HKU STAT3901 Study Notes

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[Note: Related SOA Exam: FAM (long-term)]

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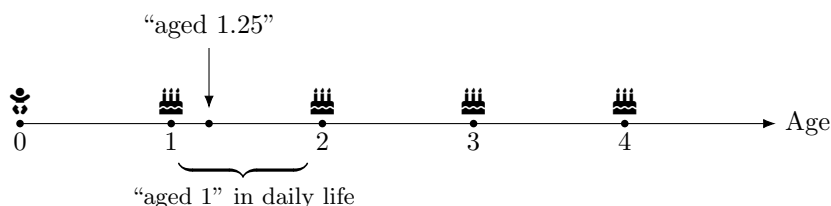
1 Survival Models

1.0.1 The study of life contingencies[1] is motivated from the needs of managing the risk arising from life insurance business 🏢. Generally speaking, life insurance products provide benefit payments that depend on the survival status of the life insured. For example, a life insurance may provide \$1 million death benefit payment at the death ☠ of the life insured.

From STAT2902, we know that the *present value* of the death benefit payment differs for different payment times. Therefore, the timing of the death of the life insured can greatly impact on the present value of the payment to be made by the insurer. If the life insured dies very early, then the insurer would need to make a rather large payment in terms of present value, hence incurring a great loss. Thus, *lifetime* of the underlying life insured is a critical element for risk management of such products.

1.0.2 Of course, no one can tell *exactly* what the lifetime of the life insured would be, so we need to deal with it using probabilistic tools: We shall model the lifetime using some probability distributions. This is the main theme of Section 1: study of survival models that are used for describing the probabilistic behaviours of the lifetime.

1.1 Future Lifetime Random Variable

1.1.1 Here, like STAT2902, we are measuring time in years unless otherwise specified. When we say that a life is aged x, it means that the life has survived for x years *exactly*. An important note is that the meaning of "aged x" here actually differs from the daily life one ⚠. In daily life language, "age" refers to the number of birthdays 🎂 passed, which is always an integer. But here the value of x can be fractional, although it is often an integer as well.

For convenience, sometimes we denote a life aged x by (x).

1.1.2 A survival model describes the probabilistic behaviours of the lifetime via a random variable known as *future lifetime random variable*. As its name suggests, this random variable measures the future lifetime of a life (aged x). Mathematically speaking, the **future lifetime random variable** for a life aged x is a <u>nonnegative continuous</u> random variable (denoted by T_x).
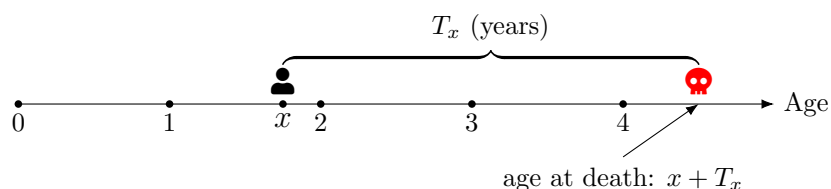
Each survival model corresponds to a probability distribution for T_x. Examples include uniform distribution, exponential distribution, etc.

1.1.3 Let us now introduce some notations that represent various kinds of probabilities about the lifetime random variable T_x:

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[1]The name "life contingencies" reflects the feature that we are studying things that are <u>contingent</u> on <u>life</u>.

Notation	Probability that (x) ...
$_tp_x = S_x(t) = \mathbb{P}(T_x > t)$	survives t years
$_tq_x = F_x(t) = \mathbb{P}(T_x \leq t)$	dies within t years
$_{u|t}q_x = \mathbb{P}(u < T_x \leq u + t)$	survives u years and dies in the subsequent t years

[Note: The "t"s in these notations can be dropped when $t = 1$.]

Examples:

- The probability that a life aged 30 survives until age 50 is $_{20}p_{30}$.
- The probability that a life aged 40 dies before age 50 is $_{10}q_{40}$.
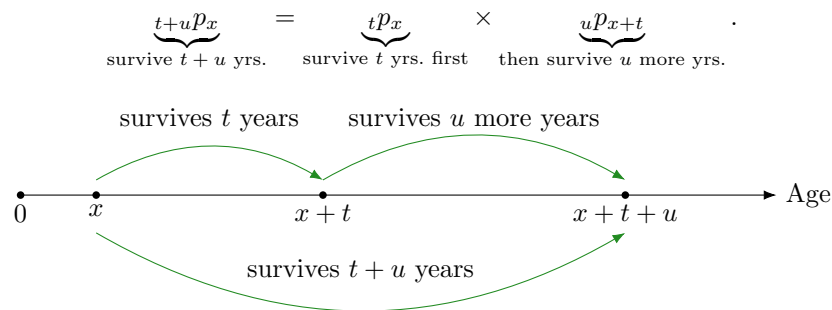- The probability that a life aged 20 dies between age 60 and 90 is $_{40|30}q_{20}$.

The notations involving "p" and "q" are the so-called *actuarial notations*, which are shorthands used by actuaries to simplify presentations[2]. It is quite normal for beginners to find them somewhat confusing and unnecessary ☺. Nonetheless, you would (hopefully!) appreciate them more as you learn more about life contingencies in STAT3901 and also STAT3909.

Remarks:

- $S_x(t)$ and $F_x(t)$ are the *survival function* and the *cumulative distribution function* (cdf) of T_x respectively.
- In general, the letter "p" is used in actuarial notations related to "survival" 🏃, and "q" is used in notations related to "death" 💀. We use "q" for the third probability as it is about <u>dying</u> in a certain time interval.

1.1.4 Next, let us introduce some useful formulas that facilitate our computations of these probabilities related to lifetimes.

(a) *Factorization formula for "p":*[3]

$$\underbrace{_{t+u}p_x}_{\text{survive } t+u \text{ yrs.}} = \underbrace{_tp_x}_{\text{survive } t \text{ yrs. first}} \times \underbrace{_up_{x+t}}_{\text{then survive } u \text{ more yrs.}} .$$

Both ways have the same starting point (age x) and the same destination (age $x+t+u$) with the same survival status (alive).

- *Special case:* $_np_x = \underbrace{p_x p_{x+1} \cdots p_{x+n-1}}_{n \text{ terms}}$

[⚠ Warning: There is <u>not</u> a "factorization formula for q"!]

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[2] Perhaps a secondary purpose is to "scare away" people that are unfamiliar with actuarial science? ☺

[3] *(If you are interested)* This can be proven mathematically by considering conditional probability, with the natural assumption that $T_0 - x | T_0 > x \overset{d}{=} T_x$, where "$\overset{d}{=}$" denotes equality in distribution. Verbally, this assumption means that for a life alive at age x, the lifetimes obtained by the following two methods should have the same distribution:

- measuring the remaining time to be lived from age x
- measuring the time lived from age 0 (newborn) and then subtracting x years

Such assumption is helpful for relating T_0 and T_x, which can in turn be utilized for relating the T's with different subscripts. For more details, see Dickson et al. (2019).

(b) *Formulas for $_{u|t}q_x$:*

- *"$p \times q$" form:*

$$_{u|t}q_x = \underbrace{_up_x}_{\text{survive } u \text{ yrs.}} \times \underbrace{_tq_{x+u}}_{\text{die in the subsequent } t \text{ yrs.}}$$
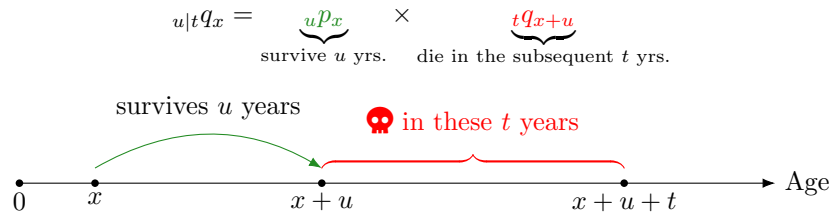
The two sides of the equation are pretty much referring to the same thing in this case.

- *"$p - p$" form:*

$$_{u|t}q_x = \underbrace{_up_x}_{\text{survive } u \text{ yrs.}} \underbrace{-_{u+t}p_x}_{\text{but not survive } u + t \text{ yrs}}$$

- *"$q - q$" form:*

$$_{u|t}q_x = \underbrace{_{u+t}q_x}_{\text{die in } u + t \text{ yrs.}} \underbrace{-_uq_x}_{\text{but not die in } u \text{ yrs.}}$$

1.2 Force of Mortality

1.2.1 The concept of *force of mortality* is analogous to *force of interest* learnt in STAT2902. Recall that for the force of interest δ_t, we can interpret "amount at time $t \times \delta_t \times \Delta t$" as the approximated amount of interest earned in the time period $[t, t + \Delta t]$, when Δt is small.

Here, instead of considering interest, which <u>increases</u> the <u>amount of money</u> 🪙, we are considering *mortality* (hence the name "force of mortality"), which <u>decreases</u> the <u>survival probability</u> 🧍.

1.2.2 The **force of mortality** at age x, denoted by μ_x, is defined by

$$\mu_x = -\frac{S_0'(x)}{S_0(x)} = -\frac{\mathrm{d}}{\mathrm{d}x}\ln S_0(x).$$

The value $\mu_x \times \Delta x$ can be interpreted as the approximated probability for (x) to die in Δx years, when Δx is small.

For example, if you are now aged x and your force of mortality at age x is μ_x, then the approximated probability for you to die 💀 within the next 24 hours would be $\mu_x \times (1/365)$ (hopefully very small!) (assuming 1 day equals $1/365$ years).

1.2.3 After defining force of mortality, let us introduce some formulas related to it:

(a) *Survival function $_tp_x \to$ force of mortality μ_{x+t}:*

$$\mu_{x+t} = -\frac{S_x'(t)}{S_x(t)} = -\frac{\mathrm{d}}{\mathrm{d}t}\ln S_x(t).$$

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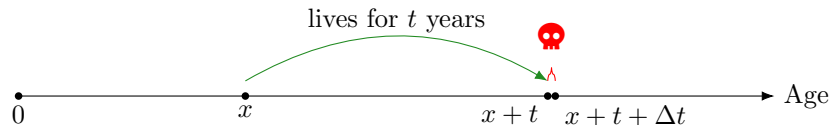
[⚠ Warning: Not differentiating with respect to x here!]

(b) *Force of mortality $\mu_{x+t} \to$ survival function $_tp_x$:*

$$_tp_x = \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right) = \exp\left(-\int_x^{x+t} \mu_r\,\mathrm{d}r\right)$$

[Note: Analogous formula for force of interest (STAT2902):

$$\frac{A(u+t)}{A(u)} = \exp\left(\int_0^t \delta_{u+s}\,\mathrm{d}s\right) = \exp\left(\int_u^{u+t} \delta_r\,\mathrm{d}r\right).$$

]

(c) *Probability density function (pdf) of T_x: $f_x(t) = {_tp_x} \times \mu_{x+t}$.*
Verbal interpretation of this formula is easier after multiplying both sides by a small Δt, resulting in:

$$\underbrace{f_x(t)\Delta t}_{\approx \text{ prob. that } T_x \in [t, t+\Delta t]} = \underbrace{_tp_x}_{\text{survive } t \text{ yrs.}} \times \underbrace{\mu_{x+t}\Delta t}_{\approx \text{ die in the subsequent } \Delta t \text{ yrs.}}.$$

(d) *Impacts on survival function after adjusting force of mortality:*

	Force of mortality	Survival function
Addition	$\mu_{x+s} \to \mu_{x+s} + \phi \quad \forall s \in [0,t]$	$_tp_x \to e^{-\phi t}\,_tp_x$
Scalar multiplication	$\mu_{x+s} \to k\mu_{x+s} \quad \forall s \in [0,t]$	$_tp_x \to \left(_tp_x\right)^k$

[Note: These formulas are helpful for handling questions involving changes in force of mortality, which usually arise from some revisions on the mortality assumption.]

1.3 Curtate Future Lifetime Random Variable

1.3.1 For now, we are exclusively dealing with the lifetime random variable T_x, which is continuous. Now, suppose that we are only interested in the remaining (integer) number of years to be lived for (x), or in other words, the number of birthdays remaining for (x). To measure that number, we can use the **curtate future lifetime random variable** for (x), denoted by K_x, which is the integer part of T_x, i.e., $\lfloor T_x \rfloor$. Note that K_x is a nonnegative discrete random variable.

1.3.2 To compute probabilities related to K_x, the typical strategy is to relate it with T_x, as we are more familiar with probability calculations for T_x. Here are some formulas: For any $k = 0, 1, , \ldots,$

- *probability mass function (pmf) of K_x:* $\mathbb{P}(K_x = k) = {_{k|}q_x} \overset{[1.1.4]b}{=} {_kp_x}\,q_{x+k}$
- *cdf of K_x:* $\mathbb{P}(K_x \le k) = {_{k+1}q_x}$ [⚠ Warning: Not $_kq_x$!]

1.4 Shortcuts for Moments of Future Lifetimes

1.4.1 Of course, moments of future lifetime random variable can be obtained by definition, but usually it is not the most efficient approach. Here, we will introduce some shortcut formulas for obtaining moments of various kinds of future lifetime random variables:

- *Continuous future lifetime: T_x*
- *Capped continuous future lifetime: $T_x \wedge n = \min\{T_x, n\}$*
- *Curtate future lifetime: K_x*
- *Capped curtate future lifetime: $K_x \wedge n$*

[Note: n is a positive integer.]

1.4.2 **Continuous future lifetime T_x.**

- *Mean (or complete expectation of life):* $\mathring{e}_x = \mathbb{E}[T_x] = \int_0^\infty {}_t p_x \, \mathrm{d}t.$

  Proof. Note that

  $$\mathbb{E}[T_x] = \mathbb{E}\left[\int_0^{T_x} 1 \, \mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{t<T_x\}} \, \mathrm{d}t\right] = \int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{t<T_x\}}\right] \mathrm{d}t = \int_0^\infty {}_t p_x \, \mathrm{d}t.$$

  $\square$

- *(Less commonly used) Second moment:* $\mathbb{E}\left[T_x^2\right] = \int_0^\infty 2t \times {}_t p_x \, \mathrm{d}t.$

  Proof. By introducing a double integral and changing order of integrations, we have

  $$\mathbb{E}\left[T_x^2\right] = \int_0^\infty s^2 f_x(s) \, \mathrm{d}s = \int_0^\infty \int_0^s 2t f_x(s) \mathrm{d}t \, \mathrm{d}s = \int_0^\infty 2t \int_t^\infty f_x(s) \, \mathrm{d}s \, \mathrm{d}t = \int_0^\infty 2t \times {}_t p_x \, \mathrm{d}t.$$

  $\square$

1.4.3 Capped continuous future lifetime $T_x \wedge n$.

- *Mean (or temporary complete expectation of life):* $\mathring{e}_{x:\overline{n}|} = \int_0^n {}_t p_x \, \mathrm{d}t.$

  Proof. Note that

  $$\mathbb{E}[T_x \wedge n] = \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{t<T_x \wedge n\}} \, \mathrm{d}t\right] = \int_0^n \mathbb{E}\left[\mathbf{1}_{\{t<T_x\}}\right] \mathrm{d}t = \int_0^n {}_t p_x \, \mathrm{d}t.$$

  $\square$

1.4.4 Curtate future lifetime K_x. The formulas are somewhat analogous to those in [1.4.2].

- *Mean (or curtate expectation of life):* $e_x = \mathbb{E}[K_x] = \sum_{k=1}^\infty {}_k p_x.$ [⚠ Warning: The sum starts at $k=1$ rather than $k=0$.]

  Proof. Note that

  $$\mathbb{E}[K_x] = \mathbb{E}\left[\sum_{k=1}^{K_x} 1\right] = \mathbb{E}\left[\sum_{k=1}^\infty \mathbf{1}_{\{k \le K_x\}}\right] = \sum_{k=1}^\infty \mathbb{E}\left[\mathbf{1}_{\{k \le T_x\}}\right] = \sum_{k=1}^\infty {}_k p_x.$$

  (Here $\sum_{k=1}^{K_x} 1$ is regarded as 0 if $K_x = 0$.) $\square$

- *(Less commonly used) Second moment:* $\mathbb{E}\left[K_x^2\right] = \sum_{k=1}^\infty (2k-1)_k p_x.$

  Proof. By introducing a double <u>sum</u> and changing order of <u>summations</u>, we have

  $$\mathbb{E}\left[K_x^2\right] = \sum_{j=0}^\infty j^2 \mathbb{P}(K_x = j) = \sum_{j=0}^\infty \sum_{k=1}^j (2k-1)\mathbb{P}(K_x = j) = \sum_{k=0}^\infty (2k-1) \underbrace{\sum_{j=k}^\infty \mathbb{P}(K_x = j)}_{\mathbb{P}(K_x \ge k)} = \sum_{k=0}^\infty (2k-1)_k p_x.$$

  [Note: Here, $\sum_{k=1}^j (2k-1) = 0$ when $j = 0$. One can prove that $\sum_{k=1}^j (2k-1) = j^2$ for any $j = 1, 2, \ldots$ by induction, or a proof without words (see this Wikipedia page).] $\square$

1.4.5 Capped curtate future lifetime $K_x \wedge n$.

- *Mean (or temporary curtate expectation of life):* $e_{x:\overline{n}|} = \sum_{k=1}^n {}_k p_x.$

  Proof. Note that

  $$\mathbb{E}[K_x \wedge n] = \mathbb{E}\left[\sum_{k=1}^{K_x \wedge n} 1\right] = \mathbb{E}\left[\sum_{k=1}^n \mathbf{1}_{\{k \le K_x\}}\right] = \sum_{k=1}^n \mathbb{E}\left[\mathbf{1}_{\{k \le T_x\}}\right] = \sum_{k=1}^n {}_k p_x.$$

 $\square$

1.5 Recursions for Expectations of Life

1.5.1 *Recursion* is a very important concept in life contingencies. This is the first appearance of recursion, and it will appear frequently throughout STAT3901 (and even STAT3909!). Time diagram is an indispensable tool for understanding the recursive formulas, and solving problems involving recursions would become much easier 👍 once you grasp the skills of drawing the appropriate pictures for recursions (or just having them in your mind 🧠).

1.5.2 The recursive formulas for different types of expectations of life are as follows.

- *Complete:* $\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$

- *Curtate:* $e_x = e_{x:\overline{n}|} + {}_np_xe_{x+n} \overset{(n=1)}{=} p_x(1 + e_{x+1})$
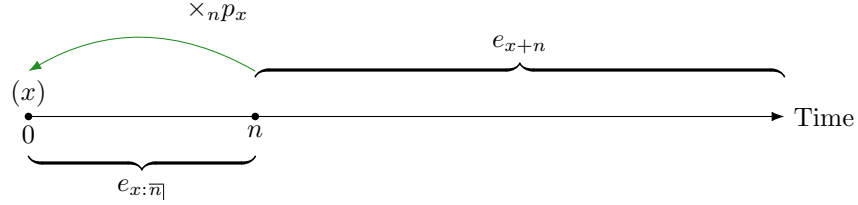
- *Temporary complete:* $\mathring{e}_{x:\overline{n}|} \overset{(m \leq n)}{=} \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n-m}|}$

- *Temporary curtate:* $e_{x:\overline{n}|} \overset{(m \leq n)}{=} e_{x:\overline{m}|} + {}_mp_xe_{x+m:\overline{n-m}|} \overset{(m=1)}{=} p_x(1 + e_{x+1:\overline{n-1}|})$
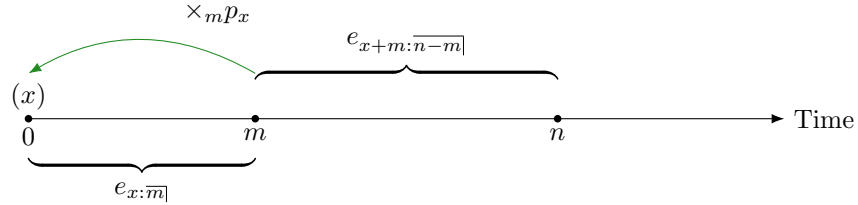
(If you are interested) The idea behind these pictures is related to how we derive the recursive formulas, namely by "splitting" the terms and factoring out some constants. For example:

$$
\begin{aligned}
e_x &= e_{x:\overline{n}|} + {}_{n+1}p_x + {}_{n+2}p_x + {}_{n+3}p_x + \cdots \\
&= e_{x:\overline{n}|} + {}_np_x \times p_{x+n} + {}_np_x \times {}_2p_{x+n} + {}_np_x \times {}_3p_{x+n} + \cdots \\
&= e_{x:\overline{n}|} + {}_np_x(p_{x+n} + {}_2p_{x+n} + {}_3p_{x+n} + \cdots) \\
&= e_{x:\overline{n}|} + \underbrace{{}_np_x}_{\text{"like" discounting factor}} e_{x+n}.
\end{aligned}
$$

1.5.3 Recursions here are helpful for recalculating expectations of life after changes in mortality assumption in early time period. For example, if there is an adjustment on the value of p_x only, then we can utilize the recursive formula $e_x = p_x(1 + e_{x+1})$ to recalculate e_x conveniently, since e_{x+1} would not be affected by this adjustment, and we just need to update the value of p_x on the RHS to get the new e_x.

1.6 Mortality Laws

1.6.1 So far we have primarily discussed calculations of probabilistic quantities given a survival model. But how should we select a survival model for modelling the lifetime? We will discuss two approaches in Section 1:

(1) Mortality laws (to be discussed here)

(2) Life tables (to be discussed in Section 1.7)

The first one is based on some parametric formulas for forces of mortality, and the second one is a data-based approach.

1.6.2 Here we will introduce three kinds of mortality laws:

(1) **Exponential** (or constant force of mortality)

(2) **Generalized uniform** (or **generalized de Moivre's law**)

(3) **Makeham's law**

The formulas for μ_x, $_tp_x$, $\mathring{e}_x$ and $\mathrm{Var}\,(T_x)$ for these mortality laws are as follows.

	Exponential	Generalized uniform/de Moivre ($\alpha = 1$: uniform/de Moivre)	Makeham ($A = 0$: Gompertz)
Parameters	$\mu > 0$	$\alpha > 0,\ \omega > x$	$A, B > 0,\ c > 1$
Distribution of T_x	$\mathrm{Exp}\,(\mu)$	$\alpha = 1$: $\mathrm{U}[0, \omega - x]$	(not standard)
μ_x	μ	$\dfrac{\alpha}{\omega - x}$	$A + Bc^x$
$_tp_x$	$e^{-\mu t}$	$\left(1 - \dfrac{t}{\omega - x}\right)^{\alpha}$	$\exp\left[-At - \dfrac{B}{\ln c}c^x(c^t - 1)\right]$
$\mathring{e}_x$	$1/\mu$	$\dfrac{\omega - x}{\alpha + 1}$	(no closed form)
$\mathrm{Var}\,(T_x)$	$1/\mu^2$	$\alpha = 1$: $\dfrac{(\omega - x)^2}{12}$	(no closed form)

[Note: Here μ is the *rate* parameter of the exponential distribution.] The formula of $_tp_x$ for Makeham's law is rather complex, but it is not necessary to memorize it because:

- *(in SOA exam FAM)* it is included in the SOA FAM-L table

- *(in STAT3901)* we can just memorize the formula of μ_x instead and use [1.2.3]b to perform calculations related to the survival function

1.7 Life Tables

1.7.1 Practically, the insurer often uses a *life table* ⊞ to model the lifetime, instead of the mortality laws in Section 1.6. Although there are some "nice" formulas available for those mortality laws, they can only describe lifetimes in certain "preconfigured" ways, which limits the flexibility. On the other hand, life table can be constructed based on historical data about real mortality experience of individuals, which allows a more accurate description of the pattern of human mortality. Particularly, as the insurer has sold many policies, a large amount of data would be available for constructing some custom life tables.

1.7.2 A basic life table looks like the following:

Age x	ℓ_x	d_x
30	10000	50.25
31	9949.75	60.11
32	9889.64	62.89
33	9826.75	70.37
$\vdots$	$\vdots$	$\vdots$

The idea behind the construction of life table is that we start with monitoring the survival status of a large group of people at a certain starting age (e.g., 30 here), and then we check the number of survivors in the group in each of the subsequent ages. Through this, we can deduce some behaviour

about the human mortality. Fractional values appear in the life table because the values are scaled such that $\ell_{30} = 10000$ exactly (for convenience). For more details about the construction of life table, see Bowers et al. (1997).

1.7.3 Theoretically, by construction, the symbol ℓ_x refers to the *expected* number of survivors to age x, and the symbol d_x refers to the *expected* number of deaths 💀 between age x and age $x + 1$. (In practice, the values of ℓ_x's and d_x's in a life table are often inserted/estimated based on some real past mortality figures.)

Practically, to use the life table for calculating probabilistic quantities about the lifetime, we would use the following formulas (which are derived based on the theoretical meanings of ℓ_x and d_x above):

Probability	Using ℓ_x	Using $d_x(= \ell_x - \ell_{x+1})$	
${}_t p_x$	$\dfrac{\ell_{x+t}}{\ell_x}$		
${}_t q_x$	$1 - \dfrac{\ell_{x+t}}{\ell_x}$	$\dfrac{\overbrace{d_x + \cdots + d_{x+t-1}}^{t \text{ terms}}}{\ell_x}$	$\overset{(t=1)}{=} \dfrac{d_x}{\ell_x}$
${}_{u\mid t} q_x$	$\dfrac{\ell_{x+u} - \ell_{x+u+t}}{\ell_x}$	$\dfrac{\overbrace{d_{x+u} + \cdots + d_{x+u+t-1}}^{t \text{ terms}}}{\ell_x}$	$\overset{(t=1)}{=} \dfrac{d_{x+u}}{\ell_x}$

1.8 Fractional Age Assumptions

1.8.1 In a life table, often the quantities (ℓ_x, d_x etc.) are given only for *integer* ages x. In such case, it is impossible to compute probabilities involving *fractional* ages and years (e.g. ${}_{0.5}p_{50.2}, {}_{0.7}q_{60}$, etc.), without further assumptions.

To improve the flexibility, we may impose an additional assumption on the behaviour of the lifetime in fractional age (called **fractional age assumption**). The fractional age assumptions to be discussed here are:

- *(most common)* **uniform distribution of deaths** (UDD):
  <u>linear</u> interpolation between ℓ_x and ℓ_{x+1}.

- **constant force of mortality** (CFM):
  <u>exponential</u> interpolation between ℓ_x and ℓ_{x+1}, i.e., linear interpolation between $\ln \ell_x$ and $\ln \ell_{x+1}$.

- *(least common)* **Balducci assumption**:
  <u>hyperbolic</u> interpolation between ℓ_x and ℓ_{x+1}, i.e., linear interpolation between $1/\ell_x$ and $1/\ell_{x+1}$.

1.8.2 The following collects some key 🔑 formulas for the fractional age assumptions. While memorizing 🧠 all the formulas below can allow you to solve problems more efficiently 👍, it is not necessary to do so. Instead, one can just memorize the more commonly used formulas below (labelled with "⭐"), because it is not too hard to derive the rest of the formulas (which are less commonly used) just by using the formulas with "⭐" (try it!).

In the table below, we assume that x is an integer age and that $s, t \geq 0$ are any values satisfying $0 \leq s + t \leq 1$.

	UDD	CFM	Balducci
ℓ_{x+t}	$= (1-t)\ell_x + t\ell_{x+1} = \ell_x - td_x$	$\ell_x^{1-t}\ell_{x+1}^t$	solve $\dfrac{1}{\ell_{x+t}} = \dfrac{1-t}{\ell_x} + \dfrac{t}{\ell_{x+1}}$
$_tp_x$		$= {}_tp_{x+s} = (p_x)^t$	
$_tq_x$	$= {}_{s|t}q_x = tq_x$		
$_tq_{x+s}$	$\dfrac{tq_x}{1 - sq_x}$		
$_{1-t}q_{x+t}$			$(1-t)q_x$
μ_{x+t}	$\dfrac{q_x}{1 - tq_x}$	$= \mu_x = -\ln p_x \quad \forall t$	
$f_x(t)$	$q_x \quad \forall t$	$\mu_x e^{-\mu_x t}$	
$\mathring{e}_x$	$e_x + 1/2$		
$\mathrm{Var}(T_x)$	$\mathrm{Var}(K_x) + 1/12$		

Remarks:

- The "1/2" and "1/12" in the UDD formulas for $\mathring{e}_x$ and $\mathrm{Var}(T_x)$ correspond to the mean and variance of $\mathrm{U}[0,1]$. The idea is that under UDD, actually we can write $T_x = K_x + U_x$, where K_x and $U_x \sim \mathrm{U}[0,1]$ are independent.

- There are no "nice" relationships between $\mathring{e}_x$ & e_x and $\mathrm{Var}(T_x)$ & $\mathrm{Var}(K_x)$ for CFM and Balducci.

For finding quantities whose formulas are not found in the table, a general strategy is to express them in terms of some quantities with known formulas. Notably, often it is helpful to express them in terms of "ℓ"s. For example:

$$_{0.7|1.4}q_{41.6} = \frac{\ell_{42.3} - \ell_{43.7}}{\ell_{41.6}} = (\text{use } \begin{cases} \ell_{x+t} = (1-t)\ell_x + t\ell_{x+1} & \text{for UDD} \\ \ell_{x+t} = \ell_x^{1-t}\ell_{x+1}^t & \text{for CFM} \\ 1/\ell_{x+t} = (1-t)/\ell_x + t/\ell_{x+1} & \text{for Balducci} \end{cases} \quad \ldots).$$

[Note: For this example, in case we are only given q_x for $x = 41, 42, 43$, we can "artificially generate" the values of "ℓ"s by setting ℓ_{41} as a certain positive value, say 100 without loss of generality[4]. After that, we can find the values of "ℓ"s by:

$$\ell_{42} = 100(1 - q_{41}), \quad \ell_{43} = 100(1 - q_{41})(1 - q_{42}), \quad \ell_{44} = 100(1 - q_{41})(1 - q_{42})(1 - q_{43}).$$

We can then compute $\ell_{42.3}$, $\ell_{43.7}$, and $\ell_{41.6}$ using appropriate formulas.]

1.9 Select Survival Models and Tables

1.9.1 The key idea for a **select survival model** can be visualized as follows. Notably, there is a **select period** of d years, after which the effect from selection disappears.

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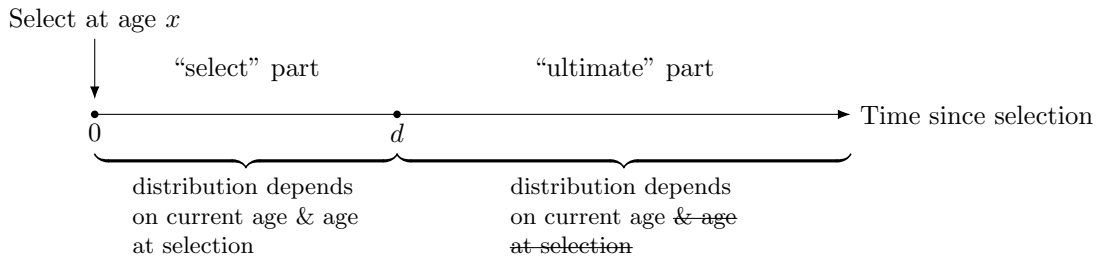
[4]The value set would not affect the result due to the cancelling effect of numerator and denominator.

A common example where "selection" arises is *underwriting* 🔍. The idea is that after performing underwriting on an individual at age x ("selection" at age x), the insurer knows more information about the individual and thus may apply a survival model that is different from the "standard" one for that individual. For example, a more "adverse" model may be used if it is known that the individual is quite unhealthy.

As time passes, the information collected from the underwriting should become more outdated and less relevant. This feature corresponds to the concept of *select period*, which specifies a time length after which the information from underwriting is completely not applicable, so the survival model becomes the "standard" one again.

1.9.2 With selection at age x, previous symbols and formulas can still be used with only one change: replacing every "x" by "$[x]$" to indicate the age at selection. Furthermore, we should simplify "$[x] + t$" (for a life aged $x + t$ and selected at age x) to just "$x + t$" if $t \geq d$, when the select period is d years, since the effect from selection has already been gone. Examples:

- "$_{u|t}q_x$" would become "$_{u|t}q_{[x]}$".
- "$\mathring{e}_x$" would become "$\mathring{e}_{[x]}$".
- The factorization formula "$_{t+u}p_x = {}_tp_x \times {}_up_{x+t}$" would become "$_{t+u}p_{[x]} = {}_tp_{[x]} \times {}_up_{[x]+t}$". [⚠ Warning: Not "$_{t+u}p_{[x]} = {}_tp_{[x]} \times {}_up_{[x+t]}$"! "$[x + t]$" means that the selection takes place at age $x + t$, which is <u>not</u> the case here.]

  Assuming $t \geq d$, we should simplify it further to "$_{t+u}p_{[x]} = {}_tp_{[x]} \times {}_up_{x+t}$"

  [⚠ Warning: Do <u>not</u> change "$[x]$" to "x"!]

1.9.3 Probabilistic calculations for select survival models are often based on *life tables*. In this case, we will consider a more advanced form of life table, called *select life table*, which gives us some quantities with square brackets, in addition to the ordinary ones.

A select life table can have the following two formats:

- *(used in AM92 table) Format 1: Current age fixed in each row.*
  For example, suppose the select period is 3 years:

Age x	$\ell_{[x]}$	$\ell_{[x-1]+1}$	$\ell_{[x-2]+2}$	$\ell_x(= \ell_{x-3+3})$
30	9950.15	9963.21	9964.27	10000
31	9901.78	9911.22	9925.12	9932.62
32	9860.23	9875.12	9882.12	9897.14
33	9825.13	9830.04	9842.11	9847.89
34	9801.43	9804.06	9812.61	9818.33
35	9762.13	9768.96	9775.37	9788.19
⋮	⋮	⋮	⋮	⋮

- *(more common form in SOA exam) Format 2: Age at selection fixed in each row.*
  For example, suppose the select period is 3 years:

Age x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	$\ell_{x+3}(= \ell_{[x]+3})$
30	9950.15	9911.22	9882.12	9847.89
31	9901.78	9875.12	9842.11	9818.33
32	9860.23	9830.04	9812.61	9788.19
⋮	⋮	⋮	⋮	⋮

Using these select tables, we can use the formulas in [1.7.3] (with some slight adjustments) to compute various quantities. For example, we have:

- $_2p_{[31]} = \dfrac{\ell_{[31]+2}}{\ell_{[31]}}$.

- $_2p_{[31]+1} = \dfrac{\ell_{34}}{\ell_{[31]+1}}$.

- $d_{[x]} = \ell_{[x]} - \ell_{[x]+1}$.

1.9.4 Lastly, let us explore several typical problems related to select survival model.

   (a) *Calculating $\ell_{[x]+t}$ with $t \in [0, d)$:* Given ℓ_{x+d} and some probabilities that allow us to compute $_{d-t}p_{[x]+t}$, we can find $\ell_{[x]+t}$ by

$$_{d-t}p_{[x]+t} = \frac{\ell_{x+d}}{\ell_{[x]+t}}.$$

   (b) *Going from $[x]$ to $[x+1]$:* An useful idea is to relate each of the select quantities for $[x]$ and $[x+1]$ with a <u>common</u> ultimate quantity, to build a "bridge" between the select quantities for $[x]$ and $[x+1]$. Once we have the "bridge", we can go from $[x]$ to $[x+1]$ via the intermediate ultimate quantity.
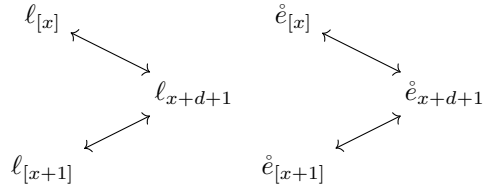
   (d: select period)

2 Life Insurance

2.0.1 As mentioned at the beginning of Section 1, a motivation for studying survival model is that the lifetime of the insured life impacts greatly the present value of the benefit payment, hence the loss for the insurer. After studying survival models in Section 1 and the probabilistic behaviours of lifetime in various models, let us investigate how the present value of the benefit payment would behave in Section 2. Such present value is called the **present value random variable** (PVRV). It is a random variable since it is a function of the lifetime, which is a random variable.

2.1 Basic Insurance Coverages

2.1.1 Let us start with studying some basic kinds of insurance coverages: (i) whole life insurance, (ii) term life insurance, (iii) pure endowment, (iv) endowment insurance, and (v) deferred whole life insurance. They will serve as our basic building blocks for all the discussions in Section 2. To guide our discussions here, for each kind of insurance coverage, we shall explore it in the following three aspects:

(1) *Definition:* meaning of the insurance coverage

(2) *Key formulas:* formulas for some key probabilistic quantities of PVRV

(3) *Recursive formulas:* formulas used for recursions (to be discussed after studying the first two aspects for all five kinds of basic insurance coverages)

For convenience, henceforth we shall assume that the amount of benefit payment is always 1, unless otherwise specified. If the actual benefit amount is not 1, then we can just multiply the PVRV by the actual benefit amount and adjust the formulas accordingly. Throughout, we shall consider the case where the insurance coverage is purchased when the life insured is aged x, so we are interested in the present value at age x.

2.1.2 Whole life insurance.

(1) *Definition:* A death benefit (DB) is paid upon the death ☠ of the life insured (lifetime coverage: payable regardless of when the insured life dies).

For all kinds of insurance coverages in Section 2.1, they can be broadly classified into two types: (i) discrete and (ii) continuous:

- *Discrete:* The benefit payment is paid *at the end of the year* of death.
- *Continuous:* The benefit payment is paid *at the moment* of death.

More specifically, if we start counting time at the age x of the life insured:

- The end of the year of death is at time $K_x + 1$.
- The moment of death is at time T_x.
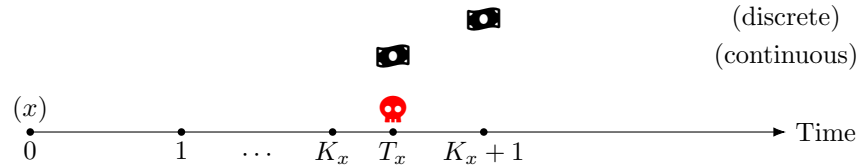
(2) *Key formulas:*

	Discrete	Continuous
PVRV	v^{K_x+1}	v^{T_x}
EPV (mean of PVRV)	$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_k p_x q_{x+k}$	$\bar{A}_x = \int_0^{\infty} e^{-\delta t} {}_t p_x \mu_{x+t} \, \mathrm{d}t$
Second moment	${}^2 A_x = \sum_{k=0}^{\infty} v^{2(k+1)} {}_k p_x q_{x+k}$	${}^2 \bar{A}_x = \int_0^{\infty} e^{-2\delta t} {}_t p_x \mu_{x+t} \, \mathrm{d}t$
Variance	${}^2 A_x - A_x^2$	${}^2 \bar{A}_x - \bar{A}_x^2$

<u>Remarks</u>:

- The second moment is actually just the EPV evaluated at double force of interest, i.e., $\delta \to 2\delta$ or $v \to v^2$.
- EPV stands for **expected present value**. Sometimes the term **actuarial present value** (APV) is used instead.

2.1.3 ⭐ **Important note.** Here we can observe that the EPV formulas take a similar "form" for both discrete and continuous cases: Both are "summing" products of terms. Generally speaking, we have the following formula for calculating EPV:

$$\text{EPV} = \underbrace{\sum}_{\text{all possible payment times}} \text{or} \int \quad \text{benefit amount} \times \text{discount factor} \times \text{prob. of triggering payment.}$$

For example, in the continuous case above:

- "All possible payment times" start at time 0 with no end, thus we have "$\int_0^\infty$".
- Benefit amount is 1, so we do not see it in the formula.
- Discount factor is v^t at time t.
- Probability of triggering payment at time t is loosely

$$\text{probability that } T_x \in [t, t+\mathrm{d}t] = \underbrace{{}_t p_x}_{\text{survive } t \text{ yrs.}} \times \underbrace{\mu_{x+t}\,\mathrm{d}t}_{\text{die in the subsequent } \mathrm{d}t \text{ yrs. (instantaneously)}} .$$

Once you understand how this formula works, there is no need to "memorize" the EPV formula for every single kind of insurance product 👍. In fact, this formula is still applicable for *life annuities* (to be discussed in Section 3) and even in STAT3909!

2.1.4 Term life insurance.

(1) *Definition:* A DB is paid upon the death of the life insured, provided that the death occurs within the policy term, say n years.

(2) *Key formulas:*

	Discrete	Continuous
PVRV	$\begin{cases} v^{K_x+1} & \text{if } K_x = 0,1,\ldots,n-1, \\ 0 & \text{if } K_x = n, n+1, \ldots \end{cases}$	$\begin{cases} v^{T_x} & \text{if } T_x \leq n, \\ 0 & \text{if } T_x > n \end{cases}$
EPV	$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_k p_x q_{x+k}$	$\bar{A}^1_{x:\overline{n}|} = \int_0^n e^{-\delta t} {}_t p_x \mu_{x+t}\,\mathrm{d}t$
Second moment	${}^2 A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{2(k+1)} {}_k p_x q_{x+k}$	${}^2 \bar{A}^1_{x:\overline{n}|} = \int_0^n e^{-2\delta t} {}_t p_x \mu_{x+t}\,\mathrm{d}t$
Variance	${}^2 A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2$	${}^2 \bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2$

[Note: The "1" on top of x suggests that the life aged x is the <u>1</u>st thing to be "gone" (dead 💀), earlier than the elapse of the n-year term, for the benefit to be triggered. More details about this kind of notation will be discussed in STAT3909.]

2.1.5 Pure endowment.

(1) *Definition:* A *survival benefit* is paid at the end of the policy term (say n years) if the life insured is still alive ⚕ at that time.

[Note: There is no distinction of "continuous" and "discrete" for pure endowment, because the only possible payment time is always precisely at the end of the policy term.]

(2) *Key formulas:*

	Formula
PVRV	$\begin{cases} 0 & \text{if } T_x < n, \\ v^n & \text{if } T_x \geq n \end{cases}$
EPV	$A_{x:\overline{n}|}^{1} = {}_nE_x = v^n{}_np_x$
Second moment	${}^2A_{x:\overline{n}|}^{1} = {}_n^2E_x = v^{2n}{}_np_x = v^n{}_nE_x$
Variance	${}_n^2E_x - ({}_nE_x)^2 = v^{2n} \times {}_np_x \times {}_nq_x$

Remarks:

- The "1" on top of $\overline{n}|$ suggests that the n-year term is the 1st thing to be "gone" (elapsed), earlier than death 💀 of the life aged x, for the benefit to be triggered. Again, more details about this kind of notation will be discussed in STAT3909.
- The notation for EPV is not "$\bar{A}_{x:\overline{n}|}^{1}$"!

2.1.6 Endowment insurance.

(1) *Definition:* A combination of a term life insurance and a pure endowment of the same term (say n-year).

[Note: In general, unless otherwise specified, we shall assume that the term life insurance and pure endowment have the same benefit amount. If they have different benefit amounts, then such insurance is said to be a special endowment insurance.]

(2) *Key formulas:*

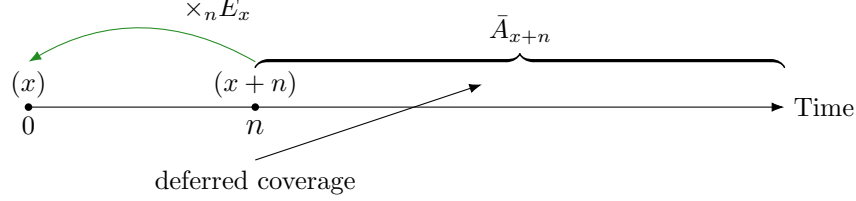|	Discrete	Continuous
PVRV	$v^{(K_x+1)\wedge n} = \begin{cases} v^{K_x+1} & \text{if } K_x = 0, 1, \ldots, n-1, \\ v^n & \text{if } K_x = n, n+1, \ldots \end{cases}$	$v^{T_x \wedge n} = \begin{cases} v^{T_x} & \text{if } T_x \leq n, \\ v^n & \text{if } T_x > n \end{cases}$
EPV	$A_{x:\overline{n}|} = A_{x:\overline{n}|}^{1} + A_{x:\overline{n}|}^{1}$	$\bar{A}_{x:\overline{n}|} = \bar{A}_{x:\overline{n}|}^{1} + A_{x:\overline{n}|}^{1}$
Second moment	${}^2A_{x:\overline{n}|} = {}^2A_{x:\overline{n}|}^{1} + {}^2A_{x:\overline{n}|}^{1}$	${}^2\bar{A}_{x:\overline{n}|} = {}^2\bar{A}_{x:\overline{n}|}^{1} + {}^2A_{x:\overline{n}|}^{1}$
Variance	${}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2$ or $\text{Var}(Z_{\text{term}}) + \text{Var}(Z_{\text{PE}}) - 2\underbrace{A_{x:\overline{n}|}^{1}A_{x:\overline{n}|}^{1}}_{\text{Cov}(Z_{\text{term}},Z_{\text{PE}})}$	${}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2$ or $\text{Var}(Z_{\text{term}}) + \text{Var}(Z_{\text{PE}}) - 2\underbrace{\bar{A}_{x:\overline{n}|}^{1}A_{x:\overline{n}|}^{1}}_{\text{Cov}(Z_{\text{term}},Z_{\text{PE}})}$

[Note: Z_{term} and Z_{PE} denote the PVRV's for (n-year) term life insurance and pure endowment respectively. Note that the PVRV for endowment insurance is $Z_{\text{term}} + Z_{\text{PE}}$.]

2.1.7 Deferred whole life insurance.

(1) *Definition:* A DB is paid upon the death of the life insured, provided that the death occurs after the deferred period (e.g., n years).

(2) *Key formulas:*

	Discrete	Continuous
PVRV	$\begin{cases} 0 & \text{if } K_x = 0, 1, \ldots, n-1, \\ v^{K_x+1} & \text{if } K_x = n, n+1, \ldots \end{cases}$	$\begin{cases} 0 & \text{if } T_x \leq n, \\ v^{T_x} & \text{if } T_x > n \end{cases}$
EPV	$_{n\|}A_x = A_x - A^1_{x:\overline{n}\|} = {}_nE_x A_{x+n}$	$_{n\|}\bar{A}_x = \bar{A}_x - \bar{A}^1_{x:\overline{n}\|} = {}_nE_x \bar{A}_{x+n}$

deferred coverage

Second moment	$_{n\|}^{2}A_x = {}_2A_x - {}^2A^1_{x:\overline{n}\|} = {}_nE_x\,{}^2A_{x+n}$	$_{n\|}^{2}\bar{A}_x = {}^2\bar{A}_x - {}^2\bar{A}^1_{x:\overline{n}\|} = {}_nE_x\,{}^2\bar{A}_{x+n}$
Variance	$_{n\|}^{2}A_x - ({}_{n\|}A_x)^2$	$_{n\|}^{2}\bar{A}_x - ({}_{n\|}\bar{A}_x)^2$

For the variance, we can also use a similar approach as the one for endowment insurance. We first let Z_{WL}, Z_{term}, and Z_{deferred} denote the PVRV's for whole life insurance, n-year term life insurance, and n-year deferred whole life insurance respectively. Then, we have $Z_{\text{WL}} = Z_{\text{term}} + Z_{\text{deferred}}$, thus
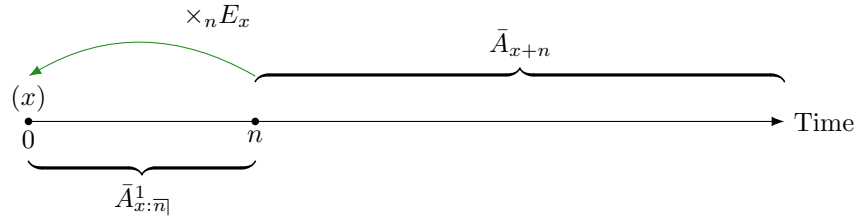
$$\text{Var}\,(Z_{\text{WL}}) = \text{Var}\,(Z_{\text{term}}) + \text{Var}\,(Z_{\text{deferred}}) + 2\,\text{Cov}\,(Z_{\text{term}}, Z_{\text{deferred}})$$

$$= \begin{cases} \text{Var}\,(Z_{\text{term}}) + \text{Var}\,(Z_{\text{deferred}}) - 2A^1_{x:\overline{n}\|} \times {}_{n\|}A_x & \text{for discrete case,} \\ \text{Var}\,(Z_{\text{term}}) + \text{Var}\,(Z_{\text{deferred}}) - 2\bar{A}^1_{x:\overline{n}\|} \times {}_{n\|}\bar{A}_x & \text{for continuous case.} \end{cases}$$

If the variances $\text{Var}\,(Z_{\text{WL}})$ and $\text{Var}\,(Z_{\text{term}})$ are known, we can then deduce the desired variance $\text{Var}\,(Z_{\text{deferred}})$.

2.1.8 **Recursive formulas.** The recursive formulas here reassemble substantially the recursions of expectations of life we have seen in Section . The intuition is pretty much the same.

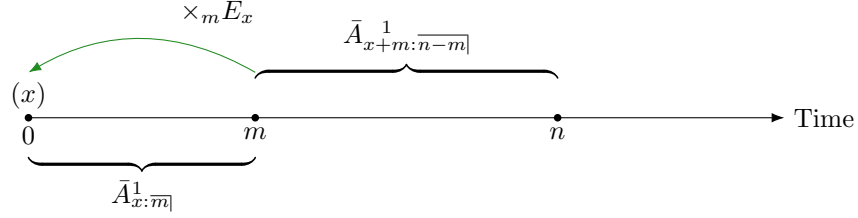- *Whole life insurance:*

  - *(discrete)* $A_x = A^1_{x:\overline{n}\|} + {}_nE_x A_{x+n} \overset{(n=1)}{=} vq_x + vp_x A_{x+1}$.
  - *(continuous)* $\bar{A}_x = \bar{A}^1_{x:\overline{n}\|} + {}_nE_x \bar{A}_{x+n}$.

- *Term life insurance:*

  - *(discrete)* $A^1_{x:\overline{n}\|} \overset{(m\leq n)}{=} A^1_{x:\overline{m}\|} + {}_mE_x A^1_{x+m:\overline{n-m}\|} \overset{(m=1)}{=} vq_x + vp_x A^1_{x+1:\overline{n-1}\|}$.
  - *(continuous)* $\bar{A}^1_{x:\overline{n}\|} \overset{(m\leq n)}{=} \bar{A}^1_{x:\overline{m}\|} + {}_mE_x \bar{A}^1_{x+m:\overline{n-m}\|}$.

- *Endowment insurance:*
  - *(discrete)* $A_{x:\overline{n}|} \overset{(m \leq n)}{=} A^1_{x:\overline{m}|} + {}_mE_x A_{x+m:\overline{n-m}|} \overset{(m=1)}{=} vq_x + vp_x A_{x+1:\overline{n-1}|}$. [⚠ Warning: On the RHS of the second expression, be careful about which "A" has a "1" on top of x, and which does not!]
  - *(continuous)* $\bar{A}_{x:\overline{n}|} \overset{(m \leq n)}{=} \bar{A}^1_{x:\overline{m}|} + {}_mE_x \bar{A}_{x+m:\overline{n-m}|}$.
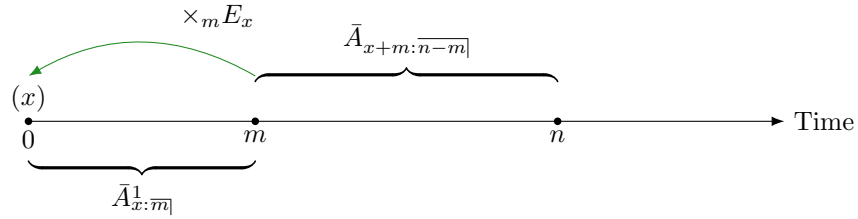
2.1.9 Apart from EPV, second moment, and variance, another probabilistic quantity of PVRV that is of interest is the *percentile*. Recall that the *pth percentile* of a random variable X is the value c that makes $\mathbb{P}(X \leq c) = p$.[5] Here we denote the pth percentile of X by $\pi_p(X)$.

2.1.10 **Percentile trick:** In our case here, the random variable of interest is the PVRV, and we denote it by Z. For many insurance coverages discussed here, we can write $Z = g(T_x)$ for some strictly *decreasing* continuous function g. Then, the *percentile trick* suggests that we actually have

$$\pi_p(Z) = \pi_p(g(T_x)) = g(\pi_{1-p}(T_x)).$$

In words, the pth percentile of $g(T_x)$ equals g evaluated at the $(1-p)$th percentile of T_x. Because percentiles of T_x are usually easier to find, this trick can allow us to compute percentiles of PVRV more efficiently.

2.2 Variations of Basic Insurance Coverages

2.2.1 We have mentioned at the beginning of Section 2.1 that the basic insurance coverages studied in Section 2.1 will serve as our building blocks. Now, let us those building blocks to construct some more advanced insurance coverages, by introducing some variants to those basic coverages. Here, we shall discuss two kinds of variants: (i) $1/m$thly insurance and (ii) variable insurance benefits. Likewise, we shall explore each of them in the following three aspects:

(1) *Definition:* meaning of the variant

(2) *Key formulas:* formulas for some key probabilistic quantities of PVRV

(3) *Recursive formulas:* formulas used for recursions

2.2.2 $1/m$**thly insurance.**

(1) *Definition:* A DB is paid at the end of the $1/m$th year of death. (For example, if $m = 12$, then a DB is paid at the end of the *month* ("1/12th year") of death.)

The $1/m$thly insurance serves as a middle ground between "discrete" and "continuous".

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[5]This is actually a simplified definition that works for some sufficiently nice random variable only. But this definition is sufficient for our purpose here.

(2) *Key formulas:* The formulas for $1/m$thly insurance can generally be obtained by modifying the respective formulas for *discrete* case suitably. Before discussing the modifications, we first need to introduce a preliminary concept, which serves as a middle ground between K_x and T_x.

The **$1/m$thly curtate future lifetime random variable** $K_x^{(m)}$ is defined by $K_x^{(m)} = \frac{1}{m}\lfloor mT_x \rfloor$. In words, $K_x^{(m)}$ is T_x *rounded down to the lower $1/m$th of a year.*

For example, $K_x^{(2)}$ is T_x rounded down to the lower half of a year:
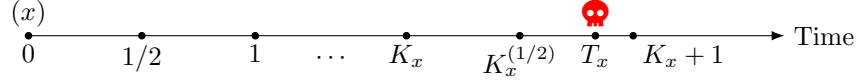
The pmf of $K_x^{(m)}$ is given by $\mathbb{P}\left(K_x^{(m)} = \frac{k}{m}\right) = \frac{k}{m}p_x \times \frac{1}{m}q_{x+\frac{k}{m}}$ for each $k = 0, 1, \ldots$.

Using the random variable $K_x^{(m)}$, we can then specify the modifications needed for obtaining the formulas applicable to $1/m$thly insurance.

- *PVRV:* Replace $K_x + 1$ by $K_x^{(m)} + 1/m$.
  Example (whole life insurance): $v^{K_x+1} \to v^{K_x^{(m)}+1/m}$.
- *EPV:*
  - *Notation:* Put "(m)" as the right subscript of A in the notation for discrete case.
  - *Formula:* Still use the general EPV calculation formula in [2.1.3].
  Example (n-year term life insurance):

$$A^{(m)\,1}_{x:\overline{n}|} = \sum_{k=0}^{mn-1} v^{\frac{k+1}{m}} \,\frac{k}{m}p_x\, \frac{1}{m}q_{x+\frac{k}{m}} = v^{\frac{1}{m}}\,\frac{1}{m}q_x + v^{\frac{2}{m}}\,\frac{1}{m}p_x\,\frac{1}{m}q_{x+\frac{1}{m}} + \cdots + v^n\,\frac{mn-1}{m}p_x\,\frac{1}{m}q_{x+\frac{mn-1}{m}}.$$

(3) *Recursive formulas:* Again the recursive formulas can be obtained by modifying the ones for discrete case in [2.1.8] suitably. Let us use whole life insurance ($A_x^{(m)}$) as an example:

$$A_x^{(m)} = A^{(m)\,1}_{x:\overline{n}|} + {}_nE_x A_{x+n}^{(m)} \overset{(n=\frac{1}{m})}{=} v^{\frac{1}{m}}\,\frac{1}{m}q_x + v^{\frac{1}{m}}\,\frac{1}{m}p_x A_{x+\frac{1}{m}}.$$
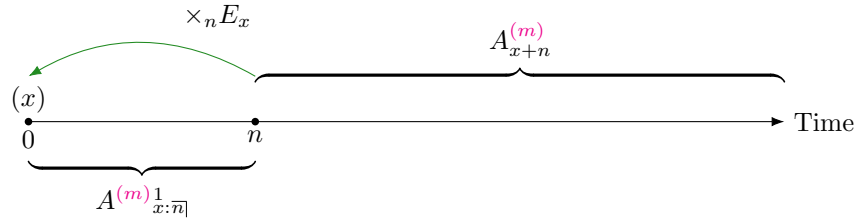
2.2.3 Variable insurance benefits.

(1) *Definition:* This variant refers generally to any kind of insurance coverage where the amount of DB can differ for different payment time points.

[Note: In this case, the amount of DB may no longer stay at the constant 1 anymore.]

(2) *Key formulas:*

- *EPV:* Use the general EPV formula in [2.1.3].
- *Second moment:* Still use the general EPV formula in [2.1.3], but square *both* benefit amount *and* discount factor.
- *Variance:* Still use the standard formula: Variance $=$ Second moment $-$ EPV^2 (nothing special).

Example (whole life insurance with DB amount being b_t when paid at time t):

	Discrete	Continuous
PVRV	$b_{K_x+1} \times v^{K_x+1}$	$b_{T_x} \times v^{T_x}$
EPV	$\displaystyle\sum_{k=0}^{\infty} b_{k+1} v^{k+1} {}_k p_x q_{x+k}$	$\displaystyle\int_0^{\infty} b_t e^{-\delta t} {}_t p_x \mu_{x+t}\, \mathrm{d}t$
Second moment	$\displaystyle\sum_{k=0}^{\infty} b_{k+1}^2 v^{2(k+1)} {}_k p_x q_{x+k}$	$\displaystyle\int_0^{\infty} b_t^2 e^{-2\delta t} {}_t p_x \mu_{x+t}\, \mathrm{d}t$
Variance	Second moment $-$ EPV^2	Second moment $-$ EPV^2

(3) *Recursive formulas:* There is not any "nice" recursive formula for this case in general. But in the following, we will study two special cases: *arithmetically increasing insurance* and *geometrically increasing insurance*. There are some "nice" recursive formulas available for the former one.
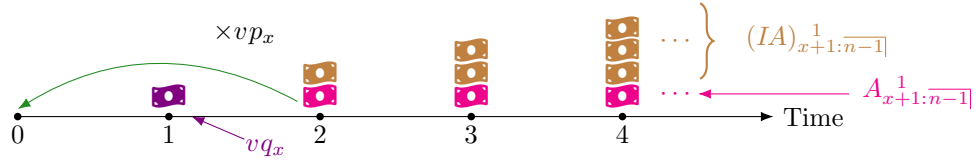
2.2.4 Arithmetically increasing insurance.

(1) *Definition:* The amount of DB paid at time t (if payable) is t.

(2) *Key formulas:* Let us use arithmetically increasing n-year term life insurance as an example.
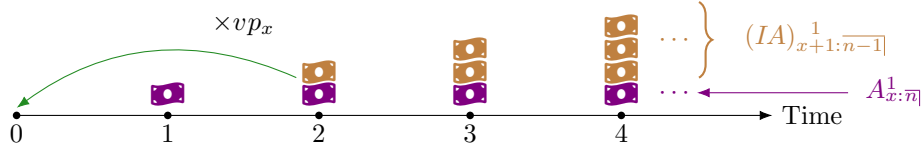
	Discrete	Continuous
PVRV	$\begin{cases}(K_x+1)v^{K_x+1} & \text{if } K_x = 0, 1, \ldots, n-1 \\ 0 & \text{if } K_x = n, n+1, \ldots\end{cases}$	$\begin{cases}T_x v^{T_x} & \text{if } T_x \leq n, \\ 0 & \text{if } T_x > n\end{cases}$
EPV	$(IA)^1_{x:\overline{n}|} = \displaystyle\sum_{k=0}^{n-1} (k+1) v^{k+1} {}_k p_x q_{x+k}$	$(\bar{I}\bar{A})^1_{x:\overline{n}|} = \displaystyle\int_0^n t e^{-\delta t} {}_t p_x \mu_{x+t}\, \mathrm{d}t$
Second moment	$\displaystyle\sum_{k=0}^{n-1} (k+1)^2 v^{2(k+1)} {}_k p_x q_{x+k}$	$\displaystyle\int_0^n t^2 e^{-2\delta t} {}_t p_x \mu_{x+t}\, \mathrm{d}t$
Variance	Second moment $-$ EPV^2	Second moment $-$ EPV^2

(3) *Recursive formulas:* There are two "nice" recursive formulas available.

- $(IA)^1_{x:\overline{n}|} = v q_x + v p_x \left((IA)^1_{x+1:\overline{n-1}|} + A^1_{x+1:\overline{n-1}|} \right).$

- $(IA)^1_{x:\overline{n}|} = A^1_{x:\overline{n}|} + v p_x (IA)^1_{x+1:\overline{n-1}|}.$

2.2.5 Geometrically increasing insurance.

(1) *Definition: (only applicable to the discrete case)* The amount of DB paid at time $k+1$ (if payable) is $(1+j)^k$ for any $k = 0, 1, \ldots$. [Note: With this definition, the amount of DB at time 1 (the first possible DB) is 1.]

(2) *Key formulas:* Let us use geometrically increasing n-year term life insurance as an example.

$$\text{EPV} = \frac{1}{1+j} A_{x:\overline{n}|i^*} \overset{(i=j)}{=} \frac{1}{1+j} {}_n q_x$$

where $i^* = (i-j)/(1+j)$.

Proof. Note that

$$\text{EPV} = \sum_{k=0}^{n-1}(1+j)^k v^{k+1} {}_kp_x q_{x+k} = \frac{1}{1+j}\sum_{k=0}^{n-1}\underbrace{\left(\frac{1+i}{1+j}\right)^{-(k+1)}}_{(1+i^*)^{-(k+1)}} {}_kp_x q_{x+k},$$

and that when $i^* = 0$, we have $A_{x:\overline{n}|i^*} = \mathbb{E}\big[\mathbf{1}_{\{T_x<n\}}\big] = {}_nq_x$, where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. $\qquad\square$

2.2.6 With selection at age x, all the previous "A" symbols and formulas can still be used, after replacing "x" by $[x]$ and simplifying "$[x]+t$" to "$x+t$" if $t \geq \underset{\text{(select period)}}{d}$.

2.3 Relating $\bar{A}$, A, and $A^{(m)}$

2.3.1 In Sections 2.1 and 2.2, we have studied formulas for EPVs of continuous ("$\bar{A}$"), discrete ("A") and $1/m$thly ("$A^{(m)}$") insurances. Here, we would like to investigate their relationship. This is of interest since in many tables (like the SOA FAM-L table or the AM92 table), only the values of discrete EPVs (those "As") can be found, but not the values of continuous EPVs (those "$\bar{A}$"s) or $1/m$thly EPVs (those "$A^{(m)}$"s).

2.3.2 Here we will discuss two kinds of formulas that relate these EPVs:

- *UDD:* Formulas that work under the UDD assumption
- *Claims acceleration approach:* <u>Approximation</u> formulas that work in general

2.3.3 **UDD.** We can organize the UDD formulas into three groups:

- $A \overset{UDD}{\to} \bar{A}$: The common feature shared by all the formulas here is that the adjustment factor i/δ is multiplied:

Type	Formula
Whole life	$\bar{A}_x = \dfrac{i}{\delta}A_x$
Term life	$\bar{A}^1_{x:\overline{n}|} = \dfrac{i}{\delta}A^1_{x:\overline{n}|}$
Deferred whole life	${}_{n|}\bar{A}_x = \dfrac{i}{\delta}{}_{n|}A_x$

[⚠ Warning: We do <u>not</u> have "$\bar{A}_{x:\overline{n}|} = \dfrac{i}{\delta}A_{x:\overline{n}|}$". The correct formula for the endowment insurance should be $\bar{A}_{x:\overline{n}|} = \dfrac{i}{\delta}A^1_{x:\overline{n}|} + \underbrace{A_{x:\overline{n}|}^{1}}_{\text{no }i/\delta!}$.]

- $A \overset{UDD}{\to} A^{(m)}$: Change $i/\delta \to i/i^{(m)}$ and $\bar{A} \to A^{(m)}$ in the formulas for $A \overset{\text{UDD}}{\to} \bar{A}$.
- ${}^2A \overset{UDD}{\to} {}^2\bar{A}$: Change $i/\delta \to i^*/\delta^*$ and $A \to {}^2A$ in the formulas for $A \overset{\text{UDD}}{\to} \bar{A}$, where $\delta^* = 2\delta$ and $i^* = 2i + i^2$ is the effective interest rate equivalent to the double force of interest 2δ.

2.3.4 **Claims acceleration approach.** This approach gives us *approximation* formulas for $A \to \bar{A}$ and $A \to A^{(m)}$. The pro 👍 of this approach is that it does not require the UDD assumption, but the con 👎 is that it is only an *approximation*! The formulas are as follows.

- $A \overset{\approx}{\to} A^{(m)}$: Change $i/i^{(m)} \to (1+i)^{(m-1)/2m}$ and replace "$=$" by "$\approx$" in the formulas for $A \overset{\text{UDD}}{\to} A^{(m)}$.

  [Note: The heuristic idea behind is that for $1/m$thly insurances, the average payment time of DB is the $\frac{m+1}{2m}$th of the year of death under UDD, which is $1 - \frac{m+1}{2m} = \frac{m-1}{2m}$ years earlier than the end of the year of death for the discrete case. Hence, multiplying $(1+i)^{(m-1)/2m}$ is an attempt to "bring the DB earlier" (or <u>accelerate</u> the <u>claim</u> payment) through reducing the magnitude of discounting.]

- $A \overset{\approx}{\to} \bar{A}$: Let $m \to \infty$, i.e., change $(1+i)^{(m-1)/2m} \to (1+i)^{1/2}$, in the formulas for $A \overset{\approx}{\to} A^{(m)}$.

3 Life Annuity

3.0.1 Apart from insurance coverages, another common type of insurance product that is offered by life insurers is *life annuity*. While life insurance is primarily about insuring against the risk of "living for too short", or dying 💀 too early (except for those (pure) endowment insurances), life annuity is about insuring against the risk of "living for too long". "Living for too long" is also a risk since, for example, the retirement savings may not be adequate for maintaining a good quality of life if the lifetime is unexpectedly long. Life annuity provides *survival* benefit payments as long as the **annuitant** (we use this term instead of "life insured" for annuity products) survives, which yields a stable stream of incomes for maintaining the quality of life after retirement.

3.0.2 Like life insurance, the present value of the benefit payments (PVRV here) directly contributes to the amount of loss for the insurer. If the lifetime is longer, more survival benefit payments would need to be made, which amplifies the insurer's loss. So in Section 3, we will investigate the probabilistic behaviours of the present value of benefit payments, like what we did in Section 2. The organization of Section 3 will be (highly) similar to Section 2.

3.1 Basic Life Annuities

3.1.1 We shall start by studying some basic life annuities: (i) whole life annuity, (ii) temporary life annuity, (iii) deferred whole life annuity, and (iv) guaranteed annuity. Like what we did in Section 2, we shall explore each kind of life annuity in the following three aspects:

(1) *Definition:* meaning of the annuity

(2) *Key formulas:* formulas for some key probabilistic quantities of PVRV

(3) *Recursive formulas:* formulas used for recursions (to be discussed after studying the first two aspects for all four kinds of basic life annuities)

Henceforth, we shall assume that the amount of *each* benefit payment is 1 (for discrete life annuities) or the *rate* of benefit payment is 1 (for continuous life annuities), unless otherwise specified. [Note: We will explain what "discrete" and "continuous" mean very soon.]

Throughout, we shall consider the case where the annuity is purchased when the annuitant is aged x, so we are interested in the present value at age x.

3.1.2 **Whole life annuity.**

(1) *Definition:* Periodic benefit payments are made whenever the annuitant is alive, for life.
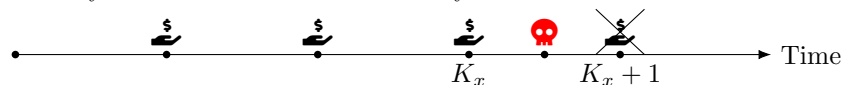
For all kinds of life annuities in Section 3.1, they can be broadly classified into two types: (i) discrete and (ii) continuous:

- *Discrete:* The benefit payments are made at *discrete time points*. More specifically, we are considering the following two cases (learnt in STAT2902):
  - *Due:* Payments are made at the beginning of year.
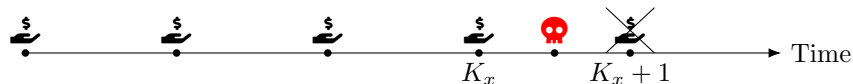
  - *Immediate:* Payments are made at the end of year.

- *Continuous:* The benefit payments are made *continuously* (recall STAT2902).

(2) *Key formulas:*

	Discrete	Continuous
PVRV	**Due:** $\ddot{a}_{\overline{K_x+1}} = \dfrac{\overbrace{1 - v^{K_x+1}}^{\text{PVRV of WL insurance}}}{d}$ **Immediate:** $a_{\overline{K_x}}$	$\bar{a}_{\overline{T_x}} = \dfrac{\overbrace{1 - v^{T_x}}^{\text{PVRV of WL insurance}}}{\delta}$
EPV	**Due:** *(annuity-insurance)* $\ddot{a}_x = \dfrac{1 - A_x}{d}$ *(summation)* $\ddot{a}_x = \displaystyle\sum_{k=0}^{\infty} v^k {}_k p_x$ **Immediate:** *(due-immediate)* $a_x = \ddot{a}_x - \underbrace{1}_{\text{time } 0}$	*(annuity-insurance)* $\bar{a}_x = \dfrac{1 - \bar{A}_x}{\delta}$ *(integral)* $\bar{a}_x = \displaystyle\int_0^{\infty} e^{-\delta t} {}_t p_x \, \mathrm{d}t$
Variance	**Due:** $\dfrac{\overbrace{{}^2 A_x - A_x^2}^{\text{Variance of WL insurance PVRV}}}{d^2}$	$\dfrac{\overbrace{{}^2 \bar{A}_x - \bar{A}_x^2}^{\text{Variance of WL insurance PVRV}}}{\delta^2}$

The general EPV calculation formula introduced in [2.1.3] is still useful for obtaining the summation/integral formula for EPV here. For example, in the continuous case above:

- "All possible payment times" start at time 0 with no end, thus we have "$\int_0^{\infty}$".
- Benefit amount at time t (amount of payment made in $[t, t + \mathrm{d}t]$) is loosely

$$\underbrace{\text{rate}}_{1} \times \underbrace{\text{time length}}_{\mathrm{d}t} = \mathrm{d}t.$$

- Discount factor is v^t at time t.
- Probability of triggering payment at time t is the probability that the annuitant is alive at that time, i.e., ${}_t p_x$.

3.1.3 **Temporary life annuity.** [Note: Here we use the term "temporary life annuity" instead of "term life annuity" to emphasize that the benefit payments are only *temporary*.]

(1) *Definition:* Periodic benefit payments are made whenever the annuitant is alive, for at most n years.

(2) *Key formulas:*

	Discrete	Continuous
PVRV	**Due:** $\ddot{a}_{\overline{(K_x+1) \wedge n}}$ **Immediate:** $a_{\overline{K_x \wedge n}}$	$\bar{a}_{\overline{T_x \wedge n}}$
EPV	**Due:** *(annuity-insurance)* $\ddot{a}_{x:\overline{n}} = \dfrac{1 - A_{x:\overline{n}}}{d}$ *(summation)* $\ddot{a}_{x:\overline{n}} = \displaystyle\sum_{k=0}^{n-1} v^k {}_k p_x$ **Immediate:** *(due-immediate)* $a_{x:\overline{n}} = \ddot{a}_{x:\overline{n}} - \underbrace{1}_{\text{time } 0} + \underbrace{{}_n E_x}_{\text{time } n}$	*(annuity-insurance)* $\bar{a}_{x:\overline{n}} = \dfrac{1 - \bar{A}_{x:\overline{n}}}{\delta}$ *(integral)* $\bar{a}_{x:\overline{n}} = \displaystyle\int_0^n e^{-\delta t} {}_t p_x \, \mathrm{d}t$
Variance	**Due:** $\dfrac{{}^2 A_{x:\overline{n}} - A_{x:\overline{n}}^2}{d^2}$	$\dfrac{{}^2 \bar{A}_{x:\overline{n}} - \bar{A}_{x:\overline{n}}^2}{\delta^2}$

3.1.4 **Deferred whole life annuity.**

(1) *Definition:* Periodic benefit payments start after the deferred period (e.g., n years), and are made whenever the annuitant is alive.

(2) *Key formulas:*

	Discrete	Continuous
PVRV	**Due:** $\begin{cases} 0 & \text{if } K_x = 0, 1, \ldots, n-1, \\ v^n \ddot{a}_{\overline{K_x+1-n\rvert}} \\ = \ddot{a}_{\overline{K_x+1\rvert}} - \ddot{a}_{\overline{n\rvert}} & \text{if } K_x = n, n+1, \ldots \end{cases}$ **Immediate:** $\begin{cases} 0 & \text{if } K_x = 0, 1, \ldots, n-1, \\ v^n a_{\overline{K_x-n\rvert}} \\ = a_{\overline{K_x\rvert}} - a_{\overline{n\rvert}} & \text{if } K_x = n, n+1, \ldots \end{cases}$	$\begin{cases} 0 & \text{if } T_x \le n, \\ v^n \bar{a}_{\overline{T_x-n\rvert}} \\ = \bar{a}_{\overline{T_x\rvert}} - \bar{a}_{\overline{n\rvert}} & \text{if } T_x > n \end{cases}$
EPV	**Due:** $(WL - temporary)$ $_{n\rvert}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n\rvert}}$ $(discounted\ EPV)$ $_{n\rvert}\ddot{a}_x = {}_nE_x \ddot{a}_{x+n}$ **Immediate:** $(WL - temporary)$ $_{n\rvert}a_x = a_x - a_{x:\overline{n\rvert}}$ $(discounted\ EPV)$ $_{n\rvert}a_x = {}_nE_x a_{x+n}$	$(WL - temporary)$ $_{n\rvert}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n\rvert}}$ $(discounted\ EPV)$ $_{n\rvert}\bar{a}_x = {}_nE_x \bar{a}_{x+n}$

### 3.1.5 Guaranteed annuity/Certain-and-life annuity.

(1) *Definition:* Periodic benefit payments are *guaranteed* for n years (i.e., made no matter the annuitant survives or not) and continue to be made afterwards whenever the annuitant survives.

(2) *Key formulas:*

	Discrete	Continuous
PVRV	**Due:** $\ddot{a}_{\overline{\max\{K_x+1,n\}\rvert}}$ **Immediate:** $a_{\overline{\max\{K_x,n\}\rvert}}$	$\bar{a}_{\overline{\max\{T_x,n\}\rvert}}$
EPV *(certain + deferred)*	**Due:** $\ddot{a}_{\overline{x:\overline{n\rvert}}} = \ddot{a}_{\overline{n\rvert}} + {}_{n\rvert}\ddot{a}_x$ **Immediate:** $a_{\overline{x:\overline{n\rvert}}} = a_{\overline{n\rvert}} + {}_{n\rvert}a_x$	$\bar{a}_{x:\overline{n\rvert}} = \bar{a}_{\overline{n\rvert}} + {}_{n\rvert}\bar{a}_x$

3.1.6 Recursive formulas.

The recursive formulas for annuity are highly similar to the ones for insurance discussed in [2.1.8].
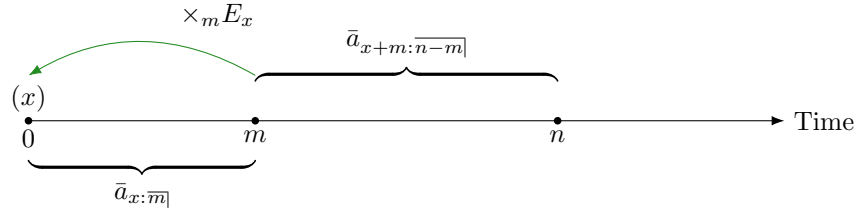
- *Whole life annuity:*
  - *(discrete, due)* $\ddot{a}_x = \ddot{a}_{x:\overline{n\rvert}} + {}_nE_x \ddot{a}_{x+n} \stackrel{(n=1)}{=} 1 + vp_x \ddot{a}_{x+1}$.
  - *(discrete, immediate)* $a_x = a_{x:\overline{n\rvert}} + {}_nE_x a_{x+n} \stackrel{(n=1)}{=} vp_x + vp_x a_{x+1}$.
  - *(continuous)* $\bar{a}_x = \bar{a}_{x:\overline{n\rvert}} + {}_nE_x \bar{a}_{x+n}$.
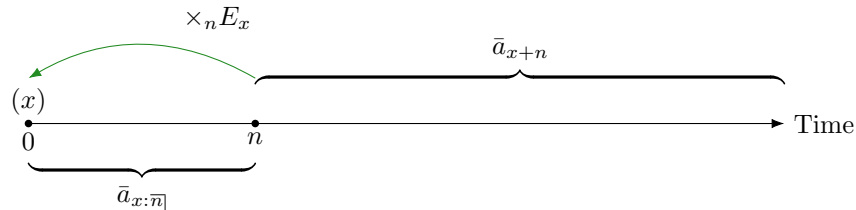
- *Temporary life annuity:*
  - *(discrete, due)* $\ddot{a}_{x:\overline{n\rvert}} \stackrel{(m \le n)}{=} \ddot{a}_{x:\overline{m\rvert}} + {}_mE_x \ddot{a}_{x+m:\overline{n-m\rvert}} \stackrel{(m=1)}{=} 1 + vp_x \ddot{a}_{x+1:\overline{n-1\rvert}}$.
  - *(discrete, immediate)* $a_{x:\overline{n\rvert}} \stackrel{(m \le n)}{=} a_{x:\overline{m\rvert}} + {}_mE_x a_{x+m:\overline{n-m\rvert}} \stackrel{(m=1)}{=} vp_x + vp_x a_{x+1:\overline{n-1\rvert}}$.

- *(continuous)* $\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{m}|} + {}_mE_x\,\bar{a}_{x+m:\overline{n-m}|}$.

3.1.7 Percentile trick:

3.1.7 **Percentile trick:** The random variable of interest here is the PVRV, denoted by Y. For many annuity products discussed here, we can write $Y = h(T_x)$ for some strictly *increasing* continuous function h. Then, the *percentile trick* suggests that we actually have

$$\pi_p(Y) = \pi_p(h(T_x)) = h(\pi_p(T_x)).$$

In words, the pth percentile of $h(T_x)$ equals h evaluated at the pth percentile of T_x. This allows us to compute percentiles of PVRV more efficiently.

3.2 Variations of Basic Life Annuities

3.2.1 Like Section 2, here we will discuss three kinds of variations of basic life annuities: (i) $1/m$thly annuity, (ii) variable annuity benefits and (iii) *(new!)* complete and apportionable annuities. We shall explore them in the following three aspects:

(1) *Definition:* meaning of the variant

(2) *Key formulas:* formulas for some key probabilistic quantities of PVRV

(3) *Recursive formulas:* formulas used for recursions

3.2.2 $1/m$**thly annuity.** ⚠ Here we shall assume that the amount of each benefit payment is $1/m$ rather than 1, so that the *total* benefit amount in a year is 1.

(1) *Definition:* Benefit payment is made at the beginning (due case) or end (immediate case) of each $1/m$th of a year in the relevant period, whenever the annuitant is alive.

(2) *Key formulas:* Like the insurance case, the formulas for $1/m$thly annuity can be obtained by modifying the respective formulas for *discrete* case suitably:
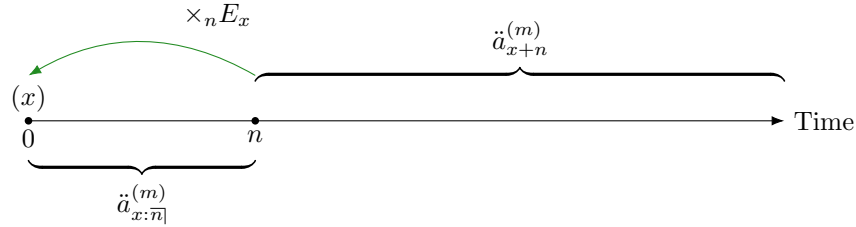
- *PVRV:*
  - *(both due and immediate)* Multiply by $1/m$.
  - *(due)* Change $K_x + 1 \to K_x^{(m)} + 1/m$ and $\ddot{a} \to \ddot{a}^{(m)}$ (recall the meaning of annuity symbol with superscript "(m)" from STAT2902).
  - *(immediate)* Change $K_x \to K_x^{(m)}$ and $a \to a^{(m)}$.
- *EPV:*
  - *Notation:* Put "(m)" as the right superscript of $\ddot{a}$ or a in the notation for discrete case.
  - *Formula:*
    * *(annuity-insurance)* Change $A \to A^{(m)}$ and $d \to d^{(m)}$.
    * *(summation)* Still use the general EPV formula in [2.1.3] (remember that the amount of each payment is $1/m$ rather than 1 ⚠).
    * *(due-immediate)* Change $a \to a^{(m)}$, $\ddot{a} \to \ddot{a}^{(m)}$ and $1 \to 1/m$ (both $-1 \to -1/m$ and $(1)_nE_x \to (1/m)_nE_x$ (if applicable)).
  - *Variance: (due only)* Change $A \to A^{(m)}$.

Examples:

	Whole life	Temporary life
PVRV	**Due:** $\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}}$ **Immediate:** $a^{(m)}_{\overline{K_x^{(m)}}}$	**Due:** $\ddot{a}_{\overline{(K_x^{(m)}+\frac{1}{m})\wedge n}}$ **Immediate:** $a_{\overline{K_x^{(m)}\wedge n}}$
EPV	**Due:** *(annuity-insurance)* $\ddot{a}^{(m)}_x = \dfrac{1-A^{(m)}_x}{d^{(m)}}$ *(summation)* $\ddot{a}^{(m)}_x = \sum\limits_{k=0}^{\infty} \dfrac{1}{m} v^{\frac{k}{m}} {}_{\frac{k}{m}} p_x$ **Immediate:** *(due-immediate)* $a^{(m)}_x = \ddot{a}^{(m)}_x - \dfrac{1}{m}$	**Due:** *(annuity-insurance)* $\ddot{a}^{(m)}_{x:\overline{n}} = \dfrac{1-A^{(m)}_{x:\overline{n}}}{d^{(m)}}$ *(summation)* $\ddot{a}^{(m)}_{x:\overline{n}} = \sum\limits_{k=0}^{mn-1} \dfrac{1}{m} v^{\frac{k}{m}} {}_{\frac{k}{m}} p_x$ **Immediate:** *(due-immediate)* $a^{(m)}_{x:\overline{n}} = \ddot{a}^{(m)}_{x:\overline{n}} - \dfrac{1}{m} + \dfrac{1}{m} {}_nE_x$
Variance	**Due:** $\dfrac{{}^2A^{(m)}_x - (A^{(m)}_x)^2}{(d^{(m)})^2}$	**Due:** $\dfrac{{}^2A^{(m)}_{x:\overline{n}} - (A^{(m)}_{x:\overline{n}})^2}{(d^{(m)})^2}$

(3) *Recursive formulas:* Modify the ones for discrete case in [3.1.6] suitably. Let us use whole life annuity-due $(\ddot{a}^{(m)}_x)$ as an example:

$$\ddot{a}^{(m)}_x = \ddot{a}^{(m)}_{x:\overline{n}} + {}_nE_x \ddot{a}^{(m)}_{x+n} \overset{(n=\frac{1}{m})}{=} \underbrace{\frac{1}{m}}_{\text{not 1} \; ⚠} + v^{\frac{1}{m}} {}_{\frac{1}{m}} p_x \ddot{a}^{(m)}_{x+\frac{1}{m}}.$$

3.2.3 Like the insurance case, $1/m$thly annuity serves as a "middle ground" between discrete annuity and continuous annuity. Note that in any of these three types of annuity, the total benefit amount in a year is 1. Their difference lies in the *payment frequency*, with the following order: discrete $\leq 1/m$thly $\leq$ continuous.

In view of this, we are then interested to know how the EPVs vary as the payment frequency changes. The results are summarized below:

- *Due:* $\ddot{a}^{(m)}_x$ *decreases* in m.

  [Intuition 💡: As m increases, the overall payment time in each year becomes *later*, resulting in a *larger* discounting, hence *smaller* EPV.]

- *Immediate:* $a^{(m)}_x$ *increases* in m.

  [Intuition 💡: As m increases, the overall payment time in each year becomes *earlier*, resulting in a *smaller* discounting, hence *larger* EPV.]

- *Comparison between different kinds of annuity:* $a_x \leq a^{(m)}_x \leq \bar{a}_x \leq \ddot{a}^{(m)}_x \leq \ddot{a}_x$.

  [Intuition 💡: The order of the overall payment time in each year for these kinds of annuities is:

$$\underset{(\text{latest})}{\text{immediate}} \geq 1/m\text{thly immediate} \geq \text{continuous} \geq 1/m\text{thly due} \geq \underset{(\text{earliest})}{\text{due}} \; .$$

Hence the reverse order applies for the EPVs.]

3.2.4 Variable annuity benefits.

(1) *Definition:* This variant refers generally to any kind of life annuity whose payment amount can vary for different payment time points.

(2) *Key formulas:*

- *EPV:* Use the general EPV formula in [2.1.3].
- *(not too important) Second moment:* Use the definition of expectation. [⚠ Warning: For life annuities, the second moment is <u>not</u> EPV at double force of interest (even if the benefit amount/rate is 1).]
- *(not too important) Variance:* Use the standard formula: Variance = Second moment $-$ EPV^2.

Example (whole life annuity with benefit amount being b_t when paid at time t):

	Discrete	Continuous
PVRV	**Due:** $\displaystyle\sum_{k=0}^{K_x} b_k v^k$ **Immediate:** $\displaystyle\sum_{k=1}^{K_x} b_k v^k$	$\displaystyle\int_0^{T_x} b_t v^t \, \mathrm{d}t$
EPV	**Due:** *(summation)* $\displaystyle\sum_{k=0}^{\infty} b_k v^k{}_k p_x$ **Immediate:** *(due-immediate)* EPV for due $- \underbrace{b_0}_{\text{time } 0}$	*(integral)* $\displaystyle\int_0^{\infty} b_t e^{-\delta t}{}_t p_x \mathrm{d}t$
Second moment *(not too important)*	**Due:** *(definition)* $\displaystyle\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} b_k v^k\right)^2 {}_j p_x q_{x+j}$ $\overset{(b_k=1)}{=}\displaystyle\sum_{j=0}^{\infty}(\ddot{a}_{\overline{j}\rceil})^2 {}_j p_x q_{x+j}$ ⚠ NOT $\displaystyle\sum_{k=0}^{\infty} b_k{}^2 v^{2k}{}_k p_x$!	*(definition)* $\displaystyle\int_0^{\infty}\left(\int_0^{s} b_t v^t \, \mathrm{d}t\right)^2 {}_s p_x \mu_{x+s} \, \mathrm{d}s$ $\overset{(b_t=1)}{=}\displaystyle\int_0^{\infty}(\bar{a}_{\overline{s}\rceil})^2 {}_s p_x \mu_{x+s} \, \mathrm{d}s$ ⚠ NOT $\displaystyle\int_0^{\infty} b_t{}^2 e^{-2\delta t}{}_t p_x \mathrm{d}t$!
Variance *(not too important)*	Second moment $-$ EPV^2	Second moment $-$ EPV^2

(3) *Recursive formulas:* Like the insurance case, "nice" recursion formulas are only available for some special cases. In the following, we will study two special cases: *arithmetically increasing annuity* and *geometrically increasing annuity*, where "nice" recursive formulas are available for the former one.
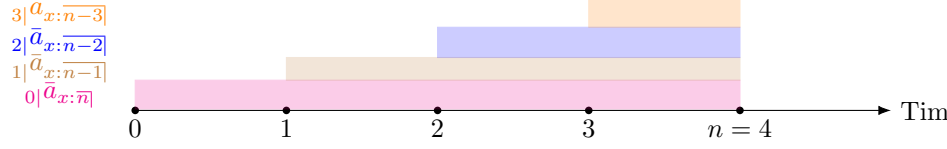
3.2.5 Arithmetically increasing annuity.

(1) *Definition:* ⚠ There are multiple ways in which the benefit amount can vary:

- *annually increasing:* benefit amount (if payable) is $k+1$ at time k, for each $k = 0, 1, \ldots$.
- *continuously increasing:* benefit amount (if payable) is t at time t.

Some typical forms of arithmetically increasing annuities:

Annuity type Benefit pattern	Discrete (due)	Continuous
Annually increasing	✔	✔
Continuously increasing	(N/A)	✔

(2) *Key formulas:* We shall use arithmetically increasing *n-year temporary life annuity* as an example.

- *EPV:*

Benefit pattern \ Annuity type	Discrete (due) ("$\ddot{a}$")	Continuous ("$\bar{a}$")
		(summation of deferred annuity EPVs)
Annually increasing ("I")	$(I\ddot{a})_{x:\overline{n}\rvert} = \sum_{k=0}^{n-1} (k+1)v^k {}_kp_x$	$(I\bar{a})_{x:\overline{n}\rvert} = \sum_{k=0}^{n-1} {}_{k\rvert}\bar{a}_{x:\overline{n-k}\rvert}$

Continuously increasing ("$\bar{I}$")	(N/A)	$(\bar{I}\bar{a})_{x:\overline{n}\rvert} = \int_0^n tv^t {}_tp_x \mathrm{d}t$

(Exercise: Can you come up with the PVRV for each of these three kinds of arithmetically increasing *n*-year temporary life annuities?)

- *Recursive formulas:* There are two nice recursive formulas for $(I\ddot{a})_{x:\overline{n}\rvert}$, which are similar to the ones in [2.2.4]:

  − $(I\ddot{a})_{x:\overline{n}\rvert} = 1 + vp_x\Big((I\ddot{a})_{x+1:\overline{n-1}\rvert} + \ddot{a}_{x+1:\overline{n-1}\rvert}\Big)$
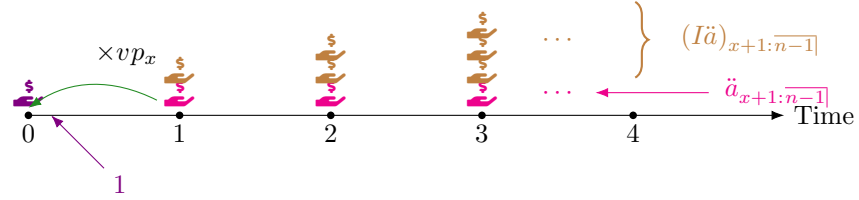
  − $(I\ddot{a})_{x:\overline{n}\rvert} = \ddot{a}_{x:\overline{n}\rvert} + vp_x(I\ddot{a})_{x+1:\overline{n-1}\rvert}$
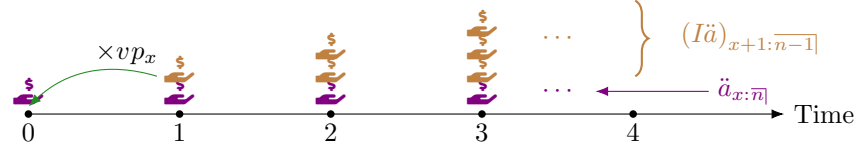
3.2.6 Geometrically increasing annuity.

(1) *Definition:* The benefit amount (if payable) is $(1+j)^k$ at time k, for each $k = 0, 1, \ldots$.

(2) *Key formulas:* We use geometrically increasing *n-year term life insurance* as an example. Like the insurance case in [2.2.5], we need to introduce an auxiliary interest rate $i^* = (i-j)/(1+j)$:

$$\text{EPV} = \ddot{a}_{x:\overline{n}\rvert} @ i^* \overset{(i=j,\, n\to\infty)}{\to} 1 + e_x.$$

Proof. Firstly, note that

$$\text{EPV} = \sum_{k=0}^{n-1}(1+j)^k v^k {}_kp_x = \sum_{k=0}^{n-1}(1+i^*)^{-k} {}_kp_x = \ddot{a}_{x:\overline{n}\rvert} @ i^*.$$

When $i = j$, we have $i^* = 0$, so

$$\ddot{a}_{x:\overline{n}\rvert} @ i^* \overset{(n\to\infty)}{\to} \ddot{a}_x^{@i^*=0} = \sum_{k=0}^{\infty} 1^k {}_kp_x = 1 + \underbrace{\sum_{k=1}^{\infty} {}_kp_x}_{e_x}.$$
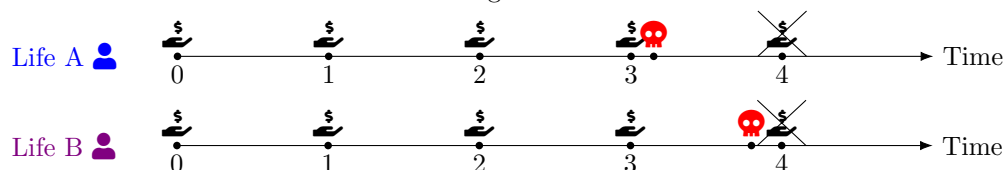
$\square$

3.2.7 Complete and apportionable annuities.

(1) *Definition:*

- *Complete:* Change $a^{(m)} \to \dfrac{\delta}{i^{(m)}} \times \bar{a}$.

  [Note: It is only applicable to $1/m$thly life annuity-<u>immediate</u>.]

- *Apportionable:* Change $\ddot{a}^{(m)} \to \dfrac{\delta}{d^{(m)}} \times \bar{a}$.

  [Note: It is only applicable to $1/m$thly life annuity-<u>due</u>.]

(If you are interested)

The idea behind. Consider the following situation:

Both lives A 👤 and B 👤 do not receive the benefit payment 💲 at time 4. But life B 👤 survives most of the last period, while life A 👤 only survives a little of the last period! So this may cause some unfairness.

To address this issue, the complete/apportionable variant converts discrete payments to a stream of *continuous* payments, which ceases upon death 💀. The unfairness can be eliminated with this conversion since the life B 👤, which survives longer, can then receive more payments. The scaling factors $\delta/i^{(m)}$ and $\delta/d^{(m)}$ are to ensure that the stream of continuous payments obtained is indeed equivalent to the original discrete payments in terms of present value. See Bowers et al. (1997) for more details.

(2) *Key formulas:* Here we use *n-year temporary life annuity* as an example.

	Complete	Apportionable
PVRV	$\dfrac{\delta}{i^{(m)}}\bar{a}_{\overline{T_x \wedge n}|}$	$\dfrac{\delta}{d^{(m)}}\bar{a}_{\overline{T_x \wedge n}|}$
EPV	$\dfrac{\delta}{i^{(m)}}\bar{a}_{x:\overline{n}|}$	$\dfrac{\delta}{d^{(m)}}\bar{a}_{x:\overline{n}|}$
Variance	$\left(\dfrac{\delta}{i^{(m)}}\right)^2 \times \underbrace{\dfrac{{}^2\bar{A}_{x:\overline{n}|} - \bar{A}^2_{x:\overline{n}|}}{\delta^2}}_{\text{Variance of } \bar{a}_{\overline{T_x \wedge n}|}}$	$\left(\dfrac{\delta}{d^{(m)}}\right)^2 \times \underbrace{\dfrac{{}^2\bar{A}_{x:\overline{n}|} - \bar{A}^2_{x:\overline{n}|}}{\delta^2}}_{\text{Variance of } \bar{a}_{\overline{T_x \wedge n}|}}$

(3) *Recursive formulas:* Since the EPVs of complete and apportionable annuities are just scalar multiples of EPVs of continuous life annuities, we can just use the recursive formulas for the continuous case mentioned in [3.1.6], with suitable scaling on the terms. For example:

$$\frac{\delta}{i^{(m)}}\bar{a}_{x:\overline{n}|} = \frac{\delta}{i^{(m)}}\bar{a}_{x:\overline{m}|} + {}_mE_x \times \frac{\delta}{i^{(m)}}\bar{a}_{x+m:\overline{n-m}|}.$$

3.2.8
With selection at age x, all the previous "a" symbols and formulas can still be used, after replacing "x" by $[x]$ and simplifying "$[x] + t$" to "$x + t$" if $t \geq \underset{\text{(select period)}}{d}$.

3.3 Relating $\bar{a}$, $\ddot{a}$ and $\ddot{a}^{(m)}$

3.3.1
Like the insurance case, there are two kinds of formulas that relate $\bar{a}$, $\ddot{a}$ and $\ddot{a}^{(m)}$:

- *UDD:* Formulas that work under the UDD assumption
- *Woolhouse's formulas:* <u>Approximation</u> formulas that work in general

Again such relationship is of interest since in many tables (e.g., SOA FAM-L table and AM92 table), only the values of "$\ddot{a}$"s are given, but not the "$\ddot{a}^{(m)}$"s and "$\bar{a}$"s.

3.3.2 UDD. The UDD formulas can be organized into two groups:

- $\ddot{a} \overset{UDD}{\rightarrow} \ddot{a}^{(m)}$:

Annuity type	UDD formula
Whole life	$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$
Temporary life	$\ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x)$
Deferred whole life	${}_{n|}\ddot{a}_x^{(m)} = \alpha(m){}_{n|}\ddot{a}_x - \beta(m){}_nE_x$

  We have $\alpha(m) = \dfrac{id}{i^{(m)}d^{(m)}}$ and $\beta(m) = \dfrac{i - i^{(m)}}{i^{(m)}d^{(m)}}$.

- $\ddot{a} \overset{UDD}{\rightarrow} \bar{a}$: Let $m \to \infty$, which changes $\ddot{a}^{(m)} \to \bar{a}$, in the UDD formulas for $\ddot{a} \overset{\text{UDD}}{\rightarrow} \ddot{a}^{(m)}$. More specifically, make the following changes on the UDD formulas for $\ddot{a} \overset{\text{UDD}}{\rightarrow} \ddot{a}^{(m)}$:

  - $\ddot{a}^{(m)} \to \bar{a}$.
  - $\alpha(m) \to \alpha(\infty) = \dfrac{id}{\delta^2}$.
  - $\beta(m) \to \beta(\infty) = \dfrac{i - \delta}{\delta^2}$.

[Note: Values of $\alpha(m)$ and $\beta(m)$ at $i = 0.05$ for some m (including $m = \infty$) can be found in the SOA FAM-L table.]

3.3.3 Woolhouse's formula. Woolhouse's formula is an approximation formula for $\ddot{a} \to \ddot{a}^{(m)}$ and $\ddot{a} \to \bar{a}$. Like the claims acceleration approach for the insurance case, the advantage 👍 of using Woolhouse's formula is that the UDD assumption is not needed, but the obvious disadvantage 👎 is that the formula only serves for approximation.

There are two forms of Woolhouse's formula, identified by the number of terms involved. Both are for $\ddot{a}_x \overset{\approx}{\rightarrow} \ddot{a}_x^{(m)}$.

- *3-term formula:* $\boxed{\ddot{a}_x^{(m)} \approx \ddot{a}_x - \dfrac{m-1}{2m} - \dfrac{m^2-1}{12m^2}(\mu_x + \delta)}$. [Note: This is available in the SOA FAM-L table.]

- *2-term formula:* Drop the last term in the 3-term formula, i.e., $\boxed{\ddot{a}_x^{(m)} \approx \ddot{a}_x - \dfrac{m-1}{2m}}$.

Remarks:

- For the 3-term formula, if μ_x is not available, we can approximate it by $\boxed{\mu_x \approx -\dfrac{1}{2}\ln\left(\dfrac{\ell_{x+1}}{\ell_{x-1}}\right)}$. This approximation comes from:

$$\frac{\ell_{x+1}}{\ell_{x-1}} = {}_2p_{x-1} = \exp\left(-\int_{x-1}^{x+1} \mu_s \, ds\right) \approx e^{-2\mu_x}.$$

- For $\ddot{a}_{x:\overline{n}|} \overset{\approx}{\rightarrow} \ddot{a}_{x:\overline{n}|}^{(m)}$ (temporary life) and ${}_{n|}\ddot{a}_x \overset{\approx}{\rightarrow} {}_{n|}\ddot{a}_x^{(m)}$ (deferred whole life), the key 🔑 trick is to express the temporary life annuity and deferred whole life annuity EPVs in terms of the whole life annuity EPVs, so that Woolhouse's formula can be applied:

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \underbrace{\ddot{a}_x^{(m)}}_{\text{apply Woolhouse}} - {}_nE_x \times \underbrace{\ddot{a}_{x+n}^{(m)}}_{\text{apply Woolhouse}} = \cdots = \text{expression in terms of } \ddot{a}_{x:\overline{n}|},$$

$$ {}_{n|}\ddot{a}_x^{(m)} = {}_nE_x \times \underbrace{\ddot{a}_{x+n}^{(m)}}_{\text{apply Woolhouse}} = \cdots = \text{expression in terms of } {}_{n|}\ddot{a}_x.$$

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- For $\ddot{a}_x \overset{\approx}{\to} \bar{a}_x$, again we would let $m \to \infty$, and then the 3-term formula would become

$$\boxed{\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)}.$$

  – *Special case:* Since $\mathring{e}_x = \int_0^\infty {}_t p_x \, \mathrm{d}t = \bar{a}_x$ with $i = 0$, the approximation formula above allows us to approximate $\mathring{e}_x$:

$$\boxed{\mathring{e}_x \approx} \underbrace{(\ddot{a}_x \text{ @ } i = 0)}_{\color{red}1+e_x} - \frac{1}{2} - \frac{1}{12}(\mu_x + 0) = \boxed{e_x + \frac{1}{2} - \frac{1}{12}\mu_x}.$$

- *(If you are interested)* For more details about how the Woolhouse's formula is derived, see Dickson et al. (2019).

- *(Not commonly used)* Applying annuity-insurance formula, we can obtain yet another formula for $A_x \overset{\approx}{\to} A_x^{(m)}$:

$$
\begin{aligned}
A_x^{(m)} &= 1 - d^{(m)} \ddot{a}_x^{(m)} \\
&\approx 1 - d^{(m)}\left(\ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x + \delta) \right) && \text{(Woolhouse)} \\
&= 1 - d^{(m)}\left(\frac{1 - A_x}{d} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x + \delta) \right).
\end{aligned}
$$

4 Premiums

4.0.1 In Sections 2 and 3, we have studied the behaviour of PVRV for life insurance and life annuity, and now we should have a better understanding on the potential loss to the insurer, arising from the benefit payments made in insurance/annuity products. Of course, the insurer would *not* offer the insurance/annuity products free-of-charge, and one of the most important tasks for pricing actuaries is to price the products appropriately:

- If the premiums charged (i.e., amounts of money to be paid by the policyholder) are too low, the potential loss could not be sufficiently covered, leading to a high risk of *bankruptcy* for the insurer!
- If the premiums charged are too high, the insurer could lose a lot of business 👥 due to market competition.

4.0.2 Pricing the products suitably is by no means an easy task. Actually, in practice the insurer often has a whole actuarial pricing team to deal with this, as there are many factors that influence the appropriate price to be set, and the decision on the actual premiums charged requires a substantial amount of actuarial judgement.

Here, we will study a somewhat simple framework for pricing the products, which is about following some simple *premium principles*, i.e., rules for determining the premiums. While it is convenient to just follow a premium principle for obtaining the premiums to be charged, the resulting value is often not so realistic for various practical reasons. (To learn more, consider having an actuarial internship in pricing team 😄.) Nevertheless, this simple framework can provide us some intuition on how different factors could influence the amount of premiums. Without further ado, let us start discussing the premium principles.

4.1 Basic Topics in Equivalence Principle

4.1.1 The first and the most frequently used premium principle to be discussed here is the *equivalence principle*. To illustrate this principle, consider the following simple example about premiums of an insurance coverage.

Suppose the insurer offers a discrete whole life insurance coverage to an individual aged x, with death benefit (or <u>s</u>um insured) S. At the beginning of each year, the policyholder would pay a fixed amount of premium P to the insurer, whenever he is alive.
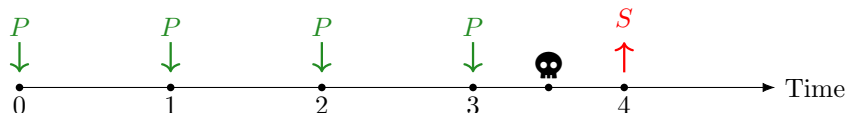
The idea of the equivalence principle is that the amount P should be set such that, in terms of EPV, the premiums received is equivalent to the DB paid. Observe that the premiums above actually form a discrete whole life annuity-due: A payment of amount P is made at the beginning of each year whenever the policyholder is alive. Here the insurer is the recipient of the payments.

Hence, we can express this equivalence of EPVs as $P\ddot{a}_x = S \times A_x$, through which we can solve for the equivalence principle premium: $P = S \times \dfrac{A_x}{\ddot{a}_x}$.

4.1.2 Actually, we have implicitly made a simplifying assumption in the example above: namely that there are no *expenses*. With this assumption, the only cash flows involved are the premiums and the DB payment. However, there are *definitely* expenses in the actual practice, such as commissions to agent, costs for processing premiums, costs for settling the DB, etc.

Thus, to reveal a fuller picture[6], expenses should be incorporated in the calculation of premiums as well. For instance, suppose that for each receipt of premium P, there is an expense of $e = 0.01P$

---

[6]Even with the inclusion of expenses, it is still not the "full picture", as there are still many other practical subtleties involved (to be learnt when you work in actuarial pricing team). But here we will not consider them for simplicity.

incurred, due to the costs for processing premiums. Note that the expenses are to be paid by the insurer, hence they are losses to the insurer.
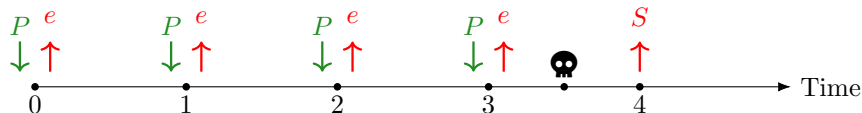
The idea of the equivalence principle remains the same. We would like to set the amount P such that, in terms of EPV, the premiums received is equivalent to the DB paid *and* the expenses paid. Noting the expenses above again form a discrete whole life annuity-due, we can express this equivalence as $P\ddot{a}_x = S \times A_x + e\ddot{a}_x = S \times A_x + 0.01P\ddot{a}_x$. From this, we get $P = S \times \dfrac{A_x}{0.99\ddot{a}_x}$.

4.1.3 With the examples above, you should (hopefully!) have a better idea about the intuitive meaning of equivalence principle. Now, let us state formally the definition of equivalence principle. The definition involves a random variable called **loss at issue random variable**, which has two forms, depending on whether the expenses are considered:

- *net loss:* $L_0^n = \text{PV}_0(\text{benefits}) - \text{PV}_0(\text{premiums})$
- *gross loss:* $L_0^g = \text{PV}_0(\text{benefits and expenses}) - \text{PV}_0(\text{premiums})$

Here $\text{PV}_0(\cdot)$ refers to the time-0 present value.

Then, the **equivalence principle** suggests that the premium is set such that:

- *(ignoring expenses)* $\mathbb{E}[L_0^n] = 0$, i.e., $\text{EPV}_0(\text{premiums}) = \text{EPV}_0(\text{benefits})$.
- *(considering expenses)* $\mathbb{E}[L_0^g] = 0$, i.e., $\text{EPV}_0(\text{premiums}) = \text{EPV}_0(\text{benefits}) + \text{EPV}_0(\text{expenses})$.

Here $\text{EPV}_0(\cdot)$ refers to the time-0 EPV.

4.1.4 Some terminologies:

- *net premium:* premium set by equivalence principle with expenses ignored
- *gross premium:* premium with expenses considered (which may or may not be set by equivalence principle)
- *fully discrete:* Both benefits and premiums are only payable at discrete time points within policy term. More specifically, the discrete time points refer to:
    - *(for premiums)* the beginning of each year, whenever the policyholder is alive
    - *(for DB)* the end of the year of death
    - *(less common) (for annuity benefits)* the beginning (for due) or end (for immediate) of each year, whenever the policyholder is alive
- *fully continuous:*
    - Premiums (or less commonly, annuity benefits) are payable continuously within policy term, whenever the policyholder is alive.
    - DB is payable at the moment of death within policy term.
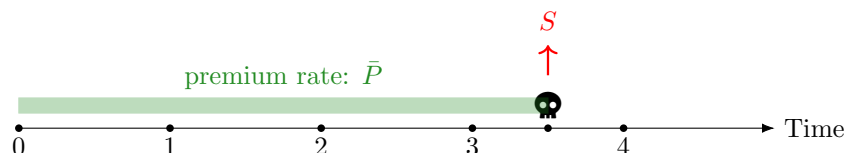
Next, we will introduce some key formulas for computing (i) net premium and (ii) gross premium set by equivalence principle.

4.1.5 Net premium.

Insurance type	Net premium (fully discrete)	Net premium (fully continuous)
Whole life	$P_x = \dfrac{A_x}{\ddot{a}_x} = \underbrace{\dfrac{dA_x}{1-A_x}}_{(A \text{ only})} = \underbrace{\dfrac{1}{\ddot{a}_x} - d}_{(\ddot{a} \text{ only})}$	(Change $A_x \to \bar{A}_x$, $\ddot{a}_x \to \bar{a}_x$ and $d \to \delta$)
Term life	$P^1_{x:\overline{n}|} = \dfrac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$ (NOT $P^1_{x:\overline{n}|} = \dfrac{A^1_{x:\overline{n}|}}{\ddot{a}^1_{x:\overline{n}|}}$ ⚠)	(Change $A^1_{x:\overline{n}|} \to \bar{A}^1_{x:\overline{n}|}$ and $\ddot{a}_{x:\overline{n}|} \to \bar{a}_{x:\overline{n}|}$)
Endowment	$P_{x:\overline{n}|} = \dfrac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \underbrace{\dfrac{dA_{x:\overline{n}|}}{1-A_{x:\overline{n}|}}}_{(A \text{ only})} = \underbrace{\dfrac{1}{\ddot{a}_{x:\overline{n}|}} - d}_{(\ddot{a} \text{ only})}$	(Change $A_{x:\overline{n}|} \to \bar{A}_{x:\overline{n}|}$, $\ddot{a}_{x:\overline{n}|} \to \bar{a}_{x:\overline{n}|}$ and $d \to \delta$)
Pure endowment	$P_{x:\overline{n}|}^{\ \ 1} = \dfrac{A_{x:\overline{n}|}^{\ \ 1}}{\ddot{a}_{x:\overline{n}|}}$	(Change $A_{x:\overline{n}|}^{\ \ 1} \to \bar{A}_{x:\overline{n}|}^{\ \ 1}$ and $\ddot{a}_{x:\overline{n}|} \to \bar{a}_{x:\overline{n}|}$)

We can observe that all the net premiums here take a simple ratio form ("$A/\ddot{a}$" or "$\bar{A}/\bar{a}$").

4.1.6 Gross premium.

- *Common types of expenses (for fully discrete case):*

Type	Definition	Example
Initial	Incurred at the policy issuance.	Commissions to agents
Renewal	Incurred at each premium payment time except the first.	Costs for issuing annual statements
Termination/Settlement	Incurred when paying DB or endowment benefit.	Costs for paperwork required to settle the claim of DB/endowment benefit

Notations for these expenses:

- I: initial expense
- e: each renewal expense
- E: termination/settlement expense

[Note: For fully continuous case, the renewal expenses would become a stream of payments, incurring simultaneously with the stream of continuous premium payments. In such case, the notation e would denote the *rate* of renewal expense instead. Also, typically the initial expense is assumed to be zero in the fully continuous case.


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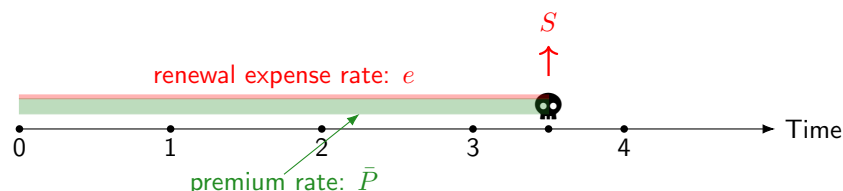
- *Calculating gross premium set by equivalence principle:* In general, the gross premium can be determined by solving $\mathrm{EPV}_0(\text{gross premiums}) = \mathrm{EPV}_0(\text{benefits}) + \mathrm{EPV}_0(\text{expenses})$.

– *Tip (for fully discrete case):* Often we can express $\mathrm{EPV}_0(\text{expenses})$ in terms of the EPV of a life annuity-due via the "decomposition":

$$(I - e) + e\ddot{a}_x \quad \text{or} \quad (I - e) + e\ddot{a}_{x:\overline{n}|}.$$

This allows us to compute $\mathrm{EPV}_0(\text{expenses})$ more conveniently.
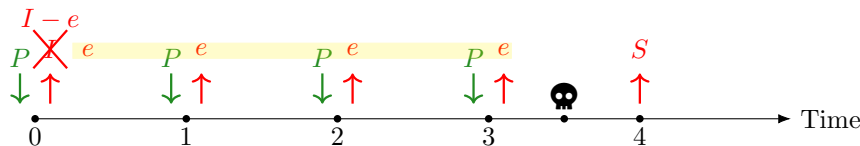
4.2 Further Topics in Equivalence Principle

4.2.1 The further topics in equivalence principle to be discussed here encompass the following three themes:

- *Theme 1: Variations on the policy.* We will introduce some variants to the basic insurance coverages mentioned in Section 4.1 and discuss how to determine the premiums via equivalence principle with such variants: (i) $1/m$thly premiums and (ii) refund of premiums.

  Similar to what was done before, we shall explore each variant in the following two aspects:

  (1) *Definition:* meaning of the variant
  (2) *Key formulas:* important formulas about the variant

- *Theme 2: Probabilistic quantities of loss at issue random variable.* Section 4.1 is primarily about calculating premiums such that the mean loss at issue random variable equals zero. Here, apart from its mean, we will explore more probabilistic quantities of the loss at issue random variable: (i) variance and (ii) probabilities.

- *Theme 3: Extra risks.* Here we will explore the impacts on the premiums with the presence of extra mortality risk, which can be incorporated in the following three ways: (i) age rating, (ii) constant addition to μ_x and (iii) constant multiple of q_x.

Theme 1: Variations on the policy

4.2.2 **$1/m$thly premiums.** We have discussed $1/m$thly insurance and $1/m$thly annuity in Sections 2.2 and 3.2. In a similar manner, we can define the notion of $1/m$thly premiums.

(1) *Definition:* Premium payment is made at beginning of each $1/m$th of a year in the relevant period, whenever the policyholder is alive.

(2) *Key formulas:* Let P be the amount of each $1/m$thly premium.

Policy type	EPV of premiums	
Whole life	$mP \times \ddot{a}_x^{(m)}$	(often use: UDD/Woolhouse ...)
n-year policy	$mP \times \ddot{a}_{x:\overline{n}|}^{(m)}$	(often use: UDD/Woolhouse ...)

4.2.3 **Refund of premiums.**

(1) *Definition:* The total amount of premium paid (with or without interest accumulation) is added on top of the DB.

Thus, the premiums are "refunded" when death 💀 occurs within the policy term. Usually this variant is applied on n-year policies.

(2) *Key formulas:*

- *Generic formula for calculating premium:* We can just insert the added benefit (premium refund) to the equation for equivalence principle and solve for the premium P:

$$\text{EPV}_0(\text{premiums}) = \text{EPV}_0(\text{original benefits}) + \text{EPV}_0(\text{premium refund}) + \text{EPV}_0(\text{expenses})$$

- *Formulas of* EPV_0 *(premium refund)* *(for n-year policies):*

	EPV_0(premium refund)		
Without interest	$P(IA)^1_{x:\overline{n}|}$		
With interest	$\underbrace{P\ddot{a}_{x:\overline{n}|}}_{\text{EPV of refund with interest}}$	$-$	$\underbrace{{}_np_x \times P\ddot{a}_{\overline{n}|}}_{\text{no refund if not dead in } n \text{ years}}$

  – *Illustration of premium refund with interest:*
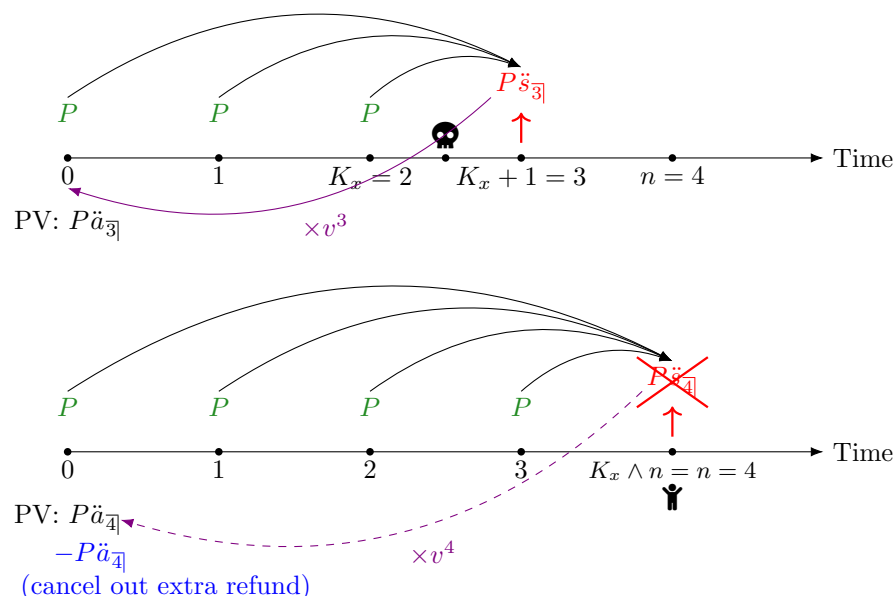
Theme 2: Probabilistic quantities of loss at issue random variable

4.2.4 Variance of L_0.

- *WL/endowment insurance:* (G: amount of each premium)

Type	Fully discrete	Fully continuous
WL	$\left(S+E+\dfrac{G-e}{d}\right)^2 \underbrace{({}^2A_x - A_x^2)}_{\text{Variance of insurance PVRV}}$ $\underset{\text{(no expenses)}}{=} \left(\dfrac{S}{1-A_x}\right)^2({}^2A_x - A_x^2)$	(Change $A_x \to \bar{A}_x$, ${}^2A_x \to {}^2\bar{A}_x$ and $d \to \delta$ on the left)
Endowment	(Change $A_x \to A_{x:\overline{n}|}$ and ${}^2A_x \to {}^2A_{x:\overline{n}|}$ above)	(Change $\bar{A}_x \to \bar{A}_{x:\overline{n}|}$ and ${}^2\bar{A}_x \to {}^2\bar{A}_{x:\overline{n}|}$ above)

- *First principle approach (for fully discrete short-term policies):* For illustration, we use a fully discrete 4-year term life insurance with expenses ignored and DB being S as an example. Let P be the amount of each premium.
  Steps:
  (1) *List all possible values of L_0 with the corresponding probabilities:*

Value of L_0	Event	Probability
$Sv - P$	$K_x = 0$	q_x
$Sv^2 - P\ddot{a}_{\overline{2|}}$	$K_x = 1$	$_{1|}q_x$
$Sv^3 - P\ddot{a}_{\overline{3|}}$	$K_x = 2$	$_{2|}q_x$
$Sv^4 - P\ddot{a}_{\overline{4|}}$	$K_x = 3$	$_{3|}q_x$
$-P\ddot{a}_{\overline{4|}}$	$K_x \geq 4$	$_4p_x$
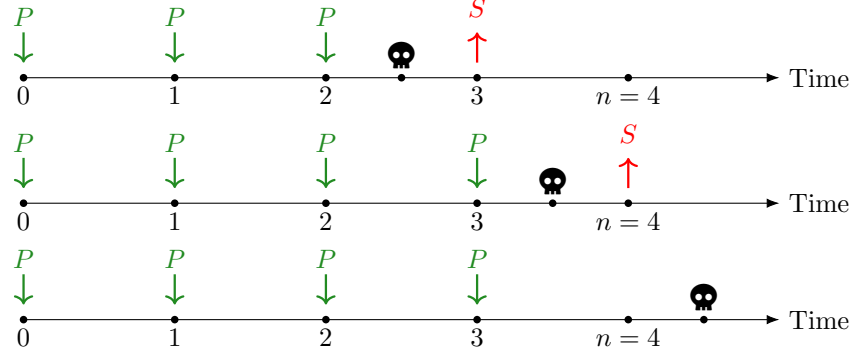
(2) *Calculate* $\mathbb{E}[L_0]$ *and* $\mathbb{E}[L_0^2]$:

- $\mathbb{E}[L_0] = \sum$ value $\times$ probability ($= 0$ if equivalence principle is used)

- $\mathbb{E}[L_0^2] = \sum$ value^2 $\times$ probability

(3) *Calculate* $\mathrm{Var}(L_0)$: $\mathrm{Var}(L_0) = \mathbb{E}[L_0^2] - (\mathbb{E}[L_0])^2 \overset{\text{(equivalence principle)}}{=} \mathbb{E}[L_0^2]$.

4.2.5 **Probabilities for** L_0. Often we are interested in *loss/profit probability:*

- loss probability $= \mathbb{P}(L_0 > 0)$
- profit probability $= \mathbb{P}(L_0 < 0)$

There are two approaches for computing probabilities for L_0. For illustration, let's say we are interested in the profit probability $\mathbb{P}(L_0 < 0)$.

- *Exact calculation:*
  (1) Write $L_0 < 0 \iff \cdots \iff$ (event in terms of K_x or T_x).
  (2) Compute the probability of the event in terms of K_x or T_x (which equals $\mathbb{P}(L_0 < 0)$).
- *Normal approximation:* If L_0 is a loss at issue random variable comprising of losses from a large number of independent policies, then we can apply normal approximation:

$$\frac{L_0 - \overbrace{\mathbb{E}[L_0]}^{=0 \text{ if equivalence principle is used}}}{\sqrt{\mathrm{Var}(L_0)}} \overset{\text{approx.}}{\sim} \mathrm{N}(0,1).$$

[Note: If L_0 is a sum of n independent and identically distributed (i.i.d.) losses, then $\mathrm{Var}(L_0) = n \times$ variance of each loss (not n^2 ⚠).]
From this, we have

$$\mathbb{P}(L_0 < 0) \approx \Phi\left(\frac{-\mathbb{E}[L_0]}{\sqrt{\mathrm{Var}(L_0)}}\right)$$

where Φ denotes the standard normal cdf.

Theme 3: Extra risks

4.2.6 **Age rating**. This approach incorporate the extra mortality risk by treating the individual as a standard but *older* life.

For example, an individual aged 40 with extra mortality risk may be treated as a standard life (i.e., having no extra mortality risk), but aged 50.

4.2.7 **Constant addition to μ_x.** [Note: We shall add a superscript $'$ to the quantities for an individual with extra mortality risk. Quantities without the superscript $'$ denote the ones for a normal individual with no extra mortality risk.]

Key formulas:

	$\mu'_{x+t}\ (0 \le t \le n)$	δ'	$\bar{a}'_{x:\overline{n}|}$	$_nE'_x$
Constant addition to μ_x	$\mu_{x+t} + \phi$	δ	$\bar{a}_{x:\overline{n}|}$ @ FOI $\delta + \phi$	$_nE_x$ @ FOI $\delta + \phi$
Constant addition to δ *(for comparison)*	μ_{x+t}	$\delta + \phi$	$\bar{a}_{x:\overline{n}|}$ @ FOI $\delta + \phi$	$_nE_x$ @ FOI $\delta + \phi$

Remarks:

- FOI: force of interest
- The result is still applicable after changing $\bar{a} \to \ddot{a}$ (with or without $'$).

This table suggests that the effects on the EPVs of temporary annuity and pure endowment are the same with either constant addition to μ_x or constant addition to force of interest. In other words, to analyze the impact on these two types of EPVs after adding a constant ϕ to μ_x, we can just increase the force of interest by ϕ instead.

[⚠ Warning: The effects may <u>not</u> be the same for other kinds of EPVs. For example, for endowment insurance EPV:

- constant addition to μ_x: $\bar{A}'_{x:\overline{n}|} = 1 - \delta(\bar{a}_{x:\overline{n}|}$ @ FOI $\delta + \phi)$
- constant addition to δ: $\bar{A}'_{x:\overline{n}|} = 1 - (\delta + \phi)(\bar{a}_{x:\overline{n}|}$ @ FOI $\delta + \phi)$

]

After obtaining the new values of EPVs based on the formulas here, the premiums can be recalculated.

4.2.8 **Constant multiple of q_x.** In this approach, we scale up the 1-year mortality probabilities: $q'_{x+k} = c \times q_{x+k}$ where $c > 1$ for each $k = 0, 1, \dots$. For this case, there is not "shortcut" formula and we would need to recalculate the premiums by repeating the whole procedure with new mortality probabilities. (This may be more practical with the aid of a spreadsheet 🗎.)

[Note: Since we are scaling 1-year mortality probabilities, when we are to calculate premiums for some $1/m$thly or continuous products, we would need to apply the UDD assumption or use approximation formula for the computations.]

4.3 Portfolio Percentile Premium Principle

4.3.1 Equivalence principle is not the only premium principle available. Another common premium principle is the *portfolio percentile principle*. As its name suggests, this premium principle is related to percentile. But what percentile are we referring to?

4.3.2 The idea of *portfolio percentile principle* is to set the premium such that the profit probability is high. Mathematically, the **portfolio percentile principle** suggests that we should set the premium such that

$$\mathbb{P}(L_0 < 0) = p \quad \text{or} \quad \underbrace{\pi_p(L_0)}_{p\text{th percentile of } L_0} = 0$$

where p is close to 1. Here, L_0 is usually the total loss for a portfolio of n independent policies with identical terms, meaning that L_0 is the sum of n i.i.d. individual loss-at-issue random variables $L_{0,1}, \ldots, L_{0,n}$, and we are setting the premium to be used for all these n policies.

4.3.3 The two approaches for probability calculations in [4.2.5] can also be used for calculating portfolio percentile premium:

- *Exact calculation ($n = 1$):* In general, we can perform similar steps as the ones in [4.2.5], with probability calculation being replaced by percentile calculation.

  But there is a shortcut based on the *percentile trick*.

  Condition: $L_0 = g(T_x)$ or $L_0 = h(T_x)$, where g and h are strictly decreasing *(more common; for insurances)* and strictly increasing continuous functions respectively.

  Shortcut:

  (1) *Finding $\pi_p(L_0)$:*

  $$\pi_p(L_0) = \begin{cases} g(\pi_{1-p}(T_x)) & \text{when } L_0 = g(T_x), \\ h(\pi_p(T_x)) & \text{when } L_0 = h(T_x). \end{cases}$$

  (2) *Calculating the portfolio percentile premium:* Solve the equation $\pi_p(L_0) = 0$ for the premium.

- *Normal approximation (for large n):*

  (1) *Finding $\pi_p(L_0)$:* After applying the normal approximation, we can treat

  $$L_0 = \mathbb{E}[L_0] + \sqrt{\text{Var}\,(L_0)}Z = n\mathbb{E}[L_{0,i}] + \sqrt{n\,\text{Var}\,(L_{0,i})}Z$$

  where $Z \sim \text{N}\,(0,1)$ and $i \in \{1, \ldots, n\}$. Then, using the percentile trick, we have

  $$\boxed{\pi_p(L_0) = \mathbb{E}[L_0] + \sqrt{\text{Var}\,(L_0)}\pi_p(Z) = n\mathbb{E}[L_{0,i}] + \sqrt{n\,\text{Var}\,(L_{0,i})}\pi_p(Z)}.$$

  [Note: The value $\pi_p(Z)$ can be found using a standard normal table *(for STAT3901)* or the Prometric's standard normal calculator *(available in SOA exam FAM)*.]

  (2) *Calculating the portfolio percentile premium:*
  - *Method 1 (portfolio basis):* Solve the equation $\pi_p(L_0) = 0$ for the premium.
  - *Method 2 (per-policy basis):* Solve the equation

  $$\frac{1}{n}\pi_p(L_0) = \boxed{\mathbb{E}[L_{0,i}] + \sqrt{\frac{\text{Var}\,(L_{0,i})}{n}}\pi_p(Z) = 0}$$

  for the premium. [⚠ Warning: Not solving $\mathbb{E}[L_{0,i}] + \sqrt{\text{Var}\,(L_{0,i})}\pi_p(Z) = 0$!]

4.3.4 **Portfolio percentile vs. equivalence principle premium.** So far we have discussed two premium principles. How do they compare? There are two main features:

- Portfolio percentile premium usually *decreases* with n and is *higher* than the equivalence principle premium. [Intuition 💡: Portfolio percentile premium principle is more "conservative" when there are not too many policies in the portfolio being considered.]

- As $n \to \infty$, the portfolio percentile premium converges to the equivalence principle premium. [Intuition 💡: As n gets larger, the stronger "diversification" among the policies reduces the gap between the two premiums.]

These two features can be explained by noting that the portfolio percentile premium is the solution to the equation $\mathbb{E}[L_{0,i}] + \sqrt{\frac{\text{Var}\,(L_{0,i})}{n}}\pi_p(Z) = 0$, while the equivalence principle premium is the solution to the equation $\mathbb{E}[L_{0,i}] = 0$.

5 Policy Values I

5.0.1 Other than pricing insurance products, another important function performed by actuaries is *reserving* or *valuation*, which is about determining a suitable amount of money 💰 to set aside for covering the potential losses. This amount would depend on the chance for the benefit payment to be triggered.

For example, consider a fully discrete whole life insurance with expenses ignored, and suppose we would like to reserve enough money now (the start of this year) for paying the potential DB at the year end (on average). Intuitively, if the life insured is still quite young (e.g., aged 20) now, then we should not need to reserve too much money, as his remaining lifetime should be quite long 🧍 and it is rather unlikely for him to die in this year. On the other hand, if the life insured is already very old (e.g, aged 99) now, then intuitively we should reserve quite a lot of money as there is a good chance that the life will die ☠ in this year.

This example suggests the characteristic that the reserve amount depends on the future lifetime, and varies as the policy progresses (where the underlying life insured would become older and older).

5.0.2 Like what was done in Section 4, we shall only discuss a somewhat simple framework for determining the reserve amount, which instructs us to set this amount as the so-called *policy value*. But of course, the process of determining suitable reserve amounts is definitely not that simple in practice, and actually the insurer often has a whole actuarial valuation team to deal with this task. (Again, to learn more about how reserving works in practice, consider having an actuarial internship in valuation team 😊.)

5.1 Basic Topics in Policy Values

5.1.1 Like *equivalence principle* discussed in Section 4, the concept of policy value is related to a random variable. Such random variable is called time-t (present value of) **future loss random variable**, which again has two forms, depending on whether the expenses are considered:

- *time-t **net future loss**:* $L_t^n = \mathrm{PV}_t(\text{future benefits}) - \mathrm{PV}_t(\text{future \textbf{net} premiums})$
- *time-t **gross future loss**:* $L_t^g = \mathrm{PV}_t(\text{future benefits and expenses}) - \mathrm{PV}_t(\text{future \textbf{gross} premiums})$

Here $\mathrm{PV}_t(\cdot)$ refers to the time-t present value, and we also assume that the policy is still in force at time t (particularly this means the underlying life is still alive at time t).[7]

[⚠ Warning: There may be an ambiguity on whether the benefits/premiums/expenses *precisely at time t* should be included as "future" benefits/premiums/expenses. For consistency, we shall follow this convention:

- <u>Benefits</u> and <u>benefit</u>-related expenses (i.e., settlement expenses) @ time t: NOT included.
- <u>Premiums</u> and <u>premium</u>-related expenses (i.e., initial and renewal expenses) @ time t: included.

With this convention, we can express the time-t gross loss as

$$L_t^g = \mathrm{PV}_t(\text{benefits and benefit-related expenses @ time} > t)$$
$$+ \mathrm{PV}_t(\text{premium-related expenses @ time} \geq t) - \mathrm{PV}_t(\textbf{gross} \text{ premiums @ time} \geq t).$$

]

5.1.2 With time-t future loss random variable, we can define the time-t **policy value**, which again has two forms, depending on whether the expenses are considered:

- *time-t **net premium policy value**:*

$$_tV^n = \mathbb{E}[L_t^n] = \mathrm{EPV}_t(\text{future benefits}) - \mathrm{EPV}_t(\text{future \textbf{net} premiums}).$$

---

[7]This extra assumption is needed here, since unlike the case for equivalence principle, the policy may have already been vanished at time t (because the underlying life has already been dead ☠ or the policy has already been expired). But in such case, it does not make sense to talk about future loss at such time, which is something that does not even exist.

- *time-t **gross premium policy value**:*

$$_tV^g = \mathbb{E}[L_t^g] = \text{EPV}_t(\text{future benefits and expenses}) - \text{EPV}_t(\text{future \textbf{gross} premiums}).$$

Here $\text{EPV}_t(\cdot)$ refers to the time-t EPV.

<u>Remarks</u>:

- We may drop the superscript (n or g) if it is clear from context which kind of policy value we are referring to.
- Intuitively, this definition suggests that we should set aside a reserve amount that equals exactly the amount of money we would expect to lose in the future.[8]

5.1.3 The basic topics in policy values to be discussed here are:

- *Boundary values of $_tV$:* We will introduce some "edge cases" in which the policy values are readily available without needing any calculation 👍.
- *Shortcut formulas for WL/endowment:* In virtue of the *annuity-insurance formulas*, we can derive some shortcut formulas that apply for whole life/endowment insurances.

5.1.4 **Boundary values of $_tV$.**

Conditions	Boundary value
(no condition)	$_0V^n = 0$
(i) gross premium is set by equivalence principle, and (ii) policy value basis = premium basis [Note: **Basis** means a set of assumptions. Note that policy value basis and premium basis *can* be different!]	$_0V^g = 0$
n-year term life insurance	$_nV^g = 0$
n-year endowment insurance	$_nV^g = S + E$
n-year deferred whole life insurance (assuming premiums are payable for at most n years)	$_nV^g = \begin{cases} (S+E) \times A_{x+n} & \text{for fully discrete case,} \\ (S+E) \times \bar{A}_{x+n} & \text{for fully continuous case} \end{cases}$

Notations: S denotes sum insured and E denotes settlement expense.

[Note: Technically, for those n-year policies, time-n policy value is *undefined* since the policy expires and is no longer in force at that time. The notation $_nV^g$ actually means $\lim_{t \to n^-} {}_tV^g$ (same for net-premium policy value).]

5.1.5 **Shortcut formulas for WL/endowment.** First, we give the shortcut formulas for fully discrete whole life insurance with sum insured S and gross premium G:

Quantity	Shortcut formula
$_tV^n$	$S\left(1 - \dfrac{\ddot{a}_{x+t}}{\ddot{a}_x}\right)_{\text{(only } \ddot{a})} = S\left(\dfrac{A_{x+t} - A_x}{1 - A_x}\right)_{\text{(only } A)}$
$\text{Var}\left(L_t^g\right)$	$\left(S + E + \dfrac{G-e}{d}\right)^2 \underbrace{\left(^2A_{x+t} - A_{x+t}^2\right)}_{\text{Variance of WL PVRV for } (x+t)}$

---

[8]This approach may be quite natural at the first glance, but actually setting the reserve in this way is somewhat risky ⚠ because if the future experience turns out to be worse than expected (not that uncommon), we would not have enough money in our reserve to cover the loss anymore, which can lead to disastrous consequences! This can explain why our framework here is more like a "toy model" than a legitimate approach to be used directly in practice. Practically, perhaps some extra "buffer" is needed, to be safe.

Proof. Here we only prove the formulas for $_tV^n$ for illustration and leave the proof for the variance formula as exercise.

-

$$_tV^n = S\underbrace{A_{x+t}}_{1-d\ddot{a}_{x+t}} - \underbrace{S\left(\frac{1}{\ddot{a}_x} - d\right)\ddot{a}_{x+t}}_{\text{net premium}} = S\left(1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right).$$

-

$$_tV^n = SA_{x+t} - \underbrace{S\cdot\frac{dA_x}{1-A_x}\overbrace{\ddot{a}_{x+t}}^{(1-A_{x+t})/d}}_{\text{net premium}} = S\left(\frac{A_{x+t}(1-A_x) - A_x(1-A_{x+t})}{1-A_x}\right) = S\left(\frac{A_{x+t} - A_x}{1-A_x}\right).$$

$\square$

To obtain the shortcut formulas for other cases, perform the following changes on the shortcut formulas above:

- *(n-year endowment)* Change $\begin{cases} A_x \to A_{x:\overline{n}|} \\ \ddot{a}_x \to \ddot{a}_{x:\overline{n}|} \end{cases}$ and $\begin{cases} A_{x+t} \to A_{x+t:\overline{n-t}|} \\ \ddot{a}_{x+t} \to \ddot{a}_{x+t:\overline{n-t}|} \end{cases}$ $(0 \le t \le n)$.

- *(1/mthly insurances and premiums)* Add superscript (m) to $\ddot{a}$ and A, and change $d \to d^{(m)}$. [Note: In this case G refers to the <u>total</u> amount of gross premiums in a year.]

- *(fully continuous)* Change $\ddot{a} \to \bar{a}$, $A \to \bar{A}$, and $d \to \delta$.

5.2 Further Topics in Policy Values

5.2.1 In Section 5.1 we have studied some relatively basic topics in policy values. Armed with our basic understanding in policy values, we can now have some further and more advanced discussions on policy values. The topics to be discussed can be organized into the following three themes:

- *Theme 1: Policy value recursions:* Since policy values are defined at different time points, we can derive some recursive formulas for policy values that illustrate the evolution of policy value over time. We will study (i) basic policy value recursion, (ii) variations of basic policy value recursion, and (iii) applications of policy value recursions.

- *Theme 2: Other types of policy values:* Apart from the net premium and gross premium policy values in Section 5.1, there are still other kinds of policy values and we will discuss two here:
  - *Retrospective policy values:* policy values from a <u>retrospective</u> view ◀◀.
  - *Expense policy values:* policy values with expenses being the only things considered.

- *Theme 3: Modified net premium reserves:* Reserves which are obtained by <u>modifying</u> the <u>net premiums</u> in the net premium policy value approach. A common example that will be discussed here is the *full preliminary term (FPT) reserves.*

Apart from these, there are still some advanced topics in policy values, but we shall leave the discussions of them to the section titled "Policy Values II" in the STAT3909 study notes.

Theme 1: Policy value recursions

5.2.2 **Basic policy value recursion.** To better understand recursions of policy values, we should bear in mind 🧠 a "slogan": "<u>w</u>hat <u>y</u>ou <u>h</u>ave <u>i</u>s <u>w</u>hat <u>yo</u>u <u>n</u>eed (expected)" (WYHIWYN). Let us use the

following basic policy value recursive formula to illustrate this "slogan":

$$\underbrace{({}_tV + G_t - e_t)(1 + i)}_{\text{what you have}} = \underbrace{q_{x+t}(S_{t+1} + E_{t+1}) + p_{x+t}({}_{t+1}V)}_{\text{what you need (expected)}}$$ [9]

(t is an integer). Notations:

- G_t: gross premium at time t
- e_t: sum of initial and renewal expenses at time t
- S_{t+1}: amount of sum insured (death benefit) at time $t + 1$
- E_{t+1}: settlement expense at time $t + 1$
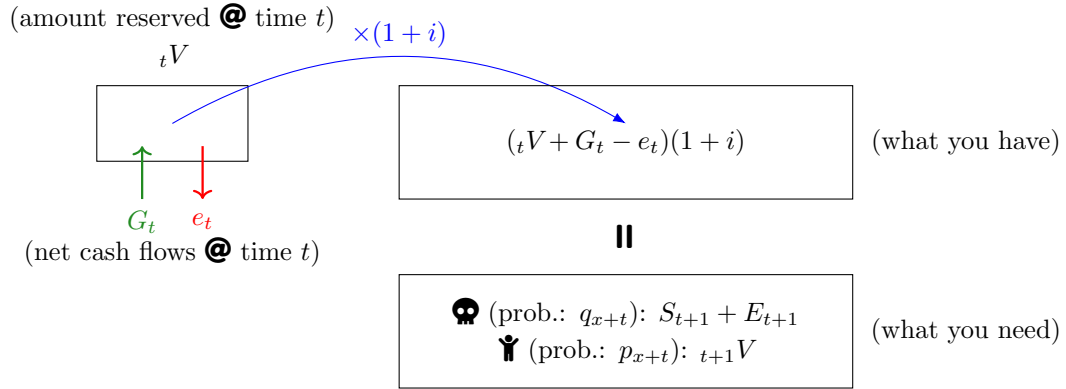
5.2.3 Variations of basic policy value recursion.

- *Net amount at risk (NAAR) form:* Reorganizing the terms on the RHS of the basic policy value recursive formula gives

$$({}_tV + G_t - e_t)(1 + i) = {}_{t+1}V + q_{x+t}(\underbrace{S_{t+1} + E_{t+1} - {}_{t+1}V}_{\text{NAAR}}).$$

with annotations: over the left side $({}_tV + G_t - e_t)(1 + i)$ is "(what you have)" and over $S_{t+1} + E_{t+1} - {}_{t+1}V$ is "(what you need)".

[Note: Since q_{x+t} only appears in one place in this formula, it can be useful for questions asking us to solve for q_{x+t} given some policy values.]

The **net amount at risk** (also known as **death strain at risk** (DSAR) or **sum at risk** (SAR)) $\underbrace{S_{t+1} + E_{t+1}}_{\text{payment needed}} - \underbrace{{}_{t+1}V}_{\text{reserve available}}$ can be interpreted as the *additional* amount of money to be paid to dying member over the reserve available.

This recursive formula in NAAR form expresses "what you need" in a different way. Instead of dividing the situation into (i) the life dies ☠ in this year and (ii) the life survives this year, it partitions "what you need" into (i) reserve amount and (ii) extra amount needed over the reserve.

[9]Here an implicit assumption is that when the life survives @ time $t + 1$, the only amount needed is the time-$t + 1$ policy value and no extra amount is required. This holds true for insurance products (even for endowment insurance with $t = n - 1$, as the policy value ${}_nV$ would include the endowment benefit by convention), but not for annuity products. But this should normally not be an issue since almost all problems requiring the usage of policy value recursions would be about insurances. In the rare case where we need to use policy value recursions for annuity products, we can use a similar intuition and adapt the recursion formula by adding the survival benefit for the "survival" case and removing the payments for the "death" case:

$$q_{x+t}(S_{t+1} + E_{t+1}) + p_{x+t}({}_{t+1}V) \to p_{x+t}({}_{t+1}V + \text{survival benefit}).$$

$_tV$

$\times(1+i)$

$$(_tV + G_t - e_t)(1+i)$$ (what you have)

$G_t \qquad e_t$

=

| $_{t+1}V$ | extra for ☠ (expected) | (what you need) |
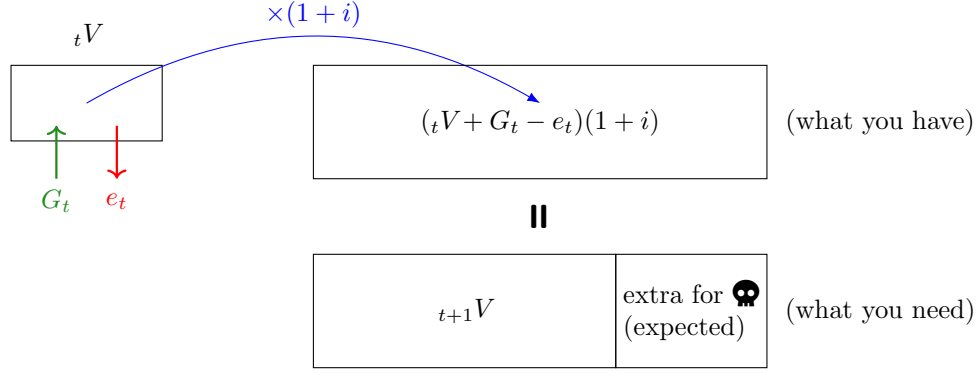
- *Interim policy values:* This variation suggests the behaviour of policy values between integer time points.
  (t is an integer and $0 < s < 1$)

  – *Relating $_tV$ and $_{t+s}V$:*

$$\underbrace{(_tV + G_t - e_t)(1+i)^s}_{\text{(what you have)}} = {}_sq_{x+t}\underbrace{\big(v^{1-s}(S_{t+1} + E_{t+1})\big) + {}_sp_{x+t}({}_{t+s}V)}_{\text{(what you need)}}.$$

The necessity of the extra discount factor v^{1-s} can be illustrated in the following picture:

$_tV$

$\times(1+i)^s$

$$(_tV + G_t - e_t)(1+i)^s$$ (what you have)

$G_t \qquad e_t$

=

☠ (prob.: ${}_sq_{x+t}$): $v^{1-s}(S_{t+1} + E_{t+1})$
☻ (prob.: ${}_sp_{x+t}$): ${}_{t+s}V$ (what you need)

$\times(1+i)^{1-s}$

☠ Payment @ time $t+1$: $S_{t+1} + E_{t+1}$
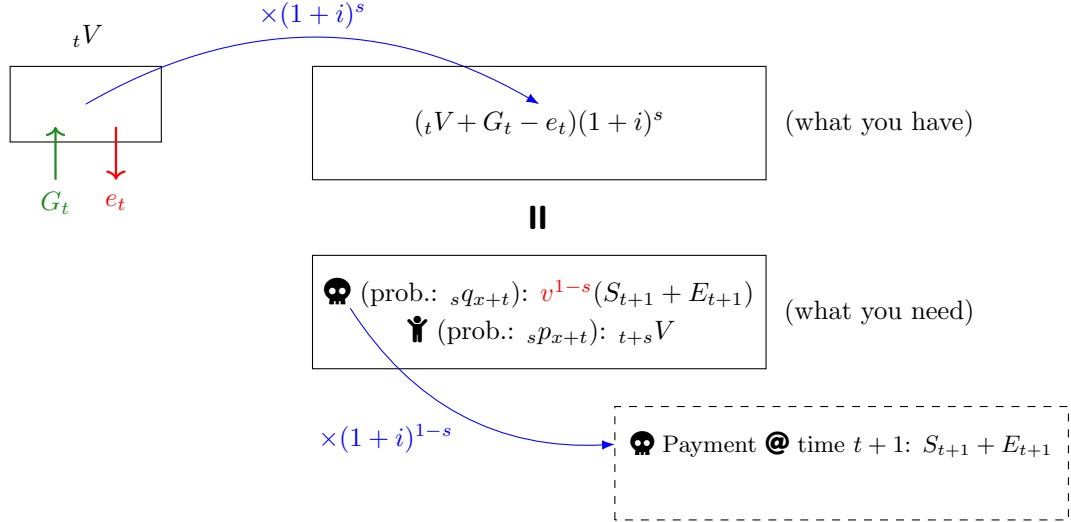
In short, the amount $v^{1-s}(S_{t+1} + E_{t+1})$ is NOT referring to the amount you need to pay @ time $t + s$ (which is not year-end), but is referring to the amount you need to <u>save</u> @ time $t + s$ for the amount that <u>will</u> surely be paid @ time $t + 1$, if death ☠ occurs in the time period $(t, t+s]$. [Note: Of course, death ☠ can also occur in the remaining period $(t+s, 1]$, but that is incorporated in the time-$t + s$ policy value $_{t+s}V$.]

  – *Relating $_{t+s}V$ and $_{t+1}V$:*

$$\underbrace{{}_{t+s}V(1+i)^{1-s}}_{\text{(what you have)}} = {}_{1-s}q_{x+t+s}(S_{t+1} + E_{t+1}) + \underbrace{{}_{1-s}p_{x+t+s}({}_{t+1}V)}_{\text{(what you need)}}$$

In this case, we <u>neither</u> need to add premium or subtract expenses on the LHS <u>nor</u> need to discount $S_{t+1} + E_{t+1}$ on the RHS. The following picture can help us to understand this better:

(amount reserved
@ time $t + s$)

$_{t+s}V$

$\times (1+i)^{1-s}$ (time $t + 1$)

$_{t+s}V(1+i)^{1-s}$ (what you have)

(no premiums/expenses
@ time $t + s$)

$\|$

☠ (prob.: $_{1-s}q_{x+t+s}$): $(S_{t+1} + E_{t+1})$
⚊ (prob.: $_{1-s}p_{x+t+s}$): $_{t+1}V$ (what you need)

(need to <u>pay</u> ☠ payment here if death occurs)
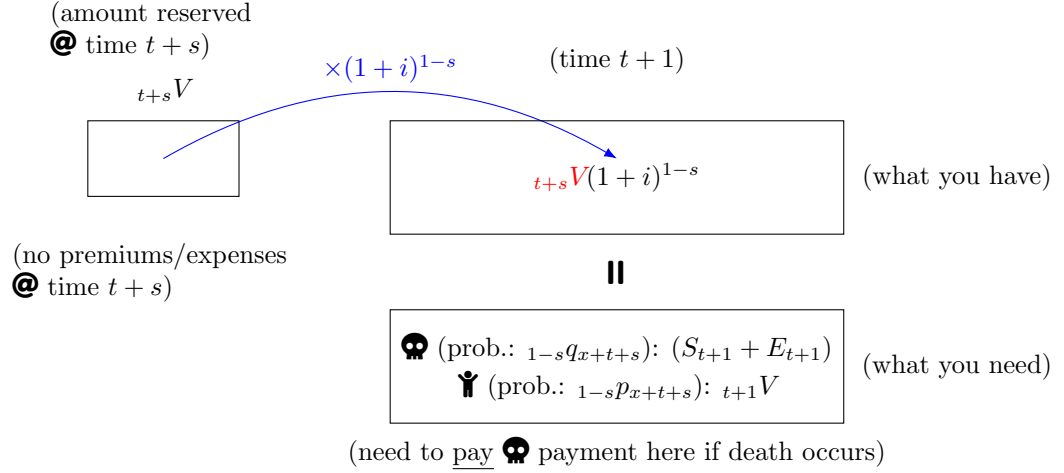
- *1/mthly version:* $(t = 0, 1/m, 2/m, \ldots)$ The idea of policy value recursion in the $1/m$thly case ($1/m$thly insurance and premiums) is pretty much the same; We are just considering $1/m$th of a year instead of a year at a time. The recursive formulas are analogous to the ones for the annual case:

$$\overset{\text{(what you have)}}{(_{t}V + G_t - e_t)(1+i)^{\frac{1}{m}}} \overset{\text{(basic)}}{=} \tfrac{1}{m}q_{x+t}\Big(S_{t+\frac{1}{m}} + E_{t+\frac{1}{m}}\Big) + \tfrac{1}{m}p_{x+t}\Big(_{t+\frac{1}{m}}V\Big)$$

$$\overset{\text{(NAAR)}}{=} {}_{t+\frac{1}{m}}V + \tfrac{1}{m}q_{x+t}\underset{\text{(what you need)}}{\Big(S_{t+\frac{1}{m}} + E_{t+\frac{1}{m}} - {}_{t+\frac{1}{m}}V\Big)}.$$

5.2.4   Applications of policy value recursions.

(1) *Characteristics of changes in policy values over time:*

- $_{t}V$ <u>jumps immediately</u> after each premium payment time, by the amount of that premium payment, less the corresponding expenses.
- $_{t}V$ <u>remains smooth</u> between two consecutive premium payment time.
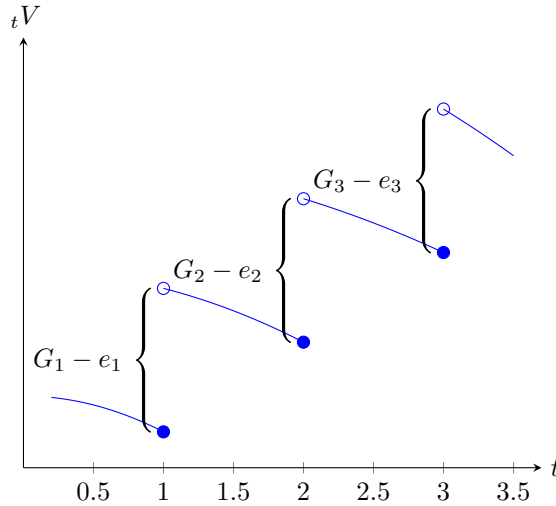
The "jumping" characteristic can be explained by considering the relationship between $_{t}V$ and $_{t+s}V$: as $s \to 0^+$, we have $_{t+s}V \to {}_{t}V + G_t - e_t$. The "smooth" characteristic can be explained via the relationship between $_{t+s}V$ and $_{t+1}V$, which suggests that $_{t+s}V$ is a smooth function in s, for any $0 < s < 1$.

(2) *Approximating $_{t+s}V$ via linear interpolation:*
   (fully discrete (annual) policy, t is an integer, and $0 < s \leq 1$)

$$\boxed{_{t+s}V \approx (1-s)(_tV + G_t - e_t) + s(_{t+1}V)}.$$
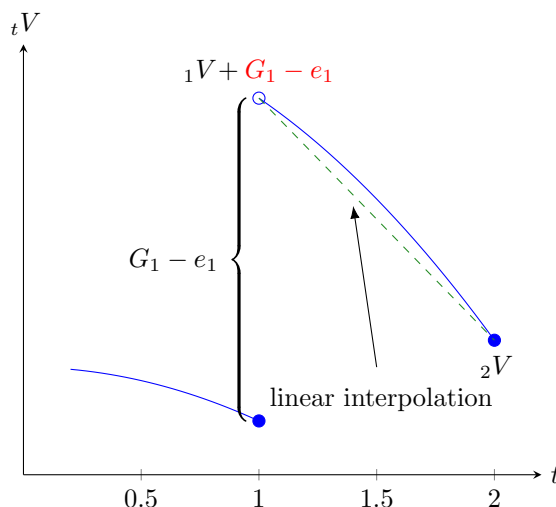
[Note: This approximation can be adapted to the $1/m$thly case, by considering the graph like above for the $1/m$thly case, where jumps occur per $1/m$ of a year rather than per year.]

(3) *Refunding policy value upon death:*

- *Settings:*
  - Ignore expenses.
  - Fully discrete (annual) n-year term life insurance policy with time-t DB $S_t = F_t + {}_tV^n$ $(t = 1, 2, \ldots, n)$. [Note: F_t is the face amount, i.e., the benefit amount before adding the net premium policy value.]
  - Constant annual net premium: P.
- *Problem-solving approaches:*
  - *Approach 1: Recursions.* Utilize boundary policy values $_0V^n = 0$ and $_nV^n$ and apply the following recursive formula (possibly multiple times):

$$\boxed{(_tV^n + P)(1 + i) = q_{x+t}F_t + {}_{t+1}V^n}$$

  $(t = 0, \ldots, n-1)$, to solve for the quantities desired.
  - *(Less commonly used) Approach 2: Using specialized formula.*

$$\boxed{_tV^n = P\ddot{s}_{\overline{t}|} - \sum_{j=1}^{t} q_{x+j-1}F_j(1+i)^{t-j}}$$

  $(t = 0, \ldots, n)$.
  How to use this formula:
  (1) Set $t = n$ and use the known boundary value $_nV^n$ to solve for the net premium P (assuming all other quantities are known).
  (2) Compute policy values at all time points of interest using the formula with the solved premium P, or by recursion.

  Proof. Applying the policy value recursive formula, for any $t = 1, 2, , \ldots, n$, we have

$$(_{t-1}V^n + P)(1+i) = q_{x+t-1}F_{t-1} + {}_tV^n,$$

which implies
$$_tV^n = (_{t-1}V^n + P)(1+i) - q_{x+t-1}F_{t-1}.$$

With this formula, we can deduce:

* $_1V^n = (\underbrace{_0V^n}_{0} + P)(1+i) - q_xF_0 = P(1+i) - q_xF_0 = P\ddot{s}_{\overline{1}|} - q_xF_0.$

* $_2V^n = (P(1+i) - q_xF_0 + P)(1+i) - q_{x+1}F_1 = P\ddot{s}_{\overline{2}|} - (q_xF_0(1+i) + q_{x+1}F_1).$

* $\vdots$

* $_nV^n = P\ddot{s}_{\overline{n}|} - \sum_{j=1}^{n} q_{x+j-1}F_j(1+i)^{n-j}.$

$\square$

Theme 2: Other types of policy values

5.2.5 Retrospective policy values.

Idea. In Section 5.1, we compute the reserve (policy value) to be set at time t by considering the time-t *future* loss. Now, instead of having such a prospective view, here we set the time-t reserve as the share 🐦 available in the "expected pool of money 🪙" (accumulated from the net cash flows received between now and time t) per expected survivor 📧 time t[10]:

$$\frac{\mathrm{EPV}_0(\overbrace{\ell_x \times \text{NCFs received between now and time } t}^{\text{all NCFs injected to the "pool"}}) \times \overbrace{(1+i)^t}^{\text{accumulate}}}{\underbrace{\ell_{x+t}}_{\text{expected number of survivors 📧 time } t}}$$

$$= \frac{\ell_x \times \mathrm{EPV}_0(\text{NCFs received}) \times (1+i)^t}{\ell_{x+t}}$$

$$= \frac{\mathrm{EPV}_0(\text{NCFs received})}{_tE_x}.$$

[Note: The value $\mathrm{EPV}_0(\text{NCFs received})/_tE_x$ is called the "expected accumulated value" or *(more common)* "**actuarial accumulated value** (AAV)" of the NCFs received. The rationale is that, as seen above, it is indeed the mean <u>accumulated value</u> of NCFs received (of ℓ_x policies with identical terms), per <u>expected</u> survivor.]

This leads to the following definition of retrospective policy values.

(1) *Definition:*

- The time-t **retrospective net premium policy value** is

$$\frac{\mathrm{EPV}_0(\text{past } \textbf{net} \text{ premiums}) - \mathrm{EPV}_0(\text{past benefits})}{_tE_x}.$$

- The time-t **retrospective gross premium policy value** is

$$\frac{\mathrm{EPV}_0(\text{past } \textbf{gross} \text{ premiums}) - \mathrm{EPV}_0(\text{past benefits and expenses})}{_tE_x}.$$

[⚠ Warning: Again, there may be an ambiguity on whether the benefits/premiums/expenses *precisely at time* t should be included as "past" benefits/premiums/expenses. For consistency, we shall follow this convention:

- <u>Benefits</u> and <u>benefit</u>-related expenses (i.e., settlement expenses) 📧 time t: included.

---

[10]We are now at time 0, and the actual number of survivors 📧 time t is not known. So we are considering expected number of survivors here.

- Premiums and premium-related expenses (i.e., initial and renewal expenses) **@** time t: NOT included.

This is opposite to the convention used for the usual policy value!

With this convention, we can express the time-t retrospective gross premium policy value as

$$_tV^R = \frac{1}{_tE_x}\big(\,\text{EPV}_0(\text{gross premiums @ time} <t) - \text{EPV}_0(\text{premium-related expenses @ time} <t)$$
$$- \text{EPV}_0(\text{benefits and benefit-related expenses @ time} \leq t)\big).$$

]

[Note: Often we just use the notation $_tV^R$ to denote both the retrospective net premium and gross premium policy values, as further adding the superscript n or g to this notation would make it clumsy. Usually we should be able to tell which kind of retrospective policy value is being considered from the context.]

(2) *Key property:* The most important result for retrospective policy values is the following, which tells us *when* the usual policy value and the retrospective one coincide.[11] In such case, we have another way to compute the usual policy value, namely via the formulas for retrospective policy value above.

It suffices to just focus on the comparison between the *gross premium* ones, since the net premium (retrospective) policy value is just a special case of the gross premium (retrospective) policy value.

Proposition 5.2.a. Let $_tV$ and $_tV^R$ be the time-t usual and retrospective gross premium policy values respectively. Then, $_tV = {}_tV^R$ if:

  i. the gross premiums are set by equivalence principle, and
  ii. bases for the premiums and both policy values are the same.

Proof. The key 🔑 is to show that $-{}_tE_x \times {}_tV^R + {}_tE_x \times {}_tV = {}_0V = 0$, by combining the terms carefully.

Note that

$$_tE_x \times {}_tV = \text{EPV}_0(\text{benefits and benefit-related expenses @ time} >t)$$
$$+ \text{EPV}_0(\text{premium-related expenses @ time} \geq t) - \text{EPV}_0(\text{gross premiums @ time} \geq t)$$

and

$$-{}_tE_x \times {}_tV^R = -\text{EPV}_0(\text{gross premiums @ time} <t) + \text{EPV}_0(\text{premium-related expenses @ time} <t)$$
$$+ \text{EPV}_0(\text{benefits and benefit-related expenses @ time} \leq t).$$

Hence,

$$-{}_tE_x \times {}_tV^R + {}_tE_x \times {}_tV = \text{EPV}_0(\text{benefits and benefit-related expenses})$$
$$+ \text{EPV}_0(\text{premium-related expenses}) - \text{EPV}_0(\text{gross premiums})$$
$$= {}_0V = 0,$$

and the result follows. $\qquad\square$

5.2.6 Expense policy values.

Throughout the discussions here, we shall make the following assumptions.

- Net/gross premiums are level.
- Premium and policy value bases are the same.
- Gross premiums are set by equivalence principle.

---

[11]These two policy values can differ!

(1) *Definitions:*

- The **expense loading** (or **expense premium**) P^e is the level amount added on top of the net premium to fund expenses, i.e., the amount that solves the following equation:

$$\underset{\text{(in terms of } P^e)}{\text{EPV}_0(\text{expense loadings})} = \underset{\text{(often in terms of gross premium } P^g)}{\text{EPV}_0(\text{expenses})}.$$

[Note: Often the "gross $-$ net" formula below is more useful for computing expense loading.]

- The time-t **expense policy value** is defined by

$$_tV^e = \text{EPV}_t(\text{future }\underline{\text{expenses}}) - \text{EPV}_t(\text{future }\underline{\text{expense}}\text{ loadings}).$$

(2) *Key formulas:*

- *Gross $-$ net formulas:*

Quantity	Formula	
P^e	$\underset{\text{(gross premium)}}{P^g}$ $-$	$\underset{\text{(net premium)}}{P^n}$
$_tV^e$	$_tV^g - {_tV^n}$	

Proof. For the formula of P^e, by equivalence principle we have

$$\text{EPV}_0(\text{gross premiums}) - \text{EPV}_0(\text{net premiums}) = \text{EPV}_0(\text{benefits}) + \text{EPV}_0(\text{expenses}) - \text{EPV}_0(\text{benefits})$$
$$= \text{EPV}_0(\text{expenses})$$
$$= \text{EPV}_0(\text{expense loadings}).$$

Since both gross and net premiums are level, we deduce that $P^e = P^g - P^n$.
Next, for the formula of $_tV^e$, since $P^e = P^g - P^n$, we have

$$_tV^e = \text{EPV}_t(\text{future expenses}) - \text{EPV}_t(\text{future expense loadings})$$
$$= \text{EPV}_t(\text{future benefits}) + \text{EPV}_t(\text{future expenses}) - \text{EPV}_t(\text{future \textbf{gross} premiums})$$
$$- [\text{EPV}_t(\text{future benefits}) - \text{EPV}_t(\text{future \textbf{net} premiums})]$$
$$= {_tV^g} - {_tV^n}.$$

$\square$

- *Signs of $_tV^e$ (discrete case):*
  - *level discrete expenses over time:*
    $\boxed{P^e = \text{amount of each expense}}$ and $\boxed{_tV^e = 0}$ (which implies that $_tV^n = {_tV^g}$[12]) for any $t \geq 0$.
  - *(more common) non-level discrete expenses over time:*
    Here we suppose that the renewal expenses are level, and the initial expense I is higher than the renewal expense e *(typically the case)*.
    Then, we have $\boxed{e < P^e < I}$ and $\boxed{_tV^e < 0}$ (which means $_tV^g < {_tV^n}$) for any $t>0$.
    [Note: The negative expense policy value (in absolute value) is sometimes called the **deferred acquisition cost** (DAC).]
- *Recursive formulas for $_tV^e$:* They can be obtained by treating $_tV^e$ as the policy value of another "policy" obtained after stripping off all non-expense-related elements:
  - remove sum insured (i.e., set it to zero)
  - $P^g \rightarrow P^e$
  - keep initial/renewal and settlement expenses as the original amounts

  Example of recursive formulas:

$$({_tV^e} + P^e - e_t)(1 + i) = q_{x+t}(E_{t+1}) + p_{x+t}({_{t+1}V^e}).$$

---

[12]It may appear counterintuitive that the net premium and gross premium policy values are identical in this case, considering that the latter include expenses but the former does not. However, this effect is compensated by the difference between gross and net premiums in the policy value calculations.

Theme 3: Modified net premium reserves

5.2.7 **Motivation.** With the presence of negative expense policy value, including expenses can *reduce* the amount of reserve to be set (policy value). This means that ignoring expenses and setting reserves as the net premium policy values is actually too conservative.

This suggests the usage of gross premium policy value is more appropriate, but there are also some disadvantages 🔖 of using that (e.g., computational burden would increase). This leads to the idea of only "modifying" the net premium policy value approach to make it less conservative, without switching entirely to the gross premium policy value approach.

5.2.8 **Definition.** The modified net premium reserves are the net premium policy values with the original level net premiums modified to non-level net premiums.

Equation of EPVs for modified net premiums:

$$\text{EPV}_0(\text{modified net premiums}) = \text{EPV}_0(\text{level net premiums before modification})$$
$$= \text{EPV}_0(\text{benefits}).$$

This holds because the net premiums must satisfy the equivalence principle before or after modification. A common example of modified net premium reserves is the *full preliminary term reserves*.

5.2.9 **Full preliminary term (FPT) reserves.**

(1) *Definition:* The full preliminary term reserves $_tV^{\text{FPT}}$ are the net premium policy values with the net premiums given by:

    i. *(beginning of first year)* first year cost of insurance:

$$\alpha = \text{EPV}_0(\text{DB } @ \text{ time} \leq 1) = \underbrace{S}_{\text{DB}} \times vq_x.$$

    ii. *(the rest)* level amount $\beta = P^{\text{FPT}}$ that solves the equivalence principle:

$$\text{EPV}_1(\text{net premiums of } P^{\text{FPT}}) = \text{EPV}_1(\text{future benefits}) = \text{EPV}_1(\text{benefits } @ \text{ time} > 1).$$
$$\text{(EPV of benefits for insurance issued to } (x+1))$$
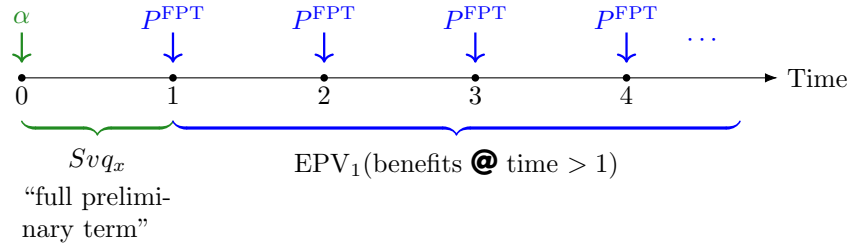
Remarks:

- Here we assume the original policy is an insurance with policy term at least 2 years.
- In case there is a selection at age x, the "insurance issued to a life aged $x+1$" would become "insurance issued to a life aged $[x]+1$" (not $[x+1]$ ⚠). The idea is that this policy issuance is only notional and would not trigger selection at age $x+1$ again.

(2) *Key formulas:*

- *Properties:*
  - $_0V^{\text{FPT}} = {}_1V^{\text{FPT}} = 0$ (by definition/equivalence principle)
  - *(less conservative)* If $\underset{\text{(first year cost of insurance)}}{Svq_x} < \underset{\text{(level net premium)}}{P}$ (often the case), then $P^{\text{FPT}} > P$ and $_tV^{\text{FPT}} < {}_tV^n$ for any $t > 0$.

– *(relationship with usual policy value)* For any $t \geq 1$, $_tV^{\mathrm{FPT}}$ is the time-$t-1$ net premium policy value for insurance issued to $(x+1)$ (providing the remaining coverage, after the first year).

- *Shortcuts for fully discrete WL/endowment insurances:* Suppose the sum insured is S.

Insurance type	$_tV^{\mathrm{FPT}} \quad (t \geq 1)$
WL	$\underset{\text{(only } \ddot{a})}{S\left(1 - \dfrac{\ddot{a}_{x+t}}{\ddot{a}_{x+1}}\right)} = \underset{\text{(only } A)}{S\left(\dfrac{A_{x+t} - A_{x+1}}{1 - A_{x+1}}\right)}$
n-year endowment	$\underset{\text{(only } \ddot{a})}{S\left(1 - \dfrac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x+1:\overline{n-1}|}}\right)} = \underset{\text{(only } A)}{S\left(\dfrac{A_{x+t:\overline{n-t}|} - A_{x+1:\overline{n-1}|}}{1 - A_{x+1:\overline{n-1}|}}\right)}$

These follow from the property that $_tV^{\mathrm{FPT}}$ is the time-$t-1$ net premium policy value for insurance issued to $(x+1)$, for any $t \geq 1$.

References

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A., & Nesbitt, C. J. (1997). *Actuarial mathematics*. Society of Actuaries.

Dickson, D. C., Hardy, M. R., & Waters, H. R. (2019). *Actuarial mathematics for life contingent risks*. Cambridge University Press.

Concepts and Terminologies

Results

Section 1

- [1.1.4]a: factorization formula for "p"

- [1.1.4]b: formulas for $_{u|t}q_x$

Section 4

- [4.2.3]: formulas for policies with premium refund

- [4.2.4]: formulas for variance of L_0

- [4.2.5]: approaches for computing probabilities for L_0

- [4.2.7]: formulas for effects of constant addition to μ_x

- [4.3.3]: approaches for computing portfolio percentile premiums

- [4.3.4]: portfolio percentile vs. equivalence principle premium

Section 5

- [5.1.4]: boundary values of $_tV$

- [5.1.5]: shortcut formulas for policy values of WL/endowment insurances

- [5.2.2]: "what you have is what you need" slogan for policy value recursions

- [5.2.3]: variations of basic policy value recursion

- [5.2.4]: applications of policy value recursions

- Proposition 5.2.a: conditions for equality of retrospective and usual policy values

- [5.2.6]: formulas and properties for expense policy values

- [5.2.9]: formulas and properties for FPT reserves