Peer-to-peer risk-sharing schemes with heterogeneity and infinite-mean losses

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Introduction

Where do infinite-mean losses arise from?

- Natural disasters, e.g., earthquakes and hurricanes
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  → catastrophic (infinite-mean) losses
- **Want:** protection against them
- **How?**

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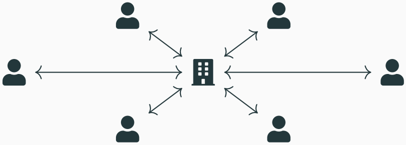
  - Centralized:



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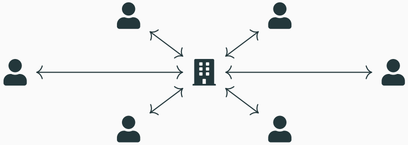

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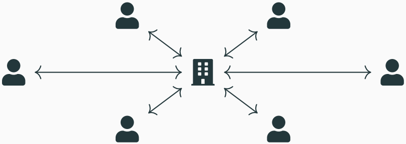

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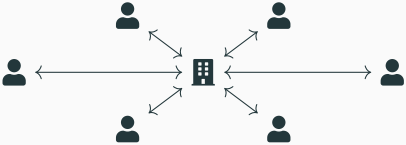

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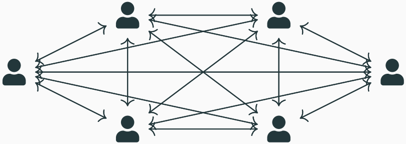

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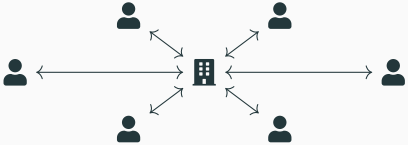

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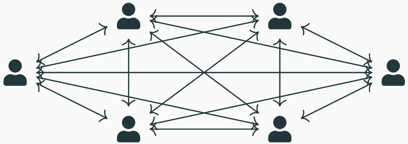

  - Does it work?

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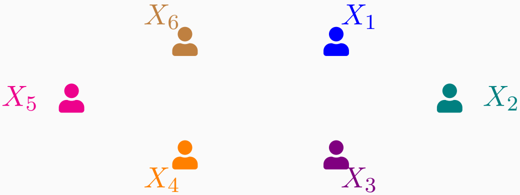
- “Conventional wisdom”: diversification is good (don’t put all your eggs in one basket ...)
- **Not quite applicable** in this setting, as “diversifying” infinite-mean losses would indeed worsen the outcome (Ibragimov et al., 2009; Chen et al., 2024)!

Preparations

Setting and notations

- n agents, with initial losses $X_1, \ldots, X_n$



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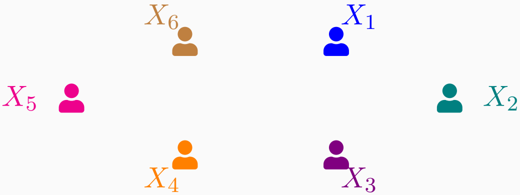
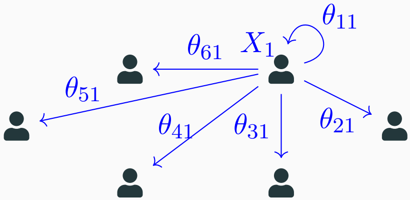


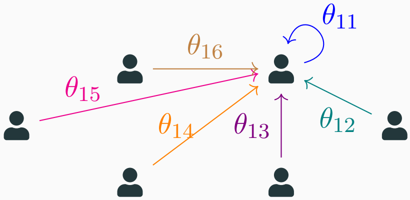
- θ_{ij} (or just θ_j): proportion of loss X_j allocated to agent i



Setting and notations

- Y_i: loss after risk sharing for agent i

$$\boxed{Y_1 = \theta_{11}X_1 + \theta_{12}X_2 + \cdots + \theta_{16}X_6}$$



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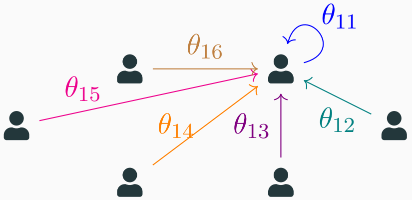

➔ All (linear) risk allocations:

$$Y_i = \sum_{j=1}^{n} \theta_{ij} X_j, \quad (\theta_{i1}, \dots, \theta_{in}), (\theta_{1j}, \dots, \theta_{nj}) \in \Delta_n,$$

where $\Delta_n := \{(\theta_1, \dots, \theta_n) \in [0,1]^n : \sum_{i=1}^n \theta_i = 1\}$.

P2P risk sharing with infinite-mean losses

Why not diversify infinite-mean losses?

- Chen et al. (2024, Theorem 1): $X_1 \leq_{\mathrm{st}} \sum_{j=1}^{n} \theta_j X_j$ for all $(\theta_1, \ldots, \theta_n) \in \Delta_n$
  - applicable to independent and identically distributed (iid) infinite-mean Pareto losses
  - Pareto losses: $X \sim \mathrm{Pareto}(\alpha)$ with shape parameter $\alpha > 0$ (and scale parameter 1), if its CDF is given by $F(x) = 1 - (1/x)^{\alpha}, \quad x \geq 1$.
  - $X \leq_{\mathrm{st}} Y$ refers to the **first-order stochastic dominance**, i.e., $\mathbb{P}(X > t) \leq \mathbb{P}(Y > t)$ for all $t \in \mathbb{R}$

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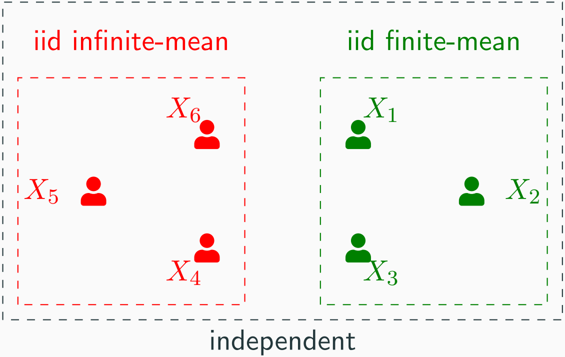
- **Idea:** How about *nonlinear* risk allocations and *heterogeneous* losses?

Two main extensions: nonlinearity and heterogeneity

- **Heterogeneity:** introduced via the **two-group conditions**:
  1. $X_1, \ldots, X_n$ are independent.
  2. First m losses $X_1, \ldots, X_m$ are iid finite-mean.
  3. Remaining $n-m$ losses $X_{m+1}, \ldots, X_n$ are iid infinite-mean.



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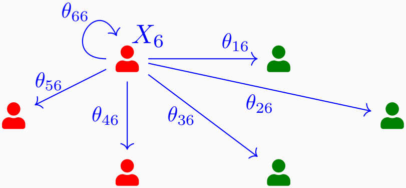

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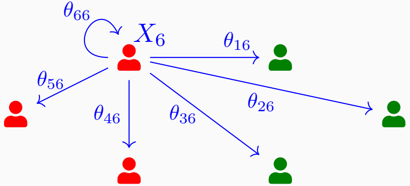
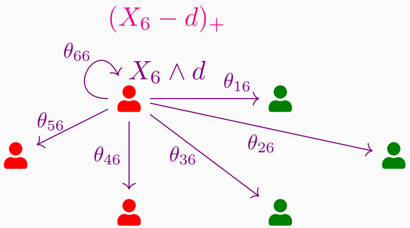


  - Nonlinear risk allocations: $X_6 = X_6 \wedge d + (X_6 - d)_+$, where $x \wedge y := \min\{x, y\}$ and $x_+ := \max\{x, 0\}$.



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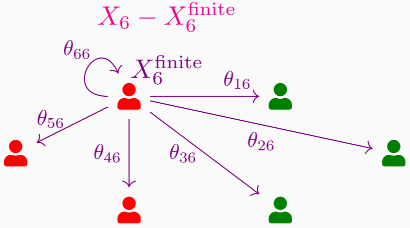
- **Nonlinearity:**
  - Nonlinear risk allocations: $X_6 = X_6^{\text{finite}} + X_6 - X_6^{\text{finite}}$, where $X_6^{\text{finite}} = F_1^{-1}(F_6(X_6))$, with F_i being the CDF of X_i and F_i^{-1} being its generalized inverse, i.e., $F_i^{-1}(p) = \inf\{x \in \mathbb{R} : F_i(x) \geq p\}$.



Three P2P risk-sharing schemes

Scheme [L]

- **Definition:** The **scheme [L]** is the set of all risk allocations taking the form $Y_i = \sum_{j=1}^{n} \theta_{ij} X_j$ for all $i = 1, \ldots, n$, where $(\theta_{i1}, \ldots, \theta_{in}) \in \Delta_n$ for all $i = 1, \ldots, n$, and $(\theta_{1j}, \ldots, \theta_{nj}) \in \Delta_n$ for all $j = 1, \ldots, n$.

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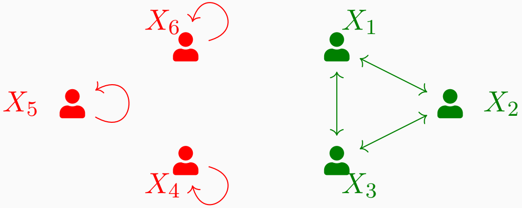
- **Risk-sharing rule in focus: Rule [L*]**, given by $Y_i = \frac{1}{m} \sum_{k=1}^m X_k$ for all $i = 1, \ldots, m$, and $Y_j = X_j$ for all $j = m+1, \ldots, n$.



Properties of the rule [L*]

- **First-order stochastic dominance under Pareto losses and two-group conditions:**

$$\theta_1 X_1 + \cdots + \theta_{m-1} X_{m-1} + (1 - \theta_1 - \cdots - \theta_{m-1}) X_m \leq_{\mathrm{st}} \sum_{i=1}^{n} \theta_i X_i$$

for all $(\theta_1, \ldots, \theta_n) \in \Delta_n$.

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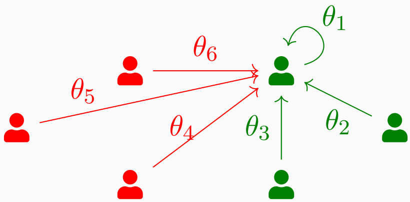

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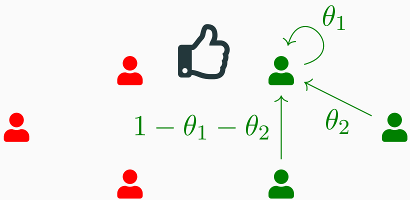

Properties of the rule [L*]

- **Pareto optimality under Pareto losses and two-group conditions:** The rule [L*] is *Pareto optimal in scheme [L] under the preference $\preceq_{\mathrm{sc}}$*, defined by $Y_i \preceq_{\mathrm{sc}} Z_i$ if

$$\begin{cases} Z_i \leq_{\mathrm{cx}} Y_i & \text{when } Y_i \text{ and } Z_i \text{ both have finite mean,} \\ Z_i \leq_{\mathrm{st}} Y_i & \text{otherwise,} \end{cases}$$

  where $\leq_{\mathrm{cx}}$ refers to the **convex order**, i.e., $X \leq_{\mathrm{cx}} Y$ if $\mathbb{E}[\varphi(X)] \leq \mathbb{E}[\varphi(Y)]$ for all convex functions φ such that both expectations are finite.

  - **Pareto optimality**: There does not exist another rule that is an improvement for all agents and a strict improvement for at least one agent.

Scheme [FR]

- **Definition:** Suppose F_j is continuous for all $j = m+1, \ldots, n$. The **scheme [FR]** is the set of all risk allocations with the following form:

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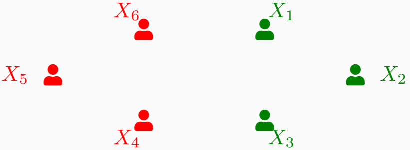
for all $i = 1, \ldots, m$, and

$$Y_j = \sum_{k=1}^{m} \theta_{jk} X_k + \sum_{k=m+1}^{n} \theta_{jk} X_k^{\text{finite}} + (X_j - X_j^{\text{finite}})$$

for all $j = m+1, \ldots, n$, where $X_k^{\text{finite}} := F_1^{-1}(F_k(X_k))$ for all $k = m+1, \ldots, n$, $(\theta_{i1}, \ldots, \theta_{in}) \in \Delta_n$ for all $i = 1, \ldots, n$, and $(\theta_{1k}, \ldots, \theta_{nk}) \in \Delta_n$ for all $k = 1, \ldots, n$.



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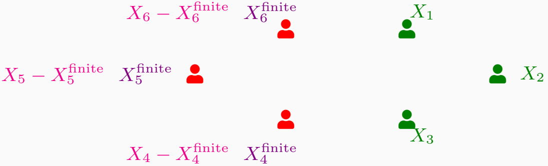

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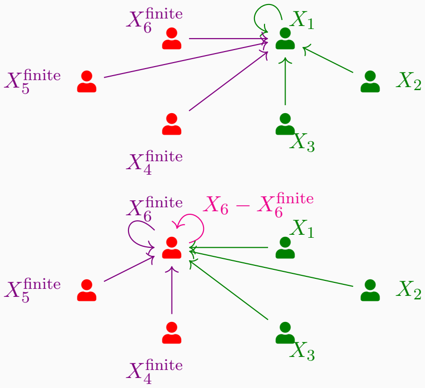

Properties of the rule [FR*]

- **Improvement for finite-mean agents under two-group conditions:**
  $(1/n)\big(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}\big) \le_{\text{cx}} (1/m)\sum_{k=1}^{m} X_k$

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- **Improvement in finite-mean portion for infinite-mean agents under two-group conditions:**
  - Rule [L*]: $Y_j = X_j = X_j^{\text{finite}} + (X_j - X_j^{\text{finite}})$
  - Rule [FR*]:
    $Y_j = \frac{1}{n}(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\text{finite}}) + (X_j - X_j^{\text{finite}})$
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- **Improvement from "diversification benefits" for infinite-mean agents under Pareto losses and two-group conditions:**
  - Simplification under Pareto losses and two-group conditions: $X_1, \ldots, X_m \overset{\text{iid}}{\sim} \text{Pareto}(\alpha)$ and $X_{m+1}, \ldots, X_n \overset{\text{iid}}{\sim} \text{Pareto}(\beta)$ with $\alpha > 1$ and $\beta \leq 1$ ➔ $X_j^{\text{finite}} = X_j^{\beta/\alpha}$ for all $j = m+1, \ldots, n$.

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  - Diversification benefits:

Rule	Finite-mean	Infinite-mean	
[L*]	$X_j^{\beta/\alpha}$	$X_j - X_j^{\beta/\alpha}$	Comonotonic
[FR*]	$\frac{1}{n}(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k^{\beta/\alpha})$	$X_j - X_j^{\beta/\alpha}$	Mixed with some independent losses

Scheme [LS]

- **Definition:** Let $d \in \mathbb{R}$ be a value such that $\mathbb{E}[X_{m+1} \wedge d] = \mathbb{E}[X_1]$. The **scheme [LS]** is the set of all risk allocations which take the following form:

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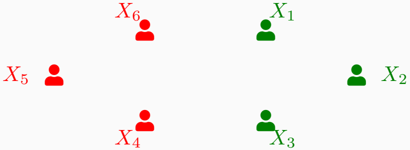

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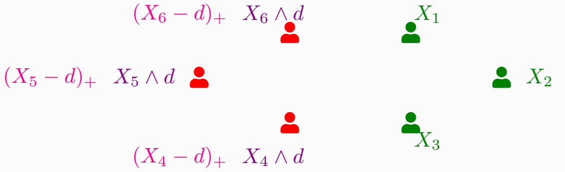

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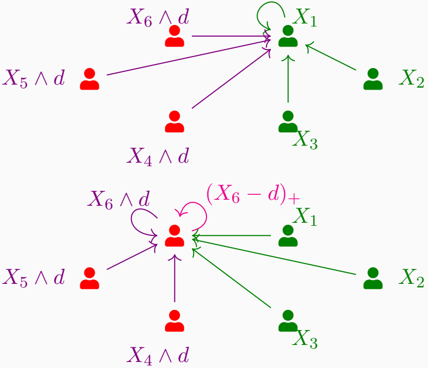

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$$\frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j \wedge d\right) \leq_{\text{cx}} \frac{1}{n}\left(\sum_{i=1}^{m} X_i + \sum_{j=m+1}^{n} X_j^{\text{finite}}\right)$$
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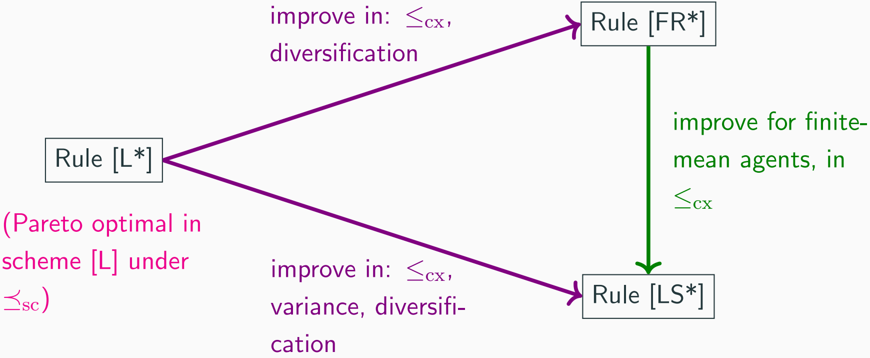
➔ [LS*] better than [FR*] better than [L*]
- **Improvement in finite-mean portion for infinite-mean agents under two-group conditions:**
  - Rule [L*]: $Y_j = X_j = X_j \wedge d + (X_j - d)_+$
  - Rule [LS*]:
    $Y_j = \frac{1}{n}(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k \wedge d) + (X_j - d)_+$
  - Improvement in finite-mean portion: Suppose $X_1 \leq_{\mathrm{st}} X_{m+1}$.
    Then $\mathrm{Var}\left(\frac{1}{n}(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k \wedge d)\right) \leq \mathrm{Var}\left(X_j \wedge d\right)$
    iff $n \geq \frac{1}{2}\left(1 + \sqrt{1 + 4m[(\mathrm{Var}\left(X_1\right))/(\mathrm{Var}\left(X_{m+1} \wedge d\right)) - 1]}\right)$
    (≈ 5.1098 when $m = 3$ and $\mathrm{Var}\left(X_1\right) = 8\,\mathrm{Var}\left(X_{m+1} \wedge d\right)$).

Properties of the rule [LS*]

- **Improvement from "diversification benefits" for infinite-mean agents under two-group conditions:**

Rule	Finite-mean	Infinite-mean	
[L*]	$X_j \wedge d$	$(X_j - d)_+$	Comonotonic
[LS*]	$\frac{1}{n}(\sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_k \wedge d)$	$(X_j - d)_+$	Mixed with some independent losses

Summary of the key results



➔ The rule [LS*] may be considered to be the best one among these three rules.

References

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