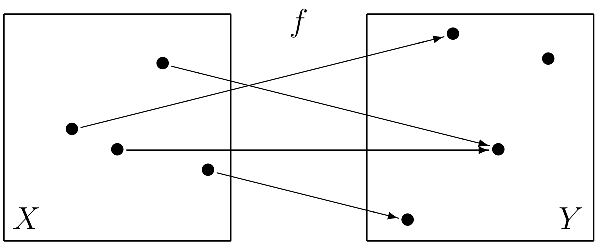
1.4 Functions

A function $f\colon X \to Y$ is a rule, which assigns every $x \in X$ of a set X exactly one element $f(x) \in Y$. This can be illustrated in accordance with:



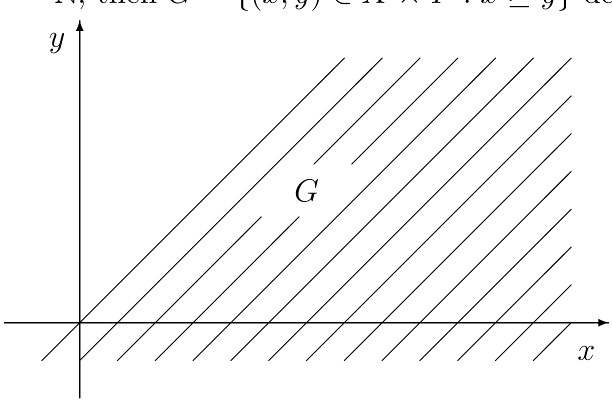
Definition 1.10 Let $X, Y \neq \emptyset$ be arbitrary sets and $G \subset X \times Y$.

1. The triple (G, X, Y) is called a **relation** between the sets X and Y.
2. A relation (G, X, Y) between the sets X and Y is a **function** $f : X \to Y$, if
   a) for each $x \in X$, there exists a $y = f(x)$ with $(x, y) \in G$, that is
   $$\forall_{x\in X}\ \exists_{y\in Y} : (x, y) \in G;$$
   b) for all $x \in X$ and $y, y' \in Y$ such that $(x, y), (x, y') \in G$, it follows that $y = y'$ is unique, that is
   $$\forall_{x\in X,\, y,y'\in Y}\big((x, y) \in G \wedge (x, y') \in G) \Rightarrow y = y'\big).$$

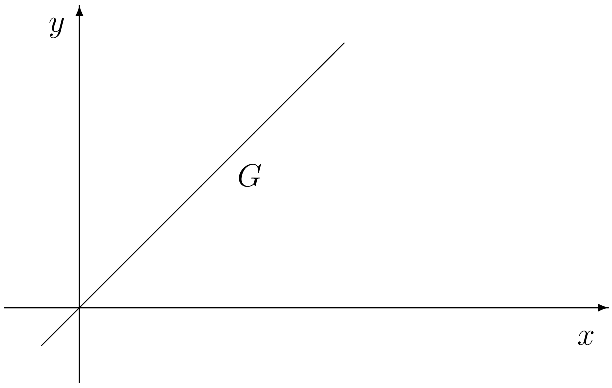
A function is often also called **mapping**.

Example 1.11

1. If we choose $X = Y = \mathbb{N}$, then $G = \{(x, y) \in X \times Y : x \geq y\}$ defines a relation.



2. If $X = Y \neq \emptyset$, then a function is given by $G = \{(x, y) \in X \times Y : x = y\}$.



△

In case of a function, we write $f\colon X \to Y$ and call X the *domain* and Y the *codomain* of f. The set $G = \{(x, y) : y = f(x),\ x \in X\}$ is the *graph* of f and it holds $y = f(x) \Leftrightarrow (x, y) \in G$. The domain and codomain of a function have to be known for the precise description of a function. Two functions $f\colon X \to Y$ and $g\colon A \to B$ are equal if and only if $A = X$, $B = Y$, and $f(x) = g(x)$ for all $x \in X$.

Definition 1.12 Let $A \subset X$ be a subset of the domain X of the function f. Then, the set $f(A) = \{y \in Y : y = f(x), x \in A\} \subset Y$ is called the **image** of A under the function f. If $B \subset Y$, then the set

$$f^{-1}(B) = \{x \in X : f(x) = y,\ y \in B\} \subset X$$

is the **preimage** of the set B under f.

Definition 1.13 The function $f\colon X \to Y$ is

1. **injective** or **one-to-one** if for all $y \in f(X)$ and $x, x' \in X$ with $f(x) = f(x') = y$ holds $x = x'$;
2. **surjective** if $f(X) = Y$ holds;
3. **bijective** if f is injective and surjective.

Given two functions $f\colon X \to Y$ and $g\colon Y \to Z$, then their *composition* $h = g \circ f\colon X \to Z$ is defined by

$$x \mapsto h(x) = (g \circ f)(x) = g\big(f(x)\big).$$

Notice that it holds in general $f \circ g \neq g \circ f$. The *identity* $\mathrm{id}\colon X \to X$ is defined by $\mathrm{id}(x) = x$

Let $f\colon X \to Y$ be a bijective function. Then, there exists for each $y \in Y$ exactly one $x \in X$ with $y = f(x)$. The function $g\colon Y \to X$ with $g(y) = x \Leftrightarrow f(x) = y$ is called *inverse function*, which we denote by $f^{-1} := g : Y \to X$. Especially, it holds $f^{-1} \circ f = f \circ f^{-1}$.

Two sets X and Y are said to be of the same *cardinality*, written as $\mathrm{card}(X) = \mathrm{card}(Y)$, if there exists a *bijective* function $f\colon X \to Y$ between both sets.

Remark Two sets A and B with *fintely* many elements are of the same cardinality, if and only if they consist of the same amount of elements.

Theorem 1.14 Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers. The Cartesian product $\mathbb{N}^2 := \mathbb{N} \times \mathbb{N}$ is of the same cardinality as $\mathbb{N}$.

Proof. Let $p, q \in \mathbb{N}$ and hence $(p, q) \in \mathbb{N}^2$ be arbitrary. We employ "Cantor's diagonal argument" by introducing the function

$$f(p, q) := \frac{(p + q - 1)(p + q - 2)}{2} + p.$$

Its name is motivated by the following two-dimensional scheme

	1	2	3	4
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2,2)	(2,3)	
3	(3,1)	(3,2)		
4	(4,1)			$\ddots$

	1	2	3	4
1	1	2	4	7
2	3	5	8	
3	6	9		
4	10			$\ddots$

For $n \in \mathbb{N}$, the sets

$$M_n = \{(1, n), (2, n - 1), \ldots, (n - 1, 2), (n, 1)\}$$

are subsets of $\mathbb{N}^2$. We infer from the above scheme that the sets M_n are pairwise disjoint and that $\bigcup_{n\in\mathbb{N}} M_n = \mathbb{N}^2$. Further, the sets

$$N_n := \left\{ \frac{(n - 1)n}{2} + 1, \frac{(n - 1)n}{2} + 2, \ldots, \frac{(n - 1)n}{2} + n \right\}$$

are subsets of $\mathbb{N}$. Because of

$$\frac{(n - 1)n}{2} + n = \frac{n(n + 1)}{2},$$

we conclude that the sets N_n are also pairwise disjoint and that

$$\bigcup_{n\in\mathbb{N}} N_n = \mathbb{N}.$$

One readily infers that the function $f : M_n \to N_n$ maps M_n onto N_n bijectively. $\square$

Definition 1.15 A set A is (at most) **countable** if A is either finite, which means that it consists of finitely many elements, or if $\mathrm{card}(A) = \mathrm{card}(\mathbb{N})$.

Corollary 1.16

1. The set of integers
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$
is countable.

2. The set of all positive fractions is countable.
3. The set of rational numbers
$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$$
is countable.

The following theorem is trivial in case of sets which consist of finitely many elements, but it is a strong result in case of sets of infinite cardinality.

Theorem 1.17 (Cantor and Russell) Let X be an arbitrary set. Then, there exists
1. *no* surjective function
$$f\colon X \to \mathcal{P}(X);$$
2. *no* injective function
$$g\colon \mathcal{P}(X) \to X.$$

Proof. 1. Let $f\colon X \to \mathcal{P}(X)$ be an arbitrary function. We will show that there always exists an element $Y \in \mathcal{P}(X)$ such that $Y \notin f(X)$, which means that it does not lie in the image of f. First note that for all $x \in X$ the image $f(x) \in \mathcal{P}(X)$ is a subset of X, that is $f(x) \subset X$. We define the set
$$A := \{x \in X : x \notin f(x)\} \in \mathcal{P}(X)$$
and we proof the assertion by contraposition. To this end, assume that
$$\text{there exists an } x \in X \text{ such that } f(x) = A.$$
Then, there are only two possibilities for an arbitrary $x \in X$:
   a) It holds $x \in f(x)$, which implies $x \notin A$. We conclude $f(x) \not\subset A$, which means that $f(x)$ is no subset of A, and thus $f(x) \neq A$.
   b) It holds $x \notin f(x)$ and hence $x \in A$. Because of $x \notin f(x)$, $A \not\subset f(x)$ is no subset of $f(x)$. Therefore, it also holds $f(x) \neq A$.

   This leads to a contradiction with the assumption. Consequently, there exists no $x \in X$ such that $f(x) = A$ and we conclude that f is not surjective.
2. We prove the second assertion similarly to the first one. Assume that there exists an injective function $g\colon \mathcal{P}(X) \to X$. We define
$$B := \{g(Y) : Y \in \mathcal{P}(X) \text{ und } g(Y) \notin Y\} \subset X,$$
and deduce that are only the following two possibilities:
   a) If $g(B) \in B$, then there exists $Y \subset X$ with $g(B) = g(Y)$. Because of the definition of B, it follows that $B \neq Y$. This results in a contradiction with the injectivity of g.
   b) The case of $g(B) \notin B$ is a contradiction by itself as $g(B) \notin B \in \mathcal{P}(X)$ implies $g(B) \in B$.

□

Corollary 1.18 The power set $\mathcal{P}(\mathbb{N})$ of the natural numbers is uncountable.

Theorem 1.19 (Schröder–Bernstein) If the functions $f\colon X \to Y$ and $g\colon Y \to X$ are both *injective*, then it holds $\operatorname{card}(X) = \operatorname{card}(Y)$, which means that there exists a bijective function

$$h : X \to Y.$$

Proof. We call $u \in X$ a *predecessor* of $y \in Y$ if $f(u) = y$. Likewise, $v \in Y$ is called a predecessor of $x \in X$ if $g(v) = x$. An element $a \in X \cup Y$ is called *ancestor* of $b \in X \cup Y$, if there are finitely many predecessors a_i, $i = 0, 1, \dots, n$, such that $a_0 = a$, $a_n = b$ and a_i is predecessor of a_{i+1}.

Let $X_0 \subset X$ be the set of elements in X with an even or an infinite number of ancestors. Let further $X_1 \subset X$ denote the set of all $x \in X$ with an odd number of ancestors and let $Y_2 \subset Y$ consist of all $y \in Y$ with an even number of ancestors. Then, it holds first $X_0 \cap X_1 = \emptyset$, $X_0 \cup X_1 = X$, and second $g\colon Y_2 \to X_1$ is a bijective function, which means that there exists a function $\gamma\colon X_1 \to Y_2$ such that $\gamma\big(g(y)\big) = y$, $y \in Y_2$, and $g\big(\gamma(x)\big) = x$, $x \in X_1$. We define

$$h(x) = \begin{cases} f(x), & x \in X_0, \\ \gamma(x), & x \in X_1. \end{cases}$$

The function h is injective and $h(X_0) \subset Y$ is the set of all elements $y \in Y$ with an odd or infinite number of ancestors. Thus, Y is the disjoint union $Y = Y_2 \cup h(X_0)$, which means that $h : X \to Y$ is also surjective. The function $h : X \to Y$ is therefore bijective, which implies the assertion. □

Let $f\colon X \to Y$ be an injective function, then there exists a *left inverse* $g\colon Y \to X$ which satisfies $(g \circ f)(x) = g\big(f(x)\big) = x$ for each $x \in X$. We can define such a function just by setting $g(y) = x$ if $y = f(x)$. Hence, if $f(X) = Y$, we already have defined g. If, however, $f(X) \neq Y$, we choose an arbitrary $a \in X$ and set $g(y) = a$ for all $y \in Y \setminus f(X)$. One readily verifies then that $g\big(f(x)\big) = x$ for all $x \in X$.

In case of a surjective function $f\colon X \to Y$, one may ask whether it is possible to construct a *right inverse* $g\colon Y \to X$ such that $f\big(g(y)\big) = y$, $y \in Y$. As $Y = f(X)$, we have to choose for each $y \in Y$ some element $g(y)$ of the preimage $f^{-1}(\{y\}) \subset X$. The question, if this is always possible for infinite sets, cannot be answered. Therefore, we shall postulate this possibility in form of the so-called *axiom of choice*.

Axiom 1.20 (Axiom of choice) For any surjective function $f\colon X \to Y$, there exists a right inverse $g\colon Y \to X$ such that $(f \circ g)(y) = f\big(g(y)\big) = y$ for all $y \in Y$.

The axiom of choice is an important but controversial tool, for example in *functional analysis*. As part of this introductory course, we will hardly use it.

Remark The "naive" notion of a set introduced in the beginning of this paragraph has its limits. Consider the so-called "universal set"

$$\mathcal{U} = \{X : X \text{ is a set}\}.$$

Then, $\mathcal{P}(\mathcal{U})$ is also a set and $\mathcal{P}(\mathcal{U}) \subset \mathcal{U}$ as $\mathcal{U}$ contains all sets. There is always an injective function $p\colon \mathcal{U} \to \mathcal{P}(\mathcal{U})$, namely $p(U) := \{U\}$ for all $U \in \mathcal{U}$, and also an injective function $g\colon \mathcal{P}(\mathcal{U}) \to \mathcal{U}$, namely $g(U) = U$ for all $U \in \mathcal{P}(\mathcal{U})$. Hence, $\operatorname{card}\big(\mathcal{P}(\mathcal{U})\big) = \operatorname{card}(\mathcal{U})$ in contradiction with the Theorem of Cantor and Russell 1.17. We infer that the naively introduced notion of a set, especially sets of sets, has to be treated somewhat carefully.

A similar example is Russell's antinomy: Let $\mathcal{B}$ the set of all sets, which do not contain themselves. This formulation is of course contradictory in itself.

The discovery of these contradictions led to the development of the the "axiomatic set theory", the detailed treatment of which, however, would go beyond the scope of this lecture.