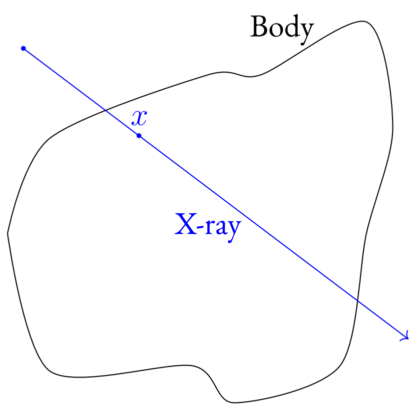
Chapter 7

The Radon transform and computerised tomography

The Radon transform provides the mathematical foundation of Computerised Tomography (CT), a discipline that provides techniques of reconstruction of functions from their integrals over lines. The greek word *tomography* is the composition of *tomos* (slice) and *graphein* (write). The goal is to obtain cross-sectional images of an obkect from its projections.

A simple physical model for computerised tomography is as follows. Let I_0 be initial intensity of an X-ray beam and let $f(x)$ denote the X-ray attenuation coefficient of the body receiving the beam at point x. Let L denote the line along which the ray propagates.



The intensity of the ray once it has gone through the body is (since $I' = -f(x)I$)

$$I = I_0 \exp\left(-\int_L f(x)\,dx\right).$$

In this way one obtains a function defined on the set of lines L going through the body:

$$g(L) = \int_L f(x)\,dx = \log\left(\frac{I_0}{I}\right).$$

This function is the Radon transform of f (denoted $g = \mathcal{R}f$), and the problem in Computerised Tomography is to reconstruct f from the values $g(L)$.

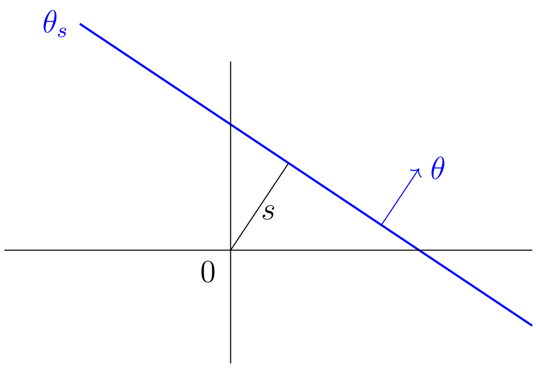
In 1917 the Austrian mathematician Johann Radon obtained an inversion formula for f in terms of g. However, in practice only a finite number of lines can be considered (depending on the scanning geometry and the scanner resolution), and Radon's formula is not suitable for such numerical computations. The subject was revitalised in the 1960's with a new inversion formula given by Allan McCornack. This lead to the construction of the first CT scanners, which started operating comercially in 1972. The original prototype took 160 parallel readings through 180 directions, each 1 degree apart.

7.1 Formal definition and main properties

The Radon transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ in the Schwarz class is a function defined on the set of hyperplanes of $\mathbb{R}^n$. A hyperplane is determined by an orthonormal vector $\theta \in \mathbb{S}^{n-1}$ (unit sphere in $\mathbb{R}^n$) and its (signed) distance from 0, denoted by s. We denote

$$\theta_s = s\theta + \theta^\perp = \{x \in \mathbb{R}^n : \langle x, \theta\rangle = s\},$$

where $\theta^\perp$ indicates the orthogonal to the space $\langle\theta\rangle$ in $\mathbb{R}^n$.



Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be integrable along all lines (something which in applications is no restriction). The *Radon transform* of f is the function

$$\mathcal{R}f(\theta, s) = \int_{\theta_s} f = \int_{\theta^\perp} f(s\theta + y)\,dy, \qquad \theta \in \mathcal{S}^{n-1},\ s \in \mathbb{R}.$$

Observe that $\mathcal{R}f(\theta, s)$ computes precisely the integral of f on the hyperplane determined by θ and s. It is clear from the definiton that $\mathcal{R}f$ is an even function defined on $\mathcal{Z} := \mathbb{S}^{n-1} \times \mathbb{R}$, in the sense that

$$\mathcal{R}f(-\theta, -s) = \mathcal{R}f(\theta, s)\,, \qquad \theta \in \mathbb{S}^{n-1},\ s \in \mathbb{R}.$$

We state, without proof a basic regularity result for the Radon transform.

Proposition 9. *The Radon transform $\mathcal{R}$ transforms $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathcal{Z}) := \mathcal{S}(\mathbb{R}^{n+1})_{|\mathcal{Z}}$.*

7.2 Radon transform and Fourier transform

Henceforth the Fourier transform and the convolution of functions defined on $\mathcal{Z}$ shall always be taken with respect to the variable s: for $g, h \in \mathcal{S}(\mathcal{Z})$ and for $\theta \in \mathbb{S}^{n-1}$, $s, \sigma \in \mathbb{R}$:

$$\hat{g}(\theta, \sigma) = \int_{\mathbb{R}} g(\theta, s)\, e^{-2\pi i \sigma s} ds \quad , \quad (g * h)(\theta, s) = \int_{\mathbb{R}} g(\theta, s-u)\, h(\theta, u)\, du.$$

The first result we shall prove relates in a precise way the Radon and the Fourier transforms of a given $f \in \mathcal{S}(\mathbb{R}^n)$: the Fourier transform of the θ-projection coincides with the θ-slice $\hat{f}(\cdot\theta)$ of the Fourier transform of f.

Theorem 17. *(Fourier projection slice theorem) For $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\widehat{\mathcal{R}f}(\theta, \sigma) = \hat{f}(\sigma\theta)\,, \qquad \theta \in \mathbb{S}^{n-1}\,,\ \sigma \in \mathbb{R}.$$

Proof. By definition

$$\hat{f}(\sigma\theta) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i \langle t, \sigma\theta\rangle} dt.$$

Write now the points in $\mathbb{R}^n$ as $t = s\theta + y$, with $y \in \theta^\perp$. Since $\langle t, \sigma\theta\rangle = s\sigma\langle\theta, \theta\rangle = s\sigma$, by Fubini's theorem

$$\begin{aligned}\hat{f}(\sigma\theta) &= \int_{\mathbb{R}} \int_{\theta^\perp} f(s\theta + y)\, e^{-2\pi i \sigma s} dy\, ds = \int_{\mathbb{R}} e^{-2\pi i \sigma s} \Big(\int_{\theta^\perp} f(s\theta + y)\, dy \Big)\, ds \\ &= \widehat{\mathcal{R}f}(\theta, \sigma)\,.\end{aligned}$$

□

Corollary 3. The map $\mathcal{R} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathcal{Z})$ is injective.

Proof. If $\mathcal{R}f \equiv 0$ then $\hat{f}(\sigma\theta) = \widehat{\mathcal{R}f}(\theta, \sigma) = 0$ for all $\theta \in \mathbb{S}^{n-1}$ and $\sigma \in \mathbb{R}$. Hence $\hat{f} \equiv 0$, and by the uniqueness theorem $f \equiv 0$. □

7.3 Uniqueness

Usually $\mathcal{R}f$ is only known on a proper, small subset of $\mathcal{Z}$. Do these data determine f, or at least the part of it we are looking at? That is, is there local uniqueness? We state next two results along these lines.

Support theorem. *(Hole theorem) Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let $K \subset \mathbb{R}^n$ be a compact and convex set (the "hole") such that $\mathcal{R}f(\theta, s) = 0$ for each θ_s not meeting K. Then $f \equiv 0$ on $\mathbb{R}^n \setminus K$.*

To prove this one first observes that it is enough to consider the case $K = \overline{B(0,R)}$, $R > 0$. Then it is enough to see that

$$\mathcal{R}(t_j f)(\theta, s) = 0 \quad \text{for } j = 1, \dots, n, \theta \in \mathbb{S}^{n-1}, s > R.$$

This is so because in this case, by linearity, $\mathcal{R}(pf)(\theta, s) = 0$ for every polynomial p and all hyperplane θ_s not meeting K, and by Weierstrass approximation theorem $f \equiv 0$ on those hyperplanes, i.e. $f \equiv 0$ on $\mathbb{R}^n \setminus K$.

The proof of the reduction above uses the hypothesis and some differentiation relations of the Radon transform (we skip this proof).

In this first result we assumed the vanishing of $\mathcal{R}f$ for $s \in \mathbb{R}$ outside a bounded interval for all $\theta \in \mathbb{S}^{n-1}$. The second result assumes that $\mathcal{R}f$ vanishes only on a certain set of directions, but for all $s \in \mathbb{R}$.

Definition 15. A set $A \subset \mathbb{S}^{n-1}$ is a uniqueness set for the homogeneous polynomials in $\mathbb{R}^n$ if for every such polynomial p the vanishing of p on A implies $p \equiv 0$.

Theorem 18. *Let $A \subset \mathbb{S}^{n-1}$ be a uniqueness set for the homogeneous polynomials in $\mathbb{R}^n$. If $f \in \mathcal{C}_c(\mathbb{R}^n)$ is such that $\mathcal{R}f(\theta, s) = 0$ for all $\theta \in A$ and for all $s \in \mathbb{R}$, then $f \equiv 0$.*

Proof. Since $f \in \mathcal{C}_c(\mathbb{R}^n)$ its Fourier transform extends to an entire function in $\mathbb{C}^n$, and can therefore be expanded as a series

$$\hat{f}(\xi) = \sum_{k=0}^{\infty} p_k(\xi),$$

where p_k is a homogeneous polynomial of degree k.

Then, by the Fourier projection slice theorem, for $\theta \in A$ and $s \in \mathbb{R}$,

$$0 = \widehat{\mathcal{R}f}(\theta, s) = \hat{f}(s\theta) = \sum_{k=0}^{\infty} p_k(s\theta) = \sum_{k=0}^{\infty} s^k p_k(\theta).$$

Looking at this as a power series on s we deduce that $p_k(\theta) = 0$ for all $\theta \in A$, which by hypothesis implies $p_k \equiv 0$, $k \geq 0$. Then $\hat{f} = 0$ and therefore $f = 0$. $\square$

7.4 Inversion formulas

The reconstruction algorithms used in applications rely on explicit inversion formulas. Here we mention two of them. The first one is the Fourier inversion formula, which is the base of the so-called "gridding algorithm". The second one involves the "backprojection" operator, which will be discussed later.

Fourier inversion formula. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$f(t) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} \Big(\int_{\mathbb{R}} |\sigma|^{n-1} e^{2\pi i\sigma\langle t,\theta\rangle}\widehat{\mathcal{R}f}(\theta,\sigma)\,d\sigma\Big)\,d\theta.$$

Proof. Using polar coordinates ($\xi = \sigma\theta$, $\sigma > 0$) and the Fourier slice theorem one has

$$\begin{aligned} f(t) &= \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i\langle t,\xi\rangle} d\xi = \int_0^\infty \int_{\mathbb{S}^{n-1}} \hat{f}(\sigma\theta)\, e^{2\pi i\sigma\langle t,\theta\rangle}\, d\theta\, \sigma^{n-1} d\sigma \\ &= \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty \sigma^{n-1}\, e^{2\pi i\sigma\langle t,\theta\rangle}\, \widehat{\mathcal{R}f}(\theta,\sigma)\,d\sigma\right)\,d\theta. \end{aligned}$$

Denote

$$F(\theta,t) = \int_0^\infty \sigma^{n-1}\, e^{2\pi i\sigma\langle t,\theta\rangle}\, \widehat{\mathcal{R}f}(\theta,\sigma)\,d\sigma$$

and observe that

$$\int_{\mathbb{S}^{n-1}} F(-\theta,t)\,d\theta = \int_{\mathbb{S}^{n-1}} F(\theta,t)\,d\theta.$$

Therefore

$$f(t) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} (F(\theta,t) + F(-\theta,t))\,d\theta.$$

By the parity of $\mathcal{R}f$,

$$\begin{aligned} F(-\theta,t) &= \int_0^\infty \sigma^{n-1}\, e^{2\pi i\sigma\langle t,-\theta\rangle}\, \widehat{\mathcal{R}f}(-\theta,\sigma)\,d\sigma = \int_0^\infty \sigma^{n-1}\, e^{2\pi i(-\sigma)\langle t,\theta\rangle}\, \widehat{\mathcal{R}f}(\theta,-\sigma)\,d\sigma \\ &= \int_{-\infty}^0 |\sigma|^{n-1}\, e^{2\pi i\sigma\langle t,\theta\rangle}\, \widehat{\mathcal{R}f}(\theta,\sigma)\,d\sigma \end{aligned}$$

we deduce that

$$\begin{aligned} f(t) &= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \left(\int_0^\infty \sigma^{n-1}\, e^{2\pi i\sigma\langle t,\theta\rangle}\, \widehat{\mathcal{R}f}(\theta,\sigma)\,d\sigma + \int_{-\infty}^0 |\sigma|^{n-1}\, e^{2\pi i\sigma\langle t,\theta\rangle}\, \widehat{\mathcal{R}f}(\theta,\sigma)\,d\sigma\right)\,d\theta \\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \left(\int_{-\infty}^\infty |\sigma|^{n-1}\, e^{2\pi i\sigma\langle t,\theta\rangle}\, \widehat{\mathcal{R}f}(\theta,\sigma)\,d\sigma\right)\,d\theta, \end{aligned}$$

as desired. □

The previous formula expresses f in terms of the Fourier transform of $\mathcal{R}f$. In order to obtain a formula in terms of $\mathcal{R}f$ itself the following operator will play a fundamental rôle.

Definition 16. Let $g \in \mathcal{S}(\mathcal{Z})$. The backprojection of g is the function $\mathcal{R}^{\#}g \in \mathcal{S}(\mathbb{R}^n)$ defined by

$$\mathcal{R}^{\#}g(t) = \int_{\mathbb{S}^{n-1}} g(\theta, \langle t, \theta\rangle)\, d\theta\,, \quad t \in \mathbb{R}^n.$$

In case $g = \mathcal{R}f$ the operation $\mathcal{R}f(\theta, \langle t, \theta\rangle)$ is the integral of f on the hyperplane passing through $t \in \mathbb{R}^n$ and orthogonal to $\theta \in \mathbb{S}^{n-1}$, so then $\mathcal{R}^{\#}g$ is the average of the integrals of f along the hyperplanes passing through t.

In the following statement we gather the some basic properties of the backprojection.

Proposition 10. 1. *If $g \in \mathcal{C}(\mathcal{Z})$ then $\mathcal{R}^{\#}g \in \mathcal{C}(\mathbb{R}^n)$.*

2. *If $g \in \mathcal{C}^1(\mathcal{Z})$ then $\mathcal{R}^{\#}g \in \mathcal{C}^1(\mathbb{R}^n)$ and*

$$\frac{\partial}{\partial t_j}(\mathcal{R}^{\#}g) = \mathcal{R}^{\#}\big(\theta_j \frac{\partial g}{\partial s}\big).$$

Analogously, if $g \in \mathcal{C}^\infty(\mathcal{Z})$ then $\mathcal{R}^{\#}g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and for any multi-index α,

$$\frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1}\cdots\partial t_n^{\alpha_n}}(\mathcal{R}^{\#}g) = \mathcal{R}^{\#}\big(\theta^\alpha \frac{\partial^{|\alpha|} g}{\partial s^{|\alpha|}}\big).$$

3. *The backprojection and the Laplacian commute. Let $\Delta = \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial t_j^2}$ be the Laplacian in $\mathbb{R}^n$ and let $\Box = \frac{\partial^2}{\partial s^2}$ be the Laplacian in $\mathbb{R}$. For $g \in \mathcal{C}^2(\mathcal{Z})$,*

$$\mathcal{R}^{\#}(\Box g) = \Delta(R^{\#}g).$$

4. *The operator $\mathcal{R}^{\#}$ is the formal adjoint of $\mathcal{R}$, in the sense that for $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathcal{Z})$*

$$\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}} \mathcal{R}f(\theta, s)\, g(\theta, s)\, ds\, d\theta = \int_{\mathbb{R}^n} f(t)\, \mathcal{R}^{\#}g(t)\, dt.$$

In particular $f * (\mathcal{R}^{\#}g) = \mathcal{R}^{\#}(\mathcal{R}f * g)$.

Proof. Properties 1 and 2 are immediate from the definition. Property 3 is an immediate consequence of 2 and the fact that $\|\theta\|^2 = \sum\limits_{j=1}^{n} \theta_j^2 = 1$:

$$\Delta(R^{\#}g)(t) = \sum_{j=1}^{n} \mathcal{R}^{\#}\big(\theta_j^2 \frac{\partial^2 g}{\partial s^2}\big) = \mathcal{R}^{\#}\big(\frac{\partial^2 g}{\partial s^2}\big).$$

Property 4 follows from Fubini's theorem: writing $t = s\theta + y$, with $y \in \theta^\perp$

$$\begin{aligned}\int_{\mathbb{R}^n} f(t)\,\mathcal{R}^\# g(t)\,dt &= \int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}^n} f(t)\,g(\theta,\langle t,\theta\rangle)\,dt\,d\theta \\ &= \int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}\int_{\theta^\perp} f(s\theta+y)\,g(\theta,s)ds\,dy\,d\theta \\ &= \int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}\Big(\int_{\theta^\perp} f(s\theta+y)\,dy\Big)\,g(\theta,s)ds\ \ d\theta.\end{aligned}$$

□

We are ready to state the backprojection inversion formula. We begin with the case n (dimension) odd, which is technically a bit easier.

Theorem 19. *(Inversion formula for n odd) Let n be odd, let $f \in \mathcal{S}(\mathbb{R}^n)$ and denote $g = \mathcal{R}f$. Then*

$$f(t) = \frac{1/2}{(2\pi i)^{n-1}}\mathcal{R}^\#\big(\frac{\partial^{n-1}g}{\partial s^{n-1}}\big)(t)\,,\quad t\in\mathbb{R}^n.$$

In particular, for $n = 3$,

$$f(t) = -\frac{1}{8\pi^2}\mathcal{R}^\#\big(\frac{\partial^2 g}{\partial s^2}\big)(t) = -\frac{1}{8\pi^2}\mathcal{R}^\#\big(\Box g)(t) = -\frac{1}{8\pi^2}\Delta(\mathcal{R}^\# g)(t)\,,\quad t\in\mathbb{R}^3.$$

Proof. By definition

$$\frac{1/2}{(2\pi i)^{n-1}}\mathcal{R}^\#\big(\frac{\partial^{n-1}g}{\partial s^{n-1}}\big)(t) = \frac{1/2}{(2\pi i)^{n-1}}\int_{\mathbb{S}^{n-1}}\frac{\partial^{n-1}g}{\partial s^{n-1}}(\theta,\langle t,\theta\rangle)\,d\theta.$$

On the other hand, by the Fourier inversion formula proved above

$$f(t) = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\Big(\int_{\mathbb{R}}|\sigma|^{n-1}e^{2\pi i\sigma\langle t,\theta\rangle}\hat{g}(\theta,\sigma)\,d\sigma\Big)\,d\theta.$$

so we shall be done as soon as we see that

$$\int_{\mathbb{R}}|\sigma|^{n-1}e^{2\pi i\sigma\langle t,\theta\rangle}\hat{g}(\theta,\sigma)\,d\sigma = \frac{1}{(2\pi i)^{n-1}}\frac{\partial^{n-1}g}{\partial s^{n-1}}(\theta,\langle t,\theta\rangle).$$

Since n is odd we have $|\sigma|^{n-1} = \sigma^{n-1}$, and therefore (see Analytic Properties 2 and the inversion formula in the Annex)

$$\begin{aligned}\int_{\mathbb{R}}\sigma^{n-1}e^{2\pi i\sigma\langle t,\theta\rangle}\hat{g}(\theta,\sigma)\,d\sigma &= \frac{1}{(2\pi i)^{n-1}}\int_{\mathbb{R}}(2\pi i\sigma)^{n-1}\hat{g}(\theta,\sigma)\,e^{2\pi i\sigma\langle t,\theta\rangle}d\sigma \\ &= \frac{1}{(2\pi i)^{n-1}}\int_{\mathbb{R}}\widehat{\frac{\partial^{n-1}g}{\partial s^{n-1}}}(\theta,s)\,e^{2\pi i\sigma\langle t,\theta\rangle}d\sigma = \frac{1}{(2\pi i)^{n-1}}\frac{\partial^{n-1}g}{\partial s^{n-1}}(\theta,\langle t,\theta\rangle).\end{aligned}$$

□

Remark 14. When n is even $|\sigma|^{n-1} = \mathrm{sg}(\sigma)\sigma^{n-1}$, where $\mathrm{sg}(\sigma)$ indicates the sign of σ, so for the proof above to work we would need to identify $\psi(\theta, \sigma)$ for which

$$\widehat{\psi}(\theta, \sigma) = \mathrm{sg}(\sigma)\hat{g}(\theta, \sigma).$$

In this situation the Hilbert transform appears naturally.

Definition 17. The Hilbert transform of $f \in \mathcal{S}(\mathbb{R})$ is the function

$$Hf(t) = \frac{i}{\pi}\, p.v. \int_{-\infty}^{\infty} \frac{f(t-s)}{s}\, ds\,, \quad t \in \mathbb{R}.$$

Proposition 11. *Let $f \in \mathcal{S}(\mathbb{R})$. Then*

$$\widehat{Hf}(\sigma) = \mathrm{sg}(\sigma)\, \hat{f}(\sigma),$$

where the identity is understood as tempered distributions.

Once this result is proved, in case n even we have,

$$\int_{\mathbb{R}} |\sigma|^{n-1} e^{2\pi i \sigma \langle t, \theta \rangle} \hat{g}(\theta, \sigma)\, d\sigma = \frac{1}{(2\pi i)^{n-1}} \frac{\partial^{n-1} Hg}{\partial s^{n-1}}(\theta, \langle t, \theta \rangle),$$

and since by definition

$$\frac{\partial^{n-1} Hg}{\partial s^{n-1}} = H\big(\frac{\partial^{n-1} g}{\partial s^{n-1}}\big)$$

we obtain the following result.

Theorem 20. *(Inversion formula for n even) Let n be even, let $f \in \mathcal{S}(\mathbb{R}^n)$ and denote $g = \mathcal{R}f$. Then*

$$f(t) = \frac{1/2}{(2\pi i)^{n-1}} \mathcal{R}^{\#}\Big(H\big(\frac{\partial^{n-1} g}{\partial s^{n-1}}\big)\Big)(t)\,, \quad t \in \mathbb{R}^n.$$

In particular, for $n = 2$,

$$f(t) = \frac{1}{4\pi i} \mathcal{R}^{\#}\Big(H\big(\frac{\partial g}{\partial s}\big)\Big)(t)\,, \quad t \in \mathbb{R}^2.$$

Proof of Proposition 11. By definition

$$p.v. \int_{-\infty}^{\infty} \frac{f(t-s)}{s}\, ds = \lim_{\epsilon \searrow 0} \int_{|s|>\epsilon} \frac{f(t-s)}{s}\, ds.$$

We omit this notation and simply write

$$\int_{-\infty}^{\infty} \frac{f(t-s)}{s}\, ds.$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$; then, using that $\widehat{\tau_s f}(u) = \hat{f}(u)e^{-2\pi i u s}$,

$$\begin{aligned}\langle \widehat{Hf}, \varphi\rangle = \langle Hf, \widehat{\varphi}\rangle &= \frac{i}{\pi}\int_{\mathbb{R}}\Big(\int_{-\infty}^{\infty}\frac{f(t-s)}{s}\,ds\Big)\Big(\int_{\mathbb{R}}\varphi(u)e^{-2\pi i u t}du\Big)\,dt\\ &= \frac{i}{\pi}\int_{\mathbb{R}}\int_{-\infty}^{\infty}\frac{1}{s}\,\varphi(u)\Big(\int_{\mathbb{R}} f(t-s)\,e^{-2\pi i u t}dt\Big)\,ds\,du\\ &= \frac{i}{\pi}\int_{\mathbb{R}}\int_{-\infty}^{\infty}\frac{1}{s}\,\varphi(u)\,\hat{f}(u)\,e^{-2\pi i u s}ds\,du = \frac{i}{\pi}\int_{\mathbb{R}}\hat{f}(u)\varphi(u)\Big(\int_{-\infty}^{\infty}\frac{e^{-2\pi i u s}}{s}\,ds\Big)\,du.\end{aligned}$$

Thus shall be done as soon as we prove that

$$p.v.\int_{-\infty}^{\infty}\frac{e^{-2\pi i u s}}{s}\,ds = \frac{\pi}{i}\,\mathrm{sg}(u).$$

We have

$$\begin{aligned}p.v.\int_{-\infty}^{\infty}\frac{e^{-2\pi i u s}}{s}\,ds &= \int_{0}^{\infty}\frac{e^{-2\pi i u s}-e^{2\pi i u s}}{s}\,ds = -2i\int_{0}^{\infty}\frac{\sin(2\pi u s)}{s}\,,ds\\ &= -4\pi i u\int_{0}^{\infty}\frac{\sin(2\pi u s)}{2\pi u s}\,,ds.\end{aligned}$$

Since the sinc function is even

$$\int_{0}^{\infty}\frac{\sin(2\pi u s)}{2\pi u s}\,ds = \frac{1}{2}\int_{\mathbb{R}}\frac{\sin(2\pi u s)}{2\pi u s}\,ds,$$

and since (see Examples 1 (1)), for $u > 0$,

$$\widehat{\frac{1}{2u}\chi_{[-u,u]}}(s) = \frac{\sin(2\pi u s)}{2\pi u s},$$

the Fourier inversion formula gives

$$\int_{\mathbb{R}}\frac{\sin(2\pi u s)}{2\pi u s}\,ds = \int_{\mathbb{R}}\widehat{\frac{1}{2u}\chi_{[-u,u]}}(s)\,e^{2\pi i 0\,s}ds = \frac{1}{2u}\chi_{[-u,u]}(0) = \frac{1}{2u}.$$

When $u < 0$ we use the parity of the cardinal sine, replace u by $-u$ and follow the argument above to deduce that

$$\int_{\mathbb{R}}\frac{\sin(2\pi u s)}{2\pi u s}\,ds = -\frac{1}{2u}.$$

All combined

$$\int_{0}^{\infty}\frac{\sin(2\pi u s)}{2\pi u s}\,ds = \mathrm{sg}(u)\,\frac{1}{4u},$$

hence

$$p.v.\int_{-\infty}^{\infty}\frac{e^{-2\pi i u s}}{s}\,ds = (-4\pi i u)\,\frac{\mathrm{sg}(u)}{4u} = \frac{\pi}{i}\,\mathrm{sg}(u),$$

as desired. $\square$

Remark 15. Note that the derivative $\frac{\partial^{n-1}g}{\partial s^{n-1}}(\theta, \langle x, \theta\rangle)$ is determined by the values $g(\theta, t)$ for t near $\langle x, \theta\rangle$. Recall also that the hyperplanes containing $x \in \mathbb{R}^n$ are those of the form θ_s, with $s = \langle x, \theta\rangle$, $\theta \in \mathbb{S}^{n-1}$. Therefore, to compute

$$\int_{\mathbb{S}^{n-1}} \frac{\partial^{n-1}g}{\partial s^{n-1}}(\theta, \langle x, \theta\rangle)\, d\theta\,, \quad x \in \mathbb{R}^n$$

we only need the values of the integrals of f along all the hyperplanes through a neighbourhood of x. In this sense the inversion of the Radon transform in odd dimensions is a local problem.

The same conclusion is not possible when n is even, since the Hilbert transform is not local, that is, the value $Hf(t)$ depends on the value of f everywhere, not just in a neighbourhood of t.

7.5 Annex: the Fourier transform in $\mathbb{R}^n$

In Chapters 2 and 4 we have dealt only with functions of one variable. All the relevant definitions and results, with essentially the same proofs, hold as well for functions of several variables. For the sake of completeness we gather here some the main definitions and results.

Given $t, \xi \in \mathbb{R}^n$ we shall use the standard notation

$$\langle t, \xi\rangle = \sum_{k=1}^{n} t_k \xi_k.$$

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is the function $\hat{f} : \mathbb{R}^n \longrightarrow \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(t)\, e^{2\pi i\langle t,\xi\rangle} d\xi.$$

Properties. *Let $f, g \in L^1(\mathbb{R}^n)$ and let $\alpha, \beta \in \mathbb{C}$, $\lambda > 0$.*

1. *The Fourier transform is linear: $\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}$.*
2. *Conjugation: $\widehat{\overline{f}}(\xi) = \overline{\hat{f}(\xi)}$.*
3. *Translations: Given $a \in \mathbb{R}^n$ let $\tau_a f(t) = f(t - a)$. Then $\widehat{\tau_a f}(\xi) = \hat{f}(\xi)e^{-2\pi i\langle a,\xi\rangle}$.*
4. *Modulations: Given $a \in \mathbb{R}^n$ let $M_a f(t) = f(t)e^{2\pi i\langle a,t\rangle}$. Then $\widehat{M_a f}(\xi) = \tau_a \hat{f}(\xi)$.*
5. *Dilations: Let $D_\lambda f(t) = \frac{1}{\lambda^n} f(\frac{t}{\lambda})$. Then $\widehat{D_\lambda f}(\xi) = \hat{f}(\lambda\xi)$.*

The corresponding analytic properties also hold.

Analytic Properties. *Let $f \in L^1(\mathbb{R}^n)$.*

1. *$\hat{f}$ is continuous and bounded, with $|\hat{f}(t)| \leq \|f\|_1$.*

2. *If $\frac{\partial f}{\partial t_j}$ is also in $L^1(\mathbb{R}^n)$, then*
$$\widehat{\frac{\partial f}{\partial t_j}}(\xi) = 2\pi i \xi_j \, \hat{f}(\xi).$$

3. *If $t_j f(t)$ is also in $L^1(\mathbb{R}^n)$, then $\hat{f}$ is differentiable with respect to ξ_j and*
$$\frac{\partial \hat{f}}{\partial \xi_j}(\xi) = (-2\pi i \, t_j f)^\wedge(\xi).$$

4. *Riemann-Lebesgue lemma:* $\lim\limits_{|\xi|\to\infty} \hat{f}(\xi) = 0.$

5. *Multiplication formula: if g is also a function in $L^1(\mathbb{R}^n)$*
$$\int_{\mathbb{R}^n} f(t)\, \hat{g}(t)\, dt = \int_{\mathbb{R}^n} \hat{f}(t)\, g(t)\, dt$$

Remark 16. Since $G(t) := e^{-\pi\|t\|^2} = \prod_{j=1}^{n} e^{-\pi t_j^2}$ we see from Examples 1(2) and Fubini's theorem that the Fourier transform of the Gaussian G is G itself.

Inversion formula. *Let $f, \hat{f} \in L^1(\mathbb{R}^n)$. Then*
$$f(t) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \langle t,\xi\rangle} d\xi \qquad a.e. \quad t \in \mathbb{R}^n.$$

Plancherel theorem. *If $f \in L^2(\mathbb{R}^n)$ then $\hat{f} \in L^2(\mathbb{R}^n)$ and $\|f\|_2 = \|g\|_2$. In particular, if $f, g \in L^2(\mathbb{R}^n)$ then*
$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(t)\, \overline{g(t)}\, dt = \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi = \langle \hat{f}, \hat{g}\rangle.$$

7.5.1 Fourier transform and convolution

Given $f, g \in L^1(\mathbb{R}^n)$ the convolution is defined as in one variable:
$$(f * g)(t) = \int_{\mathbb{R}^n} f(t-s)\, g(s)\, ds = \int_{\mathbb{R}^n} f(s)\, g(t-s)\, ds.$$

Theorem. *Let $f \in L^p(\mathbb{R}^n)$, $p = 1, 2$, and $g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^p(\mathbb{R}^n)$ and*
$$(f * g)^\wedge(\xi) = \hat{f}(\xi)\, \hat{g}(\xi).$$

7.5.2 Fourier transform and distributions

The *Schwartz class* in $\mathbb{R}^n$ is

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) : P_{m,k}(\varphi) < \infty \, \forall m, k \in \mathbb{N}\},$$

where

$$P_{m,k}(\varphi) = \sup_{\substack{t \in \mathbb{R}^n \\ |\alpha| \leq k}} (1 + |t|)^m |D^\alpha \varphi(t)|.$$

Here we use the standard notation for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = \sum_{j=1}^{n} |\alpha_j|$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial t^{\alpha_1} \cdots \partial t^{\alpha_n}}.$$

A *tempered distribution* is a continuous linear map $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$.

The set of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$. Given $T \in \mathcal{S}'(\mathbb{R}^n)$, its *Fourier transform* is the tempered distribution $\widehat{T}$ defined by

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Properties. *Let T be a tempered distribution. Then*

1. $\frac{\partial \widehat{T}}{\partial t_j} = (\widehat{-2\pi i t_j}) T,$
2. $\widehat{\frac{\partial T}{\partial t_j}} = (2\pi i t_j) \widehat{T},$
3. $\widehat{\tau_a T} = e^{-2\pi i \langle a, \xi \rangle} \widehat{T}$ *and* $\tau_a \widehat{T} = \widehat{e^{2\pi i \langle a, \xi \rangle}} T.$

Poisson summation formula. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k).$$