Advanced Mathematics for Scientific Challenges
Autumn 2022

EXERCISES 1.3

1. The goal of this exercise is to see the limitations of Newton's method, with an example in which Newton's method is divergent while a descent method is convergent to the minimum.

   Let us consider the function
   $$g(x) = -e^{-x^2},$$
   that has a unique minimum at $x = 0$. Note that $g'(x) < 0$ if $x < 0$ and $g'(x) > 0$ if $x > 0$, which implies that any reasonable descent method should be able to find the minimum, no matter the starting point. Instead, let us use a Newton's method on the function g' (i.e., to solve $g'(x) = 0$).

   (a) Let $\{x_n\}_n$ be the sequence of points produced by the Newton's method starting at the seed $x_0 = 1$. Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

   (b) Find a value $\alpha > 0$ such that, if $x_0 \in [0, \alpha)$ the Newton's method converges to 0, and if $x_0 > \alpha$ the Newton's method diverges.

2. Discuss if the following functions are unimodal:

   (a) $g(x) = x^3 - x$ on $x \in [-2, 0]$, and on $x \in [0, 2]$

   (b) $g(x) = \exp(-x)$ on $x \in [0, 1]$.

   (c) $g(x) = |x| + |x - 1|$ on $x \in [-2, 2]$.

3. **(Optional)** Look for the Golden section search method. Explain it.

Deadline: November 1, 23:59.