1. Let $f \in L^2[0, 2\pi]$. Reorganise its Fourier series to show that

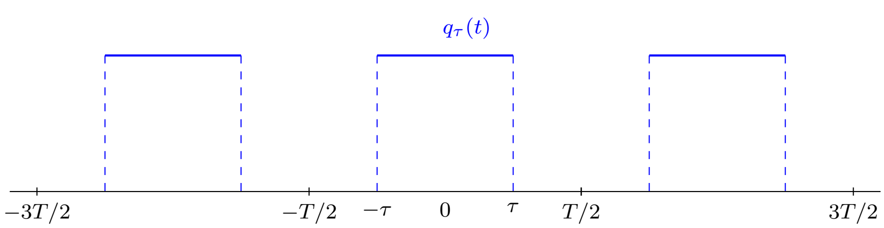
$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + i \sum_{n=1}^{\infty} b_n \sin(nt),$$

where,

$$\begin{aligned} a_0 &= \hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(nt)\, dt\, n \geq 1, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(nt)\, dt\, n \geq 1. \end{aligned}$$

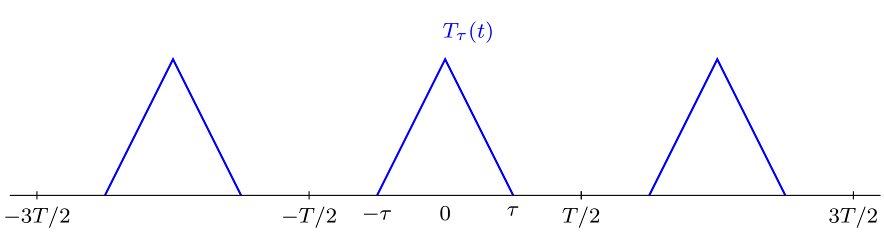
2. A *square wave* is a T-periodic function q_τ which in the interval $[-T/2, T/2]$ has the form $q_\tau = (2\tau)\chi_{(-\tau,\tau)}$, where $0 < \tau < T/2$. Explicitly,

$$q_\tau(t) = \sum_{k\in\mathbb{Z}} (2\tau)\chi_{(-\tau,\tau)}(t - Tk)\,.$$



Similarly, a *triangular wave* is a T-periodic function t_τ which in $[-T/2, T/2]$ has the form $t_\tau = (\tau - |t|)\chi_{(-\tau,\tau)}$, again with $0 < \tau < T/2$. Explicitly,

$$T_\tau(t) = \sum_{k\in\mathbb{Z}} (\tau - |t|)\chi_{(-\tau,\tau)}(t - Tk)\,.$$



(a) Find the Fourier series of $q_\tau(t)$.

(b) Find the Fourier series of $T_\tau(t)$.

(c) Compare the order of decay of the Fourier coefficients in both series. Can you explain the difference?

3. Find the Fourier series of the 2π-periodic function

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi. \end{cases}$$

What does the Fourier series at $t = 0$ converge to?

4. (a) Obtain the Fourier series of the 2π-periodic function defined in $|t| < \pi$ as $f(t) = t \sin t$.

(b) Compute the value of the numerical series

$$\text{(a)} \quad \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}, \qquad \text{(b)} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1}, \qquad \text{(c)} \quad \sum_{n=2}^{\infty} \frac{1}{(n^2 - 1)^2}.$$

5. Let f be the $2L$-periodic function in $\mathbb{R}$ defined in $(-L, L)$ as $f(t) = e^t$.

(a) Compute the Fourier series of f.

(b) Compute, for $a \in \mathbb{R}$, the value of $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+a^2}$.

6. Define the Bessel functions $J_n(x)$ through the Fourier series

$$e^{ix \sin(t)} = \sum_{n \in \mathbb{Z}} J_n(x) e^{int}.$$

Compute, for $x \in \mathbb{R}$,

$$\sum_{n \in \mathbb{Z}} |J_n(x)|^2 .$$

7. Let $f \in \mathcal{C}^k[0, 2\pi]$ with $f^{(j)}(0) = f^{(j)}(2\pi)$ for all $j \leq k$ (f is $\mathcal{C}^k$ as a 2π-periodic function in $\mathbb{R}$). Prove that $\lim\limits_{|n| \to \infty} |n|^k |\hat{f}(n)| = 0$.

8. *The isoperimetric inequality*. Let γ be a simple closed curve in $\mathbb{R}^2$ of length ℓ, and let A denote the area of the region enclosed by this curve. Then

$$A \leq \frac{\ell^2}{4\pi},$$

with equalityif and only if γ is a circle.

Notice that when γ is a circle of length ℓ then the radius is $\ell/(2\pi)$, and therefore the area i $l^2/(4\pi)$.

(a) See that it is enough to consider the case $\ell = 2\pi$ and that γ is parametrised by the arc-length, so that the inequality to be proved has the form $A \leq \pi$.

(b) Prove that
$$A = \frac{1}{2}\int_\gamma (x\,dy - y\,dx) = \frac{1}{2}\left|\int_\gamma (x\,dy - y\,dx)\right|.$$

(c) Let $\gamma(t) = (x(t), y(t))$, $t \in [0, 2\pi]$ be the parametrisation given by the arc-length and let $x(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}$, $y(t) = \sum_{n\in\mathbb{Z}} b_n e^{int}$ be the corresponding Fourier series. Prove that
$$\sum_{n\in\mathbb{Z}} |n|^2(|a_n|^2 + |b_n|^2) = 1\,.$$

(d) Prove that
$$A = \pi\left|\sum_{n\in\mathbb{Z}} n(a_n\bar{b}_n - b_n\bar{a}_n)\right|$$
and deduce that $A \leq \pi$.

(e) Assume $A = \pi$. Prove that
$$x(t) = a_{-1}e^{-it} + a_0 + a_1e^{it}, \quad y(t) = b_{-1}e^{-it} + b_0 + b_1e^{it},$$
with $a_{-1} = \bar{a}_1, b_{-1} = \bar{b}_1$, and deduce that γ is a circle.

9. Given $x \in \mathbb{R}$ let $[x]$ denote its integer part and $< x >= x - [x] \in [0, 1)$ its fractional part. *Weyl's equidistribution theorem:* if α is an irrational number, the sequence $\{< n\alpha >\}_{n\in\mathbb{N}}$ is equidistributed in $[0, 1)$, in the sense that for any interval $(a, b) \subset [0, 1)$
$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N :< n\alpha >\in (a, b)\}}{N} = |(a, b)| = b - a.$$

(a) Prove that if $\alpha \in \mathbb{Q}$ then there are finitely many distinct values $< n\alpha >$, $n \in \mathbb{N}$, and that if $\alpha \notin \mathbb{Q}$ then there are infinitely many.

(b) Show that Weyl's equidistribution theorem is equivalent to the identities
$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N \chi_{(a,b)}(n\alpha) = \int_0^1 \chi_{(a,b)}(t)\,dt,$$
where $(a, b) \subset [0, 1)$ and $\chi_{(a,b)}$ indicates the 1-periodic function in $\mathbb{R}$ such that in $[0, 1)$ has value 1 on (a, b) and 0 on $[0, 1) \setminus (a, b)$.

(c) Prove that for any 1-periodic function $f \in \mathcal{C}(\mathbb{R})$ and $\alpha \notin \mathbb{Q}$,
$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(n\alpha) = \int_0^1 f(t)\,dt.$$
(Hint: consider functions $f(t) = e^{2\pi ikt}$, $k \in \mathbb{Z}$.)

(d) Deduce Weyl's equidistribution theorem.

This result has an interpretation in terms of (discrete) dynamical systems. Let $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ denote a rotation by an irrational angle α (i.e. $\rho(\theta) = \theta + 2\pi\alpha$, $\theta \in [0, 2\pi)$). Given $f \in L^1(\mathbb{T})$, take the iterates $f^{(n)}(\theta) = f(\rho^{(n)}(\theta))$. Then the discrete dinamical system $\{f^{(n)}(\theta)\}_{n\in N}$ is *ergodic*: $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{n=1}^N f^{(n)}(\theta)$ exists for all θ and
$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f^{(n)}(\theta) = \int_0^{2\pi} f(\theta)\,\frac{d\theta}{2\pi}.$$