# FOURIER TRANSFORM IN $L^2(\mathbb{R})$

We would like to take advantage of the fact that $L^2(\mathbb{R})$ has the structure of a Hilbert space. An initial inconvenience is that $L^2$ functions are not necessarily integrable (in $L^1$), so we have to be careful.

As usual, in $L^2(\mathbb{R})$ we consider the scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,dx$$

and the associated norm $\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\int_{\mathbb{R}} |f(x)|^2\,dx\right)^{1/2}$.

In $L^2(\mathbb{R})$ the rôles of $f$ and $\hat{f}$ are equivalent, and this symmetry is often quite useful. This is clear in the following result.

<u>Plancherel theorem</u>: Let $f \in L^1 \cap L^2$. Then $\hat{f} \in L^2(\mathbb{R})$ and $\|f\|_2 = \|\hat{f}\|_2$.

In particular, if $f, g \in L^1 \cap L^2$, then

$$\int_{\mathbb{R}} f(x)\,\overline{g(x)}\,dx = \int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi \quad .$$

As we see, the Fourier transform is an isometry in $L^2(\mathbb{R})$.

The requirement $f \in L^1(\mathbb{R})$ in this statement can be removed, if we interpret $\hat{f}$ for $f \in L^2$ in the appropriate way. We shall see this later.

For the proof we will need the following lemma.

<u>Lemma:</u> If $f, g \in L^2(\mathbb{R})$ then $f * g$ is a continuous bounded function with $\|f * g\|_\infty \le \|f\|_2 \|g\|_2$.

<u>Proof:</u> That $f * g$ is bounded is just consequence of the Cauchy - Schwarz inequality:

$$|(f*g)(x)| \le \int_{\mathbb{R}} |f(t)\, g(x-t)|\,dt \le \left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} |g(x-t)|^2 dt\right)^{1/2}$$

$$= \|f\|_2 \cdot \|g\|_2 .$$

Notice also that the convolution is well-defined:

$$|f(t)\, g(x-t)| \le |f(t)|^2 + |g(x-t)|^2$$

and $f, g \in L^2(\mathbb{R})$.

In order to prove the continuity observe that, by the Cauchy - Schwarz inequality

$$|(f*g)(x+h)-(f*g)(x)| \le \int_{\mathbb{R}} |f(x+h-t)-f(x-t)|\,|g(t)|\,dt$$

$$\le \left( \int_{\mathbb{R}} |f(x+h-t)-f(x-t)|^2\,dt \right)^{1/2} \cdot \|g\|_2$$

$$= \|\tau_h f - f\|_2 \cdot \|g\|_2$$

We finish by using the continuity with respect to translations of the $L^2$-norm, as seen in a previous lemma (page 2.4 in the document on Fourier transform).

<u>Proof</u> (of Plancherel theorem)

Define $\tilde{f}(x) = \overline{f(-x)}$. This is done so that $\hat{\tilde{f}}(\xi) = \overline{\hat{f}(\xi)}$. Let also $g = f * \tilde{f}$. Then, by the Lemma, $g$ is continuous and bounded. Moreover $\hat{g}(\xi) = \hat{f}(\xi) \cdot \hat{\tilde{f}}(\xi) = |\hat{f}(\xi)|^2$.

As in the proof of the inversion formula, consider the "approximation of the identity" $G_\delta(x) = \frac{1}{\delta} G(\frac{x}{\delta})$, $\delta > 0$, where $G(t) = e^{-\pi t}$. Explicitly $G_\delta(x) = \frac{1}{\delta} e^{-\pi \frac{x^2}{\delta^2}}$.

Let also $F_\delta(t) = e^{-\pi \delta^2 t^2}$ and recall that $G_\delta = \hat{F}_\delta$.

Notice that $\displaystyle g(0) = \int_{\mathbb{R}} f(t)\, \tilde{f}(0-t)\,dt = \int_{\mathbb{R}} |f(t)|^2\,dt = \|f\|_2^2$

and that $\displaystyle \int_{\mathbb{R}} \hat{g}(\xi)\,d\xi = \int_{\mathbb{R}} |\hat{f}(\xi)|^2\,d\xi = \|\hat{f}\|_2$.

Thus, all what we need to show is the inversion formula

$$g(0) = \int_{\mathbb{R}} g(\xi)\, d\xi .$$

We do so by regularising, taking the convolution with $G_\delta$. We have

$$(g * G_\delta)(x) = \int_{\mathbb{R}} g(x-t)\, G_\delta(t)\, dt = \int_{\mathbb{R}} g(x-t)\, \hat{F}_\delta(t)\, dt$$

Apply here the multipliation formula, since for $h(t) := g(x-t)$ we have

$$\hat{h}(\xi) = \int_{\mathbb{R}} g(x-t)\, e^{-2\pi i t\xi}\, dt = \int_{\mathbb{R}} g(s)\, e^{2\pi i (s-x)\xi}\, ds$$

$$= e^{-2\pi i x} \int_{\mathbb{R}} g(s)\, e^{-2\pi i s(-\xi)}\, ds = e^{-2\pi i x\xi}\, \hat{g}(-\xi),$$

we obtain

$$(g * G_\delta)(x) = \int_{\mathbb{R}} e^{-2\pi i x\xi}\, \hat{g}(-\xi)\, F_\delta(\xi)\, d\xi$$

Evaluating at $x = 0$

$$(g * G_\delta)(0) = \int_{\mathbb{R}} \hat{g}(-\xi)\, e^{-\pi\delta^2\xi^2}\, d\xi = \int_{\mathbb{R}} \hat{g}(\xi)\, e^{-\pi\delta^2\xi^2}\, d\xi$$

We want to take the limit of this identity as $\delta \to 0$.

We start with the left hand side. In order to see that $\lim_{\delta \to 0} (g * G_\delta)(0) = g(0)$ fix $\varepsilon > 0$ and, using the lemma, take $\eta > 0$ so that

$$|t| < \eta \Rightarrow |g(t) - g(0)| < \varepsilon$$

Then, since $G_\delta \geq 0$ and $\int_{\mathbb{R}} G_\delta(t)\, dt = 1$,

$$|(g * G_\delta)(0) - g(0)| = \left| \int_{\mathbb{R}} (g(-t) - g(0))\, G_\delta(t)\, dt \right| \leq$$

$$\leq \int_{\mathbb{R}} |g(t) - g(0)|\, G_\delta(t)\, dt =$$

$$= \int_{|t|<\eta} |g(t) - g(0)|\, G_\delta(t)\, dt + \int_{|t| \geq \eta} |g(t) - g(0)|\, G_\delta(t)\, dt$$

$$\leq \varepsilon + 2\|g\|_\infty \int_{|t| \geq \eta} G_\delta(t)\, dt$$

Since

$$\int_{|t| \geq \eta} G_\delta(t)\, dt = \int_{|t| \geq \eta} \frac{1}{\delta} e^{-\pi \frac{t^2}{\delta^2}}\, dt = \int_{|s| \geq \eta/\delta} e^{-\pi s^2}\, ds.$$

there exists $\delta > 0$ so that $2\|g\|_\infty \int_{|t| \geq \eta} G_\delta(t)\, dt < \varepsilon$

and we are done.

In order to prove that

$$\lim_{\delta\to 0} \int_{\mathbb{R}} \hat{g}(\xi) e^{-\pi\delta^2\xi^2} d\xi = \int_{\mathbb{R}} \hat{g}(\xi)\, d\xi = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$$

we want to use the dominated convergence theorem. Since

$$|\hat{g}(\xi) e^{-\pi\delta^2\xi^2}| \le |\hat{g}(\xi)| = |\hat{f}(\xi)|^2$$

all we need to show is that $\hat{f} \in L^2(\mathbb{R})$.

Since $$(g * G_\delta)(0) = \int_{\mathbb{R}} \hat{g}(\xi) e^{-\pi\delta^2\xi^2} d\xi$$

and $\lim_{\delta\to 0} (g * G_\delta)(0) = g(0) = \int_{\mathbb{R}} |f(t)|^2 dt$, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$

$$\left| \int_{\mathbb{R}} \hat{g}(\xi) e^{-\pi\delta^2\xi^2} d\xi \right| \le 2 \int_{\mathbb{R}} |f(t)|^2 dt = 2\|f\|_2^2$$

This is $$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 e^{-\pi\delta^2\xi^2} d\xi \le 2\|f\|_2^2$$

Now, fixed any $R > 0$ there exists $\delta_R > 0$ small enough so that $e^{-\pi\delta^2R^2} \ge 1/2$, and therefore

$$\int_{-R}^{R} |\hat{f}(\xi)|^2 d\xi \le 2 \int_{-R}^{R} |\hat{f}(\xi)|^2 e^{-\pi\delta^2\xi^2} d\xi \le 4\|f\|_2^2 .$$

Since this holds for all $R > 0$, $\hat{f} \in L^2(\mathbb{R})$. $\boxtimes$

<u>Digression</u>. Getting rid of the condition $f \in L^1$.

Given $f \in L^2(\mathbb{R})$ it is easy to find $f_n \in L^2 \cap L^1$ and such that $\|f_n - f\|_2 \xrightarrow{n\to\infty} 0$ (for instance $f_n = f\chi_{[-n,n]}$). Then, by Plancherel $\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2 \to 0$, so $\hat{f}_n$ also converges in $L^2$. The limit of such sequence defines the Fourier transform of $f \in L^2$. This definition does not depend on the particular sequence $\{f_n\}_n$, as long as $\|f_n - f\|_2 \to 0$. If $\{f_n\}_n$, $\{g_n\}_n$ are two sequences of functions in $L^1 \cap L^2$ such that $\lim_{n\to\infty} \|f_n - f\|_2 = \lim_{n\to\infty} \|g_n - f\|_2 = 0$ then $\|f_n - g_n\|_2 = \|\hat{f}_n - \hat{g}_n\|_2$ tends to $0$ as well. We can summarize all this in the following statement.

<u>Theorem</u>: For $f \in L^2(\mathbb{R})$

(a) $\hat{f}(\xi) = \lim_{n\to\infty} \int_{-n}^{n} f(x) e^{-2\pi i x \xi}\, dx$. (Here the limit is in the $L^2$ sense, as explained above)

(b) $f$ and $\hat{f}$ satisfy the Plancherel identity.

(c) If also $g \in L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(x)\,\hat{g}(x)\,dx = \int_{\mathbb{R}} \hat{f}(\xi)\,g(\xi)\,d\xi$$

and

$$\int_{\mathbb{R}} f(x)\,\overline{g(x)}\,dx = \int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi .$$

Similarly, we can easily describe the Fourier transform of the convolution.

<u>Theorem:</u> Let $f \in L^2$ and $g \in L^1$. Then $f * g \in L^2$ and $(f*g)^\wedge(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)$.

<u>Proof:</u> That $f * g \in L^2$ is clear, by the Cauchy-Schwarz inequality:

$$\int_{\mathbb{R}} |(f*g)(x)|^2 dx \le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-t)|\,|g(t)|^{1/2}\,|g(t)|^{1/2} dt\,dx \le$$

$$\le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-t)|^2\,|g(t)|\,\|g\|_1\,dt\,dx \le$$

$$\le \|g\|_1 \int_{\mathbb{R}} |g(t)| \left( \int_{\mathbb{R}} |f(x-t)|^2 dx \right) dt = \|g\|_1^2\,\|f\|_2^2 .$$

The rest is as in the $L^1$-case. $\boxtimes$