Lesson 10

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The Itô process

Definition

An Itô process X is an stochastic process that can be written as

$$X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s, \quad 0 \leq t \leq T,$$

where

- X_0 is $\mathcal{F}_0$-measurable.
- K and H are measurable and $\mathbb{F}$-adapted.
- $\int_0^T |K_s| \mathrm{d}s < \infty$ and $\int_0^T |H_s|^2 \mathrm{d}s < \infty$, $\mathbb{P}$-a.s.

Note that it is a continuous process sum of a continuous local martingale and an absolutely continuous process.

Lemma

If M is a continuous local martingale such that $M_t = \int_0^t K_s \mathrm{d}s$, where K is (measurable and) $\mathbb{F}$-adapted then

$$M_t = 0, \text{ a.s for all } 0 \le t \le T$$

Proof.

We can assume M is a martingale: if $(\delta_n)_{n\ge 1}$ is a localizing sequence of stopping times and $M_{t\wedge\delta_n} = 0$ then

$$M_t = \lim_{n\to\infty} M_{t\wedge\delta_n} = 0, \text{a.s.}$$

We also can assume that $\int_0^T |K_s| \mathrm{d}s \le C < \infty$. Otherwise we can define the stopping time $\tau_n = \inf\left\{t, \int_0^t |K_s| \mathrm{d}s \ge n\right\}$, $\tau_n = T$ if the set is empty,and to apply the result to the martingale $(M_{t\wedge\tau_n})$. This would make $M_{t\wedge\tau_n} \equiv 0$ and we could let n go to infinity to conclude that $M_t = 0$. □

Proof.

(Continuation) Now if $\int_0^T |K_s| \mathrm{d}s$ is bounded by C and we take $t_i = T\frac{i}{n}, 0 \leq i \leq n$, we have

$$\begin{aligned} \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2 &\leq \sup_{1\leq i\leq n} |M_{t_i} - M_{t_{i-1}}| \sum_{i=1}^{n} |M_{t_i} - M_{t_{i-1}}| \\ &\leq \sup_{1\leq i\leq n} |M_{t_i} - M_{t_{i-1}}| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |K_s| \mathrm{d}s \\ &\leq C \sup_{1\leq i\leq n} |M_{t_i} - M_{t_{i-1}}| \qquad (1) \end{aligned}$$

and M is continuous, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2 = 0, \text{ a.s.,}$$

□

Proof.

(Continuation) Moreover $\sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2 \leq C^2$ by (1), so by the dominated convergence theorem $\lim_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2\right) = 0$. On the other hand, since M is a martingale

$$
\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})^2\right) &= \mathbb{E}\left(\sum_{i=1}^n (M_{t_i}^2 + M_{t_{i-1}}^2 - 2M_{t_i}M_{t_{i-1}})\right) \\
&= \mathbb{E}\left(\sum_{i=1}^n \left(M_{t_i}^2 + M_{t_{i-1}}^2 - 2M_{t_{i-1}}\mathbb{E}(M_{t_i}|\mathcal{F}_{t_{i-1}})\right)\right) \\
&= \mathbb{E}\left(\sum_{i=1}^n \left(M_{t_i}^2 + M_{t_{i-1}}^2 - 2M_{t_{i-1}}^2\right)\right) \\
&= \mathbb{E}\left(\sum_{i=1}^n \left(M_{t_i}^2 - M_{t_{i-1}}^2\right)\right) = \mathbb{E}(M_T^2 - M_0^2) \\
&= \mathbb{E}\left(M_T^2\right)
\end{aligned}
$$

consequently $M_T \equiv 0$ a.s. and so $M_t \equiv \mathbb{E}(M_T|\mathcal{F}_t) = 0$ a.s for all t. □

Theorem

The decomposition of an Itô process is unique.

Proof.

Assume that

$$X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s = X_0' + \int_0^t K_s' \mathrm{d}s + \int_0^t H_s' \mathrm{d}W_s,$$

then, taking $t = 0$, we have that $X_0 = X_0'$. Now, by the previous lemma, $\int_0^t (K_s' - K_s)\,\mathrm{d}s = 0$ and $\int_0^t (H_s' - H_s)\,\mathrm{d}W_s = 0$, now, by localizing if needed:

$$\begin{aligned} &\mathbb{E}\left(\left(\int_0^t (H_s' - H_s)\, 1_{[0,\tau_n]}(s)\mathrm{d}W_s\right)^2\right) \\ = &\int_0^t \mathbb{E}\left(\left(H_s' - H_s\right)^2 1_{[0,\tau_n]}(s)\right)\mathrm{d}s = 0, \end{aligned}$$

so $H' = H$, and $K' - K = 0$ a.s. $\mathbb{P} \otimes \mathrm{Leb}$. $\square$

Now we can extend the Itô integral w.r.t. an Itô process: If $X = \int_0^\cdot H_s \mathrm{d}W_s$ and L a (measurable) and adapted process such that $\int_0^T L_s^2 H_s^2 \mathrm{d}s < \infty$, a.s., then we can define $\int_0^\cdot L_s \mathrm{d}X_s$ following the same steps as in the case that $X = W$ and we have that $\int_0^\cdot L_s \mathrm{d}X_s = \int_0^\cdot L_s H_s \mathrm{d}W_s$. Notice that now we have that for any partition with mesh going to zero with n

$$\sum_i |X_{t_i} - X_{t_{i-1}}|^2 \xrightarrow{\mathbb{P}} \int_0^T H_s^2 \mathrm{d}s := \langle X, X \rangle_t$$

If $X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s$, we write

$$\int_0^\cdot L_s \mathrm{d}X_s := \int_0^\cdot L_s K_s \mathrm{d}s + \int_0^\cdot L_s H_s \mathrm{d}W_s.$$

Theorem

Let $X = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s$ *be an Itô process and* $f(t,x) \in C^{1,2}$ *(it is sufficient that this is true on the support of* X*) then:*

$$
\begin{aligned}
f(t, X_t) &= f(0, X_0) + \int_0^t \partial_s f(s, X_s)\mathrm{d}s + \int_0^t \partial_x f(s, X_s)\mathrm{d}X_s \\
&\quad + \frac{1}{2}\int_0^t \partial_{xx}^2 f(s, X_s)\mathrm{d}\langle X, X\rangle_s,
\end{aligned}
$$

where

$$
\int_0^t \partial_x f(s, X_s)\mathrm{d}X_s = \int_0^t \partial_x f(s, X_s) K_s \mathrm{d}s + \int_0^t \partial_x f(s, X_s) H_s \mathrm{d}W_s
$$

$$
\langle X, X\rangle_s = \int_0^t H_s^2 \mathrm{d}s.
$$

Example

Suppose we want to find a solution $S > 0$ for the stochastic differential equation (SDE)

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t), \quad S_0 = x_0 > 0.$$

S is an Itô process, then by the previous theorem

$$\mathrm{d}(\log S_t) = \frac{\mathrm{d}S_t}{S_t} - \frac{1}{2S_t^2}\mathrm{d}\langle S, S\rangle_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t - \frac{1}{2S_t^2}\sigma^2 S_t^2 \mathrm{d}t,$$

that is

$$\mathrm{d}(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W_t$$

in such a way that

$$S_t = x_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$$

Theorem

(Integration by parts formula) Let X and Y two Itô processes,
$X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s$ *and* $Y_t = Y_0 + \int_0^t K'_s \mathrm{d}s + \int_0^t H'_s \mathrm{d}W_s$.
Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \mathrm{d}Y_s + \int_0^t Y_s \mathrm{d}X_s + \langle X, Y \rangle_t$$

where

$$\langle X, Y \rangle_t = \int_0^t H_s H'_s \mathrm{d}s.$$

Proof.

By the Itô formula

$$(X_t + Y_t)^2 = (X_0 + Y_0)^2 + 2\int_0^t (X_s + Y_s)\mathrm{d}(X_s + Y_s) + \frac{1}{2}\int_0^t 2(H_s + H_s')^2 \mathrm{d}s$$

and

$$X_t{}^2 = X_0^2 + 2\int_0^t X_s \mathrm{d}X_s + \frac{1}{2}\int_0^t 2H_s^2 \mathrm{d}s,$$

$$Y_t{}^2 = Y_0^2 + 2\int_0^t Y_s \mathrm{d}Y_s + \frac{1}{2}\int_0^t 2H_s'^2 \mathrm{d}s$$

so, by subtracting the sum of these latter expressions from the first one we obtain:

$$2X_t Y_t = 2X_0 Y_0 + 2\int_0^t X_s \mathrm{d}Y_s + 2\int_0^t Y_s \mathrm{d}X_s + \int_0^t 2H_s H_s' \mathrm{d}s.$$

$\square$

Admissible strategies and arbitrage

Consider a market model with two assets. For $0 \leq t \leq T$, $S_t^0 = e^{rt}, r \geq 0$ and $S_t^1 = S_0^1 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s$, is an Itô process.

Definition

A self-financing strategy ϕ, is a measurable and adapted process, (ϕ_t^0, ϕ_t^1), two-dimensional, that satisfies

- $\int_0^T \left(|\phi_t^0| + |\phi_t^1 K_t| + \left(\phi_t^1\right)^2 H_t^2 \right) \mathrm{d}t < \infty$ $\mathbb{P}$ a.s.
- $V_t = V_0 + \int_0^t \phi_s^0 r e^{rs} \mathrm{d}s + \int_0^t \phi_s^1 \mathrm{d}S_s^1, \quad 0 \leq t \leq T.$

Denote $\tilde{S}_t = e^{-rt}S_t$, in such a way that we use the tilde as in the discrete-time setting: to indicate any discounted value.

Theorem

ϕ is self-financing strategy if and only if:

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_s$$

Proof.

If ϕ is self-financing $\mathrm{d}V_t = \phi_t^0 \mathrm{d}S_t^0 + \phi_t^1 \mathrm{d}S_t$, then since $\tilde{V}_t = e^{-rt}V_t$

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= -re^{-rt}V_t\mathrm{d}t + e^{-rt}\mathrm{d}V_t \\
&= -re^{-rt}V_t\mathrm{d}t + e^{-rt}(\phi_t^0\mathrm{d}S_t^0 + \phi_t^1\mathrm{d}S_t) \\
&= -re^{-rt}(V_t - \phi_t^0 S_t^0)\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t \\
&= -re^{-rt}\phi_t^1 S_t\mathrm{d}t + e^{-rt}\phi_t^1\mathrm{d}S_t \\
&= \phi_t^1(-re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t) \\
&= \phi_t^1\mathrm{d}\tilde{S}_t.
\end{aligned}$$

Proof.

If

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_s$$

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= \phi_t^1 \mathrm{d}\tilde{S}_t = \phi_t^1(-re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t) \\
&= -re^{-rt}V_t\mathrm{d}t + e^{-rt}\mathrm{d}V_t
\end{aligned}$$

therefore

$$\begin{aligned}
\mathrm{d}V_t &= re^{-rt}\left(V_t - \phi_t^1 S_t\right)\mathrm{d}t + \phi_t^1\mathrm{d}S_t \\
&= re^{-rt}\phi_t^0 S_t^0\mathrm{d}t + \phi_t^1\mathrm{d}S_t \\
&= \phi_t^0\mathrm{d}S_t^0 + \phi_t^1\mathrm{d}S_t.
\end{aligned}$$

□

Definition

A strategy ϕ is admissible if it is self-financing and there exists $K > 0$ such that its value $V_t \geq -K, 0 \leq t \leq T$.

Definition

An *arbitrage (opportunity)* is an admissible strategy ϕ with zero initial value and with strictly positive final value, that is

1. $V_0(\phi) = 0$,
2. $\mathbb{P}\left(V_T(\phi) \geq 0\right) = 1$,
3. $\mathbb{P}(V_T(\phi) > 0) > 0$.

Theorem

Assume that there exits $\mathbb{P}^* \sim \mathbb{P}$ such that $\tilde{S}$ is a $\mathbb{P}^*$-local martingale, then the model is free of arbitrage.

Proof.

If we consider an admissible strategy ϕ with zero initial value we have

$$\tilde{V}_t(\phi) = \int_0^t \phi_s^1 \mathrm{d}\tilde{S}_s,$$

so $\tilde{V}_t(\phi)$ is a $\mathbb{P}^*$-*local* martingale that is bounded from below, so it is a supermartingale. Consequently

$$\mathbb{E}_{\mathbb{P}^*}\left(\tilde{V}_T(\phi)\right) \leq 0,$$

now, since $\mathbb{P}\left(V_T(\phi) \geq 0\right) = 1$ and $\mathbb{P} \sim \mathbb{P}^*$ we have that $V_T(\phi) = 0$, $\mathbb{P}^*$-a.s. and $\mathbb{P}$-a.s. □

We use the notation

$$K_0 = \left\{ \int_0^T \phi_s \cdot \mathrm{d}\tilde{S}_t, \phi \text{ admissible} \right\}$$

$$C_0 = K_0 - L^0_+$$

$$C = C_0 \cap L^\infty$$

$\overline{C}$ the closure of C under L^∞

Definition

We say that the model satisfies the No Free Lunch with Vanishing Risk condition (NFLVR) if $\overline{C} \cap L^\infty_+ = \{0\}$

We have the FFTAP in continuous time:

Theorem

Let $\tilde{S}$ be a locally bounded semimartingale. There are not free lunches with vanishing risk if and only if there is probability $\mathbb{P}^* \sim \mathbb{P}$ under which $\tilde{S}$ is a local martingale.