Lesson 16

José M. Corcuera.
University of Barcelona.

Interest rates models in continuous time. Basic facts

If someone borrows one euro at time t, till maturity T, they will have to pay an amount $F(t, T)$ at time T, this is equivalent to a mean rate of continuously compounded interest $R(t, T)$ given by the equality:

$$F(t, T) = e^{(T-t)R(t,T)}.$$

If we assume that interest rates are known: $(R(t, T))_{0 \leq t \leq T}$, and there is not arbitrage then

$$F(t, s) = F(t, u)F(u, s), \forall t \leq u \leq s,$$

and from here together with the condition $F(t, t) = 1$, it follows, if $F(t, s)$ is differentiable as a function of s, that there exist a function $r(t)$ such that

$$F(t, T) = \exp\left(\int_t^T r(s)\mathrm{d}s\right).$$

In fact, let $s \geq t$

$$F(t, s+h) - F(t, s) = F(t, s)F(s, s+h) - F(t, s)$$
$$= F(t, s)(F(s, s+h) - 1),$$

$$\frac{F(t, s+h) - F(t, s)}{F(t, s)h} = \frac{F(s, s+h) - F(s, s)}{h},$$

taking $h \to 0$ we have

$$\frac{\partial_2 F(t, s)/\partial s}{F(t, s)} = \partial_2 F(s, s)/\partial s := r(s)$$

and from here

$$F(t, T) = \exp\left(\int_t^T r(s)\mathrm{d}s\right).$$

The function $r(s)$ is interpreted as an instantaneous interest rate, and it is also called *short rate*.

But look the other way round. Suppose that I want a contract to guarantee one euro at time T. We have the so called *bonds*. What is the price of a bond at time t? To receive $F(t,T)$ at time T we have to pay (put in the bank account) one euro, then, to receive one euro, we have to pay $1/F(t,T)$, so the price of the bond is

$$P(t,T) = \frac{1}{F(t,T)} = \exp\left(-\int_t^T r(s)\mathrm{d}s\right)$$

But interest rates are changing randomly with time and if we model $r(s)$ by a random process this expression does not give a formula for the price of a bond. Why?

The main object of our study is what is called the *zero coupon bond*

Definition

A zero coupon bond with maturity T is a contract that guarantees one euro at time T. Its price at time t shall be denote by $P(t,T)$.

The bonds with coupons are those that are giving certain amounts (coupons) until the maturity of the bond.

LIBOR rates (London Interbank Offer Rates)

Consider the following situation:

At time t we sell a bond with maturity S and with the money we receive, $P(t,S)$, we buy $P(t,S)/P(t,T)$ bonds with maturity T , with $T > S$. By this operation we have a contract such that we pay 1 at time S and receive $P(t,S)/P(t,T)$ at time T. This change, from 1 at S to $P(t,S)/P(t,T)$ at T, can be quoted by simple or continuously compounded interest rates in the period $[S,T]$:

- The simple *forward* interest rate (LIBOR), $L = L(t; S, T)$, which is the solution of the equation:

$$1 + (T - S)L = \frac{P(t, S)}{P(t, T)}$$

that is the simple interest rate guaranteed for the period $[S, T]$ *at time* t.

- The continuously compounded forward interest rate $R = R(t; S, T)$, solution of the equation:

$$e^{R(T-S)} = \frac{P(t, S)}{P(t, T)}.$$

analogously to the previous case, is continuously compounded interest guaranteed at time t, for the period $[S, T]$. The quotation using simple interest rates is the usual at financial markets whereas continuously compounded rates are used in theoretical frameworks.

So, in the bond market we can define different interest rates. That is the prices of the bonds can be quoted in different ways.

Definition

The simple forward rate for the interval $[S, T]$ contracted at t, is defined as

$$L(t; S, T) = -\frac{P(t, T) - P(t, S)}{(T - S)P(t, T)}.$$

Definition

The simple spot rate for $[t, T]$, spot LIBOR, is defined as

$$L(t, T) = -\frac{P(t, T) - 1}{(T - t)P(t, T)},$$

it is the previous one with $S = t$. That is to buy at time t a bond with maturity T.

Definition

The continuously compounded forward rate contracted at t for $[S, T]$ as

$$R(t; S, T) = -\frac{\log P(t, T) - \log P(t, S)}{T - S}$$

Definition

The continuously compounded spot rate for $[t, T]$ as

$$R(t, T) = -\frac{\log P(t, T)}{T - t}$$

Definition

The instantaneous forward rate at S contracted at t as (we assume smoothness in T)

$$f(t,S) = -\frac{\partial \log P(t,T)}{\partial T}\bigg|_{T=S} = \lim_{T\to S} R(t;S,T)$$

Definition

The instantaneous (spot) short rate at t

$$r(t) = f(t,t) = \lim_{S\to t} f(t,S)$$

Note that we can write

$$P(t, T) = e^{-\int_t^T f(t,u)\mathrm{d}u}$$

Fixed t, any of the rates defined previously, except the last one, allow us to recover the prices of the bonds. Then, modelling these rates is equivalent to modelling the bond prices.

Bonds with coupons.Fixed coupons bonds

The simplest of the bonds with coupons is the bond with fixed coupons. It is a bond that at some times in between 0 and T gives predetermined profits (coupons) to the owner of the bond. Its formal description is:

- Let $T_0, T_1, ..., T_n$, fixed times. T_0 is the emission time of the bond, whereas $T_1, ..., T_n$ are the payment times.
- At time T_i the owner receives the amount c_i.
- At time T_n there is an extra payment: K.

It is obvious that this bond can be replicated with a portfolio with c_i zero-coupon bonds with maturities T_i, $i = 1, .., n$ and K zero-coupon bonds with maturity T_n. So, the price at time $t \leq T_1$ will be given by

$$p(t) = KP(t, T_n) + \sum_{i=1}^{n} c_i P(t, T_i).$$

Usually the coupons are expressed in terms of certain rates r_i instead of quantities, in such a way that for instance

$$c_i = r_i(T_i - T_{i-1})K.$$

For a standard coupon the intervals of time are equal:

$$T_i = T_0 + i\delta,$$

y $r_i = r$, de manera que

$$p(t) = K\left(P(t, T_n) + r\delta\sum_{i=1}^{n} P(t, T_i)\right).$$

A mortgage can be seen as a bond with coupons bought by a bank to a particular person. The price of the bond at $t = 0$, is the loan given by the bank and the coupons are the payments of the borrower. In such a way that (taking $T_0 = 0$)

$$p(0) = \sum_{i=1}^{n} c_i P(0, T_i) = c \sum_{i=1}^{n} P(0, T_i).$$

We assume that $T_i - T_{i-1} = \delta$ (in years) and r_e is the equivalent interest rate for a period δ:

$$(1 + r_e)^{\frac{1}{\delta}} := 1 + r,$$

where r is the annual interest rate (if δ is small $r_e \approx \delta \log(1 + r)$). Then

$$P(0, T_i) = (1 + r_e)^{-i}$$

and

$$c = \frac{p(0)}{\sum_{i=1}^{n}(1 + r_e)^{-i}} = \frac{p(0) r_e}{1 - (1 + r_e)^{-n}},$$

after the first payment how much is the debt? $p(0)$ at T_n would become $p(0)(1+r_e)^n$ the quantity c paid at T_1 would become $c(1+r_e)^{n-1}$, therefore if payments would finish at T_1 the quantity, say p_1 has to satisfy the equation

$$p(0)(1+r_e)^n = c(1+r_e)^{n-1} + p_1(1+r_e)^{n-1},$$

that is

$$p_1 = p(0) - (c - p(0)r_e).$$

now if interest is revised, the new coupons are $\bar{c}$, satisfying

$$p_1 := \sum_{i=2}^{n} \bar{c} P(T_1, T_i) = \bar{c} \sum_{i=2}^{n} P(T_1, T_i).$$

Bonds with coupons. Floating rate coupon

Quite often the coupons are not fixed in advance, but rather they are updated for every coupon period. One possibility is to take $r_i = L(T_{i-1}, T_i)$ where L is the spot LIBOR. Since

$$L(T_{i-1}, T_i)(T_i - T_{i-1}) = \frac{1}{P(T_{i-1}, T_i)} - 1$$

we have (taking $K = 1$)

$$c_i = L(T_{i-1}, T_i)(T_i - T_{i-1}) = \frac{1}{P(T_{i-1}, T_i)} - 1.$$

We can replicate this amount, c_i, by selling a bond (without coupons) with maturity T_i and buying one with maturity T_{i-1} :

- With the bond sold we will have at T_i a payoff -1.
- With the bond bought, we will have 1 at T_{i-1} and we can buy bonds with maturity T_i giving a payoff $\frac{1}{P(T_{i-1},T_i)}$ at T_i.
- The total cost, at t, is $P(t, T_{i-1}) - P(t, T_i)$.

The for any time $t \leq T_0$ the price of this bond with random coupons is

$$p(t) = P(t, T_n) + \sum_{i=1}^{n} (P(t, T_{i-1}) - P(t, T_i)) = P(t, T_0)!.$$

This means that, with a unit of money at T_0, we can reproduce the cash flow of this coupon. In fact at time T_0 we can buy $\frac{1}{P(T_0,T_1)}$ bonds with maturity T_1 and then we get $\frac{1}{P(T_0,T_1)}$ units of money. Again at time T_1 we can invest one unit buying bonds with maturity T_2 and so on. The cash flow is $\frac{1}{P(T_0,T_1)} - 1$ at T_1, $\frac{1}{P(T_1,T_2)} - 1$ at T_2,...and $\frac{1}{P(T_{n-1},T_n)} - 1 + 1$ at T_n.

Forward-rate agreement

A forward-rate agreement, or FRA, is the simplest form of interest rate derivative. It is an agreement to pay a fixed simple interest rate on a fixed sum of money between two fixed dates in the future. Assume that the two times are T_1 and T_2 and that at T_1 you receive 1 and you pay a simple interest rate r between T_1 and T_2, that is you pay $1 + r(T_2 - T_1)$ at T_2. Then the price of this contract at $t \leq T_1$ is

$$p(t) := P(t, T_1) - (1 + r(T_2 - T_1)) P(t, T_2),$$

then if we want the price of the contract be zero we have the *fair* rate

$$r = -\frac{P(t, T_2) - P(t, T_1)}{(T_2 - T_1)P(t, T_2)} = L(t; T_1, T_2)!$$

that is the forward-rate defined above.

There are many types of rate swaps but all of them are basically exchanges of payments with fixed rates with random payments. We shall consider the so called *forward swaps settled in arrears*. Denote the principal by K and the swap rate (fixed rate) by R . Suppose equally spaced dates T_i, at time T_i, $i \geq 1$ we receive

$$K\delta L(T_{i-1}, T_i)$$

by paying $K\delta R$, so the cash flow at T_i is $K\delta[L(T_{i-1}, T_i) - R]$,. The value at $t \leq T_0$ of this cash flow is

$$\begin{aligned} &K(P(t, T_{i-1}) - P(t, T_i)) - K\delta RP(t, T_i) \\ &= KP(t, T_{i-1}) - K(1 + R\delta)P(t, T_i), \end{aligned}$$

so in total

$$
\begin{aligned}
p(t) &= \sum_{i=1}^{n} \left(KP(t, T_{i-1}) - K(1 + R\delta)P(t, T_i) \right) \\
&= KP(t, T_0) - K \left(P(t, T_n) + R\delta \sum_{i=1}^{n} P(t, T_i) \right). \qquad (1)
\end{aligned}
$$

Notice that $P(t, T_0)$ is the price of a coupon bond with floating rates whereas $P(t, T_n) + R\delta \sum_{i=1}^{n} P(t, T_i)$ is the price of a coupon bond with fixed rate R. This fixed rate R is usually taken in such a way that the value of the contract is zero when it is issued, this rate is named the *swap rate*. If $t \leq T_0$ is the time when it is issued, the swap rate is

$$
R = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}.
$$

Caps and Floors

A *cap* is a contract that protects you from paying more than a fixed rate in a loan, (the *cap rate*) R, even though the loan has floating rate. We can also define a *floor* that is a contract that guarantees that the rate is always above the so called *floor rate* R even for an investment with random rate. A cap is a sum of *captlets*, they consist on these basic contracts.

- The interval $[0, T]$ is divided by equidistant points: $0 = T_0, T_1, ..., T_n = T$, with distance δ. Typically $1/4$ of the year or half year.
- The cap works on a principal, say K, and the cap rate is R.
- The floating rate is for instance the LIBOR $L(T_{i-1}, T_i)$.
- The caplet i is defined as a contract with payoff *at* T_i given by

$$K\delta(L(T_{i-1}, T_i) - R)_+.$$

Theorem

The value of a cap with principal K and cap rate R is that of one portfolio with $K(1+R\delta)$ put options with maturities T_{i-1}, $i=1,...,n$ on bonds with maturities T_i and with strike $\frac{1}{1+R\delta}$.

Proof.

This is what you receive at time T_i

$$K\delta(L(T_{i-1},T_i)-R)_+ = K\left(\frac{1}{P(T_{i-1},T_i)}-1-\delta R\right)_+$$
$$= \frac{K(1+R\delta)}{P(T_{i-1},T_i)}\left(\frac{1}{1+R\delta}-P(T_{i-1},T_i)\right)_+,$$

but a payoff $\frac{1}{P(T_{i-1},T_i)}$ in T_i is equivalent to 1 at T_{i-1}. $\square$

Proof.

In other words, with the cash amount

$$K(1+R\delta)\left(\frac{1}{1+R\delta}-P(T_{i-1},T_i)\right)_+$$

at T_{i-1} I can buy

$$\frac{K(1+R\delta)}{P(T_{i-1},T_i)}\left(\frac{1}{1+R\delta}-P(T_{i-1},T_i)\right)_+$$

bonds with maturity T_i and I get this amount at T_i. □

Swaptions

It is a contract s that gives the right to enter in a swap *at the maturity time* of the *swaption*. A *payer swaption* gives the right to enter in a swap as payer of the fixed rate. A *receiver swaption* gives the right to enter as the receiver of the fixed rates.
A payer swaption has similarities with the cap contract. In the cap the owner has the right to receive a random rate and to pay a constant rate and he will exercise in each period when the random rate is greater than the fixed one. Similarly the owner of payer swaption has the right to receive a floating rate and to pay a constant rate, however in the cap you chose if paying or not at each period, in the case of a swaption te decision is taken once for ever at the maturity time of the swaption.

The value of the swap, according to (1) with principal 1, at the maturity time of the swaption, say T, is

$$P(T, T_0) - \left(P(T, T_n) + R\delta \sum_{i=1}^{n} P(T, T_i \right),$$

so the payer swaption payoff is

$$\left(P(T, T_0) - (P(T, T_n) + R\delta \sum_{i=1}^{n} P(T, T_i)) \right)_{+}$$

Then a swaption can be seen as an option to exchange a bond with fixed coupons by another with floating rate coupons. If $T = T_0$ a swaption becomes a put option with strike 1 on a bond with fixed coupons.