

CHAPTER 1

Fundamentals

1.1 Sets and Mappings

1.1.1 Sets

A set is a collection of objects.

We usually denote sets by uppercase letters, e.g. A . The elements of a set are usually denoted by lowercase letters, and we write $a \in A$ in order to indicate that a is an element of the set A . Accordingly, we write $a \notin A$ if a is not an element of A . The empty set, which does not contain any elements, is denoted by \emptyset .

We often characterise a set A by means of some property P of its elements, e.g.

$$A = \{a \mid P(a) \text{ is true}\}.$$

We say that the set A is a subset of the set B , i.e. $A \subset B$, if and only if $a \in A$ implies $a \in B$:

$$A \subset B \Leftrightarrow \forall a \in A: a \in B. \quad (1.1)$$

The empty set is a subset of any set. Two sets A, B are equal if and only if $A \subset B$ and $B \subset A$ holds. If A is not a subset of B , we write $A \not\subset B$.

We furthermore will need the intersection

$$A \cap B = \{a \mid (a \in A) \text{ and } (a \in B)\} \quad (1.2)$$

and union

$$A \cup B = \{a \mid (a \in A) \text{ or } (a \in B)\} \quad (1.3)$$

of two sets A, B .

The set $\mathcal{P}(A)$ of all subsets of A is called the power set of A

$$\mathcal{P}(A) = \{B \mid B \subset A\}. \quad (1.4)$$

We now define the Cartesian product of two sets A, B .

Definition 1.1 (Cartesian product) Let A, B be two sets and let $a \in A, b \in B$. The ordered pair of a and b is the object (a, b) (in that order), associated with a and b , such that two ordered pairs (a, b) and (a', b') are equal if and only if $a = a'$ and $b = b'$.

The set

$$A \times B = \{(a, b) \mid a \in A, b \in B\} \quad (1.5)$$

is called the Cartesian Product of A and B and we say A times B for $A \times B$.

Remark 1.1

- With the term “ordered” we distinguish the ordered pair (a, b) from the set $\{a, b\}$
- We have $(a, b) = (a', b')$ iff $a = a'$ and $b = b'$. In particular we have $(a, b) \neq (b, a)$ for $a \neq b$ and accordingly $A \times B \neq B \times A$ for $A \neq B$.
- In general, $(A \times B) \times C \neq A \times (B \times C)$.

1.1.2 Mappings

Definition 1.2 (Maps, mapping, function) Let X, Y be sets. We call a well-defined rule f , which associates to each element $x \in X$ exactly one element $f(x) = y \in Y$, a mapping (or function) and write

$$f: X \longrightarrow Y, \quad x \in X \mapsto f(x) = y \in Y.$$

Furthermore, we call the set X the domain (or definition set) of the mapping f , $X = \text{domain}(f)$, and Y the co-domain (or target set) of the mapping f , $Y = \text{codomain}(f)$.

Remark 1.2

- A mapping can be considered to be the triple X, Y, f . If any of these three objects changes, we get a different mapping.
- By definition, for every $x \in X$ there exists at least one $y \in Y$ such that $y = f(x)$.
- It is not required that for every $y \in Y$ there exists an $x \in X$ such that $y = f(x)$.
- Two different elements $x, x' \in X$, $x \neq x'$, can get assigned the same value i.e. $f(x) = f(x')$. For example, consider the constant mapping $f(x) = c$ for all $x \in X$, where $c \in Y$ is arbitrary but fixed.

Definition 1.3 (Image and preimage) Let X, Y be sets and let $f: X \longrightarrow Y$ be a mapping.

a) We call the set

$$\text{Im}(f) = \{y \mid y \in Y, \exists x \in X \text{ s.t. } y = f(x)\} \quad (1.6)$$

the image of f .

b) Let $U \subset X$. We call the set

$$f(U) = \{y \in Y \mid y = f(x), x \in U\} \quad (1.7)$$

the image of U .

c) Let $V \subset Y$. We call the set

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \quad (1.8)$$

the preimage of the set V .

Image and preimage can be viewed as set-valued mappings

$$f: \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$$

and

$$f^{-1}: \mathcal{P}(Y) \longrightarrow \mathcal{P}(X),$$

respectively. Of particular interest are the preimages $f^{-1}(\{y\})$ of the singleton sets $\{y\}$, $y \in Y$. Please note that the preimage of a singleton set can easily contain several elements or even none. As an example, take $X = Y = \mathbb{N}$ and the constant function $f(x) = c$, where $c \in \mathbb{N}$ is arbitrary but fixed. Then $f^{-1}(\{c\}) = \mathbb{N}$, but for every $c \neq c' \in \mathbb{N}$ we will have $f^{-1}(\{c'\}) = \emptyset$.

Definition 1.4 (Injective, surjective, bijective) Let X, Y be sets and let $f: X \longrightarrow Y$ be a mapping.

a) We call f injective if for all $x, x' \in X$ we have

$$x \neq x' \implies f(x) \neq f(x') \quad (1.9)$$

b) We call f surjective if for all $y \in Y$ there exists $x \in X$ such that $y = f(x)$.

c) We call f bijective if f is injective and surjective.

For bijective f , we see that for every $y \in Y$ there is one and only one $x \in X$ with $y = f(x)$, i.e. with $\{x\} = f^{-1}(\{f(x)\})$. As a consequence, the mapping $g: Y \longrightarrow X$ with

$$g(y) = g(f(x)) = x$$

is well-defined and bijective and we have for all $x \in X$ the identity

$$g(f(x)) = x$$

or

$$g \circ f = I_X,$$

where $I_X: X \longrightarrow X$, $x \mapsto x$ is the identity mapping on X . Accordingly, we have

$$f \circ g = I_Y,$$

We therefore can call g inverse function to f and write $g = f^{-1}$.

We emphasize that the inverse function f^{-1} , the value $x = f^{-1}(y) \in X$ of the inverse function for some $y \in Y$ and the pre-image $f^{-1}(V) \subset X$ of a subset $V \subset Y$ are different mathematical objects and should not be confused, even if the same symbol f^{-1} is used in all cases. For example, the preimage $f^{-1}(V) \subset X$ of a set $V \subset Y$ is a subset of X , whereas the value of the inverse function $x = f^{-1}(y) \in X$ is an element of X .

Remark 1.3 Although we define our objects precisely in writing, we often allow for some flexibility if the meaning is clear from the context or by convention. You may call this an “abuse of notation” or “abuse of terminology”. For example, you will find statements such as “Let us consider the function x^2 ”, which, strictly speaking, is also not correct.

1.2 Numbers

We denote by

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\} \quad (1.10)$$

the set of natural numbers.

We denote by

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\} \quad (1.11)$$

the set of integers.

We denote by

$$\mathbb{Q} = \{r \mid r = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}\} \quad (1.12)$$

the set of rational numbers.

We define the real numbers as the set of all (possibly infinite) decimal expansions

$$\begin{aligned} \mathbb{R} = & \{x \mid x = \sigma \cdot 10^N \cdot \sum_{i=1}^{\infty} x_i 10^{-i}, \\ & \sigma \in \{-1, 1\}, N \in \mathbb{Z}, x_i \in \{0, 1, \dots, 9\}, x_1 \neq 0\} \cup \{0\}. \end{aligned} \quad (1.13)$$

Example 1.1

- 5 is a natural number.
- -12 is an integer but not a natural number
- $\frac{1}{2}$ is a rational number, but not an integer
- $\sqrt{2}$ is a real number, but not a rational number
- -123.456 has the representation

$$\begin{aligned} -132.546 = & (-1) \cdot (1 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 10^0 \\ & + 5 \cdot 10^{-1} + 4 \cdot 10^{-2} + 6 \cdot 10^{-3}) \\ = & (-1) \cdot 10^3 \cdot (1 \cdot 10^{-1} + 3 \cdot 10^{-2} + 2 \cdot 10^{-3} \\ & + 5 \cdot 10^{-4} + 4 \cdot 10^{-5} + 6 \cdot 10^{-6}), \end{aligned}$$

i.e. we have the representation from (1.13) with $\sigma = -1$, $N = 3$, $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, $x_4 = 5$, $x_5 = 4$, $x_6 = 6$ and $x_i = 0$ for $i \geq 7$.

1.2.1 Peano Axioms

The natural numbers, which we have introduced above in Equation (1.10) in a rather naive way, can be constructed by means of the so-called Peano Axioms (Giuseppe Peano; 1858–1932).

Definition 1.5 (Peano Axioms) \mathbb{N} is a set with the following properties

- (1) \mathbb{N} has a distinguished element called 0
- (2) There exists a successor map $S: \mathbb{N} \longrightarrow \mathbb{N}$
- (3) $0 \notin S(\mathbb{N})$
- (4) If $m, n \in \mathbb{N}$, then $S(m) = S(n)$ implies $m = n$
- (5) Let E be a set with $0 \in E$ and if it always follows from $n \in E$ that $S(n) \in E$, then $\mathbb{N} \subset E$

We note that instead of 0, 1 can also be taken as the distinguished element. Axioms 2 and 4 ensure the existence and injectivity of the successor map S , which we can imagine as $S(n) = n + 1$, respectively. Axiom 3 says that 0 is not the successor of any natural number, i.e. S is not surjective. It avoids that 0 can be taken as a re-entry point. Axiom 5 is also called the principle of induction for the following reason: If 0 has the property E and if “ n has property E ” always implies that “ $S(n)$ has property E ”, then the property E follows from the property of being a natural number.

The fact that every natural number has a unique successor is a property which does not transfer to the rational or real numbers. For example, assume that $S(r)$ is the successor of a given rational or real number r and that $S(r) > r$. Then, we can compute $\frac{r+S(r)}{2}$ and we have

$$r < \frac{r+S(r)}{2} < S(r).$$

This means that $\frac{r+S(r)}{2}$ is also a possible successor to r . As this process can be repeated ad infinitum, we cannot define a successor function on \mathbb{Q} or \mathbb{R} .

Remark 1.4 We state the Peano axioms for convenience in pure text form:

- (1) Zero is a natural number.
- (2) Every natural number has a successor in the natural numbers.
- (3) Zero is not the successor of any natural number.
- (4) If the successor of two natural numbers is the same, then the two original numbers are the same.
- (5) If a set contains zero and the successor of every number is in the set, then the set contains the natural numbers.

1.3 Absolute Value

For $x \in \mathbb{R}$, the absolute value $|x|$ of x is defined as

$$|\cdot|: \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathbb{R} \ni x \mapsto |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Equivalently, we can write

$$|x| = \max(x, -x).$$

We have the following properties of $|\cdot|$:

Theorem 1.1 For the absolute value $|\cdot|: \mathbb{R} \longrightarrow \mathbb{R}$ it holds

- | | |
|---|---------------------|
| (1) We have $ x \geq 0$ for all $x \in \mathbb{R}$ and $ x = 0 \Leftrightarrow x = 0$ | definiteness |
| (2) We have $ x \cdot y = x \cdot y $ for all $x, y \in \mathbb{R}$ | homogeneity |
| (3) We have $ x + y \leq x + y $ for all $x, y \in \mathbb{R}$ | triangle inequality |

Proof We only prove the first two properties and leave the triangle inequality as a guided exercise to the reader.

Definiteness: follows immediately from the definition.

Homogeneity: For $x, y > 0$ nothing has to be shown. For the general case we write $x = \pm \hat{x}$ and $y = \pm \hat{y}$ with $\hat{x}, \hat{y} \geq 0$. By definition, $|x| = |\pm \hat{x}| = |\hat{x}|$. This gives rise to

$$|xy| = |\pm \hat{x}\hat{y}| = |\hat{x}\hat{y}| = |\hat{x}||\hat{y}|$$

□

Remark 1.5 The absolute value can also be used to measure the distance between two points $x, y \in \mathbb{R}$. To this end, we set $d(x, y) = |x - y|$ and call $d(x, y)$ the distance between x and y . We call $d(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ a metric.

In order to become familiar with the absolute value, the reader may work on the following exercises.

Exercise 1.1 To work on the following exercises, please follow the hints carefully.

- Proof **Theorem 1.1.** Hint: We have $x \leq |x|$ and $-x \leq |x|$ for all $x \in \mathbb{R}$. Moreover, we see that for showing that $|x + y| \leq |x| + |y|$, we have to show two inequalities, i.e.

$$x + y \leq |x| + |y|$$

and

$$-(x + y) \leq |x| + |y|.$$

To show the first one, exploit that $a \leq |a|$ for all $a \in \mathbb{R}$. To show the second one, exploit that $-a \leq |a|$ for all $a \in \mathbb{R}$.

- Show that $E \subset \mathbb{R}$ is bounded, iff there exists $M > 0$ such that $|x| < M$ for all $x \in E$.
- Give examples of subsets from \mathbb{R} which are bounded from above, bounded from below, or unbounded, respectively, and give respective bounds, if they exist.

- Sketch the set $E = \{x : |x - 3| \leq 6\}$.
- Let $x, y \in \mathbb{R}$ and $y \neq 0$. Show, that

$$\left| \frac{x}{y} \right| = \frac{|x|}{|y|}.$$

Hint: $x = \frac{x}{y} \cdot y$, $y \neq 0$.

1.3.1 Intervals

Let $a, b \in \mathbb{R}$. For $a \leq b$, we define the closed interval $[a, b]$ as

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

For $a = b$, the interval $[a, b]$ contains only a single point. Now let $a < b$. We define the open interval (a, b) as

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

The intervals $(a, b]$ and $[a, b)$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\} \text{ and } [a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

are called half-open. We also write

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\} \text{ and } (a, \infty) = \{x \in \mathbb{R} : x > a\}$$

as well as

$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\} \text{ and } [a, \infty) = \{x \in \mathbb{R} : x \geq a\}.$$

1.3.2 Open and Closed sets

Definition 1.6 Let $\delta > 0$ and $x \in \mathbb{R}$. Then we call the set

$$B_\delta(x) = \{z \in \mathbb{R} : |z - x| < \delta\}.$$

the open ball with radius δ around $x \in \mathbb{R}$.

Definition 1.7 (Open and Closed Sets) We call a set $N \subset \mathbb{R}$ a neighborhood of $x \in \mathbb{R}$ if it contains an open ball which contains x , i.e. if there exist a $\delta > 0$ such that $B_\delta(x) \subset N$.

A subset $O \subset \mathbb{R}$ is called open if O is a neighbourhood of each of its points. A subset $A \subset \mathbb{R}$ is called closed if its complement $\mathbb{R} \setminus A$ is open.

Remark 1.6 Open sets are important to define the convergence of sequences, continuity of functions, derivatives, and many more. So, we collect some properties of open sets.

- $U \subset \mathbb{R}$ is open iff for every $x \in U$ there exists a $\delta > 0$ such that $B_\delta(x) \subset U$.
- As an open set contains a ball with radius $\delta > 0$ around each point — it seems intuitively clear that an open set cannot contain “boundary points”. As an example, let us consider the interval $[0, 1)$ in \mathbb{R} . This

interval cannot be open, as for every $\delta > 0$, we see that $B_\delta(0) = (-\delta, \delta) \not\subset [0, 1]$. The “right” boundary of $[0, 1)$ does not cause any problems. This can be seen as follows: Let $\tilde{x} < 1$ and $|\tilde{x} - 1| \ll 1$. Here, the symbol “ \ll ” means “significantly smaller”, and this condition shall indicate that \tilde{x} is very close to 1.

Then, we can find an $\varepsilon > 0$ such that $\tilde{x} = 1 - \varepsilon$. By choosing $\delta = \varepsilon/2$, we have found $B_\delta(\tilde{x}) \subset [0, 1)$.

- A set $U \subset \mathbb{R}$ can be either open, closed, or both, or neither. Examples of subsets which are closed as well as open are the whole space \mathbb{R} and the empty set \emptyset .
- Let $a, b \in \mathbb{R}, a < b$. Then, the intervals $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ are open; the intervals $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ are closed; the intervals $(-\infty, a)$ and (a, ∞) are open (why?), and the intervals $(-\infty, a]$ and $(a, \infty]$ are closed.

Exercise 1.2 Let $O_n \subset X, n = 1, 2, \dots, N$ be open sets. Show that $O = \bigcup_{n=1}^N O_n$ is open in X . Hint: Exploit the definition of the union of sets. Given a point in O , in which set(s) can you find a neighbourhood of x ?

1.4 Some Useful Equalities and Inequalities

In this section, we present some quite useful equalities and inequalities. In order to give the possibility to practice, the proofs are put at the end of this section

Theorem 1.2 (Generalized Triangle inequality) Let $x, y \in \mathbb{R}$. Then we have

$$|x - y| \geq |x| - |y| \text{ and } |x + y| \geq |x| - |y|. \quad (1.14)$$

Proof Left as an exercise for the reader □

Exercise 1.3 Prove [Theorem 1.2](#). Hint: Set $x = (x - y) + y$ and exploit the triangle inequality for the first inequality. For the second, apply the same trick but replace y with $-y$.

Theorem 1.3 (Inequality of Bernoulli) Let $x \in \mathbb{R}, x \geq 1$. Then it holds

$$(1 + x)^n \geq 1 + nx \text{ for all } n \in \mathbb{N}. \quad (1.15)$$

Exercise 1.4 Prove [Theorem 1.3](#).

Theorem 1.4 (An Elementary Young's Inequality) Let $a, b \in \mathbb{R}$ and $\varepsilon > 0$. It holds

$$\begin{aligned} ab &\leq \frac{1}{2}(a^2 + b^2) \\ ab &\leq \frac{1}{2}\left(\frac{a^2}{\varepsilon} + \varepsilon b^2\right) \\ |ab| &\leq \frac{1}{2}(a^2 + b^2). \end{aligned} \tag{1.16}$$

Proof We show the first inequality and leave the other two as an exercise. Let now $a, b \in \mathbb{R}$ and $\varepsilon > 0$. We have

$$0 \leq (a - b)^2 = a^2 - 2ab + b^2,$$

which implies the first inequality. \square

Exercise 1.5 Prove [Theorem 1.4](#). For the proof of the third inequality, we note that by definition of the absolute value we have for $x \in \mathbb{R}$

$$x = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Thus, show and exploit ($x = ab$) that

$$\begin{aligned} ab &\leq \frac{1}{2}(a^2 + b^2) \\ -ab &\leq \frac{1}{2}(a^2 + b^2). \end{aligned}$$

Theorem 1.5 (Geometric and Arithmetic Mean) Let $a \geq 0, b \geq 0$, $a, b \in \mathbb{R}$. Then

$$\sqrt{ab} \leq \frac{1}{2}(a + b). \tag{1.17}$$

Proof Left as an exercise to the reader. \square

Exercise 1.6 Prove [Theorem 1.5](#). Hint: Use Young's inequality.

1.5 Important Functions

1.5.1 Exponential and Logarithmic Functions

Definition 1.8 (Exponential Function) For all $x \in \mathbb{R}$, we define the exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad (1.18)$$

Definition 1.9 (Euler's Number) Euler's number, denoted by the letter e , is defined as

$$e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \dots = 2.7182818\dots \quad (1.19)$$

An equivalent definition of (1.19) is

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

Definition 1.10 (Natural Logarithm) The inverse function of the exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}$ is $\log : \mathbb{R}^+ \rightarrow \mathbb{R}$, and it is called the natural logarithm. It has the following property

$$\log(xy) = \log x + \log y \text{ for all } x, y \in \mathbb{R}^+. \quad (1.20)$$

Definition 1.11 (Exponential Function for base a) For $a > 0$, we define $\exp_a : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\exp_a(x) = \exp(x \log a). \quad (1.21)$$

Definition 1.12 (Logarithmic Function for Base a) For $a > 0$, we define $\log_a : \mathbb{R}^+ \rightarrow \mathbb{R}$ as the inverse function of $\exp_a : \mathbb{R} \rightarrow \mathbb{R}$. As in Definition 1.10, we have

$$\log_a(xy) = \log_a x + \log_a y \text{ for all } x, y \in \mathbb{R}^+. \quad (1.22)$$

1.5.2 Trigonometric Functions

Consider the unit circle (also called the unit 1-sphere), in the Cartesian plane, defined by the equation

$$S^1 = \{(x, y) : (x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}. \quad (1.23)$$

Definition 1.13 A rotation in the plane is said to be of positive magnitude or “in the mathematically positive sense” if it is counterclockwise. A rotation in the plane is said to be of negative magnitude or “in the mathematically negative sense” if it is clockwise. Let a ray be drawn from the origin such that it intersects the unit sphere S^1 in the point $P = (x, y) \in S^1$.

Using angles: We denote by θ the positive angle that this ray makes with the positive x-axis, i.e. the angle measured in the direction of positive rotation. Then, we have the following definitions for the sine, $\sin: \mathbb{R} \rightarrow \mathbb{R}$, and the cosine, $\cos: \mathbb{R} \rightarrow \mathbb{R}$.

$$\sin \theta = y, \quad \cos \theta = x. \quad (1.24)$$

Here, repeated rotations lead to values of θ larger (smaller) than 360° (-360°), respectively.

Using radians: We denote by α the signed length of the segment of the unit circle, which is between the point $(1, 0)$ and the point (x, y) , measured in direction of positive rotation. Then, we have the following definitions for the sine, $\sin: \mathbb{R} \rightarrow \mathbb{R}$, and the cosine, $\cos: \mathbb{R} \rightarrow \mathbb{R}$.

$$\sin \alpha = y, \quad \cos \alpha = x. \quad (1.25)$$

Here, repeated rotations lead to values of α larger (smaller) than 2π (-2π), respectively.

Remark 1.7 The circumference of the unit circle has length 2π . Thus, we can directly see the values of sine and cosine for certain values of α . For example, we see that $0 = \sin(0) = \sin(\pi) = \sin(2\pi) = \sin(k\pi)$, $k \in \mathbb{Z}$, and that $1 = \sin(\pi/2)$ and $-1 = \sin(3/2\pi)$ or $1 = \cos(0)$ and $0 = \cos(3/2\pi)$ or $\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$ and so on. We have

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1, x \in \mathbb{R} \\ \cos(-x) &= \cos(x) \\ \sin(-x) &= -\sin(x) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y). \end{aligned}$$

As you will see in Calculus I, \sin and \cos can also be defined via the following series

$$\begin{aligned} \cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \mp \dots \\ \sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \dots \end{aligned}$$

Please note the strong similarities with the series for the exponential function (1.18)