Quantitative Finance Problem Set 1

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February 23, 2023

Exercise 1
Case 1: $\mathcal{F} = \{\emptyset, \Omega\}$
Let X be a random variable measurable with respect to $\mathcal{F}$. Then any measurable set in $\mathcal{F}$ must be either the empty set $\emptyset$ or the whole sample space Ω. Therefore, for any set B we have:

$$X^{-1}(B) = (X^{-1}(B) \cap \Omega) \cup (X^{-1}(B) \cap \emptyset) = X^{-1}(\Omega) = \Omega$$

This means that X takes every value in B with probability 1, and hence X is constant almost surely.

Case 2: $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$
Let X be a random variable measurable with respect to $\mathcal{F}$. In this case, any measurable set in $\mathcal{F}$ must be either the empty set $\emptyset$, the whole sample space Ω, A, or A^c. Therefore, for any set B we have:

$$X^{-1}(B) = (X^{-1}(B) \cap A) \cup (X^{-1}(B) \cap A^c)$$

So, X takes on the value a on A and the value b on A^c, i.e.

$$X(\omega) = \begin{cases} a & if\ \omega \in A \\ b & if\ \omega \in A^c \end{cases}$$

Since A and A^c partition Ω, X is uniquely determined by its values on A and A^c, and hence must be of the form above.

Case 3: $\mathcal{F} = \sigma(A_1, A_2, \ldots, A_n)$ with $(A_i)_{0 \le i \le n}$ a partition of Ω
Let X be a random variable measurable with respect to $\mathcal{F}$. In this case, any measurable set in $\mathcal{F}$ must be a union of some of the sets A_i (possibly including the empty set and Ω). Therefore, for any set B we have:

$$X^{-1}(B) = \bigcup_{i=1}^{n} (X^{-1}(B) \cap A_i)$$

So, X takes on the same value on each of the sets A_i, i.e.

$$X(\omega) = \begin{cases} c_1 & if\ \omega \in A_1 \\ c_2 & if\ \omega \in A_2 \\ \vdots & \\ c_n & if\ \omega \in A_n \end{cases}$$

Since the sets A_i partition Ω, X is uniquely determined by its values on the sets A_i, and hence must be constant on each of these sets. $\blacksquare$

Exercise 2
Let X be a random variable such that for every n, the conditional expectation $\mathbb{E}(X \mid \mathcal{F}_n)$ exists almost surely and $\mathbb{E}(X \mid \mathcal{F}_n)$ is $\mathcal{F}_n$-measurable. Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N$ be a filtration.
To show that $(\mathbb{E}(X \mid \mathcal{F}n))_{0 \le n \le N}$ is an $\mathbb{F}$-martingale, we need to show that it satisfies the two conditions of a martingale:

1. $\mathbb{E}(|\mathbb{E}(X \mid \mathcal{F}_n)|) < \infty$ for all n

2. $\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) = \mathbb{E}(X \mid \mathcal{F}_n)$ for all n

For the first condition, note that by the law of total expectation,

$$\mathbb{E}(|\mathbb{E}(X \mid \mathcal{F}_n)|) = \mathbb{E}(|X|) < \infty,$$

since X is a random variable.
For the second condition, we need to show that for all n,

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) = \mathbb{E}(X \mid \mathcal{F}_n).$$

We can prove this using the tower property of conditional expectation. Specifically, we have

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_m)|\mathcal{F}_n) = \mathbb{E}(X|\mathcal{F}_n) \quad \text{almost surely.}$$

So in this case the following holds:

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) = \mathbb{E}(X \mid \mathcal{F}_n) \quad \text{(by the tower property)}$$

This is because $\mathcal{F}_n \subseteq \mathcal{F}_m$, so conditioning on $\mathcal{F}_n$ first and then $\mathcal{F}_{n+1}$ does not change the conditional expectation. Therefore, the martingale property holds.
Therefore, $(\mathbb{E}(X \mid \mathcal{F}n))_{0 \leq n \leq N}$ is an $\mathbb{F}$-martingale. ■

Exercise 3
Let $X = \mathbb{1}_B$ be a random variable, where B is an element of the sigma algebra $\mathcal{F}$. Then, we want to show that $\mathbb{E}(XY|\mathcal{F}) = X\mathbb{E}(Y|\mathcal{F})$. Substituting $X = \mathbb{1}_B$ into the above equation we get:

$$\mathbb{E}(XY|\mathcal{F}) = \mathbb{E}(\mathbb{1}_B Y|\mathcal{F})$$

Applying the definition of conditional expectation to Y we obtain:

$$\mathbb{E}(\mathbb{1}_B Y|\mathcal{F}) = \mathbb{1}_B \mathbb{E}(Y|\mathcal{F}) + (1 - \mathbb{1}_B)\mathbb{E}(Y|\mathcal{F}^c)$$

Since B is an element of $\mathcal{F}$, we have that $\mathcal{F}^c$ is a subset of B^c, where B^c denotes the complement of B. Therefore, $\mathbb{E}(Y|\mathcal{F}^c)$ is constant over B, so we can write:

$$\mathbb{E}(\mathbb{1}_B Y|\mathcal{F}) = \mathbb{1}_B \mathbb{E}(Y|\mathcal{F}) + 0$$

Thus, we have:

$$\mathbb{E}(XY|\mathcal{F}) = \mathbb{E}(\mathbb{1}_B Y|\mathcal{F}) = \mathbb{1}_B \mathbb{E}(Y|\mathcal{F}) = X\mathbb{E}(Y|\mathcal{F}) \quad ■$$

To prove that $\mathbb{E}(Y|\mathcal{F}) = \mathbb{E}(Y)$ when Y is independent of $\mathcal{F}$, we need to show that for any $A \in \mathcal{F}$ the following holds:

$$\int_A \mathbb{E}(Y|\mathcal{F})d\mathbb{P} = \int_A \mathbb{E}(Y)d\mathbb{P}$$

We can start by using the definition of conditional expectation:

$$\int_A \mathbb{E}(Y|\mathcal{F})d\mathbb{P} = \int_A \left(\int_\Omega Y d\mathbb{P}(Y|\mathcal{F})\right) d\mathbb{P}$$

Since Y is independent of $\mathcal{F}$, we have $\mathbb{P}(Y|\mathcal{F}) = \mathbb{P}(Y)$, so we can substitute this in the above equation:

$$\int_A \mathbb{E}(Y|\mathcal{F})d\mathbb{P} = \int_A \left(\int_\Omega Y d\mathbb{P}(Y)\right) d\mathbb{P}$$

Since Y is a random variable, we can write the above as:

$$\int_A \mathbb{E}(Y|\mathcal{F})d\mathbb{P} = \int_A \mathbb{E}(Y)d\mathbb{P}$$

And finally, since A is an arbitrary event in $\mathcal{F}$, we can write:

$$\int_\Omega \mathbb{E}(Y|\mathcal{F})\mathbb{P}(d\omega) = \int_\Omega \mathbb{E}(Y)\mathbb{P}(d\omega)$$

which implies $\mathbb{E}(Y|\mathcal{F}) = \mathbb{E}(Y)$ almost surely.
Therefore, we have shown that if Y is independent of $\mathcal{F}$, then $\mathbb{E}(Y|\mathcal{F}) = \mathbb{E}(Y)$. ■

Exercise 4
Let $\{Y_n\}_{n\geq 1}$ with $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$, set $S_0 = 0$ and $S_n = Y_1 + \ldots + Y_n$ if $n \geq 1$
A sequence of random variables is said to be a martingale if it satisfies the following conditions:

1. The expected value of each random variable in the sequence is finite.
2. Given any index i, the conditional expected value of the next random variable in the sequence, given all the previous values up to and including i, is equal to the current value at i.

So to check if the given sequences are martingales we can check the following conditions:

1. $\mathbb{E}(|Y_n|) < \infty$ for all n
2. $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = Y_n$ for all n

Consider $M_n^{(1)} = \frac{e^{\theta S_n}}{(cosh\theta)^n}$:

1. We want to check if $\mathbb{E}(|M_n^{(1)}|) < \infty$ for all n.
$|M_n^{(1)}| = \left|\frac{e^{\theta S_n}}{\cosh(\theta)^n}\right| = \frac{e^{\theta S_n}}{\cosh(\theta)^n} = \frac{e^{\theta(Y_1\ldots Y_n)}}{\cosh(\theta)^n} = \frac{e^{\theta Y_1}\cdot\ldots\cdot e^{\theta Y_n}}{\cosh(\theta)^n} = \frac{e^{\theta Y_1}}{\cosh(\theta)}\cdot\ldots\cdot\frac{e^{\theta Y_n}}{\cosh(\theta)}$
So we have $\mathbb{E}[|M_n|] = \mathbb{E}[\frac{e^{\theta Y_1}}{\cosh(\theta)}\cdot\ldots\cdot\frac{e^{\theta Y_n}}{\cosh(\theta)}]$
Since $Y_1 \ldots Y_n$ are i.i.d. we have
$\mathbb{E}[|M_n|] = \mathbb{E}[\frac{e^{\theta Y_1}}{\cosh(\theta)}\cdot\ldots\cdot\frac{e^{\theta Y_n}}{\cosh(\theta)}] = \mathbb{E}[\frac{e^{\theta Y_1}}{\cosh(\theta)}]\cdot\ldots\cdot\mathbb{E}[\frac{e^{\theta Y_n}}{\cosh(\theta)}] = \mathbb{E}[\frac{e^{\theta Y_1}}{\cosh(\theta)}]^n =$
$= [\frac{1}{2}\frac{e^{\theta}}{\cosh(\theta)} + \frac{1}{2}\frac{e^{-\theta}}{\cosh(\theta)}]^n = [\frac{1}{2}\frac{2e^{\theta}}{e^{\theta}+e^{-\theta}} + \frac{1}{2}\frac{2e^{-\theta}}{e^{\theta}+e^{-\theta}}]^n = [\frac{e^{\theta}+e^{-\theta}}{e^{\theta}+e^{-\theta}}]^n = 1^n$
Hence, $\mathbb{E}[|M_n|]$ is finite for all n.
2. We want to check if $\mathbb{E}(M_{n+1}^{(1)}|\mathcal{F}_n) = M_n^{(1)}$ for all n
$\mathbb{E}[M_{n+1}^{(1)}|\mathcal{F}_n] = \mathbb{E}\left[\frac{e^{\theta Sn+1}}{\cosh(\theta)^{n+1}}|\mathcal{F}_n\right] = \mathbb{E}\left[\frac{e^{\theta S_n}}{\cosh(\theta)^n}\frac{e^{\theta Y_{n+1}}}{\cosh(\theta)}|\mathcal{F}_n\right] = \mathbb{E}\left[M_n^{(1)}\frac{e^{\theta Y_{n+1}}}{\cosh(\theta)}|\mathcal{F}_n\right] = M_n^{(1)}\mathbb{E}\left[\frac{e^{\theta Y_{n+1}}}{\cosh(\theta)}\right]$
Note that the Y_{n+1} is independent of the other previous n variables.
Note that Y_{n+1} is a symmetric Bernoulli random variable taking values 1 and -1 with equal probability. Therefore,
$\mathbb{E}\left[\frac{e^{\theta Y_{n+1}}}{\cosh(\theta)}\right] = \mathbb{E}\left[2\frac{e^{\theta Y_{n+1}}}{e^{\theta}+e^{-\theta}}\right] = \mathbb{E}\left[2(\frac{1}{2}\frac{e^{\theta}}{e^{\theta}+e^{-\theta}} + \frac{1}{2}\frac{e^{-\theta}}{e^{\theta}+e^{-\theta}})\right] = \mathbb{E}\left[\frac{e^{\theta}+e^{-\theta}}{e^{\theta}+e^{-\theta}}\right] = 1$
So from the previous identities, we obtain $\mathbb{E}(M_{n+1}^{(1)}|\mathcal{F}_n) = M_n^{(1)}$

Consider $M_n^{(2)} = \sum_{k=1}^{n} sign(S_{k-1})Y_k, \quad n \geq 1, M_0^{(2)} = 0$:

1. $|M_n^{(2)}| = |\sum_{k=1}^{n}\text{sign}(S_{k-1})Y_k| \leq \sum_{k=1}^{n}|Y_k|$
Since $|Y_k| = 1$ for all k, we have $|M_n^{(2)}| \leq n$
Therefore, $\mathbb{E}[|M_n^{(2)}|] \leq \mathbb{E}[n] = n < \infty$ for all n
2. $\mathbb{E}[M_{n+1}^{(2)}|\mathcal{F}_n] = \mathbb{E}\left[\sum_{k=1}^{n+1}\text{sign}(S_{k-1})Y_k|\mathcal{F}_n\right] = \sum_{k=1}^{n}\text{sign}(S_{k-1})\mathbb{E}[Y_k|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}\,\text{sign}(S_n)|\mathcal{F}_n]$
Note that Y_{n+1} is independent of $\mathcal{F}_n$, so we have $\mathbb{E}[Y_{n+1}\,\text{sign}(S_n)|\mathcal{F}_n] = \text{sign}(S_n)\mathbb{E}[Y_{n+1}] = 0$
Also, since $\mathcal{F}_n$ contains information up to time n, the conditional expectation $\mathbb{E}[Y_k|\mathcal{F}_n]$ for $k < n$ is simply Y_k
Therefore, $\mathbb{E}[M_{n+1}^{(2)}|\mathcal{F}_n] = M_n^{(2)} + 0 = M_n^{(2)}$

Consider $M_n^{(3)} = S_n^2 - n$:

1. $\mathbb{E}[|M_n^{(3)}|] = \mathbb{E}[|S_n^2 - n|] \leq \mathbb{E}[|S_n^2|] + \mathbb{E}[|n|] = \mathbb{E}[S_n^2] + n$
Since $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$, we have $S_n = Y_1 + \ldots + Y_n \leq n$ and $S_n^2 \leq n^2$
Therefore $\mathbb{E}[|M_n^{(3)}|] \leq n^2 + n < \infty$ for all n

2. $\mathbb{E}[M_{n+1}^{(3)}|\mathcal{F}_n] = \mathbb{E}[(S_{n+1}^2 - (n+1))|\mathcal{F}_n] = \mathbb{E}[(S_n + Y_{n+1})^2 - (n+1)|\mathcal{F}_n] =$
$= \mathbb{E}[S_n^2 + 2S_nY_{n+1} + Y_{n+1}^2 - (n+1)|\mathcal{F}n] = S_n^2 + 2S_n\mathbb{E}[Y_{n+1}|\mathcal{F}_n] + \mathbb{E}[Y_{n+1}^2] - (n+1) =$
$= S_n^2 - (n+1) + \mathbb{E}[Y_{n+1}^2] = M_n^{(3)} - 1 + \mathbb{E}[Y_{n+1}^2]$
Since $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$, we have $\mathbb{P}(Y_i^2 = 1) = 1$ and thus $\mathbb{E}[Y_{n+1}^2] = 1$
Therefore, $\mathbb{E}[M_{n+1}^{(3)}|\mathcal{F}_n] = M_n^{(3)}$

Exercise 5
Let us consider a portfolio of assets with weights (ϕ_1, ϕ_2, ϕ_3) invested in assets (S^1, S^2, S^3) respectively. The cost of this portfolio at time 0 is:

$$C_0 = \phi_1 S_0^1 + \phi_2 S_0^2 + \phi_3 S_0^3 = \phi_1 + \phi_2 + \phi_3$$

The value of this portfolio at time 1 is:

$$C_1 = \phi_1 S_1^1 + \phi_2 S_1^2 + \phi_3 S_1^3$$

Substituting the values of S_1^i, we get:

$$C_1 = \phi_1 x_1 \mathbb{1}_{\{\omega_1\}} + \phi_2 x_2 \mathbb{1}_{\{\omega_2\}} + \phi_3 x_3 \mathbb{1}_{\{\omega_3\}}$$

The expected value of this portfolio is:

$$\mathbb{E}(C_1) = \phi_1 x_1 \mathbb{P}(\omega_1) + \phi_2 x_2 \mathbb{P}(\omega_2) + \phi_3 x_3 \mathbb{P}(\omega_3)$$

where $\mathbb{P}(\omega_i)$ is the probability of the outcome ω_i.
Since $r = 0$, the no-arbitrage condition implies that the expected value of the portfolio must be equal to its cost at time 0:

$$E(C_1) = C_0$$

Substituting the values of C_0 and $\mathbb{E}(C_1)$, we get:

$$\phi_1 x_1 \mathbb{P}(\omega_1) + \phi_2 x_2 \mathbb{P}(\omega_2) + \phi_3 x_3 \mathbb{P}(\omega_3) = \phi_1 + \phi_2 + \phi_3$$

This equation must hold for all possible values of ϕ_1, ϕ_2, and ϕ_3. We can simplify this equation by setting $\phi_1 = 1$ and $\phi_2 = \phi_3 = 0$, which gives:

$$x_1 \mathbb{P}(\omega_1) = 1 \rightarrow x_1 = 1/\mathbb{P}(\omega_1)$$

Similarly, setting $\phi_2 = 1$ and $\phi_1 = \phi_3 = 0$, we get:

$$x_2 \mathbb{P}(\omega_2) = 1 \rightarrow x_2 = 1/\mathbb{P}(\omega_2)$$

Finally, setting $\phi_3 = 1$ and $\phi_1 = \phi_2 = 0$, we get:

$$x_3 \mathbb{P}(\omega_3) = 1 \rightarrow x_3 = 1/\mathbb{P}(\omega_3)$$

If any of these equations do not hold (i.e., if $x_i \mathbb{P}(\omega_i) \neq 1$ for any i), then there exists an arbitrage opportunity in the market. If all of these equations hold simultaneously, then the market is free of arbitrage. ■