Poincaré maps

Let X be a vector field in $M \subset \mathbb{R}^n$ (open set)

Diff. eq.: $\quad x' = X(x)$

Def. A hypersurface $\Sigma \subset \mathbb{R}^n$ is called a transversal section of X if

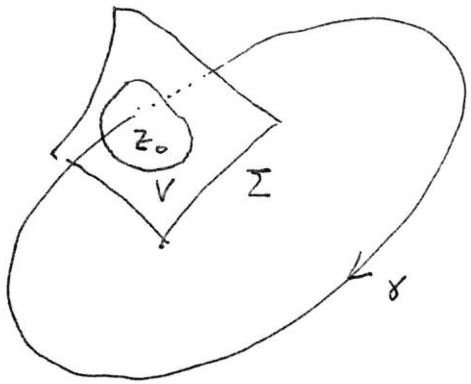
$$\langle X(z) \rangle \oplus T_z \Sigma = \mathbb{R}^n, \qquad \forall z \in \Sigma$$

Let γ be a P.O. of period T of $x' = X(x)$, φ be the flow of $x' = X(x)$

Σ a transversal section

$z_0 \in \gamma \cap \Sigma$

There $\exists$ a map



$$P : V \subset \Sigma \longrightarrow \Sigma$$
$$z \longmapsto \varphi(\tau(z), z)$$

where

- V is a nbh of z_0 in Σ
- τ is a C^0 function s.t. $\tau(z_0) = T$

  $\varphi(\tau(z), z) \in \Sigma$

$\boxed{\begin{array}{l}\underline{\text{Prop}} \quad \text{If } X \in C^r \text{ then } P \text{ is a diffeomorphism} \\ \quad\quad \text{of class } C^r\end{array}}$

It follows from:

<u>Prop</u> Let α be an orbit, Σ_1 transversal section at $\alpha(t_1)$

Σ_2 " at $\alpha(t_2)$

(both of class C^r).

Then $\exists$ V nbh of $z_1 = \alpha(t_1)$ in Σ_1 and

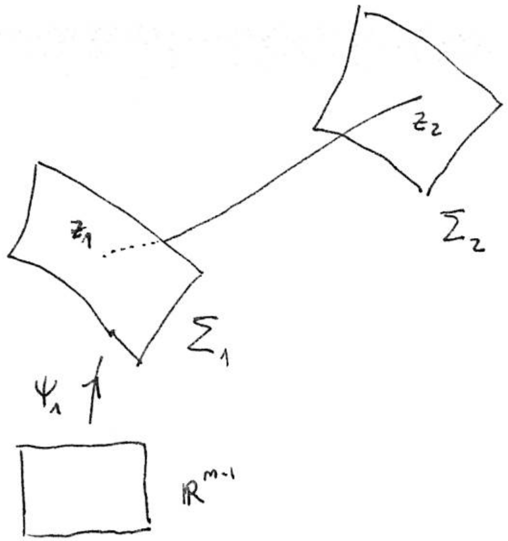
$$\begin{aligned} P_{12} : V \subset \Sigma_1 &\longrightarrow \Sigma_2 \\ z &\longmapsto \varphi(\tau(z), z) \end{aligned}$$

where τ is C^0, $\tau(z_1) = t_2 - t_1$, $\varphi(\tau(z), z) \in \Sigma_2$

Moreover P_{12} is a local diffeo of class C^r.

Proof. We take coordinates in Σ_1

$$\begin{array}{ccccc} z_1 & & \Sigma_1 & \longrightarrow & \Sigma_2 \\ \uparrow & \psi_1 & \uparrow & & \\ x_1 & & V_0 \subset \mathbb{R}^{m-1} & & \end{array}$$



and write

$$P_{12}(x) = \varphi(\bar{\tau}(x), \psi_1(x)), \qquad \bar{\tau}(x) = \tau(\psi_1(x))$$

$$\bar{\tau}(x_1) = t_2 - t_1$$

$$\varphi(\bar{\tau}(x), \psi_1(x)) \in \Sigma_2$$

We represent Σ_2 as $\Phi(\cancel{z}) = 0$ (near z_2), $\Phi \in C^r$

We have $T_{z_2}\Sigma_2 = \operatorname{Ker} D\Phi(z_2)$

Transversal. cond $\Rightarrow$ $X(z_2) \notin T_{z_2}\Sigma_2$

$\Rightarrow$ $X(z_2) \notin \operatorname{Ker} D\Phi(z_2)$

The condition for $\bar{\tau}$ can be written as

$$\Phi\left(\varphi(\bar{\tau}(x), \Psi_1(x))\right) = 0$$

To get the $\exists$ and regularity of $\bar{\tau}$ we apply the IFT to

$$F(\tau, x) = \Phi\left(\varphi(\tau, \Psi_1(x))\right)$$

- $F \in C^r$
- $F(t_2 - t_1, x_1) = \Phi\left(\varphi(t_2 - t_1, z_1)\right) = \Phi(z_2) = 0$
- $\frac{\partial F}{\partial \tau}(t_2 - t_1, x_1) = D\Phi\left(\varphi(t_2 - t_1, z_1)\right) \varphi'(t_2 - t_1, z_1)$

$$= D\Phi(z_2)\, X\big(\underbrace{\varphi(t_2 - t_1, z_1)}_{\overset{\shortparallel}{z_2}}\big) \neq 0$$

Then $\exists!\ \bar{\tau}(x)$ defined in a nbh of x_1 and

$$\bar{\tau} \in C^r$$

$$\Phi\left(\varphi(\bar{\tau}(x), \Psi_1(x))\right) = 0$$

Derivative of the Poincaré map

$P : \Sigma \longrightarrow \Sigma, \qquad P(p) = p$

Let $\Psi : U \subset \mathbb{R}^{n-1} \longrightarrow \Sigma$ be a parameterization of Σ such that $\Psi(0) = p$

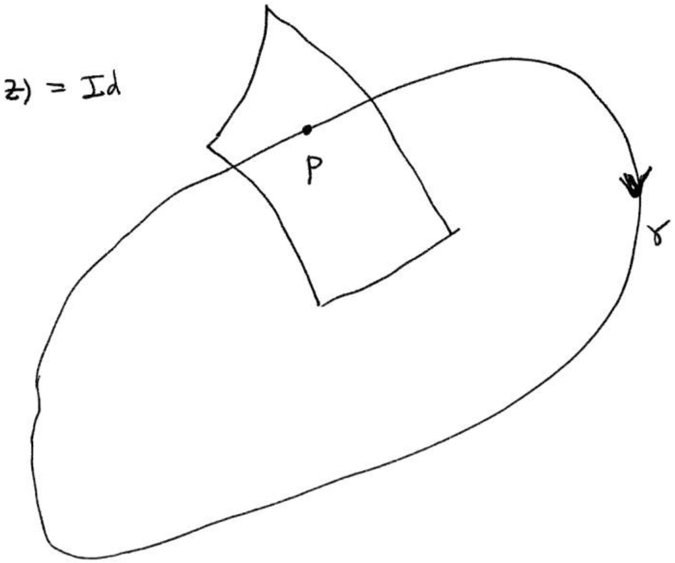
$D\Psi(0)\mathbb{R}^{n-1}$ is the tangent space $T_p\Sigma$

The columns of $D\Psi(0)$, say $v_1, \dots, v_{n-1}$, form a basis of $T_p\Sigma$. $\quad v_j = D\Psi(0)\, e_j$

Transversality $\Longrightarrow$ $\{X(p), v_1, \dots, v_{n-1}\}$ is a basis of $\mathbb{R}^n$

$D_z\varphi(t, z)$ satisfies the variational eq

$$D_z\varphi(t,z)' = DX(\varphi(t,z))\, D_z\varphi(t,z), \qquad D_z\varphi(0,z) = Id$$



(a) $\boxed{D_z\varphi(T,p)\,X(p) = X(p)} \Rightarrow 1 \in \operatorname{spec} D_z\varphi(T,p)$

Follows from $\varphi(t+T, p) = \varphi(T, \varphi(t,p))$

Differentiating w.r.t. t

$$D_t\varphi(t+T, p) = D_z\varphi(T, \varphi(t,p))\, D_t\varphi(t,p)$$

$t=0$

$$X(\underbrace{\varphi(T,p)}_{\substack{\parallel \\ p}}) = D_z\varphi(T,p)\, X(\underbrace{\varphi(0,p)}_{\substack{\parallel \\ p}})$$

(b) By definition

$$P(\Psi(x)) = \varphi(\overline{\tau}(x), \Psi(x))$$

Differentiating

$$DP(\Psi(x))\, D\Psi(x) = D_t\varphi(\overline{\tau}(x), \Psi(x))\, D\overline{\tau}(x) + D_z\varphi(\overline{\tau}(x), \Psi(x))\, D\Psi(x)$$

Evaluating at $x=0$

$$DP(p)\, D\Psi(0) = X(\underbrace{\varphi(T,p)}_{= p})\, D\overline{\tau}(0) + D_z\varphi(T,p)\, D\Psi(0)$$

$$DP(p) = \begin{pmatrix} \beta_{11} & & \beta_{1,m-1} \\ & & \\ \beta_{m-1,1} & & \beta_{m-1,m-1} \end{pmatrix}$$

$$DP(p)\, v_j = DP(p)\, D\Psi(0)\, e_j = X(p)\, \underbrace{D\overline{\tau}(0)\, e_j}_{\alpha_j} + D_z\varphi(T,p)\, v_j \longrightarrow D_z\varphi(T,p)\, v_j = -\alpha_j X(p) + \sum_{i=1}^{m-1} \beta_{ij} v_i$$

We can write the matrix of $D_z\varphi(T,p)$ with respect to the basis $(X(p), v_1, \dots, v_{m-1})$ $\searrow$

$$D_z\varphi(T,p) = \begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{m-1} \\ 0 & & & & \\ 0 & & & DP(p) & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

$$\boxed{\text{Corollary: The eigenvalues of } DP(p) \text{ are the eigenvalues of } D_z\varphi(T,p) \text{ except one } 1}$$

About $D_z\varphi(T,p)$

In general one has to integrate the variational equation numerically:

$$D_z\varphi(t,p)' = DX(\varphi(t,p))\, D_z\varphi(t,p), \qquad D_z\varphi(0,p) = Id$$

It is a T-periodic linear eq.

By Floquet theory any fundamenal solution is of the form

$$P(t)\, e^{Bt}, \qquad P(t) \text{ T-periodic}$$

In particular $\quad D_z\varphi(t,p) = P(t)\, e^{Bt}$ for some P, B

Since at $t=0$ $D_z\varphi(0,p) = Id$ we have $P(0)e^{B\cdot 0} = Id \implies P(0) = P(T) = Id$

Then

$$D_z\varphi(T,p) = e^{BT} = C \qquad \text{the monodromy matrix}$$

Computation of $DP(p)$ when P is the Poincaré map associated to a P.O. in the plane

We have that $D_z\varphi(t,z_0)$ satisfies the V.E. $\left(D_z\varphi(t,z_0)\right)' = DX(\varphi(t,z_0))\,D_z\varphi(t,z_0)$, $D_z\varphi(0,z_0)=Id$

We recall Liouville's theorem: If ϕ is a fundamental solution of $x' = A(t)x$,

$$(\det \phi(t))' = \operatorname{tr} A(t) \cdot \det \phi(t)$$

and therefore

$$\det \phi(t) = \det \phi(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(s)\, ds$$

If p belongs to a P.O. of $x' = X(x)$ in the plane with $p = \gamma(0)$ then

$$\det D_z\varphi(t,p) = \det Id \exp \int_0^t \operatorname{tr} DX(\gamma(s))\, ds$$

and then

$$DP(p) = \det D_z\varphi(T,p) = \exp \int_0^T \operatorname{tr} DX(\gamma(s))\, ds \qquad (T = \text{period})$$