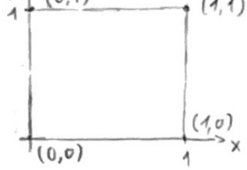
EXERCISES 1.2

1) Let us consider the convex set (polyhedron),

$$C = \{(x,y) \in \mathbb{R}^2 \text{ such that } 0 \le x \le 1, 0 \le y \le 1\}$$

Write this set in the form $\{z \in \mathbb{R}^n \text{ such that } Az = b, z \ge 0\}$, compute the basic feasible solutions and, from them, the vertices.



> Consider $C = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1\}$

The constrains $0 \le x \le 1$ and $0 \le y \le 1$ can be divided in $\begin{cases} x \ge 0 \\ x \le 1 \end{cases}$, $\begin{cases} y \ge 0 \\ y \le 1 \end{cases}$ which can be written using slack variables as $\begin{cases} x + s_1 = 1 \\ x \ge 0, s_1 \ge 0 \end{cases}$, $\begin{cases} y + s_2 = 1 \\ y \ge 0, s_2 \ge 0 \end{cases}$

obtaining the equivalent set

$$C = \{(x, y, s_1, s_2) \in \mathbb{R}^4 : x + s_1 = 1, y + s_2 = 1, x, y, s_1, s_2 \ge 0\}$$

which is of the form $C = \{z \in \mathbb{R}^4 : Az = b, z \ge 0\}$

with $z = \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Therefore we have the problem in standard form with $Az = b$, $z \ge 0$

We can compute the basic feasible solutions by the following:

$$z = \begin{bmatrix} x \\ y \\ s_1 \\ s_1 \end{bmatrix} \begin{matrix} z_B \\ \\ \\ z_N \end{matrix} \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad A_B z_B = b \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow z = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\quad A_B \quad A_N$$

repeating the same proces, which in this case is simple but normally it requires computing A_B^{-1}, we find the following basic feasible solutions:

$$z = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ corresponding to } \begin{bmatrix} x \\ y \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

By a theorem we have that for a polytope $X = \{x \in \mathbb{R}^n : Ax = b, x \ge 0\}$, and in our case C is a polyhedron which is a bounde convex polytope, the set of vertices corresponds to the set of basic feasible solutions.

Therefore the vertices are $\{(1,1), (0,1), (1,0), (0,0)\}$

2) Assume that $a_1, \dots, a_m$ are given vectors in $\mathbb{R}^3$ (all different from 0). Let $b_1, \dots, b_m$ strictly positive numbers and let us define the set

$$M = \{x \in \mathbb{R}^3 \text{ such that } a_i^T x \le b_i \text{ for } i = 1, \dots, m\}$$

(a) Show that the interior of this set is not empty

(b) We want to determine the centre and the radius of the biggest sphere contained in M. Write this problem as a linear program

(a) Proof

$a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad b > 0$ for any given $i = 1, \dots, m$

$a_i^T x = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 \le b_i \quad \forall i = 1, \dots, m$

$$\underbrace{\begin{bmatrix} \text{---} \\ a_{i1}\ a_{i2}\ a_{i3} \\ \text{---} \end{bmatrix}}_{A \in \mathbb{R}^{m \times 3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x \in \mathbb{R}^3} \le \underbrace{\begin{bmatrix} - \\ b_i \\ - \end{bmatrix}}_{b \in \mathbb{R}^m}$$

This describes M, the interior of M will be the same but $Ax < b$

Consider now the interior $\mathring{M} = \{x \in \mathbb{R}^3 : Ax < b\}$, using m variables $s_1, \dots, s_m$ with s_i strictly positive for $i = 1, \dots, m$ we can write $\mathring{M}$ as:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & \dots & 0 \\ a_{i1} & a_{i2} & a_{i3} & 0 \dots & 1 & \dots & 0 \\ \text{---} \end{bmatrix}}_{\tilde{A} \in \mathbb{R}^{m \times m+3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ \vdots \\ s_m \end{bmatrix}}_{\mathbb{R}^{m+3}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbb{R}^m}$$

which is a system of m equations in m+3 variables. Since m linearly indipendent columns ($s_i > 0 \ \forall i = 1, \dots, m$) were added to A to form $\tilde{A}$, the system admits solutions. Therefore $\mathring{M}$ is not empty. ■

(b) Maximizing the volume of a sphere is equivalent to maximizing its radius. For the sphere to be contained in M we need the points of the sphere to satisfy $a_i^T x \le b_i$ for $i = 1, \dots, m$; they all do so if the points on the border of the sphere satisfy $a_i^T x \le b_i$ for $i = 1, \dots, m$.

Therefore we can write the problem as

$$\begin{cases} \text{Max } r \\ \text{s.t. } a_i^T c \le b_i, \quad i = 1, \dots, m \\ \qquad r \le \dfrac{a_i^T c - b_i}{\|a_i\|}, \quad i = 1, \dots, m \end{cases}$$

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Ex_3_problem_a.m | Ex_3_problem_b.m | Ex_3_problem_c.m | Ex_3_problem_d.m

```
% Problem (a)
clear all
clc

x1 = optimvar( 'x1' );
x2 = optimvar( 'x2' );
x3 = optimvar( 'x3' );
prob = optimproblem;
prob.Objective = - 8*x1 - 9*x2 - 5*x3;
prob.Constraints.cons1 = x1 + x2 + 2*x3 <= 2;
prob.Constraints.cons2 = 2*x1 + 3*x2 + 4*x3 <= 3;
prob.Constraints.cons3 = 6*x1 + 6*x2 + 2*x3 <= 8;
prob.Constraints.cons4 = x1 >= 0;
prob.Constraints.cons5 = x2 >= 0;
prob.Constraints.cons6 = x3 >= 0;

sol = solve(prob)
```

Command Window

```
Solving problem using linprog.

Optimal solution found.


sol =

  struct with fields:

    x1: 1.0000
    x2: 0.3333
    x3: 0
```

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Ex_3_problem_a.m | Ex_3_problem_b.m | Ex_3_problem_c.m | Ex_3_problem_d.m

```
% Problem (b)
clear all
clc

x1 = optimvar( 'x1' );
x2 = optimvar( 'x2' );
prob = optimproblem;
prob.Objective = 5*x1 - 3*x2;
prob.Constraints.cons1 = x1 - x2 >= 2;
prob.Constraints.cons2 = 2*x1 + 3*x2 <= 4;
prob.Constraints.cons3 = -x1 + 6*x2 == 10;
prob.Constraints.cons4 = x1 >= 0;
prob.Constraints.cons5 = x2 >= 0;

sol = solve(prob)
```

Command Window

```
Solving problem using linprog.

No feasible solution found.

Linprog stopped because no point satisfies the constraints.


sol =

  struct with fields:

    x1: []
    x2: []
```

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Ex_3_problem_a.m | Ex_3_problem_b.m | Ex_3_problem_c.m | Ex_3_problem_d.m

```
% Problem (c)
clear all
clc

x1 = optimvar( 'x1' );
x2 = optimvar( 'x2' );
x3 = optimvar( 'x3' );
prob = optimproblem;
prob.Objective = - 3*x1 - 2*x2 + 5*x3;
prob.Constraints.cons1 = 4*x1 - 2*x2 + 2*x3 <= 4;
prob.Constraints.cons2 = - 2*x1 + x2 - x3 <= -1;
prob.Constraints.cons3 = x1 >= 0;
prob.Constraints.cons4 = x2 >= 0;
prob.Constraints.cons5 = x3 >= 0;

sol = solve(prob)
```

Command Window

```
Solving problem using linprog.

Problem is unbounded.


sol =

  struct with fields:

    x1: []
    x2: []
    x3: []
```

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Ex_3_problem_a.m | Ex_3_problem_b.m | Ex_3_problem_c.m | Ex_3_problem_d.m

```
% Problem (d)
clear all
clc

x1 = optimvar( 'x1' );
x2 = optimvar( 'x2' );
x3 = optimvar( 'x3' );
x4 = optimvar( 'x4' );
prob = optimproblem;
prob.Objective = - 4*x1 - 6*x2 - 3*x3 - x4;
prob.Constraints.cons1 = 1.5*x1 + 2*x2 + 4*x3 + 3*x4 <= 550;
prob.Constraints.cons2 = 4*x1 + x2 + 2*x3 + x4 <= 700;
prob.Constraints.cons3 = 2*x1 + 3*x2 + x3 + 2*x4 <= 200;
prob.Constraints.cons4 = x1 >= 0;
prob.Constraints.cons5 = x2 >= 0;
prob.Constraints.cons6 = x3 >= 0;
prob.Constraints.cons6 = x4 >= 0;

sol = solve(prob)
```

Command Window

```
Solving problem using linprog.

Optimal solution found.


sol =

  struct with fields:

    x1: 0
    x2: 25.0000
    x3: 125
    x4: 0
```

4) (OPTIONAL) Consider a linear programme (P) in standard form and its dual programme (D)

$$(P) \begin{cases} \text{Min } z = cx \\ Ax = b \\ x \geq 0 \end{cases} \qquad (D) \begin{cases} \text{Max } w = ub \\ uA \leq c \end{cases}$$

Let us denote by A_j the jth column of A. Prove that two solutions $(\bar{x}, \bar{u})$ of, respectively, (P) and (D) are optimal if and only if

$$(\bar{u} \cdot A_j - c_j)\bar{x}_j = 0, \quad \forall j = 1, \ldots, n$$

Proof.

We have shown that if x and u are solutions of respectively (P) and (D) then $z = cx \geq ub = w$

The condition $(\bar{u} \cdot A_j - c_j)\bar{x}_j = 0$, $\forall j = 1, \ldots, n$ can be written as

$(\bar{u}A - c)\bar{x} = 0$ where $\bar{u} = [\bar{u}_1, \ldots, \bar{u}_m]$ $A = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \; a_{mn} \end{bmatrix}$ $c = [c_1, \ldots c_n]$ $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$

$\bar{u}A\bar{x} - c\bar{x} = 0$

$\bar{u}\underbrace{A\bar{x}}_{=b} = c\bar{x}$ // $\bar{x}$ is a solution

$\bar{u}b = c\bar{x}$ => Proving the result is equivalent to proving the theorem of strong duality

Let $\bar{z} = c\bar{x}$, $\bar{w} = \bar{u}b$, suppose $\bar{w} < \bar{z}$, that is, suppose $\nexists u$ such that

$uA \leq c$, $ub \geq \bar{z}$ <=> $\begin{bmatrix} A \\ -b^T \end{bmatrix} x \leq \begin{bmatrix} c^T \\ -\bar{z} \end{bmatrix}$

Then for Farka's Lemma there exists a vector $\begin{bmatrix} x \\ \lambda \end{bmatrix}$, $\lambda \in \mathbb{R}$ satisfying

$\begin{bmatrix} A \\ -b^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = 0$, $\begin{bmatrix} c^T \\ -\bar{z} \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} < 0$, $\begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0$

Suppose $\lambda = 0$, then $Ax = 0$, $b^T x < 0$ and $x \geq 0$ meaning the conditions of Farkas' Lemma don't hold. Thus $\lambda > 0$

The vector $(\frac{x}{\lambda})$ is feasible, since $(\frac{x}{\lambda}) \geq 0$ and $Ax - \lambda b = 0$ => $A(\frac{x}{\lambda}) = b$

However, $cx - \lambda\bar{z} < 0$, so $c(\frac{x}{\lambda}) < \bar{z}$ which contradicts the assumption that $\bar{z}$ is the optimal value.

Therefore, if (P) and (D) admit a solution, their optimal values are equal ■