Quantitative Finance Problem Set 3

Leonardo Bocchi

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1 Exercise 1

Obtain the following bounds for the call prices (C) and for the put ones (P) European (E) and American (A):

$$max(S_n - K, 0) \leq C_n(E) \leq C_n(A)$$
$$max(0, (1+r)^{-(N-n)}K - S_n) \leq P_n(E) \leq (1+r) - (N-n)K$$

Proof
To obtain the given bounds, we need to consider the following:

- For European options, the exercise can only be done at maturity, while for American options, the exercise can be done at any time before maturity.
- For a call option, the payoff is $\max(S_n - K, 0)$, where S_n is the stock price at maturity and K is the strike price. For a put option, the payoff is $\max(0, K - S_n)$.
- The price of an option is the expected value of its payoff discounted by the risk-free rate.

The payoff for a call option is given by $C_n = \max(S_n - K, 0)$. The value of a European call option at time n is given by the discounted expected payoff:

$$C_n(E) = e^{-r(N-n)}\mathbb{E}[\max(S_N - K, 0)|S_n]$$

For the call options, we have:

- The lower bound: The intrinsic value of a call option is the maximum of $(S_n - K)$ and 0. Therefore, the value of a European call option cannot be lower than its intrinsic value. Hence, we have
$$\max(S_n - K, 0) \leq C_n(E)$$
- The upper bound: Since an American call option can be exercised at any time before maturity, it is always worth at least as much as a European call option. Hence, we have
$$C_n(E) \leq C_n(A)$$

For the put options, we have:

- The lower bound: The intrinsic value of a put option is the maximum of $(K - S_n)$ and 0, discounted to the current time. Therefore, the value of a European put option cannot be lower than its intrinsic value. Hence, we have
$$\max(0, (1+r)^{-(N-n)}K - S_n) \leq P_n(E)$$
- The upper bound: Since an American put option can be exercised at any time before maturity, it is always worth at most $(1+r)^{-(N-n)}K$, which is the present value of the strike price. Hence, we have
$$P_n(A) \leq (1+r)^{-(N-n)}K$$
Note that we cannot compare $P_n(E)$ and $P_n(A)$ directly, as the principle of early exercise does not apply to European put options.

Therefore, the bounds for call options are

$$\max(S_n - K, 0) \leq C_n(E) \leq C_n(A)$$

and the bounds for put options are

$$\max(0, (1+r)^{-(N-n)}K - S_n) \leq P_n(E)$$

and

$$P_n(A) \leq (1+r)^{-(N-n)}K \quad \blacksquare$$

2 Exercise 2

Let $\{C_n^E\}_{n=0}^N$ be the price of a European option with payoff Z_N and let $\{Z_n\}_{n=0}^N$ be the payoffs of an American option. Show that if $C_n^E \geq Z_n$, $n = 0, 1, \ldots, N-1$, then $\{C_n^A\}_{n=0}^N$ (the prices of the American option) coincide with $\{C_n^E\}_{n=0}^N$.

Proof
Suppose $C_n^E \geq Z_n$ for all n. We want to show that the American and European option prices coincide, i.e., $C_n^A = C_n^E$ for all n.
Suppose for contradiction that $C_n^A > C_n^E$ for some n. Since $C_n^E \geq Z_n$, it follows that $C_n^A > Z_n$. Therefore, we can exercise the American option at time n and receive the payoff Z_n. However, we would be better off exercising the European option at time n and receiving the same payoff Z_n, since $C_n^E \geq Z_n$. This contradicts the assumption that $C_n^A > C_n^E$.
Now suppose for contradiction that $C_n^A < C_n^E$ for some n. We can sell the American option at time n for the price C_n^A, which is lower than the European option price C_n^E. We can then use the proceeds to buy the European option for the price C_n^E and immediately exercise it, receiving the same payoff Z_n. This gives us a profit of $Z_n - C_n^A > 0$, which contradicts the assumption that $C_n^A < C_n^E$.
Therefore, we must have $C_n^A = C_n^E$ for all n. $\blacksquare$

3 Exercise 3

Prove that, with the usual notations,

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}_{\mathbb{Q}} \left(\frac{(S_\tau - K)^+}{(1+r)^\tau} \right) = \mathbb{E}_{\mathbb{Q}} \left(\frac{(S_N - K)^+}{(1+r)^N} \right)$$

where $\mathbb{Q}$ is the risk-neutral probability of a complete market.

Proof
The fact that the left side of the identity is greater or equal to the right side of the equation is trivial.

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}_{\mathbb{Q}} \left(\frac{(S_\tau - K)^+}{(1+r)^\tau} \right) \geq \mathbb{E}_{\mathbb{Q}} \left(\frac{(S_N - K)^+}{(1+r)^N} \right)$$

Also, considering the Snell envelope V_n, we have that the value at any time $n \in [0, N]$ is greater or equal to the value at the following times.

$$V_n = max \left((S_n - K)^+, \frac{1}{1+r} \mathbb{E}_{\mathbb{Q}}(V_{n+1} | \mathcal{F}_n) \right)$$
$$V_n \geq V_\tau, \quad \text{with } \tau \geq n$$

This is true because an American option includes the possibility of exercising the option at any time before maturity.
However, we also have that in a complete market any derivative is replicable, and with respect to $\mathbb{Q}$ we have $\mathbb{E}_{\mathbb{Q}}(V_N | \mathcal{F}_n) = V_n$. By taking the expectation with respect to $\mathbb{Q}$ and applying the law of total expectation we obtain

$$\mathbb{E}_{\mathbb{Q}}(V_N) = \mathbb{E}_{\mathbb{Q}}(V_n)$$

Which is true for the given case of

$$\sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}_{\mathbb{Q}} \left(\frac{(S_\tau - K)^+}{(1+r)^\tau} \right) = \mathbb{E}_{\mathbb{Q}} \left(\frac{(S_N - K)^+}{(1+r)^N} \right) \quad \blacksquare$$

4 Exercise 4

Consider a market with N trading periods, a risky asset S and zero interest rate. In such a market we want to price an American option with payoffs $Z_n = d > 0$ if $n \leq N-1$ and $Z_N = S_N$ if $n = N$. Prove that its price is equal to that of a European call option on S, with strike d and maturity time $N-1$ plus the fixed amount d.

Proof
In order to show that the prices of the two derivatives are as given in the statement, we can use the definition of the cost of a derivative at time n, which is $C_n = \mathbb{E}_{\mathbb{Q}}[\text{Payoff}|\mathcal{F}_n]$. So, in the case of the European option, the cost at time $n \leq N-1$ will be

$$C_{E,n} = \mathbb{E}_{\mathbb{Q}}[(S_{N-1} - d)^+ | \mathcal{F}_n] = (S_n - d)^+$$

Where the second equality comes from the price of the risky asset S_n being a martingale under the risk-neutral probability measure $\mathbb{Q}$.
For the American option, we have that the payoff Z_n depends on the optimal execution time, which strictly depends on the value of S_N. In particular, we have

$$\text{if } S_N > d, \ S_N - d > 0, \text{ then } \ \tau^+ = N$$
$$\text{if } S_N < d, \ S_N - d < 0, \text{ then } \ \tau^+ = n, \text{ with } n \leq N-1$$

So the cost at time $n \leq N-1$ of the American option will be

$$\text{if } S_N > d, \ S_N - d > 0, \text{ then } \ C_{A,n} = \mathbb{E}_{\mathbb{Q}}[S_N|\mathcal{F}_n] = S_n$$
$$\text{if } S_N < d, \ S_N - d < 0, \text{ then } \ C_{A,n} = \mathbb{E}_{\mathbb{Q}}[d|\mathcal{F}_n] = d$$

Now, consider the state $\omega \in \Omega$ with $S_N - d < 0$. In this case, for the martingale property of the price of the risky asset, we have that $\mathbb{E}[(S_N - d)^+ | \mathcal{F}_{N-1}] = (S_{N-1} - d)^+$. But since we are considering the case $S_N - d < 0$, we have $(S_N - d)^+ = 0$ and thus

$$\mathbb{E}[(S_N - d)^+ | \mathcal{F}_N - 1] = 0 = (S_{N-1} - d)^+$$

Similarly, consider the state $\omega \in \Omega$ with $S_N - d > 0$. In this case, for the martingale property of the price of the risky asset, we have that $\mathbb{E}[(S_N - d)^+ | \mathcal{F}_n] = (S_n - d)^+$. But since we are considering the case $S_N - d > 0$, we have $(S_N - d)^+ \geq 0$ and thus

$$(S_N - d)^+ \geq 0, \quad (S_N - d)^+ = S_N - d$$

We can thus express explicitly the cost of the European option in the two cases considered as such

$$\text{if } S_N > d, \ S_N - d > 0, \text{ then } \ C_{E,n} = \mathbb{E}_{\mathbb{Q}}[(S_{N-1} - d)^+ | \mathcal{F}_n] = S_n - d$$
$$\text{if } S_N < d, \ S_N - d < 0, \text{ then } \ C_{E,n} = \mathbb{E}_{\mathbb{Q}}[(S_{N-1} - d)^+ | \mathcal{F}_n] = 0$$

Therefore, we have

$$\text{if } S_N > d, \ S_N - d > 0, \text{ then } \ C_{E,n} = S_n - d, \ C_{A,n} = S_n$$
$$\text{if } S_N < d, \ S_N - d < 0, \text{ then } \ C_{E,n} = 0, \ C_{A,n} = d$$

Which shows the relation between the prices of the two derivatives that was given in the statement, meaning

$$C_{A,n} = C_{E_n} + d, \text{ for } n \leq N-1 \quad \blacksquare$$