Dynamical System Exercise Set Final 3,4

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Exercise Statement

3. Let f be a polynomial of degree $d \geq 2$. Suppose that $z_1, \ldots, z_d$ are the fixed points of f and assume that $f'(z_i) \neq 1$ for $i = 1, \ldots, d$ (which implies that the d fixed points are distinct). Let $\lambda_i = f'(z_i)$.

   a. Show that $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0$. Hint: Prove that the residue of $\frac{1}{z-f(z)}$ at a fixed point z_i is equal to $\frac{1}{1-\lambda_i}$, and then use the Residue Theorem.

   b. Prove that for every $i = 1, \ldots, d$, z_i is attracting or indifferent if and only if $\operatorname{Re}\left(\frac{1}{1-\lambda_i}\right) \geq \frac{1}{2}$

   c. Use these results to show that every polynomial of degree $d \geq 2$ must have at least one fixed point which is repelling or has multiplier exactly equal to 1.

4. Prove that every monic cubic polynomial is affine conjugate to one of the form
$$P_{a,b}(z) = z^3 - 3a^2 z + b$$
for some values of $a, b \in \mathbb{C}$. Compute the critical points of $P_{a,b}$. Observe that the parameter space is $\mathbb{C}^2$. To restrict to a section of (complex) dimension one we impose the following condition: we require one of the two critical points (say $-a$) to be a fixed point, and we call the other one the free critical point. We then obtain a one-parameter family of cubic polynomials, P_a, all of them having a superattracting fixed point (apart from infinity which always is one).

   a. Find an expression for $P_a(z)$.

   b. Can the Julia set of P_a be totally disconnected? And connected? And disconnected but not totally disconnected? Can it have empty interior? Justify your answers using the theorems seen in class.

   c. (Optional) Make a computer program using the escape-time algorithm, to draw the Filled Julia sets of P_a for any value of a. Make sure your program distinguishes (with different colors) the orbits that converge to $z = -a$, from those that converge to infinity, from those that do neither. The analogue of the Mandelbrot set. Make a second program to draw the parameter plane, iterating the free critical. To color the pixels, you should distinguish three different cases depending on whether the orbit of the free critical point:

a. is bounded AND is in the basin of $z = -a$ the superattracting fixed point.
b. is bounded but it is NOT in the basin of $z = -a$
c. converges to infinity; Justify that in case (a) all Fatou components are eventually mapped to the superattracting basin of $z = -a$, while in case (b) other periodic Fatou components may exist. Finally, use your first program to draw one filled Julia set in each of the three situations described

Exercise 3

Let f be a polynomial of degree $d \geq 2$. Suppose that $z_1, \ldots, z_d$ are the fixed points of f and assume that $f'(z_i) \neq 1$ for $i = 1, \ldots, d$ (which implies that the d fixed points are distinct). Let $\lambda_i = f'(z_i)$.

a. Show that $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0$. Hint: Prove that the residue of $\frac{1}{z-f(z)}$ at a fixed point z_i is equal to $\frac{1}{1-\lambda_i}$, and then use the Residue Theorem.

b. Prove that for every $i = 1, \ldots, d$, z_i is attracting or indifferent if and only if $\operatorname{Re}\left(\frac{1}{1-\lambda_i}\right) \geq \frac{1}{2}$

c. Use these results to show that every polynomial of degree $d \geq 2$ must have at least one fixed point which is repelling or has multiplier exactly equal to 1.

Proof.
a. To show that $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0$, we can use the Residue Theorem. First, let's consider the function

$$g(z) = \frac{1}{z - f(z)}.$$

The fixed points of f are the roots of the equation $z - f(z) = 0$. Since there is no fixed point of multiplier 1, all the fixed points are simple. Therefore, the fixed points z_i are simple poles of $g(z)$, and the residues at these poles are given by

$$\operatorname{Res}(g(z), z_i) = \lim_{z \to z_i} (z - z_i) g(z).$$

Now, let's calculate the residue at z_i:

$$\begin{aligned}
\operatorname{Res}(g(z, z_i) &= \lim_{z\to z_i}(z - z_i)\frac{1}{z - f(z)} \\
&= \lim_{z\to z_i}\frac{1}{1 - \frac{f(z)-z_i}{z-z_i}} \\
&= \frac{1}{1 - f'(z_i)} \\
&= \frac{1}{1-\lambda_i}.
\end{aligned}$$

Now, take a disk $D(0, r)$ such that $z_i \in D(0, r)$, $i = 1, \ldots, d$, and $r > 0$.
The residue theorem implies the theorem on the total sum of residues: *If $f(z)$ is a single-valued analytic function in the extended complex plane, except for a finite number of singular points, then the sum of all residues of $f(z)$, including the residue at the point at infinity, is zero.*
So, the sum of residues of a function in a region is equal to zero. Therefore, we have

$$\sum_{i=1}^{d}\operatorname{Res}(g(z), z_i) = 0.$$

Substitute the expression for the residue we derived earlier:

$$\sum_{i=1}^{d}\frac{1}{1-\lambda_i} = 0.$$

b. For $i = 1, \ldots, d$, a fixed point z_i is attracting if $|f'(z_i)| = |\lambda_i| < 1$. Note that the function $\frac{1}{1-z}$ maps the unit disk $\{z : |z| < 1\}$ onto the half plane $H = \left\{z : \operatorname{Re}(z) > \frac{1}{2}\right\}$. This is clear since the function $z \mapsto 1 - z$ maps the unit disk onto the disk $D = \{z : |z - 1| < 1\}$, and the disk D is mapped onto H by $\frac{1}{z}$. The converse follows. Then, $\operatorname{Re}\left(\frac{1}{1-\lambda_i}\right) \geq \frac{1}{2}$.
From what we have shown above, it follows that the residue index at an indifferent fixed point has real part equal to $\frac{1}{2}$, and that of a repelling fixed point has real part less than $\frac{1}{2}$.

c. To show that every polynomial of degree $d \geq 2$ must have at least one fixed point that is repelling or has a multiplier exactly equal to 1, we can use the results obtained in the previous parts.
Recall that for a fixed point z_i of a polynomial $f(z)$, the multiplier or the derivative at that point is given by $\lambda_i = f'(z_i)$.
From part (b), we know that for a fixed point z_i, it is attracting or indifferent if and only if $\operatorname{Re}\left(\frac{1}{1-\lambda_i}\right) \geq \frac{1}{2}$.
Now, if $\operatorname{Re}\left(\frac{1}{1-\lambda_i}\right) \geq \frac{1}{2}$, it means that z_i is not repelling (because $|\lambda_i| \leq 1$ for attracting or indifferent points).

Therefore, for every fixed point z_i, if it is not attracting or indifferent, it must be repelling. In other words, every fixed point of the polynomial $f(z)$ must be either attracting, indifferent, or repelling.
In general, since a polynomial of degree $d \geq 2$ has at least d fixed points (counting multiplicity), and each fixed point is either attracting, indifferent, or repelling, there must be at least one fixed point that is repelling. This is because, if all fixed points were attracting or indifferent, the sum $\sum_{i=1}^{d} \frac{1}{1-\lambda_i}$ would be positive (from part (b)), contradicting the result obtained in part (a), where $\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0$.
Therefore, since there is no fixed point of multiplier 1 and all the fixed points are simple (from part (a)), every polynomial of degree $d \geq 2$ must have at least one fixed point that is repelling.

Exercise 4

Prove that every monic cubic polynomial is affine conjugate to one of the form

$$P_{a,b}(z) = z^3 - 3a^2 z + b$$

for some values of $a, b \in \mathbb{C}$. Compute the critical points of $P_{a,b}$. Observe that the parameter space is $\mathbb{C}^2$. To restrict to a section of (complex) dimension one we impose the following condition: we require one of the two critical points (say $-a$) to be a fixed point, and we call the other one the free critical point. We then obtain a one-parameter family of cubic polynomials, P_a, all of them having a superattracting fixed point (apart from infinity which always is one).

a. Find an expression for $P_a(z)$.

b. Can the Julia set of P_a be totally disconnected? And connected? And disconnected but not totally disconnected? Can it have empty interior? Justify your answers using the theorems seen in class.

c. (Optional) Make a computer program using the escape-time algorithm, to draw the Filled Julia sets of P_a for any value of a. Make sure your program distinguishes (with different colors) the orbits that converge to $z = -a$, from those that converge to infinity, from those that do neither. The analogue of the Mandelbrot set. Make a second program to draw the parameter plane, iterating the free critical. To color the pixels, you should distinguish three different cases depending on whether the orbit of the free critical point:

   a. is bounded AND is in the basin of $z = -a$ the superattracting fixed point.

   b. is bounded but it is NOT in the basin of $z = -a$

   c. converges to infinity; Justify that in case (a) all Fatou components are eventually mapped to the superattracting basin of $z = -a$, while in case (b) other periodic Fatou components may exist. Finally, use your first program to draw one filled Julia set in each of the three situations described

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Proof.
Let us write a generic cubic polynomical as

$$P(z) = Az^3 + Bz^2 + Cz + D$$

for some $A, B, C, D \in \mathbb{R}$. We can conjugate $P(z)$ to

$$S(z) = z^3 + \alpha z^2 + \beta z + b$$

by a linear map to make it monic. Furthermore, we can make $S(z)$ centered (the sum of its critical points is 0) by an affine conjugacy. This way, we have the condition that $\alpha = 0$, so that we obtain the polynomial

$$R(z) = z^3 + \beta z + b$$

For this polynomial, the critical points are

$$x_1 = \frac{\sqrt{-12\beta}}{6}, \quad x_2 = -\frac{\sqrt{-12\beta}}{6}$$

and, if we choose $\beta = -3a^2$ for some $a \in \mathbb{R}$, then the critical points become

$$x_1 = a, \quad x_2 = -a$$

So we have obtained

$$P_{a,b}(z) = z^3 - 3a^2 z + b$$

which is conjugated to the initial polynomial $P(z)$.

a. Let us now impose one of the two critical points, $x_2 = -a$, to be a fixed point

$$P_{a,b}(-a) = -a = -a^3 + 3a^3 + b = 2a^3 + b$$

So

$$b = -2a^3 - a$$

and we obtain $P_a(z) = z^3 - 3a^2 z - 2a^3 - a$.

b. To solve this point, we need to translate the superattracting fixed point to 0 with the transformation $z \mapsto z + a$, and use a conjugation to get a new polynomial $P_a(z)$ such that its critical points are independent of the parameter a. Let ω be the superattracting fixed point and let γ be the other one. We will denote by $A(\omega)$ the basin of attraction of this superattracting fixed point.
From the theorem of *Connectivity of polynomial Julia Sets*, if $P_a^n(\gamma) \to \infty$ as $n \to \infty$, then the filled Julia set $K(P)$ is disconnected. Conversely, if $P_a^n(\gamma)$ is bounded for increasing n, then $K(P)$ is connected.

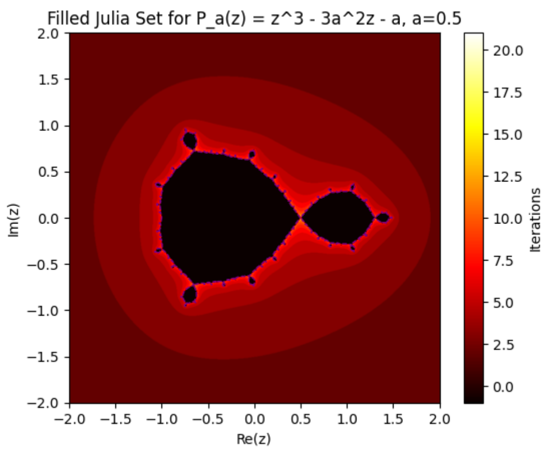
If $\gamma \in K(P)$ and $P_a^n(\gamma) \to \infty$, $K(P)$ is disconnected.
If $\gamma \in A(\omega)$ and $P_a^n(\gamma) \to \infty$, the point ω captures the orbit of γ and the set $\{P_a^n(\gamma)\}$ is bounded, but it isn't captured by the superattracting fixed point ω.

c. This point is solved using a Jupyter Notebook with Google Colab. The notebook is available here. In case there are problems accessing it, here it follows a printout of the notebook and all the outputs.

Exercise 4, point c.

(Optional) Make a computer program using the escape-time algorithm, to draw the Filled Julia sets of P_a for any value of a. Make sure your program distinguishes (with different colors) the orbits that converge to $z = -a$, from those that converge to infinity, from those that do neither. The analogue of the Mandelbrot set.

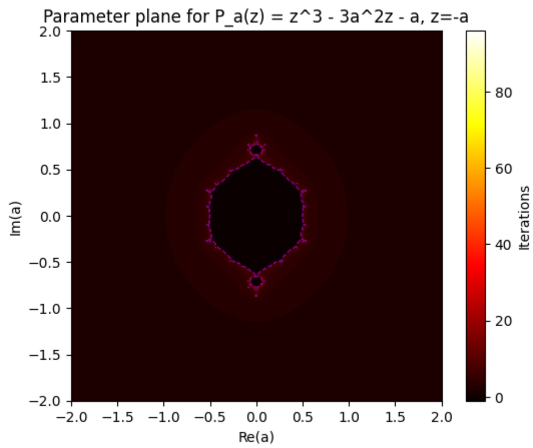
```
import numpy as np
import matplotlib.pyplot as plt

def P_a(z, a):
    return z**3 - 3*(a**2)*z - 2*(a**3) - a

def iterate_map(z, a, max_iter=1000, fp=None, inf=1e6, tol=1e-6):
    if fp==None:
        fp=-a
    for i in range(max_iter):
        z = P_a(z, a)
        if np.abs(z) > inf:  # Check if the orbit diverges to infinity
            return i
        if np.abs(z - fp) < tol:  # Check if the orbit converges to -a
            return -1
    return -1  # If max_iter reached without divergence or convergence

def plot_julia_set(a, xmin, xmax, ymin, ymax, width, height, max_iter):
    x = np.linspace(xmin, xmax, width)
    y = np.linspace(ymin, ymax, height)

    julia_set = np.zeros((height, width))

    for i in range(height):
        for j in range(width):
            zx, zy = x[j], y[i]
            result = iterate_map(complex(zx, zy), a, max_iter)
            julia_set[i, j] = result

    # Plotting the filled Julia set
    plt.imshow(julia_set, extent=(xmin, xmax, ymin, ymax), cmap='hot', interpolation='bilinear', origin='lower')

    # Adding colorbar for reference
    cbar = plt.colorbar()
    cbar.set_label('Iterations')

    # Color the points
    plt.contour(x, y, julia_set, levels=[-1], colors='purple', linewidths=1)
    plt.contour(x, y, julia_set, levels=[max_iter], colors='black', linewidths=1)

    plt.title(f'Filled Julia Set for P_a(z) = z^3 - 3a^2z - a, a={a}')
    plt.xlabel('Re(z)')
    plt.ylabel('Im(z)')
    plt.show()

# Set the parameters
a_value = 0.5
xmin, xmax, ymin, ymax = -2, 2, -2, 2
width, height = 800, 800
max_iter = 1000

# Plot the filled Julia set
plot_julia_set(a_value, xmin, xmax, ymin, ymax, width, height, max_iter)
```



Make a second program to draw the parameter plane, iterating the free critical. To color the pixels, you should distinguish three different cases depending on whether the orbit of the free critical point:

a. is bounded AND is in the basin of $z = -a$ the superattracting fixed point.
b. is bounded but it is NOT in the basin of $z = -a$
c. converges to infinity; Justify that in case (a) all Fatou components are eventually mapped to the superattracting basin of $z = -a$, while in case (b) other periodic Fatou components may exist.

```
def plot_parameter_plane(z, xmin, xmax, ymin, ymax, width, height, max_iter):
    x = np.linspace(xmin, xmax, width)
    y = np.linspace(ymin, ymax, height)

    parameter_plane = np.zeros((height, width))

    for i in range(height):
        for j in range(width):
            Re_a, Im_a = x[j], y[i]
            result = iterate_map(complex(Re_a, Im_a), complex(Re_a, Im_a), max_iter)
            parameter_plane[i, j] = result

    # Plotting the filled Julia set
    plt.imshow(parameter_plane, extent=(xmin, xmax, ymin, ymax), cmap='hot', interpolation='bilinear', origin='lower')

    # Adding colorbar for reference
    cbar = plt.colorbar()
    cbar.set_label('Iterations')

    # Color the points
    plt.contour(x, y, parameter_plane, levels=[-1], colors='purple', linewidths=1)
    plt.contour(x, y, parameter_plane, levels=[max_iter], colors='black', linewidths=1)

    plt.title(f'Parameter plane for P_a(z) = z^3 - 3a^2z - a, z=-a')
    plt.xlabel('Re(a)')
    plt.ylabel('Im(a)')
    plt.show()

# Set the parameters
z_value = 0.5
xmin, xmax, ymin, ymax = -2, 2, -2, 2
width, height = 800, 800
max_iter = 100

# Plot the filled Julia set
plot_parameter_plane(z_value, xmin, xmax, ymin, ymax, width, height, max_iter)
```



Finally, use your first program to draw one filled Julia set in each of the three situations described

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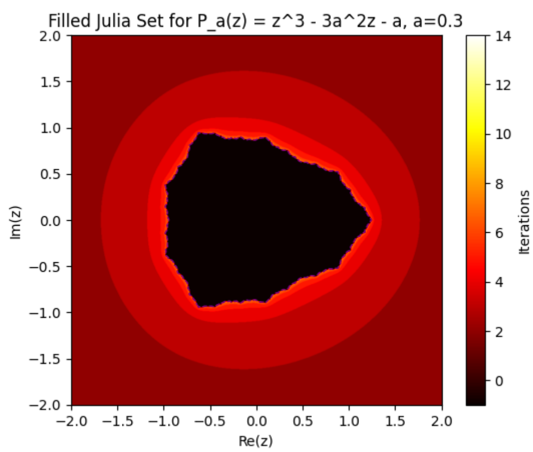
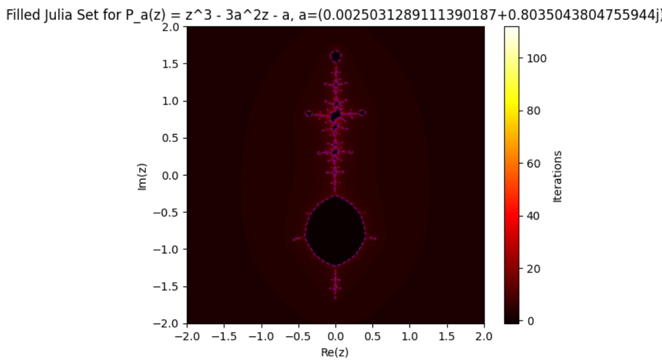
a. is bounded AND is in the basin of $z = -a$ the superattracting fixed point.

```
# Set the parameters
a_value = 0.3
xmin, xmax, ymin, ymax = -2, 2, -2, 2
width, height = 800, 800
max_iter = 1000

# Plot the filled Julia set
plot_julia_set(a_value, xmin, xmax, ymin, ymax, width, height, max_iter)
```



b. is bounded but it is NOT in the basin of $z = -a$.

```
# Set the parameters
a_value = complex(0.0025031289111390187,0.8035043804755944)
xmin, xmax, ymin, ymax = -2, 2, -2, 2
width, height = 800, 800
max_iter = 1000

# Plot the filled Julia set
plot_julia_set(a_value, xmin, xmax, ymin, ymax, width, height, max_iter)
```



c. converges to infinity.

```
# Set the parameters
a_value = complex(0.3,0.8)
xmin, xmax, ymin, ymax = -2, 2, -2, 2
width, height = 800, 800
max_iter = 1000

# Plot the filled Julia set
plot_julia_set(a_value, xmin, xmax, ymin, ymax, width, height, max_iter)
```