Lesson 3

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Theorem (FFTAP)

A financial market is viable (free of arbitrage opportunities) if and only if there exists $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$ *such that the discounted prices of the stocks* $((\tilde{S}_n^j)_{0\leq n\leq N}\ , j = 1, ..., d)$ *are* $\mathbb{P}^*$*-martingales.*

In the previous lesson we gave a proof of the "only if" part based in the asumption that

$$L \cap L_+^0 = \{0\}$$

where

$$L = \{X, X = \tilde{V}_N(\phi), (\phi_n^i)_{1 \le i \le d, 1 \le n \le N} \text{ predictable}, \tilde{V}_N = \sum_{j=1}^{N} \phi_j \cdot \Delta \tilde{S}_j\}.$$

But the no arbitrage condition is

$$\Lambda \cap L_+^0 = \{0\}, \tag{1}$$

where $\Lambda = \{X, X = \tilde{V}_N(\phi), (\phi_n)_{1 \le n \le N}$ predictable **and admissible**: $\tilde{V}_n(\phi) = \sum_{j=1}^n \phi_j \cdot \Delta \tilde{S}_j,\ \tilde{V}_n(\phi) \ge 0$ for all $1 \le n \le N\}$. We are going to see that

$$\Lambda \cap L_+^0 = \{0\} \Rightarrow L \cap L_+^0 = \{0\}$$

We have the following lemma.

Lemma

The class of **admissible** *trading strategies contains no arbitrage opportunities if and only if the class of* **self-financing** *strategies contains no arbitrage opportunities.*

Proof.

Assume that $L \cap L^0_+ \neq \{0\}$ then there exists φ predictable $V_0 = 0$ and $\tilde{V}_N(\varphi) \in L^0_+ \setminus \{0\}$. Define

$$n = \inf\{j, \tilde{V}_k(\varphi)(\omega) \geq 0 \text{ for all } k > j \text{ and } \omega \in \Omega\},$$

note that $n \leq N - 1$ since $\tilde{V}_N(\varphi) \geq 0$. Let $B = \left\{ \tilde{V}_n(\varphi) < 0 \right\}$, define the predictable vector process such that for all $i = 1, ..., d$

$$\theta^i_j = \begin{cases} 0 & j \leq n \\ \mathbf{1}_B \varphi^i_j & j > n \end{cases}$$

$\square$

Proof.

Then, $\tilde{V}_k(\theta) = 0$, for all $0 \leq k \leq n$ and for all $k > n$

$$\tilde{V}_k(\theta) = \sum_{j=n+1}^{k} \mathbf{1}_B \varphi_j \cdot \Delta \tilde{S}_j = \mathbf{1}_B \left(\sum_{j=1}^{k} \varphi_j \cdot \Delta \tilde{S}_j - \sum_{j=1}^{n} \varphi_j \cdot \Delta \tilde{S}_j \right)$$
$$= \mathbf{1}_B \left(\tilde{V}_k(\varphi) - \tilde{V}_n(\varphi) \right) \geq 0,$$

so θ is admissible and we have that $\tilde{V}_N(\theta) = \mathbf{1}_B \left(\tilde{V}_N(\varphi) - \tilde{V}_n(\varphi) \right) > 0$ in B. So $\tilde{V}_N(\theta) \in \Lambda \cap L^0_+ \setminus \{0\}$ and consequently $\Lambda \cap L^0_+ \neq \{0\}$ contradicting the no-arbitrage condition (1). $\square$

Remark

$\mathbb{P}^*$ *is named martingale measure or risk-neutral probability. Notice that the discounted values of self-financing portfolios are* $\mathbb{P}^*$*-martingales.*

We define a *European option, derivative or contingent claim* as a contract with *maturity* N and with a *payoff* $h \geq 0$, where h is $\mathcal{F}_N$-measurable.
For instance a Call is a European option with payoff $h = (S_N^1 - K)_+$, for a Put $h = (K - S_N^1)_+$, and an *Asian option* is a European one! with
$h = \left(\frac{1}{N} \sum_{j=0}^{N} S_j^1 - K \right)_+$

Put-Call parity condition

Since

$$(S_N - K)_+ - (K - S_N)_+ = S_N - K,$$

the value of a portfolio, at any time n, with a long position in a Call (that we buy, say, by C_n) and a short position in a Put (that we sell by, say, P_n) with the same strike, has to be the same as the price to buy the stock S_n and lend $\frac{K}{(1+r)^{N-n}}$ units of money.That is

$$C_n - P_n = S_n - \frac{K}{(1+r)^{N-n}}.$$

Definition

A derivative defined by h is said to be replicable if there exists an admissible strategy ϕ such that replicates h that is $V_N(\phi) = h$.

Proposition

If ϕ is a self-financing strategy that replicates h and the market is viable then it is admissible.

Proof.

$\tilde{V}_N(\phi) = \tilde{h}$ and since there exists $\mathbb{P}^*$ such that $\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_N(\phi)|\mathcal{F}_n) = \tilde{V}_n(\phi)$, we have $\tilde{V}_n(\phi) \geq 0$. $\square$

Definition

A market is said to be complete if any derivative is replicable.