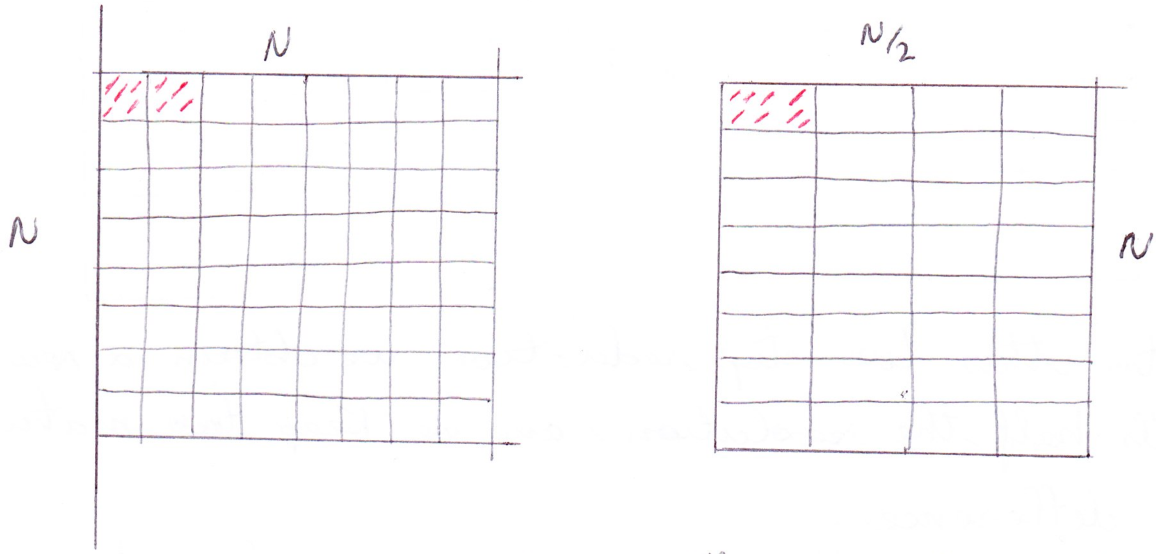
Ingrid Daubechies. Image analysis

Assume we have an image of $N \times N = N^2$ pixels. For simplicity we can assume that N is a power of 2. Each pixel has an associated grey-scale number, going from 0 = black to 255 = white.

Couple horizontally (2 by 2) the pixels and to each new 2×1 rectangle assign the average of the 2 original pixels



With this we get a $\frac{N}{2} \times N = \frac{N^2}{2}$ new matrix. In order not to loose information we also keep a $\frac{N}{2} \times N$ matrix with the differences. Since in the new 2×1 box we take the average of the 1×1 boxes, the differences in the 2 original 1×1 have the same value, but with opposite signs. Schematically

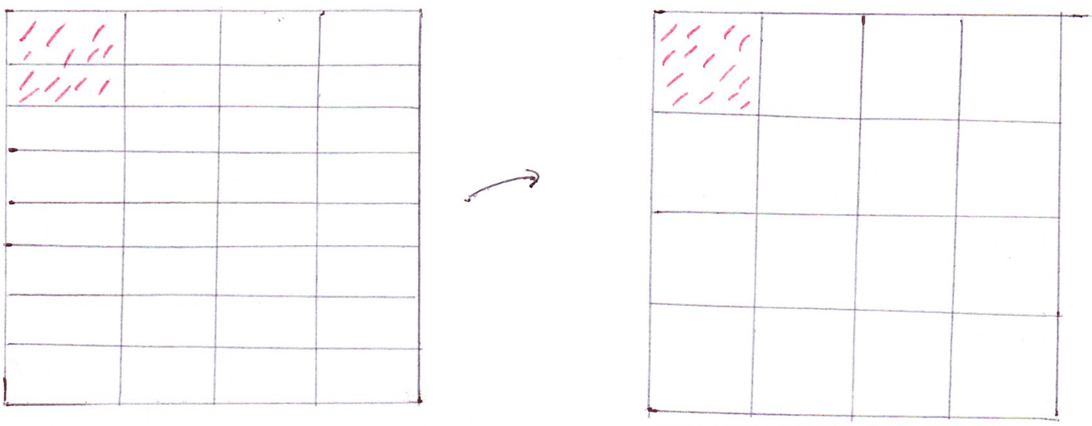
$$\boxed{a_{11} \mid a_{12}} \longrightarrow \boxed{\;h_{11}\;} \qquad \boxed{\;d_{11}\;}$$

$$a_{11} = h_{11} - d_{11} \qquad h_{11} = \frac{a_{11} + a_{12}}{2}$$

$$a_{12} = h_{11} + d_{11}$$

When d_{11} is small there is little difference between what we "see" by looking at $\boxed{h_{11}}$ and what we saw in $\boxed{a_{11} | a_{12}}$ in the original image. When d_{11} is big we loose detail.

With the new $\frac{N^2}{2}$ matrix we perform the analogous procedure, this time vertically. We get 2 new $\frac{N}{2} \times \frac{N}{2}$ matrices: one for the averages, the other for the differences



After this two step reduction we obtain a new "image" with half the resolution, and we keep two matrices with the differences.

Let $M = (a_{j,k})_{j,k=1}^{N}$ denote the original matrix. Let $M_H^1 = (h_{jk})_{\substack{j=1 \dots N/2 \\ k=1 \dots N}}$ the matrix of horizontal averages and let $D_H^1 = (d_{j,k})_{\substack{j \le N/2 \\ k \le N}}$ the matrix of horizontal differences.

We can write
$$M = M_H^1 + \Psi(D_H^1)$$
where Ψ is a function assigning the $-$ sign to the left half of the 2×1 rectangle, and the $+$ sign to the right half. Thus, we would have:

$$a_{11} = h_{11} - d_{11}$$
$$a_{12} = h_{11} + d_{11}$$
$$a_{13} = h_{12} - d_{12}$$
$$a_{14} = h_{12} + d_{12}$$
$$\vdots$$

In terms of piecewise constant functions:

$$f = f_1 + \Psi(d_1)$$

Iterating $f_1 = f_2 + \Psi(d_2)$, and therefore

$$f = f_2 + \Psi(d_1) + \Psi(d_2)$$

Notice that Ψ is the same function, just translated and dilated at different scales, i.e

$$\Psi(x) = \begin{cases} -1 & x \in [0, 1/2) \\ 1 & x \in [1/2, 1) \end{cases} \qquad \text{(Haar wavelet)}$$

Applying successively this procedure que would obtain

$$f = f_{2^k} + \sum_{j=1}^{2^k} \Psi(d_j),$$

which can be thought of as a rough image f_{2^k} plus details at different resolutions. The advantage of this representation is that it allows to discard easily the irrelevant details (which correspond to small differences d). For example, if we have a picture with a big chunk of blue sky (or sea, or a white wall) what we see at low resolution is as good (to the human eye) as the image with full details. Then we can through away all these details and keep a rough image, that needs less

memory but we see equally well.
On the other hand, in the parts of the picture where the differences are big we keep the detail.
With this we adapt the resolution of the picture to the detail of the image.