DISTRIBUTIONS AND FOURIER TRANSFORM

Distributions are objects that generalize the notion of function. They allow also the generalization of notions such as the derivative and the Fourier transform.

<u>Definition</u> A <u>distribution</u> is a linear continuous map $T: \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$

In order to understand what the continuity means we need to recall that a sequence $(\varphi_n)_n \subseteq \mathcal{C}_c^\infty(\mathbb{R})$ converges to $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ if and only if there exist $K \subset\subset \mathbb{R}$ compact and $m \geq 1$ such that:

- $\operatorname{supp} \varphi \subseteq K$; $\operatorname{supp} \varphi_n \subseteq K \quad \forall n \geq 1$
- $(\varphi_n^{(j)})_n$ converges uniformly to $\varphi^{(j)}$ in K for all

$j \leq m$, that is $\lim\limits_{n\to\infty} \sup\limits_{x\in K} |\varphi_n^{(j)}(x) - \varphi^{(j)}(x)| = 0$.

Notice that this second condition is equivalent to

$$\lim_{n\to\infty} \sup_{j\leq m} \sup_{x\in K} |\varphi^{(j)}(x)| = 0$$

Sometimes this is expressed in terms of the

seminorms

$$\|\varphi\|_{K,m} := \sup_{j\leq m} \sup_{x\in K} |\varphi^{(j)}(x)| .$$

In these terms, $T: \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ is continuous if and only if for all $K \subset\subset \mathbb{R}$ compact there exist $m = m(K) \geq 1$, $c = c(K) > 0$ such that

$$|T(\varphi)| \leq c \, \|\varphi\|_{K,m} \qquad \forall \varphi \in \mathcal{C}_c^\infty(K)$$

<u>Notation</u>: Functions φ in $\mathcal{C}_c^\infty(\mathbb{R})$ are usually called <u>test functions</u>. Sometimes $\mathcal{D}(\mathbb{R})$ is used instead of $\mathcal{C}_c^\infty(\mathbb{R})$; that's why the set of distributions is usually denoted by $\mathcal{D}'(\mathbb{R})$.

<u>Examples</u>: ① Let $f: \mathbb{R} \to \mathbb{C}$ be a measurable <u>function</u> with some mild regularity (usually $f \in L^1_{loc}$). This defines the distribution $T_f: \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ given by $T_f(\varphi) = \langle \varphi, f \rangle = \int_{\mathbb{R}} f \cdot \varphi$.

In this sense, any (reasonable) function is a distribution.

② Let μ be a <u>Borel measure</u>. Define

$$T_\mu : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C} \text{ by } T_\mu(\varphi) = \int_{\mathbb{R}} \varphi \, d\mu \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R})$$

In particular, if $a \in \mathbb{R}$ and $\mu = \delta_a$ is its associated Dirac delta measure, we have the distribution

$$\langle \delta_a, \varphi \rangle = \int_{\mathbb{R}} \varphi \, d\delta_a = \varphi(a) \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R})$$

More generally, given a sequence $(a_n)_{n\geq 1} \subseteq \mathbb{R}$ such that $\lim_{n\to\infty} |a_n| = +\infty$ and given values $\lambda_n \in \mathbb{C}$, $n \geq 1$, we can consider the <u>Dirac comb</u>

$T = \sum_n \lambda_n \delta_{a_n}$, defined by

$$\langle T, \varphi \rangle = \sum_n \lambda_n \varphi(a_n) \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R})$$

A particular case which appears in applications (see for example Shannon's theorem) is $a_n = na$, $n \in \mathbb{Z}$, for $a \in \mathbb{R}$ fixed. If $\lambda_n = 1$ for all $n \in \mathbb{Z}$, we get the distribution

$$S_a = \sum_{n\in\mathbb{Z}} \delta_{na} ,$$

the Dirac comb with equi-spaced atoms.

Several operations can be performed on distributions. For the moment we introduce two easy ones:

① Product of $\psi \in \mathcal{C}^\infty(\mathbb{R})$ with $T \in \mathcal{D}'(\mathbb{R})$. This is simply defined by

$$\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R})$$

This is well defined because $\psi\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ and because ψT is continuous. To see the continuity let $K \subset\subset \mathbb{R}$ be compact. Since $T \in \mathcal{D}'(\mathbb{R})$ there exist $m \geq 1$ and $c > 0$ so that

$$|\langle \psi T, \varphi \rangle| = |\langle T, \psi\varphi \rangle| \leq c \, \|\psi\varphi\|_{K,m} \qquad \forall \varphi \in \mathcal{C}_c^\infty(K)$$

Since K is compact and $\psi \in \mathcal{C}^\infty$, all derivatives $\psi^{(j)}$, $j \leq m$, are uniformly bounded on K. So, there exists a constant $M = M(\psi, K)$ such that $\|\psi\varphi\|_{K,m} \leq M(\psi) \, \|\varphi\|_{K,m}$. Then

$$|\langle \psi T, \varphi \rangle| \leq c M \, \|\varphi\|_{K,m} \qquad \forall \varphi \in \mathcal{C}_c^\infty(K),$$

as desired

As an example, let us take $\psi \in \mathcal{C}^\infty$ and $T = \delta_a$. Then

$$\langle \psi \delta_a, \varphi \rangle = \langle \delta_a, \psi\varphi \rangle = \psi(a)\varphi(a)$$

So $\varphi \delta_a = \varphi(a)\delta_a$. In particular

$$(x-a)\delta_a = 0 \qquad \forall a \in \mathbb{R}.$$

② <u>Differentiation:</u> Given $T \in \mathcal{D}'(\mathbb{R})$ one defines its derivative $T' \in \mathcal{D}'(\mathbb{R})$ by

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

This generalizes the formula of integration by parts for a function $f \in \mathcal{C}^1$:

$$\langle T_{f'}, \varphi \rangle = \int_{\mathbb{R}} f' \varphi = -\int_{\mathbb{R}} f \varphi' = -\langle T_f, \varphi \rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R})$$

This formula can be iterated to define $T^{(n)}, n \geq 1$:

$$\langle T^{(n)}, \varphi \rangle = (-1)^n \langle T, \varphi^{(n)} \rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

<u>Example</u> The derivative of the Dirac delta δ_a is the distribution δ_a' given by

$$\langle \delta_a', \varphi \rangle = -\varphi'(a) \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R})$$

This is an example of a distribution that is neither a function not a measure.

<u>Exercises</u>: ① Prove that if $T' \equiv 0$ then there exists a constant k such that $T = T_k$, that is

$$\langle T, \varphi \rangle = k \int \varphi \qquad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R})$$

② Let $H(x) = \mathbb{1}_{(0,+\infty)}(x)$, the Heaviside function. Prove that $H' = \delta_0$.

③ More generally, let $f \in \mathcal{C}^1(\mathbb{R} \setminus \{a\})$ such that at $a \in \mathbb{R}$ it has a jump of size s $(f(a^+) - f(a^-) = s)$. Show that $T_f' = T_{f'} + s\delta_a$.

Similarly, if $f \in \mathcal{C}^1(\mathbb{R} \setminus \{a_n\}_n)$, with $\lim_{n\to\infty} |a_n| = +\infty$, and a_n are jump discontinuities of size s_n, then

$$T_f' = T_{f'} + \sum_n s_n \delta_{a_n}.$$

In particular, the a-periodic function f with value x/a in $(0,a)$ has derivative $f' = \frac{1}{a} - \sum_{n\in\mathbb{Z}} \delta_{na}$.