

1. Basics

1.1 Propositional calculus

In the context of propositional calculus, a proposition is a statement which can uniquely be rated either as true or false. Within propositional calculus, the semantics of the statement does not matter, only its truth value, which we denote by “true” (T) or “false” (F).

Example 1.1

1. 1 plus 1 is equal to 2.
2. 1 plus 1 is equal to 1.
3. 1 plus 1 is not equal to 0.

△

Statements like “It will rain tomorrow”, “Keep quiet!”, or contradicting statements, also called antinomies, like “Epimenides the Cretan says that all the Cretans are liars” are no valid propositions.

Negation: Given a proposition A , we denote the *negation* of A by $\neg A$ (“not” A). The negation also forms a proposition. The *truth table* of $\neg A$ is given as follows:

A	$\neg A$
T	F
F	T

Conjunction: Let A and B propositions. We denote the *conjunction* A and B by $A \wedge B$. The conjunction $A \wedge B$ is also a proposition. We define that $A \wedge B$ is true, if and only if both proposition A und B are true at the same time. The respective truth table is:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction: Let A and B denote propositions. We denote the *disjunction* A or B by $A \vee B$. The latter forms a proposition. $A \vee B$ is true, if and only if at least one of the propositions A or B are true. This yields the truth table:

1.1. Propositional calculus

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Equivalence: Two propositions A and B are equivalent, if their respective truth values coincide. In this case, we write $A \Leftrightarrow B$. The corresponding truth table reads:

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Theorem 1.2 Let A , B , and C propositions. There hold the following equivalences:

1. Double negation $\neg(\neg A) \Leftrightarrow A$
2. Commutativity $A \wedge B \Leftrightarrow B \wedge A$
 $A \vee B \Leftrightarrow B \vee A$
3. Associativity $A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$
 $A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$
4. Distributivity $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
 $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
5. De Morgan's laws $\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$
 $\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$

Proof. All statements can be proven by using truth tables. For example, we find for the double negation

A	$\neg A$	$\neg(\neg A)$
T	F	T
F	T	F

The remainder of the proof is an exercise. □

Remark Note the resemblance between the statements 2 to 4 with corresponding rules for the arithmetic operations $(+, \cdot)$.

Implication: Given two propositions A and B , the *implication* $A \Rightarrow B$ (" A implies B ") is also a proposition, which is defined as $B \vee (\neg A)$. This corresponds to the truth table:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Theorem 1.3 Let A and B be propositions. Then, there holds

$$(A \Rightarrow B) \wedge (B \Rightarrow A) \Leftrightarrow (A \Leftrightarrow B)$$

Proof. The proof is left to the reader as an exercise. \square

Remark

1. The proposition $A \Leftrightarrow B$ is generally proven with the help of Theorem ???. At first, one shows that A implies B and then that B implies A .
2. If A is false, then $A \Rightarrow B$ is always true (“ex falso quodlibet”).

So far, we have seen that we can arbitrarily combine propositions using the *junctors* $\{\neg, \wedge, \vee\}$ in order to obtain new propositions. In particular, we can derive the following rules of inference, which constitute common prove techniques.

Theorem 1.4 There holds for arbitrary propositions A and B :

1. Modus ponens $(A \wedge (A \Rightarrow B)) \Rightarrow B$
2. Modus tollens $(\neg B \wedge (A \Rightarrow B)) \Rightarrow \neg A$
3. Proof by contraposition $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
4. Transitivity of implication $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

Proof. The first formula is seen as follows:

$$\begin{aligned} (A \wedge (A \Rightarrow B)) &\Leftrightarrow (A \wedge (\neg A \vee B)) \\ &\Leftrightarrow ((A \wedge \neg A) \vee (A \wedge B)) \\ &\Leftrightarrow (F \vee (A \wedge B)) \\ &\Leftrightarrow (A \wedge B). \end{aligned}$$

Hence, the left hand side is equivalent to $A \wedge B$, which indeed implies B . The other formulae are shown in a similar manner. \square

Remark The propositions in the previous theorem are called *tautologies* of the form $A \vee \neg A$, which is true for any truth value of the proposition A .

1.2 Quantifier

Let A_j , $j \in J$, be a set of propositions, for example A_1, A_2, \dots . Then, we define $\forall_{j \in J} A_j$, read as “for each A_j with j from J ”, to be true, if and only if each proposition A_j , $j \in J$, is true. If $J = \{1, \dots, N\}$ is finite, then we can express this proposition by finitely many \wedge -operations

$$\forall_{j \in J} A_j \Leftrightarrow \forall_{j=1}^N A_j \Leftrightarrow (A_1 \wedge A_2 \wedge \dots \wedge A_N).$$

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“ \forall ” is called the *universal quantifier*.

In contrast, we define that $\exists_{j \in J} A_j$, read as “there exists an A_j with j from J ”, is true, if there exists at least one true proposition A_j with $j \in J$. In case of finitely many A_j , that means $J = \{1, \dots, N\}$, we have

$$\exists_{j \in J} A_j \Leftrightarrow \exists_{j=1}^N A_j \Leftrightarrow A_1 \vee A_2 \vee \dots \vee A_N.$$

“ \exists ” is called the *existential quantifier*.

Remark: The order of the quantifiers may not be changed in general, that is

$$(\forall_{j \in J} \exists_{i \in I} A_{i,j}) \not\Leftrightarrow (\exists_{i \in I} \forall_{j \in J} A_{i,j}).$$

The first proposition means “for all $j \in J$ there exists an $i \in I$ with $A_{i,j}$ ” while the second proposition means “there exists an $i \in I$ such that for all $j \in J$ $A_{i,j}$ holds”.

Example 1.5 Compare the propositions “all students have a student identity card” and “there is a registration office for all residents of the city”. \triangle

Theorem 1.6 Given propositions A_j with an arbitrary index set $j \in J$, there hold the generalized forms of De Morgan’s laws:

$$\begin{aligned}\neg(\forall_{j \in J} A_j) &\Leftrightarrow (\exists_{j \in J} \neg A_j) \\ \neg(\exists_{j \in J} A_j) &\Leftrightarrow (\forall_{j \in J} \neg A_j)\end{aligned}$$

1.3 Set theory

Besides propositional calculus, the theory of sets belongs to the foundation of mathematics. We shall consider the naive definition of a *set*:

A set is a collection of objects of our conception.

These objects are called the *elements* of the set. This means that we precisely know the set provided we know which elements it contains. For example, $M = \{a, b, 10\}$ or $A = \{1, 2, -1\}$ are sets. Notice that the order in which the elements are mentioned is not relevant.

The most common definition of a set is via some property $P(x)$, like $P(x)$: “ x is green” for example. In this case, we write

$$M = \{x : P(x)\}$$

for the set of all x which exhibit the property $P(x)$.

Equality of sets: We write $x \in M$ if x is an element of M . If there holds $\neg(x \in M)$, we write $x \notin M$. Two sets M and N are *equal*, $M = N$ for short, if and only if they consist of the same elements.

Subsets: A set N is a *subset* of the set M , $N \subset M$ for short, if $x \in N$ always implies $x \in M$. This formally means

$$N \subset M \iff \forall_{x \in N} x \in M.$$

The abbreviation $A \not\subset B$ means that $\neg(A \subset B)$.

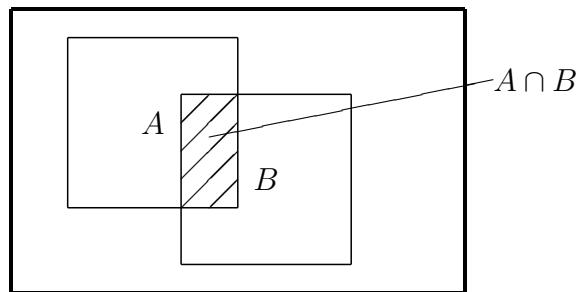
Theorem 1.7 (Transitivity of " \subset ") Let A , B , and C be sets with $A \subset B$ and $B \subset C$. Then, there also holds $A \subset C$.

Proof. Let $x \in A$. Then, it holds $x \in B$ due to $A \subset B$. Because of $x \in B$ and $B \subset C$, it also follows $x \in C$. \square

Intersection of sets: The set

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

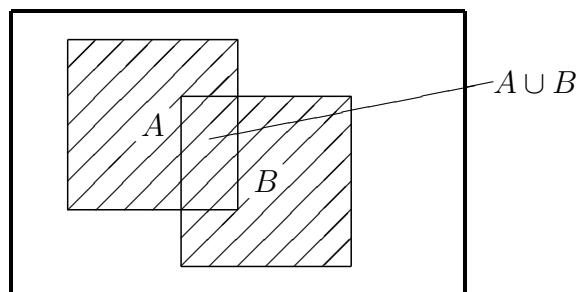
is called the *intersection* of A and B . It can be illustrated by a so-called *Venn diagram*:



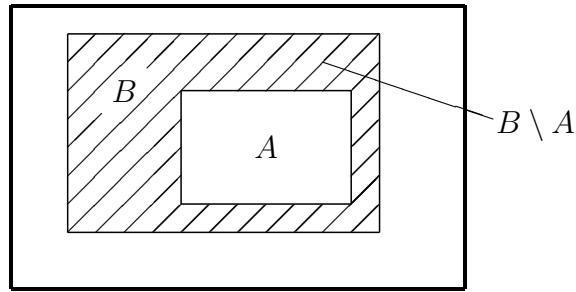
Union of sets: For two sets A and B , the set

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

is the *union* of A and B . The associated Venn diagram is:



Set-theoretic difference: Given two sets A and B , then $\{x \in B : x \notin A\}$ is also a set. It is called the *relative complement* of A with respect to B , formally denoted as $B \setminus A$:



If A is described by the property $P(x)$, then the set $\{x : \neg P(x)\}$ is called the *absolute complement* of A , \overline{A} for short.

Empty set: If $P(x)$ is a property, which is false for all $x \in A$, then the set $M := \{x \in A : P(x)\}$ contains no element. Such a set is called *empty*.

Remark Let \emptyset denote an empty set.

1. The proposition $x \in \emptyset$ is always false.
2. It holds $\emptyset \subset A$ for any set A .

Theorem 1.8 If A and B are both empty sets, then it holds $A = B$. Therefore, the empty set is unique. It is denoted by \emptyset .

Proof. We show $A \subset B$ and $B \subset A$. $A \subset B$ is equivalent to the proposition $\forall x : x \in A \Rightarrow x \in B$. This implication is true as the proposition is always false. Analogously $B \subset A$ is proven. \square

Disjoint sets: Two sets A and B are called *disjoint* if their intersection satisfies $A \cap B = \emptyset$.

Theorem 1.9 Let A , B , and C subsets of X . Then, there holds:

1. Double complement $X \setminus (X \setminus A) = A$
2. Commutativity $A \cup B = B \cup A$
 $A \cap B = B \cap A$
3. Associativity $(A \cup B) \cup C = A \cup (B \cup C)$
 $(A \cap B) \cap C = A \cap (B \cap C)$
4. Distributivity $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Absorption laws $A \cup \emptyset = A$
 $A \cap \emptyset = \emptyset$
6. De Morgan's laws $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$
 $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

Proof. The first statement is proven as follows: It holds $X \setminus (X \setminus A) = \{x \in X : x \notin (X \setminus A)\}$ by definition. From $x \in X$, we infer $x \in (X \setminus A)$ if and only if $x \notin A$. It hence follows $x \in A$ by negation, that is $x \notin (X \setminus A)$. Since $A \subset X$, we conclude $X \setminus (X \setminus A) = A$. The proof of the other statements remains as an exercise. \square

Arbitrary unions and intersections: Consider a family $\{A_j : j \in J\}$ of sets, then

$$\bigcup_{j \in J} A_j := \{x : \exists_{j \in J} x \in A_j\}$$

is the union of the sets A_j , $j \in J$, and

$$\bigcap_{j \in J} A_j := \{x : \forall_{j \in J} x \in A_j\}$$

is the intersection of the sets A_j , $j \in J$.

Cartesian product: Let $a \in A$ and $b \in B$, then (a, b) is called an *ordered pair*. Notice that $(a, b) \neq (b, a)$. The set

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

is called *Cartesian product* of A and B . We say “ A times B ” for short.

Power set: Let A be a set. Then, the set of all subsets of A

$$\mathcal{P}(A) = \{B : B \subset A\}$$

is called the *power set* of A .

Remark The following analogies hold between propositional calculus and set theory:

propositional calculus	set theory
propositions A, B	sets A, B
conjunction \wedge	intersection \cap
disjunction \vee	union \cup
negation \neg	complement \overline{A}
contradiction	empty set \emptyset
tautology	universal set X
implication $A \Rightarrow B$	subset $A \subset B$
equivalence $A \Leftrightarrow B$	equality $A = B$