# SHORT TIME FOURIER TRANSFORM

A drawback of both Fourier series and Fourier transform is that they destroy local information. Both allow reconstruction of $L^2$-functions, but the whole transform is needed to recover the function. For example, a small shift in $a$ in
$$\widehat{\chi}_{[0,a)}(\xi) = \frac{1-e^{-2\pi i a \xi}}{2\pi i \xi}$$
affects the Fourier transform at each $\xi$.

The short time Fourier intends to avoid this inconvenience by dividing a long time signal into shorter segments (of equal length) and then compute the Fourier transform separately on each shorter segment.

<u>Definition</u>: Let $f, g \in L^2(\mathbb{R})$. The <u>short time Fourier transform</u> (STFT) <u>of $f$ with window $g$</u> is
$$V_g f(t,\xi) = \int_{\mathbb{R}} f(s)\, \overline{g(s-t)}\, e^{-2\pi i \xi s}\, ds .$$
Usually $g$ is concentrated around $0$, so that $g(s-t)$ gives a "window" to measure $f$ around $t$. This is why the STFT is sometimes called "windowed Fourier transform".

Quite often it is convenient (and helpful) to think in terms of time-frequency shifts as complex shifts. Let $z = t + i\xi \in \mathbb{C}$ and define

$$\pi(z) g(s) = g(s-t) e^{2\pi i \xi s} = M_\xi \tau_t g(s) .$$

In this language:

$$V_g f(z) = \int_{\mathbb{R}} f(s) \overline{\pi(z) g(s)} \, ds = \langle f, \pi(z) g \rangle_{L^2(\mathbb{R})} .$$

<u>Example</u>: Let $g(s) = \frac{1}{2a} \chi_{[-a,a]}(s)$ (the normalisation $1/2a$ is just so that $\int_{\mathbb{R}} g = 1$). Then

$$V_g f(t) = \int_{\mathbb{R}} f(s) \frac{1}{2a} \chi_{[-a,a]}(s-t) e^{-2\pi i \xi s} ds = \frac{1}{2a} \int_{t-a}^{t+a} f(s) e^{-2\pi i \xi s} ds .$$

This is the Fourier content of $f$ in the frequency $\xi$ for time $s \in [t-a, t+a]$.

<u>Note</u>: The abrupt edges of $\chi_{[-a,a]}$ cause problems. Common windows are the Gaussian $g(s) = e^{-\pi s^2}$, or the Fourier transform of the previous one, that is $g(s) = \text{sinc}(2\pi a s)$.

Exercises: ① Let $f, g \in L^2(\mathbb{R})$. Prove the "Fundamental identity of time-frequency analysis"

$$V_g f(x,\xi) = e^{-2\pi i x\xi}\, V_{\hat{g}} \hat{f}(\xi, -x)$$

② Covariance property: for $x, u, \xi, \omega \in \mathbb{R}$

$$V_g(\tau_u M_\omega f)(x,\xi) = e^{-2\pi i u\xi}\, V_g f(x-u, \xi-\omega).$$

A fundamental result in this theory is the following.

Theorem (orthogonality)

Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then $V_{g_j} f_j \in L^2(\mathbb{R}^2)$ and

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle\, \overline{\langle g_1, g_2 \rangle}$$

In particular, for $f, g \in L^2(\mathbb{R})$ we have $\|V_g f\|_2 = \|f\|_2 \cdot \|g\|_2$, and when the window $g$ is so that $\|g\|_2 = 1$ then $V_g : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometry.

Proof: Assume that $g_1, g_2 \in (L^1 \cap L^\infty)(\mathbb{R}) \subseteq L^2(\mathbb{R})$.

Then $f_j \cdot \tau_x \bar{g}_j \in L^2(\mathbb{R}) \quad \forall x \in \mathbb{R};\ j = 1,2.$

Notice that, in general,

$$V_g f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}\, e^{-2\pi i t\omega}\, dt = (f \cdot \tau_x \bar{g})^\wedge(\omega)$$

Then, by Plancherel on the $\omega$-integral

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} (f_1 \cdot \tau_x \bar{g}_1)\hat{}(\omega) \, \overline{(f_2 \cdot \tau_x \bar{g}_2)\hat{}(\omega)} \, d\omega \, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(t) \overline{g_1(t-x)} \, \overline{f_2(t)} g_2(t-x) \, dt \, dx =$$

$$= \int_{\mathbb{R}} f_1(t) \overline{f_2(t)} \left( \int_{\mathbb{R}} \overline{g_1(t-x)} g_2(t-x) \, dx \right) dt = \langle f_1, f_2 \rangle \cdot \overline{\langle g_1, g_2 \rangle}.$$

The general case is obtained by a standard density argument: fixed $g_1 \in L^1 \cap L^\infty$, the map $g_2 \longmapsto \langle V_{g_1} f_1, V_{g_2} f_2 \rangle$ is a linear functional coinciding with $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on the dense subspace $L^1 \cap L^\infty$, so it extends to all $L^2(\mathbb{R})$. Same thing with $g_1 \longmapsto \langle V_{g_1} f_1, V_{g_2} f_2 \rangle$ $\boxtimes$

This result shows that $f$ is completely determined by its STFT $V_g f$, but it doesn't give a way to recover $f$ from $V_g f$. We need an "inversion formula" for the STFT.

<u>Inversion formula</u> Let $g, \gamma \in L^2(\mathbb{R})$ with $\langle g, \gamma \rangle \neq 0$. Then, for all $f \in L^2(\mathbb{R})$

$$f(t) = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}}\int_{\mathbb{R}} V_g f(x,\omega)\, M_\omega \tau_x \gamma(t)\, d\omega\, dx .$$

In particular, if $\gamma = g$ and we normalise $\|g\|_2 = 1$, then

$$f(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} V_g f(x,\omega)\, M_\omega \tau_x g(t)\, d\omega\, dx .$$

<u>Remarks:</u> ① This is, in a way, similar to the Fourier inversion formula: $f$ is expressed a superposition of time-frequency shifts, with $V_g f(x,\omega)$ as a weight. However, in the Fourier case the elementary functions $e_\omega(x) = e^{2\pi i x\omega}$ are not in $L^2(\mathbb{R})$, whereas $M_\omega \tau_x$ are particularly nice functions in $L^2(\mathbb{R})$.

② Assume that $\gamma$ is (essentially) supported on $T \subseteq \mathbb{R}$ and that $\hat{\gamma}$ is (essentially) supported on $\Omega \subseteq \mathbb{R}$. Then $M_\omega \tau_x \gamma$ is (essentially) supported in $x + T$, and its (essential) spectrum is $\omega + \Omega$. Thus $M_\omega \tau_x \gamma$ occupies a cell

$$(x+T) \times (\omega + \Omega)$$

in the time-frequency plane, and the size of

$V_g f(x,\omega)$ measures the contribution of this time-frequency atoms in the decomposition of $f$. The uncertainty principle limits this concentration to sets $T, \Omega$ with $|T||\Omega| \geq 1-\delta$, $\delta > 0$.

Good time resolution requires a window with small support; this comes at the price of poor frequency resolution. Similarly, good frequency resolution, by means of a band-limited window implies poor resolution in time. In practice we choose $g$ such that both $g$ and $\hat{g}$ decay rapidly. The choice of $g$ Gaussian is canonical.

<u>Proof:</u> Since $V_g f \in L^2(\mathbb{R})$,

$$\tilde{f} := \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}}\int_{\mathbb{R}} V_g f(x,\omega)\, M_\omega \tau_x \gamma \quad d\omega\, dx$$

is well defined in $L^2(\mathbb{R})$, in the following sense. For all $h \in L^2(\mathbb{R})$ the product:

$$\langle \tilde{f}, h\rangle := \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}}\int_{\mathbb{R}} V_g f(x,\omega) \langle M_\omega \tau_x \gamma, h\rangle\, d\omega\, dx \qquad (A)$$

is well defined, since by Cauchy-Schwarz inequality

$$|\langle \tilde{f}, h\rangle| = \left| \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}}\int_{\mathbb{R}} V_g f(x,\omega)\, \overline{V_\gamma h(x,\omega)}\, d\omega\, dx \right| \leq$$

$$\leq \frac{1}{|\langle \gamma, g\rangle|} \, \|V_g f\|_2^2 \, \|V_\gamma h\|^2 = \frac{1}{|\langle \gamma, g\rangle|} \|f\|_2^2 \|g\|_2^2 \|h\|_2^2 \|\gamma\|_2^2.$$

Thus, by Riesz representation theorem, $\tilde{f}$ is given by a function in $L^2(\mathbb{R})$.

Let us see finally that this function is $f$, that is, let us see that

$$\langle \tilde{f}, h \rangle = \langle f, h \rangle \qquad \forall h \in L^2(\mathbb{R}).$$

From the identity (A) above we have

$$\langle \tilde{f}, h \rangle = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(x,\omega) \left( \int_{\mathbb{R}} M_\omega \tau_x \gamma(t) \overline{h(t)}\, dt \right) d\omega\, dx$$

$$= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(x,\omega) \overline{\langle h, M_\omega \tau_x \gamma \rangle}\, d\omega\, dx$$

$$= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(x,\omega) \overline{V_\gamma h(x,\omega)}\, d\omega\, dx =$$

$$= \frac{1}{\langle \gamma, g \rangle} \langle V_g f, V_\gamma h \rangle = \frac{1}{\langle \gamma, g \rangle} \langle f, h \rangle \overline{\langle g, \gamma \rangle} = \langle f, h \rangle .$$

By duality $\tilde{f} = f$. ⊠

<u>Note</u>: In practise, given a signal $f$, the process is:

1. Analysis: $f \longrightarrow V_g f$
2. Processing: This may involve truncation, separation of signal components, compression, etc... It results in a function $F(x,\omega) \sim V_g f(x,\omega)$.
3. Synthesis: $\tilde{f}(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} F(x,\omega) M_\omega \tau_x \gamma(t)\, dx\, d\omega .$

# Discrete time-frequency representations. Gabor frames

Instead of the continuous expansion given by the inversion formula we would like to have a series expansion with respect to a countable subset of the time-frequency shifts.

Example: Let $g(t) = \chi_{[-1/2, 1/2]}(t)$. Then the system

$$\{M_k \tau_n g\}_{k,n \in \mathbb{Z}} = \{e^{2\pi i k t} g(t-n)\}_{k,n \in \mathbb{Z}}$$

is an orthonormal basis of $L^2(\mathbb{R})$. Therefore, for any $f \in L^2(\mathbb{R})$ we have the decomposition

$$f(t) = \sum_{n,k \in \mathbb{Z}} V_g f(n,k) \, M_k \tau_n g(t), \tag{D}$$

where $V_g f(n,k) = \langle f, M_k \tau_n g \rangle = \int_{n-1/2}^{n+1/2} f(t) e^{-2\pi i k t} \, dt$

Also $$\|f\|^2_{L^2(\mathbb{R})} = \sum_{n,k \in \mathbb{Z}} |V_g f(n,k)|^2 .$$

Formula (D) tells us that in order to determine (and reconstruct) $f$ it is enough to sample $V_g f(x, \omega)$ on the lattice $\mathbb{Z} \times \mathbb{Z}$.

Let us prove this. By definition,

$$V_g f(x,\xi) = \int_{\mathbb{R}} f(t)\,\chi_{[-1/2,1/2]}(t-x)\,e^{-2\pi i t\xi}\,dt$$

$$= \int_{x-1/2}^{x+1/2} f(t)\,e^{-2\pi i t\xi}\,dt$$

In particular $\quad V_g f(n,k) = \int_{n-1/2}^{n+1/2} f(t)\,e^{-2\pi i t k}\,dt,$

which can be viewed as the $k$-th coefficient of the 1-periodic function in $\mathbb{R}$ coinciding with $f$ in $[n-1/2, n+1/2]$.

Recall that, in general, given a $T$-periodic function $h$ we have

$$h(t) = \sum_{n\in\mathbb{Z}} \hat{h}(n)\,e^{i\frac{2\pi}{T}nt} \quad ; \quad \hat{h}(n) = \frac{1}{T}\int_{-T/2}^{T/2} h(t)\,e^{-i\frac{2\pi}{T}nt}\,dt$$

For $T=1$ this gives

$$h(t) = \sum_{n\in\mathbb{Z}} \hat{h}(n)\,e^{i2\pi nt} \quad ; \quad \hat{h}(n) = \int_{-1/2}^{1/2} h(t)\,e^{-2\pi i nt}\,dt.$$

Thus, for $t\in[n-\frac{1}{2}, n+\frac{1}{2}]$

$$f(t) = \sum_{k\in\mathbb{Z}} V_g f(n,k)\,e^{2\pi i k t},$$

and combining the different unit interval

$$f(t) = \sum_{n\in\mathbb{Z}} f(t)\chi_{[n-\frac{1}{2},n+\frac{1}{2}]}(t) = \sum_{n,k\in\mathbb{Z}} V_g f(n,k)\, e^{2\pi i k t}\chi_{[n-\frac{1}{2},n+\frac{1}{2}]}(t)$$

$$= \sum_{n,k\in\mathbb{Z}} V_g f(n,k)\,(M_k \tau_n g)(t).$$

That $\{e^{2\pi i n t}\chi_{[n-1/2,n+1/2]}(t)\}_{n,k\in\mathbb{Z}}$ is orthogonal is immediate: for different $n$ the supports are disjoint, and fixed $n$ the different $k$ give a Fourier basis. On the other hand, trivially $\|M_k \tau_n g\|_2 = 1$.

At this point one may wonder whether a similar basis exists with translations and modulations of the Gaussian (instead of the characteristic function). The answer is no...

Let $g\in L^2(\mathbb{R})$ be a general window, and discretize $M_w \tau_x g$ on a lattice $a\mathbb{Z}\times b\mathbb{Z}$, $ab>0$, $ab=1$ (the case $a=b=1$ is the canonical one). Let thus

$$g_{n,k} = M_{bk}\tau_{an} g \qquad n,k\in\mathbb{Z}.$$

<u>Balian-Low theorem</u> Let $g \in L^2(\mathbb{R})$, $a, b > 0$ with $ab = 1$, and let $\{g_{n,k}\}_{n,k \in \mathbb{Z}}$ as above.

If $\{g_{n,k}\}_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$, then

$$\left(\int_{\mathbb{R}} t^2 |g(t)|^2 dt\right)\left(\int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2 d\xi\right) = \infty .$$

<u>Note:</u> This is a particular manifestation of the uncertainty principle: if $\{g_{n,k}\}_{n,k}$ is a basis, the window $g$ cannot be concentrated both in time and frequency.

<u>Corollary.</u> Let $g(t) = 2^{1/4} e^{-\pi t^2}$ be the Gaussian (with $\|g\|_2 = 1$). The family $g_{n,k}(t) = M_k \tau_n g(t)$, $n,k \in \mathbb{Z}$ is not an orthonormal basis.

<u>Proof:</u> $\displaystyle\int_{\mathbb{R}} t^2 |g(t)|^2 dt = \sqrt{2} \int_{\mathbb{R}} t^2 e^{-2\pi t^2} dt < +\infty$ $\boxtimes$

Notice that in the previous case $g(t) = \chi_{[-1/2, 1/2]}(t)$ we have $\hat{g}(\xi) = \dfrac{\sin(\pi\xi)}{\pi\xi}$ and

$$\int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2 = \frac{1}{\pi^2} \int_{\mathbb{R}} |\sin(\pi\xi)|^2 d\xi = +\infty .$$

Proof: This is a sort of physics proof. We do it only for $a=b=1$ (canonical case). We have thus the system

$$g_{n,k}(t) = M_k \tau_n g(t) = e^{2\pi i k t} g(t-n).$$

Consider the following two operators:

$$Xf(x) = x f(x) \text{ Position}; \quad Pf(x) = \frac{1}{2\pi i} f'(x) \text{ Momentum}$$

These operators are self-adjoint, in the sense that

- $\langle Xf, g\rangle = \int_{\mathbb{R}} x f(x) g(x)\, dx = \langle f, Xg\rangle \qquad f \in L^2$

- $\langle Pf, g\rangle = \frac{1}{2\pi i} \int_{\mathbb{R}} f'(x) g(x)\, dx = \qquad f \in L^2$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} f(x) g'(x)\, dx = \langle f, Pg\rangle$$

Assume that $Xg(x) = x g(x) \in L^2(\mathbb{R})$ and that $Pg \in L^2(\mathbb{R})$, which, by the Fourier identity $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$, is equivalent to $\xi \hat{g}(\xi) \in L^2(\mathbb{R})$.

Since we are assuming that $\{M_k \tau_n g\}_{n,k}$ is a basis we have

$$Xg = \sum_{n,k\in\mathbb{Z}} \langle Xg, M_k \tau_n g\rangle\, M_k \tau_n g$$

$$Pg = \sum_{n',k'\in\mathbb{Z}} \langle Pg, M_{k'} \tau_{n'} g\rangle\, M_{k'} \tau_{n'} g,$$

and therefore, by the orthonormality:

$$\langle Xg, Pg\rangle = \sum_{n,k\in\mathbb{Z}} \langle Xg, M_k \tau_n g\rangle \langle M_k \tau_n g, Pg\rangle .$$

<u>Claim:</u> $\langle Xg, M_k \tau_n g\rangle = \langle M_{-k} \tau_{-n} g, Xg\rangle$

$$\langle M_k \tau_n g, Pg\rangle = \langle Pg, M_{-k} \tau_{-n} g\rangle$$

Assuming this, since $\{M_{-k}\tau_{-n} g\}_{k,n\in\mathbb{Z}}$ is the same basis

$$\langle Xg, Pg\rangle = \sum_{k,n\in\mathbb{Z}} \langle Pg, M_{-k}\tau_{-n} g\rangle \langle M_{-k}\tau_{-n} g, Xg\rangle$$

$$= \langle Pg, Xg\rangle .$$

Since both operators are self-adjoint, this gives a contradiction (provided $g \in \mathrm{Dom}(P) \cap \mathrm{Dom}(X)$); first $(PX - XP)g = P(xg(x)) - X\left(\frac{1}{2\pi i} g'(x)\right) =$

$$= \frac{1}{2\pi i}\left(g(x) + xg'(x)\right) - \frac{x}{2\pi i} g'(x) = \frac{1}{2\pi i} g(x) .$$

But the above identity gives

$$\langle PXg, g\rangle = \langle XPg, g\rangle ; \langle (PX - XP)g, g\rangle = 0 .$$

This is a contradiction:

$$0 = \langle (PX - XP)g, g\rangle = \frac{1}{2\pi i}\langle g, g\rangle .$$

It remains to prove the Claim. Let us prove the first identity; the other one is proved similarly.

By the self-adjointness
$$\langle Xg, M_k \tau_n g\rangle = \langle g, XM_k \tau_n g\rangle.$$

By definition
$$XM_k \tau_n g(x) = xe^{2\pi i k x} g(x-n) = ne^{2\pi i k x} g(x-n) + e^{2\pi i k x}(x-n)\, g(x-n)$$
$$= n M_k \tau_n g(x) + M_k \tau_n Xg(x).$$

Hence
$$\langle Xg, M_k \tau_n g\rangle = n\langle g, M_k \tau_n g\rangle + \langle g, M_k \tau_n Xg\rangle.$$

Since $g = M_0 \tau_0 g$, the orthogonality of the system $\{M_k \tau_n g\}_{k,n}$ shows that the first term in this sum is zero. Then
$$\langle Xg, M_k \tau_n g\rangle = \langle g, M_k \tau_n Xg\rangle = \int_{\mathbb{R}} g(x) e^{-2\pi i k x}\, \overline{Xg(x-n)}\, dx$$

$$(x-n=t) \qquad = \int_{\mathbb{R}} g(t+n) e^{-2\pi i k(t+n)}\, \overline{Xg(t)}\, dt$$

$$= \langle M_{-k} \tau_{-n} g, Xg\rangle \qquad \boxtimes$$

So far we have seen that a Gabor system $\{M_k \tau_n g\}_{n,k}$ cannot form an orthonormal basis unless either $g$ or $\hat{g}$ are badly localised. Take the Gaussian $g(t) = 2^{1/4} e^{-\pi t^2}$. There is really no way to recover an $f \in L^2(\mathbb{R})$ from its action on a Gabor system $g_{n,k} = M_{bk} \tau_{an} g$, $n,k \in \mathbb{Z}$, on a lattice $a\mathbb{Z} \times b\mathbb{Z}$?

The answer is of course yes - but not in the same way as it is done with an orthonormal basis. Ultimately we would like to have an expression of the form

$$f = \sum_{k,n \in \mathbb{Z}} c_{nk}(f) M_{bk} \tau_{an} g = \sum_{n,k \in \mathbb{Z}} c_{nk}(f) g_{n,k}$$

Recall that $V_g f(an, bk) = \langle f, M_{bk} \tau_{an} g \rangle = \langle f, g_{n,k} \rangle$.

If we want the system $\{g_{n,k}\}_{n,k}$ to capture most of the energy of a $f \in L^2(\mathbb{R})$ we need that

$$A \|f\|_2^2 \leq \sum_{n,k \in \mathbb{Z}} |V_g f(an, bk)|^2$$

for some $A > 0$. If we want the system not to "oversample", in the sense that too much of $\|f\|_2^2$ is given by the samples, we ask

$$\sum_{n,k \in \mathbb{Z}} |V_g f(an, bk)|^2 \leq B \|f\|_2^2$$

for some $B > 0$.

These two conditions give a notion that is more relaxed than being a basis. For example, we would double the number of $\{g_{n,k}\}_{n,k}$ and we would still get such a family. Or we could delete one $g_{n_0,k_0}$ and maybe still preserve the inequalities. This robustness, which the orthogonal bases lack, is very helpfull in applications

<u>Definition</u>: A sequence $\{e_j\}_{j\in J}$ a separable Hilbert space $H$ is called a <u>frame</u> if there exist $A, B > 0$ such that

$$(F) \qquad A\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq B\|f\|^2 \qquad \forall f \in H.$$

In this language, we are interested then in Gabor frames $\{g_{n,k}\}_{n,k}$ (as above) in $H = L^2(\mathbb{R})$. Of course an orthonormal basis is a frame, with $A, B = 1$. Also, the union of two orthonormal bases is a frame, with $A = B = 2$.

Notice that, in general, the elements in a frame are neither orthogonal nor independent.

Let us see next how the "synthesis" can be done for frames, that is, let us see how we can express $f = \sum_{n,k} c_{n,k}\, g_{n,k}$ with a canonical choice of coefficients $c_{n,k}$.

## Dual frame. Representations.

In general, given a subset $\{e_k\}_k \subseteq \mathcal{H}$, consider the operators:

$$c(f) = \{\langle f, e_k\rangle_k\}_k \quad \text{analysis } A(f)$$

$$D(c) = \sum_k c_k e_k \quad \text{for } c = \{c_k\}_k \quad \text{synthesis } S(c)$$

$$S(f) = \sum_k \langle f, e_k\rangle e_k \quad \underline{\text{frame operator}} \quad T(f)$$

<u>Proposition</u> Suppose that $\{e_k\}_k$ is a frame for $\mathcal{H}$ (with $A, B$ as in inequality (F)).

ⓐ $C : \mathcal{H} \longrightarrow \ell^2$ and it has closed range

ⓑ $D = C^*$, and consequently $D$ extends to a bounded operator from $\ell^2$ into $\mathcal{H}$ and

$$\left\| \sum_k c_k e_k \right\| \le \sqrt{B}\, \|c\|_2 \qquad c \in \ell^2$$

ⓒ $S = C^*C = DD^*$ maps $\mathcal{H}$ into $\mathcal{H}$ and is a positive invertible operator with

$$A \cdot I \le S \le B \cdot I$$

$$B^{-1} I \le S^{-1} \le A^{-1} I$$

<u>Note</u>: It can also be seen that the optimal frame bounds are $B = \|S\|$; $A = \|S^{-1}\|$.

Proof: (a) Clearly $C(f) \in \ell^2$, since

$$\sum_k |\langle f, e_k\rangle|^2 \leq B\|f\|^2 < +\infty$$

That $C$ has closed range is a consequence of the other inequality $\sqrt{A}\,\|f\| \leq \|C(f)\|$ : if $\{C(f_n)\}_n \to \alpha$ then the sequence $\{f_n\}_n$ is a Cauchy sequence and there exists $f = \lim_n f_n$ (in $\mathcal{H}$), which has necessarily $C(f) = \alpha$.

(b) Let $c = \{c_k\}_k$ be finite. Then

$$\langle C^*(c), f\rangle = \langle c, C(f)\rangle = \sum_k c_k \overline{\langle f, e_k\rangle} = \langle \sum_k c_k e_k, f\rangle$$

$$= \langle D(c), f\rangle .$$

Since $C$ is bounded and $\|C\| \leq \sqrt{B}$, it follows that $D = C^* : \ell^2 \longrightarrow \mathcal{H}$ is also bounded, with $\|D\| \leq \sqrt{B}$.

(c) By definition $S = C^*C = DD^*$ and therefore $S$ is self-adjoint and positive. Since $\langle Sf, f\rangle = \sum_k |\langle f, e_k\rangle|^2$, the frame inequality is precisely $A\cdot I \leq S \leq B\cdot I$. Also, $S$ invertible because $A > 0$. Inequalities are preserved under multiplication by positive commuting operators; therefore $AS^{-1} \leq SS^{-1} \leq BS^{-1}$.

<u>Note:</u> Given a frame $\{e_k\}_k$ and $f = \sum_k c_k e_k$, with $c = \{c_k\}_k \in \ell^2$, then $\forall \varepsilon > 0$ $\exists F_0$ finite such that

$$\left\| f - \sum_{k \in F} c_k e_k \right\| < \varepsilon \qquad \forall F \supseteq F_0 \text{ finite}$$

It is said that $\sum_k c_k e_k$ <u>converges unconditionally</u> to $f$. This is seen just by taking $F_0$ so that for $F \supseteq F_0$ $\quad \sum_{k \notin F} |c_j|^2 < \frac{\varepsilon}{\sqrt{B}}$.

<u>Corollary:</u> Let $\{e_k\}_k$ be a frame. Then $\{S^{-1} e_k\}_k$ is also a frame (called the dual frame), with constants $B^{-1}, A^{-1} > 0$, and

$$f = \sum_k \langle f, S^{-1} e_k \rangle e_k \qquad f = S(S^{-1} f) = \sum_m \langle S^{-1} f, e_m \rangle e_m$$

$$f = \sum_k \langle f, e_k \rangle S^{-1} e_k \qquad f = S^{-1}(Sf) = \sum_m \langle Sf, S^{-1} e_m \rangle S^{-1} e_m$$

Thus, we can represent $f$ through a frame $\{e_k\}_k$, but the coefficients are given by the dual frame (not the original frame). In case $\{e_k\}_k$ is an orthogonal basis, then $S = I$ and therefore the dual frame is just the original one.

- Now that we know that $\{M_k \tau_n g\}_{n,k}$, for $g$ Gaussian, is not an orthonormal basis, a natural question is:

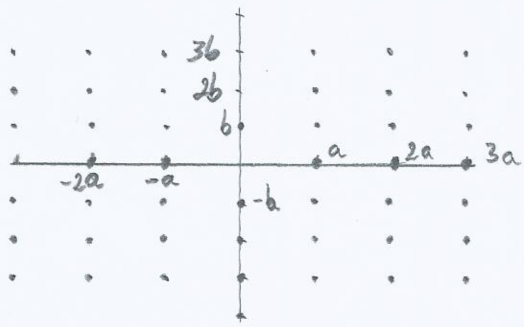
for what lattices $a\mathbb{Z} \times b\mathbb{Z}$ in the time-frequency plane the system $\{M_{bk} \tau_{an}\, g\}_{k,n}$ is a frame?

<u>Theorem</u> (Daubechies - Grossmann; Seip)
Let $g(t) = 2^{1/4} e^{-\pi t^2}$ be the (normalised) Gaussian.

The system $\{g_{n,k}\}_{n,k \in \mathbb{Z}}$, with $g_{n,k}(t) = M_{bk} \tau_{an}\, g(t)$, $a,b>0$

is a frame iff and only if $ab<1$.

<u>Note:</u> The condition $ab<1$ is a way to measure that there are enough samples of each $f \in L^2(\mathbb{R})$ taken by the frame.




The product $ab$ is the area of the lattice cell.
In a square $[0,N] \times [0,N]$ there are roughly $\frac{N}{a} \times \frac{N}{b}$ nodes of the lattice. In this sense, the "density" of the lattice (number of elements per area) is $\frac{1}{ab}$.

Actually more generally, a Gabor system $\{M_{y_k} \tau_{x_k}\, g\}_k$ is a frame if (letting $\lambda_k = x_k + i y_k$ and $\Lambda = \{\lambda_k\}_k$)

$$D(\Lambda) := \liminf_{r \to 0} \inf_{z \in \mathbb{C}} \frac{\# \Lambda \cap D(z,r)}{\pi r^2} > 1 .$$

The quantity $D(\Lambda)$ is called the <u>upper density</u> of $\Lambda$.

and it counts the infimum number of points of $\Lambda$ per unit of area. In accordance with our intuition, a sequence $\Lambda$ produces a frame if it is "dense enough".

<u>A final remark:</u> Frames are useful in many other situations. For example, Shannon's theorem for $L^2(\mathbb{R})$ functions with $\operatorname{supp} \hat{f} \subseteq [-\pi, \pi]$ says that

$$f(t) = \sum_{k \in \mathbb{Z}} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} |f(k)|^2 = \int_{\mathbb{R}} |f(t)|^2 \, dt.$$

We might want to ask whether we can have similar representations for other $\{t_k\}_k \subseteq \mathbb{R}$, in such a way that there is some stability of the norms. This leads to inequalities of the form (for $A, B > 0$),

$$A \|f\|^2 \leq \sum_k |f(t_k)|^2 \leq B \|f\|^2 \qquad f \in L^2(\mathbb{R})$$

Looking at the Fourier side, and letting $g = \hat{f}$, this is

$$A \|g\| \leq \sum_k |\langle g, \tfrac{1}{\sqrt{2\pi}} e^{-it_k \xi} \rangle|^2 \leq B \|g\|^2 \qquad g \in L^2(\mathbb{R})$$

This is precisely saying that $\left\{ \frac{1}{\sqrt{2\pi}} e^{-it_k \xi} \right\}_k$ is a frame for $L^2[-\pi, \pi]$. These are called Fourier frames.