Lesson 9

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The Itô integral

- If $H \in \mathcal{S}$ (linear space of simple predictable process),

$H_t = h_0 \mathbf{1}_{\{0\}} + \sum_{i=1}^{n} h_{i-1} \mathbf{1}_{(t_{i-1},t_i]}(t)$, h_{i-1} bounded and $\mathcal{F}_{t_{i-1}}$-measurable.

$0 = t_0 \leq t_1 \leq ... \leq t_n = T$, we define

$$I(H) := \int_0^T H_s \mathrm{d}W_s := \sum_{i=1}^{n} h_{i-1}(W_{t_i} - W_{t_{i-1}})$$

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-

$$\begin{aligned} \langle I(H), I(L) \rangle_{\mathcal{M}} \quad &: \quad = \mathbb{E}\left(\int_0^T H_s \mathrm{d}W_s \int_0^T L_s \mathrm{d}W_s \right) \\ &= \int_0^T \mathbb{E}(H_s L_s) \mathrm{d}s := \langle H, L \rangle_{\mathcal{S}} \end{aligned}$$

- Then we can extend the map from $\mathcal{S}$ to $\mathcal{H}$:

$$\mathcal{H} = \left\{ H, \text{measurable and } \mathbb{F}\text{-adapted with } \|H\|^2 := \int_0^T \mathbb{E}(H_s^2)\mathrm{d}s < \infty \right\}$$

by

$$\int_0^T H_s \mathrm{d}W_s = \mathbb{L}^2\text{-}\lim_{n\to\infty} \int_0^T H_s^n \mathrm{d}W_s$$

where $H^n \in \mathcal{S}$ and $\|H^n - H\|^2 \to 0$ when $n \to \infty$, since $\mathcal{S}$ is dense in $\mathcal{H}$ with the norm $\|\cdot\|$.

The indefinite Itô integral

The "corresponding" definition is

$$\int_0^t H_s \mathrm{d}W_s = \int_0^T H_s \mathbf{1}_{[0,t]}(s) \mathrm{d}W_s := I(H)_t,$$

for $H \in \mathcal{H}$.

Theorem

$I(H)_t$, $0 \leq t \leq T$ is a martingale.

Proof.

If $H \in \mathcal{S}$, $I(H)_t = \sum_{i=1}^{n} h_{i-1}(W_{t_i \wedge t} - W_{t_{i-1} \wedge t})$ and it is a "martingale transform". If $H \in \mathcal{H}$, $\lim_{n\to\infty} I(H^n)_t \overset{\mathbb{L}^2}{=} I(H)_t$, then, by the Jensen inequality

$$
\begin{aligned}
& \mathbb{E}\left(\left(\mathbb{E}\left(I(H)_t \middle| \mathcal{F}_s\right) - I\left(H^n\right)_s\right)^2\right) \\
= \;& \mathbb{E}\left(\left(\mathbb{E}\left(I(H)_t \middle| \mathcal{F}_s\right) - \mathbb{E}\left(I(H^n)_t \middle| \mathcal{F}_s\right)\right)^2\right) \\
\leq \;& \mathbb{E}\left(\mathbb{E}\left(\left(I(H)_t - I(H^n)_t\right)^2 \middle| \mathcal{F}_s\right)\right) \\
= \;& \mathbb{E}\left(\left(I(H)_t - I(H^n)_t\right)^2\right) \to 0.
\end{aligned}
$$

and by the a.s. uniqueness of the $\mathbb{L}^2$-limit we have that $\mathbb{E}\left(I(H)_t \middle| \mathcal{F}_s\right) = I\left(H\right)_s$, a.s. □

The Doob inequality for continuous (square integrable) martingales establishes that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} M_t^2\right) \leq 4\mathbb{E}\left(M_T^2\right)$$

and this implies that we have uniform $\mathbb{L}^2$ convergence to $(I(H)_t)_{0\leq t\leq T}$, uniform convergence in probability (by Chebyshev) and there exist a subsequence that converges uniformly and almost surely to $I(H)_t$. Now, since, as we have seen above, $I(H^n)_t$ is a continuous process we can obtain a continuous version of $I(H)_t$. (a uniform limit of a sequence of continuous function is continuous).

Lemma

If $A \in \mathcal{F}_t$ for all $H \in \mathcal{H}$

$$\int_0^T \mathbf{1}_A H_s \mathbf{1}_{(t,T]}(s) \mathrm{d}W_s = \mathbf{1}_A \int_t^T H_s \mathrm{d}W_s.$$

Proof.

If $H \in \mathcal{S}$ it is obvious. Now if $H^n \in \mathcal{S} \xrightarrow{\mathbb{L}^2} H \in \mathcal{M}$ we have that

$$\int_0^T \mathbf{1}_A H_s^n \mathbf{1}_{(t,T]}(s) \mathrm{d}W_s = \mathbf{1}_A \int_t^T H_s^n \mathrm{d}W_s \xrightarrow{\mathbb{L}^2} \mathbf{1}_A \int_0^T H_s \mathbf{1}_{(t,T]}(s) \mathrm{d}W_s$$

and by the uniqueness of the $\mathbb{L}^2$-limit we have the result. □

Theorem

Let τ be a stopping time then for all $H \in \mathcal{H}$

$$\int_0^{T\wedge\tau} H_s \mathrm{d}W_s = \int_0^T H_s \mathbf{1}_{[0,\tau]}(s) \mathrm{d}W_s.$$

Proof.

If τ is deterministic is obvious. If $\tau = \sum_{i=1}^n t_i \mathbf{1}_{A_i}$, where $\{A_i, i = 1, ..., n\}$ is a partition of Ω with A_i $\mathcal{F}_{t_i}$-measurable, we have that

$$\begin{aligned}
& \int_0^{T\wedge\tau} H_s \mathrm{d}W_s \\
= & \sum_{i=1}^n \mathbf{1}_{A_i} \int_0^{t_i} H_s \mathrm{d}W_s = \sum_{i=1}^n \mathbf{1}_{A_i} \left(\int_0^T H_s \mathrm{d}W_s - \int_0^T H_s \mathbf{1}_{(t_i,T]}(s) \mathrm{d}W_s \right) \\
= & \int_0^T H_s \mathrm{d}W_s - \int_0^T H_s \sum_{i=1}^n \mathbf{1}_{A_i} \mathbf{1}_{(t_i,T]}(s) \mathrm{d}W_s \text{ (by the Lemma)} \\
= & \int_0^T H_s \mathrm{d}W_s - \int_0^T H_s \mathbf{1}_{(\tau,T]}(s) \mathrm{d}W_s = \int_0^T H_s \mathbf{1}_{[0,\tau]}(s) \mathrm{d}W_s
\end{aligned}$$

□

Proof.

For arbitrary τ we can take $\tau_n := \sum_{k=0}^{2^n-1} \frac{(k+1)T}{2^n} \mathbf{1}_{\left\{\frac{kT}{2^n} < \tau \leq \frac{(k+1)T}{2^n}\right\}}$ then τ_n is a stopping time of the previous form and $\tau_n \downarrow \tau$, a.s., consequently, fixed ω,

$$\int_0^{T\wedge\tau} H_s \mathrm{d}W_s = \lim_{n\to\infty} \int_0^{T\wedge\tau_n} H_s \mathrm{d}W_s = \lim_{n\to\infty} \int_0^T H_s \mathbf{1}_{[0,\tau_n]}(s) \mathrm{d}W_s,$$

also, by the isometry property and by the dominated convergence theorem, we have

$$\begin{aligned} & \mathbb{E}\left(\left(\int_0^T H_s \mathbf{1}_{[0,\tau]}(s) \mathrm{d}W_s - \int_0^T H_s \mathbf{1}_{[0,\tau_n]}(s) \mathrm{d}W_s\right)^2\right) \\ = \ & \mathbb{E}\left(\int_0^T H_s^2 \mathbf{1}_{(\tau,\tau_n]}(s) \mathrm{d}s\right) \underset{n\to\infty}{\to} 0. \end{aligned}$$

So $\int_0^T H_s \mathbf{1}_{[0,\tau]}(s) \mathrm{d}W_s = \mathbb{L}^2\text{-}\lim \int_0^T H_s \mathbf{1}_{[0,\tau_n]}(s) \mathrm{d}W_s$. Now, by the uniqueness of the limit in probability we obtain the result. □

Now we want to extent the Itô integral to the set of processes

$$\tilde{\mathcal{H}} = \left\{ H, \text{measurable and } \mathbb{F}\text{-adapted with } \int_0^T H_s^2 \mathrm{d}s < \infty, \text{ a.s.} \right\}.$$

Consider the sequence of stopping times $\tau_n := \inf \left\{ 0 \leq t \leq T, \int_0^t H_s^2 \mathrm{d}s = n \right\}$, $\tau_n = T$ if this set is empty. Note that $\tau_n(\omega) = T$ on the set $A_n = \left\{ \int_0^T H_s^2 \mathrm{d}s < n \right\}$ and that $A_n \uparrow \Omega \backslash \mathcal{N}$ for all $H \in \tilde{\mathcal{H}}$.

Then we define, for $H \in \tilde{\mathcal{H}}$,

$$I(H)_t(\omega) := \left(\lim_{k\to\infty} \int_0^t H_s \mathbf{1}_{[0,\tau_k]}(s) \mathrm{d}W_s \right)(\omega).$$

Note that $\left\| H_s \mathbf{1}_{[0,\tau_k]}(s) \right\| \leq k$, that is $H.\mathbf{1}_{[0,\tau_k]}(\cdot) \in \mathcal{H}$.
We can see that this limit is well define for all ω w.p.1. Let ω be in A_n, then $\tau_n(\omega) = T$ and we have that

$$\begin{aligned} I(H)_t(\omega) \quad : \quad &= \left(\lim_{k\to\infty} \int_0^{t\wedge\tau_n(\omega)} H_s \mathbf{1}_{[0,\tau_k]}(s) \mathrm{d}W_s \right)(\omega) \\ &= \left(\lim_{k\to\infty} \int_0^t H_s \mathbf{1}_{[0,\tau_n]}(s) \mathbf{1}_{[0,\tau_k]}(s) \mathrm{d}W_s \right)(\omega) \\ &= \left(\lim_{k\to\infty} \int_0^{t\wedge\tau_k(\omega)} H_s \mathbf{1}_{[0,\tau_n]}(s) \mathrm{d}W_s \right)(\omega). \\ &= \left(\int_0^t H_s \mathbf{1}_{[0,\tau_n]}(s) \mathrm{d}W_s \right)(\omega). \end{aligned}$$

and if $m > n$ and ω is in A_n (then $\tau_n(\omega) = T$), since $\tau_m > \tau_n$

$$\begin{aligned}\left(\int_0^t H_s \mathbf{1}_{[0,\tau_m]}(s)\mathrm{d}W_s\right)(\omega) &= \left(\int_0^{t\wedge\tau_n(\omega)} H_s \mathbf{1}_{[0,\tau_m]}(s)\mathrm{d}W_s\right)(\omega) \\ &= \left(\int_0^t H_s \mathbf{1}_{[0,\tau_n]}(s)\mathbf{1}_{[0,\tau_m]}(s)\mathrm{d}W_s\right)(\omega) \\ &= \left(\int_0^t H_s \mathbf{1}_{[0,\tau_n]}(s)\mathrm{d}W_s\right)(\omega).\end{aligned}$$

and the limit is defined in a consistent way.

Finally this integral is an extension of the previous integral: if $H \in \mathcal{H}$ and $\omega \in A_n$

$$
\begin{aligned}
I(H)_t(\omega) &= \left(\int_0^t H_s \mathbf{1}_{[0,\tau_n]}(s) \mathrm{d}W_s\right)(\omega) \\
&= \left(\int_0^{t\wedge\tau_n(\omega)} H_s \mathrm{d}W_s\right)(\omega) = \left(\int_0^{t\wedge T} H_s \mathrm{d}W_s\right)(\omega) \\
&= \left(\int_0^t H_s \mathrm{d}W_s\right)(\omega).
\end{aligned}
$$

The process $I\left(H\right)_t$, $0 \leq t \leq T$ is a local martingale!!: take the sequence $\left(\tau_n\right)_{n\geq 1}$ defined above, then

$$
\begin{aligned}
I\left(H\right)_{t\wedge\tau_n} &= \left(\lim_{k\to\infty}\int_0^{t\wedge\tau_n(\omega)} H_s \mathbf{1}_{[0,\tau_k]}(s)\mathrm{d}W_s\right) \\
&= \left(\lim_{k\to\infty}\int_0^{t} H_s \mathbf{1}_{[0,\tau_n]}(s)\mathbf{1}_{[0,\tau_k]}(s)\mathrm{d}W_s\right) \\
&= \left(\int_0^{t} H_s \mathbf{1}_{[0,\tau_n]}(s)\mathrm{d}W_s\right).
\end{aligned}
$$

and since $H.\mathbf{1}_{[0,\tau_n]}(\cdot) \in \mathcal{H}$ we have that $I\left(H\right)_{\cdot\wedge\tau_n}$ is a martingale.