Dynamical System Exercise Set 1.1

Leonardo Bocchi

October 2023

1 Exercise Statement

1 (1p) Consider the following homeomorphism of the circle

$$f(x) = \begin{cases} \frac{1}{4} + 2x \pmod 1 & \text{if } x \in \left[0, \frac{1}{4}\right] \\ \frac{5}{8} + \frac{x}{2} \pmod 1 & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ x + \frac{1}{4} \pmod 1 & \text{if } x \in \left[\frac{3}{4}, 1\right] \end{cases}$$

Draw a lift of f and compute its rotation number.

2(1p) Consider $F_1(x) := x + \frac{1}{2}\sin(2\pi x)$ and $F_2(x) := x + \frac{1}{4\pi}\sin(2\pi x)$. Decide whether F_1 and F_2 are lifts of circle homeomorphisms. If so, decide whether that homeomorphism is orientation preserving. If it is, determine the rotation number.

3(2p) Let $f(\theta) = \theta + \frac{\varepsilon}{2\pi}\sin(2\pi n\theta) (\bmod 1)$ for $0 < \varepsilon < 1/n$ and $n \in \mathbb{N}$. Find an expression for the lifts F. Calculate the periodic points of f and determine their character. Draw the phase portrait of f and calculate its rotation number.

4 (2p) Let f be an orientation preserving homeomorphism of the circle. Show that all periodic orbits of f must have the same period. Is this also true for orientation reversing homeomorphisms? Prove it or give a counterexample.

5 (The Arnold family of circle maps) Given $\alpha, \epsilon \in [0, 1)$ and $\theta \in [0, 1)$, consider the circle map

$$f_{\varepsilon,\alpha}(\theta) = \theta + \alpha + \frac{\varepsilon}{2\pi}\sin(2\pi\theta) \pmod 1.$$

with one of its lifts

$$F_{\varepsilon,\alpha}(x) = x + \alpha + [\varepsilon/(2\pi)]\sin(2\pi x), \quad x \in \mathbb{R}.$$

Let $\rho(f_{\varepsilon,\alpha})$ denote the rotation number of the map $f_{\varepsilon,\alpha}$. Fixed $\epsilon \in (0, 1)$, and writing $f_\alpha = f_{\varepsilon,\alpha}$, the graph of $\alpha \mapsto \rho(f_\alpha)$ is a devil's staircase since it increases from 0 to 1 continuously, while having a derivative equal to 0 almost everywhere.

(a) (2p) Show that the map $\alpha \mapsto \rho(f_\alpha)$ is not absolutely continuous.

(b) (2p) Make a computer program (in whatever language you choose) that draws the graph of this function for different values of ϵ.

(c) (Extra credit 2p) Let T_λ denote the level set of the rotation number λ (known as the λ-Arnold Tongue). In other words,

$$T_\lambda = \{(\alpha, \epsilon) \in [0,1] \times [0,1] \mid \rho(f_{\epsilon,\alpha}) = \lambda\}.$$

Make a computer program (in whatever program you choose) to draw the tongues T_λ for $\lambda = 0, 1/2, 1/4$ and the tongue (actually curve) T_λ for $\lambda = \frac{1+\sqrt{5}}{2}$, the golden mean. For the latter, use the bisection method to locate the point in the curve for every ϵ.

2 Exercise 1

1 (1p) Consider the following homeomorphism of the circle

$$f(x) = \begin{cases} \frac{1}{4} + 2x & (\text{mod}1) & \text{if } x \in \left[0, \frac{1}{4}\right] \\ \frac{5}{8} + \frac{x}{2} & (\text{mod}1) & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ x + \frac{1}{4} & (\text{mod}1) & \text{if } x \in \left[\frac{3}{4}, 1\right] \end{cases}$$

Draw a lift of f and compute its rotation number.

Proof

Let $x \in [0, \frac{1}{4}]$, for which $f(x) = \frac{1}{4} + 2x$, and let $\pi(x) = e^{2\pi i x}$, then F is a lift of f if

$$\Pi \circ F = f \circ \Pi$$

Then $f \circ \Pi$ is

$$f \circ \Pi(x) = \frac{1}{4} + 2e^{2\pi i x}$$

Let us pose $F(x) = \frac{1}{4} + 2x + k$ with $k \in \mathbb{Z}$. Then $\Pi \circ F$ is

$$\begin{aligned} \Pi \circ F(x) &= e^{2\pi i(\frac{1}{4}+2x+k)} \\ &= e^{2\pi i(\frac{1}{4}+2x)} + e^{2\pi i k} \\ &= e^{2\pi i(\frac{1}{4}+2x)} + (e^{2\pi i})^k \\ &= e^{2\pi i(\frac{1}{4}+2x)} + 1^k \\ &= e^{2\pi i(\frac{1}{4}+2x)} \\ &= f \circ \Pi(x) \end{aligned}$$

Hence, F is a lift of f. Similarly we can compute the lift on the other intervals to obtain

$$F(x) = \begin{cases} \frac{1}{4} + 2x + k & k \in \mathbb{Z} & \text{if } x \in \left[k, k + \frac{1}{4}\right] \\ \frac{5}{8} + \frac{x}{2} + k & k \in \mathbb{Z} & \text{if } x \in \left[k + \frac{1}{4}, k + \frac{3}{4}\right] \\ x + \frac{1}{4} + k & k \in \mathbb{Z} & \text{if } x \in \left[k + \frac{3}{4}, k + 1\right] \end{cases}$$

Now, to compute the rotation number of f we can observe that $f'(x)$ is positive for all x, and thus the map is orientation preserving. Let us compute a periodic orbit of f by observing $f(x) = x$ yields $x = \frac{1}{4}$. Let's compute it's period

$$f(\frac{1}{4}) = \frac{3}{4}$$

$$f(\frac{3}{4}) = 1(\mathrm{mod} 1) = 0$$

$$f(0) = \frac{1}{4}$$

Hence, the orbit has period 3, and therefore, since the homeomorphism is orientation preserving, the rotation number of f is

$$\rho(f) = \frac{1}{3}$$

which is the average rotation of the map. ■

3 Exercise 2

2(1p) Consider $F_1(x) := x + \frac{1}{2}\sin(2\pi x)$ and $F_2(x) := x + \frac{1}{4\pi}\sin(2\pi x)$. Decide whether F_1 and F_2 are lifts of circle homeomorphisms. If so, decide whether that homeomorphism is orientation preserving. If it is, determine the rotation number.

Proof

We have seen that if F is a lift of a f then

$$F(x+m) = F(x) + m, \quad \forall m \geq 0, m \in \mathbb{Z}$$

In this case we have that F_1 and F_2 have degree 1 since

$$F_1(x+1) = F_1(x) + 1, \text{ and } F_2(x+1) = F_2(x) + 1$$

F_1 is not invertible since

$$\frac{d}{dx}F_1(x) = 1 + \pi\cos(2\pi x) = 0$$

has roots in $x_1 = \frac{(2\pi n - cos^{-1}(-1/\pi))}{(2\pi)}, \;\; n \in \mathbb{Z}$ and $x_2 = \frac{(2\pi n + cos^{-1}(-1/\pi))}{(2\pi)}, \;\; n \in \mathbb{Z}$, in which the second derivative

$$\frac{d^2}{dx^2}F_1(x_{1,2}) = -2\pi^2\sin(2\pi x_{1,2}) \neq 0$$

is not null, meaning it presents local maxima and minima, and thus it is not invertible.

On the contrary, F_2 is invertible, since it does not presents any point for which $\frac{d}{dx}F_2(x) = 0$ and thus it is monotonically increasing

$$\frac{d}{dx}F_2(x) = 1 + \frac{1}{2}cos(2\pi x) \neq 0, \quad \forall x \in \mathbb{R}$$

Therefore, F_1 is not a lift of a circle homeomorphism, while F_2 is. The circle homeomorphism with lift F_2 is orientation preserving since the degree is

$$F_2(x+1) - F_2(x) = 1$$

Its rotation number is 0 because its lift F_2 clearly presents a fixed point in $x = 0$.
■

4 Exercise 3

3(2*p*) Let $f(\theta) = \theta + \frac{\varepsilon}{2\pi}\sin(2\pi n\theta)(\text{mod}1)$ for $0 < \varepsilon < 1/n$ and $n \in \mathbb{N}$. Find an expression for the lifts F. Calculate the periodic points of f and determine their character. Draw the phase portrait of f and calculate its rotation number.

Proof

Let $f(\theta) = \theta + \frac{\varepsilon}{2\pi}\sin(2\pi n\theta)(\text{mod}1)$ for $0 < \varepsilon < 1/n$ and $n \in \mathbb{N}$.
An expression for a lift of f is

$$F_{\epsilon,n,k}(x) = x + \frac{\varepsilon}{2\pi}\sin(2\pi nx) + k \quad \text{for } k \in \mathbb{Z},\ 0 < \varepsilon < \frac{1}{n} \text{ and } n \in \mathbb{N}$$

The periodic points of f can be computed by solving

$$f(\theta) = \theta$$

$$\theta + \frac{\varepsilon}{2\pi}\sin(2\pi n\theta) = \theta + c$$

c is bound to be 0 since $\frac{\varepsilon}{2\pi}\sin(2\pi n\theta) \in [0,1]$, and thus we obtain

$$\frac{\varepsilon}{2\pi}\sin(2\pi n\theta) = 0$$

$$\sin(2\pi n\theta) = 0$$

$$2\pi n\theta = k\pi, \quad k \in \mathbb{Z}$$

$$\theta^* = \frac{k}{2n} \quad k \in \mathbb{Z}$$

To determine the characteristics of fixed points of a circle map, we need to analyze the stability and periodicity of these points. Let us compute $f'(\theta)$

$$f'(\theta) = 1 + \varepsilon n\cos(2\pi n\theta)$$

Hence, we evaluate its module at the fixed points

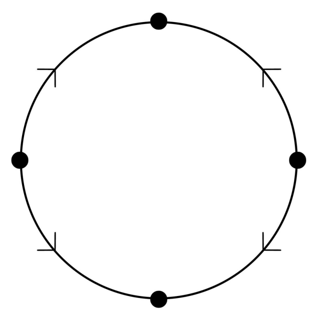
$$\begin{aligned} |f'(\theta)|\Big|_{\theta^*} &= |1 + \varepsilon n \cos(2\pi n \theta^*)| \\ &= |1 + \varepsilon n \cos(2\pi n \frac{k}{2n})| \\ &= |1 + \varepsilon n \cos(\pi k)| \end{aligned}$$

Therefore, we have

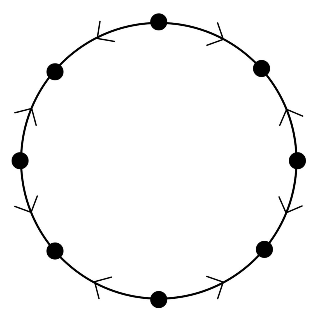
$$|f'(\theta)| < 1 \quad \text{for } k \text{ odd}$$
$$|f'(\theta)| > 1 \quad \text{for } k \text{ even}$$

So the map has respectively half of the fixed points that are attracting and half that are repelling.
Finally, the phase portrait can be represented for a fixed $n \in \mathbb{N}$ as

n = 2



n = 4



(Only the portraits for $n = 2$ and $n = 4$ are included since drawing them in LaTeXis not the quickest task)
The rotation number is the average rotation of the map, which depends directly on the periodic component $\frac{\varepsilon}{2\pi} \sin(2\pi n x)$ and is equal to

$$\rho(f) = \frac{1}{2n}, \quad \forall n \in \mathbb{N}$$

■

5 Exercise 4

4 (2p) Let f be an orientation preserving homeomorphism of the circle. Show that all periodic orbits of f must have the same period. Is this also true for orientation reversing homeomorphisms? Prove it or give a counterexample.

Proof
Let f be an orientation preserving homeomorphism

$$f : S^1 \to S^1$$

Suppose f has two periodic orbits with different periods. Then the rotation number of f computed in the two different orbits differs. However, we have shown the rotation number of f is independent of the of the point in which it is computed, thus leading to a contradiction.
If f is orientation reversing, this is not necessarily true. In fact, let

$$f(x) = [-x]$$

in this case f presents two fixed points (0 and 1/2) and all other points have period 2. (e.g. $f(0.6) = -0.4$, $f(-0.4) = 0.6$) ■

6 Exercise 5

5 (The Arnold family of circle maps) Given $\alpha, \epsilon \in [0, 1)$ and $\theta \in [0, 1)$, consider the circle map

$$f_{\varepsilon,\alpha}(\theta) = \theta + \alpha + \frac{\varepsilon}{2\pi} \sin(2\pi\theta) \pmod 1.$$

with one of its lifts

$$F_{\varepsilon,\alpha}(x) = x + \alpha + [\varepsilon/(2\pi)] \sin(2\pi x), \quad x \in \mathbb{R}.$$

Let $\rho\left(f_{\varepsilon,\alpha}\right)$ denote the rotation number of the map $f_{\varepsilon,\alpha}$. Fixed $\epsilon \in (0, 1)$, and writing $f_\alpha = f_{\varepsilon,\alpha}$, the graph of $\alpha \mapsto \rho\left(f_\alpha\right)$ is a devil's staircase since it increases from 0 to 1 continuously, while having a derivative equal to 0 almost everywhere.

(a) (2p) Show that the map $\alpha \mapsto \rho\left(f_\alpha\right)$ is not absolutely continuous.

(b) (2p) Make a computer program (in whatever language you choose) that draws the graph of this function for different values of ϵ.

(c) (Extra credit 2p) Let T_λ denote the level set of the rotation number λ (known as the λ-Arnold Tongue). In other words,

$$T_\lambda = \left\{(\alpha, \epsilon) \in [0, 1] \times [0, 1] \mid \rho\left(f_{\epsilon,\alpha}\right) = \lambda\right\}.$$

Make a computer program (in whatever program you choose) to draw the tongues T_λ for $\lambda = 0, 1/2, 1/4$ and the tongue (actually curve) T_λ for $\lambda = \frac{1+\sqrt{5}}{2}$, the golden mean. For the latter, use the bisection method to locate the point in the curve for every ϵ.

(a) (2p) Show that the map $\alpha \mapsto \rho(f_\alpha)$ is not absolutely continuous.

Proof

An equivalent definition of absolute continuity is that a function f is absolutely continuous if there is a Lebesgue integrable function g, with

$$f : [a,b] \to C, \quad g : [a,b] \to C$$

for which

$$f(x) = f(a) + \int_a^x g(t)dt$$

and equivalently $g(x) = f'(x)$ almost everywhere.
Suppose $\alpha \mapsto \rho(f_\alpha)$ is absolutely continuous.
Since the derivative of $\alpha \mapsto \rho(f_\alpha)$ exists and it is equal to 0 almost everywhere, then we have

$$f(x) = f(a) + \int_a^x g(t)dt$$

where $g(t)$ is the derivative of the map, meaning it is equal to 0 almost everywhere. Hence $\int_a^x g(t)dt = 0$ and

$$f(x) = f(a)$$

However, this is a contraddiction, since we have seen that the map $\alpha \mapsto \rho(f_\alpha)$ increases from 0 to 1 continuously. ■

(b) (2p) Make a computer program (in whatever language you choose) that draws the graph of this function for different values of ϵ.

Proof

This point is answered in a python jupyter notebook, which can be opened with Google colab. The Google Colab file can be opened from here.
In the eventuality that the file requires some sharing permissions to be opened, this file is also submitted with this document. ■

(c) (Extra credit 2p) Let T_λ denote the level set of the rotation number λ (known as the λ-Arnold Tongue). In other words,

$$T_\lambda = \{(\alpha, \epsilon) \in [0,1] \times [0,1] \mid \rho(f_{\epsilon,\alpha}) = \lambda\}.$$

Make a computer program (in whatever program you choose) to draw the tongues T_λ for $\lambda = 0, 1/2, 1/4$ and the tongue (actually curve) T_λ for $\lambda = \frac{1+\sqrt{5}}{2}$, the golden mean. For the latter, use the bisection method to locate the point in the curve for every ϵ.

Proof

This point is answered in a python jupyter notebook, which can be opened with Google colab. The Google Colab file can be opened from here.
In the eventuality that the file requires some sharing permissions to be opened, this file is also submitted with this document. ■