M01 Algorithmics

Solutions Exercise Sheet 02

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Exercises discussed in class

The exercises will be discussed in class on Tuesday, February 17, 2026 at 7:00pm in the Moodle course.

Recall. Let A be a set. We call $R \subset A \times A$ a *relation* and write xRx' if $(x, x') \in R$. The relation R is said to be:

(1) Reflexive, if: for all $x \in A$, it holds $(x, x) \in R$.
(2) Symmetric, if: for all $x, y \in A$, it holds $(x, y) \in R \iff (y, x) \in R$.
(3) Transitive, if: for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

We call R an *equivalence relation* if it is reflexive, symmetric, and transitive.

Exercise 1 *Let $A = \{1, 2, 3, 4\}$. Are the following equivalence relations? If not, do they satisfy at least one of the three properties: reflexive, symmetric, or transitive?*

a) $R_1 = \{\}$
b) $R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
c) $R_3 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (1,3), (2,3), (3,3), (4,4)\}$

Exercise 2 *On $S = \mathbb{R}$, $x \sim y$ if and only if $x^3 = y^3$. Is this an equivalence relation?*

Exercise 3 *Using proof by induction, prove that P_n holds for all $n \in \mathbb{N}$:*

$$P_n: \sum_{k=0}^{n} k = 0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Homework

The solutions of the homework are due on February 24, 2026 at 7:00pm on the Moodle page `https://moodle.fernuni.ch/course/view.php?id=XXXX`.

Homework 1 (15 points)

i. *(7.5 points) In a) and b), prove by using mathematical induction that P_n is true for all $\mathbb{N} \ni n > 0$.*

a)
$$P_n: \sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n n(n+1)}{2}.$$

b)
$$P_n: \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1.$$

ii. *(7.5 points) Let $S(\cdot)$ be a successor function. The recursion formula for addition states that:*

$$m + S(n) = S(m+n), \quad n, m \in \mathbb{N},$$

or, equivalently:

$$m + (n + 1) = (m + n) + 1, \quad n, m \in \mathbb{N}.$$

By using this result and utilizing mathematical induction, prove the following theorem:

Theorem *(Associative Law): For all* $k, n, m \in \mathbb{N}, \quad (k + m) + n = k + (m + n)$*.*

Hint: Let n *be the variable in the induction, and keep* k *and* m *fixed.*

Homework 2 (15 points) *Consider the relation* $\sim$ *on a given set* S*. Prove whether the following are equivalence relations or not:*

i. *(5 points) On* $S = \mathbb{R}$*,* $x \sim y$ *if and only if* $x^2 = y^2$*.*
ii. *(5 points) On* $S = \mathbb{R}$*,* $x \sim y$ *if and only if* $x \leq y$*.*
iii. *(5 points) On* $S = \mathcal{P}(\mathbb{N})$*,* $X \sim Y$ *if and only if* $X \subseteq Y$*.*

Homework 3 (20 points) *Consider the following sequence, called the Fibonacci sequence, in which the* n*-th component is defined as follows:*

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1, \\ f_n &= f_{n-1} + f_{n-2}, \ n \geq 3. \end{aligned}$$

In Python, implement the following:

i. *(10 points) A script that computes the* n*-th element of the Fibonacci sequence using the formula above. You can store* $f_1, f_2, \dots, f_n$ *in a vector.*
ii. *(10 points) Use the script you implemented then to compute the first 100 elements of the Fibonacci sequence. Then, use the resulting vector to compute the following ratios:*

$$r_i = \frac{f_{i+1}}{f_i}, \quad i = 1, \dots, 99.$$

Plot $r = (r_1, r_2, \dots, r_{99})$*. What do you observe? Comment on your results.*