Advanced Mathematics for Scientific Challenges
Autumn 2022

EXERCISES 1.4

1. **(Optional)** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $-f$ is also a convex function. Prove that there exist $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $f(x) = a^T x + c$.

2. Use the Kuhn-Tucker conditions to solve the following problems

   (a)

$$\left\{ \begin{array}{c} \text{Minimize } f(x) = x_1 x_2 \\ \text{subject to} \\ x_1 + x_2 \geq 2 \\ x_2 \geq x_1 \end{array} \right.$$

   (b)

$$\left\{ \begin{array}{c} \text{Minimize } f(x) = (x_1 - 1)^2 + x_2 - 2 \\ \text{subject to} \\ x_2 - x_1 = 1 \\ x_1 + x_2 \leq 2 \end{array} \right.$$

   (c)

$$\left\{ \begin{array}{c} \text{Minimize } f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\ \text{subject to} \\ x_1 - x_2 - 2x_3 \leq 12 \\ x_1 + 2x_2 - 3x_3 \leq 8 \end{array} \right.$$

3. **(Optional)** Consider the problem

$$\left\{ \begin{array}{c} \text{Minimize } f(x) \\ \text{subject to} \\ g(x) \leq 0, \\ x \in S, \end{array} \right.$$

   where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are two convex functions and $S \subseteq \mathbb{R}^n$ is a convex set. If x^* is an optimal solution of this problem such that $g(x^*) < 0$, show that x^* is also an optimal solution of the problem

$$\left\{ \begin{array}{c} \text{Minimize } f(x) \\ \text{subject to} \\ x \in S. \end{array} \right.$$

Deadline: November 15, 23:59.