Exercise 1 *We consider a market model as in the previous lessons. A* numéraire *is an adapted sequence* $Z = (Z_n)_{0 \le n \le N}$ *s.t.* $Z_0 = 1, Z_n > 0$ *for* $n = 1, ..., N$ *and* $Z_n = V_n(\varphi)$ *for some admissible strategy* φ *(*$n = 1, ..., N$*). Denote by* S^Z *the* Z*-discounted vector price process:* $S_n^Z = \frac{S_n}{Z_n}$, $n = 0, ..., N$.

1. *Prove that a predictable sequence* $\phi = (\phi_n)_{1 \le n \le N}$*, with values in* $\mathbb{R}^{d+1}$*, is self-financing iff*

$$V_n^Z(\phi) := \frac{V_n(\phi)}{Z_n} = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^Z,\ n = 1, ..., N.$$

2. *Prove that*

$$\sum_{j=1}^{n} \varphi_j \cdot \Delta S_j^Z = 0,\ n = 1, ..., N.$$

3. *Prove that for any predictable sequence* $\phi = (\phi_n)_{1 \le n \le N}$*, there exists a self-financing strategy* $\hat{\phi}$ *such that*

$$\hat{\phi}_n \cdot S_n^Z = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^Z,\ n = 1, ..., N.$$

4. *Assume that the market is viable (free of arbitrage) and let* $\mathbb{P}^*$ *be the risk-neutral probability. Define* $\mathbb{P}^Z$ *by*

$$\frac{\mathrm{d}\mathbb{P}^Z}{\mathrm{d}\mathbb{P}^*} = \frac{Z_N}{S_N^0}$$

that is

$$\mathbb{P}^Z(A) := \mathbb{E}_{\mathbb{P}^*}\left(\frac{Z_N}{S_N^0}\mathbf{1}_A\right) \text{ for all } A \in \mathcal{F}.$$

Prove that $\mathbb{P}^Z$ *is a probability equivalent to* $\mathbb{P}^*$ *and that for all* $n = 0, ..., N$

$$\mathbb{E}_{\mathbb{P}^Z}(X|\mathcal{F}_n) = \frac{\mathbb{E}_{\mathbb{P}^*}\left(X\frac{Z_N}{S_N^0}|\mathcal{F}_n\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\frac{Z_N}{S_N^0}|\mathcal{F}_n\right)}\ .$$

5. *Prove that the market is viable (free of arbitrage) iff there exists a probability* $\mathbb{P}^Z \sim \mathbb{P}$ *s.t.* S^Z *is a* $\mathbb{P}^Z$*-martingale and that in that case there is at most one deterministic numéraire.*

6. *Assume a market is viable and complete, prove that the price of a payoff* X *at time* n *is given by*

$$Z_n \mathbb{E}_{\mathbb{P}^Z}\left(\frac{X}{Z_N}\middle|\mathcal{F}_n\right).$$