Quantitative Finance

Exercise Set 5

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Exercise 1

In the Black-Scholes (BS) model compute the price and the self-financing hedging portfolios of contingent claims with payoffs:

1. $X = S_T^2$
2. $X = S_T/S_{T_0}$, $0 \leq T_0 \leq T$
3. $X = 1/S_T$.

Answer:

We have to use the following theorem:

Theorem 1. *In the BS model the price of an option with payoff $X = f(S_T) \geq 0$ and square integrable with respect to $\mathbb{P}*$, is given by*

$$C(t, S_t) = \mathbb{E}_{\mathbb{P}*}(\exp^{-r(T-t)} X|\mathcal{F}_t)$$

and if $C(t, x)$ and $C^{1,2}$, the strategy that replicates X is given by (ϕ_t^0, ϕ_t^1) with

$$\phi_t^1 = \frac{\partial C(t, S_t)}{\partial S_t}$$

$$\phi_t^0 \exp^{rt} = C(t, S_t) - \phi_t^1 S_t$$

and $C(t, S_t)$ is the solution of

$$\frac{\partial C(t, S_t)}{\partial t} + rS_t \frac{\partial C(t, S_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t, S_t)}{\partial S_t^2} = rC(t, S_t)$$

with the boundary condition $C(T, S_T) = f(S_T)$.

1) $X = S_T^2$

Since

$$S_T = S_t \exp^{\sigma(\omega_T - \omega_t) + (r - \frac{1}{2}\sigma^2)(T-t)}$$

then

$$S_T^2 = S_t^2 \exp^{2\sigma(\omega_T - \omega_t) + 2(r - \frac{1}{2}\sigma^2)(T-t)}$$

So,

$$\begin{aligned}
C(t,S_t) &= \mathbb{E}_{\mathbb{P}*}(e^{-r(T-t)}X|\mathcal{F}_t)\\
&= \mathbb{E}_{\mathbb{P}*}(e^{-r(T-t)}S_T^2|\mathcal{F}_t)\\
&= e^{-r(T-t)}\mathbb{E}_{\mathbb{P}*}(S_T^2|\mathcal{F}_t)\\
&= S_t^2 e^{r(T-t)}\mathbb{E}_{\mathbb{P}*}(e^{2\sigma(\omega_T-\omega_t)-\sigma^2(T-t)}|\mathcal{F}_t)\\
&= S_t^2 e^{r(T-t)}\mathbb{E}_{\mathbb{P}*}(e^{2\sigma(\omega_T-\omega_t)-\sigma^2(T-t)})\\
&\quad \text{since } (\omega_T-\omega_t)\sim\mathcal{N}(0,T-t), \ \text{ indipendent of } \mathcal{F}_t\\
&= S_t^2 e^{r(T-t)}\int_{-\infty}^{\infty} e^{-\sigma^2(T-t)+2\sigma\sqrt{T-t}y-\frac{1}{2}y^2}dy\frac{1}{\sqrt{2\pi}}\\
&= S_t^2 e^{r(T-t)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(4\sigma^2(T-t)+\sigma\sqrt{T-t}y+y^2)}dy\frac{1}{\sqrt{2\pi}}\\
&= S_t^2 e^{r(T-t)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-2\sigma\sqrt{T-t})^2}dy\frac{1}{\sqrt{2\pi}}\\
&= S_t^2 e^{r(T-t)}(2\sigma\sqrt{T-t})
\end{aligned}$$

Now, we can obtain

$$\phi_t^1 = \frac{\partial C(t,S_t)}{\partial S_t} = 2S_t e^{r(T-t)}(2\sigma\sqrt{T-t}) = 4S_t e^{r(T-t)}(\sigma\sqrt{T-t})$$

$$\begin{aligned}
\phi_t^0 e^{rt} &= C(t,S_t) - \phi_t^1 S_t\\
&= 2S_t^2 e^{r(T-t)}(\sigma\sqrt{T-t}) - 4S_t^2 e^{r(T-t)}(\sigma\sqrt{T-t})\\
&= -2S_t^2 e^{r(T-t)}(\sigma\sqrt{T-t})\\
\Longrightarrow \phi_t^0 &= -2S_t^2 e^{r(T-2t)}(\sigma\sqrt{T-t})
\end{aligned}$$

2) $X = S_T/S_{T_0}$, $0 \le T_0 \le T$
3) $X = 1/S_T$

Like in the point 1 of the exercise,

$$X = \frac{1}{S_T} = S_T^{-1} = S_t^{-1}\exp^{(-1)\sigma(\omega_T-\omega_t)+(-1)(r-\frac{1}{2}\sigma^2)(T_t)}$$

So,

$$
\begin{aligned}
C(t,S_t) &= \mathbb{E}_{\mathbb{P}*}(e^{-r(T-t)}X|\mathcal{F}_t)\\
&= \mathbb{E}_{\mathbb{P}*}(e^{-r(T-t)}S_T^{-1}|\mathcal{F}_t)\\
&= e^{-r(T-t)}\mathbb{E}_{\mathbb{P}*}(S_T^{-1}|\mathcal{F}_t)\\
&= S_t^{-1}e^{-2r(T-t)}\mathbb{E}_{\mathbb{P}*}(e^{-\sigma(\omega_T-\omega_t)+\frac{1}{2}\sigma^2(T-t)}|\mathcal{F}_t)\\
&= S_t^{-1}e^{-2r(T-t)}\mathbb{E}_{\mathbb{P}*}(e^{-\sigma(\omega_T-\omega_t)+\frac{1}{2}\sigma^2(T-t)})\\
&\text{ since } (\omega_T-\omega_t)\sim\mathcal{N}(0,T-t), \text{ indipendent of } \mathcal{F}_t\\
&= S_t^{-1}e^{-2r(T-t)}\int_{-\infty}^{\infty}e^{\frac{1}{2}\sigma^2(T-t)-\sigma\sqrt{T-t}y+\frac{1}{2}y^2}dy\frac{1}{\sqrt{2\pi}}\\
&= S_t^{-1}e^{-2r(T-t)}\int_{-\infty}^{\infty}e^{\frac{1}{2}(y-\sigma\sqrt{T-t})^2}dy\frac{1}{\sqrt{2\pi}}\\
&= S_t^{-1}e^{-2r(T-t)}(\sigma\sqrt{T-t})
\end{aligned}
$$

Now, we can obtain

$$
\phi_t^1 = \frac{\partial C(t,S_t)}{\partial S_t} = -S_t^{-2}e^{-2r(T-t)}(\sigma\sqrt{T-t})
$$

$$
\begin{aligned}
\phi_t^0 e^{rt} &= C(t,S_t) - \phi_t^1 S_t\\
&= S_t^{-1}e^{-2r(T-t)}(\sigma\sqrt{T-t}) + S_t^{-1}e^{-2r(T-t)}(\sigma\sqrt{T-t})\\
&= 2S_t^{-1}e^{-2r(T-t)}(\sigma\sqrt{T-t})\\
\Longrightarrow \phi_t^0 &= 2S_t^{-1}e^{-r(2T-t)}(\sigma\sqrt{T-t})
\end{aligned}
$$

Exercise 2

Show, in the BS model, that the price of an Asian option with floating strike (payoff $= \left(\frac{1}{T}\int_0^T S_u du - S_T\right)_+$) is given, at initial time, by

$$
C = e^{-rT}S_0\varphi(0,0)
$$

where φ is a solution of the equation

$$
r\varphi + \frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial x}\left(rx+\frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}\sigma^2x^2 = 0
$$

where the condition $\varphi(T,x) = (1+x)_-$.

Answer:

With reference to the slides of lesson 12, we consider the Asian options with payoff

$$
X = \left(\frac{1}{T}\int_0^T S_u du - S_T\right)_+
$$

We know that, for the given payoff

$$
\begin{aligned}
C(t,S_t) &= e^{-r(T-t)}\mathbb{E}_{\mathbb{P}*}\left(\left(\frac{1}{T}\int_0^T S_u du - S_T\right)_+ |\mathcal{F}_t\right)\\
&= e^{-r(T-t)}S_t\mathbb{E}_{\mathbb{P}*}\left(\left(\frac{1}{T}\int_0^T \frac{S_u}{S_t}du - \frac{S_T}{S_t} + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)_+ |\mathcal{F}_t\right)\\
&= e^{-r(T-t)}S_t\mathbb{E}_{\mathbb{P}*}\left(\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t}du - \frac{S_T}{S_t} - Z_t\right)_+\right)\\
&\text{where} Z_t = -\frac{1}{T}\int_0^t \frac{S_u}{S_t}du = -\frac{1}{TS_t}\int_0^t S_u du\\
&= e^{-r(T-t)}S_t\varphi(t,Z_t)
\end{aligned}
$$

As seen in the theoretical lessons, we obtain that

$$
dZ_t = \left(-\frac{1}{T}r(r-\sigma^2)Z_t\right)dt - \sigma Z_t d\omega_t
$$

We know that

$$
\widetilde{C_t} = e^{-r(T-t)}\widetilde{S}_t\varphi(t,Z_t)
$$

with $t \leq T$ is a martingale.
So, with

$$
d\varphi = \left(\left(\frac{\partial\varphi}{\partial t} + \frac{\varphi}{\partial Z_t}(\sigma^2 - r)Z_t - \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2\right)dt - \frac{\partial\varphi}{\partial Z_t}\sigma Z_t d\omega_t
$$

we can consider

$$
D\widetilde{C_t} = e^{-r(T-t)}\left(\varphi - Z_t\frac{\partial\varphi}{\partial Z_t}\right)d\widetilde{S}_t
$$

$$
r\varphi + \frac{\partial\varphi}{\partial Z_t}\left(rZ_t + \frac{1}{T}\right) + \frac{1}{2}\frac{\partial^2\varphi}{\partial Z_t^2}\sigma^2 Z_t^2 = 0
$$

As a boundary condition for t=T, we yield

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}*}\left(\left(\frac{1}{T}\int_T^T \frac{S_u}{S_T}du - \frac{S_T}{S_T} - x\right)_+\right) &= \mathbb{E}_{\mathbb{P}*}((-1-x)_+)\\
&= \mathbb{E}_{\mathbb{P}*}(-(1+x)_+)\\
&= \mathbb{E}_{\mathbb{P}*}((1+x)_-)
\end{aligned}
$$

$$
\Longrightarrow \varphi(T,x) = (1+x)_-
$$