HKU STAT3905 Study Notes

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Last Updated: 2024-11-05

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1 Introduction to Derivatives

1.1 Definition and Uses of Derivatives

1.1.1 A (financial) **derivative** is a contract (between two parties) whose value (or payoff)[1] depends on (or "is derived from") the value(s) of other more basic underlying variable(s) (e.g., total rainfall over a certain period in the future).

1.1.2 Examples of derivatives: futures, forwards, options, swaps, insurance.

1.1.3 Uses of derivatives:

- risk management: reduce the overall level of risk (example: insurance);
- speculation (risky!): make bets on various market quantities ;
- reducing transaction costs : *replicate* a series of transactions by a *single* transaction of a derivative, so that less transactions are made, thereby reducing transaction costs;
- regulatory arbitrage : circumvent regulatory restrictions [e.g., banks securitized mortgages and bought back the securitized products created, since the capital required to be kept for those securitized products used to be much less than that for the mortgages themselves (Hull, 2022, Section 8.3) (such products led to *global financial crisis* in 2008!)].

1.2 Exchange-Traded and Over-The-Counter Markets

1.2.1 An **exchange-traded market** is a market where standardized financial instruments are traded.

1.2.2 An **over-the-counter market** (OTC market) is a market where participants trade (possibly tailor-made) financial instruments directly with each other, without a central exchange.

1.2.3 For the exchange-traded market, usually there are *margins requirement* and *marking to market*, to eliminate **credit risk** (i.e., the risk that the contract will not be honored).

1.2.4 The actual implementations of margins requirement and marking to market may vary for different exchange-traded markets. But the general idea is as follows:

- margins requirement: to enter into a trade, both parties need to *deposit* a certain amount into a *margin account*;
- marking to market: as the market price $ of the traded instrument varies, the balances needed in the margin accounts also update accordingly at a certain frequency (e.g., daily).

[Note: Often there is a prespecified "minimum balance" for the margin account (called **maintenance margin**). If the account balance is lower than this value, further deposit is required.]

1.2.5 With suitable margins requirement and marking to market, even if a party does not honor the contract ("runs away"), another party can still receive a compensation (the balance in 's margin account) that covers the loss, thereby eliminating credit risk.

1.2.6 On the other hand, there are usually not margins requirement and marking to market for OTC market, so credit risk exists.

[1] We define value/payoff in section 1.6.

1.3 Types of Traders

1.3.1 There are three main types of traders in a derivatives market:

- hedgers: use derivatives to reduce risk that they face from potential future movements in market quantities;
- speculators: use derivatives to bet on future directions of market quantities — they try to make money by taking risk;
- arbitrageurs: take offsetting "positions"[2] in two or more instruments to lock in a sure (risk-free) profit.

1.3.2 A **bullish** (**bearish**) trader expects that a certain market quantity (often price of an asset) will *rise* (*drop*) in the future. With such expectation, the trader may then use appropriate strategy involving derivatives to make bet. (See section 4.5.)

[Mnemonic: A bull uses its horns in an *upward* motion to attack and a bear uses its claws in a *downward* motion to attack.]

1.3.3 The act of (possibly) making a risk-free profit without needing any (net) cash outflow (having a "free lunch") is known as **arbitrage**. Since taking offsetting positions would not lead to any (net) cash outflow, an *arbitrage opportunity* arises if such act can lead to a risk-free profit.

1.3.4 In modern financial economics, a fundamental assumption is the **no-arbitrage principle**, i.e., arbitrage opportunities *do not exist*. This assumption may be regarded as "reasonable" if the arbitrageurs "act immediately" to capture the arbitrage opportunity once it arises, making the opportunity vanish almost "instantaneously" (through market demand and supply).

In this notes, we shall assume that this principle holds true unless stated otherwise.

1.4 Buying and Selling Financial Instruments

1.4.1 To buy/sell financial instruments, typically we do so through a *broker* (e.g., banks), in an exchange-traded market. We also need to pay transaction costs, e.g., brokerage commission, taxes, etc.

1.4.2 In an exchange-traded market, participants place **orders** (i.e., instructions to buy/sell a certain number of financial instruments). There are two main kinds of orders:

- **market order**: the ordered transaction is to be executed immediately at current "market price";
- **limit order**: the instruments are to be bought (sold) at no more (no less resp.) than a specified price (called **limit price**).

1.4.3 Limit orders for a financial instrument may be visualized as follows (the numbers indicate the limit prices specified by the participants):[3]

[2] This means aggregating all those positions gives *zero* unit of instrument. In other words, the positions are completely "closed out". See section 1.6 for more details.

[3] We shall assume that there are at least one buy order and at least one sell order. (The market is not too "illiquid".)

1.4.4 In the visualization, the "top" ("bottom") price at LHS (RHS) is known as **bid price** (**ask price** or **offer price**), which is the price where a new participant can *sell* (*buy*) the instrument *immediately* (the execution price of a sell (buy) market order).

[Note: Bid price is called "bid" as it sources *from* the "bidding" (buying) side, and ask/offer price is called "ask/offer" as it sources *from* the "asking/offering" (selling) side.]

[Warning: *Buy* market order executes at *ask* price, and *sell* market order executes at *bid* price. Do not mix up them!]

1.4.5 The difference between bid price and ask price is known as **bid-ask spread**.

1.4.6 At any moment of time, ask price is greater than bid price. To see this, consider what would happen if ask price was less than or equal to bid price:[4]

[4]Here we suppose each limit order is to buy/sell one unit of the instrument.

After executing the orders, the situation becomes:

So, now the ask price is greater than bid price again. We may assume the executions of such orders happen "instantaneously", and then it is not hard to see that the only possible observation is ask price exceeding bid price.

1.5 Short Selling

1.5.1 **Short selling** means selling a financial instrument 📜 that is not owned.

1.5.2 The procedure for short selling is as follows:

(a) Borrow a number of instruments from a third party (e.g. broker).

(b) sell those instruments to the market for cash $ (creating a *short position*).

(c) Use the cash $ to buy that number of instruments some time later and return them to the lender (*closing out* or *covering* the short position).

[Note: The lender usually requires the short-seller to deposit a certain amount of money as a *collateral*, to protect against the possibility that the short-seller "runs away" and fails to return the instruments when its price surges in the future.]

1.5.3 Short selling may not be possible for certain instruments.

1.5.4 Speculators can short sell an instrument to bet its price to drop in the future. They make money if the price indeed goes down, as they first "sell high", then "buy low". But if the price goes up, they would first "sell low" then "buy high", resulting in a loss.

1.5.5 An investor with a short position is required to pay to the lender any income (e.g., dividends) that would normally be received on the "shorted" instruments, since the *owner* (the lender), not the short-seller, is entitled to those incomes.

1.6 Long and Short Positions

1.6.1 Having a **long position** (**short position**) in a financial instrument means owning a *positive* ("*negative*") amount of.

Remarks:

- Owning a "negative" amount of actually means *owing* that amount (in absolute value) of.
- Unless otherwise specified, a long (short) position in means owning (owing) *one unit* of.
- For brevity, we may simply use the word "**long** (**short**)" to mean "long (short) position in". Example: "long " means "long position in ".
- A yet another alternative expression is "**long** (**short**) a number of instrument " (treating "long" / "short" as a verb). The number specified is the increase (decrease) in amount of owned.

1.6.2 Example: in the short selling, the short-seller owes an amount of to the lender (so owns a negative amount of). Hence, is having a *short* position. (This explains why it is called "short" selling.)

1.6.3 **Closing out** or **covering** a (long or short) position in a financial instrument means doing something such that *zero* is owned ("clear" the amount of we own).

1.6.4 Example: if has a long position in , then can short one unit of to close out his long position.

1.6.5 The **value** or **payoff** of a position in a financial instrument at a certain time is the amount of money $ received *if* all positions (including those "extra" positions in other instruments created from closing out the position in) are closed out at that time.[5]

Remarks:

- Receiving a negative amount of $ means *paying* that amount (in absolute value) of $.
- "Value of an amount of " means "value of *owning* that amount of ". It also has the meaning of "(spot) *price* of that amount of " since selling that amount (yielding the (spot) *price*) closes out the position of "owning that amount of ".
- Unless stated otherwise, the time unit is always years in this notes.

[5]Implicitly we assume that this amount is *unique*, and this is justified by the no-arbitrage principle (and also the *perfect market* assumption; see [1.6.6]): Different ways of closing out those positions must yield the same amount of cash flow. Otherwise, we can perform the way yielding higher cash flow and perform the *reverse* ("opposite") of the way yielding lower cash flow to "capture" the difference risk-free (arbitrage). See [3.1.3] for more explanation on "reverse".

1.6.6 Apart from the no-arbitrage principle, another critical assumption in financial economics is the *perfect market* assumption.

In a **perfect market**,

- any asset can be freely bought or sold at a single price (called **spot price**) in any amount;
- there is no transaction cost;
- short selling is always possible;
- borrowing rate and lending rate are the same;
- credit risk does not exist (➔ no margin requirement, every loan is risk-free, etc.);

1.6.7 Due to these "nice" properties of perfect market, working with it would be convenient, and indeed many "nice" results assume the market is perfect (in addition to the no-arbitrage principle).

Unless otherwise specified, we assume that we are in a perfect market.

[⚠ Warning: However, the actual market in the real world is clearly *not* perfect. So one should be careful about the potential impacts of this when applying the results in the real world.]

1.6.8 Under the *perfect market* assumption, we have the following "linearity" property for value:

$$\text{value of owning } k \text{ 📜} = k \times \text{value of owning 1 📜}.$$

[Note: We can also phrase it in the following way, which may be more intuitive:

$$\text{price of } k \text{ 📜} = k \times \text{price of 1 📜}.$$

]

<u>Proof</u>: Firstly, the value of owning 1 📜 is the amount of cash $\$$ received from selling 1 📜 (a way of closing the long position), i.e., spot price of 📜 (denoted by S here). Now, since selling k 📜 results in a cash flow of kS (they can all be bought/sold at the spot price S by assumption), the value of owning k 📜 is kS also, by definition.

[Note: When k is negative, "selling k 📜" is supposed to mean "buying $-k$ 📜". Also, positive (negative) cash flow represents cash inflow (outflow).] □

As a special case, the value of "short 📜" is the negative of the value of "long 📜".

2 Forward and Futures Contracts

2.1 Introduction

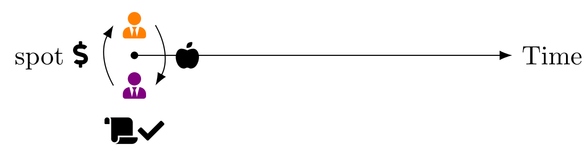
2.1.1 A **forward (contract)** is a contract between two parties (called **counterparties**) to buy or sell an asset (known as **underlying asset**, or simply **underlying**) at a certain time in the future (known as **delivery date** or **maturity date**) for a specified price (known as **forward price** or **delivery price**) $ (such that it costs nothing to enter the contract; see [2.1.3]).

Remarks:

- The contract is negotiated, agreed, and signed *today* (now). All future transaction details are then *fixed*.
- We sometimes use the phrase "forward *on* " to describe the underlying asset is .

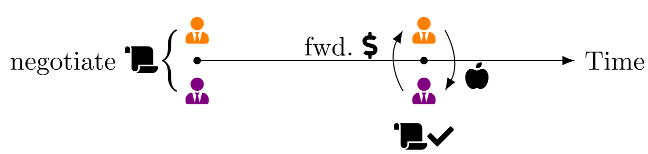
2.1.2 A forward contract may be contrasted with a *spot contract*:

- spot contract: two parties simply transact *now* at current market (spot) price (current "fair" price);



: having a long position in
: having a short position in

- forward contract: two parties transact at a certain future time point at a price that is "fair" deemed by now.



Conventionally, we regard "owning a positive/negative amount of forward (on)" as "(having the obligation for) owning the same amount, but of , *at the delivery date*".

By this convention, we know:

 – is having a long position in forward (or long forward) (as is owning a positive amount of at the delivery date);
 – is having a short position in forward (or short forward) (as is owning a negative amount of at the delivery date).

Furthermore, closing out a position in forward (i.e., doing something such that zero forward is owned) would also make the position "at the delivery date" closed out (as zero would need to be owned at that date).

2.1.3 An important feature of a forward is that there is no cost for the act of "*entering*" into a forward contract (i.e., take a long/short position in forward), ignoring transaction costs. Consequently, the *value* of a long/short position in forward is always zero *at time 0*. (But the same cannot be said for time point later than 0: See section 3.8.)

Remarks:

- This is just like the case for spot contract (i.e., contract for buying/selling *now*) — Merely "executing" orders itself would not cost anything, ignoring transaction costs.

- It turns out that under the *no-arbitrage principle* and *perfect market*, there is only one possible negotiated forward price (at least for the assets covered in section 3).

2.1.4 A forward contract is an over-the-counter instrument (traded in OTC market). A **futures (contract)** has the same contract nature as a forward contract, but it is an exchange-traded instrument (so all contract terms are standardized).

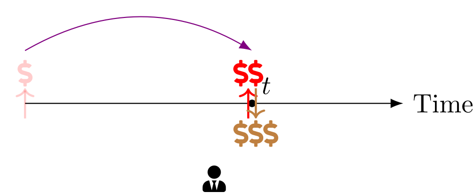
[Note: For a futures contract, the word "forward" is changed to "futures" in the terminologies. For example:

- forward price → futures price
- long/short position in forward → long/short position in futures

]

2.2 Profit and Loss of a Position in Forward/Futures

2.2.1 The **profit and loss** (**P/L** or **P&L**) of a (long/short) position at time t is the payoff of the position at time t less the *future value* of previous cash *outflows* (before time t) at time t (at the *risk-free interest rate*).



[Note: P/L at time t is the net cash flow at time t, after "accumulating" (without risk) cash flows[6] from previous time points if needed. This indicates how much *profit* can be earned from "just" that position (without "adding" extra risk).]

2.2.2 Some notations:

- S_t: (spot) price of one unit of underlying asset 🍎 at time t;
- time T (positive): delivery date;
- K: delivery price.

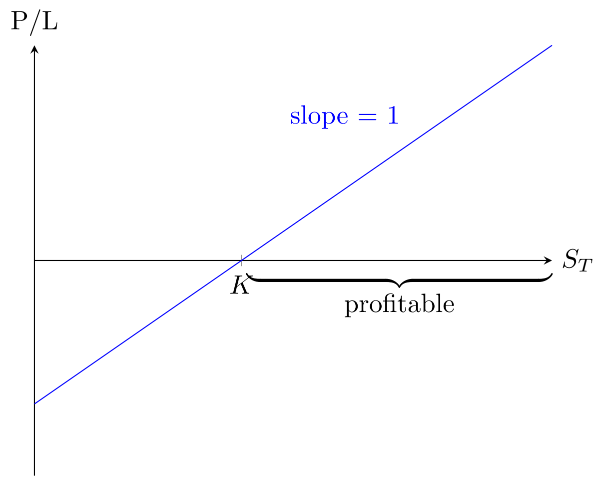
2.2.3 The P/L (at time T) of a long position in forward/futures on one unit of 🍎 is $S_T - K$ (which equals its payoff at time T as there are no cash flows before time T).

Proof: To close out all positions at time T, perform:

Transaction	Position change	Cash flow
buy 1 🍎 at the delivery price	1 📜 & 0 🍎 → 0 📜 & 1 🍎	$-K$
sell 1 🍎 at the spot price	1 🍎 → 0 🍎	$+S_T$
		Total: $S_T - K$

□

[6] For positive cash outflow (inflow) at a previous time, first borrow (lend) that amount risk-free (to "cancel out" that cash flow), and then repay (collect proceeds from) the loan at time t → yielding a cash outflow (inflow) with amount equal to the future value at time t. See section 3 for more examples like this.



When $S_T \uparrow$, the long position makes money. So, speculators can *long forward/futures* to bet the price of rises in the future.

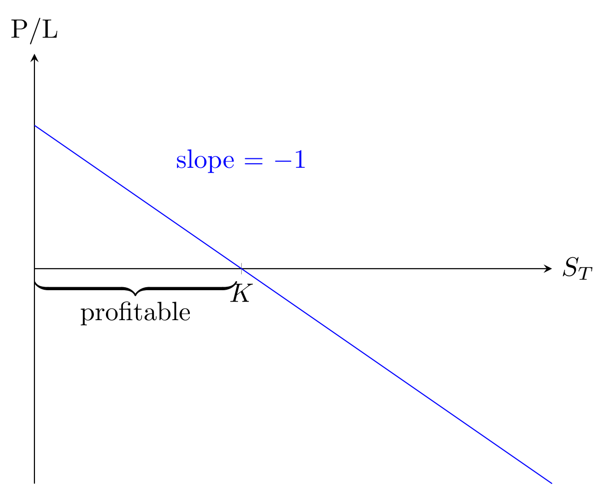
2.2.4 The P/L of a short position in forward/futures on one unit of is $K - S_T$ (which equals its payoff at time T).

Proof: An one-line proof is that it follows from the property that payoff (value) of the short forward/futures at time T is simply the negative of $S_T - K$ (payoff of the long forward/futures). Alternatively, consider the following.

To close out all positions at time T, perform:

Transaction	Position change	Cash flow
(short) sell 1 at the delivery price	-1 & 0 → 0 & -1	$+K$
buy 1 at the spot price	-1 → 0	$-S_T$
		Total: $K - S_T$

□



[Note: In general, since the payoff (and P/L also indeed) of the short position is just the negative of that for the long position, the short position payoff (or P/L) graph is just the mirror image of the long position one, across the S_T-axis.]

When $S_T \downarrow$, the short position makes money. So, speculators can *short forward/futures* to bet the price of drops in the future.

2.3 Stock Index Futures

2.3.1 A **stock index** tracks changes in the value of a hypothetical portfolio of stocks. So it can be regarded as a weighted average of prices of different stocks. Example: S&P 500.

2.3.2 A **stock index futures** is a futures on stock index. Since stock index cannot be "delivered" physically, we have **cash settlement** for those futures, i.e., investors are required to *close out* their positions in those futures and receive/pay cash at or before maturity, and there is no (physical) "delivery" at maturity).

Remarks:

- To close out a long (short) position *at* delivery date, we simply fulfill the obligation suggested in the futures, i.e., buy (sell) at the specified price.
- To close out a long (short) position in the futures *before* delivery date, we can take a short (long) position in the *same* futures. Since it is negotiated at time 0, its value at time $0 < t < T$ may *not* be zero anymore. (See section 3.8 for more details.)

2.4 Short and Long Hedge

2.4.1 A **hedge** is a trade designed to reduce risk. A **perfect hedge** is a hedge that *completely* eliminates the risk.

2.4.2 A **short hedge** (**long hedge**) is a hedge involving a short (long) position in forward/futures.

2.4.3 Situations where *short hedges* are useful:

- the hedger already owns an asset and expects to sell at a certain future time point;
- the hedger does not own now, but will own later, and then sell at a certain future time point.

In either situation, a *short hedge* can let the hedger *lock in* the selling price (namely the price specified in forward/futures) *now*, completely eliminating the uncertainty of future selling price → *perfect hedge*.

2.4.4 Situation where *long hedges* are useful:

- the hedger has to purchase at a certain future time point.

Likewise, a *long hedge* can let the hedger *lock in* the purchasing price *now*, completely eliminating the uncertainty of future purchasing price → *perfect hedge*.

3 Forward/Futures Price

3.0.1 For convenience in wordings, we shall focus on *forward* here. But all results discussed here are also applicable to futures.

3.1 Law of One Price

3.1.1 Recall the *no-arbitrage principle* and *perfect market* mentioned in [1.3.4] and [1.6.6] respectively.

3.1.2 An important consequence of the no-arbitrage principle and the perfect market assumption is the *law of one price*.

3.1.3 The law of one price (LOOP) is as follows:

Theorem 3.1.a (Law of one price)**.** Under the no-arbitrage principle and perfect market, two positions with the same payoff at any future time point must "sell" at the same price now.

Remarks:

- "Price" and "value" carry the same meaning. [Warning: "Price of a forward" ($=0$ at time 0) and "forward price" (negotiated) are not the same! "Price" here is in the former sense.]
- A position "selling" at a price $ now means that it costs $ to take that position now.
- This means there is always exactly *one price* for any instrument with a given future payoff pattern.
- For proofs involving no-arbitrage principle like this one, a general proof strategy is to use *proof by contradiction*: First assume the result is false, and then try to obtain an arbitrage strategy (which contradicts the no-arbitrage principle).

Proof: Assume to the contrary that the positions "sell" at different prices now. Let and be the "cheaper" and "more expensive" position, that "sell" at prices P and Q respectively ($P < Q$). Then we have the following arbitrage strategy:

Time	Transaction	Cash flow
0	take the position *reverse* "take the position"	$-P$ $+Q$ Total: $Q - P > 0$
any $t > 0$	close out all positions	0

[Note: There is no cash outflow and is a risk-free profit, hence it is an arbitrage strategy.]

There is zero cash flow when all positions are closed out (and for the omitted time point) since and the "reverse " always have offsetting payoffs, at any future time point. ☐

Remarks:

- For a given strategy (comprising of some transaction(s)), the **reverse** strategy is the one consisting of the "countering" transaction(s), i.e., those created by interchanging "buy" ↔ "sell", "long" ↔ "short", etc. in the original transaction(s).
- The "no transaction cost" assumption in the perfect market is useful for ensuring the cash flows at time 0 are indeed $-P$ and $+Q$.
- The "short selling is always possible" assumption is useful for ensuring that the transactions above are indeed doable.
- The rest of them of them are useful for ensuring that the cash flow when closing out all positions at time t is indeed 0.

3.1.4 Here is a corollary of LOOP:

Corollary 3.1.b. Under the no-arbitrage principle and perfect market, a position 📜 with zero payoff at any future time point must "sell" at zero price now.

Proof: Consider this position 📜 and another "position" which is about owning "nothing"/"blank paper". Clearly the latter has zero payoff at any future time also, and must "sell" at zero price now (under no-arbitrage principle). So by LOOP, the result follows. □

[Note: If the (total) cash flow at any time point $t > 0$ involving transaction(s) is 0, the payoff/value at any future time point would be zero (under no-arbitrage principle).]

3.2 No-Arbitrage Forward/Delivery Price

3.2.1 Notations:

- r: the annual risk-free interest rate compounded continuously;
- F_t: forward/delivery price for a forward (with delivery date being time T always) negotiated at time t.

[Note: **Risk-free rate** means the rate of return earned for a investment without risk, i.e., one that *guarantees* a certain future payoff pattern.]

3.2.2 In a perfect market, there is exactly one possible delivery/forward price (for a forward on 1 🍎), given by:

$$F_0 = S_0 e^{rT}.$$

Remarks:

- Although this result is for forwards on "one unit" of underlying asset 🍎, one may modify the unit to apply this result for forwards on any number of underlying asset. Example: "one unit" of 🍎 $= 1000$ apples → the result is applicable for a forward on 1000 apples (S_0 is spot price of 1000 apples).
- Furthermore, by changing the time labelling (i.e., modifying the definition of "now"), this result can still be applied for a forward negotiated at time t:

  $$F_t = S_t e^{r(T-t)}.$$

  In words,

  $$\text{negotiated forward price} = \text{“current” (wrt time } t\text{) spot price} \times e^{r(\text{time length until maturity})}.$$

- The above remarks apply similarly to later results.

Proof: We make use of the corollary 3.1.b. Consider the following strategy:

Time	Transaction	Cash flow
0	borrow F_0e^{-rT}	$+F_0e^{-rT}$
	buy 1 🍎 at spot price	$-S_0$
	short the forward	0
		Total: $F_0e^{-rT} - S_0$
T	sell 1 🍎 at delivery price	$+F_0$
	repay the loan	$-F_0$
		Total: 0

[Note: Any loan is risk-free here since we assume that credit risk does not exist in a perfect market.]

Hence we must have $F_0e^{-rT} - S_0 = 0$, as desired. □

[Note: The general idea in constructing this kind of proof is trying to design the transactions such that the (total) payoff at any future time point is 0. Then, corollary 3.1.b can be readily used.]

3.2.3 The consequence of [3.2.2] is that, in a perfect market, if the forward price F_0 is *not* S_0e^{rT}, there would be an arbitrage opportunity.

3.2.4 Naturally, we would be interested in *how* to capture such arbitrage opportunity in such case. It turns out that the strategies of capturing such opportunities are so "famous" that they have names: "cash-and-carry" and "reverse cash-and-carry".

3.2.5 **Cash-and-carry** (C&C) strategy (for the case where $F_0 > S_0e^{rT}$: forward is "overpriced"):

Time	Transaction	Cash flow
0	borrow F_0e^{-rT}	$+F_0e^{-rT}$
	buy 1 🍎 at spot price	$-S_0$
	short the forward	0
		Total: $F_0e^{-rT} - S_0 > 0$
T	sell 1 🍎 at delivery price	$+F_0$
	repay the loan	$-F_0$
		Total: 0

[Note: We call this "cash-and-carry" since at time 0 we borrow "cash", and then we "carry" 1 🍎 from time 0 to T.]

3.2.6 **Reverse cash-and-carry** (RC&C) strategy (for the case where $F_0 < S_0e^{rT}$: forward is "underpriced"):

Time	Transaction	Cash flow
0	lend F_0e^{-rT}	$-F_0e^{-rT}$
	(short) sell 1 🍎 at spot price	$+S_0$
	long the forward	0
		Total: $S_0 - F_0e^{-rT} > 0$
T	collect proceeds from the loan	$+F_0$
	buy 1 🍎 at delivery price	$-F_0$
	repay the short sale (i.e., return 1 🍎 to the lender 🏛)	0
		Total: 0

[Note: As its name suggests, this strategy is the reverse strategy for cash-and-carry. Indeed, often when we figure out an strategy to capture the arbitrage opportunity from mispricing at a specific direction, we can use its *reverse strategy* to capture the arbitrage opportunity from mispricing at *another* direction.]

3.3 Imperfect Market: Borrowing Rate > Lending Rate

3.3.1 In practice, borrowing rate often exceeds lending rate (resulting in an imperfect market).

3.3.2 Notations:

- r_B: borrowing rate (annual, compounded continuously);
- r_L: lending rate (annual, compounded continuously).

3.3.3 When $r_B > r_L$ (while other assumptions for a perfect market are satisfied), there are more than one possible forward price F_0 (here LOOP is not applicable as the market is imperfect). It can be any value lying in the price interval:

$$[S_0e^{r_LT}, S_0e^{r_BT}].$$

Proof: When $F_0 > S_0e^{r_BT}$ ($F_0 < S_0e^{r_LT}$), the C&C (RC&C resp.) strategy is an arbitrage strategy. [Note: r_B (r_L) is the rate applicable when we borrow (lend) money. So in the C&C (RC&C) strategy, the first transaction becomes "borrow $F_0e^{-r_BT}$" ("lend $F_0e^{-r_LT}$").] □

3.4 Forward on a Stock With Discrete Dividends

3.4.1 Now consider 🍎 as a stock with discrete dividends. The current stock (spot) price is S_0, and the stock 🍎 will make dividend payment 💵 D_{t_i} at *known* time t_i, $i = 1, \dots, n$, where $0 < t_1 < \cdots < t_n < T$.

3.4.2 To "incorporate" the effects of dividends, the (unique) price of forward on 🍎 in a *perfect market* needs to be adjusted to:

$$F_0 = S_0 e^{rT} - \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}.$$

[Note: In words, it is

original F_0 − FV of all dividend payments at time T (at risk-free rate).

]

Proof: Consider the following strategy:

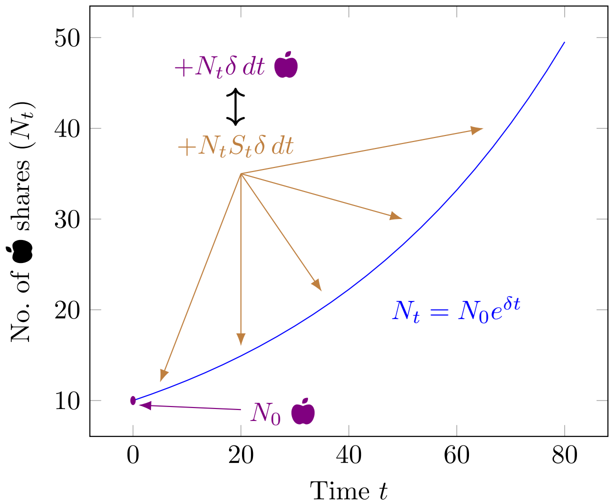
Time	Transaction	Cash flow
0	borrow $e^{-rT}\left(F_0 + \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}\right)$	$+e^{-rT}\left(F_0 + \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}\right)$
	buy 1 🍎 at spot price	$-S_0$
	short the forward	0
		Total: $e^{-rT}\left(F_0 + \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}\right) - S_0$
t_1	receive dividend payment D_{t_1}	$+D_{t_1}$
	lend D_{t_1}	$-D_{t_1}$
		Total: 0
$\vdots$	$\vdots$	$\vdots$
t_n	receive dividend payment D_{t_n}	$+D_{t_n}$
	lend D_{t_n}	$-D_{t_n}$
		Total: 0
T	sell 1 🍎 at delivery price	$+F_0$
	collect proceeds from the loans (the ones at time $t_1, \dots, t_n$)	$+\sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}$
	repay the loan (the one at time 0)	$-F_0 - \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}$
		Total: 0

Hence we must have $e^{-rT}\left(F_0 + \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}\right) - S_0 = 0$, as desired. □

3.5 Forward on a Stock With Continuous Dividends

3.5.1 For mathematical convenience, sometimes we choose to model the dividends as being paid *continuously* (at a rate proportional to the stock price) rather than in a discrete manner. Such rate is called (annual) **dividend yield**, denoted by δ.

3.5.2 The meaning of dividend yield is illustrated below:



More explanation:

- N_t: no. of shares we own at time t
- For every "infinitesimal" time interval $[t, t+dt]$, we receive dividend payment $S_t\delta\,dt$ per shares → total amount we receive is $N_tS_t\delta\,dt$.
- Reinvesting this amount in the stock adds $\dfrac{N_tS_t\delta\,dt}{S_t} = N_t\delta\,dt$ shares to the shares we own →

$$dN_t = N_t\delta\,dt \implies \frac{dN_t}{dt} = N_t\delta.$$

- Solving this ODE gives $N_t = N_0e^{\delta t}$.

[Note: We shall assume automatic dividend reinvestment in this notes. So we would own more and more shares as time passes, when has continuous dividends.]

3.5.3 In a perfect market, there is exactly one possible delivery price (for a forward on 1), given by:

$$F_0 = S_0e^{(r-\delta)T}.$$

<u>Proof</u>: Consider the following strategy:

Time	Transaction	Cash flow
0	borrow F_0e^{-rT} buy $e^{-\delta T}$ at spot price short the forward	$+F_0e^{-rT}$ $-S_0e^{-\delta T}$ 0 Total: $F_0e^{-rT} - S_0e^{-\delta T}$
$t \in (0, T)$	receive dividend payments continuously (which are being reinvested in)	0
T	sell 1 at delivery price repay the loan	$+F_0$ $-F_0$ Total: 0

[Note: The number of we own accumulates from $e^{-\delta T}$ at time 0 to 1 at time T.] This means

$$F_0e^{-rT} - S_0e^{-\delta T} = 0,$$

as desired. □

3.6 Forward Price in the Presence of Storage Cost

3.6.1 For some assets (e.g., diamond 💎), there are storage costs.

3.6.2 When an investor 👤 short sells an asset 💎 with storage costs (then 👤 needs to borrow 💎 from a third party 🏛), 🏛 needs to pay to 👤 the storage costs that would normally be incurred on the shorted 💎 (since the owner of 💎 (🏛), not the short-seller, should bear the "responsibility" of paying storage costs).

3.6.3 Consider a forward on 1 💎. Let C be *present value* of all storage costs associated with 💎 (at risk-free rate). Then, by "paying" C now, we can have just enough money to pay all the storage costs (some possibly in the future), by lending C and collecting "parts" of proceeds from the loan at suitable time points.

3.6.4 In a perfect market, there is exactly one possible delivery price (for a forward on 1 💎), given by:

$$F_0 = (S_0 + C)e^{rT}.$$

Proof: Consider the following strategy:

Time	Transaction	Cash flow
0	borrow F_0e^{-rT}	$+F_0e^{-rT}$
	buy 1 💎 at spot price	$-S_0$
	"pay" the storage costs	$-C$
	short the forward	0
		Total: $F_0e^{-rT} - (S_0 + C)$
T	sell 1 💎 at delivery price	$+F_0$
	repay the loan	$-F_0$
		Total: 0

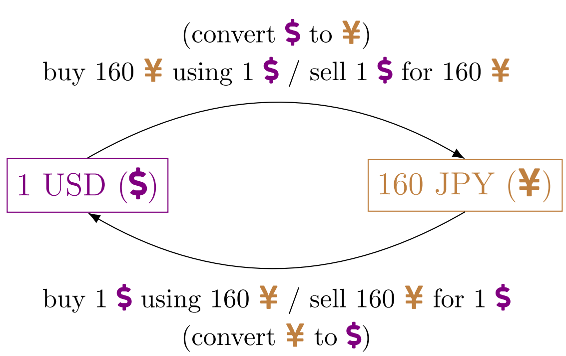
Hence, we have $F_0e^{-rT} - (S_0 + C) = 0$, as desired. □

3.7 Currency Forward

3.7.1 Sometimes the underlying asset of a forward is a certain number of currency €. Such forward is known as a **currency forward**.

3.7.2 Before proceeding further, let us first briefly introduce foreign exchange (forex/FX). The main idea of FX is illustrated below:

USD/JPY: 160 (FX quote convention)



[⚠ Warning: The expression "USD/JPY" may be a bit misleading, since the number it is referring to is indeed how many *JPY* can be converted *per USD*, but "/" is often regarded as "per" (which is not the case here!).

To avoid this, one may consider using some alternative (less popular though) quotations like "USDJPY" or "USD-JPY".]

Terminologies (based on the setting here):

- USD (\$) is the **foreign currency**.
- JPY (¥) is the **domestic currency**.

  [⚠ Warning: The definition of domestic/foreign currency is not related to where you live! Instead, it only depends on the format of the FX quotation.]
- **Exchange rate** is the price of one unit of foreign currency in terms of (or *denominated in*) domestic currency. (Here it is 160: It costs 160 ¥ to buy 1 \$.)

3.7.3 For a currency forward on a number of €, its forward/delivery price is expressed as an exchange rate (which is called **forward exchange rate**): price of 1 € (foreign currency) denominated in the domestic currency.

3.7.4 Notations:

- r_d (r_f): annual risk-free interest rate of the domestic (foreign) currency, compounded continuously
- (for emphasis) S_t: time-t spot exchange rate (time-t price of one unit of the foreign currency denominated in the domestic currency)

3.7.5 We fix our consideration on a kind of exchange rate (through which the foreign and domestic currencies are fixed), say FOR/DOM (FOR: foreign currency; DOM: domestic currency).

3.7.6 Consider a currency forward on 1 FOR. In a perfect market, there is exactly one possible forward exchange rate (FOR/DOM), given by:

$$F_0 = S_0 e^{(r_d - r_f)T}.$$

[Note: This result is applicable to currency forwards on any number of FOR after some modification. For a currency forward on k FOR, the only possible forward exchange rate (FOR/DOM) is $kS_0 e^{(r_d-r_f)T}$ (using similar argument as the proof below).]

Proof: Consider the following strategy:

Time	Transaction	Cash flow
0	borrow $F_0 e^{-r_d T}$ DOM	$+F_0 e^{-r_d T}$ DOM
	buy $e^{-r_f T}$ FOR at spot exchange rate (FOR/DOM)	$-S_0 e^{-r_f T}$ DOM $+ e^{-r_f T}$ FOR
	lend $e^{-r_f T}$ FOR	$-e^{-r_f T}$ FOR
	short the forward	0
		Total: $F_0 e^{-r_d T} - S_0 e^{-r_f T}$ DOM
T	collect proceeds from loan (lending rate: r_f)	$+1$ FOR
	sell 1 FOR at forward exchange rate (FOR/DOM)	-1 FOR $+ F_0$ DOM
	repay the loan (borrowing rate: r_d)	$-F_0$ DOM
		Total: 0

Thus, $F_0 e^{-r_d T} - S_0 e^{-r_f T} = 0$, as desired. □

3.7.7 Another perspective (as a special case of [3.5.3]): an unit of "FOR" may be viewed as a share of "stock" with continuous dividends, and the forward is denominated in DOM:

- We treat the continuous *interest* payments (from lending FOR) as "dividend" payments.

- The dividend yield is r_f since the number of "FOR" we own at time t (N_t) is $N_0 e^{r_f t}$ (the "dividends" are already in terms of FOR, hence "reinvested" automatically).
- Since the forward is denominated in DOM, the risk-free rate r in [3.5.3] is r_d.

As a result, for a forward on 1 FOR ("1 🍎"), the only possible "forward price" (forward exchange rate here) is

$$F_0 = S_0 e^{(r_d - r_f)T}.$$

[Note: An advantage of the proof approach in [3.7.6] is that it also hints how to capture arbitrage opportunity once it arises.]

3.8 Value of a Forward

3.8.1 Recall from [2.1.3] that the time-0 value of any forward negotiated at time 0 is zero. But, the value of such forward at time $t > 0$ may no longer be zero.

Motivating Example

3.8.2 We are in a perfect market and now is time 1. Suppose we longed a forward on 1 🍎 (with no dividends, no storage costs, etc.) at time 0, where $S_0 = 10$, $r = 0.05$, and $T = 10$. Then, we know the forward price negotiated was $F_0 = 10e^{0.05(10)} \approx 16.4872$. "Holding" that forward contract 📜, we are able (obligated indeed) to buy 1 🍎 at the price of 16.4872, at time $T = 10$.

Now, at time 1, the spot price of 🍎 surges 📈 to 1000. So for a forward on 1 🍎 negotiated now, the forward price is $F_1 = 1000e^{0.05(9)} \approx 1568.3122$, which is much higher than F_0.

Then, intuitively, the forward contract 📜 we currently hold (that permits us to buy 1 🍎 at time T with such a low price of 16.4872) becomes very "lucrative", and hence its value at time 1 should be quite high (i.e., we can "sell" 📜 at a high price).

3.8.3 Now consider a "parallel universe" where the spot price of 🍎 falls 📉 to 0.001 at time 1 (🍎 becomes almost worthless!). The forward price for a forward on 1 🍎 negotiated now would be $F_1 = 0.001e^{0.05(9)} \approx 0.001568$, much lower than F_0.

In such case, unfortunately the forward contract 📜 we currently hold (that requires us to buy 1 🍎 at time T with such a high price of 16.4872) becomes a burden. Intuitively, to get rid of this "burden", we need to provide compensation to others. This suggests the value of 📜 at time 1 should be negative.

Formula

3.8.4 After having some intuitive idea about the time-t value of a forward negotiated at time 0, here we give an argument to derive the formula for calculating the value.

3.8.5 Consider a forward 📜 (on 1 🍎) negotiated at time 0, and a time point $t < T$. Then, in a perfect market, the value of the forward 📜 [7] at time t is

$$(F_t - F_0)e^{-r(T-t)}.$$

[7] That is, the value of a long position in the forward.

Proof: Let V_t be its value (or "price") at time t. Then, consider the following strategy:

Time	Transaction	Cash flow
t	borrow $(F_t - F_0)e^{-r(T-t)}$[8]	$+(F_t - F_0)e^{-r(T-t)}$
	short a forward (on 1 🍎) negotiated at time t	0
	long that forward 📜 (negotiated at time 0)	$-V_t$
		Total: $(F_t - F_0)e^{-r(T-t)} - V_t$
T	buy 1 🍎 at the delivery price F_0[9]	$-F_0$
	sell 1 🍎 at the delivery price F_t	$+F_t$
	repay the loan[10]	$-(F_t - F_0)$
		Total: 0

It follows that $(F_t - F_0)e^{-r(T-t)} - V_t = 0$, as desired. □

[8] If $F_t - F_0 < 0$, this is to be replaced by "*lend* $(F_0 - F_t)e^{-r(T-t)}$".

[9] possibly on a loan that is to be repaid immediately by income(s) received from other transaction(s) at time T

[10] This is to be replaced by "collect proceeds from the loan" if $F_t - F_0 < 0$.

4 Options

4.0.1 *Options* are contracts that are somewhat similar to forward/futures. They also allow us to buy/sell an underlying asset at a specific price. The following are the main distinctions:

- right vs. obligation: An option gives us *right* to buy/sell (we have an "option" to buy/sell *or not*), but a forward/future only gives us *obligation* to buy/sell (we *must* buy/sell at delivery date no matter what).
- buy/sell timing: For some options, it is possible to buy/sell even *before* maturity.

4.1 Call and Put Options

4.1.1 There are two basic types of options: call and put options.

4.1.2 A **call option**/**put option** (or simply **call**/**put** (resp.)) gives its holder the *right* (but not the obligation) to buy/sell (resp.) an underlying asset at a specific price (known as **exercise price** or **strike price**), by a certain date (known as **expiration date** or **maturity date**).

[Note: The act of *using* the right given by the option is known as **exercising the option**.]

[Mnemonic : For the call option, we have the option to "call" from someone, and then own it. For the put option, we have the option to "put" our to someone's hand .]

4.1.3 For each of call and put options, it can be *American* or *European*. (This is referred to as its **exercise style**.) Their difference is on the buy/sell timing:

- The holder of an **American** option can exercise at any time at or before the expiration date.
- The holder of an **European** option can only exercise at the expiration date ("like" forward/futures).

[Mnemonic : American: Anytime; European: Expiration.]

[Note: Unless otherwise specified, the options mentioned here are *European*. (European options turn out to be more mathematically tractable.)]

4.1.4 Like forward/futures, we also have the long/short terminology for options. Here the meaning follows the definition in [1.6.1].

- holding/owning a positive amount of call (put) options → *long* position in call (put) option, or in short, long call (put)
- holding/owning a negative amount of call (put) options[11] → *short* position in call (put) option, or in short, short call (put)

4.1.5 Due to the "similarity" to forward/futures, we shall use the notations from [2.2.2] for strike price and expiration date:

- K: strike price ("like" delivery price for forward/futures);
- time T (positive): expiration date ("like" delivery date for forward/futures).

4.2 Payoffs of Call and Put Options

4.2.1 In this notes, we assume that every (American or European) option holder is "rational" (profit-maximizing). More precisely, we assume the right provided by the option can only be used "rationally" and cannot be "mistakenly used".[12]

This may be seen as "reasonable" since in practice, the broker usually helps the option holder to "automatically" decide whether to exercise the option, in such a way that maximizes the holder's benefit → every exercising decision is made "rationally" and no "wrong" exercising is permissible.

[11] That is, selling/**writing** that amount (in absolute value) of call (put) options .

[12] So, in some sense, there is an implicit "restriction" (by rationality) on our choice of using the right or not.

4.2.2 In the context of European options, rationality implies that a holder would exercise the option (at time T) iff a *positive* profit can be earned from exercising at that time (and then closing out all positions).

[Note: For American options, the situation is more complicated. See [7.3.5].]

4.2.3 Further elaboration on [4.2.2]: At time T the holder can choose between:

(a) do not exercise the option;

(b) exercise the option.

Rationality suggests that between these two choices, the option holder would choose the choice with higher "*value*" (cash flow obtained from "closing out" a choice, i.e., performing the (one-time only) action suggested the choice → the "choice" vanishes afterward).

[Note: In case both choices have the same value, we assume that the holder would not exercise the option.]

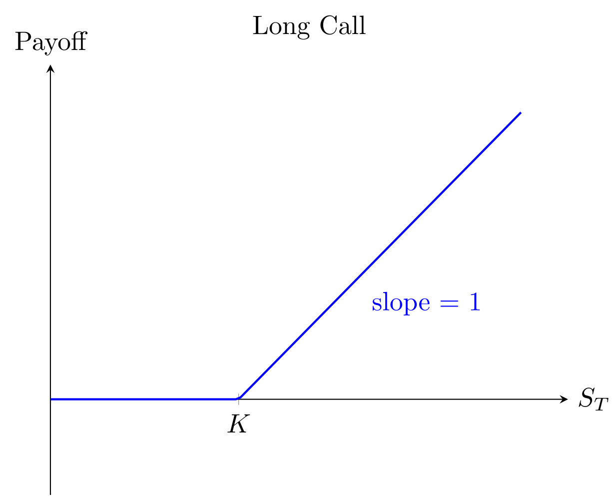
Since the former choice has *zero* value, this means the holder would choose the latter one iff it has a positive value, i.e., it results in a *positive* (> 0) profit (cash flow).

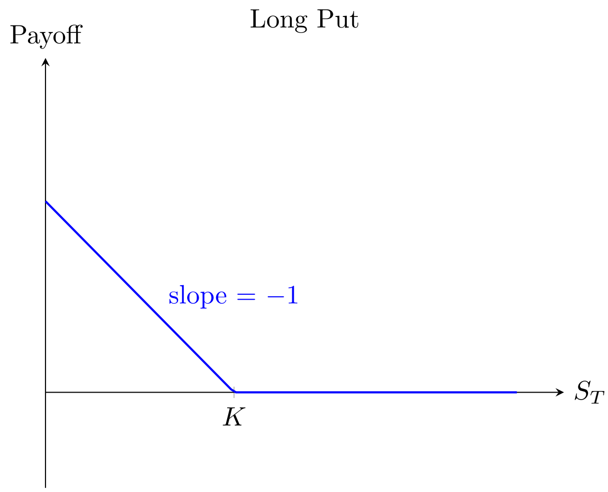
4.2.4 For a call (put) option 📜 on 1 🍎, this means the option would be exercised iff $S_T > K$ ($K > S_T$), which is positive iff $S_T > K$ ($K > S_T$).

Proof: Note that at time T, exercising the call (put) and then closing out all positions — i.e., selling (buying) 1 🍎 at the spot price in this context — results in a total cash flow of $S_T - K$ ($K - S_T$). ☐

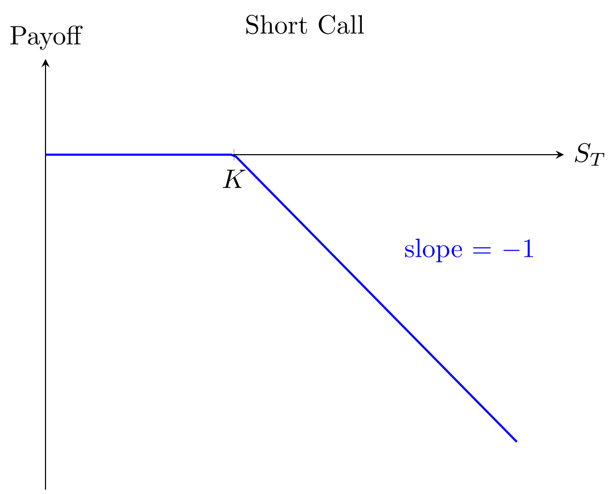
4.2.5 Consequently, the payoff of the *long* call (put) at time T is $(S_T - K)_+$ ($(K - S_T)_+$ resp.), where x_+ is the **positive part** of x, i.e., $\max\{x, 0\}$ (or $x \vee 0$).

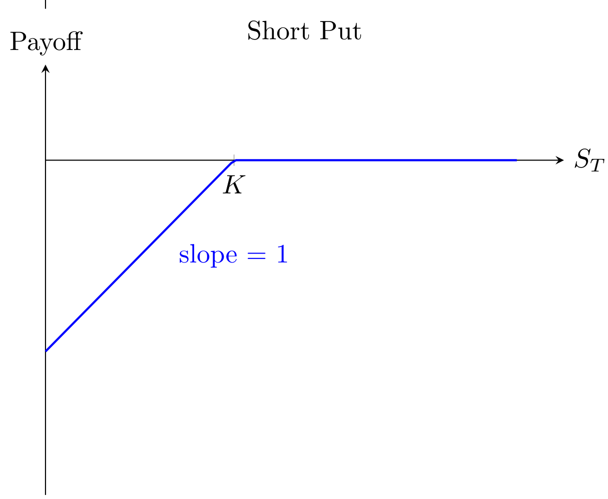
Proof: This immediately follows from the proof for [4.2.4]: The act of exercising and then closing out all positions afterward (or not exercising and letting the option expire) is what we need to close out all positions at time T, and the cash flow obtained from such act is $(S_T - K)_+$ ($(K - S_T)_+$) for call (put). ☐





4.2.6 Then, the payoff of the *short* call (put) at time T is $-(S_T - K)_+$ $(-(K - S_T)_+)$ (negative of the payoff of long call (put)).





4.3 P/L of Call and Put Options

4.3.1 Since the payoff at time T of a long call/put is always nonnegative, the (time-0) *price* (or value) of the call/put option (known as (time-0) **option price** or **option premium**) has to be positive ➔ It costs

$ to take a long call/put. Otherwise, a long call/put is already an arbitrage strategy — It can possibly make a risk-free profit without needing any cash outflow!

[Note: By changing time labelling, the same is applicable to the time-t ($t < T$) price (or value) of the option (still expiring at time T): It has to be positive.]

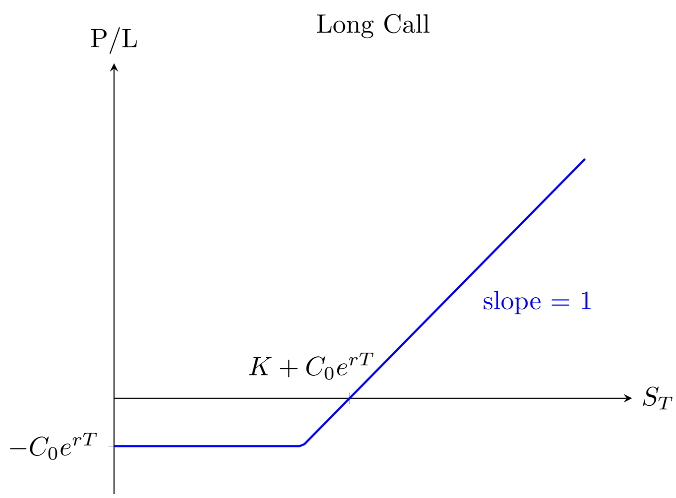
4.3.2 Notations:

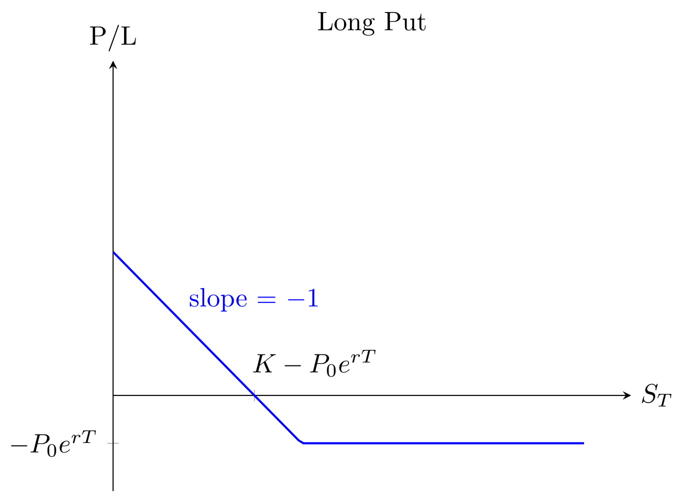
- C_t: call option price at time t;
- P_t: put option price at time t.

Then we have $C_t, P_t > 0$ for any time $t < T$.

4.3.3 Given the call (put) option price at time 0: C_0 (P_0), which gives the only cash flow before time T, we can determine the P/L of a long call (put) at time T:

$$(S_T - K)_+ - C_0 e^{rT} \quad \big((K - S_T)_+ - P_0 e^{rT}\big).$$

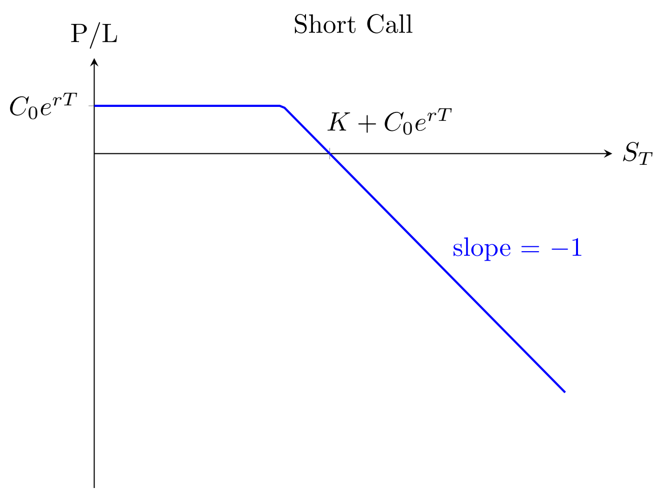


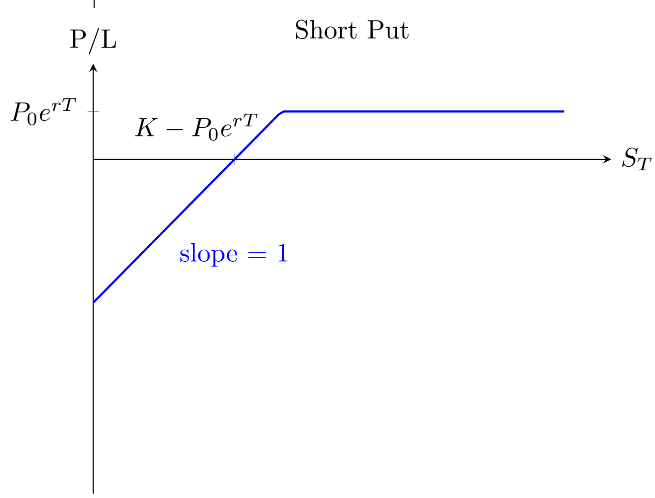


4.3.4 Likewise, the P/L of the short call (put) at time T is the *negative* of the P/L of the long call (put) at time T:

$$-(S_T - K)_+ + C_0 e^{rT} \quad \big(-(K - S_T)_+ + P_0 e^{rT}\big).$$

[Note: To take a short call (put) position (sell/write a call (put)), we *receive* the option price, not pay it. Hence, the cash *outflow* at time 0 becomes negative of the price, i.e., the time-0 cash inflow is the price.]





4.4 Moneyness of Options

4.4.1 **Moneyness** of an option describes whether its payoff (at time 0) would be positive or negative if the holder were forced to exercise the option *immediately* (i.e., at the time of purchase), i.e., the cash flow obtained after immediately exercising the option and closing out the position. [Note: This applies to European option also even though it cannot be exercised before expiration.]

4.4.2 An **in-the-money** (ITM)/**out-of-the-money** (OTM)/**at-the-money** (ATM) option is the one that would have a positive/negative/zero payoff if the holder were forced to exercise the option immediately. [Mnemonic: ITM → "getting money in our pocket"; OTM → "getting money out of our pocket"; ATM → "retaining money at our pocket (no more, no less)".]

4.4.3 For call and put options issued (written) at time 0,

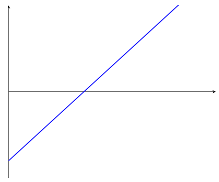
- call is ITM/OTM/ATM $\iff S_0 > K \ / \ S_0 < K \ / \ S_0 = K$ (resp.);
- put is ITM/OTM/ATM $\iff K > S_0 \ / \ K < S_0 \ / \ K = S_0$ (resp.).

[Note: The time-0 payoff if exercised immediately for the call (put) is $(S_0 - K)_+$ $((K - S_0)_+)$.]

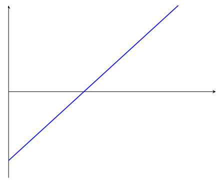
4.5 Strategies for Bullish and Bearish Speculators

4.5.1 For speculators that are *bullish* on, they can:

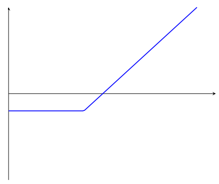
- long 🍎 (buy 🍎 now at spot price)



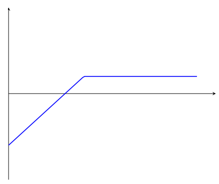
- long forward on 🍎



- long call on 🍎



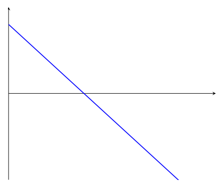
- short put on 🍎



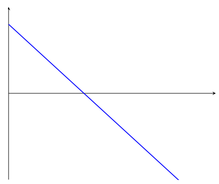
[Note: The P/L graphs of all these positions have an "increasing trend".]

4.5.2 For speculators that are *bearish* on 🍎, they can:

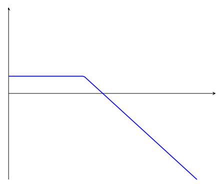
- short 🍎 (short sell 🍎 now at spot price)



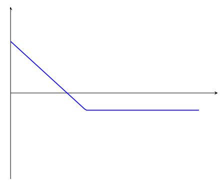
- short forward on 🍎



- short call on 🍎



- long put on 🍎



[Note: The P/L graphs of all these positions have a "decreasing trend".]

4.5.3 More option strategies are discussed in section 5.

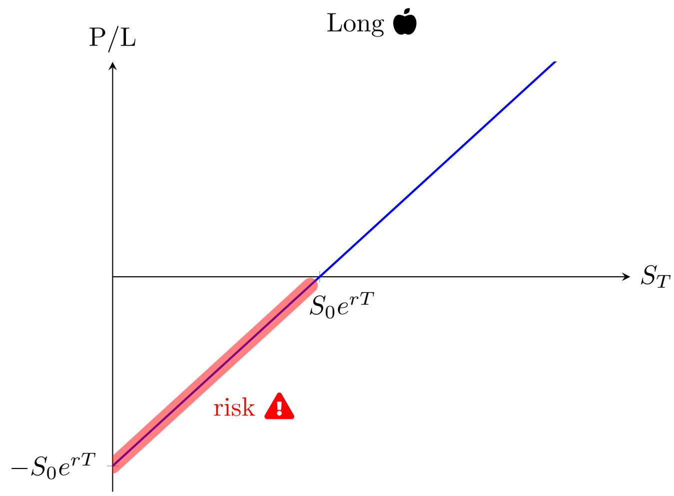
5 Option Strategies

5.0.1 In this section, we use the "basic" option positions: long call (LC), long put (LP), short call (SC), and short put (SP), to "compose" more complex option strategies.

5.1 Floors

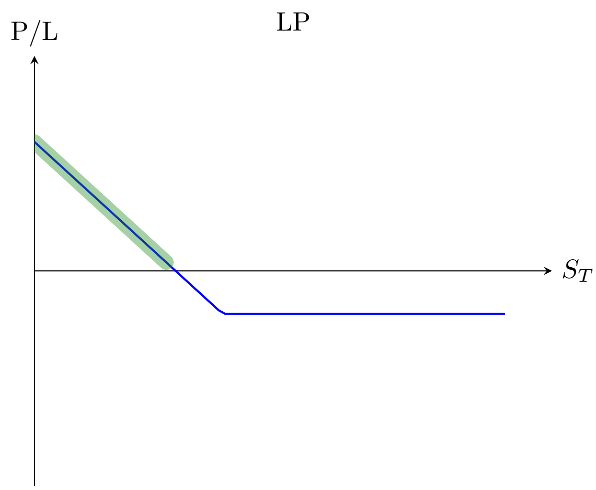
5.1.1 A *floor* is a strategy that insures a long position in an asset 🍎.

5.1.2 P/L (at time T) graph for long 🍎:

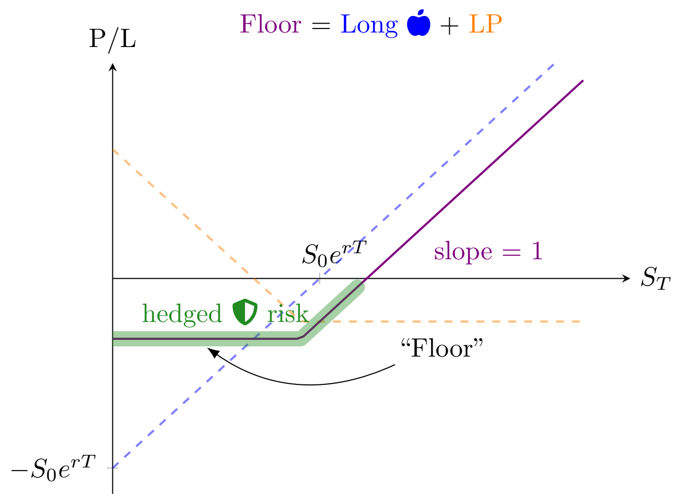


There is a risk for long 🍎: If the price of 🍎 drops 📉 significantly after T years, we would suffer a great loss.

5.1.3 To hedge (reduce) this risk 🛡, we can long a put on 🍎 since the "initial positive" part of the P/L graph for LP can help reducing the risk:



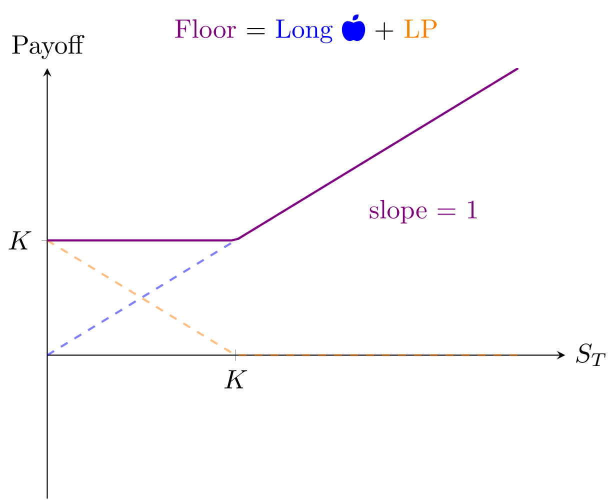
This forms a **floor** (long 🍎 and LP):



5.1.4 The P/L (at time T) of the (long) floor is given by

$$\underbrace{S_T - S_0e^{rT}}_{\text{P/L of long}} + \underbrace{(K - S_T)_+ - P_0e^{rT}}_{\text{P/L of LP}} = S_T + (K - S_T)_+ - (S_0 + P_0)e^{rT}.$$

5.1.5 Based on the P/L of the floor and the no-arbitrage principle, it turns out that we can derive a bound on the put option price P_0. Consider the payoff graph:

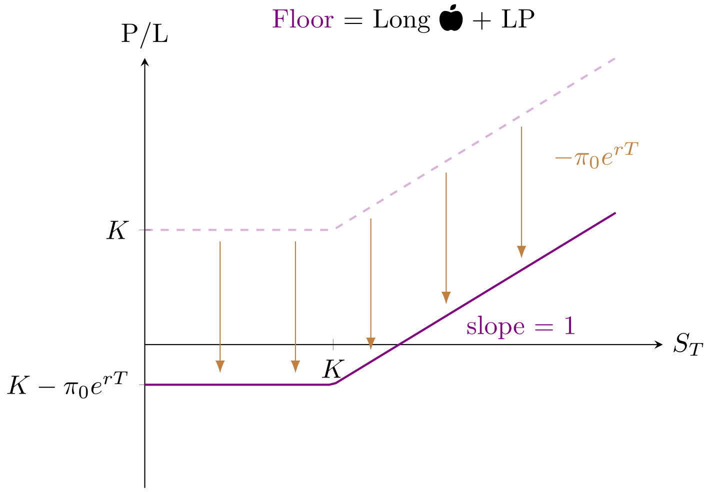


[Note: The payoff of the floor is $S_T + (K - S_T)_+$.]

5.1.6 Let π_0 be the time-0 price of the floor (which is $S_0 + P_0$). Then, its P/L can be expressed as

$$S_T + (K - S_T)_+ - \pi_0e^{rT}.$$

Since $\pi_0e^{rT} > 0$ is a constant with respect to S_T, the P/L graph can also be obtained by shifting the payoff graph *downward* by π_0e^{rT}:

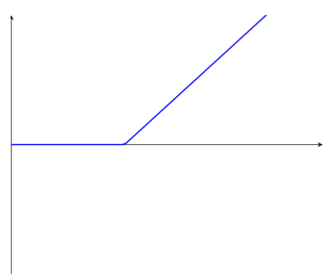


5.1.7 Under the no-arbitrage principle, the P/L cannot be always nonnegative. Hence, we must have

$$K - \pi_0 e^{rT} < 0 \implies P_0 > Ke^{-rT} - S_0,$$

yielding a lower bound of the put option price P_0.

5.1.8 The payoff graph of the floor has similar "shape" as the payoff graph for *long call*:



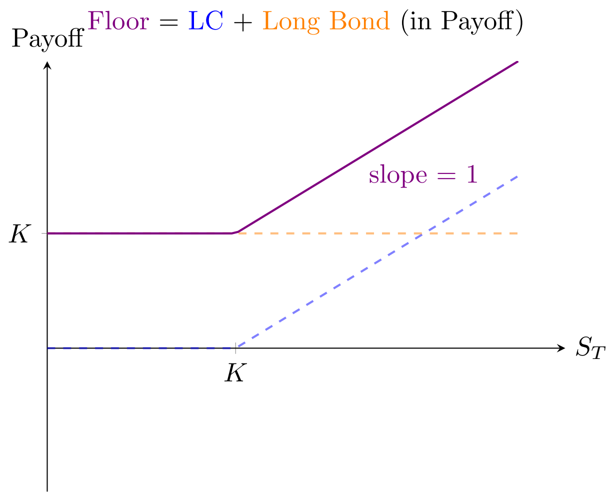
5.1.9 We can observe that the payoff graph of the floor can be obtained by shifting the payoff graph of long call *upward* by K.

Indeed, the floor can also be "composed" using a *long call* on 🍎 and a loan (lending $ risk-free at time 0 such that a cash flow of K can be collected at time T).

[Note: Sometimes the act of lending/borrowing $ (risk-free) is described as *buying*/*selling* a (risk-free zero-coupon) bond 📜 → having a *long*/*short* position in bond 📜.

The amount paid when buying the bond 📜 is the amount *lent* to the seller (borrower). Then, 📜 entitles its owner to later collect proceeds from the bond seller.]

To be more precise, we can "decompose" the payoff graph like below:



Algebraically we can write

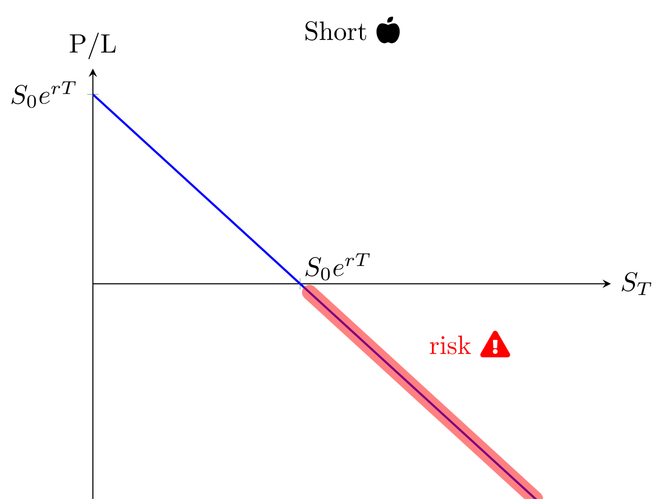
$$\underbrace{S_T + (K - S_T)_+}_{\text{floor payoff}} = \underbrace{K + (S_T - K)_+}_{\text{“LC + long bond” payoff}},$$

Since floor and “LC + long bond” have the same payoff, by the law of one price, their (time-0) prices must be the same. (This yields the *put-call parity*; See section 5.6.)

5.2 Caps

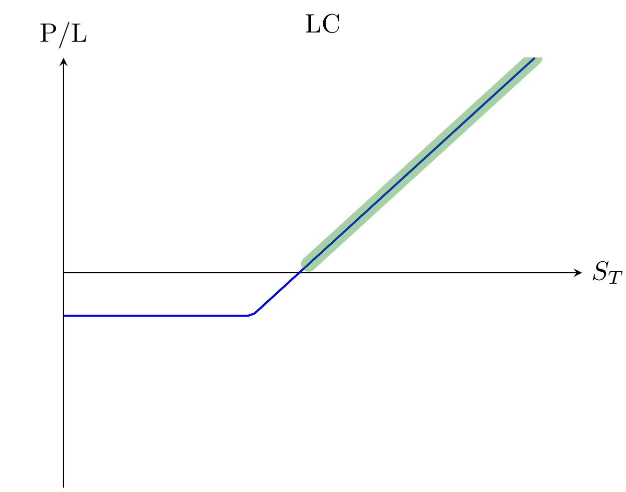
5.2.1 A *cap* is a strategy that insures a short position in an asset.

5.2.2 P/L (at time T) graph for short:

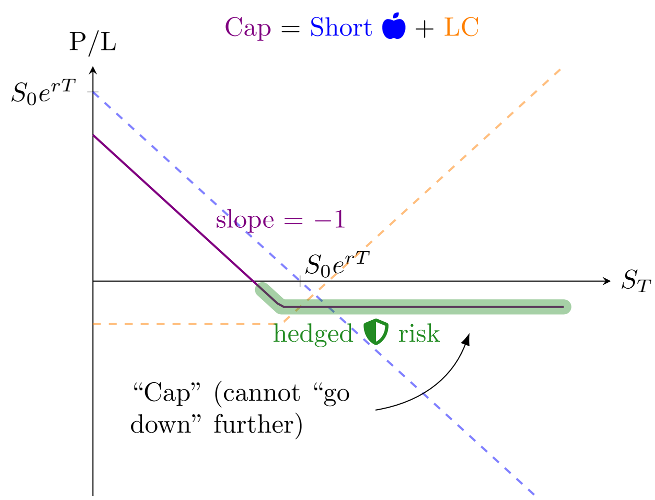


There is a risk for short: If the price of rises significantly after T years, we would suffer a great loss.

5.2.3 To hedge this risk, we can long a call on since the “later positive” part of the P/L graph for LC can help reducing the risk:



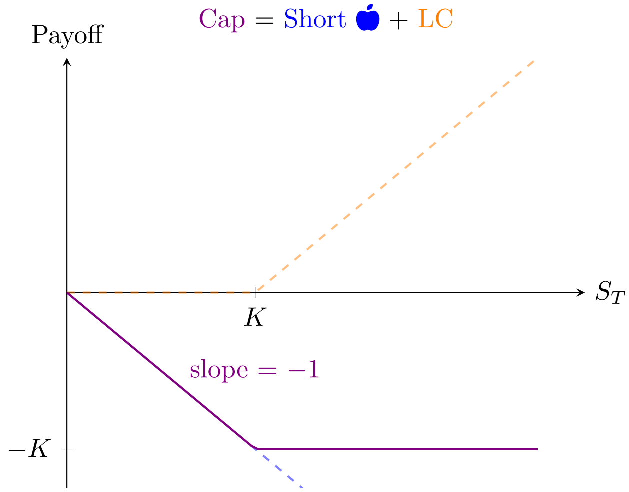
This forms a **cap** (short 🍎 and LC):



5.2.4 The P/L (at time T) of the cap is given by

$$\underbrace{-S_T + S_0e^{rT}}_{\text{long 🍎}} + \underbrace{(S_T - K)_+ - C_0e^{rT}}_{\text{LC}} = -S_T + (S_T - K)_+ - (-S_0 + C_0)e^{rT}.$$

5.2.5 Similarly, based on the P/L of the cap and the no-arbitrage principle, we can bound the call option price C_0. Consider the *payoff* graph:



[Note: The payoff of the cap is $-S_T + (S_T - K)_+$.]

5.2.6 Let π_0 be the time-0 price of the cap (which is $-S_0 + C_0$). Then, its P/L can be expressed as

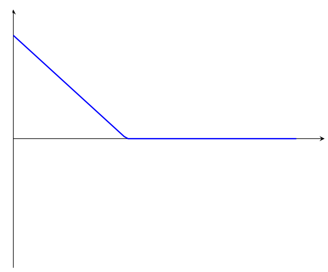
$$-S_T + (S_T - K)_+ - \pi_0 e^{rT}.$$

Under the no-arbitrage principle, the P/L cannot be always nonpositive. (Otherwise, *reverse* cap would have an always nonnegative P/L ➔ arbitrage!) Consequently, we need to shift the payoff graph *upward* to get the P/L graph. Hence, we have

$$\pi_0 < 0 \implies C_0 < S_0,$$

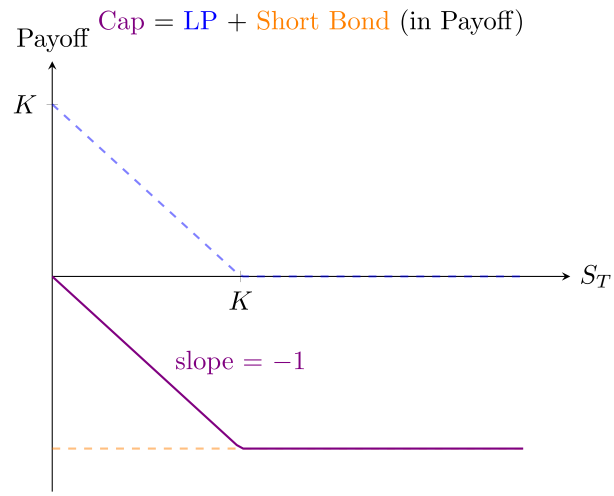
yielding an upper bound of the call option price C_0.

5.2.7 The payoff graph of the cap has similar "shape" as the payoff graph for *long put*:



5.2.8 We can observe that the payoff graph of the cap can be obtained by shifting the payoff graph of long put *downward* by K. So, the cap can also be "composed" using a *long put* on 🍎 and a short bond.

To be more precise, we can "decompose" the payoff graph like below:



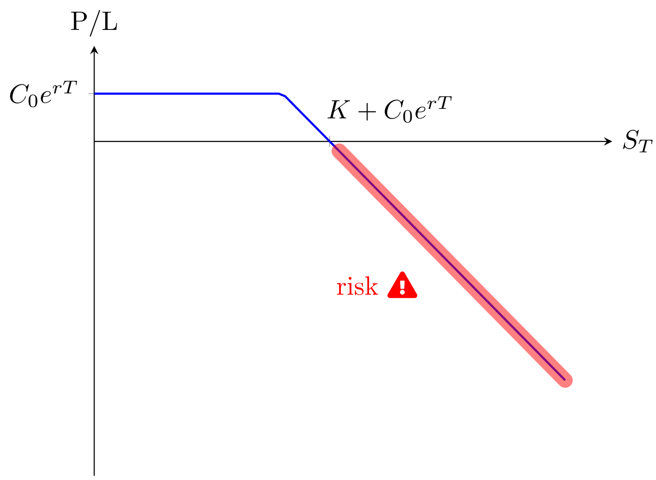
Algebraically we can write

$$\underbrace{-S_T + (S_T - K)_+}_{\text{cap payoff}} = \underbrace{-K + (K - S_T)_+}_{\text{“LP + short bond” payoff}},$$

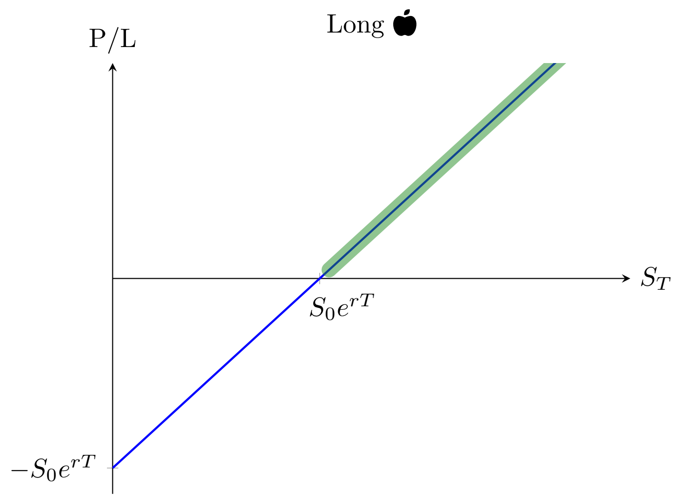
Since cap and “LP + short bond” have the same payoff, their (time-0) prices must be the same by the law of one price.

5.3 Covered Calls

5.3.1 P/L graph of writing a call on 🍎:



5.3.2 To hedge this risk 🛡, we can long 🍎 since the “later positive” part of the P/L graph for long 🍎 can help reducing the risk:



Remarks:

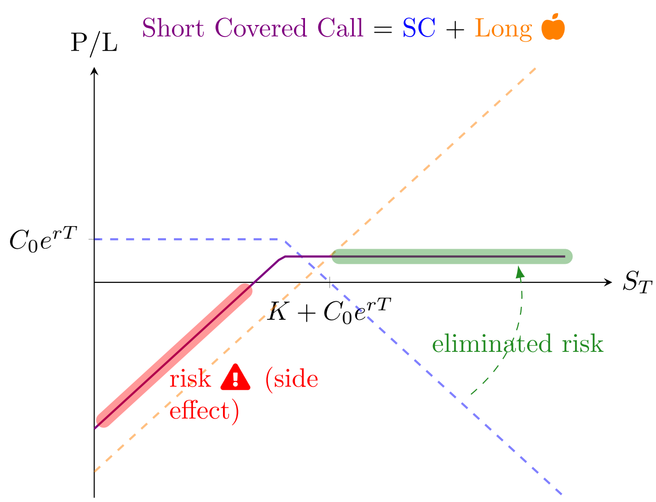
- The P/L graph of LC on 🍎 also has a "later positive" part. But if we long a call on 🍎, we simply close out the position, which is not so interesting.
- "Long forward on 🍎" has the same P/L graph as "long 🍎", so long forward is not that much "different" from long 🍎. Still, we shall consider long 🍎 here.

The act of writing the call *together with* long 🍎 is known as *writing/selling* a **covered call** (on 🍎). In other words,

$$\text{short covered call} = \text{short call} + \text{long 🍎}.$$

[Note: In contrast with *writing a covered call* on 🍎, the act of writing a call on 🍎 without having any position in 🍎 simultaneously is known as *writing* a **naked call** (or **uncovered call**). ("Naked" and "uncovered" have "similar" meaning.)]

5.3.3 The P/L graph of short covered call is:

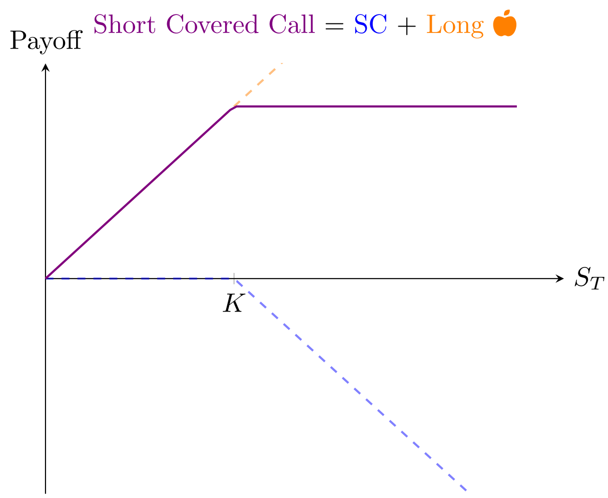


We can note that *another* risk is created as a side effect, unfortunately. However, as this risk is now "limited" (unlike the unlimited potential loss for SC), the situation may be said to be "improved". (In some sense, we are "exchanging" the risk we face for another kind of risk.)

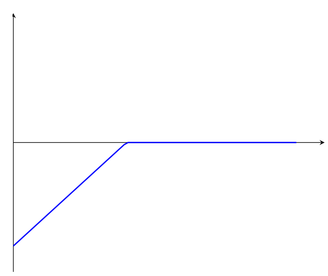
5.3.4 The P/L of the short covered call is

$$\underbrace{-(S_T-K)_+ + C_0e^{rT}}_{\text{SC}} + \underbrace{S_T - S_0e^{rT}}_{\text{long 🍎}} = S_T - (S_T-K)_+ + (C_0 - S_0)e^{rT}.$$

5.3.5 Now, consider its payoff graph:

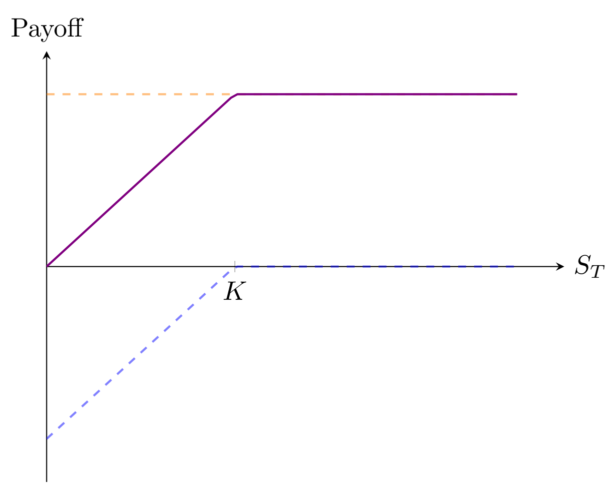


The "shape" of the graph looks like the one for the payoff graph for SP:



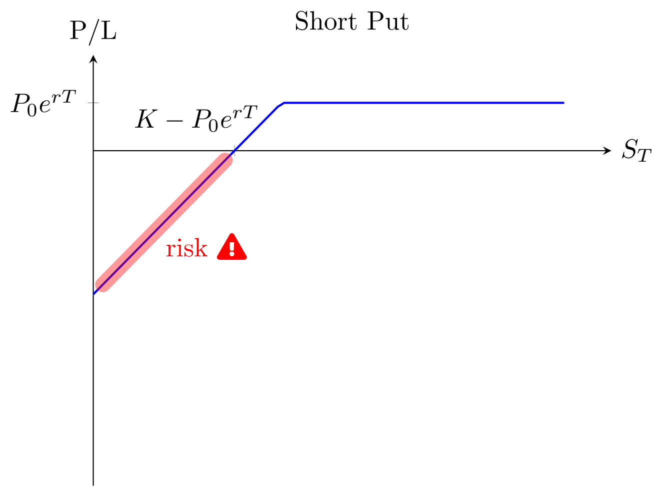
Indeed, short covered call shares the same payoff as "SP + long bond" (hence they have the same time-0 value):

Short Covered Call = SP + Long Bond (in Payoff)

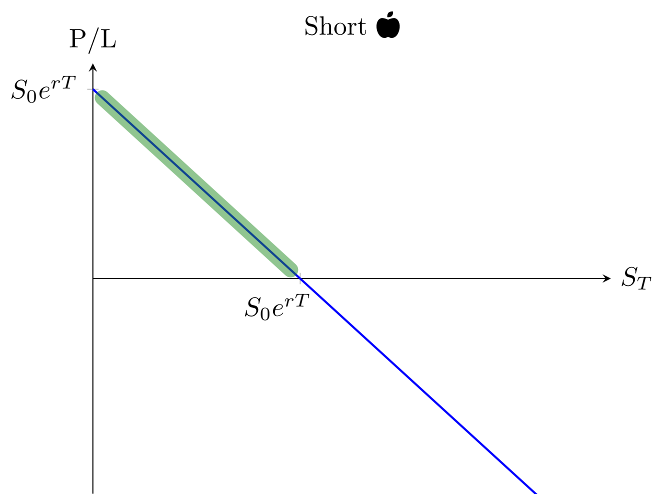


5.4 Covered Puts

5.4.1 P/L graph of writing a put on 🍎:



5.4.2 In a similar manner, to hedge this risk 🛡, we can short 🍎 since the "initial positive" part of the P/L graph for "short 🍎" can help reducing the risk:

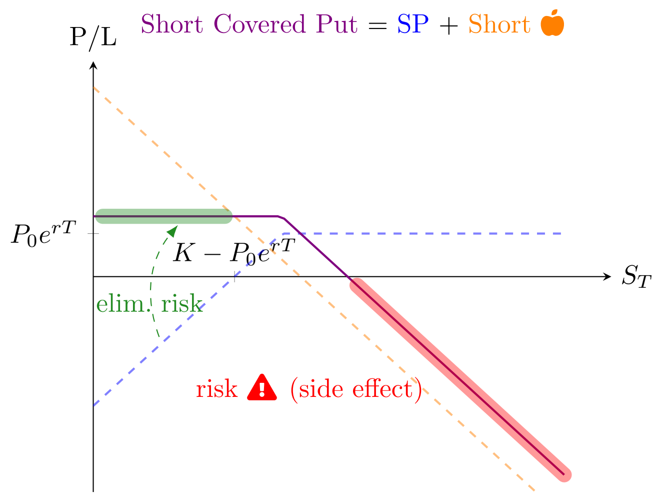


Likewise, *writing* a **covered put** (on 🍎) means writing a put on 🍎 together with short 🍎, i.e.,

$$\text{short covered put} = \text{short put} + \text{short 🍎}.$$

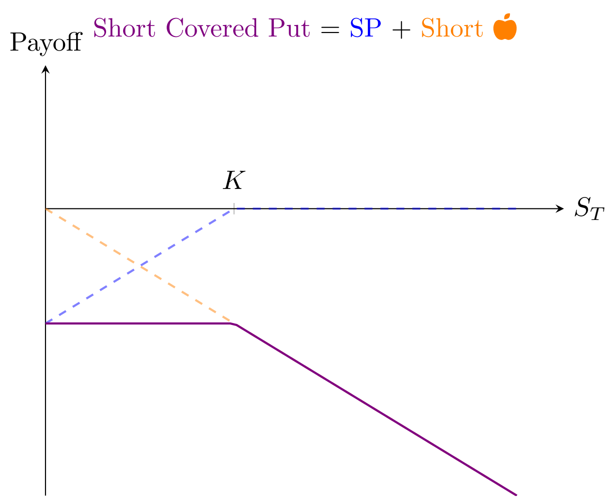
[Note: In contrast with *writing a covered put* on 🍎, the act of writing a put on 🍎 without having any position in 🍎 simultaneously is known as *writing* a **naked put** (or **uncovered put**).]

5.4.3 The P/L graph of short covered put is:

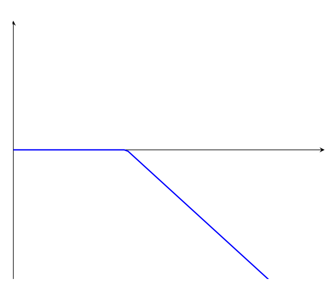


Since the risk in fact changes from *limited* (SP) to *unlimited* (here), the situation may be seen as "worsened" unfortunately. Thus, we seldom write covered put in practice.

5.4.4 Now, consider its payoff graph:

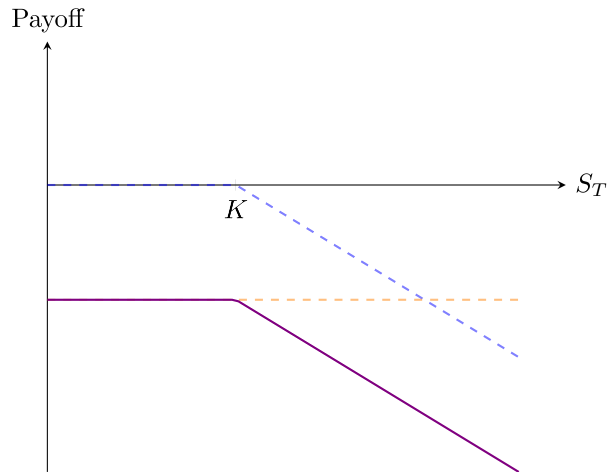


The "shape" of the graph looks like the payoff graph for SC:



Similarly, we can note that short covered put has the same payoff as "SC + short bond" (so they have the same time-0 value):

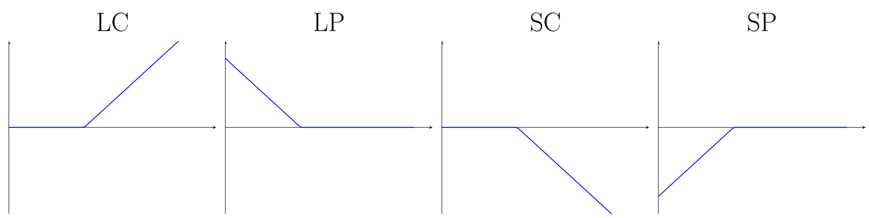
Short Covered Put = SC + Short Bond (in Payoff)



5.5 Synthetic Forwards

5.5.1 Recall that a call/put option on 🍎 gives its holder the *right* (but not the obligation) to buy/sell 🍎. But it turns out by *combining* two option positions, we can convert the "rights" to an *obligation* (forming a *synthetic forward* which mimics a *forward*).

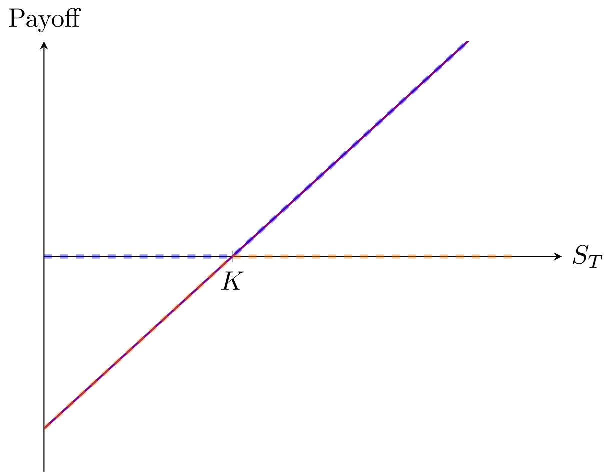
5.5.2 To construct a synthetic forward, consider the payoff graphs of the four "basic" options (LC, LP, SC, SP) on 🍎 (all with strike price K):



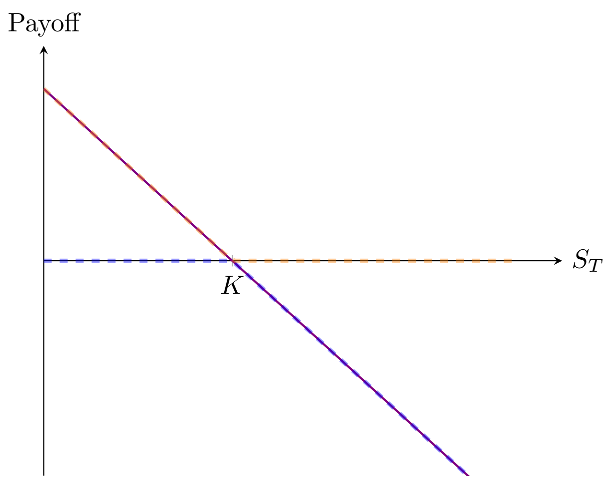
Since the payoff of a long (short) forward is upward (downward) sloping, a synthetic long (short) forward by combining option positions with "upward (downward) sloping" parts, i.e.,

- **synthetic long forward** (@ K) = LC (@ K) + SP (@ K);
- **synthetic short forward** (@ K) = SC (@ K)+ LP (@ K).

Synthetic Long Forward (@ K) = LC (@ K) + SP (@ K)



Synthetic Short Forward (@ K) = SC (@ K) + LP (@ K)



[Note: Algebraically, we can write

$$\underbrace{(S_T - K)_+}_{\text{LC}} + (\underbrace{-(K - S_T)_+}_{\text{SP}}) = \begin{cases} S_T - K + 0 & \text{if } S_T > K; \\ 0 - (K - S_T) & \text{if } S_T \le K \end{cases} = \underbrace{S_T - K}_{\text{LF}}$$

and

$$\underbrace{-(S_T - K)_+}_{\text{SC}} + \underbrace{(K - S_T)_+}_{\text{LP}} = \begin{cases} -(S_T - K) + 0 & \text{if } S_T > K; \\ 0 + (K - S_T) & \text{if } S_T \le K \end{cases} = \underbrace{K - S_T}_{\text{SF}}.$$

]

5.5.3 The synthetic long (short) forward has a "forward price" of K. As we vary the strike price K for the options, we compose synthetic long/short forwards with various "forward prices" (unlike the "genuine" long/short forward where there is only possible forward price).

5.5.4 A main difference between a *synthetic* forward and a *genuine* forward is that the time-0 value of the latter must be zero (by definition), while the time-0 value of the former may not be zero. (Prices of call and put on 🍎 with the same strike price K may not be the same.)

5.5.5 Let $C_0(K)$ ($P_0(K)$) be the time-0 call (put) price on 🍎 (with strike price K). Then, the time-0 value/price of the synthetic long forward @ K is $C_0(K) - P_0(K)$ (a function of K).

5.5.6 The relationship between call (put) price and its strike price is as follows:

Proposition 5.5.a. The call (put) price $C_0(K)$ ($P_0(K)$) is a strictly decreasing (increasing) function in K.

Proof: We focus on the call price here, and the proof for the put price is similar. Assume to the contrary that $C_0(K') \geq C_0(K)$ for some strike prices $K' > K$. Then, consider the following strategy:

Time	Transaction	Cash flow
0	long the call with strike price K	$-C_0(K)$
	short the call with strike price K'	$+C_0(K')$
		Total: $C_0(K') - C_0(K) \geq 0$

Then, the payoff at time T is given by:

Case	LC @ K	SC @ K'	Total
$S_T \leq K$	0	0	0
$K < S_T \leq K'$	$S_T - K$	0	$S_T - K > 0$
$S_T > K'$	$S_T - K$	$S_T - K'$	$K' - K > 0$

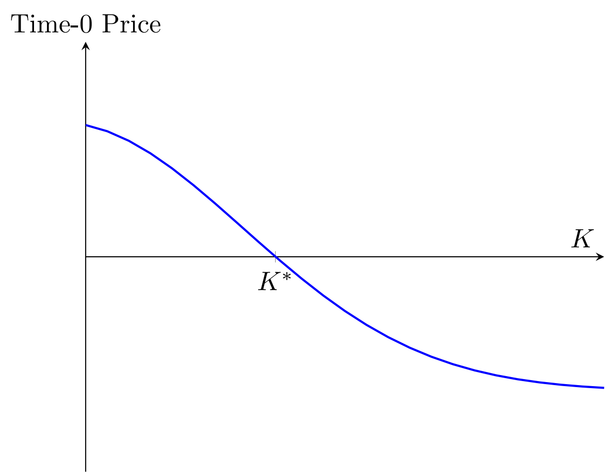
So this is an arbitrage strategy. □

5.5.7 Hence, the time-0 price $C_0(K) - P_0(K)$ is a *strictly decreasing* function of K. We also know that there is a unique solution K^* to the equation

$$C_0(K) - P_0(K) = 0,$$

namely the no-arbitrage forward price.[13]

So the graph of the function would "look like":



[Note: By the strict decreasingness, we have $C_0(K) - P_0(K) > 0$ for any $K < K^*$ and $C_0(K) - P_0(K) < 0$ for any $K > K^*$. Intuition: We need to pay (receive) $ to long a forward with delivery price less (greater) than the "fair" price K^*.]

[13] If the solution is not unique, arbitrage strategy is possible: If there was another solution $K' > K^*$, then synthetic short forward @ K' + "genuine" long forward (@ K^*) → zero price + payoff $K' - K^* > 0$ at time T → arbitrage! Similar for another case.

5.6 Put-Call Parity

5.6.1 Recall from [5.1.9] that the payoffs of a floor (long + LP) and "LC + long bond" are the same, hence they have the same (time-0) price. This yields the *put-call parity*:

Theorem 5.6.a (Put-call parity)**.** We have, under the no-arbitrage principle,

$$C_0 + Ke^{-rT} = S_0 + P_0$$

(for underlying asset with no dividends etc.).

<u>Proof</u>: Note that the price of "LC + long bond (lending Ke^{-rT})" is $C_0 + Ke^{-rT}$, while the price of the floor (long + LP) is $S_0 + P_0$. □

This equation relates prices of call and put on with the same strike price K. (But it does not suggest what their individual prices are.)

5.6.2 The put-call parity can be generalized to be applicable for *any* underlying asset (which may have dividends):

Theorem 5.6.b (Generalized put-call parity)**.** Under the no-arbitrage principle,

$$C_0 + Ke^{-rT} = F_0 e^{-rT} + P_0$$

where F_0 is the no-arbitrage forward price (for a forward on negotiated at time 0).

<u>Proof</u>: Recall that a *synthetic long forward* @ K (LC @ K + SP @ K) has the payoff $S_T - K$, i.e.,

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K.$$

Rearranging it gives

$$(S_T - K)_+ + K = (K - S_T)_+ + \underbrace{S_T}_{S_T - F_0 + F_0}.$$

Note that "LC @ K + long bond (lending Ke^{-rT})" gives the payoff on LHS, and "LP @ K + long forward + long bond (lending $F_0 e^{-rT}$)" gives the payoff on RHS. By the law of one price, their time-0 values are equal, i.e.,

$$\underbrace{C_0 + Ke^{-rT}}_{\text{former}} = \underbrace{F_0 e^{-rT} + P_0}_{\text{latter}}.$$

□

[Note: When the underlying asset does not have dividends etc., we have $F_0 = S_0 e^{rT}$, and the result reduces to theorem 5.6.a.]

5.6.3 Given a set of call and put prices, if the put-call parity is *violated*, then it means that there is an arbitrage strategy.

[Note: To obtain such arbitrage strategy, a general idea is to "buy low and sell high" at time 0. For example, if LHS is less than RHS, at time 0, we perform the transactions "associated" with LHS (LC + long bond; "buy low") and perform the *reverse* of the transactions "associated" with RHS (short bond + short forward + SP; "sell high").]

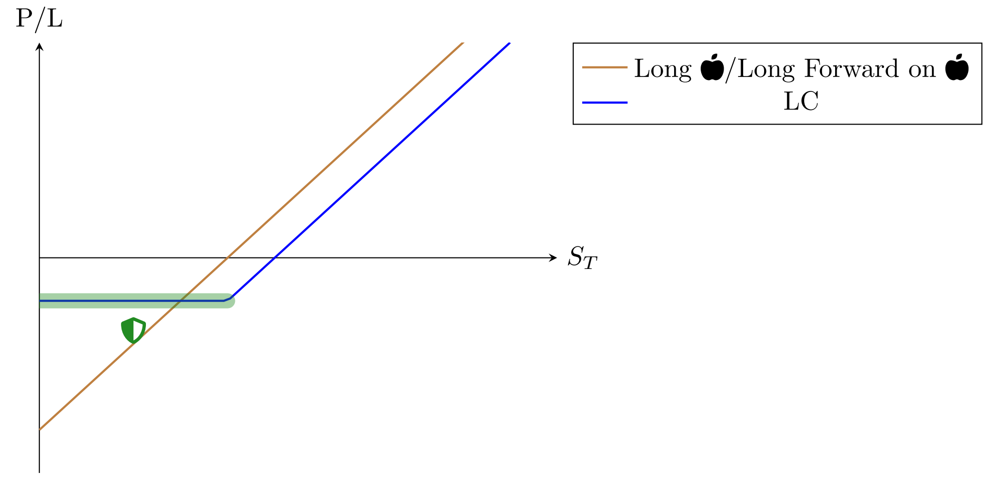
5.7 Bull Call/Put Spreads

5.7.1 An **option spread** consists of long/short options (of the same kind, i.e., call or put) on the same underlying asset , with the same expiration date and exercise style, but with different strike prices (the strike prices are "spread" out).

5.7.2 Recall the strategies for speculators bullish on in [4.5.1]:

- long
- long forward on
- LC on
- SP on

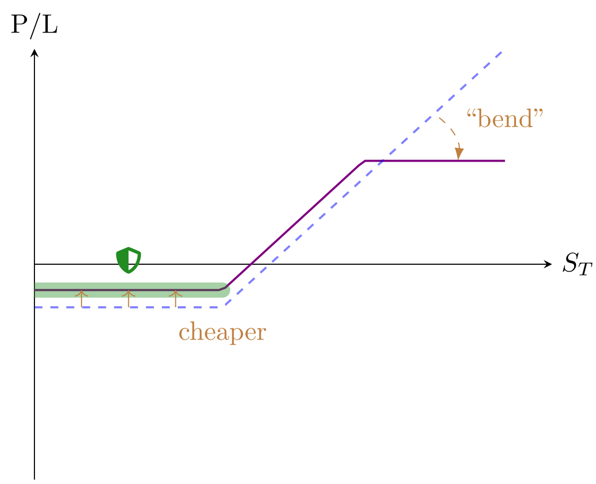
From “long /long forward on ” to “LC on ”, the bullish “extent” drops. (The profit “potential” from the rise 📈 in price of drops, but the strategy becomes “safer”.)



To obtain the protection 🛡 from LC, we need to pay a call option price C_0 at time 0.

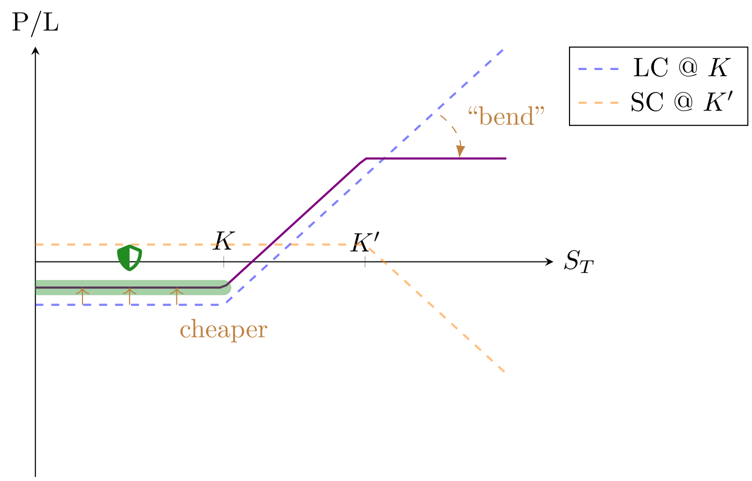
5.7.3 A *bull (call) spread* is a strategy to obtain such protection 🛡 with a *cheaper* price (at the expense of the profit “potential”).

5.7.4 Intuitively, the P/L graph of a bull call spread is obtained by “bending” a “later positive” portion for the P/L graph of LC, in exchange for a cheaper price:



5.7.5 The “bending” is performed by having a SC with a higher strike price K' than the strike price for LC (K), i.e., **bull call spread** is a combination of LC @ K and SC @ K' with $K' > K$:

(K, K')-Bull Call Spread = LC @ K + SC @ K'



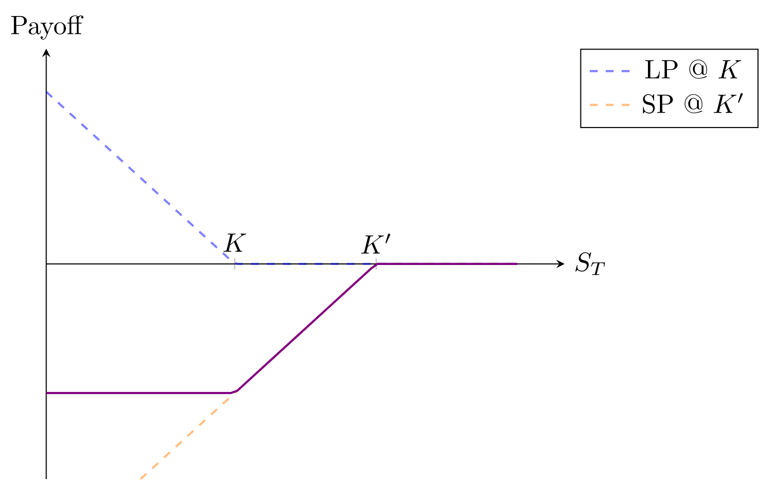
Let $C_0(K)$ and $C_0(K')$ be the prices of the call options with the strike prices K and K' respectively. Then, the price of the bull call spread is $C_0(K) - C_0(K')$ (which is positive by proposition 5.5.a) → cheaper than LC @ K.

5.7.6 The P/L of the bull call spread is

$$(S_T - K)_+ - (S_T - K')_+ - (C_0(K) - C_0(K'))e^{rT}.$$

5.7.7 It turns out that we can obtain the same P/L graph using *put* options instead, and the strategy is known as **bull put spread** (LP @ K + SP @ K'). First we consider its *payoff* graph:

Bull Put Spread = LP @ K + SP @ K'



[Note: The "shape" of this graph is the same as the one for the P/L graph of bull call spread.]

5.7.8 Let $P_0(K)$ and $P_0(K')$ be the prices of the put options with the strike prices K and K' respectively. The time-0 value of the bull put spread is $P_0(K) - P_0(K')$, which is *negative* by proposition 5.5.a.

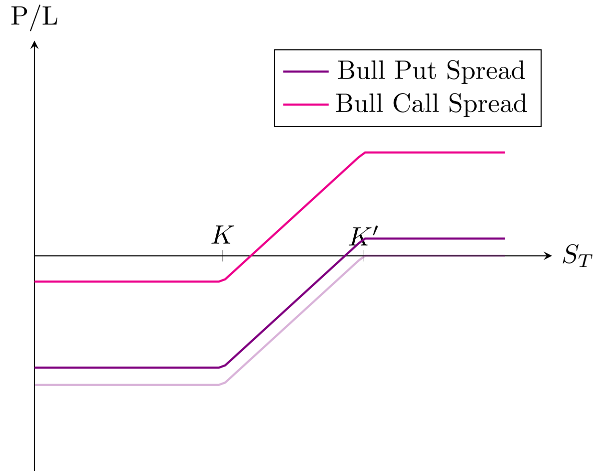
Now, since the P/L of the bull put spread is

$$(K - S_T)_+ - (K' - S_T)_+ - (P_0(K) - P_0(K'))e^{rT},$$

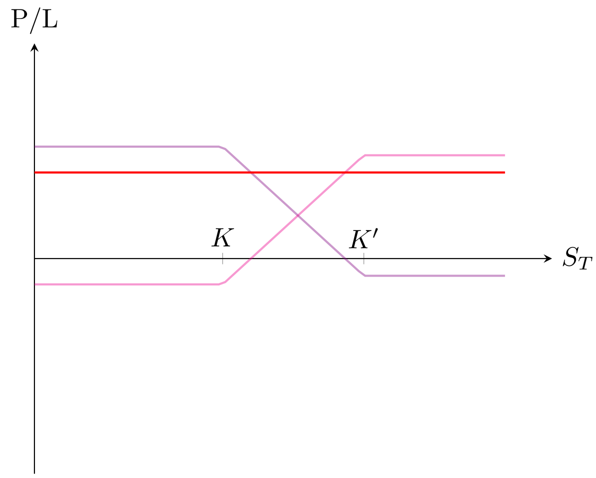
its P/L graph is obtained by from shifting *upward* the payoff graph.

5.7.9 The P/L graph obtained here *must* coincide with the P/L graph for bull call spread, or else there would be an arbitrage opportunity (short "spread with 'higher' P/L graph" and long "spread with 'lower' P/L graph" ➔ "net" P/L is a positive constant *always* ➔ arbitrage![14]):

Impossible!

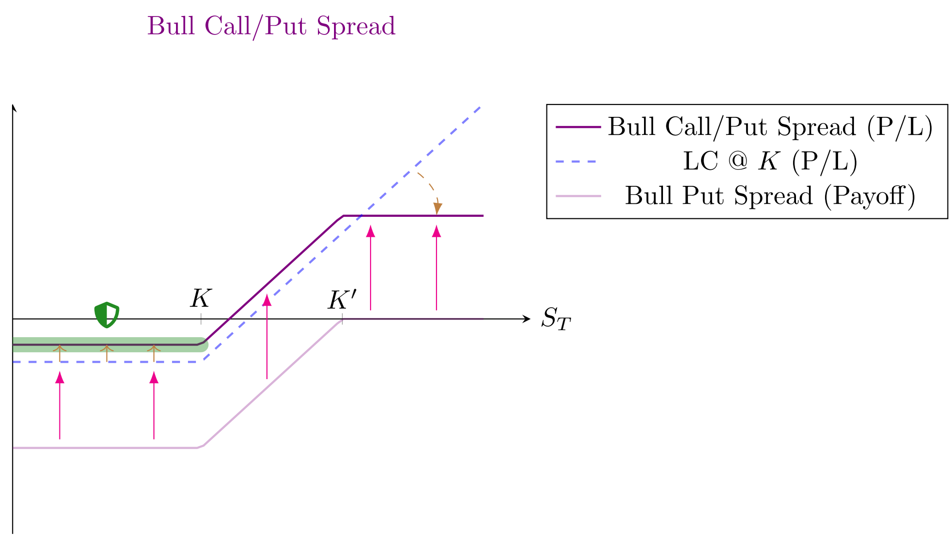


Long BCS + Short BPS = Arbitrage!



The proper P/L graph of bull put spread should coincide with the one for bull call spread:

[14]When P/L is always positive, it means the resulting cash flow at time T after accumulating all past cash flows to that time is *positive* ➔ possible to have no cash outflow but have a positive cash flow at time T ➔ arbitrage!



[Note: Since bull call spread and bull put spread have identical P/L, they are not that much "different". Thus, sometimes we just use the term **bull spread** to refer to either of them.]

5.8 Bear Call/Put Spreads

5.8.1 *Bear call/put spread* is simply *reverse* bull call/put spread, as one may expect.

[Intuition 💡: The *reverse* of a strategy for *bullish* speculators should be for *bearish* speculators.]

5.8.2 More precisely, we have:

- **bear call spread** = SC @ K + LC @ K'
- **bear put spread** = SP @ K + LP @ K'

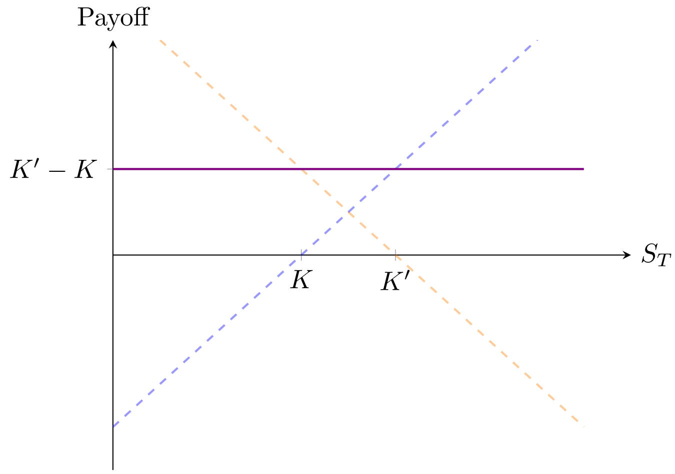
5.8.3 The payoff (P/L) of a bear call/put spread is simply the negative of the payoff (P/L) of the respective bull call/put spread. Hence, we again sometimes just use the term **bear spread** to refer to either of them. [Note: The payoff (P/L) graph of a bear spread is just the mirror image of the payoff (P/L) graph of the respective bull spread along the S_T-axis.]

5.9 Box Spreads

5.9.1 A *box spread* is a "synthetic (risk-free) bond" created using options, i.e., its payoff (at time T) is a positive constant always (and its price is payoff $\times e^{-rT}$).

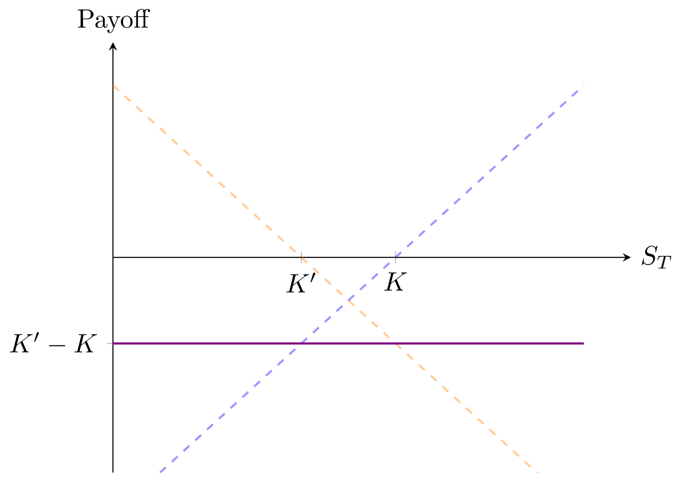
5.9.2 To get such a constant payoff, synthetic forwards are utilized:

Long Box Spread = SLF @ K + SSF @ K'



5.9.3 A **long box spread** (long position in box spread 📦) comprises of a synthetic long forward @ K and a synthetic short forward @ K', where $K' > K$. (It mimics a long position in a risk-free bond.) A **short box spread** is a short position in box spread 📦: It also consists of a synthetic long forward @ K and a synthetic short forward @ K', but $K' < K$:

Short Box Spread = SLF @ K + SSF @ K'



5.9.4 The payoff of a long/short box spread (at time T) is $K' - K$. Hence, its time-0 value is $(K' - K)e^{-rT}$ ➔ its P/L is simply zero always.

5.9.5 To be more explicit about the option positions used in a long/short box spread, we can write:

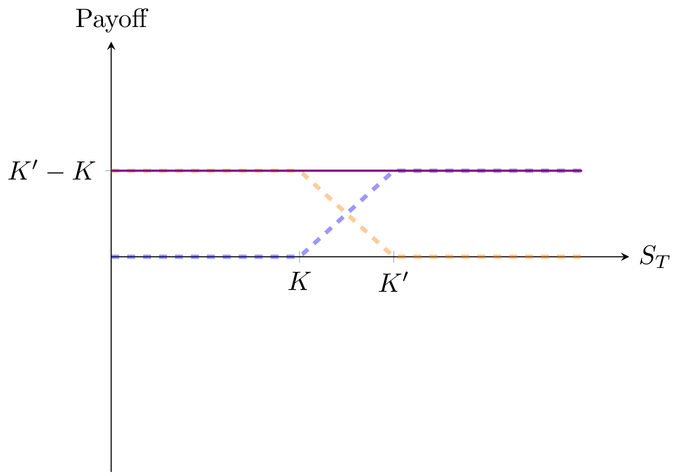
$$\text{long/short box spread} = \text{SLF @ } K + \text{SSF @ } K' = \text{LC @ } K + \text{SP @ } K + \text{LP @ } K' + \text{SC @ } K'.$$

We can collect the option positions used in the form of a table (or *box*):

Strike Price	Call	Put
K	Long	Short
K'	Short	Long

Reading it *horizontally* suggests the construction of box spread using *synthetic forwards*. Reading it *vertically* suggests another way of composing a long box spread using bull and bear spreads: bull call spread @ (K, K') + bear put spread @ (K, K').

Long Box Spread = Bull Call @ (K, K') + Bear Put @ (K, K')



5.10 Ratio Spreads

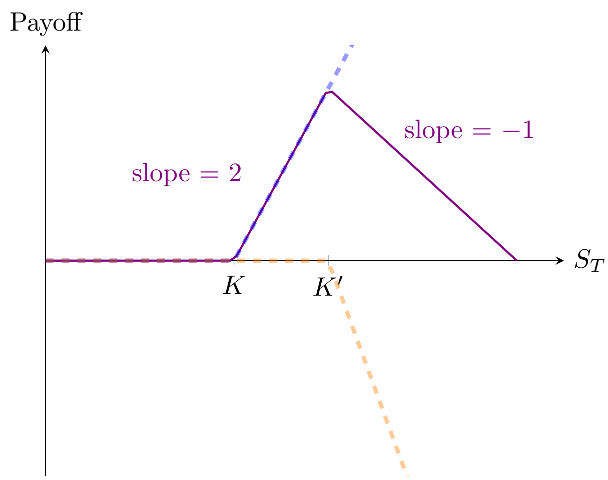
5.10.1 A **ratio spread** consists of longing m calls/puts with one strike price K and shorting n calls/puts (resp.) with another strike price $K' \neq K$, where $m \neq n$.

Remarks:

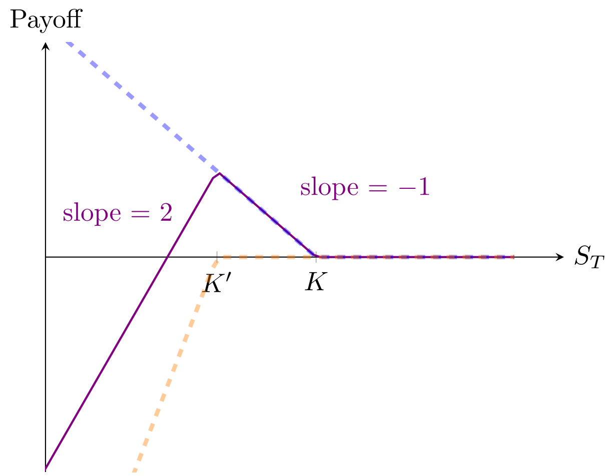
- A ratio spread is similar to bull/bear call/put spread, except that the number of calls/puts longed is different from the number of calls/puts shorted.
- $m : n$ is the "ratio" for the ratio spread.

5.10.2 A ratio spread is a very "flexible" strategy and it can have many different payoff patterns, depending on the ratio and the option type (call/put). Some examples of payoff graphs:

L2C @ K + S3C @ K'



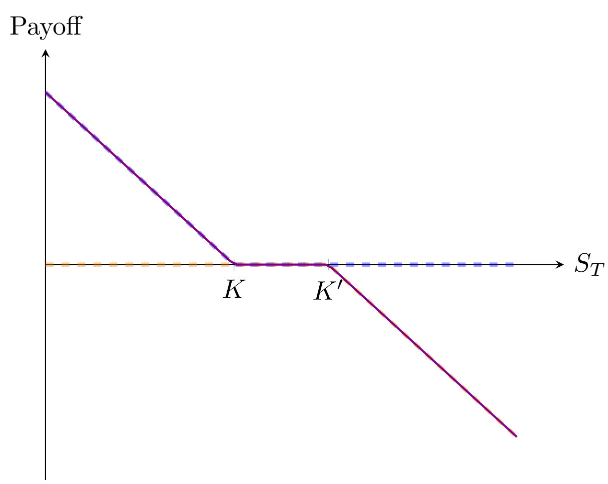
LP @ K + S3P @ K'



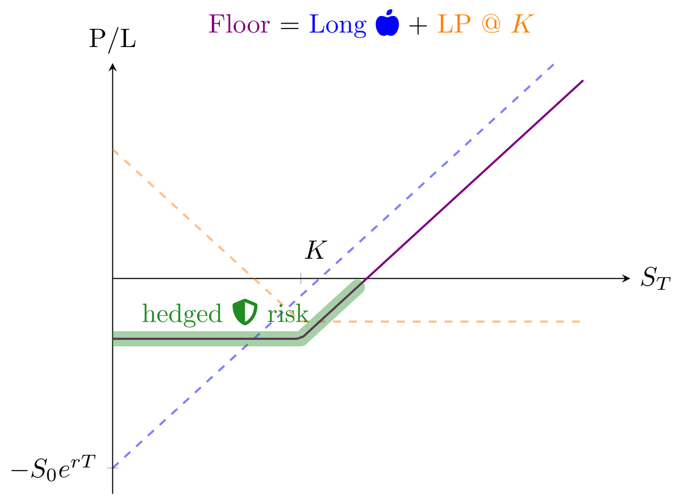
5.11 Collars

5.11.1 A **collar** consists of LP @ K + SC @ K' with $K' > K$ (both options are on the same asset and have the same expiration date):

(K, K')-Collar = LP @ K + SC @ K'



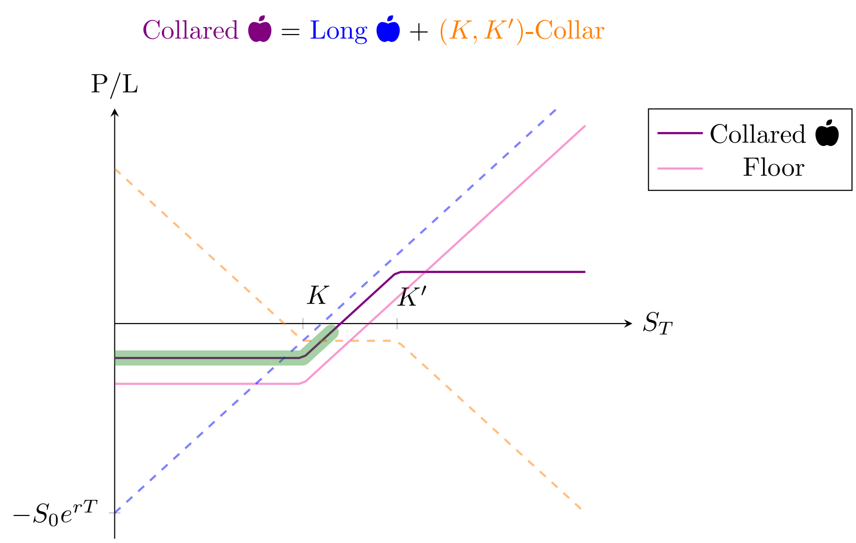
5.11.2 The main usage of collar is to insure a long position in 🍎. Recall that a *floor* is a way to hedge such risk (by adding LP on top of long 🍎). The "initial positive" part of payoff (P/L) of LP helps reducing the risk:



5.11.3 However, for the floor, we need to pay the put option price $P_0(K)$. To reduce the initial expense, a *collar* can be used instead of LP. (Note that the payoff graph of collar also has an "initial positive" part.)

[Note: We call "long 🍎 + collar" as **collared** 🍎. (We usually use this terminology for *stock*: collared stock.)]

Since the time-0 value of a collar is $P_0(K) - C_0(K')$, which is less than $P_0(K)$, this insurance is "cheaper". Of course there is no "free lunch" and the thing we give up is the profit "potential":



After taking a collar, the range of P/L gets restricted to a "narrow" range, just like a physical *collar* put around the neck of an animal that restricts its "movement". By varying K and K' (such that $K' > K$ of course), we can "place" the restriction at different "locations" and control its "strength" (how "narrow").

5.11.4 The time-0 value of a collar $P_0(K) - C_0(K')$ can be positive, negative, or zero (depending on the choice of K and K'). If it is zero, the collar is called **zero-cost collar**.

[Intuition 💡: For zero-cost collar, the "protection" sources completely from the profit potential given up, since we do not pay any money for this insurance.]

5.11.5 However, if the strike price K specified is "too high", then it is impossible to construct a zero-cost collar, as suggested by the following result:

Proposition 5.11.a. If $K \geq S_0 e^{rT}$ (the no-arbitrage forward price), then for any $K' > K$,

$$C_0(K') < P_0(K)$$

(so the time-0 value of the collar is always positive).

Proof: For any $K' > K$,

$$\begin{aligned} P_0(K) &= C_0(K) + Ke^{-rT} - S_0 && \text{(put-call parity)} \\ &> C_0(K') + Ke^{-rT} - S_0 && \text{(proposition 5.5.a)} \\ &\geq C_0(K') + S_0 e^{rT} e^{-rT} - S_0 \\ &= C_0(K'). \end{aligned}$$

□

5.12 Straddles

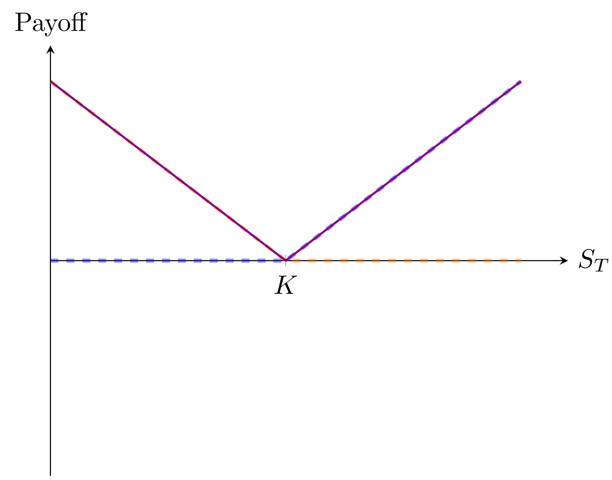
5.12.1 Sometimes speculators are *neither bullish nor bearish*, and they speculate the *volatility* instead. They are not speculating the *direction* of future price movement, but its *magnitude* (large movement → high "volatility"; small movement → low "volatility").

5.12.2 Some option strategies for volatility speculation are discussed in sections 5.12 to 5.14.

5.12.3 A **straddle** is a combination of call and put on the same asset 🍎, with the same strike price and expiration date.

5.12.4 The payoff graph of straddle:

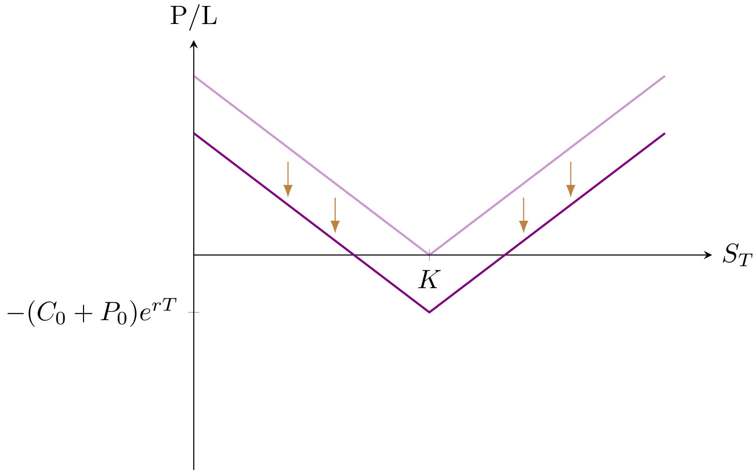
Long Straddle @ K = LC @ K + LP @ K



[Note: The "shape" of the graph looks like the act of "straddling" (sitting astride), from "top view".]

5.12.5 The time-0 price of a straddle is $C_0 + P_0$, which is positive. Hence, its P/L graph can be obtained by shifting its payoff graph downward:

Long Straddle

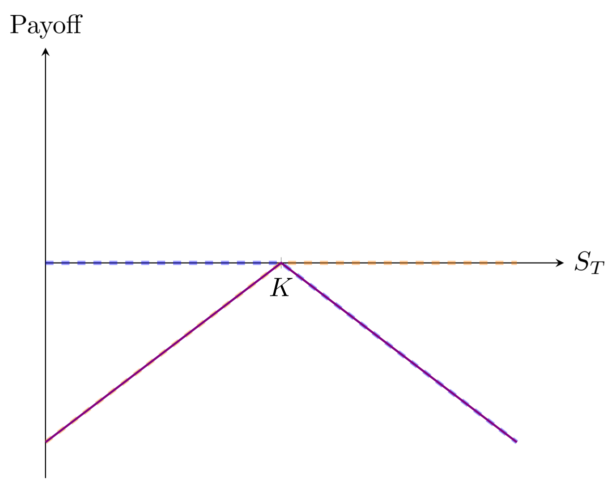


5.12.6 To speculate *high* volatility (large future price movement in either direction), one can long straddle.

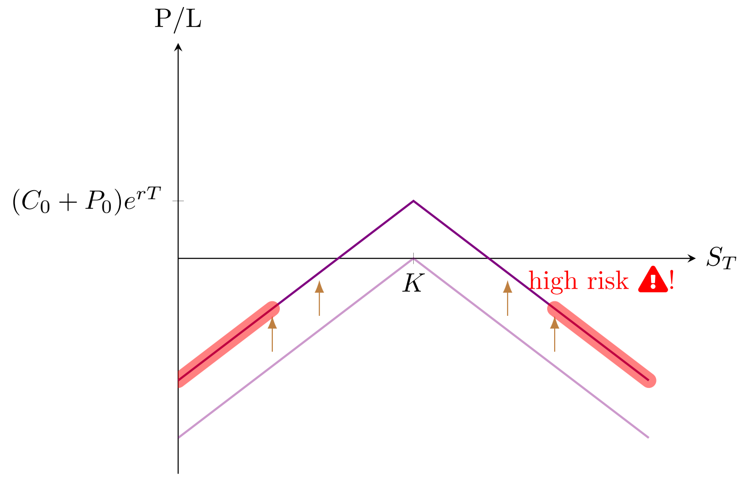
5.12.7 On the other hand, if one want to speculate *low* volatility (small future price movement in either direction), one can *short* straddle (SC @ K + SP @ K).

5.12.8 The payoff and P/L graphs of short straddle:

Short Straddle @ K = SC @ K + SP @ K



Short Straddle



[Warning: A short straddle is highly risky and has *unlimited* potential loss. This strategy caused the collapse of *Barings Bank*!]

5.13 Strangles

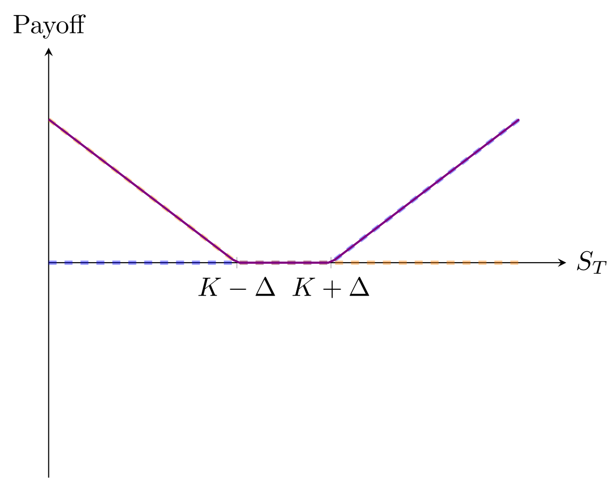
5.13.1 A *strangle* is another strategy for speculating volatility which is "cheaper" than straddle. To achieve a lower cost, call (put) option with higher (lower) strike price is used, through which some profit potential is given up.

5.13.2 A $(K - \Delta, K + \Delta)$-**strangle** is a combination of call and put with strike prices $K + \Delta$ and $K - \Delta$ respectively, on the same asset , with the same expiration date.

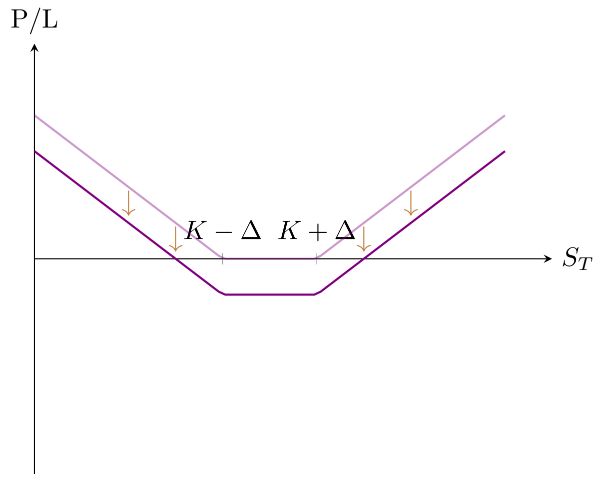
[Note: The time-0 price of the $(K - \Delta, K + \Delta)$-strangle is $C_0(K + \Delta) + P_0(K - \Delta)$, which is lower than the price of the straddle @ K: $C_0(K) + P_0(K)$ since $C_0(K + \Delta) < C_0(K)$ and $P_0(K - \Delta) > P_0(K)$ by proposition 5.5.a.]

5.13.3 The payoff and P/L graphs of long strangle:

Long $(K - \Delta, K + \Delta)$-Strangle = LC @ $K + \Delta$ + LP @ $K - \Delta$



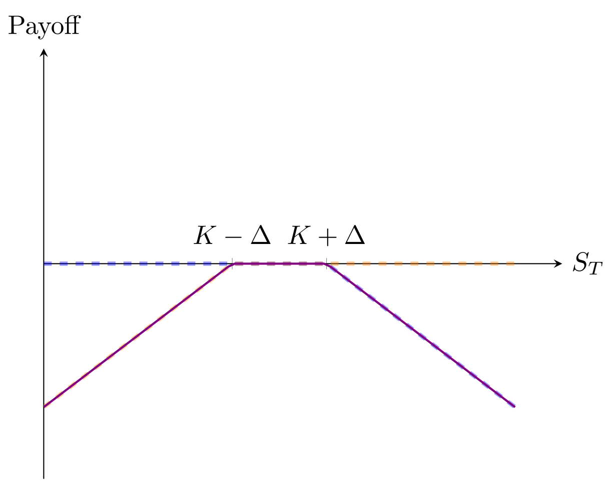
Long Strangle

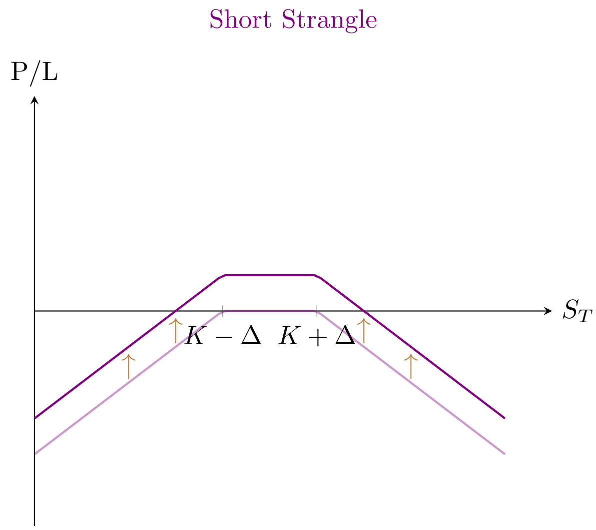


[Note: The payoff (P/L) graph of the long strangle looks like the act of "strangling". (See here for an image.)]

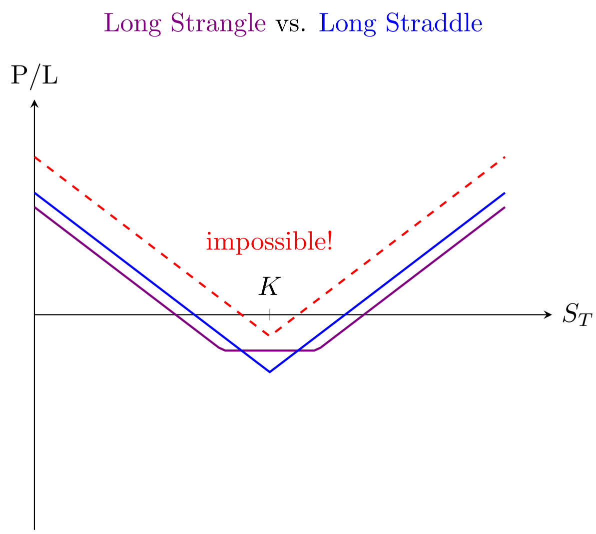
5.13.4 The payoff and P/L graphs of short strangle:

Short $(K-\Delta, K+\Delta)$-Strangle = SC @ $K+\Delta$ + SP @ $K-\Delta$





5.13.5 A comparison of P/L graphs of long strangle and long straddle:

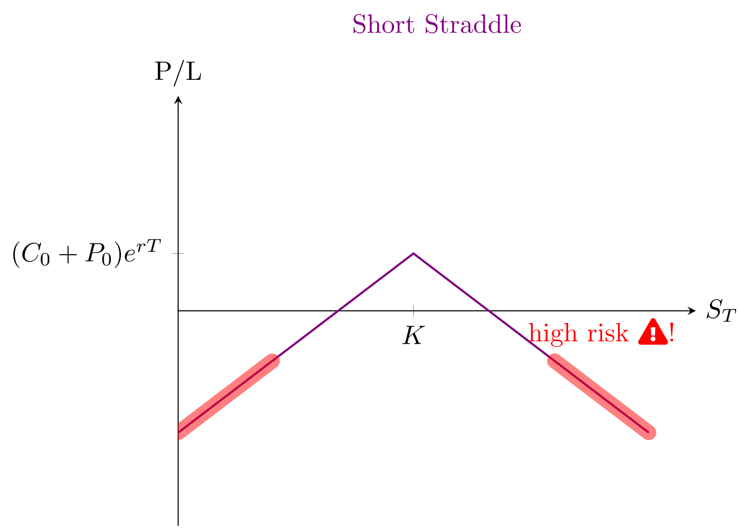


[Note: Under the no-arbitrage principle, the graphs must cross each other. Otherwise, arbitrage would be possible.]

5.14 Butterfly Spreads

5.14.1 Recall that short straddle is highly risky. This suggests a need to *insure* the short straddle, and *butterfly spread* is a strategy that does this.

5.14.2 Recall: P/L graph of short straddle:

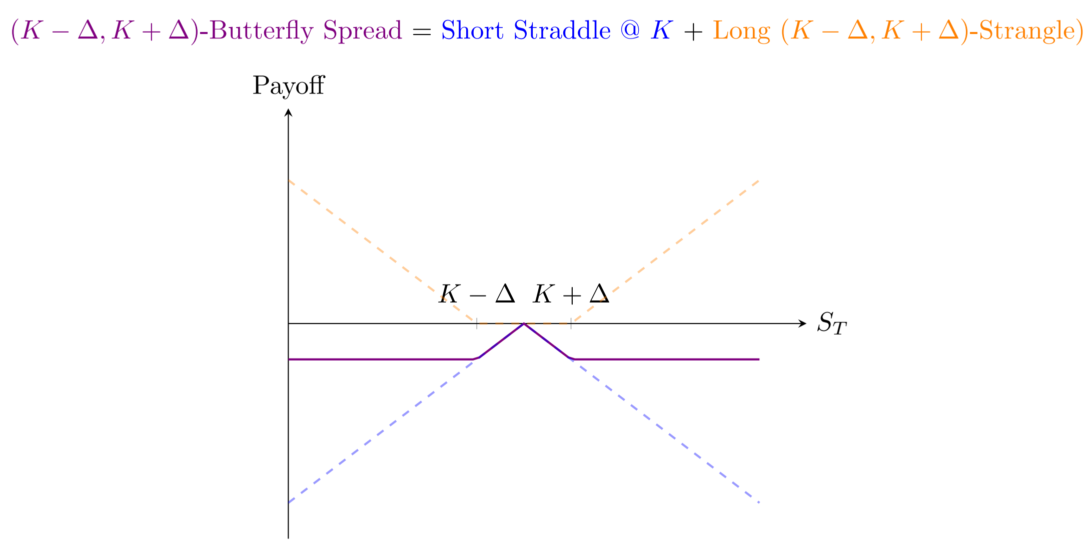


To insure the short straddle, we need *both* “initial positive” and “later positive” parts for P/L, so *long strangle* is a good candidate.

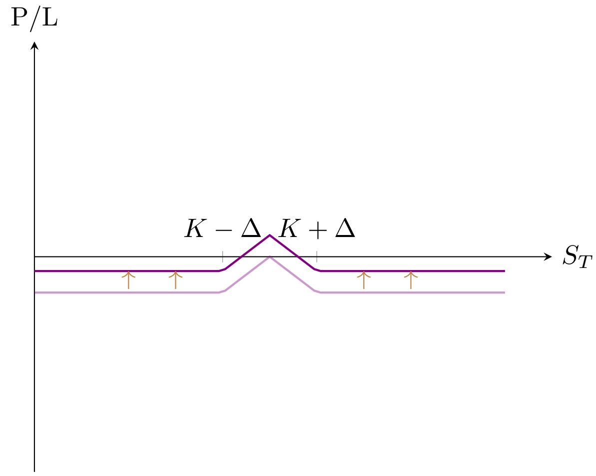
[Note: Long straddle is another choice but it would just close out the short straddle position (if they are both @ K), which is not very interesting.]

5.14.3 A $(K-\Delta, K+\Delta)$-**butterfly spread** consists of “short straddle @ K” + “long $(K-\Delta, K+\Delta)$-strangle”.

5.14.4 The payoff and P/L graphs of butterfly spread:



$(K-\Delta, K+\Delta)$-Butterfly Spread

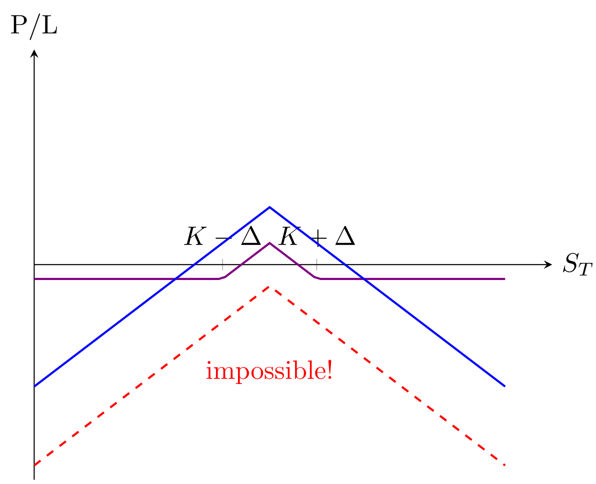


Remarks:

- Either of payoff and P/L graphs looks like a "butterfly" flying away (or towards you).
- The P/L graph is obtained by shifting the payoff graph *upward* since P/L cannot be always nonpositive (alternatively, the strangle is cheaper than the straddle → time-0 value of butterfly spread is negative).

5.14.5 A comparison of P/L graphs of butterfly spread and short straddle:

$(K-\Delta, K+\Delta)$-Butterfly Spread vs. Short Straddle @ K



[Note: Again, under the no-arbitrage principle, the graphs must cross each other (i.e., the short straddle graph cannot be "below" the "butterfly"!).]

5.14.6 A butterfly spread is called a "spread" since we can express butterfly spread as:

$$\text{butterfly spread} = \text{SC @ } K + \text{SP @ } K + \text{LC @ } K+\Delta + \text{LP @ } K-\Delta,$$

so it is indeed a combination of two option spreads:

- SC @ K + LC @ $K+\Delta$;
- SP @ K + LP @ $K-\Delta$.

6 One-Period Binomial Option Pricing Model

6.0.1 In sections 4 and 5, we discuss various aspects of options, but so far we do not have much idea about how to actually *price* the options and option strategies.

6.0.2 Starting from here, we shall discuss some models for pricing them (which is not so straightforward!).

6.1 Binomial Option Pricing Model

6.1.1 In a *binomial option pricing model*, we are assumed to be in a perfect market having (but not limited to) the following two assets:

- a risky stock which pays dividend continuously at a dividend yield δ (if $\delta = 0$, it means there is no dividend)[15];
- a (risk-free zero-coupon) bond with an annual continuously compounded risk-free rate r

[Note: Of course the no-arbitrage principle is still assumed. (It is assumed throughout this notes unless stated otherwise.)]

6.1.2 In a **one-period binomial option pricing model** whose duration is $h > 0$ (years), the spot price of the stock at the end of the (only one) period (time h) is assumed to take exactly *two* distinct possible values (hence "binomial"):

$$S_h = \begin{cases} S_u \triangleq S_0 u & \text{with probability } p; \\ S_d \triangleq S_0 d & \text{with probability } 1-p. \end{cases}$$

where $u > d > 0$, and $p \in (0,1)$[16].

<u>Remarks</u>:

- The values "u" and "d" are sometimes called **growth factors**.
- Here "u" and "d" may be interpreted as suggesting "up" and "down" respectively. But one should be careful that "down" here may not mean the price goes down, i.e., S_d is lower than S_0: It turns out that it is possible for $d \geq 1$.
- The probability measure used here is called the **true**/**real-world**/**physical** probability measure, denoted by $\mathbb{P}$. This is in contrast with the *risk-neutral probability measure* (to be discussed in section 6.3).
- At time $t \in [0, h)$, the stock price S_t is assumed to stay at S_0. When the time reaches h, the stock price "jumps" to S_u or S_d *instantaneously*. This assumption applies similarly for *multi-period binomial option pricing model*, to be studied in section 7.

6.1.3 **Binomial tree** representation of the one-period model:

$$\begin{tikzcd}
\text{Time } 0 & \text{Time } h \\
 & S_u = S_0 u \\
S_0 \arrow[ur, "\text{``up''}"] \arrow[dr, "\text{``down''}"'] & \\
 & S_d = S_0 d
\end{tikzcd}$$

6.1.4 For a **multi-period binomial option pricing model** (discussed in details in section 7) with n periods (or **n-period binomial option pricing model**),

[15] Following the previous convention, we assume the continuous dividends are reinvested into the stock automatically. So if we own N_0 shares of at time 0, the number of shares we own would accumulate to $N_0 e^{\delta t}$ at time t.

[16] The "up" probability can be neither 0 nor 1, so that the stock is indeed "risky".

- the duration of *each* period is a constant $h > 0$;
- the same growth factors u and d and "up" probability p are used in *each* period, and they are applied on the spot price at the *beginning* of the period.

6.1.5 Binomial tree representation of a two-period model:

$$
\begin{tikzcd}
\text{Time } 0 & \text{Time } h & \text{Time } 2h \\
 & & S_{uu} = S_0uu \\
 & S_u = S_0u \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & S_{ud} = S_0ud \\
 & S_d = S_0d \arrow[ur] \arrow[dr] & \\
 & & S_{dd} = S_0dd
\end{tikzcd}
$$

Remarks:

- The number of "u" and "d" in the subscript of "S" suggests the number of "up" and "down" moves (resp.) needed to get to that price (the order of the letters is not important). Sometimes to make the notation more compact, we may express the subscript using power, e.g. $S_{u^8d^2}$.
- The stock price at the latest time point in a binomial option pricing model is known as **terminal stock price**. For an n-period binomial option pricing model, there are $n+1$ distinct possible terminal stock prices.[17]
- As we can see, a multi-period binomial tree can actually be regarded as a combination of multiple one-period binomial trees. Through this connection, methods developed for one-period binomial option pricing model (in section 6) can still be utilized for the multi-period binomial option pricing model (in section 7).

6.1.6 In section 6, we focus on one-period binomial option pricing model.

6.2 Pricing by Replication

6.2.1 Consider a derivative on the stock (whose value is derived from the price of) that matures at time h, and let V_t be its value at time t.

6.2.2 Write its payoff (at time h) as

$$V_h = \begin{cases} V_u & \text{if } S_h = S_0u; \\ V_d & \text{if } S_h = S_0d. \end{cases}$$

$$
\begin{tikzcd}
\text{Time } 0 & \text{Time } h & \text{Payoff} \\
 & S_u = S_0u & V_u \\
S_0 \arrow[ur] \arrow[dr] & & \\
 & S_d = S_0d & V_d
\end{tikzcd}
$$

[17]This is because the subscript of "S" can contain $0, 1, \ldots, n$ "u"s (with the rest being "d"s) ➔ $n+1$ different subscripts.

6.2.3 We are interested in finding the time-0 price/value of the derivative : V_0. One approach for pricing the derivative is called **pricing by replication**, which involves *replicating* the payoff of using just the stock and bond (as we know the prices of these "basic" assets).

6.2.4 At time 0, we consider a portfolio consisting of Δ shares of stock and an amount B in risk-free bond (lending B).

[Note: Negative Δ or B indicates a short position (with absolute value of that amount of stock/in bond), just like the convention for "owning/lending a negative amount".]

6.2.5 Its time-0 value is $\Delta S_0 + B$ and its payoff (at time h) is

$$\begin{cases} \Delta e^{\delta h} S_0 u + Be^{rh} & \text{if } S_h = S_0 u; \\ \Delta e^{\delta h} S_0 d + Be^{rh} & \text{if } S_h = S_0 d. \end{cases}$$

6.2.6 To *replicate* the payoff of , we set the following system of linear equations (in variables Δ and B):

$$\begin{cases} \Delta e^{\delta h} S_0 u + Be^{rh} = V_u \\ \Delta e^{\delta h} S_0 d + Be^{rh} = V_d \end{cases}.$$

Since $u \neq d$, this system can be solved uniquely:

$$\Delta = e^{-\delta h} \frac{V_u - V_d}{S_0(u-d)}, \quad B = e^{-rh} \frac{uV_d - dV_u}{u-d}. \tag{1}$$

The portfolio with these values of Δ and B is called **replicating portfolio** of the derivative .

6.2.7 By the law of one price, since the replicating portfolio and the derivative have the same payoff, they have the same time-0 value:

$$V_0 = \Delta S_0 + B,$$

where Δ and B are given in eq. (1). This gives the pricing formula for the derivative .

6.2.8 The pricing formula in [6.2.7] can be rewritten in the following "discounted expectation" form:

$$\begin{aligned} V_0 &= \Delta S_0 + B \\ &= e^{-\delta h} S_0 \frac{V_u - V_d}{S_0(u-d)} + e^{-rh} \frac{uV_d - dV_u}{u-d} \\ &= e^{-rh} \frac{e^{(r-\delta)h}(V_u - V_d) + uV_d - dV_u}{u-d} \\ &= e^{-rh}[V_u q + V_d(1-q)], \end{aligned}$$

where $q = \dfrac{e^{(r-\delta)h} - d}{u-d}$.

If we "regard" q as the "up" probability (but of course q is not the *true* "up" probability), then the expression is simply the *discounted expected payoff*[18] of the derivative at the risk-free rate. More details are discussed in section 6.3.

6.2.9 Another way to develop the formula in [6.2.7] is to consider another kind of replication: replicating the payoff of a position in *bond* (i.e., a constant payoff) using the derivative and the stock instead.

[18] or expected discounted payoff (*expected present value/actuarial present value* of payoff) if we expand the terms → $V_u e^{-rh} q + V_d e^{-rh}(1-q)$

6.2.10 We consider a portfolio consisting of a *short* position in the derivative and Δ shares of stock . [Intuition : We consider a short position in since from the approach above we know, in payoff,

$$\Delta \text{ shares of } + B \text{ in } = 1 .$$

"Rearranging" it gives

$$\Delta \text{ shares of } \underbrace{-1}_{\text{short position}} = -B \text{ in } .$$

]

6.2.11 Its time-0 value is $\Delta S_0 - V_0$, and its payoff (at time h) is

$$\begin{cases} \Delta e^{\delta h} S_0 u - V_u & \text{if } S_h = S_0 u; \\ \Delta e^{\delta h} S_0 d - V_d & \text{if } S_h = S_0 d. \end{cases}$$

To replicate a *constant* payoff, we need to have

$$\Delta e^{\delta h} S_0 u - V_u = \Delta e^{\delta h} S_0 d - V_d. \tag{2}$$

Rearranging it gives

$$\Delta = e^{-\delta h} \frac{V_u - V_d}{S_0 u - S_0 d}.$$

So, the portfolio consisting of $\Delta = e^{-\delta h} \dfrac{V_u - V_d}{S_0 u - S_0 d}$ shares of and a short position in has also a constant payoff, which is the common value in eq. (2), i.e.,

$$V = \Delta e^{\delta h} S_0 u - V_u = e^{-\delta h} \frac{V_u - V_d}{S_0 u - S_0 d} e^{\delta h} S_0 u - V_u = \frac{uV_u - uV_d - uV_u + dV_u}{u - d} = \frac{dV_u - uV_d}{u - d}.$$

6.2.12 Note that the time-0 value of the position in bond with constant payoff V at time h is Ve^{-rh} (by no-arbitrage principle). Hence, by law of one price, $\Delta S_0 - V_0 = Ve^{-rh}$, which implies

$$\begin{aligned} V_0 &= e^{-\delta h} \frac{V_u - V_d}{S_0 u - S_0 d} S_0 - \frac{dV_u - uV_d}{u - d} e^{-rh} \\ &= e^{-rh} \frac{e^{(r-\delta)h}(V_u - V_d) + uV_d - dV_u}{u - d} \\ &= e^{-rh}[V_u q + V_d(1 - q)], \end{aligned}$$

which is identical to the formula in [6.2.8] (or [6.2.7]).

6.3 Risk-Neutral Pricing

6.3.1 As suggested in [6.2.8], it turns out that the formula for pricing by replication can be expressed in a "discounted expectation" form. Such formula is indeed known as the **risk-neutral pricing formula**, and q is the **risk-neutral probability** of "up".

6.3.2 Before proceeding further, let us first justify that q can indeed be a probability (i.e., its value falls in the interval $[0, 1]$):

Proposition 6.3.a. Under the no-arbitrage principle, the risk-neutral probability $q = \dfrac{e^{(r-\delta)h} - d}{u - d}$ satisfies $0 < q < 1$.

Proof: Firstly, note that

$$0 < q = \frac{e^{(r-\delta)h} - d}{u - d} < 1 \iff d < e^{(r-\delta)h} < u.$$

Now assume to the contrary that $d \geq e^{(r-\delta)h}$ or $u \leq e^{(r-\delta)}h$. We shall only give a proof for the case $d \geq e^{(r-\delta)h}$ as the proof for another case is analogous (just consider reverse of the strategy below).

In this case, we have

$$u > d \geq e^{(r-\delta)h} \implies e^{\delta h} S_u > e^{\delta h} S_d \geq S_0 e^{rh}.$$

Now consider the following strategy:

Time	Transaction	Cash flow
0	short bond (borrow S_0)	$+S_0$
	buy 1 share of stock 🍎	$-S_0$
		Total: 0

The payoffs at time h are given by:

Case	Long 🍎	Short Bond	Total
$S_h = S_u$	$e^{\delta h} S_u$	$-S_0 e^{rh}$	$e^{\delta h} S_u - S_0 e^{rh} > 0$
$S_h = S_d$	$e^{\delta h} S_d$	$-S_0 e^{rh}$	$e^{\delta h} S_d - S_0 e^{rh} \geq 0$

Hence this is an arbitrage. $\square$

[Note: This result also suggests a lower bound on u and an upper bound on d (both are of value $e^{(r-\delta)h}$): The growth factors u and d cannot be too low and too high respectively.

In other words, to construct a one-period model that is consistent with the no-arbitrage principle, these bounds on u and d need to be satisfied.]

6.3.3 Thus, it is reasonable to "treat" q as a probability. To perform risk-neutral pricing, we work in an "imaginary" *risk-neutral world* where the *actual* "up" probability is q rather than p (it is like a "parallel universe"). We shall use the notation $\mathbb{E}_{\mathbb{Q}}[\cdot]$ to denote the expectation in a risk-neutral world.

Using the notation $\mathbb{E}_{\mathbb{Q}}[\cdot]$, we can write the risk-neutral pricing formula in [6.2.8] as

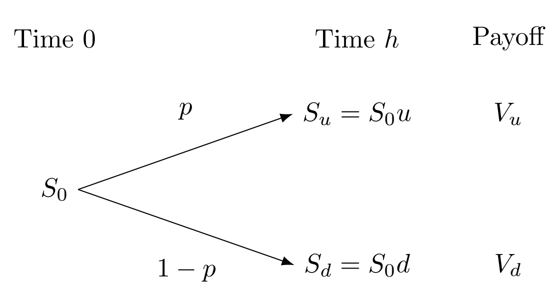
$$V_0 = e^{-rh}\mathbb{E}_{\mathbb{Q}}[\text{terminal payoff}].$$

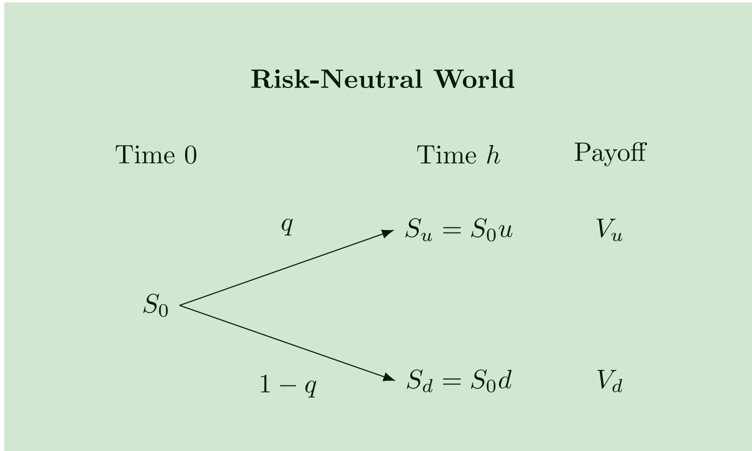
Further Explanation on Risk-Neutral Pricing

(Reference: Lo (2018, Section 4.1.2))

6.3.4 We start with the following two worlds: *real world* (the world we are living in) and *risk-neutral world*.

Real World





6.3.5 For the **risk-neutral world**,

- it consists of the same assets as the real world with identical current stock price and realizable future prices/payoffs;
- all investors are **risk-neutral**, i.e., they only care about the expected returns on their investments, but not their *riskiness* or other characteristics. In other words, the (required) *expected return rate for every asset is the risk-free rate.*

Remarks:

- The expectation is to be taken with respect to a probability measure *different* from the real-world probability measure $\mathbb{P}$, in order to (possibly) achieve such kind of risk neutrality property (every expected return rate is risk-free rate), which does not present in the real world.
- In this binomial tree case, q turns out to serve as an appropriate probability of "up" to achieve the risk neutrality property.

6.3.6 Note that the construction of replicating portfolio for the derivative 📜 does *not* depend on the "up" probability (the expressions in [6.2.6] are free of p). Hence, in *both* real and risk-neutral worlds, we would construct *the same replicating portfolio* for 📜 (i.e., the portfolio with Δ shares of 🍎 and B in bond as suggested in [6.2.6]).

6.3.7 Consequently, by the law of one price, we would have the same time-0 price for 📜 in both worlds (which equals the price of the common replicating portfolio). Therefore, to find the time-0 price of 📜 in real world (what we want), it suffices to find its price in *risk-neutral world* (which equals the real world price) — a key idea in *risk-neutral pricing.*

6.3.8 Since the expected return rate of the derivative 📜 in the risk-neutral world is the risk-free rate, the time-0 price of 📜 (in risk-neutral world) can be easily found by

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{V_h - V_0}{V_0}\right] = e^{rh} - 1 \implies V_0 = e^{-rh}\mathbb{E}_{\mathbb{Q}}[V_h] = e^{-rh}\mathbb{E}_{\mathbb{Q}}[\text{terminal payoff}].$$

This gives the *risk-neutral pricing formula.* [Note: The time-0 price of the derivative V_0 is nonrandom (it is the time-0 price of the replicating portfolio, which is deterministic).]

6.3.9 On the other hand, for the expected return rate of 📜 in the *real world*, it is *higher* than the risk-free rate due to the presence of *risk premium.* (Investors in the real world are not risk-neutral, and most are *risk-averse.* See STAT3904 for more details.)

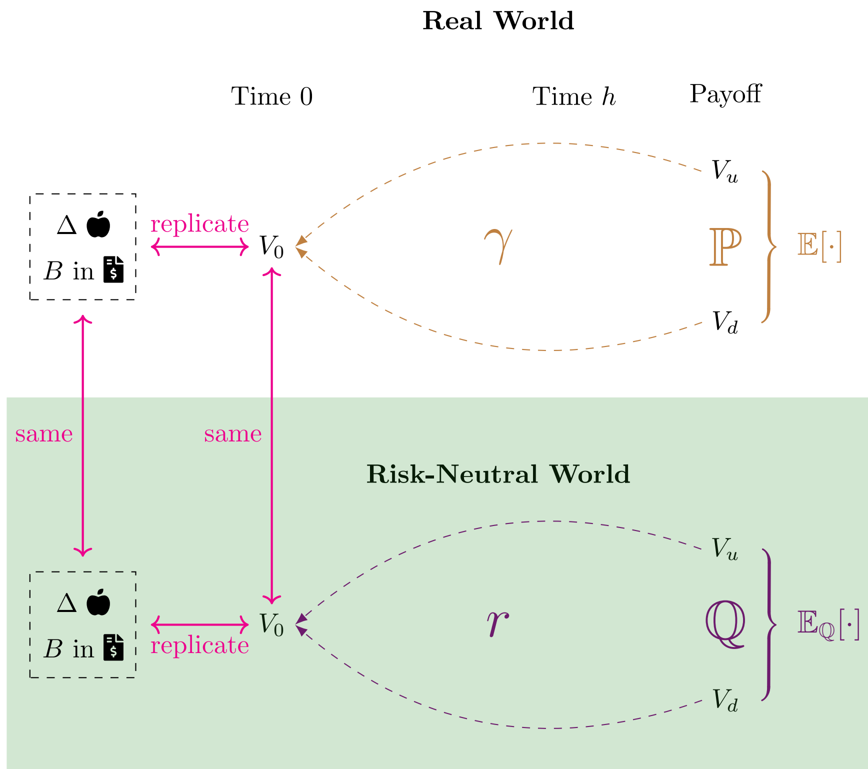
6.3.10 Let γ be the annual continuously compounded expected return rate for 📜 in the real world (which is higher than r). Then, the "real-world pricing formula" is

$$V_0 = e^{-\gamma h}\mathbb{E}[\text{terminal payoff}]$$

(derived using similar argument as [6.3.8]).

6.3.11 So, the key differences between risk-neutral pricing and real-world pricing are the following:

Kind	Expected return rate of	Probability measure	Pricing formula
risk-neutral	r	$\mathbb{Q}$	$e^{-rh}\mathbb{E}_{\mathbb{Q}}[\text{terminal payoff}]$
real-world	γ	$\mathbb{P}$	$e^{-\gamma h}\mathbb{E}[\text{terminal payoff}]$



Risk-Neutral Pricing in a More General Setting

6.3.12 To (possibly) achieve the risk neutrality property, in general we need to "calibrate" the probability measure underlying all probabilistic calculations in the world to get a **risk-neutral probability measure** $\mathbb{Q}$ (calculating expectation under $\mathbb{Q}$ → risk-neutrality property is satisfied).

6.3.13 However, in general there is no guarantee that we can *indeed* find such a risk-neutral probability measure $\mathbb{Q}$ through "calibration": risk-neutral probability measure *may not exist* (risk-neutral world may not be possible to be "constructed"!).

6.3.14 But it turns out that under an arbitrage-free[19] perfect market that consists of a risky stock (which possibly pays dividend continuously) and a risk-free (zero-coupon) bond, there must be a *unique* risk-neutral probability measure $\mathbb{Q}$. [Note: This involves substantial technicalities. See STAT3911 for more details.]

[Intuition: With such assumptions, it turns out that we can always construct replicating portfolio (involving stock and risk-free bond) for any asset. Thus, the expected return rate of any asset can be expressed as a weighted average of the expected (total) return rates of stock[20] and bond.

[19] In general case, we indeed need an even stronger condition than arbitrage-free, but for the models discussed here (binomial option pricing model/Black-Scholes model), arbitrage-free is enough.

[20] More precisely, expected return rate of a portfolio containing a stock. It incorporates both growth in stock price and no. of shares owned due to continuous dividend.

Consequently, to make the expected return rate of every asset being risk-free rate (risk neutrality property), it suffices to find a probability measure such that the expected return rate of risky stock 🍎 is risk-free rate, which turns out to be uniquely determined.]

6.3.15 For illustration purpose, we shall show that there is a *unique* risk-neutral probability measure $\mathbb{Q}$ for this binomial tree case in the following.

Proof: Firstly, in this binomial tree case we already know that replicating portfolio can be constructed for any derivative. Thus, to find a risk-neutral probability measure, it suffices to determine a probability of "up" (call it q^* at the moment) such that the expected (total) return rate of risky stock 🍎 is risk-free rate r.

For this to happen, we need to have

$$\frac{e^{\delta h}[S_0 u q^* + S_0 d(1-q^*)]}{S_0} = e^{rh} \implies (u-d)q^* = e^{(r-\delta)h} - d \implies q^* = \frac{e^{(r-\delta)h} - d}{u-d},$$

which is the q we have found before. As there is only one such possible q^*, there is a unique risk-neutral probability measure. $\square$

6.3.16 Note that a crucial basis of the risk-neutral pricing is the construction of replicating portfolio — So, the argument here is not a *replacement* of pricing by replication. Rather, it is *based on* pricing by replication.

6.4 Constructing Binomial Trees

6.4.1 So far, we treat the growth factors u and d for the binomial tree as given and perform numerous calculations based on them. But in practice we do not have a "given" binomial tree and in order to use the binomial option pricing model, we need to *construct* one (i.e. choosing values of u and d) in some way.

6.4.2 From proposition 6.3.a, we know that to be consistent with the no-arbitrage principle, the values of u and d must satisfy

$$d < e^{(r-\delta)h} < u.$$

But apart from this (and the model assumption that $u > d$), there is no other requirement on what u and d can be.

6.4.3 Since the values of u and d control how "risky"/"volatile" the stock 🍎 is (when u and d are further apart, the stock 🍎 can be seen as more "volatile"), it seems natural to choose u and d based on the inherent "riskiness" of the stock 🍎.

6.4.4 The "riskiness" of the stock 🍎 can be measured by its *volatility*, which indicates the "variability" of the stock price. The **volatility** of the stock 🍎 (denoted by σ) is the *annualized* standard deviation of its (h-year) continuously compounded *price* return[21] $\ln(S_h/S_0)$:

$$\sigma = \sqrt{\frac{1}{h}\operatorname{Var}\left(\ln\frac{S_h}{S_0}\right)}.$$

Remarks:

- The variance $\operatorname{Var}(\ln(S_h/S_0))$ is for a period with duration h years. Dividing it by h gives the *annualized* variance ("nominal" value for the variance for a 1-year period). Taking square root then gives the *annualized* standard deviation.

[21] That is, we consider only the changes in *price* of the stock and ignore the continuous dividend (growth in no. of shares owned) in the calculation.

- More generally, the **t-year volatility** (denoted by σ_t) is $\sigma_t = \sigma\sqrt{t}$. (Multiplying $\sqrt{t}$ converts the "nominal" for 1 year to "nominal" for t years.) Particularly, the h-year volatility is

$$\sigma_h = \sigma\sqrt{h} = \sqrt{\mathrm{Var}\left(\ln\frac{S_h}{S_0}\right)}.$$

- The volatility is positive since the stock 🍎 is risky $\implies$ S_h is random $\implies$ the variance is positive.

6.4.5 We can use the historical stock prices to *estimate* the stock's volatility → the (estimated) value can then be used for constructing a binomial tree. A common approach for such volatility-based construction of a binomial tree is to use a **forward tree**, which is the binomial tree with the growth factors set to be

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h-\sigma\sqrt{h}},$$

where σ, r, and δ are supposed to be known values (obtained possibly by estimation based on past data or assumption).

[Note: These choices of growth factors satisfy the bounds mentioned in [6.4.2], since when $\sigma > 0$, we have $\sigma\sqrt{h} > 0$, which implies $e^{\sigma\sqrt{h}} > 1$ and $e^{-\sigma\sqrt{h}} < 1$.]

6.4.6 To see why a forward tree is called so, first recall from [3.5.3] that the forward price for a forward on the stock 🍎 (with dividend yield δ) with delivery date h is

$$F_0 = S_0 e^{(r-\delta)h}.$$

Under the forward tree, the "up" and "down" time-h stock prices are:

- $S_u = S_0 e^{(r-\delta)h+\sigma\sqrt{h}} = F_0 e^{\sigma\sqrt{h}} = F_0 e^{\sigma_h}$;
- $S_d = S_0 e^{(r-\delta)h-\sigma\sqrt{h}} = F_0 e^{-\sigma\sqrt{h}} = F_0 e^{-\sigma_h}$.

So, the *forward* price F_0 serves as a "baseline" for time-h stock price to vary around (based on the volatility σ), and we choose e^{σ_h} and $e^{-\sigma_h}$ as the multiplicative factors for "up" and "down" respectively. [Intuition 💡: Given a log return r_{log}, the "growth factor" (in the context of rate of return) is $e^{r_{\text{log}}}$, so the prices S_u and S_d may be *informally* regarded as having "1 s.d. from baseline".]

6.4.7 For a forward tree, the risk-neutral probability (of "up") is

$$q = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} = \frac{1}{1 + e^{\sigma\sqrt{h}}}.$$

[⚠ Warning: If you are familiar with machine learning, you may notice that the final expression "seems to be" the sigmoid function with input $\sigma\sqrt{h}$. But it is <u>not</u>. The final expression here does not have minus sign!]

This formula provides a convenient way to compute the risk-neutral probability q *without* knowing u, d, r, and δ.

6.4.8 However, a disadvantage of the forward tree is that we must have $q < \frac{1}{2}$ (since $\sigma > 0 \implies e^{\sigma\sqrt{h}} > 1$), which can be seen as a *built-in bias* for a forward tree.

6.4.9 Here we give an approach to estimate the volatility σ based on past stock price data (which is also useful for the *Black-Scholes model*; see [8.1.4]).

Suppose we observe $n+1$ stock prices $S_0, S_h, S_{2h}, \ldots, S_{nh}$ (h is the time length between adjacent observations). (E.g., when $h = 1/12$, then they are observed at consecutive *monthly* intervals.) The observed (non-annualized) continuously compounded returns over the time intervals $[0,h),[h,2h),\dots,[(n-1)h,nh)$, denoted by $r_1,r_2,\dots,r_n$ (resp.), are respectively

$$r_1=\ln\frac{S_h}{S_0}, r_2=\ln\frac{S_{2h}}{S_h},\dots,r_n=\ln\frac{S_{nh}}{S_{(n-1)h}}.$$ [22]

6.4.10 Here we assume that the *returns* are independent and identically distributed (i.i.d.).[23] Having n i.i.d. observations (of returns) $r_1,\dots,r_n$, we can use standard techniques in statistics to perform the estimation.

6.4.11 Here we use a simple technique: *method of moments*. We equate the *sample variance* of the observations $r_1,\dots,r_n$ and the (theoretical) variance of h-year continuously compounded return (where the initial value is nonrandom): $\sigma_h^2=\sigma^2 h$ to get

$$\frac{1}{n-1}\sum_{i=1}^{n}(r_i-\overline{r})^2=\sigma^2 h,$$

where $\overline{r}=(1/n)\sum_{i=1}^{n} r_i$ is the sample mean of $r_1,\dots,r_n$.

6.4.12 Solving this equation in σ gives the estimate of volatility:

$$\widehat{\sigma}=\frac{1}{\sqrt{h}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i-\overline{r})^2},$$

which is known as **historical volatility**.

[22]These observations can be treated as observations from n "experiments" conducted "naturally" in the market at the past. At each initial time point i (the beginning of the time interval $[i,i+h)$), the stock price S_i is known to the market participants, while the stock price S_{i+h} at time $i+h$ is unknown (random) to them at that moment ➔ the h-year log return here "matches" with the one mentioned in [6.4.4].

Then, each observed value r_i can be seen as an "observation" from that random h-year log return in [6.4.4] in the "experiment".

[23]On the other hand, it is *unreasonable* to assume that the stock prices are i.i.d. — they are clearly dependent!

7 Multi-Period Binomial Option Pricing Model

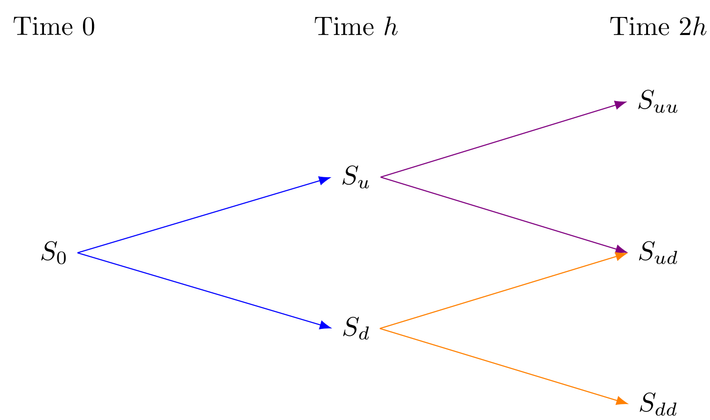
7.0.1 The main weakness of one-period binomial option pricing model is that the terminal stock price can only take *two* possible values, which is unrealistic.

7.0.2 To make the binomial option pricing model more realistic (by allowing more possible values for the terminal stock price at least), we can utilize the *multi-period binomial option pricing model.*

7.1 Pricing by Backward Induction

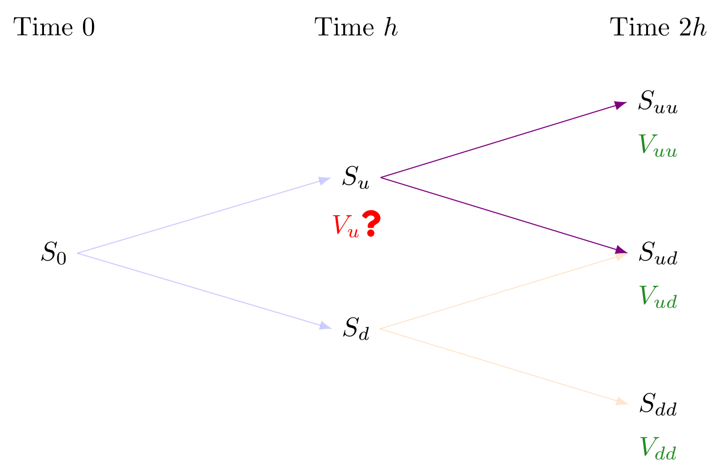
7.1.1 To apply pricing by replication in section 6 to a multi-period model, the main tool is the *backward induction.*

7.1.2 To illustrate how backward induction works, consider the following two-period binomial tree, which can be "decomposed" into three one-period binomial trees:



7.1.3 Now, we work *backward* (work from right to left), and consider the two one-period binomial trees on the right one by one.

7.1.4 Consider first the one-period binomial tree branching out of the S_u node:



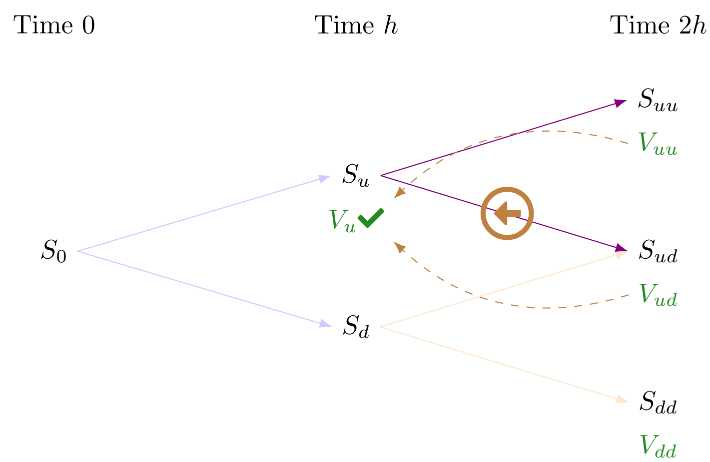
[Note: Here we assume the terminal payoffs (i.e., payoffs at the latest time point) of the derivative at every terminal node, i.e., $V_{uu}, V_{ud},$ and V_{dd} here, are well-defined and known.

This is not the case for American options. See section 7.3.]

With the terminal payoffs V_{uu} and V_{ud}, we can use pricing by replication (or risk-neutral pricing) for this one-period binomial tree to find out the (time-h) value of 📜 at the S_u node, V_u:

$$V_u = e^{-\delta h}[V_{uu}q + V_{ud}(1-q)]$$

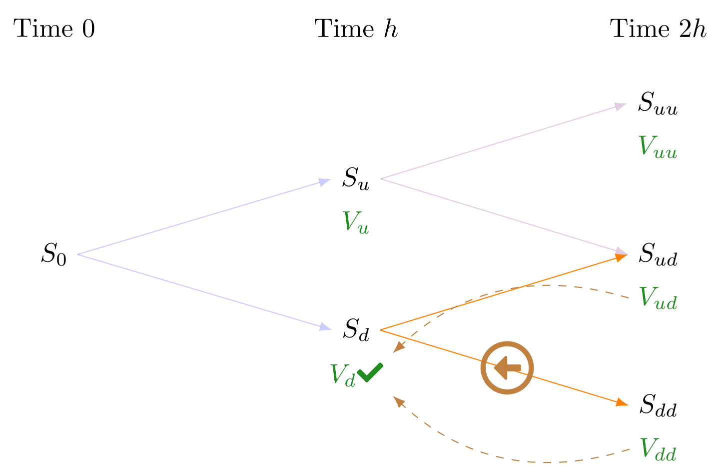
where $q = \dfrac{e^{(r-\delta)h} - d}{u - d}$.



[Note: Since the growth factors u and d are the same throughout a multi-period binomial tree, the risk-neutral probability q is also the same throughout.]

7.1.5 Now, we repeat this procedure for the one-period binomial tree branching out of the S_d node:

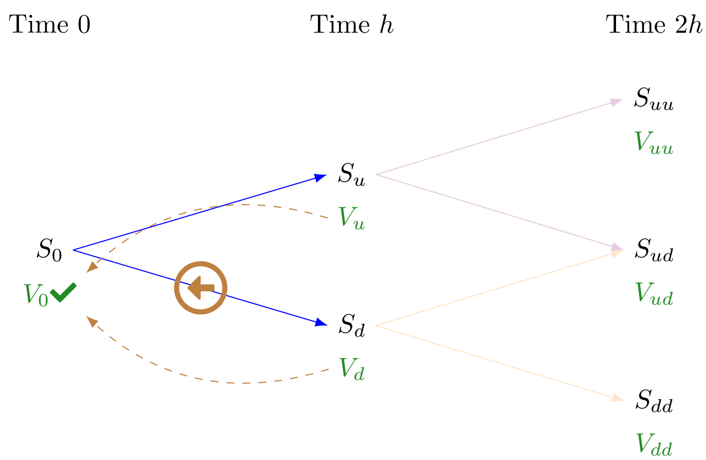
$$V_d = e^{-\delta h}[V_{ud}q + V_{dd}(1-q)].$$



7.1.6 Lastly, repeat this procedure again for the one-period binomial tree branching out of the S_0 node:

$$V_0 = e^{-\delta h}[V_u q + V_d(1-q)],$$

and we have found the time-0 value of 📜: V_0!



7.1.7 Similar approach would work for a general n-period binomial tree, and it is known as **backward induction**.

7.2 Risk-Neutral Pricing

7.2.1 As we can see, if the number of periods n involved in the tree is large, the backward induction approach would require many computations and would be rather cumbersome.

7.2.2 Fortunately, for *European option* (or any derivative with a well-defined "terminal" time point and known "terminal" payoffs) it turns out that there is an alternative and more convenient formula to find out its time-0 value, which is again somehow related to the idea of *risk-neutral pricing*.

7.2.3 To get the formula for the two-period tree, consider:

$$\begin{aligned} V_0 &= e^{-\delta h}[V_u q + V_d(1-q)] \\ &= e^{-\delta h}\left\{e^{-\delta h}[V_{uu}q + V_{ud}(1-q)]q + e^{-\delta h}[V_{ud}q + V_{dd}(1-q)](1-q)\right\} \\ &= e^{-\delta(2h)}\left[V_{uu}q^2 + V_{ud}(2q(1-q)) + V_{dd}(1-q)^2\right]. \end{aligned}$$

We can note that this formula is of the "risk-neutral pricing" form:

$$e^{-\delta(\text{total duration})}\mathbb{E}_{\mathbb{Q}}[\text{terminal payoff}],$$

by identifying that the total duration is $2h$, and the terminal payoff is

$$\begin{cases} V_{uu} & \text{with probability } q; \\ V_{ud} & \text{with probability } 2q(1-q); \\ V_{dd} & \text{with probability } (1-q)^2, \end{cases}$$

where the probabilities here are *risk-neutral*.

7.2.4 For a general n-period binomial tree, the time-0 value is

$$V_0 = e^{-\delta(nh)}\left[V_{u^n}q^n + V_{u^{n-1}d}\binom{n}{1}q^{n-1}(1-q) + \cdots + V_{d^n}(1-q)^n\right] = e^{-\delta(nh)}\sum_{i=0}^{n} V_{u^{n-i}d^i}\binom{n}{i}q^{n-i}(1-q)^i.$$

Proof: By observing the back induction process, we can see that the time-0 value has the form of

$$e^{-\delta(nh)}\sum_{\text{every terminal node}} \text{payoff at the node} \times \text{no. of paths to the node}$$
$$\times \text{ product of risk-neutral probabilities arising from the backward induction process.}$$

In an n-period binomial tree, for any $i = 0, 1, \ldots, n$, to reach the $S_{u^{n-i}d^i}$ node from the start (S_0 node), i "down" moves (and $n - i$ "up" moves) are needed.

To get a path satisfying this requirement, we need to choose i out of n periods to have "down" move ➔ $\binom{n}{i}$ distinct paths satisfying the requirement (the order of choosing does not matter).

Now note that for each of those paths, the product of the risk-neutral probabilities arising from the back induction process is $q^{n-i}(1-q)^i$ (one "up" ("down") move ⟷ one q $(1-q)$ gets multiplied). □

7.2.5 From the formula in [7.2.4], we can identify the term $\binom{n}{i}q^{n-i}(1-q)^i$ as the (unique) *risk-neutral probability* of reaching the $S_{u^{n-i}d^i}$ node (i.e., having i "down" moves) ➔ no. of "up" moves follows the *binomial distribution* (n independent trials with "up" probability q) under the unique risk-neutral probability measure $\mathbb{Q}$.

7.2.6 After identifying the risk-neutral measure $\mathbb{Q}$, we can express the formula in [7.2.4] in the risk-neutral pricing form:

$$e^{-\delta(nh)}\mathbb{E}_{\mathbb{Q}}[\text{terminal payoff}]. \tag{3}$$

7.3 American Options

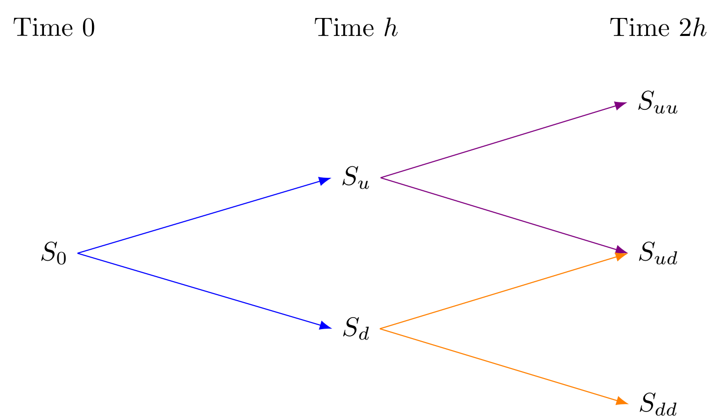
7.3.1 For an *American option*, since **early exercise** (i.e., exercising before maturity) is possible, the option can "end" early ➔ the notion of "terminal" is ill-defined (there is not a unique "terminal point"!) ➔ eq. (3) is *not applicable*.

7.3.2 To price an American option, we can use again *backward induction*, but with some "twist" not mentioned in section 7.1.

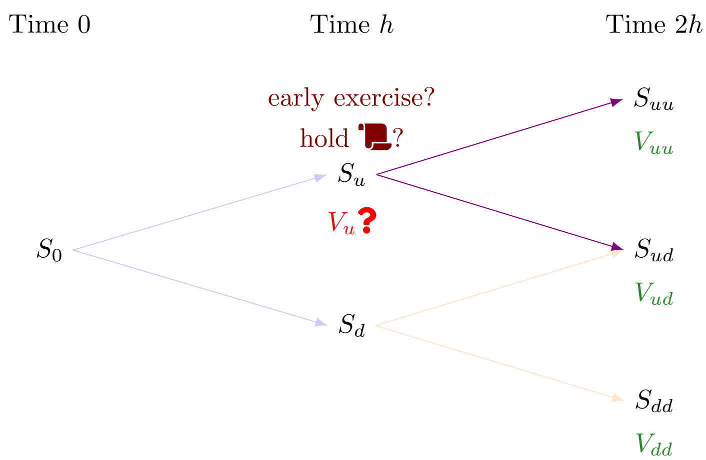
[Note: This shows an advantage of the binomial option pricing model: It can accommodate *American options*, which are less mathematically tractable than European options.]

7.3.3 The "twist" is that we need to now more consciously consider the *rationality* assumption mentioned in [4.2.1] during the backward induction process.

7.3.4 We again consider the following two-period binomial tree as an example, and we now work with an American option:



7.3.5 Consider the one-period binomial tree branching out of the S_u node:



At the S_u node, the option holder has an opportunity to choose whether to early exercise or not, i.e., choosing between early exercise and *holding* (at least) until the next time point in the tree.

By rationality, the holder would choose the one with "more positive" *value*. The values for "early exercise" and "holding" are called **immediate exercise value** and **holding value** respectively, which are given by:

- immediate exercise value: perform "early exercise" → the American option "is reduced to" an European option expiring at current time point → immediate exercise value $= (S_u - K)_+$ ($(K - S_u)_+$) for call (put);
- holding value: perform "holding" → the American option "is reduced to" a derivative with payoffs V_{uu} after "up" and V_{ud} after "down" in one-period tree[24] → holding value = value of that derivative (which can be found by pricing by replication/risk-neutral pricing).

Remarks:

- In case both values are identical, we assume the holder would *not* early exercise.
- Recall that we have assumed the stock price would stay constant from the beginning of each period until the "jump" at the very end. So, if we have shown that it is optimal to early exercise at a certain node (beginning of a period), then early exercising at a later time than the beginning in that period is *not* rational since it would result in a loss of opportunity to earn risk-free interest. Thus, it suffices to only check the possibility of early exercise at each of the non-terminal nodes.

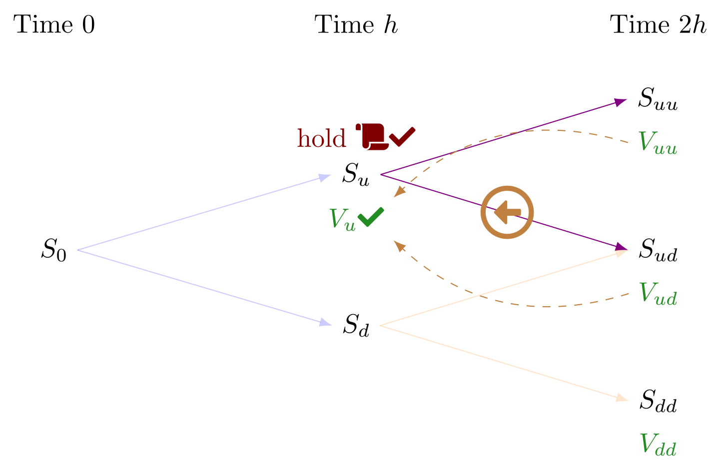
7.3.6 Consequently, when the immediate exercise value (holding value, resp.) is higher, the American option would be "reduced to" an currently expiring European option (a derivative with payoffs V_{uu} after "up" and V_{ud} after "down" in one-period tree, resp.), whose value is precisely the immediate exercise value (holding value, resp.).

In other words, the value of the American option at the S_u node (or any other non-rightmost node) is the *maximum* of the holding value and immediate exercise value.

7.3.7 Suppose the holding value is higher in this case, and we have:

$$V_u = e^{-\delta h}[V_{uu}q + V_{ud}(1-q)]$$

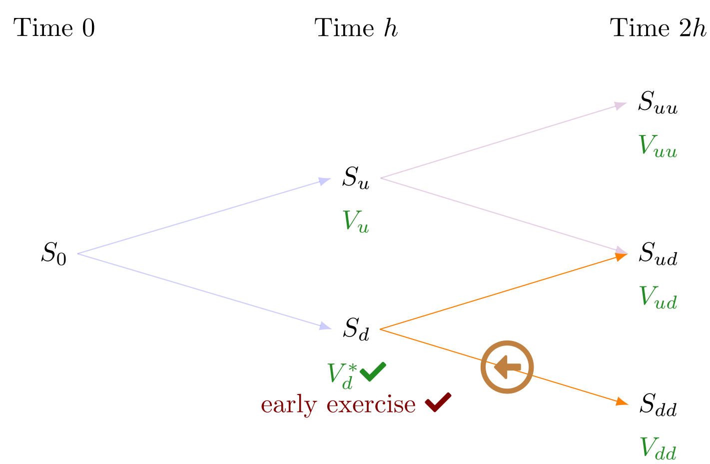
[24] The derivative is not going to "end" until the termination time point of the tree.



7.3.8 Now, we repeat this procedure for the one-period binomial tree branching out of the S_d node, and suppose the *immediate exercise value* is higher:

$$V_d^* = \begin{cases} (S_d - K)_+ & \text{for call;} \\ (K - S_d)_+ & \text{for put.} \end{cases}$$

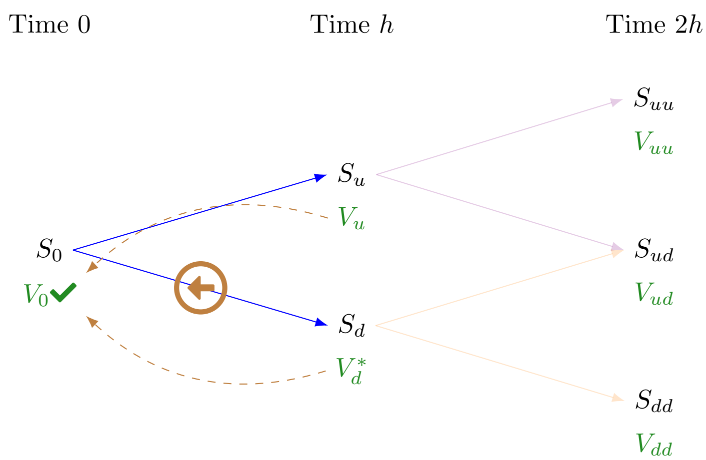
[Note: To signify that the immediate exercise value is higher, we conventionally add an asterisk $*$ as a superscript on the value of American option. This still applies when we write the actual number, e.g. we write 100^* rather than 100 (but of course the actual value represented does not change).]



7.3.9 Lastly, repeat this procedure again for the one-period binomial tree branching out of the S_0 node, and suppose the holding value is higher:

$$V_0 = e^{-\delta h}[V_u q + V_d^*(1-q)].$$

[Note: We use V_d^* in the formula since it is indeed the value (or payoff) of American option at node S_d.]



7.3.10 An important shortcut regarding American option is suggested by the result below:

Theorem 7.3.a. It is *never* rational to early exercise an *American call* on an asset with no dividends etc., when $r \geq 0$ (typically the case).

Proof: Consider an European call and an European put with the same terms (except the exercise style of course) as the American call (same underlying asset, strike price, and expiration date). Let C_t^E and P_t^E be their time-t prices respectively, and let C_t^A be the time-t price of the American call.

For any $t \in [0, T)$, by (non-generalized) put-call parity (theorem 5.6.a) (with changes in time labelling if necessary), we have

$$C_t^E + Ke^{-r(T-t)} = P_t^E + S_t.$$

Firstly, note that we must have $C_t^A \geq C_t^E$. (Otherwise, longing the American call and shorting the European call at time t would lead to arbitrage[25].) Since the European call price C_t^E is always positive, it means the American call price C_t^A is also positive.

Consequently, when $S_t \leq K$ (where early exercise is of zero value), since *selling* the American call (to receive a *positive* cash flow) is of positive value, it is not rational to early exercise.

Now consider the case where $S_t > K$. Note that

$$C_t^A \geq C_t^E > C_t^E - P_t^E = S_t - Ke^{-rT} \geq S_t - K,$$

which means *selling* the American call (to receive the price C_t^A) has a higher value than early exercise $\rightarrow$ early exercise is again not rational. $\square$

[Note: Because of this result, such American call can be readily "reduced to" the *European call* with the same terms (except exercise style), and methods for European calls can still be used, e.g. risk-neutral pricing based on terminal payoff (eq. (3)).]

[25]For such strategy, the total cash flow at time t is positive. Next, we do not early exercise our American call, and at time T, we exercise our American call if the European call is exercised $\rightarrow$ zero total cash flow.

8 Black-Scholes Model

8.0.1 The binomial option pricing model, regardless of how many periods it has, is still inherently *discrete*: There are finitely many distinct terminal stock prices and finitely many time points involved.

8.0.2 The most famous model in the *continuous* realm (where a "stream" of distinct terminal stock prices and time points is involved) is perhaps the *Black-Scholes model* — the model we study here.

8.1 Model Formulation

8.1.1 Before specifying the Black-Scholes model, we first discuss a closely related concept: *lognormal distribution*.

8.1.2 A random variable Y follows a **lognormal distribution** with parameters μ and σ^2 (denoted by $Y \sim LN(\mu, \sigma^2)$) if the "log" of Y, $\ln Y$, follows a normal distribution with mean μ and variance σ^2, i.e., $\ln Y \sim N(\mu, \sigma^2)$.

[⚠ Warning: The parameters μ and σ^2 are *not* the mean and variance of Y!]

8.1.3 To compute moments of Y, it is useful to recall that the moment generating function of a normal r.v. $X \sim N(\mu, \sigma^2)$:

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

Since $Y = e^X$, the first and second moments of Y are

$$\mathbb{E}[Y] = \mathbb{E}\left[e^X\right] = M_X(1) = e^{\mu + \sigma^2/2},$$

and

$$\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[e^{2X}\right] = M_X(2) = e^{2(\mu+\sigma^2)}.$$

Hence, the variance of Y is

$$\mathrm{Var}\,(Y) = \mathbb{E}\left[Y^2\right] - (\mathbb{E}[Y])^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) = (\mathbb{E}[Y])^2\left(e^{\sigma^2} - 1\right)$$

8.1.4 In the **Black-Scholes model** (or **Black-Scholes framework**), we are assumed to be in a perfect market having the following two assets:

- a risky stock which pays dividend continuously at a dividend yield δ, where its time-t price is

$$S_t = S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z_t\right],$$

  for some nonnegative parameters α and σ, and some *standard normal* random variable Z_t's that are "related in a certain way";
- a (risk-free zero-coupon) bond with an annual continuously compounded risk-free rate r.

Remarks:

- A more technical and precise definition (discussed in STAT3911) of the Black-Scholes model involves the notion of *geometric Brownian motion* (that describes how the Z_t's are "related"), which we shall omit here.
- As we can see, the Black-Scholes model is like a "continuous analogue" to binomial option pricing model (the setting is "similar"; just that the stock price evolves *continuously* rather than in a discrete manner). Indeed, it turns out that the Black-Scholes model can be obtained as a *limit* of multi-period binomial option pricing model (no. of periods ↑ & duration of each period ↓).
- Although the expression for S_t appears to be complex and not so intuitive, it makes the parameters α and σ more easily interpretable. (See [8.1.7].)

8.1.5 The Black-Scholes model is related to the concept of lognormal distribution, since we have

$$\ln S_t = \ln S_0 + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z_t,$$

where $Z_t \sim N(0,1)$, which implies

$$\ln S_t \sim N\left[\ln S_0 + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right],$$

or

$$S_t \sim LN\left[\ln S_0 + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right].$$

8.1.6 In the Black-Scholes model, the t-year continuously compounded rate of *price* return on the stock 🍎 (i.e., continuous dividend is not considered in the calculation of return) is

$$\ln\frac{S_t}{S_0} = \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z_t \sim N\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right].$$

So, in the Black-Scholes model, while *stock prices* are lognormally distributed, the *continuously compounded returns* are normally distributed.

8.1.7 Interpretations of α and σ:

- α: By the mean formula in [8.1.3],

$$\mathbb{E}[S_t] = \exp\left[\ln S_0 + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \frac{\sigma^2}{2}t\right] = S_0 e^{(\alpha-\delta)t},$$

which means

$$e^{\alpha t} - 1 = \mathbb{E}\left[\frac{e^{\delta t}S_t - S_0}{S_0}\right].$$

Since $(e^{\delta t}S_t - S_0)/S_0$ is the t-year rate of total return on the stock 🍎[26], it follows that α is the continuously compounded expected rate of return on the stock 🍎 (or "log expected return").
Remarks:

  - Although α should be interpreted as "log expected return", sometimes it is (improperly) called "expected log return".
  - It turns out that for *option pricing purpose*, the value of α does not matter. (See section 9.) This is analogous to the property that the *true* probability p does not affect the option pricing in binomial option pricing model.

- σ: By [8.1.6],

$$\text{Var}\left(\ln\frac{S_t}{S_0}\right) = \sigma^2 t \implies \sigma = \sqrt{\frac{1}{t}\text{Var}\left(\ln\frac{S_t}{S_0}\right)}.$$

Hence, σ is the volatility of the stock 🍎 (hence the notation σ).

8.2 Probabilistic Quantities Under Black-Scholes Model

8.2.1 Although the main usage of Black-Scholes model is option pricing, we can indeed compute some probabilistic quantities concerning the *stock* 🍎 of theoretical/practical interest, based on the specification of its stock price in the Black-Scholes model.

[26] To be more precise, it is the rate of return on a *portfolio* containing a share of stock. Both the ↑ in no. of shares owned from continuous dividend and the ↑ in stock price contribute to the ↑ in *portfolio* value.

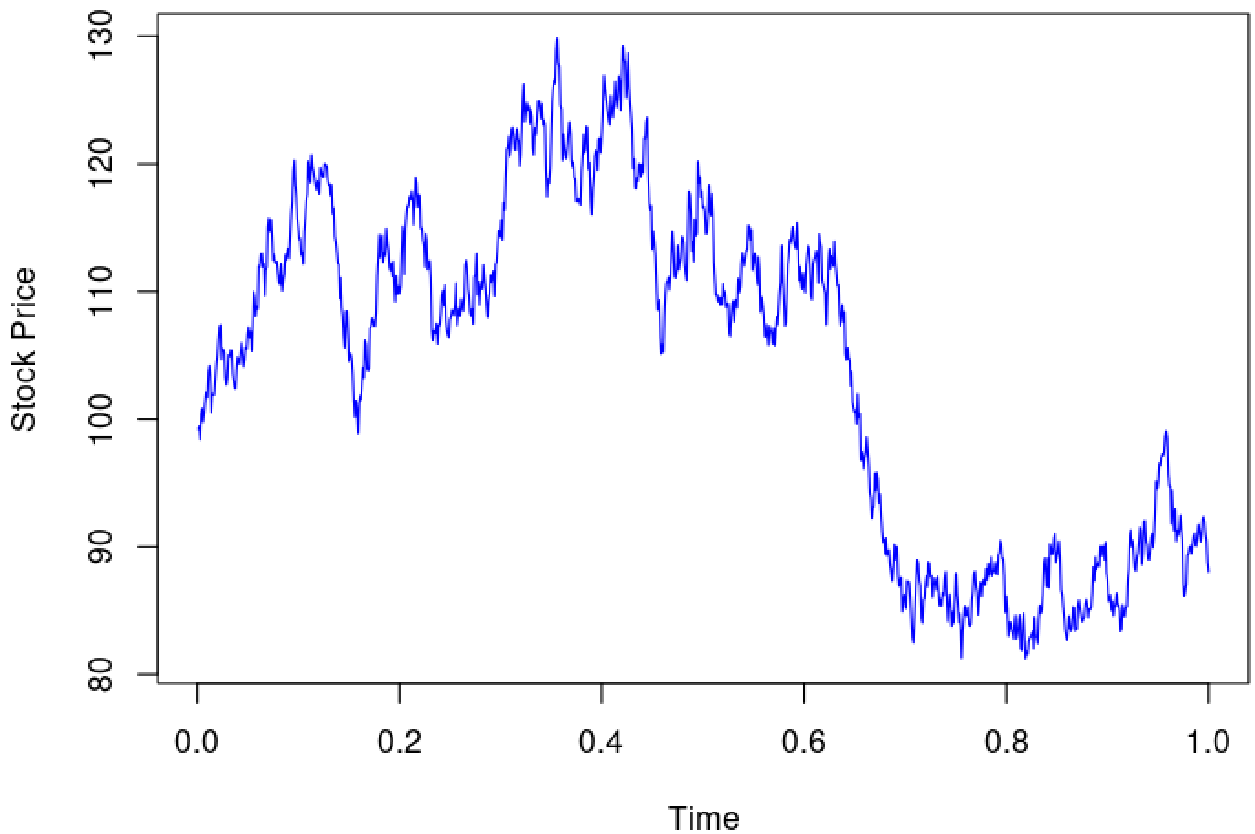


Figure 1: A simulated stock price path under the Black-Scholes model with $S_0 = 100$, $\alpha = 0.12$, $\delta = 0.02$, and $\sigma = 0.4$.

```
#R code
set.seed(1)
N <- 1000
S0 <- 100
alpha <- 0.12
delta <- 0.02
sigma <- 0.4

time_pts <- (1:N)/N
dt <- 1/N
Wt <- cumsum(rnorm(N)) * sqrt(dt)
St <- S0 * exp((alpha - delta - sigma^2/ 2) * time_pts + sigma * Wt)
plot(time_pts, St, col="blue", type="l", xlab="Time", ylab="Stock Price")
```

Exercise Probabilities

8.2.2 The first quantity is also related to option: the (true) probability that an European call on the stock will be exercised (by a rational holder) at expiration (the **exercise probability** of the call), i.e.,

$\mathbb{P}(S_T > K)$.

8.2.3 The probability is given by:

$$\begin{aligned}
\mathbb{P}(S_T > K) &= \mathbb{P}(\ln S_T > \ln K) \\
&= \mathbb{P}\left(\ln S_0 + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z_T > \ln K\right) \\
&= \mathbb{P}\left(Z_T > \frac{\ln(K/S_0) - (\alpha - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&= \mathbb{P}\left(-Z_T < \frac{\ln(S_0/K) + (\alpha - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&= \Phi(\widehat{d_2}) \qquad (-Z_T \sim N(0,1) \text{ also})
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cdf, and

$$\widehat{d_2} = \frac{\ln(S_0/K) + (\alpha - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

[Note: We shall discuss why the notation "$\widehat{d_2}$" is used here in section 9.]

8.2.4 Then, the exercise probability of an otherwise identical European *put* is

$$\mathbb{P}(S_T < K) = 1 - \mathbb{P}(S_T > K) = 1 - \Phi(\widehat{d_2}) = \Phi(-\widehat{d_2}).$$

Lognormal Prediction Intervals for Stock Prices

8.2.5 The next quantity of interest is a range of values in which the stock price has a high probability of lying (useful for "predicting" future stock price!).

8.2.6 For any $t \geq 0$ and $p \in (0,1)$ "close to" 0 (so that the probability of lying in the range is "high"), we want to find (nonrandom) constants S^L and S^U such that

$$\mathbb{P}\big(S^L < S_t < S^U\big) = 1 - p.$$

The interval $[S^L, S^U]$ is called a $100(1-p)\%$ **lognormal prediction interval** for S_t.

Remarks:

- Often we want to find an **equal-tailed** lognormal prediction interval, i.e.,

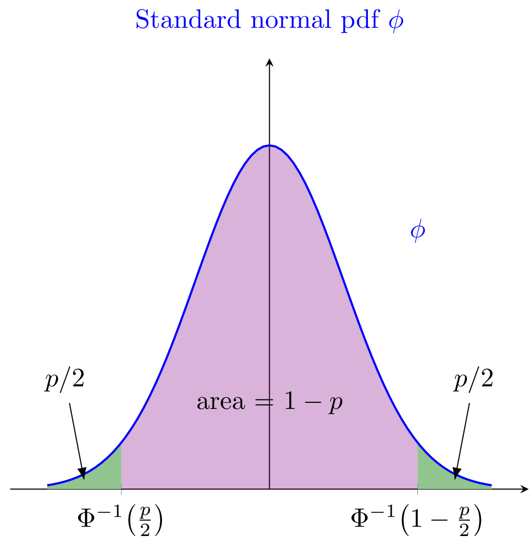
$$\mathbb{P}\big(S_t < S^L\big) = \mathbb{P}\big(S_t > S^U\big) = \frac{p}{2}.$$

- You may have learnt about the concept of *confidence interval* in your statistics course and find this concept to be "similar". The differences between the concepts of *confidence intervals* and *prediction intervals* are given below:

Type	Bounds of intervals are:	Target is:
confidence interval	random variables	(nonrandom) constant
prediction interval	(nonrandom) constants	random variable

8.2.7 To find a $100(1-p)\%$ prediction interval for S_t for any $t \geq 0$, we start with the equation

$$\mathbb{P}\left(\Phi^{-1}\left(\frac{p}{2}\right) < Z_t < \Phi^{-1}\left(1 - \frac{p}{2}\right)\right) = 1 - p.$$



After that, since

$$S_t = S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z_t\right],$$

we readily have

$$\mathbb{P}\left(S_0 e^{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}\Phi^{-1}(p/2)} < S_t < S_0 e^{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}\Phi^{-1}(1-p/2)}\right) = 1-p.$$

It follows that the $100(1-p)\%$ *equal-tailed* prediction interval for S_t is

$$\left[S_0 \exp\left\{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\Phi^{-1}\left(\frac{p}{2}\right)\right\}, S_0 \exp\left\{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\Phi^{-1}\left(1-\frac{p}{2}\right)\right\}\right].$$

8.2.8 The following are some non-equal-tailed $100(1-p)\%$ prediction intervals:

- $\left(0, S_0 \exp\left\{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\Phi^{-1}(1-p)\right\}\right]$
- $\left(S_0 \exp\left\{\left(\alpha-\delta-\frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\Phi^{-1}(p)\right\}, \infty\right)$

Proof: Start with the equations

$$\mathbb{P}\big(-\infty < Z_t < \Phi^{-1}(1-p)\big) = 1-p,$$

and

$$\mathbb{P}\big(\Phi^{-1}(p) < Z_t < \infty\big) = 1-p$$

respectively, and note that $e^x \to \infty$ (0) as $x \to \infty$ $(-\infty)$. $\square$

These prediction intervals are sometimes called **one-sided** or **one-tailed**.

Conditional Expected Stock Prices

8.2.9 This quantity is again somewhat related to option also. We know that the (unconditional) expected time-t stock price is $\mathbb{E}[S_t] = S_0 e^{(\alpha-\delta)t}$ by [8.1.7].

8.2.10 Given an European call (put) expiring at time T, we are sometimes interested in knowing the expected *terminal* (i.e., time-T) stock price, given the call (put) expires *in-the-money*, i.e., $S_T > K$ ($K > S_T$) ➔ gets exercised.[27] So we want to find the conditional expectation $\mathbb{E}[S_T|S_T > K]$ ($\mathbb{E}[S_T|S_T < K]$).

8.2.11 The following gives the formula for $\mathbb{E}[S_T|S_T > K]$ and $\mathbb{E}[S_T|S_T < K]$:

Proposition 8.2.a. Under the Black-Scholes model, we have

$$\mathbb{E}[S_T|S_T > K] = S_0 e^{(\alpha-\delta)T} \frac{\Phi(\widehat{d}_1)}{\Phi(\widehat{d}_2)}$$

and

$$\mathbb{E}[S_T|S_T < K] = S_0 e^{(\alpha-\delta)T} \frac{\Phi(-\widehat{d}_1)}{\Phi(-\widehat{d}_2)}$$

where $\widehat{d}_2 = \dfrac{\ln(S_0/K) + (\alpha - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}$ and $\widehat{d}_1 = \widehat{d}_2 + \sigma\sqrt{T}$.

[Intuition 💡: Both expressions are of the form

unconditional expected stock price $\times$ adjustment factor.

- For $\mathbb{E}[S_T|S_T > K]$, the adjustment factor is $\Phi(\widehat{d}_1)/\Phi(\widehat{d}_2) > 1$ (as $\widehat{d}_1 > \widehat{d}_2$);
- For $\mathbb{E}[S_T|S_T < K]$, the adjustment factor is $\Phi(-\widehat{d}_1)/\Phi(-\widehat{d}_2) < 1$ (as $-\widehat{d}_1 < -\widehat{d}_2$).

So, intuitively, conditioning on $S_T > K$ (S_T is "of high value")/$S_T < K$ (S_T is "of low value") increases/decreases the expected value.]

Proof: Consider first $\mathbb{E}[S_T|S_T > K]$. Here we use the following formula for conditional expectation:

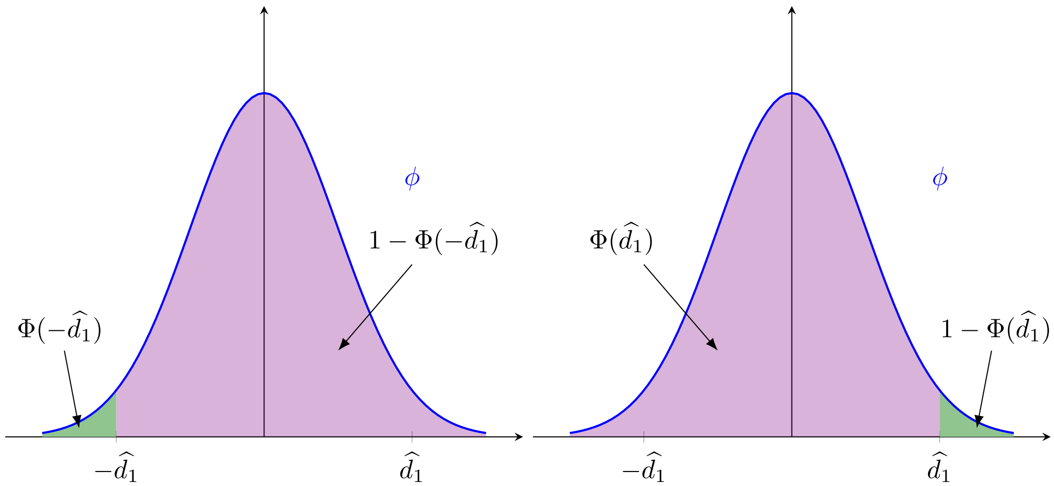
$$\mathbb{E}[S_T|S_T > K] = \frac{\mathbb{E}\left[S_T \mathbf{1}_{\{S_T>K\}}\right]}{\mathbb{P}(S_T > K)}.$$

(By [8.2.3], we have $\mathbb{P}(S_T > K) = \mathbb{P}\left(Z_T > -\widehat{d}_2\right) = \Phi(\widehat{d}_2) > 0$.) Now, we compute

$$\begin{aligned}
\mathbb{E}\left[S_T \mathbf{1}_{\{S_T>K\}}\right] &= \mathbb{E}\left[S_0 \exp\left\{\left(\alpha - \delta - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z_T\right\}\mathbf{1}_{\{Z_T>-\widehat{d}_2\}}\right] \\
&= \int_{-\infty}^{\infty} S_0 \exp\left\{\left(\alpha - \delta - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z\right\}\mathbf{1}_{\{z>-\widehat{d}_2\}}\phi(z)\,dz \\
&= S_0 e^{(\alpha-\delta)T}\int_{-\widehat{d}_2}^{\infty} e^{-(\sigma^2/2)T+\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz \\
&= S_0 e^{(\alpha-\delta)T}\int_{-\widehat{d}_2}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-(z-\sigma\sqrt{T})^2/2}\,dz \\
&= S_0 e^{(\alpha-\delta)T}\int_{-\widehat{d}_2-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}\,du & (u = z - \sigma\sqrt{T}) \\
&= S_0 e^{(\alpha-\delta)T}\underbrace{(1 - \Phi(-\widehat{d}_1))}_{\Phi(\widehat{d}_1)}
\end{aligned}$$

where $\phi(\cdot)$ is the standard normal pdf.

[27] Intuitively, this conditional expected value can measure the "extent" of the gain (loss) for the option holder (writer) on average when the option gets exercised.



It then follows that

$$\mathbb{E}[S_T|S_T > K] = S_0 e^{(\alpha-\delta)T}\frac{\Phi(\widehat{d}_1)}{\Phi(\widehat{d}_2)}.$$

Now, for $\mathbb{E}[S_T|S_T < K]$, apply law of total expectation gives

$$\mathbb{E}[S_T] = \mathbb{E}[S_T|S_T > K]\mathbb{P}(S_T > K) + \mathbb{E}[S_T|S_T < K]\mathbb{P}(S_T < K),$$

which implies

$$S_0 e^{(\alpha-\delta)T} = S_0 e^{(\alpha-\delta)T}\Phi(\widehat{d}_1) + \mathbb{E}[S_T|S_T < K]\Phi(-\widehat{d}_2),$$

and thus

$$\mathbb{E}[S_T|S_T < K] = S_0 e^{(\alpha-\delta)T}\frac{\Phi(-\widehat{d}_1)}{\Phi(-\widehat{d}_2)}.$$

(Note that $1-\Phi(\widehat{d}_1) = \Phi(-\widehat{d}_1)$.) $\square$

9 Black-Scholes Option Pricing Formula

9.1 Black-Scholes Formula

9.1.1 One major result under the Black-Scholes model is the well-known *Black-Scholes formula* which can price *European* call/put options on the stock .

9.1.2 The Black-Scholes formula is as follows:

Theorem 9.1.a (Black-Scholes formula)**.** Assume the Black-Scholes model for any time $t \in [0, T]$. Then, the time-0 price of an European option (with strike price K and expiration date T) on the stock is

$$\begin{cases} C_0 = \mathrm{BS}\,(S_0, \delta; K, r; \sigma, T) = S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2) & \text{for call;} \\ P_0 = \mathrm{BS}\,(K, r; S_0, \delta; \sigma, T) = Ke^{-rT}\Phi(-d_2) - S_0 e^{-\delta T}\Phi(-d_1) & \text{for put,} \end{cases}$$

where $d_1 = \dfrac{\ln(S_0/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T} = \dfrac{\ln(S_0/K) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}$.

<u>Remarks</u>:

- The "BS" function (Black-Scholes-type pricing function) is defined by

$$\mathrm{BS}\,(s_1, \delta_1; s_2, \delta_2; \sigma, T) = s_1 e^{-\delta_1 T}\Phi(d_1^*) - s_2 e^{-\delta_2 T}\Phi(d_2^*)$$

  where $d_1^* = \dfrac{\ln(s_1/s_2) + (\delta_2 - \delta_1 + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2^* = d_1^* - \sigma\sqrt{T} = \dfrac{\ln(s_1/s_2) + (\delta_2 - \delta_1 - \sigma^2/2)T}{\sigma\sqrt{T}}$.
- The formula can also be used to calculate the time-t ($t < T$) price of the European option, by changing time labelling ("now": time t; expiration date: time $T - t$) → replacing *all* occurrences of "T" (don't miss the ones in d_1 and d_2!) by $T - t$ gives the time-t price.

<u>Proof</u>: Here we give a (partial) proof based on risk-neutral pricing under the Black-Scholes model.

By [6.3.14], there exists a unique risk-neutral probability measure $\mathbb{Q}$ under the no-arbitrage principle and perfect market. So, henceforth we shall work in the risk-neutral world equipped with the unique risk-neutral probability measure $\mathbb{Q}$.

First of all, by risk-neutrality property, we know that the stock has an continuously compounded expected return rate r (the risk-free rate) instead of α.

Now, it "turns out" that under the risk-neutral probability measure $\mathbb{Q}$, the volatility of stock and the lognormality of stock price (at any time) are preserved. Thus,

$$S_t \sim LN\left[\ln S_0 + \left(r - \delta - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right]$$

for any $t \in [0, T]$ under the risk-neutral probability measure $\mathbb{Q}$.

Then, since the expected return rate of the European call is risk-free rate in the risk-neutral world, its time-0 price can be found by

$$\begin{aligned} C_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+] \\ &= e^{-rT}\left\{\underbrace{\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+ | S_T > K]}_{\mathbb{E}_{\mathbb{Q}}[S_T - K | S_T > K]}\mathbb{Q}(S_T > K) + \underbrace{\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+ | S_T \le K]}_{0}\mathbb{Q}(S_T \le K))\right\} \\ &= e^{-rT}(\mathbb{E}_{\mathbb{Q}}[S_T | S_T > K] - K)\mathbb{Q}(S_T > K). \end{aligned}$$

[28] In the risk-neutral world with probability measure $\mathbb{Q}$, the exercise probability for the call is

$$\mathbb{Q}(S_T > K) = \Phi\left(\frac{\ln(S_0/K) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}\right) = \Phi(d_2),$$

and the conditional expectation is

$$\mathbb{E}_{\mathbb{Q}}[S_T | S_T > K] = S_0 e^{(r-\delta)T} \frac{\Phi(d_1)}{\Phi(d_2)}$$

(as α is modified to r and σ remains unchanged).

Thus, the time-0 call price C_0 is

$$S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

Now, by the generalized put-call parity (theorem 5.6.b), we have

$$C_0 + Ke^{-rT} = S_0 e^{(r-\delta)T} e^{-rT} + P_0 \implies P_0 = Ke^{-rT}\underbrace{(1 - \Phi(d_2))}_{\Phi(-d_2)} - S_0 e^{-\delta T}\underbrace{(1 - \Phi(d_1))}_{\Phi(-d_1)}.$$

□

9.1.3 The Black-Scholes formula takes a rather complicated form, so here is a mnemonic 🧠 for the Black-Scholes formula:

For the call price formula $S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$:

- $S_0 e^{-\delta T}$: \$ needed to buy stock 🍎 *now* to have a share of 🍎 at time T (note that the no. of shares we own would accumulate between now and time T) ↔ call gives us right to get a share of 🍎 by "giving up" K at time T. ("get" → positive sign)
- Ke^{-rT}: \$ needed to lend (through bond) *now* to have K at time T ↔ call gives us right to get a share of 🍎 by "giving up" K at time T. ("giving up" → negative sign)
- "order" of terms involving d_1 and d_2: bigger one (d_1) first; smaller one (d_2) second.

For the put price formula $Ke^{-rT}\Phi(-d_2) - S_0 e^{-\delta T}\Phi(-d_1)$:

- Ke^{-rT}: \$ needed to lend (through bond) *now* to have K at time T ↔ put gives us right to get K by "giving up" a share of 🍎 at time T ("get" → positive sign).
- $S_0 e^{-\delta T}$: \$ needed to buy stock 🍎 *now* to get a share of 🍎 at time T ↔ put gives us right to get K by "giving up" a share of 🍎 at time T ("giving up" → negative sign).
- "order" of terms involving d_1 and d_2: bigger one ($-d_2$) first; smaller one ($-d_1$) second.

9.1.4 The interpretations for call and put are highly similar. Indeed, we can have a similar interpretation on the general "BS" formula:

$$\mathrm{BS}\,(s_1, \delta_1; s_2, \delta_2; \sigma, T) = s_1 e^{-\delta_1 T}\Phi(d_1^*) - s_2 e^{-\delta_2 T}\Phi(d_2^*)$$

where $d_1^* = \dfrac{\ln(s_1/s_2) + (\delta_2 - \delta_1 + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2^* = d_1^* - \sigma\sqrt{T} = \dfrac{\ln(s_1/s_2) + (\delta_2 - \delta_1 - \sigma^2/2)T}{\sigma\sqrt{T}}$.

In this case:

- s_1 (s_2): the current spot price of asset 1 (2), where asset 1 (2) is to be got (given up) at time T from exercising the option
- δ_1 (δ_2): the "growth rate" for the amount of asset 1 (2) owned[29]

Based on these, we have similar interpretations for $s_1 e^{-\delta_1 T}$ and $s_2 e^{-\delta_2 T}$. The mnemonic for orders of terms involving d_1 and d_2 can be also translated to the case for d_1^* and d_2^* here.

[28] The conditional expectations can be evaluated as follows: $\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+ | S_T > K] = \dfrac{\mathbb{E}_{\mathbb{Q}}\left[(S_T - K)_+ \mathbf{1}_{\{S_T > K\}}\right]}{\mathbb{Q}(S_T > K)} = \dfrac{\mathbb{E}_{\mathbb{Q}}\left[(S_T - K)\mathbf{1}_{\{S_T > K\}}\right]}{\mathbb{Q}(S_T > K)} = \mathbb{E}_{\mathbb{Q}}[S_T - K | S_T > K]$, and $\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+ | S_T \leq K] = \dfrac{\mathbb{E}_{\mathbb{Q}}\left[(S_T - K)_+ \mathbf{1}_{\{S_T \leq K\}}\right]}{\mathbb{Q}(S_T > K)} = \dfrac{0}{\mathbb{Q}(S_T > K)} = 0$.

[29] e.g., "risk-free rate" above may be regarded as growth rate for the amount of cash owned

9.2 European Options on a Stock With Discrete Dividends

9.2.1 In the Black-Scholes model, it is assumed that the risky stock can only possibly have *continuous* dividends, but not discrete dividends. However, in practice all dividends are discrete. Hence, we are naturally interested in whether it is possible to apply the Black-Scholes formula for an European option on a stock with *discrete* dividends.

9.2.2 Before stating a key result for dealing with this case, let us first introduce a new kind of instrument: *prepaid forward* (similar to a forward, but "prepaid").

A **prepaid forward** on the underlying asset is a contract between two parties to "buy" or "sell" for a specified price[30] (known as **prepaid forward price**, denoted by $F^P_{0,T}$) *now*, but (which is "prepaid") is to be delivered at delivery date (time T).

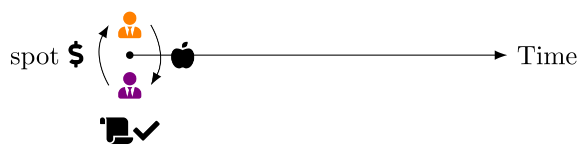
9.2.3 A prepaid forward may be treated as a "certificate of prepayment" with a fixed maturity:

- The party who pays the prepaid forward price gets to own the "certificate" → the *certificate holder* at maturity[31] can receive for free by "showing" the certificate . (Alternatively, one may view it as "exchanging" the certificate for .)
- The certificate can be sold to another person (which allows to receive for free if holds the certificate until maturity).

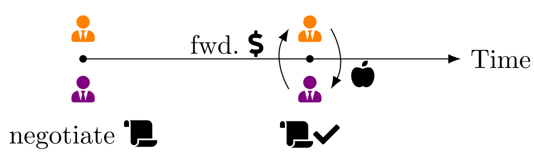
Under this interpretation, we regard "owning a positive (negative) amount of prepaid forward" as "owning the same amount of such certificate ".

9.2.4 Differences between spot contract, forward contract, and *prepaid* forward contract:

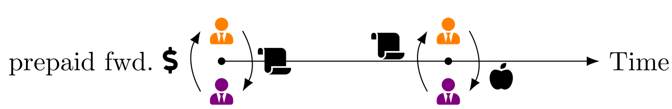
- spot contract:



- forward contract:



- *prepaid* forward contract:



9.2.5 The prepaid forward turns out to be quite useful to "get rid off" discrete dividends for the stock (between time 0 and time T)[32], while preserving the "spot payment" nature.

9.2.6 Under a perfect market, the no-arbitrage *prepaid forward* price can be easily obtained based on the formula for the no-arbitrage *forward* price (discrete dividends case): $F_0 = S_0e^{rT} - \sum_{i=1}^{n} D_{t_i}e^{r(T-t_i)}$.

It is given by

$$F^P_0 = F_0e^{-rT} = S_0 - \sum_{i=1}^{n} D_{t_i}e^{-rt_i}$$

[30] such that it costs nothing to "enter" the prepaid forward (not counting the prepaid forward price payment, of course)
[31] who is not necessarily , since can sell the certificate to others.
[32] It delays the ownership until time T → dividend payments before time T are not received.

when the stock will make dividend payment 💵 D_{t_i} at *known* time t_i, $i = 1, \ldots, n$, where $0 < t_1 < \cdots < t_n < T$.

[Note: In words, it is the spot price of 🍎 subtracted by the present value of all dividends (strictly) between time 0 and time T.]

Proof: Consider the following strategy:

Time	Transaction	Cash flow
0	short forward	0
	long prepaid forward	0
	borrow F_0e^{-rT}	$+F_0e^{-rT}$
	pay prepaid forward price F_0^P	$-F_0^P$
		Total: $F_0e^{-rT} - F_0^P$
T	accept delivery of 1 🍎 from prepaid forward	0
	sell 1 🍎 at *forward* price	$+F_0$
	repay the loan	$-F_0$
		Total: 0

It then follows that $F_0^P = F_0e^{-rT} = S_0 - \sum_{i=1}^n D_{t_i}e^{-rt_i}$, as desired. □

9.2.7 To be more explicit about the delivery date of prepaid forward, here we may use the notation $F_{t,T}^P$ to denote the (no-arbitrage) prepaid forward price for a prepaid forward on 🍎 with delivery date T, with payment made at time $t \leq T$.

Remarks:

- By changing time labelling if needed, we have

$$F_{t,T}^P = S_t - \mathrm{PV}(\text{all dividends (strictly) between time } t \text{ and } T).$$

- We can observe that the time-t price of a prepaid forward ("certificate") 📜 with delivery date being time T *is* the prepaid forward price $F_{t,T}^P$ "conveniently", by the no-arbitrage principle. To argue this, we consider the following two ways to get such a certificate 📜:
  (a) "first-hand": enter a "fresh" prepaid forward contract and pays prepaid forward price $F_{t,T}^P$ to get 📜;
  (b) "second-hand": buy a certificate 📜 from the market at its time-t price.

  Since both result in identical payoff, the time-t price must be the same as the prepaid forward price.

Proposition 9.2.a. An European call (put) on the stock 🍎 has the same (time-0) price as an otherwise identical European call (put) on the prepaid forward ("certificate") 📜 on 🍎. (Both prepaid forward and option mature at time T.)

[Intuition 💡: Getting a prepaid forward ("certificate") 📜 on 🍎 maturing *immediately* at time T is essentially the same as getting 🍎 at time T directly. (Once we get the prepaid forward "certificate", we immediately receive the delivery of 🍎 for free.)]

Proof: Firstly, note that the time-T price/value of prepaid forward is simply $F_{T,T}^P = S_T$[33].

Thus, the (time-T) payoff of the call (put) on prepaid forward is $(S_T - K)_+$ ($(K - S_T)_+$), which is identical to the payoff of the call (put) on 🍎. The result then follows by the law of one price. □

9.2.8 Proposition 9.2.a then gives us an alternative way to calculate the price of call/put on stock 🍎 with discrete dividends: calculating the price of call/put on *prepaid forward* on 🍎 instead.

[33] At time T, the "prepaid forward" is effectively just a spot contract on 🍎: the stock 🍎 is to be delivered simultaneously as the payment is made

9.2.9 The main advantage of this alternative method is that the prepaid forward price (time-t price of prepaid forward) $F_{t,T}^P$ *evolves continuously*[34] (from time 0 to T) → prepaid forward "certificate" can be treated as a *non-dividend-paying* "stock" (the "certificate" itself does not pay any dividend!) which can fit in the Black-Scholes model.

9.2.10 Hence, to price an European call/put on a stock with discrete dividends using Black-Scholes model (indirectly), we follow the steps below:

(a) Assume the Black-Scholes model for the time-t price $F_{t,T}^P$ for the prepaid forward "certificate" (for any $t \in [0,T]$).

(b) Apply the Black-Scholes formula to calculate the (time-0) price of an otherwise identical European call/put *on prepaid forward* :

- call: $C_0 = \text{BS}\left(F_{0,T}^P, 0; K, r; \sigma, T\right)$
- put: $P_0 = \text{BS}\left(K, r; F_{0,T}^P, 0; \sigma, T\right)$

[Note: The volatility σ here is the volatility for the *prepaid forward*, not for the stock with discrete dividends.]

(c) The calculated call/put price (C_0 or P_0) is the desired price.

9.3 Option Delta

9.3.1 The Black-Scholes option pricing formula takes in many parameters as inputs. Although it can calculate the time-t price of European option for any time $t \in [0,T]$, it is only based on the information available at time 0.

As time passes with further information available, we may need to adjust some model parameters and inputs in Black-Scholes model to incorporate them, which influences the calculated option prices.

9.3.2 To perform a *sensitivity analysis* that measures the impact of changes in parameters on the option prices (based on Black-Scholes model), *option Greeks* are invaluable tools.

9.3.3 **Option Greeks** are partial derivatives of the (current) option price with respect to the parameter in question, holding other inputs fixed.

9.3.4 Some common option Greeks are given below:

Option Greek	Notation	The parameter in question
Delta	Δ	current stock price (1st partial derivative)
Gamma	Γ	current stock price (2nd partial derivative)
Vega	no standard notation; sometimes $\mathcal{V}$	volatility σ
Theta	θ	length of time passed from time 0 (*not* T!)
Rho	ρ	risk-free rate r
Psi	Ψ	dividend yield δ

9.3.5 Here we focus on option *delta* (which is the most well-known option Greek). Other option Greeks are discussed in STAT3910.

9.3.6 Consider an option (call or put) on a stock . Let V and S be the *current* option price and *current* price of .

The *delta* of the option is then

$$\Delta = \frac{\partial V}{\partial S}.$$

[34] i.e., $F_{t,T}^P$ is a continuous function of t

9.3.7 Delta measures the sensitivity of option prices to changes in the price of the underlying stock. Larger Δ ➔ higher sensitivity.

9.3.8 A common, although slightly incorrect, interpretation of option delta is the *approximated* increase in the current option price per unit increase in current price of stock, *holding other inputs constant.* (This is similar to the interpretation of *duration* learnt in STAT2902.)

<u>Remarks</u>:

- It is slightly incorrect since the approximation works well only for "small" change in current stock price, and for an *one unit* increase (not so "small", mathematically), the potential error in the approximation may be "significant".
- Nonetheless, it can give us a rough idea of "sensitivity" of the option price to changes in current stock price.

9.3.9 Under the Black-Scholes model, it turns out that we have nice formulas for delta of European call and put:

Proposition 9.3.a. Let Δ_C and Δ_P be the delta of European call and put (with expiration date being time T ("current": time 0)) on a stock with current price S respectively.

Assume the Black-Scholes model for any time $t \in [0, T]$. Then, the call and put deltas are given by

$$\Delta_C = e^{-\delta T}\Phi(d_1) \quad \text{and} \quad \Delta_P = -e^{-\delta T}\Phi(-d_1).$$

<u>Proof</u>: For the call option,

$$\begin{aligned}
\frac{\partial C}{\partial S} &= \frac{\partial}{\partial S}\big(Se^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)\big) \\
&= e^{-\delta T}\Phi'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial S} && \text{(chain rule)} \\
&= e^{-\delta T}\Phi(d_1) + \big(Se^{-\delta T}\phi(d_1) - Ke^{-rT}\phi(d_2)\big)\frac{\partial d_1}{\partial S}.
\end{aligned}$$

(Note that $\dfrac{\partial d_2}{\partial S} = \dfrac{\partial}{\partial S}\left(d_1 - \underbrace{\sigma\sqrt{T}}_{\text{free of } S}\right) = \dfrac{\partial d_1}{\partial S}$.)

Now it suffices to show that

$$Se^{-\delta T}\phi(d_1) - Ke^{-rT}\phi(d_2) = 0. \tag{4}$$

To show this, consider:

$$\begin{aligned}
\frac{\phi(d_1)}{\phi(d_2)} &= \exp\left(-\frac{(d_1^2 - d_2^2)}{2}\right) \\
&= \exp\left(-\frac{(d_1 - d_2)(d_1 + d_2)}{2}\right) \\
&= \exp\left(-\frac{(\sigma\sqrt{T})(2d_1 - \sigma\sqrt{T})}{2}\right) \\
&= \exp\left[-\big(\ln(S/K) + (r - \delta + \sigma^2/2)T\big) + (\sigma^2/2)T\right] && \left(d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&= \exp\left[-\ln\frac{S}{K} - (r - \delta)T\right] \\
&= \frac{Ke^{-rT}}{Se^{-\delta T}}.
\end{aligned}$$

Hence, eq. (4) holds.

For the put option, using the generalized put-call parity (theorem 5.6.b), we have

$$C + Ke^{-rT} = P + Se^{-\delta T}.$$

Partially differentiating both sides with respect to S gives

$$\frac{\partial C}{\partial S} + 0 = \frac{\partial P}{\partial S} + e^{-\delta T} \implies \Delta_C = \Delta_P + e^{-\delta T} \implies \Delta_P = e^{-\delta T}(\Phi(d_1) - 1) = -e^{-\delta_1}\Phi(-d_1).$$

□

[Mnemonic: Equation (4) allows us to "treat" d_1 and d_2 "as if" they are free of S →

$$\Delta_C = \frac{\partial C}{\partial S} = e^{-\delta T}\Phi(d_1).$$

One-line "proof"!]

[Note: Recall that in [6.2.6] we use the notation Δ also in the replicating portfolio ($V = \Delta S + B$ using the notations here). For European call, by the Black-Scholes formula its time-0 value/price

$$C = Se^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

If we set $\Delta = e^{-\delta T}\Phi(d_1)$ (the call delta Δ_C here!) and $B = -Ke^{-rT}\Phi(d_2)$, then we can write $C = \Delta S + B$ → we find a portfolio replicating the payoff/value of the call *at time 0* (very limited; we want to replicate value at any time point within time interval $[0, T]$).

It turns out that to replicate the value at any time in $[0, T]$, one actually needs to *continuously* adjust the values of Δ and B as time passes and more information emerges → replicating portfolio is "dynamic" in nature.]

9.3.10 Apart from the "sensitivity interpretation", option delta can also be used for *hedging* purpose.

Suppose we just short (sell/write) an European call option and thus the "delta" of our position (SC) is $-\Delta_C$[35]. Since $-\Delta_C < 0$, our position value would *drop* (in approximate sense) if the stock price increases (slightly) → we face an "upside" stock price risk.

9.3.11 To protect against this risk, we can use *delta hedging*. To *delta hedge* our position, we can buy Δ_C shares of stock now → value of our portfolio becomes $V = -C + \Delta_C S$. Since

$$\frac{\partial V}{\partial S} = -\frac{\partial C}{\partial S} + \Delta_C = 0,$$

we are *locally* protected against this risk → our portfolio is *delta-neutral*.

9.3.12 In general, **delta hedging** (a portfolio) means entering some transactions such that the portfolio delta becomes zero. In such case, we call the portfolio **delta-neutral**.

[35]Technically, delta is only for an option and it is partial derivative of the option price. But here we extend the notion of "delta" a bit and define **portfolio delta** as the partial derivative of current portfolio value with respect to current stock price. A position may be regarded as a portfolio consisting only that position for our purpose here.

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Concepts and Terminologies

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