1. Recall that for $f, g \in L^2(\mathbb{R})$ and $t, \xi \in \mathbb{R}$,

$$V_g f(t,\xi) = \int_{\mathbb{R}} f(s)\overline{g(s-t)}\, e^{-2\pi i \xi s} ds.$$

Prove:

(a) $V_g f(t,\xi) = e^{-2\pi i t\xi} V_{\widehat{g}} \hat{f}(\xi, -t)$.

(b) For $t, u, \xi, \omega \in \mathbb{R}$, $V_g(\tau_u M_\omega f)(t,\xi) = e^{-2\pi i u\xi} V_g f(t-u, \xi-\omega)$.

2. Consider the Hilbert space of entire functions

$$\mathcal{F} = \Big\{ f \in H(\mathbb{C}) : \|f\|^2 := \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \frac{dm(z)}{\pi} < +\infty \Big\}.$$

This is known as the *Bargmann-Fock space*. The evaluation funcionals $E_\lambda(f) = f(\lambda)$, $\lambda \in \mathbb{C}$, are bounded in $\mathcal{F}$, so by the Riesz representation theorem there exist $K_\lambda \in \mathcal{F}$ such that

$$E_\lambda(f) = f(\lambda) = \langle f, K_\lambda \rangle, \quad \forall f \in \mathcal{F}.$$

The function K_λ is called the *reproducing kernel* of $\mathcal{F}$ at λ.

(a) Prove that $K_\lambda(z) = \sum_{n\geq 0} \overline{e_n(\lambda)} e_n(z)$, where $\{e_n\}_{n\geq 0}$ is any orthonormal basis of $\mathcal{F}$.

(b) Prove that $\{z^n/\sqrt{n!}\}_{n\geq 0}$ is an orthonormal basis and deduce the value of $K_\lambda(z)$.

(c) Let $\Lambda = \{\lambda_k\}_{k\geq 1}$ be a discrete sequence in $\mathbb{C}$. Prove that the family of normalised reproduciong kernels $k_{\lambda_k} = K_{\lambda_k}/\|K_{\lambda_k}\|$, $k \geq 1$ is a frame for $\mathcal{F}$ if and only if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\geq 1} |f(\lambda_k)|^2 e^{-|\lambda_k|^2} \leq B\|f\|^2, \quad \forall f \in \mathcal{F}.$$

3. Let $\varphi(t) = 2^{1/4} e^{-\pi t^2}$ be the normalised Gaussian (so that $\|\varphi\|_2 = 1$) and let $f \in L^2(\mathbb{R})$.

(a) Letting $z = t + i\xi$, prove that

$$V_\varphi f(t, -\xi) = e^{\pi i t\xi} Bf(z) e^{-\frac{\pi}{2}|z|^2},$$

where

$$Bf(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2} dt$$

is the so-called *Bargmann transform* of f.

(b) Check that Bf is an entire function belonging to the Bargmann-Fock space

$$\mathcal{F}_\pi = \left\{ F \in H(\mathbb{C}) : \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) < +\infty \right\}$$

and that

$$\|f\|^2 = \int_{\mathbb{C}} |Bf(z)|^2 e^{-\pi|z|^2} dm(z).$$

Therefore B is, up to a constant, an isometry from $L^2(\mathbb{R})$ to $\mathcal{F}$.

4. Let $\varphi \in L^2(\mathbb{R})$ be defined by $\widehat{\varphi}(\xi) = \chi_{[-1/2,1/2)}(\xi)$.

   (a) Prove that φ is the scaling function of a MRA. The MRA obtained in this way is called the *Shannon* MRA, and it can be viewed as the Fourier counterpart of the Haar MRA. (Hint: consider the Nyquist-Shannon formula for various dyadic bandwidths.)

   (b) Determine the detail spaces $W_n = V_{n+1} \ominus V_n$, $n \in \mathbb{Z}$ and the associated wavelet ψ. (You can use the Note in Exercise 5, if you want).

5. (a) Let φ be the scaling function of a Multi-Resolution Analysis (MRA) $\{V_n\}_{n\in\mathbb{Z}}$. Prove that there exists a 1-periodic function $H(\xi)$ such that

   i) $\widehat{\varphi}(\xi) = H(\xi/2)\,\widehat{\varphi}(\xi/2)$.

   ii) If $\widehat{\varphi}$ is continuous at 0, then $\widehat{\varphi}(\xi) = \widehat{\varphi}(0)\prod_{k=1}^{\infty} H(\xi/2^k)$.

   (Hint: write φ in the orthonormal basis of V_1). The function H is called the *refinement mask* or low-pass filter associated to the MRA.

   (b) Prove the *quadratic mirror filter* (QMF) property: in case H in (a) is a trigonometric polynomial

   $$|H(\xi)|^2 + |H(\xi + 1/2)|^2 = 1.$$

   (Hint: use the Fourier characterising condition for scaling functions φ.)

   (c) Compute the refinement mask of the Shannon wavelet (see previous exercise).

   (d) Compute the refinement mask of the Haar MRA.

   Note: it can be proved that Mallat's wavelet ψ associated to the MRA can be defined through the Fourier identity $\widehat{\psi}(\xi) = e^{i\pi\xi}\,\overline{H(\xi/2 + 1/2)}\,\widehat{\varphi}(\xi/2)$.