Bol. Soc. Esp. Mat. Apl. nº 0 (0000), 1–33

Computerized Tomography and the Radon Transform

C. Cascante, X. Massaneda,
J. Ortega-Cerdà and D. Pascuas *

Departament de Matemàtica Aplicada i Anàlisi,
Universitat de Barcelona

cascante@ub.edu, xavier.massaneda@ub.edu,
jortega@ub.edu, daniel_pascuas@ub.edu

October 27, 2007

Abstract

We give an overview of some of the mathematical foundations of Computerized Tomography as well as of some of the tools and problems that arise from the mathematical model discussed and its numerical implementation.

Key words : *Computerized tomography, Radon transform, Inversion formulas, Sampling theory.*
AMS subject classifications : *44A12,92C55,65R32,94A20*

1. Introduction

This expository paper is devoted to computerized tomography (CT), a discipline that provides techniques of reconstruction of a function from its integrals over lines. The paper is based on the lecture notes of a seminar that was run at the University of Barcelona in 2005. The main bibliographical sources were the books by Frank Natterer [13] and [14], and the references therein.

The word "tomography" is derived from the Greek words *tomos* (slice) and *graphein* (to write), and it refers to an imaging technique that allows to obtain cross-sectional images of an object from its projections.

A simple physical model for CT is as follows. If I_0 is the initial intensity of an X-ray beam, $f(x)$ denotes the X-ray attenuation coefficient of the object at the point x, L is the ray along which the radiation propagates, and I the X-ray intensity detected after having passed through the body, we have

$$I = I_0 e^{-\int_L f(x)\,dx}.$$

*The authors are supported by projects MTM2005-08984-C02-02 and 2005SGR00611.

In this integral and in all subsequent ones the measure dx corresponds to the k-Lebesgue measure where k is the dimension of underlying manifold L.

Consequently,

$$g(L) := \int_L f(x)\, dx = \log \frac{I_0}{I}.$$

The function g, defined over the set of lines going through the object, is the two dimensional *Radon transform of* f, and the main problem in CT is to reconstruct f from the values $g(L)$. In practice only a finite number of lines are considered, depending on the scanning geometry and the resolution of the scanner. In 1917, the German mathematician Johann Radon obtained an inversion formula for the reconstruction of f from the data g. However, this inversion formula is not suitable for numerical computations.

CT began in 1963 with the work of the South African (and later American) physicist Allan M. Cormack. He obtained a new inversion formula based on a development of the function f in spherical polynomials, see Section 2.5. The first commercially viable CT scanner was invented by the British engineer Godfrey N. Hounsfield at Thorn EMI Central Research Laboratories. Hounsfield conceived his scanner in 1967, but the first prototype (which used X-rays) was not built until 1971. In 1972 its existence was publicly announced, thus starting the commercial era for computerized tomography devices. The original prototype took 160 parallel readings through 180 directions, each 1 degree apart. Each scan would take a little over five minutes and once all of them were acquired, a large computer (using an algebraic reconstruction technique) needed 2.5 hours to process the final images. In 1979 the Nobel prize in Medicine was awarded to Cormack and Hounsfield.

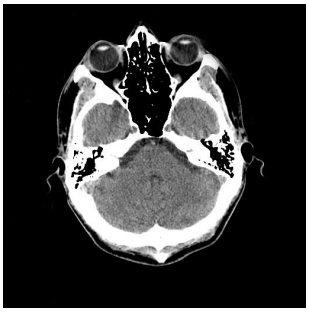

Figure 1: CT scan of a human brain after being hit by a car, by Andrew Ciscel. Original at http://www.flickr.com/photos/ciscel/124548696

In the forthcoming pages we will provide an overview of the mathematical foundations of this model.

Nevertheless there are other similar geometric transformations, with their corresponding inversion formulas, that have also been of great use in other models. Among these we mention the single photon emission computed

tomography (SPECT), the synthetic aperture radar (SAR) and the 3D dimensional models (see [14, §5.5] and the references therein).

SPECT is a medical imaging technique used in nuclear medicine which was developed in the early 1960's by David Edwards and Roy Kuhl. It is used for clinical diagnosis of brain diseases and for tumor imaging in clinical oncology. In order to conduct the scan, a short-lived radioactive isotope (usually fluorine-18) is injected in the blood or inhaled. As the radioisotope decays, it emits a positron which after travelling up to a few millimeters, encounters and annihilates with an electron, thus producing a pair of gamma photons. These photons are detected by a gamma camera collimated to detect only the photons coming in a given direction. The radiation intensity measured by the detector along the line L is given by

$$I = \int_L f(x) e^{-\int_{L(x)} \mu(y)\, dy} dx.$$

where $L(x)$ is the halfline in the direction of L starting at x, f is the distribution of the radiopharmaceutical and μ is the attenuation coefficient of the studied tissue. The value of μ is assumed to be known, and very often it is computed by a simultaneous standard tomography performed by the same machine. The goal is to compute $f(x)$, i.e. the distribution of the radioisotopes. The mathematical model of this process is a weighted Radon transform (for more information see [14, §3.2]). It is only very recently that explicit inversion formulas for this transform have been found, see [16] and [3]. The description of the range of the attenuated transform has been obtained in [15] while the range of the usual Radon transform is classical, see subsection 2.3.

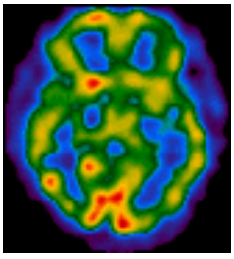

Figure 2: SPECT scan of a cocaine abused brain, Brigham and Women's Hospital, Harvard Medical School. Original at
http://brighamrad.harvard.edu/education/online/BrainSPECT

SAR is a form of radar in which sophisticated post-processing of data is used to produce a very narrow effective beam. In spite of the fact that it can only be used by moving instruments over relatively immobile targets, it has seen wide applications in remote sensing and mapping. In SAR, a plain terrain is surveyed by a single radar antenna attached to the side of an aircraft flying with constant speed along a straight trajectory. The quantity to be imaged

is the ground reflectivity, which is modelled by a function f. The strength of the radar signal reflected by a surface element dS at point x in the plane with distance r from the antenna is $1/r^2\ f(x)dS$. The reflected signal received at time t from those points x in the plane whose distance to the antenna is $tc/2$, where c denotes the speed of light. Thus, letting $y \in \mathbb{R}^2$ denote the ground coordinates of the plane at time t and $h \in \mathbb{R}$ its height, the total reflected signal at time t is

$$\frac{1}{r^2} \int_{|y-x|=\sqrt{r^2-h^2}} f(x)\, dx,\ r = \frac{tc}{2}.$$

This gives rise to an integral transform of f, the so-called average reflectivity function

$$g(y,r) = \int_{S^1} f(y + r\theta) d\theta.$$

For more details see [14, §3.6.1] and the references therein.

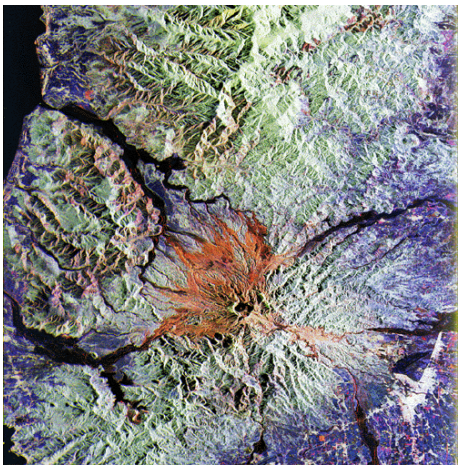

Figure 3: SAR image of Pinatubo volcano (NASA and JPL lab), original at http://veimages.gsfc.nasa.gov/526/PIA01709_md.jpg

The outline of the paper is the following. In Section 2 we give the definition and basic properties of the Radon transform. In particular, we describe its range and we prove its injectivity. We also provide three different inversion formulas. The first one is based on the Fourier transform. The second one uses the backprojection transform and fractional powers of the Laplacian. Finally, the last one (but the first ever used in CT) is obtained by expansion in spherical harmonics.

In Section 3 we deal with the problems associated to the fact that, in practice, the function must be recovered from only a finite number of values of its Radon transform. We first study how many samples are theoretically needed to recover a function with a given resolution. Then we study, in a semidiscrete

situation, how many directions are needed to determine the function with a prescribed resolution, assuming that we know the Radon transform in all the slices in the given directions.

Finally, in Section 4 we provide two numerical methods to invert the Radon transform. One is the filtered backprojection, which is the standard nowadays. The other is the so-called gridding method, which is based on the Fourier inversion formula.

2. The Radon transform

2.1. Definition and preliminaries

The Radon transform of a given function on $\mathbb{R}^n$, $n \geq 2$, is a function defined on the set of all hyperplanes of $\mathbb{R}^n$. Every hyperplane is determined by a normal vector to the hyperplane, $\theta \in S^{n-1}$, and its (signed) distance from the origin $s \in \mathbb{R}$, so that it can be written as

$$\theta_s := \{\, x \in \mathbb{R}^n \,:\, x \cdot \theta = s \,\}.$$

Note that $\theta_s = x + \theta^\perp = s\theta + \theta^\perp$, for any $x \in \theta_s$, where $\theta^\perp := \theta_0$ is the hyperplane perpendicular to θ passing through the origin.

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function which is integrable on every hyperplane of $\mathbb{R}^n$; for example, $f \in C(\mathbb{R}^n)$ satisfying the growth condition $|f(x)| = O(|x|^{-a})$, for some $a > n-1$. Then the *Radon transform of* f is the complex function $\mathcal{R}f$, defined on the set of hyperplanes of $\mathbb{R}^n$, whose value at a hyperplane equals the integral of f on that hyperplane, i.e.

$$\mathcal{R}f(\theta, s) = \mathcal{R}_\theta f(s) := \int_{\theta_s} f(x)\, dx = \int_{\theta^\perp} f(s\theta + y)\, dy \qquad (\theta \in S^{n-1}, s \in \mathbb{R}).$$

It is clear that $\mathcal{R}f$ is an even function on the cylinder $Z := S^{n-1} \times \mathbb{R}$, i.e. $\mathcal{R}f(-\theta, -s) = \mathcal{R}f(\theta, s)$, for every $(\theta, s) \in Z$.

The following result on continuity and growth of the Radon transform is easily obtained by integrating on polar coordinates over $\theta^\perp$ and using the standard continuity theorem of parametric integrals. We use the following terminology: a function $f \in C(\mathbb{R}^n)$ is *rapidly decreasing on* $\mathbb{R}^n$ if $|f(x)| = O(|x|^{-k})$ for every $k \in \mathbb{N}$. Similarly, $g \in C(Z)$ is *rapidly decreasing on* Z if $|g(\theta, s)| = O(|s|^{-k})$ for every $k \in \mathbb{N}$.

Proposition 1 *If $f \in C(\mathbb{R}^n)$ satisfies the growth condition $|f(x)| = O(|x|^{-a})$, for some $a > n-1$, then $\mathcal{R}f \in C(Z)$ and $|\mathcal{R}f(\theta, s)| = O(|s|^{n-1-a})$. In particular, if f is rapidly decreasing on $\mathbb{R}^n$, then $\mathcal{R}f$ is rapidly decreasing on Z.*

Next we consider the smoothness of the Radon transform. Let $D_j f = \frac{\partial f}{\partial x_j}$. By differentiation under the integral sign one can prove the following result.

Proposition 2 *Assume $f \in C^1(\mathbb{R}^n)$ satisfies the growth conditions*

$$|f(x)| = O(|x|^{-a}) \quad \text{and} \quad |D_j f(x)| = O(|x|^{-b}) \quad (j = 1, \dots, n),$$

for some exponents $a > n-1$ and $b > n$. Let $\mathscr{R}f$ be the extension of $\mathcal{R}f$ to $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ with homogeneity of degree -1, i.e. $\mathscr{R}f(\theta, s) := \frac{1}{|\theta|}\mathcal{R}f(\frac{\theta}{|\theta|}, \frac{s}{|\theta|})$. Then $\mathscr{R}f$ is C^1 on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$,

$$\frac{\partial \mathscr{R}f}{\partial s}(\theta, s) = \frac{1}{|\theta|^2} \sum_{j=1}^{n} \frac{\theta_j}{|\theta|} \, \mathcal{R}(D_j f)\left(\frac{\theta}{|\theta|}, \frac{s}{|\theta|}\right) \qquad (\theta \in \mathbb{R}^n \setminus \{0\},\, s \in \mathbb{R}), \tag{1}$$

and

$$\frac{\partial \mathscr{R}f}{\partial \theta_j}(\theta, s) = -\frac{\partial \mathscr{R}(x_j f)}{\partial s}(\theta, s) \qquad (\theta \in \mathbb{R}^n \setminus \{0\},\, s \in \mathbb{R},\, j = 1, \dots, n). \tag{2}$$

Most of the results on the Radon transform that we are going to discuss hold for continuous or smooth functions with some decay at infinity, but, in order to simplify the exposition, from now on we will only consider functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Recall that $\mathcal{S}(\mathbb{R}^n)$ is composed of all the functions $f \in C^\infty(\mathbb{R}^n)$ such that f and any of its partial derivatives are rapidly decreasing on $\mathbb{R}^n$. It is clear that Propositions 1 and 2 imply the following:

Corollary 3 *The Radon transform $\mathcal{R}$ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(Z) := \mathcal{S}(\mathbb{R}^{n+1})_{/Z}$.*

We consider the normalization of the Fourier transform given by:

Definition 1 *The Fourier transform of f is*

$$\widehat{f}(\zeta) = \mathcal{F}f(\zeta) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\zeta} f(x)\, dx,$$

and its inverse

$$\widetilde{f}(x) = \mathcal{F}^{-1} f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\zeta} f(\zeta)\, d\zeta.$$

Convolutions and Fourier transforms of functions in $\mathcal{S}(Z)$ are taken with respect to the second variable, that is, if $h, g \in \mathcal{S}(Z)$ then

$$(h * g)(\theta, s) := \int_{\mathbb{R}} h(\theta, s-t) g(\theta, t)\, dt \quad \text{and} \quad \widehat{h}(\theta, \sigma) := \int_{\mathbb{R}} e^{-2\pi i s\sigma} h(\theta, s)\, ds,$$

for any $\theta \in S^{n-1}$ and $s, \sigma \in \mathbb{R}$.

Next theorem states the main relationship between Fourier and Radon transforms.

Theorem 4 (Fourier projection-slice theorem) *If $f \in \mathcal{S}(\mathbb{R}^n)$ then*

$$(\mathcal{R}f)\widehat{\ }(\theta, \sigma) = \widehat{f}(\sigma\theta) \qquad (\theta \in S^{n-1}, \sigma \in \mathbb{R}). \tag{3}$$

Proof. By definition

$$(\mathcal{R}f)\widehat{\ }(\theta,\sigma) = \int_{\mathbb{R}} \left(\int_{\theta^\perp} e^{-2\pi i \sigma\theta\cdot(s\theta+y)} f(s\theta+y)\, dy \right) ds = \widehat{f}(\sigma\theta),$$

where the last identity follows by integrating on slices. □

Observe that (3) says that the Fourier transform of the "θ-projection" $\mathcal{R}_\theta f$ of f coincides with the "θ-slice" $\widehat{f}(\cdot\,\theta)$ of the Fourier transform of f.

Corollary 5 *The Radon transform $\mathcal{R} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(Z)$ is injective.*

Corollary 6 *If $f, g \in \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{R}(f * g) = (\mathcal{R}f) * (\mathcal{R}g)$.*

2.2. Uniqueness

We have just shown a "global" uniqueness result for the Radon transform: any function in $\mathcal{S}(\mathbb{R}^n)$ is determined by its Radon transform (Corollary 5). In many practical applications the Radon transform of a function is only known on a proper subset of its domain Z, and it is natural to ask whether these data determine the function, or at least its restriction to a certain subset of $\mathbb{R}^n$. In this section we present two results that deal with this kind of "local" uniqueness.

Theorem 7 (Support theorem or Hole theorem) *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let K be a convex compact set (a "hole") in $\mathbb{R}^n$. If $\mathcal{R}f(\theta, s) = 0$ for each hyperplane θ_s not meeting K, then $f \equiv 0$ on $\mathbb{R}^n \setminus K$.*

Proof. We only consider the compactly supported case, that is, $f \in C_c^\infty(\mathbb{R}^n)$. In this case, we are going to follow Strichartz's approach [22] which considerably simplifies the general proof due to Helgason [4, p.10].

Since K is a compact convex set, K is the intersection of all the closed balls which contain it, so we may assume that K is a closed ball. By making a translation we may also suppose that it is centered at the origin, i.e. $K = \overline{B}(0, R)$. Now we only have to show that $\mathcal{R}(x_j f)(\theta, s) = 0$, for $j = 1, \ldots, n$, $\theta \in S^{n-1}$ and $s > R$. This is so because then, by induction, $\mathcal{R}(pf)(\theta, s) = 0$, for every polynomial p and every hyperplane θ_s which does not meet $\overline{B}(0, R)$, and by Weierstrass approximation theorem, $f \equiv 0$ on those hyperplanes. This means that $f \equiv 0$ on $\mathbb{R}^n \setminus \overline{B}(0, R)$.

Pick $s > R$. By making a rotation, if necessary, we may assume that θ is the North Pole $\theta^* = (0, \ldots, 0, 1)$. Then

$$\mathcal{R}(x_n f)(\theta^*, s) = \int_{\theta_s^*} x_n f(x)\, dx = s \int_{\theta_s^*} f(x)\, dx = \mathcal{R}f(\theta^*, s) = 0,$$

so we only have to check that $\mathcal{R}(x_j f)(\theta^*, s) = 0$, for $1 \le j < n$, i.e.

$$\int_{\mathbb{R}^{n-1}} y_j f(y, s)\, dy = 0 \qquad (1 \le j < n). \tag{4}$$

Fix $1 \le j < n$. Since $s > R$, $\mathcal{R}f(\theta, t) = 0$, for every $\theta \in S^{n-1}$ and $t > s$, so we may differentiate and, by (1) and (2), we obtain that

$$0 = \frac{\partial \mathscr{R} f}{\partial \theta_j}(\theta^*, t) = -\mathcal{R}(x_j D_n f)(\theta^*, t) = -\int_{\mathbb{R}^{n-1}} y_j D_n f(y, t)\, dy.$$

(Observe that in the second identity we have used the fact that $f \in C_c^\infty(\mathbb{R}^n)$ and the fundamental theorem of the calculus.) Now an integration on t along the interval $(s, +\infty)$ produces (4). □

Notice that in the above theorem we assume the vanishing of the Radon transform for $s \in \mathbb{R}$ outside a bounded interval and for every direction $\theta \in S^{n-1}$. Our second uniqueness result assumes that the Radon transform vanishes only on a certain set of directions, but for every $s \in \mathbb{R}$.

Theorem 8 *Let $S \subset S^{n-1}$ be a set of unicity for the homogeneous polynomials in $\mathbb{R}^n$ (i.e. the only homogeneous polynomial p in $\mathbb{R}^n$ which vanishes identically on S is $p \equiv 0$). If $f \in C_c(\mathbb{R}^n)$ satisfies $\mathcal{R}_\theta f \equiv 0$, for every $\theta \in S$, then $f \equiv 0$.*

Proof. Since $f \in C_c(\mathbb{R}^n)$, $\widehat{f}$ extends to an entire function on $\mathbb{C}^n$, so $\widehat{f}$ is a real-analytic function on $\mathbb{R}^n$. We write its power series expansion as

$$\widehat{f}(\zeta) = \sum_{k=0}^{\infty} p_k(\zeta) \qquad (\zeta \in \mathbb{R}^n),$$

p_k being a homogeneous polynomial of degree k. Then (3) shows that

$$\sum_{k=0}^{\infty} \sigma^k p_k(\theta) = \widehat{f}(\sigma\theta) = (\mathcal{R}f)\widehat{\ }(\theta, \sigma) = 0 \qquad (\theta \in S,\, \sigma \in \mathbb{R}).$$

Therefore $p_k \equiv 0$ on S so $p_k \equiv 0$ on $\mathbb{R}^n$, since S is a set of unicity for the homogeneous polynomials. Hence $\widehat{f} \equiv 0$ and we conclude that $f \equiv 0$. □

2.3. The range of the Radon transform

We say that a rapidly decreasing continuous function g on Z satisfies the *Helgason-Ludwig consistency conditions* if for every $m \in \mathbb{N}$ there is a homogeneous polynomial p_m on $\mathbb{R}^n$ of degree m (unless $p_m \equiv 0$), such that

$$\int_{\mathbb{R}} s^m g(\theta, s)\, ds = p_m(\theta) \qquad (\theta \in S^{n-1}).$$

It turns out that these conditions essentially characterize the range of the Radon transform on $\mathcal{S}(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$.

Theorem 9 *$\mathcal{R}(\mathcal{S}(\mathbb{R}^n))$ is the space of all even functions $g \in \mathcal{S}(Z)$ that satisfy the Helgason-Ludwig consistency conditions.*

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$. We already know that g is even and belongs to $\mathcal{S}(Z)$. Let us check that it also satisfies the Helgason-Ludwig consistency conditions: For every $m \in \mathbb{N}$ and $\theta \in S^{n-1}$, we have

$$\begin{aligned}\int_{\mathbb{R}} s^m g(\theta, s)\, ds &= \int_{\mathbb{R}} \left(\int_{\theta^\perp} ((s\theta + y)\cdot\theta)^m f(s\theta + y)\, dy \right) ds \\ &= \int_{\mathbb{R}^n} (x\cdot\theta)^m f(x)\, dx = p_m(\theta),\end{aligned}$$

where

$$p_m(y) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\int_{\mathbb{R}^n} x^\alpha f(x)\, dx \right) y^\alpha \qquad (y \in \mathbb{R}^n)$$

is a homogeneous polynomial on $\mathbb{R}^n$ which is either of degree m or $p_m \equiv 0$.

The remaining part of the proof is much more involved. Let $g \in \mathcal{S}(Z)$ be an even function which satisfies the Helgason-Ludwig consistency conditions. Observe that if $f \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\mathcal{R}f = g$ then (3) implies that

$$\widehat{f}(\sigma\theta) = \widehat{g}(\theta, \sigma) \qquad (\sigma > 0,\, \theta \in S^{n-1}). \tag{5}$$

It will be enough to prove that

$$F(\sigma\theta) := \widehat{g}(\theta, \sigma) \qquad (\sigma > 0,\, \theta \in S^{n-1})$$

extends to a function $F \in \mathcal{S}(\mathbb{R}^n)$. This is so because $F = \widehat{f}$ for some $f \in \mathcal{S}(\mathbb{R}^n)$ which satisfies $\mathcal{R}f = g$, due to (5) and the fact that $\widehat{g}$ is even (because so is g).

First note that $\widehat{g} \in \mathcal{S}(Z) = \mathcal{S}(\mathbb{R}^{n+1})_{/Z}$ and so $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$. In order to show that F has a C^∞ extension to $\mathbb{R}^n$ it is enough to prove that all the partial derivatives of F are locally bounded at the origin. Let $\xi = \sigma\theta$, where $\sigma > 0$ and $\theta \in S^{n-1}$. A lengthy calculation shows that the following expression for a partial derivative with respect to ξ in terms of the coordinates σ and θ:

$$\frac{\partial^q}{\partial \xi^\alpha} = \sum_{1 \le |\beta| + k \le q} \frac{c_{\beta,k}(\theta)}{\sigma^{q-k}} \frac{\partial^{|\beta|+k}}{\partial\theta^\beta \partial\sigma^k} \qquad (\alpha \in \mathbb{N}^n,\, |\alpha| = q \ge 1), \tag{6}$$

where the $c_{\beta,k}$'s are polynomials. In order to prove the local boundedness of $\frac{\partial^q F}{\partial \xi^\alpha}$ at the origin we multiply by $g(\theta, s)$ the decomposition

$$e^{-2\pi i s} = \sum_{m=0}^{q-1} \frac{(-2\pi i s)^m}{m!} + e_q(s) \qquad (s \in \mathbb{R}),$$

and integrate with respect to s. We obtain

$$F(\sigma\theta) = P_q(\sigma\theta) + \int_{\mathbb{R}} e_q(\sigma s)\, g(\theta, s)\, ds,$$

where P_q is a polynomial of degree smaller than q, because g satisfies the Helgason-Ludwig consistency conditions. Now observe that $\frac{\partial^q P_q}{\partial \xi^\alpha} = 0$, and so, differentiating under the integral sign and using (6) we get

$$\frac{\partial^q F}{\partial \xi^\alpha}(\sigma\theta) = \sum_{1\le|\beta|+k\le q} \int_{\mathbb{R}} \frac{c_{\beta,k}(\theta)}{\sigma^{q-k}}\, s^k e_q^{(k)}(\sigma s)\, \frac{\partial^{|\beta|} g}{\partial\theta^\beta}(\theta, s)\, ds.$$

Since $e_q^{(k)}(s) = (-2\pi i)^k e_{q-k}(s)$ and $|e_q(s)| \lesssim |s|^q$ (by Taylor's formula with integral remainder), it follows that

$$\left|\frac{\partial^q F}{\partial \xi^\alpha}(\sigma\theta)\right| \lesssim 1.$$

Hence F has a C^∞ extension to $\mathbb{R}^n$. Moreover,

$$\sup_{\substack{\sigma>1\\ \theta\in S^{n-1}}} \sigma^N \left|\frac{\partial^q F}{\partial \xi^\alpha}(\sigma\theta)\right| \le \sum_{1\le|\beta|+k\le q} \sup_{\substack{\sigma>1\\ \theta\in S^{n-1}}} \sigma^N \left|\frac{\partial^{|\beta|+k}\widehat{g}}{\partial\theta^\beta \partial\sigma^k}(\theta,\sigma)\right| < \infty,$$

for every $N \in \mathbb{N}$, which shows that the extension of F belongs to $\mathcal{S}(\mathbb{R}^n)$, and the proof is complete. □

A direct consequence of the previous result and the support theorem (Theorem 7) is the following

Corollary 10 *$\mathcal{R}(C_c^\infty(\mathbb{R}^n))$ is the space of all even functions $g \in C_c^\infty(Z) := C_c^\infty(\mathbb{R}^{n+1})_{/Z}$ that satisfy the Helgason-Ludwig consistency conditions.*

2.4. Inversion formulas for the Radon transform

The inversion algorithms rely on explicit inversion formulas for the Radon transform. Among the different existing inversion formulas we mention three. The first one is the Fourier inversion formula, which is the base of the gridding algorithm that will be analyzed in subsection 4.2. The second one involves the backprojection. It is precisely the relation of the backprojection and the Fourier transform that is in the basis of the filtered backprojection algorithm developed in 4.1. Finally, in subsection 2.5 we obtain an inversion formula from the expansion of a function f and its Radon transform $\mathcal{R}f$ in spherical harmonics. These last formulas for $n = 2$ were obtained by Cormack and Kershaw (see [2] and [6]).

Theorem 11 (Fourier inversion formula) *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$. Then*

$$f(x) = \frac{1}{2}\int_{S^{n-1}} \left(\int_{-\infty}^{\infty} |\sigma|^{n-1}\, e^{2\pi i\sigma(x\cdot\theta)}\, \widehat{g}(\theta,\sigma)\, d\sigma\right) d\theta \qquad (x \in \mathbb{R}^n),$$

where $d\theta$ denotes the surface measure on S^{n-1}.

Proof. The Fourier inversion formula,

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\zeta} \widehat{f}(\zeta)\, d\zeta \qquad (x \in \mathbb{R}^n),$$

the Fourier projection-slice theorem (Theorem 4) and an integration in polar coordinates show that

$$f(x) = \int_0^\infty \sigma^{n-1} \left(\int_{S^{n-1}} e^{2\pi i x\cdot(\sigma\theta)} \widehat{f}(\sigma\theta) \right) d\sigma = \int_{S^{n-1}} F(\theta, x)\, d\theta,$$

where

$$F(\theta, x) := \int_0^\infty \sigma^{n-1}\, e^{2\pi i \sigma(x\cdot\theta)}\, \widehat{g}(\theta, \sigma)\, d\sigma.$$

Since $\int_{S^{n-1}} F(-\theta, x)\, d\theta = \int_{S^{n-1}} F(\theta, x)\, d\theta$, we have

$$f(x) = \frac{1}{2} \int_{S^{n-1}} (F(\theta, x) + F(-\theta, x))\, d\theta.$$

Moreover, it is clear that

$$F(\theta, x) + F(-\theta, x) = \int_{-\infty}^\infty |\sigma|^{n-1}\, e^{2\pi i \sigma(x\cdot\theta)}\, \widehat{g}(\theta, \sigma)\, d\sigma,$$

hence we are done. □

The previous inversion formula expresses f in terms of $(\mathcal{R}f)\widehat{\ }$, from which we are going to deduce an inversion formula expressing f in terms of $\mathcal{R}f$. In that formula the backprojection operator will play a fundamental role.

The *backprojection* of a suitable function g on $Z = S^{n-1} \times \mathbb{R}$ (for example, $g \in C(Z)$) is the function

$$\mathcal{R}^\# g(x) := \int_{S^{n-1}} g(\theta, x\cdot\theta)\, d\theta \qquad (x \in \mathbb{R}^n). \tag{7}$$

Observe that if $g = \mathcal{R}f$ then $g(\theta, x\cdot\theta)$ is the integral of f on the hyperplane passing through the point $x \in \mathbb{R}^n$ which is orthogonal to $\theta \in S^{n-1}$, so $\mathcal{R}^\# g(x)$ is the "mean" of the integrals of f on the hyperplanes passing through x. This observation suggests somehow the important role that the backprojection will play in the inversion formulas for the Radon transform.

The following statement summarizes some elementary properties of the backprojection.

Proposition 12

1. *If $g \in C(Z)$ then $\mathcal{R}^\# g \in C(\mathbb{R}^n)$.*
2. *If $g \in C^1(Z)$ then $\mathcal{R}^\# g \in C^1(\mathbb{R}^n)$ and $D_j \mathcal{R}^\# g = \mathcal{R}^\#(\theta_j \frac{\partial g}{\partial s})$, for $1 \le j \le n$.*

3. *If $g \in C^\infty(Z)$ then $\mathcal{R}^\# g \in C^\infty(\mathbb{R}^n)$ and $D^\alpha \mathcal{R}^\# g = \mathcal{R}^\#(\theta^\alpha \frac{\partial^{|\alpha|} g}{\partial s^{|\alpha|}})$, for $\alpha \in \mathbb{N}^n$.*

4. *The backprojection and the Laplacian conmute: If $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $\Box = \frac{\partial^2}{\partial s^2}$ are the usual Laplacians on $\mathbb{R}^n$, $n \geq 2$, and $\mathbb{R}$, respectively, then*

$$\mathcal{R}^\#(\Box g) = \Delta(\mathcal{R}^\# g) \qquad (g \in C^\infty(Z)).$$

5. *$\mathcal{R}^\#$ is the real formal adjoint operator of $\mathcal{R}$: if $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathcal{S}(Z)$ then*

$$\int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}f(\theta, s)\, g(\theta, s)\, ds\, d\theta = \int_{\mathbb{R}^n} f(x)\, \mathcal{R}^\# g(x)\, dx.$$

In particular,

$$f * (\mathcal{R}^\# g) = \mathcal{R}^\#(\mathcal{R}f * g) \qquad (f \in \mathcal{S}(\mathbb{R}^n), g \in \mathcal{S}(Z)).$$

Recall that the *Hilbert transform* of $\psi \in \mathcal{S}(\mathbb{R})$ is the function

$$H\psi(t) := \frac{i}{\pi}\ \mathrm{PV} \int_{-\infty}^{\infty} \frac{f(t-s)}{s}\, ds \qquad (t \in \mathbb{R})$$

It is well-known that $H\psi$ is a slowly increasing C^∞ function on $\mathbb{R}$ whose Fourier transform as a tempered distribution on $\mathbb{R}$ is $(H\psi)\widehat{\ } = \mathrm{sgn} \cdot \widehat{f}$, where $\mathrm{sgn}(t) = 1$, for $t \geq 0$, and $\mathrm{sgn}(t) = -1$, for $t < 0$. Now we have the following inversion formula.

Theorem 13 *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$. Then*

$$f(x) = \frac{1}{2\,(2\pi i)^{n-1}} \mathcal{R}^\#(\Lambda g)(x) \qquad (x \in \mathbb{R}^n), \tag{8}$$

where

$$\Lambda g = \begin{cases} \dfrac{\partial^{n-1} g}{\partial s^{n-1}}, & \text{if } n \text{ is odd,} \\ H\left(\dfrac{\partial^{n-1} g}{\partial s^{n-1}}\right), & \text{if } n \text{ is even,} \end{cases}$$

and H denotes the Hilbert transform acting on the real variable s.

Proof. According to Theorem 11 we only have to prove that

$$\int_{-\infty}^{\infty} |\sigma|^{n-1}\, e^{2\pi i \sigma (x\cdot\theta)}\, \widehat{g}(\theta, \sigma)\, d\sigma = \frac{1}{(2\pi i)^{n-1}} \Lambda g(\theta, x \cdot \theta).$$

This identity follows from the Fourier inversion formula after checking that $|\sigma|^{n-1}\, \widehat{g}(\theta, \sigma) = (2\pi i)^{1-n} (\Lambda g)\widehat{\ }(\theta, \sigma)$. □

Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$ as in Corollary 8. Note that, given $\theta \in S^{n-1}$ and $x \in \mathbb{R}^n$, the derivative $\frac{\partial^{n-1} g}{\partial s^{n-1}}(\theta, x \cdot \theta)$ is determined by the values $g(\theta, t)$,

for $t \in \mathbb{R}$ in a neighborhood of $x \cdot \theta$. But recall that the hyperplanes containing $x \in \mathbb{R}^n$ are those of the form $\theta_{x\cdot\theta}$, for $\theta \in S^{n-1}$. Therefore, to compute

$$\int_{S^{n-1}} \frac{\partial^{n-1} g}{\partial s^{n-1}}(\theta, x \cdot \theta)\, d\theta \qquad (x \in \mathbb{R}^n)$$

we only need the values of the integrals of f along the hyperplanes which meet a neighborhood of x. Hence the inversion formula (8) shows that, if n is odd, $f(x)$ can be computed just from the integrals of f along all the hyperplanes through a neighborhood of x. In this sense, the inversion of the Radon transform for odd dimensions is a local problem.

Observe that for even dimensions formula (8) does not allow the same conclusion, because the Hilbert transform is not local, that is, the value of the Hilbert transform $H\psi$ of a function ψ at t cannot be computed by using only the values of ψ on a neighborhood of t.

For historical reasons let us mention that in the even dimensional case formula (8) can be written in the following way:

Corollary 14 *Let $f \in \mathcal{S}(\mathbb{R}^n)$, where n is even, and $g = \mathcal{R}f$. Then*

$$f(x) = c_n \int_0^\infty F_x^{(n-1)}(r)\, \frac{dr}{r} \qquad (x \in \mathbb{R}^n),$$

where

$$c_n = 2\, \frac{(-1)^{\frac{n}{2}}}{(2\pi)^n} \qquad \textit{and} \qquad F_x(r) = \int_{S^{n-1}} g(\theta, x \cdot \theta + r)\, d\theta.$$

For $n = 2$ this is Radon's original inversion formula from 1917:

$$f(x) = -\frac{1}{\pi} \int_0^\infty \frac{dF_x(r)}{r} \qquad (x \in \mathbb{R}^2).$$

It is easy to check that (8) can be written in a compact way using fractional powers of $-\Box$ as

$$f = \frac{1}{2\,(2\pi)^{n-1}}\, \mathcal{R}^{\#}((-\Box)^{(n-1)/2} g).$$

On the other hand, we close this section by stating a deeper and more subtle inversion formula for the Radon transform in terms of the backprojection and fractional powers of $-\Delta$:

Theorem 15 *If $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$ then*

$$f = \frac{1}{2\,(2\pi)^{n-1}}\, (-\Delta)^{(n-1)/2} (\mathcal{R}^{\#} g).$$

For this kind of inversion formulas and all the beautiful deep theory related to them we refer the reader to [4] and [5].

2.5. Inversion formulas with spherical harmonics

We will obtain now an inversion formula for the Radon transform via the development in spherical harmonics.

A spherical harmonic Y_j of degree l is the restriction to S^{n-1} of a harmonic homogeneous polynomial of degree l on $\mathbb{R}^n$. The space $L^2(S^{n-1})$ has an orthogonal decomposition given by $L^2(S^{n-1}) = \bigoplus_{l=0}^{\infty} H_l$, where H_l denotes the vector space of spherical harmonics of degree l. If $N(n,l) = dim\, H_l$, and $\{Y_{l,k}\}_{k=1}^{N(n,l)}$ is an orthonormal basis of H_l, for $f \in L^2(S^{n-1})$ we have

$$f = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} f_{l,k} Y_{l,k},$$

where $f_{l,k} = \int_{S^{n-1}} f(w) Y_{l,k}(w)\, dw$.

In particular, if $f \in \mathcal{S}(\mathbb{R}^n)$ and $g = \mathcal{R}f$,

$$f(x) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} f_{l,k}(|x|)\, Y_{l,k}\left(\frac{x}{|x|}\right),$$

$$g(\theta, x) = \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} g_{l,k}(s) Y_{l,k}(\theta)\,.$$

The inversion formula gives an expression of $f_{l,k}$ in terms of an integral transform of $g_{l,k}$ and a Gegenbauer polynomial. For $\lambda > -1/2$, the Gegenbauer polynomial C_l^{λ} of degree l is the orthogonal polynomial in $[-1,1]$ with respect to the weight $(1-x^2)^{\lambda-1/2}$ normalized so that $C_l^{\lambda}(1) = 1$. C_l^{λ} is even if l is even and odd if l is odd.

Theorem 16 *Let $f \in \mathcal{S}(\mathbb{R}^n)$. For $r > 0$ we have*

$$f_{l,k}(r) = C(n) r^{2-n} \int_r^{\infty} (s^2 - r^2)^{(n-3)/2} C_l^{(n-2)/2}\Big(\frac{s}{r}\Big) g_{l,k}^{n-1}(s)\, ds$$

where $c(n) = \frac{(-1)^{n-1}}{2\pi^{n/2}} \frac{\Gamma((n-2)/2)}{\Gamma(n-2)}$ if $n > 2$ and $c(2) = -1/\pi$.

Proof. We briefly sketch the different steps of the proof.

1. First it can be shown that the functions $g_{l,k}$ and $f_{l,k}$ satisfy the following Abel-type integral equation, namely,

$$g_{l,k}(s) = |S^{n-2}| \int_s^{+\infty} f_{l,k}(r) r^{n-2} C_l^{(n-2)/2}\Big(\frac{s}{r}\Big)\Big(1 - \frac{s^2}{r^2}\Big)^{(n-3)/2} dr,$$

where $|S^{n-2}|$ denotes the surface area of S^{n-2} and $|S^0| = 2$. The formula is derived from the Funck-Hecke theorem, stating that for any

$h \in L^1((1-r^2)^{(n-3)/2}dr)$,

$$\int_{S^{n-1}} h(\theta w) Y_l(w) dw = C(n,l) Y_l(\theta),$$

where $C(n,l) = |S^{n-2}| \int_{-1}^{1} h(t) C_l^{(n-2)/2}(t)(1-t^2)^{(n-3)/2}\, dt$. This is a reproducing formula for the space of spherical harmonics. The above integral formula can be expressed as a multiplicative convolution. Given $\varphi, \psi \in L^1(dr/r)$ let $\varphi * \psi(s) = \int_0^\infty \varphi(r)\psi(s/r)\ dr/r$. Then $g_{l,k}(s) = (r^{n-1} f_{l,k}) * b$, where

$$b(x) = |S^{n-2}| \begin{cases} C_l^{(n-2)/2}(x)(1-x^2)^{(n-3)/2}, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

2. The Mellin transform $M\varphi : (0,\infty) \to \mathbb{R}$, given by

$$M\varphi(s) = \int_0^\infty \varphi(x) x^{s-1} dx,$$

satisfies the identity $M(\varphi * \psi) = M\varphi \cdot M\psi$. Moreover $M(r^p \varphi)(s) = M\varphi(s+p)$ for $s > 0$, $p \geq 0$. Consequently,

$$M(r^{n-2} f_{l,k})(s) = M(r^{n-1} f_{l,k})(s-1) = \frac{M(g_{l,k})(s-1)}{M(b)(s-1)}.$$

But

$$\frac{1}{M(b)(s-1)} = \frac{1}{C_1} \frac{\Gamma(s+n-2)}{\Gamma(s-1)} Ma(s),$$

where $C_1 = |S^{n-2}|\Gamma(1/2)\Gamma((n-1)/2)$, and the function a is given by

$$a(x) = c_2 \begin{cases} (1-x^2)^{(n-2)/2} C e^{(n-2)/2}(1/x), & \text{if } x < 1, \\ 0, & \text{if } x \geq 1, \end{cases}$$

$$c_2 = \begin{cases} \frac{\Gamma((n-3)/2)}{\Gamma(n-2)}, & \text{if } n > 2, \\ 2, & \text{if } n = 2. \end{cases}$$

We finally deduce that, since $M\varphi'(s) = (1-s) M\varphi(s-1)$,

$$\begin{aligned} M(r^{n-2} f_{l,k})(s) &= \frac{1}{C_1} \frac{\Gamma(s+n-2)}{\Gamma(s-1)} Ma(s) \frac{Mg'_{l,k}(s)}{1-s} \\ &= \frac{(-1)^{n-1}}{C_1} Ma(s) M(r^{n-2} g_{l,k}^{(n-1)}(s)) \\ &= \frac{(-1)^{n-1}}{C_1} M(a * r^{n-2} g_{l,k}^{(n-1)})(s)\,. \end{aligned}$$

The injectivity of the Mellin transform gives finally the desired formula.

□

3. The Radon transform discretization

In most of the applications of the Radon transform we only have a finite number of samples and we must reconstruct an arbitrary function from them. In order to perform the reconstruction we will reduce the function space we consider and we will need to introduce some tolerance error. The general scheme is as follows:

- We identify the space of functions we will deal with. The restrictions must be both, "natural" and technically suitable.
- Once we have identified the space to study, we need to consider several schemes of straight lines (which correspond to our samples) and we need some error estimates showing that we have gathered enough information to recover the function, at least theoretically.
- We need to provide some algorithms that reconstruct the function from the samples selected previously. It is not always the case that the best theoretical estimates can be achieved.
- Finally, one should make a study of the numerical stability of the algorithms used. In practice, most of the times, this is reduced to some numerical experiments.

We start by the first point. The function that we will analyze will have compact support in the unit ball $\bar{B}(0,1)$. This is a reasonable assumption in view of the applications to Computerized Tomography. We assume moreover that we are only interested in the details up to certain scale. This is usually modelled by considering only bandlimited functions, that is, functions f with Fourier transform $\widehat{f}$ supported in a ball $B(0,\Omega)$ (the smallest such Ω is called the bandwidth of f). The finest details that such functions can deal with are of scale $1/\Omega$. It is not possible for a function to be simultaneously of compact support and bandlimited, unless it vanishes identically. It is possible nevertheless to have functions essentially bandlimited in the following sense

Definition 2 *A function $f \in L^2$ is essentially Ω-bandlimited at level ε if*

$$\int_{B(0,\Omega)} |\widehat{f}|^2 \geq (1-\varepsilon)\|f\|_2^2.$$

Similarly, a function $f \in L^2$ is essentially R-supported at level ε if

$$\int_{B(0,R)} |f|^2 \geq (1-\varepsilon)\|f\|_2^2.$$

Definition 3 *The functions in L^2 that are Ω-bandlimited form a Hilbert space of functions that can be extended as entire functions, the Paley-Wiener space PW_Ω.*

There is a "folklore theorem" in signal theory that says that the dimension of the space of functions Ω-bandlimited with support in $B(0,R)$ is $|B(0,R)||B(0,\Omega)| = v_n^2(R\Omega)^n$, where $v_n = \pi^{n/2}\Gamma(1+n/2)$ is the volume of the unit ball in $\mathbb{R}^n$. This gives a lower bound on the number of samples that determine a function in this subspace.

Landau, Slepian and Pollak in a long series of papers (see [7], [8], [17], [18], [19], [20]) have formalized this result using essentially bandlimited (or essentially supported) functions. The result that we introduce derives from their work and it is the following. Let $PW^{\varepsilon}_{\Omega,R}$ denote the functions that are simultaneously essentially R-supported and essentially Ω-bandlimited both at level ε.

Theorem 17 *There are subspaces $H_{\Omega,R} \subset L^2(\mathbb{R}^n)$ with dimension satisfying*

$$\lim_{(R\Omega)\to\infty} \frac{\dim(H_{\Omega,R})}{|B(0,R)||B(0,\Omega)|} = 1$$

such that, for any $\varepsilon > 0$,

$$\limsup_{(R\Omega)\to\infty} \sup_{f\in PW^{\varepsilon}_{\Omega,R}} \frac{\operatorname{dist}(f, H_{\Omega,R})}{\|f\|_2} = 0.$$

The result is optimal, in the sense that there are no other subspaces $h_{\Omega,R}$ with the same property and such that $\liminf_{(R\Omega)\to\infty} \dim(h_{\Omega,R})/\dim(H_{\Omega,R}) < 1$.

To build the required subspaces we introduce the *concentration operator* $C_{\Omega,R}:\ PW_\Omega \to PW_\Omega$ defined as

$$C_{\Omega,R}(f) = P_\Omega[\chi_R \cdot f],$$

where χ_R stands for the characteristic function of the ball $B(0,R)$ and

$$P_\Omega(f) = \mathcal{F}^{-1}(\chi_\Omega \cdot \mathcal{F}(f)).$$

Let us see some of the properties of this operator needed in the proof of the theorem.

Proposition 18 *The operator $C_{\Omega,R}$ is bounded, compact, selfadjoint, positive and injective. Let $\lambda_k = \lambda_k(\Omega,R) > 0$, $k \in \mathbb{N}$, denote its eigenvalues arranged in decreasing order, we also have:*

(a) The trace of $C_{\Omega,R}$ is $\sum \lambda_k = |B(0,R)||B(0,\Omega)|$.

(b) The trace of $C^2_{\Omega,R} = C_{\Omega,R} \circ C_{\Omega,R}$ is $\sum \lambda_k^2 = |B(0,R)||B(0,\Omega)| \pm o((R\Omega)^n)$ as $(R\Omega) \to \infty$.

Proof. The selfadjointness and the boundedness of $C := C_{\Omega,R}$ are trivial. To prove the positivity, notice that for any function $f \in PW_\Omega$ we have

$$\langle C(f), f\rangle = \langle \chi_R \cdot f, f\rangle = \int_{B(0,R)} |f|^2. \tag{9}$$

Thus $C(f)$ estimates how much f is concentrated in $B(0,R)$. Equality (9) implies also that C is injective, because no $f \neq 0$ in PW_Ω can vanish in $B(0,R)$, due to the holomorphicity of f. Moreover, C has the following integral representation:

$$C(f)(y) = \int K(x,y) f(x)\, dx,$$

where $K(x,y) = \mathcal{J}_\Omega(y-x) \cdot \chi_R(x)$,

$$\mathcal{J}_\Omega(x) = \mathcal{F}^{-1}[\chi_\Omega](x) = \frac{\Omega^n J_{\frac{n}{2}}(2\pi|\Omega x|)}{|\Omega x|^{n/2}},$$

and J_α is the Bessel function of the first kind of order α. Then the compactness of C follows from the estimate

$$\iint K(x,y)^2\, dxdy = \iint \mathcal{J}_\Omega(x-y)^2 \chi_R(x)\, dxdy = |B(0,R)||B(0,\Omega)|. \tag{10}$$

By the spectral representation theorem, C has a complete orthonormal system of eigenvectors $f_k \in PW_\Omega$ with associated eigenvalues λ_k. Equality (9) applied to the eigenvalues gives

$$\lambda_k = \int_{B(0,R)} |f_k|^2.$$

Therefore $\lambda_k \in (0,1)$ is the relative concentration of the vector f_k in the ball $B(0,R)$. All functions $f \in PW_\Omega$ can be written as $f = \sum c_k f_k$, with $\|f\|^2 = \sum |c_k|^2$ and then

$$C(f) = \sum \lambda_k c_k f_k.$$

Since the functions $f_k(x)$ form an orthonormal basis of PW_Ω the reproducing kernel for this space is

$$\mathcal{J}_\Omega(x-y) = \sum f_k(x)\overline{f_k(y)}. \tag{11}$$

The convergence is in L^2 in each variable separately and thus uniform (since the f_k's belong to PW_Ω). Applying the operator C we have

$$C(\mathcal{J}_\Omega(\cdot - y))(x) = \sum \lambda_k f_k(x)\overline{f_k(y)},$$

and putting $y = x$ and integrating in x, we obtain

$$\int C(\mathcal{J}_\Omega(\cdot - x))(x)\, dx = \sum \lambda_k = \operatorname{trace} C.$$

Therefore the trace is

$$\begin{aligned} \operatorname{trace} C &= \iint \mathcal{J}_\Omega(y-x)\mathcal{J}_\Omega(x-y)\chi_R(x)\, dxdy \\ &= \int \mathcal{J}_\Omega(0)\chi_R(x)\, dx = |B(0,\Omega)||B(0,R)|. \end{aligned}$$

Similarly, to compute trace(C^2) we use again (11) (we denote by $C^x(f)$ the operator $C_{\Omega,R}$ acting on the function f as a function of x):

$$\begin{aligned}\sum \lambda_k^2 &= \int C^x(\overline{C^y(\mathcal{J}_\Omega(x-y))(t)})(t)\,dt \\ &= \iiint \mathcal{J}_\Omega(t-x)\mathcal{J}_\Omega(y-t)\mathcal{J}_\Omega(x-y)\chi_R(x)\chi_R(y)\,dxdydt.\end{aligned}$$

Integrating first in t and using the reproducing property of $\mathcal{J}_\Omega$ it follows that

$$\text{trace}(C^2) = \iint |\mathcal{J}_\Omega(x-y)|^2\chi_R(x)\chi_R(y)\,dxdy.$$

Let us see, finally, that

$$\text{trace}(C^2) = |B(0,\Omega)||B(0,R)| \pm o(|B(0,\Omega)||B(0,R)|).$$

Indeed, trace(C^2) $= I - J$, where

$$\begin{aligned} I &= \iint |\mathcal{J}_\Omega(x-y)|^2\chi_R(x)\,dxdy \quad \text{and} \\ J &= \iint |\mathcal{J}_\Omega(x-y)|^2\chi_R(x)(1-\chi_R(y))\,dxdy. \end{aligned}$$

Now $I = |B(0,\Omega)||B(0,R)|$, by(10). To estimate J we use a simple estimate of the Bessel functions. It follows that $|\mathcal{J}_\Omega(x)|^2 \lesssim \Omega^{2n}/(1+|\Omega x|^{n+1})$, so

$$J \lesssim \int_{|x|<R}\int_{|y|>R} \frac{\Omega^{2n}\,dydx}{1+(\Omega|x-y|)^{n+1}} \lesssim \int_0^{R\Omega} \frac{t^{n-1}\,dt}{1+(R\Omega-t)} \simeq (R\Omega)^{n-1}\log(R\Omega),$$

and the proof is complete. □

From these inequalities it can be observed that some eigenvalues are close to one (about $|B(0,R)||B(0,\Omega)|$ of them) and the remaining ones are very small. More precisely:

Proposition 19 *For any $0<t<1$ it holds*

$$\begin{aligned}(12)\quad &|B(0,R)||B(0,\Omega)| - \frac{1}{1-t}o(|B(0,R)||B(0,\Omega)|) \le \\ &\#\{k:\ \lambda_k(\Omega,R) > t\} \le |B(0,R)||B(0,\Omega)| + \frac{1}{t}o(|B(0,R)||B(0,\Omega)|),\end{aligned}$$

when $(R\Omega)\to\infty$.

Before proving the proposition, observe that it implies that, for any $0<\varepsilon<1$,

$$\#\{k:\ \varepsilon < \lambda_k(\Omega,R) < 1-\varepsilon\} = o(|B(0,R)||B(0,\Omega)|),$$

and this means that most of the eigenvalues are either very close to 1 (bigger than $1-\varepsilon$) or very close to 0 (smaller than ε). The amount of eigenvalues close

to 1 (in between $1-\varepsilon$ and 1) is of the order $|B(0,R)||B(0,\Omega)|$, the intermediate eigenvalues (in between ε and $1-\varepsilon$) are much fewer than the big ones and, on the other hand, the small ones (smaller than ε) are infinite.

Proof. To prove this result it is convenient to introduce the following notation: let $\mu = \mu_{\Omega,R}$ denote the positive measure in the interval $[0,1]$ obtained by adding the Dirac masses of the eigenvalues λ_k, i.e.

$$\mu = \sum_k \delta_{\lambda_k(\Omega,R)}.$$

In these terms the statement we want to prove is that, for any $t \in (0,1)$,

$$\mu((t,1)) = |B(0,R)||B(0,\Omega)| \pm o(|B(0,R)||B(0,\Omega)|).$$

We will use an argument that appears in [9]. By the previous proposition the moments of first and second order of μ are known:

$$\begin{aligned} &\int_0^1 x\, d\mu(x) = |B(0,R)||B(0,\Omega)|, \\ &\int_0^1 x^2\, d\mu(x) = |B(0,R)||B(0,\Omega)| \pm o(|B(0,R)||B(0,\Omega)|) \end{aligned} \tag{13}$$

Consider the parabolas $q_t(x) = \frac{x(x-t)}{1-t}$, which have $q_t(0) = q_t(t) = 0$ and $q_t(1) = 1$, and $p_t(x) = \frac{-x^2+(1+t)x}{t}$, with $p_t(t) = p_t(1) = 1$ and $p_t(0) = 0$; see the picture:

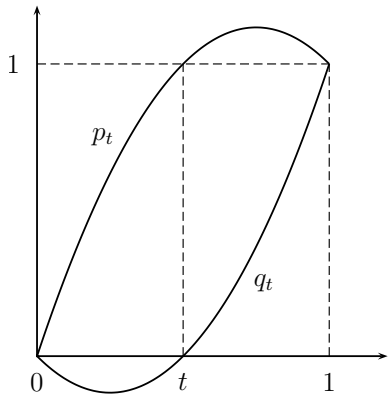

Figure 4: Auxiliary parabolas

It follows that

$$\int_0^1 q_t(x)\, d\mu(x) \le \mu([t,1]) \le \int_0^1 p_t(x)\, d\mu(x).$$

Since

$$\int_0^1 p_t(x)\,d\mu(x) = -\frac{1}{t}\int_0^1 x^2\,d\mu(x) + \frac{1+t}{t}\int_0^1 x\,d\mu(x),$$

and

$$\int_0^1 q_t(x)\,d\mu(x) = \frac{1}{1-t}\int_0^1 x^2\,d\mu(x) - \frac{t}{1-t}\int_0^1 x\,d\mu(x),$$

the estimates of the moments (13) yield (12) as desired. □

Now we prove Theorem 17. It follows from Proposition 19 that there exist $n(R\Omega), N(R\Omega) \in \mathbb{N}$, $n(R\Omega) \leq N(R\Omega)$, so that $N(R\Omega)/(|B(0,R)||B(0,\Omega)|) \to 1$, $n(R\Omega)/N(R\Omega) \to 1$, as $R\Omega \to \infty$, and simultaneously $\lambda_{N(R\Omega)}(\Omega, R) \to 0$ and $\lambda_{n(R\Omega)}(\Omega, R) \to 1$.

We take as $H_{\Omega,R}$ the vector space generated by $f_1, \dots f_{N(R\Omega)}$. Any function f of unit norm in the Paley-Wiener space can be written as $f = \sum v_k f_k$ with $\sum |v_k|^2 = 1$. On the other hand, if f is essentially R-supported at level ε and $R\Omega$ is big enough so that $\lambda_{N(R\Omega)}(\Omega, R) \leq \varepsilon$, we have

$$(1-\varepsilon) \leq \int_{B(0,R)} |f|^2 = \langle C[f], f\rangle = \sum \lambda_k |v_k|^2 \leq \varepsilon + \sum_{k<N(R\Omega)} |v_k|^2,$$

and therefore

$$\sum_{k\geq N(R\Omega)} |v_k|^2 \leq 2\varepsilon.$$

Thus any function f essentially R-supported at level ε and of norm 1 can be approximated by its Fourier sum in the basis of functions of "prolate spherical" type f_k:

$$\|f - \sum_{k<N(r,\Omega)} \langle f, f_k\rangle f_k\|^2 \leq 2\varepsilon.$$

From this we deduce the main statement, because any function $f \in PW^{\varepsilon}_{\Omega,R}$ can be approximated by a function $\widetilde{f} = P_\Omega(f) \in PW_\Omega$ which is essentially R-supported at level 2ε, and this $\widetilde{f}$ can be approximated by functions in $H_{\Omega,R}$.

To conclude we are going to see that the result is optimal. Let us denote by $G_{\Omega,R}$ the space generated by $f_1, \dots f_{n(R\Omega)}$. For any ε, take $R\Omega$ big enough so that $\lambda_{n(\Omega,R)} \geq 1-\varepsilon$. Any function $f \in G_{\Omega,R}$ is bandlimited and essentially R-supported at level ε, so the dimension of any space $h_{\Omega,R}$ that is at ε-distance from all vectors in $G_{\Omega,R}$ must be at least $n_{\Omega,R}$.

3.1. The semidiscrete case

In this section we study how many "directions" are sufficient to determine completely a function essentially bandlimited. We will assume that in each direction we know the integrals along all hyperplanes perpendicular to the given direction. In this sense, this is a semidiscrete case, as we only discretize the directions. In the next section we will consider a totally discrete case. The

semidiscrete study is due to Logan, see [11] and [12], although the sketch of the proof that we give is an original simplification different from theirs.

Let us start by a lemma in function theory of independent interest.

Lemma 20 *Let $\alpha \in \mathbb{R}$. For any $\varepsilon > 0$ there is an integer $k_0 \geq 0$ such that for any integer $k \geq k_0$ and any function g with $\operatorname{supp} \widehat{g} \subset [-1,1]$ and $g^{(j)}(0) = 0$, for $j = 0, \dots, k$, one has*

$$\int_{-(1-\varepsilon)k/2}^{(1-\varepsilon)k/2} |g(x)|^2 (1+|x|)^\alpha \, dx \leq \varepsilon \int_{\mathbb{R}} |g(x)|^2 (1+|x|)^\alpha \, dx,$$

provided the right hand side is finite.

Proof. We can assume that

$$\int_{\mathbb{R}} |g(x)|^2 (1+|x|)^\alpha \, dx = 1.$$

By the Paley-Wiener theorem we see that g is an entire function with a growth control of type $|g(z)| \leq K(1+|z|)^m e^{2\pi|\Im z|}$. As g vanishes k times at the origin we have a representation of the form

$$g(x) = \frac{1}{2\pi i} \int_{\Gamma_k} \frac{g(w) \sin^k(2\pi z/k)}{(w-x)\sin^k(2\pi w/k)} \, dw,$$

for all $x \in (-k/2, k/2)$, where Γ_k is a square centered at 0 of sidelength k with sides parallel to the axes.

In the square Γ_k the quotient $\frac{g(w)}{\sin^k(2\pi w/k)}$ is bounded by $C(1+|w|)^m$. Thus, if $x \in [(1-\varepsilon)k/2, (1-\varepsilon)k/2]$,

$$|g(x)| \leq C \left| \sin\Big(\frac{2\pi x}{k}\Big) \right|^k k^m \log\Big(\frac{1}{k/2 - |x|}\Big) \lesssim (1-\varepsilon)^k k^m \log(\epsilon k).$$

Therefore $|g(x)|$ converges uniformly to zero in the interval and the lemma is proved. □

For $m \in \mathbb{Z}$, $m \geq 0$, let H'_m be the space of all harmonic polynomials h of degree not greater than m that are even if the degree of h is even and odd if the degree of h is odd.

Definition 4 *A set $A \subset S^{n-1}$ is m-resolvent if it is a uniqueness set for H'_m.*

Theorem 21 *Let A be an m-resolvent set and let $f \in C_c^\infty(B(0,1))$. If $\mathcal{R}_\theta f \equiv 0$ for all $\theta \in A$, then*

$$\int_{|\zeta| \leq (1-\varepsilon)m/2} |\widehat{f}(\zeta)|^2 d\zeta \leq C_m \|f\|_2^2,$$

where the constant $C_m > 0$ only depends on m and satisfies $C_m \to 0$, as $m \to \infty$.

Proof. If $\mathcal{R}_\theta f \equiv 0$ for all $\theta \in A$, then for any $k \in \mathbb{N}$,

$$\mu_k(\theta) = \int_{\mathbb{R}} s^k \mathcal{R}_\theta f(s)\, ds = 0 \text{ for all } \theta \in A.$$

On the other hand, the range theorem assures that either $\mu_k \equiv 0$ or it is a homogeneous polynomial of degree k. This polynomial, when restricted to S^{n-1}, coincides with a harmonic polynomial of degree k that it is even if k is even and odd if k is odd (see [21, Theorem 2.1, p.139]). Thus, since A is an m-resolvent set, $\mu_k \equiv 0$ for $k \leq m$. This implies that

$$\frac{\partial^k}{\partial \sigma^k}(\widehat{\mathcal{R}_\theta f})(0) = 0, \quad k = 0, \ldots, m.$$

By assumption $\operatorname{supp} f \subset B(0,1)$, hence we may apply Lemma 20 with $g = \widehat{\mathcal{R}_\theta f}$ and deduce that

$$\int_{(1-\varepsilon)m/2}^{(1-\varepsilon)m/2} \sigma^{n-1} |\widehat{\mathcal{R}_\theta f}(\sigma)|^2\, d\sigma \leq C_m \int \sigma^{n-1} |\widehat{\mathcal{R}_\theta f}(\sigma)|^2\, d\sigma.$$

The proof is finished by integrating in the variable θ and using Theorem 4. □

The main application of the theorem is the following corollary that roughly speaking says that an $(m/2)$-bandlimited function is determined by its Radon transform on a set of directions that is m-resolvent.

Corollary 22 *If we know the Radon transform of an essentially bandlimited function with band $B(0,(1-\varepsilon)m/2)$ at level $r < 1 - C_m$ in an m-resolvent set of directions A, then we know it everywhere.*

Proof. Indeed, if there was another function g also essentially bandlimited with the same Radon transform in the directions of A, the difference $h = f - g$ would be essentially bandlimited and its Radon transform would vanish in the directions of A. Applying the theorem we see that $\int_{|w|<(1-\varepsilon)m/2} |\widehat{h}|^2 \leq C_m \|\widehat{h}\|^2 < \|\widehat{h}\|^2$ and since h is essentially $(1-\varepsilon)m/2$-bandlimited, we get $\int_{|w|>(1-\varepsilon)m/2} |\widehat{h}|^2 < \|\widehat{h}\|^2$. This implies that $h \equiv 0$. □

This corollary requires the exact knowledge of the Radon transform of f in all the directions of A. It would be desirable a similar result with only an "approximate" knowledge of the Radon transform in the directions of A, since, in practice, we only know its value on a finite number of slices in each of the directions of A.

Let us see how many samples we need to apply this Theorem. For a function essentially $m/2$ bandlimited we require the knowledge of its values on an m-determinant set A_m of directions. In the case we are dealing with, since the function is unidimensional and bandlimited with band $[-m/2, m/2]$, in each of the directions we need $2m$ samples to determine it in the interval $[-1,1]$ (this can be computed applying Theorem 17 to one dimensional signals

and we get $2m = \mathrm{vol}([-m/2, m/2])\,\mathrm{vol}([-1,1]))$. Therefore, all in all, we need $2m\,\mathrm{card}(A_m)$ samples. On the other hand, the dimension of H'_m is $\dim(H'_m) = 2\binom{m+n-1}{n-1} = 2\frac{m^{n-1}}{(n-1)!}(1+o(1/m))$. Therefore, we need a minimum of $\frac{4m^n}{(n-1)!}$ samples to apply the Theorem.

On the other hand, the number of needed samples according to the theorem of Landau, Pollack and Slepian (Theorem 17) is

$$\mathrm{vol}(B(0,m/2))\,\mathrm{vol}(B(0,1)).$$

In particular when the dimension is $n=2$ we need at least $(\pi^2/4)m^2 \simeq 2.467m^2$ samples. It seems that the sufficient condition that we get is of a factor $4/2.467$ redundant, therefore there is some room for improvement.

4. Reconstruction algorithms

Many algorithms have been proposed for the reconstruction of a function f from its Radon transform $g = \mathcal{R}f$, or more precisely, from a finite family of values $g_{j,l} = \mathcal{R}f(\theta_j, s_l)$. In this section we describe two reconstruction methods which are widely used in applications: the *filtered backprojection* algorithm which was proposed by Shepp and Logan in [10] and the *gridding method* (Fourier method) that was introduced in radio astronomy by [1]. Each of these schemes has several variants, depending on the scanning geometry and the specific numerical implementation.

In order to simplify the presentation we consider only the 2-dimensional case. We assume that the function f to be reconstructed is supported in $\mathbb{D} = B(0,1)$, belongs to $L^2(\mathbb{D})$ and has essential bandwidth Ω. Finally, we assume that the data are obtained scanning with *parallel standard geometry*, i.e. with a certain number p of equidistributed directions and taking in each of them $2q+1$ equispaced samples. The data have thus the following form:

(14)

$$\begin{array}{llll} g_{j,l} = g(\theta_j, s_l) & \theta_j = e^{i\varphi_j},\ \varphi_j = j\pi/p, & j = 0,\dots,p-1 & \Delta\varphi = \pi/p \\ & s_l = l/q & l = -q,\dots q & \Delta s = 1/q. \end{array}$$

Finally, it is important to remember that the mathematical models used in CT are idealizations of the complex relations between the object studied and the measured data. In numerical computations discrepancies occur for various reasons: the positive diameter of the X-ray source, the lack of accuracy of detectors, errors in the numerical approximation, etc...

4.1. The filtered backprojection

For its accuracy and fast implementation this is the reconstruction algorithm most widely used in applications. It has at least two advantages over frequency

domain interpolation schemes. First, the reconstruction procedure can be started as soon as the first projection has been measured. This clearly speeds up the process and reduces the amount of data stored at any time. The second reason is numerical: in the reconstruction process some kind of interpolation is often necessary. It turns out that interpolation in the space domain is usually simpler (linear interpolation is often enough) than in the Fourier domain, where more sophisticated methods are required.

The filtered backprojection is based in the following relation (see 5 in Proposition 12)

$$(\mathcal{R}^{\#}v) * f = \mathcal{R}^{\#}(v * \mathcal{R}f) \qquad (f \in \mathcal{S}(\mathbb{R}^n),\, v \in \mathcal{S}(Z)),$$

where $\mathcal{R}^{\#}$ is the backprojection operator defined by (7) at Section 2.4.

For $g = \mathcal{R}f$, and letting $V = \mathcal{R}^{\#}v$, the previous identity is

$$(V * f)(x) = \mathcal{R}^{\#}(v * g)(x) = \int_{S^{n-1}} (v * g)(\theta, x \cdot \theta) d\theta. \tag{15}$$

The key feature of the filtered backprojection algorithm is the choice of a so-called *point-spread function* V approximating the Dirac mass δ_0. Then the left-hand side of the identity above approximates $f(x)$ (in terms to be precised in each case). Once v is determined, using that $\mathcal{R}^{\#}v = V$, the integral on the right-hand side of the identity has to be discretized (something that depends on how the data have been obtained).

Identity (15) explains the name of the algorithm: first the data g are filtered with v (this gives $v * g$) and then the backprojection $\mathcal{R}^{\#}$ is applied.

Due to the nature of the backprojection operation, if the data were reconstructed directly, with no filtering, artifacts would appear in the reconstructed images. Moreover, because of the random nature of radioactivity, there is an inherent noise in the data that tends to make the reconstructed images rough.

As said above, the crucial point in this scheme is the choice of V. Convolution is a computationally intensive task so it is better to avoid it when possible. Since the convolution in the spatial domain is equivalent to a multiplication in the frequency domain, the filtering by V in the spatial domain can be performed by a simple multiplication by $\widehat{V}$ in the frequency domain.

Usually V is chosen so that $V * f$ deletes or de-emphasizes high frequencies, which are mostly observation noise. Since f has (essential) bandwidth Ω, one looks for V such that

$$(V * f)\widehat{\ }(\zeta) \simeq \begin{cases} \widehat{f}(\zeta), & \text{if } |\zeta| \le \Omega, \\ 0, & \text{if } |\zeta| > \Omega. \end{cases}$$

The relationship between V and v is explicit through the following distributional identity [13, Theorem 2.4]: if $g \in \mathcal{S}(Z)$ is even then

$$(\mathcal{R}^{\#}g)\widehat{\ }(\zeta) = 2|\zeta|^{1-n}\widehat{g}(\zeta/|\zeta|, |\zeta|).$$

In practice only radial symmetric functions $V(x) = V(|x|)$ are considered. Then v does not depend on θ and it is an even function of s. In this particular situation the identity above gives

$$\widehat{V}(\zeta) = 2|\zeta|^{1-n}\widehat{v}(|\zeta|), \tag{16}$$

where $\widehat{V}$ indicates the 1-dimensional Fourier transform.

In the usual cases the point-spread function V can be computed explicitly from $\widehat{V}$.

In order to reconstruct accurately functions f with essential bandwidth Ω we can take, for instance $\widehat{V}(\zeta) = \mathcal{X}_{B(0,\Omega)}(\zeta)$. More generally, consider a filter $\widehat{\phi}(\sigma)$ close to 1 when $|\sigma| \leq 1$ and with $\widehat{\phi}(\sigma) = 0$ for $|\sigma| > 1$, and define

$$\widehat{V}_\Omega(\zeta) = \widehat{\phi}\left(\frac{|\zeta|}{\Omega}\right).$$

According to (16), the corresponding function v_Ω (such that $\mathcal{R}^\# v_\Omega = V_\Omega$) is determined by the identity

$$\widehat{v}_\Omega(\sigma) = \frac{1}{2}|\sigma|^{n-1}\widehat{\phi}\left(\frac{|\sigma|}{\Omega}\right). \tag{17}$$

Multiplication by $|\sigma|^{n-1}$ in the Fourier domain corresponds (except for a constant) to the operation $\frac{\partial^{n-1}}{\partial s^{n-1}}$ when n is odd, and $H\frac{\partial^{n-1}}{\partial s^{n-1}}$ when n is even (see Theorem 13). Thus (17) can be viewed as the Fourier counterpart of the Radon reconstruction formulas (see (8)).

In applications many different $\widehat{\phi}$'s have been proposed. It seems, however, that there is no justification for any specific choice other than the expiremental results. In other words, the choice of $\widehat{\phi}$ is still more an art than a science.

Next, we show three common filters in dimension $n = 2$.

(a) *Ram-Lak filter.* Introduced in this context by Ramachandran and LakshmiNarayanan (1971). It is associated to the standard low-pass filter $\widehat{\phi}(\sigma) = \mathcal{X}_{[0,1]}(\sigma)$. Here (17) yields $\widehat{v}_\Omega(\sigma) = 1/2|\sigma|\mathcal{X}_{[0,1]}(|\sigma|/\Omega)$, hence

$$v_\Omega(s) = \int_{\mathbb{R}} \widehat{v}_\Omega(\sigma)e^{2\pi i\sigma s}d\sigma = \frac{1}{2}\int_{-\Omega}^{\Omega} |\sigma|e^{2\pi i\sigma s}d\sigma.$$

Splitting the integral for $\sigma > 0$ and $\sigma < 0$, and integrating by parts we get

$$\begin{aligned}\int_{-\Omega}^{\Omega} |\sigma|e^{2\pi i\sigma s}d\sigma &= 2\Omega^2\frac{\sin(2\pi\Omega s)}{2\pi\Omega s} + 2\frac{\cos(2\pi\Omega s) - 1}{(2\pi s)^2} \\ &= 2\Omega^2\Big(\operatorname{sinc}(2\pi\Omega s) - \frac{1}{2}\big(\operatorname{sinc}(\pi\Omega s)\big)^2\Big),\end{aligned}$$

where $\operatorname{sinc}(x) = \sin(x)/x$ is the cardinal sinus, and finally,

$$v_\Omega(s) = \Omega^2 u(2\pi\Omega s), \qquad \text{where} \quad u(s) = \operatorname{sinc}(s) - \frac{1}{2}\left(\operatorname{sinc}\left(\frac{s}{2}\right)\right)^2.$$

(b) *Cosine filter.* Here $\widehat{\phi}(\sigma) = \cos(\frac{\sigma\pi}{2})\mathcal{X}_{[0,1]}$ and the corresponding filter is

$$v_\Omega(s) = \frac{\Omega^2}{2}\left(u\left(2\pi\Omega s + \frac{\pi}{2}\right) + u\left(2\pi\Omega s - \frac{\pi}{2}\right)\right), \qquad \text{where } u \text{ is as in (a).}$$

(c) *Shepp-Logan filter.* Now $\widehat{\phi}(\sigma) = \mathrm{sinc}(\frac{\sigma\pi}{2})\mathcal{X}_{[0,1]}$ and

$$v_\Omega(s) = \frac{2\Omega^2}{\pi}u(2\pi\Omega s), \qquad \text{where} \quad u(s) = \begin{cases} \frac{\pi/2 - s\sin s}{(\pi/2)^2 - s^2}, & \text{if } s \neq \pm\pi/2, \\ 1/\pi, & \text{if } s = \pm\pi/2. \end{cases}$$

Discretization of (15). In a first instance the convolution integral of (15) has to be discretized:

$$(v_\Omega * g)(\theta, s) = \int_{\mathbb{R}} v_\Omega(s-t)g(\theta,t)\,dt = \int_{-1}^{1} v_\Omega(s-t)g(\theta,t)\,dt.$$

According to (16), v_Ω has bandwidth Ω, while g as a function of s is essentially bandlimited (by the usual Fourier relation (3)). Thus, except for a negligible error (g is only essentially bandlimited), Shannon's Theorem [13, Theorem 4.2] can be applied to $f_1(t) = v_\Omega(s-t)$, $f_2(t) = g(\theta,t)$ and the grid $(\Delta s)\mathbb{Z}$, with $\Delta s \leq 1/(2\Omega)$. This yields

$$(v_\Omega * g)(\theta, s) = \Delta s \sum_{l=-q}^{q} v_\Omega(s - s_l)\; g(\theta, s_l). \tag{18}$$

Notice that with our normalization of the Fourier transform the critical density in Shannon's theorem is $1/(2\Omega)$. Next step consists of discretizing the backprojection

$$(V * f)(x) = \mathcal{R}^{\#}(v * g)(x) = \int_0^{2\pi} (v * g)(\theta, x\cdot\theta)d\varphi, \qquad \text{where } \theta = e^{i\varphi}.$$

A computation shows that the π-periodic function $h(\varphi) = (v * g)(\theta, x\cdot\theta)$ has essential bandwidth $4\pi\Omega$, in the sense that

$$\widehat{h}(k) = \frac{1}{2\pi}\int_0^{2\pi} (v * g)(\theta, x\cdot\theta)e^{-ik\varphi}d\varphi$$

is negligible for $|k| > 4\pi\Omega$ [13, p.84-85]. Thus we can apply Shannon's theorem [13, Theorem 4.2], at the cost of only a negligible error: if $\Delta\varphi \leq 1/(2\Omega)$ we obtain the approximation

$$\begin{aligned}(V * f)(x) &= \int_0^{2\pi} (v * g)(\theta, x\cdot\theta)d\varphi = \frac{\pi}{p}\sum_{j=0}^{2p-1}(v * g)(\theta_j, x\cdot\theta_j) \\ &= \frac{2\pi}{p}\sum_{j=0}^{p-1}(v * g)(\theta_j, x\cdot\theta_j),\end{aligned}$$

where the last identity follows by π-periodicity.

This together with (18), and always taking $\max\{\Delta\varphi, \Delta s\} \leq 1/(2\Omega)$, yields

$$(V * f)(x) = \frac{2\pi}{p} \sum_{j=0}^{p-1} \Delta s \sum_{l=-q}^{q} v_\Omega(x \cdot \theta_j - s_l) g(\theta_j, s_l) \tag{19}$$
$$= \frac{2\pi}{p} \Delta s \sum_{j=0}^{p-1} \sum_{l=-q}^{q} v_\Omega(x \cdot \theta_j - s_l) g(\theta_j, s_l).$$

The algorithm, as given by (19), is computationally too demanding. It requires $O(pq)$ operations for each $f(x)$, and since f has (essential) bandwidth Ω it is necessary to compute $f(x)$ in a lattice with stepsize $1/(2\Omega)$. This gives a total number of operations of order $O(\Omega^2 pq) \simeq O(\Omega^4)$. This complexity can be reduced with a linear interpolation. Since $v_\Omega * g$ has bandwidth Ω it is determined by $(v_\Omega * g)(\theta_j, s_l)$, which can be computed with $O(pq^2)$ operations. Then the values $(v_\Omega * g)(\theta_j, x \cdot \theta_j)$ required to compute $V * f$ are obtained from the previous ones by linear interpolation. This reduces the number of operations to $O(\Omega^3)$.

Final algorithm

Step 1. For every direction θ_j, $j = 1, \ldots, p$ take the discrete convolution

$$h_{j,k} = \Delta s \sum_{l=-q}^{q} v_\Omega(s_k - s_l)\, g_{j,l} \qquad (k = -q, \ldots, q).$$

Step 2. For each x compute the discrete backprojection using a linear interpolation of the values obtained in Step 1:

$$f_A(x) = \frac{2\pi}{p} \sum_{j=0}^{2p-1} (1 - \eta) h_{j,k} + \eta h_{j,k+1},$$

where $k = k(j,x) = \left[\frac{x \cdot \theta_j}{\Delta s}\right]$, $\eta = \eta(j,x) = \frac{x \cdot \theta_j}{\Delta s} - \left[\frac{x \cdot \theta_j}{\Delta s}\right]$ and $[a]$ denotes the integer part of a.

4.2. The gridding method

This is considered at present the most accurate Fourier reconstruction method. Its main feature is the use of a weight function W, close to 1 in the reconstruction region $\mathbb{D}$, vanishing when $|x| \geq a > 1$, and with Fourier transform concentrated near 0. Then, the product $W \cdot f$ is close to f in the reconstruction region and we can therefore approximate f as soon as we can

approximate the Fourier transform

$$\begin{aligned}(W\cdot f)\widehat{\ }(\zeta) = (\widehat{W} * \widehat{f})(\zeta) &= \int_{\mathbb{R}^2} \widehat{W}(\zeta-\eta)\widehat{f}(\eta)\, d\eta \\ &= \int_0^\infty \int_{S_1} \widehat{W}(\zeta-\sigma\theta)\widehat{f}(\sigma\theta)\, \sigma d\theta\, d\sigma \\ &= \int_0^\infty \int_{S^1} \widehat{W}(\zeta-\sigma\theta)\widehat{g}(\theta,\sigma)\, \sigma d\theta\, d\sigma, \qquad (20)\end{aligned}$$

where the last identity follows from (3).

First $(W\cdot f)\widehat{\ }(\zeta)$ is evaluated in a cartesian grid; this explains the name of this method. With an inverse 2-dimensional FFT one obtains $W\cdot f$, and finally f is recovered by diving by W.

Since $(W\cdot f)\widehat{\ }$ has bandwidth 1 (f is supported in $\mathbb{D}$), in order to apply Shannon's theorem we need to consider a lattice with stepsize 1/2. Thus we want to evaluate $\{(W\cdot f)\widehat{\ }(k/2)\}_{k\in\mathbb{Z}^2}$.

The success of this method resides in an appropriate discretization of the integral (20). The values θ_j in this discretization are determined by the data (14), since the Fourier transform $\widehat{g}$ does not affect the first variable. Then the trapezoidal rule with stepsizes $\Delta\varphi = \pi/p$ in the angle and $\Delta\sigma$ in the remaining variable yields the approximation

$$(W\cdot f)\widehat{\ }(k/2) = \sum_{l=0}^{\infty}\sum_{j=0}^{2p-1} \widehat{W}(k/2 - l\Delta\sigma\theta_j)\widehat{g}(\theta_j, l\Delta\sigma)\, (l\Delta\sigma)\frac{\pi}{p}\,\Delta\sigma. \qquad (21)$$

In the derivation of this formula, however, we ignore that the integral is defined only in $[0,\infty)$, and not in the whole $\mathbb{R}$. This results in the annihilation of all the information stored in the cell represented by $l=0$ (the information given by $\widehat{g}(\theta_j,0)$). This produces artifacts and must be corrected.

Discretization of bandlimited functions defined in $[0,\infty)$. Denote $\widetilde{f} = \mathcal{F}^{-1}f$ and start with the classical *Poisson's summation formula*

$$\sum_{l\in\mathbb{Z}} f(l\Delta\sigma) = \frac{1}{\Delta\sigma}\sum_{l\in\mathbb{Z}} \widetilde{f}\big(\frac{l}{\Delta\sigma}\big) \qquad (f\in\mathcal{S}(\mathbb{R})).$$

In the second sum separate the term corresponding to $l=0$, which is $\frac{1}{\Delta\sigma}\widetilde{f}(0) = \frac{1}{\Delta\sigma}\int_{\mathbb{R}} f(\sigma)d\sigma$, so that

$$\int_{\mathbb{R}} f(\sigma)\, d\sigma = \Delta\sigma\sum_{l\in\mathbb{Z}} f(l\Delta\sigma) - \sum_{l\in\mathbb{Z}\setminus\{0\}} \widetilde{f}\big(\frac{l}{\Delta\sigma}\big). \qquad (22)$$

This general formula will be used for $f(\sigma) = g(\sigma)\ \mathrm{sgn}(\sigma)$, with g such that $g(0)=0$. As pointed out before Theorem 13, the Hilbert transform of f satisfies the relationship $(Hf)\widehat{\ }(\zeta) = \mathrm{sgn}(\zeta)\widehat{f}(\zeta)$. Therefore

$$(H\widetilde{g})\widehat{\ }(\zeta) = \mathrm{sgn}(\zeta)\widehat{\widetilde{g}}(\zeta) = f(\zeta) = \widehat{\widetilde{f}},$$

and $\widetilde{f} = H\widetilde{g}$. Then (22) yields

$$\int_{\mathbb{R}} g(\sigma)\ \mathrm{sgn}(\sigma)\ d\sigma = \Delta\sigma \sum_{l\in\mathbb{Z}} g(l\Delta\sigma)\ \mathrm{sgn}(l\Delta\sigma) + r,$$

where

$$r = \sum_{l\in\mathbb{Z}\setminus\{0\}} (H\widetilde{g})\big(\frac{l}{\Delta\sigma}\big) = \frac{i}{\pi} \sum_{l\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} \frac{\widetilde{g}(s)}{\frac{l}{\Delta\sigma} - s} ds = \frac{i}{\pi} \sum_{l=1}^{\infty} \int_{\mathbb{R}} \widetilde{g}(s) \frac{2s\ ds}{(\frac{l}{\Delta\sigma})^2 - s^2}.$$

Using that $s\widetilde{g}(s) = \frac{1}{2\pi i}\widetilde{g}'(s)$ we obtain finally

$$r = \frac{1}{\pi^2} \sum_{l=1}^{\infty} \int_{\mathbb{R}} \widetilde{g}'(s) \frac{ds}{(\frac{l}{\Delta\sigma})^2 - s^2}.$$

If g has bandwidth A and $\Delta\sigma << 1/A$ the factor s^2 in the denominator is negligible with respect to $(l/\Delta\sigma)^2$ and we obtain, approximately

$$r \simeq \frac{1}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{(l/\Delta\sigma)^2} \int_{-A}^{A} \widetilde{g}'(s) ds \simeq \frac{1}{\pi^2} (\Delta\sigma)^2 \sum_{l=1}^{\infty} \frac{1}{l^2} g'(0) = \frac{1}{6} (\Delta\sigma)^2 g'(0).$$

Hence, for g with $g(0) = 0$ we have

$$\int_{\mathbb{R}} g(\sigma)\, \mathrm{sgn}(\sigma)\, d\sigma = \Delta\sigma \sum_{l\in\mathbb{Z}} g(l\Delta\sigma)\ \mathrm{sgn}(l\Delta\sigma) + \frac{1}{6} (\Delta\sigma)^2 g'(0).$$

Use this formula for $g(\sigma) = \sigma G(\sigma)$, where $G(\sigma) = \int_{S^1} \widehat{W}(\zeta - \sigma\theta)\widehat{g}(\theta, \sigma) d\theta$ is the even function appearing in (20). Since G is even,

$$\int_0^{\infty} \sigma G(\sigma)\, d\sigma = \Delta\sigma \sum_{l=1}^{\infty} l\Delta\sigma G(l\Delta\sigma) + \frac{(\Delta\sigma)^2}{12} G(0),$$

which can be written in a unified form as

$$\int_0^{\infty} \sigma G(\sigma)\, d\sigma = \Delta\sigma \sum_{l=0}^{\infty} \sigma_l\ G(l\Delta\sigma), \qquad \text{where } \sigma_l = \begin{cases} l\Delta\sigma, & \text{if } l > 0, \\ \frac{\Delta\sigma}{12}, & \text{if } l = 0. \end{cases}$$

In our case we have thus

$$\int_0^{\infty} \int_{S^1} \widehat{W}(\zeta - \sigma\theta)\widehat{g}(\theta, \sigma)\sigma\ d\theta\ d\sigma = \Delta\sigma \sum_{l=0}^{\infty} \sigma_l \int_{S^1} \widehat{W}(\zeta - l\Delta\sigma\theta)\widehat{g}(\theta, l\Delta\sigma)\ d\theta.$$

This yields a new discretization of $(W \cdot f)\widehat{\ }(k/2)$:

$$\begin{aligned} (W \cdot f)\widehat{\ }(k/2) &= \Delta\sigma \sum_{l=0}^{\infty} \sigma_l \int_{S^1} \widehat{W}(k/2 - l\Delta\sigma\theta)\widehat{g}(\theta, l\Delta\sigma)\ d\theta \\ &= \Delta\sigma \sum_{l=0}^{\infty} \sigma_l \sum_{j=0}^{2p-1} \widehat{W}(k/2 - l\Delta\sigma\theta_j)\widehat{g}(\theta_j, l\Delta\sigma) \frac{\pi}{p}. \end{aligned}$$

Notice that this discretization differs from that in (21) only in the term corresponding to $l = 0$, which corrects the Riemann sum for the error introduced by truncating the integration domain at 0.

In order to find a good stepsize for $\Delta\sigma$ it is necessary to determine the bandwidth of the integrand in (20) as a function of σ. Let us see that the function $\sigma \mapsto \sigma\widehat{W}(\zeta - \sigma\theta)\widehat{g}(\sigma\theta)$ has bandwidth $a + 1$.

The function $\sigma \mapsto \widehat{g}(\theta, \sigma)$ has obviously bandwidth 1, hence it is enough to prove that $\sigma \mapsto \widehat{W}(\zeta - \sigma\theta)$ has bandwidth a. By the distributional identity

$$\delta(x \cdot \theta - s) = \int_{\mathbb{R}} \widehat{\delta}(\sigma) e^{2\pi i(x\cdot\theta - s)\sigma} = \int_{\mathbb{R}} e^{2\pi i(x\cdot\theta - s)\sigma} d\sigma,$$

we see that, for $|s| > a$,

$$\begin{aligned}\int_{\mathbb{R}} \widehat{W}(\zeta - \sigma\theta) e^{-2\pi i s\sigma} d\sigma &= \int_{\mathbb{R}} \Big\{ \int_{|x|\le a} W(x) e^{-2\pi i x\cdot(\zeta - \sigma\theta)} dx \Big\} e^{-2\pi i s\sigma} d\sigma \\ &= \int_{|x|\le a} W(x) e^{-2\pi i x\cdot\zeta} \int_{\mathbb{R}} e^{2\pi i(x\cdot\theta - s)\sigma} d\sigma dx \\ &= \int_{|x|\le a} W(x) e^{-2\pi i x\cdot\zeta} \delta(x\cdot\theta - s) dx = 0.\end{aligned}$$

This and the previous computations suggest the choice of a stepsize $\Delta\sigma << 1/(a+1)$.

Final algorithm

Step 1. Discretization of $\widehat{g}(\theta_j, N\Delta\sigma)$, with $\Delta\sigma << 1/(a+1)$ and $N = -Q, \dots, Q$, where Q is such that $N\Delta\sigma \subset [-\Omega, \Omega]$ (i.e. $N \le \Omega/\Delta\sigma$):

$$\widehat{g}_{j,N} = \frac{1}{q} \sum_{l=-q}^{q} e^{-2\pi i N\Delta\sigma \frac{l}{q}} g_{j,l} \qquad (j = 0, \dots, p-1 \ ; \ N = -Q, \dots, Q).$$

Step 2. For every $k \in \mathbb{Z}^2$, $|k| \le q$, compute the discretization of $(W\cdot f)\widehat{\ }(k/2)$:

$$z_k = \frac{\pi}{p}\Delta\sigma \sum_{N=0}^{Q} \sum_{j=0}^{2p-1} \sigma_N \widehat{W}(k/2 - N\Delta\sigma\theta_j)\ \widehat{g}_{j,N}.$$

Since $\widehat{W}$ decays rapidly, only few terms are relevant in this sum (those corresponding to the values with $|k/2 - N\Delta\sigma\theta_j| \le C$).

Step 3. Compute an approximation of $f(m/q)$, $m \in \mathbb{Z}^2$, $|m| \le q$ by taking the inverse Fourier transform of $(W \cdot f)\widehat{\ }$ and dividing by W:

$$f_m = \frac{1}{W(\frac{m}{q})} \sum_{|k|\le q} e^{2\pi i\pi k\cdot\frac{m}{q}} z_k.$$

References

[1] W. N. Brouw. Aperture synthesis, *Methods Comput. Phys. B*, 14:131–175, 1975.

[2] A. M. Cormack. Represenation of a function by its line integrals, with some radiological applications I, *J. Appl. Phys.*, 34:2722–2727, 1963.

[3] D. V. Finch. The attenuated x-ray transform: recent developments, *Inside out: inverse problems and applications*, 47–66, Math. Sci. Res. Inst. Publ. 47, Cambridge Univ. Press, Cambridge, 2003.

[4] S. Helgason. *The Radon Transform.* Progress in Mathematics 5, Birkhäuser, Boston, 2nd Edition, 1999.

[5] S. R. Jensen, Sufficient conditions for the inversion formula for the k-plane Radon transform in $\mathbb{R}^n$. *Math. Scand.*, 94:207–226, 2004.

[6] D. Kershaw D The determination of the density distribution of a gas flowing in a pipe from mean density measurements *J. Inst. Math. Applications*, 6(2):111-114, 1970.

[7] H. J. Landau and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty. II. *Bell System Tech. J.*, 40:65–84, 1961.

[8] _____. Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals. *Bell System Tech. J.*, 41:1295–1336, 1962.

[9] N. Lindholm. Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel. *J. Funct. Anal.*, 182(2): 390–426, 2001.

[10] L. A. Shepp. and B. F. Logan The Fourier reconstruction of a head section, *IEEE Trans. Nuclear Sci.* 21:21–43.

[11] B. F. Logan. The uncertainty principle in reconstructing functions from projections. *Duke Math. J.*, 42(4):661–706, 1975.

[12] B. F. Logan and L. A. Shepp. Optimal reconstruction of a function from its projections. *Duke Math. J.*, 42(4):645–659, 1975.

[13] F. Natterer. *The Mathematics of Computerized Tomography.* Classics in Applied Mathematics 32, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2001.

[14] F. Natterer and F. Wübbeling, *Mathematical methods in image reconstruction.* SIAM Monographs on Mathematical Modeling and Computation, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2001.

[15] R. G. Novikov. On the range characterization for the two-dimensional attenuated x-ray transformation. *Inverse Problems*, 18(3):677–700, 2002.

[16] ———. An inversion formula for the attenuated X-ray transformation. *Ark. Mat.*, 40(1):145–167, 2002.

[17] D. Slepian. Prolate spheroidal wave functions, Fourier analysis and uncertainity. IV. Extensions to many dimensions; generalized prolate spheroidal functions. *Bell System Tech. J.*, 43:3009–3057, 1964.

[18] ———. On bandwidth. *Proc. IEEE*, 64(3):292–300, 1976.

[19] ———. Some comments on Fourier analysis, uncertainty and modeling. *SIAM Rev.*, 25(3):379–393, 1983.

[20] D. Slepian and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty I. *Bell System Tech. J.*, 40:43–63, 1961.

[21] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Mathematical Series 32, Princeton University Press, Princeton, 1971.

[22] R. S. Strichartz. Radon inversion — variations on a theme. *Amer. Math. Monthly*, 89(6):377–384, 1982.