STAT3901 Study Notes

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[Note: Related SOA Exam: FAM (long-term)]

Contents

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1 Introduction to Life Contingencies

1.1 Introduction to Life Insurance Business

1.1.1 Suppose an individual (the **policyholder**) wants to buy a life insurance policy for himself. Should the insurance company accept his application and sells the policy to him? Often this is decided by performing *underwriting*.

1.1.2 During the **underwriting** process, provides some of his information to , like age, gender, occupation, etc.

1.1.3 After is finished, there are typically three outcomes:

(a) accept with standard premium charged (most common);

(b) accept with "higher than standard" premium charged (this indicates that has higher risk than usual based on the information provided);

(c) reject (the risk level suggested by the information is so high that the policyholder is *uninsurable* [e.g., it is "almost" sure that he will die very soon]).

1.1.4 Now suppose accepts his application with standard premium charged. Then, will periodically pay **premiums** to , in exchange for the life insurance coverage.

1.1.5 Note that the *life insured* for this policy is the policyholder himself. But it is possible that the life insured is someone else .

1.1.6 If the life insured dies within the term covered by the policy, then will pay the **sum insured**/**death benefit** (as specified in the policy) to the policyholder .

[Note: From this we can observe a potential issue: if the death of causes no harm to at all, then the policy may *incentivize* to *kill* ! To deal with this problem, the underwriting process would ensure the policy is only sold to that has **insurable interest**, i.e., the death of the life insured should make the policyholder quite worse off.[1]]

1.1.7 Typically, the life insurance policy would expire immediately after the sum insured is paid (if such payment exists), or when such payment is impossible to be made (e.g., see [1.2.9]).

1.1.8 The timing of the life insured's death is very important to , since:

(a) it affects the timing of the death benefit payment , hence its *present value*;

(b) it also influences how much premiums would be paid (as the premium payment ceases immediately after the death).

Both of them are crucial to the *profitability* of the life insurance business!

1.1.9 Both and depend on the survival status of the life insured , so we say that they are **life contingent** (contingent on the death/survival of). Their importance to the profitability of motivates the study of *life contingencies*.

1.2 Traditional Contracts for Life Insurance Business

1.2.1 We introduce some traditional contracts here:

(a) term life insurance |

(b) whole life insurance |

(c) endowment insurance || and pure endowment

(d) temporary life annuity |

[1] Expressing differently, it is of interest to when is alive — survival of gives some benefits to .

(e) certain-and-life (or guaranteed) annuity

(f) whole life annuity

[Note: The last two contracts are *life annuity* contracts, and are not "insurance" contracts. Although is called "life insurance" business, offers not only insurance policies, but also policies "related to life", e.g. life annuity.]

1.2.2 **Term life insurance** is a life insurance contract with a specified term — death benefit is only payable when the life insured dies *within* the term.

1.2.3 **Whole life insurance** is a life insurance contract with *indefinite* term, i.e., death benefit is always payable when the life insured dies.

1.2.4 **Endowment insurance**[2] is a term life insurance with an extra survival benefit (equal to the death benefit) payable if the life insured is still alive at the end of the term (this can be seen as a "savings element").

[Note: **Pure endowment** is an endowment insurance with the term life insurance element taken away — it only contains a survival benefit payable when the life insured is still alive at the end of the term.]

1.2.5 **Temporary life annuity** is a term annuity with periodic payments to (usually called **annuitant** in the context of life annuity), while is alive.

[Note: We use the word "temporary" instead of "term" here to emphasize that the periodic payments are only *temporary*.]

1.2.6 **Certain-and-life annuity** (or **guaranteed annuity**) is a term annuity with periodic payments to the annuitant for *at least* a guaranteed term (those payments are paid *certainly*), and the payments continue afterward as long as is alive.

1.2.7 **Whole life annuity** is a perpetuity with periodic payments to the annuitant while is alive.

1.2.8 For all the contracts here, premiums are payable. Conventionally (for "discrete" case), they are paid *at the beginning* of each period, during a time interval specified in the contract. On the other hand, the death (survival) benefit for the insurance is conventionally paid *at the end* of the period of death (last period of the term resp.) (for "discrete" case).

[Note: Unless otherwise specified, the measurement period (i.e., time unit for "periods") is years.]

1.2.9 The contracts can be **deferred**, in the sense that the "coverage" only starts a specified time length (called the **deferral period**) after the purchasing time. For example, for a deferred whole life insurance, if the death of life insured occurs during the deferral period, then no death benefit is payable (and the death benefit is impossible to be made anymore, so the policy would expire).

1.2.10 Sometimes it possible for to **lapse** or **surrender** the policy, i.e., can choose to stop paying premiums starting at a certain time, and simultaneously the policy would expire at that time. In case of policy lapse/surrender, may receive some refund (called **cash value** or **surrender value**).

[Note: Sometimes the term "lapse" is used only when there is no such refund, and the term "surrender" is used only when there is such refund.]

1.2.11 For some term contracts, it is **renewable**: the policyholder can choose to *renew* the contract at the end of the term, for a time length specified in the contract, by continuing to pay premiums (with possibly different amount from before) to receive a prolonged coverage.

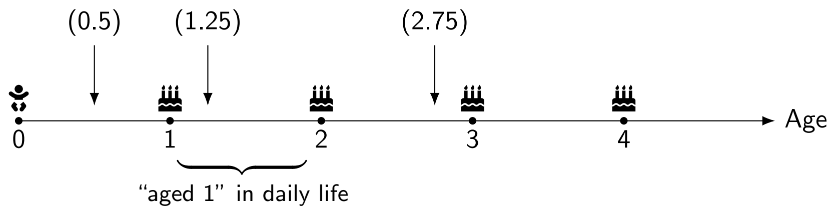
[2] Endowment means "funds invested for the support and benefit of a person". In this context, it refers to the "savings element" (to benefit the policyholder).

2 Survival Distributions

2.1 Future Lifetime Random Variable

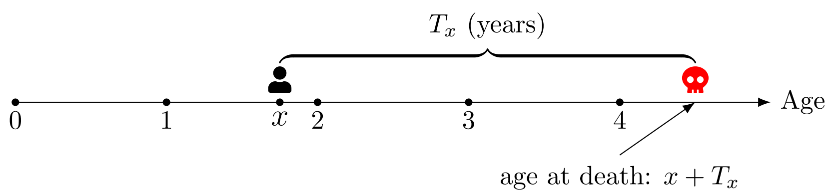
2.1.1 A life aged x (in years) is denoted by (x).

[Note: By "aged x", we mean the life's age is *exactly* x:



From here we can see that "age" in daily life language actually refers to number of birthdays passed, which is *different* from our meaning.]

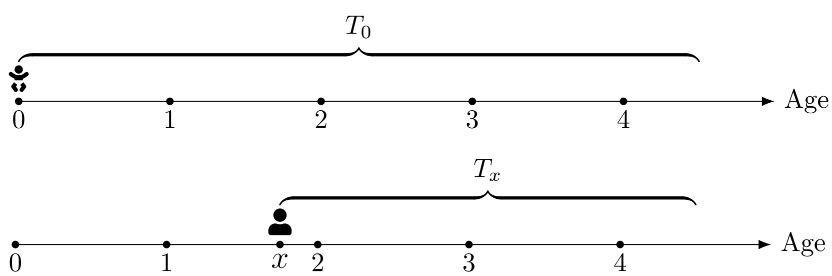
2.1.2 **Future lifetime random variable** for (x) is a continuous random variable T_x with support $[0, \infty)$. [Note: This means a life aged x cannot live forever, and the life "almost never" die exactly at age x (i.e., with probability 0).]



2.1.3 The cdf of T_x (denoted by $F_x(t)$) is $F_x(t) = \mathbb{P}(T_x \leq t)$, which gives the probability that (x) dies within t years.

2.1.4 The survival function of T_x (denoted by $S_x(t)$) is $S_x(t) = \mathbb{P}(T_x > t)$, which gives the probability that (x) survives for more than t years.

2.1.5 We would like to investigate the relationship between distributions of T_0 and T_x for the same "underlying life":



[Note: and are the same life, but at different ages.]

An "internal" information given implicitly is that survives to age x (since (x) *exists*!), i.e., the future lifetime of (T_0) is greater than x.

Given this information, one may naturally expect $T_0 - x$ and T_x to have the same distribution, as they appear to describe the same thing (how long will live for). But a rather subtle difference between them is the *modelling time* of the distribution: the distribution of $T_0 - x$ is to be modelled at age 0, while the distribution of T_x is to be modelled at age x.

With the absence of "external" information between age 0 and age x, such expectation is reasonable. However, if there is some relevant "external" information during the time interval (e.g., discovery of cancer cure), they can have different distributions.

Nonetheless, to enrich the study of life contingencies, we shall impose the assumption that for any age x,

$$(T_0 - x | T_0 > x) \stackrel{d}{=} T_x.$$

<u>Remarks</u>:

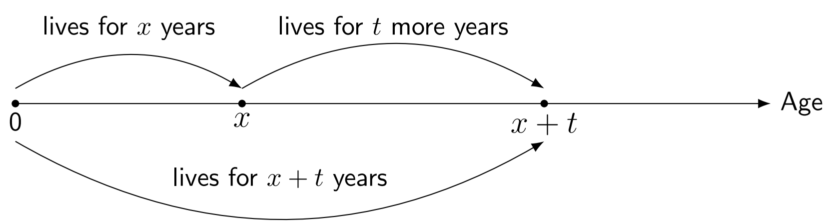
- The assumption may be understood more intuitively as stating that, for any age x, (i) T_0 and T_x have the same "underlying life"; and (ii) there is no "external" information between age 0 and age x.
- Expressing the assumption in terms of cdf, we have:

$$\mathbb{P}(T_x \le t) = \mathbb{P}(T_0 \le x + t | T_0 > x).$$

2.1.6 Having the assumption in [2.1.5], we can derive the following results:

Proposition 2.1.a. $S_0(x+t) = S_0(x)S_x(t)$ for any age x and $t \ge 0$.

[Intuition: LHS means "(0) lives for $x + t$ years", and RHS means "(0) lives for x years (becoming (x)), and then lives for t more years", so LHS and RHS "should" be the same.[3]



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[Note: This "and-then" intuition is "justified" because of the assumption in [2.1.5].]

<u>Proof</u>: Note that

$$\mathbb{P}(T_0 \le x + t | T_0 > x) = \frac{\mathbb{P}(x < T_0 \le x + t)}{\mathbb{P}(T_0 > x)} = \frac{S_0(x) - S_0(x+t)}{S_0(x)} = 1 - \frac{S_0(x+t)}{S_0(x)},$$

and

$$\mathbb{P}(T_x \le t) = 1 - S_x(t).$$

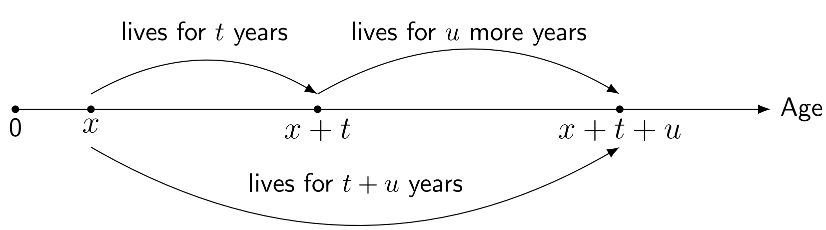
So, we have

$$\mathbb{P}(T_x \le t) = \mathbb{P}(T_0 \le x + t | T_0 > x) \implies S_x(t) = \frac{S_0(x+t)}{S_0(x)},$$

as desired. $\square$

Proposition 2.1.b. $S_x(t+u) = S_x(t)S_{x+t}(u)$ for any age x and any $t, u \ge 0$.

[Intuition: LHS means "(x) lives for $t + u$ years", and RHS means "(x) lives for t years, and then lives for u more years", so LHS and RHS "should" be the same.



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[3]Generally, this kind of "intuition" is very helpful in the study of life contingencies.

Proof: Applying the result in proposition 2.1.a to LHS and RHS gives respectively

$$\frac{S_0(x+t+u)}{S_0(x)} \text{ and } \frac{S_0(x+t)}{S_0(x)} \cdot \frac{S_0(x+t+u)}{S_0(x+t)}$$

which are clearly identical. $\square$

[Note: From here we can deduce that

$$\mathbb{P}(T_x > t+u) = \mathbb{P}(T_x > t)\mathbb{P}(T_{x+t} > u) \implies \mathbb{P}(T_x > t+u | T_x > t) = \mathbb{P}(T_{x+t} > u),$$

which implies $(T_x - t | T_x > t) \stackrel{d}{=} T_{x+t}$, a "generalization" to our assumption in [2.1.5].]

2.1.7 To model the distribution of future lifetime, we often specify a *survival function* (which can completely determine a probability distribution). Thus, we are interested in studying what conditions on a function are needed for it to qualify as a survival function for the future lifetime, so that we know whether our choice of "survival function" is reasonable or not.

Proposition 2.1.c. A function S_x defined on $[0, \infty)$ is a valid survival function for the future lifetime iff the following hold:

(S1) $S_x(0) = 1$;

(S2) $\lim_{t\to\infty} S_x(t) = 0$;

(S3) $S_x(t)$ is a nonincreasing/decreasing function of t.

[Intuition 💡: (S1) reflects that a life aged x cannot be dead at age x. (S2) means that all lives eventually die. (S3) respects the monotonicity property of probability.]

2.1.8 For mathematical convenience, we shall impose the following additional conditions on S_x:

(a) $S_x(t)$ is differentiable at any $t > 0$ (except possibly at some "edge" points)

(b) $\lim_{t\to\infty} tS_x(t) = 0$.

(c) $\lim_{t\to\infty} t^2 S_x(t) = 0$.

Remarks:

- The latter two conditions are mainly useful for ensuring the existence of mean and variance of T_x.
- For survival functions introduced here, they all satisfy these conditions as well as the properties in [2.1.7].

2.2 Force of Mortality

2.2.1 Recall that the *force of interest* at time t (δ_t) can be seen as "relative" rate of change of amount function $A(\cdot)$:

$$dA(t) = A(t)\delta_t \, \mathrm{d}t\,, \quad \text{or} \quad \delta_t = \frac{A'(t)}{A(t)},$$

where $A(t)\delta_t \, \mathrm{d}t$ may be understood intuitively as "interest earned in $[t, t + \mathrm{d}t]$".

2.2.2 Since force of mortality is also a "force", one may expect it is also defined as relative rate of change of some function. A natural choice may be the survival function S_0 (for force of mortality "modelled" at age 0). But as "mortality" is opposite to "survival", it appears that we should add a negative sign in the definition.

2.2.3 The **force of mortality (“modelled” at age 0)** at time t, denoted by μ_t (or $\mu_0(t)$), is defined as

$$\mu_t = -\frac{S_0'(t)}{S_0(t)}.$$

[Intuition 💡: We can express this as $\mathrm{d}S_0(t) = -S_0(t)\mu_t\,\mathrm{d}t$, and $S_0(t)\mu_t\,\mathrm{d}t$ may be intuitively understood as “drop in survival probability in $[t, t+\mathrm{d}t]$” or “probability of death in $[t, t+\mathrm{d}t]$”.

]

2.2.4 Denote by $f_x(t)$ the pdf of T_x. Then, we have:

Proposition 2.2.a.

$$\mu_t = \frac{f_0(t)}{S_0(t)}$$

for any $t > 0$.

[Note: This explains why the intuition in [2.2.3] works: because $S_0(t)\mu_t\,\mathrm{d}t = f_0(t)\,\mathrm{d}t$.]

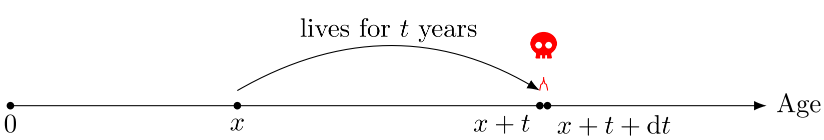
Proof: The result follows from noting that

$$-\frac{\mathrm{d}}{\mathrm{d}t}S_0(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(1 - F_0(t)) = -(-f_0(t)) = f_0(t).$$

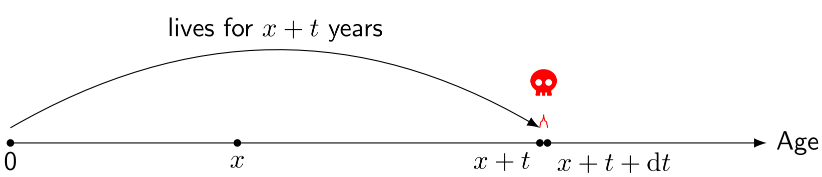
□

2.2.5 Now we would like to extend the concept of force of mortality to the case where it can be “modelled” at any age x. The definition is analogous: The **force of mortality (“modelled” at age x)** at time t, denoted by $\mu_x(t)$, is

$$\mu_x(t) = -\frac{S_x'(t)}{S_x(t)}.$$



A natural question arised is whether $\mu_x(t) = \mu_{x+t}$ or not. [Note: The latter one is to be regarded as the force of mortality at time $x+t$ (“modelled” at age 0) here:



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Fortunately, this is the case:

Proof: Firstly,

$$\frac{\mathrm{d}}{\mathrm{d}t}S_x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{S_0(x+t)}{S_0(x)} = \frac{1}{S_0(x)}\frac{\mathrm{d}}{\mathrm{d}t}S_0(\underbrace{x+t}_{u}) = \frac{S_0'(u)}{S_0(x)}\underbrace{\frac{\mathrm{d}u}{\mathrm{d}t}}_{1},$$

and

$$S_x(t) = \frac{S_0(u)}{S_0(x)}.$$

So,

$$\mu_x(t) = -\frac{S_0'(u)/S_0(x)}{S_0(u)/S_0(x)} = -\frac{S_0'(u)}{S_0(u)} = \mu_u = \mu_{x+t}.$$

□

In view of this, we can simply use the notation "μ_{x+t}" to stand for either meaning without ambiguity.

2.2.6 One would then expect the following result holds also:

Proposition 2.2.b.

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)}$$

for any age x and $t > 0$.

Proof: Replace "0" in the proof of proposition 2.2.a by x. □

2.2.7 Recall that for force of interest, we have

$$A(t) = A(0)\exp\left(\int_0^t \delta_s\,ds\right).$$

As force of mortality is defined similarly, we may anticipate similar results hold for force of mortality, and this is the case: [Note: Recall that $S_x(0) = 1$.]

Proposition 2.2.c.

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s}\,ds\right)$$

for any age x and $t > 0$.

[Intuition 💡: Firstly, since force of mortality is defined similarly as force of interest, we have analogously $S_x(t) = e^{-\mu t}$ if the force of mortality is *constant* ($\mu_x(t) = \mu_{x+t} = \mu$ for any $t > 0$). Then, when $h > 0$ is small, we would have

$$S_x(s+h) \approx S_x(s)e^{-\mu_{x+s}h}$$

(during a "small" time interval $[s, s+h]$, the force of mortality is "approximately" constant). Thus,

$$S_x(t) = \frac{S_x(h)}{S_x(0)}\frac{S_x(2h)}{S_x(h)}\frac{S_x(3h)}{S_x(2h)}\cdots\frac{S_x(t)}{S_x(t-h)} \approx \exp\left[-(\mu_x h + \mu_{x+h}h + \mu_{x+2h}h + \cdots + \mu_{x+t-h}h)\right],$$

and loosely, as $h \to 0^+$, we have

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s}\,ds\right).$$

]

Proof: Note that $\mu_x(s) = \dfrac{\mathrm{d}}{\mathrm{d}s}\ln S_x(s)$. Then, taking integral on both sides and applying fundamental theorem of calculus give the desired result. □

2.2.8 By proposition 2.2.c, if we know μ_y for any $y \in [x, x+t]$, $S_x(t)$ is determined. Hence, once μ_y is known for any $y \geq x$ (so $S_x(t)$ is determined for any $t \geq 0$), the distribution of T_x is *completely specified*.

2.2.9 In view of [2.2.8], an alternative way to model the distribution of future lifetime is to specify the force of mortality at different time. There are some typical specifications of force of mortality (which completely determine the future lifetime's distribution), called **mortality laws** (they govern the "behaviour" of mortality).

2.2.10 Here we discuss some mortality laws:

(a) Gompertz

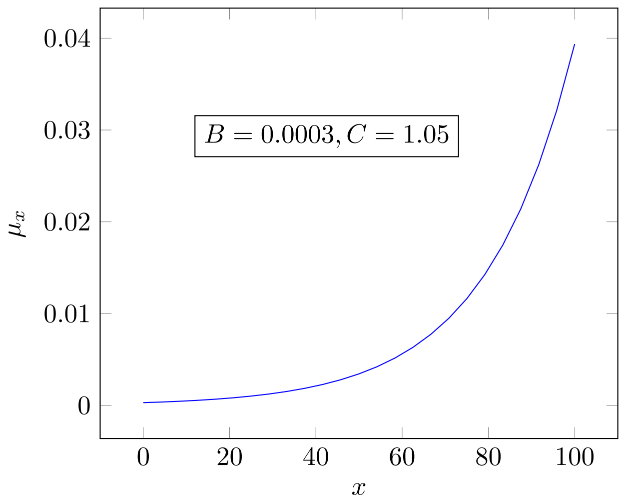
(b) Makeham

(c) (Generalized) De Movire

(d) Constant Force of Mortality

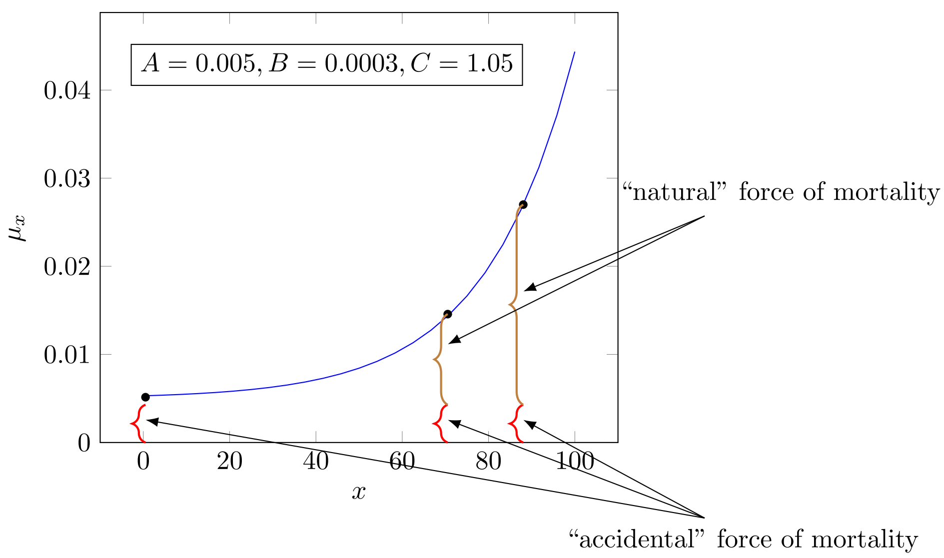
2.2.11 **Gompertz's Law** is defined by

$$\mu_x = Bc^x, \quad \text{where } B > 0, c > 1.$$



2.2.12 **Makeham's Law** (genealization to Gompertz's law) is defined by

$$\mu_x = A + Bc^x, \quad \text{where } A, B > 0, c > 1.$$



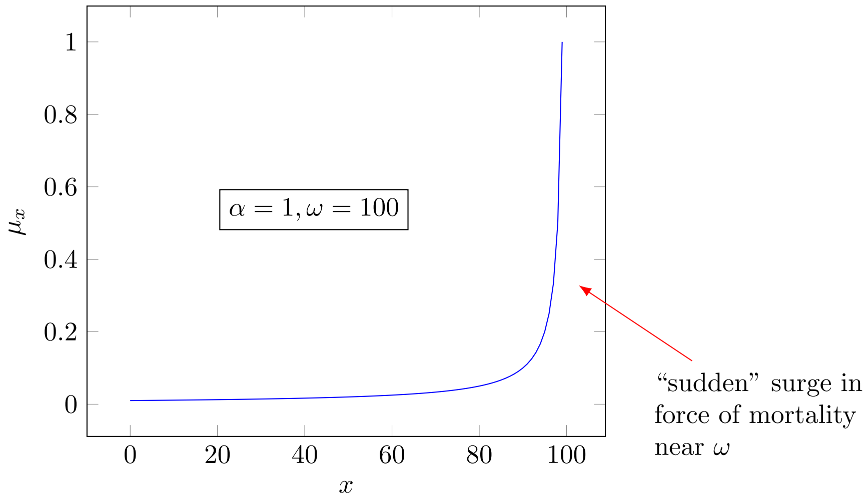
The constant A reflects the risk of *accidental death*, which is supposed to be constant for any age.

2.2.13 **De Moivre's Law** is defined by

$$\mu_x = \frac{1}{\omega - x}, \quad \text{where } 0 \le x < \omega$$

and the **generalized De Moivre's Law** is defined by

$$\mu_x = \frac{\alpha}{\omega - x}, \quad \text{where } 0 \le x < \omega, \alpha > 0.$$



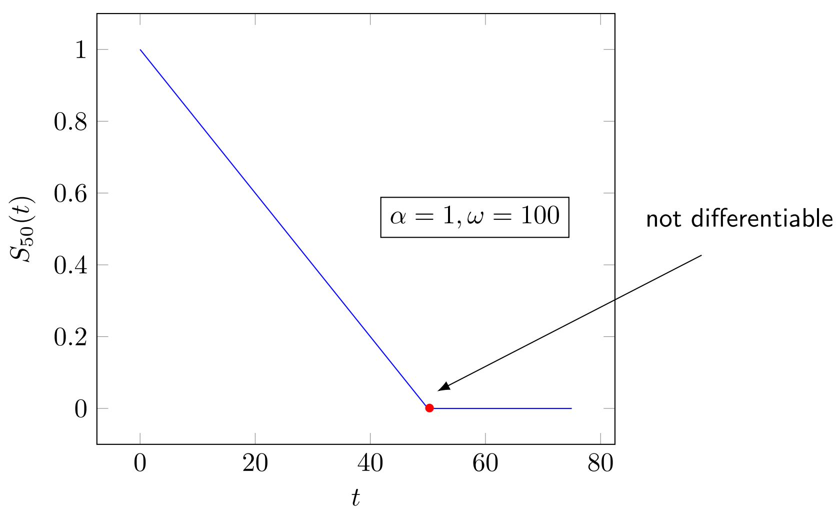
2.2.14 For De Moivre's law, the survival function is given by

$$S_x(t) = \exp\left(\int_0^t \mu_{x+s}\, ds\right) = 1 - \frac{t}{\omega - x}, \quad \text{where } 0 \le t < \omega - x.$$

Note that $\lim_{t \to \omega - x} S_x(t) = 0$, which suggests that as the age of a life appraoches ω, his surviving probability approaches 0, so ω can be seen as a "limiting age".

To ensure the survival function to be valid (i.e., satisfies the properties mentioned in [2.1.7]), we can only set $S_x(t) = 0$ for any $t \ge \omega - x$. This implies that under De Moivre's Law, the future lifetime random variable $T_x \sim U[0, \omega - x]$.

[Note: The survival function $S_x(t)$ is not differentiable at the "edge" point $t = \omega - x$.



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2.2.15 For generalized De Moivre's Law, the survival function is

$$S_x(t) = \exp\left(\alpha \int_0^t \frac{1}{\omega - x - s}\, ds\right) = \left(1 - \frac{t}{\omega - x}\right)^{\alpha}, \quad \text{where } 0 \le t < \omega - x.$$

Likewise, we can only have $S_x(t) = 0$ for any $t \ge \omega - x$.

2.2.16 Lastly, for **constant force of mortality**, as its name suggests, it assumes

$$\mu_x = \mu$$

for any age x, where $\mu > 0$ is a constant. In this case, by proposition 2.2.c, the survival function is given by

$$S_x(t) = e^{-\mu t}$$

for any $t \geq 0$, so the future lifetime random variable $T_x \sim \text{Exp}(\mu)$ (for any age x).

2.3 Actuarial Notations

2.3.1 Here we introduce actuarial notations for concepts discussed before.

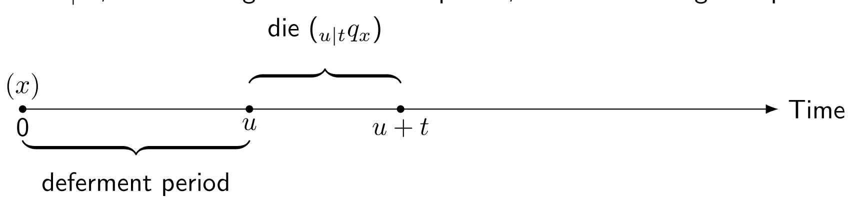
2.3.2 Two main kinds of actuarial notations here: "p" (probability related to "survival") and "q" (probability related to "death").

2.3.3 A list of actuarial notations:

- ${}_tp_x$: $S_x(t)$
- ${}_tq_x$: $F_x(t)$
- ${}_{u|t}q_x$: $\mathbb{P}(u < T_x \leq u + t) = S_x(u) - S_x(u + t)$

Remarks:

- "t" can be dropped from the notation if $t = 1$, for simplicity.
- For "$u|t$", u is the length of deferment period, and t is the length of period following the deferment.



2.3.4 Based on the definitions in [2.3.3], we can readily derive the following formulas:

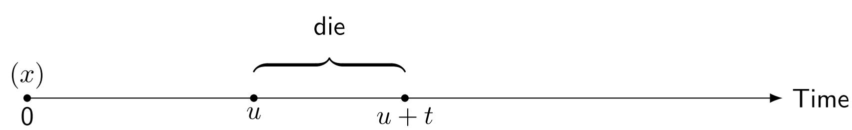
Proposition 2.3.a. For any age x and $t, u \geq 0, n \in \mathbb{N}$,

(a) ${}_tp_x + {}_tq_x = 1$. [Intuition: (x) either dies within or survive for t years.]

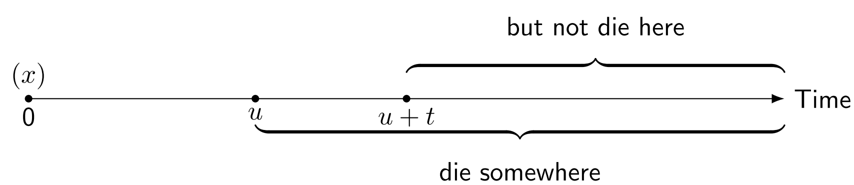
(b) ${}_{t+u}p_x = {}_tp_x\, {}_up_{x+t}$, [Note: This is just proposition 2.1.b expressed in actuarial notations.] and ${}_np_x = p_x p_{x+1} \cdots p_{x+n-1}$. [Note: This can be obtained by applying the former result repetitively.]

(c) ${}_{u|t}q_x = {}_up_x - {}_{u+t}p_x$. [Intuition:

(LHS)



(RHS)



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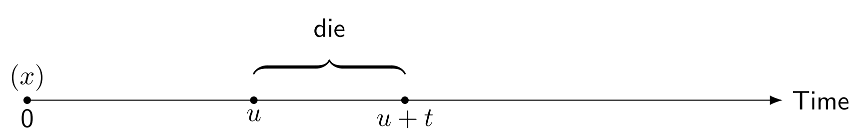
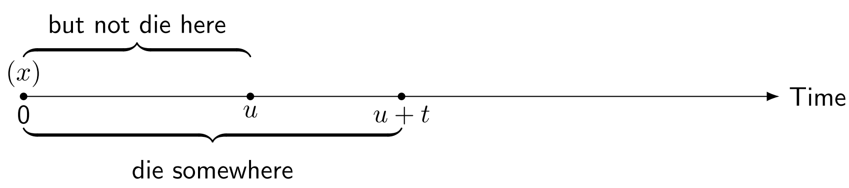
(d) ${}_{u|t}q_x = {}_{u+t}q_x - {}_uq_x$. [Intuition:

(LHS)



(RHS)



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(e) ${}_{u|t}q_x = {}_u p_x \, {}_t q_{x+u}$. [Intuition: Dies in time interval $[u, u+t]$ = First survives for u years, *and then* dies within the coming t years.]

(f) $f_x(t) = {}_t p_x \mu_{x+t}$. [Intuition: Similar to the intuition in [2.2.3], ${}_t p_x \mu_{x+t} \, \mathrm{d}t$ may be interpreted "probability of death in time interval $[t, t+\mathrm{d}t]$". On the other hand, $f_x(t)\,\mathrm{d}t$ may also be understood as "probability of death in time interval $[t, t+\mathrm{d}t]$".]

2.3.5 Intuitively, "q" and "μ" are respectively describing death "discretely" and "continuously" (analogous to "i" and "δ" in STAT2902).

2.3.6 The relationship between "q" and "μ" is given by:

Proposition 2.3.b.

$$ {}_t q_x = \int_0^t {}_s p_x \mu_{x+s} \, ds $$

for any age x and $t \geq 0$.

[Intuition: RHS "sums up" the "probabilities of death" in all "infinitesimal" time intervals within the interval $[0, t]$.]

2.4 Mean and Variance of T_x

2.4.1 The **complete expectation of life** is $\mathbb{E}[T_x]$. (Actuarial notation: $\mathring{e}_x$)

Remarks:

- "Complete" is in the sense that the "complete" ("whole") lifetime is considered in the calculation (and not, say, just the integer part of lifetime).
- By the conditions in [2.1.8], $\mathring{e}_x$ exists.

2.4.2 By definition, we have

$$ \mathring{e}_x = \int_0^\infty t \, {}_t p_x \mu_{x+t} \, \mathrm{d}t \, . $$

However, this is a bit complicated to evaluate and we almost always use the following simpler formula instead:

Proposition 2.4.a.

$$ \mathring{e}_x = \int_0^\infty {}_t p_x \, \mathrm{d}t \, . $$

Proof: Since T_x is nonnegative, we have

$$ T_x = \int_0^{T_x} 1 \, \mathrm{d}t = \int_0^\infty \mathbf{1}_{\{t < T_x\}} \, \mathrm{d}t \, . $$

[Note: $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function.]

It then follows that

$$ \mathbb{E}[T_x] = \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{t<T_x\}} \, \mathrm{d}t\right] = \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{T_x > t\}}\right] \mathrm{d}t = \int_0^\infty \mathbb{P}(T_x > t) \, \mathrm{d}t \, , $$

as desired. (The second equality holds by Fubini's theorem.) $\square$

2.4.3 To find the *variance* of T_x, we need to compute the second moment of T_x: $\mathbb{E}\left[T_x^2\right]$ (again its existence is guaranteed by the conditions in [2.1.8]).

2.4.4 Likewise we almost always use the following alternative formula instead of the definition:

Proposition 2.4.b.

$$\mathbb{E}\left[T_x^2\right] = 2\int_0^\infty t\,{}_tp_x\,\mathrm{d}t\,.$$

<u>Proof</u>: By definition, we have:

$$\mathbb{E}\left[T_x^2\right] = \int_0^\infty t^2\,{}_tp_x\mu_{x+t}\,\mathrm{d}t = -\int_0^\infty t^2\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x\,\mathrm{d}t\,.$$

Integrating it by parts gives

$$\underbrace{-[t^2\,{}_tp_x]_0^\infty}_{0} - \int_0^\infty 2t_tp_x\,\mathrm{d}t$$

where the first term is zero due to the conditions in [2.1.8]. □

Once we compute the first and second moments of T_x: $\mathbb{E}[T_x]$ and $\mathbb{E}\left[T_x^2\right]$, we can find the variance of T_x by

$$\mathrm{Var}\,(T_x) = \mathbb{E}\left[T_x^2\right] - (\mathbb{E}[T_x])^2.$$

2.4.5 Sometimes we are interested in the future lifetime random variable "capped" to n years: $T_x \wedge n$ (i.e., $\min\{T_x, n\}$) (e.g., in the context of endowment insurance; see later sections). The expectation of this capped random variable (i.e., $\mathbb{E}[T_x \wedge n]$) is called the **term expectation of life**. (Actuarial notation: $\mathring{e}_{x:\overline{n}|}$)

[Note: The notation "$x{:}\overline{n}|$" is used when the "capped" future lifetime random variable $T_x \wedge n$ is involved. Abusing the notation a bit, we may actually write $T_x \wedge n$ as "$T_{x:\overline{n}|}$", and regard the latter as "another future lifetime random variable" where "standard" actuarial notations apply (this will be useful in some later sections here, and more details will be explained in STAT3909).]

2.4.6 Similarly we have the following formula for $\mathring{e}_{x:\overline{n}|}$:

Proposition 2.4.c.

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\,\mathrm{d}t\,.$$

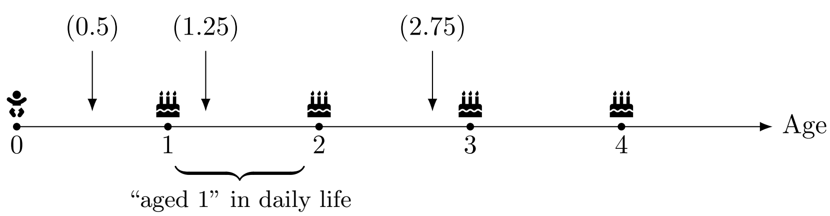
<u>Proof</u>: Replace T_x by $T_x \wedge n$ in the proof of proposition 2.4.a, and note that

$$\mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{t<T_x\wedge n\}}\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{t<T_x\}}\mathbf{1}_{\{t<n\}}\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^n \mathbf{1}_{\{t<T_x\}}\,\mathrm{d}t\right].$$

□

2.5 Curtate Future Lifetime Random Variable

2.5.1 Recall the discrepancy between the meaning of "age" here (exact) versus the daily life meaning (number of birthdays passed):



2.5.2 To capture the “daily life meaning”, we introduce the concept of curtate future lifetime. The **curtate future lifetime random variable** for (x) (denoted by K_x) is the integer part of T_x, i.e., $\lfloor T_x \rfloor$.

Remarks:

- “Curtate” means “shortened”. The lifetime expressed in “daily life age” (curtate future lifetime) is always shorter than (or equal to) the “exact” future lifetime here.
- K_x is a discrete random variable, while T_x is a continuous random variable.

2.5.3 Here is an important “bridge” between T_x and K_x:

Proposition 2.5.a. For any $k \in \mathbb{N}_0$[4], $K_x = k$ iff $k \leq T_x < k+1$ (i.e., the events $\{K_x = k\}$ and $\{k \leq T_x < k+1\}$ are equal).

[Intuition 💡: Inspect the picture above.]

Proof: “$\Leftarrow$” follows from the definition of floor function $\lfloor \cdot \rfloor$. “$\Rightarrow$” can be proved by contrapositive easily. $\square$

A corollary is the following:

Corollary 2.5.b. For any $k \in \mathbb{N}_0$, $K_x \leq k$ iff $T_x < k+1$.

Proof: The result follows from noting that $\{K_x \leq k\} = \displaystyle\bigcup_{n=1}^{k} \{K_x = n\}$. $\square$

So we have $\mathbb{P}(K_x \leq k) = {}_{k+1}q_x$. [⚠ Warning: RHS is not ${}_kq_x$.]

2.5.4 The pmf of K_x is given by:

Proposition 2.5.c. For any $k \in \mathbb{N}_0$, $\mathbb{P}(K_x = k) = {}_kp_x\, q_{x+k}$.

[Intuition 💡: $K_x = k$ means (x) dies in the time interval $[k, k+1)$, i.e., (x) survives for k years and then dies in the coming year.]

Proof: It suffices to note that $\mathbb{P}(K_x = k) = {}_{k|}q_x$[5]. $\square$

2.5.5 The **curtate expectation of life** is $\mathbb{E}[K_x]$. (Actuarial notation: e_x)

2.5.6 The following is the “discrete analogue” of proposition 2.4.a:

Proposition 2.5.d. $e_x = \displaystyle\sum_{k=1}^{\infty} {}_kp_x$.

[⚠ Warning: The sum starts at $k = 1$, not $k = 0$.]

Proof: Using a similar approach as the proof of proposition 2.4.a, we have

$$\mathbb{E}[K_x] = \mathbb{E}\left[\sum_{k=1}^{K_x} 1\right] = \mathbb{E}\left[\sum_{k=1}^{\infty} \mathbf{1}_{\{k \leq K_x\}}\right] = \sum_{k=1}^{\infty} \mathbb{P}(K_x \geq k) = \sum_{k=1}^{\infty} \mathbb{P}(T_x \geq k) = \sum_{k=1}^{\infty} {}_kp_x.$$

[Note: When $K_x = 0$, we define $\sum_{k=1}^{K_x} 1 = 0$.] $\square$

2.5.7 For the second moment, we have also the following analogous result:

Proposition 2.5.e. $\mathbb{E}[K_x^2] = \displaystyle\sum_{k=1}^{\infty} (2k-1)\, {}_kp_x$.

[4] $\mathbb{N}_0 = \{0, 1, 2 \dots\}$.

[5] Recall that ${}_{k|}q_x = {}_{k|1}q_x$

[Warning: The sum starts at $k=1$, not $k=0$.]

Proof: Note that

$$\begin{aligned}\mathbb{E}\left[K_x^2\right] &= \sum_{k=0}^{\infty} k^2({}_kp_x - {}_{k+1}p_x) \\ &= ({}_1p_x - {}_2p_x) + 4({}_2p_x - {}_3p_x) + 9({}_3p_x - {}_4p_x) + \cdots \\ &= {}_1p_x + 3\,{}_2p_x + 5\,{}_3p_x + \cdots .\end{aligned}$$

□

After computing the first and second moments: e_x and $\mathbb{E}\left[K_x^2\right]$, we can find the variance of K_x by

$$\mathrm{Var}\left(K_x\right) = \mathbb{E}\left[K_x^2\right] - e_x^2.$$

2.5.8 Likewise the **term curtate expectation of life** is $\mathbb{E}[K_x \wedge n]$, where $n \in \mathbb{N}_0$. (Actuarial notation: $e_{x:\overline{n}|}$)

2.5.9 Analogously, we have

Proposition 2.5.f.

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x.$$

Proof: Using a similar approach as the proof of proposition 2.4.c, we have

$$\mathbb{E}[K_x \wedge n] = \mathbb{E}\left[\sum_{j=0}^{\infty} \mathbf{1}_{\{j<K_x\wedge n\}}\right] = \sum_{j=0}^{n-1} \mathbb{P}(K_x > j) = \sum_{j=0}^{n-1} \mathbb{P}(T_x \geq j+1) = \sum_{k=1}^{n} {}_kp_x.$$

□

2.6 Recursions for Expectations of Life

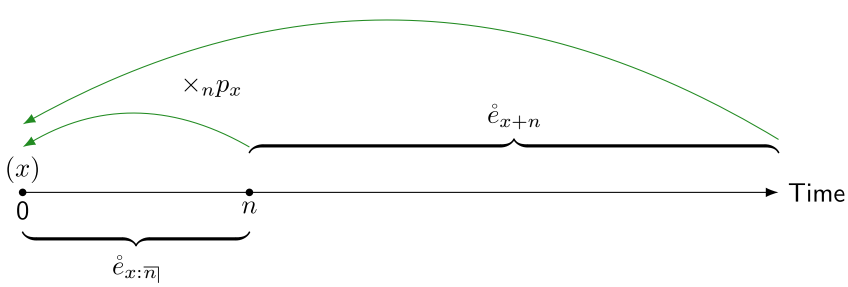
2.6.1 *Recursion* is a very important concept in life contingencies, and is often best understood by *intuition*.

2.6.2 The recursive formulas are collected in the following:

Proposition 2.6.a. For any age x and $n \in \mathbb{N}$,

(a) $\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x \mathring{e}_{x+n}$.

[Intuition:



]

Proof: Note that

$$\int_0^{\infty} {}_tp_x \,\mathrm{d}t = \int_0^{n} {}_tp_x \,\mathrm{d}t + \underbrace{\int_n^{\infty} {}_tp_x \,\mathrm{d}t}_{{}_np_x \int_0^{\infty} {}_tp_{x+n} \mathrm{d}t} = \mathring{e}_{x:\overline{n}|} + {}_np_x \mathring{e}_{x+n}.$$

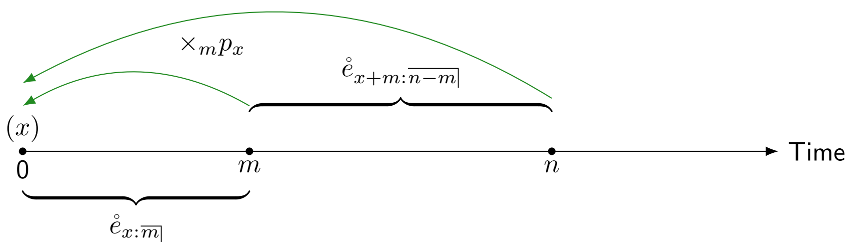
$\square$

[Note: Generally, for recursion of this "form", a proof can be done by "splitting" the integral/sum like above — "factoring out" ${}_np_x$ corresponds to the "discounting" in the intuition.]

(b) $e_x = e_{x:\overline{n}|} + {}_np_x e_{x+n}$. [Note: Special case ($n = 1$): $e_x = p_x(1 + e_{x+1})$.]

(c) $\mathring{e}_{x:\overline{n}|} = \mathring{e}_{x:\overline{m}|} + {}_mp_x \mathring{e}_{x+m:\overline{n-m}|}$ for any nonnegative integer $m \leq n$.

[Intuition 💡:

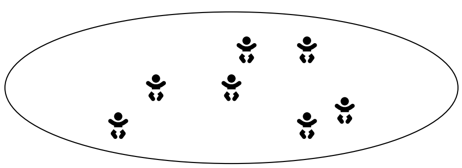


]

(d) $e_{x:\overline{n}|} = e_{x:\overline{m}|} + {}_mp_x e_{x+m:\overline{n-m}|}$ for any nonnegative integer $m \leq n$.

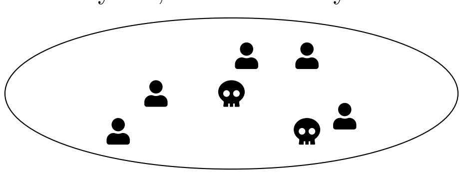
2.7 Life Tables

2.7.1 To construct a life table, we first consider $\ell_0 \in \mathbb{N}$ newborns (lives aged 0):



[Note: ℓ_0 is known as **radix** ("base" number) of the life table. A common choice is 10000.]

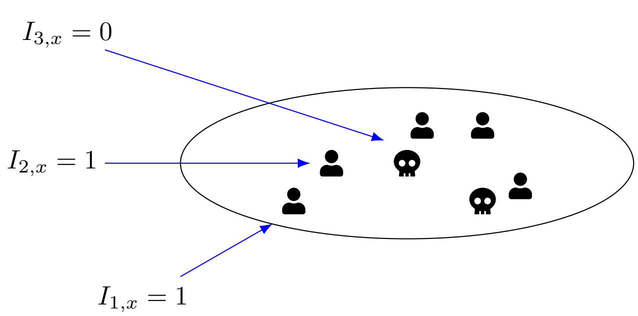
2.7.2 After x years, some lives may be dead:



2.7.3 Let $\mathcal{L}_x$ be the number of survivors to age x (which is a random variable). To determine $\mathcal{L}_x$, we shall use an *indicator* approach.

2.7.4 First we label the ℓ_0 newborns as newborn $1, 2, \ldots, \ell_0$. Then, define the indicator (random variable) for the survival to age x for newborn j by:

$$I_{j,x} = \begin{cases} 1 & \text{if newborn } j \text{ survives to age } x, \\ 0 & \text{otherwise.} \end{cases}$$



Then, we have $\mathcal{L}_x = \sum_{j=1}^{\ell_0} I_{j,x}$.

2.7.5 For any age x, the quantity of most interest is the *expected* number of survivors to age x: $\mathbb{E}[\mathcal{L}_x]$ (Actuarial notation: ℓ_x).

2.7.6 We shall assume that every newborn has the same lifetime distribution. Then, without any ambiguity we can write the probability of any newborn surviving to age x by ${}_x p_0$. Thus, we have

$$\ell_x = \sum_{j=1}^{\ell_0} {}_x p_0 = {}_x p_0 \ell_0,$$

which implies the following:

Lemma 2.7.a. $\ell_x = \ell_0 \, {}_x p_0$ for any age x.

2.7.7 Indeed, we have the following more general result:

Proposition 2.7.b. $\ell_{x+t} = \ell_x \, {}_t p_x$ for any age x and $t \geq 0$.

Proof: By lemma 2.7.a, we have $\ell_{x+t} = \ell_0 \, {}_{x+t} p_0$ and $\ell_x = \ell_0 \, {}_x p_0$. Then the result follows since ${}_{x+t} p_0 = {}_x p_0 \, {}_t p_x$. □

Typically ℓ_x for different ages x are given in a life table, and hence this result is useful for computing various probabilities based on life table, e.g.:

(a) ${}_t p_x = \dfrac{\ell_{x+t}}{\ell_x}$

(b) ${}_t q_x = 1 - \dfrac{\ell_{x+t}}{\ell_x}$

(c) ${}_{u|t} q_x = \dfrac{\ell_{x+u} - \ell_{x+u+t}}{\ell_x}$

2.7.8 A life table sometimes also provides values for "d", which can be seen as "dual" of "ℓ":

	Survivors	Deaths
number	$\mathcal{L}$	$\mathcal{D}$
exp. number	ℓ	d

2.7.9 The number of deaths between ages x and $x+n$ is ${}_n\mathcal{D}_x$ (a random variables). [Note: "between" here can be understood in either non-strict or strict sense — it does not affect the probability as the lifetime random variable is continuous.]

2.7.10 The expected number of deaths between ages x and $x+n$ is $\mathbb{E}[{}_n\mathcal{D}_x]$ (Actuarial notation: ${}_n d_x$; "n" can be dropped if $n = 1$).

2.7.11 We can have a similar indicator-based development here. Define the indicator for the death between ages x and $x+n$ for newborn j by

$${}_n I_{j,x} = \begin{cases} 1 & \text{if newborn } j \text{ dies between ages } x \text{ and } x+n, \\ 0 & \text{otherwise.} \end{cases}$$

Then similarly we have ${}_n\mathcal{D}_x = \sum_{j=1}^{\ell_0} {}_n I_{j,x}$. Hence, ${}_n d_x = \sum_{j=1}^{\ell_0} ({}_x p_0 - {}_{x+n} p_0) = \ell_x - \ell_{x+n}$.

2.7.12 We can readily deduce the following results:

Proposition 2.7.c. For any age x and $n \in \mathbb{N}$,

(a) ${}_nq_x = {}_nd_x/\ell_x$.

(b) ${}_nd_x = d_x + d_{x+1} + \cdots + d_{x+n-1}$.

2.8 Fractional Age Assumptions

2.8.1 In a life table, often the quantities (ℓ_x, d_x etc.) are given only for *integer* ages. In such case, it is impossible to compute probabilities involving *fractional* ages and years (e.g. ${}_{0.5}p_{50.2}, {}_{0.7}q_{60}$ etc.), without further assumptions.

2.8.2 To improve the flexibility, we may impose some additional assumptions. The three most common *fractional age assumptions* are:

(a) uniform distribution of deaths (UDD) (the most common one);

(b) constant force of mortality

(c) Balducci assumption (not very common; not inside SOA Exam FAM syllabus currently)

2.8.3 The **uniform distribution of deaths** (UDD) assumption is given by: for any integer age $x \in \mathbb{N}_0$, and any $t \in [0, 1)$, assume that ${}_tq_x = tq_x$. [⚠ Warning: This is not ${}_tp_x = tp_x$.]

2.8.4 Under UDD, to compute quantities involving fractional ages and years, a general approach is to express them in terms of ${}_tq_x$ where $x \in \mathbb{N}_0$ and $t \in [0, 1)$, using, e.g., properties in proposition 2.3.a.

2.8.5 Here we highlight two properties under UDD:

(a) For any integer age x and any $u, t \geq 0$ such that $u + t \in [0, 1)$, ${}_{u|t}q_x = tq_x$.
Proof: Note that ${}_{u|t}q_x = {}_{u+t}q_x - {}_uq_x = (u + t)q_x - uq_x = tq_x$. □
[Note: This means as long as $u + t \in [0, 1)$ (next integer time is not "crossed"), the "q" probability depends only on the *length* of the time interval covered ("location" of the interval *within* the fraction does not matter).]

(b) For any integer age x and any $t \in [0, 1)$, $\mu_{x+t} = \dfrac{q_x}{1 - tq_x}$.

Proof: Note that

$$\mu_{x+t} = \frac{f_x(t)}{{}_tp_x} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}{}_tq_x}{1 - {}_tq_x} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}tq_x}{1 - tq_x} = \frac{q_x}{1 - tq_x}.$$

□

2.8.6 Under UDD, the mean and variance of T_x and K_x are related in a rather simple way. First we shall state a lemma:

Lemma 2.8.a. The UDD assumption we state is *equivalent* to the following: For any integer age x, writing $T_x = K_x + U_x$, we have:

(a) K_x and U_x are independent;

(b) $U_x \sim U[0, 1)$ (hence the name "uniform distribution of deaths").

[Note: This result is seldom used directly in solving problems.]

Proof: Omitted. (It is non-trivial, but also not too complex. See Dickson et al. (2019, Section 3.3.1).) □

Proposition 2.8.b. Under UDD, for any integer age x,

(a) $\mathring{e}_x = e_x + 1/2$;

(b) $\mathrm{Var}\,(T_x) = \mathrm{Var}\,(K_x) + 1/12$.

Proof: Using lemma 2.8.a, the result follows readily as $\mathbb{E}[U_x] = 1/2$ and $\mathrm{Var}(U_x) = 1/12$. $\square$

2.8.7 The **constant force of mortality** assumption is given by: for any integer age x and any $t \in [0, 1)$, assume that $\mu_{x+t} = \mu_x^*$, where $\mu_x^* > 0$ depends only on the fixed integer age x.

2.8.8 The following is the key property for constant force of mortality assumption:

Proposition 2.8.c. ${}_tp_x = (p_x)^t$ for any integer age x and any $t \in [0, 1)$.

Proof: We have

$$ {}_tp_x = \exp\left(-\int_0^t \underbrace{\mu_{x+s}}_{\mu_x^*} \, ds\right) = e^{-t\mu_x^*} = (p_x)^t. $$

$\square$

2.8.9 Like [2.8.4], a general approach under constant force of mortality assumption is to express the quantities to be computed in terms of ${}_tp_x$ where $x \in \mathbb{N}_0$ and $t \in [0, 1)$. For example,

$$ \mu_x^* = \mu_{x+t} = -\frac{1}{{}_tp_x}\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x = -\frac{1}{(p_x)^t}\frac{\mathrm{d}}{\mathrm{d}t}{}^t = -\frac{(p_x)^t \ln p_x}{(p_x)^t} = -\ln p_x. $$

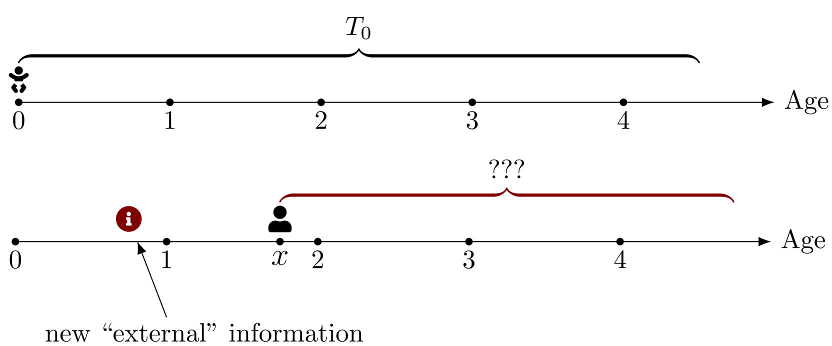
This also shows how we calculate the constant force of mortality based on life table (recall $p_x = \ell_{x+1}/\ell_x$).

2.8.10 The **Balducci assumption** is given by: for any integer age x and any $t \in [0, 1)$, assume that ${}_{1-t}q_{x+t} = (1 - t)q_x$. [Note: It is like the "tail" version of UDD.]

2.8.11 Again, like [2.8.4], a general approach under Balducci assumption is to express the quantities to be found in terms of ${}_{1-t}q_{x+t}$ where $x \in \mathbb{N}_0$ and $t \in [0, 1)$.

2.9 Select and Ultimate Survival Model

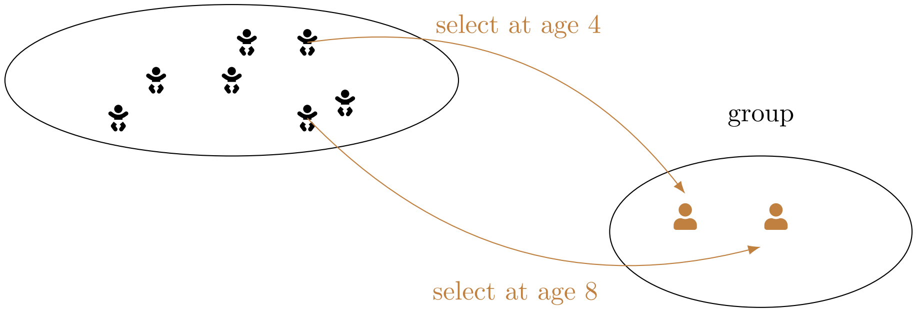
2.9.1 Recall the assumption in [2.1.5]. It is reasonable if there is no "external" information between age 0 and age x. But what if there is indeed new "external" information during that interval?



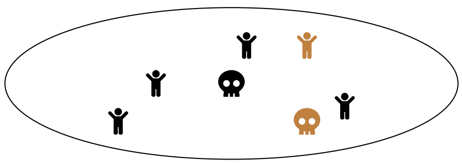
2.9.2 A common source of new "external" information is *underwriting*. When the life purchases an insurance policy between age 0 and age x, an underwriting on him is often triggered. As learns more about the health status of the life (the "external" information), the lifetime distribution modelled for him may be adjusted.

2.9.3 To incorporate this effect, we shall use the *select and ultimate survival model*. This is a generalization to the model discussed in section 2.7 (called **aggregate survival model**), which allows flexibility for incorporating the "external" information coming from underwriting or something similar in nature (this is known as **selection** in the model).

2.9.4 Illustration of the selection process:



At age 10:

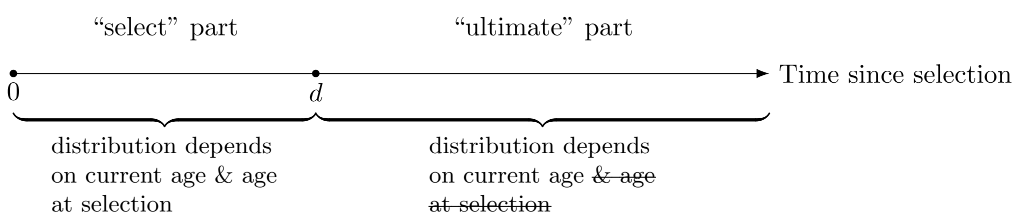


The selected lives would have different survival distributions from others.

2.9.5 The **select and ultimate survival model** is an aggregate survival model with the following features added:

(a) The survival distribution of a life can depend on the age at which the life was *selected*. (So different lives at the same age can be have different survival distributions, unlike plain aggregate survival model.)

(b) The effect on the survival distribution by the selection will *disappear* after $d > 0$ years (often an integer), called **select period**. That is, the survival distribution of a life selected *at least* d years ago is the same as the distribution for a life at the same age but never selected before.



2.9.6 In the select and ultimate survival model, a life *selected* at age x and *currently* aged $x + s$ ($s \geq 0$) is denoted by $([x] + s)$ (rarely written directly like this). Then other notations apply similarly, e.g.,

- future lifetime random variable: $T_{[x]+s}$
- ${}_tp_{[x]+s} = S_{[x]+s}(t) = \mathbb{P}(T_{[x]+s} > t)$
- ${}_tq_{[x]+s} = F_{[x]+s}(t) = \mathbb{P}(T_{[x]+s} \leq t)$
- $\mathring{e}_{[x]+s} = \mathbb{E}[T_{[x]+s}]$
- etc.

Also, we have $T_{[x]+s} \stackrel{d}{=} T_{x+s}$ for any $s \geq d$, by the definition of select period. So in such case, "$[x]$" in the notations can be replaced by "x".

2.9.7 We shall impose a similar assumption as the one in [2.1.5] for the "select" life: For any age at selection $[x]$ and any $t \geq 0$,

$$(T_{[x]} - t | T_{[x]} > t) \stackrel{d}{=} T_{[x]+t}.$$

This assumption makes sense since we are considering time periods *after* selection, so there should not be *new* "external" information in those periods (the "external" information from selection has already been incorporated).

2.9.8 As a result, previous formulas "modelled"/"started" at age x can be readily adapted (change "x" to "$[x]$") for the "select" life. [⚠ Warning: However, formulas that "pass through" the selection age x generally do not apply anymore. For example, we do not have "${}_3p_{x-1} = p_{x-1}\ {}_2p_{[x]}$".]

2.9.9 We can also impose fractional age assumption for the "select" life (assuming the select period $d \geq 1$), where "x" is changed to "$[x]$" in the assumptions: For any $x \in \mathbb{N}_0$ and any $t \in [0, 1)$,

(a) UDD: assume ${}_tq_{[x]} = tq_{[x]}$.

(b) constant force of mortality: assume $\mu_{[x]+t} = \mu^*_{[x]}$.

(c) Balducci assumption: assume ${}_{1-t}q_{[x]+t} = (1-t)q_{[x]}$.

Consequently, we have analogous properties for fractional age assumptions for the "select" life.

2.10 Select Life Tables

2.10.1 Computations of quantities in the select and ultimate survival model are usually based on a **select life table**, which includes also the life table functions (e.g. "ℓ" and "d") for lives selected at different ages.

2.10.2 It turns out that the "bottom-up" approach for constructing the tables like section 2.7 is not very feasible as there are many complications when age at selection can also impact the survival distribution.

2.10.3 We shall instead construct the tables such that an analogous result, which holds in section 2.7, is satisfied: For any age x and any $t, u \geq 0$,

$$\ell_{[x]+t+u} = \ell_{[x]+t}\ {}_up_{[x]+t}. \tag{1}$$

Besides, due to the select period d, a natural requirement is

$$\ell_{[x]+t} = \ell_{x+t} \tag{2}$$

for any $t \geq d$.

2.10.4 It turns out that, after imposing both eqs. (1) and (2), there is exactly one way to construct the tables (based on some pre-specified values for "select" survival probabilities, as well as "ultimate" life table function: ℓ_x's):

Proposition 2.10.a. Both eqs. (1) and (2) hold true iff

$$\ell_{[x]+t} = \frac{\ell_{x+d}}{{}_{d-t}p_{[x]+t}}$$

for any $0 \leq t < d$, and

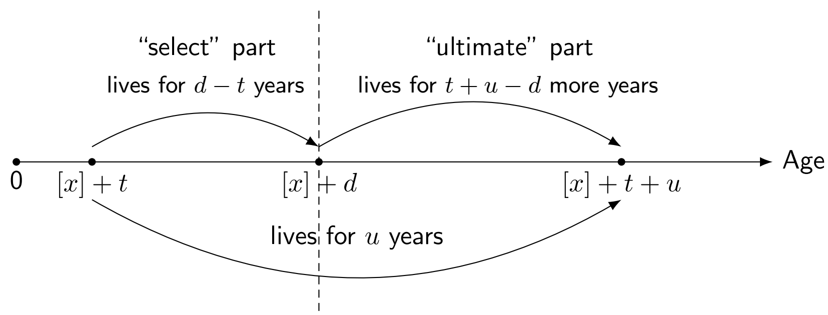
$$\ell_{[x]+t} = \ell_{x+t}$$

for any $t \geq d$, where d is the select period.

Proof: "$\Rightarrow$": Equation (2) forces $\ell_{[x]+t} = \ell_{x+t}$ for any $t \geq d$. After that, eq. (1) implies that $\ell_{[x]+d} = \ell_{x+d} = \ell_{[x]+t}\ {}_{d-t}p_{[x]+t}$ for any $0 \leq t < d$.

"$\Leftarrow$": Under the assumption, for any age x and any $t, u \geq 0$, firstly eq. (2) immediately holds. Next, consider:

- Case 1: $t+u<d$
  We have
  $$\ell_{[x]+t+u} = \frac{\ell_{x+d}}{{}_{d-t-u}p_{[x]+t+u}} \quad \text{and} \quad \ell_{[x]+t} = \frac{\ell_{x+d}}{{}_{d-t}p_{[x]+t}},$$
  so
  $$\frac{\ell_{[x]+t+u}}{\ell_{[x]+t}} = \frac{{}_{d-t}p_{[x]+t}}{{}_{d-t-u}p_{[x]+t+u}} = {}_{u}p_{[x]+t}.$$
- Case 2: $t \geq d$ (and $t+u \geq d$)
  In this case we have $\ell_{[x]+t+u} = \ell_{x+t+u}$ and $\ell_{[x]+t} = \ell_{x+t}$, thus
  $$\frac{\ell_{[x]+t+u}}{\ell_{[x]+t}} = \frac{\ell_{x+t+u}}{\ell_{x+t}} = {}_{u}p_{x+t} = {}_{u}p_{[x]+t}.$$
- Case 3: $t<d$ and $t+u \geq d$
  Here we have $\ell_{[x]+t+u} = \ell_{x+t+u}$ and $\ell_{[x]+t} = \ell_{x+d}/{}_{d-t}p_{[x]+t}$. Hence,
  $$\frac{\ell_{[x]+t+u}}{\ell_{[x]+t}} = \frac{\ell_{x+t+u}}{\ell_{x+d}}\,{}_{d-t}p_{[x]+t} = {}_{t+u-d}p_{[x]+d}\,{}_{d-t}p_{[x]+t} = {}_{u}p_{[x]+t}.$$



So in any of the cases, eq. (1) holds. □

3 Life Insurance

3.0.1 The decision of whether a policy should be sold the policyholder or not depends critically on the *present value* for the cash flows involved in (including both benefits and premiums).

3.0.2 To improve the tractability, we shall assume that the annual interest rate i is a fixed constant, so that the "randomness" of the present value (random variable) would only source from the modelled survival distribution for — the present value is *life contingent*.

3.0.3 Often the insurance business is of very large scale and many identical policies may be sold to lives modelled by the same survival distribution (the "ordinary" one, say).

3.0.4 In such case, the *average* present value of those policies is a key metric for the profitability per policy sold. By virtue of *law of large number* (assuming the applicability), the average would converge to the *expected present value* as the number of such policies sold goes to infinity. Hence, the study of life contingencies focuses a lot on the expected present value (also known as **actuarial present value** (APV)).

3.0.5 Sections 3 and 4 mainly focus on determining the APV of the benefit payments of the contracts mentioned in section 1.2.

3.0.6 The contracts can be classified into two big categories, by the payment "timing":

(a) Discrete time: insurance $\to$ payable at the end of period; life annuity $\to$ payable at the beginning of period. (K_x is involved mainly.)

(b) Continuous time: insurance $\to$ payable "immediately"; life annuity $\to$ payable "continuously" (*continuous annuity* in STAT2902). (T_x is involved mainly.)

3.0.7 It turns out that working in continuous time (working with T_x) is mathematically more "convenient". However, practical contracts are always discrete in nature. So, sometimes quantities for a practical "discrete" contract are *approximated* through working in continuous time instead. (The approximation works "better" if the contract's measurement period is shorter.)

3.0.8 ⭐An "intuition-based" general APV calculation formula for insurance/life annuity:

$$\underbrace{\sum \text{ or } \int}_{\text{all possible payment times}} \text{benefit amount} \times \text{discount factor} \times \text{prob. of triggering payment.}$$

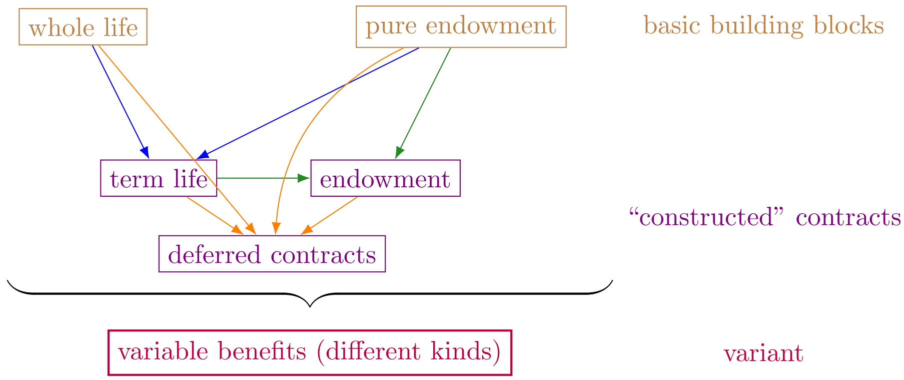
[Note: For continuous case with "$\int$", the terms above can be in "infinitesimal" sense (e.g., involving "$\mathrm{d}t$"), loosely.]

[Intuition 💡: It "sums up" the "expected present value of benefit payment":

$$\underbrace{\text{benefit amount} \times \text{discount factor}}_{\text{PV of payment}} \times \text{prob. of triggering payment}$$

in all time intervals.]

3.0.9 The following shows the relationship between different kinds of insurance (in an approach of "construction"):



3.1 Whole Life Insurance

Continuous Case

3.1.1 Consider a *continuous* whole life insurance with sum insured 1 and life insured (x). The sum insured 1 is paid *exactly at* time of death of (x), i.e., time T_x (when the life is aged x, it is time 0).

3.1.2 The **present value random variable** (p.v.r.v.) (i.e., present value of benefits outgo) of the policy is

$$Z = v^{T_x} = e^{-\delta T_x}.$$

where δ is the annual force of interest equivalent to the annual rate i. The APV is:

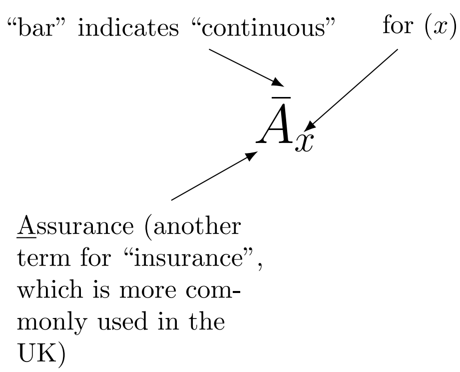
$$\mathbb{E}[Z] = \mathbb{E}\left[e^{-\delta T_x}\right] = \int_0^\infty e^{-\delta t} \underbrace{{}_tp_x \mu_{x+t}}_{f_x(t)} \, \mathrm{d}t$$

[Intuition: RHS "sums up" the "expected present value of death benefit":

$$\underbrace{e^{-\delta t}}_{\text{PV of payment}} \times \underbrace{{}_tp_x \mu_{x+t} \, \mathrm{d}t}_{\text{death prob. in } [t, t+\mathrm{d}t]}$$

($e^{-\delta(t+\mathrm{d}t)}$ and $e^{-\delta t}$ are "the same" in "infinitesimal" sense) in all "infinitesimal" time intervals.]

3.1.3 Actuarial notation for the APV:



3.1.4 Sometimes we are also interested in the *variability* of the p.v.r.v. Z. To measure this, we calculate the variance $\mathrm{Var}(Z)$.

3.1.5 Consider first the second moment of Z, which is given by

$$\mathbb{E}\left[Z^2\right] = \mathbb{E}\left[e^{-2\delta T_x}\right] = \bar{A}_x @ \, 2\delta,$$

i.e., the APV evaluated at force of interest 2δ (Actuarial notation: ${}^2\bar{A}_x$). Then the variance is

$$\mathrm{Var}\,(Z) = {}^2\bar{A}_x - (\bar{A}_x)^2.$$

3.1.6 When the sum insured for the insurance becomes S, the p.v.r.v. becomes $S \cdot Z$. Then, the APV becomes $\mathbb{E}[SZ] = S\bar{A}_x$, the second moment becomes $\mathbb{E}\big[(SZ)^2\big] = S^2 \cdot {}^2\bar{A}_x$, and the variance is $\mathrm{Var}\,(SZ) = S^2\,\mathrm{Var}\,(Z)$. [Warning: Do not miss the square for the second moment and variance!]

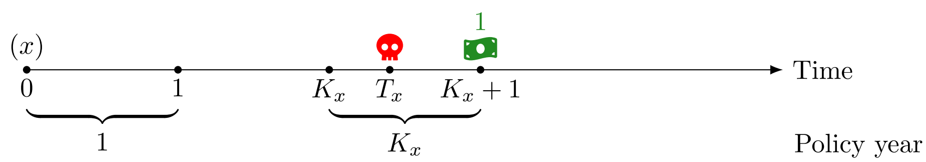
3.1.7 To compute *probability* related to the p.v.r.v. in the form of $\mathbb{P}(Z \leq ?)$, we try to find an expression of the form "$T_x \geq ?$"[6] that is *equivalent* to "$Z \leq ?$". Then, we instead compute the probability $\mathbb{P}(T_x \geq ?)$ (which equals $\mathbb{P}(Z \leq ?)$). We have similar approaches for other "forms" of the probability. [Note: This works generally for all kinds of insurance introduced in section 3.]

3.1.8 Summary:

	p.v.r.v.	APV	2nd moment	variance
expression	$e^{-\delta T_x}$	$\int_0^\infty e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$	$\bar{A}_x@\,2\delta$	${}^2\bar{A}_x - (\bar{A}_x)^2$
notation	Z	$\bar{A}_x$	${}^2\bar{A}_x$	$\mathrm{Var}\,(Z)$

Annual Case

3.1.9 Consider a *discrete* whole life insurance with sum insured 1 and life insured (x). The sum insured 1 is paid *at the end of (policy) year* of death of (x), i.e., time $K_x + 1$.



3.1.10

	p.v.r.v.	APV	2nd moment	variance
expression	v^{K_x+1}	$\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x q_{x+k}$	$A_x@\,2\delta$	${}^2A_x - (A_x)^2$
notation	Z	A_x	${}^2A_x$	$\mathrm{Var}\,(Z)$

Remarks:

- The actuarial notation A_x has no "bar" as the insurance is no longer continuous.
- We can change "v" to "$e^{-\delta}$" above, where δ is the equivalent annual force of interest.

[Intuition: For the APV formula, it sums up the expected present value of death benefit:

$$\underbrace{v^{k+1}}_{\text{PV of payment}} \times \underbrace{{}_kp_x q_{x+k}}_{\text{death prob. in } [k,k+1)}$$

in all "unit" time intervals.]

$1/m$thly Case

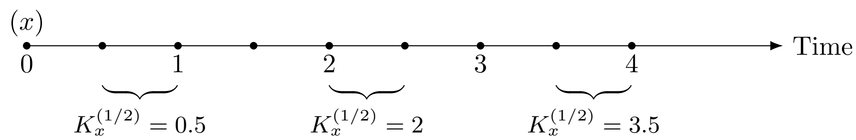
3.1.11 Consider a *discrete* whole life insurance with sum insured 1 and life insured (x). The sum insured 1 is paid *at the end of $1/m$th of a (policy) year* of death of (x).

[6] The direction of inequality is given by "$\geq$" since, loosely, $T_x \uparrow \iff Z \downarrow$, for a fixed insurance contract.

3.1.12 Since "$1/m$th of a (policy) year" is involved, it would be more convenient to develop a similar concept as the curtate future lifetime random variable K_x but with time unit being $1/m$th of a year.

3.1.13 The **$1/m$thly curtate future lifetime random variable** (denoted by $K_x^{(m)}$) is T_x rounded down to $1/m$th of a year, i.e.,

$$K_x^{(m)} = \frac{1}{m}\lfloor mT_x \rfloor.$$



3.1.14 The random variable $K_x^{(m)}$ is still discrete, and its pmf is:

$$\mathbb{P}\left(K_x^{(m)} = \frac{k}{m}\right) = {}_{\frac{k}{m}|\frac{1}{m}}q_x = {}_{\frac{k}{m}}p_x \, {}_{\frac{1}{m}}q_{x+\frac{k}{m}},$$

for any $k \in \mathbb{N}_0$.

3.1.15

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{K_x^{(m)}+1/m}$	$\sum_{k=0}^{\infty} v^{\frac{k+1}{m}} {}_{\frac{k}{m}}p_x \, {}_{\frac{1}{m}}q_{x+\frac{k}{m}}$	$A_x^{(m)} @ 2\delta$	${}^2A_x^{(m)} - \left(A_x^{(m)}\right)^2$
notation	Z	$A_x^{(m)}$	${}^2A_x^{(m)}$	$\mathrm{Var}(Z)$

3.2 (Pure) Endowment and Term Life Insurances

3.2.1 Consider an n-year pure endowment contract with survival benefit 1, i.e., 1 is payable at time n if the life insured is alive at that time.

[Note: n is usually an integer, but not necessarily. This applies to all n-year contracts.]

3.2.2

	p.v.r.v.	APV	2nd moment	variance
expression	$v^n \mathbf{1}_{\{T_x > n\}}$	$v^n {}_np_x$	${}_nE_x @ 2\delta$	${}_n^2E_x - ({}_nE_x)^2$
notation	Z	${}_nE_x$ or $A_{x:\overset{1}{\overline{n}|}}$	${}_n^2E_x$ or ${}^2A_{x:\overset{1}{\overline{n}|}}$	$\mathrm{Var}(Z)$

Remarks:

- The "1" on top of $\overline{n}|$ suggests that the n-year term is the *1st* thing to be "gone" (earllier than (x) being "gone"/dead) for the benefit to be triggered[7]. More details about this kind of notation will be discussed in STAT3909.
- There are not concepts like "continuous" and "discrete" for pure endowment — the benefit is always paid (potentially) at time n for an n-year pure endowment. Conventionally we do *not* write "$\bar{A}_{x:\overset{1}{\overline{n}|}}$".

3.2.3 The following provides some "shortcuts" for the formulas in [3.2.2]:

Proposition 3.2.a. For any age x and any $n \geq 0$,

(a) ${}_n^2E_x = v^n {}_nE_x$;

(b) $\mathrm{Var}(Z) = v^{2n} {}_np_x \, {}_nq_x$.

Proof: Firstly, ${}_n^2E_x = v^{2n} {}_np_x = v^n {}_nE_x$. Next,

$$\mathrm{Var}(Z) = {}_n^2E_x - ({}_nE_x)^2 = v^{2n} {}_np_x(1 - {}_np_x) = v^{2n} {}_np_x \, {}_nq_x.$$

□

[7] In other words, the life insured must be alive at time n to trigger the benefit.

Term Life Insurance

3.2.4 Consider an n-year term life insurance with death benefit 1, i.e., 1 is only payable if the life insured dies within n years.

3.2.5 Continuous case:

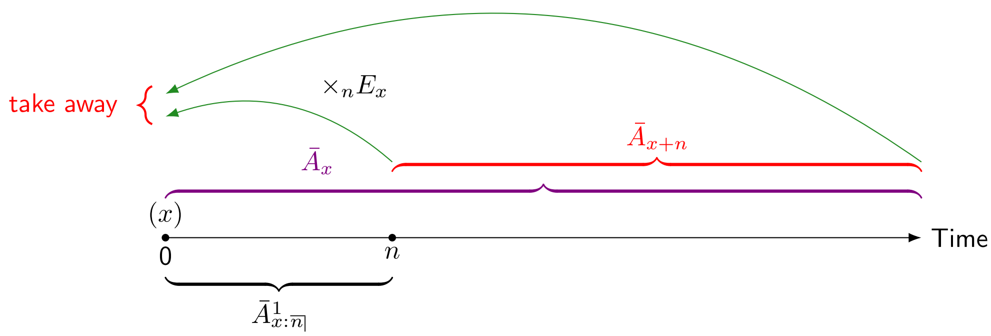
	p.v.r.v.	APV	2nd moment	variance
expression	$e^{-\delta T_x}\mathbf{1}_{\{T_x\le n\}}$	$\bar{A}_x - {}_nE_x\bar{A}_{x+n}$ or $\int_0^n e^{-\delta t}{}_tp_x\mu_{x+t}\,\mathrm{d}t$	$\bar{A}^{1}_{x:\overline{n}\rvert}@\,2\delta$	${}^2\bar{A}^{1}_{x:\overline{n}\rvert} - \left(\bar{A}^{1}_{x:\overline{n}\rvert}\right)^2$
notation	Z	$\bar{A}^{1}_{x:\overline{n}\rvert}$	${}^2\bar{A}^{1}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

Proof: To get the first APV formula, note that $Z = \underbrace{e^{-\delta T_x}}_{\text{whole life}} - v^n e^{-\delta(T_x-n)}\mathbf{1}_{\{T_x>n\}}$, so

$$\mathbb{E}[Z] = \bar{A}_x + v^n\mathbb{P}(T_x>n)\mathbb{E}\left[e^{-\delta(T_x-n)}\middle| T_x>n\right] = \bar{A}_x + v^n{}_np_x\underbrace{\mathbb{E}\left[e^{-\delta(T_{x+n})}\right]}_{\bar{A}_{x+n}}.$$

For the integral APV formula, it follows directly from the expression for p.v.r.v. ☐

[Intuition 💡: ${}_nE_x$ is like an **"actuarial discount factor"**:



]

[Note: As we shall see in section 3.3, ${}_nE_x\bar{A}_{x+n}$ is the APV of a continuous n-year deferred whole life insurance. So this formula for finding APV of term life insurance also suggests another formula for finding APV of *deferred* whole life insurance, by rearranging the terms.]

3.2.6 Annual case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{K_x+1}\mathbf{1}_{\{K_x\le n-1\}}$	$A_x - {}_nE_xA_{x+n}$ or $\sum_{k=0}^{n-1} v^{k+1}{}_kp_xq_{x+k}$	$A^{1}_{x:\overline{n}\rvert}@\,2\delta$	${}^2A^{1}_{x:\overline{n}\rvert} - \left(A^{1}_{x:\overline{n}\rvert}\right)^2$
notation	Z	$A^{1}_{x:\overline{n}\rvert}$	${}^2A^{1}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

Proof: Similar to the proof for [3.2.5]. ☐

3.2.7 $1/m$thly case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{K_x^{(m)}+\frac{1}{m}}\mathbf{1}_{\left\{K_x^{(m)}\le n-\frac{1}{m}\right\}}$	$A_x^{(m)} - {}_nE_xA_{x+n}^{(m)}$ or $\sum_{k=0}^{mn-1} v^{\frac{k+1}{m}}{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}$	$A^{1}_{x:\overline{n}\rvert}@\,2\delta$	${}^2A^{(m)}{}^{1}_{x:\overline{n}\rvert} - \left(A^{(m)}{}^{1}_{x:\overline{n}\rvert}\right)^2$
notation	Z	$A^{(m)}{}^{1}_{x:\overline{n}\rvert}$	${}^2A^{(m)}{}^{1}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

Proof: Similar to the proof for [3.2.5]. ☐

Endowment Insurance

3.2.8 A key observation is that n-year endowment insurance is just n-year term life insurance + n-year pure endowment.

3.2.9 As a result, the p.v.r.v.'s in different cases can be written as:

case	p.v.r.v.
continuous	$e^{-\delta T_x}\mathbf{1}_{\{T_x\le n\}}+e^{-\delta n}\mathbf{1}_{\{T_x>n\}}$
annual	$v^{K_x+1}\mathbf{1}_{\{K_x\le n-1\}}+v^n\mathbf{1}_{\{K_x\ge n\}}$
$1/m$thly	$v^{K_x^{(m)}+\frac{1}{m}}\mathbf{1}_{\left\{K_x^{(m)}\le n-\frac{1}{m}\right\}}+v^n\mathbf{1}_{\left\{K_x^{(m)}\ge n\right\}}$

[Note: Both events $\{K_x \geq n\}$ and $\left\{K_x^{(m)} \geq n\right\}$ have the same probability as the event $\{T_x > n\}$.]

3.2.10 But actually they can be written more compactly as follows:

case	p.v.r.v.
continuous	$e^{-\delta(T_x\wedge n)}$
annual	$v^{(K_x+1)\wedge n}$
$1/m$thly	$v^{\left(K_x^{(m)}+\frac{1}{m}\right)\wedge n}$

3.2.11 Continuous case:

	p.v.r.v.	APV	2nd moment	variance
expression	$e^{-\delta(T_x\wedge n)}$	$\bar{A}^{1}_{x:\overline{n}\vert}+{}_nE_x$	$\bar{A}_{x:\overline{n}\vert}@\,2\delta$	${}^2\bar{A}_{x:\overline{n}\vert}-\left(\bar{A}_{x:\overline{n}\vert}\right)^2$
notation	Z	$\bar{A}_{x:\overline{n}\vert}$	${}^2\bar{A}_{x:\overline{n}\vert}$	$\operatorname{Var}(Z)$

[Note: When there is not "1" on top of either x or $\overline{n}\vert$, it indicates that a benefit payment is triggered when *any* one of them is "gone". Again more details will be discussed in STAT3909.]

3.2.12 Annual case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{(K_x+1)\wedge n}$	$A^{1}_{x:\overline{n}\vert}+{}_nE_x$	$A_{x:\overline{n}\vert}@\,2\delta$	${}^2A_{x:\overline{n}\vert}-\left(A_{x:\overline{n}\vert}\right)^2$
notation	Z	$A_{x:\overline{n}\vert}$	${}^2A_{x:\overline{n}\vert}$	$\operatorname{Var}(Z)$

3.2.13 $1/m$thly case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{\left(K_x^{(m)}+\frac{1}{m}\right)\wedge n}$	$A^{(m)}{}^{1}_{x:\overline{n}\vert}+{}_nE_x$	$A^{(m)}_{x:\overline{n}\vert}@\,2\delta$	${}^2A^{(m)}_{x:\overline{n}\vert}-\left(A^{(m)}_{x:\overline{n}\vert}\right)^2$
notation	Z	$A^{(m)}_{x:\overline{n}\vert}$	${}^2A^{(m)}_{x:\overline{n}\vert}$	$\operatorname{Var}(Z)$

3.2.14 Due to the property in [3.2.8], we have an alternative method for calculating the *variance* for p.v.r.v. for endowment insurance, as follows:

Proposition 3.2.b. Write $Z = Z_1 + Z_2$ where Z, Z_1, Z_2 are the p.v.r.v.'s of the n-year endowment, term life, and pure endowment insurances respectively. Then,

- (continuous case) $\operatorname{Var}(Z)=\operatorname{Var}(Z_1)+\operatorname{Var}(Z_2)-2\bar{A}^{1}_{x:\overline{n}\vert}\,{}_nE_x$;
- (annual case) $\operatorname{Var}(Z)=\operatorname{Var}(Z_1)+\operatorname{Var}(Z_2)-2A^{1}_{x:\overline{n}\vert}\,{}_nE_x$;
- ($1/m$thly case) $\operatorname{Var}(Z)=\operatorname{Var}(Z_1)+\operatorname{Var}(Z_2)-2A^{(m)}{}^{1}_{x:\overline{n}\vert}\,{}_nE_x$.

Proof: First note that in any situation, one of Z_1 and Z_2 would be 0 (a life either dies within or survive for n years). So, the product $Z_1 Z_2$ is always 0, and hence the covariance term

$$\mathrm{Cov}(Z_1, Z_2) = \mathbb{E}[Z_1 Z_2] - \mathbb{E}[Z_1]\mathbb{E}[Z_2] = -\mathbb{E}[Z_1]\mathbb{E}[Z_2].$$

Now the result follows from the formula

$$\mathrm{Var}(Z) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2) + 2\,\mathrm{Cov}(Z_1, Z_2).$$

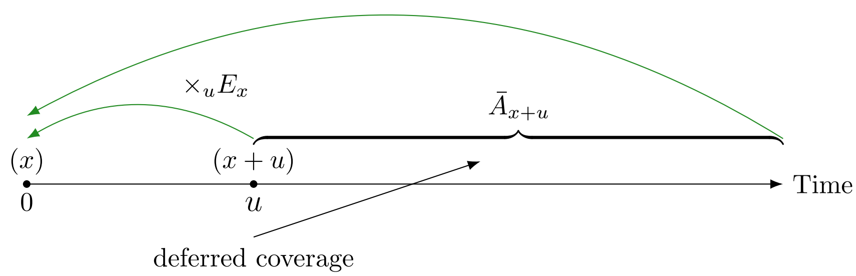
□

3.3 Deferred Life Insurance

3.3.1 There are 3 main types of deferred insurances of interest here:

- deferred whole life
- deferred term life
- deferred endowment

3.3.2 In general, the APV formulas here can be developed using the "actuarial discount factor" intuition, as we can simply "actuarially discount" the coverage for a life aged $x+u$ back now to get "deferred coverage" for the same life but aged x.



But of course, one can also use the general APV formula in [3.0.8] to develop them.

3.3.3 For a u-year deferred whole life insurance, we have:

(a) continuous case:

	p.v.r.v.	APV	2nd moment	variance
expression	$e^{-\delta T_x}\mathbf{1}_{\{T_x>u\}}$	${}_uE_x\,\bar{A}_{x+u}$ or $\displaystyle\int_u^\infty e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,\mathrm{d}t$	${}_{u\mid}\bar{A}_x @ 2\delta$	${}^2_{u\mid}\bar{A}_x - ({}_{u\mid}\bar{A}_x)^2$
notation	Z	${}_{u\mid}\bar{A}_x$	${}^2_{u\mid}\bar{A}_x$	$\mathrm{Var}(Z)$

(b) annual case:

	p.v.r.v.	APV	2nd moment	variance	
expression	$v^{K_x+1}\mathbf{1}_{\{K_x\geq u\}}$	${}_uE_x\,A_{x+u}$ or $\displaystyle\sum_{k=u}^\infty v^{k+1}\,{}_kp_x\,q_{x+k}$	${}_{u\mid}A_x @ 2\delta$	${}^2_{u\mid}A_x - ({}_{u\mid}A_x)^2$	${}_{u\mid}A_x$
notation	Z	${}_{u\mid}A_x$	${}^2_{u\mid}A_x$	$\mathrm{Var}(Z)$	

(c) $1/m$thly case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{K_x^{(m)}+\frac{1}{m}}\mathbf{1}_{\left\{K_x^{(m)}\geq u\right\}}$	${}_uE_x\,A_{x+u}^{(m)}$ or $\displaystyle\sum_{k=mu}^\infty v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}$	${}_{u\mid}A_x^{(m)} @ 2\delta$	${}^2_{u\mid}A_x^{(m)} - \left({}_{u\mid}A_x^{(m)}\right)^2$
notation	Z	${}_{u\mid}A_x^{(m)}$	${}^2_{u\mid}A_x^{(m)}$	$\mathrm{Var}(Z)$

3.3.4 For a u-year deferred n-year term life insurance, we have:

(a) continuous case:

	p.v.r.v.	APV	2nd moment	variance
expression	$e^{-\delta T_x}\mathbf{1}_{\{u<T_x\le u+n\}}$	${}_uE_x\,\bar{A}_{\overset{1}{(x+u)}:\overline{n}\rvert}$ or $\int_u^{u+n} e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$	${}_{u\rvert}\bar{A}^{1}_{x:\overline{n}\rvert}$@ 2δ	${}^2_{u\rvert}\bar{A}^{1}_{x:\overline{n}\rvert}-({}_{u\rvert}\bar{A}^{1}_{x:\overline{n}\rvert})^2$
notation	Z	${}_{u\rvert}\bar{A}^{1}_{x:\overline{n}\rvert}$	${}^2_{u\rvert}\bar{A}^{1}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

(b) annual case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{K_x+1}\mathbf{1}_{\{u\le K_x\le u+n-1\}}$	${}_uE_x\,A_{\overset{1}{(x+u)}:\overline{n}\rvert}$ or $\sum_{k=u}^{u+n-1} v^{k+1}\,{}_kp_x\,q_{x+k}$	${}_{u\rvert}A^{1}_{x:\overline{n}\rvert}$@ 2δ	${}^2_{u\rvert}A^{1}_{x:\overline{n}\rvert}-({}_{u\rvert}A^{1}_{x:\overline{n}\rvert})^2$
notation	Z	${}_{u\rvert}A^{1}_{x:\overline{n}\rvert}$	${}^2_{u\rvert}A^{1}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

(c) $1/m$thly case:

	p.v.r.v.	APV
expression	$v^{K_x^{(m)}+\frac{1}{m}}\mathbf{1}_{\left\{u\le K_x^{(m)}\le u+n+\frac{1}{m}\right\}}$	${}_uE_x\,A^{(m)}{}_{\overset{1}{(x+u)}:\overline{n}\rvert}$ or $\sum_{k=mu}^{mu+mn-1} v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}$
notation	Z	${}_{u\rvert}A^{(m)}{}^{1}_{x:\overline{n}\rvert}$

	2nd moment	variance
expression	${}_{u\rvert}A^{(m)}{}^{1}_{x:\overline{n}\rvert}$@ 2δ	${}^2_{u\rvert}A^{(m)}{}^{1}_{x:\overline{n}\rvert}-\left({}_{u\rvert}A^{(m)}{}^{1}_{x:\overline{n}\rvert}\right)^2$
notation	${}^2_{u\rvert}A^{(m)}{}^{1}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

3.3.5 For a u-year deferred n-year endowment insurance, we have:

(a) continuous case:

	p.v.r.v.	APV	2nd moment	variance
expression	$e^{-\delta T_x}\mathbf{1}_{\{u<T_x\le u+n\}}+e^{-\delta(u+n)}\mathbf{1}_{\{T_x>u+n\}}$	${}_{u\rvert}\bar{A}^{1}_{x:\overline{n}\rvert}+{}_{u+n}E_x$	${}_{u\rvert}\bar{A}_{x:\overline{n}\rvert}$@ 2δ	${}^2_{u\rvert}\bar{A}_{x:\overline{n}\rvert}-({}_{u\rvert}\bar{A}_{x:\overline{n}\rvert})^2$
notation	Z	${}_{u\rvert}\bar{A}_{x:\overline{n}\rvert}$	${}^2_{u\rvert}\bar{A}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

(b) annual case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{K_x+1}\mathbf{1}_{\{u\le K_x\le u+n-1\}}+v^{u+n}\mathbf{1}_{\{K_x\ge u+n\}}$	${}_{u\rvert}A^{1}_{x:\overline{n}\rvert}+{}_{u+n}E_x$	${}_{u\rvert}A_{x:\overline{n}\rvert}$@ 2δ	${}^2_{u\rvert}A_{x:\overline{n}\rvert}-({}_{u\rvert}A_{x:\overline{n}\rvert})^2$
notation	Z	${}_{u\rvert}A_{x:\overline{n}\rvert}$	${}^2_{u\rvert}A_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

(c) $1/m$thly case:

	p.v.r.v.	APV	2nd moment	variance
expression	$v^{K_x^{(m)}+\frac{1}{m}}\mathbf{1}_{\left\{u\le K_x^{(m)}\le u+n-\frac{1}{m}\right\}}+v^{u+n}\mathbf{1}_{\left\{K_x^{(m)}\ge u+n\right\}}$	${}_{u\rvert}A^{(m)}{}^{1}_{x:\overline{n}\rvert}+{}_{u+n}E_x$	${}_{u\rvert}A^{(m)}_{x:\overline{n}\rvert}$@ 2δ	${}^2_{u\rvert}A^{(m)}_{x:\overline{n}\rvert}-\left({}_{u\rvert}A^{(m)}_{x:\overline{n}\rvert}\right)^2$
notation	Z	${}_{u\rvert}A^{(m)}_{x:\overline{n}\rvert}$	${}^2_{u\rvert}A^{(m)}_{x:\overline{n}\rvert}$	$\mathrm{Var}\,(Z)$

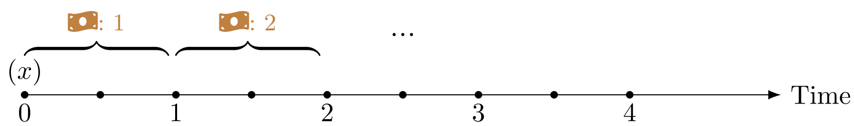
3.4 Life Insurance With Variable Benefits

3.4.6 Generally, for insurances with variable benefits, we use the general APV formula in [3.0.8]. Here we shall discuss two special cases:

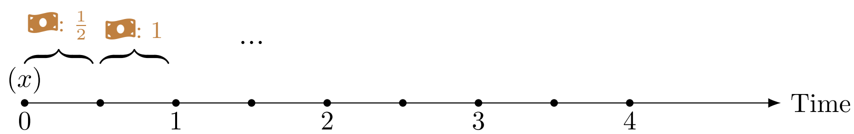
- arithmetically increasing/decreasing insurances
- geometrically increasing/decreasing insurances

3.4.7 For arithmetically increasing/decreasing insurances, there are some actuarial notations designed for them. To understand them, let us first explore different *kinds* of arithmetically increasing/decreasing insurances:

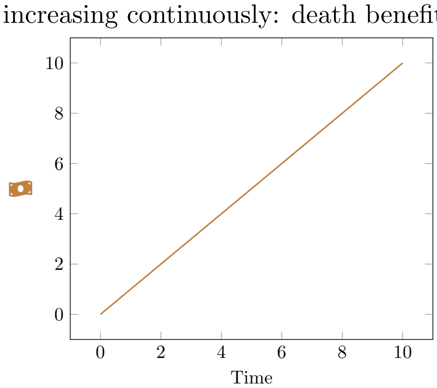
(a) increasing annually: death benefits in policy years $1, 2, \ldots$ are $1, 2, \ldots$ respectively;



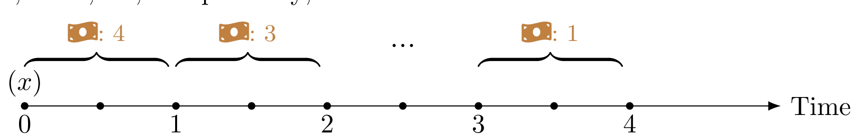
(b) increasing $1/m$thly: death benefits in time intervals $[0, 1/m), [1/m, 2/m), \ldots$ are $1/m, 2/m, \ldots$ respectively;



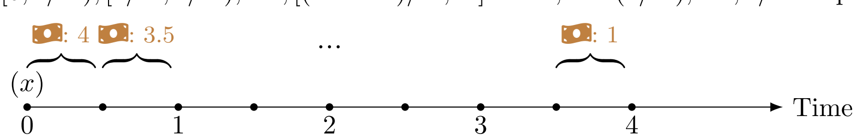
(c) increasing continuously: death benefit to be paid at any time t is t;



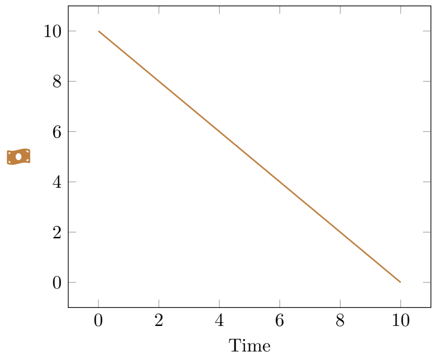
(d) decreasing annually (for n-year insurance): death benefits in policy years $1, 2, \ldots, n$ are $n, n-1, \ldots, 1$ respectively;



(e) decreasing $1/m$thly (for n-year insurance): death benefits in time intervals $[0, 1/m), [1/m, 2/m), \ldots, [(mn-1)/m, m]$ are $n, n-(1/m), \ldots, 1/m$ respectively;



(f) decreasing continuously (for n-year insurance): death benefit to be paid at time t is $n - t$, for any $t \in [0, n]$.



Remarks:

- These are understood to be only applicable for the period covered by the insurance. There is not death benefit for the period outside insurance coverage.
- The continuous cases can be seen as limits of the $1/m$thly cases as $m \to \infty$.

3.4.8 The designed actuarial notations are as follows:

insurance / kind	continuous	$1/m$thly	annual
increasing continuously	$(\bar{I}\bar{A})$	/	/
increasing $1/m$thly	$(I^{(m)}\bar{A})$	$(I^{(m)}A^{(m)})$	/
increasing annually	$(I\bar{A})$	$(IA^{(m)})$	(IA)
decreasing continuously	$(\bar{D}\bar{A})$	/	/
decreasing $1/m$thly	$(D^{(m)}\bar{A})$	$(D^{(m)}A^{(m)})$	/
decreasing annually	$(D\bar{A})$	$(DA^{(m)})$	(DA)

Remarks:

- These are only "cores" for the notations, and subscripts/superscripts can be added on them to represent a variety of insurances (but of course the term must be finite for decreasing insurances).
- Here we do not include the case where the benefit varying frequency is *higher* than the insurance itself since in such case it may not be very clear how changes in death benefits (strictly) *between* the dying time and the death benefit payment time (which may exist) should be handled.
- The kinds of insurance usually encountered here are the ones where the benefit varying and insurance "frequencies" are the same (located at the "main diagonal" for each of increasing & decreasing case) — the APV formulas for them are "nicer" also.

3.4.9 Example: The following gives the APV formulas for arithmetically increasing/decreasing n-year term life insurances, developed by the general APV formula in [3.0.8]:

notation	APV
$(I\bar{A})^1_{x:\overline{n}\rceil}$	$\displaystyle\int_0^1 e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\int_1^2 2e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\cdots+\int_{n-1}^n ne^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$
$(I^{(m)}\bar{A})^1_{x:\overline{n}\rceil}$	$\displaystyle\int_0^{1/m}\frac{1}{m}e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\int_{1/m}^{2/m}\frac{2}{m}e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\cdots+\int_{(mn-1)/m}^n ne^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$
⭐$(\bar{I}\bar{A})^1_{x:\overline{n}\rceil}$	$\displaystyle\int_0^n te^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$
$(D\bar{A})^1_{x:\overline{n}\rceil}$	$\displaystyle\int_0^1 ne^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\int_1^2(n-1)e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\cdots+\int_{n-1}^n e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$
$(D^{(m)}\bar{A})^1_{x:\overline{n}\rceil}$	$\displaystyle\int_0^{1/m} ne^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\int_{1/m}^{2/m}\left(n-\frac{1}{m}\right)e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t+\cdots+\int_{(mn-1)/m}^n \frac{1}{m}e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$
⭐$(\bar{D}\bar{A})^1_{x:\overline{n}\rceil}$	$\displaystyle\int_0^n (n-t)e^{-\delta t}\,{}_tp_x\mu_{x+t}\,\mathrm{d}t$
$(IA^{(m)})^1_{x:\overline{n}\rceil}$	$\displaystyle\sum_{k=0}^{m-1}v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}+\cdots+\sum_{k=m(n-1)}^{mn-1}nv^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}$
⭐$(I^{(m)}A^{(m)})^1_{x:\overline{n}\rceil}$	$\displaystyle\sum_{k=0}^{mn-1}\frac{k+1}{m}v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}$
$(DA^{(m)})^1_{x:\overline{n}\rceil}$	$\displaystyle\sum_{k=0}^{m-1}nv^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}+\cdots+\sum_{k=m(n-1)}^{mn-1}v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}$
⭐$(D^{(m)}A^{(m)})^1_{x:\overline{n}\rceil}$	$\displaystyle\sum_{k=0}^{mn-1}\left(n-\frac{k}{m}\right)v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{k}{m}}$
⭐$(IA)^1_{x:\overline{n}\rceil}$	$\displaystyle\sum_{k=0}^{n-1}(k+1)v^{k+1}\,{}_kp_x\,q_{x+k}$
⭐$(DA)^1_{x:\overline{n}\rceil}$	$\displaystyle\sum_{k=0}^{n-1}(n-k)v^{k+1}\,{}_kp_x\,q_{x+k}$

[Note: ⭐ = benefit varying and insurance "frequencies" are the same.]

3.4.10 For arithmetically *decreasing* insurances in particular, we have an "interesting" result:

Proposition 3.4.a. We have

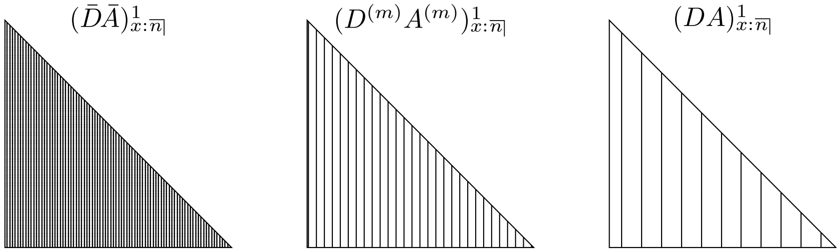
(a) $(\bar{D}\bar{A})^1_{x:\overline{n}\rceil}=\displaystyle\int_0^n \bar{A}^1_{x:\overline{n-t}\rceil}\,\mathrm{d}t$;

(b) $(D^{(m)}A^{(m)})^1_{x:\overline{n}\rceil}=\displaystyle\sum_{k=0}^{mn-1}A^{(m)\,1}_{x:\overline{n-\frac{k}{m}}\rceil}$;

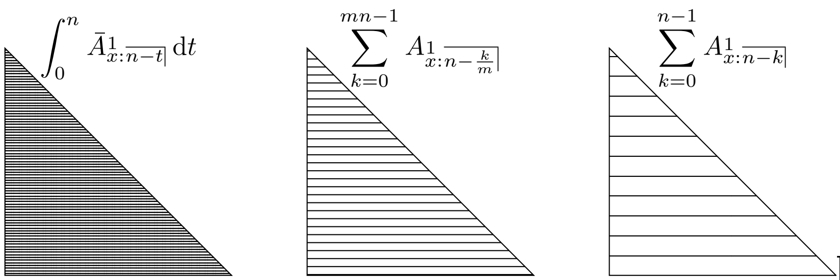
(c) $(DA)^1_{x:\overline{n}\rceil}=\displaystyle\sum_{k=0}^{n-1}A^1_{x:\overline{n-k}\rceil}$.

[Intuition 💡:

LHS: "slicing vertically"



RHS: "slicing horizontally"



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Proof: For the continuous case we have

$$(\bar{D}\bar{A})^{1}_{x:\overline{n}|}=\int_0^n (n-s)\,{}_sp_x\mu_{x+s}\,\mathrm{d}s=\int_0^n\left(\int_0^{n-s}1\,\mathrm{d}t\right){}_sp_x\mu_{x+s}\,\mathrm{d}s=\int_0^n\underbrace{\left(\int_0^{n-t}{}_sp_x\mu_{x+s}\,\mathrm{d}s\right)}_{\bar{A}^{1}_{x:\overline{n-t}|}}\mathrm{d}t\,.$$

Next, for the $1/m$thly case we have

$$\begin{aligned}(D^{(m)}A^{(m)})^{1}_{x:\overline{n}|}&=\sum_{j=0}^{mn-1}\left(n-\frac{j}{m}\right)v^{\frac{j+1}{m}}\,{}_{\frac{j}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{j}{m}}\\&=\sum_{j=0}^{mn-1}\left(\sum_{k=0}^{n-\frac{j}{m}}1\right)v^{\frac{j+1}{m}}\,{}_{\frac{j}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{j}{m}}\\&=\sum_{k=0}^{mn-1}\underbrace{\sum_{j=0}^{n-\frac{k}{m}}v^{\frac{j+1}{m}}\,{}_{\frac{j}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{j}{m}}}_{A^{1}_{x:\overline{n-\frac{k}{m}}|}}\,.\end{aligned}$$

The proof for the annual case is similar. □

3.4.11 For geometrically increasing/decreasing insurances, we focus on the annual case (where the benefit varying and insurance "frequencies" are the same) here: death benefits in policy years $1,2,3,\ldots$ are $1,(1+j),(1+j)^2,\ldots$ respectively, where $-1<j<1$.

[Note: For the $1/m$thly case where the benefit varying and insurance "frequencies" are still the same, the results here can be easily translated.]

3.4.12 Generally, the APV can be computed by considering the general APV formula in [3.0.8] and the formula for geometric series. But the following gives a shortcut formula:

Proposition 3.4.b. The APV of an n-year term life insurance (annual case) with such geometric sequence in the death benefit amounts is given by:

$$\frac{1}{1+j}A^{1}_{x:\overline{n}|i^*}.$$

where $i^* = (i - j)/(1 + j)$.

Proof: Note that the APV is

$$\sum_{k=0}^{n-1} (1+j)^k v^{k+1} {}_k p_x \, q_{x+k} = \frac{1}{1+j} \sum_{k=0}^{n-1} \left(\frac{1+i}{1+j} \right)^{-(k+1)} {}_k p_x \, q_{x+k},$$

and that

$$1 + i^* = \frac{1 + j + i - j}{1 + j} = \frac{1+i}{1+j}.$$

□

[Note: For the special case where $i = j$, we have $i^* = 0$ and hence the APV can be simplified to ${}_n q_x/(1+j)$ as the p.v.r.v. for an n-year term life insurance issued to (x) with zero interest rate is simply $\mathbf{1}_{\{T_x \le n\}}$.]

3.5 Recursions for APVs

3.5.1 There are two main reasons for considering recursions of APVs here:

(a) It provides insight on the relationships of APVs at different ages.

(b) It allows quick computations based on limited amount of information.

3.5.2 Most recursion formulas can be intuitively understood through "actuarial discounting".

3.5.3 Recursions for whole life insurance:

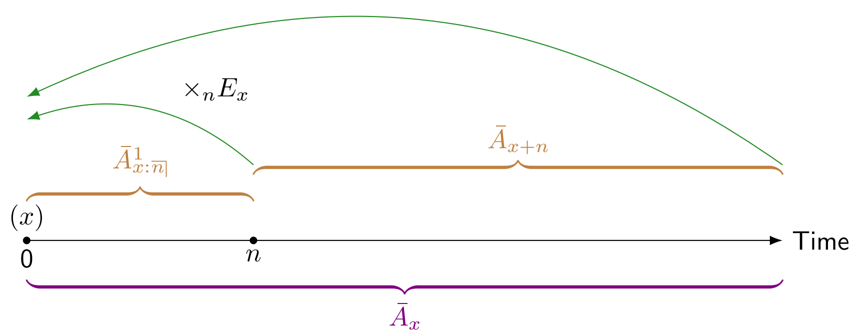
Proposition 3.5.a. For any age x and any $n \in \mathbb{N}$,

(a) $\bar{A}_x = \bar{A}^{\,1}_{x:\overline{n}|} + {}_n E_x \, \bar{A}_{x+n}$;

(b) $A_x = A^{\,1}_{x:\overline{n}|} + {}_n E_x \, A_{x+n}$;

(c) $A^{(m)}_x = A^{(m)\,1}_{\;x:\overline{n/m}|} + {}_n E_x \, A^{(m)}_{x+n/m}$.

[Intuition:



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Proof: The result follows from the proofs in [3.2.5] to [3.2.7]. □

[Note: As a special case, when $n = 1$, we have:

- $A_x = v q_x + v p_x A_{x+1}$;
- $A^{(m)}_x = v^{\frac{1}{m}} \, {}_{\frac{1}{m}} q_x + v^{\frac{1}{m}} \, {}_{\frac{1}{m}} p_x \, A^{(m)}_{x+\frac{1}{m}}$.

]

3.5.4 Recursions for term life insurance:

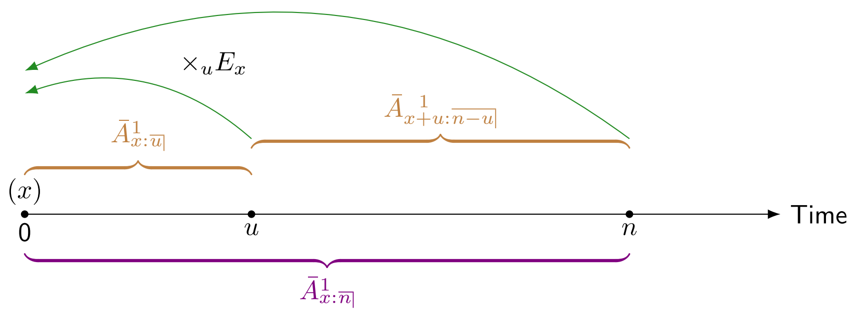
Proposition 3.5.b. For any age x and any $n \in \mathbb{N}$,

(a) $\bar{A}^{1}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{u}|} + {}_uE_x\,\bar{A}_{x+u:\overline{n-u}|}^{\,1}$ (for any $u \in \mathbb{N}$ with $u \leq n$);

(b) $A^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{u}|} + {}_uE_x\,A_{x+u:\overline{n-u}|}^{\,1}$ (for any $u \in \mathbb{N}$ with $u \leq n$);

(c) $A^{(m)}{}^{1}_{x:\overline{n}|} = A^{(m)}{}^{1}_{x:\overline{u/m}|} + {}_{u/m}E_x\,A^{(m)}{}_{(x+u/m):\overline{n-u/m}|}^{\,1}$ (for any $u \in \mathbb{N}$ with $u \leq mn$).

[Intuition:



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Proof: Similar to the proof for proposition 3.5.a. □

3.5.5 Recursions for endowment insurance:

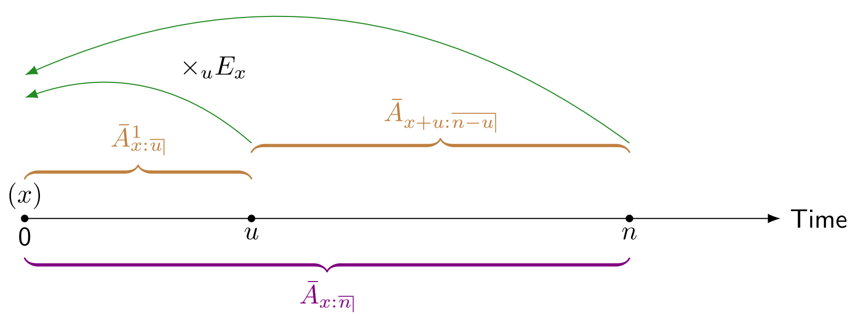
Proposition 3.5.c. For any age x and any $n \in \mathbb{N}$,

(a) $\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{u}|} + {}_uE_x\,\bar{A}_{x+u:\overline{n-u}|}$ (for any $u \in \mathbb{N}$ such that $u \leq n$);

(b) $A_{x:\overline{n}|} = A^{1}_{x:\overline{u}|} + {}_uE_x\,A_{x+u:\overline{n-u}|}$ (for any $u \in \mathbb{N}$ such that $u \leq n$);

(c) $A^{(m)}{}_{x:\overline{n}|} = A^{(m)}{}^{1}_{x:\overline{u/m}|} + {}_{u/m}E_x\,A^{(m)}_{(x+u/m):\overline{n-u/m}|}$ (for any $u \in \mathbb{N}$ such that $u \leq mn$).

[Intuition:



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Proof: Similar to the proof for proposition 3.5.a. □

3.5.6 Recursions for arithmetically increasing/decreasing insurances $(IA)/(DA)$:

Proposition 3.5.d. We have

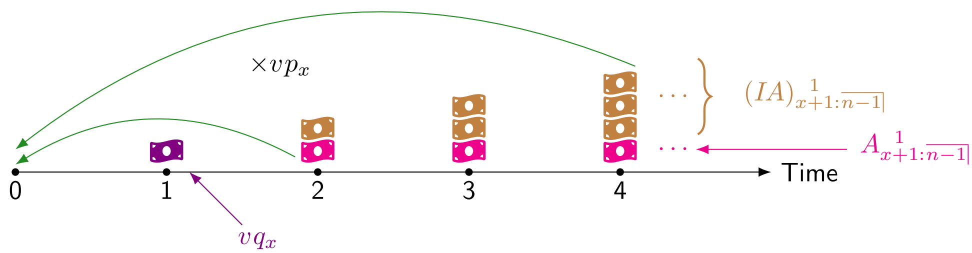
(a) $(IA)^{1}_{x:\overline{n}|} = vq_x + vp_x\left[(IA)_{x+1:\overline{n-1}|}^{\,1} + A_{x+1:\overline{n-1}|}^{\,1}\right]$ and $(IA)_x = vq_x + vp_x[(IA)_{x+1} + A_{x+1}]$;

(b) $(DA)^{1}_{x:\overline{n}|} = nvq_x + vp_x(DA)_{x+1:\overline{n-1}|}^{\,1}$;

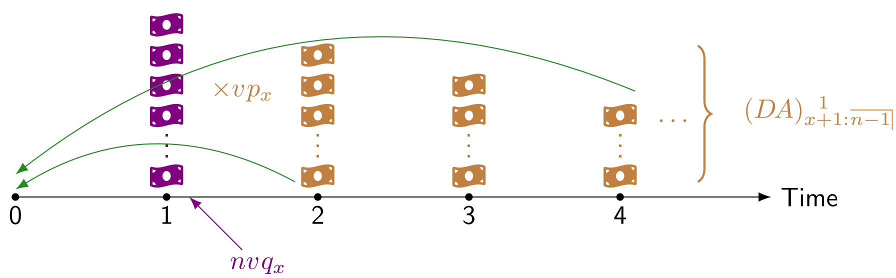
(c) $(IA)^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + vp_x(IA)_{x+1:\overline{n-1}|}^{\,1}$ and $(IA)_x = A_x + vp_x(IA)_{x+1}$.

[Intuition 💡:

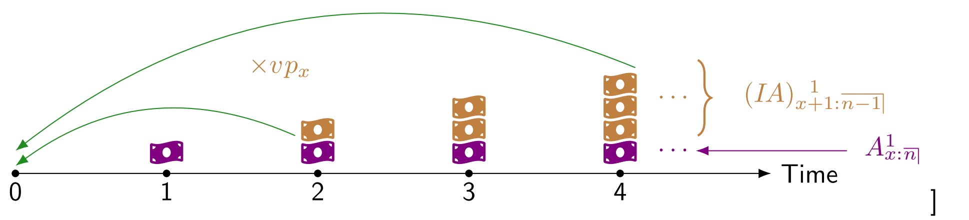
[3.5.6]a:



[3.5.6]b:



[3.5.6]c:



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<u>Proof</u>: Exercise. (The intuition already illustrates the key idea in the proof — one just needs to "split" the terms appropriately.) □

3.6 Relating $\bar{A}$, A and $A^{(m)}$

3.6.1 In a life table, we often only have the values for "A" but not for "$\bar{A}$" and "$A^{(m)}$". So, we are interested in the relationship between them to see how we can get "$\bar{A}$" and "$A^{(m)}$" from "A".

3.6.2 It turns out that in order to obtain a "nice" relationship between them, we need to impose the UDD assumption:

Proposition 3.6.a. Under UDD assumption,

(a) $\bar{A}_* = \dfrac{i}{\delta} A_*$

(b) $A_*^{(m)} = \dfrac{i}{i^{(m)}} A_*$

where A_* denotes any of $A_x, A^{1}_{x:\overline{n}|}, {}_{u|}A_x, {}_{u|}A^{1}_{x:\overline{n}|}, (IA)_x, (IA)^{1}_{x:\overline{n}|}, (DA)^{1}_{x:\overline{n}|}$ (i.e., any whole life/term life insurance with possibly deferral or "arithmetically increasing/decreasing (annual)" variant).

[⚠ Warning: A_* does <u>not</u> include $A_{x:\overline{n}|}$. Indeed, under UDD assumption we have

$$\bar{A}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + {}_nE_x$$

which can be seen by writing $\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + {}_nE_x$ (similar for $A^{(m)}_{x:\overline{n}|}$).]

<u>Proof</u>: We shall only give some key idea and concentrate on whole life insurance here. See Bowers et al. (1997, Section 4.4) for more details.

For the continuous case, consider lemma 2.8.a, and note that $v^{T_x} = v^{K_x+1}(1+i)^{1-U_x}$. It can be shown by integration that

$$\mathbb{E}\big[(1+i)^{1-U_x}\big] = i/\delta.$$

For the $1/m$thly case, consider

$$\sum_{j=0}^{\infty} v^{\frac{j+1}{m}}\,{}_{\frac{j}{m}}p_x\,{}_{\frac{1}{m}}q_{x+\frac{j}{m}} = \sum_{k=0}^{\infty}\sum_{u=0}^{m-1} v^{k+\frac{u+1}{m}}\,{}_{k+\frac{u}{m}}p_x\left({}_{\frac{1}{m}}q_{x+k+\frac{u}{m}}\right)$$

[Intuition 💡:

$$\begin{array}{cl} \dfrac{j}{m} & : \underbrace{0, \dfrac{1}{m}, \dfrac{2}{m}, \dots, \dfrac{m-1}{m}}_{0+\frac{u}{m}}, \underbrace{1, \dfrac{m+1}{m}, \dfrac{m+2}{m}, \dots, \dfrac{2m-1}{m}}_{1+\frac{u}{m}}, \dots, \\ k+\frac{u}{m} & \end{array}$$

]

By considering $a^{(m)}_{\overline{1}|}$, we have

$$\sum_{u=0}^{m-1} \frac{1}{m} v^{\frac{u+1}{m}} = \frac{i}{i^{(m)}}.$$

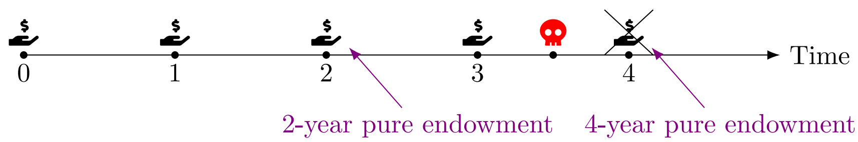
□

3.7 Incorporating Selection

3.7.1 All previous developments also apply to "select" lives (just change "x" to "$[x]$" in the notations).

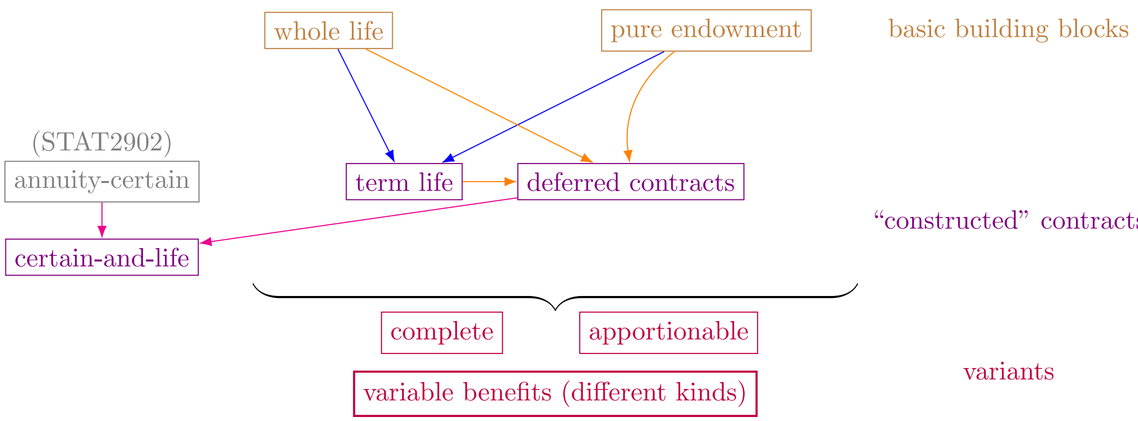
4 Life Annuity

4.0.1 A life annuity can be seen as a combination of multiple pure endowment, by considering each potential payment to the annuitant at time t as a t-year pure endowment with survival benefit equal to that payment amount issued to the annuitant. This provides one approach for developments of life annuity, and forms the basis for the general APV formula in [3.0.8] *for life annuity*.



4.0.2 Another approach for the developments is based on STAT2902. Recall that we have learned various *annuities-certain* in STAT2902. By making their terms to depend on the future lifetime (payments are only made when the annuitant is alive), they become *life annuities*.

4.0.3 Like [3.0.9], the following shows the relationship between different kinds of life annuities and pure endowment (in a "construction" approach):



[Note: The complete and apportionable variants are not inside SOA exam FAM syllabus currently.]

4.1 Whole Life Annuity

4.1.1 Like STAT2902, for *discrete* life annuity, there is a distinction between annuity-due and annuity-immediate (where payments are made at the *beginning* and *end* of each period covered, respectively). Remarks:

- It turns out that life annuity-*due* is more frequently considered.
- For life annuity-immediate, the end-of-period payment is *not* made for the period in which the life dies (since *at the time of payment* the life is *not* alive).

4.1.2 Like section 3, we also have three "frequencies" for life annuities: (i) continuous, (ii) annual, and (iii) $1/m$thly. (They correspond to "$\bar{a}$", "a", and "$a^{(m)}$" in STAT2902 respectively.)

4.1.3 Continuous case (annual payment rate: 1):

	p.v.r.v.	APV	variance
expression	$\bar{a}_{\overline{T_x}\rceil}$ or $\int_0^\infty e^{-\delta t}\mathbf{1}_{\{T_x>t\}}\,\mathrm{d}t$	$\frac{1-\bar{A}_x}{\delta}$ or $\int_0^\infty e^{-\delta t}\,{}_tp_x\,\mathrm{d}t$	$\frac{{}^2\bar{A}_x-(\bar{A}_x)^2}{\delta^2}$
notation	Y	$\bar{a}_x$	$\mathrm{Var}\,(Y)$

[Warning: For life annuities, the 2nd moment is not the APV at double force of interest.]

[Intuition: For the p.v.r.v. expression $\int_0^\infty e^{-\delta t}\mathbf{1}_{\{T_x>t\}}\,\mathrm{d}t$, it can be understood intuitively as "sum" of pure endowment with "infinitesimal" survival benefit "$\mathrm{d}t$". Based on this understanding, the APV formula $\int_0^\infty e^{-\delta t}\mathbf{1}_{\{T_x>t\}}\,\mathrm{d}t$ is "summing up" the expected present value of every t-year "infinitesimal" pure endowment:

$$\underbrace{\mathrm{d}t\, e^{-\delta t}}_{\text{PV of payment}} \times \underbrace{{}_t p_x}_{\text{prob. of surviving at time } t}.$$

]

[Note: Note that $\int_0^\infty e^{-\delta t}\mathbf{1}_{\{T_x>t\}}\,\mathrm{d}t = \int_0^{T_x} e^{-\delta t}\,\mathrm{d}t = \bar{a}_{\overline{T_x}|}$, so both p.v.r.v. expressions are indeed equivalent.]

Proof: To get the first APV formula, note that

$$\mathbb{E}\left[\bar{a}_{\overline{T_x}|}\right] = \mathbb{E}\left[\frac{1-e^{-\delta T_x}}{\delta}\right] = \frac{1-\mathbb{E}\left[e^{-\delta T_x}\right]}{\delta} = \frac{1-\bar{A}_x}{\delta}.$$

For the second APV formula, note that

$$\mathbb{E}\left[\int_0^\infty e^{-\delta t}\mathbf{1}_{\{T_x>t\}}\,\mathrm{d}t\right] = \int_0^\infty \mathbb{E}\left[e^{-\delta t}\mathbf{1}_{\{T_x>t\}}\right]\mathrm{d}t = \int_0^\infty e^{-\delta t}\,{}_t p_x\,\mathrm{d}t$$

where the first equality follows from Fubini's theorem.

Lastly, for the variance formula, we have

$$\mathrm{Var}\left(\frac{1-e^{-\delta T_x}}{\delta}\right) = \frac{1}{\delta^2}\underbrace{\mathrm{Var}\left(e^{-\delta T_x}\right)}_{{}^2\bar{A}_x-\left(\bar{A}_x\right)^2}$$

□

4.1.4 Annual case (amount of each payment: 1):

	p.v.r.v.	APV	variance
expression	due: $\ddot{a}_{\overline{K_x+1}|}$ or $\sum_{k=0}^{\infty} v^k \mathbf{1}_{\{T_x>k\}}$ immediate: $a_{\overline{K_x}|}$ or $\sum_{k=1}^{\infty} v^k \mathbf{1}_{\{T_x>k\}}$	due: $\frac{1-A_x}{d}$ or $\sum_{k=0}^{\infty} v^k\,{}_k p_x$ immediate: $\sum_{k=1}^{\infty} v^k\,{}_k p_x$	due: $\frac{{}^2A_x-(A_x)^2}{d^2}$ immediate: same as due
notation	Y	due: $\ddot{a}_x$ immediate: a_x	$\mathrm{Var}\,(Y)$

[Note: The variance for the immediate p.v.r.v. equals the one for the due p.v.r.v. since the p.v.r.v.'s just differ by a constant ($v^0 = 1$).]

4.1.5 $1/m$thly case (amount of each payment: $1/m$; total amount of payments in each year: 1):

	p.v.r.v.	APV	variance
expression	due: $\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|}$ or $\sum_{k=0}^{\infty} \frac{1}{m} v^{\frac{k}{m}} \mathbf{1}_{\{T_x>\frac{k}{m}\}}$ immediate: $a^{(m)}_{\overline{K_x^{(m)}}|}$ or $\sum_{k=1}^{\infty} \frac{1}{m} v^{\frac{k}{m}} \mathbf{1}_{\{T_x>\frac{k}{m}\}}$	due: $\frac{1-A_x^{(m)}}{d^{(m)}}$ or $\sum_{k=0}^{\infty} \frac{1}{m} v^{\frac{k}{m}}\,{}_{\frac{k}{m}} p_x$ immediate: $\sum_{k=1}^{\infty} \frac{1}{m} v^{\frac{k}{m}}\,{}_{\frac{k}{m}} p_x$	due: $\frac{{}^2A_x^{(m)}-\left(A_x^{(m)}\right)^2}{(d^{(m)})^2}$ immediate: same as due
notation	Y	due: $\ddot{a}_x^{(m)}$ immediate: $a_x^{(m)}$	$\mathrm{Var}\,(Y)$

4.1.6 For discrete life annuity-immediate, we often use the following alternative formulas to compute APV instead:

Proposition 4.1.a. For any age x,

(a) $a_x = \ddot{a}_x - 1$;

(b) $a_x^{(m)} = \ddot{a}_x^{(m)} - 1/m$.

Proof: The result follows easily from considering the summation APV formulas. $\square$

4.1.7 The APVs of whole life annuities of different frequencies can be ordered as follows:

$$a_x \le a_x^{(m)} \le \bar{a}_x \le \ddot{a}_x^{(m)} \le \ddot{a}_x,$$

for any age x and $m \in \mathbb{N}$.

[Intuition 💡: As we go from the life annuity on LHS to RHS one by one, all potential payments (where some may get "split") "shift" *earlier* (or do not "move"), so the present value gets higher (or at least does not decrease) *always*, regardless of when the life dies.]

Also, $\ddot{a}_x^{(m)}$ decreases in m while $a_x^{(m)}$ increases in m. This can be understood via similar intuition — as m increases, all potential payments "shift" later (for former) or earlier (for latter).

4.2 Temporary Life Annuity

4.2.1 Consider an n-year temporary life annuity.

4.2.2 Continuous case (annual payment rate: 1):

	p.v.r.v.	APV	variance
expression	$\bar{a}_{\overline{T_x \wedge n}\rceil}$ or $\displaystyle\int_0^n e^{-\delta t}\mathbf{1}_{\{T_x > t\}}\,\mathrm{d}t$	$\dfrac{1-\bar{A}_{x:\overline{n}\rceil}}{\delta}$ or $\bar{a}_x - {}_nE_x\,\bar{a}_{x+n}$ or $\displaystyle\int_0^n e^{-\delta t}\,{}_tp_x\,\mathrm{d}t$	$\dfrac{{}^2\bar{A}_{x:\overline{n}\rceil} - \left(\bar{A}_{x:\overline{n}\rceil}\right)^2}{\delta^2}$
notation	Y	$\bar{a}_{x:\overline{n}\rceil}$	$\mathrm{Var}\,(Y)$

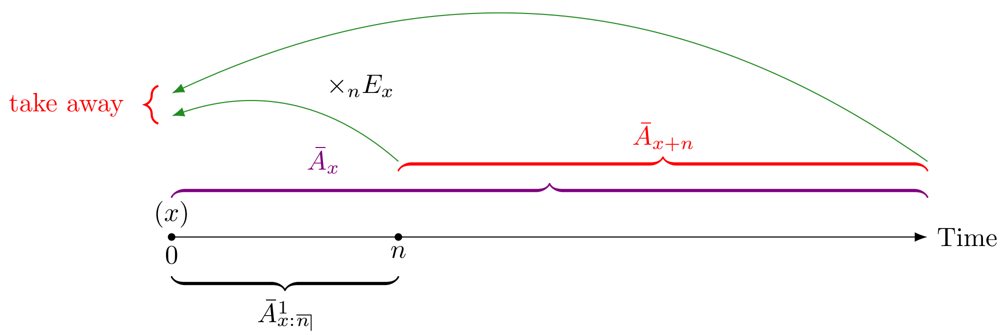
4.2.3 Annual case (amount of each payment: 1):

	p.v.r.v.	APV	variance
expression	due: $\ddot{a}_{\overline{(K_x+1)\wedge n}\rceil}$ or $\displaystyle\sum_{k=0}^{n-1} v^k \mathbf{1}_{\{T_x > k\}}$ immediate: $a_{\overline{K_x \wedge n}\rceil}$ or $\displaystyle\sum_{k=1}^{n} v^k \mathbf{1}_{\{T_x > k\}}$	due: $\dfrac{1-A_{x:\overline{n}\rceil}}{d}$ or $\ddot{a}_x - {}_nE_x\,\ddot{a}_{x+n}$ or $\displaystyle\sum_{k=0}^{n} v^k\,{}_kp_x$ immediate: $\displaystyle\sum_{k=1}^{n} v^k\,{}_kp_x$ or $a_x - {}_nE_x\,a_{x+n}$	due: $\dfrac{{}^2A_{x:\overline{n}\rceil} - \left(A_{x:\overline{n}\rceil}\right)^2}{d^2}$ immediate: omitted
notation	Y	due: $\ddot{a}_{x:\overline{n}\rceil}$ immediate: $a_{x:\overline{n}\rceil}$	$\mathrm{Var}\,(Y)$

4.2.4 $1/m$thly case (amount of each payment: $1/m$; total amount of payments in each year: 1):

	p.v.r.v.	APV	variance
expression	due: $\ddot{a}_{\overline{(K_x^{(m)}+\frac{1}{m})\wedge n}\rvert}$ or $\sum_{k=0}^{mn-1} \frac{1}{m} v^{\frac{k}{m}} \mathbf{1}_{\{T_x > \frac{k}{m}\}}$ immediate: $a_{\overline{K_x^{(m)}\wedge n}\rvert}$ or $\sum_{k=1}^{mn} \frac{1}{m} v^{\frac{k}{m}} \mathbf{1}_{\{T_x > \frac{k}{m}\}}$	due: $\frac{1-A_{x:\overline{n}\rvert}^{(m)}}{d^{(m)}}$ or $\ddot{a}_x^{(m)} - {}_nE_x \ddot{a}_{x+n}^{(m)}$ or $\sum_{k=0}^{mn-1} \frac{1}{m} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_x$ immediate: $\sum_{k=1}^{mn} \frac{1}{m} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_x$ or $\ddot{a}_x^{(m)} - {}_nE_x \ddot{a}_{x+n}^{(m)}$	due: $\frac{{}^2A_{x:\overline{n}\rvert}^{(m)} - \left(A_{x:\overline{n}\rvert}^{(m)}\right)^2}{\left(d^{(m)}\right)^2}$ immediate: omitted
notation	Y	due: $\ddot{a}_{x:\overline{n}\rvert}^{(m)}$ immediate: $a_{x:\overline{n}\rvert}^{(m)}$	$\mathrm{Var}(Y)$

4.2.5 We can similarly use the "actuarial discounting" intuition to develop the following formula (analogous to the term life insurance case):



4.2.6 Again for discrete life annuity-immediate, we usually use an alternative formula for calculating APV:

Proposition 4.2.a. For any age x,

(a) $a_{x:\overline{n}\rvert} = \ddot{a}_{x:\overline{n}\rvert} - 1 + {}_nE_x$;

(b) $a_{x:\overline{n}\rvert}^{(m)} = \ddot{a}_{x:\overline{n}\rvert}^{(m)} - 1/m + (1/m)\,{}_nE_x$.

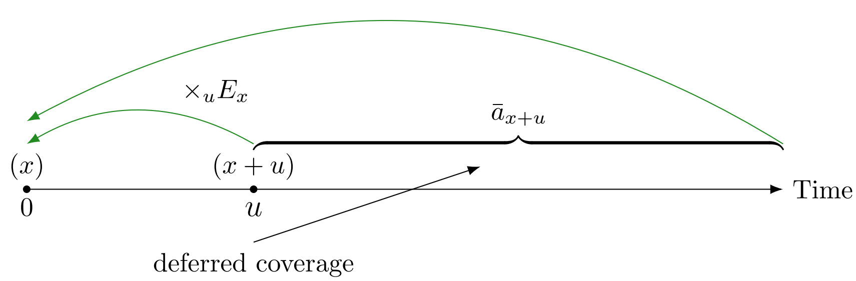
Proof: Similar to the proof for proposition 4.1.a. □

4.3 Deferred Life Annuity

4.3.1 Like section 3.3, we have different types of deferred life annuities:

- deferred whole life annuity
- deferred temporary life annuity

4.3.2 Again like section 3.3, the APV formulas can be developed using the "actuarial discount factor" intuition:



One can also use the general APV formula in [3.0.8] to develop them.

4.3.3 Here we shall focus only on deferred *whole life* annuity — similar developments can be done for deferred temporary life annuity.

4.3.4 Continuous case (annual payment rate: 1):

	p.v.r.v.	APV
expression	${}_{u\|}\bar{a}_{\overline{T_x-u}\|}\mathbf{1}_{\{T_x>u\}}$ or $\int_u^\infty e^{-\delta t}\mathbf{1}_{\{T_x>t\}}\,\mathrm{d}t$	$\bar{a}_x - \bar{a}_{x:\overline{u}\|}$ or ${}_uE_x\bar{a}_{x+u}$ or $\int_u^\infty e^{-\delta t}{}_tp_x\,\mathrm{d}t$
notation	Y	${}_{u\|}\bar{a}_x$

[Note: We have ${}_{u|}\bar{a}_{\overline{n}|} = \bar{a}_{\overline{u+n}|} - \bar{a}_{\overline{u}|}$. (Similar for "$a$" and "$a^{(m)}$".)]

4.3.5 Annual case (amount of each payment: 1):

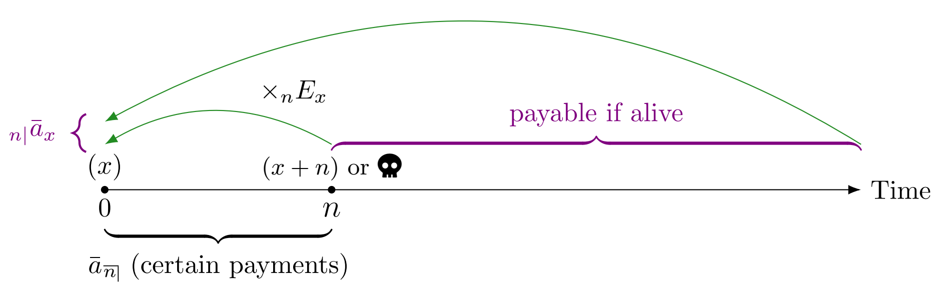
	p.v.r.v.	APV
expression	due: ${}_{u\|}\ddot{a}_{\overline{K_x+1-u}\|}\mathbf{1}_{\{T_x>u\}}$ or $\sum_{k=u}^{\infty} v^k\mathbf{1}_{\{T_x>k\}}$ immediate: ${}_{u\|}a_{\overline{K_x-u}\|}\mathbf{1}_{\{T_x>u\}}$ or $\sum_{k=u+1}^{\infty} v^k\mathbf{1}_{\{T_x>k\}}$	due: $\ddot{a}_x - \ddot{a}_{x:\overline{u}\|}$ or ${}_uE_x\ddot{a}_{x+u}$ or $\sum_{k=u}^{\infty} v^k{}_kp_x$ immediate: $a_x - a_{x:\overline{u}\|}$ or ${}_uE_x a_{x+u}$ or $\sum_{k=u+1}^{\infty} v^k{}_kp_x$
notation	Y	due: ${}_{u\|}\ddot{a}_x$ immediate: ${}_{u\|}a_x$

4.3.6 $1/m$thly case (amount of each payment: $1/m$; total amount of payments in each year: 1):

	p.v.r.v.	APV
expression	due: ${}_{u\|}\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}-u}\|}\mathbf{1}_{\{T_x>u\}}$ or $\sum_{k=u}^{\infty}\frac{1}{m}v^{\frac{k}{m}}\mathbf{1}_{\{T_x>\frac{k}{m}\}}$ immediate: ${}_{u\|}a^{(m)}_{\overline{K_x^{(m)}-u}\|}\mathbf{1}_{\{T_x>u\}}$ or $\sum_{k=u+1}^{\infty}\frac{1}{m}v^{\frac{k}{m}}\mathbf{1}_{\{T_x>\frac{k}{m}\}}$	due: $\ddot{a}_x^{(m)} - \ddot{a}^{(m)}_{x:\overline{u}\|}$ or ${}_uE_x\ddot{a}^{(m)}_{x+u}$ or $\sum_{k=u}^{\infty}\frac{1}{m}v^{\frac{k}{m}}{}_{\frac{k}{m}}p_x$ immediate: $a_x^{(m)} - a^{(m)}_{x:\overline{u}\|}$ or ${}_uE_x a^{(m)}_{x+u}$ or $\sum_{k=u+1}^{\infty}\frac{1}{m}v^{\frac{k}{m}}{}_{\frac{k}{m}}p_x$
notation	Y	due: ${}_{u\|}\ddot{a}_x^{(m)}$ immediate: ${}_{u\|}a_x^{(m)}$

4.4 Certain-And-Life/Guaranteed Annuity

4.4.1 A key observation is that an n-year certain-and-life annuity is indeed just a combination of an n-year annuity-certain and an n-year deferred whole life annuity:



4.4.2 Once we are aware of this, the developments for certain-and-life annuity become quite simple.

4.4.3 Continuous case (annual payment rate: 1):

	p.v.r.v.	APV
expression	Y in [4.3.4] $+\ \bar{a}_{\overline{n}\rvert}$	$\bar{a}_{\overline{n}\rvert} + {}_{n\rvert}\bar{a}_x$
notation	Y	$\bar{a}_{\overline{x:\overline{n}\rvert}}$

[Note: "$\overline{x:\overline{n}\rvert}$" suggests that the payments continue until the *last* of (x) and n-year term is "gone". More details will be discussed in STAT3909.]

4.4.4 Annual case (amount of each payment: 1):

	p.v.r.v.	APV
expression	due: Y in [4.3.5] (due) $+\ \ddot{a}_{\overline{n}\rvert}$ immediate: Y in [4.3.5] (immediate) $+\ a_{\overline{n}\rvert}$	due: $\ddot{a}_{\overline{n}\rvert} + {}_{n\rvert}\ddot{a}_x$ immediate: $a_{\overline{n}\rvert} + {}_{n\rvert}a_x$
notation	Y	due: $\ddot{a}_{\overline{x:\overline{n}\rvert}}$ immediate $a_{\overline{x:\overline{n}\rvert}}$

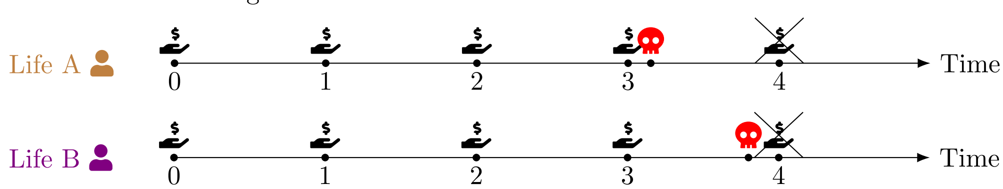
4.4.5 $1/m$thly case (amount of each payment: $1/m$; total amount of payments in each year: 1):

	p.v.r.v.	APV
expression	due: Y in [4.3.6] (due) $+\ \ddot{a}^{(m)}_{\overline{n}\rvert}$ immediate: Y in [4.3.6] (immediate) $+\ a^{(m)}_{\overline{n}\rvert}$	due: $\ddot{a}^{(m)}_{\overline{n}\rvert} + {}_{n\rvert}\ddot{a}^{(m)}_x$ immediate: $a^{(m)}_{\overline{n}\rvert} + {}_{n\rvert}a^{(m)}_x$
notation	Y	due: $\ddot{a}^{(m)}_{\overline{x:\overline{n}\rvert}}$ immediate $a^{(m)}_{\overline{x:\overline{n}\rvert}}$

4.5 Complete and Apportionable Life Annuities

[Note: This topic is not inside SOA exam FAM syllabus currently.]

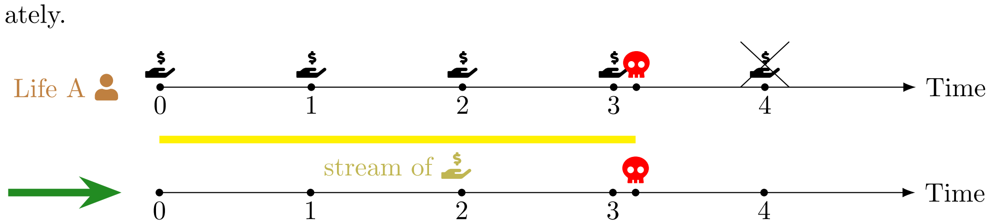
4.5.1 Consider the following situation:

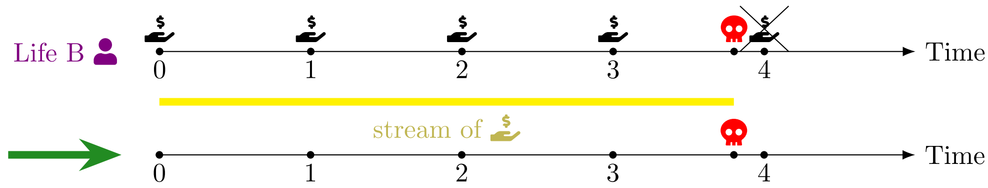


For both lives A and B, they do not receive the benefit payment at time 4. But life B survives most of the last period, while life A only survives a little of the last period! So this may cause some "unfairness". [Note: This "unfairness" only arises for discrete cases.]

4.5.2 To address this issue, the "complete" and "apportionable" variants are developed, for life annuity-immediate and life annuity-due respectively.

4.5.3 The common idea in the two variants is to convert the potential "discrete" payments in life annuity-due/immediate into *continuous* streams of payments — the streams would cease upon death immediately.

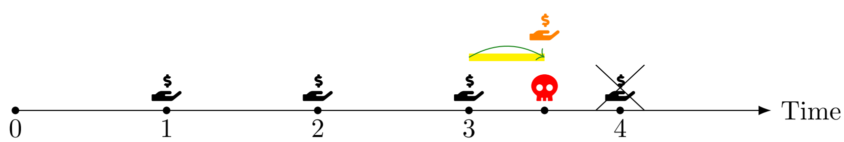




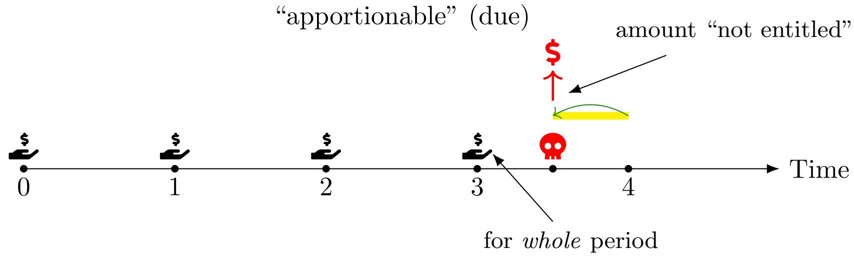
4.5.4 [4.5.3] describes the conceptual framework for developing the complete and apportionable variants. In the actual implementation, the amount and frequency of the payments payable originally (those before the death) remain unchanged.

4.5.5 The twist is that in the "last" time interval (the time interval between the original final payment and the scheduled next payment), the annuitant's account may receive (pay) an additional amount of money from (to) the insurer at the time of death , for the "complete" ("apportionable") variant, where the amount is the value of the "residual payment stream" accumulated to the time of death :

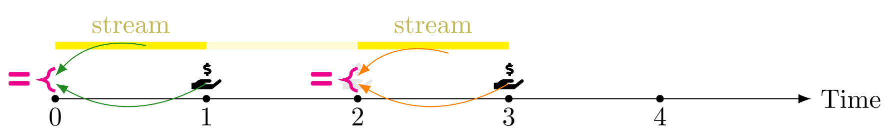
"complete" (immediate)



"apportionable" (due)



4.5.6 To preserve the "worth" of the original payments (both monetary and time values), in each ($1/m$th of) period we require the "present value" (at the beginning of period) of the "discrete" payment to be the same as the whole payment stream (in the framework) in that period:



Then, the original payments together with the additional amount at time of death would have the same present value as the payment stream, regardless of when the annuitant dies. So we can (conveniently) carry out the calculations of APV and other related quantities in the continuous framework.

4.5.7 Now we consider the general setting ($1/m$thly). In each $1/m$th of period (year), there is a single payment of $1/m$ (at the beginning or the end of the interval, depending on whether the life annuity is due or immediate). We then want to find the amount k such that the "present value" of the stream with annual rate k equals the "present value" of the single payment .

4.5.8 For **complete $1/m$thly life annuity-immediate**, we have

$$k\bar{a}_{\overline{1/m}|} = (1/m)(1+i)^{-1/m} \implies k = \frac{1/m}{\bar{s}_{\overline{1/m}|}} = \frac{1}{m}\frac{\delta}{(1+i)^{1/m}-1} = \frac{\delta}{i^{(m)}}.$$

After getting this amount, we can obtain the following[8]:

[8] Here we just include whole life and n-year temporary life as they are more often considered in this context. But in principle similar developments can be done for other kinds of life annuities.

(a) whole life:

	p.v.r.v.	APV	variance
expression	$\dfrac{\delta}{i^{(m)}}\bar{a}_{\overline{T_x}\rceil}$	$\dfrac{\delta}{i^{(m)}}\bar{a}_x$	$\left(\dfrac{\delta}{i^{(m)}}\right)^2 \times (\mathrm{Var}\,(Y)$ in [4.1.3])
notation	Y	$\mathring{a}_x^{(m)}$	$\mathrm{Var}\,(Y)$

(b) n-year temporary life:

	p.v.r.v.	APV	variance
expression	$\dfrac{\delta}{i^{(m)}}\bar{a}_{\overline{T_x \wedge n}\rceil}$	$\dfrac{\delta}{i^{(m)}}\bar{a}_{x:\overline{n}\rceil}$	$\left(\dfrac{\delta}{i^{(m)}}\right)^2 \times (\mathrm{Var}\,(Y)$ in [4.2.2])
notation	Y	$\mathring{a}_{x:\overline{n}\rceil}^{(m)}$	$\mathrm{Var}\,(Y)$

[Note: Since the APV (whole life/temporary life) is just a constant times a previously discussed "standard" APV notation, many previous results hold analogously for the "complete" APV notation. (It is similar for the apportionable case; see below.)]

4.5.9 For **apportionable $1/m$thly life annuity-due**, we have

$$k\bar{a}_{\overline{1/m}\rceil} = 1/m \implies k = \frac{1/m}{\bar{a}_{\overline{1/m}\rceil}} = \frac{1}{m}\frac{\delta}{1-(1+i)^{-1/m}} = \frac{\delta}{d^{(m)}}.$$

After getting this amount, we can obtain the following:

(a) whole life:

	p.v.r.v.	APV	variance
expression	$\dfrac{\delta}{d^{(m)}}\bar{a}_{\overline{T_x}\rceil}$	$\dfrac{\delta}{d^{(m)}}\bar{a}_x$	$\left(\dfrac{\delta}{d^{(m)}}\right)^2 \times (\mathrm{Var}\,(Y)$ in [4.1.3])
notation	Y	$\ddot{a}_x^{\{m\}}$	$\mathrm{Var}\,(Y)$

(b) n-year temporary life:

	p.v.r.v.	APV	variance
expression	$\dfrac{\delta}{d^{(m)}}\bar{a}_{\overline{T_x \wedge n}\rceil}$	$\dfrac{\delta}{d^{(m)}}\bar{a}_{x:\overline{n}\rceil}$	$\left(\dfrac{\delta}{d^{(m)}}\right)^2 \times (\mathrm{Var}\,(Y)$ in [4.2.2])
notation	Y	$\ddot{a}_{x:\overline{n}\rceil}^{\{m\}}$	$\mathrm{Var}\,(Y)$

[Mnemonic: "$i^{(m)}$" for complete life annuity-immediate, and "$d^{(m)}$" for apportionable life annuity-due.]

4.6 Life Annuity With Variable Payments

4.6.1 Like section 3.4, the general APV formula in [3.0.8] is applicable in the case where the life annuity has varying payment amounts. [Note: For the continuous case, the "benefit amount" part in the general formula would involve "dt" — the benefit paid in an "infinitesimal" time interval is "infinitesimal" for the continuous life annuity case.]

4.6.2 Again we shall discuss two special cases:

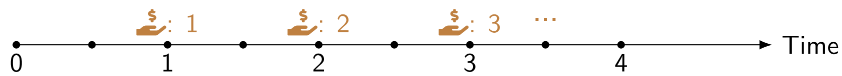
- arithmetically increasing/decreasing life annuities
- geometrically increasing/decreasing life annuities

4.6.3 Likewise, for arithmetically increasing/decreasing life annuities, there are some actuarial notations designed for them. The kinds are similar to [3.4.7] (with the same definition, but for benefit payment rather than death benefit):

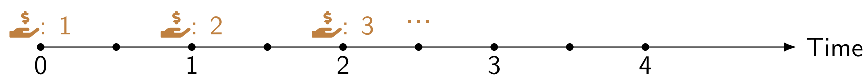
(a) increasing annually
(b) increasing $1/m$thly
(c) increasing continuously
(d) decreasing annually
(e) decreasing $1/m$thly
(f) decreasing continuously

[Note: For life annuity-immediate (-due), the payment amount "for" each period refers to the amount for the payment at the end (beginning) of that period. E.g.,

- annually increasing annual life annuity-immediate (payment amounts for policy years 1,2,3,... are 1,2,3,...):



- annually increasing annual life annuity-due (payment amounts for policy years 1,2,3,... are 1,2,3,...):



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4.6.4 The designed actuarial notations for life annuity are analogous to [3.4.8]: Continuous: $\bar{A} \to \bar{a}$; annual: $A \to \ddot{a}$ or a (due/immediate resp.); $1/m$thly: $A^{(m)} \to \ddot{a}^{(m)}$ or $a^{(m)}$ (due/immediate resp.).

4.6.5 We can develop APV formulas for arithmetically increasing/decreasing n-year temporary life annuities like [3.4.9], using the general APV formula in [3.0.8]:

- continuous life annuity: ${}_tp_x\mu_{x+t}$ ➔ ${}_tp_x$;
- annual life annuity: $(k+1)v^{k+1}{}_kp_xq_{x+k}$ ➔ $(k+1)v^k{}_kp_x$ or $(k+1)v^{k+1}{}_{k+1}p_x$ (due/immediate resp.) (similar for others);
- $1/m$thly life annuity: $\dfrac{k+1}{m}v^{\frac{k+1}{m}}{}_{\frac{k}{m}}p_xq_{x+\frac{k}{m}}$ ➔ $\dfrac{k+1}{m}v^{\frac{k}{m}}{}_{\frac{k}{m}}p_x$ or $\dfrac{k+1}{m}v^{\frac{k+1}{m}}{}_{\frac{k+1}{m}}p_x$ (due/immediate resp.) (similar for others).

Example: We have

$$(\bar{I}\bar{a})_{x:\overline{n}|} = \int_0^n te^{-\delta t}{}_tp_x\mathrm{d}t$$

($t\,\mathrm{d}t$: "benefit payment"; $e^{-\delta t}$: "discount factor"; ${}_tp_x$: "prob. of triggering payment", for every "infinitesimal" time interval $[t, t+\mathrm{d}t]$).

4.6.6 Like [3.4.11], for geometrically increasing/decreasing life annuities, we shall focus on the annual case (where the benefit varying and life annuity "frequencies" are the same) here: benefit payments for policy years $1, 2, 3, \ldots$ are $1, (1+j), (1+j)^2, \ldots$ respectively, where $-1 < j < 1$. Again one can use the general APV formula in [3.0.8] in general.

4.6.7 The following is a shortcut formula of computing the APV for an n-year temporary life annuity:

Proposition 4.6.a. The APV of an n-year temporary life annuity (annual case) with such geometric sequence in the benefit payments is given by:

$$\begin{cases} \ddot{a}_{x:\overline{n}|i^*} & \text{(due);} \\ \dfrac{1}{1+j}a_{x:\overline{n}|i^*} & \text{(immediate)} \end{cases}$$

where $i^* = (i-j)/(1+j)$.

Proof: For the "due" case, note that the APV is

$$\sum_{k=0}^{n-1}(1+j)^k v^k {}_kp_x = \sum_{k=0}^{n-1}\left(\frac{1+i}{1+j}\right)^{-k} {}_kp_x.$$

For the "immediate" case, note that the APV is

$$\sum_{k=0}^{n-1}(1+j)^k v^{k+1} {}_{k+1}p_x = \frac{1}{1+j}\sum_{k=0}^{n-1}\left(\frac{1+i}{1+j}\right)^{-(k+1)} {}_{k+1}p_x.$$

Also recall that

$$1+i^* = \frac{1+j+i-j}{1+j} = \frac{1+i}{1+j}.$$

□

[Note: Special case: If $i=j$ (then $i^*=0$), for the "due" case the APV is

$$\sum_{k=0}^{n-1} {}_kp_x = 1 + e_{x:\overline{n-1}|}$$

by proposition 2.5.f. If we further replace n-year temporary life annuity by whole life annuity (i.e., let $n\to\infty$), then the APV becomes $1+e_x$.]

4.7 Recursions for APVs

4.7.1 Again, most recursion formulas here can be intuitively understood through "actuarial discounting".

4.7.2 Due to the "conceptual similarity" of insurance and life annuity (general APV formula in [3.0.8] is applicable to both), the recursion formulas for life annuity have a similar "form" as the ones for life insurance.

4.7.3 Recursions for whole life annuity:

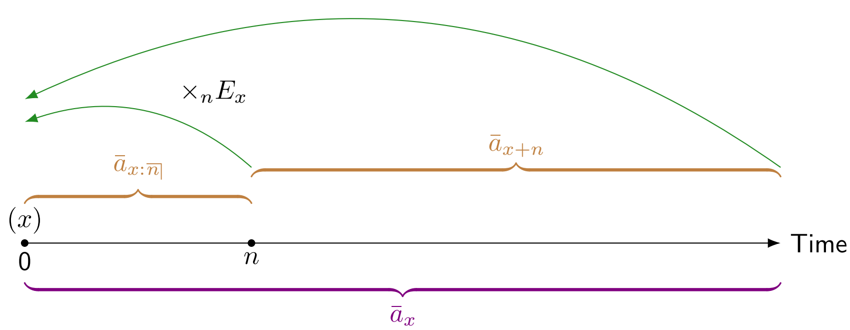
Proposition 4.7.a. For any age x and any $n\in\mathbb{N}$,

(a) $\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_nE_x\bar{a}_{x+n}$;

(b) $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x\ddot{a}_{x+n}$ (due) and $a_x = a_{x:\overline{n}|} + {}_nE_x a_{x+n}$ (immediate);

(c) $\ddot{a}_x^{(m)} = \ddot{a}^{(m)}_{x:\overline{n/m}|} + {}_{n/m}E_x\ddot{a}^{(m)}_{x+n/m}$ (due) and $a_x^{(m)} = a^{(m)}_{x:\overline{n/m}|} + {}_{n/m}E_x a^{(m)}_{x+n/m}$ (immediate).

[Intuition 💡:



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Proof: Similar to the proofs in [3.2.5] to [3.2.7]. □

[Note: Special case: when $n=1$, we have:

- $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$ (due) and $a_x = vp_x + vp_x a_{x+1}$ (immediate);
- $\ddot{a}_x^{(m)} = \dfrac{1}{m} + v^{\frac{1}{m}}\,{}_{\frac{1}{m}}p_x\ddot{a}_{x+\frac{1}{m}}^{(m)}$ (due) and $a_x^{(m)} = \dfrac{1}{m}v^{\frac{1}{m}}\,{}_{\frac{1}{m}}p_x + v^{\frac{1}{m}}\,{}_{\frac{1}{m}}p_x a_{x+\frac{1}{m}}^{(m)}$ (immediate).

]

4.7.4 Recursions for temporary life annuity:

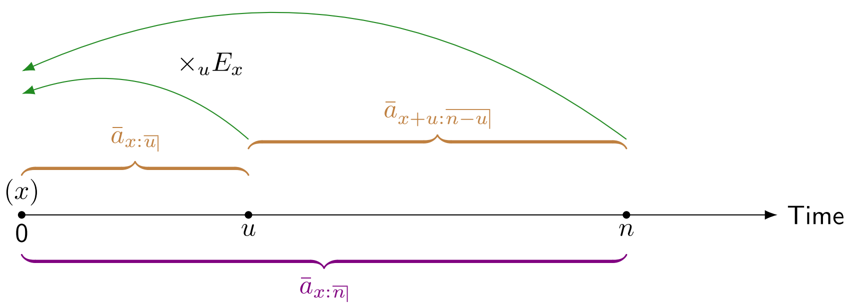
Proposition 4.7.b. For any age x and any $n \in \mathbb{N}$,

(a) $\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{u}|} + {}_uE_x\bar{a}_{x+u:\overline{n-u}|}$ (for any $u \in \mathbb{N}$ with $u \leq n$);

(b) $\ddot{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{u}|} + {}_uE_x\ddot{a}_{x+u:\overline{n-u}|}$ (due) and $a_{x:\overline{n}|} = a_{x:\overline{u}|} + {}_uE_x a_{x+u:\overline{n-u}|}$ (for any $u \in \mathbb{N}$ with $u \leq n$) (immediate);

(c) $\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{u/m}|}^{(m)} + {}_{u/m}E_x\ddot{a}_{(x+u/m):\overline{n-u/m}|}^{(m)}$ (due) and
$a_{x:\overline{n}|}^{(m)} = a_{x:\overline{u/m}|}^{(m)} + {}_{u/m}E_x a_{(x+u/m):\overline{n-u/m}|}^{(m)}$ (immediate) (for any $u \in \mathbb{N}$ with $u \leq mn$).

[Intuition 💡:



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Proof: Similar to the proof for proposition 4.7.a. □

4.7.5 Recursions for arithmetically increasing/decreasing life annuities $(I\ddot{a})/(D\ddot{a})$:

Proposition 4.7.c. We have

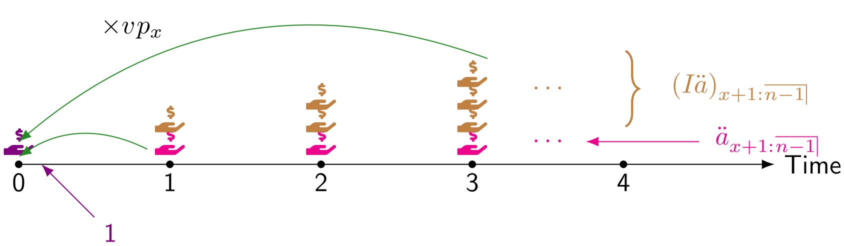
(a) $(I\ddot{a})_{x:\overline{n}|} = 1 + vp_x\left[(I\ddot{a})_{x+1:\overline{n-1}|} + \ddot{a}_{x+1:\overline{n-1}|}\right]$ and $(I\ddot{a})_x = 1 + vp_x[(I\ddot{a})_{x+1} + \ddot{a}_{x+1}]$;

(b) $(D\ddot{a})_{x:\overline{n}|} = n + vp_x(D\ddot{a})_{x+1:\overline{n-1}|}$;

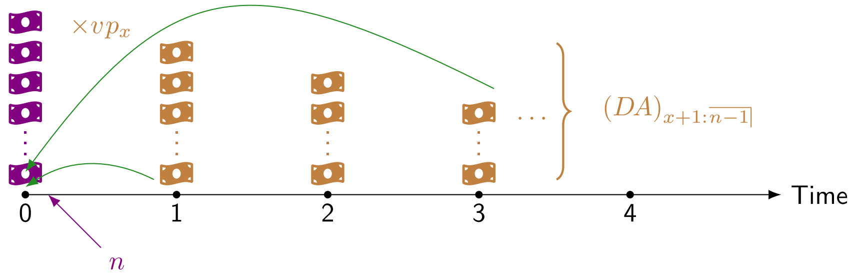
(c) $(I\ddot{a})_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} + vp_x(I\ddot{a})_{x+1:\overline{n-1}|}$ and $(I\ddot{a})_x = \ddot{a}_x + vp_x(I\ddot{a})_{x+1}$.

[Intuition 💡:

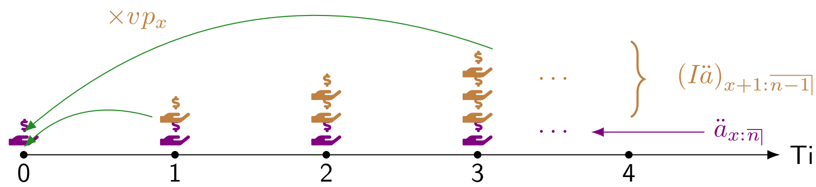
[4.7.5]a:



[4.7.5]b:



[4.7.5]c:



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Proof: Exercise. (The intuition already illustrates the key idea in the proof — one just needs to "split" the terms appropriately.) □

4.8 Relating $\bar{a}$, $\ddot{a}$ and $\ddot{a}^{(m)}$

4.8.1 In a life table, we often only have the values for "$\ddot{a}$"[9] but not for "$\bar{a}$" and "$\ddot{a}^{(m)}$". So, again we are interested in the relationship between them to see how we can get "$\bar{a}$" and "$\ddot{a}^{(m)}$" from "$\ddot{a}$".

4.8.2 To get a nice relationship, again we need to impose UDD assumption (so that proposition 3.6.a can be utilized):

Proposition 4.8.a. Under UDD assumption, we have

(a) $\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$ where $\alpha(m) = \dfrac{id}{i^{(m)}d^{(m)}}$ and $\beta(m) = \dfrac{i - i^{(m)}}{i^{(m)}d^{(m)}}$;

(b) $\bar{a}_x = \dfrac{id}{\delta^2}\ddot{a}_x - \dfrac{i-\delta}{\delta^2}$.

Proof: It suffices to prove the $1/m$thly case since the continuous case can simply be obtained from the $1/m$thly case by letting $m \to \infty$ (note that $\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} d^{(m)} = \delta$).

Now consider:

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1 - \left(i/i^{(m)}\right)A_x}{d^{(m)}} = \frac{i^{(m)} - i(1 - d\ddot{a}_x)}{i^{(m)}d^{(m)}} = \alpha(m)\ddot{a}_x - \beta(m).$$

□

Remarks:

- The result for $1/m$thly case is more often used.
- Sometimes values of $\alpha(m)$ and $\beta(m)$ for different m (at some interest rate i) are given also in a life table, for convenience.
- This result is only for *whole life* annuity, unlike proposition 3.6.a. But we can still get APVs for temporary life and deferred life annuities easily, by expressing them in terms of APVs of whole life annuities. For example:

[9] Usually, "due" quantities, instead of "immediate" quantities, are given in a life table as they are more frequently used.

- n-year temporary life:

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}_x - {}_nE_x\,\ddot{a}_{x+n} = \alpha(m)\ddot{a}_x - \beta(m) - {}_nE_x(\alpha(m)\ddot{a}_{x+n} - \beta(m)) = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x).$$

- deferred whole life:

$$_{u|}\ddot{a}^{(m)}_x = {}_uE_x\,\ddot{a}_{x+u} = {}_uE_x[\alpha(m)\ddot{a}_{x+u} - \beta(m)] = \alpha(m)\,{}_{u|}\ddot{a}_x - \beta(m)\,{}_uE_x.$$

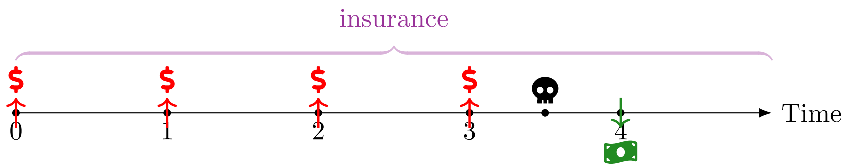
4.9 Incorporating Selection

4.9.1 Again, all previous developments also apply to "select" lives (just change "x" to "$[x]$" in the notations).

5 Premiums

5.0.1 As mentioned in [1.1.4], premiums are charged for providing a life insurance (same for life annuity).

5.0.2 Typically premiums are *series* of *level* payments made at equal interval, as long as the policyholder is alive (possibly with a maximum number of payments to be made). Hence, the premiums can actually be seen as a *life annuity* (but payments are given to the insurer, not to the policyholder).



5.0.3 Due to the importance of premiums to the insurer's profitability, charging suitable amounts of premiums is crucial. If the amounts of premiums are too low, can go bankrupt easily; But if the amounts are too high, would be not "competitive enough" and perhaps lose a lot of clients. Hence, the insurer (more specifically, *actuarial pricing team*) needs to consider many factors when determining the premiums.

5.0.4 A way to determine the premiums (called a **premium principle**) is the *equivalence principle*, which is quite "simple" and makes the calculations mathematically convenient. Based on the equivalence principle, the amounts of (level) premiums are set such that the APV of premiums equals the APV of the benefits (and expenses, if considered). In other words, the expected *present value of future loss at issue* [10] is zero.

[Note: But of course, in the actual practice, determining premiums is far from just merely following a "premium principle" — it is much more complex. To learn more, consider having an actuarial internship in pricing team ☺.]

5.0.5 The present value of future loss at issue (which is a life contingent random variable) serves as an important basis for determining premiums and assessing the insurer's profitability.

5.1 Present Value of Future Loss at Issue

5.1.1 **Present value of future loss at issue**, from the insurer's perspective, can be expressed as

$$\begin{aligned}&\text{PV of "future" benefits outgo}\\&\quad+\text{PV of "future" expenses (if considered)}\\&\quad-\text{PV of "future" premium incomes}\end{aligned}$$

(which is a random variable).

[Warning: One should be careful that, *conventionally*, "future" *insurance* benefits and the related expenses do not include the ones at time 0, while "future" premiums and premium-related expenses do include the ones at time 0.[11]

But for *life annuity* benefit payments and the related expenses, there is not a standard convention, so more specifications are needed in case they appear in this context. (Fortunately it is rather uncommon to encounter such situation. Often we are interested in finding premiums for insurance policies.)]

5.1.2 When the expenses are considered, the present value of future loss at issue is known as **gross loss at issue**. Otherwise it is known as **net loss at issue**.

[10] "Future loss" here means future "net" cash outflows from the *insurer*'s perspective.

[11] This applies similarly in a more general situation (see section 6) where "future" is taken with respect a time t (just replace "time 0" by "time t" here in such case).

5.1.3 The gross (net) loss at issue is denoted by L_0^g (L_0^n resp.).

Remarks:

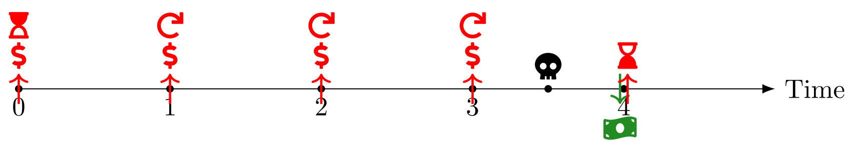
- If there is no risk of ambiguity or one would like to refer to either of gross and net losses at issue simultaneously, one may drop the superscript and just write L_0.
- The "0" indicates the present value is at time 0.

5.1.4 Three common types of expenses:

(a) **initial expenses**: expenses incurred initially (at time of policy issue, i.e., time 0);

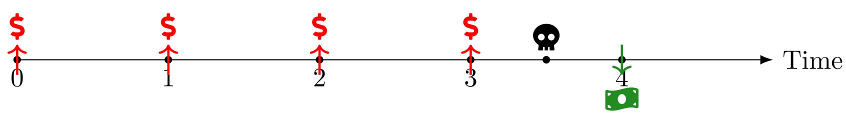
(b) **renewal expenses**: expenses incurred each time a premium (or a life annuity benefit payment) is paid (except the first one, if it is paid at time 0);

(c) **termination expenses** (or **settlement expenses**): expenses incurred when a policy terminates (e.g., death benefit/endowment survival benefit is paid).

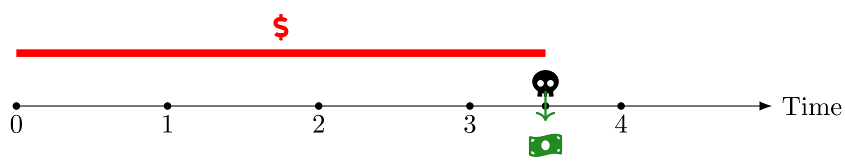


5.1.5 Two commonly seen terminologies (for life insurance):

- **fully discrete**: both death benefit and premiums are made at discrete time points (and so do the associated expenses), where the timing follows the convention in [1.2.8];



- **fully continuous**: death benefit is payable at the moment of death and premiums are payable continuously (and so do the associated expenses).



5.1.6 For the **equivalence principle**,

- the **net premium** (i.e., premium set without considering expenses) is set such that $\mathbb{E}[L_0^n] = 0$, i.e.,

  APV of "future" benefits outgo = APV of "future" premium incomes;

- the **gross premium** (i.e., premium set with expenses considered) is set such that $\mathbb{E}[L_0^g] = 0$, i.e.,

  APV of "future" benefits outgo+APV of "future" expenses = APV of "future" premium incomes.

[Note: Sometimes for *net premium*, its *definition* also states that it is determined via equivalence principle, e.g., Dickson et al. (2019). In such case, there is no need to specify whether equivalence principle is used or not for *net premium*. (For gross premium, one still needs to specify whether equivalence principle is used.)]

5.2 Actuarial Notations

5.2.1 Some actuarial notations are designed for amounts of *net* annual premiums (determined via equivalence principle) for *fully discrete* "standard" policies (i.e., survival/death benefit is of amount 1):

(a) whole life insurance: P_x;

(b) term life insurance: $P^{1}_{x:\overline{n}|}$;

(c) endowment insurance: $P_{x:\overline{n}|}$;

(d) pure endowment: $P_{x:\overset{1}{\overline{n}|}}$.

5.2.2 By equivalence principle and some results in section 4, we can easily get:

(a) $P_x = A_x/\ddot{a}_x = (dA_x)/(1-A_x) = 1/\ddot{a}_x - d$;

(b) $P^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|}/\ddot{a}_{x:\overline{n}|}$;

(c) $P_{x:\overline{n}|} = A_{x:\overline{n}|}/\ddot{a}_{x:\overline{n}|} = (dA_{x:\overline{n}|})/(1-A_{x:\overline{n}|}) = 1/\ddot{a}_{x:\overline{n}|} - d$;

(d) $P_{x:\overset{1}{\overline{n}|}} = A_{x:\overset{1}{\overline{n}|}}/\ddot{a}_{x:\overline{n}|}$.

Remarks:

- By changing $A \to \bar{A}$, $\ddot{a} \to \bar{a}$ and $d \to \delta$ in the formulas above, we can obtain the formulas for the respective *fully continuous* policies. (But there are not specific actuarial notations designed for fully continuous policies.)
- To get the net premiums when the survival/death benefit is of amount S, multiply the corresponding notation by S (since the APV of "future" benefits outgo would just be S times greater).

5.2.3 Apart from the annual net premium case, there are also actuarial notations for the $1/m$thly net premium case (while the insurance benefits are still in the annual case). To get the *total amount* of premiums in each year, we just need to add the superscript "(m)" on each notation introduced in [5.2.1].

[Warning: This means the amount of *each* premium payment (at the beginning of each $1/m$th of a year) is given by the corresponding notation *divided by* m.]

For obtaining formulas analogous to [5.2.2] in this case, we change $\ddot{a} \to \ddot{a}^{(m)}$ and $d \to d^{(m)}$.

5.3 Probability and Variance Calculations for L_0

5.3.1 To calculate probabilities and variances for L_0, generally the first step is to write down an expression for L_0.

5.3.2 For example, for a whole life insurance with sum insured S, termination expense E, initial and renewal expenses e (each), and annual (level) gross premiums G (each), the gross loss at issue is

$$L_0^g = (S+E)v^{K_x+1} - (G-e)\ddot{a}_{\overline{K_x+1}|},$$

or

$$L_0^g = (S+E)Z - (G-e)Y,$$

where Z and Y are the p.v.r.v's for annual whole life insurance and annual whole life annuity-due (with benefits being 1 [each]) respectively.

5.3.3 With the expression of L_0^g, we can easily derive a formula for $\mathrm{Var}\,(L_0^g)$:

Proposition 5.3.a. For the whole life insurance with setting described in [5.3.2], we have

$$\mathrm{Var}\,(L_0^g) = \left(S+E+\frac{G-e}{d}\right)^2 ({}^2A_x - A_x^2)$$

Proof: Since $Y = (1 - Z)/d$, we can actually express L_0^g as:

$$L_0^g = \left(S + E + \frac{G - e}{d}\right)Z - \frac{G - e}{d}.$$

The result then follows. □

5.3.4 By removing all expenses in the setting in [5.3.2] and changing the annual gross premiums G (each) to the annual net premiums SP_x (each) (determined via equivalence principle), we can derive the following expression for net loss at issue:

$$L_0^n = SZ - SP_xY.$$

We can then derive a formula for $\text{Var}(L_0^n)$:

Proposition 5.3.b. For the whole insurance with the setting here, we have

$$\text{Var}(L_0^n) = \left(\frac{S}{1 - A_x}\right)^2 ({}^2A_x - A_x^2).$$

Proof: Note that

$$L_0^n = \left(S + \frac{SP_x}{d}\right)Z - \frac{SP_x}{d} = S\left(1 + \frac{A_x}{1 - A_x}\right)Z - \frac{SP_x}{d} = \frac{S}{1 - A_x}Z - \frac{SP_x}{d}.$$

□

5.3.5 Using similar arguments, one can adapt propositions 5.3.a and 5.3.b for

- endowment insurance: change $A_x \to A_{x:\overline{n}|}$ and ${}^2A_x \to {}^2A_{x:\overline{n}|}$;
- $1/m$thly case (for both insurance and premiums): change $A \to A^{(m)}$ and $d \to d^{(m)}$;
- fully continuous case: change $A \to \bar{A}$ and $d \to \delta$.

[Note: This is possible mainly because under each of these cases, we have a similar relationship between the p.v.r.v.'s Z and Y, and a similar expression for "P". So, the proofs can go through with only a little change.]

5.3.6 An alternative method for calculating variance is the so-called *first principle* approach, which is sometimes useful for *short-term* fully discrete insurance policies. The steps are as follows:

(a) Identify all possible values of L_0 with associated probabilities. [Note: A typical approach to find the associated probabilities is to "convert" each event $\{L_0 = j\}$ to another event $\{K_x = k\}$ which has *the same probability*. (Finding the probability for the latter event is simple.)]

(b) Calculate the first and second (raw) moments of L_0:

- $\mathbb{E}[L_0] = \sum \text{value} \times \text{probability}$ (which is always 0 if equivalence principle is used);
- $\mathbb{E}\left[L_0^2\right] = \sum \text{value}^2 \times \text{probability}$.

(c) Calculate the variance

$$\text{Var}(L_0) = \mathbb{E}\left[L_0^2\right] - \mathbb{E}[L_0]^2.$$

5.3.7 Now for calculating probabilities for L_0, we often also use the "conversion" approach suggested above: "converting" the event of interest in terms of L_0 to another event in terms of K_x or T_x which has the same probability.

5.3.8 Another approach is to *approximate* the probabilities for L_0 through *normal approximation.* If L_0 is loss at issue for a "large block" of independent policies (i.e., a sum of independent losses at issue for "many" policies), then $L_0 \overset{\text{approx}}{\sim} N(\mathbb{E}[L_0], \operatorname{Var}(L_0))$.

5.3.9 Then for the normal approximation, we *treat* L_0 as such a normally distributed random variable, and finding the approximated probabilities by standardization, i.e., "converting" to the event of interest involving L_0 to another event involving $\dfrac{L_0 - \mathbb{E}[L_0]}{\sqrt{\operatorname{Var}(L_0)}} \overset{\text{this situation}}{\sim} N(0,1)$ which has the same probability.

6 Policy Values

6.0.1 After a life insurance policy is sold, the insurance company is not going to completely ignore it. Indeed, to ensure that "puts aside" (or *reserves*) sufficient money for paying the potential future benefits and expenses, should monitor the sold policies periodically. (In practice, this is often required by regulators to make sure has a good "financial health" .)

6.0.2 To monitor the "status" of the policies sold, the insurer (more specifically, *actuarial valuation team*) would periodically carry out *valuation* on those policies (based on the most "updated" information). A fundamental quantity of interest is how much money the insurer needs to reserve for a given policy, at a specified time point.

6.0.3 Here we shall discuss a relatively "simple" framework for determining such reserve, which revolves around the concept of *policy value*.

[Note: Again, the process is much more sophisticated in the actual practice, and readers may consider having an actuarial internship in valuation team to learn more about it ☺.]

6.1 Present Value of Future Loss at Time t

6.1.1 To define policy value, we need to first introduce **present value of future loss at time t**, which has a definition analogous to the present value of future loss at issue (indeed this is a generalization to it):

"PV" of "future" benefits outgo @ time t
+ "PV" of "future" expenses (if considered) @ time t
− "PV" of "future" premium incomes \$ @ time t,

and it is only defined when the policy is *in force* (i.e., is not terminated) at time t.

Remarks:

- Here the "in force" condition is added since it does not make much sense to talk about "future" benefits/expenses/premium incomes when they *do not even exist* ("no future" for a terminated policy)!
- We shall assume here that no policy lapse is possible. So the "in force" condition is equivalent to the life insured is still alive at time t ($T_x > t$).
- "PV" @ time t means present value with time t (instead of time 0) treated as "present". More specifically, we convert "T_x" in the expression for loss at issue (in [5.1.1]) to "$T_x - t$" (the "future lifetime" is taken with respect to time t) to get the expression here.
- Since the expression here is only defined when $T_x > t$, it is natural to identify its distribution as the *conditional distribution* of the expression involving $T_x - t$ *given* $T_x > t$ (which is assumed to be the same as the *unconditional* distribution of T_{x+t}). So, we can replace "$T_x - t$" in the expression by T_{x+t} (and drop the extra condition) without affecting the distribution.
- As we are almost always only interested in studying the *distribution* of the expression, writing the expression in terms of either "$T_x - t$" (with the condition) or "T_{x+t}" (with the condition dropped) is fine. ⭐Here, we shall write the expression in terms of T_{x+t} (with the condition dropped) for "simplicity". (Note that this means the curtate lifetime random variables ("K") would also have subscript "$x + t$".)

6.1.2 Likewise, when the expenses are considered, the present value of future loss at time t is known as **gross loss at time t**. Otherwise it is known as **net loss at time t**.

6.1.3 The gross (net) loss at time t is denoted by L_t^g (L_t^n resp.).

Remarks:

- Again, if there is no risk of ambiguity or one would like to refer to either of gross and net losses at time t simultaneously, one may drop the superscript and just write L_t.

- The "t" indicates the present value is at time t.

6.1.4 The **net premium policy value at time** t and **gross premium policy value at time** t, denoted by ${}_tV^n$ and ${}_tV^g$ respectively, are given by (when the policy is in force at time t)

$$ {}_tV^n = \mathbb{E}[L_t^n] \quad \text{and} \quad {}_tV^g = \mathbb{E}[L_t^g], $$

that is,

$$ \begin{aligned} {}_tV^n = {} & \text{"APV" of "future" benefits outgo @ time } t \\ & - \text{"APV" of "future" premium incomes @ time } t, \end{aligned} $$

and

$$ \begin{aligned} {}_tV^g = {} & \text{"APV" of "future" benefits outgo @ time } t \\ & + \text{"APV" of "future" expenses @ time } t \\ & - \text{"APV" of "future" premium incomes @ time } t. \end{aligned} $$

[Intuition : The definition is "similar" to the definition for "ordinary" value. Value of a "nonrandom" project at time t may be defined as the present value[12] of present/future (nonrandom) net cash (in)flows at time t . This is also the amount of received when the project is "fairly sold" at time t (intuitively, this is its "worth" at time t).

On the other hand, the policy value at time t is the *actuarial* present value of "future" cash *outflows* (from the insurer's perspective). It may be understood intuitively (and informally) as "amount of *paid* (by) when the policy is 'fairly transferred' to another party ". As a "simple" framework for reserving, this amount may be treated as the amount of reserve held at time t.]

Remarks:

- If there is no risk of ambiguity or one would like to refer to either of gross and net premium policy values at time t simultaneously, one may drop the superscript and just write ${}_tV$.
- For an n-year term policy, technically ${}_nV$ is *undefined* by definition (as the policy terminates precisely at time n). But still it is common to use the notation ${}_nV$ to refer to the *limit* $\lim_{t\to n^-} {}_tV$ (the value approached as $t < n$ gets closer and closer to n).

6.1.5 Boundary values of ${}_tV$:

- ${}_0V^n = 0$ if net premium is determined via equivalence principle;
- ${}_0V^g = 0$ if (i) gross premium is determined via equivalence principle, and (ii) policy value basis and premium basis are the same; [Note: **Basis** means a set of assumptions. This means the sets of assumptions used for calculating policy value and premium are the same.]
- $$ {}_nV = \begin{cases} 0 & \text{for } n\text{-year term life insurance;} \\ S + E & \text{for } n\text{-year endowment insurance,} \end{cases} $$

  where S is the sum insured (amount of death/survival benefit) and E is the termination expense.

[Note: The boundary values are useful for *recursions* (see section 6.3).]

[12] The interest rate used is "risk-free rate".

6.2 Calculations of Policy Values and Variances of L_t

6.2.1 We shall again consider the whole life insurance in [5.3.2]. The gross loss at time t is

$$L_t^g = (S+E)v^{K_{x+t}+1} - (G-e)\ddot{a}_{\overline{K_{x+t}+1}|}.$$

or

$$L_t^g = (S+E)Z - (G-e)Y,$$

where Z and Y are the p.v.r.v's for annual whole life insurance and annual whole life annuity-due (with benefits being 1 [each]) issued to $(x+t)$ respectively.

6.2.2 With the expression of L_t^g, we readily get the following formula for ${}_tV^g$:

$${}_tV^g = (S+E)A_{x+t} - (G-e)\ddot{a}_{x+t}.$$

6.2.3 Likewise we can get the expression for net loss at time t by removing all expenses (the annual net premiums are SP_x each, determined by equivalence principle):

$$L_t^n = SZ - SP_xY = SZ - \frac{SA_x}{\ddot{a}_x}Y.$$

6.2.4 We can then obtain the following formulas for ${}_tV^n$:

Proposition 6.2.a. For the whole life insurance with setting described in [5.3.2], we have

$${}_tV^n = S\left(1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right) = S\left(\frac{A_{x+t} - A_x}{1 - A_x}\right).$$

Proof: Firstly, we have

$${}_tV^n = SA_{x+t} - SP_x\ddot{a}_{x+t} = S\left[1 - d\ddot{a}_{x+t} - \left(\frac{1}{\ddot{a}_x} - d\right)\ddot{a}_{x+t}\right] = S\left(1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right).$$

Next, we have

$${}_tV^n = SA_{x+t} - SP_x\ddot{a}_{x+t} = S\left[\frac{A_{x+t}(1-A_x)}{1-A_x} - \frac{dA_x}{1-A_x}\cdot\frac{1-A_{x+t}}{d}\right] = S\left(\frac{A_{x+t}-A_x}{1-A_x}\right).$$

$\square$

6.2.5 For the variance formulas, using similar arguments as the proofs for propositions 5.3.a and 5.3.b, we get:

- $\mathrm{Var}\left(L_t^g\right) = \left(S + E + \dfrac{G-e}{d}\right)^2 ({}^2A_{x+t} - A_{x+t}^2)$;
- $\mathrm{Var}\left(L_t^n\right) = \left(\dfrac{S}{1-A_x}\right)^2 ({}^2A_{x+t} - A_{x+t}^2)$.

6.2.6 Using similar arguments, we can again adapt proposition 6.2.a and formulas in [6.2.5] for

- endowment insurance: change
  - $A_x \to A_{x:\overline{n}|}$ and $\ddot{a}_x \to \ddot{a}_{x:\overline{n}|}$;
  - $A_{x+t} \to A_{x+t:\overline{n-t}|}$ and $\ddot{a}_{x+t} \to \ddot{a}_{x+t:\overline{n-t}|}$;
  - ${}^2A_{x+t} \to {}^2A_{x+t:\overline{n-t}|}$;
- $1/m$thly case (for both insurance and premiums): change $\ddot{a} \to \ddot{a}^{(m)}$, $A \to A^{(m)}$, and $d \to d^{(m)}$;
- fully continuous case: change $\ddot{a} \to \bar{a}$, $A \to \bar{A}$, and $d \to \delta$.

6.3 Annual Recursion

6.3.1 Again there are two main reasons for studying recursions of policy values here:

(a) It provides insight on how the policy values change over time.

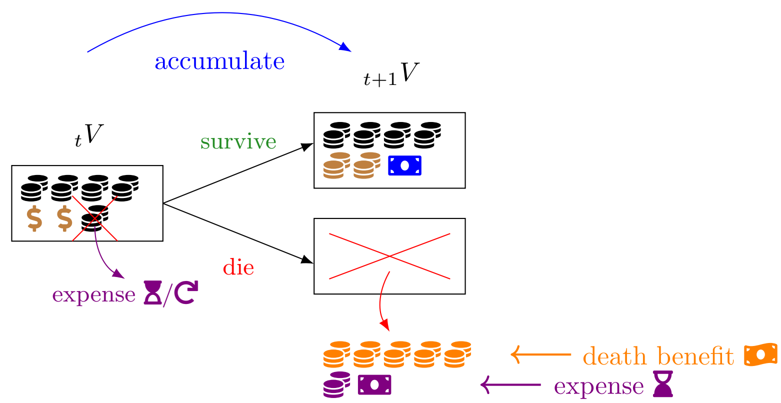
(b) It allows quick computations based on limited amount of information.

6.3.2 To intuitively understand recursions for policy values, it would be helpful to regard policy value at time t as the amount of reserve held at time t (i.e., work in the "simple" framework for reserving).

6.3.3 Here we shall focus on *annual* recursion. For other kinds of recursions and further topics, see STAT3909.

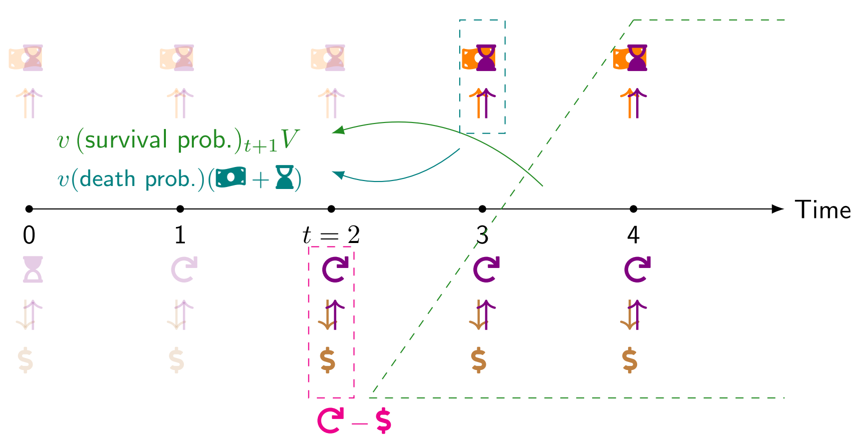
6.3.4 An "intuition-based" annual recursion formula for policy values (insurance) is as follows (where $t \in \mathbb{N}_0$):

$$\begin{aligned}&({}_tV + \text{premium } \$ - \text{initial/renewal expense } /)(1+i)\\ &\quad = {}_{t+1}V(\text{survival prob.}) + (\text{death benefit } + \text{termination expense })(\text{death prob.}).\end{aligned}$$



<u>Proof</u>:

(actuarial discounting)



[Note: We treat cash outflow as positive and cash inflow as negative, since we are dealing with *losses* for policy values (so we consider net cash *out*flows).]

The figure above illustrates the key idea in the proof ("splitting" the cash flows appropriately). Using this idea, we get:

$${}_tV = -\$ + v(\text{death prob.})(+) + v(\text{survival prob.})\,{}_{t+1}V.$$

When $t=0$, "" is replaced by "". So the result follows, by rearranging the terms. □

6.4 Retrospective Policy Value

[Note: This topic is not inside SOA exam FAM syllabus currently.]

6.4.1 Recall that the policy value at time t is defined as the actuarial present value of future loss at time t. Sometimes this policy value is called **prospective policy value** since the definition is inherently forward looking and prospective.

6.4.2 Now we are interested in studying another approach for determining policy value: *retrospective approach*. Loosely, it calculates the *retrospective policy value*, which is the "actuarial accumulated value" of past gain.

[Note: The retrospective policy value is kind of a "dual" of the prospective policy value:

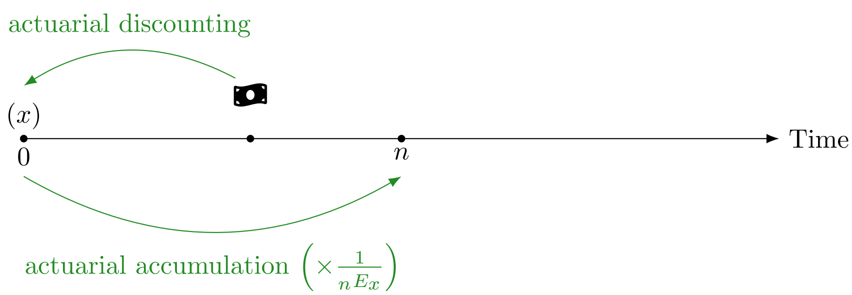
- actuarial accumulated value ↔ actuarial present value;
- past ↔ future;
- gain ↔ loss.

]

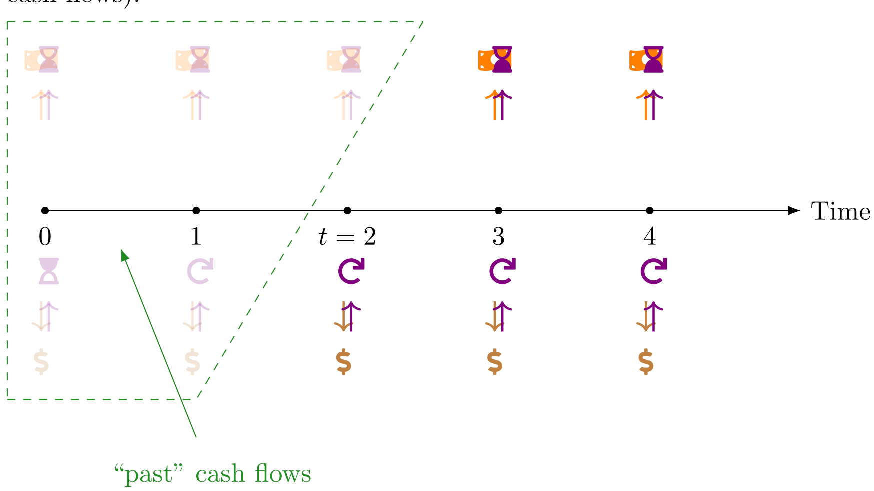
6.4.3 Before proceeding further, we should define what "actuarial accumulated value" is. Recall that actuarial present value can be obtained through actuarial discounting ("multiply ${}_nE_x$" to actuarially discount back n years).

Following the same spirit, actuarial accumulated value is defined such that an analogous "actuarial accumulation" works. The **actuarial accumulated value** (AAV) of a cash flow (for aged x at time 0), at time n, is

$$\text{APV of (at time 0)} \times \frac{1}{{}_nE_x}.$$



6.4.4 It is natural to regard the pale cash flows below as "past" cash flows (serving as "dual" for "future" cash flows):



For "past" *gain*, we should treat cash inflow as positive and cash outflow as negative. It is negative of "past" loss.

6.4.5 Fix any time $t \in \mathbb{N}_0$. Let $L_{0,t}$ be the present value of the "past" loss (at time 0). Then the **retrospective policy value** at time t, denoted by ${}_tV^R$, is given by

$$ {}_tV^R = \frac{-\mathbb{E}[L_{0,t}]}{{}_tE_x} $$

(AAV of the "past" gain).

6.4.6 Naturally we would like to know when the prospective and retrospective policy values at the same time point would coincide. (We are more interested in studying retrospective policy values in such case.) The following result suggests sufficient conditions for that:

Proposition 6.4.a. Let ${}_tV^P$ and ${}_tV^R$ be the prospective and retrospective policy values at time t respectively. Then, we have ${}_tV^P = {}_tV^R$ if

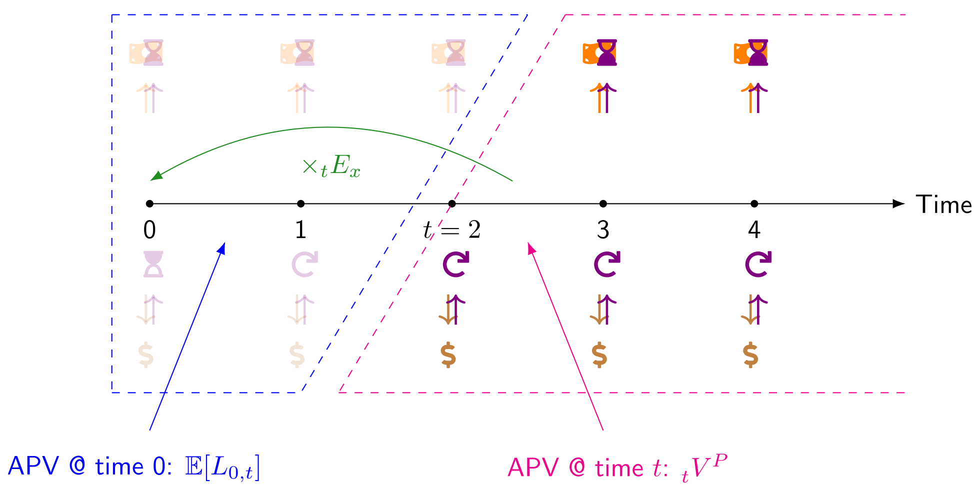
(a) the premiums are determined by equivalence principle;

(b) premium, prospective policy value, and retrospective policy value bases are the same.

Proof: Assume the two conditions hold. Firstly, by equivalence principle we have ${}_0V^P = 0$. Now, as the bases are the same, we have

$$ \underbrace{{}_0V^P}_{0} = \mathbb{E}[L_{0,t}] + {}_tE_x\,{}_tV^P $$

by "splitting" the cash flows into "past" and "future" portions, and "actuarial discounting".



The result then follows by rearranging the terms. $\square$

References

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A., & Nesbitt, C. J. (1997). *Actuarial mathematics.* Society of Actuaries.

Dickson, D. C., Hardy, M. R., & Waters, H. R. (2019). *Actuarial mathematics for life contingent risks.* Cambridge University Press.

Concepts and Terminologies

Results

[⚠ Warning: Do <u>not</u> just *regurgitate* all the results! Try your best to *understand* (most of) them intuitively!]

Section 2

- proposition 2.1.a: "(0) survives x years and then survives for t more years"
- proposition 2.1.b: "(x) survives t years and then survive for u more years"
- proposition 2.1.c: necessary and sufficient conditions for being a valid survival function
- proposition 2.2.a: relationship between μ_t and $f_0(t)$
- proposition 2.2.b: relationship between μ_{x+t} and $f_x(t)$
- proposition 2.2.c: integral formula for $S_x(t)$
- [2.2.14]: distribution and survival function for De Moivre's Law
- [2.2.15]: survival function for De Moivre's Law
- proposition 2.3.a: various properties concerning actuarial notations
- proposition 2.3.b: integral formula for ${}_tq_x$
- proposition 2.4.a: formula for $\mathring{e}_x$
- proposition 2.4.b: formula for $\mathbb{E}\left[T_x^2\right]$
- proposition 2.4.c: formula for $\mathring{e}_{x:\overline{n}|}$
- proposition 2.5.a: connection between T_x and K_x
- corollary 2.5.b: connection between T_x and K_x with inequalities
- proposition 2.5.c: pmf of K_x
- proposition 2.5.d: formula for e_x
- proposition 2.5.e: formula for $\mathbb{E}\left[K_x^2\right]$
- proposition 2.5.f: formula for $e_{x:\overline{n}|}$
- proposition 2.6.a: recursion formulas for e_x's and $\mathring{e}_x$'s
- lemma 2.7.a: relationship between ℓ_x and ℓ_0
- proposition 2.7.b: relationship between ℓ_x's and ${}_tp_x$
- proposition 2.7.c: relationship between various life table functions and probabilities $(q_x, {}_nd_x, \ell_x$ etc.)
- [2.8.5]a: property that "q" probability only depends on the length of time interval covered under UDD
- [2.8.5]b: formula for μ_{x+t} in terms of q_x under UDD
- lemma 2.8.a: an equivalent definition for UDD
- proposition 2.8.b: relationship between mean and variance of T_x and K_x under UDD
- proposition 2.8.c: key property for constant force of mortality
- proposition 2.10.a: formulas for constructing select life tables

Section 3

- [3.0.8]: "intuition-based" general APV calculation formula for insurance/life annuity
- [3.1.8]: formulas for continuous whole life insurance
- [3.1.10]: formulas for discrete whole life insurance (annual case)
- [3.1.15]: formulas for discrete whole life insurance ($1/m$thly case)
- [3.2.2]: formulas for pure endowment
- proposition 3.2.a: shortcut formulas for pure endowment
- [3.2.5]: formulas for continuous term life insurance
- [3.2.6]: formulas for discrete term life insurance (annual case)
- [3.2.7]: formulas for discrete term life insurance ($1/m$thly case)
- [3.2.11]: formulas for continuous endowment insurance
- [3.2.12]: formulas for discrete endowment insurance (annual case)
- [3.2.13]: formulas for discrete endowment insurance ($1/m$thly case)
- proposition 3.2.b: alternative formula for variance for endowment insurance
- [3.4.9]: formulas for APVs of arithmetically increasing/decreasing n-year term life insurances
- proposition 3.4.a: alternative formulas for APVs of arithmetically decreasing insurances
- proposition 3.4.b: formula for APV of a geometrically increasing/decreasing n-year term life insurance (annual case)
- proposition 3.5.a: recursion formulas for whole life insurance
- proposition 3.5.b: recursion formulas for term life insurance
- proposition 3.5.c: recursion formulas for endowment insurance
- proposition 3.5.d: recursion formulas for arithmetically and annually increasing/decreasing insurances
- proposition 3.6.a: relationship between $\bar{A}$, A and $A^{(m)}$ under UDD assumption

Section 4

- [4.1.3]: formulas for continuous whole life annuity
- [4.1.4]: formulas for discrete whole life annuity (annual case)
- [4.1.5]: formulas for discrete whole life annuity ($1/m$thly case)
- proposition 4.1.a: due-immediate formulas for discrete whole life annuity
- [4.2.2]: formulas for continuous temporary life annuity
- [4.2.3]: formulas for discrete temporary life annuity (annual case)
- [4.2.4]: formulas for discrete temporary life annuity ($1/m$thly case)
- [4.3.4]: formulas for continuous deferred whole life annuity
- [4.3.5]: formulas for discrete deferred whole life annuity (annual case)

- [4.3.6]: formulas for discrete deferred whole life annuity ($1/m$thly case)
- [4.4.3]: formulas for continuous certain-and-life annuity
- [4.4.4]: formulas for discrete certain-and-life annuity (annual case)
- [4.4.5]: formulas for discrete certain-and-life annuity ($1/m$thly case)
- [4.5.8]: formulas for complete $1/m$thly life annuity-immediate
- [4.5.9]: formulas for apportionable $1/m$thly life annuity-due
- [4.6.5]: formulas for APVs of arithmetically increasing/decreasing n-year temporary life annuities
- proposition 4.6.a: formulas for APVs of geometrically increasing/decreasing n-year temporary life annuity (annual case)
- proposition 4.7.a: recursion formulas for whole life annuity
- proposition 4.7.b: recursion formulas for temporary life annuity
- proposition 4.7.c: recursion formulas for arithmetically and annually increasing/decreasing life annuities
- proposition 4.8.a: relationship between $\bar{a}_x$, $\ddot{a}_x$ and $\ddot{a}_x^{(m)}$ under UDD assumption

Section 5

- [5.2.2]: formulas for net premiums under various special cases
- proposition 5.3.a: formula for $\mathrm{Var}\left(L_0^n\right)$ for fully discrete whole life insurance
- proposition 5.3.b: formula for $\mathrm{Var}\left(L_0^g\right)$ for fully discrete whole life insurance
- [5.3.6]: procedure for calculating $\mathrm{Var}\left(L_0\right)$ by first principle
- [5.3.7]: approach for calculating probabilities for L_0
- [5.3.8]: approach for approximating probabilities for L_0

Section 6

- [6.1.5]: boundary values of ${}_tV$
- [6.2.2]: formula for ${}_tV^g$ for fully discrete whole life insurance
- proposition 6.2.a: formulas for ${}_tV^n$ for fully discrete whole life insurance
- [6.2.5]: formulas for $\mathrm{Var}\left(L_t\right)$
- [6.3.4]: annual recursion formula for policy values
- proposition 6.4.a: sufficient conditions for the equality of the prospective and retrospective policy values