HKU STAT3909 Study Notes

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[Note: Related SOA Exam: ALTAM]

Contents

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1 Policy Values II

1.0.1 Building upon the knowledge of policy values learnt in STAT3901 (see the section titled "Policy Values I" in the STAT3901 study notes), here we will explore some further topics:

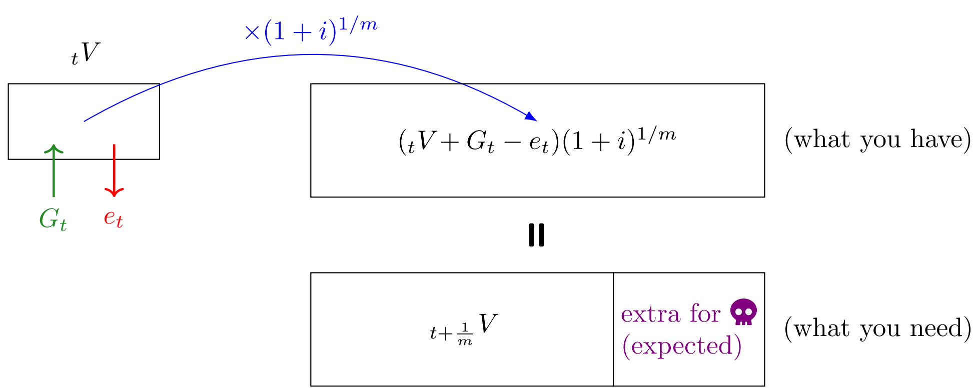
- *Continuous policy value recursion:* For each of life insurance and life annuity, we have explored the *annual*, $1/m$*thly*, and *continuous* cases in STAT3901. But for policy value recursions, we have only discussed the *annual* and $1/m$*thly* cases. The continuous case is to be discussed here.
- *Asset shares:* In STAT3901 we have learnt the concept of *retrospective* policy value, which can be interpreted as the share available in the *expected* "pool of money" at time t per *expected* survivor at time t. Here, we are going to work with *actual* quantities instead of those expected ones. In other words, we consider the share available in the *actual* "pool of money" (the asset) at time t per *actual* survivor at time t.
- *Practical issues:* Some practical issues about policy values will be discussed, namely:
  - *Dealing with policy alterations:* A policyholder may be given the flexibility to alter his policy, e.g., change from a term life insurance to a whole life insurance. The concept of policy value will be used in determining the suitable adjustments needed on the benefit amounts.
  - *Negative policy value:* While negative policy value appears to suggest that there would be expected future *gain*, prudent actuaries would avoid assigning a positive value to the policy. We will study why that is the case and some pitfalls of having negative policy value.

1.1 Continuous Policy Value Recursion: Thiele's Differential Equation

1.1.1 Recall the $1/m$thly version of policy value recursion in the net amount at risk (NAAR) form:

$$\underbrace{({}_tV + G_t - e_t)(1+i)^{\frac{1}{m}}}_{\text{(what you have)}} = \underbrace{{}_{t+\frac{1}{m}}V + {}_{\frac{1}{m}}q_{x+t}\overbrace{\Big(S_{t+\frac{1}{m}} + E_{t+\frac{1}{m}} - {}_{t+\frac{1}{m}}V\Big)}^{\text{(NAAR)}}}_{\text{(what you need)}}$$

$(t = 0, 1/m, 2/m, \ldots)$.



We can then observe that changes in policy values are influenced by multiple factors: (i) interest, (ii) premiums & renewal expenses, and (iii) NAAR $\times$ death prob. (it reduces the amount in "what we have" that can be "sent" to the policy value ${}_{t+\frac{1}{m}}V$).

1.1.2 Now, let us state the recursive formula for policy values in the continuous case, which is known as the **Thiele's differential equation**:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}{}_tV = {}_tV\delta_t + (G_t - e_t) - \underbrace{(S_t + E_t - {}_tV)}_{\text{(NAAR)}}\mu_{x+t}}$$

where:

- G_t and e_t denote the time-t *rate* of premiums and renewal expenses respectively;
- S_t and E_t denote the time-t sum insured and settlement expense respectively.

Remarks:

- Here we are in the fully continuous case, and we assume that there is no initial expense.
- Loosely, the differential equation relates "${}_tV$ and ${}_{t+\mathrm{d}t}V$", hence it is a "continuous recursion".

While we can find some "familiar" terms like $G_t - e_t$ and $S_t + E_t - {}_tV$ (NAAR) in the equation, it is not very clear what intuitive meaning this differential equation has. To understand it more intuitively, let us integrate both sides from 0 to a positive value s, which gives

$$ {}_sV - {}_0V = \int_0^s \frac{\mathrm{d}}{\mathrm{d}t}{}_tV \,\mathrm{d}t = \int_0^s \left[{}_tV\delta_t + (G_t - e_t) - (S_t + E_t - {}_tV)\mu_{x+t}\right] \mathrm{d}t\,. $$

Abusing notations a bit, we may write

$$ {}_sV = {}_0V + \int_0^s {}_tV\delta_t \,\mathrm{d}t + (G_t - e_t)\,\mathrm{d}t - (S_t + E_t - {}_tV)\mu_{x+t}\,\mathrm{d}t. $$

We can then gain some intuition out of it:

(1) Starting (time 0) reserve: ${}_0V$.

(2) In every "infinitesimal" time interval $[t, t + \mathrm{d}t]$ between time 0 and time s:
   - ⊕ *(interest)* ${}_tV\delta_t \,\mathrm{d}t$ is earned and added to the reserve.
   - ⊕ *(premium less expense)* $(G_t - e_t)\,\mathrm{d}t$ is received and added to the reserve.
   - ⊖ *(NAAR × death prob.)* $\underbrace{(S_t + E_t - {}_tV)}_{\text{NAAR}} \underbrace{\mu_{x+t}\,\mathrm{d}t}_{\text{death prob.}}$ is subtracted from the reserve.

(3) "Summing" up all these "infinitesimal" changes and adding it to the starting reserve (${}_0V$) gives the ending (time s) reserve: ${}_sV$.

[Note: *(If you are interested)* See Dickson et al. (2019) for a mathematical derivation of Thiele's differential equation.]

1.1.3 While Thiele's differential equation is certainly of theoretical interest, it is difficult to be directly used in practice, due to the high difficulty of solving this differential equation. In view of this, often *numerical* methods for solving differential equation are utilized to allow practical usages of Thiele's differential equation. One notable numerical method is **Euler's method**.

Key Formulas: (h: **step size**)

- ***Forward approximation:*** Change $\frac{\mathrm{d}}{\mathrm{d}t}{}_tV \to \frac{{}_{t+h}V - {}_tV}{h}$. This gives

$$ \frac{{}_{t+h}V - {}_tV}{h} \approx {}_tV\delta_t + (G_t - e_t) - (S_t + E_t - {}_tV)\mu_{x+t}. $$

- ***Backward approximation:*** Change $\frac{\mathrm{d}}{\mathrm{d}t}{}_tV \to \frac{{}_{t+h}V - {}_tV}{h}$, <u>and change $t \to t+h$ on the RHS.</u> This gives

$$ \frac{{}_{t+h}V - {}_tV}{h} \approx {}_{t+h}V\delta_{t+h} + (G_{t+h} - e_{t+h}) - (S_{t+h} + E_{t+h} - {}_{t+h}V)\mu_{x+t+h}. $$

These formulas allow us to perform recursions in a similar way as the $1/m$thly case, but via the Thiele's differential equation.

1.2 Asset Shares

1.2.1 *Definition:* The time-t **asset share** is the share available in the "actual pool of money" (accumulated from the actual net cash flows received before time t) per actual survivor at time t:

$$\begin{aligned}
\mathrm{AS}_t &= \frac{\mathrm{PV}_0(\ell_x \times \underline{\text{actual}}\ \text{NCFs received before time } t) \times (1+i^{\text{actual}})^t}{\underbrace{\ell_{x+t}^{\text{actual}}}_{\underline{\text{actual}}\ \text{number of survivors at time } t}} \\
&= \frac{\ell_x \times \mathrm{PV}_0(\underline{\text{actual}}\ \text{NCFs received}) \times (1+i^{\text{actual}})^t}{\ell_{x+t}^{\text{actual}}} \\
&= \frac{\mathrm{PV}_0(\underline{\text{actual}}\ \text{NCFs received})}{(v^{\text{actual}})^t\, {}_tp_x^{\text{actual}}}
\end{aligned}$$

where:

- ${}_tp_x^{\text{actual}} = \ell_{x+t}^{\text{actual}}/\ell_x$ is the "actual survival probability", i.e., the actual fraction of survivors at time t, among the people initially aged x. [Note: Similar meanings apply for "${}_tq_x^{\text{actual}}$".]
- i^{actual} is the actual effective interest rate per annum. When the actual annual interest rates are different for different years, we can modify the accumulation factor $(1+i^{\text{actual}})^t$ accordingly.

Remarks:

- We have the following convention about whether a cash flow precisely at time t should be included as cash flow "before" time t: (i) end-of-period payments (with time t being the end of that period) are *included* while (ii) start-of-period payments (with time t being the start of that period) are *excluded*. Very often, this means that (i) benefits and benefit-related expenses (i.e., settlement expenses) at time t are included, and (ii) premiums and premium-related expenses (i.e., initial and renewal expenses) at time t are excluded.
- Since actual quantities are to be used, the earliest time at which we can *possibly* compute time-t asset share is time t.

1.2.2 *Key formulas:*

(a) *(zero initial asset share)* We always have $\mathrm{AS}_0 = 0$, whenever there is no benefit at time 0 (always the case virtually).

(b) *(recursive formula)* When t is an integer time,

$$(\mathrm{AS}_t + G_t - e_t^{\text{actual}})(1+i_t^{\text{actual}}) = q_{x+t}^{\text{actual}}(S_{t+1} + E_{t+1}^{\text{actual}}) + p_{x+t}^{\text{actual}}\,\mathrm{AS}_{t+1}$$

where the quantities with "actual" superscript are the actual ones (e.g., i_t^{actual} refers to the actual time-t annual interest rate). It takes a very similar form as the usual policy value recursion.

1.3 Practical Issues

1.3.1 **Policy alteration.** Some terminologies:

- A policy is **surrendered** when some of the policy value is paid to the withdrawing policyholder, and is **lapsed** if no payment is made.
- The amount paid upon policy surrender is called the **surrender value**.
- A **paid-up policy** is a policy where premium payments stop early (they are paid up) but remains in force. Because of this, the benefit is usually reduced, and the reduced value is called the **paid-up sum insured**.

[Note: Unless otherwise specified, we shall assume that there are no policy lapses/surrenders.]

1.3.2 *Basic principle for valuation upon policy alteration:*

EPV of *original* future loss at alteration time = EPV of *revised* future loss at alteration time,

or

$$\boxed{\text{Policy value of \textit{original} policy at alteration time} = \text{Policy value of \textit{revised} policy at alteration time}}.$$

For example, if a policyholder requests to alter his term life insurance of 1 000 000 to a whole life insurance, when he is aged 65 and the term life insurance has 10 years policy term remaining, then the benefit amount W for the whole life insurance would be given by

$$1\,000\,000 A^{1}_{65:\overline{10}|} = W A_{65}$$

according to the basic principle.

1.3.3 In practice, there may be various costs and charges upon policy alteration. Such costs and charges may be regarded as a loss incurred for the revised policy, at the policy alteration time. In this example, if there is a cost of 5% of the original policy value incurred for policy alteration, then the equation would become

$$1\,000\,000 A^{1}_{65:\overline{10}|} = W A_{65} + \underbrace{0.05 \times \left(1\,000\,000 A^{1}_{65:\overline{10}|}\right)}_{\text{EPV of extra loss for revised policy}},$$

which results in a smaller benefit W.

1.3.4 **Negative policy value.** Negative policy value often arises in early time period, which is typically caused by a relatively large initial expense. To illustrate this, suppose the conditions for the equality of retrospective and usual policy values are satisfied. Then, consider ${}_tV$ for some very small t, say $t = 0.0001$. By the retrospective approach,

$$\begin{aligned} {}_tV &= \frac{\text{EPV}_0(\text{past gross premiums}) - \overbrace{\text{EPV}_0(\text{past benefits and expenses})}^{\text{involving initial expense}}}{{}_tE_x} \\ &\overset{(\text{very small } t,\ \text{very large initial expense})}{\approx} -\text{initial expense} < 0. \end{aligned}$$

It is not prudent to assign positive value to a policy having negative policy value, despite the apparent expected future gain, because the policyholder can **lapse** the policy before such gain is captured! Hence, prudent actuaries would suggest to just assign a *zero* value to such policy.

2 Multiple State Models

2.0.1 If Section 1 is described as the "appetizer", then what we are going to cover in Sections 2 to 4 would be the "main course", with Section 5 being the "dessert". Sections 2 to 4 are essentially about *generalizing* what we learnt in STAT3901 in several ways:

- *Section 2 (multiple state models):* Generalization by classifying people into multiple states.
- *Section 3 (multiple decrement models):* Generalization by classifying people *dead* through multiple causes of death (or more properly, decrement[1]).
- *Section 4 (multiple life models):* Generalization by incorporating multiple lives in a single policy.

So, in some sense, we are repeating the STAT3901 course for *three times*, one for each kind of the models here![2] While we have spent *one whole semester* for going through the STAT3901 course, we will only spend *less than one* semester for going through these three sections ("$3 \times$ STAT3901 course"). Thus, as you can expect, the pace in STAT3909 will be *much quicker* than the one in STAT3901, which should be manageable *if* you are familiar with the STAT3901 content. (Hopefully this is the case... ; if not, it is better for you to take a moment and review what you have learnt in STAT3901 first, before proceeding further.)

Section 5 is about a more practically useful topic, namely analyzing *profits*. The ultimate goal of selling insurance products is to make profits (right?), so certainly this is a topic deserving attention. We put it at the last as we are going to apply some concepts learnt in the prior sections to the profit analysis. This section is perhaps (?) the easiest part in STAT3909 and can be your "grade saver" .[3]

2.0.2 To guide our discussions about the generalizations in Sections 2 to 4, we will organize the contents covered in each of Sections 2 to 4 in the following three aspects:

(1) *Probabilistic calculations:* We will discuss formulas of various probabilistic quantities for the new models.

(2) *Insurance and annuity EPV calculations:* Armed with the formulas for computing probabilistic quantities, we will introduce EPV calculation formulas for different types of *state-contingent* (more general than "life contingent"!) insurances and annuities. As you will see, the general EPV calculation formula introduced in STAT3901 is still very useful here!

(3) *Premium and policy value calculations:* With the knowledge about insurance and annuity EPVs, we will explore premium and policy value calculations under these new models, and also calculations of some probabilistic quantities about the loss random variables underlying the premiums and policy values.

2.1 Probabilistic Calculations

2.1.1 **Motivation.** In STAT3901, we have exclusively distinguished lives according to their survival status: A life is either *alive* or *dead*. However, real life is not that binary. For instance, a life can be healthy , sick , disabled , retired, etc. Intuitively, these kinds of lives can have rather different "characteristics", so it makes sense to model differently for lives in different *states*. Mathematically, we are going to use a *multiple state model* to do this.

2.1.2 The simplest multiple state model is the one we have always been studying in STAT3901, which is called the **alive-dead model**. It can be graphically represented as follows.

[1]Analogous to the remark for asset share in Section 1.2, it is possible for a policyholder to exit from the policy without actually dying, e.g., through lapsing the policy. So the term "decrement" (which refers generally to any kind of "quitting the policy") is used instead of "death" here.

[2]Perhaps this is a bit exaggerated, as we will only repeat part of (but still quite a lot of!) the topics covered in STAT3901 for each type of the models here.

[3]Note however that "easiest" $\neq$ "easy". Certainly, you still need to pay (substantial) effort on studying this section in order for it to "save your grade" (if needed).

$$\boxed{\text{Alive (0)}} \xrightarrow{\text{die}} \boxed{\text{Dead (1)}}$$

The model has two *states*: alive (labelled as "state 0") and dead (labelled as "state 1"). The rightward arrow suggests the possible direction of transitions in states: Certainly it is only possible to go/transit from state 0 to 1, but not the reverse!

2.1.3 **Terminologies.** A multiple state model is:

- **discrete-time** if transition can only take place at the end of periods (often years).
- **continuous-time** if transition can take place at any time.

2.1.4 **Notations.** We need some new probability notations for multiple state model.

Notation	Probability that (x) ...
${}_tp_x^{ij}$	is currently in state i and will be in state j after t years.
${}_tp_x^{\overline{ii}}$	is currently in state i and will stay in state i for t years (i.e., without leaving state i in between).

Remarks:

- The notations here are not standard actuarial notations.
- ${}_tp_x^{ij}$ and ${}_tp_x^{\overline{ii}}$ are called **transition probability** and **occupancy probability** respectively.
- The "t"s in the notations can be dropped when $t = 1$.

Examples: In the alive-dead model,

- ${}_{10}p_{30}^{01}$ is the probability that a life aged 30 will be dead at time 10. Using the notations in STAT3901, it is "${}_{10}q_{30}$".
- ${}_{10}p_{30}^{00} = {}_{10}p_{30}^{\overline{00}}$ is the probability that a life aged 30 will be alive at time 10 (equivalently, be always alive for 10 years). Using the notations in STAT3901, it is "${}_{10}p_{30}$".

  [Note: Here the occupancy probability may not appear to be useful, as we are working in a multiple state model where it is not possible to go back from state 1 to state 0, so ${}_tp_x^{00} = {}_tp_x^{\overline{00}}$ always. But as we will see, in some more advanced multiple state models, the occupancy probability ${}_tp_x^{\overline{ii}}$ can be *different* from ${}_tp_x^{ii}$.]

2.1.5 **General probability calculation formula.** Like the general EPV calculation formula we learn in STAT3901, there is a general *probability* calculation formula available, which is helpful for dealing with probability calculations for multiple state model in general.

$$\boxed{\text{Desired probability} = \underbrace{\sum \text{ or } \int}_{\text{all desired transition paths}} \text{probability of going through the path}}.$$ [4]

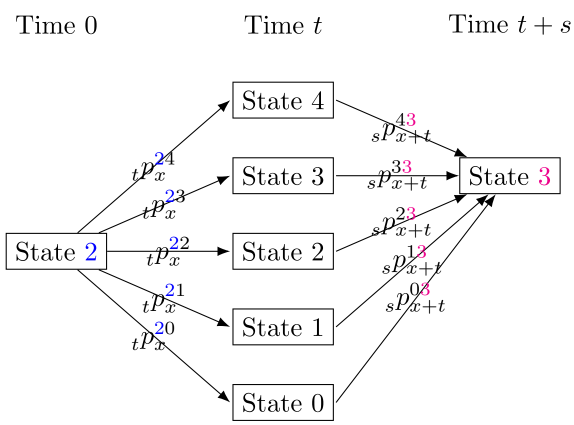
Discrete-Time Multiple State Models

2.1.6 To illustrate the general probability calculation formula, we consider a special case that is helpful for probability computations in discrete-time multiple state models. It is known as the **Chapman-Kolmogorov equation**: For any states i and j,

$$ {}_{t+s}p_x^{ij} = \sum_{\text{all states } k} {}_tp_x^{ik} \times {}_sp_{x+t}^{kj}, $$

where t and s are any nonnegative integers.

[4] *(If you are interested)* The justification of this formula is related to the mathematical formulation of multiple state model as a *Markov chain* (STAT3903). For more details about the mathematical formulation, see Dickson et al. (2019).



From this picture, we can see that summing all the states k would involve all the paths desired. Each term ${}_tp_x^{ik} \times {}_sp_{x+t}^{kj}$ in the sum corresponds to the probability of going through a path.

2.1.7 If you still recall the linear algebra knowledge from MATH1821/MATH2822 (perhaps not), you may recognize that the sum is somewhat similar to a matrix multiplication. In fact, we can indeed express the Chapman-Kolmogorov equation in a matrix form.

First we define the **k-step transition matrix** (k is a positive integer) for a multiple state model with $n+1$ states $0, 1, \ldots, n$ as the following $(n+1) \times (n+1)$ matrix:

$$P^{(k)} = \begin{bmatrix} {}_kp_x^{00} & {}_kp_x^{01} & \cdots & {}_kp_x^{0n} \\ {}_kp_x^{10} & {}_kp_x^{11} & \cdots & {}_kp_x^{1n} \\ \vdots & \vdots & \ddots & \vdots \\ {}_kp_x^{n0} & {}_kp_x^{n1} & \cdots & {}_kp_x^{nn} \end{bmatrix}$$

[Note: Usually we just call a 1-step transition matrix simply as **transition matrix** and denote it by P.] Then, we can express the Chapman-Kolmogorov equation in the following matrix form:

$$P^{(t+s)} = P^{(t)}P^{(s)}.$$

where $P^{(t)}P^{(s)}$ denotes the matrix product of $P^{(t)}$ and $P^{(s)}$. Particularly, it implies that the k-step transition matrix is indeed the kth power of the (1-step) transition matrix: $P^{(k)} = P^k$.

2.1.8 **First step analysis.** Apart from the Chapman-Kolmogorov equation, another useful tool for calculating probabilistic quantities in discrete-time multiple state models is the so-called *first-step analysis*. Essentially it is an application of the conditional variant of law of total probability/expectation:

- *(probability)*

$$\boxed{p_{ij} := \mathbb{P}(\text{ever visit state } j|\text{start in state } i) = \sum_{\text{all states } k} \mathbb{P}(\text{first step } i \to k) \times p_{kj}}.$$

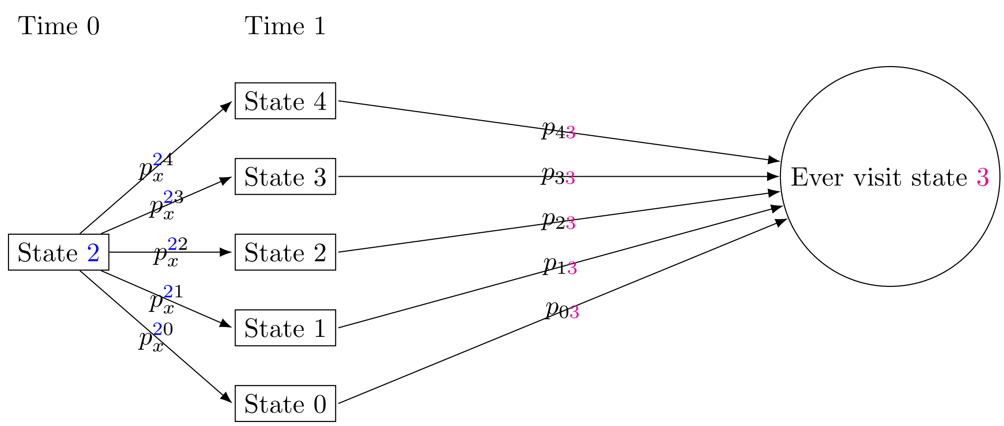
- *(expectation)*

$$\begin{aligned} \boxed{e_{ij}} &:= \mathbb{E}[\text{time spent in state } j|\text{start in state } i] \\ &= \boxed{\sum_{\text{all states } k} \mathbb{P}(\text{first step } i \to k)(e_{kj} + \delta_{ij})} \end{aligned}$$

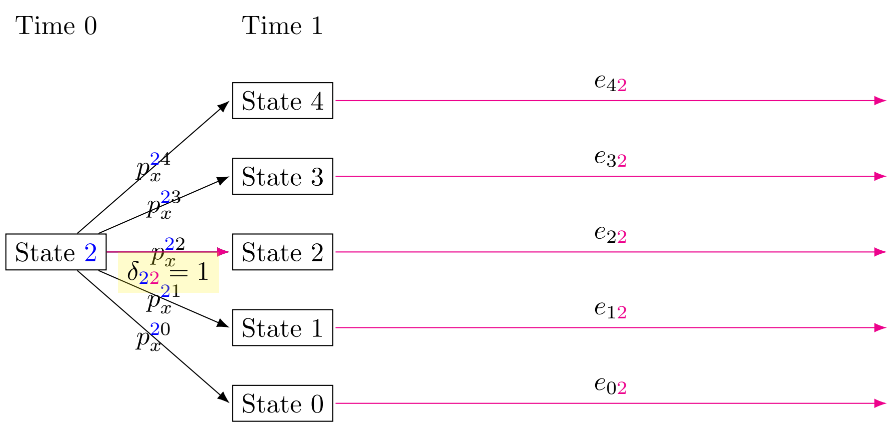
where δ_{ij} equals 1 if $i = j$, and 0 otherwise. [Note: δ_{ij} is included to count also the first period spent in state i, in case when $i = j$. Note that e_{kj} in the expression is measuring the mean time spent in state j *after* the first period, when the life is in state k at the start of the second period (time 1).]

Here $\mathbb{P}(\text{first step } i \to k)$ refers to the transition probability p_x^{ik}.

The probability formula is actually an instance of the general probability calculation formula in [2.1.5] again.



For the expectation formula, it can be understood via a similar intuition:



Continuous-Time Multiple State Models

2.1.9 **Force of transition.** The concept of *force of transition* is exclusive to *continuous-time* multiple state models. As one may expect, it has a certain similarity to the *force of mortality* learnt in STAT3901.

Recall from STAT3901 that the force of mortality μ_x is given by

$$\mu_x = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}{}_xp_0}{{}_xp_0} = \lim_{h\to 0^+}\frac{1}{{}_xp_0}\frac{{}_xp_0 - {}_{x+h}p_0}{h} = \lim_{h\to 0^+}\frac{1-{}_hp_x}{h} = \lim_{h\to 0^+}\frac{{}_hq_x}{h} \overset{\text{(alive-dead model)}}{=} \lim_{h\to 0^+}\frac{{}_hp_x^{01}}{h}.$$

This leads us to define the **force transition** from state i to state $j \neq i$, for a life aged x, by:

$$\mu_x^{ij} = \lim_{h\to 0^+}\frac{{}_hp_x^{ij}}{h}.$$

Here, like the force of mortality, we can interpret $\mu_x^{ij} \times \Delta x$ as the approximated probability for (x) to transit from state i to state $j \neq i$ in Δx years, when Δx is small. Symbolically, we can write ${}_{\Delta x}p_x^{ij} \approx \mu_x^{ij} \times \Delta x$ when Δx is small.

2.1.10 **Occupancy probability formula.** In discrete-time multiple state model, we only have "nice" formulas for transition probability, but not occupancy probability. It turns out that in *continuous-time* multiple state model, there is a "nice" formula available for occupancy probability, in terms of forces of transition:

$$\boxed{{}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij}\,\mathrm{d}s\right)}.$$

<u>Remarks</u>:

- This formula is not obtained based on the general probability formula in [2.1.5], so it deserves some more attention.
- Intuitively, $\sum_{j\neq i} \mu_{x+s}^{ij}$ sums up all the forces "driving" the life to leave state i. The larger the sum, the smaller the occupancy probability ${}_tp_x^{\overline{ii}}$.
- *(If you are interested)* The derivation of the occupancy probability formula involves some regularity assumptions on continuous-time multiple state model, which are omitted (and implicitly imposed) here. For the purpose of STAT3909, it is more important to know when and how to apply the formulas, and their proofs are not required. Nonetheless, in case you are still interested in the omitted assumptions and the derivations, see Dickson et al. (2019).

The occupancy probability formula can be seen as a generalization to the following $\mu_{x+t} \to {}_tp_x$ formula from STAT3901:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right).$$

Actually, this is a special case of the occupancy probability formula here since we can write

$${}_tp_x^{\overline{00}} = \exp\left(-\int_0^t \mu_{x+s}^{01}\,\mathrm{d}s\right)$$

using the alive-dead model notations.

2.1.11 **Kolmogorov[5] forward equation.** Like *Thiele's differential equation* in Section 1.1, *Kolmogorov forward equation* is also a differential equation, but it is about *transition probabilities*. Nonetheless, it admits a similar form of intuitive interpretation as Thiele's differential equation.

First we state the **Kolmogorov forward equation**:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x^{ij} = \sum_{\text{state } k\neq j} \left({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}\right)}$$

for any states i and j.[6]

Again, to understand it more intuitively, we integrate both sides from 0 to a positive value s:

$${}_sp_x^{ij} - {}_0p_x^{ij} = \int_0^s \sum_{\text{state } k\neq j} \left({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}\right)\mathrm{d}t\,.$$

After some rearrangements and slight abuse of notations, we get

$${}_sp_x^{ij} = \underbrace{{}_0p_x^{ij}}_{\delta_{ij}} + \int_0^s \sum_{k\neq j} {}_tp_x^{ik}\mu_{x+t}^{kj}\,\mathrm{d}t - \sum_{k\neq j} {}_tp_x^{ij}\mu_{x+t}^{jk}\,\mathrm{d}t.$$

From here we can see that there are two *opposing* "forces" active between time 0 and s that influence the chance for the life to be in state j after s years. More specifically, we can interpret it as follows:

[5] The same Kolmogorov as the one in Chapman-Kolmogorov equation!

[6] *(If you are interested)* To learn more about the derivation of Kolmogorov forward equation, see Dickson et al. (2019).

(1) "Starting" probability: ${}_0p_x^{ij}$, which serves as a baseline on which changes are continuously made to finally result in the s-year transition probability ${}_sp_x^{ij}$.

The probability is 1 if $i = j$ and 0 otherwise. Intuitively, this means that if the life *starts* at state j, then there is a somewhat large "bonus" on the chance for the life to be in state j after s years (the "starting line" is at 1).

(2) In every "infinitesimal" time interval $[t, t + \mathrm{d}t]$ between time 0 and time s:

- ⊕ *(incoming probability)* $\sum_{k \neq j} \underbrace{{}_tp_x^{ik}}_{i \to k \text{ after } t \text{ years}} \times \underbrace{\mu_{x+t}^{kj}\,\mathrm{d}t}_{\text{enter state } j \text{ in } [t, t+\mathrm{d}t]}$
- ⊖ *(outgoing probability)* $\sum_{k \neq j} \underbrace{{}_tp_x^{ij}}_{i \to j \text{ after } t \text{ years}} \times \underbrace{\mu_{x+t}^{jk}\,\mathrm{d}t}_{\text{leave state } j\ [t, t+\mathrm{d}t]}$

In short, the incoming probability (outgoing probability) sums up the probabilities corresponding to all possible paths for entering (leaving) state j in the time interval $[t, t+\mathrm{d}t]$, and we are adding incoming probability and subtracting outgoing probability. Note that **entering state** j means transiting to state j from a *different* state.

(3) "Summing" up all these "infinitesimal" contributions to the value ${}_0p_x^{ij}$, we get the "ending" (final) probability: ${}_sp_x^{ij}$.

2.1.12 **General transition probability formula.** Lastly, let us consider a general transition probability formula in continuous-time multiple state model, which is based on the general probability calculation formula in [2.1.5]: For any states i and j,

$$
{}_tp_x^{ij} = \int_0^t \left(\sum_{k \neq j} {}_sp_x^{ik}\, \mu_{x+s}^{kj} \right) \underbrace{{}_{t-s}p_{x+s}^{\overline{jj}}}_{\text{don't miss ⚠}} \,\mathrm{d}s\,.
$$
[7]

This seemingly complicated formula is indeed an instance of the general probability calculation formula. To see this, we first abuse notations a bit and write

$$
{}_tp_x^{ij} = \int_0^t \left(\sum_{k \neq j} {}_sp_x^{ik}\, \mu_{x+s}^{kj}\,\mathrm{d}s \right) {}_{t-s}p_{x+s}^{\overline{jj}}.
$$

Then, for every time point s between time 0 and time t,

- $\sum_{k \neq j} \underbrace{{}_sp_x^{ik}}_{i \to k \text{ after } s \text{ years}} \times \underbrace{\mu_{x+s}^{kj}\,\mathrm{d}s}_{\text{enter state } j \text{ in } [s, t+\mathrm{d}s]}$ is the probability for entering state j in the time interval $[s, s+\mathrm{d}s]$ (the "incoming probability" in the intuitive interpretation of Kolmogorov forward equation).
- ${}_{t-s}p_{x+s}^{\overline{jj}}$ is the probability for the life to *stay* in the state j for the remaining $t - s$ years (i.e., until time t).

Hence, $\left(\sum_{k \neq j} {}_sp_x^{ik}\, \mu_{x+s}^{kj}\,\mathrm{d}s \right)_{t-s} p_{x+s}^{\overline{jj}}$ is the probability of being in the state j at time t, by entering state j in the time interval $[s, s + \mathrm{d}s]$ and then staying in the state j until time t. "Summing" over

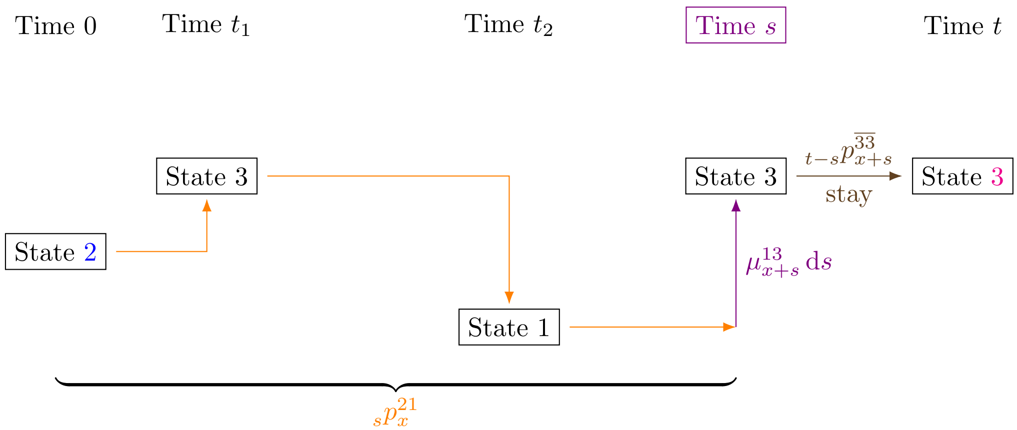
[7] *(If you are interested)* This formula is indeed obtained by solving the Kolmogorov forward equation and applying the occupancy probability formula. Solving the Kolmogorov forward equation is manageable (but not too simple still) since it can be rewritten in the form

$$
\frac{\mathrm{d}}{\mathrm{d}t}\, {}_tp_x^{ij} + \left(\sum_{k \neq j} \mu_{x+t}^{jk} \right) {}_tp_x^{ij} = \sum_{k \neq j} {}_tp_x^{ik}\, \mu_{x+t}^{kj},
$$

which is a first order linear ordinary differential equation (ODE) (learnt in MATH1821). Probably you already forget what this means and how to solve this kind of ODE, but it turns out that this form of ODE can be solved using the method of "integration factor". (Frankly speaking, I have to look this up also... 😅)

every time point s between time 0 and time t would then involve all the paths for being in the state j at time t.

[Note: For paths which enter the state j multiple times between time 0 and time t, they would be covered by the probability expression for time point s which equals the *last* time of entry to the state j.]



Several specialized and simplified versions of this general transition probability formula will appear in Sections 3 and 4.

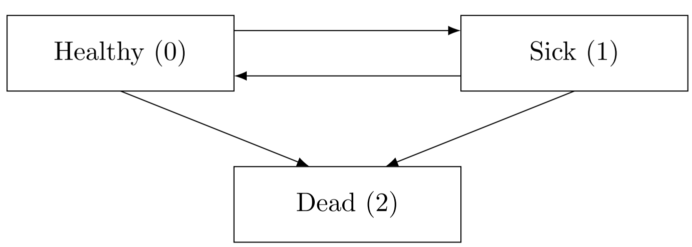
2.2 Insurance and Annuity EPV Calculations

2.2.1 After having the (somewhat lengthy) discussion about probabilistic calculations in multiple state models, we are now prepared for calculating insurance and annuity EPVs. Here we avoid using the terms "life insurance" and "life annuity" because the insurance and annuity in multiple state model are *not necessarily* life contingent anymore. We can only say that they are state contingent in general.

2.2.2 **Applications of multiple state model.** An *insurance* here may provide a benefit[8] at the end of year of each entry to a state k. Depending on what state k represents, the insurance can be interpreted differently, e.g.:

- *State* k = *Dead*: Life insurance
- *State* k = *Permanently Disabled:* Disability insurance
- *State* k = *Sick:* Health insurance
- $\vdots$

A typical example of multiple state model used in SOA exam is the *Standard Sickness-Death Model*, which looks like the following:



Likewise, an *annuity* here may provide level benefit payments at the start of each year whenever the annuitant is in a state j, and there is a high flexibility in the nature of annuity, based on the setting of state j. For example:

[8] Note that we do not use the term "death benefit" or "survival/endowment benefit" here!

- *State* $j =$ *Alive:* Life annuity
- *State* $j =$ *Temporarily Disabled:* Disability income benefits
- *State* $j =$ *Retired:* Pension (more in STAT3956)
- $\vdots$

From this, we can see that multiple state model provides a general framework for working with many types of insurance and annuity products, through "life contingencies" concepts and methods.

2.2.3 **Terminologies.** Like STAT3901, the two main types of state-contingent insurances and annuities are (i) *discrete* and (ii) *continuous*, which are defined by:

- **Discrete insurance**: Benefit is paid at the end of year of **each** entry to a state, within policy term.
- **Continuous insurance**: Benefit is paid at **each** entry to a state, within policy term.
- **Discrete annuity-due**: Benefit is paid at the start of each year whenever the annuitant is in a state, within policy term.
- **Continuous annuity**: Benefit is paid continuously while the annuitant is in a state, within policy term.

[⚠ Warning: There can be *multiple* insurance benefit payments in multiple state model, unlike the case in STAT3901!]

For simplicity, here we will not deal with the $1/m$thly case, although insurance and annuity EPVs in such case can be developed in a similar fashion.

2.2.4 **EPV notations.** Like STAT3901, we have some notations for EPV of various insurances and annuities. Again, we shall assume that the amount of each benefit payment (for discrete case) or the rate of benefit payment (for continuous case) is 1.

Comparing with the EPV notations in STAT3901, the EPV notations here have extra *superscripts* which carry information about the *states* involved. For example, the notation A_x^{ik} denotes the EPV of discrete permanent insurance for a life aged x currently in state i, which pays a benefit of 1 at the end of year of **each** entry to state k; and the notation $\ddot{a}_x^{ik}$ denotes the EPV of discrete permanent annuity-due for a life aged x currently in state i, which pays a benefit of 1 at the start of each year whenever the life is in state j at that time.

2.2.5 **EPV formulas.** The EPV formulas can all be developed using the general EPV calculation formula we learn in STAT3901, demonstrating its high utility 👍:

$$\text{EPV} = \underbrace{\sum \text{ or } \int}_{\text{all possible payment times}} \text{benefit amount} \times \text{discount factor} \times \text{prob. of triggering payment}.$$

Based on this general formula, we can obtain the following more frequently used EPV formulas:

Type	Discrete	Continuous
Permanent insurance	$A_x^{ik} = \sum_{t=0}^{\infty} v^{t+1}\left(\sum_{j\neq k} {}_tp_x^{ij}\, p_{x+t}^{jk}\right)$	$\bar{A}_x^{ik} = \int_0^{\infty} e^{-\delta t} \sum_{j\neq k} {}_tp_x^{ij}\, \mu_{x+t}^{jk}\, \mathrm{d}t$
Permanent annuity	(due) $\ddot{a}_x^{ij} = \sum_{t=0}^{\infty} v^t\, {}_tp_x^{ij}$	$\bar{a}_x^{ij} = \int_0^{\infty} e^{-\delta t}\, {}_tp_x^{ij}\, \mathrm{d}t$
n-year term insurance	$A_{x:\overline{n}\rvert}^{ik} = \sum_{t=0}^{n-1} v^{t+1}\left(\sum_{j\neq k} {}_tp_x^{ij}\, p_{x+t}^{jk}\right)$	$\bar{A}_{x:\overline{n}\rvert}^{ik} = \int_0^{n} e^{-\delta t} \sum_{j\neq k} {}_tp_x^{ij}\, \mu_{x+t}^{jk}\, \mathrm{d}t$
n-year temporary annuity	(due) $\ddot{a}_{x:\overline{n}\rvert}^{ij} = \sum_{t=0}^{n-1} v^t\, {}_tp_x^{ij}$	$\bar{a}_{x:\overline{n}\rvert}^{ij} = \int_0^{n} e^{-\delta t}\, {}_tp_x^{ij}\, \mathrm{d}t$

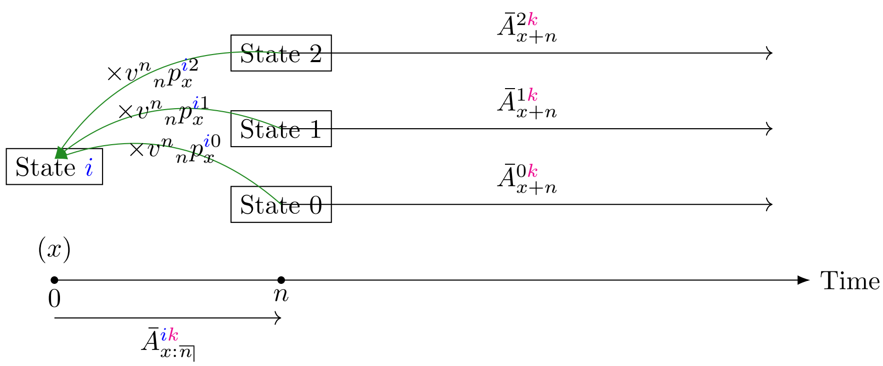
Remarks:

- Recall that *entering* a state k means transiting to state k from a *different* state!
- There is not a "1" on top of x in the notations for n-year term insurance, to avoid making the notations clumsy. Note that in a general multiple state model (not necessarily alive-dead model), we do not have a notation for denoting the EPV of an "n-year endowment insurance". We do have "annuity-insurance formula" either.

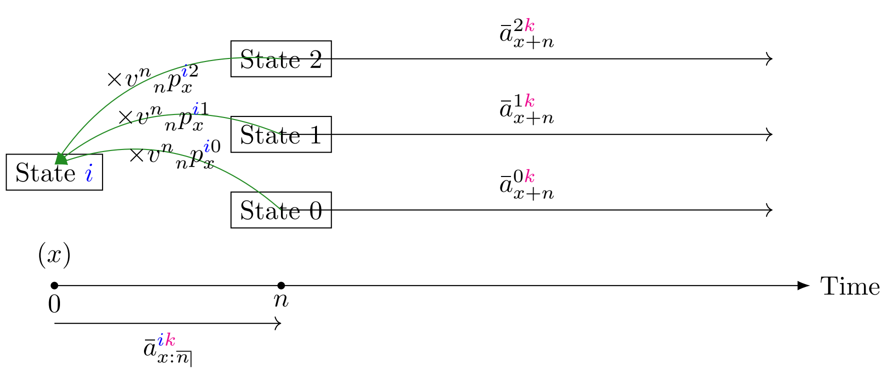
  In general, for calculating EPV of an "unfamiliar" policy (not in the list above), we need to start with the general EPV calculation formula above.
- Since multiple insurance benefit payments are possible, it is possible for the insurance EPV to *exceed* 1, unlike the case in STAT3901.

2.2.6 **Recursive formulas for EPVs.** Apart from the EPV calculation formulas above, another type of formula that is of interest is the *recursive formulas* for EPVs. The good 👍 thing is that the recursive formulas here do have similar form as the ones we learn in STAT3901. But the "bad" 👎 thing is that care must be taken when writing down the recursive formulas, to ensure that the transitions between states are correctly handled. We can't simply write down "${}_nE_x$" without thought anymore! Nonetheless, this should not be an issue if you can gain an *intuitive* understanding of the recursive formulas below. [Note: In the following, $\sum_j$ denotes the sum over every state j.]

- *(more commonly used) Permanent insurance:*
  - *(discrete)* $A_x^{ik} = A_{x:\overline{n}|}^{ik} + \sum_j v^n {}_np_x^{ij} A_{x+n}^{jk} \overset{(n=1)}{=} vp_x^{ik} + \sum_j vp_x^{ij} A_{x+1}^{jk}$.
  - *(continuous)* $\bar{A}_x^{ik} = \bar{A}_{x:\overline{n}|}^{ik} + \sum_j v^n {}_np_x^{ij} \bar{A}_{x+n}^{jk}$.



- *(more commonly used) Permanent annuity:*
  - *(discrete, due)* $\ddot{a}_x^{ik} = \ddot{a}_{x:\overline{n}|}^{ik} + \sum_j v^n {}_np_x^{ij} \ddot{a}_{x+n}^{jk} \overset{(n=1)}{=} \delta_{ik} + \sum_j vp_x^{ij} \ddot{a}_{x+1}^{jk}$.
  - *(continuous)* $\bar{a}_x^{ik} = \bar{a}_{x:\overline{n}|}^{ik} + \sum_j v^n {}_np_x^{ij} \bar{a}_{x+n}^{jk}$.



- *n-year term insurance:*
  - *(discrete)* $A^{ik}_{x:\overline{n}|} \overset{(m\leq n)}{=} A^{ik}_{x:\overline{m}|} + \sum_j v^m\, {}_mp^{ij}_x\, A^{jk}_{x+m:\overline{n-m}|} \overset{(m=1)}{=} vp^{ik}_x + \sum_j vp^{ij}_x\, A^{jk}_{x+1:\overline{n-1}|}$.
  - *(continuous)* $\bar{A}^{ik}_{x:\overline{n}|} \overset{(m\leq n)}{=} \bar{A}^{ik}_{x:\overline{m}|} + \sum_j v^m\, {}_mp^{ij}_x\, \bar{A}^{jk}_{x+m:\overline{n-m}|}$.
- *n-year temporary annuity:*
  - *(discrete, due)* $\ddot{a}^{ik}_{x:\overline{n}|} \overset{(m\leq n)}{=} \ddot{a}^{ik}_{x:\overline{m}|} + \sum_j v^m\, {}_mp^{ij}_x\, \ddot{a}^{jk}_{x+m} \overset{(m=1)}{=} \delta_{ik} + \sum_j vp^{ij}_x\, \ddot{a}^{jk}_{x+1:\overline{n-1}|}$.
  - *(continuous)* $\bar{a}^{ik}_{x:\overline{n}|} \overset{(m\leq n)}{=} \bar{a}^{ik}_{x:\overline{m}|} + \sum_j v^m\, {}_mp^{ij}_x\, \bar{a}^{jk}_{x+m:\overline{n-m}|}$.

2.2.7 **Waiting period.** For annuity benefits like disability income benefits, the benefits are often not paid *immediately* at the onset of the disability (the event covered). The time between the event onset and the start of benefit payments is called the **waiting period**. One reason for having waiting period is to give the insurer some time to verify the policyholder's eligibility for the benefit payments (e.g., check the documents to see whether he is really disabled).

With the presence of waiting period, there would be a time delay between the time of entry to the target state (e.g., temporarily disabled) and the start of the benefit payments. So, we cannot use the ordinary annuities and formulas discussed previously to perform calculations about the benefit payments of this kind of products.

2.2.8 **Continuous sojourn annuity.** To deal with waiting periods for continuous annuities in continuous-time multiple state model, we need to introduce a new concept called *continuous sojourn annuity*. It is a special annuity designed mainly for theoretical purpose (such kind of annuity can hardly be found in practice). The idea is that it serves as a "one-time-only" <u>continuous</u> annuity, lasting only for a <u>sojourn</u> ("short staying period").

A **continuous sojourn annuity** issued to (x) currently in state j pays benefits (e.g., at rate 1) continuously while (x) stays in state j, within policy term; Once (x) leaves state j, the benefits cease **forever**, i.e., there would not be any benefits anymore, even if (x) later goes back to state j.

For such continuous sojourn annuity with benefit rate 1, the notations are:

- *permanent version:* $\bar{a}^{\overline{jj}}_x$.
- *n-year temporary version:* $\bar{a}^{\overline{jj}}_{x:\overline{n}|}$.

Applying the general EPV calculation formula, we get

- $\bar{a}^{\overline{jj}}_x = \displaystyle\int_0^\infty e^{-\delta t}\, {}_tp^{\overline{jj}}_x \,\mathrm{d}t$.
- $\bar{a}^{\overline{jj}}_{x:\overline{n}|} = \displaystyle\int_0^n e^{-\delta t}\, {}_tp^{\overline{jj}}_x \,\mathrm{d}t$.

2.2.9 **Incorporating waiting periods.** Using the concept of *continuous sojourn annuity*, we can apply the general EPV calculation formula (so useful!) to incorporate waiting periods. First we consider a continuous permanent annuity issued to (x) currently in state i, with benefit rate 1 payable continuously while (x) is in state j, and waiting period w years (applicable for **each** entry to state j).

When $i \neq j$, we have

$$\boxed{\text{EPV} = \int_0^\infty \left(\bar{a}^{\overline{jj}}_{x+t} - \bar{a}^{\overline{jj}}_{x+t:\overline{w}|}\right) e^{-\delta t} \sum_{k\neq j} {}_tp^{ik}_x\, \mu^{kj}_{x+t}\,\mathrm{d}t}.$$

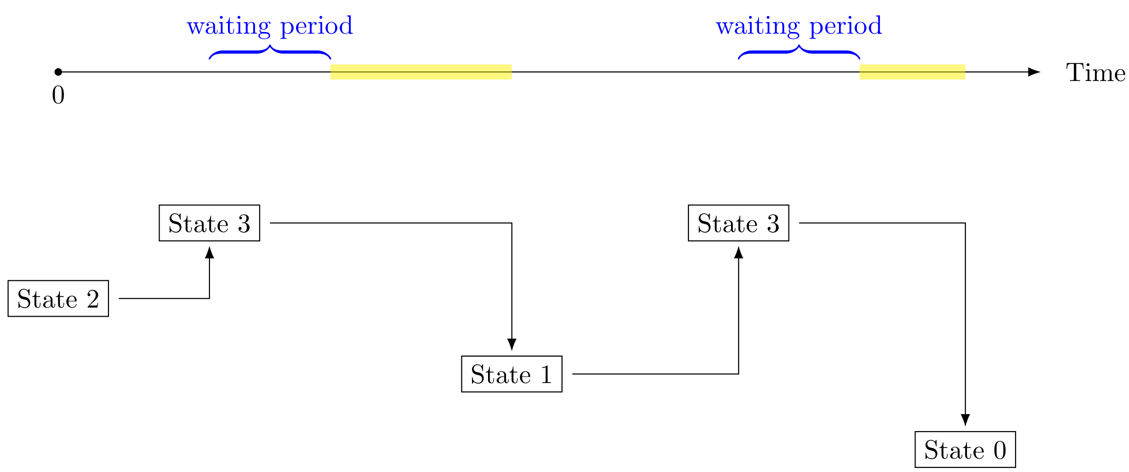
To better understand this, consider:

- "All possible payment times" start at time 0 with no end, so we have "$\int_0^\infty$".
- Discount factor at time t is $e^{-\delta t}$.

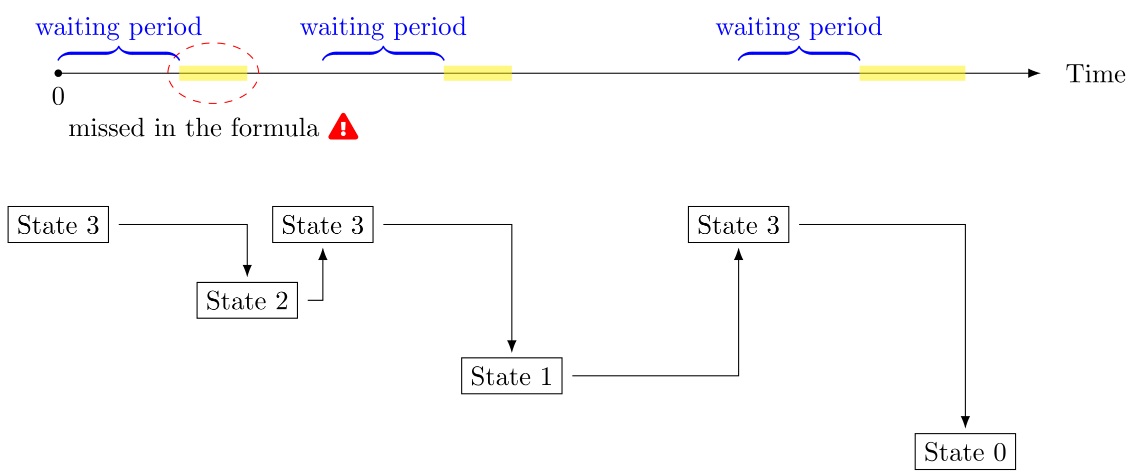
- Probability of triggering payment[9] at time t is loosely $\sum_{k\neq j} {}_tp_x^{ik}\mu_{x+t}^{kj}\,\mathrm{d}t$, which is the probability of entering state j in $[t, t+\mathrm{d}t]$, which "generates" a fresh permanent continuous sojourn annuity.
- Benefit amount at time t is the time-t EPV of the "newly generated" continuous sojourn annuity at time t, namely $\bar{a}^{\overline{jj}}_{x+t} - \underbrace{\bar{a}^{\overline{jj}}_{x+t:\overline{w}|}}_{\text{no benefits in the waiting period}}$.

  [Note: Since we are computing EPV, it would not make any difference to replace the continuous stream of payments from the continuous sojourn annuity by a single payment of amount equal to the EPV of that payment stream at time t. This is exactly the replacement we have done here.]

When we change the policy term to n years, we can simply change $\int_0^\infty \to \int_0^n$ in the formula above.



2.2.10 **Adjustment for $i = j$ case.** When $i = j$, more care is needed ⚠. Since the formula above focuses on continuous sojourn annuities "newly generated" at each entry to state j, it would miss the potential payments made immediately after the first waiting period starting now, which do not require any *entry* to state j (as the life is already in state j at the start). This issue is illustrated below.



Consequently, to calculate the correct EPV when $i = j$, we need to add an extra term $\bar{a}^{\overline{jj}}_{x} - \bar{a}^{\overline{jj}}_{x:\overline{w}|}$, which accounts for the EPV of the missed potential payments. That is, the formula becomes:

[9] Here the "payment" is indeed possibly zero due to the presence of waiting period.

- *(permanent annuity)*

$$\boxed{\text{EPV} = \bar{a}_x^{\overline{jj}} - \bar{a}_{x:\overline{w}|}^{\overline{jj}} + \int_0^\infty \left(\bar{a}_{x+t}^{\overline{jj}} - \bar{a}_{x+t:\overline{w}|}^{\overline{jj}}\right) e^{-\delta t} \sum_{k\neq j} {}_tp_x^{ik}\mu_{x+t}^{kj}\,\mathrm{d}t}.$$

- *(n-year annuity)*

$$\boxed{\text{EPV} = \bar{a}_x^{\overline{jj}} - \bar{a}_{x:\overline{w}|}^{\overline{jj}} + \int_0^n \left(\bar{a}_{x+t}^{\overline{jj}} - \bar{a}_{x+t:\overline{w}|}^{\overline{jj}}\right) e^{-\delta t} \sum_{k\neq j} {}_tp_x^{ik}\mu_{x+t}^{kj}\,\mathrm{d}t}.$$

The formula can still be explained using the terminologies in the general EPV calculation formula, since we can treat the added term as representing a single payment made at time 0 with amount equal to $\bar{a}_x^{\overline{jj}} - \bar{a}_{x:\overline{w}|}^{\overline{jj}}$ (possibly zero). The discount factor is 1 and the probability of having this payment is 1 (as the life is known to be in state j now).

2.3 Premium and Policy Value Calculations

2.3.1 Using the formulas for various insurance and annuity EPVs in Section 2.2, we can compute net/gross premiums via equivalence principle, like what we did in STAT3901. On the other hand, for policy values, although the idea behind is still similar to the STAT3901 case, adjustments are needed to incorporate the effects on policy values from different *states*. So we will focus on policy values from now on.

2.3.2 In STAT3901, the time-t policy value is defined by

- *time-t net premium policy value:*

$${}_tV^n = \text{EPV}_t(\text{future benefits}) - \text{EPV}_t(\text{future } \textbf{net} \text{ premiums}).$$

- *time-t gross premium policy value:*

$${}_tV^g = \text{EPV}_t(\text{future benefits and expenses}) - \text{EPV}_t(\text{future } \textbf{gross} \text{ premiums}).$$

Particularly, it is assumed that the policy is still in force at time t, i.e., the underlying life is alive at time t. Using the alive-dead model terminologies, it means that we assume the life is in state 0 at time t.

2.3.3 **Motivation.** For the policies in STAT3901, being in force at time t is equivalent to being in state 0 at time t. But in a more general multiple state model, this is not necessarily the case. When we just know that the policy is in force at time t, there can still be multiple possible states for the life. This would pose difficulties and complications in our calculations as the formulas discussed previously depend on a single starting state only.

Therefore, to be more compatible with our previous formulas and discussions, here we shall also define time-t policy value to be dependent on the starting state.

2.3.4 **Definition.** The time-t policy value for life in state j at time t is defined by:

- *time-t **net premium policy value**:*

$$\text{EPV}_t^{(j)}(\text{future benefits}) - \text{EPV}_t^{(j)}(\text{future } \textbf{net} \text{ premiums}).$$

- *time-t **gross premium policy value**:*

$$\text{EPV}_t^{(j)}(\text{future benefits and expenses}) - \text{EPV}_t^{(j)}(\text{future } \textbf{gross} \text{ premiums}).$$

Here $\mathrm{EPV}_t^{(j)}(\cdot)$ denotes the time-t EPV with the starting (time-t) state being state j. To avoid complicating the notations, we usually just use the notation ${}_tV^{(j)}$ to denote both the net premium and gross premium policy values. We should be able to tell which is being considered from the context (usually we are considering gross premium policy value).

Intuitively, we would expect that ${}_tV^{(j)}$ could be quite different for different states j, even if the time t remains fixed. For instance, for a whole life insurance, if we set state 0 as "healthy" and state 1 as "ill", it is natural to expect that ${}_tV^{(1)}$ would be higher than ${}_tV^{(0)}$, since a sick person should be more likely to die early than a healthy person, which raises the reserve for sick individual.

2.3.5 **Basic policy value recursion.** Like STAT3901, we are interested in developing some recursive formulas for policy values. To start with, let us consider the basic policy value version, which involves only annual time points and is for *discrete* policies. The "slogan" introduced in the STAT3901 study notes ("what you have is what you need", or WYHIWYN) still applies, and leads to the following recursive formula:

$$\boxed{\underbrace{\left({}_tV^{(j)} + G_t^{(j)} - e_t^{(j)}\right)(1+i)}_{\text{what you have}} = \underbrace{\sum_{\text{all states } k} p_{x+t}^{jk}\left(B_{t+1}^{(k)} + E_{t+1}^{(k)} + {}_{t+1}V^{(k)}\right)}_{\text{what you need (expected)}}}$$

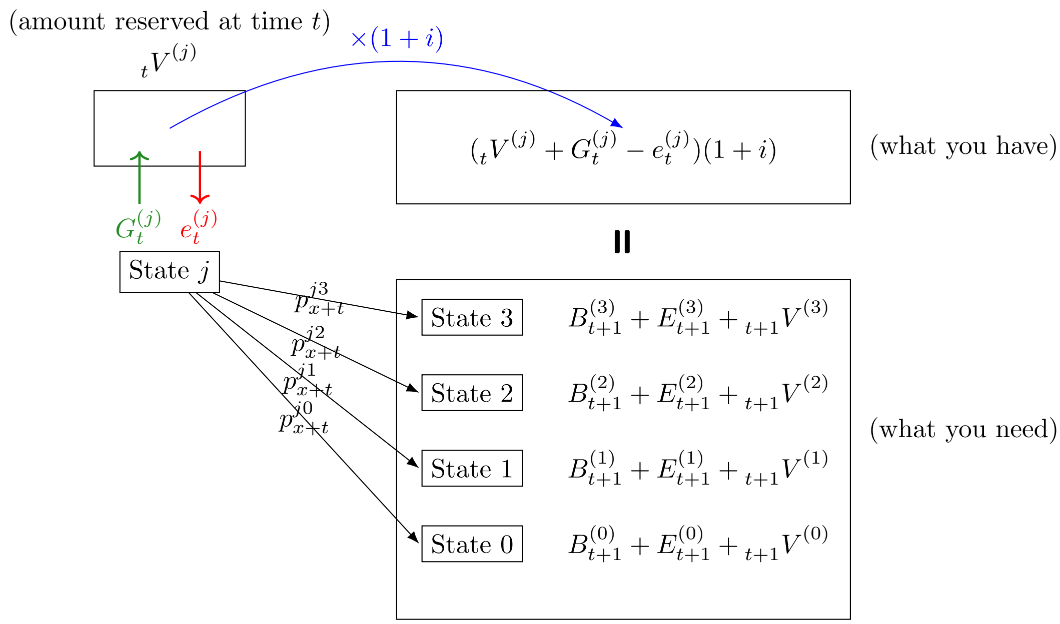
(t is an integer time point). Clumsier notations appear here as the quantities depend on state:

- $G_t^{(j)}$: gross premium at time t, for life in state j at that time
- $e_t^{(j)}$: sum of initial and renewal expenses at time t, for life in state j at that time
- $B_{t+1}^{(k)}$: amount of benefits payable at time $t+1$ for life in state k at that time, including:
  - *(insurance type benefit)* benefit paid because the life enters state k in the year $t+1$
  - *(annuity type benefit)* benefit paid because the life is in state k at time $t+1$ (start of year $t+2$) [Note: In case there is expense associated to the annuity type benefit *(seldom the case)*, we can incorporate the expense by adding it on top of the amount of annuity type benefit.]
- $E_{t+1}^{(k)}$: amount of the settlement expense associated to the insurance type benefit payable at time $t+1$ for life in state k at that time

This generalizes the basic policy recursive formula $({}_tV + G_t - e_t)(1+i) = q_{x+t}(S_{t+1} + E_{t+1}) + p_{x+t}({}_{t+1}V)$ (for life insurance) in STAT3901, since by considering the alive-dead model, we can express it as

$$({}_tV^{(0)} + G_t^{(0)} - e_t^{(0)})(1+i) = p_{x+t}^{01}(B_{t+1}^{(1)} + E_{t+1}^{(1)} + \underbrace{{}_{t+1}V^{(1)}}_{0}) + p_{x+t}^{00}(\underbrace{B_{t+1}^{(0)} + E_{t+1}^{(0)}}_{0} + {}_{t+1}V^{(0)}).$$

The following picture illustrates the basic policy value recursive formula in multiple state model setting above.



2.3.6 **"NAAR" form of basic policy value recursion.** Similar to STAT3901, we can reorganize the terms in the basic policy value recursive formula above to obtain its "net amount at risk (NAAR)"[10] form:

$$\overbrace{\left({}_tV^{(j)} + G_t^{(j)} - e_t^{(j)}\right)(1+i)}^{\text{(what you have)}} = \overbrace{{}_{t+1}V^{(j)} + B_{t+1}^{(j)} + E_{t+1}^{(j)} + \sum_{k\neq j} p_{x+t}^{jk} \underbrace{\left(B_{t+1}^{(k)} + E_{t+1}^{(k)} + {}_{t+1}V^{(k)} - {}_{t+1}V^{(j)}\right)}_{\text{"NAAR"}}}^{\text{(what you need)}},$$

which follows from rewriting

$$p_{x+t}^{jj}\left({}_{t+1}V^{(j)}\right) = \left(1 - \sum_{k\neq j} p_{x+t}^{jk}\right)\left({}_{t+1}V^{(j)}\right) = {}_{t+1}V^{(j)} - \sum_{k\neq j} p_{x+t}^{jk}\left({}_{t+1}V^{(j)}\right)$$

in the sum on the RHS of the original recursive formula.

The "NAAR" can be interpreted as the additional amount of money needed over the reserve available for the case when the life goes to state j (${}_{t+1}V^{(j)}$), in case the life goes to state $k \neq j$ rather than state j "surprisingly".

Thus, this recursive formula in "NAAR" form partitions "what you need" into (i) amount needed (payment and reserve) when the life goes to state j and (ii) extra amount needed over the reserve for going to state j, as a provision for the scenarios where the life goes to another state $k \neq j$.

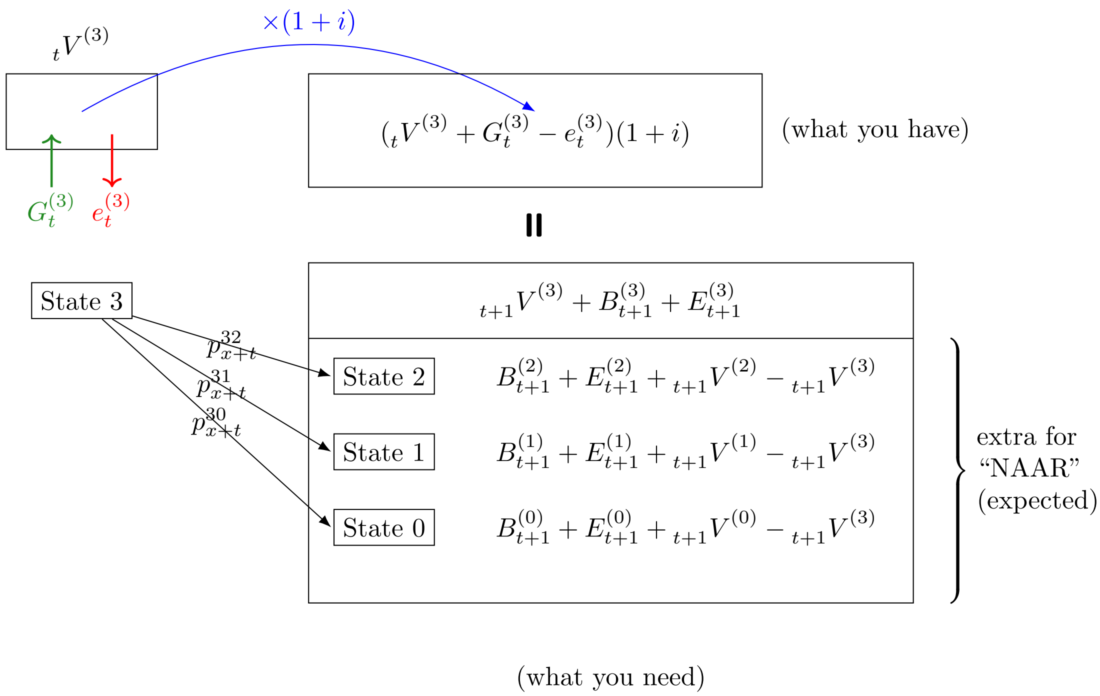
Again, this generalizes the NAAR form of basic policy recursive formula in STAT3901, namely $({}_tV + G_t - e_t)(1+i) = {}_{t+1}V + q_{x+t}(S_{t+1} + E_{t+1} - {}_{t+1}V)$, because we can express it as

$$({}_tV^{(0)} + G_t^{(0)} - e_t^{(0)})(1+i) = \underbrace{B_{t+1}^{(0)} + E_{t+1}^{(0)}}_{0} + {}_{t+1}V^{(0)} + p_{x+t}^{01}(B_{t+1}^{(1)} + E_{t+1}^{(1)} + \underbrace{{}_{t+1}V^{(1)}}_{0} - {}_{t+1}V^{(0)})$$

by considering the alive-dead model.

The following picture illustrates this recursive formula in "NAAR" form (with $j = 3$):

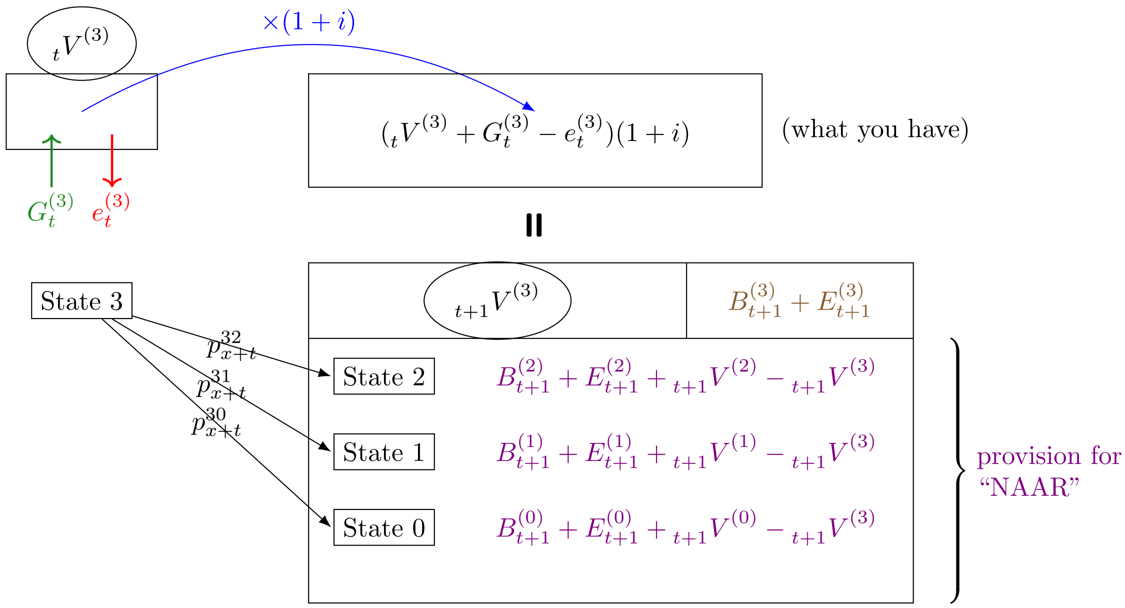
[10] Strictly speaking, in general multiple state model, the previous interpretation for NAAR may not apply anymore, as we may not be considering "dying".



2.3.7 **Continuous policy value recursion.** Again, for the sake of simplicity, we will not deal with the $1/m$thly case here, though recursions in such case can be developed in a similar way. Nevertheless, here we will discuss continuous policy value recursion in multiple state model (perhaps more complex than the $1/m$thly case? 😅). We have learnt the continuous policy value recursion in the alive-dead model in Section 1.1, which utilizes the *Thiele's differential equation*. So, as one may expect, here we are going to generalize the Thiele's differential equation to the multiple state model setting.

Like what we did in Section 1.1, we are going to study continuous policy value recursion in three steps: (i) motivation, (ii) formula, and (iii) intuitive interpretation.

2.3.8 **Motivation.** From the recursive formula in “NAAR” form above, we can observe that changes in policy values for state j ${}_tV^{(j)}$ are influenced by multiple factors: (i) interest, (ii) premiums & renewal expenses, (iii) provision for “NAAR”, and *(new!)* (iv) payments $B_{t+1}^{(j)} + E_{t+1}^{(j)}$ needed upon going to state j. So, as one may expect, these four factors would come into play in the Thiele's differential equation.



(what you need)

2.3.9 **Thiele's differential equation in multiple state model setting.** We shall consider continuous-time multiple state model and fully continuous policies here. The **Thiele's differential equation** in multiple state model setting is given by

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}{}_tV^{(j)}={}_tV^{(j)}\delta_t+G_t^{(j)}-e_t^{(j)}-R_t^{(j)}-\sum_{k\neq j}\left(S_t^{(jk)}+E_t^{(jk)}+{}_tV^{(k)}-{}_tV^{(j)}\right)\mu_{x+t}^{jk}}$$

where:

- $R_t^{(j)}$ is the time-t rate of annuity type benefit payments (with the associated expenses added, if any) while the life in state j.
- $G_t^{(j)}$ and $e_t^{(j)}$ are the time-t rates of premium payments and renewal expense payments while the life is in state j, respectively.
- $S_t^{(jk)}$ and $E_t^{(jk)}$ are the amounts of time-t insurance type benefit and the associated settlement expense, made when the life enters state j at time t, respectively.

[Note: Here we assume there is no initial expense.]

2.3.10 **Intuitive interpretation.** Like Section 1.1, we integrate both sides from 0 to a positive value s:

$$_sV^{(j)}-{}_0V^{(j)}=\int_0^s\left({}_tV^{(j)}\delta_t+G_t^{(j)}-e_t^{(j)}-R_t^{(j)}-\sum_{k\neq j}\left(S_t^{(jk)}+E_t^{(jk)}+{}_tV^{(k)}-{}_tV^{(j)}\right)\mu_{x+t}^{jk}\right)\mathrm{d}t\,.$$

Abusing notations slightly, we write

$$_sV^{(j)}={}_0V^{(j)}+\int_0^s{}_tV^{(j)}\delta_t\,\mathrm{d}t+(G_t^{(j)}-e_t^{(j)})\,\mathrm{d}t-R_t^{(j)}\,\mathrm{d}t-\sum_{k\neq j}\left(S_t^{(jk)}+E_t^{(jk)}+{}_tV^{(k)}-{}_tV^{(j)}\right)\mu_{x+t}^{jk}\,\mathrm{d}t.$$

The expression in the integral incorporates the effects from all the four factors discussed above. Let us elaborate more about the intuition:

(1) Starting (time 0) reserve for state j: ${}_0V^{(j)}$.

(2) In every "infinitesimal" time interval $[t, t+\mathrm{d}t]$ between time 0 and time s:

- ⊕ *(interest)* ${}_tV^{(j)}\delta_t\,\mathrm{d}t$ is earned and added to the reserve for state j.
- ⊕ *(premium less expense)* $(G_t^{(j)} - e_t^{(j)})\,\mathrm{d}t$ is received and added to the reserve for state j.
- ⊖ *(annuity benefit payment)* $R_t^{(j)}\,\mathrm{d}t$ is paid in the time interval $[t, t+\mathrm{d}t]$ and subtracted from the reserve for state j.
- ⊖ *(provision for "NAAR")*
  $\sum_{j\neq i}\underbrace{\left(S_t^{(jk)} + E_t^{(jk)} + {}_tV^{(k)} - {}_tV^{(j)}\right)}_{\text{"NAAR"}}\underbrace{\mu_{x+t}^{jk}\,\mathrm{d}t}_{\text{"prob." for } j\to k \text{ in } [t,t+\mathrm{d}t]}$ is subtracted from the reserve for state j.

(If you are interested) Remarks:

- The reason why only insurance type benefit "$S_t^{(jk)}$" appears in the "NAAR", but not the total benefit $B_t^{(k)}$, is that the amount of annuity type benefit within the "infinitesimal" time interval $[t, t+\mathrm{d}t]$ is "extremely small". When multiplied by the also "extremely small" quantity $\mu_{x+t}^{jk}\,\mathrm{d}t$, its contribution would be negligible.
- The "payments" part (the fourth "factor") includes only the annuity type benefit $R_t^{(j)}\,\mathrm{d}t$, without the insurance type benefit and the associated settlement expense. This is because for the insurance type benefit to be triggered in the "infinitesimal" time interval $[t, t+\mathrm{d}t]$, at least *two* changes in states need to occur within the interval: The life currently in state j needs to at least first go to another state, and then enter back state j. The chance of having such event is "negligible"[11], so the terms for insurance type benefit and its associated settlement expense can be dropped.

2.3.11 **Euler's method.** Again, we can apply the *Euler's method* to apply the Thiele's differential equation here in practice. Recall the formulas are:

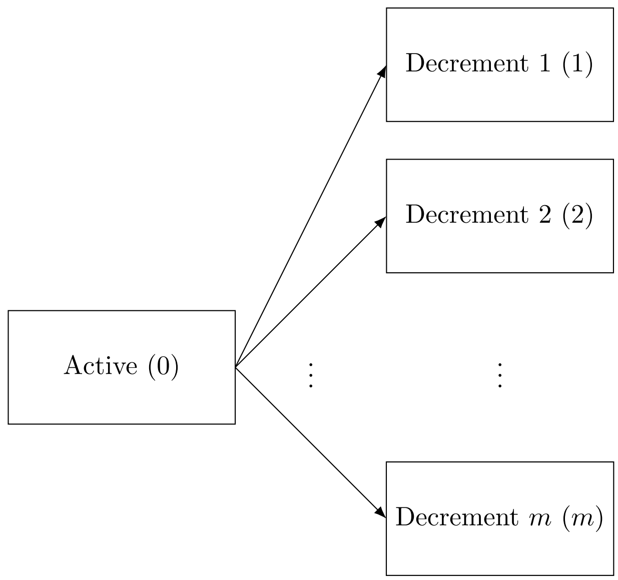
- *Forward approximation:* Change $\frac{\mathrm{d}}{\mathrm{d}t}{}_tV \to \frac{{}_{t+h}V - {}_tV}{h}$.
- *Backward approximation:* Change $\frac{\mathrm{d}}{\mathrm{d}t}{}_tV \to \frac{{}_{t+h}V - {}_tV}{h}$, <u>and change $t \to t+h$ on the RHS</u> (quite many changes would be needed, be careful ⚠).

[11] This is indeed related to an omitted assumption for the continuous-time multiple state model. But this should be "reasonable" as it is "extremely unlikely" for two changes in states to occur "simultaneously" (within the same "infinitesimal" time interval).

3 Multiple Decrement Models

3.0.1 Multiple decrement model is about generalization by classifying different ways of death/decrement. This generalization can be helpful as some policies may provide different benefit amounts for deaths due to different causes. The good thing is that multiple decrement model can actually be seen as a special case of multiple state model, so results in Section 2 can be recycled. Hence, we will concentrate on some concepts and formulas specialized for the multiple decrement model here.

3.0.2 A multiple decrement model can be graphically represented as follows.



[Note: "Active" means that the life is still "active" in the policy.]

For obvious reasons, all the arrows here are one-directional. Note also that there can only be *at most one* change in states for a multiple decrement model, namely transition from state 0 (active) to one of the decrement states. When the life transits from state 0 to state $j \neq 0$ (decrement j), we say that **decrement j occurs**. Throughout Section 3, we will work in *continuous-time* multiple decrement models (i.e., decrement can occur at any time), unless otherwise specified.

3.1 Probabilistic Calculations

3.1.1 As multiple decrement model aims to classify different ways of "death", terminologies and concepts exclusively related to death and survival from STAT3901 would make sense in multiple decrement models generally, unlike the multiple state model case. Therefore, we can now generalize concepts related to life table in STAT3901, such as those "ℓ"s and "d"s, and fractional age assumptions.

3.1.2 **Multiple decrement tables.** Naturally, the first thing to be generalized is the concept of *life table* itself. Here, instead of considering "life table", we will consider *multiple decrement table*[12], which takes the following form:

Age x	ℓ_x^{00}	d_x^{01}	d_x^{02}	$d_x^{0\bullet}$
30	10000	50.25	40	90.25
31	9909.75	60	35.5	95.5
32	9814.25	70	40	110
33	9704.25	100	45	145
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

[12] As suggested by the name of this model, we care more about decrements than "life", so we use this terminology.

The meanings of the notations are:

- ℓ_x^{00}: expected number of active lives at age x.
- d_x^{0j}: expected number of decrements j between ages x and $x+1$.
- $d_x^{0\bullet}$: expected number of decrements between ages x and $x+1$, i.e., sum of all d_x^{0j}'s. Note that $d_x^{0\bullet} = \ell_x^{00} - \ell_{x+1}^{00}$.

3.1.3 **Formulas of probabilistic quantities based on multiple decrement table.** Like life table in STAT3901, the primary purpose of multiple decrement table is to allow us to compute probabilistic quantities based on the data given in the table. Here are some key formulas ($j \neq 0$ denotes a label for decrement state):

Quantity	using ℓ_x^{00}	Using d_x^{0j}	Using $d_x^{0\bullet}$
${}_tp_x^{00} = {}_tp_x^{\overline{00}}$	$\dfrac{\ell_{x+t}^{00}}{\ell_x^{00}}$	/	/
${}_tp_x^{0j}$	/	$\dfrac{\overbrace{d_x^{0j} + \cdots + d_{x+t-1}^{0j}}^{t\text{ terms}}}{\ell_x^{00}} \overset{(t=1)}{=} \dfrac{d_x^{0j}}{\ell_x^{00}}$	/
${}_tp_x^{0\bullet}$	$1 - \dfrac{\ell_{x+t}^{00}}{\ell_x^{00}}$	/	$\dfrac{\overbrace{d_x^{0\bullet} + \cdots + d_{x+t-1}^{0\bullet}}^{t\text{ terms}}}{\ell_x^{00}} \overset{(t=1)}{=} \dfrac{d_x^{0\bullet}}{\ell_x^{00}}$
${}_{u\vert t}p_x^{0j}$	/	$\dfrac{\overbrace{d_{x+u}^{0j} + \cdots + d_{x+u+t-1}^{0j}}^{t\text{ terms}}}{\ell_x^{00}} \overset{(t=1)}{=} \dfrac{d_{x+u}^{0j}}{\ell_x^{00}}$	/
${}_{u\vert t}p_x^{0\bullet}$	$\dfrac{\ell_{x+u}^{00} - \ell_{x+u+t}^{00}}{\ell_x^{00}}$	/	$\dfrac{\overbrace{d_{x+u}^{0\bullet} + \cdots + d_{x+u+t-1}^{0\bullet}}^{t\text{ terms}}}{\ell_x^{00}} \overset{(t=1)}{=} \dfrac{d_{x+u}^{0\bullet}}{\ell_x^{00}}$
μ_x^{0j}	$-\dfrac{\frac{\mathrm{d}}{\mathrm{d}x}\ell_x^{0j}}{\ell_x^{00}}$	/	/
$\mu_x^{0\bullet}$	$-\dfrac{\frac{\mathrm{d}}{\mathrm{d}x}\ell_x^{00}}{\ell_x^{00}}$	/	/

Remarks:

- ℓ_x^{0j} denotes the expected number of lives who (i) are active at age x and (ii) *ultimately* fall into decrement j. Note that ℓ_x^{00} is the sum of all ℓ_x^{0j}'s.
- We always have ${}_tp_x^{\overline{00}} = {}_tp_x^{00}$ in multiple decrement model, because it is impossible to go back to state 0 once the life leaves it.

Here, ${}_tp_x^{00}$, ${}_tp_x^{0j}$, and μ_x^{0j} still possess the same meanings from Section 2. The notation $\mu_x^{0\bullet}$ is the sum of all μ_x^{0j}'s. [Note: Sometimes the force of transition μ_x^{0j} is called **force of decrement** here.]

For the meanings of the rest of probability notations, see the table below.

Notation with formulas	Probability that (x) ...
${}_tp_x^{0\bullet} = \sum_j {}_tp_x^{0j} = 1 - {}_tp_x^{00}$	has any decrement within t years
${}_{u\vert t}p_x^{0j} = {}_up_x^{00} \times {}_tp_{x+u}^{0j} = {}_{u+t}p_x^{0j} - {}_up_x^{0j}$	stays active for u years and has decrement j in the subsequent t years
${}_{u\vert t}p_x^{0\bullet} = {}_up_x^{00} \times {}_tp_{x+u}^{0\bullet} = {}_{u+t}p_x^{0\bullet} - {}_up_x^{0\bullet} = {}_up_x^{00} - {}_{u+t}p_x^{00}$	stays active for u years and has any decrement in the subsequent t years

[Note: The "t"s in these notations can be dropped when $t = 1$.]

3.1.4 **Specialized transition probability formula.** The general transition probability formula in [2.1.12] can be simplified substantially in the multiple decrement model setting. First recall the general formula:

$$ {}_tp_x^{ij} = \int_0^t \left(\sum_{k \neq j} {}_sp_x^{ik} \mu_{x+s}^{kj} \right) {}_{t-s}p_{x+s}^{\overline{jj}} \, \mathrm{d}s \, . $$

Certainly, when $i \neq 0$, the transition probability ${}_tp_x^{ij}$ always 1 if $j = i$ and always 0 otherwise (we do not even need to use the general formula above to tell this!). This is not too interesting, so we focus on the case when $i = 0$.

Since ${}_tp_x^{00} = {}_tp_x^{\overline{00}}$, we can just apply the occupancy probability formula to compute ${}_tp_x^{00}$. So henceforth we focus on on the transition probability of the form ${}_tp_x^{0j}$ where $j \neq 0$. Note that:

- $\mu_{x+s}^{kj} = 0$ whenever $k \neq 0$, since it is impossible to transit from one decrement state to another decrement state.
- ${}_{t-s}p_{x+s}^{\overline{jj}} = 1$ because it is impossible to leave a decrement state.

Therefore, we can simplify the general formula as:

$$ \boxed{{}_tp_x^{0j} = \int_0^t {}_sp_x^{00} \mu_{x+s}^{0j} \, \mathrm{d}s}. $$

3.1.5 **Fractional age assumptions.** Like life table, multiple decrement table often contains values for *integer* ages only. Hence, to compute probabilistic quantities involving fractional ages and years based on multiple decrement table, a *fractional age assumption* is needed. Here we will discuss two common ones[13]:

- **uniform distribution of decrements** (UDD again): ${}_tp_x^{0j} = tp_x^{0j}$ for $\underline{\text{any decrement } j}$ and any $0 \leq t \leq 1$.
- **constant forces of decrement** (CF): $\mu_{x+t}^{0j} = \mu_x^{0j}$ for $\underline{\text{any decrement } j}$ and any $0 \leq t < 1$.

3.1.6 **Key formulas for fractional age assumptions.** Below, we assume that x is an integer age and t is any value in $[0, 1]$.

	UDD	CF
${}_tp_x^{00}$	—	$= {}_tp_{x+s}^{00} = (p_x^{00})^t$
${}_tp_x^{0j}$	$= {}_{u\vert t}p_x^{0j} = tp_x^{0j}$	$\dfrac{p_x^{0j}}{p_x^{0\bullet}} \times {}_tp_x^{0\bullet}$ ("share" of j)
${}_tp_x^{0\bullet}$	$= {}_{u\vert t}p_x^{0\bullet} = tp_x^{0\bullet}$	—
μ_{x+t}^{0j}	$\dfrac{p_x^{0j}}{1 - tp_x^{0\bullet}}$	$\dfrac{p_{x+t}^{0j}}{p_{x+t}^{0\bullet}} \times \mu_{x+t}^{0\bullet}$ ("share" of j)

For finding quantities whose formulas are not found in the table, a general strategy is to express them in terms of some quantities with known formulas. For example:

- UDD: ${}_tp_x^{00} = 1 - {}_tp_x^{0\bullet} = 1 - t \cdot p_x^{0\bullet}$.

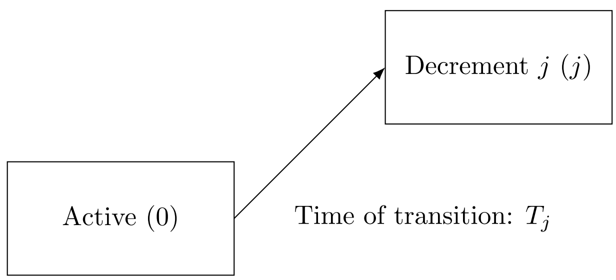
[13] Let's forget about the somewhat "unpleasant" and seldom used *Balducci assumption*...😅

- UDD: $(0 \le s+t \le 1)$ ${}_tp_{x+s}^{0j} = \dfrac{{}_{s|t}p_x^{0j}}{{}_sp_x^{00}} = \dfrac{tp_x^{0j}}{1 - sp_x^{0\bullet}}$.
- CF: ${}_tp_x^{0\bullet} = 1 - {}_tp_x^{00} = 1 - (p_x^{00})^t$.

3.1.7 **Random variable approach to multiple decrement model.** So far we have dealt with multiple decrement model as a special case of multiple state model. But this is not the only way of handling multiple decrement model. Another notable approach is to introduce suitable *random variables* for a life aged x that is currently active:

- $T = T_x$: time of occurrence of decrement.
- $J = J_x$: label of the decrement occurred (or cause of the eventual decrement). For a multiple decrement model with m decrements, it is a discrete random variable taking value in $\{1, \dots, m\}$.

3.1.8 We can define T more precisely as follows. First, for any $j = 1, \dots, m$ (assuming there are m decrements), let T_j denote the "notional" time of occurrence of decrement j in isolation, i.e., the time of transition from state 0 to state j after "extracting" these two states from the multiple decrement model (without changing the probabilistic behaviour of transition from state 0 to state j):



Then, the random variable T is defined by $T = \min\{T_1, \dots, T_m\}$. In other words, the observed decrement in the multiple decrement model is seen as the decrement with the shortest "notional" time of occurrence in isolation, and such time in isolation is taken as the occurrence time in the multiple decrement model. One (reasonable) assumption to be imposed here is that $T_1, \dots, T_m$ are independent.

By default, the multiple decrement model is continuous-time, thus $T_1, \dots, T_m$ are all continuous. In [3.1.14], we will deal with a case where some of $T_1, \dots, T_m$ are *discrete*, which is trickier.

3.1.9 **Formulas for probabilistic quantities based on random variable approach.** In the following table, we list some key formulas for probabilistic quantities under this random variable approach. (Here we assume there are m decrements.)

Quantity	Formula
marginal density function $f_T(t)$	$\sum_{j=1}^{m} f_{T,J}(t,j)$
marginal mass function $f_J(j)$	$\int_0^\infty f_{T,J}(t,j)\,\mathrm{d}t = \lim_{t\to\infty} {}_tp_x^{0j}$
${}_tp_x^{00} = {}_tp_x^{\overline{00}}$	$\mathbb{P}(T>t) = \int_t^\infty f_T(s)\,\mathrm{d}s$
${}_tp_x^{0j}$	$\mathbb{P}(T\le t\cap J=j) = \int_0^t f_{T,J}(s,j)\,\mathrm{d}s$
${}_tp_x^{0\bullet}$	$\mathbb{P}(T\le t) = \int_0^t f_T(s)\,\mathrm{d}s$
${}_{u\vert t}p_x^{0j}$	$\mathbb{P}(u<T\le u+t\cap J=j) = \int_u^{u+t} f_{T,J}(s,j)\,\mathrm{d}s$
${}_{u\vert t}p_x^{0\bullet}$	$\mathbb{P}(u<T\le u+t) = \int_u^{u+t} f_T(s)\,\mathrm{d}s$
μ_{x+t}^{0j}	$\frac{f_{T,J}(t,j)}{\mathbb{P}(T>t)} = \frac{f_{T,J}(t,j)}{{}_tp_x^{00}}$
$\mu_{x+t}^{0\bullet}$	$\frac{f_T(t)}{\mathbb{P}(T>t)} = \frac{f_T(t)}{{}_tp_x^{00}} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x^{00}}{{}_tp_x^{00}}$
$\mathring{e}_x^{00} = \mathbb{E}[T]$	$\int_0^\infty {}_tp_x^{00}\,\mathrm{d}t$

[Note: Here $f_{T,J}$ denotes the joint probability function of T and J.]

One can also define the "curtate" time of occurrence of decrement $K := \lfloor T\rfloor$ and develop the corresponding formulas. But things are starting to feel overwhelming, let us stop here for now...😅 (See [3.3.2] for an example usage of K.)

3.1.10 **Associated single decrement model.** In [3.1.8], we have "extracted" models containing only single decrement from a multiple decrement model. Each such single decrement model obtained in this way is called an **associated single decrement model**. One important property is that every associated single decrement model preserves the probabilistic behaviour of transition from the active state (state 0) to the decrement state in the multiple decrement model. More specifically, we require the force of decrement μ_x^{0j} to be always the same for both the multiple decrement model and the associated single decrement model (for decrement j).

Transition probabilities in the associated single decrement model for decrement j are defined based on the random variable T_j (time of occurrence of decrement j in the associated single decrement model) and have special notations:

- *Transition probability from state 0 to state 0:* ${}_tp_x^{\prime 00(j)} = \mathbb{P}(T_j > t)$.
- *Transition probability from state 0 to state j:* ${}_tp_x^{\prime 0j} = \mathbb{P}(T_j \le t)$.

[Note: The decrement probability ${}_tp_x^{\prime 0j}$ is sometimes called **independent probability**, and the decrement probability ${}_tp_x^{0j}$ is sometimes called **dependent probability**. To better understand these terminologies, consider:

$$
{}_tp_x^{0j} = \int_0^t \underbrace{{}_tp_x^{00}}_{{}_tp_x^{\overline{00}}} \mu_{x+s}^{0j}\,\mathrm{d}s = \int_0^t \exp\left(-\int_0^s \mu_{x+r}^{0\bullet}\,\mathrm{d}r\right)\mu_{x+s}^{0j}\,\mathrm{d}s\,.
$$

From this expression, we can see that ${}_tp_x^{0j}$ is <u>dependent</u> on the forces of transition for other decrements, not just decrement j. On the other hand, ${}_tp_x^{\prime 0j}$ is <u>independent</u> from the forces of transition for decrements other than decrement j.]

3.1.11 **Relationships between multiple decrement model and associated single decrement model.** We can relate the transition probability in the multiple decrement model with the ones here via the following equation (assuming there are m decrements):

$$\boxed{{}_tp_x^{00} = \prod_{j=1}^{m} {}_tp_x^{\prime 00(j)}}.$$

Proof. Note that

$$ {}_tp_x^{00} = \mathbb{P}(T > t) = \mathbb{P}(\min\{T_1, \dots, T_m\} > t) = \mathbb{P}(T_1 > t \cap \cdots \cap T_m > t) = \prod_{j=1}^{m} \mathbb{P}(T_j > t) = \prod_{j=1}^{m} {}_tp_x^{\prime 00(j)}.$$

□

One consequence of this relationship is that ${}_tp_x^{0j} < {}_tp_x^{\prime 0j}$ when there are more than one decrement in effect for the multiple decrement model (i.e., there is a positive force of decrement other than μ_x^{0j}). Intuitively, this happens due to the presence of *competing* decrements in multiple decrement model: If a life has a decrement $k \neq j$, then he cannot have the decrement j anymore. So in this sense, decrements j and k are "competing".

Proof. When there are more than one decrement in effect, ${}_tp_x^{\prime 00(k)} < 1$ for some $k \neq j$. Hence, ${}_tp_x^{00} < {}_tp_x^{\prime 00(j)}$. Thus,

$$ {}_tp_x^{0j} = \int_0^t {}_sp_x^{00} \mu_{x+s}^{0j} \, ds < \int_0^t {}_sp_x^{\prime 00(j)} \mu_{x+s}^{0j} \, ds = {}_tp_x^{\prime 0j}.$$

□

3.1.12 **Further relationships under UDD.** Without further assumptions, it is not possible to directly obtain ${}_tp_x^{0j}$ from ${}_tp_x^{\prime 0j}$, and vice versa. So here, we will impose a further assumption, namely UDD, which yields some nice formulas that allow us to go from one type of quantity to another. Depending on whether we assume UDD in the multiple decrement model (known as **MUDD**) or in all the associated single decrement models (known as **SUDD**), the formulas would be different.

MUDD is just the usual UDD assumption discussed previously (namely assuming ${}_tp_x^{0j} = tp_x^{0j}$ for every decrement j and any $0 \le t \le 1$). SUDD means imposing the UDD assumption on every associated single decrement model, i.e., assuming ${}_tp_x^{\prime 0j} = tp_x^{\prime 0j}$ for every decrement j and any $0 \le t \le 1$.

Formulas for relating p_x^{0j} and $p_x^{\prime 0j}$ under MUDD and SUDD are as follows ($0 \le t \le 1$).

	MUDD	SUDD
$p_x^{0j} \to p_x^{\prime 0j}$	${}_tp_x^{\prime 0j} = 1 - ({}_tp_x^{00})^{\overbrace{p_x^{0j}/p_x^{0\bullet}}^{(\text{“share” of } j)}}$	(solve system of equations from below numerically)
$p_x^{\prime 0j} \to p_x^{0j}$	(solve the equation above)	${}_tp_x^{0j} = p_x^{\prime 0j}\left(t - \dfrac{t^2}{2}\sum_{k \neq j} p_x^{\prime 0k} + \dfrac{t^3}{3}\sum_{k,\ell \neq j} p_x^{\prime 0k} p_x^{\prime 0\ell} - \cdots\right)$

Often we are using these formulas with $t = 1$, but not always.

3.1.13 **More about formulas under SUDD.** Using the formulas under SUDD is a bit tricky in practice. First, let us consider the formula for $p_x^{\prime 0j} \overset{\text{SUDD}}{\to} p_x^{0j}$. It would become more complex (rather quickly!) as there are more decrements. In practice, this formula is usually used when there are 2 or 3 decrements, so let us write down the formulas more explicitly in these cases:

- *2 decrements (1, 2):*

$$\begin{cases} {}_tp_x^{01} = p_x'^{01}\left(t - \dfrac{t^2}{2}p_x'^{02}\right) \overset{(t=1)}{=} p_x'^{01}\left(1 - \dfrac{1}{2}p_x'^{02}\right), \\ {}_tp_x^{02} = p_x'^{02}\left(t - \dfrac{t^2}{2}p_x'^{01}\right) \overset{(t=1)}{=} p_x'^{02}\left(1 - \dfrac{1}{2}p_x'^{01}\right). \end{cases}$$

- *3 decrements (1, 2, 3):*

$$\begin{cases} {}_tp_x^{01} = p_x'^{01}\left(t - \dfrac{t^2}{2}(p_x'^{02} + p_x'^{03}) + \dfrac{t^3}{3}(p_x'^{02}p_x'^{03})\right) \overset{(t=1)}{=} p_x'^{01}\left(1 - \dfrac{1}{2}(p_x'^{02} + p_x'^{03}) + \dfrac{1}{3}(p_x'^{02}p_x'^{03})\right), \\ {}_tp_x^{02} = p_x'^{02}\left(t - \dfrac{t^2}{2}(p_x'^{01} + p_x'^{03}) + \dfrac{t^3}{3}(p_x'^{01}p_x'^{03})\right) \overset{(t=1)}{=} p_x'^{02}\left(1 - \dfrac{1}{2}(p_x'^{01} + p_x'^{03}) + \dfrac{1}{3}(p_x'^{01}p_x'^{03})\right), \\ {}_tp_x^{03} = p_x'^{03}\left(t - \dfrac{t^2}{2}(p_x'^{01} + p_x'^{02}) + \dfrac{t^3}{3}(p_x'^{01}p_x'^{02})\right) \overset{(t=1)}{=} p_x'^{03}\left(1 - \dfrac{1}{2}(p_x'^{01} + p_x'^{02}) + \dfrac{1}{3}(p_x'^{01}p_x'^{02})\right). \end{cases}$$

Next, regarding the "formula" for $p_x^{0j} \overset{\text{SUDD}}{\to} p_x'^{0j}$, let us explain more about what is meant by "numerically solving the system of equations". Let us use the 2 decrements case with $t = 1$ as an example (almost always the case when we are to apply this method in practice).

From above, we have a system of equations for this case:

$$\begin{cases} p_x^{01} = p_x'^{01}\left(1 - \dfrac{1}{2}p_x'^{02}\right), \\ p_x^{02} = p_x'^{02}\left(1 - \dfrac{1}{2}p_x'^{01}\right), \end{cases}$$

which can be rearranged as

$$\begin{cases} p_x'^{01} = \dfrac{p_x^{01}}{1 - (1/2)p_x'^{02}}, \\ p_x'^{02} = \dfrac{p_x^{02}}{1 - (1/2)p_x'^{01}}. \end{cases}$$

It is difficult to solve this system analytically in general, so we would apply the following *iterative* process to solve it *numerically*:

(1) *(Initialization)* Assign ${}^{(0)}p_x'^{01} \leftarrow p_x^{01}$ and ${}^{(0)}p_x'^{02} \leftarrow p_x^{02}$ initially.

(2) *(Iteration)* For $i = 1, \ldots, N$ (N: number of iterations):

$$\begin{cases} {}^{(i)}p_x'^{01} \leftarrow \dfrac{p_x^{01}}{1 - (1/2)\left({}^{(i-1)}p_x'^{02}\right)}, \\ {}^{(i)}p_x'^{02} \leftarrow \dfrac{p_x^{02}}{1 - (1/2)\left({}^{(i-1)}p_x'^{01}\right)}. \end{cases}$$

(3) *(Return final results)* The numerical solution is given by ${}^{(N)}p_x'^{01}$ and ${}^{(N)}p_x'^{02}$.

[Note: Instead of fixing a specific number of iterations N to be performed, one can also iterate until a certain stopping criterion is met (e.g., the assigned values of $p_x'^{01}$ and $p_x'^{02}$ remain unchanged up to a specific number of decimal places after an iteration).]

3.1.14 **Presence of discrete decrements.** So far we have exclusively discussed multiple decrement model in the *continuous-time* setting. Now, in the last part of Section 3.1, we will discuss calculations outside this setting. More specifically, we consider the case where some decrements are discrete, i.e., some of $T_1, \ldots, T_m$ (as defined in [3.1.8]) are discrete. Here we will introduce formulas for computing dependent decrement probability ${}_tp_x^{0j}$ in terms of independent decrement probabilities:

- *Decrement j is continuous (i.e., T_j is continuous):*

$$ {}_tp_x^{0j} = \int_0^t {}_sp_x^{00}\mu_{x+s}^{0j}\,\mathrm{d}s = \boxed{\int_0^t \left(\prod_{k=1}^{m} {}_sp_x^{\prime 00(k)}\right)\mu_{x+s}^{0j}\,\mathrm{d}s}. $$

  - *Special case:* If we further assume SUDD for decrement j and $0 \le t \le 1$, then

$$ \boxed{{}_tp_x^{0j} = p_x^{\prime 0j}\int_0^t \prod_{k\neq j} {}_sp_x^{\prime 00(j)}\,\mathrm{d}s} $$

  since ${}_sp_x^{\prime 00(j)}\mu_{x+s}^{0j} = p_x^{\prime 0j}$ in this case (by [3.1.6]).

- *(Tricky ⚠) Decrement j is discrete (i.e., T_j is discrete):*

  (1) For each possible time of decrement s (e.g., $t = 1, 2, , \ldots$ if decrement j can only occur at the end of each year), the **dependent** probability of having decrement j at time s is

$$ \begin{aligned} \mathbb{P}(T = s \cap J = j) &= \mathbb{P}\left(\left(\bigcap_{k\neq j} T_k > s\right) \cap T_j = s\right) \\ &= \left(\prod_{k\neq j}\mathbb{P}(T_k > s)\right)\mathbb{P}(T_j = s) \\ &= \boxed{\left(\prod_{k\neq j} {}_sp_x^{\prime 00(k)}\right) \times \textbf{independent}\text{ probability of having decrement } j \text{ at time } s}. \end{aligned} $$

  Intuitively, the first term $\prod_{k\neq j} {}_tp_x^{\prime 00(k)}$ is the probability of "surviving through" all decrements other than decrement j, so the idea is that the life needs to both "survive through" all other decrements for s years, and has decrement j at time s.

  (2) We then have

$$ {}_tp_x^{0j} = \mathbb{P}(T \le t \cap J = j) = \boxed{\sum_{s\le t}\mathbb{P}(T = s \cap J = j)}. $$

3.2 Insurance and Annuity EPV Calculations

3.2.1 While we have spent quite a lot of time to discuss probabilistic calculations in multiple decrement model, we will go through the remaining parts of Section 3 in a much shorter time (fortunately!), as there are not much "new" things to be discussed: Most results are just specialized versions of what have seen in Section 2.

3.2.2 **EPV formulas.** EPV formulas in multiple decrement model are essentially special cases of the ones in [2.2.5], and can again be developed based on the famous general EPV calculation formula. So we will just list some more frequently used EPV formulas below for reference (assuming amount of each benefit payment/benefit rate is 1):

Type	Discrete	Continuous
Permanent insurance	$A_x^{0k} = \sum_{t=0}^{\infty} v^{t+1} {}_tp_x^{00} p_{x+t}^{0k}$	$\bar{A}_x^{0k} = \int_0^{\infty} e^{-\delta t} {}_tp_x^{00} \mu_{x+t}^{0k} \,\mathrm{d}t$
Permanent annuity	(due) $\ddot{a}_x^{00} = \sum_{t=0}^{\infty} v^t {}_tp_x^{00}$	$\bar{a}_x^{00} = \int_0^{\infty} e^{-\delta t} {}_tp_x^{00} \mathrm{d}t$
n-year term insurance	$A_{x:\overline{n}\vert}^{0k} = \sum_{t=0}^{n-1} v^{t+1} {}_tp_x^{00} p_{x+t}^{0k}$	$\bar{A}_{x:\overline{n}\vert}^{0k} = \int_0^{n} e^{-\delta t} {}_tp_x^{00} \mu_{x+t}^{0k} \,\mathrm{d}t$
n-year temporary annuity	(due) $\ddot{a}_{x:\overline{n}\vert}^{00} = \sum_{t=0}^{n-1} v^t {}_tp_x^{00}$	$\bar{a}_{x:\overline{n}\vert}^{00} = \int_0^{n} e^{-\delta t} {}_tp_x^{00} \mathrm{d}t$

3.3 Premium and Policy Value Calculations

3.3.1 **Loss random variable.** Here we are going to discuss the only "new" thing, namely *loss random variable*. We have not discussed it in the multiple state model setting, because dealing with it in such a general setting is somewhat complicated and would not be too fruitful. But in the multiple decrement model setting, the concept of loss random variable would become more manageable and interesting.

A natural way to develop the loss random variable in the multiple decrement setting is to utilize the random variable approach: The time-t (present value of) **future loss random variable** is defined by:

- *time-t* ***net future loss:*** $L_t^n = \mathrm{PV}_t(\text{future benefits}) - \mathrm{PV}_t(\text{future } \textbf{net} \text{ premiums})$
- *time-t* ***gross future loss:*** $L_t^g = \mathrm{PV}_t(\text{future benefits and expenses}) - \mathrm{PV}_t(\text{future } \textbf{gross} \text{ premiums})$

Here we assume that the life (aged x at time 0) is active at time t.

As one may expect, the terms would involve the time of occurrence of decrement $T = T_{x+t}$ and the cause-of-decrement random variable $J = J_{x+t}$. Indeed, the future loss often takes the following form (we use L_t^g as an example):

$$L_t^g = \begin{cases} \underbrace{\mathrm{PV}_t(\text{future benefits and expenses } \textbf{for decrement } 1)}_{\text{(expression in terms of } T)} - \underbrace{\mathrm{PV}_t(\text{future gross premiums})}_{\text{(expression in terms of } T)} & \text{if } J = 1, \\ \vdots & \\ \underbrace{\mathrm{PV}_t(\text{future benefits and expenses } \textbf{for decrement } m)}_{\text{(expression in terms of } T)} - \underbrace{\mathrm{PV}_t(\text{future gross premiums})}_{\text{(expression in terms of } T)} & \text{if } J = m, \end{cases}$$

where PV_t(future gross premiums) is independent from J as premiums are only payable at state 0 (active state).

3.3.2 **Case Study: Whole life double indemnity insurance.** To illustrate how we work with loss random variable in the multiple decrement model, let us use a *whole life double indemnity insurance* as an example.

First we consider the fully continuous case, with the following terms:

- A benefit of $2S$ is paid immediately at the occurrence of accidental death (decrement 1).
- A benefit of S is paid immediately at the occurrence of non-accidental death (decrement 2).
- Net premiums of rate P are paid continuously until (accidental or non-accidental) death.

Then, the time-t net future loss of this policy would be

$$L_t^n = \begin{cases} 2Sv^{T-t} - P\bar{a}_{\overline{T-t}\vert} & \text{if } J = 1, \\ Sv^{T-t} - P\bar{a}_{\overline{T-t}\vert} & \text{if } J = 2. \end{cases}$$

On the other hand, for the fully discrete case with the following terms:

- A benefit of $2S$ is paid at the end of year of accidental death (decrement 1).
- A benefit of S is paid at the end of year of non-accidental death (decrement 2).
- A net premium P is paid at the beginning of each year whenever the life insured is active.

Then, the time-t net future loss of this policy would be

$$L_t^n = \begin{cases} 2Sv^{K+1-t} - P\ddot{a}_{\overline{K+1-t|}} & \text{if } J = 1, \\ Sv^{K+1-t} - P\ddot{a}_{\overline{K+1-t|}} & \text{if } J = 2, \end{cases}$$

where t is an integer time and $K \triangleq \lfloor T \rfloor$ is the "curtate" time of occurrence of decrement.

With the future loss random variables, we can then perform calculations about their probabilistic quantities, premiums, and policy values, like what we did in STAT3901. (You may want to review the corresponding parts in STAT3901, and try to "translate" the tasks there to the case here.) Here the time-t policy value is given by:

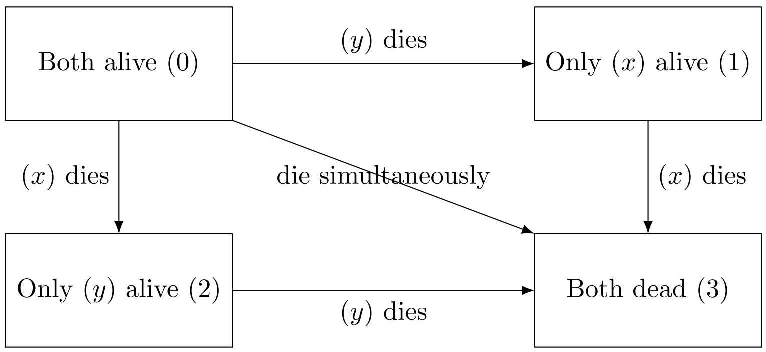
- *time-t net premium policy value:* ${}_tV^n = \mathbb{E}[L_t^n]$.
- *time-t gross premium policy value:* ${}_tV^g = \mathbb{E}[L_t^g]$.

The time-t policy value here coincides with the "${}_tV^{(0)}$" for the multiple state model.

4 Multiple Life Models

4.0.1 Multiple life model is a generalization by increasing the number of lives involved in a single policy. So far, we have only considered the case with only one life in a single policy, but in practice, a policy can involve more than one lives, e.g., couple insurance (2 lives) and business partner insurance (2 or more lives). How should we model such kind of products? The good news is that multiple life model can again be seen as a special case of multiple state model, which allows us to recycle results in Section 2. So, like what we did in Section 3, we will focus on concepts and formulas specialized for the multiple life model here.

4.0.2 To illustrate how the multiple life model is a special case of multiple state model, consider the case with 2 lives *(main focus in Section 4)*: (x) (a life aged x) and (y) (another life aged y). In this case, the multiple life model can be graphically represented as follows.



[Note: By regarding "(x)" and "(y)" in the state names as labels for lives originally aged x and y respectively, the state names here can still be used at later time (where the two lives are not aged x and y respectively anymore).]

[Warning: There are different ways of labelling the states, so be careful about the definition being used. Here we shall always adopt the labelling above for multiple life model with 2 lives.]

For obvious reasons, all the directions here are one-directional. The direct transition from state 0 to state 3 (simultaneous death) may occur when, e.g., both lives die together in a car crash.

Unless otherwise specified, throughout Section 4, we will work in *continuous-time* multiple life models with *2 lives*, where transitions can take place at any time.

4.1 Probabilistic Calculations

4.1.1 **Revised multiple state model notations.** Since there are more than one lives involved in a multiple life model, we need to use a *revised* version of the previous multiple state model notations (like "${}_tp_x^{ij}$", "μ_{x+t}^{ij}", "$\bar{A}_x^{ij}$", etc.), so that the notations can carry the age information from all the lives involved.

To make the changes properly, the key is to look at the starting state:

- *State 0:* Both lives are alive and involved, so we need to include the ages for both of them in the post-subscript.
- *State 1:* Only the life originally aged x is alive and involved, so we only need to include the age for that life in the post-subscript.
- *State 2:* Only the life originally aged y is alive and involved, so we only need to include the age for that life in the post-subscript.

(We are not interested in studying quantities with starting state being 3.)

Examples:

- ${}_tp_x^{0j} \to {}_tp_{xy}^{0j}$, $\mu_x^{0j} \to \mu_{xy}^{0j}$.
  [Note: When we replace x and y by actual numbers, we should add a colon to separate them. For example, we should write ${}_tp_{20:30}^{0j}$ rather than "${}_tp_{2030}^{0j}$" $\leftarrow$ a life aged 2030???]
- ${}_sp_{x+t}^{0j} \to {}_sp_{x+t:y+t}^{0j}$, $\mu_{x+t}^{0j} \to \mu_{x+t:y+t}^{0j}$.
- ${}_tp_x^{2j} \to {}_tp_y^{2j}$, $\mu_x^{2j} \to \mu_y^{2j}$.
- ${}_sp_{x+t}^{2j} \to {}_sp_{y+t}^{2j}$, $\mu_{x+t}^{2j} \to \mu_{y+t}^{2j}$.
- ${}_tp_x^{\overline{00}} \to {}_tp_{xy}^{\overline{00}}$, ${}_tp_x^{0\bullet} \to {}_tp_{xy}^{0\bullet}$, $\mu_{x+t}^{0\bullet} \to \mu_{x+t:y+t}^{0\bullet}$.
  [Note: Here, "$0\bullet$" quantity refers to the sum of all corresponding "$0j$" quantities with $j = 1, 2, 3$.]
- $\bar{A}_{x:\overline{n}|}^{03} \to \bar{A}_{xy:\overline{n}|}^{03}$, $\ddot{a}_x^{00} \to \ddot{a}_{xy}^{00}$.

4.1.2 **Specialized occupancy probability formulas.** Recall the occupancy probability formula from [2.1.10]:

$${}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij}\,\mathrm{d}s\right).$$

In the multiple life model setting, we can simplify it to get:

- ${}_tp_{xy}^{00} = {}_tp_{xy}^{\overline{00}} = \exp\left(-\int_0^t \mu_{x+s:y+s}^{0\bullet}\,\mathrm{d}s\right)$.
- ${}_tp_x^{11} = {}_tp_x^{\overline{11}} = \exp\left(-\int_0^t \mu_{x+s}^{13}\,\mathrm{d}s\right)$.
- ${}_tp_y^{22} = {}_tp_y^{\overline{22}} = \exp\left(-\int_0^t \mu_{y+s}^{23}\,\mathrm{d}s\right)$.

4.1.3 **Specialized transition probability formulas.** Simplifying the general transition probability formula from [2.1.12]:

$${}_tp_x^{ij} = \int_0^t \left(\sum_{k\neq j} {}_sp_x^{ik}\mu_{x+s}^{kj}\right) {}_{t-s}p_{x+s}^{\overline{jj}}\,\mathrm{d}s\,,$$

we get:

- ${}_tp_{xy}^{01} = \int_0^t {}_sp_x^{00} \times \mu_{x+s:y+s}^{01} \times {}_{t-s}p_{x+s}^{11}\,\mathrm{d}s$.
- ${}_tp_{xy}^{02} = \int_0^t {}_sp_x^{00} \times \mu_{x+s:y+s}^{02} \times {}_{t-s}p_{x+s}^{22}\,\mathrm{d}s$.
- ${}_tp_{xy}^{03} = \int_0^t \left({}_sp_{xy}^{00}\mu_{x+s:y+s}^{03} + {}_sp_{xy}^{01}\mu_{x+s}^{13} + {}_sp_{xy}^{02}\mu_{y+s}^{23}\right)\mathrm{d}s$.
- ${}_tp_x^{13} = \int_0^t {}_sp_x^{11}\mu_{x+s}^{13}\,\mathrm{d}s$.
- ${}_tp_y^{23} = \int_0^t {}_sp_y^{22}\mu_{y+s}^{23}\,\mathrm{d}s$.

Remarks:

- It is impossible to leave state 3 once it is entered.
- Be careful about the post-subscripts of the notations.

4.1.4 **Random variable approach to multiple life model.** Like multiple decrement model, multiple life model can be handled using a random variable approach. The idea is to (as expected) introduce multiple future lifetime random variables, one for each life.

Let T_x and T_y denote the (not necessarily independent) future lifetime random variables of (x) and (y) respectively. When considering T_x and T_y *individually* (marginal distributions), we are back to the case in STAT3901 where the notations there would apply (e.g., ${}_tp_x = \mathbb{P}(T_x > t)$, ${}_tq_y = \mathbb{P}(T_y \leq t)$, etc.). But here, in multiple life model, we are more interested in their *joint* behaviour and *joint* distribution. Particularly, we are going to study two ways of "combining" the two lifetimes together: (i) joint-life status and (ii) last-survivor status.

4.1.5 **Joint-life status.** The idea of *joint-life status* is to "join" or "link" the two lives together. The death of one of the lives would "break the link", and in such case the joint-life status is deemed to "fail".

The **joint-life status** of two lives (x) and (y), denoted by (xy) (or $(x{:}y)$ if it looks better), is said to *fail* when one of the lives dies. Let T_{xy} denote the time to failure of the joint-life status (xy). By treating "fail" as "die" and "not fail" as "survive", we may regard (xy) as a "special life" and T_{xy} as its "future lifetime". By definition, we have $T_{xy} = T_x \wedge T_y = \min\{T_x, T_y\}$.

Using the concept of joint-life status, we can (finally!) explain the rationale behind the seemingly strange "$x{:}\overline{n}|$" notations appearing in STAT3901 (e.g., endowment insurance EPV $\bar{A}_{x:\overline{n}|}$ and temporary life annuity EPV $\bar{a}_{x:\overline{n}|}$). The underlying idea is to treat "$(\overline{n}|)$" as a "life" whose "lifetime" is *certainly* n years, i.e., "$T_{\overline{n}|} = n$" always. Hence, the "future lifetime" of the "joint-life status" $(x{:}\overline{n}|)$ is $T_{x:\overline{n}|} = T_x \wedge T_{\overline{n}|} = T_x \wedge n$. Then, for example, we can treat an n-year endowment insurance issued to (x) as a whole life insurance issued to "$(x{:}\overline{n}|)$", which explains the notation "$\bar{A}_{x:\overline{n}|}$".

4.1.6 **Last-survivor status.** The idea of *last-survivor status* is to, as its name suggests, represent the status of last survivor. When the last survivor dies, the status is deemed to "fail".

The **last-survivor status** of two lives (x) and (y), denoted by $(\overline{xy})$ (or $(\overline{x{:}y})$ if it looks better), is said to *fail* when all lives die (equivalently, the last survivor dies). Let $T_{\overline{xy}}$ denote the time to failure of the last-survivor status. Like joint-life status, $(\overline{xy})$ can be treated as a "special life" with "future lifetime" being $T_{\overline{xy}}$, when we regard "fail" as "die" and "not fail" as "survive". By definition, we have $T_{\overline{xy}} = T_x \vee T_y = \max\{T_x, T_y\}$.

4.1.7 **Joint-life and last-survivor statuses for more than two lives.** Joint-life and last-survivor statuses can be naturally extended to the case with more than two lives. Suppose there are m lives $(x_1), \dots, (x_m)$. Their joint-life status is denoted by $(x_1 \cdots x_m)$ and we have $T_{x_1 \cdots x_m} = \min\{T_{x_1}, \dots, T_{x_m}\}$; Their last-survivor status is denoted by $(\overline{x_1 \cdots x_m})$ and we have $T_{\overline{x_1 \cdots x_m}} = \max\{T_{x_1}, \dots, T_{x_m}\}$.

Although this works for any number of lives, very often the maximum number of lives we are asked to deal with is 3.

4.1.8 **Combining multiple joint-life and last-survivor statuses.** Since joint-life and last-survivor statuses can be treated as "special lives", it is possible to combine multiple statuses by considering joint-life/last-survivor status of two "special lives". For example:

- Joint-life status of two "special lives" (xy) and (wz):
  - *Notation:* $(xy{:}wz)$ (or $(x{:}y{:}w{:}z)$).
  - *Time to failure:* $T_{xy:wz} = \min\{T_{xy}, T_{wz}\} = \min\{T_x, T_y, T_w, T_z\}$.

  Note that we can "simplify" this bulky joint-life status to a "simpler" joint-life status of 4 lives: (x), (y), (w), and (z).
- Last-survivor status of two "special lives" $(\overline{xy})$ and $(\overline{wz})$:
  - *Notation:* $(\overline{\overline{xy}{:}\overline{wz}})$ (or $(\overline{\overline{x{:}y}{:}\overline{w{:}z}})$).
  - *Time to failure* $T_{\overline{\overline{xy}:\overline{wz}}} = \max\{T_{\overline{xy}}, T_{\overline{wz}}\} = \max\{T_x, T_y, T_w, T_z\}$.

  Note that we can "simplify" this bulky last-survivor status to a "simpler" last-survivor status of 4 lives: (x), (y), (w), and (z).

- Last-survivor status of two "special lives" $(\overline{xy})$ and (wz):
  - *Notation:* $(\overline{\overline{xy}:wz})$
  - *Time to failure:* $T_{\overline{\overline{xy}:wz}} = \max\{T_{\overline{xy}}, T_{wz}\} = \max\{\max\{T_x, T_y\}, \min\{T_w, T_z\}\}$.

Things would get complicated quickly as we combine more statuses together, so let us stop here.

Revisiting Survival Model Topics for Joint-Life and Last-Survivor Statuses

4.1.9 As we associate joint-life and last-survivor statuses with "special lives", and associate time to failure with "future lifetime", we can study numerous aspects about the joint-life and last-survivor statuses by revisiting the following topics about survival models from STAT3901 (with slight changes in names):

- probabilities about ~~future lifetime~~ time to failure
- force of ~~mortality~~ failure
- curtate ~~future lifetime~~ time to failure
- moments of ~~future lifetimes~~ time to failure
- recursions for expectations of ~~life~~ time to failure

[Note: We will focus on the case with 2 lives for simplicity, but similar developments can be done for the cases with more lives.]

4.1.10 **Probabilities about time to failure.**

(1) *Notations:* The probability notations from STAT3901 can be recycled for the joint-life and last-survivor statuses, by treating them as "special lives":

Notation	Probability that ...
${}_tp_{xy} = S_{xy}(t) = \mathbb{P}(T_{xy} > t)$	(xy) "survives" t years
${}_tq_{xy} = F_{xy}(t) = \mathbb{P}(T_{xy} \le t)$	(xy) "dies" within t years
${}_{u\|t}q_{xy} = \mathbb{P}(u < T_{xy} \le u+t)$	(xy) "survives" u years and "dies" in the subsequent t years
${}_tp_{\overline{xy}} = S_{\overline{xy}}(t) = \mathbb{P}(T_{\overline{xy}} > t)$	$(\overline{xy})$ "survives" t years
${}_tq_{\overline{xy}} = F_{\overline{xy}}(t) = \mathbb{P}(T_{\overline{xy}} \le t)$	$(\overline{xy})$ "dies" within t years
${}_{u\|t}q_{\overline{xy}} = \mathbb{P}(u < T_{\overline{xy}} \le u+t)$	$(\overline{xy})$ "survives" u years and "dies" in the subsequent t years

(2) *Key formulas:*

(i) *Factorization formula for "p" for* ***joint-life status****:*

$$\underbrace{{}_{t+u}p_{xy}}_{\text{"survive" } t+u \text{ yrs.}} = \underbrace{{}_tp_{xy}}_{\text{"survive" } t \text{ yrs. first}} \times \underbrace{{}_up_{x+t:y+t}}_{\text{then "survive" } u \text{ more yrs.}}.$$ [14]

[⚠ Warning: We do NOT have a corresponding formula for last-survivor status like below:

$${}_{t+u}p_{\overline{xy}} = {}_tp_{\overline{xy}} \times {}_up_{\overline{x+t:y+t}}.$$

The key issue is that, after $(\overline{xy})$ "survives" t years, it may NOT become $(\overline{x+t:y+t})$ ⚠. This is because "surviving" t years here just means the last-survivor status $(\overline{xy})$ does *not* fail for t years, i.e., *not all* lives die within t years. It is possible for *exactly one* life to die ☠ within t years! In such a situation, "$(\overline{x+t:y+t})$" does not make sense anymore as it is defined on the condition that both lives are alive (NOT the case here!).
On the other hand, the joint-life status does not have such an issue, since after (xy) "survives" t years, it indeed becomes $(x+t:y+t)$, like a "normal" life. This is because not failing for t years really means both lives survive t years in the joint-life status case.]

[14] *(If you are interested)* Like the case for STAT3901, this can be proven mathematically by considering conditional probability, with the natural assumption that $T_{xy} - t | T_{xy} > t \overset{d}{=} T_{x+t:y+t}$, where "$\overset{d}{=}$" denotes equality in distribution. Note that it does NOT make sense to impose an analogous definition for last-survivor status ⚠, due to the reason discussed in the "warning" part.

(ii) *Formulas for "${}_{u|t}q_{\square}$":*
Joint-life status:

- *"$p \times q$" form:*
$$ {}_{u|t}q_{xy} = \underbrace{{}_{u}p_{xy}}_{\text{"survive" } u \text{ yrs.}} \times \underbrace{{}_{t}q_{x+u:y+u}}_{\text{"die" in the subsequent } t \text{ yrs.}} $$
- *"$p - p$" form:*
$$ {}_{u|t}q_{xy} = \underbrace{{}_{u}p_{xy}}_{\text{"survive" } u \text{ yrs.}} \underbrace{- {}_{u+t}p_{xy}}_{\text{but not "survive" } u+t \text{ yrs.}} $$
- *"$q - q$" form:*
$$ {}_{u|t}q_{xy} = \underbrace{{}_{u+t}q_{xy}}_{\text{"die" in } u+t \text{ yrs.}} \underbrace{- {}_{u}q_{xy}}_{\text{but not "die" in } u \text{ yrs.}} $$

Last-survivor status:

- *"$p - p$" form:*
$$ {}_{u|t}q_{\overline{xy}} = \underbrace{{}_{u}p_{\overline{xy}}}_{\text{"survive" } u \text{ yrs.}} \underbrace{- {}_{u+t}p_{\overline{xy}}}_{\text{but not "survive" } u+t \text{ yrs.}} $$
- *"$q - q$" form:*
$$ {}_{u|t}q_{\overline{xy}} = \underbrace{{}_{u+t}q_{\overline{xy}}}_{\text{"die" in } u+t \text{ yrs.}} \underbrace{- {}_{u}q_{\overline{xy}}}_{\text{but not "die" in } u \text{ yrs.}} $$

[⚠ Warning: Due to the same reason as above, we do NOT have "$p \times q$" form for last-survivor status.]

(iii) *(New!) Joint-life + Last-survivor = Individual sum:*

- ${}_{t}p_{xy} + {}_{t}p_{\overline{xy}} = {}_{t}p_{x} + {}_{t}p_{y}$.
- ${}_{t}q_{xy} + {}_{t}q_{\overline{xy}} = {}_{t}q_{x} + {}_{t}q_{y}$.
- *(density functions)* $f_{xy}(t) + f_{\overline{xy}}(t) = f_{x}(t) + f_{y}(t)$.

[Note: These follow from the inclusion-exclusion principle of probability.]

(iv) *(New!) Inclusion-exclusion type formulas:* Here we consider the case with 3 lives (x), (y), and (z). Denote ${}_{t}p_{xyz} = \mathbb{P}(T_{xyz} > t)$, ${}_{t}q_{xyz} = \mathbb{P}(T_{xyz} \le t)$, ${}_{t}p_{\overline{xyz}} = \mathbb{P}(T_{\overline{xyz}} > t)$, and ${}_{t}q_{\overline{xyz}} = \mathbb{P}(T_{\overline{xyz}} \le t)$. Then we have:

- ${}_{t}q_{xyz} = {}_{t}q_{x} + {}_{t}q_{y} + {}_{t}q_{z} - {}_{t}q_{\overline{xy}} - {}_{t}q_{\overline{xz}} - {}_{t}q_{\overline{yz}} + {}_{t}q_{\overline{xyz}}$.
- ${}_{t}p_{xyz} = {}_{t}p_{x} + {}_{t}p_{y} + {}_{t}p_{z} - {}_{t}p_{\overline{xy}} - {}_{t}p_{\overline{xz}} - {}_{t}p_{\overline{yz}} + {}_{t}p_{\overline{xyz}}$.
- ${}_{t}q_{\overline{xyz}} = {}_{t}q_{x} + {}_{t}q_{y} + {}_{t}q_{z} - {}_{t}q_{xy} - {}_{t}q_{xz} - {}_{t}q_{yz} + {}_{t}q_{xyz}$.
- ${}_{t}p_{\overline{xyz}} = {}_{t}p_{x} + {}_{t}p_{y} + {}_{t}p_{z} - {}_{t}p_{xy} - {}_{t}p_{xz} - {}_{t}p_{yz} + {}_{t}p_{xyz}$.

[Note: Not surprisingly, these again follow from the inclusion-exclusion principle of probability.]

4.1.11 Force of failure.

(1) *Definitions:* The **force of failure** at time t is defined by:

- *Joint-life status:* $\mu_{xy}(t) = -\dfrac{S'_{xy}(t)}{S_{xy}(t)} = -\dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t}p_{xy}}{{}_{t}p_{xy}}$.
- *Last-survivor status:* $\mu_{\overline{xy}}(t) = -\dfrac{S'_{\overline{xy}}(t)}{S_{\overline{xy}}(t)} = -\dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{t}p_{\overline{xy}}}{{}_{t}p_{\overline{xy}}}$.

(2) *Interpretations:* The forces of failure here share similar interpretations to the force of mortality:

- $\mu_{xy}(t)\Delta t$: approximated probability for (xy) to fail in the time interval $[t, t + \Delta t]$, when Δt is small.
- $\mu_{\overline{xy}}(t)\Delta t$: approximated probability for $(\overline{xy})$ to fail in the time interval $[t, t + \Delta t]$, when Δt is small.

(3) *Key formulas:*

(i) *Force of failure* $\mu_\square \to$ *survival function* ${}_tp_\square$*:*

- *(joint-life status)*
$$ {}_tp_{xy} = \exp\left(-\int_0^t \mu_{xy}(s)\,\mathrm{d}s\right). $$
- *(last-survivor status)*
$$ {}_tp_{\overline{xy}} = \exp\left(-\int_0^t \mu_{\overline{xy}}(s)\,\mathrm{d}s\right). $$

(ii) *Density functions:*

- *(joint-life status)* $f_{xy}(t) = {}_tp_{xy} \times \mu_{xy}(t)$.
- *(last-survivor status)* $f_{\overline{xy}}(t) = {}_tp_{\overline{xy}} \times \mu_{\overline{xy}}(t)$.

4.1.12 **Curtate time to failure.**

(1) *Definitions:* The **curtate time to failure** is defined by:

- *Joint-life status:* $K_{xy} = \lfloor T_{xy} \rfloor$.
- *Last-survivor status:* $K_{\overline{xy}} = \lfloor T_{\overline{xy}} \rfloor$.

(2) *Key formulas:*

(i) *Mass functions:*

- *(joint-life status)* $\mathbb{P}(K_{xy} = k) = {}_{k|}q_{xy} = \begin{cases} \text{“}p \times q\text{” form ...} \\ \text{“}p - p\text{” form ...} \\ \text{“}q - q\text{” form ...} \end{cases}.$
- *(last-survivor status)* $\mathbb{P}(K_{\overline{xy}} = k) = {}_{k|}q_{\overline{xy}} = \begin{cases} \cancel{\text{“}p \times q\text{” form ...}} \\ \text{“}p - p\text{” form ...} \\ \text{“}q - q\text{” form ...} \end{cases}.$

(ii) *Cumulative distribution functions:*

- *(joint-life status)* $\mathbb{P}(K_{xy} \le k) = {}_{k+1}q_{xy}$.
- *(last-survivor status)* $\mathbb{P}(K_{\overline{xy}} \le k) = {}_{k+1}q_{\overline{xy}}$.

4.1.13 **Moments of time to failure.**

(1) *Notations:*

Notation	Meaning	Notation	Meaning
$\mathring{e}_{xy}$	$\mathbb{E}[T_{xy}]$	$\mathring{e}_{xy:\overline{n}\rvert}$	$\mathbb{E}[T_{xy} \wedge n]$
$\mathring{e}_{\overline{xy}}$	$\mathbb{E}[T_{\overline{xy}}]$	$\mathring{e}_{\overline{xy}:\overline{n}\rvert}$	$\mathbb{E}[T_{\overline{xy}} \wedge n]$
e_{xy}	$\mathbb{E}[K_{xy}]$	$e_{xy:\overline{n}\rvert}$	$\mathbb{E}[K_{xy} \wedge n]$
$e_{\overline{xy}}$	$\mathbb{E}[K_{\overline{xy}}]$	$e_{\overline{xy}:\overline{n}\rvert}$	$\mathbb{E}[K_{\overline{xy}} \wedge n]$

(2) *Key formulas:* ($\square = xy$ or $\overline{xy}$)

(i) *Shortcut formulas for expectation:*

- $\mathring{e}_\square = \displaystyle\int_0^\infty {}_tp_\square\,\mathrm{d}t.$
- $e_\square = \displaystyle\sum_{k=1}^\infty {}_kp_\square.$

(ii) *Shortcut formulas for temporary expectation:*

- $\mathring{e}_{\square:\overline{n}\rvert} = \displaystyle\int_0^n {}_tp_\square\,\mathrm{d}t.$
- $e_{\square:\overline{n}\rvert} = \displaystyle\sum_{k=1}^n {}_kp_\square.$

(iii) *(Less commonly used) Shortcut formulas for second moment:*
   - $\mathbb{E}\left[T_{\square}^2\right] = \int_0^{\infty} 2t \times {}_tp_{\square}\,\mathrm{d}t.$
   - $\mathbb{E}\left[K_{\square}^2\right] = \sum_{k=1}^{\infty}(2k-1)_kp_{\square}.$

(iv) *(New!) Joint-life + Last-survivor = Individual sum:*
   - $\mathring{e}_{xy} + \mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y.$
   - $e_{xy} + e_{\overline{xy}} = e_x + e_y.$

   [Note: These follow from the identity $T_{xy} + T_{\overline{xy}} \equiv T_x + T_y$.]

(v) *(New!) Covariance formula:*

$$\mathrm{Cov}\,(T_{xy}, T_{\overline{xy}}) = \mathrm{Cov}\,(T_x, T_y) + (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy})$$

Proof. (Sketch) The key is to first note the identity $T_{xy}T_{\overline{xy}} \equiv T_xT_y$, and then write

$$\mathrm{Cov}\,(T_{xy}, T_{\overline{xy}}) = \underbrace{\mathbb{E}[T_xT_y] - \mathbb{E}[T_x]\mathbb{E}[T_y]}_{\mathrm{Cov}\,(T_x,T_y)}\underbrace{+\mathbb{E}[T_x]\mathbb{E}[T_y] - \mathbb{E}[T_{xy}]\mathbb{E}[T_x + T_y - T_{xy}]}_{(\mathring{e}_x-\mathring{e}_{xy})(\mathring{e}_y-\mathring{e}_{xy})}.$$

□

4.1.14 **Recursions for expectations of time to failure.** In general, recursions only work for *joint-life status*, but NOT for last-survivor status ⚠, due to the reason discussed before in [4.1.10].

Recursive formulas:

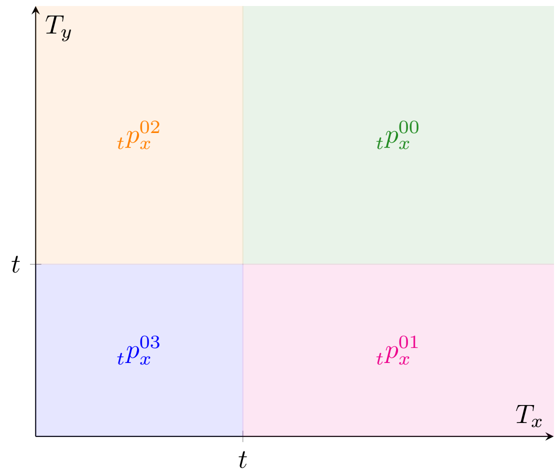
- $\mathring{e}_{xy} = \mathring{e}_{xy:\overline{n}|} + {}_np_{xy}\,\mathring{e}_{x+n:y+n}$
- $e_{xy} = e_{xy:\overline{n}|} + {}_np_{xy}e_{x+n:y+n} \overset{(n=1)}{=} p_{xy}(1 + e_{x+1:y+1})$
- $\mathring{e}_{xy:\overline{n}|} \overset{(m\leq n)}{\equiv} \mathring{e}_{xy:\overline{m}|} + {}_mp_{xy}\,\mathring{e}_{x+m:y+m:\overline{n-m}|}$
- $e_{xy:\overline{n}|} \overset{(m\leq n)}{\equiv} e_{x:\overline{m}|} + {}_mp_{xy}e_{x+m:y+m:\overline{n-m}|} \overset{(m=1)}{=} p_{xy}(1 + e_{x+1:y+1:\overline{n-1}|})$

[Note: $\mathring{e}_{x+m:y+m:\overline{n-m}|}$ can be treated as $\mathbb{E}[T_{x+m:y+m} \wedge (n-m)]$ or $\mathbb{E}\left[T_{x+m:y+m:\overline{n-m}|}\right]$ without ambiguity, as both of them actually mean the same.]

Translating between multiple state and random variable approach notations

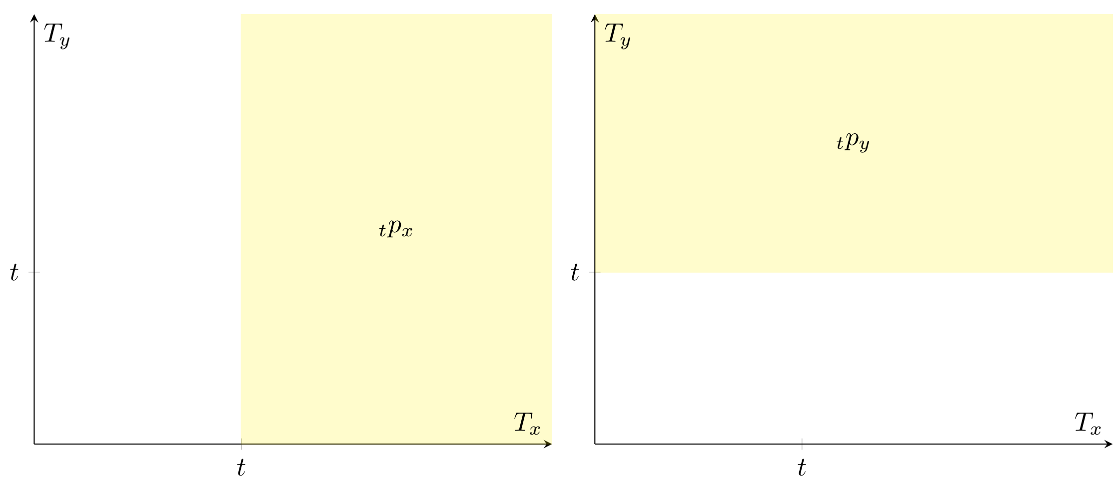
4.1.15 After revisiting the survival model topics, we have accumulated enough "ingredients" to discuss how we can "translate" between the multiple state notations and the notations based on random variable approach.

4.1.16 **Translation between probability notations.** A handy visual tool that helps us to do the translation in this case is the following picture:



This suggests the correspondence between multiple state transition probabilities and the events for the random variable approach. Using this picture, we can deduce, for example:

- ${}_tp_x = \mathbb{P}(T_x > t) = {}_tp_x^{00} + {}_tp_x^{01}$.
- ${}_tp_y = \mathbb{P}(T_y > t) = {}_tp_x^{00} + {}_tp_x^{02}$.
- ${}_tp_{xy} = \mathbb{P}(T_x > t \cap T_y > t) = {}_tp_x^{00}$.
- ${}_tq_{\overline{xy}} = \mathbb{P}(T_x \le t \cap T_y \le t) = {}_tp_x^{03}$.



4.1.17 **Translation between forces of transition/failure.** In multiple life model, there are two types of "forces": (i) force of transition (for multiple state approach) and (ii) force of failure (for random variable approach). One would naturally expect them to have some relationships, and that is indeed the case. First we introduce some extra notations:

$$\mu_{x+t:y+t} := \mu_{x+t:y+t}(0), \quad \mu_{\overline{x+t:y+t}} := \mu_{\overline{x+t:y+t}}(0).$$ [15]

Particularly, these notations can be interpreted as follows:

[15] *(If you are interested)* To better understand the rationale behind these notations, let us consider the more familiar case for *force of mortality*. Mimicking the notations here, we write $\mu_x(s) = -\frac{S_x'(s)}{S_x(s)}$. Then, note that

$$\mu_{x+t}(0) = -\frac{S_{x+t}'(0)}{S_{x+t}(0)} = -\frac{1}{1}\,\frac{\mathrm{d}}{\mathrm{d}s}{}_sp_{x+t}\bigg|_{s=0} = -\,\frac{\mathrm{d}}{\mathrm{d}s}\frac{{}_{x+t+s}p_0}{{}_{x+t}p_0}\bigg|_{s=0} = -\frac{1}{{}_{x+t}p_0}\,\frac{\mathrm{d}}{\mathrm{d}s}{}_{x+t+s}p_0\bigg|_{s=0} = -\frac{S_0'(x+t)}{S_0(x+t)}$$

where the final expression is precisely the definition for "μ_{x+t}" from STAT3901.

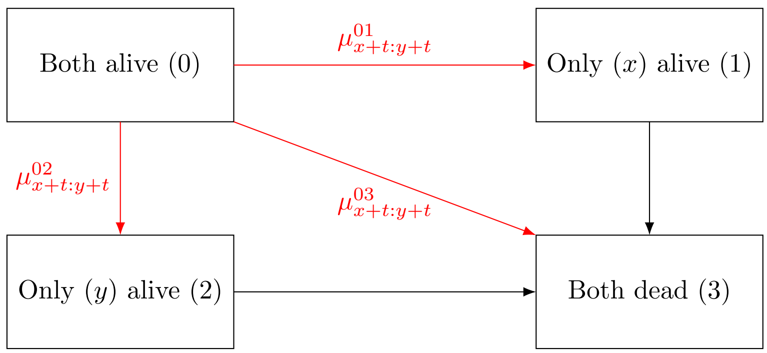
- $\mu_{x+t:y+t}\Delta t$: approximated probability for $(x+t:y+t)$ to fail in Δt years, when Δt is small.
- $\mu_{\overline{x+t:y+t}}\Delta t$: approximated probability for $(\overline{x+t:y+t})$ to fail in Δt years, when Δt is small.

Now we are ready to state the relationships between forces of transition and failure:

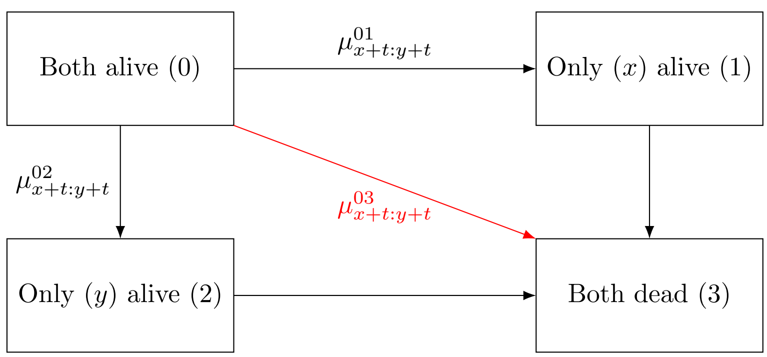
- $\mu_{xy}(t) = \mu_{x+t:y+t} = \mu^{0\bullet}_{x+t:y+t}$.
- ⚠ $\mu_{\overline{xy}}(t) \neq \mu_{\overline{x+t:y+t}} = \mu^{03}_{x+t:y+t}$.

These relationships can be intuitively understood via the interpretations of the forces:

- $\mu_{x+t:y+t} = \mu^{0\bullet}_{x+t:y+t}$:
  Having $(x+t:y+t)$ to fail in Δt years is equivalent to the occurrence of any one of the three red state transitions below within Δt years, with approximated probability $\mu^{0\bullet}_{x+t:y+t}\Delta t$.



- $\mu_{\overline{x+t:y+t}} = \mu^{03}_{x+t:y+t}$:
  Having $(\overline{x+t:y+t})$ to fail (i.e., both lives die ☠) in Δt years is equivalent to the occurrence of the red state transition below within Δt years, with approximated probability $\mu^{03}_{x+t:y+t}\Delta t$. [Note: When Δt is very small, the chance of having two transitions in such a short time frame is negligible, so the *direct* transition from state 0 to state 3 has to occur.]



Regarding $\mu_{xy}(t) = \mu_{x+t:y+t}$ and $\mu_{\overline{xy}}(t) \neq \mu_{\overline{x+t:y+t}}$, they serve as reminders to us for treating the forces of failure carefully. While we do have the intuitively appealing equality $\mu_{xy}(t) = \mu_{x+t:y+t}$ for *joint-life status* (verify that $\mu_{xy}(t) = \mu_{x+t:y+t}(0)$!), similar equality does NOT hold for last-survivor status ⚠. The reason is again the one we have discussed before in [4.1.10], which leads to the breakdown of "factorization formula for p" for last-survivor status.

4.1.18 **Probabilities about order of death.** Before proceeding to Section 4.2, let us introduce one more kind of probability related to the joint behaviour of T_x and T_y, which is about *order of death*. This type of probability is of interest since some insurance products may provide benefits contingent on the order of death, e.g., a couple insurance may only provide a death benefit when the husband dies before the wife. (See [4.2.13] for more discussions on this.)

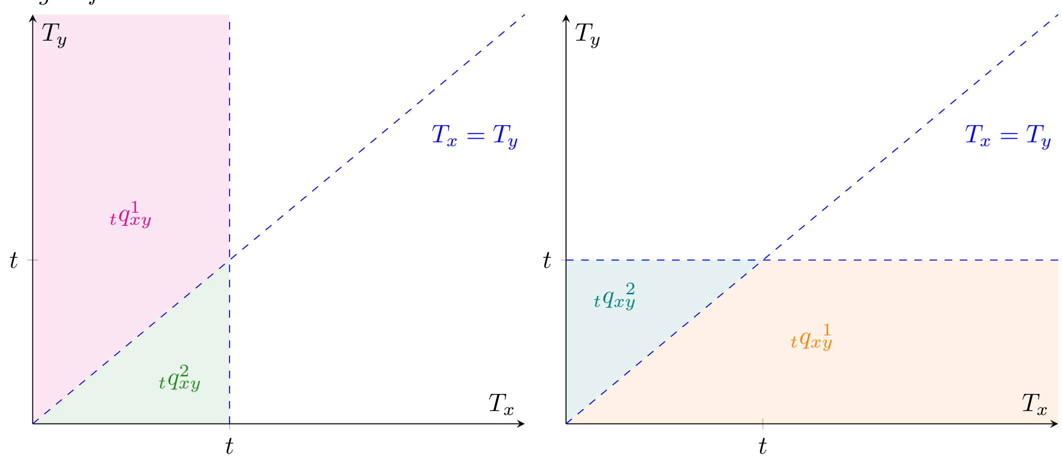
(1) *Notations:*

Notation	Probability that ...
${}_tq_{\overset{1}{x}y} = \mathbb{P}(T_x < T_y \cap T_x \le t)$	(x) dies before (y) and (x) dies within t years
${}_tq_{x\overset{1}{y}} = \mathbb{P}(T_y < T_x \cap T_y \le t)$	(y) dies before (x) and (y) dies within t years
${}_tq_{\overset{2}{x}y} = \mathbb{P}(T_x > T_y \cap T_x \le t)$	(x) dies after (y) and (x) dies within t years
${}_tq_{x\overset{2}{y}} = \mathbb{P}(T_y > T_x \cap T_y \le t)$	(y) dies after (x) and (y) dies within t years

[⚠ Warning: Be careful about the difference between ${}_tq_{\overset{1}{x}y}$ & ${}_tq_{x\overset{2}{y}}$, and ${}_tq_{x\overset{1}{y}}$ & ${}_tq_{\overset{2}{x}y}$.]

(2) *Key formulas (based on random variable approach):*

(i) *Integral formulas:*



- ${}_tq_{\overset{1}{x}y} = \int_0^t \int_u^\infty f_{T_x,T_y}(u,v)\,\mathrm{d}v\,\mathrm{d}u.$
- ${}_tq_{\overset{2}{x}y} = \int_0^t \int_0^u f_{T_x,T_y}(u,v)\,\mathrm{d}v\,\mathrm{d}u.$
- ${}_tq_{x\overset{1}{y}} = \int_0^t \int_v^\infty f_{T_x,T_y}(u,v)\,\mathrm{d}u\,\mathrm{d}v.$
- ${}_tq_{x\overset{2}{y}} = \int_0^t \int_0^v f_{T_x,T_y}(u,v)\,\mathrm{d}u\,\mathrm{d}v.$

[Note: f_{T_x,T_y} is the joint density function of T_x and T_y.]

(ii) *Conditioning formulas:*

- $${}_tq_{\overset{1}{x}y} = \int_0^t \underbrace{\mathbb{P}(T_y > u | T_x = u)}_{(y)\text{ survives longer}} \underbrace{{}_up_x\mu_{x+u}\,\mathrm{d}u}_{(x)\text{ dies in }[u,u+\mathrm{d}u]}.$$
- $${}_tq_{\overset{2}{x}y} = \int_0^t \underbrace{\mathbb{P}(T_y < u | T_x = u)}_{(y)\text{ died earlier}} \underbrace{{}_up_x\mu_{x+u}\,\mathrm{d}u}_{(x)\text{ dies in }[u,u+\mathrm{d}u]}.$$
- $${}_tq_{x\overset{1}{y}} = \int_0^t \underbrace{\mathbb{P}(T_x > u | T_y = u)}_{(x)\text{ survives longer}} \underbrace{{}_up_y\mu_{y+u}\,\mathrm{d}u}_{(y)\text{ dies in }[u,u+\mathrm{d}u]}.$$
- $${}_tq_{x\overset{2}{y}} = \int_0^t \underbrace{\mathbb{P}(T_x < u | T_y = u)}_{(x)\text{ died earlier}} \underbrace{{}_up_y\mu_{y+u}\,\mathrm{d}u}_{(y)\text{ dies in }[u,u+\mathrm{d}u]}.$$

Proof. (Sketch) Consider the first formula as an example. We can use the law of total expectation as follows:

$$\begin{aligned}
{}_tq^{1}_{xy} &= \mathbb{P}(T_x < T_y \cap T_x \le t) \\
&= \mathbb{E}\big[\mathbf{1}_{\{T_x<T_y\}}\mathbf{1}_{\{T_x\le t\}}\big] \\
&= \mathbb{E}\big[\mathbb{E}\big[\mathbf{1}_{\{T_x<T_y\}}\mathbf{1}_{\{T_x\le t\}}|T_x\big]\big] \\
&= \mathbb{E}\big[\mathbb{E}\big[\mathbf{1}_{\{T_x<T_y\}}|T_x\big]\mathbf{1}_{\{T_x\le t\}}\big].
\end{aligned}$$

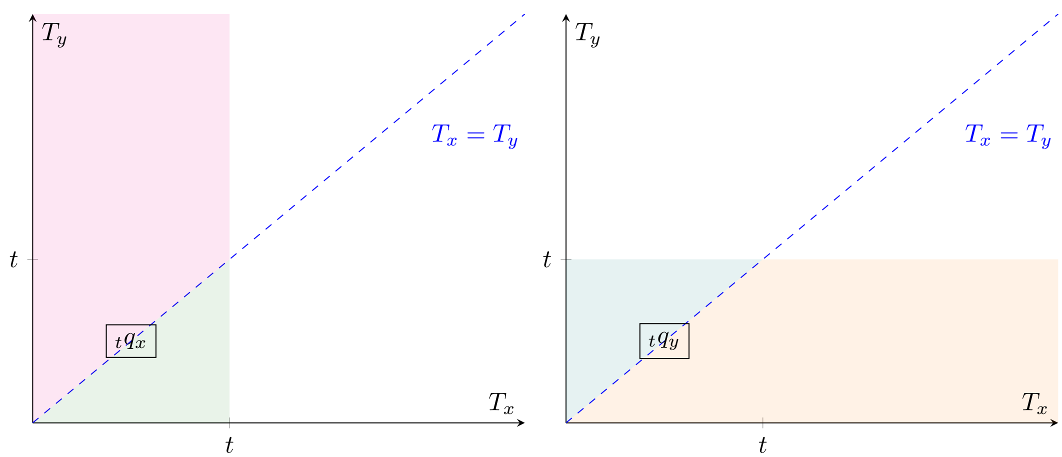
Now define $g(T_x) := \mathbb{E}\big[\mathbf{1}_{\{T_x<T_y\}}|T_x\big]$, and then we have

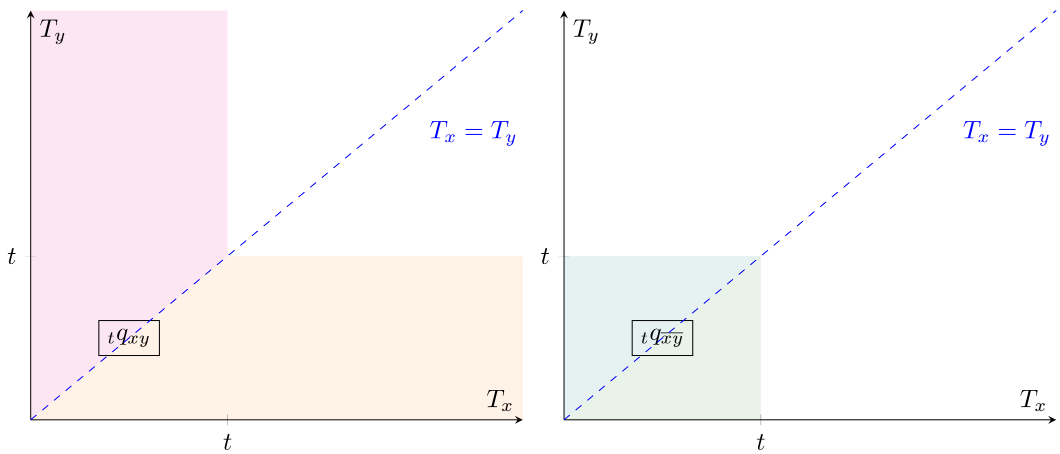
$${}_tq^{1}_{xy} = \int_0^t g(u)\underbrace{{}_up_x\mu_{x+u}}_{f_x(u)}\,\mathrm{d}u = \int_0^t \underbrace{\mathbb{E}\big[\mathbf{1}_{\{T_x<T_y\}}|T_x=u\big]}_{=\mathbb{E}\left[\mathbf{1}_{\{u<T_y\}}|T_x=u\right]=:\mathbb{P}(T_y>u|T_x=u)}\,{}_up_x\mu_{x+u}\,\mathrm{d}u\,.$$

$\square$

(iii) *Relationships:* Assume that simultaneous death is impossible, i.e., $\mu^{03}_{x+t:y+t}\equiv 0$, so that $\mathbb{P}(T_x=T_y)=0$.

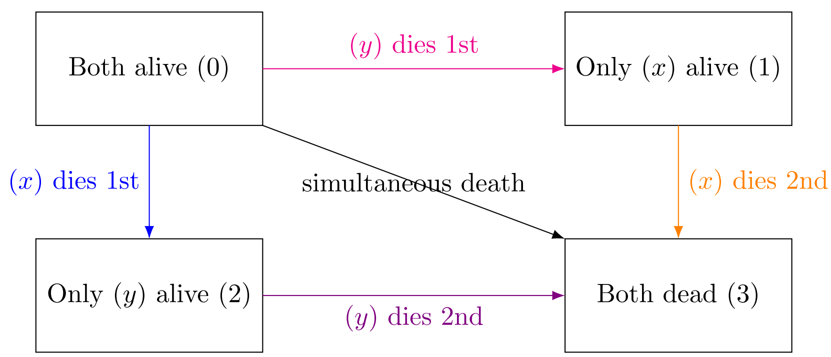
- $$\underbrace{{}_tq^{1}_{xy}}_{(x)\text{ dies 1st}} + \underbrace{{}_tq^{2}_{xy}}_{(x)\text{ dies 2nd}} = \underbrace{{}_tq_x}_{(x)\text{ dies}}\,.$$
- $$\underbrace{{}_tq_{x\overset{1}{y}}}_{(y)\text{ dies 1st}} + \underbrace{{}_tq_{x\overset{2}{y}}}_{(y)\text{ dies 2nd}} = \underbrace{{}_tq_y}_{(y)\text{ dies}}\,.$$
- $$\underbrace{{}_tq^{1}_{xy}}_{(x)\text{ dies 1st}} + \underbrace{{}_tq_{x\overset{1}{y}}}_{(y)\text{ dies 1st}} = \underbrace{{}_tq_{xy}}_{\text{1st death}}\,.$$
- $$\underbrace{{}_tq^{2}_{xy}}_{(x)\text{ dies 2nd}} + \underbrace{{}_tq_{x\overset{2}{y}}}_{(y)\text{ dies 2nd}} = \underbrace{{}_tq_{\overline{xy}}}_{\text{2nd death}}\,.$$





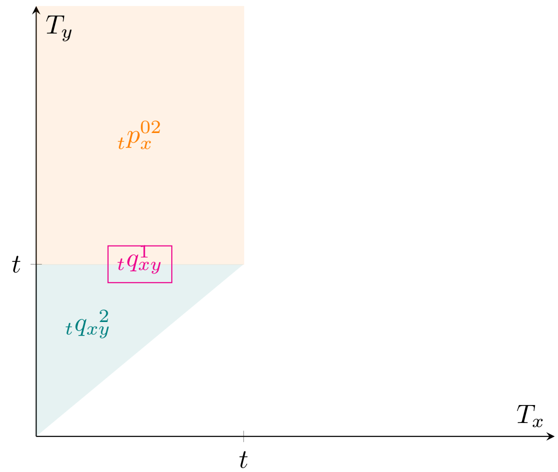
4.1.19 **Specialized probability calculation formulas based on multiple state model.** Several more formulas for probabilities about order of death can be obtained by viewing them from a multiple state model perspective, and specializing the general probability calculation formula in [2.1.5]:

- $_tq^1_{xy} = \int_0^t {}_up^{00}_{xy}\mu^{02}_{x+u:y+u}\,\mathrm{d}u \neq {}_tp^{02}_{xy}$ ⚠.
- $_tq^2_{xy} = \int_0^t {}_up^{01}_{xy}\mu^{13}_{x+u:y+u}\,\mathrm{d}u \neq {}_tp^{03}_{xy}$ or $_tp^{13}_{xy}$ ⚠.
- $_tq_{xy}{}^{1} = \int_0^t {}_up^{00}_{xy}\mu^{01}_{x+u:y+u}\,\mathrm{d}u \neq {}_tp^{01}_{xy}$ ⚠.
- $_tq_{xy}{}^{2} = \int_0^t {}_up^{02}_{xy}\mu^{23}_{x+u:y+u}\,\mathrm{d}u \neq {}_tp^{03}_{xy}$ or $_tp^{23}_{xy}$ ⚠.



[Note: When simultaneous death occurs, (x) dies neither before nor after (y). (We are using "before" and "after" in the strict sense.)]

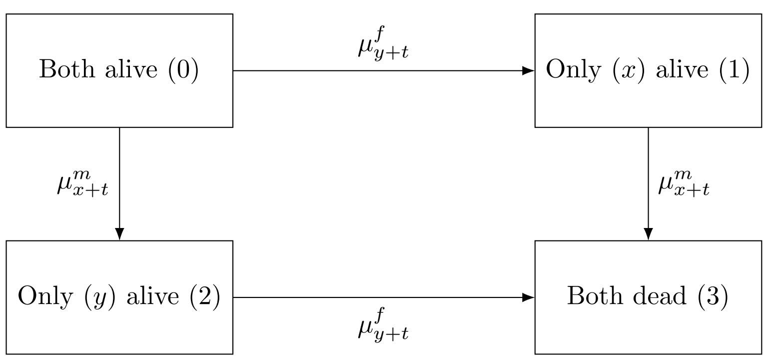
Although the equation $_tq^1_{xy} = {}_tp^{02}_{xy}$ may appear to be intuitively appealing, it is actually wrong ⚠. After thinking more carefully, we can observe that the latter ($_tp^{02}_{xy}$) requires the system to be in state 2 at time t, while the former ($_tq^1_{xy}$) does not: We just need to have a transition from state 0 to state 2 within t years, and we do not care about what happens next (staying in state 2 or transiting further to state 3 are both fine). In fact, we have $_tq^1_{xy} = \underbrace{{}_tq_{xy}{}^{2}}_{\text{go to state 3 next}} + \underbrace{{}_tp^{02}_{xy}}_{\text{stay in state 2}}$ instead, which can be intuitively understood based on the following picture:



4.1.20 **Independent model.** Many formulas discussed previously can be simplified a lot when we assume *independent lifetimes* in the multiple life model (perhaps not so realistic in practice). Of course, we can characterize the independent lifetimes assumption by "T_x and T_y are independent". This is a natural characterization based on the random variable approach. Another less natural but equivalent characterization based on the multiple state model approach is as follows:

- *((y)'s mortality does not depend on (x)'s status)* $\mu^{01}_{x+t:y+t} = \mu^{23}_{y+t} = \mu^{f}_{y+t}$
- *((x)'s mortality does not depend on (y)'s status)* $\mu^{02}_{x+t:y+t} = \mu^{13}_{x+t} = \mu^{m}_{x+t}$
- *(no simultaneous death)* $\mu^{03}_{x+t:y+t} = 0$

[Note: By treating the lives (x) and (y) as "husband (male)" and "wife (female)" respectively, we can understand why the notations μ^{m}_{x+t} and μ^{f}_{y+t} are used.]



Under this independence assumption, we have

$$\boxed{{}_tp_{xy} = {}_tp_x \cdot {}_tp_y} \quad \text{and} \quad \boxed{{}_tq_{\overline{xy}} = {}_tq_x \cdot {}_tq_y}.$$

[⚠ Warning: We do NOT have "${}_tq_{xy} = {}_tq_x \cdot {}_tq_y$" and "${}_tp_{\overline{xy}} = {}_tp_x \cdot {}_tp_y$"! For joint-life status, we can only "split p"; For last-survivor status, we can only "split q".]

These allow us to simplify many formulas discussed before.

Of course, we can extend this assumption to the case with 3 or more lives, but in such case we often only use the characterization based on random variable approach, due to the complexity of multiple state model when handling 3 or more lives.

4.2 Insurance and Annuity EPV Calculations

4.2.1 After completing the (lengthy!) discussions on probabilistic calculations for multiple life model, it is time to apply the concepts and formulas learnt there to calculate EPVs (and related quantities like variances of PVRVs) for various kinds of insurance and annuity products issued to multiple lives (ultimately, the existence of these products is the reason why the topic of multiple life model appears in STAT3909!).

4.2.2 Certainly, concepts and formulas from Section 2 can be applied here to help us calculating the EPVs. However, some products to be discussed here have special features that cannot be described using just the terminologies in Section 2.2, which forces us to develop some new concepts and formulas to deal with those products. In particular, as we know from Section 4.1, in multiple life model we have two kinds of "special lives": joint-life and last-survivor statuses. One natural strategy is then to "revisit" topics about life insurance and life annuity from STAT3901 for these "special lives", which can yield numerous formulas and tools for analyzing the products.

So, as you can expect after reading the paragraph above, Section 4.2 will again be rather lengthy...[16] Nonetheless, the good 👍 news is that the versatile *general EPV calculation formula* will continue to be of great use.

Revisiting Insurance and Annuity Topics for Joint-Life and Last-Survivor Statuses

4.2.3 Instead of revisiting the *enormous* amount of life insurance and annuity formulas from STAT3901 *one by one*, here we will just study some key points and formulas (and also some pitfalls ⚠), because many formulas there are just specialized versions of the *general EPV calculation formula*.

4.2.4 **Insurances.**

(1) *Notations:* Insurance notations from STAT3901 still apply for joint-life and last-survivor statuses and carry the same meaning (with the future lifetime random variable T being the "lifetime" of one of these "special lives"); We just need to change the age "x" in the notation to "xy" or "$\overline{xy}$". The only special thing is that for term life insurance notation, we would add a hat on top of xy (joint-life status only) to make the notations better-looking.
Examples:

$$A_x \to \begin{cases} A_{xy} \\ A_{\overline{xy}} \end{cases} \qquad A^{1}_{x:\overline{n}|} \to \begin{cases} A^{1}_{\widehat{xy}:\overline{n}|} \\ A^{1}_{\overline{xy}:\overline{n}|} \end{cases} \qquad A_{x:\overline{n}|} \to \begin{cases} A_{xy:\overline{n}|} \\ A_{\overline{xy}:\overline{n}|} \end{cases} \qquad A_{x:\overset{1}{\overline{n}|}} \to \begin{cases} A_{xy:\overset{1}{\overline{n}|}} \\ A_{\overline{xy}:\overset{1}{\overline{n}|}} \end{cases} \qquad {}_nE_x \to \begin{cases} {}_nE_{xy} \\ {}_nE_{\overline{xy}} \end{cases}$$

(2) *Formulas (examples):*

(i) *Specialized EPV calculation formulas:*

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k:y+k} \qquad \bar{A}_{xy} = \int_0^{\infty} e^{-\delta t}\,{}_tp_{xy}\,\underbrace{\mu_{x+t:y+t}}_{\text{or } \mu_{xy}(t)}\,\mathrm{d}t$$

$$A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1}\cancel{{}_kp_{\overline{xy}}\,q_{x+k:y+k}}\,{}_{k|}p_{\overline{xy}} \qquad \bar{A}_{\overline{xy}} = \int_0^{\infty} e^{-\delta t}\,{}_tp_{\overline{xy}}\cancel{\mu_{x+t:y+t}}\,\mu_{\overline{xy}}(t)\,\mathrm{d}t$$

(ii) *Variance formulas:*

$$\mathrm{Var}\,(\mathrm{PVRV}) = {}^2A_{\square} - A_{\square}^2$$

where $\square$ is the appropriate symbol for the insurance and status being considered (e.g., $xy:\overline{n}|$ for an n-year endowment insurance "issued to" (xy)). Change $A \to \bar{A}$ for the continuous case (except pure endowment).

[16] It is quite normal to feel overwhelmed ☹ by the *sheer* amount of terms and formulas you have seen so far. (If you do *not* feel that, you would probably get A+ in STAT3909 easily ☺.) Section 4 is indeed a rather challenging part in STAT3909, and it does take time to digest such a large amount of content here.

(iii) *Relationships:*

- *(endowment = term + pure endowment)*

$$A_{\square:\overline{n}|} = A^{\;1}_{\square:\overline{n}|} + A_{\square:\overline{n}|}^{\;\;\;\;\;1}.$$

- *(deferred = WL − term)*

$${}_{n|}A_{\square} = A^{1}_{\square} - A^{\;1}_{\square:\overline{n}|}.$$

- *(discounted EPV formula for **joint-life status**)*

$${}_{n|}A_{xy} = {}_nE_{xy}\,A_{x+n:y+n}.$$

[⚠ Warning: We do NOT have "${}_{n|}A_{\overline{xy}} = {}_nE_{\overline{xy}}\,A_{\overline{x+t:y+t}}$"!]

- *(second moment of pure endowment)*

$${}_nE_{\square} = v^n\,{}_nE_{\square}.$$

Remarks:

- $\square = xy$ or $\overline{xy}$.
- Change $A \to \bar{A}$ for the continuous case.
- Use the notation $A^{\;1}_{\widehat{xy}:\overline{n}|}$ instead of $A^{\;1}_{xy:\overline{n}|}$.

4.2.5 Annuities.

(1) *Notations:* Annuity notations from STAT3901 still apply for joint-life and last-survivor statuses; Again we just need to change the age "x" in the notation to "xy" or "$\overline{xy}$".
Examples:

$$\ddot{a}_x \to \begin{cases}\ddot{a}_{xy}\\ \ddot{a}_{\overline{xy}}\end{cases} \qquad \ddot{a}_{x:\overline{n}|} \to \begin{cases}\ddot{a}_{xy:\overline{n}|}\\ \ddot{a}_{\overline{xy}:\overline{n}|}\end{cases} \qquad {}_{n|}\ddot{a}_x \to \begin{cases}{}_{n|}\ddot{a}_{xy}\\ {}_{n|}\ddot{a}_{\overline{xy}}\end{cases} \qquad \ddot{a}_{\overline{x:\overline{n}|}} \to \begin{cases}\ddot{a}_{\overline{xy:\overline{n}|}}\\ \ddot{a}_{\overline{\overline{xy}:\overline{n}|}}\end{cases}$$

← read carefully ⚠

(2) *Formulas (examples):*

(i) *Specialized EPV calculation formulas:*

$$\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k\,{}_kp_{xy} \qquad \bar{a}_{xy} = \int_0^{\infty} e^{-\delta t}\,{}_tp_{xy}\,\mathrm{d}t$$

$$\ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} v^k\,{}_kp_{\overline{xy}} \qquad \bar{a}_{\overline{xy}} = \int_0^{\infty} e^{-\delta t}\,{}_tp_{\overline{xy}}\,\mathrm{d}t$$

(ii) *Variance formulas (for WL or temporary life annuity):*

$$\mathrm{Var}\,(\mathrm{PVRV}) = \frac{{}^2A_{\square} - A^2_{\square}}{d^2}$$

where $\square = xy$, $\overline{xy}$, $xy{:}\overline{n}|$, or $\overline{xy}{:}\overline{n}|$. For the continuous case, change $A \to \bar{A}$ and $d \to \delta$.

(iii) *Relationships:*

- *(deferred = WL − temporary)*

$${}_{n|}\ddot{a}_{\square} = \ddot{a}_{\square} - \ddot{a}_{\square:\overline{n}|}$$

- *(guaranteed = certain + deferred)*

$$\ddot{a}_{\overline{\square:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_{\square}$$

- *(discounted EPV formula for **joint-life status**)*

$${}_{n|}\ddot{a}_{xy} = {}_nE_{xy}\,\ddot{a}_{x+n:y+n}$$

[⚠ Warning: We do NOT have "${}_{n|}\ddot{a}_{\overline{xy}} = {}_nE_{\overline{xy}}\,\ddot{a}_{\overline{x+n:y+n}}$"!]

Remarks:

- $\square = xy$ or $\overline{xy}$.
- Change $\ddot{a} \to \bar{a}$ for the continuous case.

4.2.6 **Recursions for insurance and annuity EPVs.** Regarding *recursions*, one important thing to bear in mind is that we do NOT have any recursive formula for last-survivor status, due to the breakdown of the "factorization formula for p" (which is foundational to recursions) for last-survivor status. On the other hand, we do have recursive formulas available for joint-life status, which can be obtained by adapting the recursive formulas from STAT3901:

- *Whole life insurance:*
  - *(discrete)* $A_{xy} = A^{\ 1}_{\widehat{xy}:\overline{n}|} + {}_nE_{xy}A_{x+n:y+n} \overset{(n=1)}{=} vq_{xy} + vp_{xy}A_{x+1:y+1}$.
  - *(continuous)* $\bar{A}_{xy} = \bar{A}^{\ 1}_{\widehat{xy}:\overline{n}|} + {}_nE_{xy}\bar{A}_{x+n:y+n}$.
- *Term life insurance:*
  - *(discrete)* $A^{\ 1}_{\widehat{xy}:\overline{n}|} \overset{(m\leq n)}{=} A^{\ 1}_{\widehat{xy}:\overline{m}|} + {}_mE_{xy}A_{\overset{1}{\widehat{x+m:y+m}}:\overline{n-m}|} \overset{(m=1)}{=} vq_{xy} + vp_{xy}A_{\overset{1}{\widehat{x+1:y+1}}:\overline{n-1}|}$.
  - *(continuous)* $\bar{A}^{\ 1}_{\widehat{xy}:\overline{n}|} \overset{(m\leq n)}{=} \bar{A}^{\ 1}_{\widehat{xy}:\overline{m}|} + {}_mE_{xy}\bar{A}_{\overset{1}{\widehat{x+m:y+m}}:\overline{n-m}|}$.
- *Endowment insurance:*
  - *(discrete)* $A_{xy:\overline{n}|} \overset{(m\leq n)}{=} A^{\ 1}_{\widehat{xy}:\overline{m}|} + {}_mE_{xy}A_{x+m:y+m:\overline{n-m}|} \overset{(m=1)}{=} vq_{xy} + vp_{xy}A_{x+1:y+1:\overline{n-1}|}$.
  - *(continuous)* $\bar{A}^{\ 1}_{\widehat{xy}:\overline{n}|} \overset{(m\leq n)}{=} \bar{A}^{\ 1}_{\widehat{xy}:\overline{m}|} + {}_mE_{xy}\bar{A}_{x+m:y+m:\overline{n-m}|}$.
- *Whole life annuity:*
  - *(discrete, due)* $\ddot{a}_{xy} = \ddot{a}_{xy:\overline{n}|} + {}_nE_{xy}\ddot{a}_{x+n:y+n} \overset{(n=1)}{=} 1 + vp_{xy}\ddot{a}_{x+1:y+1}$.
  - *(continuous)* $\bar{a}_{xy} = \bar{a}_{xy:\overline{n}|} + {}_nE_{xy}\bar{a}_{x+n:y+n}$.
- *Temporary life annuity:*
  - *(discrete, due)* $\ddot{a}_{xy:\overline{n}|} \overset{(m\leq n)}{=} \ddot{a}_{xy:\overline{m}|} + {}_mE_{xy}\ddot{a}_{x+m:y+m:\overline{n-m}|} \overset{(m=1)}{=} 1 + vp_{xy}\ddot{a}_{x+1:y+1:\overline{n-1}|}$.
  - *(continuous)* $\bar{a}_{xy:\overline{n}|} = \bar{a}_{xy:\overline{m}|} + {}_mE_{xy}\bar{a}_{x+m:y+m:\overline{n-m}|}$.

4.2.7 **Covariance between WL PVRVs.** With the presence of *two* random variables T_x and T_y in multiple life model, *covariance* between PVRVs would be of more interest. Recall the covariance formula discussed previously in [4.1.13]:

$$\mathrm{Cov}\left(T_{xy}, T_{\overline{xy}}\right) = \mathrm{Cov}\left(T_x, T_y\right) + (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}).$$

For whole life insurance PVRV, we have a formula having similar form:

$$\mathrm{Cov}\left(v^{T_{xy}}, v^{T_{\overline{xy}}}\right) = \boxed{\mathrm{Cov}\left(v^{T_x}, v^{T_y}\right) + (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})},$$

which can be proved in a similar way as the proof for the covariance formula above, utilizing the identity $v^{T_{xy}}v^{T_{\overline{xy}}} \equiv v^{T_x}v^{T_y}$ (try it!).

4.2.8 **Joint-life + Last-survivor = Individual sum.** In Section 4.1, we have seen this type of relationship for "${}_tp_\square$", "${}_tq_\square$", "$e_\square$", "$\mathring{e}_\square$", and $f_\square(t)$ (density function). Using a similar argument, we can derive relationships of this kind for insurance and annuity EPVs:

- *Insurance EPV:*

$$A_{xy} + A_{\overline{xy}} = A_x + A_y.$$

  Possible variations:
  - *(continuous)* $A \to \bar{A}$
  - *(term life)* WL subscript $\to$ term life subscript

  - *(endowment)* WL subscript → endowment subscript
  - *(pure endowment)* $A_{xy} \to {}_nE_{xy}$, etc.

- *Annuity EPV:*

$$\ddot{a}_{xy} + \ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y.$$

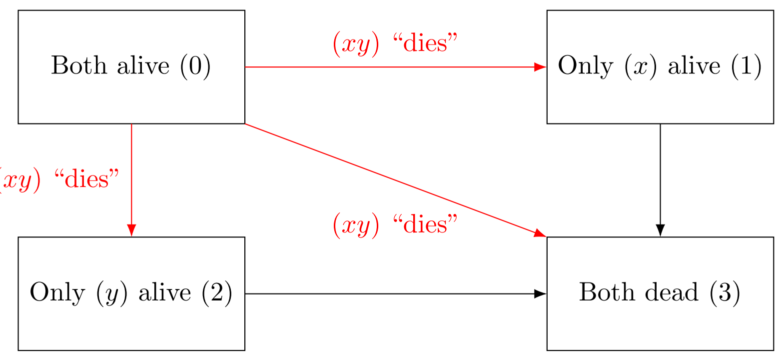
  Possible variations:

  - *(continuous)* $\ddot{a} \to \bar{a}$
  - *(temporary life)* WL subscript → temporary life subscript

Connections with multiple state model

4.2.9 Apart from developing the formulas based on the random variable approach like what we have done above, we can also obtain some formulas by viewing the insurance and annuity products here as *state-contingent* products, and applying the formulas developed in Section 2.2. (Recall how we revise the multiple state EPV notations from [4.1.1].)

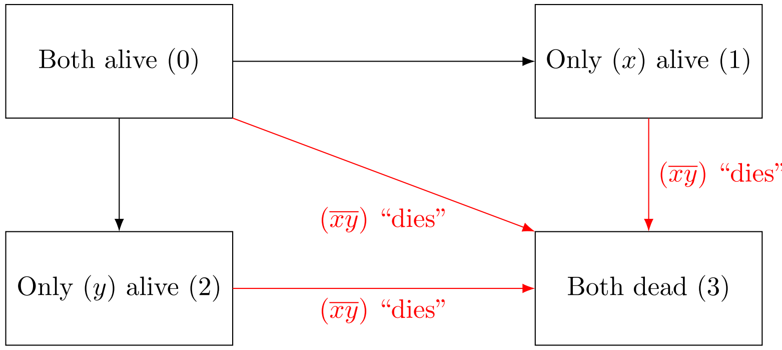
4.2.10 **Insurances and annuities for joint-life status.**



Based on this picture and the general EPV calculation formula, we can obtain the following formulas. [Note: In some formulas below, we need to assume that it is impossible to directly transit from state 0 to state 3 (simultaneous death, or **common shock**). We will abbreviate this assumption as "NCS" (no common shock).]

Type	Discrete	Continuous
WL insurance	$A_{xy} \overset{\text{(NCS)}}{=} A^{01}_{xy} + A^{02}_{xy}$ $\neq \sum_{t=0}^{\infty} v^{t+1}\,{}_tp^{00}_{xy}\,p^{0\bullet}_{x+t:y+t}$ without NCS	$\bar{A}_{xy} = \overbrace{\int_0^\infty e^{-\delta t}\,{}_tp^{00}_{xy}\,\mu^{0\bullet}_{x+t:y+t}\,dt}^{\text{NOT } \bar{A}^{01}_x + \bar{A}^{02}_x + \bar{A}^{03}_x \ ⚠}$ $\overset{\text{(NCS)}}{=} \bar{A}^{01}_{xy} + \bar{A}^{02}_{xy}$
WL annuity	(due) $\ddot{a}_{xy} = \ddot{a}^{00}_{xy}$	$\bar{a}_{xy} = \bar{a}^{00}_{xy}$
n-year term life insurance	$A^{\;1}_{\widehat{xy}:\overline{n}\rvert} \overset{\text{(NCS)}}{=} A^{01}_{xy:\overline{n}\rvert} + A^{02}_{xy:\overline{n}\rvert}$	$\bar{A}^{\;1}_{\widehat{xy}:\overline{n}\rvert} = \int_0^n e^{-\delta t}\,{}_tp^{00}_{xy}\,\mu^{0\bullet}_{x+t:y+t}\,dt$ $\overset{\text{(NCS)}}{=} \bar{A}^{01}_{xy:\overline{n}\rvert} + \bar{A}^{02}_{xy:\overline{n}\rvert}$
n-year temporary annuity	(due) $\ddot{a}_{xy:\overline{n}\rvert} = \ddot{a}^{00}_{xy:\overline{n}\rvert}$	$\bar{a}_{xy:\overline{n}\rvert} = \bar{a}^{00}_{xy:\overline{n}\rvert}$

4.2.11 **Insurances and annuities for last-survivor status.**



Based on this picture and the general EPV calculation formula, we can obtain the following formulas.

Type	Discrete	Continuous
WL insurance	$A_{\overline{xy}} = A^{03}_{xy}$ (not "$A^{03}_{\overline{xy}}$"!)	$\bar{A}_{xy} = \bar{A}^{03}_{xy}$
WL annuity	(due) $\ddot{a}_{\overline{xy}} = \ddot{a}^{00}_{xy} + \ddot{a}^{01}_{xy} + \ddot{a}^{02}_{xy}$	$\bar{a}_{\overline{xy}} = \bar{a}^{00}_{xy} + \bar{a}^{01}_{xy} + \bar{a}^{02}_{xy}$
n-year term life insurance	$A^{\;1}_{\overline{xy}:\overline{n}\rvert} = A^{03}_{xy:\overline{n}\rvert}$	$\bar{A}^{\;1}_{\overline{xy}:\overline{n}\rvert} = \bar{A}^{03}_{xy:\overline{n}\rvert}$
n-year temporary annuity	(due) $\ddot{a}_{\overline{xy}:\overline{n}\rvert} = \ddot{a}^{00}_{xy:\overline{n}\rvert} + \ddot{a}^{01}_{xy:\overline{n}\rvert} + \ddot{a}^{02}_{xy:\overline{n}\rvert}$	$\bar{a}_{xy:\overline{n}\rvert} = \bar{a}^{00}_{x:\overline{n}\rvert} + \bar{a}^{01}_{x:\overline{n}\rvert} + \bar{a}^{02}_{x:\overline{n}\rvert}$

Special Insurance and Annuity Products

4.2.12 Some insurance and annuity products have special features that cannot be captured using just the ordinary insurance/annuity products from STAT3901 and joint-life/last-survivor status. In general, we would need to use the *general EPV calculation formula* to deal with these special products. Here, we will have several case studies about these special products, and develop some specialized formulas for them.

4.2.13 **Case Study 1: Insurances contingent on order of death.** Not surprisingly, the probabilities about order of death (learnt in [4.1.18]) would be helpful for dealing with insurances contingent on order of death. Let us study the case of permanent insurances with benefits payable when either (x) dies before (y) or (x) dies after (y).

(1) *Notations:* (The insurance benefits are assumed to be 1 below.)

Paid when... \ Type	Discrete	Continuous
(x) dies before (y)	$A_{\overset{1}{x}y}$	$\bar{A}_{\overset{1}{x}y}$
(x) dies after (y)	$A_{\overset{2}{x}y}$	$\bar{A}_{\overset{2}{x}y}$
(y) dies before (x)	$A_{x\overset{1}{y}}$	$\bar{A}_{x\overset{1}{y}}$
(y) dies after (x)	$A_{x\overset{2}{y}}$	$\bar{A}_{x\overset{2}{y}}$

[Warning: Note that $\bar{A}_{\overset{1}{x}y} \neq \bar{A}_{x\overset{2}{y}}$. While this pair of insurance products look quite similar, the subtle yet critical difference between them is that, given (x) dies before (y):

- the former insurance pays benefit upon the <u>first</u> death (i.e., death of (x)), and
- the latter insurance pays benefit upon the <u>second</u> death (i.e., death of (y)).

More specifically, the PVRVs of the former and latter insurances are respectively

$$\begin{cases} v^{T_x} & \text{if } T_x < T_y, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \begin{cases} v^{T_y} & \text{if } T_x < T_y, \\ 0 & \text{otherwise.} \end{cases}$$

]

(2) *Key formulas:*

(i) *Relationships:* Assume that there is no common shock, i.e., $\mu^{03}_{x+t:y+t} \equiv 0$.

- $$\underbrace{A^{1}_{xy}}_{(x)\text{ dies 1st}} + \underbrace{A^{2}_{xy}}_{(x)\text{ dies 2nd}} = \underbrace{A_{x}}_{(x)\text{ dies}}.$$

- $$\underbrace{A_{x\overset{1}{y}}}_{(y)\text{ dies 1st}} + \underbrace{A_{x\overset{2}{y}}}_{(y)\text{ dies 2nd}} = \underbrace{A_{y}}_{(y)\text{ dies}}.$$

- $$\underbrace{A^{1}_{xy}}_{(x)\text{ dies 1st}} + \underbrace{A_{x\overset{1}{y}}}_{(y)\text{ dies 1st}} = \underbrace{A_{xy}}_{\text{1st death}}.$$

- $$\underbrace{A^{2}_{xy}}_{(x)\text{ dies 2nd}} + \underbrace{A_{x\overset{2}{y}}}_{(y)\text{ dies 2nd}} = \underbrace{A_{\overline{xy}}}_{\text{2nd death}}.$$

For the continuous case, change $A \to \bar{A}$. Due to the availability of these relationships, henceforth we will focus only on developing the formulas for A^{1}_{xy} and $\bar{A}^{1}_{xy}$.

(ii) *EPV formulas:*

- *(discrete)*

$$A^{1}_{xy} = \sum_{k=0}^{\infty} v^{k+1}{}_{k}p_{xy}\,q^{\;1}_{x+k:y+k} = \sum_{k=0}^{\infty} v^{k+1}{}_{k}p^{00}_{xy}p^{02}_{x+k:y+k}.$$

- *(continuous)*

$$\begin{aligned}\bar{A}^{1}_{xy} &= \int_{0}^{\infty}\int_{s}^{\infty} e^{-\delta t} f_{T_x,T_y}(s,t)\,\mathrm{d}t\,\mathrm{d}s\\ &= \int_{0}^{\infty} e^{-\delta s}\mathbb{P}(T_y > s | T_x = s)\,{}_{s}p_{x}\mu_{x+s}\,\mathrm{d}s\\ &= \int_{0}^{\infty} e^{-\delta s}{}_{s}p^{00}_{xy}\mu^{02}_{x+s:y+s}\,\mathrm{d}s\\ &= \bar{A}^{02}_{xy}.\end{aligned}$$

4.2.14 Case Study 2: Reversionary annuities.

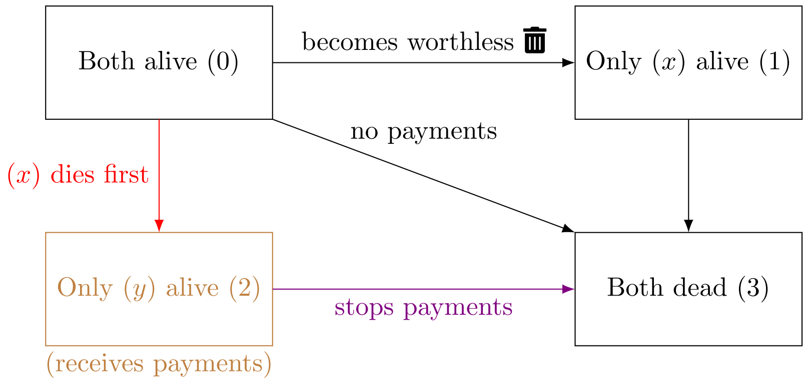
(1) *Definition:* A **reversionary annuity** is an annuity paying level benefits to one individual, say (y), after the death of the other, say (x), for the rest of (y)'s lifetime. If (y) dies before (x), then the reversionary annuity becomes worthless.

(2) *Key formulas:* (Continuous case)

- *PVRV:*

$$Y = \begin{cases} {}_{T_x|}\bar{a}_{\overline{T_y - T_x}|} & \text{if } T_x \le T_y,\\ 0 & \text{otherwise}\end{cases} = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|} & \text{if } T_x \le T_y,\\ \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_y}|} & \text{otherwise}\end{cases} = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}.$$

- *EPV :*
  - *(RHS – LHS:RHS)* $\bar{a}_{x|y} = \mathbb{E}[Y] = \boxed{\bar{a}_{y} - \bar{a}_{xy}}$.
  - *(multiple state formula)* $\boxed{\bar{a}_{x|y} = \bar{a}^{02}_{xy}}$.



[Note: Formulas for discrete (due) case can be developed similarly; We just need to change: $T_x \to K_x + 1$, $T_y \to K_y + 1$, $T_{xy} \to K_{xy} + 1$, and $\bar{a} \to \ddot{a}$. Food for thought: How to describe a discrete reversionary annuity-due in words?]

4.2.15 **Case Study 3: Variants of reversionary annuities.** The terms in a reversionary annuity can be modified in various ways, such as:

(1) Starting payments also when (x) is old enough (e.g., reaches age $x + n$):

$$\text{EPV} = \bar{a}_{x:\overline{n}|\,|y} \overset{\text{(RHS}-\text{LHS:RHS)}}{=} \bar{a}_y - \bar{a}_{xy:\overline{n}|}.$$

(2) Starting payments only when (x) dies early enough (e.g., within n years):

$$\text{EPV} = \bar{a}_{x|y} \underbrace{- {}_nE_{xy}\bar{a}_{x+n|y+n}}_{\text{excluding payments originally made when } (x) \text{ survives } n \text{ years}}.$$

(3) Becoming worthless also when (y) is old enough (e.g., reaches age $x + n$):

$$\text{EPV} = \bar{a}_{x|(y:\overline{n}|)} \overset{\text{(RHS}-\text{LHS:RHS)}}{=} \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}.$$

(4) Becoming worthless also when payments last long enough (e.g., for n years):

$$\text{EPV} = \int_0^\infty \underbrace{\bar{a}_{y+t:\overline{n}|}}_{\text{or } \bar{a}^{22}_{y+t:\overline{n}|}} e^{-\delta t} {}_tp^{00}_{xy}\mu^{02}_{x+t:y+t}\,\mathrm{d}t.$$

[Note: Compare this with the EPV of usual reversionary annuity, which can be expressed as:

$$\bar{a}_{x|y} = \int_0^\infty \underbrace{\bar{a}_{y+t}}_{\text{or } \bar{a}^{22}_{y+t}} e^{-\delta t} {}_tp^{00}_{xy}\mu^{02}_{x+t:y+t}\,\mathrm{d}t.$$

]

For (1), the annuity would pay benefits "more leniently". For (2) to (4), the annuity would pay benefits "more strictly".

[Warning: Instead of just "memorizing" the formulas here, try your best to understand the intuition behind these formulas, as you may be asked to calculate EPV of reversionary annuity variants that do not appear here (or even some unfamiliar special insurances and annuities that are not at all related to the case studies here)! One piece of advice I would give is that the *general EPV calculation formula* is always your (best) friend!]

4.3 Premium and Policy Value Calculations

4.3.1 Premiums and policy values can be computed based on the formulas for insurance and annuity EPVs in Section 4.2. The concept of state-dependent policy values from Section 2.3 also applies here as the multiple life model is a special case of multiple state model. As there is not really "new" thing for multiple life model to be discussed here, let us (finally!) end Section 4 now, and progress to the (much) more friendly Section 5 ☺.

5 Profit Analysis

5.0.1 The profit analysis in Section 5 can be divided into two types:

(1) *Classifying profits by sources:* While the profit amount can be obtained by a simple "revenue − cost", the insurer would also like to know the *contributions* to the profits from different factors, e.g. interest and mortality. Here we will discuss a method to do that.

(2) *(Main focus) Testing the profits under different scenarios:* Also known as **profit testing**, here we are investigating the impacts on the evolution of future net cash flows and emergence of profits upon changes in assumptions made (e.g., interest and mortality assumptions). Profit testing is an application of *scenario analysis* (learnt in STAT3904) in a life contingencies context.

While the primary purpose of profit testing is to determine the profitability of various products offered by the insurer, it can actually help us to perform core actuarial functions, namely *pricing* and *reserving*, which raises actuaries' interest in this topic.

[Note: Throughout Section 5, we shall focus on discussing the profit analysis of insurance products.]

5.1 Classification of Profits by Sources

5.1.1 Recall the basic policy value recursion formula we learn from STAT3901:

$$\underbrace{({}_tV + G_t - e_t)(1+i)}_{\text{what you have}} = \underbrace{q_{x+t}(S_{t+1} + E_{t+1}) + p_{x+t}({}_{t+1}V)}_{\text{what you need (expected)}}$$

(t is an integer time). Intuitively, it can be interpreted as saying that "what you have is what you *expect to* need". In practice, "what you *actually* need" at time $t+1$ would almost always differ from your expectation, like many things in real life. In fact, since "what you have" is calculated based on assumptions (e.g., interest rate), even "what you *actually* have" could be different from what you calculated, when your assumptions do not match exactly with the actual experience!

The difference "what you *actually* have" − "what you *actually* need" is indeed the *actual* profit from the policy; It is good 👍 for "what you *actually* have" to exceed "what you *actually* need", as you can keep some amount of "what you have" as profit 😊. On the other hand, it is bad 👎 for "what you *actually* need" to exceed "what you *actually* have", since it means that you have no choice but to use "your own money" to cover the shortfall, resulting in a loss ☹!

5.1.2 **Notations.** Let us now introduce some notations related to profit calculations:

Notation	Meaning
$\ell_{x+t}^{\text{actual}}$	actual number of "survivors" at time t
i_t^{actual}	actual annual effective interest rate at time t
e_t^{actual}	sum of actual initial and renewal expenses (per policy) at time t
E_{t+1}^{actual}	actual settlement expense (per policy) at time $t+1$

Remarks:

- Like the asset shares case in Section 1.2, "survivors" actually mean policies in force, or "surviving policies". But sometimes we still use the term "survivors" to simplify wordings.
- For other types of quantities like sum insured and premiums, we assume that they are fixed throughout the whole policy term and do not depend on the actual experience.

5.1.3 **Total amount of profits.** Before performing the classification, we need to first calculate the total amount of profits, which is given by:

$$\ell_{x+t}^{\text{actual}}\left(\underbrace{({}_tV + G_t - e_t^{\text{actual}})(1+i_t^{\text{actual}})}_{\text{what you actually have}} - \underbrace{q_{x+t}^{\text{actual}}(S_{t+1} + E_{t+1}^{\text{actual}}) + p_{x+t}^{\text{actual}}({}_{t+1}V)}_{\text{what you actually need}}\right)$$

where $p_{x+t}^{\text{actual}} = \ell_{x+t+1}^{\text{actual}}/\ell_{x+t}^{\text{actual}}$ and $q_{x+t}^{\text{actual}} = 1 - p_{x+t}^{\text{actual}}$.

5.1.4 **Classifying profits by sources.** Now we introduce one method to decompose the total profit into several components, each for one source. The key idea is as follows:

$$\text{Profit due to } \square = \text{Profit with actual } \square - \text{Profit with assumed } \square$$

where $\square$ stands for "interest", "mortality", or "expense".

More specifically, we are decomposing the total profit as follows:

$$\begin{aligned}\text{Total profit} &= \text{Profit with all actual quantities} - \overbrace{\text{Profit with all assumed quantities}}^{\text{always } 0}\\ &= \text{Profit with actual interest, expenses \& mortality} - \text{Profit with actual interest \& expenses}\\ &\quad + \text{Profit with actual interest \& expenses} - \text{Profit with actual interest}\\ &\quad + \text{Profit with actual interest} - \text{Profit with all assumed quantities}\\ &= \text{Profit due to mortality}\\ &\quad + \text{Profit due to expenses}\\ &\quad + \text{Profit due to interest.}\end{aligned}$$

While this decomposition is intuitively appealing, it actually has one major limitation ⚠, namely that the profits due to various sources obtained *depend on the "order" of classifications*. In the case above, the "order" would be

$$\text{Interest} \to \text{Expenses} \to \text{Mortality}$$

which refers to the order for converting the assumed quantity to the actual one, starting from the "all assumed quantities" case:

- Profit due to interest = Profit with actual interest − Profit with all assumed quantities (we convert assumed interest to actual interest here).
- Profit due to expenses = Profit with actual interest & expenses − Profit with actual interest (we convert assumed expenses to actual expenses here). [⚠ Warning: Make sure you consider <u>all</u> kinds of expenses: initial, renewal, and settlement expenses.]
- Profit due to mortality = Profit with actual int., exp. & mortality−Profit with actual int. & exp. (we convert assumed mortality to actual mortality here).

The values obtained could be different if we decompose the profits in the following order instead (Mortality → Expenses → Interest):

$$\begin{aligned}\text{Total profit} &= \text{Profit with actual interest, expenses \& mortality} - \text{Profit with actual mortality \& expenses}\\ &\quad + \text{Profit with actual mortality \& expenses} - \text{Profit with actual mortality}\\ &\quad + \text{Profit with actual mortality} - \text{Profit with all assumed quantities.}\end{aligned}$$

5.1.5 **Shortcuts for decomposing profits.** Some shortcuts are available when we decompose the profits as above, due to cancellations of terms. To illustrate them, let us use the decomposition in the order Interest → Expenses → Mortality as an example:

- *Profit due to interest:*

$$\begin{aligned}&\text{Profit with actual interest} - \overbrace{\text{Profit with all assumed quantities}}^{0}\\ =\quad &\ell_{x+t}^{\text{actual}}\left(({}_tV + G_t - e_t)(1 + i_t^{\text{actual}}) - q_{x+t}(S_{t+1} + E_{t+1}) + p_{x+t}({}_{t+1}V)\right).\end{aligned}$$

- *Profit due to expenses:*

$$
\begin{aligned}
&\text{Profit with actual interest \& expenses} - \text{Profit with actual interest} \\
= \quad & \ell_{x+t}^{\text{actual}}\Big(({}_tV + G_t - e_t^{\text{actual}})(1+i_t^{\text{actual}}) - q_{x+t}(S_{t+1} + E_{t+1}^{\text{actual}}) + p_{x+t}({}_{t+1}V) \Big) \\
& - \ell_{x+t}^{\text{actual}}\Big(({}_tV + G_t - e_t)(1+i_t^{\text{actual}}) - q_{x+t}(S_{t+1} + E_{t+1}) + p_{x+t}({}_{t+1}V) \Big) \\
= \quad & \ell_{x+t}^{\text{actual}}\Big[-(e_t^{\text{actual}} - e_t)(1+i_t^{\text{actual}}) - q_{x+t}(E_{t+1}^{\text{actual}} - E_{t+1}) \Big].
\end{aligned}
$$

- *Profit due to mortality:*

$$
\begin{aligned}
&\text{Profit with actual interest, expenses \& mortality} - \text{Profit with actual interest \& expenses} \\
\overset{\text{(NAAR form)}}{=} \quad & \ell_{x+t}^{\text{actual}}\Big(({}_tV + G_t - e_t^{\text{actual}})(1+i_t^{\text{actual}}) - q_{x+t}^{\text{actual}}(S_{t+1} + E_{t+1}^{\text{actual}} - {}_{t+1}V) + {}_{t+1}V \Big) \\
& - \ell_{x+t}^{\text{actual}}\Big(({}_tV + G_t - e_t^{\text{actual}})(1+i_t^{\text{actual}}) - q_{x+t}(S_{t+1} + E_{t+1}^{\text{actual}} - {}_{t+1}V) + {}_{t+1}V \Big) \\
= \quad & -\ell_{x+t}^{\text{actual}}(q_{x+t}^{\text{actual}} - q_{x+t})(S_{t+1} + E_{t+1}^{\text{actual}} - {}_{t+1}V).
\end{aligned}
$$

[Note: Here we always multiply the per-policy profit by the actual number of policies at time t: "Actual" or "assumed" mortality is actually referring whether p_{x+t}^{actual} & q_{x+t}^{actual} or p_{x+t} & q_{x+t} should be used.]

5.2 Profit Testing

5.2.1 To conduct profit testing, we will make extensive use of the following notations:

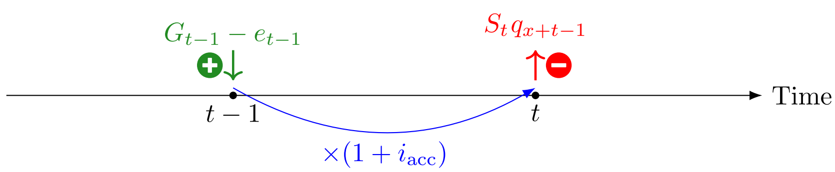
- E_0: **pre-contract expense**. This is given by initial expense less renewal expense and is conventionally treated as a payment made "at the end of year 0"[17].
- e_{t-1}: renewal expense made at the beginning of year t (i.e., time $t-1$). Note that the initial expense is decomposed into pre-contract expense E_0 and "renewal expense" e_0 made at the beginning of year 1 here.
- G_{t-1}: gross premium payable at the beginning of year t.
- S_t: death benefit payable at the end of year t if death occurs in that year.
- i_{acc}: annual effective interest rate (or "accumulation rate") used in the profit testing.

Armed with these notations, we can now discuss several important quantities involved in profit testing.

5.2.2 **Expected net cash flows.** There are two kinds of expected net cash flows, which can be distinguished by whether we are considering *per policy in-force* or *per policy issued*.

- *Per policy in-force (at the start of year t):* The **in-force expected net cash flow** (or **in-force emerging surplus**) for year t, denoted by CF_t, is the expected net cash flow obtained by accumulating (at i_{acc}) all the (expected) cash flows in year t to the year end:

$$\boxed{CF_0 = -E_0} \quad \text{and} \quad \boxed{CF_t = (G_{t-1} - e_{t-1})(1+i_{\text{acc}}) - S_t q_{x+t-1}} \text{ for any } t = 1, 2, \ldots.$$



[17] This is in contrast with the first premium payment time, which is at the beginning of year 1. Mathematically we can treat them as referring to the same time point, but for some terms appearing later, it is essential to distinguish them carefully.

Here we have $CF_0 = -E_0$ since the only (negative) cash flow in "year 0" is $-E_0$, corresponding to the pre-contract expense "at the end of year 0".

- *Per policy issued:* The **expected net cash flow per policy issued** for year t, denoted by EC_t, is given by

$$\boxed{EC_0 = CF_0} \quad \text{and} \quad \boxed{EC_t} = \frac{\overbrace{\ell_{x+t-1}CF_t}^{\text{total expected NCF}}}{\underbrace{\ell_x}_{\text{no. of policies issued}}} = \boxed{{}_{t-1}p_x \times CF_t} \text{ for any } t = 1, 2, \ldots.$$

5.2.3 **Expected profits.** Again there are two kinds of expected profits, which can similarly distinguished by whether *per policy in-force* or *per policy issued* is considered.

- *Per policy in-force:* The **in-force expected profit** for year t, denoted by PR_t, is like the profit we have seen in Section 5.1, and can be expressed as "what you have" − "what you need", with the latter being expected value:

$$\boxed{PR_0 = -\underbrace{E_0}_{\text{(what you need)}}} \quad \text{and} \quad \boxed{PR_t = \underbrace{({}_{t-1}V + G_{t-1} - e_{t-1})(1 + i_{\text{acc}})}_{\text{(what you have)}} - \underbrace{q_{x+t-1}(S_t + E_t) + p_{x+t-1}({}_tV)}_{\text{(what you need)}}}$$

for any $t = 1, 2, \ldots$. While it appears that PR_t is always zero for any $t = 1, 2, \ldots$ due to the basic policy value recursion formula, it is actually not the case ⚠. The reason is that the profit testing interest rate i_{acc} may NOT be the same as the interest rate used for computing policy values (the one appearing in the recursive formula).

- *Per policy issued:* The **expected profit per policy issued** for year t, denoted by σ_t, is given by

$$\boxed{\sigma_0 = PR_0} \quad \text{and} \quad \boxed{\sigma_t} = \frac{\overbrace{\ell_{x+t-1}PR_t}^{\text{total expected profit}}}{\underbrace{\ell_x}_{\text{no. of policies issued}}} = \boxed{{}_{t-1}p_x \times PR_t} \text{ for any } t = 1, 2, \ldots.$$

After collecting the in-force expected profits and expected profits per policy issued for different years into vectors, there are special names for the vectors:

- **Profit vector**: $(PR_0, PR_1, \ldots, PR_n)$.
- **Profit signature**: $(\sigma_0, \sigma_1, \ldots, \sigma_n)$.

[Note: Here we suppose that a policy with n years term is being considered, so we only include the values up to year n.]

5.2.4 **Relationship between expected net cash flows and profits.** There is an interesting relationship between the in-force expected NCF CF_t and the in-force expected profit PR_t as follows:

$$PR_t = CF_t - {}_{t-1}V(1 + i_{\text{acc}}) - p_{x+t-1}({}_tV).$$

It is not hard to verify this directly but checking the definitions (try this!). The interesting thing about this relationship is that a rather intuitive interpretation can be obtained by noting that the blue expression is the (expected) increase in reserve per policy in-force. Denoting it by IR_t, we can write $\boxed{PR_t = CF_t - IR_t}$, which can be intuitively understood as saying that:

$$\text{profit you have} = \underbrace{\text{amount \$ going into your pocket}}_{\text{(from the NCF)}} - \underbrace{\text{amount \$ going out of your pocket}}_{\text{(for increasing reserve)}}.$$

[Note: To see why the blue expression is the increase in reserve per policy in-force, write

$${}_{t-1}V(1 + i_{\text{acc}}) - p_{x+t-1}({}_tV) = \frac{\overbrace{\ell_{x+t-1} \times {}_{t-1}V(1 + i_{\text{acc}}) - \ell_{x+t} \times {}_tV}^{\text{total (expected) increase in reserve}}}{\underbrace{\ell_{x+t-1}}_{\text{no. of policies in-force}}}$$

where we note that the start-of-year reserve ${}_{t-1}V$ can earn interest at rate i_{acc} in year t, and at the end of year t, the reserve amount ${}_tV$ is only for each policy in-force at that time.]

5.2.5 **Profit measures.** After learning about these important quantities involved in profit testing, it is time to actually apply them in profit testing. Here we will do that by using some *profit measures* which, as its name suggests, measure profits. In fact, you should have seen some of them back in STAT3904. But we will go through all of them in details here, in case you already forget the STAT3904 content (pretty normal, I guess).

Since our goal here is to measure profits per policy issued[18], we will make extensive use of the expected profits *per policy issued* σ_t's. Four profit measures will be discussed here:

- Net present value (NPV)
- *(new!)* Profit margin
- Discounted payback period (DPP)
- Internal rate of return (IRR)

[Note: Henceforth we shall consider a policy with n years term.]

5.2.6 **Net present value.**

(1) *Definition:* The **net present value** (NPV) of a policy is the sum of all discounted σ_t's (at an interest rate r, known as the **required rate of return**, **hurdle rate**, or **risk discount rate**):

$$\boxed{\text{NPV @ } r = \sum_{t=0}^{n} \sigma_t (1+r)^{-t}}.$$

(2) *Discussion:* We have the following relationships between how we set reserves and the emerging pattern of expected profits:

- Setting higher reserves earlier ➔ Higher σ_t's emerging later
- Setting higher reserves later ➔ Higher σ_t's emerging earlier

Often, the hurdle rate exceeds the profit testing interest rate, i.e., $r > i_{\text{acc}}$. In such case, higher σ_t's emerging earlier leads to a higher NPV. Hence, setting higher reserves later would usually result in a higher NPV.

5.2.7 **Profit margin.** The **profit margin** of a policy at the hurdle rate r is the NPV per expected PV of premiums:

$$\text{Profit margin @ } r = \frac{\overbrace{\ell_x \times \text{NPV @ } r}^{\text{(total NPV)}}}{\underbrace{\sum_{t=0}^{n-1} G_t \times \ell_{x+t} \times (1+r)^{-t}}_{\text{(total expected PV of premiums)}}} = \boxed{\frac{\text{NPV @ } r}{\sum_{t=0}^{n-1} G_t \times {}_tp_x \times (1+r)^{-t}}}.$$

Intuitively, profit margin tells the proportion of premiums earned as profits.

5.2.8 **Discounted payback period.** The **discounted payback period** (DPP) is the earliest time at which the NPV is nonnegative:

$$\text{DPP} = \min\{k \in \{0,1,2,\dots\} : \text{NPV}_k \geq 0\}$$

where $\text{NPV}_k = \sum_{t=0}^{k} \sigma_t (1+r)^{-t}$ is the **partial NPV up to time k**, for any $k = 0, 1, \dots, n$.

DPP *does not exist* when $\text{NPV}_k < 0$ for any $k = 0, 1, , \dots, n$. In such case, it means that the policy "never pays back" and thus it is probably not a good idea to offer such policy.

[18]We want to analyze the profitability right at the onset, to decide whether we should offer those products or not at the very beginning.

5.2.9 **Internal rate of return.** The **internal rate of return** (IRR) is any interest rate at which the NPV is zero, i.e., any value i_{IRR} such that NPV @ $i_{\text{IRR}} = 0$.

[Note: There can be no IRR or multiple IRRs. But in general, if the profit signature $(\sigma_0, \sigma_1, \ldots, \sigma_n)$ has exactly one sign change from negative to positive, i.e., all negative terms before positive terms, then the IRR is uniquely determined.]

5.2.10 **Pricing by profit testing.** Now let us start discussing how we can use profit testing to set premiums (pricing) and reserves (reserving). For pricing, there can be many ways to utilize the results from profit testing. For instance, the insurer may compute the NPV at a fixed hurdle rate for various premiums, and set the premium as the *breakeven premium*, i.e., the one that leads to a zero NPV. In practice, it may be more feasible with the help of a spreadsheet.

5.2.11 **Reserving by profit testing.** The discussion about using profit testing to set reserves is more interesting since there is a special method to do that, which is known as *zeroization*. As you may expect, this method involves some "zeros".

Zeroization is an iterative process which computes the reserves to be set in a backward fashion. Basically, zeroization determines reserves that leads to zero profits in later years. Intuitively, this process sets "just enough" reserves to avoid having excessively high reserves in earlier years, which would lead to a lower NPV as suggested in [5.2.6].

More specifically, the zeroization process is as follows (for an n-year policy):

- Solve $PR_n = 0$ for ${}_{n-1}V$ [19] and label it as ${}_{n-1}V^Z$.
- Solve $PR_{n-1} = 0$ with ${}_{n-1}V \to {}_{n-1}V^Z$ for ${}_{n-2}V$, and label it as ${}_{n-2}V^Z$.
- $\vdots$
- Solve $PR_1 = 0$ with ${}_1V \to {}_1V^Z$ for ${}_0V$ and label it as ${}_0V^Z$. [Warning: The time-0 reserve is not necessarily zero!]

Remarks:

- Often the formula $PR_t = CF_t - IR_t$ is helpful here.
- In case a negative solution is obtained when solving $PR_t = 0$, we should set the time-t reserve ${}_tV^Z$ as 0 (because it does not make sense to have "negative reserve"), and 0 is the smallest possible reserve.
- The value ${}_tV^Z$ is known as the time-t **zeroized reserve**.

5.2.12 **Profit testing for multiple state model.** The last (finally! ☺) topic to be discussed in Section 5 relates profit testing with what we have learnt in Section 2: multiple state model. Here we will *generalize* the quantities for profit testing from [5.2.2] and [5.2.3], and the generalized quantities can be applied in profit testing through profit measures.

(1) *Notations:* In multiple state model, we would add superscripts to the notations previously used (like what we have done in Section 2.3), to indicate the states in which the quantities are applicable:
   - *(pre-contract expense)* E_0 remains unchanged
   - *(renewal expense)* $e_{t-1} \to e_{t-1}^{(j)}$
   - *(gross premium)* $G_{t-1} \to G_{t-1}^{(j)}$
   - *(death benefit)* $S_t \to S_t^{(j)}$
   - i_{acc} remains unchanged (assuming it is the same for every state)
   - $CF_t \to CF_t^{(j)}$
   - EC_t remains unchanged
   - $PR_t \to PR_t^{(j)}$

[19] It is possible since the boundary value ${}_nV$ is known.

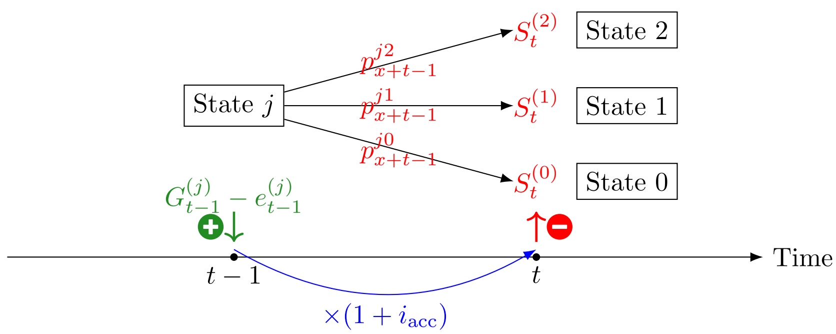
- σ_t remains unchanged

[Note: We assume the starting state is fixed and clear from context, so we do not add superscripts to E_0, EC_t, and σ_t as they all only depend on the starting state. Without loss of generality, we suppose the starting state is 0 henceforth.]

(2) *Expected net cash flow...*

- *per policy in-force (in state j):*

$$CF_0^{(0)} = -E_0 \quad \text{and} \quad CF_t^{(j)} = (G_{t-1}^{(j)} - e_{t-1}^{(j)})(1+i_{\text{acc}}) - \sum_k p_{x+t-1}^{jk} S_t^{(k)} \text{ for any } t = 1, 2, \ldots.$$



- *per policy issued:*

$$EC_0 = CF_0^{(0)} \quad \text{and} \quad EC_t = \frac{\overbrace{\sum_j \ell_{x+t-1}^{(j)} CF_t^{(j)}}^{\text{total expected NCF}}}{\underbrace{\ell_x^{(0)}}_{\text{no. of policies issued}}} = \sum_j {}_{t-1}p_x^{0j} \times CF_t^{(j)} \text{ for any } t = 1, 2, \ldots.$$

[Note: Here $\ell_{x+t-1}^{(j)}$ denotes the expected number of policies in-force in state j at time $t-1$.]

(3) *(More important) Expected profit...*

- *per policy in-force (in state j):*

$$PR_0^{(0)} = -\underbrace{E_0}_{\text{(what you need)}} \quad \text{and}$$

$$\boxed{PR_t^{(j)} = \underbrace{({}_{t-1}V^{(j)} + G_{t-1}^{(j)} - e_{t-1}^{(j)})(1+i_{\text{acc}})}_{\text{(what you have)}} - \underbrace{\sum_k p_{x+t-1}^{jk}\left(B_t^{(k)} + E_t^{(k)} + {}_tV^{(k)}\right)}_{\text{(what you need)}}}$$

for any $t = 1, 2, \ldots$, where $B_t^{(k)}$ and $E_t^{(k)}$ carry the meanings from [2.3.5].

- *per policy issued:*

$$\sigma_0 = PR_0^{(0)} \quad \text{and} \quad \boxed{\sigma_t} = \frac{\overbrace{\sum_j \ell_{x+t-1}^{(j)} PR_t^{(j)}}^{\text{total expected profit}}}{\underbrace{\ell_x^{(0)}}_{\text{no. of policies issued}}} = \boxed{\sum_j {}_{t-1}p_x^{0j} \times PR_t^{(j)}} \text{ for any } t = 1, 2, \ldots.$$

References

Dickson, D. C., Hardy, M. R., & Waters, H. R. (2019). *Actuarial mathematics for life contingent risks.* Cambridge University Press.

Links to Definitions

Results

Section 1

- [1.2.2]b: recursive formula for asset shares
- [1.3.2]: basic principle for valuation upon policy alteration

Section 2

- [2.1.5]: general probability calculation formula
- [2.1.6]: Chapman-Kolmogorov equation
- [2.1.8]: first step analysis
- [2.1.10]: occupancy probability formula
- [2.1.11]: Kolmogorov forward equation
- [2.1.12]: general transition probability formula
- [2.2.6]: EPV recursive formulas in multiple state model
- [2.2.9]: EPV formula for continuous annuity with waiting period when $i \neq j$
- [2.2.10]: EPV formula for continuous annuity with waiting period when $i = j$
- [2.3.5]: basic policy value recursion in multiple state model
- [2.3.9]: Thiele's differential equation in multiple state model setting

Section 3

- [3.1.3]: formulas for converting multiple decrement table quantities to probabilistic quantities
- [3.1.6]: formulas for fractional age assumptions
- [3.1.9]: formulas for probabilistic quantities based on random variable approach
- [3.1.11]: relationships between multiple decrement model and associated single decrement models
- [3.1.12]: further relationships between multiple decrement model and associated single decrement models, under MUDD/SUDD
- [3.1.14]: formula for ${}_tp_x^{0j}$ with the presence of discrete decrements

Section 4

- [4.1.10]: probabilities about T_{xy} and $T_{\overline{xy}}$
- [4.1.13]: moments and covariances about T_{xy} and $T_{\overline{xy}}$
- [4.1.18]: probabilities about order of death
- [4.1.20]: key formulas under independent lifetime assumption
- [4.2.7]: covariance between whole life PVRVs
- [4.2.8]: "Joint-life + Last-survivor = Individual sum" formulas for EPVs
- [4.2.13]: EPV formulas for insurances contingent on order of death
- [4.2.14]: EPV formulas for reversionary annuities

Section 5

- [5.1.4]: method for classifying profits by sources
- [5.2.2]: formulas for expected net cash flow (per policy in-force/issued)
- [5.2.3]: formulas for expected profit (per policy in-force/issued)
- [5.2.6]: formula for NPV
- [5.2.7]: formula for profit margin
- [5.2.8]: formula for DPP
- [5.2.9]: formula for IRR
- [5.2.11]: zeroization process for setting reserves by profit testing
- [5.2.12]: formulas for quantities involved in profit testing in multiple state model