Exercise 1

Let A, B and C be propositions. Prove

$$(A \wedge B \Rightarrow C) \Leftrightarrow (A \Rightarrow (B \Rightarrow C))$$

a) using truth tables.

A	B	C	$A \wedge B$	$A \wedge B \Rightarrow C$	$B \Rightarrow C$	$(A \Rightarrow (B \Rightarrow C))$
F	T	T	F	T	T	T
F	T	F	F	T	F	T
F	T	F	F	T	T	T
F	T	F	F	T	T	T
T	F	T	F	T	T	T
T	F	T	F	T	F	T
T	F	F	F	T	T	T
T	F	F	F	T	T	T

$(\neg A)$ $(\neg B)$ $\neg(A \wedge B)$ $\neg(A \wedge B) \vee C$ $(\neg B) \vee C$ $((\neg A) \vee (B \Rightarrow C))$

$(A \wedge B \Rightarrow C) \Leftrightarrow (A \Rightarrow (B \Rightarrow C))$

b) using operational calculus: $(A \wedge B \Rightarrow C) \Leftrightarrow (A \Rightarrow (B \Rightarrow C))$

$$\begin{aligned}
A \wedge B \Rightarrow C &\Leftrightarrow (\neg(A \wedge B)) \vee C \\
&\Leftrightarrow (\neg A \vee \neg B) \vee C \\
&\Leftrightarrow \neg A \vee (\neg B \vee C) \\
&\Leftrightarrow (\neg A) \vee (B \Rightarrow C) \\
&\Leftrightarrow (A \Rightarrow (B \Rightarrow C)).
\end{aligned}$$

Exercise 2

Let E be an arbitrary set and $A, B, C \in \mathcal{P}(E)$.

a) Show that if $A \cap B = A \cup B$ then $A = B$.

$A \subset B$: Let $x \in A$, Then, $x \in A \Rightarrow x \in A \cup B$
And, $A \cap B = A \cup B \Rightarrow x \in A \cap B \Rightarrow x \in B$
ie. $x \in A \wedge x \in B$
It show that if $x \in A$ then $x \in B$. ie. $A \subset B$.

$B \subset A$: Let $x \in B$, Then, $x \in B \Rightarrow x \in A \cup B$
And, $A \cap B = A \cup B \Rightarrow x \in A \cap B \Rightarrow x \in A$
ie. $x \in B \wedge x \in A$
It show that if $x \in B$ then $x \in A$. ie. $B \subset A$.

Since $A \subset B$ and $B \subset A$, we conclude that $A = B$.

b) Show that if $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$. Is one of the two conditions sufficient?

$B \subset C$: Let $x \in B$, then:

Case 1: $x \in A$
$x \in A \Rightarrow x \in A \cap B$.
And $A \cap B = A \cap C \Rightarrow x \in A \cap C$
Then, $x \in A \cap C \Rightarrow x \in C$
i.e. $x \in B \wedge x \in C$

Case 2: $x \notin A$
$x \in B \Rightarrow x \in A \cup B$.
And $A \cup B = A \cup C \Rightarrow x \in A \cup C$
Then, $(x \in A \cup C) \wedge (x \notin A) \Rightarrow x \in C$
i.e. $x \in B \wedge x \in C$

$C \subset B$: Let $x \in C$, then:

Case 1: $x \in A$
$x \in A \Rightarrow x \in A \cap C$.
And $A \cap B = A \cap C \Rightarrow x \in A \cap B$
Then, $x \in A \cap B \Rightarrow x \in B$
i.e. $x \in C \wedge x \in B$

Case 2: $x \notin A$
$x \in C \Rightarrow x \in A \cup C$.
And $A \cup B = A \cup C \Rightarrow x \in A \cup B$
Then, $(x \in A \cup B) \wedge (x \notin A) \Rightarrow x \in B$
i.e. $x \in C \wedge x \in B$

Since $B \subset C$ and $C \subset B$, we can conclude that $B = C$.
"Is one of the two conditions sufficient?"
NO, Let $A = \emptyset$, $B = \{1\}$, and $C = \{2\}$, then $A \cap B = A \cap C = \emptyset$ but $B \neq C$
and $A \cup B = B$ and $A \cup C = C$, but $B \neq C$
Let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$, then $A \cup B = A \cup C = \{1, 2\} = A$, but $B \neq C$
and $A \cap B = B$ and $A \cap C = C$, but $B \neq C$

Exercise 3

Let E be arbitrary and $A, B, C \in \mathcal{P}(E)$
symmetric difference Δ:

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

then, $x \in A \Delta B \Leftrightarrow x \in (A \setminus B) \cup (B \setminus A)$
$\Leftrightarrow x \in (A \setminus B) \vee x \in (B \setminus A)$
$\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$
$\Leftrightarrow (x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A)$
$\Leftrightarrow (x \in A \cup B) \wedge (x \notin A \cap B)$

$P_1 := (x \in A \Delta B \Rightarrow x \in (A \cup C) \Delta (B \cup C))$,
$P_2 := (x \in A \Delta B \Rightarrow x \in (A \cap B) \Delta (B \cap C))$.

a) $x \in A \Delta B \Rightarrow x \in (A \cup C) \Delta (B \cup C) \Leftrightarrow (x \in A \cup B) \wedge (x \notin A \cap B) \Rightarrow (x \in (A \cup C) \cup (B \cup C)) \wedge (x \notin (A \cup C) \cap (B \cup C))$
$\Leftrightarrow (x \in A \cup B) \wedge (x \notin A \cap B) \Rightarrow (x \in A \cup B \cup C) \wedge (x \notin C \cup (A \cap B))$
$\Leftrightarrow (x \in A \cup B) \wedge (x \notin A \cap B) \Rightarrow (x \in A \cup B \cup C) \wedge (x \notin A \cap B) \wedge (x \notin C)$

$P_1 := ((x \in A \cup B) \wedge (x \notin A \cap B) \Rightarrow (x \in A \cup B \cup C) \wedge (x \notin A \cap B) \wedge (x \notin C))$

Conclusion:
The condition implies that no element of $A \Delta B$ can be in C.
ie. the condition for P_1: $x \notin C$.
x can not be in C otherwise P_1 can not hold.

b)

$P_2 := (x \in A \Delta B \Rightarrow x \in (A \cap C) \Delta (B \cap C))$.

$x \in A \Delta B \Rightarrow x \in (A \cap B) \Delta (A \cap C) \Leftrightarrow (x \in A \cup B) \wedge (x \notin A \cap B) \Rightarrow (x \in (A \cap C) \cup (B \cap C)) \wedge (x \notin (A \cap C) \cap (B \cap C))$
$\Leftrightarrow$ ——— " ——— $\Rightarrow (x \in C \cap (A \cup B)) \wedge (x \notin A \cap B \cap C)$
$\Leftrightarrow$ ——— " ——— $\Rightarrow (x \in C) \wedge (x \in A \cup B) \wedge (x \notin A \cap B \cap C)$

Conditions:
$(x \in C) \wedge (x \in A \cup B)$ and $x \notin A \cap B \cap C$

$P_2 := ((x \in A \cup B) \wedge (x \notin A \cap B) \Rightarrow (x \in C) \wedge (x \in A \cup B) \wedge (x \notin A \cap B \cap C))$.

ie. the condition to P_2 is that x is an element of C ($x \in C$) and x can not belong to the intersection of A, B and C ($x \notin (A \cap B \cap C)$ which is automatically true because $x \in A \Delta B \Rightarrow x \notin A \cap B$.

Conclusion:
Since $x \in A \Delta B \Rightarrow x \in C$, the condition is: $A \Delta B \subseteq C$

c) N = "the set of element of E, which do not satisfy at least one of the propositions P_1 and P_2"
Let $A \Delta B$ be false then, P_1 and P_2 are true. ($\in$ false quodlibet)

$A \Delta B$	P_1	$A \Delta B \Rightarrow P_1$
T	T	T
T	F	F
F	T	T
F	F	T

$A \Delta B$	P_2	$A \Delta B \Rightarrow P_2$
T	T	T
T	F	F
F	T	T
F	F	T

Let $A \Delta B$ be true then, P_1 or P_2 can not be satisfy.

$A \Delta B$	P_1	$A \Delta B \Rightarrow P_1$
T	T	T
T	F	F
F	T	T
F	F	T

$A \Delta B$	P_2	$A \Delta B \Rightarrow P_2$
T	T	T
T	F	F
F	T	T
F	F	T

ie. $x \in A \Delta B$, P_1 say $x \notin C$ and P_2 say $x \in C$
But if P_1 is satisfied P_2 can not and otherwise if P_2 is satisfied P_1 can not be.

Conclusion: $N = A \Delta B$ do not satisfy at least one of the propositions P_1 and P_2.

Exercise 4

$A = \{(x, y) \mid (x, y) \in \mathbb{R}^2\}$

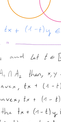
$A \subset \mathbb{R}^2$ is convex if

$$\forall t \in [0, 1] : \forall_{x, y \in A} : tx + (1-t)y \in A$$

Let $C \in \mathcal{P}(\mathbb{R}^2)$ denote the set of all convex subsets of $\mathbb{R}^2$.
Prove or disprove

a)

$\forall A_1, A_2 \in \mathcal{P}(\mathbb{R}^2) : A_1, A_2 \in C \Rightarrow A_1 \cap A_2 \in C$.



$\underbrace{\forall t \in [0, 1], tx + (1-t)y \in A_1}_{A_2}$ by definition

Let $x, y \in A_1 \cap A_2$ and let $t \in [0, 1]$

- Since $x, y \in A_1 \cap A_2$ then, $x, y \in A_1$ and $x, y \in A_2$
- Since A_1 is convex, $tx + (1-t)y \in A_1$
- Since A_2 is convex, $tx + (1-t)y \in A_2$

Conclusion: Since the $tx + (1-t)y$ is in A_1 and A_2, it belongs to $A_1 \cap A_2$.
$tx + (1-t)y \in A_1 \cap A_2$

b)

$\forall A_1, A_2 \in \mathcal{P}(\mathbb{R}^2) \mid A_1 \cap A_2 \in C \Rightarrow A_1, A_2 \in C$

Let $A_1 \in C$ and $A_2 \notin C$ (shaped like a PacMan)
Their intersection can be convex. $A_1 \cap A_2 \in C$ but $A_2 \notin C$ (is not convex)

c) $\forall A_1, A_2, \ldots, A_n \in \mathcal{P}(\mathbb{R}^2) : A_1, A_2, \ldots, A_n \in C \Rightarrow \bigcap_{i=1}^{n} A_i \in C$

$A_1 \cap A_2$ is convex (a)
ie. $D = A_1 \cap A_2$ is convex,
$E = D \cap A_3$ —— " ——,
$F = E \cap A_4$ —— " ——,
$\vdots$
$N = M \cap A_n$ is also convex!

$\bigcap_{i=1}^{n} A_i = \{x \mid \forall_{i \in I} : x \in A_i\}$

$tx + (1-t)y \in \bigcap_{i=1}^{n} A_i$

ie. $\bigcap_{i=1}^{n} A_i \in C$

Let $S_k = \bigcap_{i=1}^{k} A_i$ (convex by hypothesis)
Then, $\bigcap_{i=1}^{k+1} A_i = S_k \cap A_{k+1}$.
intersection of S_k and A_{k+1}, two convex sets, is convex according to (a).
Conclusion: By induction, the intersection of any convex sets is convex.