Very short review on O.D.E.

An ode is a (formal) expression $\dot{x} = f(t, x)$, $\dot{x} = x' = \frac{dx}{dt}$

with $f : U \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and U open set.

A solution is a function $\varphi : I \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ (I interval) s.t.

$$(t, \varphi(t)) \in U, \qquad \forall t \in I$$
$$\dot{\varphi}(t) = f(t, \varphi(t)), \qquad \forall t \in I$$

Given $\dot{x} = f(t,x)$ and $(t_0, x_0) \in U$, the initial value problem (I.V.P) or Cauchy problem consists ~~in~~ finding a solution φ of the equation s.t. $\varphi(t_0) = x_0$.

Symbolically we represent the I.V.P. by

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

Peano theorem

If f is continuous then $\forall (t_0, x_0) \in M$ $\exists$ a solution φ such that $\varphi(t_0) = x_0$

Remark The solution may not be unique

Picard - Lindelöf theorem

If f is continuous and locally Lipschitz w.r.t. x (uniformly in t) $\exists$ a unique (local) solution of the I.V.P. $\dot{x} = f(t,x)$, $x(t_0) = x_0$.

Maximal solutions

Assume that $\forall (t_0, x_0) \in M$ the I.V.P. has a unique solution

We say that $\varphi : J \subset \mathbb{R} \to \mathbb{R}^n$ is a maximal solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$

if any other solution $\psi : I \subset \mathbb{R} \to \mathbb{R}^n$ satisfies $I \subset J$.

Under the previous conditions of uniqueness, $\forall (t_0, x_0) \in M$ $\exists !$ maximal solution.

We denote it by

$$\varphi(t, t_0, x_0)$$

We denote its domain (for t) by (ω_-, ω_+). Of course ω_-, ω_+ depend on (t_0, x_0)

the equations may depend on parameters $(\lambda_1, .., \lambda_p) = \lambda \in \mathbb{R}^p$. We write $\boxed{\dot{x} = f(t, x, \lambda)}$

with $f: \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$, $\mathcal{U}$ open set.

The maximal solutions are given by $\varphi(t, t_0, x_0, \lambda)$, where φ is defined on

$D \subset \mathbb{R} \times \mathcal{U}$. Actually $D = \bigcup_{(t_0, x_0, \lambda) \in \mathcal{U}} (\omega_-(t_0, x_0, \lambda), \omega_+(t_0, x_0, \lambda)) \times \{(t_0, x_0, \lambda)\}$

<u>Theorem</u> Assume $f \in C^r(\mathcal{U})$. Then

(1) D is an open set

(2) $\varphi \in C^r(D)$ and $\dot{\varphi} \in C^r(D)$

<u>Properties of maximal solutions</u>

(1) If $\omega_+ < \infty$ then $(t, \varphi(t, t_0, x_0))$ leaves any compact set contained in $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$

(2) Assume $\mathcal{U} = (a, \infty) \times V$, $V \subset \mathbb{R}^n$ open set.

If $\{\varphi(t, t_0, x_0) \mid t_0 \leq t < \omega_+\} \subset K \subset V$, K compact set, then $\omega_+ = \infty$

(3) Assume $\mathcal{U} = \mathbb{R} \times \mathbb{R}^n$ and $\| f(t, x) - f(t, y) \| \leq L \|x - y\|$, $\forall t, x, y$, then $(\omega_-, \omega_+) = \mathbb{R}$

In particular, for $\dot{x} = A(t)\,x$ with $A : \mathbb{R} \longrightarrow \mathbb{R}^{n \times m}$ continuous

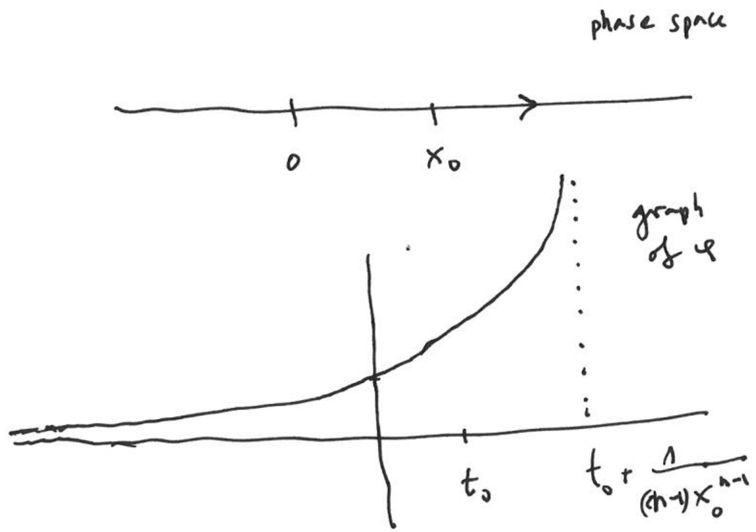
the solutions are defined for all $t \in \mathbb{R}$

<u>Example</u>

$$\dot{x} = x^m \,, \qquad m \in \mathbb{N}, \quad m > 1$$

The solution of the I.V.P. is

$$\varphi(t, t_0, x_0) = \frac{x_0}{\left[1 + (1-m)\, x_0^{n-1} (t - t_0)\right]^{1/m-1}}$$



If $x_0 > 0$ $\qquad \omega_- = -\infty, \qquad \omega_+ = t_0 + \dfrac{1}{(m-1)\, x_0^{m-1}}$

If $x_0 < 0$ depends on whether m is even or odd.

Equilibrium points (also called fixed points or singular points)

x_0 is an equilibrium point if $f(t, x_0) = 0 \quad \forall t$

Then $\psi(t) = x_0$ is a solution : $\dot{\psi}(t) = f(t, \psi(t))$

Periodic solutions

A non-constant solution

$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ is a periodic solution if

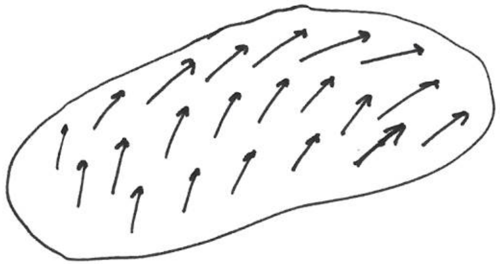
$\exists\, T > 0$ s.t. $\gamma(t+T) = \gamma(t), \quad \forall t$

The infimum of such T is called period of γ.

Autonomous equations

Are eq. of the form $\dot{x} = f(x)$. Sometimes we write $\dot{x} = X(x)$

X is called vector field



We assume we have uniqueness of solutions

In this case $\varphi(t, t_0, x_0) = \varphi(t - t_0, 0, x_0)$

We simply write $\varphi(t, x)$

Flow property

(1) $\varphi(0, x) = x$

(2) $\varphi(t, \varphi(s, x)) = \varphi(t+s, x)$, if $t, t+s \in (\omega_-(x), \omega_+(x))$

Orbit

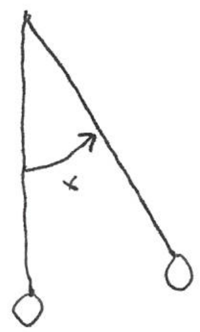
If $x \in M$, the orbit of x is $O(x) = \{\varphi(t, x) \mid t \in (\omega_-, \omega_+)\} \subset \mathbb{R}^n$

$\dot{x} = x^n$ has ∞ many solutions but only 3 orbits

Phase space : domain of f, M

Phase portrait : M together with the set of orbits

Ex: The mathematical pendulum

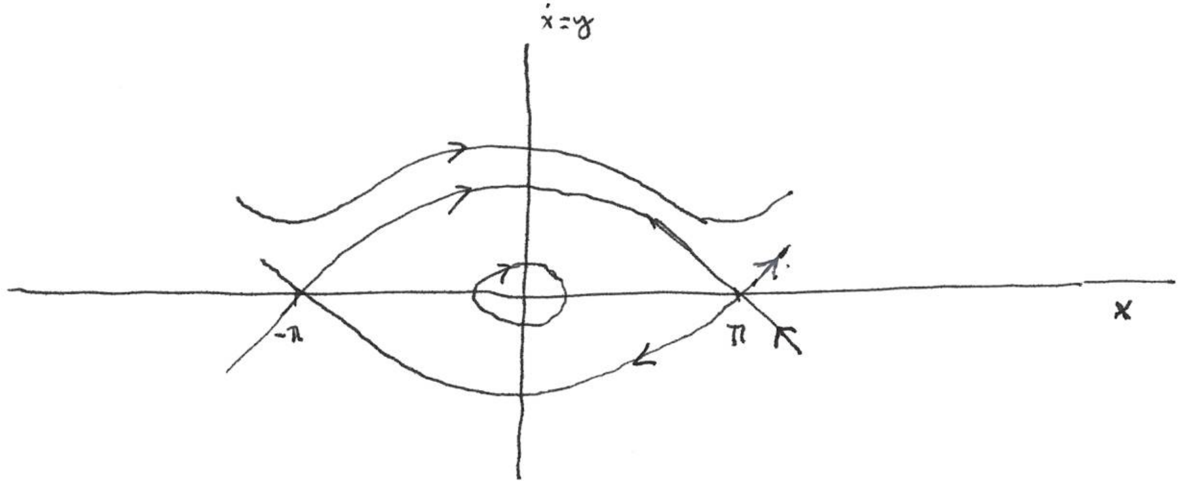


$$\ddot{x} + \sin x = 0$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$$

Fixed points: $\begin{cases} y = 0 \\ \sin x = 0 \end{cases}$

$\rightarrow (0, k\pi)$, $k \in \mathbb{Z}$



First integral

$$H(x,y) = \frac{1}{2} y^2 - (\cos x - 1)$$

Stability of equilibrium points

$\dot{x} = f(t,x)$, $f(t, x_0) = 0$

$f : [a,\infty) \times V \subset \mathbb{R}^{1+m} \longrightarrow \mathbb{R}^m$

- x_0 is stable if

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \text{s.t. if } \| x - x_0 \| < \delta \;\; \left\{ \begin{array}{l} \varphi(t, t_0, x) \text{ is defined } \forall t \geq t_0 \\ \\ \| \varphi(t, t_0, x) - x_0 \| < \varepsilon , \;\; \forall t \geq t_0 \end{array} \right.$$

- x_0 is asymptotically stable if it is stable and

$$\exists \eta > 0 \;\; \text{s.t. if } \| x - x_0 \| < \eta , \qquad \lim_{t \to \infty} \varphi(t, t_0, x) = x_0$$

(for some t_0)

- x_0 is unstable if it is not stable

Stability for linear equations with ctant coefficients

In this case we have a complete characterization of stability

First we recall some basic properties concerning linear systems

Consider $\dot{x} = Ax$

The general solution is $\varphi(t,x) = e^{At}x$, $e^{At} = I + At + 1/2\, t^2 A^2 + \dots$

the origin is an equilibrium point. If $\det A \neq 0$ it is the unique equilibrium point

A matrix function ϕ is called fundamental solution of $\dot{x} = Ax$

if $\dot{\phi}(t) = A\phi(t)$, and $\det \phi(t) \neq 0$

$\boxed{\begin{array}{l}\text{Given } A,\ \forall \mu > \max\{\operatorname{Re}\lambda \mid \lambda \in \operatorname{spec} A\} \text{ there exists } K \geq 1 \text{ s.t.} \\[1em] \qquad \| e^{At} \| \leq K e^{\mu t}, \quad \forall t \geq 0 \\[1em] \text{Moreover if } \mu = \max\{\operatorname{Re}\lambda \mid \lambda \in \operatorname{spec} A\} \text{ and the Jordan boxes} \\ \text{associated to all } \lambda_0 \text{ such that } \operatorname{Re}\lambda_0 = \mu \text{ diagonalize then} \\[1em] \qquad \| e^{At} \| \leq K e^{\mu t}, \quad \forall t \geq 0\end{array}}$

Ex. $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e^{At} = \mathbf{I} + At = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$,

$$\left\| \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\| = \sup_{\|v\| = 1} \left\| \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 1 \\ t \end{pmatrix} \right\| = \sqrt{1 + t^2} \quad \text{in Euclidean norm}$$

Prop Given A, $\forall \mu > \max\{\mathrm{Re}\lambda \mid \lambda \in \mathrm{spec}\, A\}$

$\exists K \geq 1$ s.t.

$\|e^{At}\| \leq K e^{\mu t}, \forall t \geq 0$

We write A in Jordan form $B = C^{-1}AC$, where C is the matrix of the change of basis. We have

$$B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_m \end{pmatrix} = \mathrm{diag}\,(B_1, \dots, B_m),$$

where B_j is either of the form

$$\lambda I = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \text{ or } \lambda I + N = \begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$ (we use the complex Jordan form)

since $B^2 = \begin{pmatrix} B_1^2 & & & \\ & B_2^2 & & \\ & & \ddots & \\ & & & B_m^2 \end{pmatrix}$ etc we have $\exp(Bt) = \begin{pmatrix} e^{B_1 t} & & \\ & \ddots & \\ & & e^{B_m t} \end{pmatrix}$

and moreover

$$e^{Bt} = C^{-1} e^{At} C$$

The previous matrix decomposition is associated to a decomposition

$$\mathbb{R}^n = E_1 \oplus \ldots \oplus E_m$$

We choose the norm in $\mathbb{R}^n$ $\|v\| = \max(\|v_1\|, \ldots, \|v_m\|)$ if $v = v_1 + \ldots + v_m$

Note that $\|v_j\| \leq \|v\|$, $\forall j$, In $E_j = \mathbb{R}^{d_j}$ we also choose the max norm.

Then

$$\|e^{Bt}\| = \sup_{\|v\|\leq 1} \|e^{Bt} v\| = \sup_{\|v\|\leq 1} \max \left\{ \|e^{B_1 t} v_1\|, \ldots, \|e^{B_m t} v_m\| \right\}$$

$$\leq \max \left\{ \|e^{B_1 t}\|, \ldots, \|e^{B_m t}\| \right\}$$

<u>Computation of $\|e^{B_jt}\|$</u> we write $w \in E_j$; $d = \dim E_j$, $B_j = \begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$

$$e^{B_jt} = e^{\lambda I t + Nt} = e^{\lambda I t} e^{Nt} = \begin{pmatrix} e^{\lambda t} & & \\ & \ddots & \\ & & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & & & \\ t & \ddots & & \\ t^2/2 & & \ddots & \\ \vdots & & & 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & & & & \\ t & \ddots & & & \\ t^2/2 & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ \frac{t^{d-1}}{(d-1)!} & \cdots & \cdots & t & 1 \end{pmatrix}$$

$$e^{B_jt} w = e^{\lambda t} \begin{pmatrix} w_1 \\ t w_1 + w_2 \\ t^2/2\, w_1 + t w_2 + w_3 \\ \\ \frac{t^{d-1}}{(d-1)!} w_1 + \ldots + w_d \end{pmatrix}$$

we recall that $|w_k| \leq \|w\|$.

$$\|e^{B_jt} w\| \leq \underbrace{|e^{\lambda t}|}_{e^{\mathrm{Re}\lambda t}} P_d(t) \|w\| , \quad \text{where } P_d(t) = 1 + t + \ldots + \frac{t^{d-1}}{(d-1)!} , \text{ for } t \geq 0$$

Then

$$\|e^{B_j t}\| < P_d(t)\, e^{Re\lambda t} = e^{\mu t} P_d(t)\, e^{(Re\lambda - \mu)t}.$$

Since $\lim\limits_{t \to \infty} P_d(t)\, e^{(Re\lambda - \mu)t} = 0$

there exists a constant $M_j = \sup\limits_{t \in [0,\infty)} P_d(t)\, e^{(Re\lambda - \mu)t} < \infty$

and since

$$\|e^{B_j t}\| \leq M_j e^{\mu t}$$

$$\Rightarrow \|e^{Bt}\| \leq M e^{\mu t}, \quad M = \max\{M_j\}$$

$$\Rightarrow \|e^{At}\| \leq \|C\| \|C^{-1}\| \|e^{Bt}\|$$

If $\|\cdot\|_*$ is another norm, $\exists \alpha, \beta > 0$ s.t. $\alpha \|\cdot\| \leq \|\cdot\|_* \leq \beta \|\cdot\|$

$$\|e^{At}\|_* = \sup_{\|v\|_* \leq 1} \|e^{At} v\|_* \leq \sup_{\|v\|_* \leq 1} \beta \|e^{At} v\| \leq \sup_{\|v\|_* \leq 1} \beta \|e^{At}\| \, \|v\|$$

$$\leq \beta \sup_{\|v\|_* \leq 1} \|e^{At}\| \frac{1}{\alpha} \|v\|_* \leq \frac{\beta}{\alpha} \|e^{At}\|$$

__Thm__ Given $\dot{x} = Ax$

(i) 0 is asymptotically stable if and only if $\operatorname{Spec} A \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$

(ii) 0 is stable $\Leftrightarrow$ $\operatorname{Spec}(A) \subset \{x \in \mathbb{C} \mid \operatorname{Re} x \leq 0\}$ and the Jordan boxes associated to eigenvalues with $\operatorname{Re}\lambda = 0$, if any, diagonalize.

__Prof__

(i) $\boxed{\Rightarrow}$ Assume 0 is asymptotically stable

Suppose that $\exists \lambda \in \operatorname{Spec} A$ s.t. $\operatorname{Re}\lambda \geq 0$

Let $v \neq 0$ be an eigenvector of λ. Then, $\forall \delta \in \mathbb{C}$ $\varphi(t) = \delta e^{\lambda t} v$ is a (maybe complex) solution of $\dot{x} = Ax$,

Indeed: $\varphi'(t) = \delta \lambda e^{\lambda t} v$

$A\varphi(t) = \delta e^{\lambda t} A v = \delta e^{\lambda t} \lambda v$

(The Re and Im parts of $e^{\lambda t} v$ also are solutions)

Then given any nbh of 0 there is an initial condition δv s.t.

$$|\varphi(t,\delta v) - 0| = |\delta e^{\lambda t} v| = |\delta v| e^{Re\lambda t}$$

If $Re\lambda = 0$ the solution is bounded, but does not converge to zero

If $Re\lambda > 0$ the solution is unbounded

In both cases the Re and Im parts of $e^{\lambda t} v$ do not converge to zero simultaneously.

$\boxed{\Leftarrow}$ Let $b = \max\{Re\lambda \mid \lambda \in \operatorname{spec} A\} < 0$

Then there exists $b < \mu < 0$ and $K \geq 1$ s.t. $\|e^{At}\| \leq Ke^{\mu t}$, $\forall t \geq 0$

$\forall \varepsilon > 0$ $\exists \delta = \frac{\varepsilon}{K}$ s.t. if $\|x - 0\| < \delta$, $\|e^{At}x - 0\| \leq \underbrace{Ke^{\mu t}\|x\|}_{\substack{\wedge \\ K\frac{\varepsilon}{K} = \varepsilon}}$, $\forall t \geq 0$

Moreover $e^{At}x \to 0$ when $t \to \infty$

(ii) $\boxed{\Rightarrow}$ If $\exists$ Jordan box such that doesnot diagonalize then

$$\operatorname{Ker}(A-\lambda I)^2 \setminus \operatorname{Ker}(A-\lambda I) \neq \emptyset$$

$\exists\, w \in \operatorname{Ker}(A-\lambda I)^2$ s.t. $\underbrace{(A-\lambda I)w}_{\shortparallel\, v} \neq 0$ $\quad Aw - \lambda w = v \to Aw = \lambda w + v$

Then for all $\delta \in \mathbb{C}$ $\varphi(t) = \delta e^{\lambda t}(w + vt)$ is a solution:

$$\varphi' = \delta\lambda e^{\lambda t}(w + vt) + \delta e^{\lambda t} v;$$

$$A\varphi = \delta e^{\lambda t}(Aw + Avt) = \delta e^{\lambda t}(\lambda w + v + \lambda v t)$$

If $\lambda \notin \mathbb{R}$, $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ are solutions, and both can not be bounded, since $v \neq 0$

$\boxed{\Leftarrow}$ $\|\varphi(t,x)\| \leq \|e^{At}x\| \leq K\|x\|.$

Routh - Hurwitz criterion

To study the stability of linear systems we don't need to know the eigenvalues of the matrix, only the sign of its real part.

The eigenvalues are the roots of the characteristic equation

$$\det(A-\lambda\, Id)=0 \qquad \leftrightarrow \qquad \lambda^{n}+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\dots+a_{n-1}\lambda+a_n=0 \qquad (*)$$

Construct the $n\times n$ matrix

$$H=\begin{pmatrix} a_1 & 1 & & & & \\ a_3 & a_2 & a_1 & 1 & & \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 \\ & & & \ddots & & \\ & & & & a_{n-1} & a_{n-2} \\ & & & & 0 & a_n \end{pmatrix}$$

- the terms in the diagonal are the coefficients of the polynomial
- In the first row there are the coefficients of odd order, up to the maximum, for the ones corresponding to indices bigger than n we put 0.
- Along rows we put coefficients in decreasing order and 1 at the end. After we put 0.

Criterion: All roots of the characteristic equation (*) have negative real parts if and only if all the principal diagonal minors of H are strictly positive

Example Markus - Yamabe

$$x' = A(t)x, \qquad A(t) = \begin{pmatrix} -1 + 3/2\cos^2 t & 1 - 3/2 \sin t \cos t \\ -1 - 3/2 \sin t \cos t & -1 + 3/2 \sin^2 t \end{pmatrix}$$

$\det A(t) = 1/2, \quad \operatorname{tr} A(t) = -1/2$

Then the eigenvalues of $A(t)$ are the solutions of $\lambda^2 + 1/2\,\lambda + 1/2 = 0$

which are

$$\lambda = \frac{-1/2 \pm \sqrt{1/4 - 2}}{2} = -1/4 \pm \sqrt{7}/4\, i$$

but the system is unstable because for any δ

$$\varphi(t) = \delta e^{t/2} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

is a solution

Linear eq. with periodic coefficients

Floquet theory

Let $x' = A(t)x$, A continuous and T-periodic

Let $\phi(t)$ be a fundamental matrix ($\phi'(t) = A(t)\phi(t)$, $\phi(t_0)$ invertible for some (and hence all) t_0)

Then there exists a matrix function $P(t)$, C^1 and T-periodic and $\exists B$ s.t.

$$\phi(t) = P(t)e^{Bt}, \quad \forall t \in \mathbb{R}$$

Remark: By the periodicity $A(t)$ is bounded and therefore the solutions exist for all t.

Proof

$\phi(t+T)$ is also a fundamental matrix

We will prove that $\exists C$ s.t. $\phi(t+T) = \phi(t)\, C$, $\forall t$

Indeed,

(i) $\phi^{-1}(t)\phi(t) = I \Rightarrow (\phi^{-1}(t))'\phi(t) + \phi^{-1}(t)\phi'(t) = 0 \Rightarrow \phi^{-1}(t) = -\phi(t)^{-1}\phi'(t)\phi^{-1}(t)$

(ii) $[\phi^{-1}(t)\phi(t+T)]' = (\phi^{-1}(t))'\phi(t+T) + \phi^{-1}(t)(\phi(t+T))'$

$= -\phi^{-1}(t)\phi'(t)\phi^{-1}(t)\phi(t+T) + \phi^{-1}(t)A(t+T)\phi(t+T)$

$= -\phi^{-1}(t)A(t)\phi(t)\phi^{-1}(t)\phi(t+T) + \phi^{-1}(t)A(t)\phi(t+T) = 0$

Note that $\det C \neq 0$

We write $C = e^{BT}$

We introduce $P(t) = \phi(t)e^{-Bt}$. It is C^1 and

$P(t+T) = \phi(t+T)e^{-B(T+t)} = \phi(t)\,C\,e^{-BT}e^{-Bt} = \phi(t)e^{-Bt} = P(t)$

We recall $C = e^{BT}$

Notation $\quad C$ = monodromy matrix

characteristic multipliers = eigenvalues of C

characteristic exponents = $\quad$ " $\quad$ of B

Prop $\quad$ The change $x = P(t)y$ transforms the equation $x' = A(t)x$ to $y' = By$

Prop $\quad x' = P'(t)y + P(t)y' \longrightarrow y' = P(t)^{-1}[x' - P'(t)y] = P^{-1}(t)[A(t)x - P'(t)y]$

$$= P^{-1}(t)[A(t)P(t) - P'(t)]y$$

On the other hand $P(t)e^{Bt}$ fundamental matrix implies

$$P'e^{Bt} + PBe^{Bt} = AP\,e^{Bt} \longrightarrow P' + PB = AP \longrightarrow AP - P' = PB \Rightarrow \boxed{y' = By}$$

First thm of Liapunov

Thm let $U \subset \mathbb{R}^n$ be an open set, $0 \in U$, $g : \mathbb{R} \times U \to \mathbb{R}^n$. Let

$$\dot{x} = Ax + g(t,x)$$

Such that

(1) Spec $A \subset \{\operatorname{Re} z < 0\}$

(2) g is C^0 and $o(x)$ uniformly in t, for $t \in [0,\infty)$

(3) The i.v.p. has unique solution

Then 0 is asymptotically stable

Note 1

$g(t,x) = o(x)$ means $\displaystyle\lim_{x\to 0} \frac{g(t,x)}{\|x\|} = 0$

Note 2

If we have $x' = F(x)$ and $x=0$ is a fixed pt, and $F \in C^1$

Then

$$x' = F(x) = DF(0)x + R(x), \quad R(x) = o(x)$$

Prof

(1) $\Rightarrow \exists K \geq 1,\ \mu > 0$ s.t. $\|e^{At}\| \leq K e^{-\mu t},\ t \geq 0$

(2) $\Rightarrow$ I take $\varepsilon = \dfrac{\mu}{2K}$ on the def. of limit: $\exists \delta_1 > 0$ s.t.

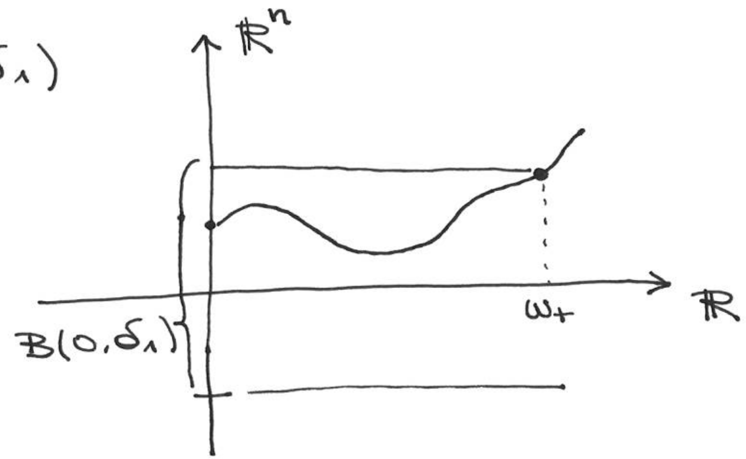
$$\text{if } \|x\| < \delta_1 \qquad \frac{\|g(t,x)\|}{\|x\|} \leq \frac{\mu}{2K} \qquad \forall t \geq 0$$

we assume that $B(0,\delta_1) \subset \mathcal{U}$

Given $\varepsilon > 0$ we take $\delta < \min\left(\dfrac{\varepsilon}{K}, \dfrac{\delta_1}{K}\right)$

$\forall x \in B(0,\delta)$ we consider $\varphi(t)$ the maximal solution $\varphi(t,0,x)$, $t \in (\omega_-, \omega_+)$

for the equation restricted to $\mathbb{R} \times B(0,\delta_1)$



We write

$$\varphi(t) = e^{At}x + \int_0^t e^{A(t-s)} g(s,\varphi(s))\,ds, \qquad t \in (\omega_-, \omega_+)$$

For $t \in [0, \omega_+)$ we bound

$$\|\varphi(t)\| \leq \|e^{At}\| \; \|x\| + \int_0^t \|e^{A(t-s)}\| \; \|g(s,\varphi(s))\|\,ds$$

$$\leq Ke^{-\mu t} \|x\| + \int_0^t Ke^{-\mu(t-s)} \frac{\mu}{2K} \|\varphi(s)\|\,ds$$

$$\leq e^{-\mu t} \left[K\|x\| + \int_0^t \frac{\mu}{2} e^{+\mu s} \|\varphi(s)\|\,ds \right]$$

Let $\psi(t) = e^{\mu t}\|\varphi(t)\|$

$$\psi(t) \leq K\|x\| + \frac{\mu}{2} \int_0^t \psi(s)\,ds$$

Gronwall's lemma $\left(\begin{array}{l} u : [0, \omega) \longrightarrow \mathbb{R}, \quad u \in C^0, \quad b \geq 0, \\ u(t) \leq a + b \int_0^t u(s)\, ds \longrightarrow u(t) \leq a e^{bt} \end{array} \right)$

implies

$$\Psi(t) \leq K \|x\| e^{\mu/2 t} \Rightarrow \|\varphi(t)\| \leq K e^{-\mu/2 t} \|x\|, \quad t < \omega_+$$

The consequences of this bound are

(1) since $\|x\| < \delta$ $\Rightarrow \|\varphi(t)\| < K\delta < \delta_1$ $\quad t < \omega_+ \Rightarrow \omega_+ = \infty$

(2) $\|\varphi(t)\| < \varepsilon, \quad t < \omega_+ \Longrightarrow$ stability

(3) $\|\varphi(t)\| < K e^{-\mu/2 t} \|x\| \Rightarrow \lim_{t \to \infty} \varphi(t) = 0$