1,2}$ *map such that*

1. $U(t, \cdot)$ is estrictly increasing and strictly concave
2. $U'(t, c) = \frac{\partial}{\partial c} U(t, c)$ satisfies

$$\lim_{c \to \infty} U'(t, c) = 0$$
$$\lim_{c \to 0^+} U'(t, c) = \infty$$

for all $t \geq 0$. Notice that $U'(t, c)$ is estrictly decreasing and continuous with range equal to $(0, \infty)$ so there exist an inverse function $I(t, c) : [0, \infty) \times (0, \infty) \to (0, \infty)$ such that

$$I(t, U'(t, c)) = c = U'(t, I(t, c))$$

for all $c \in (0, \infty)$.

We want to find

$$\max_{\substack{c,\theta \\ V_0 = x}} \mathbb{E}\left(\int_0^T F(t, c_t)\mathrm{d}t + G(V_T)\right),$$

where F and G are utility functions (the latter independent of t) and $c_t \geq 0$, for all $t \geq 0$. Let define *the optimal value function*

$$H(t, x) := \max_{\substack{c,\theta \\ V_t = x}} \mathbb{E}\left(\int_t^T F(t, c_s)\mathrm{d}s + G(V_T) \middle| \mathcal{F}_t\right),$$

so we assume a Markovian behaviour of $\mathbb{E}\left(\int_t^T F(t,c_s)\mathrm{d}s + G(V_T)\Big|\,\mathcal{F}_t\right)$. We have that

$$\mathrm{d}V_t = V_t\{(r+\theta_t(\mu-r))\mathrm{d}t + \theta_t\sigma\mathrm{d}B_t\} - c_t\mathrm{d}t.$$

$$\begin{aligned}
H(t,x) &= \max_{\substack{c,\theta\\V_t=x}} \mathbb{E}\left(\int_t^T F(s,c_s)\mathrm{d}s + G(V_T)\Bigg|\,\mathcal{F}_t\right)\\
&= \max_{\substack{c,\theta\\V_t=x}} \mathbb{E}(\left(\left(\int_t^{t+h} F(s,c_s)\mathrm{d}s + \int_{t+h}^T F(s,c_s)\mathrm{d}s + G(V_T)\right)\Bigg|\,\mathcal{F}_t\right)\\
&= \max_{\substack{c,\theta\\V_t=x}} \mathbb{E}\left(\left(\int_t^{t+h} F(s,c_s)\mathrm{d}s + \mathbb{E}\left(\int_{t+h}^T F(s,c_s)\mathrm{d}s + G(V_T)\Bigg|\,\mathcal{F}_{t+h}\right)\right)\Bigg|\,\mathcal{F}_t\right)\\
&= \max_{\substack{c,\theta\\V_t=x}} \mathbb{E}\left(\int_t^{t+h} F(s,c_s)\mathrm{d}s + H(t+h,V_{t+h})\Bigg|\,\mathcal{F}_t\right).
\end{aligned}$$

Assume that $H(t,x)$ is $C^{1,2}$ to apply the Itô formula, then, with $V_t = x$,

$$\begin{aligned}
H(t+h,V_{t+h}) = H(t,x) &+ \int_t^{t+h} (\frac{\partial H}{\partial s} + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}V_s^2\theta_s^2\sigma^2)\mathrm{d}s\\
&+ \int_t^{t+h} \frac{\partial H}{\partial x}\mathrm{d}V_s\\
&= H(t,x) + \int_t^{t+h}\left(\frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}(V_s(r+\theta_s(\mu-r)) - c_s) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}V_s^2\theta_s^2\sigma^2\right)\mathrm{d}s\\
&+ \int_t^{t+h}\frac{\partial H}{\partial x}V_s\theta_s\sigma\mathrm{d}B_s.
\end{aligned}$$

Now if $\int_0^T \mathbb{E}\left(\frac{\partial H}{\partial x}V_s\theta_s\right)^2\mathrm{d}t < \infty$, we have

$$\begin{aligned}
\mathbb{E}(H(t+h,V_{t+h})|\mathcal{F}_t) &= H(t,x)\\
&+ \mathbb{E}\left(\int_t^{t+h}(\frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}(V_s(r+\theta_s(\mu-r)) - c_s) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}V_s^2\theta_s^2\sigma^2)\mathrm{d}s\Bigg|\,\mathcal{F}_t\right).
\end{aligned}$$

Then, for all $h \geq 0$

$$\begin{aligned}
0 &= \max_{\substack{c,\theta\\V_t=x}} \mathbb{E}\left(\int_t^{t+h}(F(s,c_s) + \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}(V_s(r+\theta_s(\mu-r)) - c_t) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}V_s^2\theta_s^2\sigma^2)\mathrm{d}s\Bigg|\,\mathcal{F}_t\right)\\
&= \mathbb{E}\left(\int_t^{t+h}\max_{c,\theta}\left((F(s,c_s) + \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}(V_s(r+\theta_s(\mu-r)) - c_t) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}V_s^2\theta_s^2\sigma^2)\right)\mathrm{d}s\Bigg|\,\mathcal{F}_t\right)_{V_t=x},
\end{aligned}$$

since this is true for all $h \geq 0$,

$$\int_0^t \max_{c,\theta}\left((F(s,c_s) + \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}(V_s(r+\theta_s(\mu-r)) - c_t) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}V_s^2\theta_s^2\sigma^2)\right) \mathrm{d}s,\ 0 \leq t \leq T,$$

is an $(\mathcal{F}_t)_{0\leq t\leq T}$ -martingale with initial value equal to zero and since it is abolutely continuos it is zero almost surely. That is

$$\max_{\substack{c,\theta\\ V_t=x}}\left(\frac{\partial H}{\partial t} + F(t,c_t) + \frac{\partial H}{\partial x}(V_t(r+\theta_t(\mu-r)) - c_t) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}V_t^2\theta_t^2\sigma^2\right)$$

$$= \frac{\partial H}{\partial t} + \max_{c_t,\theta_t}(F(t,c_t) + \frac{\partial H}{\partial x}(x(r+\theta_t(\mu-r)) - c_t) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}x^2\theta_t^2\sigma^2) = 0.\text{(HJB equation)}\ ,$$

with the boundary condition $H(T,x) = G(x)$.

Example 3.10.1 *Assume utility functions $F(t,x) = e^{-\delta t}\log x$ and $G(x) = \log x$. The HJB equation is*

$$\frac{\partial H}{\partial t} + \max_{c,\theta}(F(t,c) + \frac{\partial H}{\partial x}(x(r+\theta(\mu-r) - c) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}x^2\theta^2\sigma^2) = 0, \quad (3.19)$$

first we solve the static optimization problem:

$$\max_{c,\theta}(F(t,c) + \frac{\partial H}{\partial x}(x(r+\theta(\mu-r)) - c) + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}x^2\theta^2\sigma^2),$$

and we obtain the first order conditions:

$$F'(t,\hat{c}) = \frac{\partial H}{\partial x} \Longleftrightarrow \hat{c} = I(t, \frac{\partial H}{\partial x})$$
$$\hat{\theta} = -\frac{\frac{\partial H}{\partial x}}{x\frac{\partial^2 H}{\partial x^2}}\frac{\mu - r}{\sigma^2},$$

and in our case $F(t,x) = e^{-\delta t}\log x$ and $G(x) = \log x$, so

$$\hat{c} = \frac{1}{e^{\delta t}\frac{\partial H}{\partial x}} \quad (3.20)$$
$$\hat{\theta} = -\frac{\frac{\partial H}{\partial x}}{x\frac{\partial^2 H}{\partial x^2}}\frac{\mu - r}{\sigma^2}. \quad (3.21)$$

As ansatz we can use a function of the form

$$H(t,x) = a(t)\log x + b(t),$$

with

$$a(T) = 1, b(T) = 0,$$

by the boundary condition. Then we have by substituting in (3.20,3.21)

$$\hat{c} = \frac{e^{-\delta t}}{a(t)} x$$
$$\hat{\theta} = \frac{\mu - r}{\sigma^2},$$

moreover

$$\frac{\partial H}{\partial t} = \dot{a}(t) \log x + \dot{b}(t)$$
$$\frac{\partial H}{\partial x} = \frac{a(t)}{x}$$
$$\frac{\partial^2 H}{\partial x^2} = -\frac{a(t)}{x^2}.$$

Taking all of thsi into account, the equation (3.19) can be written as

$$0 = \dot{a}(t) \log x + \dot{b}(t) + e^{-\delta t} \log \frac{e^{-\delta t}}{a(t)} x + a(t)(r + \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2}) - e^{-\delta t} = 0$$

and we obtain the conditions for $a(t)$ and $b(t)$. In particular

$$\dot{a}(t) = -e^{-\delta t}$$

so

$$a(t) = 1 + \frac{e^{-\delta t} - e^{-\delta T}}{\delta}.$$

and the optimal consumption-investment strategy is given by

$$c_t = \frac{\delta e^{-\delta t}}{\delta + e^{-\delta t} - e^{-\delta T}} V_t,$$
$$\theta = \frac{\mu - r}{\sigma^2}.$$

3.10.3 The martingale method

Proposition 3.10.1 *Let $\xi \geq 0$ be a positive random variable variable $\mathcal{F}_T$-measurable. And let $(c_s)_{0 \leq s \leq T}$ a positive adapted process. Assume that*

$$\mathbb{E}_{\mathbb{P}^*} \left(\tilde{\xi} + \int_0^T \tilde{c}_s \mathrm{d}s \right) = x$$

and that $\mathbb{E}_{P^*} \left(\tilde{\xi}^2 + \int_0^T \tilde{c}_s^2 \mathrm{d}s \right) < \infty$, then there exist an admissible strategy, with initial value x, that replicates the consumption process c and the final wealth ξ.

Proof. Consider the random variable $\mathcal{F}_T$-measurable $\xi + \int_0^T c_s e^{r(T-s)} \mathrm{d}s \geq 0$. We now that, since it is square integrable, there exist and admissible strategy replicating $\xi + \int_0^T c_s e^{r(T-s)} \mathrm{d}s$, then there exist an admissible strategy $\left(\phi_t^0, \phi_t^1\right)_{0 \leq t \leq T}$ such that

$$\mathbb{E}_{\mathbb{P}^*}\left(\tilde{\xi} + \int_0^T \tilde{c}_s \mathrm{d}s \middle| \mathcal{F}_t\right) = x + \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_t,$$

this strategy allows a consumption $c_s \mathrm{d}s$ at any time s, in such a way that we have to give back $\int_0^T c_s e^{r(T-s)} \mathrm{d}s$ to the bank. ∎

Remark 3.10.1 *The self-financing strategy "with consumption" coincides with that in the previous proposition in the investment in the risky asset, the part in the bank account is diferent in such a way that discounted value at time t is given by*

$$x + \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_t - \int_0^t \tilde{c}_s \mathrm{d}s = \mathbb{E}_{\mathbb{P}^*}\left(\tilde{\xi} + \int_t^T \tilde{c}_s \mathrm{d}s \middle| \mathcal{F}_t\right) \tag{3.22}$$

Consider then the problem

$$\max_{\substack{(c,\phi) \\ V_0 = x}} \mathbb{E}\left(\int_0^T F(t, c_t)\mathrm{d}t + G(V_T)\right),$$

the corresponding *Lagrangian* is

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T F(t, c_t)\mathrm{d}t + G(V_T)\right) - \lambda\left(\mathbb{E}_{\mathbb{P}^*}\left(\tilde{V}_T + \int_0^T \tilde{c}_t \mathrm{d}t\right) - x\right) \\
&= \mathbb{E}\left(\int_0^T F(t, c_t)\mathrm{d}t + G(V_T) - \lambda\left(\tilde{V}_T Z_T + \int_0^T \tilde{c}_t Z_t \mathrm{d}t - x\right)\right) \\
&= \mathbb{E}\left(\int_0^T F(t, c_t)\mathrm{d}t + G(V_T) - \lambda(V_T N_T + \int_0^T c_t N_t \mathrm{d}t - x\right),
\end{aligned}$$

with $Z_t := \mathbb{E}\left(\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} | \mathcal{F}_t\right)$ and $N_t := e^{-rt} Z_t, 0 \leq t \leq T$. The first order conditions lead us to

$$\begin{aligned}
F'(t, c_t) &= \lambda N_t \\
G'(V_T) &= \lambda N_T \\
\mathbb{E}\left(V_T N_T + \int_0^T c_t N_t \mathrm{d}t\right) &= x
\end{aligned}$$

Then, by writing $I_1(t, x)$ for the inverse function of $F'(t, x)$ and $I_2(x)$ the inverse of $G'(x)$, we have

$$\begin{aligned}
c_t &= I_1(t, \lambda N_t) \\
V_T &= I_2(\lambda N_T)
\end{aligned}$$

Example 3.10.2 *Consider the previous example. First*

$$c_t = \frac{e^{-\delta t}}{\lambda N_t}$$
$$V_T = \frac{1}{\lambda N_T},$$

with λ satisfying

$$\frac{1}{\lambda} + \frac{1}{\lambda}\int_0^T e^{-\delta t}\mathrm{d}t = x,$$

so

$$\lambda = \frac{\delta + 1 - e^{-\delta T}}{\delta x}.$$

By (3.22)

$$\begin{aligned}
\tilde{V}_t &= \mathbb{E}_{\mathbb{P}^*}\left(\tilde{V}_T + \int_t^T \tilde{c}_s \mathrm{d}s \middle| \mathcal{F}_t \right) \\
&= \frac{\mathbb{E}\left(V_T N_T + \int_t^T c_s N_s \mathrm{d}s \middle| \mathcal{F}_t \right)}{Z_t} \\
&= \frac{\frac{1}{\lambda} + \frac{1}{\lambda}\int_t^T e^{-\delta s}\mathrm{d}s}{Z_t} \\
&= \frac{\gamma(t)x}{Z_t},
\end{aligned}$$

that is

$$V_t = \frac{\gamma(t)x}{N_t}$$

with

$$\gamma(t) = \frac{\delta + e^{-\delta t} - e^{-\delta T}}{1 + \delta - e^{-\delta T}},$$

and we conclude that

$$\begin{aligned}
c_t &= \frac{e^{-\delta t}}{\lambda N_t} = \frac{e^{-\delta t}}{\gamma(t)\lambda x} V_t \\
&= \frac{\delta e^{-\delta t}}{\delta + e^{-\delta t} - e^{-\delta T}} V_t,
\end{aligned}$$

that si what we obtain by the dynamic programming method. If we want to obtain the optimal strategy by this method, we nee to write V_t in terms of S_t and to make explicit the process (N_t). From (2.9) we have that

$$Z_t = \mathbb{E}\left(\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} \middle| \mathcal{F}_t \right) = \exp\left\{ \frac{r-\mu}{\sigma} B_t - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t \right\}, 0 \le t \le T$$

and since $S_t = S_0 \exp\left\{\mu t - \frac{\sigma^2}{2}t + \sigma B_t\right\}$, *we have*

$$\begin{aligned} Z_t &= \left(\frac{S_t}{S_0}\exp\left\{-\mu t + \frac{\sigma^2}{2}t\right\}\right)^{\frac{r-\mu}{\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t\right\} \\ &= \left(\frac{S_t}{S_0}\exp\left\{-\mu t + \frac{\sigma^2}{2}t\right\}\right)^{\frac{r-\mu}{\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t\right\} \\ &= S_t^{\frac{r-\mu}{\sigma^2}} a e^{bt}, \end{aligned}$$

Example 3.10.3 *for certain constants* a *and* b. *Since*

$$\tilde{V}_t = \frac{\gamma(t)x}{Z_t} = \frac{1}{a}\gamma(t)xe^{-bt}S_t^{\frac{\mu-r}{\sigma^2}}$$

and we look for ϕ *such that*

$$\tilde{V}_t = x + \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_s - \int_0^t \tilde{c}_s \mathrm{d}s$$

it turns out

$$\begin{aligned} \phi_t^1 = \frac{\partial \tilde{V}_t}{\partial \tilde{S}_t} &= \frac{\mu - r}{\sigma^2}\frac{e^{rt}}{a}\gamma(t)xe^{-bt}S_t^{\frac{\mu-r}{\sigma^2}-1} \\ &= \frac{\mu - r}{\sigma^2}\frac{e^{rt}\tilde{V}_t}{S_t} = \frac{\mu - r}{\sigma^2}\frac{V_t}{S_t}, \end{aligned}$$

and

$$\frac{\phi_t^1 S_t}{V_t} = \frac{\mu - r}{\sigma^2}.$$

as we obtained by the previous method.

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