HKU STAT2902 Study Notes

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[Note: Related SOA Exam: FM]

Contents

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1 The Measurement of Interest

1.1 Origin of Interest

1.1.1 We start with two people: 👤 and 🚶. Suppose 👤 lends his 🍎 to 🚶.

1.1.2 Terminologies:

- 👤: **lender**
- 🚶: **borrower**
- 🍎: **capital** (owned by lender 👤)

1.1.3 In order to provide *incentive* for the lender 👤 to lend his 🍎 to 🚶, the borrower 🚶 needs to give something to 👤, which is known as **interest** → main thing to be studied in this notes!

1.1.4 Usually, both capital and interest are expressed in terms of *money* 💲, which makes the calculations convenient.

1.2 Terminologies for Investment

1.2.1 Here *investment* is in the broad sense of putting aside (*invest*/*lend*) a sum of money 🪙 (*initial investment*) into an "investment fund" 🐖 (playing the role of "borrower") to get more money 💲 (*interest*) in return (amount of 💲 in 🐖 grows over time).

[Note: The potential of earning more money in return by putting aside money for some time is sometimes called **time value of money**.]

1.2.2 For the investment fund 🐖, we shall impose the following assumptions:

(a) There is neither *contribution* nor *withdrawal* after the initial investment, so that changes in "value" of (amount of money in) the fund 🐖 are only due to *interest*. [Note: In section 3.3, this assumption will be relaxed, and there we will develop some ways to deduce "interest rate" with possibly intermediate contributions and withdrawals.]

(b) The "growth rate" for money in the fund 🐖 is constant over time (regardless of the amount in the fund) and is independent from the amount and timing of initial investment.

1.2.3 **Principal** is the amount of initial investment made now (at time 0).

1.2.4 **Accumulated value** (AV) at time t is the total amount (principal + extra money in return) received when we withdraw all money from the fund 🐖 at time t .

Remarks:

- Accumulated value at time 0 = principal, since we simply receive back our initial investment if we withdraw immediately from 🐖 at time 0 (there is no opportunity for "growth"!).
- Sometimes we just simply call "accumulated value at time t" as "value at time t". (There is another kind of value; See section 1.4.)

1.2.5 **Amount of interest** (between now and time t) is the extra money in return (when withdrawing from 🐖 at time t), which is time-t accumulated value − principal.

1.2.6 **Measurement period** is the time *unit* for "time-related" phrase like "time t", "tth period" etc.

[Note: Unless otherwise specified, we assume the measurement period is *years* here → e.g., "1 period" = "1 year".]

1.2.7 **Accumulation function** (denoted by $a(t)$) gives the accumulated value of principal of 1 at time t, for any $t \geq 0$.

[Note: Special case: $a(0)$ = principal = 1.]

1.2.8 **Amount function** (denoted by $A(t)$) gives the accumulated value of principal of k at time t, for any $t \geq 0$.

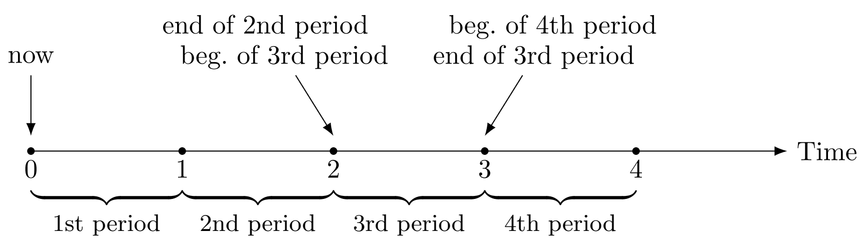
Remarks:

- Special case: $A(0) =$ principal $= k$.
- We have $A(t) = ka(t)$ for any $t \geq 0$ as the "growth rate" is independent from the amount of initial investment.

1.2.9 **nth period** is the time period from time $n-1$ to time n, for any $n \in \mathbb{N}$.

[⚠ Warning: It is not the time period from time n to time $n+1$!]

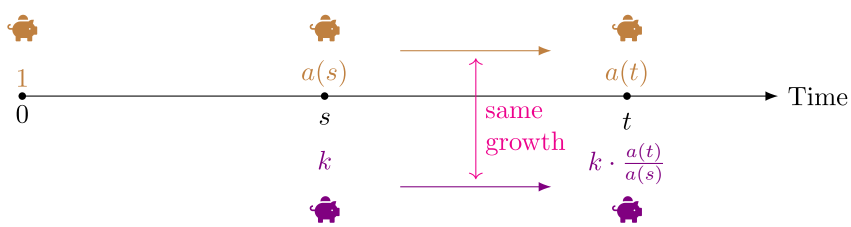
Furthermore, time n $(n-1)$ is called the **end of nth period** (**beginning of nth period**)



1.2.10 The **amount of interest earned during nth period** (denoted by I_n) is the increase in accumulated value during nth period (accumulated value at the end of nth period subtracted by that at the beginning of nth period):

$$I_n = A(n) - A(n-1).$$

1.2.11 Time-t accumulated value for an initial investment made at time $s > 0$ (where $t \geq s$) is $k \cdot \dfrac{a(t)}{a(s)}$.



1.3 Interest Rates

1.3.1 Interest rate describes the amounts of interest $ earned during different periods, and there are different kinds of interest rates.

1.3.2 **Effective interest rate for nth period** (denoted by i_n) is the ratio of interest earned during nth period to the accumulated value at the *beginning* of nth period, i.e. $I_n/A(n-1)$.

1.3.3 The effective interest rate i_n can be expressed in the following forms:

- $i_n = \dfrac{A(n) - A(n-1)}{A(n-1)}$ or $A(n) = A(n-1)(1+i_n)$ (in terms of $A(\cdot)$'s)
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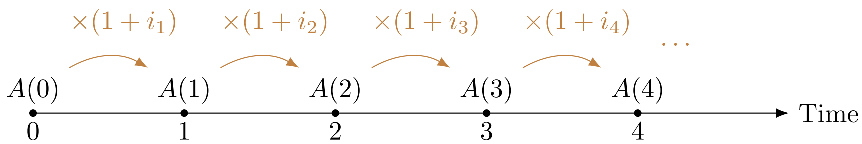


1.3.4 For *simple interest*, the amount of interest earned during each period is a constant:

$$\underbrace{\text{principal}}_{A(0)} \times \underbrace{\text{simple interest rate}}_{i}.$$

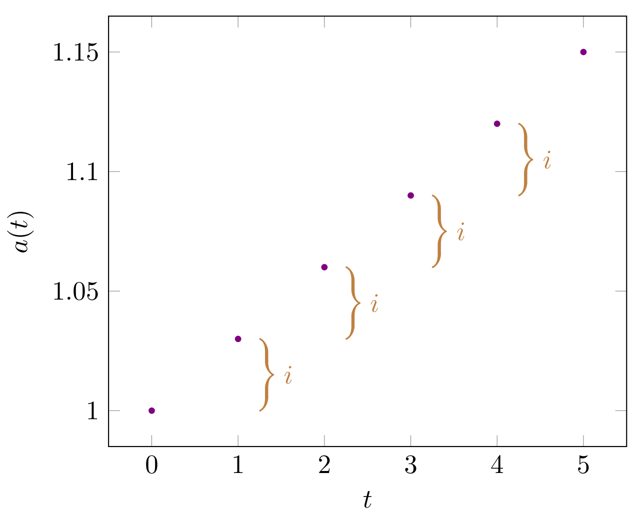
1.3.5 Now fix a measurement period (time unit). Since a constant amount of interest ($A(0)i$) is earned during each period, we can deduce the accumulated value at the end of each period:

- $A(1) = A(0) + A(0)i = A(0)(1+i)$
- $A(2) = \underbrace{A(1)}_{A(0)(1+i)} + A(0)i = A(0)(1+2i)$
- $A(3) = \underbrace{A(2)}_{A(0)(1+2i)} + A(0)i = A(0)(1+3i)$
- etc.

1.3.6 Generally, we have

- $A(t) = A(0)(1+it) = k(1+it)$ for any $t \in \mathbb{N}_0$[1];
- special case ($k = 1$): $a(t) = 1 + it$ for any $t \in \mathbb{N}_0$.

$a(t) = 1 + it$ for any $t \in \mathbb{N}_0$



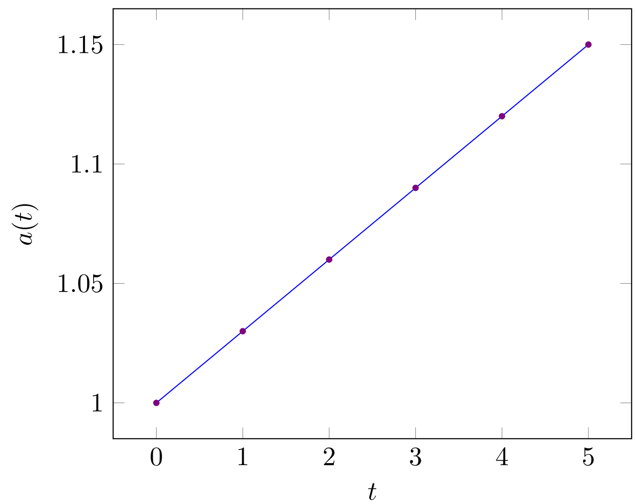
<u>Remarks</u>:

- The plot only contains discrete dots since only values at integer time points are deduced.
- When the measurement period chosen is shorter, the deduced values would be more "close in time" (but they are still "discrete" in nature).

1.3.7 We can observe that the dots in the plot are kind of "linear", so it seems to be natural to join them by a straight line:

[1] $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

$a(t) = 1 + it$ for any $t \geq 0$



From this we obtain the general definition of **simple interest**:

$$a(t) = 1 + it$$

for any $t \geq 0$, where i is the **simple interest rate** ➔ amount of interest earned during any time interval is *proportional* to the length of interval.

1.3.8 For *compound interest*, the amount of interest earned during each period is given by

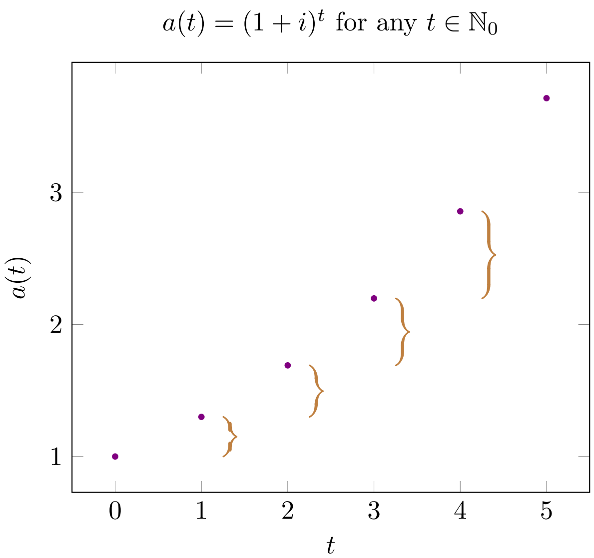
$$\text{accumulated value at the beginning of period} \times \underbrace{\text{compound interest rate}}_{i}.$$

1.3.9 Similarly, we can deduce the accumulated value at the end of each period:

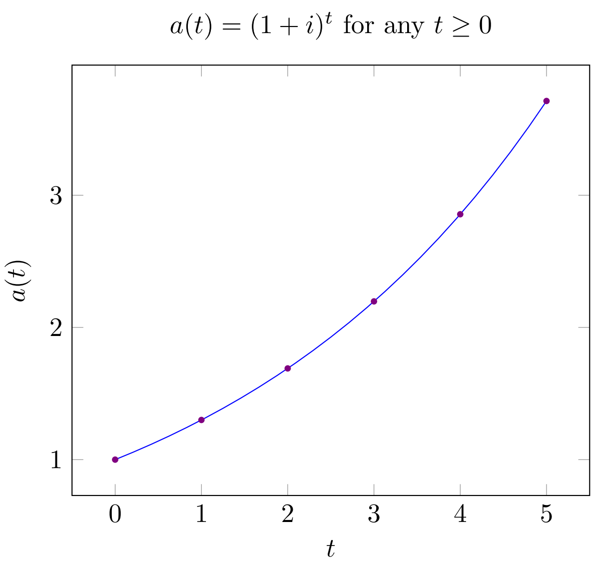
- $A(1) = A(0) + A(0)i = A(0)(1+i)$
- $A(2) = A(1) + A(1)i = \underbrace{A(1)}_{A(0)(1+i)}(1+i) = A(0)(1+i)^2$
- $A(3) = A(2) + A(2)i = \underbrace{A(2)}_{A(0)(1+i)^2}(1+i) = A(0)(1+i)^3$
- etc.

1.3.10 Generally, we have

- $A(t) = A(0)(1+i)^t = k(1+i)^t$ for any $t \in \mathbb{N}_0$;
- special case ($k = 1$): $a(t) = (1+i)^t$ for any $t \in \mathbb{N}_0$.



1.3.11 Since the dots in the plot are kind of "exponential", it is natural to join them by an exponential function:



This gives the general definition of **compound interest**:

$$a(t) = (1+i)^t$$

for any $t \geq 0$, where i is the **compound interest rate**.

1.3.12 A remarkable property for compound interest is that *effective* interest rate and *compound* interest rate coincide: Under compound interest with compound interest rate i, the effective interest rate

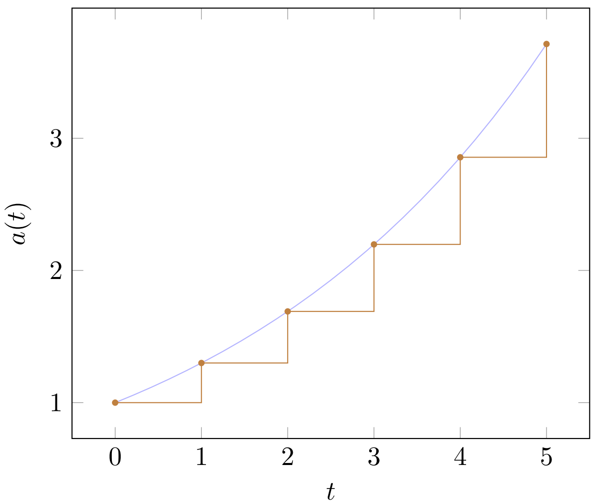
$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = 1 + i - 1 = i.$$

for any $n \in \mathbb{N}$.

Remarks:

- Because of this "nice" property, compound interest is often used.
- Here, unless otherwise specified, we shall assume compound interest, and use the notation i to denote the constant (annual) compound interest rate (which is also the effective interest rate for any year).

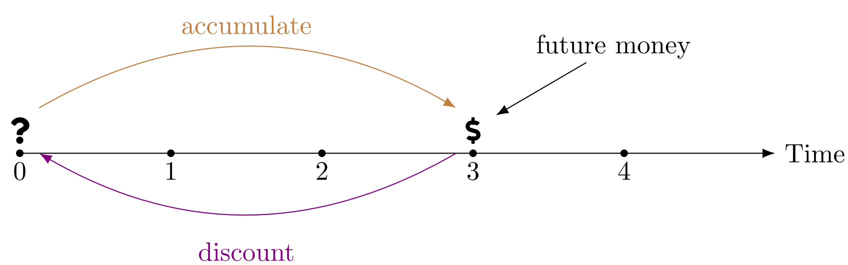
1.3.13 In practice, interest (amount for whole period) is often only paid at the end of each period → "jumps" of amount of $ in at discrete time points, rather than in a "continuous manner" (interests paid/added to the fund simultaneously as they are earned → amount of $ in grows "continuously").



1.4 Present Value

1.4.1 *Present value* captures the idea of "current worth of future money". This concept is useful for answering questions like "How much money do we need to invest in now to have an accumulated value of 1000 at time 5?"

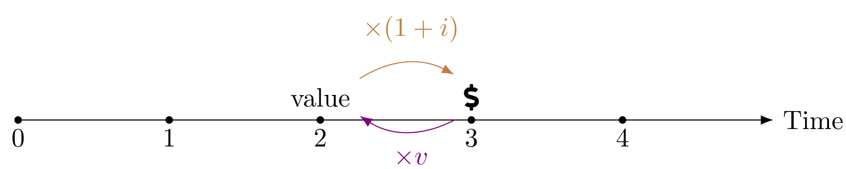
1.4.2 To find out the current worth, generally *discounting* needs to be performed, which can be understood as the "reverse process" of accumulating (i.e., earning interest over time).



1.4.3 Given an effective interest rate i, the **discount factor** (denoted by v) is given by $v = 1/(1+i)$.

[Note: Multiplying the discount factor v = dividing by $1+i$ = reverse process of multiplying by $1+i$ (accumulating for one period) → "discount" for one period.]

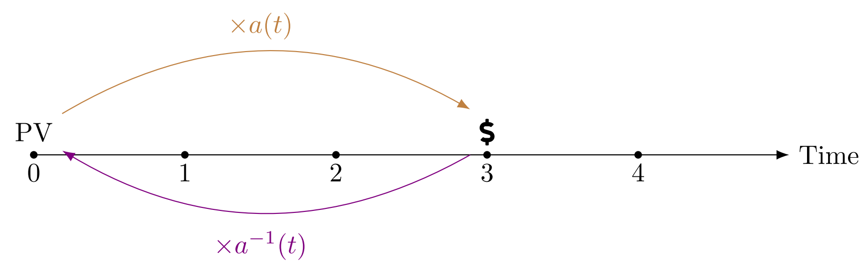
1.4.4 Multiplying the discount factor v to an amount $ (or cash inflow of that amount) at the end of a period gives its *value* (worth) at the beginning of the period (i.e., the amount needed at that time in order to accumulate to $ at the end of the period).



1.4.5 The *time-0 value* of an amount $\$$ is known as its **present value** (PV).

1.4.6 More generally, we can discount for any number of periods. A "more general" discount factor is given by the inverse function $a^{-1}(t)$ → discounting for t periods: discount an amount $\$$ at time t back to time 0.

[Note: Still, this discounting serves as the reverse process of accumulating (for t years). This applies similarly to other cases (discounting from a time t to another time $s < t$ ↔ reverse process of accumulating from time s to time t etc.)]



1.4.7 The expressions for $a^{-1}(t)$ under simple and compound interest are respectively:

- simple interest: $a^{-1}(t) = 1/(1+it)$;
- compound interest: $a^{-1}(t) = (1+i)^{-t} = v^t$ (frequently used! ⭐)

1.5 Discount Rates

1.5.1 *Discount rate*, as its name suggests, is somewhat related to the process of discounting.

[⚠ Warning: Discount *rate* and discount *factor* are not the same!]

1.5.2 Discount rate may be seen as "dual" of interest rate, and we also have three kinds of discount rates:

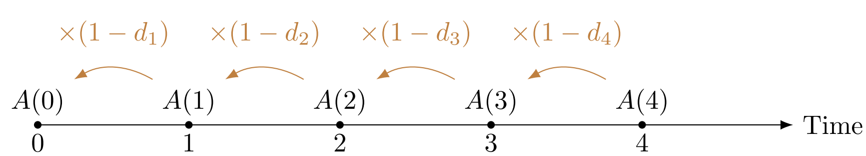
- effective discount rate ↔ effective interest rate
- simple discount rate ↔ simple interest rate
- compound discount rate ↔ compound interest rate

[Note: Nonetheless, there are some subtleties in developing discount rates that do not present in the development of interest rates.]

1.5.3 **Effective discount rate for nth period** (denoted by d_n) is the ratio of interest earned (or "discount earned") during nth period to the accumulated value at the *end* of nth period, i.e. $I_n/A(n)$.

1.5.4 The effective discount rate d_n can be expressed in the following forms:

- $d_n = \dfrac{A(n) - A(n-1)}{A(n)}$ or $A(n-1) = A(n) - \underbrace{A(n)d_n}_{\text{"discount earned"}} = A(n)(1-d_n)$ (in terms of $A(\cdot)$'s)
- $d_n = \dfrac{a(n) - a(n-1)}{a(n)}$ or $a(n-1) = a(n) - \underbrace{a(n)d_n}_{\text{"discount earned"}} = a(n)(1-d_n)$ (in terms of $a(\cdot)$'s)



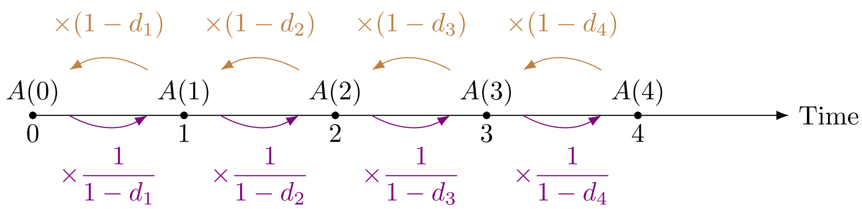
[Intuition 💡: About the term "discount earned": In each period the beginning value may be seen as the ending value with "discount" (an amount subtracted). As time passes, the "discount" is gradually "earned back".]

Remarks:

- From this, we can see that the effective discount rate is inherently "backward" → "discounting" process.
- An implicit requirement on d_n is that it must be less than 1 (for $A(n-1)$ to be positive → "sensible").

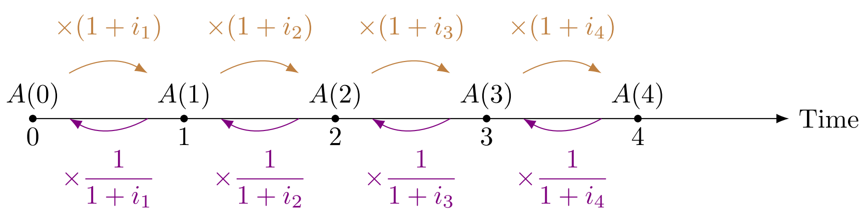
1.5.5 Of course one can reverse this "reverse process" and use the effective discount rate d_n to "move forward":

- $A(n) = A(n-1) \cdot \dfrac{1}{1-d_n}$
- $a(n) = a(n-1) \cdot \dfrac{1}{1-d_n}$



Compare this with the case for effective interest rate:

- $A(n-1) = A(n) \cdot \dfrac{1}{1+i_n}$
- $a(n-1) = a(n) \cdot \dfrac{1}{1+i_n}$



1.5.6 For *simple discount*, the amount of "discount earned" during each period is constant.

[⚠ Warning: The constant amount is not "$A(0)i$" or "$A(0)d$".]

To determine the constant amount, first we need to fix an $n \in \mathbb{N}$ as the "last" time point in our consideration. Then, the constant amount of "discount earned" in each period is given by

$$A(n) \times \underbrace{\text{simple discount rate}}_{d}$$

(where $d < 1$ to ensure positive accumulated values).

1.5.7 To deduce the accumulated values at different time points, we need to work *backward* from time n ("reverse process" of the process for the case of effective interest rate):

- $A(n-1) = A(n) - A(n)d = A(n)(1-d)$

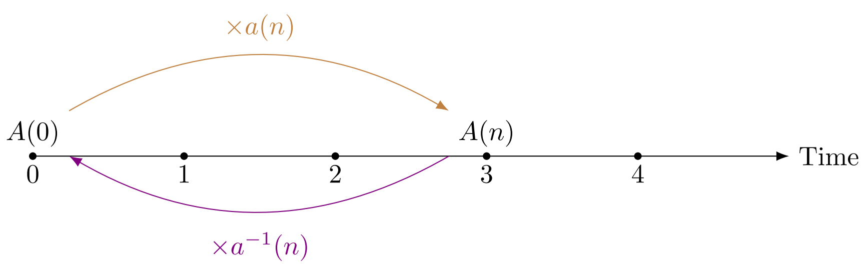
- $A(n-2) = \underbrace{A(n-1)}_{A(n)(1-d)} - A(n)d = A(n)(1-2d)$
- $\vdots$
- $A(n-p) = A(n)\underbrace{(1-pd)}_{>0}$ ✓
- $A(n-p-1) = A(n)\underbrace{(1-(p+1)d)}_{\leq 0}$ ✗

[Note: The equations are only valid ("sensible") when RHS is positive, so we stop this process once RHS becomes nonpositive.]

1.5.8 The general form is thus $A(n-k) = A(n)(1-kd)$ for any $k = 0, 1, \ldots, n$ where RHS is positive (i.e., $k < 1/d$).

1.5.9 Particularly, when $n < 1/d$, then we have

$$A(0) = A(n)\underbrace{(1-nd)}_{a^{-1}(n)}.$$

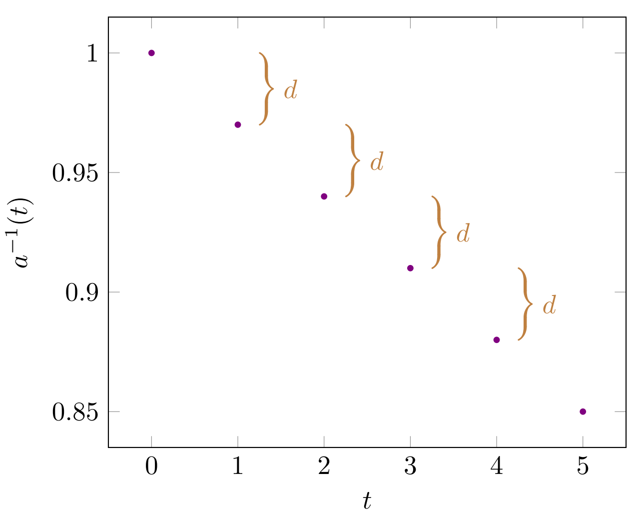


Consequently, as we fix different $n \in \mathbb{N}$ satisfying $n < 1/d$, we have

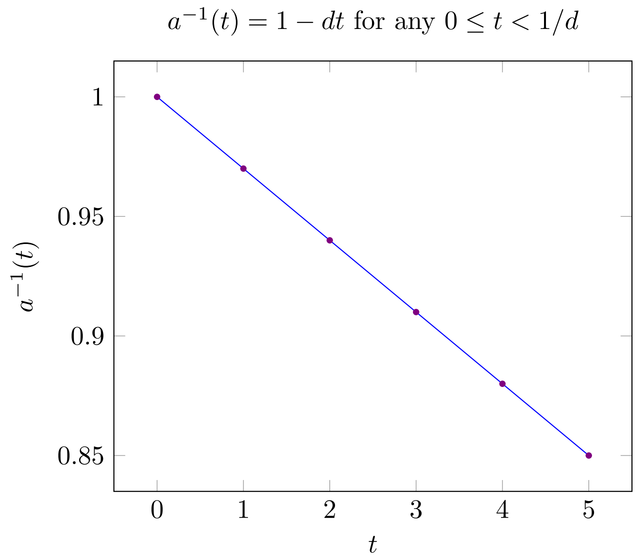
$$a^{-1}(n) = 1 - nd$$

for any $n \in \mathbb{N}$ with $n < 1/d$. (Of course the equation also holds for $n = 0$ since $a^{-1}(0) = 1$ always.)

$a^{-1}(t) = 1 - dt$ for any $t \in \mathbb{N}_0$ with $t < 1/d$



1.5.10 It is again natural to join the dots by a straight line:



This gives the general definition of **simple discount**:

$$a^{-1}(t) = 1 - dt$$

for any $0 \leq t < 1/d$, where d is the **simple discount rate**.

[⚠ Warning: It is not $a(t) = 1 - dt$ in the definition! We consider the *inverse function* $a^{-1}(t)$ instead. Alternatively, one may express the definition as

$$a(t) = \frac{1}{1 - dt}$$

for any $0 \leq t < 1/d$ → $a(t)$ is *nonlinear*!]

1.5.11 For *compound discount*, the amount of "discount earned" each period is

$$\text{accumulated value at the } \textit{end} \text{ of period} \times \underbrace{\text{compound discount rate}}_{d}$$

(where $d < 1$).

1.5.12 Likewise, we work backward from a fixed time n:

- $A(n-1) = A(n) - A(n)d = A(n)(1-d)$
- $A(n-2) = A(n-1) - A(n-1)d = \underbrace{A(n-1)}_{A(n)(1-d)}(1-d) = A(n)(1-d)^2$
- $\vdots$
- $A(0) = A(n)(1-d)^n$

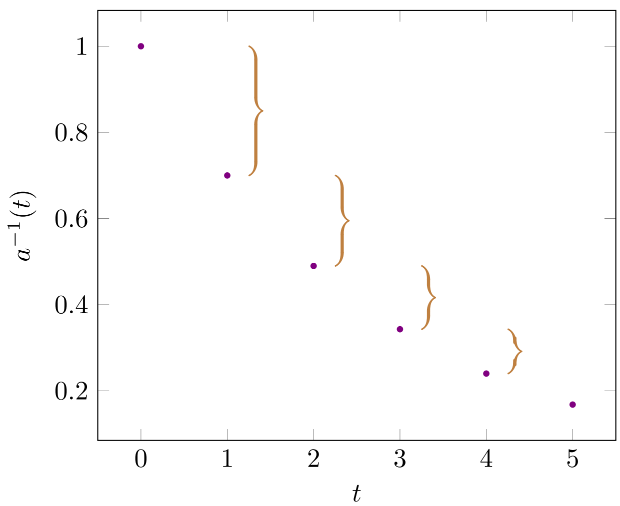
(RHS is always positive since $1 - d > 0$.)

1.5.13 Using a similar argument as [1.5.9], we have

$$a^{-1}(t) = (1-d)^t$$

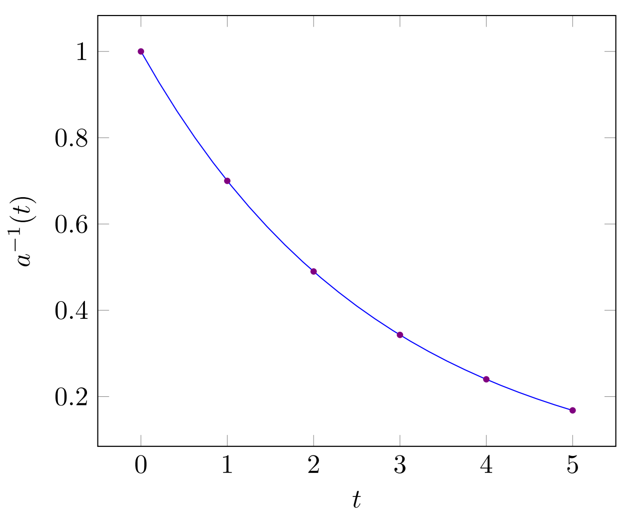
for any $t \in \mathbb{N}_0$.

$a^{-1}(t) = (1-d)^t$ for any $t \in \mathbb{N}_0$



1.5.14 Like the case for compound interest rate, it is natural to join the dots by an exponential function:

$a^{-1}(t) = (1-d)^t$ for any $t \geq 0$



This leads to the general definition of **compound discount**:

$$a^{-1}(t) = (1-d)^t$$

for any $t \geq 0$.

Remarks:

- Alternatively, we can express the definition as

$$a(t) = \frac{1}{(1-d)^t}$$

  for any $t \geq 0$.
- Like interest rate, unless otherwise specified, we shall assume *compound discount* and use the notation d to denote the (annual) compound discount rate.

1.6 Relationship Between Interest and Discount Rates

1.6.1 We have separately discussed interest and discount rates in sections 1.3 and 1.5. So naturally, the next question to ask is how are they *related*.

1.6.2 An important concept that connects two rates is the concept of *equivalency*.

1.6.3 Two interest/discount rates are **equivalent** if a fixed principal $ invested at each of the rates gives the *same accumulated value* at any fixed time point.

[Note: More explicitly, let r_1 and r_2 be the two rates. Then it means

$$a(t) \text{ @ } r_1 = a(t) \text{ @ } r_2$$

for any $t \geq 0$.]

1.6.4 Example: For compound interest rate i and compound discount rate d, they are equivalent when

$$(1+i)^t = \frac{1}{(1-d)^t} \quad \text{for any } t \geq 0,$$

which implies $1+i = \dfrac{1}{1-d}$.

[Note: Unless otherwise specified, we shall assume the rates i and d are equivalent (when we use the notations "i" and "d"). (This applies similarly for other interest-related notations when we just write them out.)]

1.6.5 The following are some results under the equivalency between i and d:

- $i = \dfrac{d}{1-d}$
- $d = \dfrac{i}{1+i}$
- $v = 1 - d$

(All of them follow readily from rearranging the equation $1 + i = 1/(1-d)$.)

1.6.6 We can observe from section 1.5 that the formulas for discount rates appear to be "more messy", hence the equivalency between i and d should be the main (but not only!) tool you are using for calculations involving discount rates.

1.7 Nominal Interest and Discount Rates

1.7.1 *Nominal rate*, as suggested by its name, is only nominal ("in name only") and is not an *actual* rate that is directly used in the calculations. It serves as a *reference* for computing *actual* interests.

1.7.2 Example: When a bank 🏛 offers 12% loan interest rate (per annum) "compounded monthly", "12%" serves as a reference only: It does not simply mean the annual effective interest rate is 12%. Rather, it means

- the measurement period is months (signified by "monthly"), while the "base" time unit is years (signified by "per annum");
- compound interest is assumed (signified by "compounded");
- the compound interest rate (for each month) is 12%/12 = 1%. (Number of months in a period in "base" time unit = 12.)

[Note: Unless otherwise specified, we assume that the "base" time unit is years for any nominal rate.]

1.7.3 More generally, we can define nominal interest rate as follows. For the case of **compounded $m$thly** with the (annual) **nominal interest rate compounded $m$thly** (denoted by $i^{(m)}$),

- the measurement period is $1/m$ of years;
- compound interest is assumed;
- the compound interest rate is $i^{(m)}/m$.

In this case, the accumulation function is given by

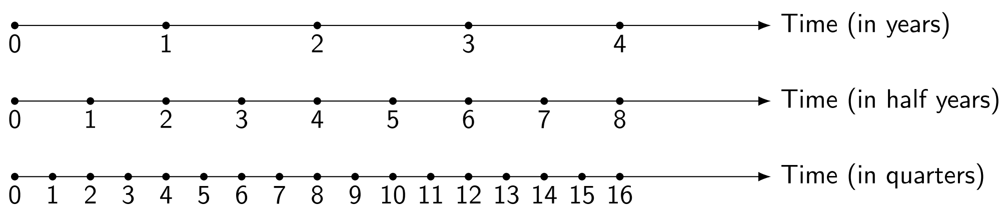
$$a(s) = \left(1 + \frac{i^{(m)}}{m}\right)^{s}$$

for any $s \geq 0$, *where the time unit of s is $1/m$ of years*, or

$$a(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{mt}$$

for any $t \geq 0$, *where the time unit of t is years.*

[Note: Time t in years is time mt in $1/m$ of years.



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[⚠ Warning: Make sure that you are clear about the time unit being used!]

1.7.4 For the case of *compounded mthly* with the (annual) **nominal discount rate compounded $m$thly** (denoted by $d^{(m)}$),

- the measurement period is $1/m$ of years;
- compound interest is assumed;
- the compound discount rate is $d^{(m)}/m$.

In this case, the accumulation function is

$$a(s) = \left(1 - \frac{d^{(m)}}{m}\right)^{-s}$$

for any $s \geq 0$, *where the time unit of s is $1/m$ of years*, or

$$a(t) = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}$$

for any $t \geq 0$, *where the time unit of t is years.*

1.7.5 We can also talk about the concept of equivalency for nominal rates:

- when compound effective interest rate i and nominal interest rate $i^{(m)}$ are equivalent,

$$(1+i)^t = \left(1 + \frac{i^{(m)}}{m}\right)^{mt} \quad \forall t \geq 0 \implies \boxed{1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^{m}}.$$

  [⚠ Warning: We need to use the same time unit for both sides!]

- when compound effective discount rate d and nominal interest rate $d^{(m)}$ are equivalent,

$$(1-d)^{-t} = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt} \quad \forall t \geq 0 \implies \boxed{1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^{m}}.$$

1.8 Force of Interest and Discount

1.8.1 Loosely, (annual) force of interest (discount) is the "nominal rate" "$i^{(\infty)}$" ("$d^{(\infty)}$" resp.):

$$\text{compounded } 1/m\text{thly} \xrightarrow{\quad m\to\infty \quad} \text{compounded "continuously"}$$

$$i^{(m)}\ (d^{(m)}) \xleftrightarrow[\text{same numerical value}]{} i^{(\infty)}\ (d^{(\infty)})$$

1.8.2 More precisely, we fix the nominal interest and discount rates compounded $m$thly at some values $i^{(\infty)}$ and $d^{(\infty)}$ respectively. Then, consider the limits of the accumulation function $a(t)$ for the compounded $m$thly case at each of the rates as $m \to \infty$: For any $t \geq 0$,

$$\lim_{m\to\infty}\left(1+\frac{i^{(\infty)}}{m}\right)^{mt} = e^{i^{(\infty)}t},$$

and

$$\lim_{m\to\infty}\left(1-\frac{d^{(\infty)}}{m}\right)^{-mt} = e^{-d^{(\infty)}(-t)} = e^{d^{(\infty)}t}.$$

1.8.3 When the values $i^{(\infty)}$ and $d^{(\infty)}$ are such that they are equivalent asymptotically ("at limit"), i.e.,

$$e^{i^{(\infty)}t} = e^{d^{(\infty)}t} \quad \text{for any } t \geq 0 \implies i^{(\infty)} = d^{(\infty)},$$

the common value $i^{(\infty)} = d^{(\infty)}$ is denoted by δ and called **force of interest** (or **force of discount**). For the **compounded continuously** case with (annual) force of interest δ (or interest rate δ *compounded continuously*), the accumulation function is given by

$$a(t) = e^{\delta t} \quad \text{for any } t \geq 0.$$

1.8.4 Then, when the effective interest rate i and force of interest δ are equivalent, we have

$$(1+i)^t = e^{\delta t}\ \forall t \geq 0 \implies \boxed{\delta = \ln(1+i)}.$$

Remarks:

- By writing $i^{(m)} = m\Big[(1+i)^{1/m}-1\Big] = \dfrac{(1+i)^{1/m}-1}{1/m}$ (the nominal interest rate compounded $m$thly *that is equivalent to* the rate i), we can actually obtain the force of interest δ (equivalent to the rate i) by

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty}\frac{(1+i)^{1/m}-1}{1/m} = \lim_{x\to 0}\frac{(1+i)^x-1}{x} = \ln(1+i) = \delta.$$

- Furthermore, since $\left(1-\dfrac{d^{(m)}}{m}\right)^{-m} = 1+i \implies d^{(m)} = m\Big[1-(1+i)^{-1/m}\Big]$ (the nominal discount rate compounded $m$thly *that is equivalent to* the rate i), we have also

$$\lim_{m\to\infty} d^{(m)} = \lim_{x\to 0}\frac{1-(1+i)^{-x}}{x} = -\lim_{x\to 0}\frac{v^x-1}{x} = -\ln v = \ln(1+i) = \delta.$$

1.8.5 [1.8.3] gives the definition of *constant* force of interest. Generally, in the compounded continuously case, we can indeed allow (possibly) *varying* force of interest, whose definition can be motivated through an "infinitesimal" argument (see [1.8.6]).

1.8.6 For the compounded continuously case with constant force of interest δ, we have for any $t \geq 0$, $a(t) = e^{\delta t}$, which implies $A(t) = A(0)e^{\delta t}$. Hence, the interest earned during the time interval $[t, t+h]$ is

$$A(t+h) - A(t) = \underbrace{A(t)}_{A(0)e^{\delta t}}(e^{\delta h} - 1) \approx A(t)(1 + \delta h - 1) = A(t)\delta h.$$

(The approximation works better when h is smaller.) After that, loosely speaking, the interest earned in the infinitesimal time interval $[t, t + \mathrm{d}t]$ is $A(t)\delta\,\mathrm{d}t$.

Now, for the case with (possibly) *varying* force of interest δ_t (as a function of time t), the interest earned in $[t, t + \mathrm{d}t]$ becomes $A(t)\delta_t\,\mathrm{d}t$.[2] Thus, we have

$$\mathrm{d}A(t) = \underbrace{A(t + \mathrm{d}t) - A(t)}_{\text{interest earned in } [t, t+\mathrm{d}t]} = A(t)\delta_t\,\mathrm{d}t \implies \frac{\mathrm{d}A(t)}{\mathrm{d}t} = A(t)\delta_t.$$

It follows that

$$\delta_t = \frac{A'(t)}{A(t)}$$

which gives the definition of **force of interest at time t**.

1.8.7 From this definition of varying force of interest, we can derive the following result.

Proposition 1.8.a. For any time t,

(a) $a(t) = \exp\left(\int_0^t \delta_s\,\mathrm{d}s\right)$;

(b) $A(t) = A(0) + \int_0^t A(s)\delta_s\,\mathrm{d}s$.

[Intuition 💡:

- For the first formula, firstly for any time s we have $A(s+h) \approx A(s)e^{\delta_s h}$ when h is small. Hence,

$$a(t) = \frac{A(t)}{A(0)} = \frac{A(h)}{A(0)}\frac{A(2h)}{A(h)} \cdots \frac{A(t)}{A(t-h)} \approx \exp(\delta_0 h + \delta_1 h + \cdots + \delta_{t-h} h)$$

  Then, loosely, letting $h \to 0$ gives

$$a(t) = \exp\left(\int_0^t \delta_s\,\mathrm{d}s\right).$$

- For the second formula, $A(s)\delta_s\,\mathrm{d}s$ can be regarded as "interest earned in infinitesimal time interval $[s, s + \mathrm{d}s]$", and thus $\int_0^t A(s)\delta_s\,\mathrm{d}s$ can be interpreted as "sum" of all such interests ➔ amount of interest earned in time interval $[0, t]$.

]

Proof: Firstly, by chain rule we have $\delta_s = \frac{\mathrm{d}}{\mathrm{d}s}\ln A(s)$. Thus,

$$\int_0^t \delta_s\,\mathrm{d}s = \ln A(t) - \ln A(0) = \ln\left(\frac{A(t)}{A(0)}\right) = \ln a(t),$$

and then the first formula follows.

[2] Loosely speaking, during the infinitesimal time interval $[t, t + \mathrm{d}t]$, the force of interest can be "regarded" as staying at its time-t value δ_t since the time length involved is "infinitesimally small" ➔ "constant" force of interest during the interval.

Next, note that $A(s)\delta_s = A'(s)$, so

$$\int_0^t A(s)\delta_s \,\mathrm{d}s = A(t) - A(0),$$

proving the second formula. $\square$

[Note: As a special case, when $\delta_s = \delta$ for any $s \in [0, t]$, the first formula becomes

$$a(t) = e^{\delta t},$$

which is reduced to the case in [1.8.3].]

2 Interest Problems and Annuities

2.1 Interest Problems

2.1.1 An interest problem (involving a single investment) generally involves four basic quantities:

(a) principal;

(b) length of investment period;

(c) interest rate (or force of interest);

(d) accumulated value at the end of the investment period,

where three of them are given, and we are asked to find the remaining one (target quantity).

2.1.2 To find the target quantity, we need to:

(a) set up an *equation of value*, i.e., equation in the form of

$$\text{principal} \times \text{accumulation factor} = \text{accumulated value at the end}$$

where "accumulation factor" incorporates the interest effect over the investment period;

(b) solve the equation for the target quantity, either analytically or numerically.

In simple cases, analytical method is possible for obtaining the solution. But occasionally, numerical method is needed (e.g., when analytical method is too "complex" or does not exist).

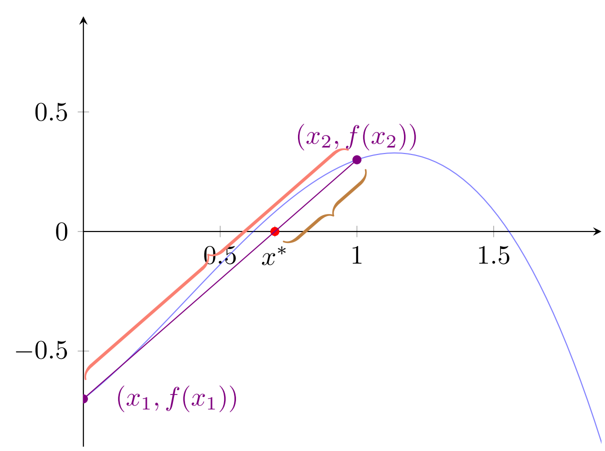
2.1.3 Here we introduce a basic numerical method: *linear interpolation*. The process is as follows:

(a) Set up the equation of value in the form of $f(x) = 0$ where x is the unknown target quantity, for some function f. (So, now we want to find a x such that the equation holds, i.e., a *root* of the equation.)

(b) Find out two pairs of numbers $(x_1, f(x_1))$ and $(x_2, f(x_2))$ such that $f(x_1) < 0 < f(x_2)$. (They may be given in the question sometimes.)

(c) Perform *linear interpolation* between coordinates $(x_1, f(x_1))$ and $(x_2, f(x_2))$: Joining them by a line segment and the intersection of the segment and x-axis gives an *approximated* root x^*, which is obtained by solving

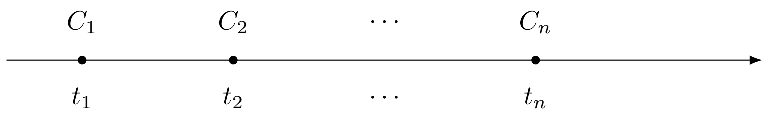
$$\frac{f(x_2) - 0}{x_2 - x^*} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$



2.2 Method of Equated Time

2.2.1 The *method of equated time* is also an approach for *approximating* an unknown quantity, but in a setting where *multiple* investments are involved.

2.2.2 The setting is as follows. Suppose there are cash flows (e.g., amounts of money withdrawn from several investments) of amounts $C_1, \dots, C_n$ at time $t_1, \dots, t_n$ respectively.



[Note: Conventionally, "cash flow" of a nonnegative (negative) amount refers to cash <u>in</u>flow (<u>out</u>flow) of that amount in absolute value.]

2.2.3 Sometimes we want to have a "sense" of "overall timing" of these cash flows (CFs). A metric for that is a time t such that

$$\text{PV of a single CF } C_1 + \cdots + C_n \text{ @ time } t = \text{PV of above CFs @ time } t_1, \dots, t_n\text{[3]},$$

i.e.,

$$(C_1 + \cdots + C_n)v^t = C_1 v^{t_1} + \cdots + C_n v^{t_n}.$$

Here of course one can solve analytically for t:

$$t = \frac{1}{\ln v} \ln\left(\frac{C_1 v^{t_1} + \cdots + C_n v^{t_n}}{C_1 + \cdots + C_n}\right).$$

However, the expression on RHS may appear to be "too complex" for some, and the *method of equated time* provides an approximation method that yields a "simpler" expression.

2.2.4 For **method of equated time**, the solution t is approximated by

$$\bar{t} = \frac{C_1 t_1 + \cdots + C_n t_n}{C_1 + \cdots + C_n},$$

which is a weighted average of time according to the amounts of cash flows.

[Note: A "better" approximation that takes *time value of money* into account will be discussed in ??.]

2.3 Annuity-Immediate and Annuity-Due

2.3.1 An **annuity** is a series of cash inflows made at *equal* intervals of time (i.e., "made regularly").

<u>Remarks</u>:

- An annuity whose payments are made *certainly* is called **annuity-certain**.
- An annuity where *uncertainties* are involved in the payments is called **contingent annuity**. (Special case: when payments are only made while the recipient is *alive*, the annuity is called **life annuity** (to be studied in STAT3901).)
- We shall focus on annuity-certain in STAT2902: "Annuity" refers to annuity-certain here.

2.3.2 An **n-period annuity-immediate** (**n-period annuity-due**) is an annuity whose cash inflows are made at the *end* (*beginning*) of each period, for n periods.

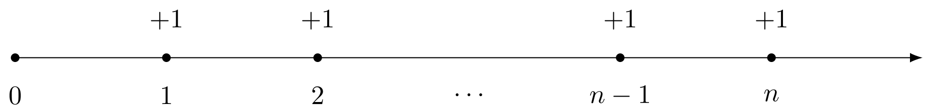
[Note: Here "immediate" is in the sense of "immediately after current *period*" (<u>not</u> "immediately after now"!): We need to wait till the end of current period to receive the first cash inflow.

For "due", it is in the sense of "due now": The first cash flow is made now (time 0).]

[3]PV of multiple CFs is obtained by summing the PV of each of them, since PV of multiple CFs means the total amount needed now to accumulate (separately) to yield each of those CFs.

2.3.3 The PV of an n-period annuity-immediate (amount of each CF = 1) is

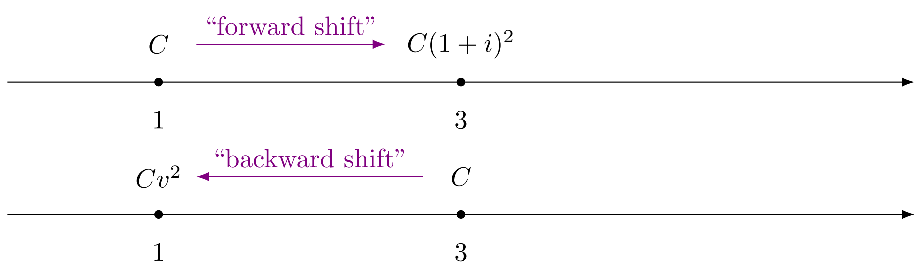
$$a_{\overline{n}|} = v + v^2 + \cdots + v^n = \frac{v}{1-v}(1-v^n) = \boxed{\frac{1-v^n}{i}}.$$



[Note: In general, for an n-period annuity-immediate (amount of each CF = k), its PV is $ka_{\overline{n}|}$. (Similar for other kinds of annuities.)]

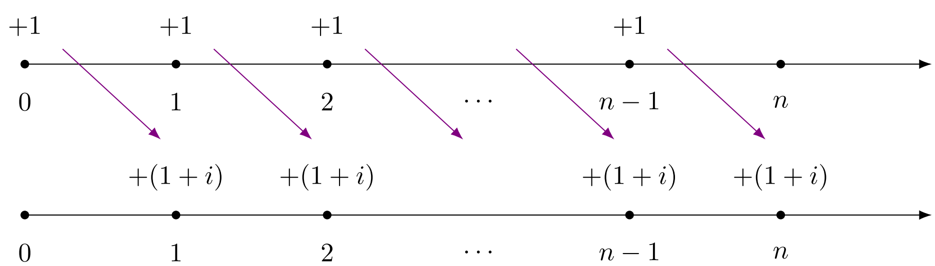
2.3.4 Before proceeding further, let us introduce a key "trick" that is useful for deriving many upcoming formulas: Accumulating/discounting ("shifting") CFs at the given interest rate ➔ *amount* of shifted CF becomes the *value* of original CF at that time point.

The usefulness of this trick is that the value of the shifted CF turns out to be identical to the one of the original CF (under *compound* interest)! This helps us "reducing" various situations to a familiar case.



2.3.5 For the PV of an n-period annuity-due (amount of each CF = 1), we can develop its formula based on $a_{\overline{n}|}$ through "shifting":

Original (annuity-due)



Shifted (annuity-immediate)

Since value remains unchanged after "shifting", it follows that the desired PV is

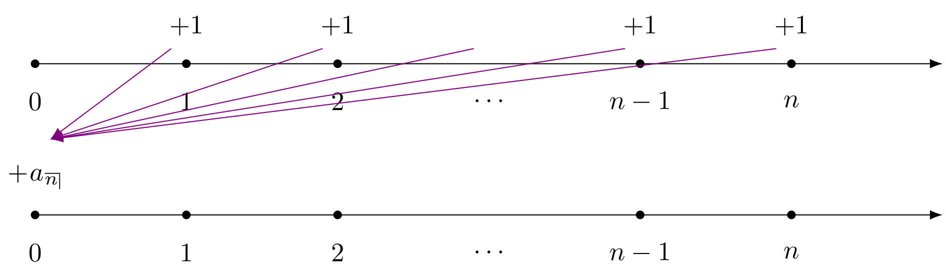
$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|} = (1+i)\frac{1-v^n}{i} = \boxed{\frac{1-v^n}{d}}.$$

[Mnemonic: The PV of an n-period annuity immediate (due) is $\frac{1-v^n}{i}$ $(\frac{1-v^n}{d})$.]

2.3.6 After obtaining the formulas for the PV of n-period annuity-immediate and annuity-due, getting the formulas for their *accumulated value* at time n becomes straightforward (through "shifting").

2.3.7 For the accumulated value of annuity-immediate:

Original (annuity-immediate)



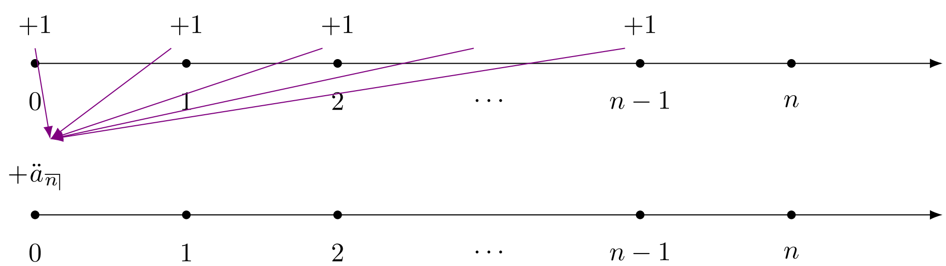
Shifted (single CF)

From this we know that the accumulated value of an n-period annuity-immediate at time n is the same as the one for the single CF:

$$s_{\overline{n}|} = a_{\overline{n}|}(1+i)^n = \boxed{\frac{(1+i)^n - 1}{i}}.$$

2.3.8 For the accumulated value of annuity-due:

Original (annuity-due)



Shifted (single CF)

Likewise, the accumulated value of an n-period annuity-due at time n is

$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n = \boxed{\frac{(1+i)^n - 1}{d}}.$$

2.4 Perpetuities

2.4.1 Simply speaking, a *perpetuity* is an "∞-period" annuity.

2.4.2 A **perpetuity-immediate** (**perpetuity-due**) is an annuity-immediate (annuity-due) whose cash inflows last forever.

2.4.3 The PV of a perpetuity-immediate is then the limit

$$a_{\overline{\infty}|} = \sum_{n=1}^{\infty} v^n = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \boxed{\frac{1}{i}}.$$

2.4.4 Likewise, the PV of a perpetuity-due is

$$\ddot{a}_{\overline{\infty}|} = \sum_{n=0}^{\infty} v^n = \lim_{n\to\infty} \ddot{a}_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{d} = \boxed{\frac{1}{d}}.$$

2.5 Annuities With CFs Less Frequent Than "Each Period"

2.5.1 Consider here annuities with CFs made *each k periods*, where $k > 1$ is an integer. In this case, we can also develop formulas for annuity/perpetuity-immediate and annuity/perpetuity-due, by "adjusting" the formulas mentioned in sections 2.3 and 2.4.

2.5.2 First, we discuss n-period annuities with such CF frequency. Here we shall assume that n is divisible by k (so that $n/k \in \mathbb{N}$).

2.5.3 For an n-period annuity-immediate (due) with such CF frequency, the cash inflows are made at the end (beginning) of *each k periods*, for n periods. (So there are $n/k \in \mathbb{N}$ cash inflows in total.)

[Note: For a perpetuity-immediate (due) with such CF frequency, it is defined in a similar manner as before: the respective annuity-immediate (due) with cash inflows lasting forever.]

2.5.4 An important trick to derive formulas in this kind of case (different CF frequencies) is called **j-method**. The procedure is as follows:

(a) Find the *k-period* interest rate j that is equivalent to the (1-period) interest rate i (by solving $1 + j = (1 + i)^k$ for j).

(b) Change the measurement period to k periods (1 unit of time $= k$ periods now).

(c) Apply the formulas derived in sections 2.3 and 2.4 with interest rate j. [⚠ Warning: Check the amount of each CF to see if multiplication of a constant is needed!]

A key benefit of this method is that it does *not* require us to memorize another set of formulas to handle calculations in this kind of situation. Hence, this should be the main approach to be used in solving problems of this kind.

2.5.5 Occasionally, specific "formulas" for this kind of situation are useful (and provide some "shortcuts"). So, henceforth we will derive some formulas (still based on j-method mainly) for the sake of completeness.

2.5.6 For the PV of an n-period annuity-immediate with such CF frequency (amount of each CF = 1):

interest rate per unit of time: i

Now, using j-method, we change the measurement period (divide each of the time above by k) and get:

interest rate per unit of time: j

Applying the equivalency relation between i and j, we can get its PV:

$$a_{\overline{n/k}|j} = \frac{1-(1+j)^{-n/k}}{j} = \frac{1-(1+i)^{-n}}{(1+i)^k-1} = \frac{1-v^n}{i}\cdot\frac{i}{(1+i)^k-1} = \boxed{\frac{a_{\overline{n}|i}}{s_{\overline{k}|i}}}.$$

[Note: The "j" in the notation is to signify the interest rate (per unit of time) used is j.]

2.5.7 For the PV of an n-period annuity-due with such CF frequency (amount of each CF = 1), applying the j-method and then using the previous relation with annuity-immediate ("shifting") gives its PV:

$$\ddot{a}_{\overline{n/k}|j} = (1+j)a_{\overline{n/k}|j} = (1+i)^k\frac{a_{\overline{n}|i}}{s_{\overline{k}|i}} = \boxed{\frac{a_{\overline{n}|i}}{a_{\overline{k}|i}}}.$$

2.5.8 Next, we can easily obtain formulas for PVs of perpetuity-immediate and perpetuity-due with such CF frequency (amount of each CF = 1):

- perpetuity-immediate: $\text{PV} = \lim_{n\to\infty}\frac{a_{\overline{n}|i}}{s_{\overline{k}|i}} = \frac{a_{\overline{\infty}|i}}{s_{\overline{k}|i}} = \boxed{\frac{1}{i s_{\overline{k}|i}}}$;
- perpetuity-due: $\text{PV} = \lim_{n\to\infty}\frac{a_{\overline{n}|i}}{a_{\overline{k}|i}} = \frac{a_{\overline{\infty}|i}}{a_{\overline{k}|i}} = \boxed{\frac{1}{i a_{\overline{k}|i}}}$.

2.6 Annuities With CFs More Frequent Than "Each Period"

2.6.1 Consider here annuities with CFs made *each* $1/m$ *of a period*, where $m > 1$ is an integer. Likewise we can use j-method in this case [Warning: Be careful about the amount of each CF!], and we shall derive some formulas based on it (which are more commonly used than the ones in section 2.5).

2.6.2 For an n-period annuity-immediate (due) with such CF frequency, the cash inflows are made at the end (beginning) of *each* $1/m$ *of a period*, for n periods. (So there are $mn \in \mathbb{N}$ cash inflows in total.)

2.6.3 For an n-period annuity-immediate with such CF frequency (amount of each CF = $1/m$):

$\boxed{\text{interest rate per unit of time: } i}$

Now again we change the measurement period:

$\boxed{\text{interest rate per unit of time: } j}$

[Note: To get the interest rate j, we solve $1+j=(1+i)^{1/m}$ for j.]

Hence, the PV is

$$a^{(m)}_{\overline{n}|} = \frac{1}{m}a_{\overline{mn}|j} = \frac{1}{m}\cdot\frac{1-(1+j)^{-mn}}{j} = \frac{1-(1+i)^{-n}}{m\big[(1+i)^{1/m}-1\big]} = \boxed{\frac{1-v^n}{i^{(m)}}}.$$

Thus, by "shifting", the time-n accumulated value is

$$s_{\overline{n}|}^{(m)} = a_{\overline{n}|}^{(m)} \underbrace{(1+i)^n}_{\text{or } (1+j)^{mn}} = \boxed{\frac{(1+i)^n - 1}{i^{(m)}}}.$$

2.6.4 For an n-period annuity-due with such CF frequency (amount of each CF $= 1/m$), we can likewise apply the j-method and use the previous relation with annuity-immediate to get its PV:

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}\ddot{a}_{\overline{mn}|j} = \frac{1}{m}(1+j)a_{\overline{mn}|j} = \underbrace{(1+j)}_{(1+i)^{1/m}} \cdot \frac{1-v^n}{i^{(m)}} = \frac{1-v^n}{(1+i)^{-1/m} \cdot \underbrace{m[(1+i)^{1/m} - 1]}_{i^{(m)}}} = \boxed{\frac{1-v^n}{d^{(m)}}}$$

since $\left(1 - \dfrac{d^{(m)}}{m}\right)^{-m} = 1 + i \implies d^{(m)} = m\Big[1 - (1+i)^{-1/m}\Big]$.

Similarly, its time-n accumulated value is

$$\ddot{s}_{\overline{n}|}^{(m)} = \ddot{a}_{\overline{n}|}^{(m)}(1+i)^n = \boxed{\frac{(1+i)^n - 1}{d^{(m)}}}.$$

[Mnemonic: The PV of an n-period annuity immediate (due) (amount of each CF $= 1/m$) with such CF frequency is $\dfrac{1-v^n}{i^{(m)}}$ $\Big(\dfrac{1-v^n}{d^{(m)}}\Big)$.]

2.6.5 Next, we can similarly obtain formulas for PVs of perpetuity-immediate and perpetuity-due with such CF frequency (amount of each CF $= 1/m$):

- perpetuity-immediate: PV $= \lim\limits_{n\to\infty} a_{\overline{n}|}^{(m)} = \boxed{\dfrac{1}{i^{(m)}}}$;
- perpetuity-due: PV $= \lim\limits_{n\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \boxed{\dfrac{1}{d^{(m)}}}$.

2.7 Continuous Annuities

2.7.1 Loosely speaking, a *continuous annuity* is an annuity with "infinitesimal" CFs made "*each* $1/\infty$ *of a period*" (*continuously*). The "total amount" of those "infinitesimal" CFs for one period is the **rate of CFs**.

2.7.2 Under *constant* force of interest δ (equivalent to the rate i), for an n-period continuous annuity (rate of cash inflows $= 1$), the PV can be obtained as the limit

$$\bar{a}_{\overline{n}|} = \lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \frac{1-v^n}{i^{(m)}} = \boxed{\frac{1-v^n}{\delta}}.$$

Remarks:

- The nominal rate $i^{(m)}$ is equivalent to the rate i for every m, so it approaches δ as $m \to \infty$.
- This is also equal to the limit $\lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)}$ as we also have $\lim_{m\to\infty} d^{(m)} = \delta$ (where the nominal rate $d^{(m)}$ is equivalent to the rate i for every m).

For the time-n accumulated value, applying "shifting" argument to the expression in the limit gives

$$\bar{s}_{\overline{n}|} = \lim_{m\to\infty} a_{\overline{n}|}^{(m)}(1+i)^n = \bar{a}_{\overline{n}|}(1+i)^n = \boxed{\frac{(1+i)^n - 1}{\delta}}.$$

2.7.3 In general, under *varying* force of interest δ_t (as a function of time t), the formula for PV of an n-period continuous annuity (rate of cash inflows $= 1$) is given by

$$\int_0^n a^{-1}(t)\,\mathrm{d}t = \int_0^n \exp\left(-\int_0^t \delta_s\,\mathrm{d}s\right)\mathrm{d}t$$

(which may be seen as following from the definition).

[Intuition: In each infinitesimal time interval $[t, t+\mathrm{d}t]$, the PV of the "stream" of cash inflows during the interval (total amount: rate $\times$ time length $= 1 \times \mathrm{d}t = \mathrm{d}t$) is

$$\text{total amount} \times \text{discount factor} = a^{-1}(t)\mathrm{d}t.$$

"Summing" all the PVs up then gives the total PV:

$$\int_0^n a^{-1}(t)\,\mathrm{d}t\,.$$

]

2.8 Annuities With CFs Varying in Arithmetic Sequence

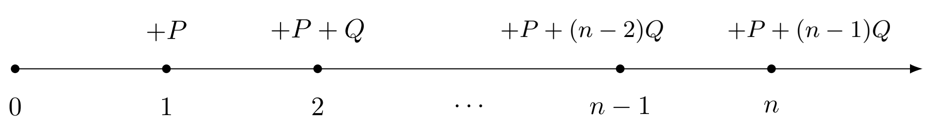
2.8.1 Even among annuities with CFs varying in arithmetic sequence, we can divide them into different groups (by the frequency of CFs). In this section we will discuss the following groups:

- "each period" (one CF per period)
- "each k periods" ("less frequent")
- "each $1/m$ of a period" ("more frequent")

We shall focus on *annuity-immediate* here. [Note: The formulas (notations) can be easily translated for the annuity-due case, by "shifting" (adding two dots on top of a).]

"Each Period"

2.8.2 In this case, for an n-period annuity-immediate with such CF pattern, the cash inflows at time $1, \dots, n$ are $P, P+Q, P+2Q, \dots, P+(n-1)Q$ respectively (where the constants P and Q are such that all quantities involved are nonnegative).



2.8.3 To find its PV, we use the following result

Proposition 2.8.a. The PV of this kind of n-period annuity-immediate is

$$Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i}.$$

<u>Proof</u>: Firstly, the PV is

$$\begin{aligned} &Pv + (P+Q)v^2 + (P+2Q)v^3 + \cdots + (P+(n-1)Q)v^n \\ &\quad = P(v + v^2 + v^3 + \cdots + v^n) + Q(\underbrace{v^2 + 2v^3 + \cdots + (n-1)v^n}_{u}) \\ &\quad = Pa_{\overline{n}|} + Qu. \end{aligned}$$

It then suffices to show that $u = \dfrac{a_{\overline{n}|} - nv^n}{i}$. Now, we use the following trick:

$$\begin{aligned}(1+i)u - u &= v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1} \\ &\quad - v^2 - 2v^3 - \cdots - (n-2)v^{n-1} - (n-1)v^n \\ &= v + v^2 + v^3 + \cdots + v^{n-1} + v^n - nv^n = a_{\overline{n}|} - nv^n,\end{aligned}$$

as desired. □

2.8.4 Now we introduce some special cases (with names and special notations):

- **n-period increasing annuity** ($P = 1, Q = 1$):

$$\text{PV} = (Ia)_{\overline{n}|} = a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i} = \frac{(1+i)a_{\overline{n}|} - nv^n}{i} = \boxed{\frac{\ddot{a}_{\overline{n}|} - nv^n}{i}},$$

and (by "shifting")

$$\text{time-}n\text{ AV} = (Is)_{\overline{n}|} = (Ia)_{\overline{n}|}(1+i)^n = \boxed{\frac{\ddot{s}_{\overline{n}|} - n}{i}}.$$

- **n-period decreasing annuity** ($P = n, Q = -1$):

$$\text{PV} = (Da)_{\overline{n}|} = n\underbrace{a_{\overline{n}|}}_{(1-v^n)/i} - \frac{a_{\overline{n}|} - nv^n}{i} = \frac{n - nv^n - a_{\overline{n}|} + nv^n}{i} = \boxed{\frac{n - a_{\overline{n}|}}{i}},$$

and (by "shifting")

$$\text{time-}n\text{ AV} = (Ds)_{\overline{n}|} = (Da)_{\overline{n}|}(1+i)^n = \boxed{\frac{n(1+i)^n - s_{\overline{n}|}}{i}}.$$

2.8.5 We can obtain another set of PV formulas for the increasing and decreasing annuities using "horizontal view" and "splitting".

2.8.6 For increasing annuity:

From this we can express the PV of an n-period increasing annuity as

$$a_{\overline{n}|} + va_{\overline{n-1}|} + \cdots + +v^{n-1}a_{\overline{2}|} + v^n a_{\overline{1}|} = \sum_{t=0}^{n-1} v^t a_{\overline{n-t}|}$$

(by "shifting" cash inflows in each row such that the first one is at time 1).

2.8.7 For decreasing annuity:

From this we can likewise express the PV of an n-period decreasing annuity as

$$a_{\overline{n}|} + a_{\overline{n-1}|} + \cdots + + a_{\overline{2}|} + a_{\overline{1}|} = \sum_{t=0}^{n-1} a_{\overline{n-t}|}.$$

[Note: Here we do not even need "shifting"!]

2.8.8 We can also develop a formula for *perpetuity-immediate* with the CF pattern in [2.8.2], i.e., cash inflows at time $1, 2, \ldots,$ are $P, P+Q, \ldots$ respectively:

$$\mathrm{PV} = \lim_{n\to\infty}\left(Pa_{\overline{n}|} + Q\cdot\frac{a_{\overline{n}|} - nv^n}{i}\right) = \frac{P}{i} + \frac{Qa_{\overline{\infty}|}}{i} = \boxed{\frac{P}{i} + \frac{Q}{i^2}}.$$

2.8.9 As a special case, the PV of an **increasing perpetuity** (increasing annuity whose cash inflows last forever) is

$$(Ia)_{\overline{\infty}|} = \lim_{n\to\infty}\frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \frac{\ddot{a}_{\overline{\infty}|}}{i} = \frac{1}{id}.$$

Remarks:

- An intuitive reason for $\lim_{n\to\infty} nv^n = 0$ is that as $n \to \infty$, the "speed" at which v^n drops to zero (exponential) is higher than the "speed" at which n approaches infinity (linear). Formally, one can use L'Hôpital's rule to show this.
- There is not "decreasing perpetuity" since negative cash inflow is not allowed.

"Each k Periods"

2.8.10 Here consider an n-period annuity-immediate where the cash inflows at time $k, 2k, \ldots, n$ are $1, 2, \ldots, n/k$ respectively (assuming n is divisible by k).

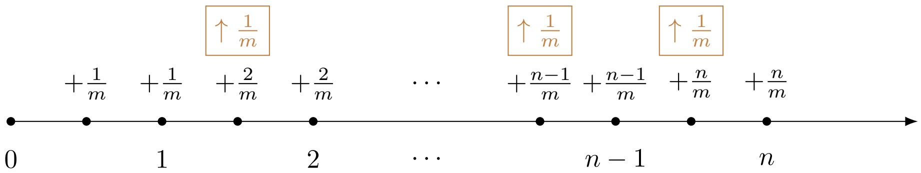
2.8.11 To determine its PV, we can use j-method (the k-period rate j is found by solving $1 + j = (1+i)^k$) and apply the formula for PV of n-period increasing annuity in [2.8.4]:

$$\mathrm{PV} = (Ia)_{\overline{n/k}|j} = \frac{\ddot{a}_{\overline{n/k}|j} - (n/k)(1+j)^{-n/k}}{j} = \boxed{\frac{\frac{a_{\overline{n}|i}}{a_{\overline{k}|i}} - \frac{n}{k}v^n}{(1+i)^k - 1}}.$$

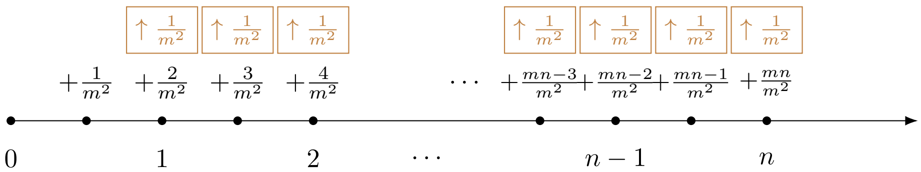
"Each $1/m$ of a Period"

2.8.12 We shall analyze two kinds of annuity-immediate under this case, that have different CF patterns (despite having the same CF frequency):

(a) amount of CF increases by $1/m$ per period:

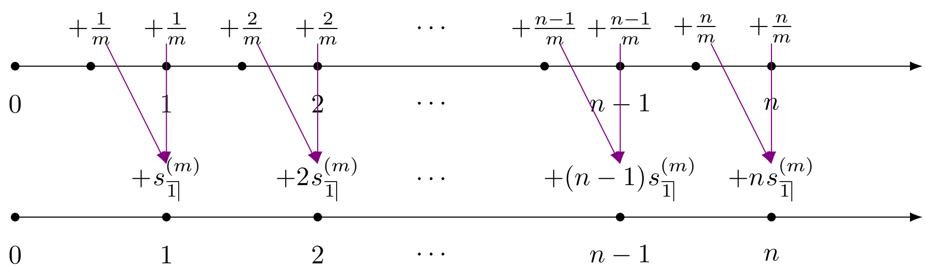


(b) amount of CF increases by $1/m^2$ per $1/m$ of a period:



2.8.13 For the first kind where the amount of CF increases by $1/m$ per period, we can perform a "shifting" trick to derive its PV formula. First we "shift" the CFs in the following way:

Original



Shifted

From this, we know that its PV is

$$(Ia)^{(m)}_{\overline{n}|} = s^{(m)}_{\overline{1}|}(Ia)_{\overline{n}|} = \frac{(1+i)-1}{i^{(m)}}(Ia)_{\overline{n}|} = \frac{i}{i^{(m)}}\,\frac{\ddot{a}_{\overline{n}|}-nv^n}{i} = \boxed{\frac{\ddot{a}_{\overline{n}|}-nv^n}{i^{(m)}}}.$$

[Mnemonic: The expression is $(Ia)_{\overline{n}|}$ with $i \to i^{(m)}$.]

2.8.14 For the second kind where the amount of CF increases by $1/m^2$ per $1/m$ of a period, we can use j-method to derive its PV formula (the rate j for $1/m$ of a period can be found by solving $1+j = (1+i)^{1/m}$ for j):

$$\text{PV} = (I^{(m)}a)^{(m)}_{\overline{n}|} = \frac{1}{m^2}(Ia)_{\overline{mn}|j} = \frac{1}{m^2}\cdot\frac{\overbrace{\ddot{a}_{\overline{mn}|j}}^{m\ddot{a}^{(m)}_{\overline{n}|i}} - mn(1+j)^{-mn}}{\underbrace{j}_{i^{(m)}/m}} = \boxed{\frac{\ddot{a}^{(m)}_{\overline{n}|}-nv^n}{i^{(m)}}}.$$

[Mnemonic: The expression is $(Ia)^{(m)}_{\overline{n}|}$ with $\ddot{a}_{\overline{n}|} \to \ddot{a}^{(m)}_{\overline{n}|}$.]

2.9 Continuous Annuities With CFs Varying Continuously

2.9.1 In section 2.7, we assume that the rate of cash inflows for n-period continuous annuity stays at a constant 1. Here we consider a more general case where the rate can vary as a function of time t ("vary continuously").

2.9.2 We first consider a special case where the rate is t at time t and we have a constant force of interest δ. We can observe that this case can be regarded as the limit of [2.8.12]b as $m \to \infty$. Hence, its PV is

$$(\bar{I}\bar{a})_{\overline{n}|} = \lim_{m\to\infty} (I^{(m)}a)^{(m)}_{\overline{n}|} = \lim_{m\to\infty} \frac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{i^{(m)}} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}.$$

2.9.3 The general case is that the rate is $g(t)$ at time t for some function g, and we have a varying force of interest δ_t (as a function of t). In this case, the PV is

$$\int_0^n g(t)a^{-1}(t)\,\mathrm{d}t = \int_0^n g(t)\exp\left(-\int_0^t \delta_s\,\mathrm{d}s\right)\mathrm{d}t$$

(which may be seen as following from the definition).

[Intuition 💡: In each infinitesimal time interval $[t, t+\mathrm{d}t]$, the PV of the "stream" of cash inflows during the interval (total amount: rate $\times$ time length $= g(t)\,\mathrm{d}t$) is

$$\text{total amount} \times \text{discount factor} = g(t)a^{-1}(t)\mathrm{d}t.$$

"Summing" all the PVs up then gives the total PV:

$$\int_0^n g(t)a^{-1}(t)\,\mathrm{d}t\,.$$

]

2.10 Annuities With CFs Varying in Geometric Sequence

2.10.1 Apart from arithmetic sequence, the CFs in an annuity can also vary in a *geometric sequence*. But in such case, for most calculations, using the *geometric series formula* would be sufficient:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \begin{cases} \dfrac{a(1-r^n)}{1-r} & \text{if } r \neq 1; \\ an & \text{if } r = 1. \end{cases}$$

(One just needs to be careful about what the common ratio r is.)

2.10.2 But for the sake of completeness, here we will derive some formulas in this case. Consider an n-period annuity-immediate where the cash inflows at time $1, 2, \ldots, n$ are $1, (1+k), \ldots, (1+k)^{n-1}$ with $k > -1$ (to avoid negative cash inflow) respectively. Then, its PV is

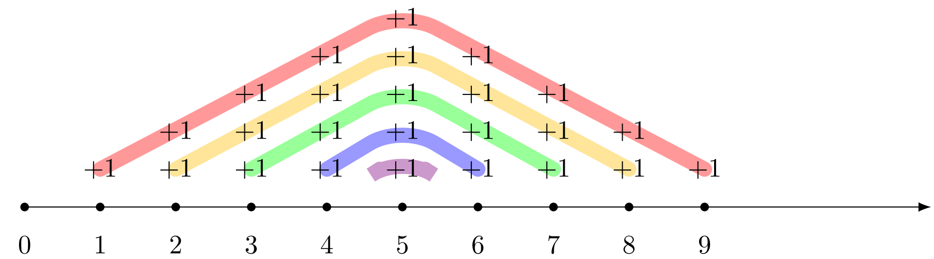
$$v + (1+k)v^2 + \cdots + (1+k)^{n-1}v^n = \begin{cases} nv & \text{if } i = k; \\ \dfrac{v[1-(v(1+k))^n]}{1-v(1+k)} & \text{if } i \neq k. \end{cases}$$

2.10.3 For *perpetuity-immediate* whose cash inflows at time $1, 2, \ldots,$ are $1, (1+k), \ldots$ (assuming $i \neq k$ and $k > -1$) respectively, its PV is

$$\lim_{n\to\infty} \frac{v[1-(v(1+k))^n]}{1-v(1+k)} = \frac{v}{1-v(1+k)} = \frac{1}{1+i-1-k} = \frac{1}{i-k}.$$

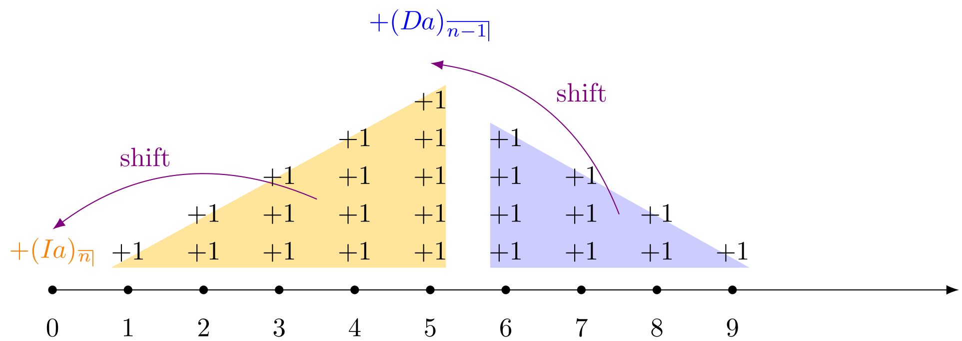
2.11 Rainbow Immediate

2.11.1 Starting from here, we will discuss some annuities that are "exotic" in terms of CF pattern. The first one is known as *rainbow immediate*, whose CF pattern looks like a "rainbow":



2.11.2 More precisely, an $2n-1$)**-period rainbow immediate** is an $2n-1$-period annuity-immediate whose CFs at time $1,\ldots,n,n+1,\ldots,2n-1$ are $1,\ldots,n,n-1,\ldots,1$ respectively.

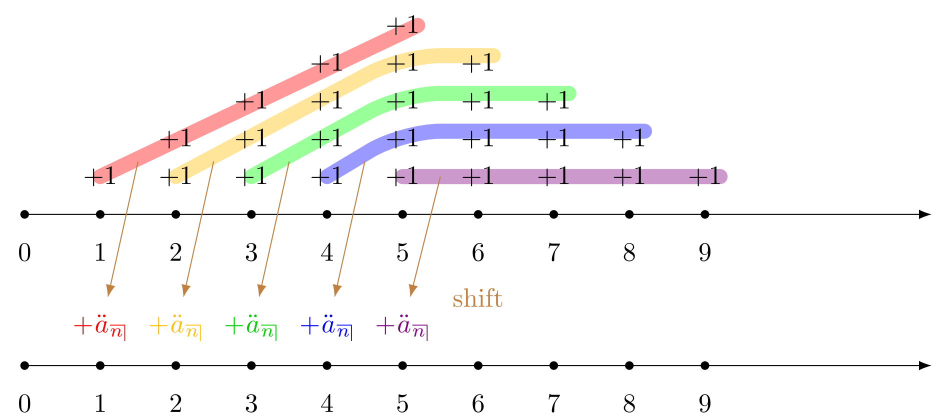
2.11.3 There are two ways to derive its PV formula. The first way is the "splitting" and "shifting" approach:



From this, the PV is

$$(Ia)_{\overline{n}|} + v^n (Da)_{\overline{n-1}|}.$$

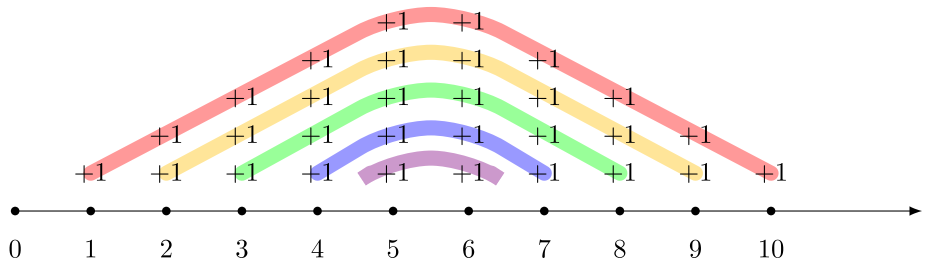
2.11.4 Another way is to have a "diagonal & horizontal" view and "shifting":



From this, we know that the PV is $\ddot{a}_{\overline{n}|} a_{\overline{n}|}$.

2.12 Paused Rainbow Immediate

2.12.1 The next annuity is *paused rainbow immediate*, whose CF pattern looks like a "rainbow" with a "pause" at the middle:



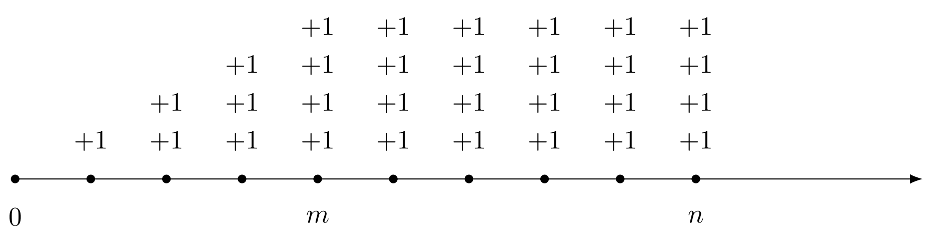
2.12.2 More precisely, an **$2n$-period paused rainbow immediate** is an $2n$-period annuity-immediate whose CFs at time $1, \ldots, n, n+1, \ldots, 2n-1$ are $1, \ldots, n, n, \ldots, 1$ respectively.

2.12.3 In a similar manner as [2.11.3] and [2.11.4], we can derive two formulas for PV of an n-period paused rainbow immediate:

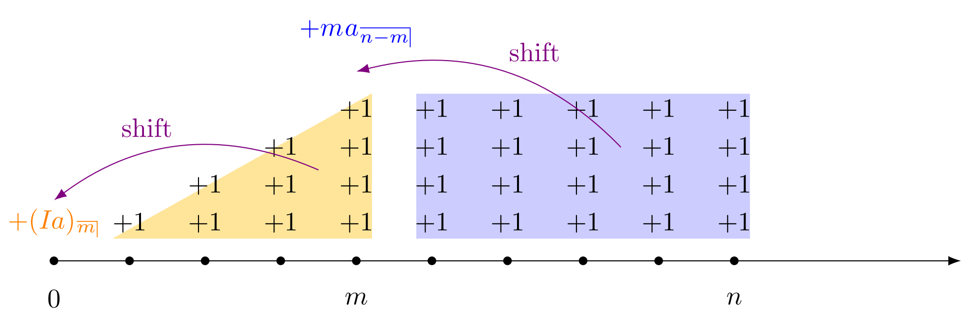
- $\text{PV} = (Ia)_{\overline{n}|} + v^n (Da)_{\overline{n}|}$;
- $\text{PV} = \ddot{a}_{\overline{n+1}|} a_{\overline{n}|}$.

2.13 "Paused-From-Time-m" Increasing Annuity

2.13.1 The last "exotic" annuity discussed here is the so-called n-period "paused-from-time m" increasing annuity (this name is not standard), where $m < n, m, n \in \mathbb{N}$. Simply speaking, it is an n-period increasing annuity but the CF stays at m ("paused") starting from time m until the end.



2.13.2 To get its PV, we can use the "splitting" and "shifting" approach:



Hence, the PV is

$$(I_{\overline{m}|} a)_{\overline{n}|} = (Ia)_{\overline{m}|} + v^m (m a_{\overline{n-m}|}).$$

[Note: "$I_{\overline{m}|}$" represents "increasing for m periods (up to time m)".]

2.14 Summary of Tricks and Intuition

2.14.1 The following are tricks/intuition mentioned for deriving various formulas, and they are also useful in solving problems sometimes:

(a) "splitting" and "shifting" (✔ useful generally)

(b) j-method (✔ "more frequent"/"less frequent" case)

(c) think in "infinitesimal" (✔ continuous case)

(d) "horizontal" and "diagonal" views (✔ varying CFs case)

2.15 Varying Effective Interest Rate

2.15.1 Lastly, we discuss a general setting where the effective interest rate can vary and compound interest may *not* be assumed throughout. In this case, we generally need to use "first principle" to compute the PV/AV of an annuity, namely summing the PV/AV of each CF one by one.

2.15.2 To be more explicit, we have the following (general) formulas, using the effective interest rate notation:

- n-period annuity-immediate:

$$\begin{aligned}\mathrm{PV} &= a^{-1}(1) + a^{-1}(2) + \cdots + a^{-1}(n) \\ &= \boxed{(1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_1)^{-1}\cdots(1+i_n)^{-1}}.\end{aligned}$$

and

$$\begin{aligned}\text{time-}n\text{ AV} &= \mathrm{PV}\cdot(1+i_1)(1+i_2)\cdots(1+i_n) \\ &= \boxed{1 + (1+i_n) + (1+i_{n-1})(1+i_n) + \cdots + (1+i_2)\cdots(1+i_n)}.\end{aligned}$$

- n-period annuity-due

$$\begin{aligned}\mathrm{PV} &= a^{-1}(0) + a^{-1}(1) + \cdots + a^{-1}(n-1) \\ &= \boxed{1 + (1+i_1)^{-1} + \cdots + (1+i_1)^{-1}\cdots(1+i_{n-1})^{-1}}.\end{aligned}$$

and

$$\begin{aligned}\text{time-}n\text{ AV} &= \mathrm{PV}\cdot(1+i_1)(1+i_2)\cdots(1+i_n). \\ &= \boxed{(1+i_n) + (1+i_{n-1})(1+i_n) + \cdots + (1+i_1)\cdots(1+i_n)}.\end{aligned}$$

3 Discounted Cash Flow Analysis

3.0.1 The *discounted cash flow analysis* starts with the concept of *net cash flow* (NCF). **Net cash flow** at time t (denoted by C_t) is

$$(\text{total}) \text{ cash inflow @ time } t - (\text{total}) \text{ cash outflow @ time } t.$$

Remarks:

- So, to be more precise, NCF is net cash *inflow*.
- Sometimes we drop the word "net" and just call NCF as "cash flow" (like in section 2).
- **Discounted cash flow analysis** refers generally to any analysis involving discounting CFs. We will discuss two kinds of discounted cash flow analysis: project analysis in section 3.1 and fund return analysis in section 3.3.

3.1 Project Analysis With Investment Criteria

3.1.1 Here we consider a project with NCFs (at time $0, 1, \ldots, n$) $C_0, C_1, , \ldots, C_n$ (NCF at time point other than $0, 1, \ldots, n$ is zero). (So the project is *terminated* at time n since from that time point onward there is no more NCF.)

[Note: Theoretically, a project can *never terminate* (having infinitely many positive NCFs). But here we shall focus on a project with finitely many positive NCFs like above.]

3.1.2 Here we will introduce several investment criteria to *analyze* the project:

- net present value (NPV);
- internal rate of return (IRR) or yield rate;
- discounted payback period (DPP).

3.1.3 **Net present value** (NPV) of the project at an interest rate i is the PV of the NCFs at rate i:

$$\text{NPV} = P(i) = \sum_{t=0}^{n} C_t v^t.$$

The solution to the equation $P(i) = 0$ (unknown: i) is called **internal rate of return** (IRR) or **yield rate** of the project.

Remarks:

- "Internal" suggests that the calculation of IRR itself does not consider "external factors" like inflation. (But of course, one may already incorporate some "external information" into the NCFs by "adjusting" them based on some "external factors").
- There is no general "formula" to find IRR, so typically we need to use numerical method to find IRR. (But we need to be careful that there may be zero or multiple IRRs. For a sufficient condition to ensure uniqueness of IRR, see [3.1.12].)

3.1.4 Before proceeding further, let us clarify on the meaning of accumulating/discounting positive and negative NCF.

- Accumulating
  - positive NCF: lending/investing that amount for a time length (earning *lending/investing (interest) rate*) and then collect proceeds from the loan later ➔ becomes a positive NCF at a later time after accumulation;
  - negative NCF: borrowing/financing that amount (in absolute value) for a time length (earning *borrowing/financing (interest) rate*) and then repay the loan later ➔ becomes a negative NCF at a later time after accumulation;

- Discounting
  - positive NCF: deducing NCF at a previous time that would accumulate to this positive NCF at this time (discount at *lending/investing rate*);
  - negative NCF: deducing NCF at a previous time that would accumulate to this negative NCF at this time (discount at *borrowing/financing rate*).

Remarks:

- We can see that a minor difference between accumulating/discounting positive and negative NCFs is that different *kinds* of interest rate are used. When the lending and borrowing rates differ, we need to be careful about the appropriate interest rate to be used. For example, when we calculate NPV, positive (negative) NCF needs to be discounted at lending (borrowing) rate.
- Unless stated otherwise, we shall assume that lending and borrowing rates coincide for convenience.

3.1.5 Now we discuss *practical meanings* of NPV and IRR (which are important for us to analyze the project using these criteria).

3.1.6 For NPV, let i_0 be the lending (or borrowing) rate. Then, consider the following cases:

- (NPV > 0) NPV is the amount we need to *invest/lend* now to accumulate (separately) to yield *all* the NCFs in the project (i.e., to "get" all the project NCFs) ➔ the project NCFs have a "positive worth" as a whole ("like" an asset).
- (NPV < 0) NPV (in absolute value) is the amount we need to *borrow/finance* now to "get" all the project NCFs ➔ the project NCFs have a "negative worth" as a whole ("like" a debt).
- (NPV $= 0$) We do not need to invest anything to "get" all the project NCFs ➔ the project NCFs have "zero worth" as a whole. (The project *breaks even.*)

3.1.7 Based on the interpretation in [3.1.6], we have the following practical meanings of NPV:

- A project is *profitable* (has "positive worth") if and only if its NPV > 0.
- Project A is *more profitable* ("worth more") than project B if and only if the NPV of project A $>$ that of project B.

3.1.8 For IRR, it is instructive to "associate" the project with a "fund" where cash inflow corresponds to *withdrawal* from the fund, and cash outflow corresponds to *contribution* to the fund.

Then, for an interest rate i to qualify as a constant "rate of return" for the project, *accumulating* all the withdrawals and contributions at that interest rate i to the end of project should give the "final fund value"[4] , i.e.,

- (when the value is positive) amount we get if we withdraw all money from the fund, or
- (when the value is negative) amount we need to further contribute to the fund for "clearing the debt".

(In either case, the value is equal to the time-n NCF C_n.) In other words, the rate i should satisfy

$$\sum_{t=0}^{n-1} C_t(1+i)^{n-t} = C_n \iff P(i) = 0 \iff i \text{ is IRR.}$$

Hence, IRR can be interpreted as a *possible* constant rate of return for the project, as one may expect.

Remarks:

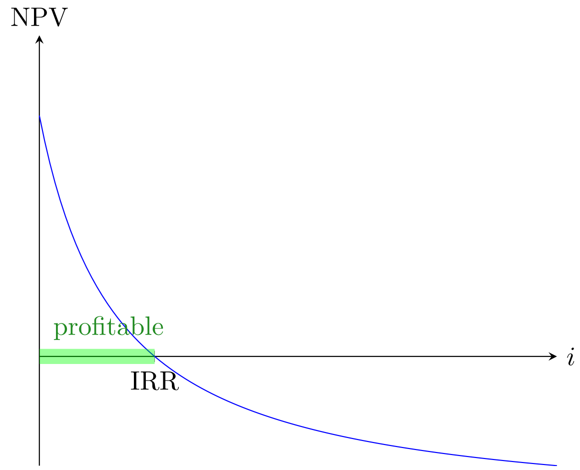
- Note that the return rate is only "possible": The project may not even have constant rate of return!

[4] Intuitively speaking, we keep accumulating "remaining balance" (even for negative balance; may be seen as "accumulating overdraft charges" in this case) in the fund . To interpret the accumulation more formally, one may convert "withdrawals" and "contributions" to "positive NCFs" and "negative NCFs" respectively, then refer to [3.1.4].

- Sometimes there are multiple IRRs, indicating that there are several *possible* constant rates of return for the project, just by observing the NCFs.

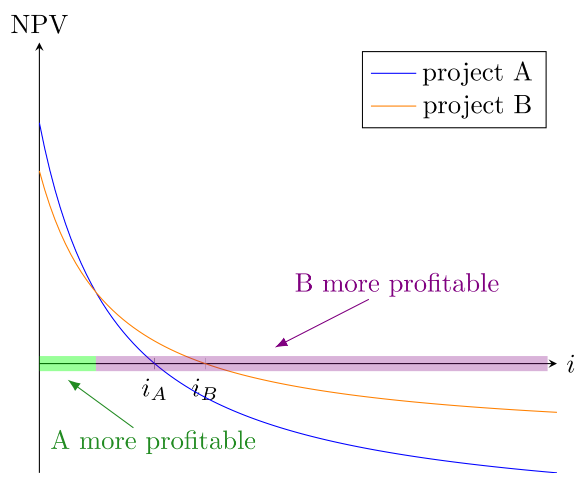
3.1.9 A "typical" NPV graph looks like the following — a strictly decreasing function in i which crosses x-axis exactly once:

[Intuition 💡: Usually this happens when the early CFs are relatively more negative, and the later CFs are relatively more positive (which are "typical"). Then, as the interest rate i rises, those relatively more positive CFs get discounted more heavily ➔ reducing NPV.]



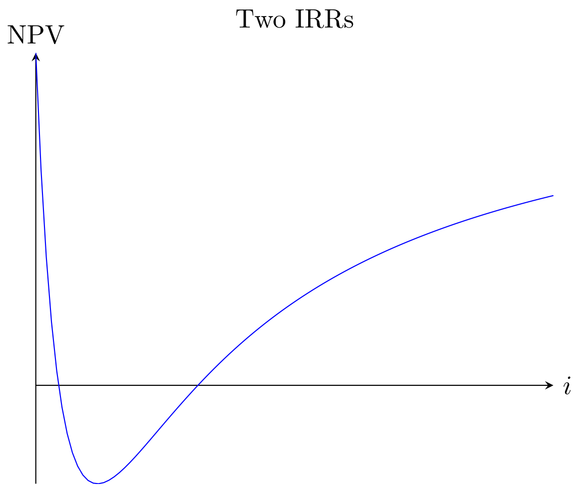
Under this "typical" case, the project is profitable (i.e., NPV > 0) if and only if the interest rate $i <$ IRR, giving a profitability criterion based on IRR.

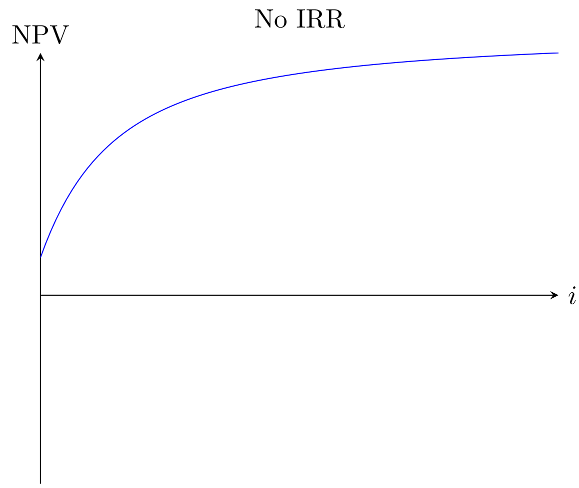
3.1.10 However, unlike NPV, IRR *cannot* be used for comparing profitability of different projects ⚠. To see this, consider the following graph:



As seen in the graph, IRR of project B (i_B) is greater than IRR of project A (i_A), but project A is more profitable than project B for some i!

3.1.11 In some "non-typical" case, the NPV graph can look weird and it can have *multiple roots* or even *no root*! ➔ IRR may *not* be unique and may *not* exist!





3.1.12 Now we are interested in knowing when IRR exists uniquely. Here we give a sufficient condition for its uniqueness, based on *Descartes' rule of signs*: Consider the project NCFs $C_0, C_1, \ldots, C_n$ as a sequence (in this order). Then, IRR of the project exists *uniquely* if there is *exactly one* sign change in the sequence (ignoring any zero NCF in the sequence, if exist).

[Note: This means when the sequence of NCFs (ignoring zero NCFs; in terms of sign) is of the form "$+ + \cdots + - - \cdots -$" or "$- - \cdots - + + \cdots +$", then IRR is guaranteed to exist uniquely.]

3.1.13 Now we discuss the last investment criteria here: *discounted payback period*. The **discounted payback period** (DPP) of the project at an interest rate i is

$$\text{DPP} = \min\{m \in \mathbb{N}_0 : \text{NPV}_m \geq 0\}$$

where $\text{NPV}_m = \sum_{t=0}^{m} C_t v^t$ is the **partial NPV up to time** m.

3.1.14 DPP can be regarded as the time where the project *first* "pays back" or "breaks even". If $\text{NPV}_n < 0$ for any $n \in \mathbb{N}_0$, then DPP *does not exist*.

3.1.15 After removing "v^t" in [3.1.13], the resulting time becomes **payback period** of the project at the interest rate i.

3.2 Reinvestment Rates

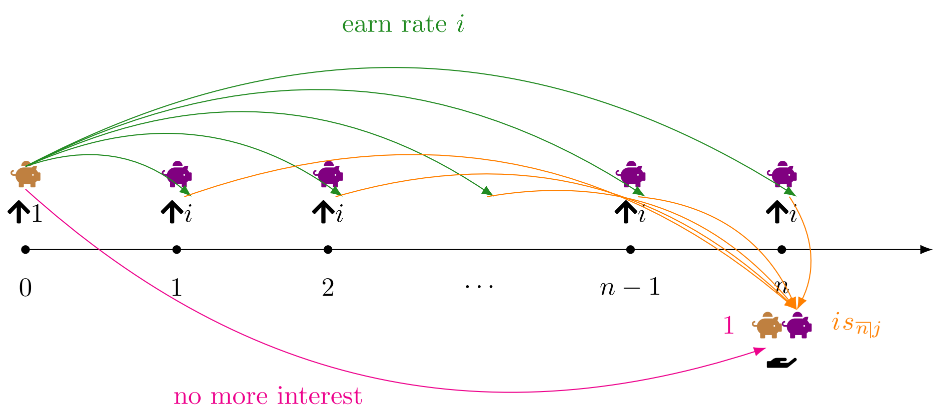
3.2.1 In section 3.1, the investment (or lending) rate is assumed to stay constant throughout the project. But this is not necessarily the case in practice: While *initial investment* may "capture good opportunities" and hence earn a relatively high rate, such "good opportunities" may be already gone for later *reinvestments* of interests, so those reinvestments may earn a *lower* rate (*reinvestment rate*).

3.2.2 Here we will derive some formulas with reinvestment rate j being distinct from the initial investment (interest) rate i. To be more precise,

- the initial investment earns interests (payable at the end of each period) at the rate i;
- interests paid are reinvested (immediately) at the rate j (earning further interests).

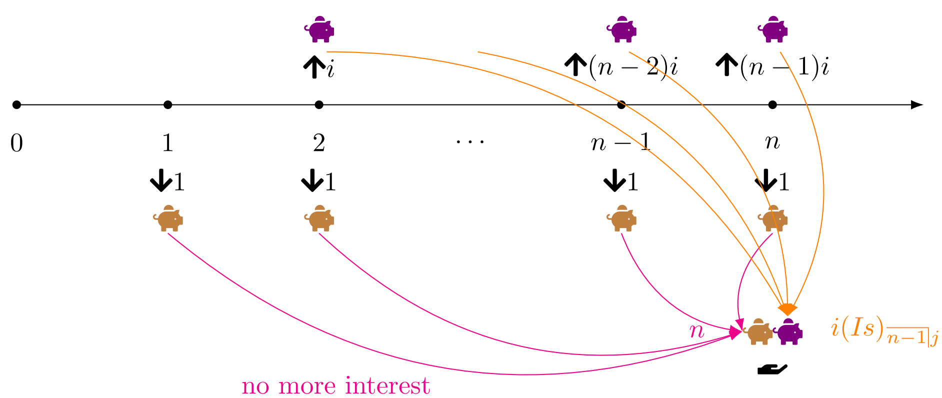
3.2.3 In this case, the accumulated value of 1 at time n is

$$1 + i s_{\overline{n}|j}.$$



3.2.4 The accumulated value of an n-period annuity-immediate (amount of each CF = 1) at time n is

$$n + i(Is)_{\overline{n-1}|j}.$$



[Note: The interest payment at time m is

$$i \times \text{amount in at time } m-1$$

(which is to be reinvested into immediately).]

3.3 Measures of Fund Return Rate

3.3.1 Consider here a fund where we invest money $\$$ in at time 0 and "clear" the balance in at time 1[5], with possibly additional contributions/withdrawals between time 0 and 1.

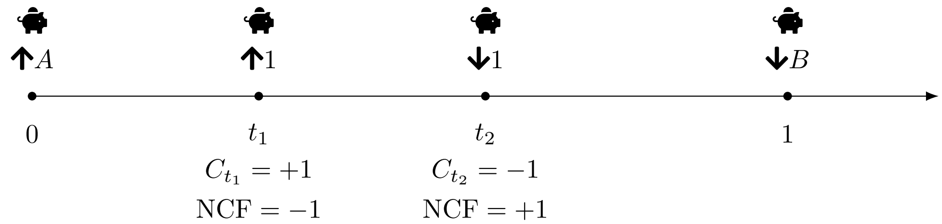
3.3.2 We introduce the following notations:

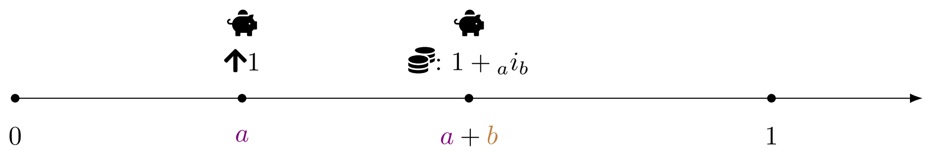
- A: amount in at time 0 → we invest A into at time 0
- B: amount in at time 1 (just before the "clearing") → we withdraw B out of at time 1[6]
- I: amount of interest earned between time 0 and 1
- C_t: net contribution to (net cash <u>out</u>flow from our perspective) at time t, where $0 < t < 1$ (finitely many) [Note: Here we regard withdrawal (cash <u>in</u>flow) as negative contribution to . (Note that contribution is cash <u>out</u>flow.)] [Warning: In this context, C_t is <u>no longer</u> NCF at time t! Indeed, it is *negative* of NCF at time t!]

[5] i.e., withdraw $\$$ when the balance is positive, or contribute $\$$ when the balance is negative (to make the fund balance zero)

[6] Here withdrawing a negative amount means contributing that amount (in absolute value) into .

- C: total net contribution to in time interval $(0,1)$: $C = \sum_t C_t$
- ${}_a i_b$: amount of interest earned between time a and $a+b$, when 1 is invested into at time a





3.3.3 Here we discuss the following measures of fund return rate:

- IRR
- dollar-weighted rate of return/interest (DWRR)
- "simplified" DWRR
- time-weighted rate of return/interest (TWRR)

3.3.4 Let i be the IRR. Then, by definition we have

$$\begin{aligned} & \text{NPV @ } i = 0 \\ \Leftrightarrow \quad & -A + \sum_t (-C_t) v^t + Bv = 0 \\ \Leftrightarrow \quad & B = A(1+i) + \sum_t C_t \underbrace{(1+i)^{1-t}}_{1 + {}_t i_{1-t}}. \end{aligned} \tag{1}$$

So, IRR is an interest rate i such that eq. (1) holds.

3.3.5 Now consider the DWRR. First we shall assume *simple interest* temporarily here. Then, eq. (1) becomes

$$B = A(1+i) + \sum_t C_t \underbrace{(1 + (1-t)i)}_{1 + {}_t i_{1-t}},$$

which implies

$$B - A - C = i\left[A + \sum_t C_t (1-t)\right]. \tag{2}$$

[Note: Recall that $C = \sum_t C_t$.]

Since changes in the amount in the fund are only due to the net contributions and interest, we have $B = A + \underbrace{C}_{\text{total net contrib.}} + \underbrace{I}_{\text{interest}}$, and hence can rewrite eq. (2) as

$$i = \boxed{\frac{I}{A + \sum_t C_t (1-t)}}.$$

This interest rate i is called **dollar-weighted rate of return** (DWRR), sometimes denoted by i^{DW}.

Remarks:

- From this, we see that DWRR is essentially IRR but assuming simple interest.
- The rate is "dollar-weighted" in the sense that the net contributions ("dollars" \$) influence the rate i (contribute to "weight" for i in eq. (2)).

3.3.6 Now, for "simplified" DWRR, it is simplified from DWRR in the sense that it assumes contributions/withdrawals can only possibly occur at time 0.5 ("middle")[7]. Due to this assumption, we have

$$\sum_t C_t(1-t) = 0.5C_{0.5} = 0.5C.$$

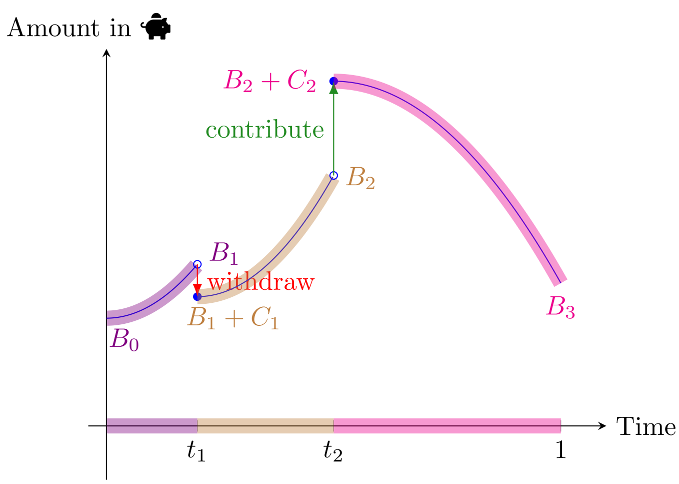
Hence, eq. (2) becomes

$$i = \boxed{\frac{2I}{A+B-I}}$$

and this interest rate i is **"simplified" DWRR** (this name is nonstandard).

[Note: The expression is *independent* from C, which is a remarkable property for "simplified" DWRR.]

3.3.7 The final measure to be discussed is TWRR. For this one, it is *not* related to IRR, unlike (simplified) DWRR. To understand TWRR, consider the following graph:



Here we let j_1, j_2, j_3 be the effective rate for the (whole) time intervals $[0, t_1], [t_1, t_2], [t_2, 1]$ respectively. Then,

$$B_0(1+j_1) = B_1, \quad (B_1 + C_1)(1+j_2) = B_2, \quad (B_2 + C_2)(1+j_3) = B_3.$$

In this case, the TWRR is

$$i^{\text{TW}} = (1+j_1)(1+j_2)(1+j_3),$$

which is "time-weighted" in the sense that the quantity only depends on the "inherent growth rate" of fund over *time*, but not the actual amount in . (We just deduce the "inherent growth rates" *based on* the observed amounts in , and calculate TWRR using those rates.)

3.3.8 In general, suppose there are $m-1$ time points $t_1, \ldots, t_{m-1}$ where the net contribution is nonzero (we shall denote them by $C_1, \ldots, C_{m-1}$ respectively). Then, using similar notations as above[8], we have

$$\begin{aligned} B_0(1+j_1) &= B_1 \\ (B_1+C_1)(1+j_2) &= B_2 \\ &\vdots \\ (B_{m-1}+C_{m-1})(1+j_m) &= B_m, \end{aligned}$$

[7] It may be deemed "reasonable" when the net contributions spread quite "uniformly" → we "take average" in some sense.

[8] For example, $B_1, \ldots, B_{m-1}$ are amounts/balances in "*just before*" time $t_1, \ldots, t_{m-1}$ respectively.

and the **time-weighted rate of return** (TWRR), sometimes denoted by i^{TW}, is

$$i^{\mathrm{TW}} = \boxed{(1+j_1)\cdots(1+j_m) - 1}.$$

4 Amortization Schedules and Sinking Funds

4.0.1 There are two methods for repaying a *loan*:

(a) **amortization method**: The loan is repaid by installments (this process is referred to as **amortization** of the loan).

→ Each installment has a portion for "repaying principal" and the rest is for "repaying interest".

(b) **sinking fund method**: The *interest on loan* (always on the *whole* loan amount → constant) is repaid by installments, and the whole principal is repaid by a *lump-sum* payment at the *end* of the term of loan, sourced from making *deposits* into a fund (called *sinking fund*) which also earns interest but possibly at a different rate from loan.

→ Each installment (not including the final lump-sum payment) is *only* for repaying interest.

4.1 Amortization Method

4.1.1 We consider two cases for amortization method:

(a) level installments are made at regular intervals;

(b) (possibly) non-level installments are made at regular intervals.

4.1.2 For the first case with level installments, we introduce the following notations:

- L: amount borrowed (principal)
- n: number of installments
- R: amount of each installment
- i: (annual effective) interest rate for the loan

Here we assume WLOG that the time between two successive installments is a year (to match with the measurement period for i). If this is not the case, we can use j-method, and then replace i by j in the following. (Similar for the non-level installments case.)

4.1.3 For an **amortized loan**, we have by definition

$$\text{PV of all installments} = L,$$

or symbolically,

$$L = Ra_{\overline{n}|i},$$

which implies that the amount of each installment is

$$R = \frac{L}{a_{\overline{n}|i}} = \frac{Li}{1-v^n}.$$

4.1.4 Now, consider the following *amortization schedule* (decomposition of each installment into "repayment of principal" and "repayment of interest":

Time k	Installment amount	Interest repaid $I_k = B_{k-1}i$	Principal repaid $P_k = R - I_k$	Outstanding balance B_k
0	0	0	0	$L\ (= B_0)$
1	R	$B_0 i$	$R - I_1$	$B_0 - P_1$
2	R	$B_1 i$	$R - I_2$	$B_1 - P_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	R	$B_{n-1} i$	$R - I_n$	$B_{n-1} - P_n = 0$

[Note: The term of the loan represented here is n periods, so by definition, the outstanding balance at time n has to be zero.]

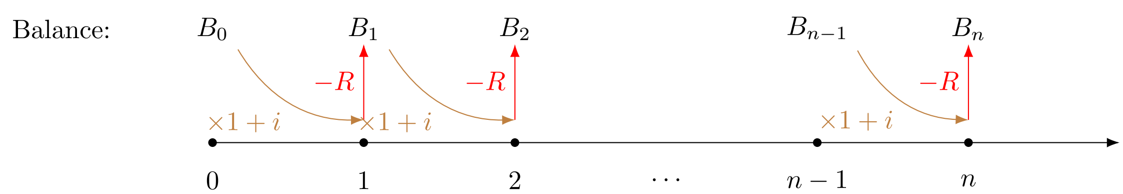
4.1.5 From [4.1.4], we can observe the following pattern:

- $B_1 = B_0 - P_1 = B_0 - R - B_0 i = B_0(1+i) - R$
- $B_2 = B_1 - P_2 = B_1 - R - B_1 i = B_1(1+i) - R$
- $\vdots$

In general, for any $k = 1, \ldots, n$, we have

$$B_k = \boxed{B_{k-1}(1+i) - R}.$$

This is a *recursive* formula.



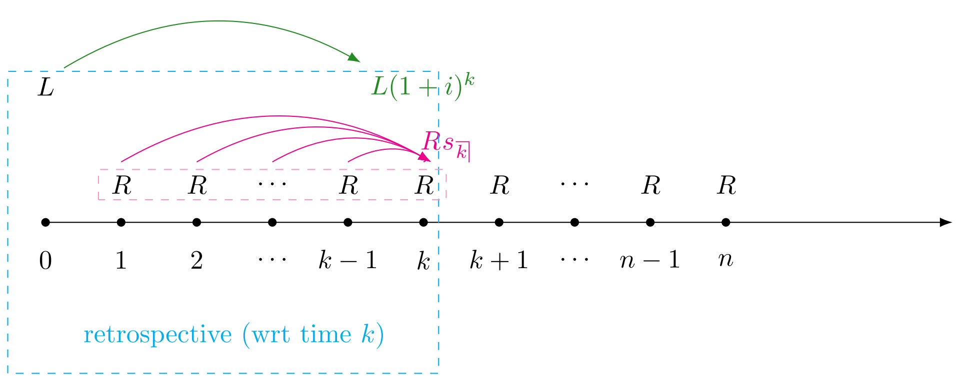
4.1.6 Now, based on [4.1.5], we can deduce:

- $B_0 = L$
- $B_1 = B_0(1+i) - R = L(1+i) - R$
- $B_2 = B_1(1+i) - R = L(1+i)^2 - R(1+i) - R = L(1+i)^2 - Rs_{\overline{2}|i}$
- $\vdots$

In general, for any $k = 0, \ldots, n$, we have

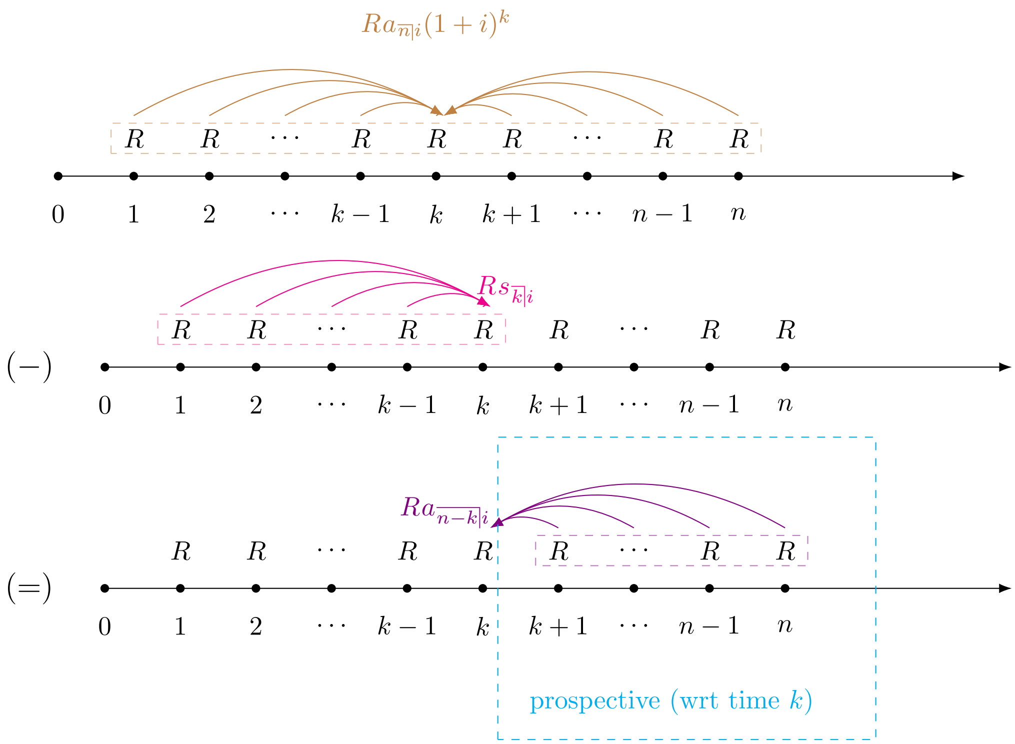
$$B_k = \boxed{L(1+i)^k - Rs_{\overline{k}|i}}.$$

This is a *retrospective* formula.



4.1.7 Based on [4.1.6], we can derive a *prospective* formula. Since $L = Ra_{\overline{n}|i}$, for any $k = 0, \ldots, n$, we have

$$B_k = Ra_{\overline{n}|i}(1+i)^k - Rs_{\overline{k}|i} = \boxed{Ra_{\overline{n-k}|i}}.$$



4.1.8 From [4.1.7], we can obtain formulas for interest repaid I_k and principal P_k: For any $k = 0, \ldots, n$,

- $I_k = B_{k-1}i = Ra_{\overline{n-k}|i}\, i = \boxed{R(1 - v^{n-k-1})}$;
- $P_k = R - I_k = R - R(1 - v^{n-k-1}) = \boxed{Rv^{n-k-1}}$.

These are useful for *directly* computing P_k and I_k for any k, without performing iterative computations like the amortization schedule in [4.1.4].

[Note: The assumption that the installments are *level* is important for deriving these formulas. We do not have formulas like these when the installments are non-level.]

4.1.9 Now we consider the second case with (possibly) non-level installments. The notations are largely similar to the ones in [4.1.2], except that we denote the amount of installment at time k by R_k, for any $k = 1, \ldots, n$ (where $R_1, \ldots, R_n$ may be distinct).

4.1.10 Following the definition in [4.1.3], we have

$$L = R_1 v + R_2 v^2 + \cdots + R_n v^n.$$

4.1.11 For this case, the amortization schedule can be obtained by modifying [4.1.4] slightly:

Time k	Installment amount	Interest repaid $I_k = B_{k-1}i$	Principal repaid $P_k = R - I_k$	Outstanding balance B_k
0	0	0	0	$L\ (= B_0)$
1	R_1	$B_0 i$	$R_1 - I_1$	$B_0 - P_1$
2	R_2	$B_1 i$	$R_1 - I_2$	$B_1 - P_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	R_n	$B_{n-1}i$	$R_n - I_{n-1}$	$B_{n-1} - P_n = 0$

4.1.12 In a similar manner as before, we can derive the following formulas:

(a) (recursive) $B_k = B_{k-1}(1+i) - R_k$;

(b) (retrospective) $B_k = L(1+i)^k - R_1(1+i)^{k-1} - R_2(1+i)^{k-2} - \cdots - R_k$;

(c) (prospective) $B_k = R_{k+1}v + R_{k+2}v^2 + \cdots + R_n v^{n-k}$,

for any $k = 1, \ldots, n$.

4.2 Sinking Fund Method

4.2.1 Here we shall focus on the case where the sinking fund deposits are *level*, and we introduce the following notations:

- L: amount borrowed (principal)
- n: number of installments
- D: amount of each sinking fund deposit
- R: total amount of payment made at each time point (which is the installment for interest on loan $ + D, i.e., $Li + D$)
- i: (annual effective) interest rate for the loan
- j: (annual effective) interest rate earned on sinking fund

4.2.2 For a **sinking fund**, we have by definition

$$\text{AV of all sinking fund deposits at time } n \text{ (@ rate } j) = L.$$

Symbolically, in this case we have

$$L = Ds_{\overline{n}|j}.$$

4.2.3 Now, by [4.2.2], we have

$$R = Li + D = \boxed{L\left(i + \frac{1}{s_{\overline{n}|j}}\right)},$$

which gives a formula for computing the *total amount of payment* R.

[⚠ Warning: R here is <u>not</u> the amount of installment for interest on loan! That installment is *always* Li (constant) for the sinking fund method, by definition.]

4.2.4 We can express the formula in [4.2.3] more compactly by defining the notation $a_{\overline{n}|i\&j}$ in the following way:

$$\frac{1}{a_{\overline{n}|i\&j}} = i + \frac{1}{s_{\overline{n}|j}}.$$

[Intuition 💡: To intuitively understand this choice of notation, note that we have

$$\frac{1}{a_{\overline{n}|i}} = i + \frac{1}{s_{\overline{n}|i}}$$

since

$$\text{LHS} = \frac{i}{1-v^n} = \frac{i(1+i)^n}{(1+i)^n - 1} = \frac{i[(1+i)^n - 1] + i}{(1+i)^n - 1} = i + \frac{i}{(1+i)^n - 1} = \text{RHS}.$$

]

Using this notation, we can write

$$R = \frac{L}{a_{\overline{n}|i\&j}}.$$

Particularly, when $i = j$, we have

$$R = \frac{L}{a_{\overline{n}|i\&i}} = \frac{L}{a_{\overline{n}|i}},$$

which turns out to be the same as the expression of "R" for amortization method (though the meanings of two "R"s are different).

5 Bonds and Other Securities

5.0.1 A **security** is a tradable financial asset.

5.0.2 Here, we will discuss the following securities:

- bonds (main focus)
- common stocks
- preferred stocks

For other kinds of securities like forwards, futures, and options, see STAT3905.

5.1 Bonds

5.1.1 A **bond** is an interest-bearing security which promises to pay stated amount(s) of money at some future date(s).

Remarks:

- This definition is rather general, so there are indeed many kinds of bonds. But here we shall focus on some "simple" bonds. (Nonetheless, we will briefly discuss some variants in sections 5.5 and 5.6.)
- **Coupons** are periodic *level* payments promised in a bond, which are made at the end of each period (year unless otherwise specified).
- A bond with nonzero (zero) coupons is called a **coupon-paying bond** (**zero-coupon bond**).

5.1.2 For simplicity, we shall make the following assumptions here:

(a) all obligations for paying money are met — there is no *default* ("breaking promise") [Note: See STAT3904 for a way to handle the *default risk*.];

(b) bond has a fixed **maturity** (i.e., time at which all promised payments are made) [Note: This will be loosened when we discuss *callable bonds* later in section 5.6.];

(c) we are only interested in finding bond price (given a "yield rate"[9]) *immediately after* a "coupon" payment date, or at time 0 [Note: Here time 0 is the *purchasing time* of the bond.].

[Note: The **term** of a bond is the time length between time 0 and the maturity date. For a bond with n periods term, we call it **n-period bond**.]

5.1.3 The main questions of interest here are:

(a) Given a specific yield rate, what is the bond price (at time 0, without loss of generality[10])?

(b) Given a time-0 bond price (purchasing price), what is the (implied) yield rate? [Note: Loosely, this means "How high is the return if the bond is purchased at this price?"]

5.1.4 To answer these questions, it is useful to introduce some notations for describing a bond:

- F: **face value/par value/nominal value** (value "printed" on the "face" of bond, used as reference for calculating payment amount(s))
- P: purchasing price (time-0 price) of the bond
- A: purchasing price per unit nominal (i.e., expressed as a unit where 1 unit $= F$)
  [Note: A units $= P$ by definition, so $A = P/F$.]
- r: **coupon rate** (or **nominal yield**), which is the *total* amount of coupon(s) in each year expressed as a fraction of face value: $r = \dfrac{\text{annual coupon amount}}{F}$

[9] This is related to the concept of IRR in section 3.

[10] For time-k price of an n-year bond (where "time k" here is supposed to be the time immediately after payment of kth coupon), it is the same as the time-0 price of an otherwise identical bond, but with term $n - k$ years.

- n: total number of coupons to be paid throughout
- C: redemption value, i.e., amount of payment promised to be made at *maturity* (not including coupon at that time!)
- g: **modified coupon rate**, i.e., annual coupon amount expressed as a fraction of *redemption value*: $g = \dfrac{\text{annual coupon amount}}{C}$
- R: redemption value per unit nominal: $R = C/F$; the bond is
  - **redeemable at par** if $R = 1$ (redemption value = par value);
  - **redeemable above par** if $R > 1$ (redemption value > par value);
  - **redeemable below par** if $R < 1$ (redemption value < par value)
- i: **yield rate/yield to maturity/yield to redemption** — By treating initial payment of purchasing price P as cash outflow and the rest of incomes from bond as cash inflows, we can regard the bond as a "project", and then i is simply the yield rate (IRR) of the "project"

  [Note: The terms "to maturity" and "to redemption" are related to the interpretation of IRR: rate of return for a "fund" which accumulates *to maturity/redemption*.]
- K: present value of C (treated as an amount at *maturity*) at yield rate i
- G: **base amount**, i.e., the amount we need to invest into a fund at rate i such that the periodic interest payments (assumed to be paid at the end of each period) are identical to the coupon payments from the bond ➔ $Gi = Fr$.

Graphical Illustration

	$+Gi$	$+Gi$	$\cdots$	$+Gi$	$+Gi + C$
	or	or		or	or
	$+Cg$	$+Cg$	$\cdots$	$+Cg$	$+Cg + C$
	or	or		or	or
$-P$	$+Fr$	$+Fr$	$\cdots$	$+Fr$	$+Fr + C$
0	1	2	$\cdots$	$n-1$	n

5.1.5 Now we are ready to answer the first question in [5.1.3]. To determine the time-0 bond price given a specific yield rate i, we have the following four formulas:

(a) basic formula: $P = Fra_{\overline{n}|i} + Cv^n$

(b) premium/discount formula: $P = C + C(g - i)a_{\overline{n}|i}$

(c) base amount formula: $P = G + (C - G)v^n$

(d) Makeham's formula: $P = K + \dfrac{g}{i}(C - K)$

5.1.6 For basic formula, it is given by

$$P = Fra_{\overline{n}|i} + \underbrace{Cv^n}_{K},$$

which directly follows from the fact that i is the IRR of the "project" and the definition of IRR.

5.1.7 To derive premium/discount formula, we start from the basic formula:

$$P = Fra_{\overline{n}|i} + Cv^n = Fra_{\overline{n}|i} + C(1 - ia_{\overline{n}|i}) = C + (\underbrace{Fr}_{Cg} - Ci)a_{\overline{n}|i} = \boxed{C + C(g - i)a_{\overline{n}|i}}.$$

To understand why the formula is called "premium/discount formula", we first introduce the following terminologies: The bond is

- **sold at premium** if $P > C$ (purchasing price is "in excess of" redemption value ➔ "premium");
- **sold at discount** if $P < C$ (purchasing price is "cheaper than" redemption value ➔ "discount").

When the bond is sold at premium (discount), the **premium** (**discount**) is given by $P - C$ ($C - P$) (the difference between purchasing price and redemption value).

Thus, the term $C(g - i)a_{\overline{n}|i}$ in the premium/discount formula is the premium or discount (in absolute value) ➔ hence "premium/discount".

[Note: By premium/discount formula, we can see that the bond is sold at premium (discount) iff $g > i$ ($g < i$).]

5.1.8 For deriving base amount formula, consider:

$$P = Fra_{\overline{n}|i} + Cv^n = G\underbrace{ia_{\overline{n}|i}}_{1-v^n} + Cv^n = \boxed{G + (C - G)v^n}.$$

The main feature of base amount formula is that there is not "$a_{\overline{n}|i}$" and we just have "v^n" ➔ relatively simpler to compute.

5.1.9 Finally, Makeham's formula can be derived as follows:

$$P = Fra_{\overline{n}|i} + Cv^n = Cg\underbrace{a_{\overline{n}|i}}_{\frac{1-v^n}{i}} + K = \boxed{K + \frac{g}{i}(C - K)}.$$

[Intuition 💡: By comparing with basic formula, we see that $\frac{g}{i}(C - K)$ is the PV of all coupons ($Fra_{\overline{n}|i}$). Then, consider:

- $g > i$ ➔ sold at premium ➔ PV of coupons is relatively "higher" as $g/i > 1$. (We "pay extra for it")
- $g < i$ ➔ sold at discount ➔ PV of coupons is relatively "lower" as $g/i < 1$. (We "pay less, so have less".)

]

5.1.10 Answering the second question in [5.1.3] is quite straightforward since essentially we are just finding IRR of a "project", so standard approaches (e.g. numerical) apply.

5.2 Incorporating Income and Capital Gains Taxes

5.2.1 In practice, there are taxes charged on the bond payments. Two typical taxes are income and capital gain taxes. Here we consider the situation that *income* tax is payable for each coupon payment (treated as "income"), and *capital gain* tax is payable (at the maturity) *in case* the redemption value (C) exceeds purchasing price (P) (the difference $C - P$ is treated as "capital gain").

5.2.2 First consider the case with income tax (and no capital gains tax). Let t_{inc} be the income tax rate. Then, after charging income tax on each coupon payment, the amount of each coupon decreases from Fr to $Fr(1 - t_{\text{inc}})$. After making such modification on the coupon amount, we can obtain the following bond pricing formulas:

(a) basic formula: $P = Fr(1 - t_{\text{inc}})a_{\overline{n}|i} + Cv^n$

(b) premium/discount formula: $P = C + C[g(1 - t_{\text{inc}}) - i]a_{\overline{n}|i}$

(c) Makeham's formula: $P = K + \dfrac{g(1 - t_{\text{inc}})}{i}(C - K)$

5.2.3 To incorporate capital gains tax, it is more tricky since we also need to decide *whether* there is a capital gain (and hence capital gains tax). As mentioned previously, there is a capital gain iff $C > P$. Based on premium/discount formula, we know that the condition is equivalent to:

- $g < i$ (no income tax), or
- $g(1 - t_{\text{inc}}) < i$ (income tax rate: t_{inc}).

[Note: This equivalent criterion is helpful for deciding whether there is capital gains tax payable.]

5.2.4 Now, suppose that capital gains tax is payable, and let t_{cap} be the capital gains tax rate. Then, the capital gains tax payable at the maturity is $(C - P)t_{\text{cap}}$, which can be treated as an extra cash outflow (from the perspective of bond purchaser) at the maturity. Hence, we just need to add $(C - P)t_{\text{cap}}v^n$ at the RHS of bond pricing formulas in [5.1.5] to handle the capital gains tax.

5.3 Incorporating Inflation

5.3.1 Another practical element involved is *inflation*, which affects the *real* amounts of future cash flows.

5.3.2 To track the inflation, we usually utilize a *price level index*, denoted by $Q(t)$ for time-t value. The defining property of a **price level index** is that the **real present value** of a time-t cash flow $Q(t)(1+i)^t$ is $Q(0)$ (so both interest and inflation effects are incorporated). Expressing differently, the real present value of a time-t cash flow C is

$$C \cdot \underbrace{\frac{Q(0)}{Q(t)}}_{\text{inflation}} \cdot \underbrace{v^t}_{\text{interest}}.$$

5.3.3 Thus, given a price level index, we can simply modify present value to *real* present value to incorporate inflation. For example, basic formula becomes

$$P = Fr\left(\frac{Q(0)}{Q(1)}v + \frac{Q(0)}{Q(2)}v^2 + \cdots + \frac{Q(0)}{Q(n)}v^n\right) + C \cdot \frac{Q(0)}{Q(n)}v^n$$

(assuming absence of taxes here; but income and capital gains taxes can also be incorporated in a similar manner as before).

5.4 Bond Amortization

5.4.1 Since bond *seller* is effectively borrowing money from bond *buyer*, and the loan is repaid by installments (in the form of coupon payments and final payment of redemption value), we can also do *loan amortization* for a *bond* (from the perspective of bond seller), like section 4.1.

5.4.2 For bond amortization, an important quantity is **book value** of a bond at time k ($k = 0, \ldots, n$), which is the time-k price of the bond (i.e., time-0 price of an otherwise identical bond but with $n - k$ periods term). Here, denote the time-k book value by B_k. Then, by basic formula, for any $k = 0, \ldots, n$,

$$B_k = Fra_{\overline{n-k|}} + Cv^{n-k}. \tag{3}$$

5.4.3 To relate book value and loan amortization, the time-k book value can actually be understood as the time-k *outstanding balance* for the loan (implied by the bond), by comparing eq. (3) with the prospective method in loan amortization.

[⚠ Warning: Equation (3) is <u>not</u> *completely* identical to the formula for prospective method. They are the same for any $k = 0, \ldots, n - 1$, but when $k = n$, the time-n book value here is C (while the time-n outstanding balance is zero for loan amortization).]

5.4.4 With this understanding, we can also develop similar formulas corresponding to the recursive and retrospective methods in loan amortization: [Note: We need to have some special treatment for time-n book value, due to the distinction mentioned above.] For any $k = 0, \ldots, n$,

- (recursive) $B_k = B_{k-1}(1 + i) - Fr$ [⚠ Warning: We do <u>not</u> have $B_n = B_{n-1}(1 + i) - Fr - C$ because of the special treatment!]

- (retrospective) $B_k = P(1+i)^k - Frs_{\overline{k}|}$ [Note: P is the "amount of loan" in this context.] [⚠ Warning: We do <u>not</u> have $B_n = P(1+i)^k - Frs_{\overline{n}|} - C$ because of the special treatment!]

5.4.5 Now we introduce a notation to denote changes in book values: $\Delta B_k = B_{k+1} - B_k$, which is given by $B_k i - Fr$ by recursive formula.

5.4.6 By premium/discount formula, for any $k = 0, \dots, n$, we have

$$B_k = \underbrace{C}_{B_n} + \underbrace{C(g-i)a_{\overline{n-k}|}}_{\bigstar}.$$

From here we can observe that:

- When the bond is sold at premium ($g > i$), $\bigstar$ gradually $\downarrow$ to zero (from a positive value) as k $\uparrow$. Hence, $\Delta B_k < 0$ for any $k = 1, \dots, n$.
  <u>Remarks</u>:
  - $\bigstar$ is amount of premium (with respect to time-k price).
  - This process is known as **amortization of premium** or **writing down**.
- When the bond is sold at discount ($g < i$), $\bigstar$ gradually $\uparrow$ to zero (from a negative value) as k $\uparrow$. Hence, $\Delta B_k > 0$ for any $k = 1, \dots, n$.
  <u>Remarks</u>:
  - $\bigstar$ (in absolute value) is amount of discount (with respect to time-k price).
  - This process is known as **accumulation of discount** or **writing up**.

In either case, the time-k book value gets closer and closer to the time-n book value C as k $\uparrow$. Intuitively, this is because the premium is "amortized", or the discount is "cancelled out" by "accumulation".

5.4.7 We can also develop a *bond amortization schedule*, like [4.1.4]:

Time k	"Installment" (coupon) amount	Interest repaid $I_k = B_{k-1}i$	Principal repaid $P_k = Cg - I_k$	Book value B_k
0	0	0	0	$C + C(g-i)a_{\overline{n}\|}$
1	Cg	$B_0 i$	$Cg - I_1$	$C + C(g-i)a_{\overline{n-1}\|}$
2	Cg	$B_1 i$	$Cg - I_2$	$C + C(g-i)a_{\overline{n-2}\|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	Cg	$B_{n-1}i$	$Cg - I_n$	C

5.4.8 From [5.4.7], we can derive a formula for P_k for any $k = 1, \dots, n$:

$$P_k = Cg - iB_{k-1} = Cg - i[C + C(g-i)a_{n-k+1}] = C(g-i)(1 - ia_{n-k+1}) = \boxed{C(g-i)v^{n-k+1}}.$$

Since $P_k = -\Delta B_{k-1}$, this also suggests a formula for ΔB_k for any $k = 0, \dots, n-1$.

5.5 Serial Bonds

5.5.1 A **serial bond** is a series of (possibly) coupon-paying ("ordinary") bonds with the same g and i, but possibly different n (maturity dates differ) and C (redemption values differ).

[Note: Thus, a serial bond is essentially a portfolio of (possibly) coupon-paying bonds.]

5.5.2 To price a serial bond, we need to price the bonds in the series individually and sum them up. To do this in a convenient way, we use *Makeham's formula*.

5.5.3 Suppose that there are m bonds in the series, with maturity dates $n_1, \ldots, n_m$ and redemption values $C_1, \ldots, C_m$ respectively. We also denote the present values of redemption values by $K_1, \ldots, K_m$, and the (time-0) prices of m bonds by $P_1, \ldots, P_m$ respectively. Then, the (time-0) price of serial bond is

$$\sum_{i=1}^{m} P_i = \sum_{i=1}^{m} \left(K_i + \frac{g}{i}(C_i - K_i)\right) = \sum_{i=1}^{m} K_i + \frac{g}{i}\left(\sum_{i=1}^{m} C_i - \sum_{i=1}^{m} K_i\right) = \boxed{K' + \frac{g}{i}(C' - K')}$$

where $K' = \sum_{i=1}^{m} K_i$ and $C' = \sum_{i=1}^{m} C_i$.

[Note: The expression is still quite like the form in Makeham's formula: We just need to use K' and C' instead of simply K and C.]

5.6 Callable Bonds

5.6.1 A **callable bond** is a bond where the bond issuer (seller) has an *option* to redeem (i.e., pay the redemption value to buyer) early to force the bond to mature at that time ("call" the bond).

[Note: Since there are different possible time for the issuer to call the bond, the bond term (time length from time 0 to maturity) is *not fixed* for a callable bond.]

5.6.2 To make the analysis of callable bond "more tractable", one way is to assume that the option is utilized such that the result is *optimal for the issuer* (and hence worst for the buyer[11]). So, from the perspective of *bond buyer*, we carry out calculations in "worst-case scenario".

[Note: More precisely, "worst result" for bond buyer means the present value of cash inflows (from the perspective of bond buyer) is the lowest.[12]]

5.6.3 A general first step for performing such calculations is to compare the hypothetical purchasing prices that *would* result under different scenarios [⚠ Warning: Those hypothetical purchasing prices should not be confused with the *actual* purchasing price!] → *lowest one* is the worst for bond buyer since it implies that under that scenario, the present value of all future cash inflows is the lowest.

After that, the lowest value is the price of the callable bond since it corresponds to the worst-case scenario for bond buyer.

5.7 Preferred Stocks

5.7.1 A **preferred stock** is a security which provides fixed dividend payments at the end of each period forever.

[Note: A preferred stock is like a bond, but it is an *ownership security* rather than a *debt security*. Also, since the payment continues forever, it has no maturity date.]

5.7.2 Given a specific yield rate i, the purchasing price P of a preferred stock with dividend amount D can be found using a similar way as basic formula:

$$P = Da_{\overline{\infty}|i} = \boxed{\frac{D}{i}}.$$

5.8 Common Stocks

5.8.1 A **common stock** a security which provides dividend payments (which are not fixed) at the end of each period forever.

[11] This is because bond transaction can be regarded as a *zero-sum game* (ignoring transaction costs etc.).

[12] Note that the initial cash outflow (payment of bond price) can be ignored since it is the same no matter when the issuer calls the bond: It has already been paid, and changes in call timing would not affect that amount!

5.8.2 The future dividends for a common stock are influenced by many factors (e.g. profitability, economic environment etc.). Different people may have different "views" on future dividends, even with the knowledge of the same *current information*. Still, an "overall" view from the market is reflected by the current market price of the common stock. (See STAT3904 for more discussions about this.)

5.8.3 Suppose that a specific yield rate i is given. Then, consider a common stock where the amounts of dividends at time 1, 2, 3, ... are in a geometric sequence: D, $D(1+k)$, $D(1+k)^2, \ldots$ (resp.) with $-1 < k < i$ (so that dividends are all positive and the geometric sum below is finite).

The purchasing price P of the common stock is

$$P = Dv\left[1 + \frac{1+k}{1+i} + \left(\frac{1+k}{1+i}\right)^2 + \cdots\right] = \frac{Dv}{1 - \frac{1+k}{1+i}} = \boxed{\frac{D}{i-k}}.$$

6 Evolution of Interest Rates

6.0.1 In this section, we will focus on studying how interest rates/yield rates *evolve*, both over different *bond terms* (sections 6.1 and 6.2) and over *time* (in a random manner) (section 6.3).

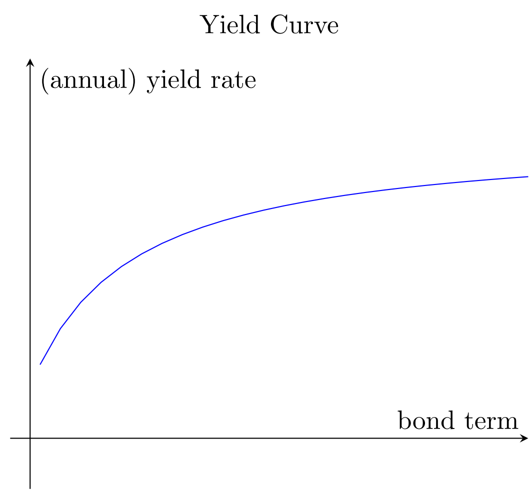
6.1 Term Structure of Interest Rates

6.1.1 In section 5, we have assumed that yield rate stays constant throughout, regardless of bond term.

However, this is clearly *not* true in practice: We can observe that for otherwise identical bonds with different terms, the (implied) yield rates differ. This suggests that yield rate appears to have some relationship with bond term ➔ a single value is *not* sufficient to describe yield rate!

[Note: The phenomenon that yield rate varies according to bond term is called **term structure of interest rates**.]

6.1.2 To represent the term structure of interest rates, we can use a **yield curve** (a function mapping from term to respective yield rate).



6.1.3 In attempt to *explain* this phenomenon, there are three typical theories:

(a) expectation theory

(b) liquidity preference

(c) market segmentation

6.1.4 For **expectation theory**:

- expected ↓ in future interest rate
  - ➔ we want to "capture" high interest rate available now for *longer time*
  - ➔ short-term (long-term) bond becomes *less attractive* (*more attractive*)
  - ➔ purchasing price ↓ (↑)
  - ➔ implied yield rate ↑ (↓)
- expected ↑ in future interest rate
  - ➔ we do not want to "lock in" a low interest rate available now for too long
  - ➔ short-term (long-term) bond becomes *more attractive* (*less attractive*)
  - ➔ purchasing price ↑ (↓)
  - ➔ implied yield rate ↓ (↑)

6.1.5 For **liquidity preference**: Investors prefer to be "liquid" and want to have free access to their funds

- ➔ *naturally* inclined towards short-term rather than long-term bonds
- ➔ *higher* yield rate required for *long-term* bonds to compensate for this preference.

6.1.6 For **market segmentation**: Bonds of *different* terms are attractive for *different* investors (as they are used for different purposes)

- ➔ "different markets" for bonds of different terms (market "divided" into multiple segments)
- ➔ bonds of different terms are subject to different forces of supply and demand
- ➔ they have different prices and different yield rates.

6.2 Spot Rates, Forward Rates, and Par Yields

6.2.1 Apart from providing whole yield curve, there are other methods to describe term structure of interest rates (partially), using the following terminologies:

(a) spot rates

(b) forward rates

(c) par yields

[Note: The "underlying" measurement period for these rates is years by convention.]

6.2.2 A **spot rate** for a term of n periods (or n-period spot rate) is the (constant) *annual* interest rate equivalent to the n-period interest rate (both rates are applicable from time 0 to time n). (Equivalently, it is the annual yield rate of an n-period zero-coupon bond purchased at time 0[13].) The notation for n-period spot rate is s_n.

[Note: Typically n is an integer, but it does not have to be.]

6.2.3 For example, if we are given n-period spot rate for every $n \in \mathbb{N}$ (for describing the term structure of interest rates), we can find the purchasing price of a coupon-paying bond by basic formula as follows:

$$P = Fr\big[(1+s_1)^{-1} + (1+s_2)^{-2} + \cdots + (1+s_n)^{-n}\big] + C(1+s_n)^{-n}.$$

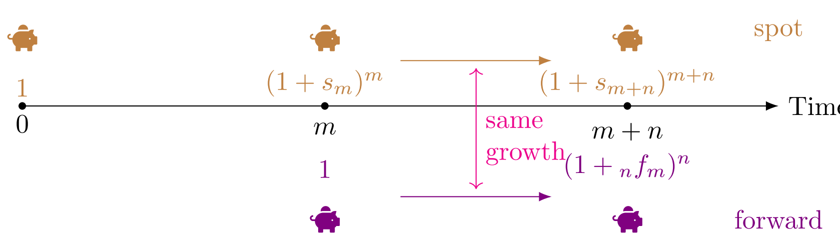
6.2.4 An m-period deferred n-period **forward rate** is the *annual* interest rate equivalent to the n-period interest rate (both rates are applicable from time m (starting time deferred for m periods) to time $m+n$), denoted by ${}_nf_m$. [⚠ Warning: This is <u>not</u> to be confused with ${}_mf_n$! The right subscript denotes the *starting time* and the left subscript denotes the *time length*.]

[Note: Typically m and n are integers, but they do not have to be.]

6.2.5 Consider spot rates and forward rates for the *same* term structure of interest rates. Since the "growth rate" is supposed to be independent from timing of initial investment (as in [1.2.2]b), we have the following relationship between spot and forward rates:

$$(1+{}_nf_m)^n = \frac{(1+s_{m+n})^{m+n}}{(1+s_m)^m},$$

for any m and n.



[13] under the condition that the interest rate earned in the zero-coupon bond is the same as the "market" interest rate (which should be assumed by default)

6.2.6 As a corollary of [6.2.5], we have

$$(1+s_n)^n = (1+{}_1f_0)(1+{}_1f_1)\cdots(1+{}_1f_{n-1})$$

for any $n \in \mathbb{N}$.

6.2.7 Finally, for *par yield*, it is related to a bond *redeemable at par* (i.e., $C = F$). To motivate the definition of par yield, we shall consider the following result.

Proposition 6.2.a. For a bond redeemable at par, its coupon rate r equals its yield rate i (assumed to be nonzero and exist uniquely) iff its purchasing price P equals its par value F.

Proof: "$\Rightarrow$": Assume that $r = i$. Then, by basic formula,

$$P = Fia_{\overline{n}|i} + Fv^n = F(1-v^n) + Fv^n = F.$$

"$\Leftarrow$": Assume that $P = F$. Then, by definition of IRR, we have

$$-F + Fra_{\overline{n}|i} + Fv^n = 0 \implies \frac{r}{i}(1-v^n) + v^n = 1 \implies r(1-v^n) = i(1-v^n) \implies r = i.$$

$\square$

6.2.8 An **n-period par yield** is a coupon rate for an n-period bond that is redeemable at par such that its price equals its par value ("sold at par"), denoted by i_{p_n} ($n \in \mathbb{N}$).

[Note: By proposition 6.2.a, equivalently, it is a coupon rate for a bond redeemable at par such that the coupon rate and yield rate coincide (when yield rate exists uniquely) ➔ hence named "par yield".]

6.2.9 Following the definition of par yield, given spot rates, we have by basic formula

$$F = Fi_{p_n}\left[(1+s_1)^{-1} + \cdots + (1+s_n)^{-n}\right] + F(1+s_n)^{-n},$$

which implies

$$i_{p_n} = \frac{1-(1+s_n)^{-n}}{(1+s_1)^{-1} + \cdots + (1+s_n)^{-n}}.$$

From here we can see that par yield also suggests relationship of a kind of "rate" with bond term n, "through" spot rates ➔ it also describes term structure of interest rates.

6.3 Stochastic Approach to Interest

6.3.1 So far all interest-related quantities are deterministic (non-random) ➔ *deterministic* approach to interest.

6.3.2 Here we are interested in investigating the behaviour of various quantities when we incorporate *randomness* to interest ➔ *stochastic* approach to interest, which may be seen as "more realistic".

6.3.3 We shall start by introducing randomness to *effective* interest rate for nth period: i_n, for every $n \in \mathbb{N}$. In stochastic approach to interest, we shall suppose the effective interest rate for nth period is a *random variable* I_n, for any $n \in \mathbb{N}$.

[Note: $I_1, I_2, \ldots$ may or may not be independent and they can have different distributions.]

6.3.4 Denote the mean and variance of I_n by j_n and s_n^2 respectively. Now, we are interested in what happens if we do the work in section 1 under this stochastic setting.

6.3.5 For example, for the time-n accumulated value of 1 invested at time $n-1$:

- its mean is $\mathbb{E}[1+I_n] = \boxed{1+j_n}$;
- its variance is $\mathrm{Var}\,(1+I_n) = \mathrm{Var}\,(I_n) = \boxed{s_n^2}$;

- its second moment is $\mathbb{E}\big[(1+I_n)^2\big] = \mathrm{Var}\,(1+I_n) + (\mathbb{E}[1+I_n])^2 = \boxed{s_n^2 + (1+j_n)^2}$.

6.3.6 For calculations involving *multiple* periods, they are affected by the *dependence structure* of $I_1, I_2, \ldots$. As a simple case, we focus on the time interval $[0, n]$ and $I_1, \ldots, I_n$ are *independent*.

Denote the time-n accumulated value of principal of 1 by S_n (this is $a(n)$). Then, by definition of effective interest rate, we have

$$S_n = (1+I_1)\cdots(1+I_n).$$

Due to independence, we can compute the following probabilistic quantities easily:

- mean: $\mathbb{E}[S_n] = \mathbb{E}[1+I_1]\cdots\mathbb{E}[1+I_n] = \boxed{(1+j_1)\cdots(1+j_n)}$
- second moment: $\mathbb{E}\big[S_n^2\big] = \mathbb{E}\big[(1+I_1)^2\big]\cdots\mathbb{E}\big[(1+I_n)^2\big] = \boxed{\big[s_1^2 + (1+j_1)^2\big]\cdots\big[s_n^2 + (1+j_n)^2\big]}$
- variance: $\mathrm{Var}\,(S_n) = \mathbb{E}\big[S_n^2\big] - (\mathbb{E}[S_n])^2 = \boxed{\big[s_1^2 + (1+j_1)^2\big]\cdots\big[s_n^2 + (1+j_n)^2\big] - [(1+j_1)\cdots(1+j_n)]^2}$

6.3.7 For further specification of the behaviour of stochastic interest, we may impose a class of distribution on each of $I_1, I_2, \ldots$, and a common one is *lognormal distribution*.

6.3.8 A random variable Y follows a **lognormal distribution** with parameters μ and σ^2 (denoted by $Y \sim LN(\mu, \sigma^2)$) if the "log" of Y, $\ln Y$, follows a normal distribution with mean μ and variance σ^2, i.e., $\ln Y \sim N(\mu, \sigma^2)$.

[⚠ Warning: The parameters μ and σ^2 are *not* the mean and variance of Y!]

6.3.9 To compute moments of Y, it is useful to recall that the moment generating function of a normal r.v. $X \sim N(\mu, \sigma^2)$:

$$M_X(t) = \mathbb{E}\big[e^{tX}\big] = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

Since $Y = e^X$, the first and second moments of Y are

$$\mathbb{E}[Y] = \mathbb{E}\big[e^X\big] = M_X(1) = \boxed{e^{\mu+\sigma^2/2}},$$

and

$$\mathbb{E}\big[Y^2\big] = \mathbb{E}\big[e^{2X}\big] = M_X(2) = \boxed{e^{2(\mu+\sigma^2)}}.$$

Hence, the variance of Y is

$$\mathrm{Var}\,(Y) = \mathbb{E}\big[Y^2\big] - (\mathbb{E}[Y])^2 = \boxed{e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)}.$$

6.3.10 To impose lognormal distribution here, we shall assume $(1+I_n) \sim LN(\mu_n, \sigma_n^2)$ (or equivalently, $\ln(1+I_n) \sim N(\mu_n, \sigma_n^2)$) for any $n \in \mathbb{N}$.

[Note: Usage of lognormal distribution on $1+I_n$ is "reasonable" here since $1+I_n$ should be positive (effective interest rate should be larger than -100%, even if negative interest rate is possible!).]

6.3.11 Again, we shall also assume that $I_1, \ldots, I_n$ are independent. Then, as functions of $I_1, \ldots, I_n$,

$$\ln(1+I_1), \ldots, \ln(1+I_n)$$

are also independent. Then, we have

$$\ln S_n = \ln\left(\prod_{t=1}^{n}(1+I_t)\right) = \sum_{t=1}^{n}\ln(1+I_t) \sim N\left(\sum_{t=1}^{n}\mu_t, \sum_{t=1}^{n}\sigma_t^2\right),$$

so in this case S_n is also lognormally distributed.

[Note: After knowing the distribution of S_n, we can use the formulas in [6.3.9] to compute its mean and variance.]

7 Duration, Convexity, and Immunization

7.1 Duration

7.1.1 Recall that we have discussed an approximation of "overall timing" of cash flows $C_1, \ldots, C_n$ (at time $1, \ldots, n$) in section 2.2, namely *method of equated time*:

$$\bar{t} = \frac{C_1 t_1 + \cdots + C_n t_n}{C_1 + \cdots + C_n},$$

7.1.2 Here we shall discuss a "better" approximation that takes into account time value of money: **Macaulay duration** (denoted by D_{mac}), defined by

$$D_{\text{mac}} = \frac{C_1 v^{t_1} t_1 + \cdots + C_n v^{t_n} t_n}{C_1 v^{t_1} + \cdots + C_n v^{t_n}}.$$

This approximates "overall timing" ("*duration*") of the cash flows.

7.1.3 It turns out that there is a connection between D_{mac} (about "timing") and *interest rate sensitivity*. Consider a project with those CFs, thus having NPV (at rate i):

$$P(i) = \sum_{t=1}^{n} C_t (1+i)^{-t}.$$

The interest rate sensitivity of a project can be loosely understood as how "sensitive" $P(i)$ is in respond to changes in rate i.

7.1.4 Mathematically, sensitivity is measured in an "infinitesimal" setting (like force of interest). We have:

$$\mathrm{d}P(i) = P(i) \times \text{sensitivity} \times \mathrm{d}i\,,$$

or

$$\text{sensitivity} = \frac{1}{P(i)} \cdot \frac{\mathrm{d}P(i)}{\mathrm{d}i}.$$

[Note: So, $P(i+h) - P(i) \approx P(i) \times \text{sensitivity} \times h$ for small h.] Typically, NPV is a strictly decreasing function of i, thus

$$\frac{\mathrm{d}P(i)}{\mathrm{d}i} \leq 0,$$

and this means the "sensitivity" above is often *negative* due to the typical inverse relationship between NPV and interest rate.

7.1.5 Conventionally, we want to have a (usually) positive number that measures "volatility" (higher ➔ more volatile). Hence, we define the **volatility** (or **modified duration**) by

$$-\frac{1}{P(i)} \cdot \frac{\mathrm{d}P(i)}{\mathrm{d}i}.$$

It is denoted by $\overline{v}$ (when the term "volatility" is used) or D_{mod} (when the term "modified duration" is used).

7.1.6 The reason why the volatility is also called *modified duration* is that the volatility can be expressed as:

$$-\frac{1}{P(i)} \cdot \frac{\mathrm{d}P(i)}{\mathrm{d}i} = -\frac{1}{\sum_{t=1}^{n} C_t (1+i)^{-t}} \cdot \frac{\mathrm{d}}{\mathrm{d}i} \sum_{t=1}^{n} C_t (1+i)^{-t} = \frac{1}{\sum_{t=1}^{n} C_t (1+i)^{-t}} \cdot \sum_{t=1}^{n} t(1+i)^{-t-1} C_t = \boxed{\frac{D_{\text{mac}}}{1+i}}.$$

[Note: This suggests the connection between D_{mac} and interest rate sensitivity.]

7.2 Convexity

7.2.1 The concept of convexity is somewhat related to interest rate sensitivity also. For *volatility*, we can observe that it is related to the first-order term in Taylor expansion. For *convexity*, it is related to the *second-order term* in Taylor expansion.

7.2.2 For first-order Taylor expansion, we have

$$P(i+h) \approx P(i) + P'(i)h = \boxed{P(i) - P(i)\overline{v}h}$$

when h is small.

[Note: This provides a formula for approximating NPV after changes in interest rate using volatility $\overline{v}$.]

7.2.3 For second-order Taylor expansion, we have

$$P(i+h) \approx P(i) + P'(i)h + \frac{1}{2}P''(i)h^2 = \boxed{P(i) - P(i)\overline{v}h + \frac{1}{2}P(i) \times \text{convexity} \times h^2}$$

when h is small.

[Note: This provides a formula for approximating NPV after changes in interest rate using both volatility and convexity (yielding a better approximation).]

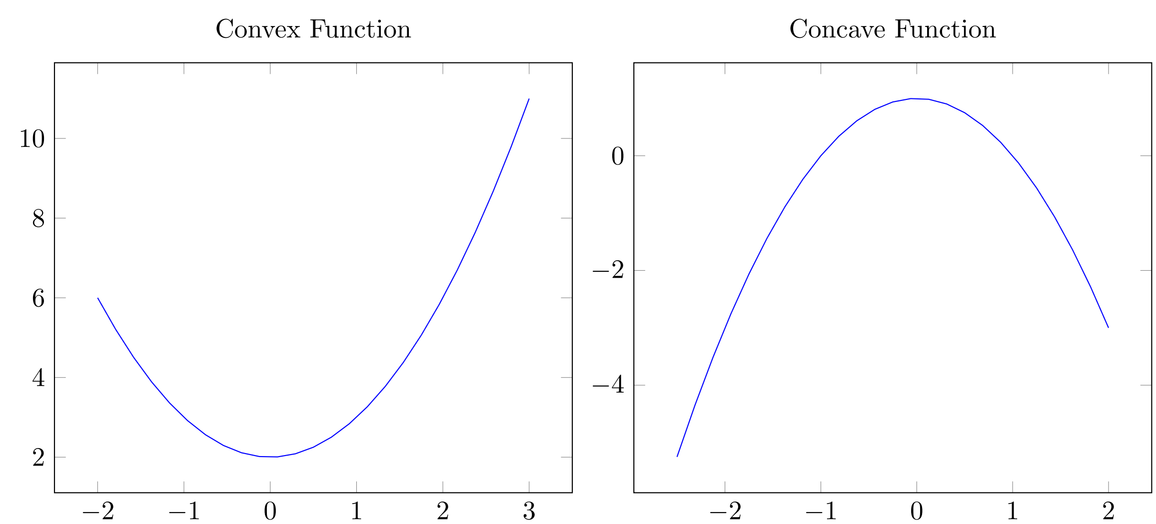
Here suggests the definition of convexity, denoted by $\overline{c}$:

$$\overline{c} = \frac{P''(i)}{P(i)}.$$

[Note: There is not negative sign in the definition of convexity, mainly to "match" with the mathematical notion of *convex*:

$$P(\cdot) \text{ is convex } \iff P''(i) \geq 0 \quad \forall i \iff \text{convexity always nonnegative.}$$

]



7.3 Immunization

7.3.1 We have discussed quantities related to interest rate sensitivity, so we are now interested in knowing how to *reduce* or *eliminate* the impact from interest rate movements (reducing/eliminating sensitivity).

7.3.2 **Immunization** is a technique to reduce or even eliminate the impact of interest rate movements on a project.

7.3.3 We introduce the following notations for the project:

- $P_A(i)$: NPV of all cash inflows (positive CFs) at rate i (from "assets")
- $P_L(i)$: NPV (in absolute value) of all cash outflows (negative CFs) at rate i (from "liabilities")

To perform immunization, we usually want to "match" assets and liabilities in some sense.

7.3.4 A simple immunization approach is known as *Redington immunization*. First, we assume that yield curve stays flat (i.e., yield rate is constant for any bond term) ➔ interest movement only leads to upward/downward shift in yield curve (not affecting its *shape*).

Note that the NPV of project at rate i is $P(i) = P_A(i) - P_L(i)$. To perform **Redington immunization**, we "match" assets and liabilities in our project to produce CFs such that

- at *current* interest rate i_0, $P(i_0) = 0$ (or $P_A(i_0) = P_L(i_0)$);
- $P'(i_0) = 0$ (or $P'_A(i_0) = P'_L(i_0)$);
- $P''(i_0) > 0$ (or $P''_A(i_0) > P''_L(i_0)$).

In this case, $P(i)$ would have a *local minimum* at $i = i_0$ (by second derivative test).[14] Hence, "small" movements of interest rate from i_0 in *either* direction would not ↓ the NPV[15] ➔ providing a "local" protection against interest rate risk.

[14]This means that there exists $\delta > 0$ such that $P(i_0) \leq P(i)$ for any $i \in (i_0 - \delta, i_0 + \delta)$.
[15]The resulting NPV is then nonnegative, since $P(i_0) = 0$.

References

Kellison, S. G. (2008). *Theory of interest* (3rd ed.). McGraw-Hill Education.

Concepts and Terminologies

Results

Section 1

- [1.3.3]: formulas for effective interest rate
- [1.5.4]: formulas for effective discount rate
- [1.6.5]: relationships between i and d when they are equivalent
- [1.7.5]: relationship between nominal and effective interest/discount rates under equivalency
- [1.8.4]: relationship between force of interest and effective interest rate under equivalency
- proposition 1.8.a: formulas regarding varying force of interest

Section 2

- [2.2.4]: approximation formula for method of equated time

  [Note: Amount of each CF is 1 for the following annuities unless stated otherwise.]

- [2.3.3]: PV of an n-period annuity-immediate
- [2.3.5]: PV of an n-period annuity-due
- [2.3.7]: time-n AV of an n-period annuity-immediate
- [2.3.8]: time-n AV of an n-period annuity-due
- [2.4.3]: PV of a perpetuity-immediate
- [2.4.4]: PV of a perpetuity-due
- [2.5.6]: PV of an n-period annuity-immediate ("CFs less frequent")
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- [2.8.6]: alternative formula for PV of an n-period increasing annuity
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- [2.8.14]: PV of an n-period annuity-immediate ("CFs more frequent" & "$\uparrow$ per $1/m$ of a period")
- [2.10.2]: PV of an n-period annuity-immediate with CFs varying in geometric sequence
- [2.11.3]: PV formula for an $2n-1$-period rainbow immediate based on "splitting" and "shifting"
- [2.11.4]: PV formula for an $2n-1$-period rainbow immediate based on "horizontal" and "diagonal" view
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- [2.13.2]: PV formula for an n-period "paused-from-time-m" increasing annuity based on "splitting" and "shifting"
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- [3.1.12]: a sufficient condition for the uniqueness of IRR
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Section 7

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