FOURIER TRANSFORM AND DISTRIBUTIONS

For $f \in L^1(\mathbb{R})$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ the multiplication formula gives

$$\int_{\mathbb{R}} \hat{f}(x)\,\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\,\hat{\varphi}(x)\,dx.$$

We would be tempted to define therefore the Fourier transform of $T \in \mathcal{D}'(\mathbb{R})$ by

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

This is not possible, because $\hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$ when $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$.

The way to circumvent this problem is to consider only distribution T for which the right hand side above behaves well (i.e. distributions which are well defined for a bigger space than $\mathcal{C}_c^\infty(\mathbb{R})$). Thus, $\hat{T}$ is not defined for all distributions but only for some "good" ones.

Definition: The Schwartz class consists of the functions $\varphi \in \mathcal{C}^\infty$ such that for all $m, k \in \mathbb{N}$

$$P_{m,k}(\varphi) = \sup_{\substack{x \in \mathbb{R} \\ i \leq k}} (1+|x|)^m \,|\varphi^{(i)}(x)| < +\infty .$$

We denote this class by $\mathcal{S}$.

Notice that $\mathcal{C}_c^\infty(\mathbb{R}) \subseteq \mathcal{S}$ and that there are functions in $\mathcal{S}$ which are not in $\mathcal{C}_c^\infty(\mathbb{R})$ (for example $f(x)=e^{-x^2}$). Functions in $\mathcal{S}$ have fast decay, but are not necessarily compactly supported.

<u>Note</u>: The topology of $\mathcal{S}$ is given in terms of the seminorms $P_{m,k}$ above. Thus $(\varphi_n)_n \longrightarrow 0$ in $\mathcal{S}$ if and only if $(P_{m,k}(\varphi_n))_n \longrightarrow 0$ $\forall m,k \in \mathbb{N}$.

Let us see first that the Fourier transform behaves well in $\mathcal{S}$.

<u>Lemma</u>: If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$. In particular, by Plancherel, $\|\hat{f}\|_2 = \|f\|_2$.

<u>Proof</u>: Using that

$$(\hat{f})^{(i)}(\xi) = [(-2\pi i x)^i f]^\wedge(\xi) \quad ; \quad \widehat{f^{(m)}}(\xi) = (2\pi i \xi)^m \hat{f}(\xi)$$

we have

$$\begin{aligned}
(1+|\xi|)^m |(\hat{f})^{(i)}(\xi)| &= (1+|\xi|)^m |[(-2\pi i x)^i f]^\wedge(\xi)| \\
&\lesssim \left| \left[\frac{\partial^m}{\partial x^m} ((-2\pi i x)^i f) \right]^\wedge (\xi) \right| \\
&\lesssim \left\| \frac{\partial^m}{\partial x^m} [(-2\pi i x)^i f(x)] \right\|_{L^1(\mathbb{R})}
\end{aligned}$$

This L^1 norm is finite, since by hypothesis $p_{M,K}(\varphi) < +\infty$ for all M, K. $\boxtimes$

Definition: A tempered distribution is a linear continuous map $T : \mathcal{S} \to \mathbb{C}$.
The space of tempered distributions is denoted by $\mathcal{S}'$.

Remarks: ① A tempered distribution T restricted to $\mathcal{C}_c^\infty(\mathbb{R}) \subseteq \mathcal{S}$ is a distribution.

② If $\psi \in \mathcal{S}$ and T is a tempered distribution, then $\psi T \in \mathcal{S}'$ as well:

$< \psi T, \varphi > = < T, \psi\varphi >$ and $\psi\varphi \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$.

③ If $T \in \mathcal{S}'$ then also $T' \in \mathcal{S}'$:

$< T', \varphi > = - < T, \varphi' >$ and $\varphi' \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$.

Examples: ⓐ $L^1(\mathbb{R}), L^2(\mathbb{R}) \subseteq \mathcal{S}'$. Let $f \in L^p(\mathbb{R}), p = 1, 2$.
By Hölder's inequality

$$|< T_f, \varphi >| = \left| \int_{\mathbb{R}} f(x) \varphi(x)\, dx \right| \leq \|f\|_p \, \|\varphi\|_q \qquad \frac{1}{p} + \frac{1}{q} = 1$$

and this is finite because $\|\varphi\|_q < +\infty$ for all $q > 0$ and $\varphi \in \mathcal{S}$.

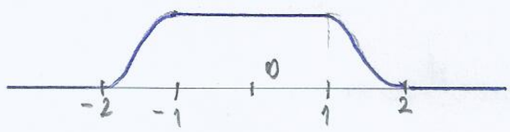
ⓑ Let $f(x)=e^x$. Here $T_f \in \mathcal{D}'(\mathbb{R}) \setminus \mathcal{S}'$

T_f is a distribution, because $f \in L^1_{loc}(\mathbb{R})$. But it is not a tempered distribution. To prove this let us see that there exists $\varphi \in \mathcal{S}$ such that for some m, k there is no $C>0$ with

$$|\langle T_f, \varphi\rangle| = \left| \int_{\mathbb{R}} e^x \varphi(x)\, dx \right| \leq C\, P_{m,k}(\varphi) \qquad (E)$$

Let $\varphi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ be a smoothed out version of the indicator $\chi_{(-1,1)}$, that is $0 \leq \varphi_0 \leq 1$ with

$$\varphi_0(x) = \begin{cases} 1 & x \in [-1,1] \\ 0 & x \in [-2,2] \end{cases}$$



Consider the functions $\varphi_n(x) = \varphi_0(x-n)$. On the one hand

$$|\langle T, \varphi_n\rangle| = \int_{\mathbb{R}} e^x \varphi_n(x-n)\, dx \sim e^n$$

since

$$\int_{n-1}^{n+1} e^x dx \leq \int_{\mathbb{R}} e^x \varphi_n(x-n)\, dx \leq \int_{n-2}^{n+2} e^x\, dx$$

On the other hand

$$P_{m,k}(\varphi_n) = \sup_{x\in\mathbb{R}} (1+|x|)^m |\varphi_n^{(i)}(x)| = \sup_{n-2\leq x\leq n+2} (1+|x|)^m |\varphi_0^{(i)}(x-n)|$$

$$\leq \|\varphi_0^{(i)}\|_\infty\, C\, n^m$$

Thus, the estimate (E) above would mean $e^n \lesssim n^m$, which clearly not possible for all n.

Exercise: Let T be a Dirac comb. Prove that T is a tempered distribution. More generally, let $(a_n)_n \in \mathbb{R}$ be such that $\lim_{n\to\infty} |a_n| = +\infty$ and let $\alpha_n \in \mathbb{R}$ be such that for some $m \geq 1$ $\left\{ \frac{|\alpha_n|}{|a_n|^m} \right\}_{n\geq 1}$ is bounded. Prove that $T = \sum_{n\in\mathbb{Z}} \alpha_n \delta_{a_n}$ is a tempered distribution.

Definition: Let $T \in \mathcal{S}'$. Its Fourier transform is the tempered distribution $\hat{T}$ defined by

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \qquad \forall \varphi \in \mathcal{S}.$$

Examples: ⓐ $\hat{\delta}_a = e^{-2\pi i t a}$. In particular $\hat{\delta}_0 \equiv 1$.
To see this take $\varphi \in \mathcal{S}$; then

$$\langle \hat{\delta}_a, \varphi \rangle = \hat{\varphi}(a) = \int_{\mathbb{R}} \varphi(t) e^{-2\pi i a t}\, dt = \langle e^{-2\pi i a t}, \varphi \rangle .$$

ⓑ $\widehat{e^{2\pi i a t}} = \delta_a$. This is proved similarly, using the inversion formula.

ⓒ $\widehat{\delta_0'} = 2\pi i t$. Let $\varphi \in \mathcal{S}$; then

$$\langle \widehat{\delta_0'}, \varphi \rangle = \langle \delta_0', \hat{\varphi} \rangle = -\hat{\varphi}'(0) = -[(-2\pi i t)\varphi]^\wedge(0)$$

$$= -\int_{\mathbb{R}} (-2\pi i t)\varphi(t) e^{-2\pi i t \cdot 0}\, dt = \int_{\mathbb{R}} (2\pi i t)\varphi(t)\, dt$$

$$= \langle 2\pi i t, \varphi \rangle .$$

ⓓ Let $(a_n)_n \subseteq \mathbb{R}$ be such that $\lim_{n\to\pm\infty} |a_n| = +\infty$ and let $(\alpha_n)_n$ be such that $|\alpha_n| = O(|a_n|^m)$ for some $m \geq 1$ (as in the previous exercise). Then

$$\widehat{\sum_{n\in\mathbb{Z}} \alpha_n \delta_{a_n}} = \sum_{n\in\mathbb{Z}} \alpha_n \hat{\delta}_{a_n} = \sum_{n\in\mathbb{Z}} \alpha_n e^{-2\pi i t a_n} \in \mathcal{S}'.$$

In particular, for a Dirac comb $T = \sum_{n\in\mathbb{Z}} \delta_{na}$ we have $\hat{T} = \sum_{n\in\mathbb{Z}} e^{-2\pi i t a n}$.

Many properties of the Fourier transform for functions are also valid for distributions. These follow directly from the definition.

<u>Properties</u>: Let $T \in \mathcal{S}'$. Then:

ⓐ $\hat{T}^{(k)} = \widehat{(-2\pi i t)^k T}$

ⓑ $\widehat{T^{(k)}} = (2\pi i s)^k \hat{T}$

ⓑ $\widehat{\tau_a T} = e^{-2\pi i a s} \hat{T}$; $\widehat{e^{2\pi i a t} T} = \tau_a \hat{T}$.

Let us finish this section with a classical result.

<u>Poisson summation formula</u>. Let $f \in \mathcal{S}$. Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Sometimes this is given in the apparently more general form:

$$\sum_{n\in\mathbb{Z}} f(n)e^{-2\pi i n\xi} = \sum_{n\in\mathbb{Z}} \hat{f}(\xi+n)$$

This just follows from applying the identity above to $M_{-\xi}f(t) = f(t)e^{-2\pi i\xi t}$.

<u>Proof</u>: Let $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$, which is a 1-periodic function and can therefore be expanded as a Fourier series in $[0,1]$:

$$F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x} \qquad c_n = \int_0^1 F(t)e^{-2\pi i n t}\,dt$$

By definition of F

$$c_n = \sum_{m\in\mathbb{Z}} \int_0^1 f(t+m)e^{-2\pi i n t}\,dt = \qquad t+m=s$$

$$= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(s)e^{-2\pi i n s}\,ds = \int_{\mathbb{R}} f(s)e^{-2\pi i n s}\,ds = \hat{f}(n)$$

Thus, actually, $F(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n x}$

Evaluating at 0 we get the identity:

$$F(0) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \quad . \quad \boxtimes$$

<u>Note</u>: Similarly one can prove a Poisson formula for the nodes $\{na\}_{n\in\mathbb{Z}}$, where $a\in\mathbb{R}$ is given. Then the Fourier side is given in terms of the "dual" net $\{n/a\}_{n\in\mathbb{Z}}$. Explicitly:

$$\sum_{n\in\mathbb{Z}} f(x+an) = \frac{1}{a}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{n}{a}\right) e^{2\pi i \frac{n}{a} x}$$

<u>Exercise</u>: Consider the tempered distributions

$$\langle T_1, \varphi\rangle = \sum_{n\in\mathbb{Z}} \varphi(n) \quad , \quad \langle T_2, \varphi\rangle = \sum_{n\in\mathbb{Z}} \hat{\varphi}(n) \quad ; \varphi\in S.$$

ⓐ Prove that for all $k\in\mathbb{Z}$ the distributions $T_j, j=1,2$ are invariant by the translation τ_k and the modulation M_k, that is

$$\langle M_k T_j, \varphi\rangle = \langle T_j, \varphi\rangle \quad ; \langle \tau_k T_j, \varphi\rangle = \langle T_j, \varphi\rangle \quad \varphi\in S.$$

ⓑ Observe that for the Gaussian $G(t) = e^{-\pi t^2}$

$T_1(G) = T_2(G)$.

(These two properties actually imply that $T_1 = T_2$).