Contents

1 Introduction

Blackjack (or "21") is a casino banking game that has long attracted interest from mathematicians and gamblers alike. The appeal lies in the interplay between luck (the draw of the cards) and strategy (the choices made by the player). Unlike many casino games, blackjack offers the player decision options that can influence the outcome, prompting questions about optimal play and the house advantage. The foundational work of Baldwin *et al.* [1] used early computers to establish the first basic strategy for blackjack, defining the optimal player decisions for every situation. A few years later, Edward Thorp's *Beat the Dealer* [2] famously introduced card-counting techniques that can swing the long-term expectation in the player's favor. These efforts demonstrated that blackjack, unlike games of pure chance, can be modeled and analyzed to a high degree of precision.

In this thesis, we build a rigorous model of blackjack as a *stochastic process*. In particular, we develop a Markov chain analysis of a single blackjack hand played with basic strategy. Markov chains provide an ideal framework for this problem: the progression of a hand (drawing cards, transitioning between totals, and ending in win/lose outcomes) can be seen as a sequence of state-to-state transitions with probabilistic outcomes depending only on the current state (the Markov property). By enumerating the possible states of the game for both the dealer and player and assigning transition probabilities according to the rules of play, we obtain a complete probabilistic model of the hand. Using this model, we compute the *player's expected profit or loss per hand* (i.e. the player's advantage or disadvantage). The analysis accounts for all standard rules, including blackjacks, busts, doubling down, and splitting pairs, under house rules that are typical of casino blackjack (e.g. dealer stands on soft 17, no re-splitting beyond one split, etc., as detailed later).

We begin with a brief overview of the relevant Markov chain theory in Section 2, focusing on finite absorbing chains. Next, in Section 3 we summarize the rules of blackjack and the basic strategy for the player, since these define the structure of our model. Section 4 constitutes the core of the thesis: we develop the Markov chain model of the game. We construct the *dealer's model* (who follows fixed rules) and the *player's model* (who follows basic-strategy decisions) and then integrate them to compute outcome probabilities and expected values. Finally, Section 5 concludes with a discussion of the implications of this modeling, confirming known results and providing a baseline for more advanced analyses such as card counting, and suggestions for further extensions.

Tables and figures are included to clarify the Markov chain state structure and to visualize key findings. All numerical results are generated by the Python code that can be found at `https://github.com/leocarte/Stochastic-Modeling-of-Blackjack` and are cross-verified with the literature where possible.

2 Markov Chain Theory Background

In this section we summarize the concepts from probability theory and stochastic processes that will be used to model the game of blackjack. Specifically, we focus on discrete-time Markov chains with a finite state space, including the special case of absorbing Markov chains. This will set the stage for constructing a Markov chain representation of blackjack in the next sections.

2.1 Definition of Markov Chains and Transition Matrices

A *Markov process* is a type of stochastic process (a collection of random variables) that evolves through a sequence of states satisfying the *Markov property*: given the present state, the future evolution is independent of the past states [4]. Formally, let $X_0, X_1, X_2, \ldots$ be a sequence of random variables representing the state of a system at each time step. The process is a Markov process if for all times n and for all states i, j in the state space S (which is the set of all N possible states the process can be in), we have:

$$\Pr\bigl(X_{n+1} = j \mid X_n = i,\ X_{n-1} = x_{n-1}, \ldots, X_0 = x_0\bigr) = \Pr\bigl(X_{n+1} = j \mid X_n = i\bigr),$$

i.e. the probability of moving to state j at the next step depends only on the current state i, and not on the earlier history of the process [4]. If these transition probabilities do not depend on n so that

$$\Pr\bigl(X_{n+1} = j \mid X_n = i\bigr) = \Pr\bigl(X_1 = j \mid X_0 = i\bigr),$$

the process is called *time-homogeneous*. In this case, we call the Markov process a *Markov chain* [4]. A Markov chain is typically characterized by a set of transition probabilities $P_{ij} = \Pr(X_1 = j \mid X_0 = i)$ for each $i, j \in S$. These define the dynamics: from state i, the process will go to state j with probability P_{ij} in one step. We can arrange the transition probabilities into an $N \times N$ transition matrix $P = [P_{ij}]$ [4]. The ith row of P gives the distribution of the next-state if the current state is i. By the laws of probability, each row of P sums to 1 (since from state i, the process must go to something in the next step, even if it might be itself). The initial state distribution can be described by a vector $\boldsymbol{\pi}^{(0)}$; the state distribution n steps later is

$$\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(0)} P^n,$$

where P^n is the n-step transition matrix (the n-fold matrix product, or equivalently the matrix of transition probabilities after n steps). The entry $(P^n)_{ij}$ then gives the probability of moving from state i to state j in exactly n steps [4]:

$$(P^n)_{ij} = \Pr\big(X_n = j \mid X_0 = i\big).$$

2.2 Absorbing Markov Chains and Absorption Probabilities

A crucial subset of Markov chains for our analysis is the class of absorbing Markov chains. A state $a \in S$ is called an *absorbing* state if once the chain enters a, it remains there forever; in terms of the transition matrix, a is absorbing if $P_{aa} = 1$ and thus $P_{aj} = 0$ for all $j \neq a$ [4]. If a Markov chain has at least one absorbing state and if from every non-absorbing, or *transient*, state there is a nonzero-probability path to some absorbing state, then the chain is called an *absorbing Markov chain* [4]. In such chains, as $n \to \infty$, the process will end up absorbed in one of the absorbing states with probability 1.

By suitably ordering the states so that the first t states are transient and the remaining r states are absorbing, the transition matrix P can be written in *canonical form*:

$$P = \begin{pmatrix} T & A \\ 0 & I \end{pmatrix},$$

where:

- T is the submatrix of transitions among transient states (of size $t \times t$ if there are t transient states).
- A is the submatrix of transitions from transient states to absorbing states (size $t \times a$ if there are a absorbing states).
- I is an identity matrix of size $a \times a$ for the absorbing states, reflecting that each absorbing state transitions to itself with probability 1.

- The zero block is an $a \times t$ matrix of zeros, indicating no transitions from absorbing back to transient states [4].

For such a system, one can define the *fundamental matrix*

$$F = (I - T)^{-1}.$$

The entry F_{ij} of this matrix gives the expected number of times the chain visits transient state j if it started in transient state i [4]. The *absorbing matrix* (or matrix of absorption probabilities) is then

$$N = F\,A,$$

where N_{ij} is the probability that the chain, starting in transient state i, is eventually absorbed in absorbing state j [4]. In other words, the values of each row of N tell us the distribution of ending up in each absorbing outcome starting from a specific transient state. Equivalently, one may raise the transition matrix P to a sufficiently large power n, since the rows corresponding to transient states converge to their absorption probabilities [3]:

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 0 & N \\ 0 & I \end{pmatrix}.$$

This framework will be used in Section 4 to determine the probabilities of each terminal outcome in a blackjack hand. The absorbing states correspond to terminal outcomes of the hand (player busts, player stands with certain total, etc., as we will define), and the transient states correspond to intermediate totals or situations where play is still ongoing. We will use the above theory in practice by constructing the transition matrices for the dealer and the player and then effectively computing P^n for a large n to derive the absorbing-state distributions. In this way, the Markov chain framework will allow us to compute the probability of each possible outcome (win, lose, tie, etc.) for any given initial situation in blackjack.

3 Blackjack Rules and Basic Strategy

Before constructing the stochastic model, we outline the rules of blackjack and the optimal basic strategy for the player. The rules determine the state space and transition probabilities of our Markov chain model, and the basic strategy provides the decision policy for the player's moves. We assume a typical set of rules (that match the conditions under which basic strategy is usually computed) and an infinite-deck shoe [3]. The infinite deck assumption is used in our Markov model for mathematical convenience, effectively meaning that card probabilities remain constant between draws.

3.1 Game Objective and Card Values

Blackjack is a comparing card game between a player and a dealer. The objective is for the player to accumulate a hand of cards with a total value higher than the dealer's hand without exceeding 21. If the player's total exceeds 21, the hand is declared a *bust* and the player immediately loses the round. If the dealer busts (exceeds 21) and the player has not busted, the player wins. If neither busts, the higher total wins; a tie (equal totals of 21 or less) results in a *push* (no money exchanged) [1].

Card values are as follows:

- Number cards 2–9 are worth their face value (2 through 9 points respectively).
- All face cards (Jack, Queen, King) and Tens count as 10 points. We will refer to any 10, J, Q, or K generically as a "10-value card" or simply "10" for ease of modeling.
- Aces can count as either 1 point or 11 points, whichever yields the highest total not exceeding 21. An Ace that is being counted as 11 is often called a *soft Ace*. A hand that contains an Ace counted as 11 is called a *soft hand*, since the Ace can revert to 1 if needed. A hand in which all Aces are counted as 1 or there are no Aces is called a *hard hand*. For example, Ace–6 is "soft 17" (counted as $11 + 6 = 17$), whereas Ace–6–10 would become "hard 17" because if the Ace remained 11 the total would be 27 (bust), so it is counted as 1 making the total 17.

A two-card hand that totals 21 (i.e. an Ace and a 10-value card) is called a *blackjack* or *natural*. A blackjack is a special win for the player, paying a bonus (traditionally, the player wins 1.5 times the bet, or "3:2 payout") unless the dealer also has a blackjack, in which case the outcome is a push.

3.2 Dealing Procedure and Payouts

At the start of each round, the player places a bet. The dealing then proceeds as follows:

- The player is dealt two cards face up.
- The dealer is dealt two cards: one face up (the "up-card" visible to the player) and one face down (the "hole card").

If the dealer's up-card is an Ace, the player is typically offered the option of *insurance*, a side bet up to half the original bet that pays 2:1 if the dealer's hole card is a 10 (i.e. if the dealer has a blackjack). We assume the player does not take insurance under basic strategy because insurance has a negative expected value unless card counting information is available [2] (as it pays 2:1 on a roughly 9:4 against event).

After the initial deal, any immediate outcomes are resolved:

- If the dealer has a blackjack and the player does not, the dealer wins and the player loses their bet.
- If both player and dealer have blackjack, it is a push (tie).
- If the player has blackjack and the dealer does not, the player is immediately paid 3:2 on the bet.

If neither has a blackjack, the round continues with the player's turn. The player now makes decisions to try to improve their hand, guided by basic strategy:

- The player may *hit* (take another card) to increase the total, and can continue hitting as long as desired. If the player's total exceeds 21 at any point, the player busts and immediately loses the bet regardless of the dealer's subsequent action [1].
- The player may *stand* (take no more cards), freezing the total at its current value.
- If the player's first two cards have the same value (a pair, e.g. 8-8 or K-10), the player may opt for a *split*. When the player splits a pair, the two cards are separated into two independent hands ("twin" hands). An additional bet equal to the original is placed on the second hand. The dealer gives each of the two new hands a second card (so each hand has two cards), and the player then plays out both hands independently, following the same rules (hit/stand/etc.) on each. Splitting has some special restrictions: if the pair was two Aces, most casinos allow only one additional card to be drawn on each Ace and then the hand automatically stands (denoted in the code by `houseRules.HASAA = 0`, that is "hit after split aces" not allowed). Also, if an Ace is drawn on a split ten (or vice versa), it is typically

counted as 21 but not as a blackjack for payout purposes. Furthermore, casinos often limit re-splitting: in our assumptions, if a split hand is dealt a card matching it, the player cannot split again (`houseRules.MSA = 0`, that is "multiple splits" not allowed). This is consistent with Baldwin *et al.* [1].

- The player may choose to *double down* on the first two cards if desired. Doubling down means the player doubles the original bet (placing an additional equal bet) and commits to drawing exactly one more card, after which the hand will automatically stand. After splitting, except if the split was Aces, the player may also double down on the new hands (`houseRules.DASA = 1`, "double after split" allowed).

- Some games offer the option to *surrender*: the player can concede the hand immediately after the initial deal (and after checking for dealer blackjack) by forfeiting half the bet and not playing the hand. In our model we include a surrender option even though we assume it's not allowed (`houseRules.SRA = 0`).

If the player has not busted on all their hands (since it would result in the immediate loss of all bets) and has stood on some total, the dealer then plays their hand. The dealer has no choices to make, since their strategy is fixed:

- The dealer will reveal the hole card and sum the two-card total.

- The dealer must hit until the total is 17 or higher, and must stand on 17 or above. If the dealer has a "soft 17" (i.e. a total of 17 with an Ace counted as 11, such as Ace-6), the house rules specify whether the dealer hits or stands. In our model, we assume the common rule that the dealer stands on soft 17 (`houseRules.DSSS = 1`).

- The dealer never doubles or splits and does not have the surrender option; those options apply only to the player.

Once the dealer stands or busts, the final totals are compared:

- If the dealer busts, the player wins and is paid $+1$ times the bet for each surviving hand (or $+2$ if the player had doubled down on that hand).

- If the dealer does not bust and the player has a higher total than the dealer, the player wins and is paid $+1$ times the bet (or $+2$ if doubled).

- If the dealer's total is higher than the player's (and neither busts), the player loses the bet (or twice the bet if doubled).

- If player and dealer totals are equal, the result is a push, and the player's bet is returned with no gain or loss.

These rules define all possible outcomes of a hand and also their probabilities if we know the underlying card distribution. In our model, we assume an infinite deck model for simplicity: at each draw, the probability of each card does not depend on the cards previously dealt but is fixed at the baseline shoe distribution [3]. Specifically, if we assume a standard infinite shoe, the probability of drawing any particular rank 2–9 is $1/13 \approx 7.69\%$, and the probability of a 10-value card (10, J, Q, or K) is $4/13 \approx 30.8\%$. The probability of Ace is $1/13$ as well. In a finite deck, the probabilities change slightly after each card is dealt, but this approximation is very accurate for multi-deck play (obviously, the greater the number of decks used, the better the approximation of an infinite deck will be), and it makes the stochastic process a Markov process.

3.3 Basic Strategy for the Player

Basic strategy is the term for the optimal decision policy for the player if the goal is to maximize expected value (minimize losses) without card counting, given the rules in place. Basic strategy tells the player for each possible combination of player hand and dealer up-card whether to hit, stand, double, split, or surrender. This strategy has been derived through exhaustive analysis by Baldwin *et al.* [1] and later refined; it is well-known and often summarized in strategy tables. We assume throughout that the player follows basic strategy perfectly.

In our implementation, basic strategy is encoded as a set of lookup tables indexed by the player's total or pair and the dealer's up-card. These tables are provided in the Appendix and cover three main hand types: hard totals (Table 1), soft totals (Table 2), and pairs. For pairs, two versions are included, depending on whether doubling after splitting is allowed (Table 3) or not (Table 4). Each table entry encodes a specific move: "S" (Stand), "H" (Hit), "DB" (Double if allowed, otherwise Hit), "DS" (Double if allowed, otherwise Stand), "SP" (Split), or "SR" (Surrender if allowed, otherwise Hit).

These tables capture the key principles of basic strategy across different hand types. For instance, soft totals are played more aggressively than hard totals, since there is no risk of immediate bust. Pair splitting follows well-established rules: for example, always split Aces and 8s, while never split 5s (because 10 is a good total to double down on instead) or 10s (since 20 is already a strong hand). Surrender is recommended only in specific high-risk scenarios but, while our model supports surrender decisions, we assume it is not allowed by default.

In the next section, we incorporate this strategy into the player's Markov chain model.

4 Stochastic Modeling of a Blackjack Hand

We now construct the Markov chain model for a single hand of blackjack, divided into two parts: the dealer's model and the player's model. We will model each component separately as a Markov chain and then combine them to compute the probabilities of each outcome (player win, loss, or push) and the expected value for the player's bet. To do this effectively, we need to define the state spaces for the dealer and player Markov chains. Intuitively, a state should capture all information needed to decide the next step of play. We will use a systematic labeling of states to encode this information.

4.1 Dealer's Markov Chain Model

We define the dealer's state space Ψ_D as follows (adapted from Wakin & Rozell [3]):

- $\texttt{first}_i$: States representing the situation where the dealer has exactly one card, with value i (where i can be 2 through 11, using 11 to represent an Ace counted as 11). These correspond to the dealer's up-card. Once the dealer draws the second card, the state will transition out of this category.
- $\texttt{hard}_t$: Transient states where the dealer's total is t and the hand is hard. We include $t = 4, 5, \ldots, 17$. ($\texttt{hard}_{17}$ is included as a possible transient state before overriding $\texttt{stand}_{17}$ to stand.)
- $\texttt{soft}_t$: Transient states where the dealer's total is t and the hand is soft, for $t = 12, 13, \ldots, 17$. ($\texttt{soft}_{17}$ is included as a possible transient state before overriding $\texttt{stand}_{17}$ to stand.)
- $\texttt{stand}_t$: Absorbing states where the dealer has finalized their hand by standing on total t, for $t = 17, 18, 19, 20, 21$. (Note: $\texttt{stand}_{21}$ would occur only when drawing to 21 without it being a blackjack, that we treat separately as `bj` state.)
- `bj`: An absorbing state for dealer blackjack (a two-card 21).

- `bust`: An absorbing state for dealer bust (total > 21).

In total, the dealer's state space has $|\Psi_D| = 37$ states by the above enumeration [3].

From each transient state, the dealer draws one card of value i with probability d_i, under the infinite-deck assumption:

$$d_2 = d_3 = \cdots = d_9 = d_{11} = \tfrac{1}{13}, \quad d_{10} = \tfrac{4}{13}.$$

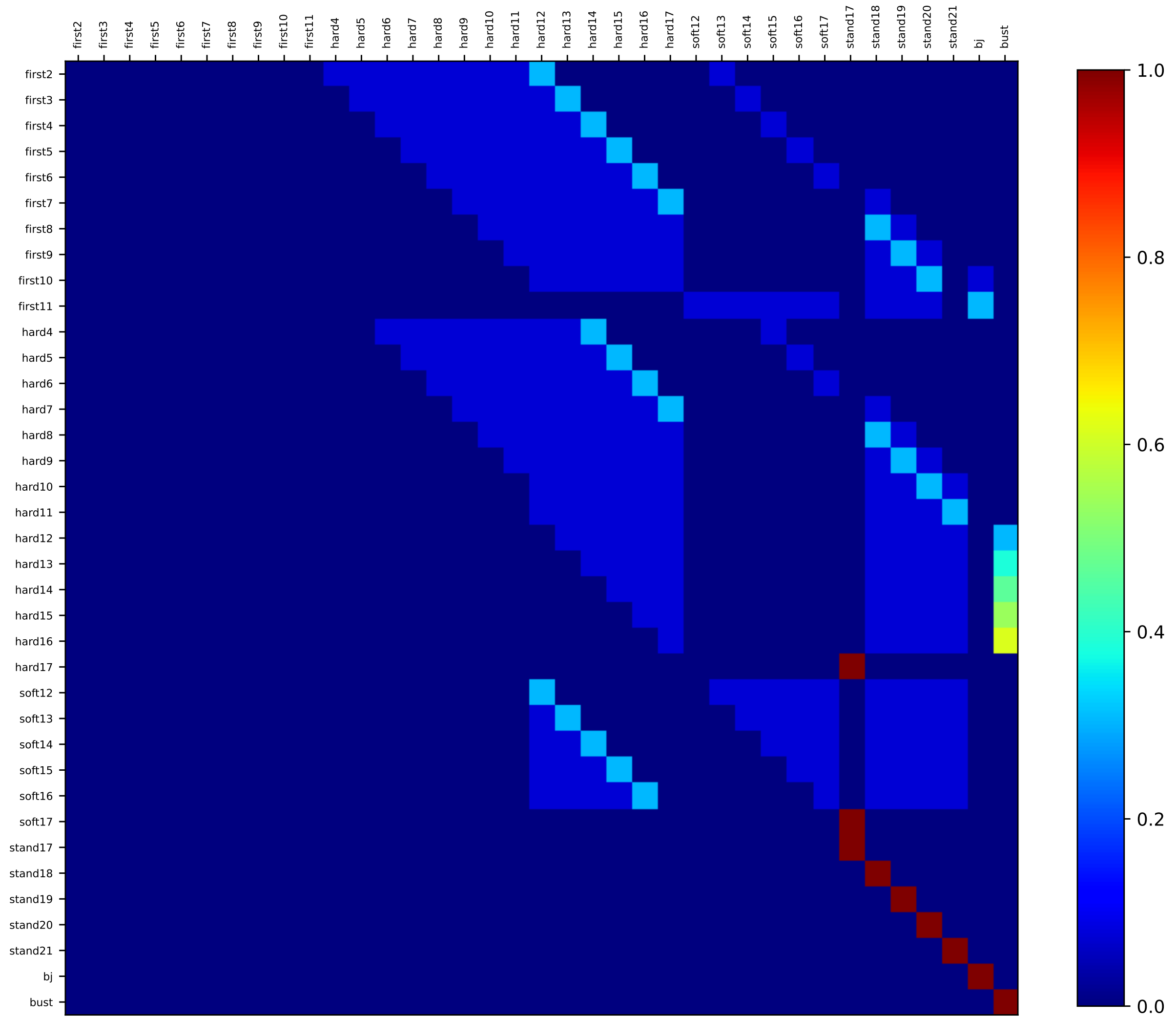
These probabilities remain constant for each draw.

We can construct the dealer's transition matrix D through the function `calc_dealer_trans` in the following way:

- From $\texttt{first}_i$, draw a second card k:
  - If neither i nor k is an Ace, let $t = i+k$. If $t \leq 16$, move to $\texttt{hard}_t$; if $17 \leq t \leq 20$, move to $\texttt{stand}_t$.
  - If exactly one of i, k is an Ace and the other is $m \leq 9$, total $m + 11$ is soft: if ≤ 16, go to $\texttt{soft}_{m+11}$; if $17 \leq m + 11 \leq 20$, go to $\texttt{stand}_{m+11}$.
  - If $\{i, k\} = \{\text{Ace}, 10\}$, transition to `bj`.
  - If $i = k = \text{Ace}$, convert the Ace to 1 and treat as soft total 12: go to $\texttt{soft}_{12}$.
- From $\texttt{hard}_t$ ($t \leq 16$), draw another card k:
  - If $k \neq 11$, let $u = t + k$. If $u \leq 16$, go to $\texttt{hard}_u$; if $17 \leq u \leq 21$, go to $\texttt{stand}_u$; if $u > 21$, go to `bust`.
  - If $k = 11$, then if $t + 11 \leq 21$, go to $\texttt{soft}_{t+11}$; otherwise count the Ace as 1 and let $u = t + 1$, applying the same rules for u.
- From $\texttt{soft}_t$ ($t \leq 16$), draw another card k:
  - If $k \neq 11$ and $t + k \leq 21$, let $s = t + k$. If $s \leq 16$, go to $\texttt{soft}_s$; if $17 \leq s \leq 21$, go to $\texttt{stand}_s$.
  - If $k \neq 11$ and $t + k > 21$, convert the Ace to 1: let $u = t + k - 10$ and treat as a hard total.
  - If $k = 11$, let $s = t + 1$; if $s \leq 16$, go to $\texttt{soft}_s$, else to $\texttt{stand}_s$.

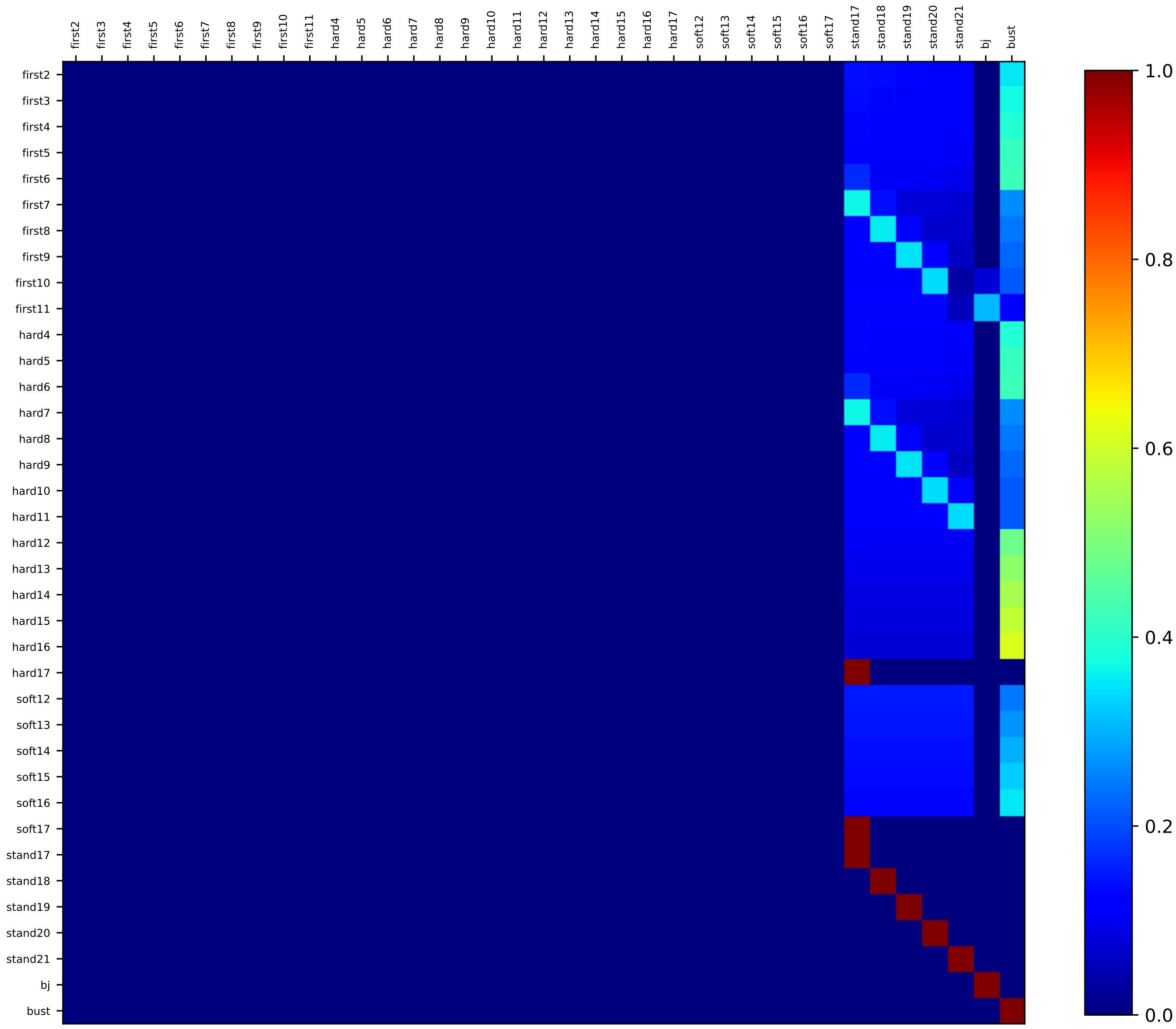
Figure 1 below shows a representation of the dealer's transition matrix.

Figure 1: Dealer's transition matrix



Ultimately, the dealer's chain will end in one of the absorbing states: $\texttt{stand}_{17}$, $\texttt{stand}_{18}$, $\texttt{stand}_{19}$, $\texttt{stand}_{20}$, $\texttt{stand}_{21}$, `bj`, or `bust`. By raising D to a sufficiently large power (i.e. 17, as within 17 transitions any random walk will necessarily reach one of these absorbing states [3]), the transient-state rows converge to absorption probabilities into these states. The matrix $D_\infty = D^{17}$ is represented in Figure 2 below.

Figure 2: Dealer's outcome distribution matrix



By examining the appropriate row of D_∞ corresponding to the initial state $\texttt{first}_i$, we get the distribution over the dealer's possible outcomes given that up-card i is shown [3].

4.2 Player's Markov Chain Model under Basic Strategy

Modeling the player is more complex because the player has decision options determined by basic strategy and the dealer's up-card. Essentially, we will have a state for each possible situation of the player's hand, and from each state the transitions will reflect the basic strategy move: some moves lead to drawing a card (like hit or double), while others lead to absorption (like stand or split). We therefore construct 10 different Markov chains [3], one for each dealer up-card $\gamma \in \{2, 3, \ldots, 11\}$ and each one having its own transition matrix P_γ, whose common state space Ψ_P encodes every relevant player situation [3]:

- `first`$_i$: The state where the player has just one card valued i, for $i = 2, \ldots, 11$. This state will automatically transition to a two-card state without any decision (since the second card will always be drawn).

- `twoHard`$_t$: Transient states where the player has a two-card hard total t, for $t = 4, 5, \ldots, 21$.

- `hard`$_t$: Transient states where the player has more than two cards for a hard total t, for $t = 5, 6, \ldots, 21$.

- `twoSoft`$_t$: Transient states where the player has a two-card soft total t, for $t = 12, 13, \ldots, 21$.

- `soft`$_t$: Transient states where the player has more than two cards for a soft total t, for $t = 13, 14, \ldots, 21$.

- `pair`$_i$: Transient states where the player has a pair, with each card of value i, for $i = 2, \ldots, 11$. In this state the player has the option to split.

- `stand`$_t$: Absorbing states where the player stands on total t, for $t = 4, 5, \ldots, 21$.

- `doubStand`$_t$: Absorbing states where the player stands with a doubled bet on total t, for $t = 6, \ldots, 21$.

- `split`$_i$: Absorbing states representing that the player splits a pair, with each card of value i, for $i = 2, \ldots, 11$. We treat the act of splitting as transitioning to a terminal state that encodes what was split because we assume that, after the split, the player starts two new hands, each beginning in the `first`$_i$ state. To represent how one of these two hands is played, we build a separate Markov chain (similar to the original one used for the player's actions) but tailored to follow the specific rules that apply after a hand is split [3].

- `bj`: An absorbing state for the player's blackjack.

- `surrender`: An absorbing state for the player's surrender, even though we assume this is not allowed by default.

- `bust`: An absorbing state for the player's bust without doubling.

- `doubBust`: An absorbing state for the player's bust after doubling.

In total $|\Psi_P| = 122$ distinct states are enumerated.

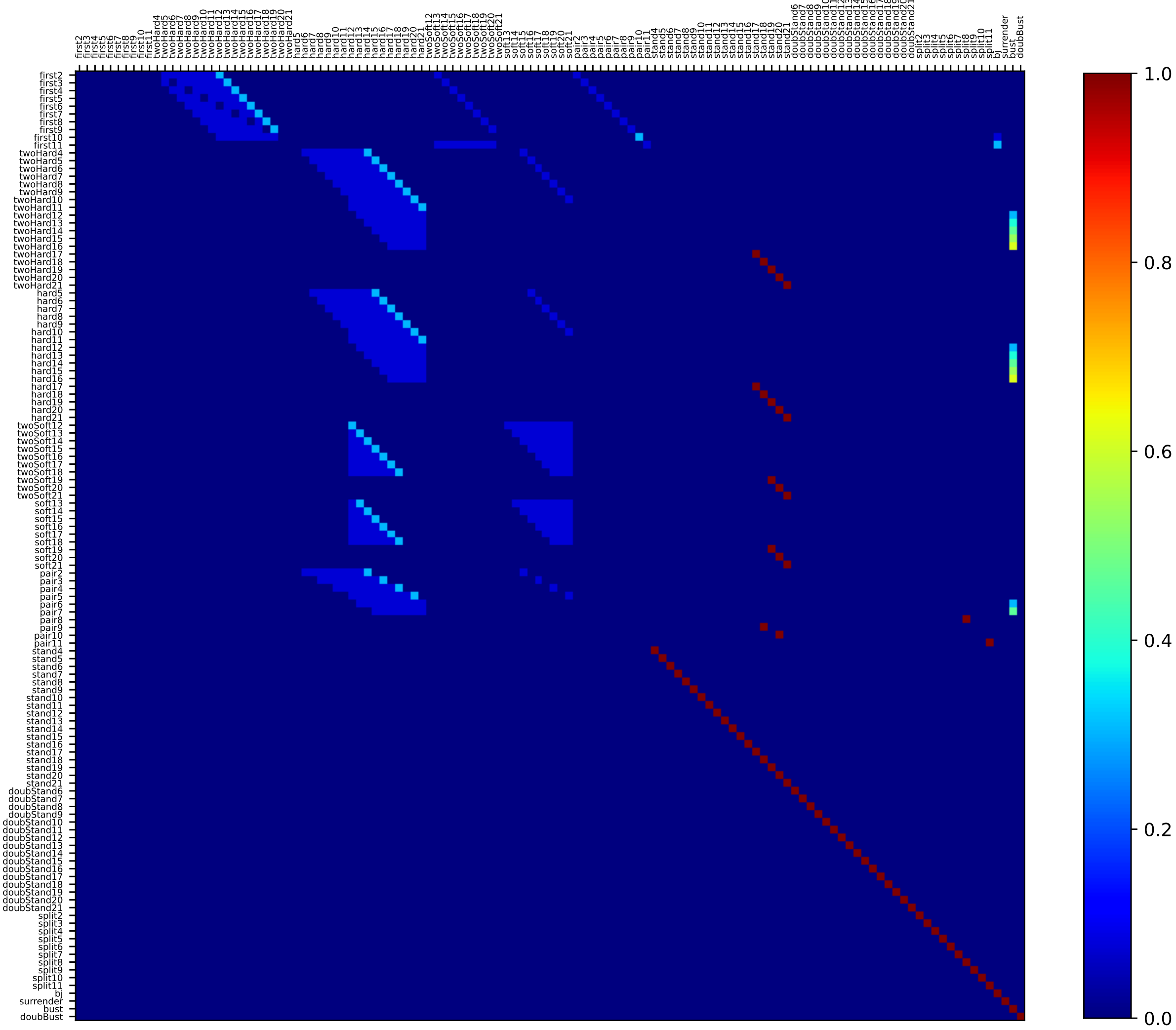
Each transition matrix P_γ for the player is constructed as follows through the function `calc_player_trans`:

- From $\texttt{first}_i$, a second card j is drawn with probability d_j. If $\{i, j\} = \{\text{Ace}, 10\}$, move to `bj`. Else if $i = j$, move to $\texttt{pair}_i$. Otherwise if one card is an Ace, go to $\texttt{twoSoft}_t$, else to $\texttt{twoHard}_t$, where $t = i + j$.
- From a two-card state $\texttt{twoHard}_t$, $\texttt{twoSoft}_t$, or $\texttt{pair}_i$, follow the basic strategy move (S, H, DB, DS, SP, or SR) for that combination of the player's hand against the dealer's up-card γ:
  - If the decision is Stand: transition to $\texttt{stand}_t$.
  - If the decision is Hit: draw another card k with probability d_k, yielding a new $\texttt{hard}_u$ or $\texttt{soft}_u$ (where $u = j + k$, unless an Ace is reevaluated at 1, in which case $u = j + k - 10$) or `bust`.
  - If the decision is Double: draw exactly one additional card k; if the resulting total $u = j + k \leq 21$, transition to $\texttt{doubStand}_u$, else to `doubBust`.
  - If the decision is Split (obviously only in the case of $\texttt{pair}_i$): transition to $\texttt{split}_i$; the two new hands are handled in the separate Markov submodel by the function `calc_player_trans_split`.
- From a multi-card state $\texttt{hard}_t$ or $\texttt{soft}_t$, basic strategy prescribes only Hit or Stand; doubling and splitting are unavailable. Hits follow the same draw logic described above, stands go to $\texttt{stand}_t$.

For example, in Figure 3 is represented the player's transition matrix in case the dealer shows an Ace as up-card ($\gamma = 11$).

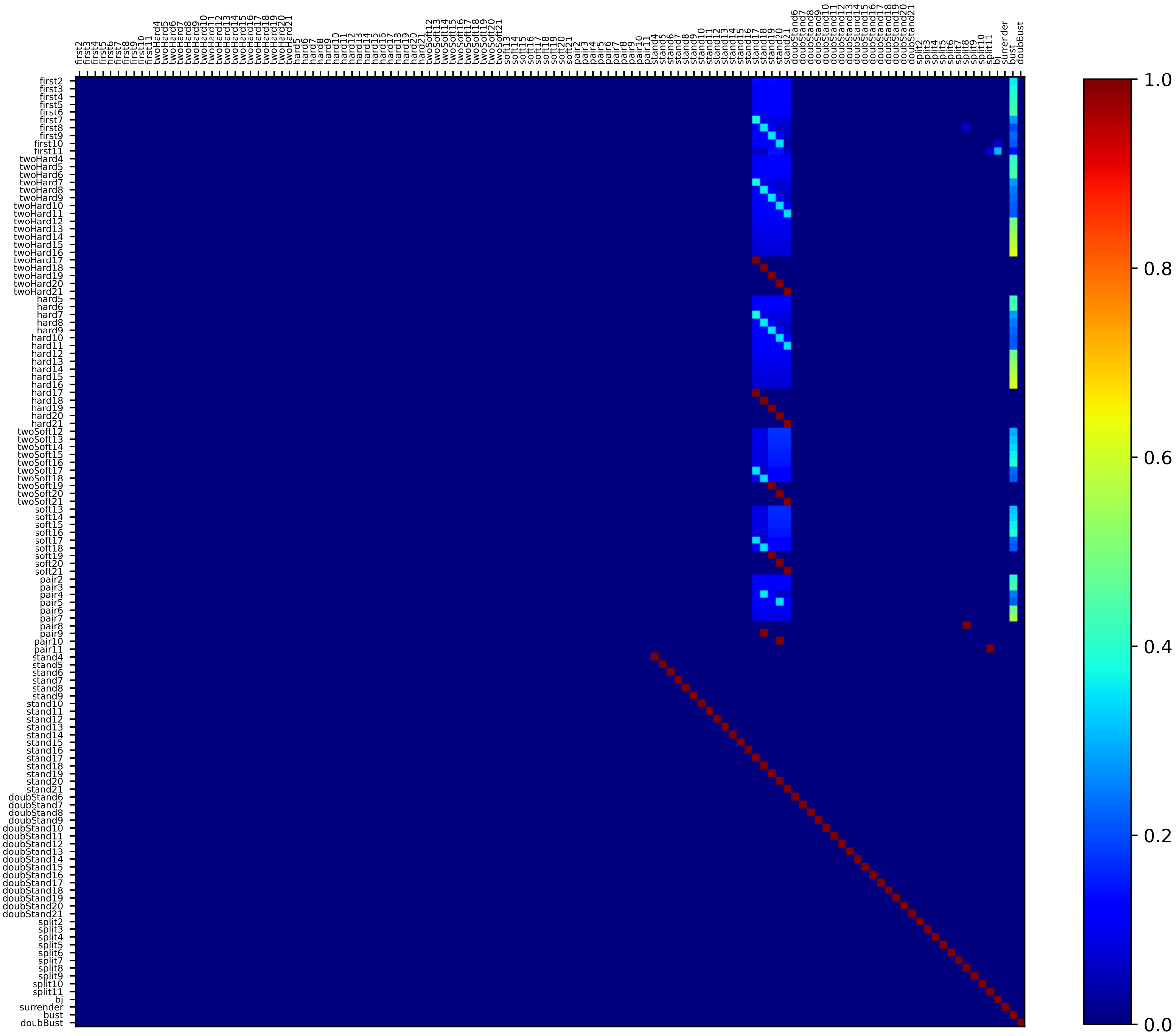
By raising each P_γ to a sufficiently large power (i.e. 21, as within 21 transitions any random walk will necessarily reach one of these absorbing states [3]), we obtain the absorption probabilities into all final states, conditional on dealer up-card γ. Figure 4 shows a representation of the matrix $P_{\gamma,\infty} = P_\gamma^{21}$ with an Ace as the dealer's up-card. Again, row $\texttt{first}_i$ of $P_{\gamma,\infty}$ will yield the distribution over the possible outcomes, assuming the player begins with card i [3].

Figure 3: Player's transition matrix for $\gamma = 11$



Through a similar process in `calc_player_trans_split`, a PS_∞ matrix for post-split hands is also separately computed. A split hand starts with one card of the pair and then a second card is drawn. The `calc_player_trans_split` function constructs transition probabilities similarly to `calc_player_trans` but for starting states representing this situation and specific rules (`houseRules.HASAA = 0`, `houseRules.MSA = 0`, and `houseRules.DASA = 1`). Ultimately, PS_∞ gives the distribution of outcomes for a hand that results from a split and is represented in Figure 5 in the Appendix.

Figure 4: Player's outcome distribution matrix for $\gamma = 11$



4.3 Computing the Player's Advantage

From the Markov chains for dealer and player, we can now compute the expected value for the player's bet through the functions `compute_term_profit` and `adv_single_hand`. The player's advantage (or disadvantage) is defined as the expected net profit per initial bet, assuming the player follows the basic strategy and the dealer follows the house rules. A positive advantage means the player is favored, while a negative advantage means the house is favored. To compute this, we need to combine the probabilities of each outcome with their respective payoffs. We do this conditional on each possible dealer up-card and then weight by the probability of each dealer up-card.

Suppose the dealer's up-card is γ, which occurs with probability

$$d_\gamma = \begin{cases} \frac{1}{13}, & \gamma = 2, 3, \ldots, 9, 11 \\ \frac{4}{13}, & \gamma = 10. \end{cases}$$

To find the player's advantage in this situation, we first compute the probability distributions on the terminal states of the dealer and of the player.

The dealer's absorbing-state distribution

$$u_\gamma(i) \quad \big(i \in U = \{\, i \in \Psi_D : D_{ii} = 1\}\big)$$

is read off from row $\texttt{first}_\gamma$ of D_∞ and represents the probability that the dealer's chain is absorbed in terminal state i (where U is the set of dealer terminal states) [3]. Similarly, the player's absorbing-state distribution

$$v_\gamma(j) \quad \big(j \in V_\gamma = \{\, j \in \Psi_P : (P_\gamma)_{jj} = 1\}\big)$$

comes from $P_{\gamma,\infty}$ and represents the probability that the player's chain is absorbed in terminal state j (where V_γ is the set of player terminal states) [3]. Since dealer and player chains are independent once γ is fixed, the joint probability of any combination of dealer and player outcome is $u_\gamma(i)\, v_\gamma(j)$.

Assuming a unit bet, for any pair of $i \in U$ and $j \in V_\gamma$ we define the profit function as:

$$\pi(i,j) = \begin{cases} +1.5, & j = \texttt{bj},\ i \neq \texttt{bj} \\ 0, & j = \texttt{bj},\ i = \texttt{bj} \\ -1, & j = \texttt{bust} \\ -2, & j = \texttt{doubBust} \\ +1, & j = \texttt{stand}_t,\ i \in \{\texttt{bust}\} \cup \{\texttt{stand}_u : u < t\} \\ -1, & j = \texttt{stand}_t,\ i \in \{\texttt{bj}\} \cup \{\texttt{stand}_u : u > t\} \\ 0, & j = \texttt{stand}_t,\ i = \texttt{stand}_t \\ +2, & j = \texttt{doubStand}_t,\ i \in \{\texttt{bust}\} \cup \{\texttt{stand}_u : u < t\} \\ -2, & j = \texttt{doubStand}_t,\ i \in \{\texttt{bj}\} \cup \{\texttt{stand}_u : u > t\} \\ 0, & j = \texttt{doubStand}_t,\ i = \texttt{stand}_t \\ 2\,\pi_{\text{split}(j)}, & j = \texttt{split}_j. \end{cases}$$

Here, $\pi_{\text{split}(j)}$ is the expected profit of one of the two obtained hands after splitting a pair with each card valued j, and computed similarly via the post-split Markov chain (where each new hand has its own probability distribution from PS_∞).

Weighing each $\pi(i, j)$ by its probability, the conditional expected value given up-card γ is

$$\sum_{i \in U} \sum_{j \in V_\gamma} u_\gamma(i)\, v_\gamma(j)\, \pi(i, j).$$

Finally, averaging over all possible up-cards, the overall player advantage (expected net profit per unit bet) is [3]

$$Adv \;=\; \sum_{\gamma=2}^{11} d_\gamma \sum_{i \in U} \sum_{j \in V_\gamma} u_\gamma(i)\, v_\gamma(j)\, \pi(i, j) \approx -0.0057,$$

i.e. a house edge of approximately 0.57%. This means that a basic strategy player will only lose about half a cent per dollar bet, or about 50 cents for every $100 bet, on average. It's important to highlight that this small remaining house edge is entirely due to the rules favoring the house in certain tie-breaking situations, notably the fact that if both player and dealer bust, the player loses immediately [1]. On the other side, the 3:2 payoff for blackjack helps the player; without it, the house edge would be much higher.

5 Conclusion

In this thesis, we developed a comprehensive Markov chain model of a single blackjack hand played under optimal basic strategy and we computed the player's advantage (or, more accurately, disadvantage) relative to the house.

By defining a finite set of states for both the dealer and player (distinguishing among hard and soft totals, pairs, and all decision options) and by encoding the transition probabilities under the infinite-deck assumption, we obtained exact absorption probabilities into all final outcomes. In this way, we were able to calculate the probabilities of every outcome (win, loss, or tie) in a single hand and, combining them with a carefully defined payoff function, we computed the player's expected value per hand under standard casino rules.

This resulted house edge of approximately 0.57% closely matches the classical values of 0.6% reported by Baldwin *et al.* [1] and of 0.62% calculated by Thorp [2], as well as the value of 0.52% computed by Wakin & Rozell, under their assumed rules [3].

Our analysis did not assume any card counting or variation in bet size. It was a one-hand, fixed-bet model. Card counting can shift the odds to the player's favor by capitalizing on rich or poor shoe compositions [3]. In Markov chain terms, counting introduces additional states related to the deck composition (e.g. using the High-Low Index developed by Thorp [2]), which would expand the state space greatly. Wakin & Rozell extended a similar Markov approach to include a state for the High-Low count and showed that the player's advantage can become positive when the count is high [3].

Beyond validating known results, this work illustrates the power and flexibility of Markov chain modeling for casino games. Rule variations, such as hitting soft 17, restricting doubling, altering blackjack payouts, or implementing surrender, can be accommodated by adjusting the transition rules and recomputing the chains. Moreover, extensions to card-counting analysis involve augmenting the state space with a count variable, yielding a higher-dimensional Markov chain whose equilibrium reveals the long-term player advantage when varying bets [3].

In conclusion, stochastic modeling via absorbing Markov chains provides a rigorous and transparent method for deriving exact probabilities and expected values in blackjack. It not only confirms the efficacy of optimal basic strategy in minimizing losses but also lays the groundwork for analyzing advanced topics such as card counting, side bets, and game-rule impacts within a unified probabilistic framework.

6 Appendix

Table 1: Basic Hard Strategy

Player\Dealer	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**A**
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	DB	DB	DB	DB	H	H	H	H	H
10	DB	DB	DB	DB	DB	DB	DB	DB	H	H
11	DB	DB	DB	DB	DB	DB	DB	DB	DB	H
12	H	H	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	SR	H
16	S	S	S	S	S	H	H	SR	SR	SR
17	S	S	S	S	S	S	S	S	S	S
18	S	S	S	S	S	S	S	S	S	S
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

Table 2: Basic Soft Strategy

Player\Dealer	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**A**
12	H	H	H	H	DB	H	H	H	H	H
13	H	H	H	H	DB	H	H	H	H	H
14	H	H	H	DB	DB	H	H	H	H	H
15	H	H	DB	DB	DB	H	H	H	H	H
16	H	H	DB	DB	DB	H	H	H	H	H
17	H	DB	DB	DB	DB	H	H	H	H	H
18	S	DS	DS	DS	DS	S	S	H	H	H
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

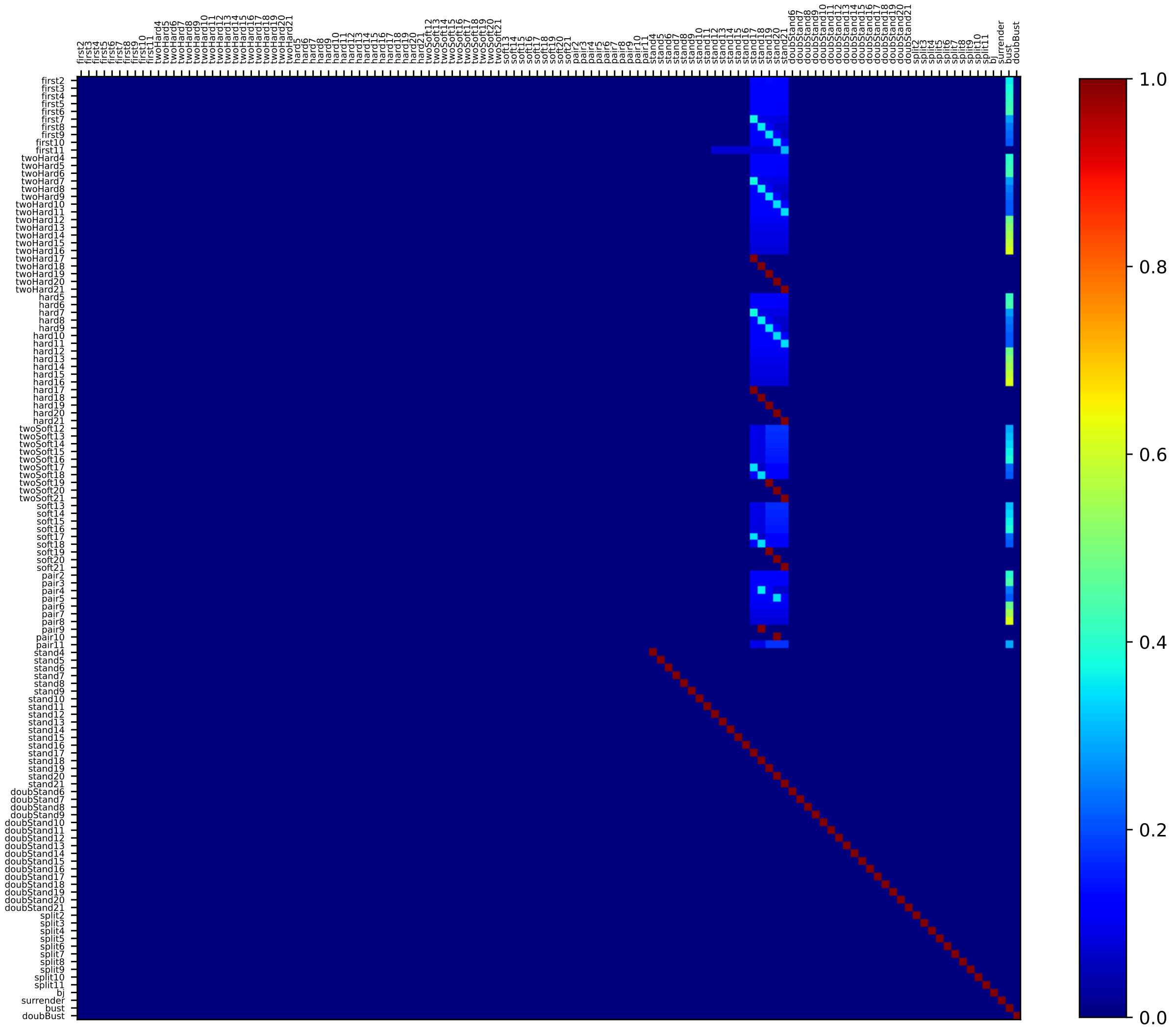
Table 3: Basic Pair Strategy (DASA)

Player\Dealer	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**A**
2–2	SP	SP	SP	SP	SP	SP	H	H	H	H
3–3	SP	SP	SP	SP	SP	SP	H	H	H	H
4–4	H	H	H	SP	SP	H	H	H	H	H
5–5	DB	DB	DB	DB	DB	DB	DB	DB	H	H
6–6	SP	SP	SP	SP	SP	H	H	H	H	H
7–7	SP	SP	SP	SP	SP	SP	H	H	H	H
8–8	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
9–9	SP	SP	SP	SP	SP	S	SP	SP	S	S
10–10	S	S	S	S	S	S	S	S	S	S
A–A	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP

Table 4: Basic Pair Strategy (No DASA)

Player\Dealer	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**A**
2–2	H	H	SP	SP	SP	SP	H	H	H	H
3–3	H	H	SP	SP	SP	SP	H	H	H	H
4–4	H	H	H	H	H	H	H	H	H	H
5–5	DB	DB	DB	DB	DB	DB	DB	DB	H	H
6–6	H	SP	SP	SP	SP	H	H	H	H	H
7–7	SP	SP	SP	SP	SP	SP	H	H	H	H
8–8	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP
9–9	SP	SP	SP	SP	SP	S	SP	SP	S	S
10–10	S	S	S	S	S	S	S	S	S	S
A–A	SP	SP	SP	SP	SP	SP	SP	SP	SP	SP

Figure 5: Player's outcome distribution matrix from a split for $\gamma = 11$



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