Exercise 1 *Let B be a Brownian motion, prove, without using the Itô formula, that the following stochastic processes are martingales respect to $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$,*

$$
\begin{aligned}
X_t &= t^2 B_t - 2\int_0^t s B_s ds \\
X_t &= e^{t/2}\cos B_t \\
X_t &= e^{t/2}\sin B_t \\
X_t &= (B_t + t)\exp(-B_t - \frac{1}{2}t) \\
X_t &= B_t^1 B_t^2.
\end{aligned}
$$

In the last case B_t^1 y B_t^2 are two independent Brownian motions and $\mathcal{F}_t = \sigma(B_s^1, B_s^2, 0 \leq s \leq t)$.

Exercise 2 *Show that in the Bachelier model with $r = 0$ and $S_t = S_0 + +\sigma W_t, t \in [0, T]$ the price of a call option is given by*

$$
\begin{aligned}
C_t &= f(t, S_t) = \mathbb{E}((S_T - K)_+ | S_t) = (S_t - K)\Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \\
&\quad + \sigma\sqrt{T-t}\,\phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

where Φ and ϕ are, respectively, the cumulative distribution function and the density of a standard normal distribution.