ASSIGNMENT REPORT

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EEE

Sep 21, 2017

Contents

1 Algorithm Description

1.1 Bisection method

The bisection method in mathematics is a root-finding method that repeatedly bisects an interval and then selects a subinterval in which a subinterval in which a root must lie for further processing.

1.2 Newton-Raphson method

The Newton-Raphson method is a method for finding successively better approximations to the roots of a real-valued function. The formula of Newton-Raphson method is

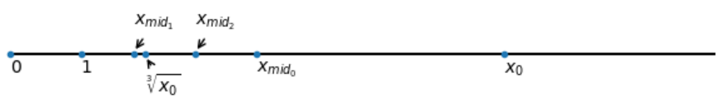
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

2 Proof

2.1 Bisection method

Firstly I assume the given number X_0 is positive. Because the symmetry of the positive and negative number on number axis, the algorithm should work for negative number as long as it works for positive number.

I. If $x_0 > 1$, let $x_{high_0} = x_0$, $x_{low_0} = 0$, $x_{mid_0} = \frac{x_{high_0}+x_{low_0}}{2}$



Step A: If $x_{mid_0}^3 > x_0 \Rightarrow x_{mid_0} > \sqrt[3]{x}$,

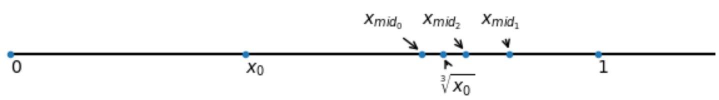
then let $x_{high_1} = x_{mid_0}$, $x_{low_1} = x_{low_0} = 0$, $x_{mid_1} = \frac{x_{high_1}+x_{low_1}}{2}$

Step B: If $x_{mid_1}^3 < x_0 \Rightarrow x_{mid_1} < \sqrt[3]{x}$,

then let $x_{low_2} = x_{mid_1}$, $x_{high_2} = x_{high_1} = x_{mid_0}$, $x_{mid_2} = \frac{x_{high_2}+x_{low_2}}{2}$

Repeat step A or B until $|x_{mid_{n+1}} - x_{mid_n}| < precision\ error$

II. If $x_0 < 1$, let $x_{high_0} = 1$, $x_{low_0} = x_0$, $x_{mid_0} = \frac{x_{high_0}+x_{low_0}}{2}$



Step A: If $x_{mid_0}^3 < x_0 \Rightarrow x_{mid_0} < \sqrt[3]{x}$,

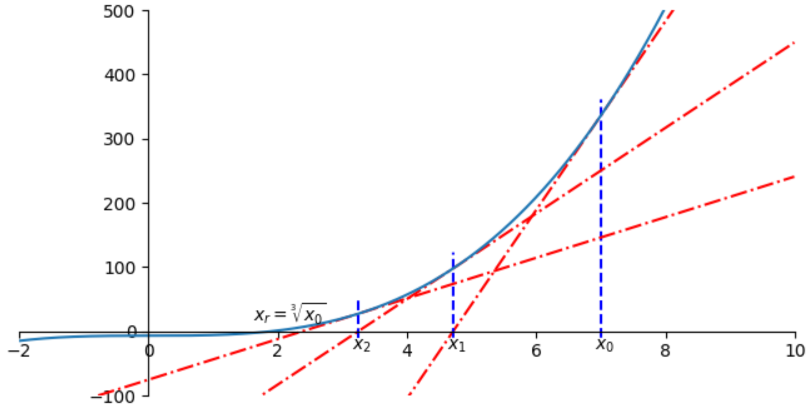
then let $x_{low_1} = x_{mid_0}$, $x_{high_1} = x_{high_0} = 1$, $x_{mid_1} = \frac{x_{high_1}+x_{low_1}}{2}$

Step B: If $x_{mid_1}^3 > x_0 \Rightarrow x_{mid_1} > \sqrt[3]{x}$,

then let $x_{high_2} = x_{mid_1}$, $x_{low_2} = x_{low_1} = x_{mid_0}$, $x_{mid_2} = \frac{x_{high_2}+x_{low_2}}{2}$

Repeat step A or B until $|x_{mid_{n+1}} - x_{mid_n}| < precision\ error$

2.2 Newton-Raphson method



Let $x_r = \sqrt[3]{x_0}$, so $f(x_r) = 0$ $----------------------------$ $Eq.1$

Given $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $----------------------------$ $Eq.2$

By Taylor Series,

$$f(x_r) = f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(x_n)}{2!}(x_r - x_n)^2 + \frac{f'''(x_n)}{3!}(x_r - x_n)^3 + \cdots$$

Here the previous three items are used as the dominant polynomials.

Let $f(x_r) \approx f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(x_n)}{2!}(x_r - x_n)^2$ $------------$ $Eq.3$

Substitute Eq.1 and Eq.2 into Eq.3,

$\Rightarrow 0 = f'(x_n)(x_n - x_{n+1}) + f'(x_n)(x_r - x_n) + \frac{f''(x_n)}{2!}(x_r - x_n)^2$

$\Rightarrow x_{n+1} - x_r = \frac{f''(x_n)}{2!}(x_r - x_n)^2$

Let error $e_n = x_n - x_r$,

$\Rightarrow e_{n+1} = \frac{f''(x_n)}{2f'(x_n)}e_n^2$ $--------------------------------$ $Eq.4$

The last equation shows that after each iteration, the error in new estimate is proportional to the square of the old estimate. That means the precision digits will double for each iteration.

3 Complexity Analysis

3.1 Bisection method

Here I define the time complexity is dependent on the required precision digit N. For example, 1.12345 with precision digit N=3 means the first 3 digits after decimal point .123 are accurate.

Considering the worst case, to determine one precision digit, for example, the current state is $x_{low_0} = 1.0$, $x_{high_0} = 2.0$ and the expected first accurate digit is 1.1. By definition $x_{mid_n} = \frac{x_{high_n} + x_{low_n}}{2}$, we can find $x_{mid_0} = 1.5$. After several iterations we can get $x_{mid_1} = 1.25$, $x_{mid_2} = 1.125$, $x_{mid_3} = 1.0625$, $x_{mid_4} = 1.09375$, $x_{mid_5} = 1.109375$.

This shows that the algorithm needs to iterate at most 6 times to determine the interval where the accurate digit lies for the worst case. If the required precision digits is N, then it needs at most 6N times to find the required precision interval, which means the time complexity of bisection method is $O(N)$.

3.2 Newton-Raphson method

The Eq.4 $e_{n+1} = \frac{f''(x_n)}{2f'(x_n)} e_n^2$ shows that the precision digits will double after each iteration. For example, if the error for the 1st time iteration is 0.1(N=1), the error for the 2nd and 3rd time iteration will be $0.1^2 = 0.01$(N=2) and 0.0001(N=2^2). So after t times of iteration, it shows $N \propto 2^t$. If ignoring the affect of the initial value x_0, the time complexity of Newton-Raphson method is $O(\log_2 N)$

4 Conclusion

After several times of experiment, the results show that with the increasing number of precision digits N, the total number of iterations for bisection method is proportional to N.

For Newton-Raphson method, the number of iterations does not increase significantly, which verifies its complexity is proportional to $\log_2 N$. However, the Eq.4 shows that the error of next iteration is also dependent on the previous input value x_n, which means the initial value has a significant influence on the time complexity of Newton-Raphson method. Since it is impossible to determine the optimal value for x_0, we cannot explicitly define the time complexity of Newton-Raphson method.