DYNAMICAL SYSTEMS
MÀSTER EN MATEMÀTICA AVANÇADA
Fall semester, 2023

Final exercise set

Due January 23rd, 2024

Please upload your answers in the virtual campus, using the allocated tasks. Please create TWO pdf files: one for problems 1 and 2, and a second one for problems 3 and 4.

1. Given the map

$$f(x, y, z) = A(x, y, z)^\top + (yz, x^2, y^2)^\top$$

with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

1) Compute the approximation of the center manifold of the origin up to order 4 (included),
2) compute the reduction of the map to the center manifold,
3) study the stability of the origin.

2. Let $X(x)$ a vector-field of $\mathbb{R}^2$ such that $X(0) = 0$ and with 0 a non-hyperbolic equilibrium point such that

$$DX(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Prove that a normal form in a neighborhood of the origin, up to order 2, is

$$\begin{aligned} x' &= y + O_3 \\ y' &= ax^2 + bxy + O_3. \end{aligned}$$

3. Let f be a polynomial of degree $d \geq 2$. Suppose that $z_1, \ldots, z_d$ are the fixed points of f and assume that $f'(z_i) \neq 1$ for $i = 1, \ldots, d$ (which implies that the d fixed points are distinct). Let $\lambda_i = f'(z_i)$.

(a) Show that $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0$. *Hint: Prove that the residue of* $\frac{1}{z-f(z)}$ *at a fixed point* z_i *is equal to* $\frac{1}{1-\lambda_i}$, *and then use the Residue Theorem.*

(b) Prove that for every $i = 1, \ldots, d$, z_i is attracting or indifferent if and only if $\mathrm{Re}(\frac{1}{1-\lambda_i}) \geq \frac{1}{2}$.

(c) Use these results to show that every polynomial of degree $d \geq 2$ must have at least one fixed point which is repelling or has multiplier exactly equal to 1.

4. Prove that every **monic** cubic polynomial is affine conjugate to one of the form

$$P_{a,b}(z) = z^3 - 3a^2 z + b$$

for some values of $a, b \in \mathbb{C}$. Compute the critical points of $P_{a,b}$.

Observe that the parameter space is $\mathbb{C}^2$. To restrict to a section of (complex) dimension one we impose the following condition: we require one of the two critical points (say $-a$) to be a fixed point, and we call the other one the *free critical point*. We then obtain a one-parameter family of cubic polynomials, P_a, all of them having a superattracting fixed point (apart from infinity which always is one).

(a) Find an expression for $P_a(z)$.

(b) Can the Julia set of P_a be totally disconnected? And connected? And disconnected but not totally disconnected? Can it have empty interior? Justify your answers using the theorems seen in class.

(c) (Optional) Make a computer program using the escape-time algorithm, to draw the Filled Julia sets of P_a for any value of a. Make sure your program distinguishes (with different colors) the orbits that converge to $z = -a$, from those that converge to infinity, from those that do neither.

The analogue of the Mandelbrot set. Make a second program to draw the parameter plane, iterating the free critical. To color the pixels, you should distinguish three different cases depending on whether the orbit of the free critical point:

(a) is bounded AND is in the basin of $z = -a$ the superattracting fixed point.

(b) is bounded but it is NOT in the basin of $z = -a$

(c) converges to infinity;

Justify that in case (a) all Fatou components are eventually mapped to the superattracting basin of $z = -a$, while in case (b) other periodic Fatou components may exist.

Finally, use your first program to draw one filled Julia set in each of the three situations described