Lesson 20

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Defaultable bonds

Definition

A zero-coupon bond with *default* is a contract with maturity time T and payoff

$$X = \mathbf{1}_{\{\tau > T\}},$$

where τ is the (random) *default time*.

Then the arbitrage price of this bond at time t will be given by

$$D(t,T) := \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_s \mathrm{d}s} \mathbf{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right),$$

where $(\mathcal{G}_t)_{0 \le t \le T}$ represents the flow of the *total* information we have in the market. So far we use the letter $\mathcal{F}$ to indicate the available information in the market but now we will consider different kind of information and we will use $(\mathcal{F}_t)_{0 \le t \le T}$ for *the default free market information* that includes the short rate process. Obviously $D(t,T)$ will depend on the model for τ. There are different approaches.

Merton's approach to pricing defaultable bonds

In the Merton approach there is a firm's value process $(V_t)_{t\geq 0}$, that evolves as a geometric Brownian motion under the risk-neutral martingale measure $\mathbb{P}^*$, specifically

$$\mathrm{d}V_t = V_t((r-\kappa)\mathrm{d}t + \sigma_V \mathrm{d}W_t^*).$$

where r and κ are the interest and dividend rate respectevely. That is

$$V_t = V_0 \exp\left(\left(r-\kappa-\frac{1}{2}\sigma_V^2\right)t + \sigma_V W_t^*\right).$$

Therefore, we have that

$$V_T = V_t \exp\left(\left(r-\kappa-\frac{1}{2}\sigma_V^2\right)(T-t) + \sigma_V (W_T^* - W_t^*)\right),$$

Then if at the maturity T the total value V_T of the firm's assets is less than the total notional value L of the firm's debt, the firm defaults. Otherwise, the firm does not default, and the debt, that is the L zero-coupon bonds, is paid. Consequently, the default time τ is defined as

$$\tau := T\mathbf{1}_{\{V_T<L\}} + \infty\mathbf{1}_{\{V_T\geq L\}}$$

and

$$\mathbf{1}_{\{\tau>T\}} = \mathbf{1}_{\{V_T\geq L\}}.$$

It correspond to a digital option in a Black-Scholes model, so

$$D(t,T) = \Phi(d_-)e^{-r(T-t)},$$

where

$$d_- = \frac{\log(\frac{V_t}{L}) + (r-\kappa-\frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{(T-t)}}.$$

In his approach Merton also considers the possibility of a *partial recovery* in case of default in such a way that set of bond holders receive V_T if $V_T < L$. So, the price of this bond with this recovery rule will be

$$D(t,T) + \frac{1}{L}\mathbb{E}_{\mathbb{P}*}\left(\frac{V_T \mathbf{1}_{\{V_T<L\}}}{e^{r(T-t)}} \middle| \mathcal{G}_t \right) = e^{-r(T-t)} - \frac{1}{L}P_t,$$

where P_t is the price of a put option with strike L sold by the bond holders to the owners of the firm. To see that note that in that case the payoff for a holder of a zero-coupon bond is

$$\begin{aligned}
\mathbf{1}_{\{\tau>T\}} + \frac{V_T}{L}\mathbf{1}_{\{\tau\leq T\}} &= \mathbf{1}_{\{V_T\geq L\}} + \frac{V_T}{L}\mathbf{1}_{\{V_T<L\}} \\
&= 1 - \mathbf{1}_{\{V_T<L\}} + \frac{V_T}{L}\mathbf{1}_{\{V_T<L\}} \\
&= 1 - \left(1 - \frac{V_T}{L}\right)\mathbf{1}_{\{V_T<L\}} \\
&= 1 - \left(1 - \frac{V_T}{L}\right)_+ \\
&= 1 - \frac{1}{L}\left(L - V_T\right)_+ .
\end{aligned}$$

Hazard process approach

In this approach the total information available for the investors is given by a filtration $(\mathcal{G}_t)_{0\leq t\leq T}$ and $\mathcal{G}_t = \sigma(\mathbf{1}_{\{\tau\leq s\}}, 0 \leq s \leq t) \vee \mathcal{F}_t$, where $\mathcal{F}_t$ is the information of the *default free market* that includes the short rate process. On the other hand τ is not necessarily an $(\mathcal{F}_t)$-stopping time, but it is assumed that there exists a non negative $(\mathcal{F}_t)$-adapted process such that

$$\mathbb{P}^*(\tau > t|\mathcal{F}_t) = e^{-\int_0^t \lambda_s \mathrm{d}s} > 0 \text{ for all } t \geq 0.$$

where $\mathbb{P}^*$ is the risk neutral probability. Under this framework we have the following proposition.

Proposition

$$D(t,T) = \mathbf{1}_{\{t<\tau\}} \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T (\lambda_s + r_s)\mathrm{d}s} \middle| \mathcal{F}_t \right).$$

The proof of this proposition is based in the following lemma.

Lemma

For any random variable X, integrable w.r.t. $\mathbb{P}^*$,

$$\mathbf{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{P}^*}\left(X \middle| \mathcal{G}_t \right) = \mathbf{1}_{\{t<\tau\}} \frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}} \middle| \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}} \middle| \mathcal{F}_t \right)}.$$

Proof.

We have to prove that

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}X\mathbf{1}_A\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right)}\mathbf{1}_A\right), \qquad (1)$$

for all $A \in \mathcal{G}_t$. Then it is enough to consider sets of the form $A = \{\tau \le s\} \cap B$, $B \in \mathcal{F}_t$ and $0 \le s \le t$ or $A \in \mathcal{F}_t$. If $A \in \mathcal{F}_t$

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right)}\mathbf{1}_A\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right)\frac{\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right)}\mathbf{1}_A\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\mathbf{1}_A\middle|\mathcal{F}_t\right)\right) = \mathbb{E}_{\mathbb{P}^*}\left(X\mathbf{1}_{\{t<\tau\}}\mathbf{1}_A\right).
\end{aligned}$$

If $A = \{\tau \le s\} \cap B, B \in \mathcal{F}_t$, $\mathbf{1}_{\{t<\tau\}}\mathbf{1}_A = 0$, so both sides of (1) vanish. □

Proof.

Notice that we also need to prove that $\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\middle|\mathcal{F}_t\right) > 0$ when $\{t < \tau\}$. But in fact, set $Y_t := \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\middle|\mathcal{F}_t\right)$, we have

$$\begin{aligned}
\mathbb{P}^*\left(\{\tau > t\} \cap \{Y_t > 0\}\right) &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{Y_t>0\}}\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{Y_t>0\}}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\middle|\mathcal{F}_t\right)\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(Y_t\mathbf{1}_{\{Y_t>0\}}\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(Y_t\right),
\end{aligned}$$

where the last equality is due to the fact that $Y_t \geq 0$. Finally

$$\begin{aligned}
\mathbb{P}^*\left(\{\tau > t\} \cap \{Y_t > 0\}\right) &= \mathbb{E}_{\mathbb{P}^*}\left(Y_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\middle|\mathcal{F}_t\right)\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\right) = \mathbb{P}^*\left(\{\tau > t\}\right)
\end{aligned}$$

□

Proof.

(of the theorem)

$$
\begin{aligned}
D(t,T) &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}}\, e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{G}_t\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\mathbf{1}_{\{T<\tau\}}\, e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{G}_t\right) \\
&= \mathbf{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}}\, e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{G}_t\right) \\
&= \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}}\mathbf{1}_{\{T<\tau\}}\, e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}} \,\middle|\, \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}} \,\middle|\, \mathcal{F}_T\right) e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\}} \,\middle|\, \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T \lambda_s \mathrm{d}s} e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{F}_t\right)}{e^{-\int_0^t \lambda_s \mathrm{d}s}}
\end{aligned}
$$

$\square$

Note that we do not need the Key Lemma to calculate the price at time zero, because in such a situation $\mathcal{G}_0 = \mathcal{F}_0 = \{\phi, \Omega\}$, then

$$
\begin{aligned}
D(0,T) &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}} e^{-\int_0^T r_s \mathrm{d}s}\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{T<\tau\}} \middle| \mathcal{F}_T\right) e^{-\int_0^T r_s \mathrm{d}s}\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T \lambda_s \mathrm{d}s} e^{-\int_0^T r_s \mathrm{d}s}\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T (\lambda_s + r_s) \mathrm{d}s}\right).
\end{aligned}
$$

Credit default swaps (CDS)

CDS is a credit derivative that offers protection against default of a bond. Assume that the nominal of the bond is N and the recovery rate is $R < 1$, in such a way that the owner of the bond receives only NR in case of default.

Then the buyer of the CDS, in order to ensure the total nominal N, pays at time T_i

$$sN(T_i - T_{i-1})$$

provided the default time $\tau > T_i$, $i = 1, ..., n$ and receives

$$N(1 - R)$$

at time τ if $\tau \leq T_n$.
We shall omit the so-called *accrual premium*: $sN(\tau - T_{\lfloor \tau \rfloor})$. Where

$$\lfloor \tau \rfloor := \min\{T_j \in \{T_0, T_1, T_2, ...\} : T_j \leq \tau < T_{j+1}\}, \qquad \tau > T_0 = 0.$$

In this way the discounted price of the CDS at time zero will be

$$\mathbb{E}_{\mathbb{P}^*}\left(N(1-R)\mathbf{1}_{\{\tau\le T_n\}}e^{-\int_0^\tau r_s\mathrm{d}s}-\sum_{i=1}^n sN(T_i-T_{i-1})\mathbf{1}_{\{\tau>T_i\}}e^{-\int_0^{T_i}r_s\mathrm{d}s}\right)$$

$$=N(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau\le T_n\}}e^{-\int_0^\tau r_s\mathrm{d}s}\right)$$

$$-\sum_{i=1}^n sN(T_i-T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>T_i\}}e^{-\int_0^{T_i}r_s\mathrm{d}s}\right)$$

$$=N(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau\le T_n\}}e^{-\int_0^\tau r_s\mathrm{d}s}\right)$$

$$-\sum_{i=1}^n sN(T_i-T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}(r_s+\lambda_s)\mathrm{d}s}\right).$$

We can write, for $t_i = i\frac{T_n}{k}$,

$$\mathbf{1}_{\{\tau \le T_n\}} e^{-\int_0^\tau r_s \mathrm{d}s} = \lim_{k\to\infty} \sum_{i=1}^{k} \mathbf{1}_{\{t_{i-1}<\tau\le t_i\}} e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s},$$

and

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau \le T_n\}} e^{-\int_0^\tau r_s \mathrm{d}s}\right) \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t_{i-1}<\tau\le t_i\}} e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s}\right) \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \mathbb{E}_{\mathbb{P}^*}\left(\left(\mathbf{1}_{\{\tau > t_{i-1}\}} - \mathbf{1}_{\{\tau > t_i\}}\right) e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s}\right) \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \mathbb{E}_{\mathbb{P}^*}\left(\left(e^{-\int_0^{t_{i-1}} \lambda_s \mathrm{d}s} - e^{-\int_0^{t_i} \lambda_s \mathrm{d}s}\right) e^{-\int_0^{t_{i-1}} r_s \mathrm{d}s}\right) \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\int_0^{T_n} e^{-\int_0^s (r_u+\lambda_u)\mathrm{d}u} \lambda_s \mathrm{d}s\right).
\end{aligned}$$

Since

$$
\begin{aligned}
& \lim_{k\to\infty} \sum_{i=1}^{k} \left(e^{-\int_0^{t_{i-1}} \lambda_s \mathrm{d}s} - e^{-\int_0^{t_i} \lambda_s \mathrm{d}s} \right) e^{-\int_0^{t_{i-1}} \mathrm{d}s} \\
= & \lim_{k\to\infty} \sum_{i=1}^{k} \left(1 - e^{-\int_{t_{i-1}}^{t_i} \lambda_s \mathrm{d}s} \right) e^{-\int_0^{t_{i-1}} (\lambda_s + r_s) \mathrm{d}s} \\
= & \lim_{k\to\infty} \sum_{i=1}^{k} \lambda_{t_{i-1}} \left(t_i - t_{i-1} \right) e^{-\int_0^{t_{i-1}} (\lambda_s + r_s) \mathrm{d}s}
\end{aligned}
$$

Therefore the price of the CDS is

$$N(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\int_0^{T_n} e^{-\int_0^s (r_u+\lambda_u)\mathrm{d}u}\lambda_s \mathrm{d}s\right)$$
$$-\sum_{i=1}^n sN(T_i - T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}(r_s+\lambda_s)\mathrm{d}s}\right).$$

s is chosen in such a way that the price of this contract is zero:

$$s = \frac{(1-R)\mathbb{E}_{\mathbb{P}^*}\left(\int_0^{T_n} e^{-\int_0^s (r_u+\lambda_u)\mathrm{d}u}\lambda_s \mathrm{d}s\right)}{\sum_{i=1}^n (T_i - T_{i-1})\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}(r_s+\lambda_s)\mathrm{d}s}\right)}.$$

If the intensity λ is deterministic

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}(r_s+\lambda_s)\mathrm{d}s}\right) &= e^{-\int_0^{T_i}\lambda_s\mathrm{d}s}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^{T_i}r_s\mathrm{d}s}\right) \\ &= \mathbb{P}^*\left(\tau > T_i\right)P(0,T_i)\end{aligned}$$

and

$$\mathbb{E}_{\mathbb{P}^*}\left(\int_0^{T_n} e^{-\int_0^s(r_u+\lambda_u)\mathrm{d}u}\lambda_s\mathrm{d}s\right) = \int_0^{T_n} e^{-\int_0^s\lambda_u\mathrm{d}u}\lambda_s P(0,s)\mathrm{d}s$$

then

$$s = \frac{(1-R)\int_0^{T_n} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s P(0,s)\mathrm{d}s}{\sum_{i=1}^n (T_i - T_{i-1})\mathbb{P}^*(\tau > T_i)\, P(0,T_i)}.$$

and

$$\begin{aligned} s_j &= \frac{(1-R)\int_0^{T_j} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s P(0,s)\mathrm{d}s}{\sum_{i=1}^j (T_i - T_{i-1})\mathbb{P}^*(\tau > T_i)\, P(0,T_i)} \\ &\approx \frac{(1-R)\sum_{i=1}^j \left(\mathbb{P}^*(\tau > T_{i-1}) - \mathbb{P}^*(\tau > T_i)\right) P(0,T_i)}{\sum_{i=1}^j (T_i - T_{i-1})\mathbb{P}^*(\tau > T_i)\, P(0,T_i)}. \end{aligned}$$

These values s_j, $j = 1, ..., n$ are observed in the market and can be used to deduce $\mathbb{P}^*(\tau > T_i)$, $i = 1, ..., n$ and obtain a piecewise constant hazard rate.

Consider a *payer swaption* with maturity $T < T_0$, *tenor structure* $T_1, T_2, ..., T_n$, and *swap rate* R. Its payoff is

$$(S(T) - Z(T))_+$$

con

$$S(T) = P(T, T_0) - P(T, T_n)$$

that is the value of the floating payments minus the last fixed payment and

$$Z(T) = R\delta \sum_{i=1}^{n} P(T, T_i)$$

the value of payments with fixed rate minus the last payment.

We can take $Z(t)$ as *numeraire* and the price will be

$$Z(t)\mathbb{E}_{\mathbb{P}^{(Z)}}\left(\left.\frac{(S(T)-Z(T))_+}{Z(T)}\right|\mathcal{F}_t\right) = Z(t)\mathbb{E}_{\mathbb{P}^{(Z)}}\left(\left.\left(\frac{S(T)}{Z(T)}-1\right)_+\right|\mathcal{F}_t\right).$$

Then, if we assume that under $\mathbb{P}$, or $\mathbb{P}^*$ we have an evolution

$$\mathrm{d}\left(\frac{S(t)}{Z(t)}\right) = \frac{S(t)}{Z(t)}\left(\mu\mathrm{d}t + \sigma\mathrm{d}W_t\right),$$

with σ constant, it turns out that, under $\mathbb{P}^{(Z)}$

$$\mathrm{d}\left(\frac{S(t)}{Z(t)}\right) = \frac{S(t)}{Z(t)}\sigma\mathrm{d}W_t^Z,$$

so

$$\frac{S(T)}{Z(T)} = \frac{S(t)}{Z(t)}\exp\left\{\int_t^T \sigma\mathrm{d}W_s^Z - \frac{1}{2}\int_t^T \sigma^2\mathrm{d}s\right\},$$

and we obtain the Black-Scholes formula of a call with strike 1 and $r = 0$, multiplied by $Z(t)$:

$$Z(t)\left(\frac{S(t)}{Z(t)}\Phi(d_+) - \Phi(d_-)\right) = S(t)\Phi(d_+) - Z(t)\Phi(d_-),$$

with

$$\Phi(d_\pm) = \frac{\log\frac{S(t)}{Z(t)} \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

This formula is known as *the Margrabe formula* . Remember that the *swap rate* is given by

$$R(t) = \frac{P(t,T_0) - P(t,T_n)}{\delta\sum_{i=1}^n P(t,T_i)},$$

so

$$\frac{S(t)}{Z(t)} = \frac{P(t,T_0) - P(t,T_n)}{R\delta\sum_{i=1}^n P(t,T_i)} = \frac{R(t)}{R}.$$

Therefore the volatility σ corresponds to the volatility of $R(t)$. The previous formula can be written more explicitly as

$$\text{Swaption}_t = (P(t, T_0) - P(t, T_n))\,\Phi(d_+) - \left(R\delta \sum_{i=1}^{n} P(t, T_i) \right) \Phi(d_-),$$

where

$$\Phi(d_\pm) = \frac{\log\left(P(t, T_0) - P(t, T_n)\right) - \log\left(R\delta \sum_{i=1}^{n} P(t, T_i)\right) \pm \frac{1}{2}\sigma^2 (T - t)}{\sigma\sqrt{T - t}}.$$

Definition

Let X be a payoff at T. A forward contract on X with delivering time T is a contract established at $t < T$ that specifies a forward price $f(t;T)$ **that will be paid at** T for receiving X. The price $f(t;T)$ is fixed in such a way that the contract price at t is zero.

Theorem

$$\begin{aligned} f(t;T) &= \frac{1}{P(t,T)} \mathbb{E}_{\mathbb{P}^*} \left(X \exp\left\{ -\int_t^T r_s ds \right\} \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{P}^T}(X|\mathcal{F}_t). \end{aligned}$$

Proof.

The total payoff of this contract at T is

$$X - f(t;T),$$

so if $\mathbb{P}^*$ is the risk neutral probability the price of this contract will be zero at t if and only if

$$\mathbb{E}_{\mathbb{P}^*}\left((X - f(t;T)) \exp\left\{ -\int_0^T r_s ds \right\} \middle| \mathcal{F}_t \right) = 0,$$

therefore

$$\mathbb{E}_{\mathbb{P}^*}\left(X \exp\left\{ -\int_0^T r_s ds \right\} \middle| \mathcal{F}_t \right) = f(t;T)\mathbb{E}_{\mathbb{P}^*}\left(\exp\left\{ -\int_0^T r_s ds \right\} \middle| \mathcal{F}_t \right).$$

$\square$

Proof.

If we use $\mathbb{P}^T$

$$\mathbb{E}_{\mathbb{P}^T}\left(\left. \frac{X - f(t;T)}{P(T,T)} \right| \mathcal{F}_t \right) = 0,$$

Therefore

$$f(t;T) = \mathbb{E}_{\mathbb{P}^T}\left(X | \mathcal{F}_t \right).$$

$\square$

Definition

Let X a payoff at T. A contract of futures on X and delivering time T is a financial asset with the following properties

- There exist a *future price* $F(t;T)$ on X at each time t.
- At T the owner of the contract pays $F(T;T)$ and receives X.
- For any arbitrary interval $(s,t]$ the owner receives $F(t;T)-F(s;T)$.
- At each time the price of the contract is zero.

Theorem

Let $\mathbb{P}^*$ be a risk neutral probability measure such that the discounted value of a self-financing portfolio with one future contract is a $\mathbb{P}^*$-martingale, then

$$F(t;T) = \mathbb{E}_{\mathbb{P}^*}(X|\mathcal{F}_t).$$

Proof.

Let V_t be the value of a self-financing portfolio formed by a bank account and one future contract

$$\begin{aligned} V_t &= \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 \times 0 \\ &= \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} \end{aligned}$$

with $\phi_t^1 = 1$, but

$$\begin{aligned} \mathrm{d}V_t &= r_t \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} \mathrm{d}t + \phi_t^1 \mathrm{d}F(t; T) \\ &= r_t V_t \mathrm{d}t + \phi_t^1 \mathrm{d}F(t; T), \end{aligned}$$

so

$$\mathrm{d}\tilde{V}_t = e^{-\int_0^t r_s \mathrm{d}s} \phi_t^1 \mathrm{d}F(t; T),$$

with $F(T; T) = X$ and since $\tilde{V}$ is a martingale (a Brownian one), with respect to $\mathbb{P}^*$, it turns out that $F(\cdot; T)$ is a martingale and therefore

$$F(t; T) = \mathbb{E}_{\mathbb{P}^*}(F(T; T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(X|\mathcal{F}_t).$$

Corollary

Future prices and forward prices coincide if and only if interest rates are deterministic.

Proof.

$f(t, T) = F(t; T)$ if and only if

$$\mathbb{P}^* = \mathbb{P}^T.$$

We know that

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}^*} = \frac{e^{-\int_0^T r_s \mathrm{d}s}}{P(0, T)},$$

then $\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}^*} = 1$ if and only if

$$P(0, T) = e^{-\int_0^T r_s \mathrm{d}s}.$$

□