Lesson 5

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American options

An American option is a derivative that can be exercised at *any* time between 0 and N, and consequently it has an associated payoff sequence defined by an $(\mathcal{F}_n)$-adapted positive process $(Z_n)_{0\leq n\leq N}$ to indicate the immediate payoff when it is exercised at time n. In case of an American call $Z_n := (S_n - K)_+$ and in the case of an American put $Z_n := (K - S_n)_+$.
To obtain the price, U_n, at time n, *we assume that the market is complete*. If the payoff is Z_τ at the random time τ and this random time is the time when the option is exercised then its price at time $n \leq \tau$ is

$$U_n = (1+r)^n \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau | \mathcal{F}_n).$$

In fact we can always assume that the payoff is paid at time N by taking into account the interest rates, that is

$$Z_\tau(1+r)^{N-\tau},$$

then we can use the pricing formula for the European options

$$U_n = (1+r)^n \mathbb{E}_{\mathbb{P}^*}\left(\left. \frac{Z_\tau(1+r)^{N-\tau}}{(1+r)^N} \right| \mathcal{F}_n \right) = (1+r)^n \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau | \mathcal{F}_n).$$

If the option can be exercised at any time we will have that, to cover any potential payoff, the discounted price should be

$$\tilde{U}_n = \sup_{\tau \geq n} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau | \mathcal{F}_n).$$

In particular

$$U_0 = \sup_{\tau} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau).$$

If the random time τ takes the value n, that is the exercice time is n, then this event, $\{\tau = n\}$, has to be part of the information at n, in other words $\{\tau = n\} \in \mathcal{F}_n$. By this reason we introduce the following definition.

Definition

A random variable ν taking values in $\{0, 1, ..., N\}$ is a *stopping*, or Markov, time if

$$\{\nu = n\} \in \mathcal{F}_n, \quad 0 \leq n \leq N$$

Remark

Equivalently ν *is a stopping time if* $\{\nu \leq n\} \in \mathcal{F}_n, \quad 0 \leq n \leq N$, *definition that can be extended to the continuous time setting. We write* $\tau_{0,N}$ *for the set of stopping times and we denote* $\tau_{n,N}$ *the set of stopping times with values in* $\{n, n+1, ..., N\}$.

Then the initial price of an American option with payoff $(Z_n)_{0\leq n\leq N}$ will be given by

$$U_0 = \sup_{\tau\in\tau_{0,N}} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau).$$

Then one goal will be to find $\nu \in \tau_{0,N}$ such that

$$\sup_{\tau\in\tau_{0,N}} \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\tau) = \mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_\nu).$$

These times will called *optimal stopping times.* In order to characterize the optimal stopping times we need to introduce some definitions and to obtain some results.

Let $(X_n)_{0\leq n\leq N}$ be an adapted stochastic process and ν a stopping time, then we define

$$X_n^{\nu} := X_{n\wedge\nu},\ 0\leq n\leq N.$$

Note that

$$X_n^{\nu}(\omega) = \begin{cases} X_n(\omega) & \text{si } n\leq \nu(\omega) \\ X_{\nu(\omega)}(\omega) & \text{si } n > \nu(\omega) \end{cases}$$

Proposition

Let $(X_n)_{0\leq n\leq N}$ be adapted, then $(X_n^\nu)_{0\leq n\leq N}$ is adapted and if $(X_n)_{0\leq n\leq N}$ is a (sup, super) martingale, then $(X_n^\nu)_{0\leq n\leq N}$ is a (sub, super) martingale.

Proof.

$$\begin{aligned} X_n^\nu = X_{n\wedge\nu} &= X_0 + \sum_{j=1}^{n\wedge\nu}(X_j - X_{j-1}) \\ &= X_0 + \sum_{j=1}^{n} \mathbf{1}_{\{j\leq\nu\}}(X_j - X_{j-1}), \end{aligned}$$

but $\{j \leq \nu\} = \overline{\{\nu \leq j-1\}} \in \mathcal{F}_{j-1}$, since ν is a stopping time, and consequently $\mathbf{1}_{\{j\leq\nu\}}$ is $\mathcal{F}_{j-1}$-measurable and the sequence $(\phi_j)_{1\leq j\leq N}$ con $\phi_j = \mathbf{1}_{\{j\leq\nu\}}$ is predictable. Obviously X_n^ν es $\mathcal{F}_n$-measurable and $\square$

Proof.

$$\mathbb{E}(X^{\nu}_{n+1} - X^{\nu}_{n}|\mathcal{F}_n) = \mathbb{E}(\mathbf{1}_{\{n+1\leq\nu\}}(X_{n+1} - X_n)|\mathcal{F}_n)$$

$$= \mathbf{1}_{\{n+1\leq\nu\}}\mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) \lesseqgtr 0 \text{ if } (X_n) \text{ is } \begin{array}{l}\text{super}\\ \text{martingala}\\ \text{sub}\end{array}$$

$\square$

The Snell envelope

Let $(Y_n)_{0\leq n\leq N}$ be an adapted process (to $(\mathcal{F}_n)_{0\leq n\leq N}$), define

$$\begin{aligned} X_N &= Y_N \\ X_n &= \max(Y_n, E(X_{n+1}|\mathcal{F}_n)), \quad 0 \leq n \leq N-1, \end{aligned}$$

we say that $(X_n)_{0\leq n\leq N}$ is the *Snell envelope* of $(Y_n)_{0\leq n\leq N}$.

Proposition

The process $(X_n)_{0\le n\le N}$ is the smallest supermartingale that dominates the process $(Y_n)_{0\le n\le N}$.

Proof.

$(X_n)_{0\le n\le N}$ is adapted and by construction

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) \le X_n.$$

Let $(T_n)_{0\le n\le N}$ be another supermartingale that dominates $(Y_n)_{0\le n\le N}$, then $T_N \ge Y_N = X_N$. Assume that $T_{n+1} \ge X_{n+1}$. Then, by the monotonicity of the expectation and since $(T_n)_{0\le n\le N}$ is a supermartingale:

$$T_n \ge \mathbb{E}(T_{n+1}|\mathcal{F}_n) \ge \mathbb{E}(X_{n+1}|\mathcal{F}_n),$$

moreover $(T_n)_{0\le n\le N}$ dominates $(Y_n)_{0\le n\le N}$, so $T_n \ge Y_n$

$$T_n \ge \max(Y_n, \mathbb{E}(X_{n+1}|\mathcal{F}_n)) = X_n$$

Remark

Fixed ω if X_m is strictly greater than Y_m for all $m \leq n$, $X_m(\omega) = \mathbb{E}(X_{m+1}|\mathcal{F}_m)(\omega)$ so X_m behaves, until this n, as a martingale, this indicates that if we "stop" the supermartingale X properly we can obtain a martingale.

Proposition

The random variable

$$\nu = \inf\{n \geq 0, X_n = Y_n\}$$

is a stopping time and $(X_n^\nu)_{0 \leq n \leq N}$ is a martingale.

Proof.

$$\{\nu = n\} = \{X_0 > Y_0\} \cap \ldots \cap \{X_{n-1} > Y_{n-1}\} \cap \{X_n = Y_n\} \in \mathcal{F}_n.$$

And

$$X_n^\nu = X_0 + \sum_{j=1}^{n} \mathbf{1}_{\{j \le \nu\}} (X_j - X_{j-1}).$$

Therefore

$$X_{n+1}^\nu - X_n^\nu = \mathbf{1}_{\{n+1 \le \nu\}} (X_{n+1} - X_n),$$

but on the set $\{n+1 \le \nu\}$ we have that $X_n > Y_n$ so $X_n = \mathbb{E}(X_{n+1}|\mathcal{F}_n)$ on this set, and we can write

$$X_{n+1}^\nu - X_n^\nu = \mathbf{1}_{\{n+1 \le \nu\}} (X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n)).$$

$\square$

Proof.

Consequently

$$\begin{aligned}
\mathbb{E}(X_{n+1}^{\nu} - X_n^{\nu}|\mathcal{F}_n) &= \mathbb{E}\left(\mathbf{1}_{\{n+1\leq\nu\}}\left(X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n)\right)\middle|\,\mathcal{F}_n\right) \\
&= \mathbf{1}_{\{n+1\leq\nu\}}\mathbb{E}\left(X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n)\middle|\,\mathcal{F}_n\right) \\
&= 0.
\end{aligned}$$

$\square$

The following corollary establishes that ν is an optimal stopping time.

Corollary

$$X_0 = \mathbb{E}(Y_\nu|\mathcal{F}_0) = \sup_{\tau\in\tau_{0,N}} \mathbb{E}(Y_\tau|\mathcal{F}_0)$$

Proof.

$(X_n^\nu)_{0\leq n\leq N}$ is a martingale and consequently

$$\begin{aligned} X_0 &= \mathbb{E}(X_N^\nu|\mathcal{F}_0) = \mathbb{E}(X_{N\wedge\nu}|\mathcal{F}_0) \\ &= \mathbb{E}(X_\nu|\mathcal{F}_0) = \mathbb{E}(Y_\nu|\mathcal{F}_0). \end{aligned}$$

On the other hand $(X_n)_{0\leq n\leq N}$ is supermartingale and then $(X_n^\tau)_{0\leq n\leq N}$ as well for all $\tau \in \tau_{0,N}$, so

$$X_0 \geq \mathbb{E}(X_N^\tau|\mathcal{F}_0) = \mathbb{E}(X_\tau|\mathcal{F}_0) \geq \mathbb{E}(Y_\tau|\mathcal{F}_0),$$

therefore

$$\mathbb{E}(Y_\nu|\mathcal{F}_0) \geq \mathbb{E}(Y_\tau|\mathcal{F}_0), \quad \forall\tau \in \tau_{0,N}.$$

Remark

Analogously we could prove

$$X_n = \mathbb{E}(Y_{\nu_n}|\mathcal{F}_n) = \sup_{\tau \in \tau_{n,N}} \mathbb{E}(Y_\tau|\mathcal{F}_n),$$

where

$$\nu_n = \inf\{j \geq n, X_j = Y_j\}.$$

We say that ν_n are optimal stopping times.

Another expression for the price of American options

Now we can apply these results to give a more explicit expression for the price of American options. In fact if

- We take as probability, the risk-neutral one, $\mathbb{P}^*$,
- The process $(Y_n)_{0\le n\le N}$ as $(\tilde{Z}_n)_{0\le n\le N}$, and
- $$\nu_n := \inf\{j \ge n, X_j = \tilde{Z}_j\},$$

we will have that

$$X_n = \mathbb{E}_{\mathbb{P}^*}\left(\tilde{Z}_{\nu_n}|\mathcal{F}_n\right) = \sup_{\tau\in\tau_{n,N}} \mathbb{E}_{\mathbb{P}^*}\left(\tilde{Z}_\tau|\mathcal{F}_n\right) = \tilde{U}_n,$$

and the discounted prices of the American options, $\left(\tilde{U}_n\right)_{0\leq n\leq N}$, will be given by $(X_n)_{0\leq n\leq N}$, in other words, $\left(\tilde{U}_n\right)_{0\leq n\leq N}$ is the Snell envelope of the discounted payoffs $\left(\tilde{Z}_n\right)_{0\leq n\leq N}$, and we can write

$$\begin{cases} U_N = Z_N \\ U_n = \max(Z_n, S_n^0 \mathbb{E}_{\mathbb{P}^*}(\tilde{U}_{n+1}|\mathcal{F}_n)) & \text{if } 0 \leq n \leq N-1. \end{cases}$$

This formula for the price can be obtained, as well, by doing a backward induction. Define $U_N = Z_N$. At time $N-1$, owners of the option can choose between receiving Z_{N-1} or the *equivalent* amount to Z_N at time $N-1$ that is the amount that replicates Z_N at $N-1$ given by $S^0_{N-1}\mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_N|\mathcal{F}_{N-1})$. Obviously they will choose the maximum of the two values, so we have

$$U_{N-1} = \max(Z_{N-1}, S^0_{N-1}\mathbb{E}_{\mathbb{P}^*}(\tilde{Z}_N|\mathcal{F}_{N-1}))$$

and, by backward induction,

$$U_n = \max(Z_n, S^0_n\mathbb{E}_{\mathbb{P}^*}(\tilde{U}_{n+1}|\mathcal{F}_n)),\ 0 \leq n \leq N-1$$