Quick review of Hilbert spaces

A Hilbert space is a vector space H on $\mathbb{C}$ with a Hermitian product $\langle x, y\rangle$, $x, y \in H$ such that

(i) $\langle x, y\rangle$ is linear in both variables (bilinear),

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$,

(iii) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

With the norm $\|x\| = \langle x, x\rangle^{1/2}$ the space H has the structure of Banach space.

Basic estimates. *(a) Cauchy-Schwartz inequality:*

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \quad x, y \in H.$$

(b) Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|, \quad x, y \in H.$$

Examples. 1. Let

$$\ell^2(\mathbb{N}) = \big\{(a_n)_{n=1}^{\infty} \subset \mathbb{C} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\big\}$$

with the Hermitian product

$$\langle a, b\rangle = \sum_{n=1}^{\infty} a_n \bar{b}_n \,, \quad a = (a_n)_n,\ b = (b_n)_n.$$

Here the norm takes the form $\|a\| = \big(\sum\limits_{n=1}^{\infty} |a_n|^2\big)^{1/2}$.

2. More generally, let (X, μ) be a measure space and let $A \subset X$ be a measurable set. Consider

$$H = L^2(A, \mu) = \big\{f : X \longrightarrow \mathbb{C} : \int_A |f|^2 d\mu < +\infty\big\},$$

with the Hermitian product

$$\langle f, g\rangle = \int_A f\,\overline{g}\,d\mu.$$

Two particular cases will appear often during this course:

The first one is a model for T-periodic signals with finite energy; given $T > 0$ let

$$L^2[0,T] = \big\{ f : [0,T] \longrightarrow \mathbb{C} : \int_0^T |f(t)|^2 dt < +\infty \big\}$$

with the inner product

$$\langle f, g\rangle = \frac{1}{T}\int_0^T f(t)\,\overline{g(t)}\,dt.$$

The factor $1/T$ is just a normalisation which makes the formulas cleaner. Then, the norm of $f \in L^2[0,T]$ is given by

$$\|f\|_2 = \Big(\frac{1}{T}\int_0^T |f(t)|^2\,dt\Big)^{1/2}.$$

The second one corresponds to L^2 functions in the whole real line, that is

$$L^2(\mathbb{R}) = \big\{ f : \mathbb{R} \longrightarrow \mathbb{C} : \|f\|_2^2 := \int_{\mathbb{R}} |f(t)|^2 dt < +\infty \big\},$$

equipped with the Hermitian product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\,dt, \quad f, g \in L^2(\mathbb{R}).$$

Closed subspaces and projections

Given a closed subspace $M \subset H$ consider its orthogonal space:

$$M^\perp = \{x \in H : \langle x, y\rangle = 0\, \forall y \in M\}.$$

Projection theorem. *Let $M \subset H$ be closed. Then $H = M \oplus M^\perp$, that is, for any $x \in H$ there exist unique $y \in M$ and $z \in M^\perp$ such that $x = y + z$. Moreover*

$$d(x, M) = \|x - y\| = \|z\| = \sup_{w \in M^\perp, \|w\|=1} \|\langle x, w\rangle|.$$

In this situation the projection operators $P_M : H \longrightarrow M$ and $Q_M : H \longrightarrow M^\perp$, defined respectively by $P(x) = y$ and $Q(x) = z$, are linear, continuous and satisfy the Pythagorean identity

$$\|x\|^2 = \|P_M(x)\|^2 + \|Q_M(x)\|^2, \quad x \in H.$$

Hilbert bases

A Hilbert basis is a complete orthonormal system $\{e_i\}_{i\in I}$ of elements of H such that the closure of its span $\overline{\langle e_i\rangle_{i\in I}}$ is the whole H. That the span $V = \langle e_i\rangle_{i\in I}$ is dense in H is equivalent to $V^\perp = \{0\}$, that is, that if $\langle x, e_i\rangle = 0$ for all $i \in I$, then $x = 0$ necessarily.

We shall always assume that H is *separable*, that is, that there exists a countable orthonormal basis $\{e_n\}_{n=1}^\infty$.

Theorem. *Let $\{e_n\}_{n=1}^\infty$ be a countable orthonormal system and let $V = \langle e_n\rangle_{n=1}^\infty$ be its span. Given $x \in H$,*

(a) $P_{\overline{V}}(x) = \sum\limits_{n=1}^\infty \langle x, e_n\rangle\, e_n\,,$

(b) $\sum\limits_{n=1}^\infty |\langle x, e_n\rangle|^2 \le \|x\|^2$ *(Bessel's inequality).*

Corollary. *Let $\{e_n\}_{n=1}^\infty$ be a countable orthonormal system. The following are equivalent:*

(a) $\{e_n\}_{n=1}^\infty$ *is a Hilbert basis.*

(b) If $\langle x, e_n\rangle = 0$ *for all* $n \ge 1$*, then* $x = 0$*.*

(c) $\|x\|^2 = \sum\limits_{n=1}^\infty |\langle x, e_n\rangle|^2$ *(Parseval's identity).*

Observe that when $\{e_n\}_{n=1}^\infty$ is an orthonormal basis the map

$$\begin{aligned} S : \ell^2(\mathbb{N}) &\longrightarrow H \\ (a_n)_{n=1}^\infty &\mapsto \sum_{n=1}^\infty a_n\, e_n \end{aligned}$$

is an isometric isomorphism with inverse

$$\begin{aligned} A : H &\longrightarrow \ell^2(\mathbb{N}) \\ x &\mapsto \ (\langle x, e_n\rangle)_{n=1}^\infty . \end{aligned}$$