HKU STAT3956 Study Notes

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1 Pension Mathematics

1.0.1 Building upon our knowledge in life contingencies from STAT3901 and STAT3909, in STAT3956 we will delve into the topic of *pension*, which is a fund with contributions made throughout an employee's career, designed for providing retirement income to the employee.

1.0.2 **Defined benefit and defined contribution pensions.** There are two main types of pension plans: (i) defined contribution (DC) and (ii) defined benefit (DB).

- The **defined contribution** plan specifies ("defines") the amount of contribution, as a function of salary into the pension. The contributions are invested and may be determined to meet a certain target benefit level. However, the actual retirement income could differ from the target, depending on the investment returns.
- The **defined benefit** plan specifies ("defines") the amount of benefit, typically related to salary near retirement or salary through employment. Again, the contributions are invested to meet the target benefit. If the investment or mortality experience turns out to be adverse, then the contributions can be increased. On the other hand, if the investment or mortality experience turns out to be favorable, then the contributions can be decreased. This gives *pension actuaries* (possibly including you in the future!) an important task of monitoring pension plans regularly and assessing whether the contributions need to be adjusted. In view of this, the DB plan is of more interest than the DC plan in actuarial science, although the former is more mathematically complicated than the latter.

1.0.3 **Pay-as-you-go vs. fully funded pensions.** Apart from DB vs. DC, another aspect that distinguishes pensions is the way the pension fund is built up. There are two major mechanisms for that: (i) pay-as-you-go (PAYG) and (ii) fully funded. The **pay-as-you-go** basis works by requiring the active workers, who are still working and not yet retired, to pay the income of the retirees, through taxation. On the other hand, the **fully funded** basis works by ensuring that the contributions previously made to the fund are enough for paying the retirement benefits required.

1.1 DC Pension Plans

1.1.1 **Basic terms and notations about salary.** Let us start studying pension mathematics by discussing the mathematically simpler *DC pension plan*. Like what we did in STAT3901/STAT3909, we first introduce some basic terms and (actuarial) notations about *salary*, which is the main driver for the amount of contribution in a DC pension plan.

- S_x: salary received in year of age x to $x+1$, assuming the individual works full time throughout the period from age x to $x+1$.
- $\{s_y\}_{y\geq x_0}$: **salary scale**, where each s_y is interpreted as a scaled version of the salary received in year of age y to $y+1$, and s_{x_0} is an arbitrary positive value (often 1), with x_0 being a suitable initial age (often positive integer, e.g., 20 or 25).

  Specifically, we have $s_y = kS_y$ for some constant scaling factor $k > 0$, for all $y \geq x_0$. Thus, we often use the salary scale to perform calculations via the following ratio formula:

$$\boxed{\frac{s_y}{s_x} = \frac{\text{salary received in year of age } y \text{ to } y+1}{\text{salary received in year of age } x \text{ to } x+1}}$$

  for all $y > x \geq x_0$, assuming the individual works full time from age x to $y+1$.
- o: **offset**, which decreases the *pensionable salary* from S_x to $S_x - o$, where **pensionable salary** refers to the salary amount actually used for the pension benefit calculations. This offset is associated with *social security*. To see this, consider the following simple example where the pension benefit is targeted to be 70% of the final salary. With the social security and the offset, the (targeted) total retirement income is social security $+ 0.7$(final salary $- o$). To make this total retirement income 70% of the final salary, we need to choose the offset to be $10/7 \times$ social security.

  Remarks:

  - Often, the value of offset ($10/7 \times$ social security here) is specified as the *minimum wage*. Then, final salary $- o$ is always nonnegative, and the retirement income is just the social security if the individual earns just the minimum wage as their final salary.
  - For simplicity, we shall assume that the offset is 0 unless otherwise specified. In case offset is positive, we can simply subtract it from every original salary received before applying formulas on them.

- R: **replacement ratio**, which is given by

$$R = \frac{\text{pension income in the year after retirement}}{\text{salary in the year before retirement}},$$

  assuming the pension plan member survives the year following retirement.

- $\bar{S}_x$: **salary rate** at exact age x, which means that $\bar{S}_x \Delta t$ is the salary received in the period from age x to $x + \Delta t$, for small $\Delta t > 0$.[1]

- $\{\bar{s}_y\}_{y \geq x_0}$: **rate of salary function**, where each $\bar{s}_y$ is interpreted as a scaled version of the salary rate at exact age y. Again, the value $\bar{s}_{x_0}$ is an arbitrary positive value (often 1) with x_0 being a suitable initial age.

  Like the salary scale $\{s_y\}_{y \geq x_0}$, we have $\bar{s}_y = k\bar{S}_y$ for some constant scaling factor $k > 0$, for all $y \geq x_0$. So, again we often use the following ratio formula for performing calculations:

$$\boxed{\frac{\bar{s}_y}{\bar{s}_x} = \frac{\text{salary rate at age } y}{\text{salary rate at age } x}}$$

  for all $y > x \geq x_0$, assuming the individual works full time from age x to y.

1.1.2 **Relationships between salaries received and salary rates.** To obtain the salary received S_x (discrete) from the salary rate $\bar{S}_x$ (continuous), we can use the following integral formula:

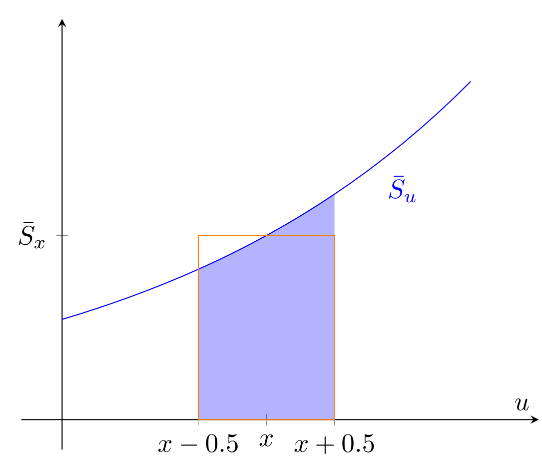
$$\boxed{S_x = \int_0^1 \bar{S}_{x+t}\,\mathrm{d}t}$$

where "$\bar{S}_{x+t}\,\mathrm{d}t$" can be interpreted as the salary received in the "infinitesimal" period $[x+t, x+t+\mathrm{d}t]$, and "$\int_0^1$" sums over all such salaries received.

However, the integral above is generally hard to compute and may require the usage of numerical integration, which is undesirable in an exam context ✎ particularly. In such a case, the following *approximation* can be helpful:

$$\boxed{S_{x-0.5} \approx \bar{S}_x}.$$

It can be graphically understood as follows:



[1] When working with salary rates, an implicit assumption is that salaries are paid *continuously* (so that it makes sense to talk about "rate").

The blue area is the exact value of $S_{x-0.5}$, and the area of the orange rectangle is $\bar{S}_x \times 1 = \bar{S}_x$. These two areas are approximately equal, and are exactly equal if the salary rate is linear in the period $[x-0.5, x+0.5]$.

1.1.3 **From contributions to pension benefit.** After making numerous contributions to a pension fund throughout the entire career, the individual can enjoy a pension benefit payable regularly (e.g., monthly or annually) following retirement, whose amount is determined based on the final accumulated value in the fund, which sources from both the contributions (certain) and investment returns (often uncertain); as we can see here, while the contributions are defined and specified in a DC pension plan, the *benefit amount* is uncertain. In Section 1.2, we will study the DB pension plan, which has the opposite nature: The benefit amount is defined and specified, but the contributions are uncertain.

To convert the final accumulated value in the pension fund into pension benefit, the fund is used to purchase a life annuity of appropriate type at the time of conversion, depending on the feature of pension benefit. Here are two examples:

- *(Monthly benefits payable starting from retirement)* If the pension benefit is paid to the individual at the start of each month from age 65 (the age of retirement), then the corresponding life annuity is the whole life month annuity-due issued to a life aged 65, with expected present value (EPV) given by
$$\text{total annual benefit} \times \ddot{a}_{65}^{(12)}.$$
- *(Spouse pension)* Another common type of pension benefit is known as **spouse pension**, which is a reversionary annuity for the individual's spouse (recall STAT3909). Assuming that the reversionary annuity benefit is payable at the start of each month when the individual is dead while their spouse is alive, and that the individual and their spouse are respectively aged 65 and 60 at retirement, the EPV of that annuity is given by
$$\text{total annual benefit} \times \ddot{a}_{65|60}^{(12)}.$$

In general, we let a_y^r denote the EPV of a life annuity with annual benefit 1 for a pension member that retires at age y. In the examples above, we have $a_y^r = \ddot{a}_{65}^{(12)}$ and $a_y^r = \ddot{a}_{65|60}^{(12)}$ respectively.

Using this notation, we can determine the amount of pension benefit via the equation below:

$$\underbrace{\text{final accumulated value in the pension fund}}_{\text{sourced from contributions and fund returns}} = \text{annual benefit} \times a_y^r \tag{1}$$

where y is the age of retirement of the individual (often 65). In particular, sometimes we have a *target* annual benefit for achieving a certain replacement ratio R (so the target amount is $R \times S_{y-1}$, where y is the age of retirement), and from there we would then find out the required contributions (e.g., a certain percentage of salary) to meet such a target.

1.1.4 **DC capital accrual.** To monitor the status of a DC pension plan, it is of interest to keep track of the capital accrual (accumulated value) in the pension fund. A generic formula for capital accrual in a DC pension plan is:

$$\boxed{\text{DCCAP}_{k+1} = \text{DCCAP}_k(1+r_k) + \text{Pr}_k, \quad k = 1, 2, \dots},$$

where DCCAP_k and Pr_k are respectively the DC pension fund capital and the contribution made (premium) at the end of year k (time k), and r_k is the fund return rate from time k to $k+1$ (year $k+1$). [Note: Contributions are assumed to be made at the *end* of each year, where the information about the amount of salary received in that year is known.]

In particular, the values of DCCAP_k's facilitate the calculation of a quantity known as **current accrued benefit** (CAB), which refers to the amount of annual pension benefit to be received (starting from a certain future time point) *if* the individual stopped working from now on and the fund was used to purchase the (deferred) life annuity now; the benefit amount is only based on the contributions from the *past* service. This quantity is helpful for understanding better how much the pension plan is "worth" currently (somewhat analogous to the idea of *reserve/policy value* learnt in STAT3901/STAT3909). For example, if the individual starts working at age 20 (time 0 for the DC pension fund) and the pension benefit is to be paid at the start of every year from age 65, then the current accrued benefit at time 10 (when the individual is aged 30) is

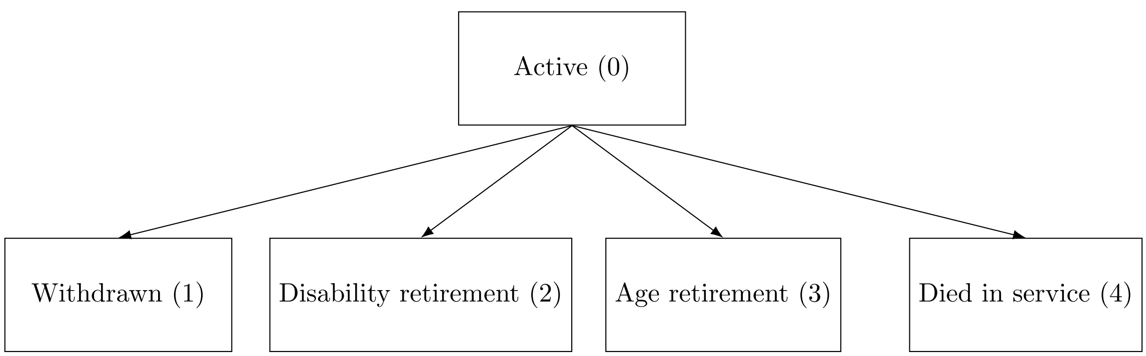
$$\mathrm{CAB}_{10} = \frac{\mathrm{DCCAP}_{10}}{{}_{35|}\ddot{a}_{30}} = \frac{\mathrm{DCCAP}_{10}}{{}_{35}E_{30}\,\ddot{a}_{65}},$$

where ${}_{35|}\ddot{a}_{30} = {}_{35}E_{30}\,\ddot{a}_{65}$ is the EPV of the pension benefit (35-year deferred whole life annuity-due) with annual amount 1 purchased at age 30.

1.2 DB Pension Plans

1.2.1 After studying the relatively simple DC pension plan in Section 1.1 to gain a sense of how pension works, we now proceed to DB pension plan which is more mathematically complicated and thus requires more actuarial analysis. Unlike DC pension plan where the contributions are defined in a straightforward manner (just a constant percentage of salary, or perhaps a percentage that varies across age), the benefits in DB pension plan are often defined in a more complicated way, e.g., different benefits may be payable when the person "exits" due to different reasons. The dependency of the benefit on the type of "exit" calls for the need of *multiple decrement model* learnt in STAT3909.

1.2.2 **The service table.** A standard decrement model used for this kind of pension plan calculations is the following, which incorporates four types of exits:



Here:

- "active" means that the person is still in employment and in the pension plan,
- "withdrawn" means that the person withdraws from the pension plan,
- "disability retirement" means that the person retires due to disability/illness,
- "age retirement" means that the person retires due to ageing, and
- "died in service" means that the person dies during work (service).

The probabilistic information about this multiple decrement model is usually summarized in tabular form at integer ages (multiple decrement table from STAT3909). Such a table is called the **service table**, which starts at a certain age x_0 (radix), and looks like the following (an excerpt of the Standard Service Table from Dickson et al. (2019)):

x	ℓ_x	w_x	i_x	r_x	d_x
20	1 000 000	95 104	951	0	237
21	903 707	85 946	859	0	218
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
59	95 351	1884	94	0	288
60^-	93 085	0	0	27 926	0
60^+	65 160	0	62	6188	210
61	58 700	0	56	5573	212
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
64	42 805	0	41	4061	215
65^-	38 488	0	0	38 488	0

where:

- the radix x_0 is 20,
- ℓ_x is the expected number of active people at age x (STAT3909 notation: ℓ_x^{00}),
- w_x is the expected number of people that withdraw from the pension plan between ages x and $x+1$ (STAT3909 notation: d_x^{01}),
- i_x is the expected number of people that retire due to disability/illness between ages x and $x+1$ (STAT3909 notation: d_x^{02}), and
- d_x is the expected number of people that die during work between ages x and $x+1$ (STAT3909 notation: d_x^{04}).

Special attention should be given to r_x. In general, it is the expected number of people that retire due to ageing between ages x^+ (an age that is "just after" age x) and $x+1^-$ (an age that is "just before" age $x+1$). For every age x where age retirement *exactly at age x or $x+1$* is impossible, this is the same as the expected number of people that retire due to aging between ages x and $x+1$ (STAT3909 notation: d_x^{03}).

Indeed, here we assume that age retirement can occur exactly at age 60 or age 65 only, so we only need to pay special attention when $x = 59, 60, 64, \text{or } 65$. When $x = 60$ or 65, the value r_x does not consider age retirement occurring *exactly* at age x (60 or 65); when $x = 59$ or $x = 64$, the value r_x does not consider age retirement occurring *exactly* at age $x+1$ (60 or 65). The exact-age retirements are instead considered in r_{60^-} and r_{65^-}: The value r_{60^-} (r_{65^-}) refers to the expected number of people that retire due to aging *exactly* at age 60 (65), or "between ages 60^- and 60 (65^- and 65)".

Due to the presence of age retirement exactly at age 60 or 65, there are also special notations for "ℓ". The value ℓ_{60^-} (ℓ_{65^-}) is the expected number of active people at age 60^- (65^-), i.e., "just before" age 60 (65). The value ℓ_{60^+} is the expected number of active people at age 60^+, i.e., "just after" age 60; this refers to the expected number of active people *immediately after* the exact-age retirement at age 60 occurs. For notational convenience, we often just write ℓ_{60} in place of ℓ_{60^+}. For $i_{60^+}, w_{60^+}, r_{60^+}$, they can just be understood as i_{60}, w_{60}, r_{60} respectively, and we usually use the latter set of notations.

Based on our STAT3909 knowledge, we have the following formulas: For all $k = 0, 1, \ldots,$

- $w_{x_0+k} = \ell_{x_0} \times {}_kp_{x_0}^{00} \times p_{x_0+k}^{01}$.
- $i_{x_0+k} = \ell_{x_0} \times {}_kp_{x_0}^{00} \times p_{x_0+k}^{02}$.
- $r_{x_0+k} = \ell_{x_0} \times {}_kp_{x_0}^{00} \times p_{x_0+k}^{03}$.
- $d_{x_0+k} = \ell_{x_0} \times {}_kp_{x_0}^{00} \times p_{x_0+k}^{04}$.

For all integer ages x except $x = 59, 60, 64, 65$, we have $\ell_{x+1} = \ell_x - w_x - i_x - r_x - d_x$. For $x = 59$ and $x = 64$, we have $\ell_{60^-} = \ell_{59} - w_{59} - i_{59} - r_{59} - d_{59}$ and $\ell_{65^-} = \ell_{64} - w_{64} - i_{64} - r_{64} - d_{64}$. For $x = 60$ and $x = 65$, we have $\ell_{60} = \ell_{60^-} - r_{60^-}$ and $\ell_{65} = \ell_{65^-} - r_{65^-}$.

1.2.3 **Final salary DB pension.** While a DB pension plan can get quite mathematically involved, let us start with studying a relatively simple kind of DB pension plan, whose (annual) benefit amount is a function of the *final salary*, i.e., the salary received in the *last* working year.

Assuming that the individual starts working at age x_e and retires at age y, the annual benefit amount of such a pension plan is given by

$$B = \alpha \times S_y \times (y - x_e)$$

where α is called the **accrual percentage**, which can be interpreted as the <u>percentage</u> of pensionable salary <u>accrued</u> to the pension fund per year of employment.

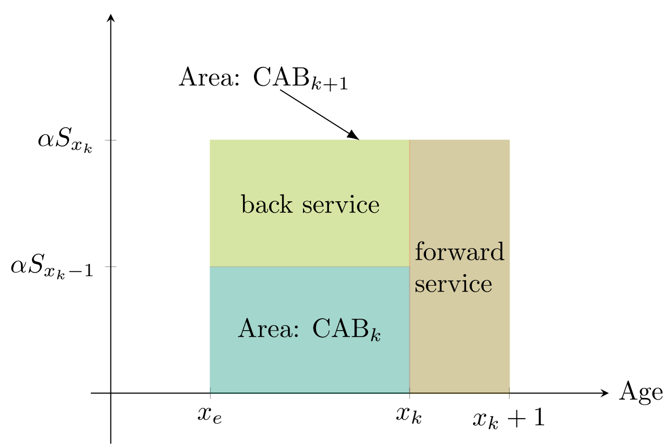
1.2.4 **Current accrued benefit for final salary DB pension.** To examine the current "worth" of such a pension plan, it is helpful to compute the *current accrued benefit*. Letting $x_k \le y$ denote the age of the individual at time k, the CAB at time k is

$$\boxed{\mathrm{CAB}_k = \alpha \times S_{x_k - 1} \times (x_k - x_e)},$$

because *if* the individual stopped working at time k, then the salary received in their last working year (age $x_k - 1$ to x_k) would be S_{x_k-1} and the number of years of employment would be $x_k - x_e$. Hence, the increase in CAB from time k to $k+1$ is

$$\begin{aligned}\mathrm{CAB}_{k+1} - \mathrm{CAB}_k &= \alpha \times S_{x_k} \times (x_k + 1 - x_e) - \alpha \times S_{x_k-1} \times (x_k - x_e)\\ &= \underbrace{\alpha(S_{x_k} - S_{x_k-1})(x_k - x_e)}_{\text{(back service)}} + \underbrace{\alpha S_{x_k}}_{\text{(forward service)}}.\end{aligned}$$

This increase is contributed by two factors: (i) **back service** $\alpha(S_{x_k} - S_{x_k-1})(x_k - x_e)$ which reflects the growth in CAB due to the past work of $x_k - x_e$ years (looking <u>back</u>) and (ii) **forward service** αS_{x_k} which reflects the growth in CAB due to the work in the year "<u>forward</u>": time k to $k+1$. The decomposition of the increase into back and forward services can also be graphically understood below:



1.2.5 **Projected accrued benefit for final salary DB pension.** While current accrued benefit seems to be a neat way to capture the current "worth" of the pension plan, the usage of *current* salary in its calculation may receive some criticisms. Specifically, the pension benefit is supposed to be paid out after the person's retirement, with the benefit amount based on the *final* salary rather than the *current* one. Especially when the current time is still far from the time of the person's retirement, the difference between these two salaries can be huge and materially influence the value calculated.

In view this, there is an alternative way to examine the current "worth" of the pension plan, which uses the *projected* salary rather than the *current* salary in the calculation. The resulting value is called **projected accrued benefit**, which is obtained based on the projected final salary and (still) the length of past service. Assuming the individual starts working at age x_e and retires at age y, and letting $\widehat{S}_{y-1}$ denote the projected value of S_{y-1} (final salary), the PAB at time k is

$$\boxed{\mathrm{PAB}_k = \alpha \times \widehat{S}_{y-1} \times (x_k - x_e)}.$$

where x_k is the age of the individual at time k. This reduces to CAB_k if the projected value $\widehat{S}_{y-1}$ is simply taken as the current salary S_{x_k-1}.

[Note: Sometimes, the length of past service "$x_k - x_e$" is replaced by the length of both past and future services "$y - x_e$" in the definition of PAB (we use asterisked notation to denote such a PAB for clarity; see [1.3.7]). We should be careful about how PAB is defined in different contexts.]

1.2.6 **Final average salary DB pension.** While final salary pension is easy to explain and understand, one of its common criticisms is that the pension benefit amount can change abruptly simply due to a sudden change in the salary received in the last working year, e.g., if the final salary received is somehow much lower than before, then the benefit amount can drop sharply. To reduce the sensitivity of the benefit on the very last salary received, an alternative design of DB pension plan is to specify the benefit amount as a function of the *average* of *last few* salaries received (e.g., for three to five years).

For an individual that starts working at age x_e and retires at age y, the benefit amount of such a pension plan is given by

$$B = \alpha \times S_y^F \times (y - x_e)$$

where S_y^F is the **final average salary** evaluated at age y, which is the average of a prespecified number of salaries received in the past few (say m) years ("final"): $S_y^F = (S_{y-1} + S_{y-2} + \cdots + S_{y-m})/m$. [⚠ Warning: Not $S_y^F = (S_y + S_{y-1} + \cdots + S_{y-m+1})/m$!]

Letting x be the current age of the individual, the final average salary S_y^F can be expressed in terms of the most recent salary received S_{x-1} and salary scales:

$$S_y^F = \frac{S_{y-1} + S_{y-2} + \cdots + S_{y-m}}{m} = S_{x-1} \times \frac{\overbrace{(s_{y-1} + s_{y-2} + \cdots + s_{y-m})/m}^{=:z_y}}{s_{x-1}}.$$

From this, we see that S_y^F can be expressed as $S_{x-1} \cdot z_y / s_{x-1}$ where z_y is the corresponding **salary averaging function** for the *salary scale*. Note that this only gives the formula for z_y for *integer* ages y. In case the subscript of z is fractional (e.g., $z_{y+1/2}$), the value can be determined by applying *linear interpolation* between the two nearest integers, e.g.,

$$\begin{aligned} z_{y+1/2} = \frac{1}{2} z_y + \frac{1}{2} z_{y+1} &= \frac{1}{2} \frac{s_{y-1} + s_{y-2} + \cdots + s_{y-m}}{m} + \frac{1}{2} \frac{s_y + s_{y-1} + \cdots + s_{y-m+1}}{m} \\ &= \frac{s_{y-1} + s_{y-2} + \cdots + s_{y-m+1}}{m} + \frac{1}{2} \frac{s_y + s_{y-m}}{m}. \end{aligned}$$

So, correspondingly, the final average salary at fractional age is obtained by applying linear interpolation between final average salaries at the two nearest integer ages, e.g., $S_{y+1/2}^F = (1/2) S_y^F + (1/2) S_{y+1}^F$.

1.2.7 **CAB and PAB for final average DB pension.** Similar to the developments in [1.2.4] and [1.2.5], the CAB and PAB at time k for this kind of pension are respectively given by

$$\boxed{\text{CAB}_k = \alpha \times S_{x_k}^F \times (x_k - x_e)} \quad \text{and} \quad \boxed{\text{PAB}_k = \alpha \times \widehat{S}_y^F \times (x_k - x_e)}$$

where the individual is aged $x_k \le y$ at time k, starts working at age x_e, and retires at age y; and $\widehat{S}_y^F$ denotes the projected value of S_y^F, obtained based on the projected z_y, or in other words, the average of projected salary scales. [Note: For final average DB pension, we usually focus on PAB rather than CAB. Also, for notational convenience, we do not add a "$\widehat{\ }$" on z_y and the associated salary scales even if they are projected values; this should be clear from context.]

1.2.8 **Career average earnings DB pension.** The third type of DB pension to be discussed here, known as career average earnings (CAE) DB pension, takes the averaging of salaries to the extreme, and uses the average of the salaries received *throughout the whole career* as the basis for pension benefit calculations.

For an individual that starts working at age x_e and retires at age y, the pension benefit amount is given by

$$B = \alpha \times \left(\frac{1}{y - x_e} \sum_{j=x_e}^{y-1} S_j \right) \times (y - x_e) = \sum_{j=x_e}^{y-1} \alpha S_j = \alpha \mathrm{TPE}_y$$

where TPE_y denotes the *total pensionable earnings* up to age y (throughout the entire career), which is $\sum_{j=x_e}^{y-1} S_j$ here. Notice that for a CAE DB pension plan, a constant proportion α of the salary received accrues to the pension benefit amount for each year.

1.2.9 **CAB for career average earnings DB pension.** For the CAE DB pension here, we focus on the CAB calculation only. Projection of salary can be handled in the *career average revalued earnings* pension plan ([1.2.10]), which is a popular variant of this pension plan that takes inflation into account.

The CAB at time k for this kind of pension is given by

$$\boxed{\mathrm{CAB}_k = \sum_{j=x_e}^{x_k-1} \alpha S_j = \alpha \mathrm{TPE}_{x_k}}$$

where the individual is aged $x_k \leq y$ at time k, starts working at age x_e, and retires at age y. Note that the increase in CAB from time k to $k+1$ is

$$\mathrm{CAB}_{k+1} - \mathrm{CAB}_k = \sum_{j=x_e}^{x_k} \alpha S_j - \sum_{j=x_e}^{x_k-1} \alpha S_j = \alpha S_{x_k},$$

which reinforces the previously mentioned observation that a constant proportion α of the salary received accrues to the pension benefit amount for each year.

1.2.10 **Career average revalued earnings pension plan.** To reflect more accurately the *purchasing power* of the salaries received (rather than their nominal values) throughout the career, we may incorporate an *indexation* into our benefit calculations for the CAE plan, which adjusts the salaries based on inflation.

Let j be the constant annual inflation rate based on the index. For an individual that starts working at age x_e and retires at age y (with $y > x_e$ being integers), the pension benefit amount is given by

$$B = \alpha \times \left(\frac{1}{y - x_e} \sum_{\ell=x_e}^{y-1} S_\ell (1+j)^{y-(\ell+1)} \right) \times (y - x_e) = \alpha \mathrm{TPRE}_y$$

where TPRE_y denotes the *total pensionable revalued earnings* up to age y (throughout the entire career), which is $\sum_{\ell=x_e}^{y-1} S_\ell (1+j)^{y-(\ell+1)}$ here. Note that the starting time point for the inflation adjustment is at the *end* of the year corresponding to each salary S_ℓ received (age $\ell + 1$ here); we can interpret this as suggesting that each S_ℓ is contributed to the pension benefit amount at the end of the year.

[Note: For simplicity, we assume a constant inflation rate here; the idea here can be extended to the case with varying inflation rates without much difficulty.]

1.2.11 **CAB and PAB for career average revalued earnings DB pension.** Projection of salary for CARE DB pension works by adjusting the salaries received *further* according to the index. For this pension plan, the CAB and PAB at time k are respectively

$$\boxed{\mathrm{CAB}_k = \alpha \times \left(\frac{1}{x_k - x_e} \sum_{\ell=x_e}^{x_k-1} S_\ell (1+j)^{x-(\ell+1)} \right) \times (x_k - x_e) = \alpha \mathrm{TPRE}_{x_k}}$$

and

$$\boxed{\mathrm{PAB}_k = \alpha \times \left(\frac{1}{x_k - x_e} \sum_{\ell=x_e}^{x_k-1} S_\ell (1+j)^{y-(\ell+1)} \right) \times (x_k - x_e) = \alpha \mathrm{TPRE}_{x_k} (1+j)^{y-x_k}}$$

where the individual is aged x_k (integer) at time k, starts working at age x_e (integer), and retires at age y (integer), with $x_e < x_k \le y$. Expressing the CAB formula in recursive form, we have

$$\boxed{\text{CAB}_{k+1} = \text{CAB}_k(1+j) + \alpha S_{x_k}},$$

which clarifies how the inflation rate influences the increases in CAB.

1.2.12 **Valuation of pension benefit.** After discussing how we obtain CAB and PAB for various types of DB pension, which are about the *amount* of annual pension benefit, we now proceed to studying the EPV-based *valuation* of pension benefit with such an annual amount, which involves also (i) annuity factors (depending on the payment pattern for the pension benefit) and (ii) probabilities of various "exits" at different time points (computed based on the service table). Sometimes the resulting EPV is called **actuarial liability**. The general EPV formula learnt in STAT3901/STAT3909 is still of great use here:

$$\text{EPV} = \underbrace{\sum \text{ or } \int}_{\text{all possible payment times}} \text{benefit amount} \times \text{discount factor} \times \text{prob. of triggering payment}.$$

[Note: Here the "$\sum$" version is more useful.]

1.2.13 **General EPV formulas for accrued age retirement benefit.** Let a_y^r denote the EPV of a life annuity with annual benefit 1 for a pension member that retires (enters the state "age retirement") at age y. Let x_r denote the latest possible age of retirement (often 65). Suppose that the individual is now active and aged x, who is denoted by "(x)".

For simplicity, we shall assume that all exits (entering the state "age retirement" here) can only possibly happen at the mid-point of each year, except for the ages having a superscript "$-$" in the service table (see [1.2.2]), where exits can happen *exactly* at those ages. [Note: Exact-age exits are considered for age retirement since there is a period in which retirement is allowed, and quite a lot of people are going to retire exactly at the *start* of that period (often age 60), and also many people will be *required* to retire exactly at the *end* of that period (often age 65).]

Let t be the current time, at which the individual is aged x. Then:

- The (time-t) EPV of (x)'s *projected* accrued *age retirement* benefit is

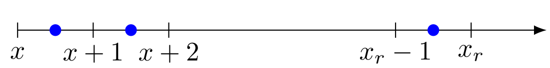
$$\boxed{\underbrace{\sum_{y=x}^{x_r-1}}_{\text{(mid-year exits)}} \text{PAB}_{t,y+1/2} \times a_{y+1/2}^r \times v^{y+1/2-x} \times \frac{r_y}{\ell_x} + \underbrace{\sum_{y=x+1}^{x_r}}_{\text{(exact-age exits)}} \text{PAB}_{t,y} \times a_y^r \times v^{y-x} \times \frac{r_{y^-}}{\ell_x}.}$$

  [Note: We add "$, y$" to the subscript of the notation PAB_k to emphasize the dependence of projected accrued benefit on the retirement age y, via the salary projection.]

- The (time-t) EPV of (x)'s *current* accrued *age retirement* benefit is

$$\boxed{\underbrace{\sum_{y=x}^{x_r-1}}_{\text{(mid-year exits)}} \text{CAB}_t \times a_{y+1/2}^r \times v^{y+1/2-x} \times \frac{r_y}{\ell_x} + \underbrace{\sum_{y=x+1}^{x_r}}_{\text{(exact-age exits)}} \text{CAB}_t \times a_y^r \times v^{y-x} \times \frac{r_{y^-}}{\ell_x}}.$$

In both formulas above, we handle the mid-year exits and exact-age exits separately through two sums. Now, let us focus on elaborating more about the formula for the *projected* case (the formula for the *current* case can be understood in a similar way). For the first sum about mid-year exits, it is summing from $y = x$ to $y = x_r - 1$, covering all the possible mid-year exit ages $x + 1/2, x + 3/2, \ldots, x_r - 1/2$ (blue dots below):



At each mid-year exit age $y+1/2$, the EPV of the benefit at that time is $\mathrm{PAB}_{k,y+1/2}\times a^r_{y+1/2}$, and the discounted factor (back to age x) is $v^{y+1/2-x}$. The probability of triggering this benefit at the mid-year exit age $y+1/2$ is

$$\begin{aligned} r_y/\ell_x &= \text{probability for } (x) \text{ to retire between ages } y \text{ and } y+1^- \\ &\overset{\text{(only exit mid-year)}}{=} \text{probability for } (x) \text{ to retire at age } y+1/2. \end{aligned}$$

Next, consider the second sum about exact-age exits. It is summing from $y=x+1$ to $y=x_r$, covering all the possible *exact-age* exit ages $x+1, x+2, \ldots, x_r$ (*not* including age x since the individual is active at age x). While quite a number of exact-age exit ages are considered here, there are often at most two ages where exact-age exits are possible here, namely 60 and 65 (shown in the service table in [1.2.2]). At each exact-age exit age y, the EPV of the benefit at that time is $\mathrm{PAB}_{t,y}\times a^r_y$, and the discounted factor (back to age x) is v^{y-x}. The probability of triggering this benefit at the exact-age exit age y is

$$\begin{aligned} r_{y^-}/\ell_x &= \text{probability for } (x) \text{ to retire ``between ages } y^- \text{ and } y\text{''} \\ &= \text{probability for } (x) \text{ to retire exactly at age } y; \end{aligned}$$

here "age y^-" may be intuitively treated as an age that is "just before" age y.

1.2.14 **EPV formula for accrued age retirement benefit: Examples.** Now, let us illustrate the general EPV formulas in [1.2.13] more concretely by applying them to some pension plans we have studied here.

(a) *Final average salary DB pension:* Let z_y be the salary averaging function used for projection, and x_e be the age at which (x) starts working. Then, the EPV of (x)'s *projected* accrued age retirement benefit is

$$\alpha\times S_{x-1}\times(x-x_e)\left(\sum_{\substack{y=x\\ \text{(mid-year exits)}}}^{x_r-1}\frac{z_{y+1/2}}{s_{x-1}}\times a^r_{y+1/2}\times v^{y+1/2-x}\times\frac{r_y}{\ell_x}+\sum_{\substack{y=x+1\\ \text{(exact-age exits)}}}^{x_r}\frac{z_y}{s_{x-1}}\times a^r_y\times v^{y-x}\times\frac{r_{y^-}}{\ell_x}\right),$$

and the EPV of (x)'s *current* accrued age retirement benefit is

$$\alpha\times S^F_x\times(x-x_e)\left(\sum_{\substack{y=x\\ \text{(mid-year exits)}}}^{x_r-1}a^r_{y+1/2}\times v^{y+1/2-x}\times\frac{r_y}{\ell_x}+\sum_{\substack{y=x+1\\ \text{(exact-age exits)}}}^{x_r}a^r_y\times v^{y-x}\times\frac{r_{y^-}}{\ell_x}\right).$$

(b) *Career average earnings DB pension:* The EPV of (x)'s current accrued age retirement benefit is

$$\alpha\times\mathrm{TPE}_x\left(\sum_{\substack{y=x\\ \text{(mid-year exits)}}}^{x_r-1}a^r_{y+1/2}\times v^{y+1/2-x}\times\frac{r_y}{\ell_x}+\sum_{\substack{y=x+1\\ \text{(exact-age exits)}}}^{x_r}a^r_y\times v^{y-x}\times\frac{r_{y^-}}{\ell_x}\right).$$

(c) *Career average revalued earnings DB pension:* The EPV of (x)'s projected accrued age retirement benefit is

$$\alpha\times\mathrm{TPRE}_x\left(\sum_{\substack{y=x\\ \text{(mid-year exits)}}}^{x_r-1}(1+j)^{y+1/2-x}a^r_{y+1/2}\times v^{y+1/2-x}\times\frac{r_y}{\ell_x}+\sum_{\substack{y=x+1\\ \text{(exact-age exits)}}}^{x_r}(1+j)^{y-x}a^r_y\times v^{y-x}\times\frac{r_{y^-}}{\ell_x}\right).$$

1.2.15 **General EPV formulas for accrued withdrawal benefit.** Similar to [1.2.13], we can develop general EPV formulas for accrued *withdrawal* benefit. Let a^w_y denote the EPV of a (deferred) life annuity with annual benefit 1 for a pension member that withdraws from the pension plan at age y.

Let x_r denote the latest possible age of retirement (often 65). Suppose that the individual is now active and aged x, who is denoted by "(x)".

For simplicity, we shall assume that all exits (entering the state "withdrawn" here) can only possibly happen at the mid-point of each year; note that exact-age exits do not apply for withdrawal here.

Let t be the current time, at which the individual is aged x. Then:

- The (time-t) EPV of (x)'s *projected* accrued *withdrawal* benefit is

$$\boxed{\sum_{y=x}^{x_r-1} \text{PAB}_{t,y+1/2} \times a^w_{y+1/2} \times v^{y+1/2-x} \times \frac{w_y}{\ell_x}}.$$

- The (time-t) EPV of (x)'s *current* accrued *withdrawal* benefit is

$$\boxed{\sum_{y=x}^{x_r-1} \text{CAB}_t \times a^w_{y+1/2} \times v^{y+1/2-x} \times \frac{w_y}{\ell_x}}.$$

The only differences between the formulas here and the respective formulas in [1.2.13] are that every "r" is changed to "w", and the sums for exact-age exits are dropped.

1.3 Funding DB Pension Benefits

1.3.1 Following the valuation of DB pension benefits as studied in Section 1.2, the next step is to determine appropriate amounts of contributions for *funding* those pension benefits, which are computed based on how the EPVs of pension benefits vary over time. Here, we shall retain the assumption from Section 1.2 that only mid-year or exact-age exits are possible, with the latter applicable only for age retirement. Also, we assume that contributions are made at the *start* of each year.

Before going through the formulas of the contribution amounts, let us introduce some related terms and notations in [1.3.2] to [1.3.4], for facilitating our later discussion.

1.3.2 **Actuarial liability/reserve.** Throughout our discussion in Section 1.3, we assume that the pension member is aged x and active at an integer time t, which is the start of year $t+1$; our analysis here will primarily focus on the year $t+1$, i.e., the time period from age x to $x+1$.

The notation ${}_kV^{(0j)}$ denotes the time-k actuarial liability (or **reserve** or EPV) of the member's accrued benefits for decrement j, if the member is active at time k. If the member enters the decrement j at or before time k, then this actuarial liability/EPV is denoted by ${}_kV^{(j)}$ instead. Particularly, we write $j=r$ if decrement j is "age retirement", and $j=w$ is decrement j is "withdrawn".

[Note: The notion of actuarial liability/reserve is indeed closely related to the idea of state-dependent policy value learnt in STAT3909 (we just do not have "premiums" here), explaining why we also use "${}_tV$" in the notation here. As we shall see, the recursive formulas for the actuarial liabilities here have highly similar structure as those for state-dependent policy values.]

1.3.3 **Normal contribution.** The **normal contribution**/**normal cost** at time t is the amount required to be paid at time t such that, together with the time-t reserve, they exactly match with the EPV of benefits due mid-year, together with the EPV of new actuarial liability at the year end (if exact-age exit is not possible at that time), or just before the year end (if exact-age exit is possible at that time). Briefly speaking, the purpose of normal contribution is to pay for *additional* benefits accrued from the service in the upcoming year, or in the upcoming half-year for exits occurring at the upcoming mid-year; this idea will be again illustrated in [1.3.5].

Often, the time-t normal contribution is expressed as a percentage of the salary in the upcoming year (year $t+1$), and that percentage is called the **normal contribution rate**. If the benefits in consideration are for decrement j, then that normal contribution is denoted by C_t^j. The definition of

C_t^j can also be understood via the **funding equation**, which equates "what we have" and the EPV of "what we need" (which closely resembles the idea for developing policy value recursions!).

The funding equations for age retirement and withdrawal benefits are respectively given by:

- *(Age retirement benefits)*

$$\boxed{\substack{\text{(what we have)}\\ {}_tV^{(0r)}+C_t^r}} = \overset{\text{(what we need on average)}}{\underbrace{v^{1/2}\times {}_{1^-}p_x^{0r}\times {}_{t+1/2}V^{(r)}}_{\text{time-}t\text{ EPV of benefits for mid-year exits}} + \underbrace{v\times {}_{1^-}p_x^{00}\times {}_{t+1^-}V^{(0r)}}_{\text{time-}t\text{ EPV of new actuarial liability}}}$$

$$\overset{\text{(service table notations)}}{=} \boxed{v^{1/2}\times\frac{r_x}{\ell_x}\times {}_{t+1/2}V^{(r)}+v\times\frac{\ell_{x+1^-}}{\ell_x}\times {}_{t+1^-}V^{(0r)}}$$

  where:

  - ${}_{1^-}p_x^{0r}$ is the probability for (x) to be in the state "age retirement" after "1^-" years, i.e., have age retirement strictly within one year. Equivalently, this is the probability for (x) to have age retirement exactly at the upcoming mid-year due to our assumption.
  - ${}_{1^-}p_x^{00}$ is the probability for (x) to be in the state "active" after "1^-" years, i.e., be still active just before the year end.

- *(Withdrawal benefits)*

$$\boxed{\substack{\text{(what we have)}\\ {}_tV^{(0w)}+C_t^w}} = \overset{\text{(what we need on average)}}{\underbrace{v^{1/2}\times {}_{1}p_x^{0w}\times {}_{t+1/2}V^{(w)}}_{\text{time-}t\text{ EPV of benefits for mid-year exits}} + \underbrace{v\times {}_{1}p_x^{00}\times {}_{t+1}V^{(0w)}}_{\text{time-}t\text{ EPV of new actuarial liability}}}$$

$$\overset{\text{(service table notations)}}{=} \boxed{v^{1/2}\times\frac{w_x}{\ell_x}\times {}_{t+1/2}V^{(w)}+v\times\frac{\ell_{x+1}}{\ell_x}\times {}_{t+1}V^{(0w)}}$$

  where ${}_1p_x^{0w}$ is the probability for (x) to withdraw from the pension plan within one year, and ${}_1p_x^{00}$ is the probability for (x) to be still active at the year end. Here, we use "1" rather than "1^-" because exact-age exits are not applicable for withdrawal.

[Note: When working with normal contribution, we often consider the case $t = 0$.]

1.3.4 **Traditional/Projected Unit Credit funding method.** As seen in [1.3.3], the funding process depends heavily on the values of actuarial liabilities (those "V"s). There are two funding methods, depending on whether the *accrued benefits* underlying the calculations of actuarial liabilities are *projected* accrued benefits (PAB) or *current* accrued benefits (CAB). When PAB is used, the funding method is called **Projected Unit Credit** (PUC); when CAB is used, the funding method is called **Traditional Unit Credit** (TUC), or **Current Unit Credit**.

In general, the *funding equations* introduced in [1.3.3] are very useful for finding normal contributions under PUC and TUC funding, with those "V"s being computed as actuarial liabilities based on suitable CAB/PAB. Nonetheless, there are indeed fairly simple shortcut formulas available under the special case with no *mid-year exits*, which will be discussed in [1.3.5] and [1.3.6]; we will focus only on the formulas for age retirement benefits determined based on final average salary (final average salary DB pension).

1.3.5 **Normal contribution shortcut formula under PUC funding.** Rearranging the funding equation for normal contribution in [1.3.3] gives

$$C_t^r = v^{1/2}\times\frac{r_x}{\ell_x}\times {}_{t+1/2}V^{(r)}+v\times\frac{\ell_{x+1^-}}{\ell_x}\times {}_{t+1^-}V^{(0r)}-{}_tV^{(0r)}. \tag{2}$$

Under PUC funding, applying the formula for *projected* accrued age retirement benefit in [1.2.14]a gives

$$
\begin{aligned}
{}_{t}V^{(0r)} &= \alpha \times S_x \times (x - x_e)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} \frac{z_{y+1/2}}{s_x} \times a^r_{y+1/2} \times v^{y+1/2-(x)} \times \frac{r_y}{\ell_x} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \frac{z_y}{s_x} \times a^r_y \times v^{y-(x)} \times \frac{r_{y^-}}{\ell_x} \Bigg), \\
{}_{t+1^-}V^{(0r)} &= \alpha \times S_x \times (x + 1 - x_e)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} \frac{z_{y+1/2}}{s_x} \times a^r_{y+1/2} \times v^{y+1/2-(x+1)} \times \frac{r_y}{\ell_{x+1^-}} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \frac{z_y}{s_x} \times a^r_y \times v^{y-(x+1)} \times \frac{r_{y^-}}{\ell_{x+1^-}} \Bigg).
\end{aligned}
$$

Also, we have

$$
{}_{t+1/2}V^{(r)} = \alpha \times S_x \times (x + 1/2 - x_e) \times \frac{z_{x+1/2}}{s_x} \times a^r_{x+1/2}
$$

[Note: As S_y/s_y is constant for all $y \geq x_e$, we can indeed modify the subscript x of the pair "S_x" and "s_x" without affecting the result, as long as the corresponding values are available (here we implicitly assume that S_x and s_x are given).]

Putting these expressions into Equation (2) yields

$$
\begin{aligned}
C^r_t &= \alpha \times S_x \Bigg(\frac{1}{2}\frac{r_x}{\ell_x} \times \frac{z_{x+1/2}}{s_x} \times a^r_{x+1/2} + \underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} \frac{z_{y+1/2}}{s_x} \times a^r_{y+1/2} \times v^{y+1/2-(x)} \times \frac{r_y}{\ell_x} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \frac{z_y}{s_x} \times a^r_y \times v^{y-(x)} \times \frac{r_{y^-}}{\ell_x} \Bigg) \quad \text{(verify this!)}.
\end{aligned}
$$

The expression here illustrates that the normal contribution is to pay for *additional* benefits accrued from the service in the upcoming year (so there is no "$\times n$" here), or in the upcoming half-year for exits occurring at the upcoming mid-year (so there is a "1/2" for the term corresponding to this exit).

It can be simplified to

$$
C^r_t = \frac{{}_{t}V^{(0r)}}{n} - \alpha \times S_x \times \frac{1}{2}\frac{r_x}{\ell_x} \times \frac{z_{x+1/2}}{s_x} \times a^r_{x+1/2}
$$

where $n = x - x_e$ is the number of years of past service for the member at time t (assumed to be positive).

If exits are not allowed for the upcoming mid-year, then it can be further simplified to just

$$
\boxed{C^r_t = \frac{{}_{t}V^{(0r)}}{n}},
$$

the time-t actuarial liability averaged over all years of past service.

1.3.6 **Normal contribution shortcut formula under TUC funding.** Under TUC funding, applying the formula for *current* accrued age retirement benefit in [1.2.14]a gives

$$
\begin{aligned}
{}_{t}V^{(0r)} &= \alpha \times S^F_x \times (x - x_e)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} a^r_{y+1/2} \times v^{y+1/2-(x)} \times \frac{r_y}{\ell_x} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} a^r_y \times v^{y-(x)} \times \frac{r_{y^-}}{\ell_x} \Bigg), \\
{}_{t+1^-}V^{(0r)} &= \alpha \times S^F_{x+1} \times (x + 1 - x_e)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} a^r_{y+1/2} \times v^{y+1/2-(x+1)} \times \frac{r_y}{\ell_{x+1^-}} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} a^r_y \times v^{y-(x+1)} \times \frac{r_{y^-}}{\ell_{x+1^-}} \Bigg).
\end{aligned}
$$

Also, we have

$$
{}_{t+1/2}V^{(r)} = \alpha \times S^F_{x+1/2} \times (x + 1/2 - x_e) \times a^r_{x+1/2}
$$

Putting these expressions into Equation (2) yields

$$
\begin{aligned}
C^r_t &= v^{1/2} \times \frac{r_x}{\ell_x} \times \alpha \times S^F_{x+1/2} \times (x + 1/2 - x_e) \times a^r_{x+1/2} \\
&\quad + \alpha \times S^F_{x+1} \times (x + 1 - x_e)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} a^r_{y+1/2} \times v^{y+1/2-(x)} \times \frac{r_y}{\ell_x} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} a^r_y \times v^{y-(x)} \times \frac{r_{y^-}}{\ell_x} \Bigg) \\
&\quad - \alpha \times S^F_x \times (x - x_e)\Bigg(a^r_{x+1/2} \times v^{1/2} \times \frac{r_x}{\ell_x} + \underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} a^r_{y+1/2} \times v^{y+1/2-(x)} \times \frac{r_y}{\ell_x} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} a^r_y \times v^{y-(x)} \times \frac{r_{y^-}}{\ell_x} \Bigg). \\
&= \underbrace{\frac{1}{2} \times \alpha \times S^F_{x+1/2} \times v^{1/2} \times \frac{r_x}{\ell_x} \times a^r_{x+1/2}}_{\text{additional half-year accrual for age } x+1/2 \text{ retirement}} + \underbrace{\alpha \times (S^F_{x+1/2} - S^F_x) \times n \times \frac{r_x}{\ell_x} \times v^{1/2} \times a^r_{x+1/2}}_{\text{back service for half-year service for age } x+1/2 \text{ retirement}} \\
&\quad \underbrace{+ \alpha \times S^F_{x+1}\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} a^r_{y+1/2} \times v^{y+1/2-(x)} \times \frac{r_y}{\ell_x} + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} a^r_y \times v^{y-(x)} \times \frac{r_{y^-}}{\ell_x} \Bigg)}_{\text{additional full-year accrual for retirements at or after age } x+1} \\
&\quad + \underbrace{\left(\frac{S^F_{x+1}}{S^F_x} - 1\right) \times \left({}_{t}V^{(0r)} - \alpha \times S^F_x \times n \times a^r_{x+1/2} \times v^{1/2} \times \frac{r_x}{\ell_x}\right)}_{\text{back service for full-year service for retirements at or after age } x+1}
\end{aligned}
$$

where $n = x - x_e$ is the number of years of past service for the member at time t. While the resulting expression offers insights on different components that influence the normal contribution C^r_t, it is rather lengthy. To make it more compact, we can instead write:

$$C_t^r = \alpha \times v^{1/2} \times \frac{r_x}{\ell_x} \times a_{x+1/2}^r \left(\frac{1}{2} \times S_{x+1/2}^F \times (n+1/2) - S_x^F \times n \right) + \left(\frac{S_{x+1}^F}{S_x^F} \times \frac{n+1}{n} - 1 \right) {}_tV^{(0r)}$$

$$\overset{\text{(no age } x+1/2 \text{ retirement)}}{=} \boxed{\left(\frac{S_{x+1}^F}{S_x^F} \times \frac{n+1}{n} - 1 \right) {}_tV^{(0r)}}.$$

1.3.7 **Projected benefit cost method.** For TUC and PUC funding methods, each contribution is determined based on the additional benefit accrual within the upcoming *one*-year period (hence "unit credit"). In the following, we will study an alternative and more "prospective" funding method, known as *projected benefit cost* method, which determines the appropriate amount of *level* contribution at the current time point based on future benefit accrual in its entirety, not only for one-year period. Here we will only discuss this method briefly, since this funding method is indeed rather uncommon and often not allowed in practice[2]. The primary purpose of including it here is to demonstrate that unit credit method is not the only possible way for funding DB pension benefits.

The **projected benefit cost** (PBC) method works conceptually as follows. Let C_t^{PBC} be the *level* annual contribution determined by the PBC method at time t. Suppose that the only possible and mandatory retirement age is x_r and the member can only exit the active state by either age retirement (at age x_r) or death. Let t_r denote the time at which the member is aged x_r.

Let PAB_t^* denote the time-t PAB but based on the length of both past and future service (assuming the member works from starting age x_e to the retirement age x_r), i.e., "$x - x_e$" (length of past service only) is replaced by "$x_r - x_e$" (length of both past and future services) in the expression.

- The EPV of future projected pension benefits at time t is $\mathrm{PVB}_t = \mathrm{PAB}_t^* \times {}_{x_r-x|}\ddot{a}_x$.
- The EPV of future level contributions at time t is $\mathrm{PVC}_t = C_t^{\mathrm{PBC}} \times \ddot{a}_{x:\overline{x_r-x}|}$.
  [Note: The contributions, made before retirement, are payable only when the member is active (equivalent to "alive" here).]

The value of C_t^{PBC} is the one that satisfies the following budgeting equation, which equates the amount we have and the amount we need (both on average):

$$\underset{\text{(amount we have)}}{\mathrm{Reserve}_t + \mathrm{PVC}_t} = \underset{\text{(amount we need)}}{\mathrm{PVB}_t}.$$

Here, $\mathrm{Reserve}_t$ is the reserve at time t sourced from past contributions (details omitted).

1.3.8 **Formulas for contributions by the PBC method.** To compute contributions determined by the PBC method, the following two formulas are helpful. We assume that the member is aged x_e at time 0.

(a) *(Time-0 contribution)*

$$\boxed{C_0^{\mathrm{PBC}} = \mathrm{PAB}_0^* \times \frac{{}_{x_r-x_e|}\ddot{a}_{x_e}}{\ddot{a}_{x_e:\overline{x_r-x_e}|}}}$$

which can be derived based on the fact that $\mathrm{Reserve}_0 = 0$ (intuitively, there is no reserve at time 0 since there are no "past contributions" at that time).

(b) *(Recursive formula)* For every time $k = 1, \ldots, t_r - 1$, we have

$$\boxed{C_k^{\mathrm{PBC}} = C_{k-1}^{\mathrm{PBC}} + \underbrace{\overbrace{(\mathrm{PAB}_k^* - \mathrm{PAB}_{k-1}^*)}^{\text{increase in projected benefit}} \times \frac{{}_{x_r-x_e-k|}\ddot{a}_{x_e+k}}{\ddot{a}_{x_e+k:\overline{x_r-x_e-k}|}}}_{\text{(adjustment due to change in projection)}}}.$$

[2] The reason is that the contributions by the PBC method are determined based on the assumptions they are *level*, which can lead to substantial "overpayment", especially during the early time period where the benefit accrual is of limited size.

1.4 Retiree Health Benefits

1.4.1 Previously, we have focused on analyzing pension plan that provides regular income to the individual after retirement. Conceptually, it works by accumulating the contributions in a pension fund, which will be utilized to purchase *annuity* product upon retirement "on your behalf". In Section 1.4, we will study another type of pension, which will purchase *health insurance* product upon retirement "on your behalf", thereby providing health benefits to the individual after retirement.

1.4.2 **EPV of health insurance benefit.** Let $B(x,s)$ denote the annual premium payable for health insurance under a pension plan (providing health insurance benefits after retirement), for a life aged x at time s. Suppose that a life retires at age y at time t. To fund the health insurance coverage for the life after retirement until their death, the annual premiums are payable at the start of each year whenever the life is alive, beginning at the retirement. Hence, the time-t EPV of the health insurance benefits, which equals the time-t EPV of the annual health insurance premiums (by assumption: equivalence principle), is

$$\begin{aligned}
&B(y,t) + vp_y B(y+1,t+1) + v^2{}_2p_y B(y+2,t+2) + \cdots \\
&= B(y,t)\left(1 + vp_y\frac{B(y+1,t+1)}{B(y,t)} + v^2{}_2p_y\frac{B(y+2,t+2)}{B(y,t)} + \cdots\right) \\
&= \boxed{B(y,t)a^B_{y,t}}
\end{aligned}$$

where

$$a^B_{y,t} := 1 + vp_y\frac{B(y+1,t+1)}{B(y,t)} + v^2{}_2p_y\frac{B(y+2,t+2)}{B(y,t)} + \cdots$$

is the time-t annuity factor for the health insurance premiums for a life retiring at age y at time t, which are scaled such that the first premium is 1.

1.4.3 **EPV of health insurance benefit: Geometrically increasing premiums.** While the EPV formula in [1.4.2] is quite general, it is often hard to employ in practice since the value of $B(x,s)$ can vary rather irregularly as x and s change. To make the EPV formula "nicer" and easier to be used in practice, we usually make the following assumptions about how $B(x,s)$ varies. Suppose that (i) the annual inflation rate (growth rate with respect to *time*) for the health insurance premiums is j, and that (ii) the premiums grow exponentially with *age* by a constant factor c. Specifically, for all ages x, time s, and $k \geq 0$:

- (i) means that $B(x,s+k) = B(x,s)(1+j)^k$.
- (ii) means that $B(x+k,s) = B(x,s)c^k$.

This implies that $B(x+k_1,s+k_2) = B(x,s)c^{k_1}(1+j)^{k_2}$ for all $k_1,k_2 \geq 0$.

Therefore, under this assumption we can write, for all ages y and time t,

$$\begin{aligned}
a^B_{y,t} &= 1 + vp_y c(1+j) + v_2 p_y c^2(1+j)^2 + \cdots \\
&= 1 + (1+i^*)^{-1}p_y + (1+i^*)^{-2}{}_2p_y + \cdots \\
&= \ddot{a}_{y|i^*}
\end{aligned}$$

where i^* is the modified interest rate satisfying that

$$(1+i^*)^{-1} = (1+i)^{-1}c(1+j) \implies i^* = \frac{1}{(1+i)^{-1}c(1+j)} - 1 = \boxed{\frac{1+i}{c(1+j)} - 1}.$$

Since the resulting expression does not depend on t, we usually just write $a^B_y = \ddot{a}_{y|i^*}$ (dropping the ", t" in the notation $a^B_{y,t}$). This allows us to simplify the EPV formula in [1.4.2] as:

$$\boxed{B(y,t)\ddot{a}_{y|i^*}}.$$

[Note: It is possible (indeed quite common) that $i^* < 0$.]

In the following discussion, we will always impose such assumptions so that we have this simplified EPV formula.

If we additionally assume that $q_x = q$ for all ages $x = y, y+1, \ldots$ *(constant mortality rate after retirement)*, then we have

$$\ddot{a}_{x|i^*} = 1 + \frac{1-q}{1+i^*} + \frac{(1-q)^2}{(1+i^*)^2} + \cdots = \frac{1}{1-(1-q)/(1+i^*)} = \boxed{\frac{1+i^*}{q+i^*}},$$

and hence the EPV formula can be written as

$$\boxed{B(y,t)\frac{1+i^*}{q+i^*}}.$$

1.4.4 **Actuarial value of total health benefits.** The EPV formula in [1.4.2] is applicable at the time t of retirement of the individual (and indeed also works at later time: $t+1, t+2, \ldots$, as long as the life is still alive at that point), but not at the earlier time points, where the individual is still active. To calculate EPVs at those points, we can use a similar approach as [1.2.13].

For an active life currently aged x at time s, the **actuarial value of total health benefits**, or AVTHB in short, is given by

$$\begin{aligned}
&\underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} \overbrace{B(y+1/2, s+y+1/2-x)}^{B(x,s)(c(1+j))^{y+1/2-x}} a^B_{y+1/2} \times v^{y+1/2-x} \times \frac{r_y}{\ell_x} \\
&+ \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \underbrace{B(y, s+y-x)}_{B(x,s)(c(1+j))^{y-x}} a^B_y \times v^{y-x} \times \frac{r_{y-}}{\ell_x} \\
&= B(x,s)\left(\underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} a^B_{y+1/2} \times v^{y+1/2-x}(c(1+j))^{y+1/2-x} \times \frac{r_y}{\ell_x} + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} a^B_y \times v^{y-x}(c(1+j))^{y-x} \times \frac{r_{y-}}{\ell_x}\right) \\
&= \boxed{B(x,s)\left(\underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} \ddot{a}_{y+1/2|i^*} \times (1+i^*)^{-(y+1/2-x)} \times \frac{r_y}{\ell_x} + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \ddot{a}_{y|i^*} \times (1+i^*)^{-(y-x)} \times \frac{r_{y-}}{\ell_x}\right)}
\end{aligned}$$

where some notations from [1.4.3] and [1.2.13] are used here. [Note: We often take $s = 0$.]

It turns out that the expression inside the parenthesis has a highly similar structure as the previous EPV formulas for accrued age retirement benefit in [1.2.13], just with the interest rate i replaced by i^*!

1.4.5 **Funding of health insurance benefits.** After discussing the valuation of health insurance benefits (finding their EPVs), we proceed to study the *funding* of those benefits: determining appropriate amounts of contributions to fund the benefits. Although we have discussed funding in Section 1.3 also, the approach used here is quite different.

The funding methods in Section 1.3 are *accruals-based methods*, which are based on actuarial liabilities, obtained by considering *accruals* from past service. However, for retiree health benefit, there is no accrual based on past service, since the health benefit amount does not depend on the length of service.

[Note: Typically, the health insurance benefit is only eligible for a retiree with sufficiently long service. Here, we shall always assume that the length of service is long enough for the retiree to qualify for the health insurance benefit; otherwise, we do not need to consider its funding!]

Instead of considering accrual based on past service, our approach here is to assume that the health benefit accrues *linearly* over the retiree's employment period (service), such that the accrued benefit at the retirement time matches with the health insurance benefit needed.

1.4.6 **Actuarial liability under linear accrual.** Throughout our discussion of funding of health insurance benefit, we assume that the employee is aged x and active at an integer time s, which is the start of year $s+1$; our analysis here will primarily focus on the year $s+1$. Let x_e denote the age at which the employee starts working, and x_r denote the latest possible age of retirement.

To better understand the linear accrual, consider the following. Suppose that the employee retires at age y. Then the EPV of the health insurance benefit at retirement is $B(y, s+y-x)a_y^B$. Accruing linearly to this amount from age x_e to age y, the time-s accrued benefit for the employee aged x is then

$$B(y, s+y-x)a_y^B \times \frac{\overbrace{x-x_e}^{\text{length of past service}}}{\underbrace{y-x_e}_{\text{length of employment period}}}.$$

In other words, an amount of $B(y, s+y-x)a_y^B/(y-x_e)$ is accrued in each year of the employment period.

Based on the linear accrual, the time-s **actuarial liability** (or reserve or EPV) of the retiree health benefit is

$$\begin{aligned} {}_sV^{(0B)} = B(x,s)\Bigg(&\underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} \ddot{a}_{y+1/2|i^*} \times \frac{x-x_e}{y+1/2-x_e} \times (1+i^*)^{-(y+1/2-x)} \times \frac{r_y}{\ell_x} \\ &+ \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \ddot{a}_{y|i^*} \times \frac{x-x_e}{y-x_e} \times (1+i^*)^{-(y-x)} \times \frac{r_{y^-}}{\ell_x}\Bigg). \end{aligned}$$

[Note: We often take $s=0$.]

We use the notation ${}_kV^{(0B)}$ to denote the time-k actuarial liability/EPV of the retiree health benefit if the individual is active at time k, and the notation ${}_kV^B$ is used if the individual is retired at time k; if the individual is aged x_k at time k, then we have ${}_kV^B = B(x_k,k)a_{x_k}^B$.

1.4.7 **Normal contribution for health insurance benefit.** The calculation of normal contribution in this context follows the same principle suggested in [1.3.3]. For retiree health benefit, the funding equation is given by

$$\begin{aligned} \overset{\text{(what we have)}}{{}_sV^{(0B)} + C_s^B} &= \overset{\text{(what we need on average)}}{\underbrace{v^{1/2} \times {}_{1^-}p_x^{0r} \times {}_{s+1/2}V^{(B)}}_{\text{time-}t\text{ EPV of benefits for mid-year exits}} + \underbrace{v \times {}_{1^-}p_x^{00} \times {}_{s+1^-}V^{(0B)}}_{\text{time-}s\text{ EPV of new actuarial liability}}} \\ &\overset{\text{(service table notations)}}{=} v^{1/2} \times \frac{r_x}{\ell_x} \times {}_{s+1/2}V^{(B)} + v \times \frac{\ell_{x+1^-}}{\ell_x} \times {}_{s+1^-}V^{(0B)} \end{aligned}$$

where C_s^B denotes the normal contribution for the health insurance benefit.

From this funding equation, we can obtain the following formula of the normal contribution C_s^B:

$$\begin{aligned} C_s^B = B(x,s)\Bigg(&\ddot{a}_{x+1/2|i^*} \times \frac{1/2}{x+1/2-x_e} \times (1+i^*)^{-1/2} \times \frac{r_x}{\ell_x} \\ &+ \underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} \ddot{a}_{y+1/2|i^*} \times \frac{1}{y+1/2-x_e} \times (1+i^*)^{-(y+1/2-x)} \times \frac{r_y}{\ell_x} \\ &+ \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \ddot{a}_{y|i^*} \times \frac{1}{y-x_e} \times (1+i^*)^{-(y-x)} \times \frac{r_{y^-}}{\ell_x}\Bigg), \end{aligned}$$

which simplifies to

$$\boxed{C_s^B = \frac{{}_sV^{(0B)}}{n}}$$

with $n = x - x_e > 0$ being the number of years of past service for the member at time s (similar to [1.3.5]), if exits are not allowed for the upcoming mid-year.

Proof. We have:

$$\begin{aligned}
{}_sV^{(0B)} &= B(x,s)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x}^{x_r-1}} \ddot{a}_{y+1/2|i^*} \times \frac{x-x_e}{y+1/2-x_e} \times (1+i^*)^{-(y+1/2-x)} \times \frac{r_y}{\ell_x} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \ddot{a}_{y|i^*} \times \frac{x-x_e}{y-x_e} \times (1+i^*)^{-(y-x)} \times \frac{r_{y^-}}{\ell_x}\Bigg), \\
{}_{s+1/2}V^{(B)} &= B(x+1/2, s+1/2)a^B_{x+1/2} = B(x,s)(c(1+j))^{1/2}\ddot{a}_{x+1/2|i^*}, \\
{}_{s+1^-}V^{(0B)} &= B(x+1,s+1)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} \ddot{a}_{y+1/2|i^*} \times \frac{x+1-x_e}{y+1/2-x_e} \times (1+i^*)^{-(y+1/2-x-1)} \times \frac{r_y}{\ell_{x+1^-}} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \ddot{a}_{y|i^*} \times \frac{x+1-x_e}{y-x_e} \times (1+i^*)^{-(y-x-1)} \times \frac{r_{y^-}}{\ell_{x+1^-}}\Bigg) \\
&= B(x,s)c(1+j)\Bigg(\underset{\text{(mid-year exits)}}{\sum_{y=x+1}^{x_r-1}} \ddot{a}_{y+1/2|i^*} \times \frac{x+1-x_e}{y+1/2-x_e} \times (1+i^*)^{-(y+1/2-x-1)} \times \frac{r_y}{\ell_{x+1^-}} \\
&\quad + \underset{\text{(exact-age exits)}}{\sum_{y=x+1}^{x_r}} \ddot{a}_{y|i^*} \times \frac{x+1-x_e}{y-x_e} \times (1+i^*)^{-(y-x-1)} \times \frac{r_{y^-}}{\ell_{x+1^-}}\Bigg).
\end{aligned}$$

Then, the target expression of C_s^B can be directly derived from the formula

$$C_s^B = v^{1/2} \times \frac{r_x}{\ell_x} \times {}_{s+1/2}V^{(B)} + v \times \frac{\ell_{x+1^-}}{\ell_x} \times {}_{s+1^-}V^{(0B)} - {}_sV^{(0B)},$$

which is obtained by rearranging the funding equation. □

1.5 Spouse Pension

1.5.1 In [1.1.3], we have encountered a type of pension benefit known as *spouse pension*, which offers regular benefit payments to an individual's spouse in case the individual is dead while their spouse is alive. In Section 1.5, we will analyze this kind of benefit in more details and investigate the extra contributions needed to fund the addition of a spouse pension element on top of the existing pension benefit.

1.5.2 **Three ways for funding spouse pension.** Here, we will discuss three methods for funding spouse pension:

(1) *Accrual:* Similar to the *fully funded* basis, the amounts of contributions are determined such that they are enough for covering the accrued spouse pension benefit.

(2) *Risk coverage:* Similar to the *pay-as-you-go* basis, the amount of each contribution (made at the start of year) is determined such that it is enough for covering the spouse pension benefit for the *upcoming year*.

(3) *Hybrid:* A combination of *accrual* and *risk coverage*, e.g., using the *risk coverage* method for the part of spouse pension benefits paid before age 65, and the *accrual* method for the part of spouse pension benefits paid starting from age 65.

1.5.3 **Accrual method.** Here, we use a similar approach as the projected benefit cost method ([1.3.7]) for determining suitable contributions for funding the accrued spouse pension benefit, mainly for convenience. Particularly, we will just focus on the simple and special case $t = 0$, in which we can find out the amount of annual contribution C_0 by directly equating the EPV of future level contributions (made while the individual is active) and the EPV of future benefits, like the *equivalence principle* learnt in STAT3901.

Let PAB_t^* denote the time-t PAB but based on the length of both past and future service. Suppose that:

- The individual is aged x and active, and their spouse is aged y at time 0.
- The time-t accrued spouse pension benefit is given by $U_t^{\mathrm{SP}} = s_p \cdot \mathrm{PAB}_t^*$, where s_p is a fixed percentage which specifies the portion of the accrued (annual) pension benefit to be included in the accrued (annual) spouse pension benefit.
- The only possible and mandatory retirement age is x_r and the member can only exit the active state by either age retirement (at age x_r) or death.
- The spouse pension benefit is of the form of a reversionary annuity, whose benefit is payable at the start of each year, when the individual is dead while their spouse is alive. [Note: We can modify the reversionary annuity to another type, e.g., a monthly one, but then the annuity factor below needs to be adjusted accordingly.]

Then, the amount of annual contribution C_0 is determined by

$$\boxed{\underset{\text{(amount we have)}}{C_0\,\ddot{a}_{x:\overline{x_r-x}|}} = \underset{\text{(amount we need)}}{U_0^{\mathrm{SP}}\,\ddot{a}_{x|y}}}.$$

1.5.4 **Risk coverage method.** For the risk coverage method, it may be viewed as a "one-year" version of accrual method. Specifically, we find the amount of time-0 contribution C_0 by equating the EPV of the contribution within 1 year and the EPV of future benefits within 1 year. Here we impose an additional assumption that the lifetimes of the individual and their spouse are *independent*. Then, the amount of annual contribution C_0 is

$$\boxed{\underset{\text{(amount we have within 1 year)}}{C_0} = \underset{\text{(amount we need within 1 year)}}{vq_x p_y U_x^{\mathrm{SP}}\,\ddot{a}_{y+1}}}.$$

1.5.5 **Hybrid method.** Consider the example where we use the *risk coverage* method for the part of spouse pension benefits paid before age x_r, and the *accrual* method for the part of spouse pension benefits paid starting from age x_r. Then, by [1.5.3] and [1.5.4] (with the independent lifetimes assumption imposed still), the amounts of annual contribution based on the accrual and risk coverage methods are respectively given by

$$\begin{aligned} C_0^{\text{accrual}} &= \frac{U_0^{\mathrm{SP}}\,{}_{x_r-x|}\ddot{a}_{x|y}}{\ddot{a}_{x:\overline{x_r-x}|}}, \\ C_0^{\text{risk}} &= vq_x p_y U_0^{\mathrm{SP}}\,\ddot{a}_{y+1}, \end{aligned}$$

where ${}_{x_r-x|}\ddot{a}_{x|y}$ denotes the EPV of a $(x_r - x)$-year deferred reversionary annuity, whose benefit is payable at the start of each year when the individual is dead while their spouse is alive, *starting from the time at which the individual is aged x_r*; its EPV is given by ${}_{x_r-x|}\ddot{a}_{x|y} = {}_{x_r-x}E_x\,\ddot{a}_{x_r|y+(x_r-x)}$.

Then, the total amount of annual contribution is given by $C_0 = C_0^{\text{accrual}} + C_0^{\text{risk}}$.

2 Universal Life Insurance

2.0.1 In STAT3901 and STAT3909, we have primarily focused on the analysis of *traditional* life insurance, which purely offers insurance benefits. However, in modern ages, insurance companies offers not only these traditional policies, but also some more "modern" policies which has some additional features over the policies we have studied before. In Section 2, we will study a kind of "modern policies" common in the market, known as *universal life (UL) insurance*.

2.0.2 **Universal life insurance in a nutshell.** The key features of UL insurance are:

(a) It incorporates *profit sharing*. Apart from the insurance benefits, there are also benefits sourced from the profits of investments made by the insurer.

(b) It allows *more flexible* payment schedules. Instead of requiring the policyholder to pay a level and fixed premium regularly, UL insurance permits the policyholder to sometimes pay a reduced amount of premium or even skip a payment, without facing the risk of the policy being lapsed.

With these two "nice" features, it is not surprising that UL insurance is a rather popular product in the market. The core of UL insurance that supports these two features is the *notional account*, where inflows and outflows occur (notionally); our analysis of UL insurance in Section 2 also places a lot of emphasis on this notional account.

2.1 Structure of Universal Life Insurance

2.1.1 **Basic terms.** Let us start by going through various terms that describe a universal life insurance:

- The **account value** AV_t is the balance of funds in the notional account, at time t (integer).
- At the end of the year of policyholder's death, the total death benefit payable is the account value at the payment time, plus a nonnegative **additional death benefit** (ADB). The **corridor factor requirement** refers to the guaranteed minimum value for the ratio of the total death benefit (account value plus ADB) to the account value, which is at least 1.

  There are two common types of UL insurance, which offer different kinds of death benefits:

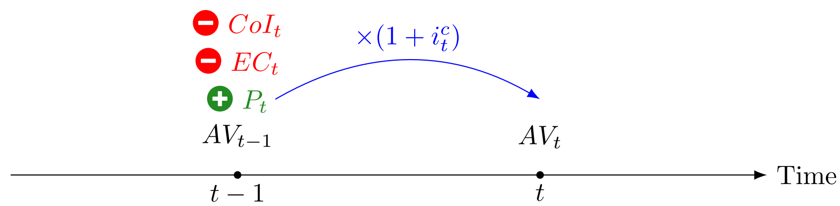
  - **Type A** specifies a level *total death benefit* ("on paper"), which is known as the **face amount**. So, the ADB drops as the account value rises. Nevertheless, with the corridor factor requirement, the ADB cannot become too small and it is possible that the *actual* total death benefit *exceeds the face amount* due to the requirement; it is important to consider the corridor factor requirement for this kind of UL insurance.
  - **Type B** specifies a level *ADB*. So, the total death benefit is the account value plus the level ADB specified; it is implicitly assumed that the level ADB specified is large enough, so that the corridor factor requirement is never violated. Hence, we can ignore the corridor factor requirement for this kind of UL insurance (this makes the analysis of type B UL simpler! 😜).
- A premium P_t is paid at the start of the tth year. It is sometimes subject to a minimum level and payment term, but is otherwise flexible.
- An **expense charge** EC_t is deducted from the account value at the start of the tth year. This is at the insurer's discretion for covering the expenses associated with the UL policy, typically with a maximum.
- A **credited interest rate** i_t^c is the annual effective interest rate applied for accumulating account value during the tth year. This is again at the insurer's discretion, typically with a minimum guarantee. In STAT3956, we often focus on the case where i_t^C is fixed always.
- A **Cost of Insurance** (CoI) CoI_t is deducted from the account value at the start of the tth year. This is for supporting the ADB, and is computed based on (i) the amount of ADB, (ii) a mortality rate (one-year death probability) known as the **CoI (mortality) rate**, and (iii) an interest rate known as the **CoI interest rate**. The approach for calculating the CoI is similar to the *risk coverage method* introduced in [1.5.4].

- A **surrender value/cash value** CV_t is paid on surrender at the end of the tth year (assuming surrender can only occur at the end of each year). It is given by the account value AV_t minus a **surrender charge**.

2.1.2 **Fundamental equation of UL insurance.** To incorporate elements mentioned in [2.1.1] in the computation/projection of account values at different time points, we always use the following recursive formula, which is fundamental for a UL insurance:

$$(AV_{t-1} + P_t - EC_t - CoI_t)(1 + i_t^c) = AV_t,$$

for all $t = 1, 2, \ldots$, with $AV_0 = 0$ (there is no balance in the notional account at the outset).



This formula is essentially just accounting all the inflows and outflows for the notional account in the tth year; there is not even any life-contingent element included directly!

[Note: The only purpose of EC_t, CoI_t, and i_t^c is to derive the account values; they do not correspond to "real" cash flows.]

2.1.3 **CoI calculations for type B UL.** To use the fundamental equation in [2.1.2], we need to compute the value of CoI_t. The calculations are quite straightforward for *type B* UL.

Let ADB be the level amount of ADB, q^*_{x+t-1} be the CoI mortality rate for the tth year, and i_q be the CoI interest rate. Then, for type B UL we have

$$\boxed{CoI_t = v_q q^*_{x+t-1} ADB}$$

where $v_q := (1 + i_q)^{-1}$.

2.1.4 **CoI calculations for type A UL.** The CoI calculations are a bit more complex for type A UL, since we need to consider the *corridor factor requirement*, which can potentially influence the amount of ADB.

Let FA be the level face amount specified. *If* there were no corridor factor requirement, then the total death benefit at time t would indeed be FA and thus the ADB at time t would simply be given by $FA - AV_t$. Things get more complicated with the presence of the corridor factor requirement. With a corridor factor γ_t at time t, the total death benefit at time t becomes $\max\{FA, \gamma_t AV_t\}$:

- If $FA < \gamma_t AV_t$, then simply having FA as the total death benefit violates the corridor factor requirement. Hence, the total death benefit is adjusted upwards to $\boxed{\gamma_t AV_t}$ to fulfill the corridor factor requirement.
- If $FA \geq \gamma_t AV_t$, having FA as the total death benefit fulfills the corridor factor requirement, so we can simply take $\boxed{FA}$ as the total death benefit.

Hence, the ADB at time t is

$$ADB_t = \underbrace{\max\{FA, \gamma_t AV_t\}}_{\text{total death benefit at time } t} - AV_t = \boxed{\max\{FA - AV_t, (\gamma_t - 1)AV_t\}}.$$

This expression suggests that ADB_t indeed depends on AV_t, but the fundamental equation in [2.1.2] indicates that AV_t in turn depends on ADB_t, via the CoI_t term. To handle this rather complex relationship, we perform a case-by-case analysis:

(1) *Case 1: The total death benefit is the face amount (the corridor factor is not binding).* Let ADB_t^f and AV_t^f respectively denote the time-t ADB and account value in this case. Then, we have $ADB_t^f = FA - AV_t^f$, and accordingly the CoI in this case is $CoI_t^f = v_q q^*_{x+t-1}(FA - AV_t^f)$. Hence, by the fundamental equation in [2.1.2], we have

$$\begin{aligned} AV_t^f &= \Big(AV_{t-1} + P_t - EC_t - v_q q^*_{x+t-1}\Big(FA - AV_t^f\Big)\Big)(1 + i_t^c) \\ \implies AV_t^f &= \frac{(AV_{t-1} + P_t - EC_t - v_q q^*_{x+t-1} FA)(1 + i_t^c)}{1 - v_q q^*_{x+t-1}(1 + i_t^c)}. \end{aligned}$$

(2) *Case 2: The total death benefit is $\gamma_t AV_t$ (the corridor factor is binding).* Let ADB_t^c and AV_t^c respectively denote the time-t ADB and account value in this case. Then, we have $ADB_t^c = \gamma_t AV_t^c - AV_t^c = (\gamma_t - 1)AV_t^c$, and accordingly the CoI in this case is $CoI_t^c = v_q q^*_{x+t-1}(\gamma_t - 1)AV_t^c$). Hence, by the fundamental equation in [2.1.2], we have

$$\begin{aligned} AV_t^c &= \big(AV_{t-1} + P_t - EC_t - v_q q^*_{x+t-1}(\gamma_t - 1)AV_t^c\big)(1 + i_t^c) \\ \implies AV_t^c &= \frac{(AV_{t-1} + P_t - EC_t)(1 + i_t^c)}{1 + v_q q^*_{x+t-1}(\gamma_t - 1)(1 + i_t^c)}. \end{aligned}$$

[Note: There is no need to memorize the final formulas above, we can just use the fundamental equation to solve for AV_t^f and AV_t^c.]

(3) *Choosing the right account value.* Since $ADB_t = \max\{ADB_t^f, ADB_t^c\}$ and AV_t is a strictly decreasing function of ADB_t[3], we have $AV_t = \min\{AV_t^f, AV_t^c\}$. It follows that the *smaller* one between AV_t^f and AV_t^c should be chosen to be the actual account value at time t (and the *larger* one between ADB_t^f and ADB_t^c serves as the actual ADB at time t). If AV_t^c is chosen, then the corridor factor is binding; otherwise, the corridor factor is not binding.

In short, the procedure above utilizes the fact that the actual situation falls in one of the two cases above (depending on whether the corridor factor is binding). By determining the account value in each of the cases, we deduce that the actual situation corresponds to the case that generates a *smaller* account value (or equivalently, a *larger* ADB), and hence the actual account value is the smaller one between the two candidates above.

2.1.5 **No lapse guarantee.** A UL insurance sometimes has a **no lapse guarantee**, which ensures that the total death benefit remains payable even if the account value is inadequate for providing the coverage, subject to some conditions, e.g., a minimum premium is paid at each premium date.

When discussing no lapse guarantee here, we will focus on type A UL without corridor factor requirement, so that the total death benefit is fixed. Furthermore, we will not consider no lapse guarantee when performing profit testing in Section 2.2.

Let FA be the face amount of a type A UL (without corridor factor requirement). At time t, the account value is AV_t. Assuming that no more premiums is payable starting from time t, the reserve (policy value) for death benefit coverage is given by the EPV of the total death benefit: $(FA)A_{x+t}$. [Note: If premiums remain payable, then we should include them in the reserve calculation here.]

If $(FA)A_{x+t} \le AV_t$, then the account value is adequate for providing the death benefit coverage and the no lapse guarantee is not triggered. Otherwise, the account value is inadequate and additional reserve is needed to fill up the shortfall, for providing the *no lapse guarantee*. In other words, the reserve for the no lapse guarantee at time t is:

$${}_tV^{\text{nlg}} = \max\{(FA)A_{x+t} - AV_t, 0\}.$$

[Note: Sometimes the death benefit coverage has a term, say lasting until time n. In that case, we replace A_{x+t} by $A^{\,1}_{x+t:\overline{n-t}|}$ (assuming that $t < n$). Similar adjustments can be made in response to other kinds of modifications.]

[3] From the fundamental equation in [2.1.2], we see that this is the case if the CoI mortality rate is positive, which is always the case here (it does not make much sense to have zero CoI mortality rate anyway).

2.2 Profit Testing of Type A and Type B Universal Life Insurance

2.2.1 After projecting account values based on the fundamental equation in [2.1.2] (more convenient in a spreadsheet environment), the next step in the analysis of UL insurance is to have *profit testing* on it. Before proceeding further, an important thing to keep in mind is that the expense charge EC_t, the Cost of Insurance CoI_t, and the credited interest rate i_t^c are used for projecting account values only, and are *irrelevant* for the profit testing procedure once the account values are determined. For the purpose of profit testing, another set of quantities is used for capturing the expense, mortality, and the interest earned.

2.2.2 **Relevant cash flows.** To perform profit testing, the first step is to correctly determine the relevant (expected) cash flows for the UL insurance. To facilitate our discussion, we list all the relevant cash flows for the profit test in the tth year below, where $t \in \{1, 2, \dots\}$:

- AV_{t-1}: the account value at the start of the tth year (time $t-1$).
- P_t: the premium payable at the start of the tth year.
- E_t: the expense incurred at the start of the tth year.

  [Note: We use the notation E_0 to denote the pre-contract expense (expense incurred at inception), which is handled *differently* from the expense E_1 incurred at the start of the first year in profit testing. Make sure to distinguish these two carefully !]
- I_t: the interest earned during the tth year on the invested fund associated with the UL insurance.
- EDB_t: the expected cost of death benefits paid at the end of the tth year, given that the policy is in force at the start of the tth year.

  Letting p_{x+t-1}^{0d} be the death probability in the tth year for an individual being active at the start of the tth year, the value of EDB_t is given by:

  $$\boxed{EDB_t = p_{x+t-1}^{0d}(\text{total death benefit at time } t + \text{associated settlement expense at time } t)};$$

  here settlement expense may present, depending on the question setting. Depending on whether the UL insurance is of type A or type B, we calculate the total death benefit at time t differently:

  - *Type B:* The total death benefit at time t is $AV_t + ADB$, where ADB is the level ADB specified.
  - *Type A:* Following the procedure mentioned in [2.1.4], we determine the correct AV_t and ADB_t by considering cases, and the total death benefit is their sum. [Warning: The sum is not necessarily the face amount when the corridor factor presents!]
- ESB_t: the expected cost of surrender benefits at the end of the tth year, given that the policy is in force at the start of the tth year.

  Letting p_{x+t-1}^{0w} be the withdrawal probability in the tth year for an individual being active at the start of the tth year, the value of ESB_t is given by:

  $$\boxed{ESB_t = p_{x+t-1}^{0w}(CV_t + \text{associated settlement expense at time } t)}.$$
- EAV_t: the expected cost of setting up the account value at the end of the tth year (only needed if the policy remains in force at that time), given that the policy is in force at the start of the tth year.

  Letting p_{x+t-1}^{00} be the probability that an individual being active at the start of the tth year remains active at the end of the tth year, the value of EAV_t is given by:

  $$\boxed{EAV_t = p_{x+t-1}^{00} AV_t}.$$

2.2.3 **Profit vectors.** After determining the relevant cash flows, the next step is to convert them to *in-force expected profits*, which form a profit vector. [Note: This concept is covered in STAT3909.]

For all $t = 1, 2, , \ldots,$ let Pr_t denote the expected profit emerging at time t (the end of the tth year), given that the policy is in-force at time $t-1$ (the start of the tth year). Furthermore, Pr_0 involves only the *pre-contract expense* E_0, which is incurred "slightly before" the start of the policy (or "the end of 0th year"): $\mathrm{Pr}_0 = -E_0$.

Then, for all $t = 1, 2, \ldots,,$ we have

$$\boxed{\mathrm{Pr}_t = AV_{t-1} + P_t - E_t + I_t - EDB_t - ESB_t - EAV_t},$$

or alternatively, letting $i_t := I_t/(AV_{t-1} + P_t - E_t)$ be the interest rate earned during the tth year, we can obtain the following formula which may look more familiar:

$$\boxed{\mathrm{Pr}_t = (AV_{t-1} + P_t - E_t)(1 + i_t) - EDB_t - ESB_t - EAV_t}.$$

2.2.4 **Profit signatures.** After obtaining the in-force expected profits, we then convert them to *expected profits per policy issued*, which form a profit signature. [Note: This concept is again covered in STAT3909.]

- For all $t = 1, 2, , \ldots,$ the expected profit emerging at time t per policy issued (or: given that the policy is in-force at time 0) is $\Pi_t = {}_{t-1}p_x^{00} \times \mathrm{Pr}_t$.
- For $t = 0$, we have $\Pi_0 = \mathrm{Pr}_0 = -E_0$.

2.2.5 **Profit measures.** After getting the profit signature, we can then compute various profit measures to assess the profitability of the UL insurance, e.g., net present value (NPV), internal rate of return (IRR), discounted payback period (DPP), and profit margin. These are all covered in STAT3909, but for completeness, let us review their formulas below (we assume that the UL policy in consideration has an n-year term, i.e., it remain in force for a maximum of n years and the policyholder must surrender at time n if not dead at that time):

(a) *NPV:* At a risk discount rate r, the **net present value** of the UL policy is

$$\mathrm{NPV} = \sum_{t=0}^{n} \Pi_t (1+r)^{-t}.$$

The *partial* NPV at time $k \le n$ (integer) is

$$\mathrm{NPV}(k) = \sum_{t=0}^{k} \Pi_t (1+r)^{-t}.$$

(b) *IRR:* **Internal rate of return** is an interest rate j such that the NPV at that rate is zero, i.e.,

$$\sum_{t=0}^{n} \Pi_t (1+j)^{-t} = 0.$$

(c) *DPP:* **Discounted payback period** is the earliest time at which the partial NPV becomes nonnegative: $\min\{k \ge 0 : \mathrm{NPV}(k) \ge 0\}$.

(d) *Profit margin:* At a risk discount rate r, the **profit margin** is the NPV expressed as a proportion of the EPV of the premiums:

$$\frac{\sum_{t=0}^{n} \Pi_t (1+r)^{-t}}{\sum_{t=0}^{n-1} P_t \times {}_t p_x^{00} (1+r)^{-t}}.$$

3 Embedded Options

3.0.1 In Section 2, we have discussed *universal life insurance*, which is a form of insurance incorporating some profit sharing in its design. Some investments are made on the insurer's side, and part of the profits from them are shared to the policyholder's notional account via the credited interest rate. The policyholder can then benefit from the upside potential of investments via the growth in their account value due to the credited interest rate.

3.0.2 **Equity-linked contract.** While the policyholder of a universal life insurance can enjoy investment returns, the benefit comes in a rather indirect manner. To make the product more directly linked to (equity) investment returns, *equity-linked contracts* are developed, which are quite similar to mutual funds. In our context here, an **equity-linked contract** (or **variable annuity**) specifies an equity fund to which the initial single premium (paid by the policyholder) is invested, with a fixed term to maturity. On survival to the maturity or death during the policy term, whichever is earlier, a benefit is paid, whose amount is linked to the fund value at the payment time.

[Note: Sometimes there are multiple premium payments and hence multiple investments. But for simplicity, here we focus on the case with just a single premium and a single investment.]

Here, we will discuss equity-linked contracts with some forms of *guarantees* (insurance elements), which offer protections against adverse investment outcomes. They can often be viewed as extra options embedded to the contracts, and thus can be analyzed using option pricing theory learnt in STAT3905/STAT3910. We will consider two types of guarantees in Section 3: (i) **guaranteed minimum maturity benefit** (GMMB) and (ii) **guaranteed minimum death benefit** (GMDB). As suggested by their names, the former provides a guaranteed minimum benefit amount for the case where the benefit is paid on survival to the maturity, and the latter provides a guaranteed minimum benefit amount for the case where the benefit is paid on death during the policy term. Either or both types of guarantees can present for an equity-linked contract.

3.1 Guaranteed Minimum Maturity Benefit

3.1.1 **Notations.** Let us first introduce the following notations, which are used throughout Section 3.

- F_t: time-t value of the policyholder's fund.
- P: the single premium.
- n: the policy term.
- $h(n)$: the GMMB option payoff at time n (the maturity).
- x: the age of the policyholder at the purchasing time (time 0).
- r: the continuously compounded annual risk-free rate.
- Φ: the standard normal cumulative distribution function (CDF).

3.1.2 **GMMB that guarantees the single premium.** A common type of GMMB for an equity-linked contract guarantees that the benefit amount is at least the single premium P, if the benefit is paid on survival to the maturity. Then, *on survival to the maturity*, the GMMB payoff at the maturity is

$$h(n) = \begin{cases} P - F_n & \text{if } F_n < P, \\ 0 & \text{otherwise} \end{cases} = (P - F_n)_+,$$

since an extra benefit of $P - F_n$ is needed in case the fund value F_n at the maturity is less than the single premium P (to honour the guarantee), and *no extra benefit is needed otherwise.* Here we can observe that the expression for the GMMB payoff resembles the payoff of an *European put option* on the fund, with strike price P (recall this from STAT3905/STAT3910 😉?). Consequently, we can handle the GMMB in the same way we handle European put options.

3.1.3 **The general case of GMMB pricing.** Recall from STAT3905/STAT3910 that a convenient method for pricing options (or derivatives in general) is *risk-neutral pricing*. For a payoff $h(n)$ at time n, its time-t price is

$$e^{-r(n-t)}\mathbb{E}_t^{\mathbb{Q}}[h(n)]$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ denotes the expectation taken under the risk-neutral measure $\mathbb{Q}$ and based on the information available at time t (in particular, the time-t fund value F_t), with $0 \leq t \leq n$.

For the case $t = 0$, this suggests that the current (time-0) price of a payoff $h(n)$ at the maturity is $e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[h(n)]$. However, this quantity is not directly taken as the current price of the GMMB, because the GMMB is only triggered when the policyholder *survives to the maturity*. If the policyholder dies during the policy term, no guarantee is offered by the GMMB, resulting a payoff of 0 for GMMB, and hence the corresponding price is 0.

In view of this, the actual time-0 price/value of the GMMB is a *weighted average* of the prices $e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[h(n)]$ and 0, where the weights source from the *survival probability* (having a life-contingent element here!). That is, the time-0 price/value of the GMMB is

$$\boxed{\pi(0) = {}_np_x \times e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[h(n)] + {}_nq_x \times 0 = {}_np_x \times e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[h(n)]}.$$

Using a similar argument, the time-t price/value of the GMMB is

$$\boxed{\pi(t) = {}_{n-t}p_{x+t} \times e^{-r(n-t)}\mathbb{E}_t^{\mathbb{Q}}[h(n)]},$$

provided that the policyholder is still alive at time t, with $0 \leq t \leq n$.

The time-t value can be interpreted as the insurer's *reserve* for the GMMB at that time. More precisely, the time-t value can be viewed as the (approximated) cost of setting up replicating portfolio for the guarantee at time t, averaged over a large number of policies in-force at that time. With a large number of policies, the proportion of policies with survival to the maturity (and hence requiring a replicating portfolio that is worth $e^{-r(n-t)}\mathbb{E}_t^{\mathbb{Q}}[h(n)]$ at time t) approaches ${}_{n-t}p_{x+t}$, assuming the applicability of law of large numbers.

In short, while the time-t value is not necessarily the cost of setting up a portfolio that perfectly replicates the payoff of the GMMB for an *individual* policy, it can be understood as the *average* cost of setting up a portfolio that replicates the payoff of the GMMB, for many policies.

[Note: Here we assume that no surrenders/lapses are possible for convenience, so that we can simply use "${}_{n-t}p_{x+t}$" here. In case surrenders/lapses are possible, "being alive" should be changed to "being active" and the probability "${}_{n-t}p_{x+t}$" should be changed to "${}_{n-t}p_{x+t}^{00}$" (for being active at the maturity).]

3.1.4 **Further assumptions on the fund.** While the risk-neutral pricing formulas for GMMB in [3.1.3] are general, it is difficult to utilize them to actually compute the prices without further assumptions. Here we will impose some assumptions about the fund, so that the situation is "close" to the Black-Scholes model, while realistic elements (expenses specifically) are incorporated:

- *Initial expense:* An initial expense is deducted from the fund, as a proportion e of the single premium P. That is, only $(1-e)P$ is effectively invested into the fund initially.
- *Management charges:* At the start of each year except the first, a management charge is deducted from the fund, as a proportion m of the fund value at that time.

  [Note: The time points at which management charges are applicable can differ (e.g., at the start of each year including the first, etc.). Particularly, the first-year management charge is excluded here mainly due to the presence of the initial expense. If there is no initial expense ($e = 0$), then we often still assume that management charge is applicable for the first year.]
- *Dynamics of the fund price process:* Before considering any expenses, the dynamics of the fund price process is described by a non-dividend-paying index (stock) price process $\{S_t\}_{t\geq 0}$, with $S_0 = 1$, where $\{S_t\}_{t\geq 0}$ follows a geometric Brownian motion with volatility parameter σ. That is,

with an initial investment of P into the fund, the time-t fund value would be $F_t = P(S_t/S_0) = PS_t$, ignoring expenses. Furthermore, we are assumed to be in a perfect market (no transaction cost, possible to freely buy/sell any assets, etc.).

With the presence of these expenses, the fund values at integer time points are given by:

$$
\begin{aligned}
F_0 &= (1-e)P, \\
F_1 &= F_0 \frac{S_1}{S_0} = P(1-e)S_1, \\
F_2 &= F_1 \underbrace{(1-m)}_{\text{deduct at the start of year 2}} \frac{S_2}{S_1} = P(1-e)(1-m)S_1 \frac{S_2}{S_1} = P(1-e)(1-m)S_2, \\
&\vdots \\
F_n &= F_{n-1} \underbrace{(1-m)}_{\text{deduct at the start of year } n} \frac{S_n}{S_{n-1}} = P(1-e)(1-m)^{n-1}S_n.
\end{aligned}
$$

[Warning: Note that the fund value F_k at time k does not consider expense deduction at the start of year $k+1$ (if any); we may view that deduction as occurring "slightly after" time k. On the other hand, F_k considers the expense deduction at the end of year k (if any); we may view it as occurring "slightly before" time k.]

3.1.5 **Time-0 value of the GMMB under further assumptions.** Under the assumptions in [3.1.4], the time-0 value of the GMMB guaranteeing a benefit amount of at least the premium P can be expressed as

$$
\boxed{\pi(0) = P\,{}_np_x \xi e^{-rn}\mathbb{E}_0^{\mathbb{Q}}\big[(\xi^{-1} - S_n)_+\big] = P\,{}_np_x\big(e^{-rn}\Phi(-d_2) - \xi\Phi(-d_1)\big)}
$$

where $\xi := (1-e)(1-m)^{n-1} > 0$ is the *expense factor*,

$$
d_1 = \frac{\ln \xi + (r + \sigma^2/2)n}{\sigma\sqrt{n}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{n}.
$$

Proof. With the presence of the expenses specified in [3.1.4], we have $F_n = P(1-e)(1-m)^{n-1}S_n = P\xi S_n$. Hence, the time-0 value of the GMMB can be expressed as

$$
\begin{aligned}
\pi(0) &= {}_np_x e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[h(n)] = {}_np_x e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[(P - F_n)_+] = {}_np_x e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[(P - P\xi S_n)_+] \\
&= P\,{}_np_x e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[(1 - \xi S_n)_+] \overset{((ax)_+ = a(x)_+ \text{ if } a>0)}{=} P\,{}_np_x \xi e^{-rn}\mathbb{E}_0^{\mathbb{Q}}\big[(\xi^{-1} - S_n)_+\big].
\end{aligned}
$$

To obtain the latter expression, we apply the *Black-Scholes formula* on $e^{-rn}\mathbb{E}_0^{\mathbb{Q}}\big[(\xi^{-1} - S_n)_+\big]$, which is the time-0 price of an n-year European put on a unit of the non-dividend-paying underlying index (stock) with strike price ξ^{-1}:

$$
e^{-rn}\mathbb{E}_0^{\mathbb{Q}}\big[(\xi^{-1} - S_n)_+\big] = \xi^{-1}e^{-rn}\Phi(-d_2) - S_0\Phi(-d_1) \overset{(S_0=1)}{=} \xi^{-1}e^{-rn}\Phi(-d_2) - \Phi(-d_1)
$$

where

$$
d_1 = \frac{\ln(S_0/\xi^{-1}) + (r + \sigma^2/2)n}{\sigma\sqrt{n}} = \frac{\ln \xi + (r + \sigma^2/2)n}{\sigma\sqrt{n}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{n}.
$$

Plugging this back to the expression above, we have

$$
\pi(0) = P\,{}_np_x\xi\big(\xi^{-1}e^{-rn}\Phi(-d_2) - \Phi(-d_1)\big) = P\,{}_np_x\big(e^{-rn}\Phi(-d_2) - \xi\Phi(-d_1)\big).
$$

□

[Warning: It is more important to understand the *derivation process* of this formula rather than just memorizing this formula, because questions can be about GMMBs with (slightly) different kinds of guarantee (not exactly the ones discussed here). The key is to rewrite the expression so that an European put payoff appears, and then the Black-Scholes formula can be applied.]

3.1.6 **Time-t value of the GMMB under further assumptions.** In a similar way, we can show that under the assumptions in [3.1.4], the time-t value of the GMMB guaranteeing a benefit amount of at least the premium P can be expressed as

$$\boxed{\pi(t) = P_{n-t}p_{x+t}\xi e^{-r(n-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(\xi^{-1} - S_n)_+\right] = P_{n-t}p_{x+t}\left(e^{-r(n-t)}\Phi(-d_2(t)) - \xi S_t\Phi(-d_1(t))\right)}$$

where $\xi := (1-e)(1-m)^{n-1} > 0$ is the expense factor,

$$d_1(t) = \frac{\ln(\xi S_t) + (r + \sigma^2/2)(n-t)}{\sigma\sqrt{n-t}}, \quad \text{and} \quad d_2(t) = d_1(t) - \sigma\sqrt{n-t},$$

provided that the policyholder is still alive at time t.

Proof. Exercise. (Make sure you know how to do this ⚠.) □

3.1.7 **Funding the guarantee.** So far we have discussed how to compute the time-0 value of the GMMB, and that can be viewed as the cost of setting up the guarantee by the insurer. However, where is this cost sourced from and how should we fund the guarantee? There are two common methods:

(a) *Front-end-loading:* We can have a relatively large expense deduction initial (in the form of the initial expense), part of which is to be used to cover the cost of the guarantee. This is known as a **front-end-load**, which is common for UK policies but less common in North America.

(b) *Risk premiums:* Instead of sourcing the cost from initial expense at once, an alternative way is to source the cost of from regularly incurred expenses (management charges). For simplicity, here we assume that there is no initial expense, and accordingly, a management charge is deducted from the fund at the start of each year (including the first).

At the start of each year, a *risk premium* is a proportion $c < m$ of the fund value at that time (before the deduction of management charge, if any) used for funding the guarantee (provided that the policyholder is still alive at that time); the risk premiums are considered as sourced from the management charges and are *not deducted* from the fund. This approach is more common in North America and is our main focus here.

[Note: If we had $c > m$, then the management charge deducted from the fund would be insufficient for covering the risk premium, which is used for funding the guarantee. An implication is that the insurer would need to "take some money out of its pocket" to sufficiently fund the guarantee, which is undesirable 👎. Here we also do not allow the case $c = m$, since a portion of the management charge should be reserved for other purposes (e.g., profits and other administrative expenses). In either case, the insurer should consider raising the management charge m, so that $c < m$ can be achieved.]

The amount of **risk premium** c is determined such that the time-0 value of the risk premiums equals the cost of the guarantee (similar to the *equivalence principle* from STAT3901).

[Note: Similar to the management charges, the time points at which the risk premiums are applicable can differ, but we will still compute the risk premium c in the way mentioned above.]

3.1.8 **Risk premiums.** Let us illustrate how to compute the risk premiums below. Assume that a risk premium c is charged at the start of each year, a management charge m is applicable at the start of each year (including the first), and there is no initial expense. Then, the time-0 value of the risk premiums is the "expected present value":

$$\begin{aligned}\sum_{k=0}^{n-1} e^{-rk}\mathbb{E}_0^{\mathbb{Q}}[cF_k] \times {}_kp_x &= \sum_{k=0}^{n-1} cP(1-m)^k \underbrace{e^{-rk}\mathbb{E}_0^{\mathbb{Q}}[S_k]}_{S_0=1} \times {}_kp_x = \sum_{k=0}^{n-1} cP(1-m)^k \times {}_kp_x \\ &= \sum_{k=0}^{n-1} cP(1+i^*)^{-k} \times {}_kp_x = cP\ddot{a}_{x:\overline{n}|i^*},\end{aligned}$$

with $i^* = m/(1-m)$ so that $(1+i^*)^{-1} = 1-m$. So, the risk premium c is given by

$$\boxed{c = \frac{\pi(0)}{P\ddot{a}_{x:\overline{n}|i^*}} = \frac{{}_np_x \times e^{-rn}\mathbb{E}_0^{\mathbb{Q}}[h(n)]}{P\ddot{a}_{x:\overline{n}|i^*}}}.$$

[Note: If the risk premium c obtained is somehow not less than m, then the value is invalid and it suggests that a higher management charge m may be needed.]

3.2 Guaranteed Minimum Death Benefit

3.2.1 For equity-linked contracts offered by insurers, GMDB is more popular than GMMB since it has a stronger "insurance flavour" than GMMB (some guarantees/protections are available upon death). While GMDB is analyzed in a similar way as GMMB to some degree, the analysis of GMDB is somewhat more complicated since the guarantee can be triggered at any time point during the policy term (when the policyholder dies), rather than only at the maturity.

3.2.2 **The general case of GMDB pricing.** Let $h(t)$ denote the GMDB payoff at time t. By risk-neutral pricing, the time-0 price of a payoff $h(t)$ at time t is

$$v(0,t) := e^{-rt}\mathbb{E}_0^{\mathbb{Q}}[h(t)].$$

Assuming that the benefit is payable at the moment of death (so the GMDB is triggered at that moment also), the time-0 price/value of the GMDB is:

$$\boxed{\pi(0) = \int_0^n v(0,t)\,{}_tp_x\mu_{x+t}\,\mathrm{d}t}.$$

[Note: The structure is quite similar to the EPV of a continuous n-year term life insurance with unit benefit learnt in STAT3901; we just replace v^t by $v(0,t)$ here.]

This quantity can again be interpreted as the *average* cost of setting up an aggregate replicating portfolio. Intuitively, "${}_tp_x\mu_{x+t}\,\mathrm{d}t$" may be viewed as the "proportion" of policies where death occurs at time t, and thus a replicating portfolio that is worth $v(0,t)$ needs to be set up at time 0; "Summing" $v(0,t)\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$ from $t=0$ to $t=n$ then gives the "weighted average" cost.

The formula is analogous for the discrete case. For example, if the benefit payable at the end of the year of death, then the time-0 value of the GMDB is:

$$\boxed{\pi(0) = \sum_{k=0}^{n-1} v(0,k+1)\,{}_kp_x q_{x+k}};$$

we can obtain those formulas by simply replacing v^t by $v(0,t)$ in the respective EPV formulas for n-year term life insurance with unit benefit.

The idea is similar for the pricing of the GMDB at a general time $s \in [0,n]$. By risk-neutral pricing, the time-s price of a payoff $h(t)$ at time t is $v(s,t) := e^{-r(t-s)}\mathbb{E}_s^{\mathbb{Q}}[h(t)]$, with $0 \le s \le t \le n$. Then, for the continuous case, the time-s value of the GMDB is

$$\boxed{\pi(s) = \int_0^{n-s} v(s,s+u)\,{}_up_{x+s}\mu_{x+s+u}\,\mathrm{d}u},$$

provided that the policyholder is alive at time s.

For the discrete case with the benefit payable at the end of the year of death, the time-s value of the GMDB is:

$$\boxed{\pi(s) = \sum_{k=0}^{n-1-s} v(s,s+k+1)\,{}_kp_{x+s}q_{x+s+k}}$$

provided that the policyholder is alive at time s (integer).

Again, these formulas can be obtained by replacing v^u by $v(s, s+u)$ in the respective time-s EPV formulas for n-year term life insurance with unit benefit.

[Note: Again, here we assume that no surrenders/lapses are possible for convenience, so that we can simply use "p" here. In case surrenders/lapses are possible, we should replace "alive" by "active" and each "p" by "p^{00}" (for being active), while those "q"s should still be death probabilities.]

3.2.3 **GMDB that guarantees the single premium.** To illustrate the valuation of the GMDB, here we focus on a specific type of GMDB, which guarantees a benefit amount of at least the premium P, payable at the end of the year of death within the policy term. Hence, if the policyholder's year of death is $k+1 \in \{1, \dots, n\}$, then the GMDB payoff at time $k+1$ is

$$h(k+1) = \begin{cases} P - F_{k+1} & \text{if } F_{k+1} < P, \\ 0 & \text{otherwise} \end{cases} = (P - F_{k+1})_+.$$

3.2.4 **Time-0 value of the GMDB under further assumptions.** To obtain a more explicit formula, we impose the assumptions in [3.1.4]. Then, we have

$$\begin{aligned} v(0, k+1) &= e^{-r(k+1)} \mathbb{E}_0^{\mathbb{Q}}[(P - F_{k+1})_+] = e^{-r(k+1)} \mathbb{E}_0^{\mathbb{Q}}\big[(P - P(1-e)(1-m)^k S_{k+1})_+\big] \\ &= Pe^{-r(k+1)} \mathbb{E}_0^{\mathbb{Q}}\big[(1 - (1-e)(1-m)^k S_{k+1})_+\big] = \boxed{P\xi_k e^{-r(k+1)} \mathbb{E}_0^{\mathbb{Q}}\big[(\xi_k^{-1} - S_{k+1})_+\big]} \\ &= P\xi_k \Big(\xi_k^{-1} e^{-r(k+1)} \Phi(-d_2(0, k+1)) - S_0 \xi_k \Phi(-d_1(0, k+1))\Big) \\ &= \boxed{P\Big(e^{-r(k+1)} \Phi(-d_2(0, k+1)) - \xi_k \Phi(-d_1(0, k+1))\Big)} \end{aligned}$$

where $\xi_k := (1-e)(1-m)^k > 0$,

$$d_1(0, k+1) = \frac{\ln \xi_k + (r + \sigma^2/2)(k+1)}{\sigma\sqrt{k+1}}, \quad \text{and} \quad d_2(0, k+1) = d_1(0, k+1) - \sigma\sqrt{k+1}.$$

Hence, the time-0 value of the GMDB is

$$\boxed{\pi(0) = \sum_{k=0}^{n-1} P\Big(e^{-r(k+1)} \Phi(-d_2(0, k+1)) - \xi_k \Phi(-d_1(0, k+1))\Big) {}_k p_x \, q_{x+k}}.$$

[Note: This formula generally requires substantial computations, especially when n is large. In an exam environment, it is likely that n is very small (say 2 or 3) to make the computation workload reasonable.]

3.2.5 **Time-s value of the GMDB under further assumptions.** The argument is similar for the time-s value of the GMDB (with s being an integer). Under the assumptions in [3.1.4], we have

$$\begin{aligned} v(s, s+k+1) &= e^{-r(k+1)} \mathbb{E}_s^{\mathbb{Q}}[(P - F_{s+k+1})_+] = e^{-r(k+1)} \mathbb{E}_s^{\mathbb{Q}}\big[(P - P(1-e)(1-m)^{s+k} S_{s+k+1})_+\big] \\ &= Pe^{-r(k+1)} \mathbb{E}_s^{\mathbb{Q}}\big[(1 - (1-e)(1-m)^{s+k} S_{s+k+1})_+\big] = \boxed{P\xi_{s+k} e^{-r(k+1)} \mathbb{E}_s^{\mathbb{Q}}\big[(\xi_{s+k}^{-1} - S_{s+k+1})_+\big]} \\ &= P\xi_{s+k} \Big(\xi_{s+k}^{-1} e^{-r(k+1)} \Phi(-d_2(s, s+k+1)) - S_s \Phi(-d_1(s, s+k+1))\Big) \\ &= \boxed{P\Big(e^{-r(k+1)} \Phi(-d_2(s, s+k+1)) - \xi_{s+k} S_s \Phi(-d_1(s, s+k+1))\Big)} \end{aligned}$$

where $\xi_{s+k} := (1-e)(1-m)^{s+k} > 0$,

$$d_1(s, s+k+1) = \frac{\ln(\xi_{s+k} S_s) + (r + \sigma^2/2)(k+1)}{\sigma\sqrt{k+1}}, \quad \text{and} \quad d_2(s, s+k+1) = d_1(s, s+k+1) - \sigma\sqrt{k+1}.$$

Hence, the time-s value of the GMDB is

$$\boxed{\pi(s) = \sum_{k=0}^{n-1-s} P\Big(e^{-r(k+1)}\Phi(-d_2(s, s+k+1)) - \xi_{s+k} S_s \Phi(-d_1(s, s+k+1))\Big)\, {}_kp_{x+s}\, q_{x+s+k}},$$

provided that the policyholder is alive at time s.

3.2.6 **Risk premiums.** The risk premiums for GMDB are computed using the same principle as the ones for GMMB. For illustration, assume that a risk premium c is charged at the start of each year, a management charge m is applicable at the start of each year (including the first), there is no initial expense, and the benefit is payable at the end of the year of death. Then, the time-0 value of the risk premiums is $cP\ddot{a}_{x:\overline{n}|i^*}$ where $i^* = m/(1-m)$ (derived in [3.1.8]). Hence, the risk premium c is given by

$$\boxed{c = \frac{\pi(0)}{P\ddot{a}_{x:\overline{n}|i^*}} = \frac{\sum_{k=0}^{n-1} v(0, k+1)\, {}_kp_x\, q_{x+k}}{P\ddot{a}_{x:\overline{n}|i^*}}}.$$

3.3 Hedging of GMMB and GMDB

3.3.1 Similar to the option pricing theory learnt in STAT3905/STAT3910, the prices/values of GMMB and GMDB we have computed in Sections 3.1 and 3.2 can be interpreted as the costs of setting up *hedge portfolios*, consisting of suitable positions in stock and risk-free (zero-coupon) bond, for the respective guarantees. In STAT3910, we have discussed (i) formulas for computing the appropriate positions in stock and bond for delta-hedging an option, and (ii) how dynamic hedging works. These two aspects of hedging will also be covered in Section 3.3, and the content here will be applied in profit testing (see Section 3.4).

Remarks:

- To manage the risk of GMMB and GMDB offered in an equity-linked contract, there are two common approaches for the insurer: (i) *buying options from external market* ("*externally hedging*") and (ii) *internally hedging*. Theoretically, both approaches would cost the same to the insurer, namely the price/value computed in Sections 3.1 and 3.2. For the approach (i), the insurer needs not have expertise in hedging (the content of Section 3.3 can be skipped!) and just needs to pay some money **$**, and the payoffs from the purchased options are assumed to exactly cover all the guarantee payments (the sellers in the market would handle the hedging themselves). On the other hand, the approach (ii) requires the insurer to have expertise in hedging (the content in Section 3.3 is important!).
- Throughout Sections 3.3 and 3.4, we again assume that no surrenders/lapses are possible for convenience. If they are possible, then we just replace "alive" by "active" and "p" by "p^{00}".

3.3.2 **Idea.** Recall from STAT3910 that to perform delta-hedging for an option, the number of shares of stock to be purchased is given by its *time-t delta*, which is calculated by partially differentiating the time-t value of that option with respect to the time-t stock price S_t. Then, the amount to be invested into risk-free bond can be obtained by

$$\text{time-}t \text{ value of the option} - \text{time-}t \text{ delta} \times S_t.$$

The delta-hedging for GMMB/GMDB works in a similar way, but we need to make some suitable modifications to incorporate the life-contingent elements via some kinds of "weighted averages", just like what we did in the valuation of GMMB/GMDB in Sections 3.1 and 3.2.

3.3.3 **Delta-hedging of GMMB.** Recall from [3.1.3] that the time-t value of the GMMB is

$$\pi(t) = {}_{n-t}p_{x+t} \times e^{-r(n-t)}\mathbb{E}_t^{\mathbb{Q}}[h(n)] = {}_{n-t}p_{x+t} \times v(t, n),$$

provided that the policyholder is alive at time t, with $0 \le t \le n$, where the notation $v(t, n)$ comes from Section 3.2 (notice how the expression "looks like" the time-t EPV of an n-year pure endowment).

This quantity can be interpreted as the average cost of setting up aggregate replicating portfolio for a large number of policies: A proportion ${}_{n-t}p_{x+t}$ of policies requires a replicating portfolio worth $v(t,n)$ at time t, and the rest of the policies (with proportion ${}_{n-t}q_{x+t}$) does not require that (the corresponding stock and bond parts both have 0 values). Accordingly, the values of the stock and bond parts in this case are *weighted averages* of the quantities determined by the STAT3910 approach, where the weights come from the survival probabilities.

Hence, provided that the policyholder is alive at time t, the value of the stock part of the hedge portfolio for the GMMB at time t is

$$ {}_{n-t}p_{x+t} \times \left(\frac{\partial}{\partial S_t} v(t,n)\right) \times S_t + {}_{n-t}q_{x+t} \times 0 = \boxed{{}_{n-t}p_{x+t} \times \left(\frac{\partial}{\partial S_t} v(t,n)\right) \times S_t} =: \Psi_t S_t. $$

The value of the bond part of the hedge portfolio for the GMMB at time t is then

$$ {}_{n-t}p_{x+t} \times \left(v(t,n) - \left(\frac{\partial}{\partial S_t} v(t,n)\right) \times S_t\right) + {}_{n-t}q_{x+t} \times 0 = \boxed{\pi(t) - {}_{n-t}p_{x+t} \times \left(\frac{\partial}{\partial S_t} v(t,n)\right) \times S_t} =: \Upsilon_t. $$

[Note: Indeed, the value of the bond part can just be determined by subtracting the value of the stock part from the value of the GMMB, rather than computing the weighted average. This is also true for GMDB.]

3.3.4 **Delta-hedging of GMDB.** Recall from [3.2.2] that for the continuous case, the time-s value of the GMDB is

$$ \pi(s) = \int_0^{n-s} v(s,s+u)\,{}_u p_{x+s}\mu_{x+s+u}\,\mathrm{d}u, $$

provided that the policyholder is alive at time s, with $0 \le s \le n$. Intuitively, "${}_u p_{x+s}\mu_{x+s+u}\,\mathrm{d}u$" may be viewed as the "proportion" of policies where death occurs at time $s+u$, and thus a replicating portfolio that is worth $v(s,s+u)$ needs to be set up at time s. We can then follow the STAT3910 approach to compute the values of the stock and bond parts for each possible replicating portfolio, and compute the weighted averages of these values to determine the values of the stock and bond parts of the hedge portfolio for the GMDB.

Hence, the value of the stock part of the hedge portfolio for the GMDB at time s is

$$ \begin{aligned} &\boxed{\int_0^{n-t} \left(\frac{\partial}{\partial S_s} v(s,s+u)\right) S_s \times {}_u p_{x+s}\mu_{x+s+u}\,\mathrm{d}u} \\ &= \left[\int_0^{n-t} \left(\frac{\partial}{\partial S_s} v(s,s+u)\right) \times {}_u p_{x+s}\mu_{x+s+u}\,\mathrm{d}u\right] \times S_s =: \Psi_s S_s \end{aligned} $$

and the value of the bond part of the hedge portfolio for the GMDB at time s is

$$ \begin{aligned} &\int_0^{n-t} \left[v(s,s+u) - \left(\frac{\partial}{\partial S_s} v(s,s+u)\right) S_s\right] \times {}_u p_{x+s}\mu_{x+s+u}\,\mathrm{d}u. \\ &= \boxed{\pi(s) - \int_0^{n-t} \left(\frac{\partial}{\partial S_s} v(s,s+u)\right) S_s \times {}_u p_{x+s}\mu_{x+s+u}\,\mathrm{d}u} =: \Upsilon_s, \end{aligned} $$

provided that the policyholder is alive at time s.

Analogously, for the discrete case (the benefit is payable at the end of the year of death), the values of the stock and bond parts of the hedge portfolio for the GMDB at time s are:

$$ \begin{aligned} &\boxed{\sum_{k=0}^{n-1-s} \left(\frac{\partial}{\partial S_s} v(s,s+k+1)\right) S_s \times {}_k p_{x+s} q_{x+s+k}} \\ &= \left[\sum_{k=0}^{n-1-s} \left(\frac{\partial}{\partial S_s} v(s,s+k+1)\right) \times {}_k p_{x+s} q_{x+s+k}\right] \times S_s =: \Psi_s S_s \end{aligned} $$

and

$$\boxed{\pi(s) - \sum_{k=0}^{n-1-s} \left(\frac{\partial}{\partial S_s} v(s, s+k+1) \right) S_s \times {}_k p_{x+s} q_{x+s+k}} =: \Upsilon_s,$$

respectively, provided that the policyholder is alive at time s (integer).

3.3.5 **Specialized formulas under further assumptions.** Imposing the assumptions in [3.1.4], we can obtain more explicit formulas of the values of the stock and bond parts of the hedge portfolio for GMMB/GMDB.

(a) *GMMB:* From [3.1.6], we know that

$$\pi(t) = {}_{n-t}p_{x+t} \underbrace{P\xi\Big(\xi^{-1}e^{-r(n-t)}\Phi(-d_2(t)) - S_t\Phi(-d_1(t))\Big)}_{v(t,n)}.$$

From our STAT3910 knowledge, the partial derivative $\partial/\partial S_t\ v(t,n)$ is then given by $-P\xi\Phi(-d_1(t))$, which is the "coefficient" of S_t. [Note: As we know from STAT3910, actually both $d_1(t)$ and $d_2(t)$ are functions of S_t, so computing the partial derivative takes some work, but ultimately the result is quite "intuitive".]

Substituting this partial derivative into the formulas in [3.3.3] then gives the values of the stock and bond parts.

(b) *GMDB:* We take the discrete case as an example. From [3.2.5], we know that

$$v(s, s+k+1) = P\xi_{s+k}\Big(\xi_{s+k}^{-1}e^{-r(k+1)}\Phi(-d_2(s, s+k+1)) - S_s\Phi(-d_1(s, s+k+1))\Big).$$

Hence, the partial derivative $\partial/\partial S_s\ v(s, s+k+1)$ is given by $-P\xi_{s+k}\Phi(-d_1(s, s+k+1))$, which is the "coefficient" of S_s. Substituting this partial derivative into the formulas for the discrete case in [3.3.4] then gives the values of the stock and bond parts.

3.3.6 **Dynamic hedging.** Theoretically, the delta hedging suggested above requires *continuous rebalancing*. However, this is clearly not practical and we can only perform the rebalancing at discrete time points. This is similar to the situation investigated in STAT3910, where we consider the net cash flows arising from the regular rebalancing at discrete time points *(dynamic hedging)*, as differences between the cost of setting up new hedge portfolio and the value of the old hedge portfolio brought forward. The net cash flows may be viewed as the *rebalancing costs*: Negative net cash flow indicates that there is a positive rebalancing cost; the insurer needs to inject additional money **$** to perform the rebalancing. Positive cash flow indicates that the rebalancing cost is negative, i.e., the insurer can extract some money **$** during the rebalancing. As you may expect, we again need to make some suitable adjustments to incorporate the life-contingent elements.

3.3.7 **Rebalancing costs.** For the hedge portfolio for GMMB and/or GMDB at time t, let $\Psi_t S_t$ and Υ_t denote the values of the stock and bond parts, respectively. [Note: To determine the values of the stock and bond parts for a combination of GMMB and GMDB, we just need to add up the respective values computed according to the previous formulas.]

Suppose that the rebalancing occurs every $1/m$ years, the benefit (and GMDB) is also payable at the end of $1/m$ year of death, and the current time is $t \in \{0, 1/m, \dots, n-2/m\}$. Provided that the policyholder is alive at time t, the cost of setting up the new hedge portfolio at time $t+1/m$ is

$$\boxed{{}_{1/m}p_{x+t} \times \pi(t+1/m) = {}_{1/m}p_{x+t} \times (\Psi_{t+1/m}S_{t+1/m} + \Upsilon_{t+1/m})}$$

and the value of the old hedge portfolio brought forward is

$$\boxed{\pi^{\text{bf}}(t+1/m) = \Psi_t S_{t+1/m} + \Upsilon_t e^{r/m}}$$

where r is the continuously compounded risk-free rate per annum. Here, the cost of setting up the new hedge portfolio incorporates a life-contingent element and can be interpreted as the *average* cost of setting up new hedge portfolio for a large number of policies in-force at time t:

- For the individuals that survive to time $t+1/m$ (with proportion ${}_{1/m}p_{x+t}$), each requires a new hedge portfolio worth $\pi(t+1/m) = \Psi_{t+1/m}S_{t+1/m} + \Upsilon_{t+1/m}$ at time $t+1/m$.
- For the rest of them (who die within $1/m$ years), we do not set up "new hedge portfolio" (so the associated cost is 0). Instead, we need to pay a cost that fulfills the GMDB payoff at time $t+1/m$ (if any), and that cost of GMDB at time $t+1/m$ is given by $h^{\text{GMDB}}(t+1/m)_{1/m}q_{x+t}$, where $h^{\text{GMDB}}(t+1/m)$ denotes the GMDB payoff at time $t+1/m$.

Therefore, for each $t \in \{0, 1/m, \ldots, n-2/m\}$, the rebalancing cost at time $t+1/m$ is

$$\boxed{{}_{1/m}p_{x+t} \times \pi(t+1/m) - \pi^{\text{bf}}(t+1/m)}.$$

The calculation of the rebalancing cost at time n requires special attention. In this case, the cost of setting up new hedge portfolio at time n is always 0 (provided that the policyholder is alive at time $n-1$). This is because for a large number of policies in-force at time $n-1/m$:

- For individuals that survive to time n (with proportion ${}_{1/m}p_{x+n-1/m}$), we do not set up "new hedge portfolio". Instead, we need to pay a cost that fulfills the *GMMB* payoff at time n (if any), and that cost of GMMB at time n is $h^{\text{GMMB}}(n)_{1/m}p_{x+n-1/m}$, where $h^{\text{GMMB}}(n)$ denotes the GMMB payoff at time n.
- For the rest of them (who die with $1/m$ years), we also do not set up "new hedge portfolio", and we need to pay a cost that fulfills the *GMDB* payoff at time n (if any). That cost of GMDB at time n is $h^{\text{GMDB}}(n)_{1/m}q_{x+n-1/m}$, where $h^{\text{GMDB}}(n)$ denotes the GMDB payoff at time n.

On the other hand, the value of the old hedge portfolio brought forward can still be computed by the usual formula:

$$\boxed{\pi^{\text{bf}}(n) = \Psi_{n-1/m}S_n + \Upsilon_{n-1/m}e^{r/m}}.$$

Consequently, the rebalancing cost at time n is just

$$\boxed{-\pi^{\text{bf}}(n)}.$$

[Note: While the costs of GMMB and GMDB are not directly related to the computations of the rebalancing costs for the dynamic hedging, it plays an important role in the *profit testing* of GMMB and GMDB (see Section 3.4).]

3.4 Profit Testing of GMMB and GMDB

3.4.1 The core of profit testing is to identify all the relevant cash flows, similar to the treatment for universal life insurance in Section 2. Once all the relevant cash flows are correctly identified, the computation of the profit measures becomes a routine task. Therefore, let us start with determining all the cash inflows and outflows associated with the GMMB and GMDB for the insurer here.

3.4.2 **Setup.** For consistency, our discussion of profit testing will focus on the setting described below:

- There is no initial front-end-load expense.
- A management charge is deducted from the fund at the start of every $1/m$ years, as a proportion m of the fund value at that time.
- The benefit (and GMDB) is payable at the end of $1/m$ year of death.
- *(For internally hedging)* Rebalancing occurs every $1/m$ years.

- *(For externally hedging)* At the start of every $1/m$ years, a risk premium is a proportion c of the fund value at that time, before the deduction of management charge. It is sourced from the management charge (not deducted from the fund), and is used for purchasing options from external market at that time; the options purchased using all the risk premiums are supposed to generate payoffs that exactly cover all the guarantee payments.
- At the start of every $1/m$ years, an expense is incurred to the insurer, as a proportion ε of the fund value at that time, after the deduction of management charge.

[Note: Of course, different setups for the profit testing are possible and we can adjust the profit testing accordingly; we should always read the question setting carefully.]

3.4.3 **Cash flows for externally hedging.** When the insurer "hedges externally", i.e., buys options from external market for hedging the guarantees (possibly at multiple time points), we can ignore all rebalancing costs and costs of GMMB and GMDB, since once the options are purchased from the external market, their payoffs are supposed to exactly cover all the guarantee payments, without inducing any extra cash flows for the insurer. The relevant inflows and outflows for the insurer are:

- Inflows:
  - Regular management charge income.
  - Return on income over each $1/m$-year period.
- Outflows:
  - Expenses.
  - Costs of purchasing options from external market (which equal the risk premiums according to the setup).

More precisely, the cash flows at different time points are as follows:

Time	Management charge	Expense	Cost of buying options
0	mF_0	$\varepsilon(1-m)F_0$	cF_0
$1/m$	$mF_{1/m}$	$\varepsilon(1-m)F_{1/m}$	$cF_{1/m}$
$2/m$	$mF_{2/m}$	$\varepsilon(1-m)F_{2/m}$	$cF_{2/m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1/m$	$mF_{n-1/m}$	$\varepsilon(1-m)F_{n-1/m}$	$cF_{n-1/m}$
n	0	0	0

3.4.4 **Cash flows for internally hedging.** When the insurer hedges internally, the rebalancing costs and the costs of GMMB and GMDB from Section 3.3 become relevant, and the relevant inflows and outflows for the insurer are:

- Inflows:
  - Regular management charge income.
  - Return on income over each $1/m$-year period.
- Outflows:
  - Expenses.
  - *(Pre-contract)* Cost of setting up the initial hedge portfolio.
  - *(For $1/m \le t \le n$)* Rebalancing cost.
  - *(For $1/m \le t \le n$)* Cost of GMDB.
  - *(For $t = n$)* Cost of GMMB.

More precisely, the cash flows at different time points are as follows:

Time	Management charge	Expense	Cost of setting up the initial hedge portfolio	Rebalancing cost	Cost of GMMB	Cost of GMDB
Pre-contract	0	0	$\pi(0)$	0	0	0
0	mF_0	$\varepsilon(1-m)F_0$	0	0	0	0
$1/m$	$mF_{1/m}$	$\varepsilon(1-m)F_{1/m}$	0	${}_{1/m}p_x \times \pi(1/m) - \pi^{\mathrm{bf}}(1/m)$	0	$h^{\mathrm{GMDB}}(1/m) \times {}_{1/m}q_x$
$2/m$	$mF_{2/m}$	$\varepsilon(1-m)F_{2/m}$	0	${}_{1/m}p_{x+1/m} \times \pi(2/m) - \pi^{\mathrm{bf}}(2/m)$	0	$h^{\mathrm{GMDB}}(2/m) \times {}_{1/m}q_{x+1/m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1/m$	$mF_{n-1/m}$	$\varepsilon(1-m)F_{n-1/m}$	0	${}_{1/m}p_{x+n-2/m} \times \pi(n-1/m) - \pi^{\mathrm{bf}}(n-1/m)$	0	$h^{\mathrm{GMDB}}(n-1/m) \times {}_{1/m}q_{x+n-2/m}$
n	0	0	0	$-\pi^{\mathrm{bf}}(n)$	$h^{\mathrm{GMMB}}(n) \times {}_{1/m}p_{x+n-1/m}$	$h^{\mathrm{GMDB}}(n) \times {}_{1/m}q_{x+n-1/m}$

3.4.5 **Profit vectors for externally hedging.** To perform profit testing, we need to convert the cash flows obtained above into a profit vector, consisting of in-force expected profits emerging at different time points. For each $t \in \{1/m, 2/m, \ldots, n\}$, let Pr_t denote the expected profit emerging at time t, given that the policyholder is alive at time $t-1/m$.

For Pr_0, we only consider the initial front-end-load expense as the relevant outflow, which is assumed to be 0 here:

$$\mathrm{Pr}_0 = -\text{initial front-end-load expense} = 0.$$

For all $t = 1/m, 2/m, \ldots, n$, we have

$$\begin{aligned}\mathrm{Pr}_t = (&\text{Management charge at time } t-1/m - \text{Expense at time } t-1/m \\ &- \text{Cost of buying options at time } t-1/m)(1+i_{\mathrm{acc}})^{1/m}\end{aligned}$$

where i_{acc} is the annual effective return/accumulation rate on the income for the insurer.

3.4.6 **Profit vectors for internally hedging.** Expected profits Pr_t for the internally hedging case are computed in a similar way. Here, Pr_0 considers the initial front-end-load expense and the cost of setting up the initial hedge portfolio as the relevant outflows, and so we have:

$$\mathrm{Pr}_0 = -\text{initial front-end-load expense} - \text{cost of setting up the initial hedge portfolio} = -\pi(0).$$

For all $t = 1/m, 2/m, \ldots, n$, we have

$$\begin{aligned}\mathrm{Pr}_t = &(\text{Management charge at time } t-1/m - \text{Expense at time } t-1/m)(1+i_{\mathrm{acc}})^{1/m} \\ &\underbrace{-\text{Rebalancing cost at time } t - \text{Cost of GMDB at time } t - \text{Cost of GMMB at time } t}_{\text{expected cash flows at time } t \text{ given survival at time } t-1/m},\end{aligned}$$

where i_{acc} is the annual effective return/accumulation rate on the income for the insurer.

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- [1.1.2]: relationship between salaries received and salary rates (exact and approximate)
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Section 3

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- [3.4.5]: profit vectors for externally hedging
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References

Dickson, D. C. M., Hardy, M. R., & Waters, H. R. (2019). *Actuarial mathematics for life contingent risks* (3rd ed.). Cambridge University Press.