TWO APPLICATIONS OF FOURIER ANALYSIS

Fourier analysis was born in the study of the heat equation, so one could say, at least from a historical perspective, that differential equations are its more important application. Here we illustrate the power of Fourier analysis with two famous results.

Ⓐ THE HEISENBERG UNCERTAINTY PRINCIPLE

We have already noticed that time and frequency cannot be localised simultaneously. Here we have a precise statement.

<u>Theorem</u> Let $f \in L^2(\mathbb{R})$ and let $a, b \in \mathbb{R}$

Then $$\left(\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 dx\right)\left(\int_{\mathbb{R}} (\xi-b)^2 |\hat{f}(\xi)|^2 d\xi\right) \geq \frac{\|f\|_2^4}{16\pi^2},$$

and the identity holds if $f(x) = ce^{ibx}e^{-\gamma|x-a|^2}$, for $c \in \mathbb{C}$ and $\gamma > 0$ (i.e, if $f(x)$ is a Gaussian).

<u>Note:</u> In quantum mechanics $f(x)$ is the wave function of a particle. The condition $f \in L^2$ expresses that it has finite energy.

The <u>position operator</u>

$$P(f) = x\, f(x)$$

indicates the density of probability of the position of the particle.

The operator $\quad Q(f) = \frac{1}{2\pi i} f' \quad$ is the moment operator (derivative of the state).

In this language, and by Plancherel,

$$\int_{\mathbb{R}} |Pf(x)|^2 dx = \int_{\mathbb{R}} x^2 |f(x)|^2 dx$$

$$\int_{\mathbb{R}} |Qf(x)|^2 dx = \int_{\mathbb{R}} \left|\frac{1}{2\pi i} f'(x)\right|^2 dx = \int_{\mathbb{R}} \left|\frac{1}{2\pi i} \widehat{f'}(\xi)\right|^2 d\xi$$

$$= \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi .$$

So the theorem above shows that there is a limit to localising both position and momentum, and that the best compromise is obtained with the eigenfunctions of the "annihilation operator" $P + iQ$ (the so-called coherent states).

Proof: By the basic identities on translations and modulations, we can assume that $a=b=0$. Assume also that $xf(x), \xi\hat{f}(\xi) \in L^2$, otherwise the inequality is obvious. Notice that this implies that $f, \hat{f} \in L^1$: by the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}} |f(x)|\,dx = \int_{\mathbb{R}} (1+|x|)\,|f(x)|\,\frac{dx}{1+|x|} \leq$$

$$\leq \left(\int_{\mathbb{R}} (1+|x|)^2 |f(x)|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} \frac{dx}{(1+|x|)^2}\right)^{1/2} < +\infty$$

In particular, by the Riemann-Lebesgue lemma applied to $\hat{f}(\xi)$, we deduce that f is continuous and $\lim\limits_{|x|\to\infty} f(x) = 0$.

Also, $f' \in L^2$, because $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi) \in L^2$ and we can apply Plancherel's identity.

Now, since $(|f|^2)' = (f\cdot\bar{f})' = 2\,\mathrm{Re}(f\cdot\bar{f}')$, for $c<d$ we have

$$2\,\mathrm{Re}\left(\int_c^d x\, f(x)\,\overline{f'(x)}\,dx\right) = \int_c^d x\; 2\,\mathrm{Re}\left(f(x)\,\overline{f'(x)}\right) dx =$$

$$= \left[x\,|f(x)|^2\right]_c^d - \int_c^d |f(x)|^2 dx$$

Since $f, xf, f' \in L^2$, there exist sequences $\{c_n\} \searrow -\infty$; $\{d_n\} \nearrow +\infty$, with

$$\lim_{n\to\infty} d_n |f(d_n)|^2 = \lim_{n\to\infty} c_n |f(c_n)|^2 = 0.$$

Thus, using that $f'(x) = (2\pi i \xi \hat{f})^\vee(x)$, we have

$$\int_{\mathbb{R}} |f(x)|^2 dx = -2 \operatorname{Re} \int_{\mathbb{R}} x f(x) \overline{(2\pi i \xi \hat{f})^\vee(x)} \, dx =$$

$$= 4\pi \operatorname{Im} \int_{\mathbb{R}} x f(x) (\xi \hat{f})^\vee(x) \, dx .$$

Squaring and applying successively Cauchy-Schwarz inequality and Plancherel's identity

$$\|f\|_2^4 \leq 16\pi^2 \left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx\right) \left(\int_{\mathbb{R}} |(\xi \hat{f})^\vee(x)|^2 dx\right)$$

$$\leq 16\pi^2 \left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx\right) \left(\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi\right),$$

as desired.

The identity holds only when $xf(x) = \gamma f'(x)$, that is, when $f(x) = \kappa e^{\gamma x^2}$. The condition $\gamma < 0$ is necessary so that $f \in L^2$ $\boxtimes$

(B) THE KOTELNIKOV-SHANNON SAMPLING THEOREM

This is a fundamental result in digital signal processing, establishing a sufficient condition for a sample rate to recover the whole information of a continuous time signal of finite band-width.

Assume that $f(t)$ is a continuous signal (a sound, for example) of finite energy, i.e. $f \in L^2(\mathbb{R})$. Assume that f has finite band-width (range of frequencies): there exists $\tau > 0$ so that $\operatorname{supp} \hat{f} \subseteq [-\tau, \tau]$. This is a natural assumption for at least two reasons: 1) the range of frequencies perceived by the human ear is limited (between 20 Hz and 20.000 Hz); 2) transporting media attenuate extreme frequencies.

<u>Theorem</u>: Let $f \in L^2(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subseteq [-\tau, \tau]$. Then f can be completely recovered from the samples $\{f(\frac{k}{2\tau})\}_{k \in \mathbb{Z}}$ through the so-called "cardinal series"

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2\tau}\right) \cdot \operatorname{sinc}\left[2\tau\left(t - \frac{k}{2\tau}\right)\right].$$

(Here $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ denotes the cardinal sine).

Moreover $\int_{\mathbb{R}} |f(t)|^2 dt = \frac{1}{2c} \sum_{k \in \mathbb{Z}} |f(\frac{k}{2c})|^2$

Remarks: ① This is sometimes referred to as the "Fundamental theorem in information theory". It allows to encode a signal through a sequence of numbers (digitalisation) from which one can completely reconstruct it.

② The sampling rate $\frac{1}{2c}$ is called the Nyquist rate (Harry Nyquist was a communications engineer working first for AT&T and later for Bell Telephone Laboratories).

③ The result was first proved by Vladimir A. Kotelnikov, a pioneer in information theory and radar astronomy, in 1933. He worked at the Moscow Energy Institute. Independently, it was proved also by Claude Shannon, an electrical engineer, and by Edmund Whittaker, a mathematician.

<u>Proof:</u> By Plancherel

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{-\tau}^{\tau} |\hat{f}(\xi)|^2 d\xi = \|f\|_2^2 < +\infty,$$

thus $f \in L^2[-\tau,\tau]$. Since $\left\{\frac{1}{\sqrt{2\tau}} e^{i\pi \frac{k}{\tau}\xi}\right\}_{k\in\mathbb{Z}}$ is an

orthonormal basis of $L^2[-\tau,\tau]$, we can write

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}} \langle \hat{f}, e_k\rangle e_k(\xi) \quad , \text{ where } e_k(\xi) = \frac{1}{\sqrt{2\tau}} e^{i\pi\frac{k}{\tau}\xi}.$$

Then, by the inversion formula:

$$(1) \quad f(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i t\xi} d\xi = \int_{-\tau}^{\tau} \sum_{k\in\mathbb{Z}} \langle \hat{f}, e_k\rangle e_k(\xi) e^{2\pi i t\xi} d\xi$$

$$= \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{2\tau}} \langle \hat{f}, e_k\rangle \int_{-\tau}^{\tau} e^{2\pi i(\frac{k}{2\tau}+t)\xi} d\xi .$$

Here

$$\frac{1}{\sqrt{2\tau}}\langle \hat{f}, e_k\rangle = \frac{1}{2\tau}\int_{-\tau}^{\tau} \hat{f}(\xi) e^{-i\pi\frac{k}{\tau}\xi} d\xi = \frac{1}{2\tau}\int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i(\frac{-k}{2\tau})\xi} d\xi$$

$$= \frac{1}{2\tau} f\left(\frac{-k}{2\tau}\right) \qquad \text{(by the inversion formula)}$$

Also

$$\int_{-\tau}^{\tau} e^{2\pi i(t+\frac{k}{2\tau})\xi} d\xi = \left[\frac{e^{2\pi i(t+\frac{k}{2\tau})\xi}}{2\pi i(t+\frac{k}{2\tau})}\right]_{\xi=-\tau}^{\xi=\tau} =$$

$$= \frac{e^{2\pi i\tau(t+\frac{k}{2\tau})} - e^{-2\pi i\tau(t+\frac{k}{2\tau})}}{2\pi i(t+\frac{k}{2\tau})} = \frac{\sin(2\pi\tau(t+\frac{k}{2\tau}))}{\pi(t+\frac{k}{2\tau})} =$$

$= 2\tau \ \mathrm{sinc}\left[2\tau\left(t+\frac{k}{2\tau}\right)\right].$

Pluggins these equalities into formula (1) above we finally get (changing k by $-k$):

$$f(t) = \sum_{k\in\mathbb{Z}} \frac{1}{2\tau} f\left(-\frac{k}{2\tau}\right) 2\tau \ \mathrm{sinc}\left[2\tau\left(t+\frac{k}{2\tau}\right)\right]$$

$$= \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2\tau}\right) \mathrm{sinc}\left[2\tau\left(t-\frac{k}{2\tau}\right)\right].$$

It remains to see that $\|f\|_2^2 = \frac{1}{2\tau}\sum_{k\in\mathbb{Z}} \left|f\left(\frac{k}{2\tau}\right)\right|^2$.

By the formula we have just proved, it will be enough to show that the system

$$\left\{\sqrt{2\tau}\ \mathrm{sinc}\left[2\tau\left(t-\frac{k}{2\tau}\right)\right]\right\}_{k\in\mathbb{Z}}$$

is orthonormal in $L^2(\mathbb{R})$. To see this just notice that

$$\mathrm{sinc}\left[2\tau\left(t-\frac{k}{2\tau}\right)\right]^\wedge(\xi) = e^{\pi i k \frac{\xi}{\tau}} \left[\mathrm{sinc}(2\tau t)\right]^\wedge(\xi)$$

$$= e^{i\pi k\frac{\xi}{\tau}} \frac{1}{2\tau} \chi_{[-\tau,\tau]}(\xi)$$

Then, by Plancherel,

$$\left\langle \mathrm{sinc}\left[2\tau\left(t-\frac{k}{2\tau}\right)\right], \mathrm{sinc}\left[2\tau\left(t-\frac{m}{2\tau}\right)\right]\right\rangle = \frac{1}{(2\tau)^2}\int_{-\tau}^{\tau} e^{\pi i(m-k)\frac{\xi}{\tau}}\,d\xi = \frac{\delta_{km}}{2\tau}.$$

<u>Remark.</u> This last part of the proof also shows that for any sequence $\{a_k\}_{k\in\mathbb{Z}} \in \ell^2$ the function $f(t) := \sum_{k\in\mathbb{Z}} a_k \operatorname{sinc}[2\tau(t-\frac{k}{2\tau})]$ defines $f\in L^2(\mathbb{R})$ with $\operatorname{supp}\hat{f} \subseteq [-\tau,\tau]$ and $f(\frac{k}{2\tau}) = a_k$; $k\in\mathbb{Z}$.

Digression: Fourier transform and analytic functions

For the sake of simplicity let us momentarily reverse the rôles of f and $\hat{f}$ (which, by Plancherel, are equivalent). Let $f\in L^2(\mathbb{R})$ supported in $[-\tau,\tau]$ and consider

$$F(\zeta) := \hat{f}(\zeta) = \int_{\mathbb{R}} f(t) e^{-2\pi i\zeta t}\, dt = \int_{-\tau}^{\tau} f(t) e^{-2\pi i \zeta t}\, dt .$$

We already saw that this is analytic. Actually this defines an entire function ($F\in H(\mathbb{C})$) of exponential type ($\exists A,B>0 : |F(\zeta)| \le A e^{B|\zeta|}, \zeta\in\mathbb{C}$). Let us see this. A direct estimate and Cauchy-Schwarz's inequality yield:

$$|F(\zeta)| \le \int_{-\tau}^{\tau} |f(t)|\, e^{-2\pi t \,\mathrm{Re}(i\zeta)}\, dt \le e^{2\pi\tau|\mathrm{Im}\,\zeta|} \int_{-\tau}^{\tau} |f(t)|\, dt$$

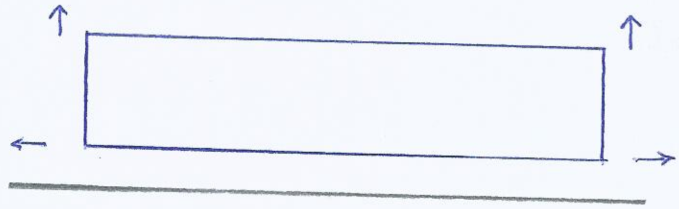
$$\le e^{2\pi\tau|\mathrm{Im}\,\zeta|}\, \|f\|_2 \sqrt{2\tau} .$$

It remains to see that F is holomorphic. This follows easily from Morera's theorem: F is continuous and for any $\mathcal{C}^1$ closed curve γ

$$\int_\gamma F(\zeta)\,d\zeta = \int_{-\tau}^{\tau} f(t)\left(\int_\gamma e^{-2\pi i \zeta t}\,d\zeta\right)dt = 0,$$

since, by Cauchy's theorem, $\int_\gamma e^{-2\pi i \zeta t}\,dt = 0$.

<u>Note:</u> The reciprocal is also true: if $F(\zeta) = \hat{f}(\zeta)$ belongs to $L^2(\mathbb{R})$ and extends to an entire function of exponential type, then $\operatorname{supp} f \subseteq [-\tau, \tau]$. The proof goes along the same lines, defining $f(x) \text{“}=\text{”} \int_{\mathbb{R}} F(\zeta) e^{-2\pi i \zeta x}\,d\zeta$ and applying the residue theorem to a rectangle



Similarly, one can prove that for $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp}\phi \subseteq [-A, A]$ the Fourier transform $\hat{\phi}$ can be extended to an entire function $F(z) = \hat{\phi}(z)$, $z \in \mathbb{C}$ such that

$$\forall m \in \mathbb{N}\ \exists C_m > 0 : \quad |\hat{\phi}(z)| \leq C_m (1+|z|)^{-m} e^{2\pi A |\operatorname{Im} z|}, \quad z \in \mathbb{C}.$$

<u>Note:</u> If we go back to the original situation (reversing again the rôles of f and $\hat{f}$) we see that when $f \in L^2(\mathbb{R})$ is band-limited, it can be extended to an entire function $f(z)$, $z \in \mathbb{C}$. In particular f can only vanish on a discrete set without accumulation points on $\mathbb{C}$. Thus, the signal $f(t)$ has to be not zero for (almost) all t. This seems to contradict intuition. Here we just copy Joseph Slepian's reflections: "it makes no sense to discuss whether real life functions are band-limited or time-limited, since this would mean to measure the signal in remote and future times with arbitrarily high precision".

The Paley-Wiener space,

$$PW_\tau = \{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subseteq [-\tau, \tau] \}$$

is just a mathematical model.