Simulation Methods

Computation of Lyapunov exponents

Joan Carles Tatjer

Departament de Matemàtiques i Informàtica

Universitat de Barcelona

Outline

1. Lyapunov exponents

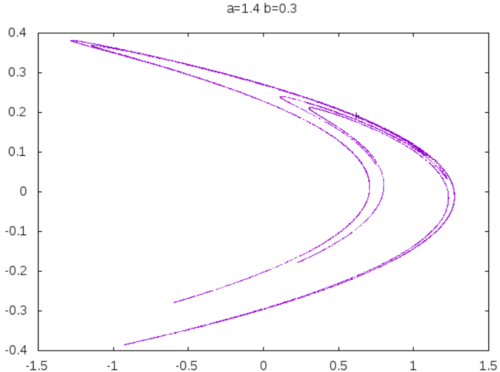
2. Numerical computation of Lyapunov exponents

Introduction I

Consider the map $H_{a,b}(x, y) = (1 + y - ax^2, bx)$. It is called the **Hénon map**.

If we take $a = 1.4$ and $b = 0.3$, and take an initial value (x_0, y_0) two things can happen (numerically):

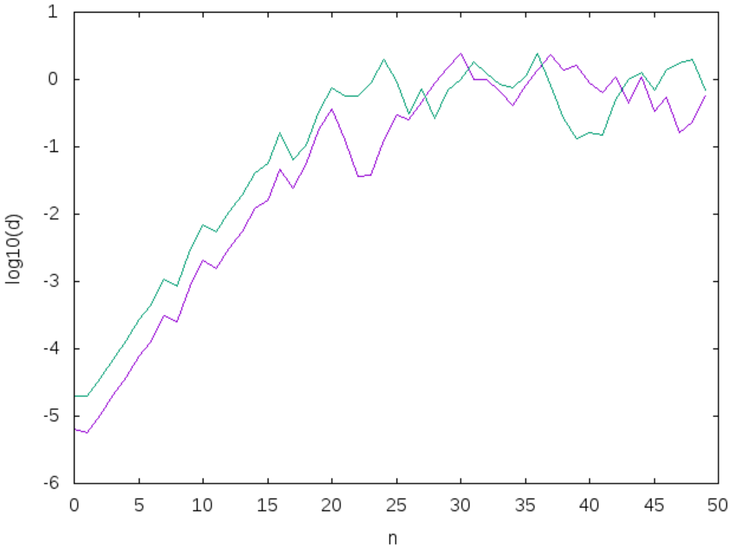
- $H^n_{a,b}(x_0, y_0)$ is not bounded when $n \to \infty$.
- $H^n_{a,b}(x_0, y_0)$ tends to a strange invariant set (Hénon attractor).



Experiment

- Take $(x_0, y_0) = (0, 0)$ and perform $n = 10^6$ iterates. We can suppose that $(x_1, y_1) = H^n_{a,b}(x_0, y_0)$ belongs to the Hénon attractor S.
- Take $(x_2, y_2) = H^n_{a,b}(x_1, y_1) \in S$. If $n = 128791$ then $\|(x_1, y_1) - (x_2, y_2)\|_2 \approx 1.466 \cdot 10^{-5}$, for $n = 128791$. If $n = 3068338$ then $\|(x_1, y_1) - (x_2, y_2)\|_2 \approx 2.824 \cdot 10^{-6}$.
- Perform 50 iterates of (x_1, y_1) in both cases. We have drawn the graph of $\log_{10}(d(n))$ in the next figure, where

$$d = d(n) = \|H^n_{a,b}((x_1, y_1)) - H^n_{a,b}((x_2, y_2))\|_2$$


1
0
-1
-2
-3
-4
-5
-6
log10(d)
0
5
10
15
20
25
30
35
40
45
50
n

Introduction II

Lyapunov charateristic exponents (LCE) of a trajectory of a dynamical system measure the mean exponential rate of divergence (convergence) of trajectories surronding it.
Consider a differential equation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n$$

with flow φ_t. If $\epsilon \in \mathbb{R}$, $\xi, v \in \mathbb{R}^n$ and $\eta = \xi + \epsilon v$ then

$$\varphi_t(\xi + \epsilon v) - \varphi_t(\xi) = \epsilon D\varphi_t(\xi)v + O(\epsilon^2).$$

We know that $D\varphi_t(\xi)$ is the principal fundamental matrix solution at $t = 0$ of

$$\dot{W} = Df(\varphi_t(\xi))W$$

along the solution of the original system starting at ξ.
If $\mathrm{Lip}(f)$ is the Lipschitz constant of f then using the Gronwall lemma

$$|\varphi_t(\xi + \epsilon v) - \varphi_t(\xi)| \leq \epsilon |v| e^{t\mathrm{Lip}(f)}.$$

This motivates the definition:

Definition

Suppose that $\xi, v \in \mathbb{R}^n$, $v \neq 0$, and $\varphi_t(\xi)$ is defined for all $t \geq 0$. The Lyapunov exponent at ξ in the direction of v for the flow φ_t is defined to be

$$\chi(\xi, v) = \limsup_{t\to\infty} \frac{1}{t} \ln\left(\frac{|D\varphi_t(\xi)v|}{|v|}\right).$$

Example

Given $a, b > 0$, consider the system

$$\left.\begin{array}{rcr} \dot{x} & = & -ax \\ \dot{y} & = & by \end{array}\right\}$$

If $v = (w, z)$ then

$$\chi(\xi, v) = \left\{\begin{array}{rll} b & \text{if} & z \neq 0 \\ -a & \text{if} & z = 0 \text{ and } w \neq 0 \end{array}\right.$$

Proof.

We have that $\varphi_t(\xi) = (e^{-at}\xi_1, e^{bt}\xi_2)$, where $\xi = (\xi_1, \xi_2)$ and

$$D\varphi_t(\xi)v = \begin{pmatrix} e^{-at} & 0 \\ 0 & e^{bt} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} e^{-at}w \\ e^{bt}z \end{pmatrix}.$$

$$|D\varphi_t(\xi)v| = \sqrt{e^{-2at}w^2 + e^{2bt}z^2}.$$

If $z \neq 0$:

$$|D\varphi_t(\xi)v| = e^{bt}|z|\sqrt{e^{-2(a+b)t}w^2z^{-2} + 1},$$

$$\lim_{t\to\infty} \frac{1}{t} \ln |D\varphi_t(\xi)v| = b + \lim_{t\to\infty} \frac{\ln|z|}{t} + \lim_{t\to\infty} \frac{1}{2t} \ln(e^{-2(a+b)t}w^2z^{-2} + 1) = b.$$

If $z = 0$, $w \neq 0$:

$$|D\varphi_t(\xi)v| = e^{-at}|w|,$$

$$\lim_{t\to\infty} \frac{1}{t} \ln |D\varphi_t(\xi)v| = -a.$$

$\square$

Proposition

Suppose, as in the definition, that $\xi \in \mathbb{R}^n$ *and* $\varphi_t(\xi)$ *is defined for all* $t \geq 0$. *If* $\omega(\xi)$ *is not an equilibrium point and there exists a compact subset* $K \subset \mathbb{R}^n$ *such that* $\varphi_t(\xi) \in K$ *for all* $t \geq 0$ *then*

$$\chi(\xi, f(\xi)) = 0.$$

Proof.

For all $s \geq 0$:

$$D\varphi_s(\xi)f(\xi) = D\varphi_s(\xi)\frac{d}{dt}\varphi_t(\xi)_{|t=0} =$$

$$\frac{d}{dt}(\varphi_s(\varphi_t(\xi))_{|t=0} = \frac{d}{dt}\varphi_{s+t}(\xi) = f(\varphi_s(\xi)).$$

As $\varphi_t(\xi)$ is bounded for $t \geq 0$ then $\exists\, M > 0$ s.t. $|D\varphi_t(\xi)f(\xi)| \leq M$. This implies that $\chi(\xi) \leq 0$.

Moreover, as $\omega(\xi)$ is not an equilibrium point then there exists a sequence $(t_k)_{k\geq 0} \to \infty$ s.t. $p = \lim_{k\to\infty} \varphi_{t_k}(\xi)$ is not an equilibrium point. Then $f(p) \neq 0$ and $\lim_{k\to\infty} \frac{1}{t_k} \ln\left(\frac{|D\varphi_{t_k}(\xi)f(\xi)|}{|f(\xi)|}\right) = 0 \leq \chi(\xi)$. □

Comment

If we impose that the positive orbit of ξ is contained in a compact set, it implies that the orbit is defined for all $t \geq 0$.

In the discrete case we have

Definition

Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a smooth n-dimensional map. Suppose that $\xi, v \in \mathbb{R}^n$, $v \neq 0$, and that $f^m(\xi)$ is defined for all $m \geq 0$. The Lyapunov exponent at ξ in the direction of v for the map f is defined to be

$$\chi(\xi, v) = \limsup_{m \to \infty} \frac{1}{m} \ln \left(\frac{|Df^m(\xi)v|}{|v|} \right).$$

Proposition

The Lyapunov exponent satisfies the following properties:

1. $\chi(\xi, v+w) \leq \max(\chi(\xi, v), \chi(\xi, w))$, *for all* $v, w \in \mathbb{R}^n$,
2. $\chi(\xi, cv) = \chi(\xi, v)$, *for all* $v \in \mathbb{R}^n$ *and any* $c \in \mathbb{R}$, $c \neq 0$.

Proof.

1. We use

$$\ln|Df^m(\xi)(v+w)| \leq \ln(2\max(|Df^m(\xi)v|, |Df^m(\xi)w|)) =$$

$$\ln 2 + \max(\ln(|Df^m(\xi)v|), \ln(|Df^m(\xi)w|)).$$

2. Immediate.

$\square$

Comment

Recall that given a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$,

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 0} \sup_{m\geq n} a_m,$$

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 0} \inf_{m\geq n} a_m = -\limsup_{n\to\infty}(-a_n).$$

These limits always exist but they can be $\pm\infty$. *Moreover:*

- $(a_n)_n$ *converges in* $\overline{\mathbb{R}}$ *iff* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.
- *If* $a_n \leq b_n$ *for all* $n \geq n_0$ *then* $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.
- $\limsup_{n\to\infty} a_n = \sup\{\xi \in \overline{\mathbb{R}} \mid \exists (a_{n_k})_k \to \xi\}$.
- $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, *if the right side of the inequality is defined.*
- $\limsup_{n\to\infty} \max(a_n, b_n) \leq \max(\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n)$.

Comment

From now on, we consider the discrete case. We can extend the definition of Lyapunov exponent to the case $v = 0$:

$$\chi(\xi, 0) = -\infty.$$

In order to have $\chi(\xi, v) < \infty$, *we will asumme* $\limsup_{m\to\infty} \frac{1}{m} \ln |Df^m(\xi)| < \infty$.

Comment

Suppose that a map has a compact invariant set that contains an orbit which is dense in the invariant set. The existence of a positive Lyapunov exponent for this orbit ensures that nearby orbits tend to separate exponentially fast from the dense set orbit. But as it is bounded, this suggests that each small neighbourhood in the invariant set undergoes both stretching an folding as it evolves. This complicated behaviour is often taken as a signature of chaos.

We can generalize de definition of Lyapunov exponent to a subspace:

Definition

Let $E^p \subset \mathbb{R}^n$ a subspace of dimension p and $v_1, \ldots, v_p$, p vectors generating E^p. Then

$$\chi(\xi, E^p) = \limsup_{m\to\infty} \frac{1}{m} \ln \mathrm{Vol}^p(Df^m(\xi)U),$$

where U is the parallelepiped generated by $v_1, \ldots, v_p$, is called the LCE of order p.

Proposition

The previous definition does not depend on the vectors v_i, $i = 1, \ldots, p$, generating E^p.

Proof.

We recall that if U is a parallelepiped generated by $v_1, \ldots, v_p$ and $A = (v_1 \cdots v_p)$ is a $n \times p$ matrix then $\mathrm{Vol}^p(U) = \sqrt{\det(A^T A)}$.
If $A = (v_1 \cdots v_p)$, $B = (w_1 \cdots w_p)$, such that $v_1, \ldots, v_p$ and $w_1, \ldots, w_p$ generate E^p, then $\exists$ a non-singular $p \times p$ matrix C such that $B = AC$.
Then, if U_1 is the parallelepiped generated by $w_1, \ldots, w_p$:

$$\mathrm{Vol}^p(Df^m(\xi)U_1) = \sqrt{\det(B^T Df^m(\xi)^T Df^m(\xi)B)} =$$

$$\sqrt{\det(C^T A^T Df^m(\xi)^T Df^m(\xi)AC)} = |\det(C)|\mathrm{Vol}^p(Df^m(\xi)U).$$

Then

$$\limsup_{m\to\infty} \frac{1}{m} \ln \mathrm{Vol}^p(Df^m(\xi)U) = \limsup_{m\to\infty} \frac{1}{m} \ln \mathrm{Vol}^p(Df^m(\xi)U_1).$$

$\square$

Example

Consider $f(x) = \Lambda x$, where $\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, and $|\lambda_1| \geq |\lambda_2| > |\lambda_3|$. Let E^2 be a subspace generated by the orthonormal vectors $q_1^T = (q_{11}, q_{21}, q_{31})$ and $q_2^T = (q_{12}, q_{22}, q_{32})$. Define $Q = (q_1\ q_2)$, $\tilde{Q} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, $\tilde{q}^T = (q_{31}, q_{32})$, $\tilde{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, and suppose that $\det \tilde{Q} \neq 0$. Then

Example

Consider $f(x) = \Lambda x$, where $\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, and $|\lambda_1| \geq |\lambda_2| > |\lambda_3|$. Let E^2 be a subspace generated by the orthonormal vectors $q_1^T = (q_{11}, q_{21}, q_{31})$ and $q_2^T = (q_{12}, q_{22}, q_{32})$. Define $Q = (q_1 \ q_2)$, $\tilde{Q} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, $\tilde{q}^T = (q_{31}, q_{32})$, $\tilde{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, and suppose that $\det \tilde{Q} \neq 0$. Then

$$\chi(\xi, E^p) = \limsup_{m\to\infty} \frac{1}{m} \ln\left(\sqrt{\det(\tilde{Q}^T\tilde{\Lambda}^{2m}\tilde{Q} + \lambda_3^{2m}\tilde{q}\tilde{q}^T)}\right) =$$

$$\limsup_{m\to\infty} \frac{1}{m} \ln\left(\sqrt{\det(\tilde{Q}^T\tilde{\Lambda}^{2m}\tilde{Q})\det(I + \lambda_3^{2m}\tilde{Q}^{-1}\tilde{\Lambda}^{-2m}(\tilde{Q}^T)^{-1}\tilde{q}\tilde{q}^T)}\right) =$$

$$\limsup_{m\to\infty} \frac{1}{m} \left(\ln(|\det(\tilde{Q}|)| + \ln(|\det \tilde{\Lambda}^m|)\right) = \ln(|\det(\tilde{\Lambda})|) = \ln|\lambda_1| + \ln|\lambda_2|.$$

Proposition

The following properties hold:

1. *If* $v, w \in \mathbb{R}^n$ *are such that* $\chi(\xi, v) \neq \chi(\xi, w)$ *then*

$$\chi(\xi, v + w) = \max(\chi(\xi, v), \chi(\xi, w)).$$

2. *If* $v_1, \ldots, v_m \in \mathbb{R}^n$ *and* $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$ *then*

$$\chi(\xi, \alpha_1 v_1 + \cdots + \alpha_m v_m) \leq \max\{\chi(\xi, v_i) \mid 1 \leq i \leq m\}.$$

   If, in addition, there exists i *such that* $\chi(\xi, v_i) > \chi(\xi, v_j)$*, for all* $j \neq i$ *then*

$$\chi(\xi, \alpha_1 v_1 + \cdots + \alpha_m v_m) = \chi(\xi, v_i).$$

3. *If for some* $v_1, \ldots, v_m \in \mathbb{R}^n \setminus \{0\}$ *the numbers* $\chi(\xi, v_1), \ldots, \chi(\xi, v_m)$ *are distinct, then* $v_1, \ldots, v_m$ *are linearly independent.*
4. *The function* $\chi(\xi, \cdot)$ *attains no more than* n *distinct finite values.*

Proof.

1. If $\chi(\xi, v) < \chi(\xi, w)$ then

$$\chi(\xi, v+w) \leq \chi(\xi, w) = \chi(\xi, v+w-v) \leq \max\{\chi(\xi, v+w), \chi(\xi, v)\}.$$

Then, $\chi(\xi, v+w) \geq \chi(\xi, v)$. If not, $\chi(\xi, w) \leq \chi(\xi, v)$, which contradicts the assumption. Then all the inequalities are equalities, and in particular,

$$\chi(\xi, v+w) = \chi(\xi, w).$$

2. It is consequence of 1. and the previous proposition on Lyapunov exponents.
3. Assume that $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$, with some $\alpha_i \neq 0$. Then

$$-\infty = \chi\left(\xi, \sum_{i=1}^{m} \alpha_i v_i\right) = \max\{\chi(\xi, v_i) \mid 1 \leq i \leq m \text{ and } \alpha_i \neq 0\} \neq \infty,$$

which yields a contradiction.

4. It holds since there are no more than n linearly independent vectors.

Let $\nu_1 > \nu_2 \cdots > \nu_s$ be the possible values of $\chi(\xi, v)$. We define $L_i = \{v \in \mathbb{R}^n : \chi(\xi, v) \leq \nu_i\}$.

Proposition

1. *The sets L_i are linear subspaces.*
2. *If $L_{s+1} = \{0\}$, then $L_{s+1} \subset \cdots \subset L_1 = \mathbb{R}^n$, with $L_{i+1} \neq L_i$.*
3. $\chi(\xi, v) = \nu_i$ *if* $v \in L_i \setminus L_{i+1}$.

Proof.

1. This comes from item 2. of the previous proposition.
2. It is obvious that $L_{i+1} \subset L_i$. Let v be a vector s.t. $\chi(\xi, v) = \nu_i$. Then $v \in L_i$ but $v \notin L_{i+1}$.
3. For any $v \in L_i \setminus L_{i+1}$ we have $\nu_{i+1} < \chi(\xi, v) \leq \nu_i$. Then $\chi(\xi, v) = \nu_i$.

□

We call the collection of subspaces $(L_i)_i$ a filtration of $\mathbb{R}^n$ and $k_i = \dim L_i - \dim L_{i+1}$ the multiplicity of ν_i.

Definition

Let $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$ s.t. $\chi(\xi, v_1) \geq \cdots \geq \chi(\xi, v_n)$. It is called an (ordered) normal basis (with respect to ξ) if

$$\sum_{i=1}^{n} \chi(\xi, v_i) \leq \sum_{i=1}^{n} \chi(\xi, w_i),$$

where $w_1, \ldots, w_n$ is any basis of $\mathbb{R}^n$.

Proposition

Let $v_1, \ldots, v_n$ an ordered normal basis and $w_1, \ldots, w_n$ a basis such that $w_{\sum_{j=1}^{i-1} k_j + 1}, \ldots, w_n$ is a basis of L_i, for $i = 2, \ldots, s$. Then $\chi(\xi, w_i) = \chi(\xi, v_i)$, $i = 1, \ldots, n$.

Proof.

Firstly, we see that $\chi(\xi, w_i) \leq \chi(\xi, v_i)$, since

$$\chi(\xi, w_i) = \nu_s \leq \chi(\xi, v_i), \qquad \sum_{j=1}^{s-1} k_j + 1 \leq i \leq n,$$

and, in general,

$$\chi(\xi, w_k) = \nu_i \leq \chi(\xi, v_k), \qquad \sum_{j=1}^{i-1} k_j + 1 \leq k \leq \sum_{j=1}^{i} k_j.$$

Finally, the normality of $v_1, \ldots, v_n$ implies that

$$\sum_{i=1}^{n} \chi(\xi, v_i) \leq \sum_{i=1}^{n} \chi(\xi, w_i) \leq \sum_{i=1}^{n} \chi(\xi, v_i).$$

Then, $\chi(\xi, v_i) = \chi(\xi, w_i)$, for all i. □

As a consequence of the previous proposition we have

Theorem

If $v_1, \ldots, v_n$ is an ordered normal basis then $v_{\sum_{j=0}^{i-1} k_j+1}, \ldots, v_n$ is a basis of L_i, $i = 2, \ldots, s$. Moreover if $w_1, \ldots, w_n$ is another ordered normal basis then $\chi(\xi, w_i) = \chi(\xi, v_i)$, $i = 1, \ldots, n$.

From this theorem we can define:

Definition

Let $v_1, \ldots, v_n$ be an orderered normal basis. We call $\chi_i(\xi) = \chi(\xi, v_i)$, $i = 1, \ldots, n$ the Lyapunov Characteristic Exponents (LCEs) of ξ, and the set of all LCEs the spectrum of $(Df^m(\xi))_m$.

Example

Consider, as before, $f(x) = \Lambda x$, $x \in \mathbb{R}^3$, such that $|\lambda_1| = |\lambda_2| > |\lambda_3|$. Take $v = (\alpha_1, \alpha_2, \alpha_3)$.

$$\chi(\xi, v) = \begin{cases} \ln|\lambda_1| & \text{if} \quad \alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0, \\ \ln|\lambda_3| & \text{if} \quad \alpha_1 = \alpha_2 = 0 \text{ and } \alpha_3 \neq 0 \end{cases}$$

Moreover, $L_1 = \mathbb{R}^3$, $L_2 = \langle e_3 \rangle$ and e_1, e_2, e_3 is a normal basis.

Indeed,

$$|Df^m(\xi)v| = |(\lambda_1^m \alpha_1, \lambda_2^m \alpha_2, \lambda_3^m \alpha_3)| = \sqrt{\lambda_1^{2m}\alpha_1^2 + \lambda_2^{2m}\alpha_2^2 + \lambda_3^{2m}\alpha_3^2}.$$

If $\alpha_1 = \alpha_2 = 0$ then

$$|Df^m(\xi)v| = |\lambda_3|^m |\alpha_3|.$$

If $\alpha_1^2 + \alpha_2^2 \neq 0$ then

$$|Df^m(\xi)v| = |\lambda_1|^m \sqrt{\alpha_1^2 + \alpha_2^2 + \lambda_3^{2m}\lambda_1^{-2m}\alpha_3^2}.$$

From this, we obtain the result.

Definition

The family $(Df^m(\xi))_m$ is called regular if all the mappings $Df^m(\xi)$ are invertible, $\lim_{m\to\infty}\frac{1}{m}\ln|\det Df^m(\xi)|$ exists, it is finite and there exists a basis $v_1,\ldots,v_n$ of $\mathbb{R}^n$ such that

$$\lim_{m\to\infty}\frac{1}{m}\ln|\det Df^m(\xi)| = \sum_{i=1}^{n}\chi(\xi,v_i).$$

Lemma

The basis of the previous definition is normal, and $\chi(\xi, v_i)$ is finite for all $1 \leq i \leq n$.

Proof: Let $w_1, \ldots, w_n$ a basis of $\mathbb{R}^n$. By the Hadamard's inequality:

$$\ln|\det Df^m(\xi)| \leq \sum_{i=1}^{n} \ln|Df^m(\xi)w_i| - \ln|\det(w_1, \ldots, w_n)|.$$

Then $\sum_{i=1}^{n} \chi(\xi, v_i) = \lim_{m\to\infty} \frac{1}{m} \ln|\det Df^m(\xi)| \leq \sum_{i=1}^{n} \chi(\xi, w_i)$.

Definition

When in the definition of LCEs $\limsup_{m\to\infty}$ can be replaced by $\lim_{m\to\infty}$ then we say that exact LCE exist.

Theorem

Let $(Df^m(\xi))_m$ *a regular family. Then*

1. *The exact LCEs of any order exist: In particular, for any* $0 \neq v \in \mathbb{R}^n$,

$$\chi(\xi, v) = \lim_{m\to\infty} \frac{1}{m} \ln |Df^m(\xi)v|.$$

2. *For any p-dimensional subspace* $E^p \subset \mathbb{R}^n$ *one has*

$$\chi(\xi, E^p) = \sum_{k=1}^{p} \chi_{i_k}(\xi),$$

   with a suitable sequence $1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq n$.

3. *For any p-dimensional subspace* $E^p \subset \mathbb{R}^n$ *one has* $\chi(\xi, E^p) = \min \sum_{i=1}^{p} \chi(\xi, w_i)$, *where the minimum is taken over all the bases* $w_1, \ldots, w_p$ *of* E^p.

Comment

If one could know a priori the filtration $(L_i)_{0\leq i\leq s}$ *($s \leq n$) and perform exact computations, then one would be able to estimate all LCEs, i.e.* $\nu_1,\ldots,\nu_n$*. If* $v \in L_i \setminus L_{i+1}$ *then*

$$\nu_i = \lim_{m\to\infty} \frac{1}{m} \ln |Df^m(\xi)v|, \quad 1 \leq i \leq s.$$

But we know that $\chi(\xi, \alpha_1 v_1 + \cdots + \alpha_n v_n) = \chi(\xi, v_i)$ *if* $\chi(\xi, v_i) > \chi(\xi, v_j)$ *for* $j \neq i$ *and* $\alpha_i \neq 0$*. Then if we take at random a vector* v *and* $v_1,\ldots,v_n$ *is a normal basis then* $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ *with* $\alpha_1 \neq 0$*, which implies that* $\chi(\xi, v) = \chi(\xi, v_1) = \nu_1$*.*

The key result for the numerical computation of the LCEs is

Theorem

Let $E^p \subset \mathbb{R}^n$ a vectorial subspace of dimension p. If for all j, $2 \leq j \leq s$,

$$\dim(E^p \cap L_j) = \max\left(0, p - \sum_{i=1}^{j-1} k_i\right), \quad \textit{(condition R)}$$

and $(Df^m(\xi))_m$ is a regular family then

$$\chi(\xi, E^p) = \sum_{i=1}^{p} \chi_i(\xi).$$

Proof.

Define $j_0 \geq 1$ such that $p \leq \sum_{i=1}^{j-1} k_i$ if $j > j_0$ and $p > \sum_{i=1}^{j-1} k_i$ if $j \leq j_0$. Then

$$\dim(E^p \cap L_j) = \begin{cases} 0 & \text{if} \quad j > j_0, \\ p - \sum_{i=1}^{j-1} k_i & \text{if} \quad j \leq j_0 \end{cases}$$

This implies that $\dim L_{j_0+1} = n - \sum_{i=1}^{j_0} k_i \leq n - p$. Then we can obtain a basis of E^p $v_1, \ldots, v_p$ s.t. $v_{\sum_{i=1}^{j-1} k_i+1}, \ldots, v_p$ is a basis of $E^p \cap L_j$, $j \leq j_0$. If we extend this basis to a basis of L_1, we have that $v_{p+1}, \ldots, v_n \in L_{j_0}$. Then $v_{\sum_{i=0}^{j-1} k_i+1}, \ldots, v_{p+1}, \ldots, v_n$ is a basis of L_j, $j \leq j_0$. As it is an ordered normal basis, applying item 3 of the theorem, we obtain the result. □

Comment

We note that a subspace taken at random satisfies condition R, since with probabilty 1, the dimension of the intersection of a subspace of dimension p with a subspace of dimension r is $\max(0, p + r - n)$. *In our case,* $r = \dim L_j = n - \sum_{i=1}^{j-1} k_i$.

Let $f : M \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 map such that M is compact, connected and with non-empty interior, and $f(M) \subset M$. Let μ be and invariant regular Borel measure, that is $\mu(A) = \mu(f^{-1}(A))$ for all borelian $A \subset M$ and $\mu(K) < \infty$ if K is compact.

Theorem (Oseledec)

There exists a measurable subset $M_1 \subset M$, $\mu(M_1) = 1$, such that for every $x \in M_1$ the family $(Df^m(x))_m$ is regular.

Now, applying the previous results, we have:

Theorem

$\exists$ measurable $M_1 \subset M$, $\mu(M_1) = 1$ such that, if $x \in M_1$, $1 \leq p \leq n$, and $v_1, \ldots, v_p \in \mathbb{R}^n$ satisfy condition R with respect to $(Df^m(x))_m$, one has

$$\lim_{m\to\infty} \frac{1}{m} Vol^p([Df^m(x)v_1, \ldots, Df^m(x)v_p]) = \sum_{i=1}^{p} \chi_i(x),$$

where $[Df^m(x)v_1, \ldots, Df^m(x)v_p]$ denotes the open parallelepiped generated by $Df^m(x)v_1, \ldots, Df^m(x)v_p$.

Numerical computation of Lyapunov exponents

Let $f : M \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $f(M) \subset M$. We use the following method: Given n initial vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ chosen at random, one has to evaluate:

$$\lim_{m\to\infty} \frac{1}{m} \ln \mathrm{Vol}^p([Df^m(x)v_1, \ldots, Df^m(x)v_p]) = \chi_1(x) + \cdots + \chi_p(x),$$

for $p = 1, \ldots, n$.

Comment

This procedure has two dificulties:

1. *If $\chi_1(x) > 0$ then $|Df^m(x)v|$ increases exponentially.*
2. *The angle between the directions of $|Df^m(x)v_1|$ and $|Df^m(x)v_2|$ in general becomes very small.*

Computation of the maximal Lyapunov exponent

We fix $s \in \mathbb{N}$ and a random $v \in \mathbb{R}^n$ with $|v| = 1$ and define recursively:

$$\begin{aligned} w_0 &= v, \\ \alpha_k &= |Df^s(f^{(k-1)s}(x))w_{k-1}|, \\ w_k &= \frac{Df^s(f^{(k-1)s}(x))w_{k-1}}{\alpha_k}, \end{aligned}$$

for $k \geq 1$. Then

$$|Df^{ks}(x)v| = |Df^s(f^{(k-1)s}(x))Df^s(f^{(k-2)s}(x)) \cdots Df^s(x)v| = \alpha_1 \cdots \alpha_k.$$

Indeed,

$$Df^{ks}(x)v = \alpha_1 \cdots \alpha_{k-1} Df^s(f^{(k-1)s}(x))w_{k-1}.$$

It is obviously true if $k = 1$ and if it is true for k then

$$Df^{(k+1)s}(x)v = Df^s(f^{ks}(x))Df^{ks}(x)v = \alpha_1 \cdots \alpha_{k-1}\alpha_k Df^s(f^{ks}(x))w_k.$$

Then taking norms we obtain the desired result.

Then $\chi_1 = \lim_{k\to\infty} \frac{1}{ks} \sum_{i=1}^{k} \ln \alpha_i$. Now, if s is not too large, as α_i are uniformly bounded, we can compute an approximation of this limit.

Comment (The power method)

Take $f(x) = Ax$, with eigenvalues λ_i, $i = 1, \dots, n$ such that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$. If we take $s = 1$:

$$\alpha_k = |Aw_{k-1}|, \quad w_k = \frac{Aw_{k-1}}{|Aw_{k-1}|}.$$

This means that when $k \to \infty$, w_k tends to an eigenvector of eigenvalue λ_1.

Quasi-periodically forced maps

Let $F(x,\theta) = (f(x,\theta), \theta + \omega \pmod{2\pi})$, where $f : \mathbb{R}^n \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^n$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $\omega/(2\pi)$ is an irrational constant. We say that $x = x(\theta)$ is an invariant curve if

$$x(\theta + \omega) = f(x(\theta), \theta),\ \forall\, \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

Note that $(x(\theta), \theta)$ is an invariant curve of F, that we call Γ. Let μ be the Lebesgue measure for $\mathbb{R}/2\pi\mathbb{Z}$. Let $C \subset \Gamma$ be a mesurable set. We define $\mu_\Gamma(C) = \mu(\Pi_2(C))$, where Π_2 is the projection of C on $\mathbb{R}/2\pi\mathbb{Z}$. Then μ_Γ is invariant and ergodic (the F-invariant sets have zero or full measure).

Computation of Lyapunov exponents for the case $n = 1$

We define $a(\theta) = D_1 f(x(\theta), \theta) \in \mathbb{R}$. Given $v = (1, 0) \in \mathbb{R}^2$ and $(x(\theta), \theta) \in \Gamma$ regular:

$$\chi((x(\theta), \theta), v) = \lim_{m \to \infty} \frac{1}{m} \ln |DF^n(x(\theta), \theta) v| =$$

$$\lim_{m \to \infty} \frac{1}{m} \ln |a(\theta + (m-1)\omega) a(\theta + (m-2)\omega) \cdots a(\theta)| =$$

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=0}^{m-1} \ln |a(\theta + i\omega)| = \frac{1}{2\pi} \int_0^{2\pi} \ln(|a(\theta)|)\, d\mu,$$

by the Birkhoff Ergodic Theorem.

If $v = (0, 1)$ then

$$\chi((x(\theta), \theta), v) = 0.$$

This means that the Lyapunov exponents are 0 and $\frac{1}{2\pi} \int_0^{2\pi} \ln(|a(\theta)|)\, d\mu$ for almost all θ. We can obtain approximations of the second Lyapunov exponents by using the trapezoidal rule.

The method for all exponents

Idea: In order to compute the kth order LCE, we replace at each step, the image of the vectors by an orthonormalized basis: Let $v_1, \ldots, v_p \in \mathbb{R}^n$ a collection of p orthonormal vectors. We define recursively:

$$\begin{aligned} w_0^{(j)} &= v_j, \quad j = 1, \ldots, p, \\ \beta_k^{(p)} &= \mathrm{Vol}^p([Df^s(f^{(k-1)s}(x))w_{k-1}^{(1)}, \ldots, Df^s(f^{(k-1)s}(x))w_{k-1}^{(p)}]), \end{aligned}$$

where $\{w_k^{(j)}\}_{1 \le j \le p}$ is an arbitrary orthonormalization of $\{Df^s(f^{(k-1)s}(x))w_{k-1}^{(j)}\}_{1 \le j \le p}$. Then, using the next lemma, we immediate prove the following theorem:

Theorem

$Vol^p([Df^{ks}(x)v_1, \ldots, Df^{ks}(x)v_p]) = \beta_1^{(p)} \cdots \beta_k^{(p)}$ *and*

$$\sum_{i=1}^{p} \chi_i(x) = \lim_{k \to \infty} \frac{1}{ks} \sum_{i=1}^{k} \ln \beta_i^{(p)}.$$

Lemma

If $A_k = (w_k^{(1)}, \dots, w_k^{(p)})$, *for all* $k \geq 1$, *then there exist* $p \times p$ *nonsingular matrices* B_k *such that*

1. $(Df^s(f^{(k-1)s}(x))A_{k-1} = A_k B_k$.
2. $\beta_k^{(p)} = |\det(B_k)|$.
3. $Df^{ks}(x)A_0 = A_k B_k B_{k-1} \cdots B_1$,

Proof:

1. The existence of B_k is due to the fact that $\{w_k^{(j)}\}_{1 \leq j \leq p}$ is an orthonormalization of $\{Df^s(f^{(k-1)s}(x))w_{k-1}^{(j)}\}_{1 \leq j \leq p}$.
2. $\beta_k^{(p)} = \sqrt{\det(B_k^T A_k^T A_k B_k)} = \sqrt{\det(B_k^T B_k)} = |\det(B_k)|$.
3. It is obviously true if $k = 1$ and if it is true for k then

$$Df^{(k+1)s}(x)A_0 = Df^s(f^{ks}(x))Df^{ks}(x)A_0 =$$

$$Df^s(f^{ks}(x))A_k B_k \cdots B_1 = A_{k+1} B_{k+1} \cdots B_k.$$

Comment

An efficient tool for the orthonomalization is the QR factorization. Indeed, if we call $Q_0 = A_0$, *then we can define* $A_k = Q_k$ *such that* Q_k *is an* $n \times p$ *matrix with orthonormal columns, and* $Df^s(f^{(k-1)s}(x))Q_{k-1} = Q_k R_k$, *where* $Q_k R_k$ *is the reduced QR factorization of* $Df^s(f^{(k-1)s}(x))Q_{k-1}$, *where* R_k *is a* $p \times p$ *upper triangular matrix. If* $R_k = (r_{ij}^{(k)})_{1 \le i,j \le p}$ *then* $\beta_k^{(p)} = |r_{11}^{(k)} \cdots r_{pp}^{(k)}|$. *Therefore*

$$\sum_{i=1}^{p} \chi_i(x) = \lim_{k \to \infty} \frac{1}{ks} \sum_{i=1}^{k} \sum_{j=1}^{p} \ln |r_{j,j}^{(i)}|.$$

Then

$$\chi_p(x) = \lim_{k \to \infty} \frac{1}{ks} \sum_{i=1}^{k} \ln |r_{p,p}^{(i)}|.$$

Computation of Lyapunov exponents of attractors

Let us suppose that $f : M \subset \mathbb{R}^n \longrightarrow f(M) \subset \mathbb{R}^n$ is a C^1 diffeomorphism such that M is a compact set such that $f(M) \subset M$. We define $\Lambda = \bigcap_{i \geq 0} f^m(M)$

Proposition

Suppose that there exists $b \in \mathbb{R}$ such that $0 < b < 1$ and $\det Df(x) = b$, for all $x \in M$. Then Λ is an invariant set ($f(\Lambda) = \Lambda$), and $Vol^n(\Lambda) = 0$.

Proof: The invariance of Λ is immediate, because $f^i(M) \subset f^{i-1}(M)$, for all $i \geq 0$.
On the other hand, if $|\det Df(x)| = b < 1$ for all $x \in M$, then, applying the theorem of change of variables for integrals we have $\mathrm{Vol}^n(f^m(M)) = b^m \mathrm{Vol}^n(M)$, which implies that $\mathrm{Vol}^n(\Lambda) = 0$.

Definition

In the hypotheses of the previous proposition, we say that f is a dissipative diffeomorphism, and that Λ is an attractor.

Comment

If Λ is a hyperbolic attracting periodic orbit then $\chi(x, v) < 0$ for all $x \in \Lambda$ and $v \in \mathbb{R}^n$. If there exists $x \in \Lambda$ and $v \in \mathbb{R}^n$ such that $\chi(x, v) > 0$ then Λ cannot be a periodic point. If the orbit of x is dense in Λ then we say that Λ is a strange attractor.

Example

Let $H_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$H_{a,b}(x, y) = (1 + y - ax^2, bx) \quad \text{(Hénon map)}.$$

We have $\det H_{a,b}(x, y) = -b$, for all $(x, y) \in \mathbb{R}^2$.

If we take $a = 1.4$ and $b = 0.3$ then

1. $H_{a,b}$ is dissipative.
2. $H_{a,b}$ has two saddle fixed points p_- and p_+.
3. $\Lambda = \overline{W^u(p_+)}$ is a compact invariant set of zero measure (Hénon attractor).

Computation of the Lyapunov exponents of Λ

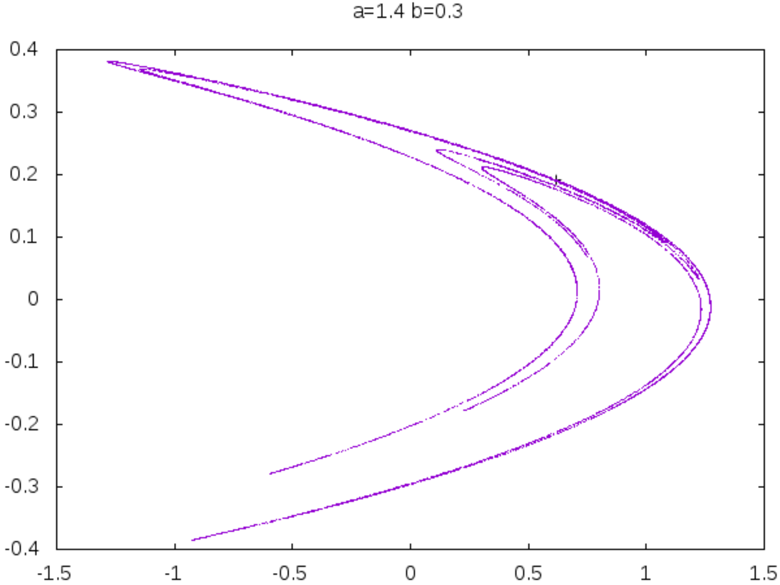
It is not known if Λ is a strange attractor. In order to compute the Lyapunov exponent we have to take a point in Λ. The algorithm is the following:

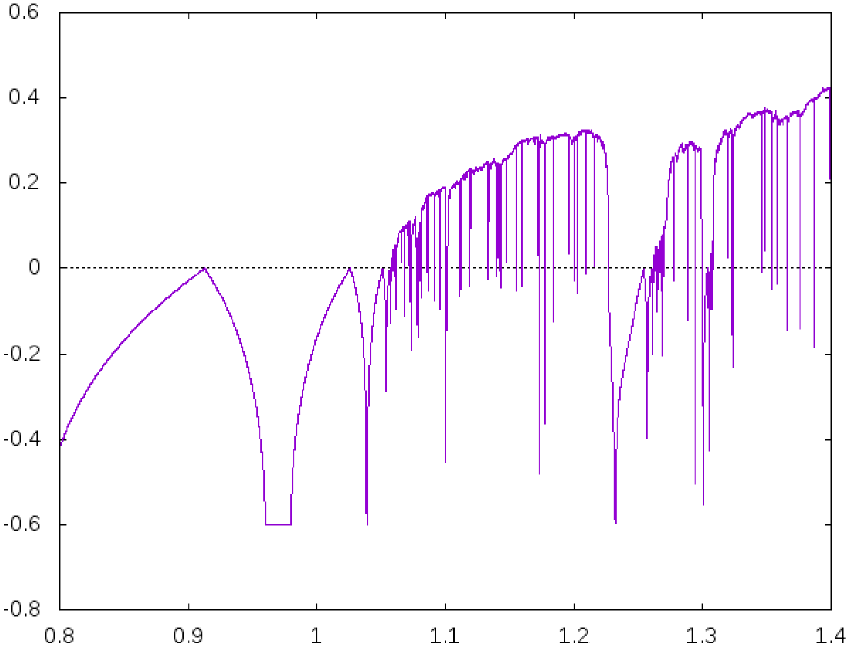
1. Take an initial point in the basin of attraction of Λ. For example $(x_0, y_0) = (0, 0)$
2. Compute $H^n_{a,b}(0, 0) = (x_n, y_n)$ for n large, for example $n = 10,000$. Then, we can suppose that $(x_n, y_n) \in \Lambda$.
3. Compute m iterates more, to see if they fill the attractor. For example $m = 10,000$.
4. Take $v \in \mathbb{R}^2$, choose $s \in \mathbb{N}$ and compute $\lambda_k = \frac{1}{ks} \log(|Df^{ks}(x_{n+m}, y_{n+m})v|)$, using the method of the slide 20, until $|\lambda_k - \lambda_{k-1}|$ is less than some threshold.
5. Repeat the computation for several (x, y), v and s.
6. We have obtained the maximal Lyapunov exponent χ_1. If we assume that $\chi_1 + \chi_2 = \ln |b|$ (see the theory), we have also the second Lyapunov exponent.

Comment

1. *The speed of convergence of the limit defining the Lyapunov exponents is slow in general.*
2. *The strange attractors for the Hénon map are not robust. This means that if H_{a_0,b_0} has a strange attractor, one can find parameters $(a_n, b_n) \to (a_0, b_0)$ such that H_{a_n,b_n} has an attracting periodic orbit. In practice this means that a numerically found strange attractor possibly is not a real strange attractor. For example, if $a = 1.4000000009849371$ and $b = 0.3000000019143266$ there exists an attracting periodic orbit of period 31.*
3. *However, if an approximate positive Lyapunov exponent exists, we can talk about 'practical' strange attractor.*

The Hénon attractor




0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0.8
0.9
1
1.1
1.2
1.3
1.4

Example

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\det Df(x,y,z) = c \ \forall (x,y,z) \in \mathbb{R}^3$. We want to compute the Lyapunov exponents of F at $\mathbf{x} = (x,y,z)$, where (x,y,z) is a regular point. We know that if $v \in \mathbb{R}^3 \setminus \{0\}$ and

$$\chi_1(\mathbf{x}) \geq \chi_2(\mathbf{x}) \geq \chi_3(\mathbf{x})$$

are the Lyapunov exponents of F at $\mathbf{x}$ then

$$\chi_1(\mathbf{x}) + \chi_2(\mathbf{x}) + \chi_3(\mathbf{x}) = \ln |c|.$$

Computation of the Lyapunov exponents:

Take at random two orthonormal vectors $v_1^{(0)}, v_2^{(0)} \in \mathbb{R}^3$. Then

- $A_0 = Q_0 = (v_1^{(0)}\ v_2^{(0)})$ is a 3×2 matrix with orthonormal columns,
- $A_1 = Df^s(\mathbf{x})Q_0 = (v_1^{(1)}\ v_2^{(1)})$.
- $Q_1 = (w_1^{(1)}\ w_2^{(1)})$ such that the vectors

$$w_1^{(1)} = \frac{v_1^{(1)}}{|v_1^{(1)}|}, \quad w_2^{(1)} = \frac{v_2^{(1)} - \langle w_1^{(1)}, v_2^{(1)}\rangle w_1^{(1)}}{|v_2^{(1)} - \langle w_1^{(1)}, v_2^{(1)}\rangle w_1^{(1)}|} \text{ are orthonormal.}$$

- $A_1 = Q_1 R_1$ where $R_1 = \begin{pmatrix} r_{11}^{(1)} & r_{12}^{(1)} \\ 0 & r_{22}^{(1)} \end{pmatrix}, \begin{array}{lcl} r_{11}^{(1)} & = & |v_1^{(1)}|, \\ r_{12}^{(1)} & = & \langle v_2^{(1)}, w_1^{(1)}\rangle, \\ r_{22}^{(1)} & = & \langle w_2^{(1)}, v_2^{(1)}\rangle \end{array}.$
- $\beta_1 = |r_{11}^{(1)} r_{22}^{(1)}|$, and $\alpha_1 = |r_{11}^{(1)}| = |Df^s(\mathbf{x})v_1^{(0)}|$.
- Replacing $\mathbf{x}$ by $f^s(\mathbf{x})$ and iterating the process we obtain that

$$\chi_1(\mathbf{x}) = \lim_{k\to\infty} \frac{1}{ks} \sum_{i=1}^{k} \ln |r_{11}^{(i)}|, \qquad \chi_2(\mathbf{x}) = \lim_{k\to\infty} \frac{1}{ks} \sum_{i=1}^{k} \ln |r_{22}^{(i)}|.$$

QR factorization: the Gram-Schmidt Algorithm

It is known that if $A = [a_1 \cdots a_m]$ is an $m \times n$ matrix $(m \geq n)$ $\exists$ $Q = [q_1 \cdots q_m]$ $m \times n$ with orthogonal columns and a triangular matrix $R = (r_{ij})_{1 \leq i,j \leq n}$ s.t. $A = QR$. If rang $A = n$ and diag $R > 0$ then the factorization is unique. Then

$$q_k = \frac{1}{r_{kk}} \left(a_k - \sum_{i=1}^{k-1} r_{ik} q_i \right), \quad 1 \leq k \leq n.$$

Therefore, for a fixed $k \leq n$

$$r_{jk} = a_k^T q_j, \quad j < k, \qquad r_{kk} = \left\| a_k - \sum_{i=1}^{k-1} r_{ik} q_i \right\|_2.$$

This is the **Gram-Schmidt algorithm**. At each step k it computes both the k columns of Q and R.

Warning. This algorithm is numerically unstable.

The Modified Gram-Schmidt Algorithm

1. First step: $r_{11} = \|a_1\|_2$, $q_1 = a_1/r_{11}$, $r_{1,j} = q_1^T a_j$, $j = 2, \ldots, n$.
2. k-th step: we know $q_1, \ldots, q_{k-1}$ and $(r_{ij})_{1\le i\le k-1, 1\le j\le n}$. Then

$$r_{kk} = \left\| a_k - \sum_{i=1}^{k-1} r_{ik} q_i \right\|_2, \qquad q_k = \frac{a_k - \sum_{i=1}^{k-1} r_{ik} q_i}{r_{kk}},$$

$$r_{k,j} = q_k^T a_j, \qquad k+1 \le j \le n.$$

Comment

In the modified Gram-Schmidt (MGS), the kth column of Q (q_k) and the kth row of R (r_k^T) are determined. From the numerical point of view MGS is more stable than the classical one.

For more information:

1. G. Benettin, L. Galgani, A. Giorgilli, J.-M. Strelcyn: *Lyapunov Characteristic Exponents for smooth dynamical systems and for hamiltonian systems; a method for computing all of them. Part 1: Theory*, Meccanica **15**, 9-20(1980).
2. G. Benettin, L. Galgani, A. Giorgilli, J.-M. Strelcyn: *Lyapunov Characteristic Exponents for smooth dynamical systems and for hamiltonian systems; A method for computing all of them. Part 2: Numerical application*, Meccanica **15**, 21-30(1980).