HKU MATH3401 Study Notes

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Last Updated: 2024-12-16

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1 Metric Spaces

1.0.1 In MATH2241, we have been doing analysis in *real numbers*, which are something we are quite familiar with. Now, in MATH3401, we attempt to *generalize* the ideas there to a more *abstract* setting.

1.0.2 Two "core" features that can be extracted from the setting in MATH2241 are *points* • and *distances*. Every real number may be regarded as a point • in the real number line, and the absolute value function $|\cdot|$ serves as a way to measure distance.

The two notions *points* and *distances* form the foundation for a *metric space* (a generalization to the setting or "space" we work in MATH2241).

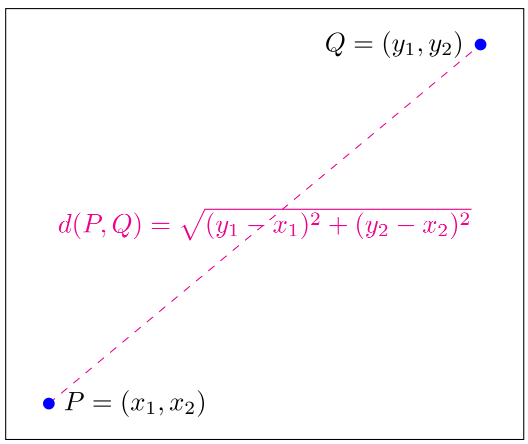
1.0.3 A high-level overview of MATH3401 is that we are studying *continuous functions* between *metric spaces*. (Note that the definition of continuous functions in MATH2241 are only specific to the case studied there, and we will define the notion of continuity more generally later.)

To start with, we shall discuss some concepts related to *metric space*.

1.1 Definition of a Metric Space

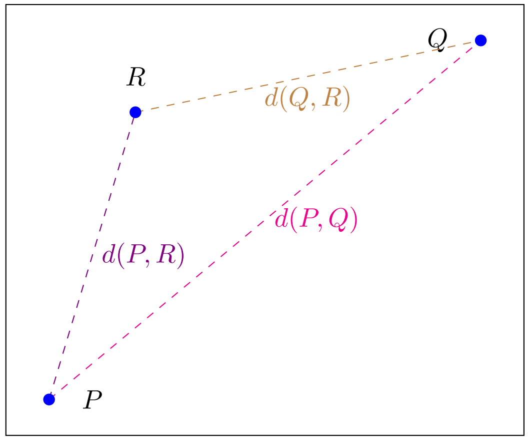
1.1.1 To motivate the definition of a metric space, we consider a typical way to measure distance in $\mathbb{R}^2$.

1.1.2 The following is a familiar way to measure the distance between two points in $\mathbb{R}^2$



Note that it carries the following properties which are natural for measuring distance:

- $d(P,Q) \geq 0$ for any points $P, Q \in \mathbb{R}^2$, and $d(P,Q) = 0$ iff $P = Q$ [Intuition: Distance should be nonnegative. Also, given a point •, the only point having *zero* distance from (the same "position" as) • is the point • itself.]
- $d(P,Q) = d(Q,P)$ for any points $P, Q \in \mathbb{R}^2$ [Intuition: Distance between P and Q = distance between Q and P.]
- (*triangle inequality*) $d(P,R) \leq d(P,Q) + d(Q,R)$ for any points $P, Q, R \in \mathbb{R}^2$.



These properties define a *metric*.

1.1.3 Let X be a nonempty set. Then, a **metric** is a function $d : X \times X \to \mathbb{R}$ satisfying:

(M1) For any $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$.

(M2) For any $x, y \in X$, $d(x, y) = d(y, x)$.

(M3) For any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

We also call the ordered pair (X, d) a **metric space**.

[Note: When the choice of the metric d is clear from context, we may simply write X instead of (X, d). For example, unless stated otherwise, the metric chosen when $X = \mathbb{R}^n$ would the Euclidean distance.]

This definition of metric space captures the idea of *points* and *distances*. For any $x, y \in X$, $d(x, y)$ is the **distance between x and y with respect to d**. Elements in X are said to be **points**.

1.1.4 Sometimes we only want to focus on a certain "part" of a metric space (without altering the way of measuring distance). This leads to the notion of *metric subspace*. Let (X, d) be a metric space and let Y be a nonempty subset of X. Define $d_Y : Y \times Y \to \mathbb{R}$ by

$$d_Y(x, y) = d(x, y)$$

for any $x, y \in Y$. Then, (Y, d_Y) is called a **metric subspace** of (X, d) and d_Y is called as the **relative metric induced by d on Y**.

1.1.5 In fact, (Y, d_Y) is also a metric space.

Proof. Since (X, d) is a metric space, (M1)–(M3) hold for all the points in X. Now, as $Y \subseteq X$, every point in Y is also a point in X, so immediately (M1)–(M3) are satisfied for all the points in Y also. $\square$

1.2 Examples of Metric Spaces

1.2.1 The definition of metric space is quite general. Indeed, many kinds of sets (equipped with a certain metric d) can be metric spaces. We will give some examples of metric spaces in Section 1.2 and *prove* that some are indeed metric spaces.

1.2.2 (a familiar one) Let $X = \mathbb{R}^n$. For any points $P = (x_1, \dots, x_n)$ and $Q = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, define $d(P, Q) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$ (Euclidean distance). Then, (X, d) is a metric space.

Proof. For (M1), $d(P, Q) \geq 0$ follows from the nonnegativity of square root function. Also, we have

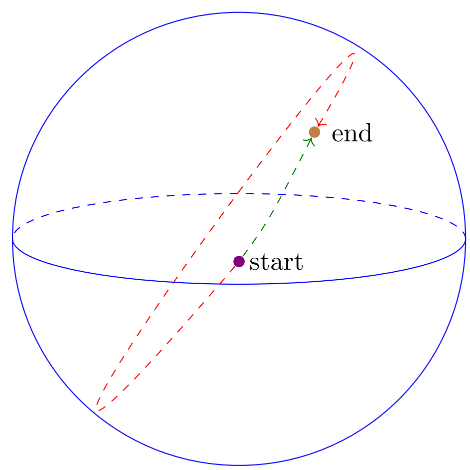
$$d(P,Q) = 0 \iff (y_1 - x_1)^2 + \cdots + (y_n - x_n)^2 = 0 \iff \begin{cases} y_1 = x_1 \\ y_2 = x_2 \\ \quad \vdots \\ y_n = x_n \end{cases} \iff P = Q.$$

(M2) follows from the property that $(y_i - x_i)^2 = (x_i - y_i)^2$ for any $i = 1, \ldots, n$.

(M3) can be proven by using some algebraic tricks and Cauchy-Swartz inequality, but we shall omit the details here. □

1.2.3 Let $X = S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3$, an unit sphere in $\mathbb{R}^3$.

To motivate the definition of the following metric, suppose that the unit sphere represents the "Earth" 🌍, and the two points (• and •) on it represent two places. Now, suppose that you want to travel ✈ from • to •. Intuitively, the shortest path for the travel is represented by the green dashed arrows (one cannot "drill through the Earth straightly" to go to •!). It then appears that the arc length of that path should be the distance between the two points.



Mathematically, we define $d(x, y) =$ the length of the smaller arc on the unique great circle (a planar circle on S^2 with unit radius) joining the two points x and y. Then, (X, d) is again a metric space.

Proof. Omitted. □

1.2.4 It turns out that one can form a metric space from *any* set, using the *discrete metric*. Let X be any set. Then, for any $x, y \in X$, define

$$d(x, y) = 1 - \delta_{xy}$$

where $\delta_{xy} = \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{otherwise.} \end{cases}$ This metric is called **discrete metric**, under which the distance between two different points is *always* one, and the distance between two identical points are zero. (X, d) is a metric space, which is called **discrete metric space**.

Proof. Fix any $x, y \in X$. Then, since $d(x,y) = 0$ or 1, it must be nonnegative. Furthermore, we have $d(x,y) = 0 \iff \delta_{xy} = 1 \iff x = y$, so (M1) is satisfied.

(M2) follows from the fact that δ_{xy} is symmetric (i.e., $\delta_{xy} = \delta_{yx}$).

For (M3), we prove by cases.

- Case 1: $x = z$. Then, we have $d(x,z) = 0 \leq d(x,y) + d(y,z)$ (as RHS must be nonnegative).
- Case 2: $x \neq z$. Then, $d(x,z) = 1$. Now, consider:
  (a) Subcase 1: $x \neq y$. Then, $d(x,y) + d(y,z) = (1+0)$ or $(1+1)$.
  (b) Subcase 2: $z \neq y$. Then, $d(x,y) + d(y,z) = (0+1)$ or $(1+1)$.

  In either subcase, we have $d(x,y) + d(y,z) \geq 1 = d(x,z)$, as desired.

□

1.2.5 As mentioned earlier, MATH3401 is about studying continuous functions on metric space. So, here we consider an example of a metric space for a set of *continuous functions*. Let $X = C([a,b])$, the set of all real-valued continuous functions with domain $[a,b]$. In this case, each "point" is indeed a function, rather than a number. How should we measure the distance between two *functions*?

There are multiple ways to do so, but the following three are relatively more famous. For any functions $f, g \in X$:

- L^2 *norm:*
$$d_2(f,g) = \left[\int_a^b [f(x) - g(x)]^2 \,\mathrm{d}x\right]^{1/2} \triangleq \|f - g\|_2.$$
- L^1 *norm:*
$$d_1(f,g) = \int_a^b |f(x) - g(x)| \,\mathrm{d}x \triangleq \|f - g\|_1.$$
- L^∞ *norm:*
$$d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\} \triangleq \|f - g\|_\infty.$$

  [Note: In general, for any $p \geq 1$, we have the L^p *norm*:
$$d_p(f,g) = \left[\int_a^b [f(x) - g(x)]^p \,\mathrm{d}x\right]^{1/p} \triangleq \|f - g\|_p.$$
  It turns out that the limit of $d_p(f,g)$ as $p \to \infty$ is $\sup\{|f(x) - g(x)| : x \in [a,b]\}$, hence the name "$L^\infty$ norm".]

  [Intuition 💡: L^1 and L^2 norms capture the intuitive idea that two functions f and g are "close" if for "most" input x, $f(x)$ and $g(x)$ are "close".]

1.2.6 The last example we give here concerns a metric space with *finite* set. Let $X = \{a, b, c\}$ (where a, b, and c are distinct). Define the function $d : X \times X \to \mathbb{R}$ by

d	a	b	c
a	0	2	3
b	2	0	2
c	3	2	0

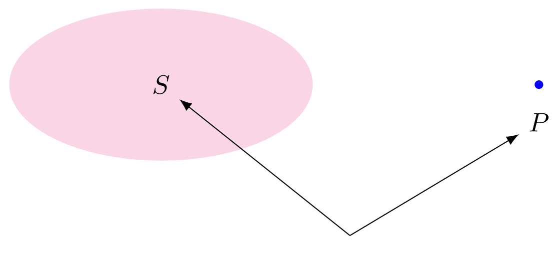
Then, (X, d) is a metric space.

Proof. (M1) follows since every entry in the table is nonnegative and all the diagonal entries are zero. (M2) follows since the table is symmetric along its diagonal.

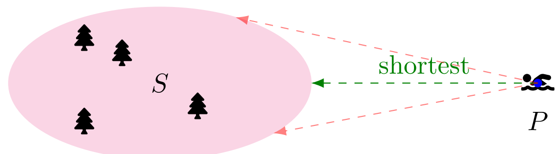
For (M3), we can exhaust all the possibilities (since there are only finitely many possible pairs of points) and verify it. □

1.3 Distance Between a Point and a Set

1.3.1 Apart from distance between two points, sometimes we are interested in knowing distance between a point and a set. How should we define it?



1.3.2 An intuitive way to measure the distance between the point P and the set S is to use the length of the "*shortest*" path for "travelling" from point P ("your current location") to set S ("island").

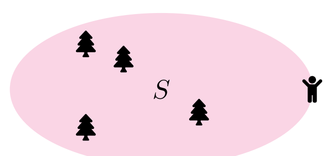


1.3.3 Mathematically, in a metric space (X, d), given any point $P \in X$ and any nonempty set $S \subseteq X$, the **distance from point P to set S** is

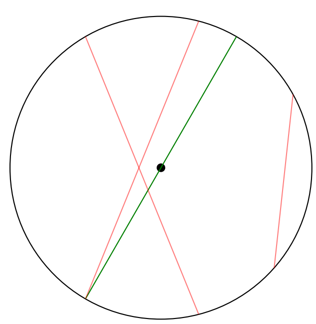
$$d(P, S) = \inf_{x \in S} d(P, x).$$

[Note: The notation $\inf_{x \in S} d(P, x)$ is another way to write $\inf\{d(P, x) : x \in S\}$. This applies to "sup" similarly.]

1.3.4 Now, after "arriving" the set S ("island"), we may start "exploring" 🔍 it. We would then like to know how "large" the "island" is, which may be measured by its "diameter".



1.3.5 Recall the notion of *diameter* for a circle. It is the length of a line segment passing through the center and whose endpoints lie on the circle. It is also the *maximum distance* between two points lying on the circle.



In a similar manner, we may define the notion of *diameter* for other kinds of geometrical objects as follows.

Let (X,d) be a metric space. For any nonempty set $S \subseteq X$, the **diameter** of S is

$$D(S) = \sup_{P,Q \in S} \{d(P,Q)\}.$$

We have $D(S) = \infty$ when the set $\{d(P,Q) : P,Q \in S\}$ is not bounded above. For convenience, we extend the notion of *diameter* to an empty set by defining $D(\varnothing) = -\infty$.

1.3.6 The set S is said to be **bounded** if $D(S) \neq \infty$ (equivalently, $\{d(P,Q) : P,Q \in S\}$ is bounded above, or there exists $M > 0$ such that $d(P,Q) \leq M$ for any $P,Q \in S$). A function from any non-empty set to X is said to be **bounded** if its range is bounded.

1.3.7 Examples: Consider the metric space $\mathbb{R}$ (equipped with Euclidean distance). Then:

- the diameter of the closed interval $[0,1]$ is 1
- the diameter of the open interval $(0,1)$ is 1 [Note: If we use "max" instead of "sup" in the definition for diameter, then this is *undefined*, which is unsatisfactory. Hence, in the definition we use the notion of supremum instead of maximum.]

1.4 Topology of Metric Spaces

1.4.1 Roughly speaking, *topology* studies the properties of a geometric object that are preserved under "continuous deformations". Here, we will introduce some important notions related to metric space topology.

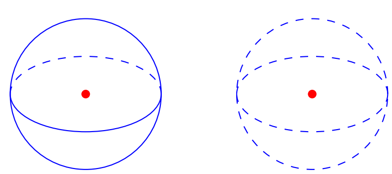
1.4.2 For the case of $\mathbb{R}$, we are familiar with the notions of *open interval* (interval not including its endpoints) and *closed interval* (interval including its endpoints). We can generalize these notions to *open sets* and *closed sets* respectively, which are fundamental to metric space topology.

1.4.3 To define open and closed sets, we need to introduce some preliminary notions. Consider a metric space (X,d) throughout. Let $a \in X$ and $r > 0$. Then, the **open ball** in X with center a and radius r is

$$B(a,r) = \{x \in X : d(a,x) < r\},$$

and the **closed ball** in X with center a and radius r is

$$\overline{B}(a,r) = \{x \in X : d(a,x) \leq r\},$$



closed ball   open ball

[Note: Let S be a nonempty subset of X. Considering (S,d) as a metric subspace of (X,d) (which is itself a metric space), an open ball in S with center a and radius r can be expressed as

$$B_S(a,r) = \{x \in S : d(a,x) < r\} = B_X(a,r) \cap S$$

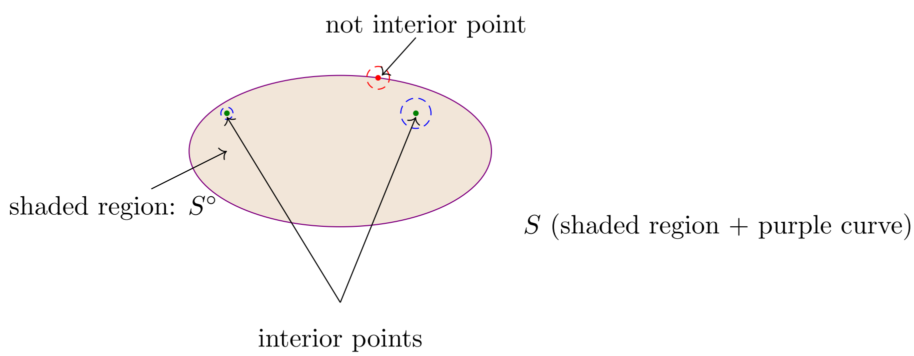
where $B_X(a,r)$ is the open ball with same center and radius, but in X.]

1.4.4 Let $S \subseteq X$. Then, a point $a \in S$ is an **interior point** of S if there exists $r > 0$ such that

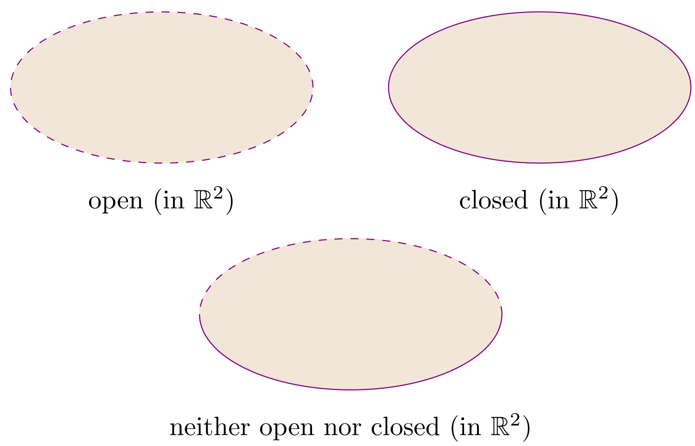
$$B(a, r) \subseteq S.$$

The **interior** of S, denoted by S° or $\operatorname{int} S$, is the set of all interior points of S.

[Intuition 💡: Viewing S as an "island", an interior point of S is a location at the "inner part" of "island", and the interior of S is the whole "inner part" of "island".]



1.4.5 Now we are ready to define open and closed sets. A set $S \subseteq X$ is **open** in X if $S = S^\circ$, and it is **closed** in X if $X \setminus S$ is open in X.



<u>Remarks</u>:

- For convenience in wordings, sometimes we write "S is an open (closed) subset/set of X" or "S is an open (closed) set in X" to mean that $S \subseteq X$ is open (closed) in X.
- When the set S is empty, there is no interior point of S (since S does not even have element!). Thus, the interior of S is empty as well. Hence, we have $S = S^\circ = \varnothing$, meaning that empty set is open in X. (This holds true for any metric space (X, d)!)
- The concepts of "open" and "closed" are with respect to the metric space (X, d). So, we should specify what metric space (X, d) we are referring to when talking about open and closed sets, unless it is clear from context.

1.4.6 We always have $S \supseteq S^\circ$ since every interior point of S must be an element of S (by definition). Hence, to show that a set S is open in X, it suffices to show that $S \subseteq S^\circ$, i.e., every point in S is an interior point of S.

More explicitly, this means that S is open in X iff for any $x \in S$, there exists $r > 0$ such that $B(x, r) \subseteq S$. This may serve as a definition of openness that is more convenient to be checked.

1.4.7 The concepts of open and closed sets are not mutually exclusive. Indeed, a set can be *both open and closed*.

Proposition 1.4.a. Let (X, d) be a metric space. Then, $\varnothing$ and X are both open and closed in X.

Proof. Firstly, from the remark above we know that $\varnothing$ is open in X. Now we will show that X is also open in X. Consider any point $a \in X$, and choose any $r > 0$. Then, we immediately have

$$B(a, r) = \{x \in X : d(a, x) < r\} \subseteq X,$$

meaning that the point a is an interior point of X. Thus, we have $X \subseteq X^\circ$ (which implies that $X = X^\circ$) and hence X is open in X.

Next, note that $\varnothing = X \setminus X$ and $X = X \setminus \varnothing$ are open in X. Thus, X and $\varnothing$ are closed in X. $\square$

1.4.8 For different metric spaces, the same set can have different properties regarding openness and closedness. The following example illustrates this phenomenon.

Let $X = \mathbb{R}$ and equip it with the Euclidean distance metric d (defined by $d(x, y) = |x - y|$ for any $x, y \in X$). Let $Y = [0, 1) \cup (2, 3) \subseteq X$, which induces a metric subspace of (X, d): (Y, d_Y). Then, $S = [0, 1)$ is neither open nor closed in X, but is both open and closed in Y.

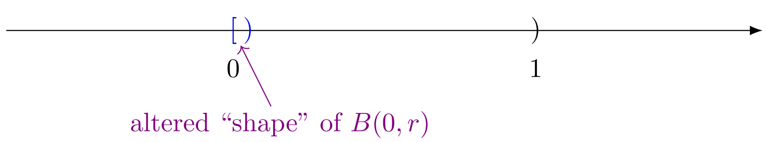
Proof. Firstly, $S = [0, 1)$ is not open in X since $0 \in S$ is *not* an interior point of S (for any $r > 0$, the open ball $B(0, r)$ contains elements outside S). It is also not closed in X since $X \setminus S = (-\infty, 0) \cup [1, \infty)$ is not open in X (we can similarly show that 1 is not an interior point of $X \setminus S$). This shows S is neither open nor closed in X.

Next, we will prove that S is both open and closed in Y. For the openness, we will only show that 0 is an interior point of Y. (Every other point in S is clearly an interior point of Y, by choosing a sufficiently small $r > 0$ (e.g., $r = \frac{1}{2} \cdot \max\{\text{distance between the point and } 0, \text{distance between the point and } 1\}$).)

We first need to choose a small $r > 0$, say $r = 0.1$. Then, we have

$$B(0, r) = \{y \in Y : |y - 0| < r\} = \{y \in [0, 1) \cup (2, 3) : y < 0.1\} = [0, 0.1) \subseteq Y$$

(which is <u>not</u> $(-0.1, 0.1)$ ⚠). Thus, 0 is an interior point of Y, and hence S is open in Y.



For the closedness, note that $Y \setminus S = (2, 3)$. Every point in $Y \setminus S$ can be shown to be an interior point of Y (by choosing a sufficient small $r > 0$, like what was mentioned above). So, $Y \setminus S$ is open in Y, thus S is closed in Y. $\square$

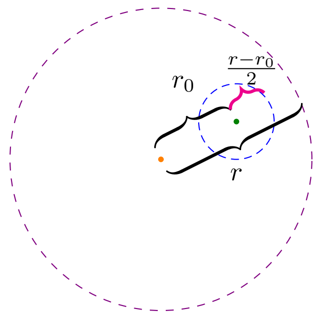
1.4.9 Next, we will show that any open (closed) ball in X is open (closed) in X, as suggested by its name. It turns out that it is not that trivial to prove this!

1.4.10 For any $a \in X$ and $r > 0$, the open ball $S = B(a, r)$ is open in X.

Proof. For any $x \in S$, let $r_0 = d(x, a)$, which is less than r by the definition of open ball. Now, choose $r_1 = \frac{r - r_0}{2}$. Then, for any $y \in B(x, r_1)$, we have

$$d(y, a) \leq \underbrace{d(y, x)}_{< r_1} + \underbrace{d(x, a)}_{r_0} < \frac{r - r_0}{2} + r_0 = \frac{r + r_0}{2} < r \quad (\text{as } r_0 < r),$$

thus $y \in S = B(a,r)$. This means that $B(x,r_1) \subseteq S$, i.e., x is an interior point of S. Since x is arbitrary, it follows that every point in S is an interior point of S, so S is open in X. $\square$



1.4.11 For any $a \in X$ and $r > 0$, the closed ball $S = \overline{B}(a,r)$ is closed in X.

Proof. Firstly, if $S = X$, there is nothing to prove (we know that X is both open and closed in X). Thus henceforth we will assume that $S = \overline{B}(x,r)$ is a proper subset of X (we know that it is a subset of X, by definition). Then, $X \setminus S$ is nonempty.

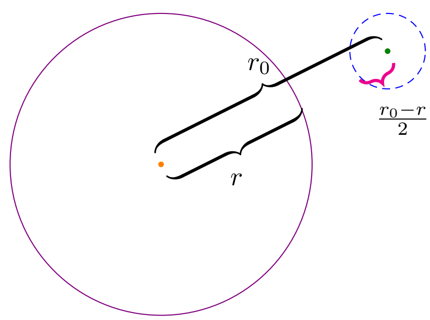
Now, for any $x \in X \setminus S = X \setminus \overline{B}(a,r)$, we let $r_0 = d(x,a) > r$. Now, choose $r_1 = \frac{r_0 - r}{2}$. Then, for any $y \in B(x,r_1)$, rearranging the inequality in (M3) gives

$$d(y,a) \geq \underbrace{d(x,a)}_{r_0} - \underbrace{d(x,y)}_{<r_1} > r_0 - \frac{r_0 - r}{2} = \frac{r_0 + r}{2} > r.$$

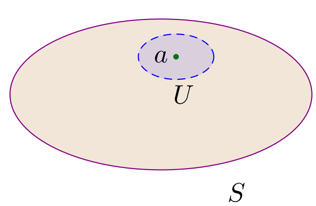
This implies that $y \notin \overline{B}(a,r)$, and hence $y \in X \setminus \overline{B}(a,r)$. Thus, we have

$$B(x,r_1) \subseteq X \setminus S = X \setminus \overline{B}(a,r),$$

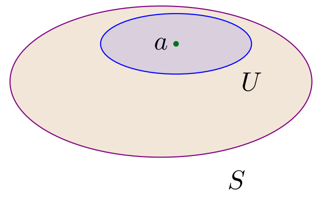
which means that $X \setminus S$ is open in X, and so S is closed in X. $\square$

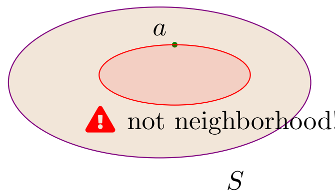


1.4.12 Apart from *open ball*, we also have the notion of *open neighborhood*. Consider a metric space (X,d). Let S be a nonempty subset of X and $a \in S$. Then, an **open neighborhood** of a in S is a set $U \subseteq S$ which contains a and is open in X.



More generally, a **neighborhood** of a in S is a set $U \subseteq S$ which contains a in its interior.





1.5 Properties for Open and Closed Sets

1.5.1 Here, we introduce several results that tell us some properties for the openness and closedness.

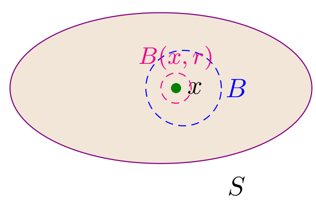
1.5.2 The first one is about an criterion for showing that a set S is open, which is somewhat "similar" to $S \subseteq S^\circ$.

Proposition 1.5.a. Let (X, d) be a metric space. A set $S \subseteq X$ is open in X iff for any $x \in S$, there exists an open ball (in X) B (*not necessarily centered at* x) such that $x \in B \subseteq S$.

[Note: Compare this with the condition that utilizes $S \subseteq S^\circ$: A set $S \subseteq X$ is open in X if for any $x \in S$, there exists $r > 0$ such that $B(x, r) \subseteq S$ (the open ball is necessarily centered at x).]

Proof. "$\Rightarrow$": Immediate by choosing $B = B(x, r)$ where the latter is the open ball $B(x, r)$ mentioned in the note above.

"$\Leftarrow$":



Assume that for any $x \in S$, we have such open ball B where $x \in B \subseteq S$. Then, we can always choose a sufficiently small $r > 0$ that makes $B(x, r) \subseteq B \subseteq S$ (see figure above). It follows that S is open in X. $\square$

1.5.3 Next, we consider the openness/closedness of union/intersection of open/closed sets. First, we consider open sets.

Proposition 1.5.b. Let (X, d) be a metric space.

(a) The union of *any collection* of open sets in X is open in X.

(b) The intersection of *finitely many* (⚠ finitely many only!) open sets in X is open in X.

Proof.

(a) We write $S = \bigcup_{\lambda \in \Lambda} S_\lambda$ where S_λ is open in X (for any $\lambda \in \Lambda$), and Λ is any index set. Then, we want to show that S is open in X also. Consider any $a \in S$. Note that $a \in S_{\lambda^*}$ for some $\lambda^* \in \Lambda$ by definition. Thus, by the openness of S_{λ^*}, there exists $r > 0$ such that

$$B(a, r) \subseteq S_{\lambda^*} \subseteq S.$$

Since a is arbitrary, this implies that S is open in X.

(b) We write $S = \bigcap_{i=1}^n S_i$ where S_i is open in X for any $i = 1, \dots, n$. Then, we want to show that S is open in X. Consider any $a \in S$. By definition, $a \in S_i$ for any $i = 1, \dots, n$, and so by the openness there exist $r_1, \dots, r_n > 0$ such that $B(a, r_i) \subseteq S_i$ for any $i = 1, \dots, n$.
Now, let $r = \min\{r_1, \dots, r_n\} > 0$, and then we have

$$B(a,r) \subseteq B(a,r_i) \subseteq S_i$$

for any $i = 1, \dots, n$. Thus, $B(a,r) \subseteq S$. Since a is arbitrary, this implies that S is open in X.

□

1.5.4 After establishing the results for open sets, we can derive the following results for closed sets.

Corollary 1.5.c. Let (X, d) be a metric space.

(a) The union of *finitely many* (⚠ finitely many only!) closed sets in X is closed in X.
(b) The intersection of *any collection* of closed sets in X is closed in X.

Proof.

(a) Let $E_1, \dots, E_n$ be closed subsets of X. Then, $X \setminus E_1, \dots, X \setminus E_n$ are all open sets in X. Thus, by [1.5.3]b, the set

$$\bigcap_{i=1}^n (X \setminus E_i) = X \setminus \left(\bigcup_{i=1}^n E_i \right) \quad \text{(De Morgan's law)}$$

is open in X, so $\bigcup_{i=1}^n E_i$ is closed in X, as desired.

(b) Fix any index set Λ, and let $E_\lambda \subseteq X$ be a closed set in X for any $\lambda \in \Lambda$. Then, for any $\lambda \in \Lambda$, $X \setminus E_\lambda$ is open in X. Thus, by [1.5.3]a, the set

$$\bigcup_{\lambda \in \Lambda} (X \setminus E_\lambda) = X \setminus \left(\bigcap_{\lambda \in \Lambda} E_\lambda \right) \quad \text{(De Morgan's law)}$$

is open in X, so $\bigcap_{\lambda \in \Lambda} E_\lambda$ is closed in X, as desired.

□

1.5.5 Now, we consider again the concept of *metric subspace*. Let (X, d) be a metric space. For any nonempty $S \subseteq X$, it induces a metric space (S, d_S) (metric subspace of (X, d)). We are then interested in investigating the relationship between openness/closedness *in* S and openness/closedness *in* X. We have the following relations:

(a) $T \subseteq S$ is open in S iff $T = U \cap S$ for some $U \subseteq X$ which is open in X.
Proof. "$\Rightarrow$": Assume that $T \subseteq S$ is open in S. Then, for any $x \in T$, we can choose $r_x > 0$ such that $B_S(x, r_x) \subseteq T$. This implies that

$$\bigcup_{x \in T} B_S(x, r_x) \subseteq T.$$

Now, let $U = \bigcup_{x \in T} B_X(x, r_x)$, which is open in X by [1.5.3]a (as the open ball $B_X(x, r_x)$ is always open in X).
Since $\{x\} \subseteq B_S(x, r_x)$, we have

$$T = \bigcup_{x \in T} \{x\} \subseteq \bigcup_{x \in T} B_S(x, r_x) \subseteq T,$$

which forces

$$T = \bigcup_{x \in T} B_S(x, r_x).$$

Since $B_S(x, r_x) = \{y \in S : d(y, x) < r_x\} = \{y \in X : d(y, x) < r_x\} \cap S = B_X(x, r_x) \cap S$ (as $S \subseteq X$), by distributivity we have

$$T = \left(\bigcup_{x \in T} B_X(x, r_x)\right) \cap S = U \cap S,$$

as desired.

"$\Leftarrow$": Assume that $T = U \cap S$ for some $U \subseteq X$ which is open in X. Then, for any $x \in T$ (and hence $x \in U$), by the openness of U there exists $r_x > 0$ such that $B_X(x, r_x) \subseteq U$. Thus,

$$B_S(x, r_x) = B_X(x, r_x) \cap S \subseteq U \cap S = T,$$

and so x is an interior point of T. Since x is arbitrary, T is open in S. $\square$

(b) $T \subseteq S$ is closed in S iff $T = E \cap S$ for some $E \subseteq X$ which is closed in X.

Proof.

$$\begin{aligned}
&T \subseteq S \text{ is closed in } S \\
\iff &S \setminus T \text{ is open in } S \\
\iff &S \setminus T = U \cap S \text{ where } U \subseteq X \text{ is open in } X \\
\iff &T = S \setminus (S \setminus T) \\
&\quad = S \setminus (U \cap S) \\
&\quad = (X \cap S) \setminus (U \cap S) && \text{(since } S \subseteq X) \\
&\quad = (X \setminus U) \cap S \\
&\quad = E \cap S && (E \subseteq X \text{ is closed in } X).
\end{aligned}$$

$\square$

1.5.6 Here we consider a result regarding relative complement:

Proposition 1.5.d. Let (X, d) be a metric space. Let $U \subseteq X$ and $E \subseteq X$ be open and closed sets in X respectively. Then, $U \setminus E$ is open in X and $E \setminus U$ is closed in X.

Proof. Firstly, $X \setminus E$ is open in X. Thus, by [1.5.3]b,

$$U \cap (X \setminus E) = U \setminus E \quad (\text{as } U \subseteq X)$$

is open in X. On the other hand, note that $X \setminus U$ is closed in X (as $X \setminus (X \setminus U) = U$ is open in X). Hence, by [1.5.4]b,

$$E \cap (X \setminus U) = E \setminus U \quad (\text{as } E \subseteq X)$$

is closed in X. $\square$

1.5.7 The following result connects *open set* and *union of open balls*.

Proposition 1.5.e. Let (X, d) be a metric space, and $U \subseteq X$ be a nonempty set. Then, U is open in X iff U is a union of open balls.

Proof. "$\Rightarrow$": Assume that U is open in X. Then for any $x \in U$, there exists an open ball $B(x, r_x)$ such that $x \in B(x, r_x) \subseteq U$. Hence, we have

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B(x, r_x) \subseteq U,$$

which forces

$$U = \bigcup_{x \in U} B(x, r_x),$$

being a union of open balls.

"$\Leftarrow$": Assume that U is a union of open balls. Then, we can write

$$U = \bigcup_{\lambda\in\Lambda} B_\lambda$$

where B_λ is an open ball in X (which is also open in X) for any $\lambda \in \Lambda$, for some index set Λ. Thus, by [1.5.3]a, U is open in X. $\square$

1.5.8 The next result is a rather remarkable one, which is called *Lindelöf's theorem* (or *Lindelöf's lemma*).

Theorem 1.5.f (Lindelöf's theorem)**.** Let U be an open subset of $\mathbb{R}^n$. Write $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ where Λ is an index set, and U_λ is open in $\mathbb{R}^n$ for any $\lambda \in \Lambda$. Then, there is a *countable subset* $\{U_i : i \in \Gamma\} \subseteq \{U_\lambda : \lambda \in \Lambda\}$ such that $U = \bigcup_{i\in\Gamma} U_i$ where Γ is a countable (finite or countably infinite) set of indices.

[Intuition 💡: This result tells us that in $\mathbb{R}^n$ we can always "simplify" an arbitrary union of open sets to a *countable* union of open sets.]

Proof. Note first that the family $\mathcal{U} = \left\{B\left(q, \frac{1}{k}\right) : q \in \mathbb{Q}^n, k \in \mathbb{N}\right\}$ is countable.

Then, fix any $x \in U \subseteq \mathbb{R}^n$. Since we have $U = \bigcup_{\lambda\in\Lambda} U_\lambda$, there exists $\lambda \in \Lambda$ such that $x \in U_\lambda$. Furthermore, since this U_λ is open in $\mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that $x \in B\left(x, \frac{1}{N}\right) \subseteq U_\lambda$.

Next, by the *density* of $\mathbb{Q}^n$ on $\mathbb{R}^n$[1], there exists $q_x \in \mathbb{Q}^n$ such that $d(q_x, x) < \frac{1}{2N}$. Now consider the open ball $Q_x \triangleq B\left(q_x, \frac{1}{2N}\right) \in \mathcal{U}$. Firstly, since $d(q_x, x) < \frac{1}{2N}$, we have $x \in Q_x$, thus $\{x\} \subseteq Q_x$.

Secondly, for any $y \in Q_x$, by triangle inequality we have

$$d(x, y) \le d(x, q_x) + d(q_x, y) < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N},$$ [2]

which means $y \in B\left(x, \frac{1}{N}\right)$. Thus, we have

$$Q_x \subseteq B\left(x, \frac{1}{N}\right) \subseteq U_\lambda.$$

Now, collect all these Q_x's into the set $\{Q_x : x \in U\} \subseteq \mathcal{U}$. Since $\mathcal{U}$ is countable, it follows that $\{Q_x : x \in U\}$ must also be countable. For each of the (at most countably many) *distinct* open balls in the set, from the above argument we know that it is a subset of U_λ for some $\lambda \in \Lambda$ (i.e., all points in the open ball are contained in U_λ).

We are thus able to index/relabel these (at most countably many) U_λ's using a countable index set Γ. Then, collect all these relabelled U_λ's into the (countable) set $\{U_i : i \in \Gamma\} \subseteq \{U_\lambda : \lambda \in \Lambda\}$. With these notations, we can write

$$\bigcup_{x\in U} Q_x \subseteq \bigcup_{i\in\Gamma} U_i.$$

(In words, every point falling in one of the (distinct) open balls is contained in one of the U_i's.)

Lastly, note that

$$U = \bigcup_{x\in U} \{x\} \subseteq \bigcup_{x\in U} Q_x \subseteq \bigcup_{i\in\Gamma} U_i \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda = U,$$

which implies that

$$\bigcup_{i\in\Gamma} U_i = U,$$

as desired. $\square$

[1] This means that, for every $x \in \mathbb{R}^n$, the following holds: For any $\varepsilon > 0$, there exists $q_x \in \mathbb{Q}^n$ such that $d(x, q_x) < \varepsilon$.

[2] We have $d(x, q_x) < \frac{1}{2N}$ by above, and $d(q_x, y) < \frac{1}{2N}$ since $y \in Q_x = B\left(q_x, \frac{1}{2N}\right)$.

1.5.9 By Proposition 1.5.e, every open set $U \subseteq \mathbb{R}^n$ is a union of open balls:

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda \quad (B_\lambda \text{ denotes an open ball}).$$

Then, applying Theorem 1.5.f on this suggests that U is indeed a *countable union* of open balls (we set $\{U_\lambda : \lambda \in \Lambda\} = \{B_\lambda : \lambda \in \Lambda\}$, which only contains open balls). So we can actually say more about the characteristic of an open set U when it is a subset of $\mathbb{R}^n$.

1.5.10 When $n = 1$, it turns out that we can say even more about it. We first state the following lemma that is useful for proving the main result later on.

Lemma 1.5.g. Let $U \subseteq \mathbb{R}$ be a nonempty set which is open in $\mathbb{R}$. For any $x \in U$, let

$$I_x = \bigcup_{\lambda \in \Lambda_x} B_\lambda$$

where $B_\lambda = (a_\lambda, b_\lambda)$ (with $a_\lambda, b_\lambda \in \mathbb{R}$) and $\Lambda_x = \{\lambda : B_\lambda \text{ is an open interval with } x \in B_\lambda \subseteq U\}$.

Then, we have:

(a) $I_x = \left(\inf_{\lambda \in \Lambda_x}\{a_\lambda\}, \sup_{\lambda \in \Lambda_x}\{b_\lambda\}\right)$.

(b) (maximal property) I_x is the largest open interval containing x which is a subset of U. (That is, (i) $x \in I_x \subseteq U$ and (ii) for any open interval J with $x \in J \subseteq U$, $J \subseteq I_x$.)[3]

(c) For any $x, y \in U$, either $I_x \cap I_y = \varnothing$ (i.e., I_x and I_y are disjoint) or $I_x = I_y$.

Proof.

(a) "$\subseteq$": For any $\lambda \in \Lambda_x$, we have

$$\inf_{\lambda \in \Lambda_x}\{a_\lambda\} \le a_\lambda \le b_\lambda \le \sup_{\lambda \in \Lambda_x}\{b_\lambda\},$$

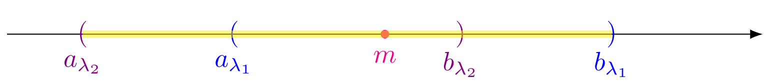
and hence $B_\lambda = (a_\lambda, b_\lambda) \subseteq \left(\inf_{\lambda \in \Lambda_x}\{a_\lambda\}, \sup_{\lambda \in \Lambda_x}\{b_\lambda\}\right)$.
Since this holds for any $\lambda \in \Lambda_x$, we have

$$I_x = \bigcup_{\lambda \in \Lambda_x} B_\lambda \subseteq \left(\inf_{\lambda \in \Lambda_x}\{a_\lambda\}, \sup_{\lambda \in \Lambda_x}\{b_\lambda\}\right).$$

"$\supseteq$": For notational convenience, write $a = \inf_{\lambda \in \Lambda_x}\{a_\lambda\}$ and $b = \sup_{\lambda \in \Lambda_x}\{b_\lambda\}$. For any $m \in (a, b)$, there exist $\lambda_1, \lambda_2 \in \Lambda_x$ such that $m > a_{\lambda_1}$ and $m < b_{\lambda_2}$.[4]
Then, consider the union

$$(a_{\lambda_1}, b_{\lambda_1}) \cup (a_{\lambda_2}, b_{\lambda_2}) = (\min\{a_{\lambda_1}, a_{\lambda_2}\}, \max\{b_{\lambda_1}, b_{\lambda_2}\}).$$

Since $a_{\lambda_1} < m < b_{\lambda_2}$, m belongs to this union. Furthermore, since $(a_{\lambda_1}, b_{\lambda_1})$ and $(a_{\lambda_2}, b_{\lambda_2})$ are both open intervals containing a common point x, their union is also an open interval containing x.



[3] Such open interval must also be unique, since when there are two such intervals I and J, we would have $J \subseteq I$ (by the "largest" property of I) and $I \subseteq J$ (by the "largest" property of J), which implies that $I = J$.

[4] First consider the case that a and b are both finite. For the former, we utilize the result that $\ell = \inf C$ iff for any $\varepsilon > 0$, there exists $c \in C$ such that $c < \ell + \varepsilon$. In this case, we have $C = \{a_\lambda : \lambda \in \Lambda_x\}$, and $\ell = a$. Then, we set $\varepsilon = m - a$, and hence there exists $\lambda_1 \in \Lambda_x$ such that $a_{\lambda_1} < a + (m - a) = m$ ("a_{λ_1}" plays the role of "c"). The reasoning is similar for the latter.

Now, suppose a or b is not finite. Assuming $a = -\infty$, the set $\{a_\lambda : \lambda \in \Lambda_x\}$ is not bounded below, so there always exists $\lambda_1 \in \Lambda_x$ such that $a_{\lambda_1} < m$ (regardless of how small/negative m is). (The proof is similar when $b = \infty$.)

Since both $(a_{\lambda_1}, b_{\lambda_1})$ and $(a_{\lambda_2}, b_{\lambda_2})$ are subsets of U, their union is also a subset of U. Hence, together with the result above, we know that the union is an open interval containing x which is a subset of U, so the union can actually be expressed as B_{λ_0} for some $\lambda_0 \in \Lambda_x$.

Recall that m belongs to this union, i.e., $m \in B_{\lambda_0}$. As m is an arbitrary element in (a, b), we conclude that $(a, b) \subseteq B_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda_x} B_\lambda = I_x$.

(b) It is quite straightforward to prove the maximal property. First of all, since $x \in B_\lambda \subseteq U$ for any $\lambda \in \Lambda_x$, we have

$$x \in I_x = \bigcup_{\lambda \in \Lambda_x} B_\lambda \subseteq U.$$

Next, for any open interval J with $x \in J \subseteq U$, it can be written as B_{λ_J} for some $\lambda_J \in \Lambda_x$. Then, we readily have

$$B_{\lambda_J} \subseteq \bigcup_{\lambda \in \Lambda_x} B_\lambda = I_x.$$

(c) First we shall prove the following claim: For any $x, y \in U$, if $y \in I_x$, then $I_x = I_y$.

Proof. "$\subseteq$": By (b), I_y is the largest open interval containing y, that is a subset of U. Here, by assumption we have $y \in I_x$, and we know that $I_x \subseteq U$. Hence, by the maximal property of I_y,

$$I_x \subseteq I_y.$$

"$\supseteq$": Since $I_x \subseteq I_y$ and $x \in I_x$, we must have $x \in I_y$ (by definition of subset). Hence, repeating the argument for "$\subseteq$" part suggests that

$$I_y \subseteq I_x.$$

□

Now, to prove (c), we will prove that $I_x \cap I_y \neq \varnothing \implies I_x = I_y$ (which is logically equivalent to $I_x \cap I_y = \varnothing$ or $I_x = I_y$). Assume that $I_x \cap I_y \neq \varnothing$. Then, there exists $u \in I_x \cap I_y$, i.e., $u \in I_x$ and $u \in I_y$. By the claim above, it implies that $I_x = I_u$ and $I_y = I_u$ (respectively), and thus $I_x = I_y$ as desired.

□

While the proof for Lemma 1.5.g is rather lengthy, it paves the way for us to prove the following theorem more easily.

Theorem 1.5.h. Every nonempty open subset U of $\mathbb{R}$ is a union of countably many (pairwise) disjoint open intervals in $\mathbb{R}$. Furthermore, such a collection of intervals is *unique*.

Proof. Consider any nonempty open subset U of $\mathbb{R}$. For any $x \in U$, let $I_x = \bigcup_{\lambda \in \Lambda_x} B_\lambda$ (which carries the same meaning as that in Lemma 1.5.g).

By Lemma 1.5.g, we have $I_x \cap I_y = \varnothing$ or $I_x = I_y$ for any $x, y \in U$. Thus, $\mathcal{I} = \{\text{the distinct } I_x\text{'s} : x \in U\}$ is a collection (or set) of pairwise disjoint open intervals whose union is U (as the collection is just essentially obtained from removing redundancies in $\{I_x : x \in U\}$ whose union is U[5], and removing redundancies would not affect the outcome for the union).

Then, by Lindelöf's theorem (Theorem 1.5.f), there is a *countable* sub-collection of $\mathcal{I}$ having union equal to U. But when one removes an interval from $\mathcal{I}$ (which is pairwise disjoint from each of the other open intervals in $\mathcal{I}$), the union of the remaining intervals in $\mathcal{I}$ would *not* be U anymore. This forces the sub-collection to be $\mathcal{I}$ itself. Hence, $\mathcal{I}$ is countable. This shows the existence part of Theorem 1.5.h.

For the uniqueness part of Theorem 1.5.h, suppose that there are two such countable collections of pairwise disjoint open intervals $\mathcal{I} = \{I_\gamma : \gamma \in \Gamma\}$ and $\mathcal{J} = \{J_\theta : \theta \in \Theta\}$ (with some relabelling on the indices), where each of the collections has a union equal to U, i.e.,

$$\bigsqcup_{\gamma \in \Gamma} I_\gamma = \bigsqcup_{\theta \in \Theta} J_\theta = U.$$

[5] Since $x \in I_x$ (or $\{x\} \subseteq I_x$) and $I_x \subseteq U$ for any $x \in U$, we have $U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} I_x \subseteq U$.

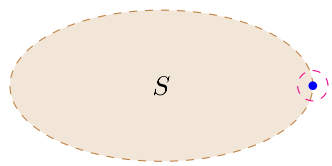
[Note: The notation $\bigsqcup$ denotes *disjoint union*, which carries the same meaning as $\bigcup$, but it emphasizes that the sets involved in the union are pairwise disjoint.] Due the pairwise disjoint property, for any $x \in U$, there exists a *unique* I_{γ_x} and a *unique* J_{θ_x} (in $\mathcal{I}$ and $\mathcal{J}$ respectively) such that $x \in I_{\gamma_x}$ and $x \in J_{\theta_x}$. This implies that $I_{\gamma_x} = I_x = J_{\theta_x}$ (see the claim in the proof of Lemma 1.5.g).

As we consider all $x \in U$, the equality would involve all pairs of intervals in $\mathcal{I}$ and $\mathcal{J}$, and so we conclude that $\mathcal{I} = \mathcal{J}$, establishing the uniqueness. $\square$

1.6 Adherent, Accumulation, and Boundary Points

1.6.1 We have introduced the notion of *openness* and *closedness* in Section 1.4, which describe *sets*. In Section 1.6, we will introduce some notions that describe *points* (in addition to *interior points* in Section 1.4). Throughout Section 1.6, we shall work in a metric space (X, d).

1.6.2 Let S be a subset of X. A point $x \in X$ is an **adherent point** of S if $B_X(x, r) \cap S \neq \varnothing$ for any $r > 0$. The set of all adherent points of S (in X) is called the **closure** of S in X, and is denoted by $\overline{S}$.



Remarks:

- Roughly speaking, an adherent point of S is a point that can be "stuck" to the set S ("touches" or "barely touches" S).
- Since every point $x \in S$ must be an adherent point of S ($B_X(x, r) \cap S$ contains the point x at least, for any $r > 0$), it follows that $S \subseteq \overline{S}$.
- Since every adherent point of S must belong to X, it follows that $\overline{S} \subseteq X$.

1.6.3 The following result explains why $\overline{S}$ is called "closure" of S: Because it is the smallest *closed* subset of X which contains all points in S.

Theorem 1.6.a. The closure $\overline{S}$ is the smallest closed subset of X which contains all points in S.[6]

Proof. Firstly, we will show that $\overline{S}$ is closed in X. Fix any $x \in X \setminus \overline{S}$ (i.e., any point in X that is not an adherent point of S). Then, there exists $r > 0$ such that $B_X(x, r) \cap S = \varnothing$ by definition. This means that all points in $B_X(x, r)$ do not belong to S, which implies that $B_X(x, r) \subseteq X \setminus S$.

Now, take any $y \in B_X(x, r)$. Then, by triangle inequality,

$$B_X(y, r - d(x, y)) = \{z \in X : d(y, z) < r - d(x, y)\} \subseteq \{z \in X : d(x, z) < r\} = B_X(x, r) \subseteq X \setminus S,$$

which means that $B_X(y, r - d(x, y)) \cap S = \varnothing$, and thus $y \in X \setminus \overline{S}$. Since y is arbitrarily chosen from $B_X(x, r)$, we conclude that $B_X(x, r) \subseteq X \setminus \overline{S}$, and so $X \setminus \overline{S}$ is open in X, i.e., $\overline{S}$ is closed in X.

Secondly, we will show that $\overline{S}$ is the *smallest* closed subset of X which contains all points in S. Let C be a closed subset of X which contains all points in S, i.e., $C \supseteq S$. We then have $X \setminus C \subseteq X \setminus S$. Now, fix any $x \in X \setminus C$. Since $X \setminus C$ is open, there exists $r > 0$ such that $x \in B_X(x, r) \subseteq X \setminus C \subseteq X \setminus S$. This implies that $B_X(x, r) \cap S = \varnothing$ for this r, and so x is not an adherent point of S, i.e., $x \in X \setminus \overline{S}$.

Since x is arbitrarily chosen from $X \setminus C$, we conclude that $X \setminus C \subseteq X \setminus \overline{S}$, or $\overline{S} \subseteq C$. $\square$

1.6.4 Another way to view the concept of adherent point is to consider the notion of *distance between a point and a set* (discussed in Section 1.3). We have the following criterion for adherent point:

Proposition 1.6.b. Let S be a subset of X. A point $x \in X$ is an adherent point of S iff $d(x, S) = 0$.

[6] This means that for any closed subset C of X which contains all points in S, we have $\overline{S} \subseteq C$.

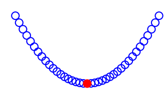
Proof. Note that

$$\begin{aligned}
& x \text{ is an adherent point of } S \\
\iff & \text{for any } r > 0,\ B_X(x, r) \cap S \neq \varnothing \\
\iff & \text{for any } r > 0, \text{ there exists } s_r \in S \text{ such that } d(x, s_r) < r \\
\iff & \inf_{s \in S} d(x, s) = 0 \\
\iff & d(x, S) = 0.
\end{aligned}$$

□

1.6.5 Next, we will introduce the notion of *accumulation point*. Let S be a subset of X. A point $x \in X$ is an **accumulation point** of S (or **limit point** of S) if it is an adherent point of $S \setminus \{x\}$, i.e., $B(x, r) \cap (S \setminus \{x\}) \neq \varnothing$ for any $r > 0$. The set of all accumulation points of S (in X) is called the **derived set** of S in X, and is denoted by S'.

[Intuition: A point is an accumulation point when there are *other* points which are "arbitrarily close" to it ("accumulating" to that point).

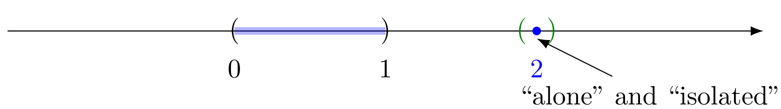


]

1.6.6 The concepts of accumulation point and adherent point are similar, and the only difference is "$\setminus\{x\}$". Since $B(x, r) \cap (S \setminus \{x\}) \neq \varnothing$ implies $B_X(x, r) \cap S \neq \varnothing$, it follows that every accumulation point of S must also be an adherent point of S (but not vice versa), i.e., $S' \subseteq \overline{S}$ (but the equality may not hold).

Example: Let $S = (0, 1) \cup \{2\}$. Then, $S' = [0, 1]$ while $\overline{S} = [0, 1] \cup \{2\}$. [Note: 2 is not an accumulation point of S since $B(2, r) \cap (S \setminus \{2\})$ is empty when $r = 1/2$, but 2 is an adherent point of S as $B(2, r) \cap S \supseteq \{2\}$ for any $r > 0$.] For this example, we can observe that 2 is "isolated" from other points in S. This motivates the definition of *isolated point*.

1.6.7 Let S be a subset of X. A point $x \in S$ is an **isolated point** of S if there exists $r > 0$ such that $B(x, r) \cap S = \{x\}$.



1.6.8 We can express the closure $\overline{S}$ as a union of the derived set S' and the set S itself:

Proposition 1.6.c. Let S be a subset of X. Then, $\overline{S} = S' \cup S$.

Proof. "$\supseteq$": First, we have $S' \subseteq \overline{S}$ and $S \subseteq \overline{S}$. Thus, $S' \cup S \subseteq \overline{S}$.

"$\subseteq$": For any $x \in \overline{S}$, if $x \in S$, we immediately have $x \in S' \cup S$. So henceforth we assume $x \notin S$. Then, since x is an adherent point of S, we have for any $r > 0$,

$$B(x, r) \cap S \neq \varnothing.$$

Furthermore, since $x \notin S$, we have $S = S \setminus \{x\}$, so we indeed have

$$B(x, r) \cap (S \setminus \{x\}) \neq \varnothing$$

for any $r > 0$. This means that $x \in S' \subseteq S' \cup S$. □

1.6.9 Note that an isolated point $x \in S$ is the same as an adherent point of S which is *not* an accumulation point of S.

Proof. "$\Rightarrow$": Since $x \in S$, x must be an adherent point of S. Also, there exists $r > 0$ such that $B(x,r) \cap S = \{x\}$, so x is not an accumulation point of S.

"$\Leftarrow$": When a point x is an adherent point of S but is not an accumulation point of S, we have:

- $B(x,r) \cap S \neq \varnothing$ for any $r > 0$, and
- $B(x,r_0) \cap (S \setminus \{x\}) = \varnothing$ for some $r_0 > 0$.

This then forces that $B(x,r_0) \cap S = \{x\}$ for the r_0, so x is an isolated point of S. $\square$

1.6.10 Therefore, we can indeed express the closure $\overline{S}$ as a *disjoint union* of the derived set S' and the *set of all isolated points of* S, denoted by S^i:

$$\overline{S} = S' \sqcup S^i.$$

1.6.11 It turns out that we can give an criterion for being an accumulation point by considering *cardinality*.

Proposition 1.6.d. Let S be a subset of X and $x \in X$. Then, x is an accumulation point of S iff $B(x,r) \cap S$ is an infinite set for any $r > 0$.

Proof. "$\Rightarrow$": Assume to the contrary that x is an accumulation point of S while $B(x,r) \cap S$ is a finite set for some $r > 0$. Since $B(x,r) \cap S$ is a finite set, we can pick a $y \in B(x,r) \cap S$ which has the *minimum* distance from x (call it r_0).

Then, consider the open ball $B(x,r_0/2)$. By construction of r_0, $B(x,r_0/2) \cap (S \setminus \{x\})$ must be empty. This contradicts to the assumption that x is an accumulation point of S.

"$\Leftarrow$": Straightforward since when $B(x,r) \cap S$ is an *infinite* set for any $r > 0$, it must still contain elements after taking away x (for any $r > 0$), i.e., $B(x,r) \cap (S \setminus \{x\}) \neq \varnothing$ for any $r > 0$. $\square$

1.6.12 Next, we will introduce a result that gives criteria for *closedness* of a set, using the notion of closure and derived set.

Proposition 1.6.e. Let S be a subset of X. Then, the following are equivalent.

(a) S is closed in X.

(b) $S' \subseteq S$.

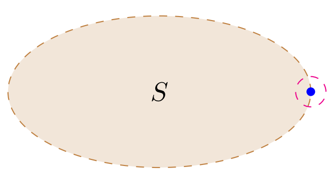
(c) $\overline{S} = S$.

Proof. (a) $\implies$ (b): Assume to the contrary that S is closed in X and S' is *not* a subset of S. Then, there exists $x \in S' \setminus S \subseteq X \setminus S$. But since S is closed in X, i.e., $X \setminus S$ is open in X, there must exist $r > 0$ such that $B(x,r) \subseteq X \setminus S$, which means $B(x,r) \cap S = \varnothing$. This then contradicts to $x \in S'$ (x is an accumulation point, hence adherent point of S).

(b) $\implies$ (c): Assume that $S' \subseteq S$. Then, by Proposition 1.6.c, we have $\overline{S} = S' \cup S = S$.

(c) $\implies$ (a): Assume that $\overline{S} = S$. By Theorem 1.6.a, we know that $\overline{S}$ is closed in X, so does S. $\square$

1.6.13 Then, we will introduce the notion of *boundary point*.



We would like to capture the intuitive notion of "a point located at the 'boundary' of S" by *boundary point*. From the picture above, we observe that for a point that lies on the "boundary", every open ball centered at that point contains some elements in S as well as some elements *not* in S, no matter how small the ball is. This leads to the following definition of boundary point.

1.6.14 Let $S \subseteq X$ be a subset. A point $x \in X$ is a **boundary point** of S if $B(x,r) \cap S \neq \varnothing$ and $B(x,r) \cap (X \setminus S) \neq \varnothing$ for any $r > 0$. The set of all boundary points of S is called the **boundary** of S, and is denoted by ∂S.

Remarks:

- By definition, we can write $\partial S = \overline{S} \cap \overline{X \setminus S} \subseteq \overline{S}$. This implies that $\partial S = \partial(X \setminus S)$ and ∂S is closed in X (as an intersection of two closed subsets in X).
- Note that $S^\circ \cap \partial S = \varnothing$. This is because for any interior point x of S, we need $B(x,r) \subseteq S$ for any $r > 0$, so the ball $B(x,r)$ cannot possibly contain elements outside S, failing a requirement for being a boundary point of S. This shows that a point cannot possibly be simultaneously interior point and boundary point of S.

The notion of boundary here coincides with our usual understanding of "boundary". For example, when $X = \mathbb{R}^n$, the boundary of an open ball $B(a,r)$ ("solid" n-ball) is

$$\partial B(x,r) = \{y \in \mathbb{R}^n : d(x,y) = r\},$$

an *n-sphere* (which is the "boundary"/"surface" of the ball in our usual understanding).

1.6.15 Recall from [1.6.10] that we can express the closure $\overline{S}$ as the disjoint union of a set containing all accumulation points and another set containing all isolated points.

Using the concept of *boundary points* and *interior points*, we can express the closure $\overline{S}$ in a similar fashion as follows:

$$\overline{S} = \partial S \sqcup S^\circ.$$

Proof. From the remarks above, we know that $S^\circ \cap \partial S = \varnothing$. So it suffices to show that $\overline{S} = \partial S \cup S^\circ$.

"$\supseteq$": We have $\partial S \subseteq S$ as noted in the previous remarks, and also we have $S^\circ \subseteq S \subseteq \overline{S}$. Thus, the union $\partial S \cup S^\circ \subseteq S$.

"$\subseteq$": Fix any $x \in \overline{S}$. If $x \in S^\circ$, then we are done. So henceforth assume that $x \notin S^\circ$. In this case, we must have $B(x,r) \cap (X \setminus S) \neq \varnothing$ (i.e., $B(x,r)$ is not a subset of S) for any $r > 0$. Also, since $x \in \overline{S}$, we have $B(x,r) \cap S \neq \varnothing$ for any $r > 0$. It follows that $x \in \partial S \subseteq \partial S \cup S^\circ$. □

1.7 Properties of Interiors, Closures, Derived Sets, and Boundaries

1.7.1 So far we have introduced multiple kinds of *points*, including:

(a) interior points ⟷ interior S°

(b) adherent points ⟷ closure $\overline{S}$

(c) accumulation points ⟷ derived set S'

(d) boundary points ⟷ boundary ∂S

In Section 1.7 we will discuss some set-related properties about them. Throughout we let (X,d) be a metric space and S,T be arbitrary subsets of X.

1.7.2 Properties for interior S°:

(a) (preserving subset inclusion) $S \subseteq T \implies S^\circ \subseteq T^\circ$.

Proof. Assume $S \subseteq T$. For any $s \in S^\circ$, there exists $r > 0$ such that $B(s,r) \subseteq S \subseteq T$, so $s \in T^\circ$. Hence $S^\circ \subseteq T^\circ$. □

(b) (idempotence) $(S^\circ)^\circ = S^\circ$.

Proof. We need to show that S° is open in X. Fix any $x \in S^\circ$. Then there exists $r > 0$ such that $B(x,r) \subseteq S$.

It then suffices to show that $B(x,r) \subseteq S^\circ$. First fix any $y \in B(x,r)$. By the openness of $B(x,r)$, there exists $\delta > 0$ such that $B(y,\delta) \subseteq B(x,r) \subseteq S$, which implies that y is also an interior point of S, i.e., $y \in S^\circ$. This shows $B(x,r) \subseteq S^\circ$. □

(c) $S^\circ \subseteq T \subseteq S \implies T^\circ = S^\circ$.
Proof. By [1.7.2]a and [1.7.2]b,

$$S^\circ \subseteq T \subseteq S \implies S^\circ = (S^\circ)^\circ \subseteq T^\circ \subseteq S^\circ \implies T^\circ = S^\circ.$$

□

(d) (commutativity with intersection) $S^\circ \cap T^\circ = (S \cap T)^\circ$.
Proof. "$\subseteq$": Fix any $x \in S^\circ \cap T^\circ$. Then there exist $r_1, r_2 > 0$ such that $B(x, r_1) \subseteq S$ and $B(x, r_2) \subseteq T$. Let $r = \min\{r_1, r_2\} > 0$, and we have $B(x, r) \subseteq S \cap T$. Thus, $x \in (S \cap T)^\circ$.
"$\supseteq$": Note that $S \cap T \subseteq S$ and $S \cap T \subseteq T$. Thus, by [1.7.2]a, we have

$$(S \cap T)^\circ \subseteq S^\circ \quad \text{and} \quad (S \cap T)^\circ \subseteq T^\circ.$$

This means $(S \cap T)^\circ \subseteq S^\circ \cap T^\circ$. □

(e) $S^\circ \cup T^\circ \subseteq (S \cup T)^\circ$.
Proof. We have $S \subseteq S \cup T$ and $T \subseteq S \cup T$. Thus, by [1.7.2]a, $S^\circ \subseteq (S \cup T)^\circ$ and $T^\circ \subseteq (S \cup T)^\circ$. Hence,

$$S^\circ \cup T^\circ \subseteq (S \cup T)^\circ.$$

□

[⚠ Warning: It is <u>not</u> true that $S^\circ \cup T^\circ \supseteq (S \cup T)^\circ$. For example, consider $S = [0, 1]$ and $T = [1, 2]$ in $\mathbb{R}$.]

1.7.3 Properties for closure $\overline{S}$:

(a) (preserving subset inclusion) $S \subseteq T \implies \overline{S} \subseteq \overline{T}$.
Proof. Assume $S \subseteq T$. For any $x \in \overline{S}$ and any $r > 0$, we have

$$B(x, r) \cap T \supseteq B(x, r) \cap S \neq \varnothing,$$

which implies $B(x, r) \cap T \neq \varnothing$, so $x \in \overline{T}$. □

(b) (idempotence) $\overline{(\overline{S})} = \overline{S}$.
Proof. Note that $\overline{S}$ is closed, and hence the smallest closed set in X containing all elements in $\overline{S}$ is precisely $\overline{S}$ itself. This means that $\overline{(\overline{S})} = \overline{S}$ by Theorem 1.6.a. □

(c) $S \subseteq T \subseteq \overline{S} \implies \overline{T} = \overline{S}$.
Proof. By [1.7.3]a and [1.7.3]b, we have

$$S \subseteq T \subseteq \overline{S} \implies \overline{S} \subseteq \overline{T} \subseteq \overline{(\overline{S})} = \overline{S} \implies \overline{T} = \overline{S}.$$

□

(d) (commutativity with union) $\overline{S} \cup \overline{T} = \overline{S \cup T}$.
Proof. "$\subseteq$": We have $S \subseteq S \cup T$ and $T \subseteq S \cup T$. Thus, by [1.7.3]a, $\overline{S} \subseteq \overline{S \cup T}$ and $\overline{T} \subseteq \overline{S \cup T}$. Hence,

$$\overline{S} \cup \overline{T} \subseteq \overline{S \cup T}.$$

"$\supseteq$": Fix any $x \in \overline{S \cup T}$ and any $r > 0$. Then we have

$$[B(x, r) \cap S] \cup [B(x, r) \cap T] = B(x, r) \cap (S \cup T) \neq \varnothing. \tag{1}$$

If $B(x, r) \cap S \neq \varnothing$ for any $r > 0$, then $x \in \overline{S} \subseteq \overline{S} \cup \overline{T}$ and we are done. So henceforth assume that $B(x, r_0) \cap S = \varnothing$ for some $r_0 > 0$. Then,

$$B(x, r) \cap S = \varnothing$$

for any $0 < r \leq r_0$. By Equation (1), we must have $B(x, r) \cap T \neq \varnothing$ for any $0 < r \leq r_0$. Also, for any $r > r_0$ we have

$$B(x, r) \cap T \supseteq B(x, r_0) \cap T \neq \varnothing.$$

Hence, we conclude that $B(x, r) \cap T \neq \varnothing$ for any $r > 0$, thus $x \in \overline{T} \subseteq \overline{S} \cup \overline{T}$. □

(e) $\overline{S} \cap \overline{T} \supseteq \overline{S \cap T}$.

Proof. Note that $S \cap T \subseteq S$ and $S \cap T \subseteq T$. Hence, by [1.7.3]a, we have $\overline{S \cap T} \subseteq \overline{S}$ and $\overline{S \cap T} \subseteq \overline{T}$. Thus,

$$\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}.$$

□

[⚠ Warning: It is <u>not</u> true that $\overline{S} \cap \overline{T} \subseteq \overline{S \cap T}$. For example, consider $S = [0, 1)$ and $T = (1, 2]$ in $\mathbb{R}$.]

1.7.4 Properties for derived set S':

(a) (preserving subset inclusion) $S \subseteq T \implies S' \subseteq T'$.

Proof. Assume $S \subseteq T$. For any $x \in S'$ and any $r > 0$, we have

$$B(x, r) \cap T \setminus \{x\} \supseteq B(x, r) \cap S \setminus \{x\} \neq \varnothing,$$

which implies $B(x, r) \cap T \setminus \{x\} \neq \varnothing$, so $x \in T'$. □

(b) $(S')' \subseteq S'$.

Proof. Fix any $x \in (S')'$ and any $r > 0$. Then there exists $y \in B(x, r) \cap S' \setminus \{x\}$. Let $\delta = r - d(x, y) > 0$. Then,

$$B(y, \delta) \subseteq B(x, r).$$

Since $y \in S'$, by Proposition 1.6.d we know that $B(y, \delta) \cap S \setminus \{y\}$ is infinite, so we can find an element other than x in it. In other words, there exists

$$z \in B(y, \delta) \cap S \setminus \{x, y\} \subseteq B(x, r) \cap S \setminus \{x\},$$

which implies that $B(x, r) \cap S \setminus \{x\}$ is nonempty, so $x \in S'$. □

[⚠ Warning: It is <u>not</u> true that $(S')' \supseteq S'$. For example, consider $S = \{1/n : n \in \mathbb{N}\}$ in $\mathbb{R}$.]

(c) (commutativity with union) $S' \cup T' = (S \cup T)'$.

Proof. Similar to the proof for the respective property for closure. □

(d) $S' \cap T' \supseteq (S \cap T)'$.

Proof. Similar to the proof for the respective property for closure. □

[⚠ Warning: It is <u>not</u> true that $S' \cap T' \subseteq (S \cap T)'$. For example, consider $S = [0, 1)$ and $T = (1, 2]$ in $\mathbb{R}$.]

1.7.5 Properties for boundary ∂S:

(a) $\partial(\partial S) \subseteq \partial S$.

Proof. Firstly, since $\partial S = \overline{S} \cap \overline{X \setminus S}$ and closure is closed, we know that ∂S is closed in X, which implies that $\overline{\partial S} = \partial S$. Thus,

$$\partial(\partial S) = \overline{\partial S} \cap \overline{X \setminus \partial S} \subseteq \overline{\partial S} = \partial S$$

□

[⚠ Warning: It is <u>not</u> true that $\partial(\partial S) \supseteq \partial S$. For example, consider $S = \mathbb{Q}$ in $\mathbb{R}$.]

(b) $S \subseteq T \not\Rightarrow \partial S \subseteq \partial T$. Example: $S = [1, 2]$ and $T = [0, 3]$ in $\mathbb{R}$.

(c) $\partial S \cup \partial T \supseteq \partial(S \cup T)$.

Proof. Note that

$$\begin{aligned}
\partial(S \cup T) &= \overline{S \cup T} \cap \overline{X \setminus (S \cup T)} \\
&= \overline{S \cup T} \cap \overline{(X \setminus S) \cap (X \setminus T)} \\
&= (\overline{S} \cup \overline{T}) \cap \overline{(X \setminus S) \cap (X \setminus T)} \\
&\subseteq (\overline{S} \cup \overline{T}) \cap (\overline{X \setminus S} \cap \overline{X \setminus T}) \\
&= [\overline{S} \cap (\overline{X \setminus S} \cap \overline{X \setminus T})] \cup [\overline{T} \cap (\overline{X \setminus S} \cap \overline{X \setminus T})] \\
&\subseteq [\overline{S} \cap \overline{X \setminus S}] \cup [\overline{T} \cap \overline{X \setminus T}] \\
&= \partial S \cup \partial T.
\end{aligned}$$

$\square$

[⚠ Warning: It is <u>not</u> true that $\partial S \cup \partial T \subseteq \partial(S \cup T)$. For example, consider $S = [0, 2]$ and $T = \{1\}$ in $\mathbb{R}$.]

(d) i. $\partial S \cap \partial T \not\subseteq \partial(S \cap T)$. Example: $S = [0, 1)$ and $T = (1, 2]$ in $\mathbb{R}$.
 ii. $\partial S \cap \partial T \not\supseteq \partial(S \cap T)$. Example: $S = [1, 2]$ and $T = [0, 3]$ in $\mathbb{R}$.

1.7.6 The following summarizes properties about interiors, closures, derived sets and boundaries.

type	idempotent?	preserve subset?	commute with $\cup$?	commute with $\cap$?
interior S°	✔	✔	✘($S^\circ \cup T^\circ \subseteq (S \cup T)^\circ$ only)	✔
closure $\overline{S}$	✔	✔	✔	✘($\overline{S} \cap \overline{T} \supseteq \overline{S \cap T}$ only)
derived set S'	✘($\subseteq$ only)	✔	✔	✘($S' \cap T' \supseteq (S \cap T)'$ only)
boundary ∂S	✘($\subseteq$ only)	✘	✘($\partial S \cup \partial T \supseteq \partial(S \cup T)$ only)	✘

1.8 Compactness

1.8.1 We have introduced the concept of *open set* and *closed set*. They can be seen as generalizations to the concept of *open interval* and *closed interval* in $\mathbb{R}$ respectively. However, when we consider a more general notion like open or closed set instead of a more "specialized" notion like open or closed interval, some properties may be lost.

1.8.2 For example, consider the case in $\mathbb{R}$. For a *closed interval* $[a, b]$ $(a, b \in \mathbb{R})$, we have the *extreme value theorem* which states that for any function $f \in C[a, b]$ ($C[a, b]$ denotes the set of all continuous functions from $[a, b]$ to $\mathbb{R}$), f attains its maximum and minimum. More specifically, there exist $c, d \in [a, b]$ such that

$$f(c) = \min\{f(x) : x \in [a, b]\}$$

and

$$f(d) = \max\{f(x) : x \in [a, b]\}.$$

This property may *not* hold for closed set (even in $\mathbb{R}$) ⚠. For example, $[0, \infty)$ is a closed set in $\mathbb{R}$ (but *not* a closed interval). A continuous function defined on $[0, \infty)$ can be *unbounded* (so such maximum and minimum *do not even exist*!).

1.8.3 To preserve this property from extreme value theorem, we thus need to enforce additional constraint on closed sets. It turns out that by restricting our attention to *compact sets*, the property is preserved. We shall discuss the notion of *compactness* in Section 1.8.

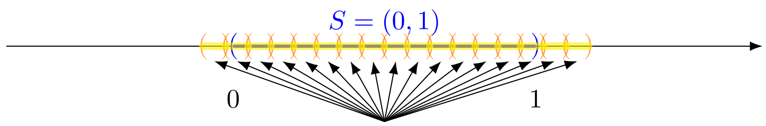
1.8.4 Before defining compactness, we need to introduce some preliminary terminologies. Let (X, d) be a metric space throughout Section 1.8. Let S be a subset of X. Then, a family $\mathcal{F}$ of subsets of X is said to be a **cover** of S (in X) if

$$\bigcup_{F \in \mathcal{F}} F \supseteq S,$$

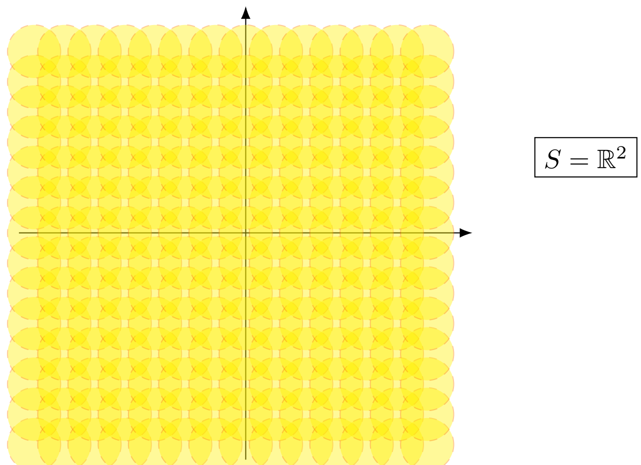
i.e., the union of all the family members in $\mathcal{F}$ contains all elements in S. If every $F \in \mathcal{F}$ is open in X, then $\mathcal{F}$ is called an **open cover** of S.

[Intuition 💡: All the sets in a cover $\mathcal{F}$ of S altogether provide a complete "coverage" of S, hence the name "cover".]

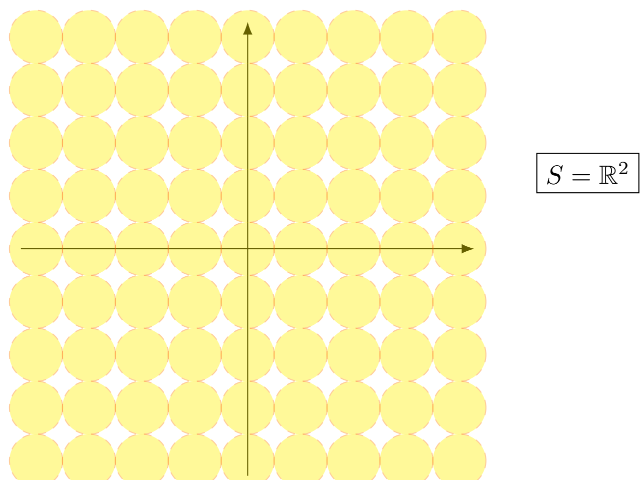
1.8.5 The following pictures illustrate graphically what *covers* look like:



forming an open cover of S



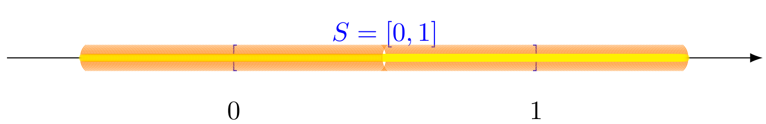
forming an open cover of S



not forming an open cover of S (there are some "gaps")

1.8.6 The following gives some concrete examples of covers. Let $X = \mathbb{R}$ here (and d be the standard Euclidean distance).

(a) $\mathcal{F} = \{(x - 1/2, x + 1/2) : x \in [0,1]\}$ is an open cover of $[0,1]$.



It is an *uncountable* cover of $[0,1]$. From the picture we can observe that there are indeed many *redundancies* in the family. We do not really need uncountably many members to cover $[0,1]$! In fact, having only *two* of them is sufficient to cover $[0,1]$: We can pick $(1/4 - 1/2, 1/4 + 1/2) = (-1/4, 3/4)$ and $(3/4 - 1/2, 3/4 + 1/2) = (1/4, 5/4)$. Then, we have

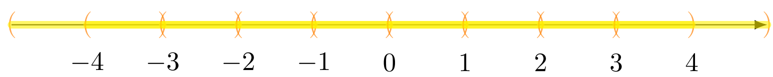
$$(-1/4, 3/4) \cup (1/4, 5/4) = (-1/4, 5/4) \supseteq [0,1].$$

Thus, the *finite* set $\mathcal{F}_0 = \{(-1/4, 3/4), (1/4, 5/4)\}$ can already serve as an open cover of $[0, 1]$. [Note: Since $\mathcal{F}_0$ is formed by members picked from the original cover $\mathcal{F}$, sometimes we call $\mathcal{F}_0$ as a *subcover* of $\mathcal{F}$.

More generally, given any cover $\mathcal{F}$ of a set S, a family $\mathcal{G}$ is said to be a **subcover** of $\mathcal{F}$ if $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{G}$ is also a cover of S.]

(b) $\mathcal{F} = \{(n-1, n+1) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$.

$\boxed{S = \mathbb{R}}$



1.8.7 Now, we are ready to introduce the notion of *compactness*. A set $S \subseteq X$ is said to be **compact** if every open cover of S has a finite subcover.

Examples:

- The open cover $\mathcal{F}$ in [1.8.6]a has a finite subcover, namely $\mathcal{F}_0 = \{(-1/4, 3/4), (1/4, 5/4)\}$.
- The open cover $\mathcal{F}$ in [1.8.6]b does *not* have a finite subcover.

  Proof. Assume to the contrary that it has a finite subcover $\mathcal{F}_0 = \{(n-1, n+1) : n \in S\}$ where S is a finite set of integers. Let M be the maximum integer in the finite set S. Then, note that the real number $M + 1$ is not contained in any of the open intervals in $\mathcal{F}_0$, so $\mathcal{F}_0$ is not a cover of $\mathbb{R}$, contradiction. □

  This shows that $\mathbb{R}$ is not compact (when $X = \mathbb{R}$ and the metric is the standard Euclidean distance).

1.8.8 The definition of a compact set may appear to be a bit technical and does not convey too much information on the "nature" of the set. So in the following we are going to prove some results that tell us more about what a compact set "looks like". The first result suggests the *closedness* and *boundedness* of a compact set.

Theorem 1.8.a. Every compact subset of X is closed (in X) and bounded.

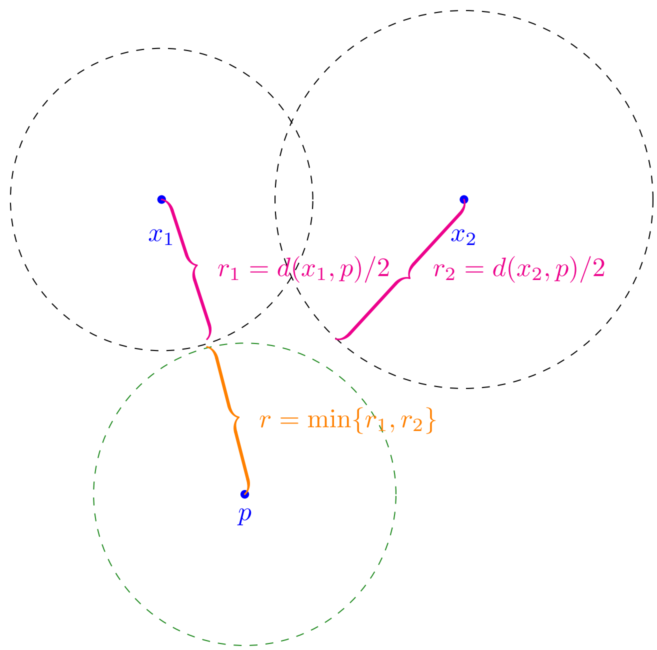
Proof. Let $S \subseteq X$ be any compact set. If S is empty, then its diameter is conventionally set as $-\infty$, which is not ∞. Thus, it is bounded. Also, we know from Proposition 1.4.a that empty set must be closed in X. So we are done with this case.

Henceforth, we assume that $S \neq \varnothing$. Fix any $x \in S$. We first show the <u>boundedness</u>. Consider the open cover $\mathcal{F} = \{B(x, n) : n \in \mathbb{N}\}$ of S. [Note: We have $B(x, 1) \subseteq B(x, 2) \subseteq B(x, 3) \subseteq \cdots$.] By the compactness, $\mathcal{F}$ has a finite subcover $\mathcal{F}_0$. Then, we have $\bigcup_{F \in \mathcal{F}_0} F \supseteq S$. By the finiteness, we know that the union $\bigcup_{F \in \mathcal{F}_0} F$ must be a subset of an open ball $B(x, M)$ for some $M \in \mathbb{N}$. Hence, for this M, we have $d(P, Q) \leq M$ for any $P, Q \in S$, establishing the boundedness.

Next, we show the <u>closedness</u>. By definition, it suffices to show that $X \setminus S$ is open in X. Fix any $p \in X \setminus S$. We would like to find a $r > 0$ such that $B(p, r) \subseteq X \setminus S$, i.e., $B(p, r) \cap S = \varnothing$.

Since S is compact, the open cover $\mathcal{F} = \left\{B\left(x, \frac{1}{2}d(x, p)\right) : x \in S\right\}$ of S has a finite subcover, written as $\{B(x_i, r_i) : i = 1, \ldots, n\}$. Then we have $\bigcup_{i=1}^{n} B(x_i, r_i) \supseteq S$.

Let $r = \min\{r_1, \ldots, r_n\}$. Then, we can observe that $B(p, r) \cap B(x_i, r_i) = \varnothing$ for any $i = 1, \ldots, n$, by the construction of the open cover $\mathcal{F}$.



Since $\bigcup_{i=1}^n B(x_i, r_i) \supseteq S$, it follows that $B(p, r) \cap S = \varnothing$ also. Thus, we conclude that S is closed in X. $\square$

1.8.9 The next one is about the so-called *Boltzano-Weierstrass property.* A subset S of a metric space X has the **Boltzano-Weierstrass property**, or is **limit point compact** if every infinite subset of S has an accumulation point in S (where S is considered as a metric subspace).

Theorem 1.8.b. Every compact subset S of X has the Boltzano-Weierstrass property.

Proof. Assume to the contrary that $T \subseteq S$ is an infinite set with no accumulation point in S. Fix any $s \in S$. Since s is *not* an accumulation point of T, there exists an open ball $B(s, r_s)$ containing s such that $B(s, r_s) \cap T = \varnothing$ (when $s \notin T$) or $B(s, r_s) \cap T = \{s\}$ (when $s \in T$)

Now, as S is compact, the open cover $\{B(s, r_s) : s \in S\}$ of S has a finite subcover $\mathcal{F}_0$. Note that we then have

$$\bigcup_{F \in \mathcal{F}_0} F \supseteq S \supseteq T.$$

Since the union $\bigcup_{F \in \mathcal{F}_0} F$ is finite (as each family member contains either zero or one element), it implies that T is finite also, contradiction. $\square$

1.8.10 The following result describes the behaviour of *closed subsets* of a compact set.

Theorem 1.8.c. Every closed subset of a compact set S is compact.

Proof. Let $S \subseteq X$ be a compact set and $T \subseteq S$ be a closed set in S. Let $\mathcal{F} = \{U_\lambda : \lambda \in \Lambda\}$ be any open cover of T *in* S.

As T is closed in S, by definition $S \setminus T$ is open in S. Hence, $\mathcal{F} \cup (S \setminus T)$ is an open cover of S in S, thus it has a finite sub-cover $\mathcal{F}_0$. By adding the set $S \setminus T$ into $\mathcal{F}_0$ if needed, we can assume that $S \setminus T \in \mathcal{F}_0$, so $\mathcal{F}_0$ takes the form $\{U_1, \ldots, U_n, S \setminus T\}$ (with possibly relabelling for the indices). Hence, we have $T \subseteq S \subseteq (\bigcup_{i=1}^n U_i) \cup (S \setminus T)$.

Since $T \cap (S \setminus T) = \varnothing$, it follows that $T \subseteq \bigcup_{i=1}^n U_i$. Thus, the family $\{U_1, \ldots, U_n\}$ serves as a finite subcover of $\mathcal{F} = \{U_\lambda : \lambda \in \Lambda\}$, as desired. $\square$

[Note: Since every subset $T \subseteq S$ which is closed in X can be written as $T = T \cap S$, it is also closed in S. Hence, we can still apply Theorem 1.8.c for such closed subsets.]

1.8.11 We now consider the compactness of union/intersection of compact sets:

(a) The union of *finitely many* compact sets in X is compact in X.

(b) The intersection of *any collection* of compact sets in X is compact in X.

Proof.

(a) Write $S = \bigcup_{i=1}^{n} S_i$ where S_i is compact for each $i = 1, \ldots, n$. Fix any open cover $\mathcal{F}$ of the union $\bigcup_{i=1}^{n} S_i$. Note that for every $i = 1, \ldots, n$, $\mathcal{F}$ also covers S_i, thus there exists a finite subcover $\mathcal{F}_i$ that covers S_i by the compactness of S_i.
Let $\mathcal{G} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$. Since $\mathcal{F}_i$ is finite for each $i = 1, \ldots, n$, and $\bigcup_{F \in \mathcal{G}} F \supseteq \bigcup_{i=1}^{n} S_i = S$, the union $\mathcal{G}$ is a finite subcover of $\mathcal{F}$ that covers $\bigcup_{i=1}^{n} S_i$.

(b) Write $S = \bigcap_{\lambda \in \Lambda} S_\lambda$ where S_λ is compact, hence closed (by Theorem 1.8.a), for any $\lambda \in \Lambda$. By [1.5.4]b, the intersection $S = \bigcap_{\lambda \in \Lambda} S_i$ is also closed in X. Now pick any set in the collection of compact sets, say S_{λ^*}, and we can see that $\bigcap_{\lambda \in \Lambda} S_i \subseteq S_{\lambda^*}$. Thus, as a closed subset of a compact set, we conclude by Theorem 1.8.c that the intersection is compact in X.

□

1.9 Compactness in $\mathbb{R}^n$

1.9.1 When we focus on the case where the underlying metric space is $X = \mathbb{R}^n$ *equipped with standard Euclidean metric* d, we can have more fruitful results regarding compactness. The key goal of Section 1.9 is to prove the equivalence of the following statements for a subset S of $\mathbb{R}^n$:

(a) S is compact.

(b) S is closed and bounded.

(c) S has the Boltzano-Weierstrass property.

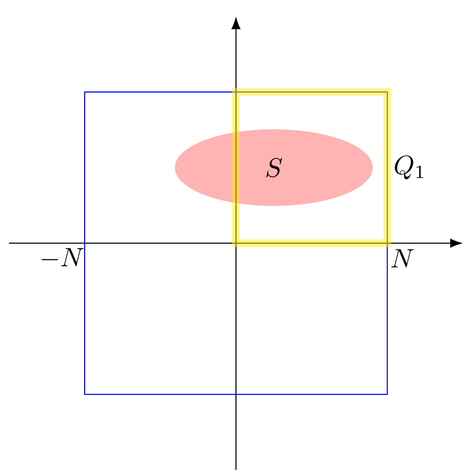
[⚠ Warning: This equivalence may *not* hold if the metric used is not standard Euclidean metric.]

Note that we have shown (a) $\implies$ (b) in Theorem 1.8.a and (a) $\implies$ (c) in Theorem 1.8.b. Thus, it suffices to show that, *under the metric space* $\mathbb{R}^n$, we have (b) $\implies$ (a) and (c) $\implies$ (b) also.

1.9.2 To establish these two implications, we need to prove several results first. The first one is the *Boltzano-Weierstrass theorem*, which is a generalized version of the theorem with the same name studied in MATH2241.

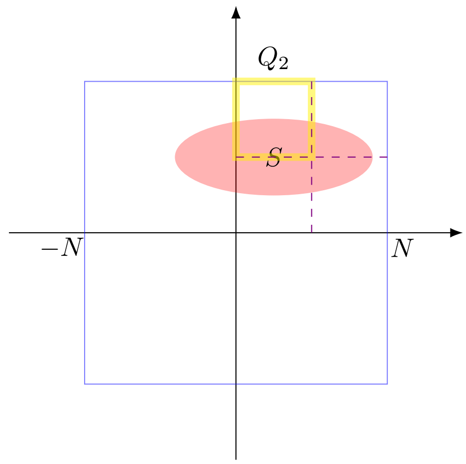
Theorem 1.9.a (Boltzano-Weierstrass theorem)**.** Every bounded infinite set $S \subseteq \mathbb{R}^n$ has an accumulation point in $\mathbb{R}^n$.

Proof. We prove only the case $n = 2$. Proofs for other cases are analogous. Firstly, due to the boundedness of S, we are able to choose a sufficiently large N such that the square $[-N, N] \times [-N, N]$ fully contains S, i.e., $S \subseteq [-N, N]^2$.



Since S is infinite, among the four quadrants of the square, there exists a quadrant that contains infinitely many points of S. Denote it by Q_1.

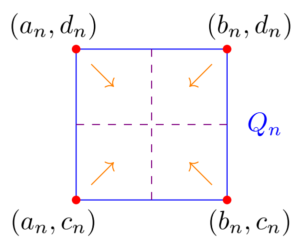
Next, further divide Q_1 into four quadrants. Similarly, one of them must contain infinitely many points of S. Call it Q_2. Note that $Q_2 \subseteq Q_1$, and also the side length of Q_2 is half of that of Q_1.



Continuing in this fashion, we can obtain a nested sequence of quadrants $\{Q_n\}_{n=1}^{\infty}$ such that each Q_n contains infinitely many points of S, and for any $n \in \mathbb{N}$, Q_{n+1} is a subset of Q_n with the side length being half of that of Q_n.

Claim: $\{Q_n\}_{n=1}^{\infty}$ converges to a point (x_0, y_0), for some $x_0, y_0 \in \mathbb{R}$.

Proof. Fix any $n \in \mathbb{N}$ and consider the four corners of the square Q_n.



No matter which quadrant Q_{n+1} is, the four corners must "move" in the direction suggested by the orange arrows (or do not "move"). More precisely, we have:

- $\{a_n\}$ and $\{c_n\}$ are increasing; and
- $\{b_n\}$ and $\{d_n\}$ are decreasing.

Considering the initial large square $[-N, N]^2$, we can see that $\{a_n\}$ and $\{c_n\}$ are bounded above, and the sequences $\{b_n\}$ and $\{d_n\}$ are bounded below. Since the side lengths of the quadrants are shrinking (by half each time), we can apply monotone convergence theorem to argue that there exist $x_0, y_0 \in \mathbb{R}$ such that:

- $\{a_n\} \to x_0$, $\{b_n\} \to x_0$; and
- $\{c_n\} \to y_0$, $\{d_n\} \to y_0$.

$\square$

Claim: The point $p = (x_0, y_0)$ is an accumulation point of S.

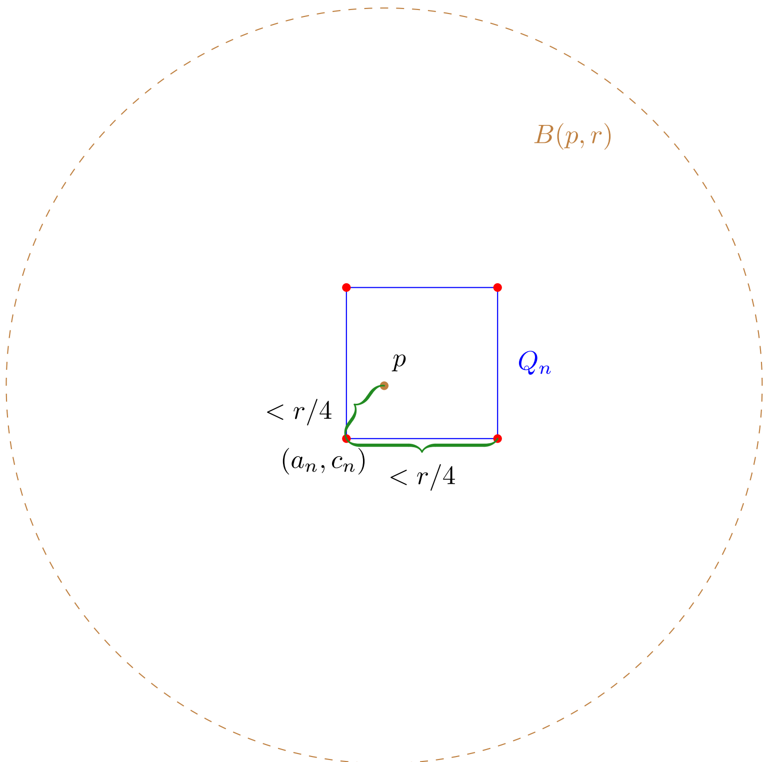
Proof. It suffices to show that for any $r > 0$, $B(p, r) \supseteq Q_n$ for some $n \in \mathbb{N}$, as this would then imply that $B(p, r)$ contains infinitely many points of S, and thus p is an accumulation point of S by Proposition 1.6.d.

Fix any $r > 0$. Since $\{(a_n, c_n)\} \to p$, there exists $M_1 \in \mathbb{N}$ such that $d(p, (a_n, c_n)) < r/4$ for any positive integer $n \geq M_1$. Also, there exists $M_2 \in \mathbb{N}$ such that the side length of Q_n: $\ell(Q_n) < r/4$ for any $n \geq M_2$, as the side lengths are shrinking.

Take $M = \max\{M_1, M_2\}$, and choose any $n \geq M$. We then have for any $q \in Q_n$,

$$d(q, p) \leq d(q, (a_n, c_n)) + d((a_n, c_n), p) \leq \sqrt{2}\ell(Q_n) + \frac{r}{4} \leq \frac{\sqrt{2}}{4}r + \frac{r}{4} = \frac{\sqrt{2}+1}{4}r < r.$$

So, $Q_n \subseteq B(p, r)$ as desired.



$\square$

The result then follows by this claim. $\square$

[Note: It is crucial that the metric d is Euclidean distance in this proof.]

1.9.3 The next theorem is also a generalization to a theorem studied in MATH2241. This time it generalizes *nested interval theorem*, and it is known as *Cantor intersection theorem*. Boltzano-Weierstrass theorem in needed in its proof below.

Theorem 1.9.b (Cantor intersection theorem)**.** Every decreasing nest of closed and bounded nonempty subsets of $\mathbb{R}^n$ has nonempty intersection. Symbolically, if $\{Q_k\}_{k=1}^{\infty}$ is a sequence of non-empty closed subsets of $\mathbb{R}^n$ satisfying

(a) $Q_{k+1} \subseteq Q_k$ for any $k \in \mathbb{N}$, and

(b) Q_1 is bounded,

then $\bigcap_{k=1}^{\infty} Q_k \neq \varnothing$.

Proof. Assume to the contrary that

$$\bigcap_{k=1}^{\infty} Q_k = \varnothing. \tag{2}$$

Then consider the following.

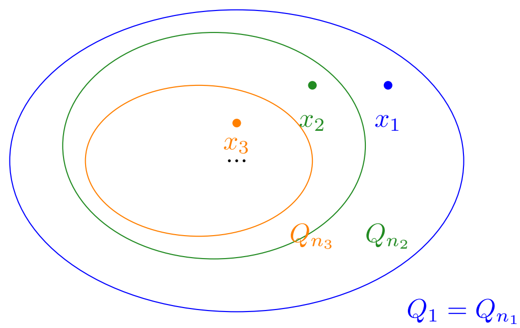
(1) Since Q_1 is nonempty by assumption, there exists $x_1 \in Q_1 \triangleq Q_{n_1}$. By Equation (2), there exists $n_2 \in \mathbb{N}$ such that $x_1 \notin Q_{n_2}$; Otherwise, the intersection would contain at least x_1.

(2) Since Q_{n_2} is nonempty, there exists $x_2 \in Q_{n_2}$. By Equation (2), there exists $n_3 \in \mathbb{N}$ such that $x_2 \notin Q_{n_3}$.

(3) Since Q_{n_3} is nonempty, there exists $x_3 \in Q_{n_3}$. By Equation (2), there exists $n_4 \in \mathbb{N}$ such that $x_3 \notin Q_{n_4}$.

(4) ...

Continuing in this way, we would get a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_i \notin Q_{n_i}$ for any $i \in \mathbb{N}$.



By construction of the sequence $\{x_i\}$, we have the following properties.

(a) The set $\{x_i : i \in \mathbb{N}\}$ is bounded, since Q_1 is bounded.

(b) The set $\{x_i : i \in \mathbb{N}\}$ contains infinitely many elements since the sequence contains distinct terms, i.e., $x_i \neq x_j$ for any $i \neq j$.

Then, by Boltzano-Weierstrass theorem (Theorem 1.9.a), the set $\{x_i : i \in \mathbb{N}\}$, being an infinite bounded subset of $\mathbb{R}^n$, has an accumulation point p in $\mathbb{R}^n$. We then want to show that $p \in \bigcap_{k=1}^{\infty} Q_k$ to arrive at a contradiction.

Fix any $k \in \mathbb{N}$. By assumption, Q_k is closed, so $\overline{Q}_k = Q_k$. Hence, it suffices to show that p is an adherent point of Q_k, which would then imply that $p \in \overline{Q}_k = Q_k$. We shall prove this by definition.

Fix any $r > 0$. Then we need to show that $B(p, r) \cap Q_k \neq \varnothing$. Since p is an accumulation point of the set $\{x_i : i \in \mathbb{N}\}$, the open ball $B(p, r)$ contains infinitely many points from $\{x_i : i \in \mathbb{N}\}$ by Proposition 1.6.d. Hence, there exists $m > k$ such that $B(p, r)$ contains x_m.

By construction, we also have $x_m \in Q_{n_m}$. So, we actually have $x_m \in B(p, r) \cap Q_{n_m}$, which implies that $B(p, r) \cap Q_{n_m} \neq \varnothing$. Since $n_m \geq m > k$, we have $Q_{n_m} \subseteq Q_k$, thus $B(p, r) \cap Q_k \neq \varnothing$.

Hence, $p \in \overline{Q}_k = Q_k$. As $k \in \mathbb{N}$ is arbitrary, it follows that $p \in \bigcap_{k=1}^{\infty} Q_k$, which contradicts Equation (2). □

1.9.4 Using *Cantor intersection theorem* and *Lindelöf's theorem*, we can prove the implication (b) $\implies$ (a) under the metric space $\mathbb{R}^n$. This implication is also known as *Heine-Borel theorem*.

Theorem 1.9.c (Heine-Borel theorem)**.** Every closed and bounded subset of $\mathbb{R}^n$ is compact.

Proof. Consider any closed and bounded set $S \subseteq \mathbb{R}^n$. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be any open cover of S, i.e., $\bigcup_{\lambda \in \Lambda} U_\lambda \supseteq S$ where U_λ is open (in $\mathbb{R}^n$) for any $\lambda \in \Lambda$. To show the compactness, we want to find a finite subcover of $\mathcal{U}$.

We first use Lindelöf's theorem (Theorem 1.5.f). It suggests that there exists a countable family $\{U_k\}_{k=1}^{\infty}$ such that for any $k \in \mathbb{N}$, $U_k = U_\lambda$ for some $\lambda \in \Lambda$, and it satisfies $\bigcup_{k=1}^{\infty} U_k = \bigcup_{\lambda \in \Lambda} U_\lambda \supseteq S$

Next, let $Q_k = \bigcup_{i=1}^{k} U_i$, which is open since it is an union of open sets. Then, we have $\bigcup_{k=1}^{\infty} Q_k = \bigcup_{i=1}^{\infty} U_i \supseteq S$.

Claim: There exists $N \in \mathbb{N}$ such that $Q_N = \bigcup_{i=1}^{N} U_i \supseteq S$.

Proof. Assume to the contrary that for any $k \in \mathbb{N}$, S contains some elements not in Q_k, i.e., $(X \setminus Q_k) \cap S \neq \varnothing$. For any $k \in \mathbb{N}$, note that:

- $(X \setminus Q_k) \cap S$ is closed (in $\mathbb{R}^n$) since $X \setminus Q_k$ and S are both closed, so do their intersection.
- $(X \setminus Q_k) \cap S$ is bounded since S is bounded.
- $(X \setminus Q_{k+1}) \subseteq (X \setminus Q_k)$.

Then, applying Cantor intersection theorem (Theorem 1.9.b),

$$\bigcap_{k=1}^{\infty} [(X \setminus Q_k) \cap S] \neq \varnothing.$$

But on the other hand, we have

$$\begin{aligned}
\bigcap_{k=1}^{\infty} [(X \setminus Q_k) \cap S] &= \bigcap_{k=1}^{\infty} [(X \cap S) \setminus (Q_k \cap S)] \\
&= \bigcap_{k=1}^{\infty} [S \setminus (Q_k \cap S)] \\
&= S \setminus \left[\bigcup_{k=1}^{\infty} (Q_k \cap S)\right] \\
&= S \setminus \left[\left(\bigcup_{k=1}^{\infty} Q_k\right) \cap S\right] \\
&= S \setminus S & \left(\text{since } \textstyle\bigcup_{k=1}^{\infty} Q_k \supseteq S\right) \\
&= \varnothing,
\end{aligned}$$

contradiction. □

By the claim, we know that $\{U_1, \ldots, U_N\}$ is a finite subcover of $\mathcal{U}$, as desired. □

1.9.5 Now, we are ready to prove the main result in Section 1.9 as follows.

Theorem 1.9.d. Let S be a subset of $\mathbb{R}^n$. Then the following are equivalent.

(a) S is compact.

(b) S is closed and bounded.

(c) S has the Boltzano-Weierstrass property.

Proof. So far we have proved that (a) $\iff$ (b) (Theorems 1.8.a and 1.9.c) and (a) $\implies$ (c) (Theorem 1.8.b). Hence, it suffices to prove that (c) $\implies$ (b).

First, assume that S has the Boltzano-Weierstrass property. We will first prove the boundedness of S. Assume to the contrary that S is not bounded (which implies that it is infinite). Then, for any $k \in \mathbb{N}$ we can pick $x_k \in S$ such that $x_k \geq k$. But then the infinite set $\{x_k : k \in \mathbb{N}\}$ cannot possibly have an accumulation point. For example, every open ball with radius $1/2$ cannot possibly contain infinitely many x_n's. Contradiction.

Next, we will prove the closedness of S. By Proposition 1.6.e, it suffices to prove that $\overline{S} = S$. We know that $S \subseteq \overline{S}$ always, so it actually suffices to show that $\overline{S} \subseteq S$. By [1.6.10], every point in $\overline{S}$ is either an accumulation point or isolated point of S. Since isolated point of S must belong to S by definition, it remains to show that *every accumulation point of S belongs to S*.

Let $x \in \mathbb{R}^n$ be any accumulation point of S. Then, for any $k \in \mathbb{N}$, setting the radius $r = 1/k$, there exists $x_k \in B(x, 1/k) \cap S$. Since each open ball $B(x, 1/k)$ contains infinitely many points in S, we can require the x_n's to be all distinct, i.e., $x_k \neq x_m$ for any positive integers $n \neq m$. This then forms an infinite subset $\{x_k : k \in \mathbb{N}\}$ of S. By the Boltzano-Weierstrass property of S, the set $\{x_k : k \in \mathbb{N}\}$ has an accumulation point $y \in S$.

Claim: We have $x = y \in S$.

Proof. Assume to the contrary that $x \neq y$. Then, let $r_0 = d(y, x) > 0$. Consider the open ball $B(y, r_0/2)$. Since y is an accumulation point of $\{x_k : k \in \mathbb{N}\}$, $B(y, r_0/2)$ contains infinitely many x_n's by Proposition 1.6.d. Choose a large $N \in \mathbb{N}$ such that $1/N < r_0/2$. Then, there exists $k > N$ such that $d(y, x_k) < r_0/2$. Also, for this k, we have $d(x_k, x) < 1/k < 1/N$ as $x_k \in B(x, 1/k)$. Hence,

$$d(y, x) \leq d(x_k, x) + d(y, x_k) < \frac{1}{N} + \frac{r_0}{2} < \frac{r_0}{2} + \frac{r_0}{2} = r_0,$$

contradicting to the fact that $r_0 = d(y, x)$. $\square$

By the claim, the arbitrarily chosen accumulation point x belongs to S, as desired. $\square$

1.9.6 For a general metric space, Theorem 1.9.d is only "partially" true. To be more precise, we have:

- (a) $\implies$ (b) by Theorem 1.8.a
- (a) $\implies$ (c) by Theorem 1.8.b

Indeed, it can still be shown that (c) $\implies$ (a), but in general (b) $\not\Rightarrow$ (a).

Hence, for a general metric space, we have (a) $\iff$ (c) $\implies$ (b). Particularly, this shows that a criterion for compactness is the Boltzano-Weierstrass property.

For a more general result concerning different criteria for compactness, see Theorem 4.5.a.

2 Limits and Continuity

2.0.1 In MATH2241, we have learnt the concept of *limits and continuity* in $\mathbb{R}$. Here, we would extend the notion to a more general *metric space*. As we will see, some definitions and terminologies are analogous to the ones we learnt in MATH2241.

2.1 Convergence in a Metric Space

2.1.1 Throughout Section 2.1, we shall consider an arbitrary metric space (X, d).

2.1.2 A sequence $\{x_n\}$ in X **converges** to $a \in X$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any positive integer $n \geq N$,

$$d(x_n, a) < \varepsilon,$$

i.e., $x_n \in B(a, \varepsilon)$. In this case, we write $\lim_{n\to\infty} x_n = a$, $\{x_n\} \to a$, or $\{x_n\}_{n=1}^{\infty} \to a$. We call a as **limit** of $\{x_n\}$. If the sequence $\{x_n\}$ converges to *some* $a \in X$, we say that the sequence **converges**/is **convergent**. Otherwise, we say that it **diverges**/is **divergent**.

[Note: The definition of a **sequence** is analogous to the one in MATH2241: It is a function from $\mathbb{N}$ to X.]

2.1.3 We can relate the notion of convergence in metric space with that in $\mathbb{R}$ as follows:

$$\{x_n\} \to a \iff \{d(x_n, a)\} \to 0.$$

Proof. Note that

$$\begin{aligned} & \{x_n\} \to a \\ \iff & \text{for any } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } |d(x_n, a) - 0| = d(x_n, a) < \varepsilon \text{ for any } n \geq N \\ \iff & \{d(x_n, a)\} \to 0. \end{aligned}$$

□

2.1.4 Like the case for $\mathbb{R}$, the limit here is unique also (if exists). This property can be proved using the same approach as the one in MATH2241 (essentially just replacing some symbols).

Proposition 2.1.a. A sequence $\{x_n\}$ in X can converge to at most one point in X.

Proof. Assume that $\{x_n\} \to a$ and $\{x_n\} \to b$ for some $a, b \in X$. Fix any $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$d(x_n, a) < \frac{\varepsilon}{2} \quad \text{for any } n \geq N_1,$$

and

$$d(x_n, a) < \frac{\varepsilon}{2} \quad \text{for any } n \geq N_2.$$

Then, choose $N = \max\{N_1, N_2\}$. By triangle inequality (M3), for any $n \geq N$,

$$d(a, b) \leq d(x_n, a) + d(x_n, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

2.1.5 The following result gives some properties of a convergent sequence.

Proposition 2.1.b. Let $\{x_n\}$ be a sequence in X and $a \in X$. Suppose that $\{x_n\} \to a$. Then,

(a) $\{x_n : n \in \mathbb{N}\}$ is bounded in X.

(b) a is an adherent point of $\{x_n : n \in \mathbb{N}\}$.

(c) If $\{x_n : n \in \mathbb{N}\}$ is an infinite set, then a is an accumulation point of $\{x_n : n \in \mathbb{N}\}$.

Proof.

(a) When $\{x_n\} \to a$, there exists $N \in \mathbb{N}$ such that $d(x_n, a) < 1$ for any $n \geq N$. Then, by setting $M = \max\{d(x_1, a), \ldots, d(x_{N-1}, a), 1\}$, we have $d(x_n, a) < M$ for any $n \in \mathbb{N}$. Applying triangle inequality (M3), we can then show that $\{x_n : n \in \mathbb{N}\}$ is bounded in X.

(b) Fix any $r > 0$. By assumption there exists $N \in \mathbb{N}$ such that $x_n \in B(a, r)$ for any $n \geq N$. Thus, $B(a, r) \cap \{x_n : n \in \mathbb{N}\} \neq \varnothing$, so a is an adherent point of $\{x_n : n \in \mathbb{N}\}$.

(c) Fix any $r > 0$. By assumption there exists $N \in \mathbb{N}$ such that $x_n \in B(a, r)$ for any $n \geq N$. Since $\{x_n : n \in \mathbb{N}\}$ is infinite, the intersection $B(a, r) \cap \{x_n : n \in \mathbb{N}\} \supseteq \{x_n : n \geq N\}$ is also infinite. Hence, by Proposition 1.6.d, a is an accumulation point of $\{x_n : n \in \mathbb{N}\}$.

□

2.1.6 Using the concept of *sequences*, we can obtain an useful characterization of adherent points and accumulation points as follows:

Proposition 2.1.c. Let $S \subseteq X$ and $a \in X$. Then,

(a) $a \in \overline{S}$ iff there is a sequence $\{x_n\}$ in S such that $\{x_n\} \to a$.

(b) $a \in S'$ iff there is an infinite sequence $\{x_n\}$ of distinct points in S such that $\{x_n\} \to a$.

Proof.

(a) "$\Rightarrow$": Suppose that $a \in \overline{S}$. Then for any $n \in \mathbb{N}$, there exists $x_n \in S$ such that $0 \leq d(x_n, a) < 1/n$. We can see that the sequence $\{x_n\}$ constructed in this way converges to a, since $\{d(x_n, a)\} \to 0$ by sandwich theorem.

"$\Leftarrow$": Assume that there is a sequence $\{x_n\}$ in S converging to a. By Proposition 2.1.b, we know that a is an adherent point of $\{x_n : n \in \mathbb{N}\}$, i.e., $a \in \overline{\{x_n : n \in \mathbb{N}\}}$. Since the sequence $\{x_n\}$ is in S, we have $\{x_n : n \in \mathbb{N}\} \subseteq S$. It then follows by [1.7.3]a that

$$a \in \overline{\{x_n : n \in \mathbb{N}\}} \subseteq \overline{S}.$$

(b) "$\Rightarrow$": Suppose that $a \in S'$. Note first that for any $n \in \mathbb{N}$, by Proposition 1.6.d, $B(a, 1/n) \cap S$ is an infinite set. Thus, it is possible to choose $x_n \in S \setminus \{x_1, \ldots, x_{n-1}\}$ with $0 \leq d(x_n, a) < 1/n$ for every $n \in \mathbb{N}$. This constructs an infinite sequence $\{x_n\}$ of distinct points in S with $\{x_n\} \to a$.

"$\Leftarrow$": Assume there is an *infinite* sequence $\{x_n\}$ of distinct points in S such that $\{x_n\} \to a$. By Proposition 2.1.b, we know that $a \in \{x_n : n \in \mathbb{N}\}'$. Since $\{x_n : n \in \mathbb{N}\} \subseteq S$, it follows by [1.7.4]a that

$$a \in \{x_n : n \in \mathbb{N}\}' \subseteq S'.$$

□

2.1.7 With this sequential characterization, we can get yet another criterion for closedness.

Proposition 2.1.d. Let $S \subseteq X$ and $a \in X$. Then, the following are equivalent.

(a) S is closed in X.

(b) If $\{x_n\}$ is a sequence in S with $\{x_n\} \to a$, then $a \in S$.

Proof. (a) $\implies$ (b): Suppose that S is closed in X and $\{x_n\}$ is a sequence in S with $\{x_n\} \to a$. Then by Proposition 2.1.c and Proposition 1.6.e, we have $a \in \overline{S} = S$.

(b) $\implies$ (a): Assume that (b) holds. Fix any $a \in \overline{S}$. By Proposition 2.1.c, there is a sequence $\{x_n\}$ in S such that $\{x_n\} \to a$. Applying (b) then gives $a \in S$. This shows $\overline{S} \subseteq S$. Since we must have $S \subseteq \overline{S}$, it follows that $\overline{S} = S$, hence S is closed in X by Proposition 1.6.e. □

2.1.8 Next, we will discuss a result about *subsequences*, which is a generalization to the respective result in MATH2241. It can be proved using the same approach as the one in MATH2241.

Proposition 2.1.e. Let $\{x_n\}$ be a sequence in X and $a \in X$. Then $\{x_n\} \to a$ iff $\{x_{n_k}\}_{k=1}^{\infty} \to a$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

[Note: The concept of *subsequence* is defined in the same manner as the one for MATH2241: We first let $n_1 < n_2 < n_3 < \cdots$ be positive integers sorted in strictly increasing order, and then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is a **subsequence** of $\{x_n\}$.]

Proof. "$\Leftarrow$" is immediate since $\{x_n\}$ is a subsequence of itself. So it remains to prove "$\Rightarrow$". Assume that $\{x_n\} \to a$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon$ for any $n \geq N$. Fix any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then for any $k \geq N$, since $n_k \geq k \geq N$, we have $d(x_{n_k}, a) < \varepsilon$. □

2.2 Complete Metric Spaces

2.2.1 Throughout Section 2.2, we shall consider an arbitrary metric space (X, d).

2.2.2 To motivate the notion of *complete metric spaces*. We consider the set $\mathbb{Q}$ of rational numbers. It is well-known that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ is *irrational*. Nonetheless, using decimal representation, we can obtain better and better *approximations* to $\sqrt{2}$, using just rational numbers.

More specifically, we can express $\sqrt{2}$ as $1.414213562373095\cdots$. From this we can construct a sequence $\{x_n\}$ in $\mathbb{Q}$ as follows:

$$x_1 = 1, \quad x_2 = 1.4, \quad x_3 = 1.41, \quad x_4 = 1.414, \quad x_5 = 1.4142, \quad x_6 = 1.41421, \quad x_7 = 1.414214, \quad \cdots$$

by rounding $\sqrt{2}$ in increasing number of decimal places. We can show that $\{x_n\} \to \sqrt{2} \notin \mathbb{Q}$. So, this sequence $\{x_n\}$ does *not* converge to any number in $\mathbb{Q}$.

However, it is not hard to observe that the terms in this sequence get "closer and closer" together, so intuitively it seems that the sequence should converge to a certain "point". Here that "point" is $\sqrt{2}$, which is *not* contained in $\mathbb{Q}$. So in this sense, $\mathbb{Q}$ appears to have something "missing" at $\sqrt{2}$ and it is not so "complete".

2.2.3 This kind of sequence with terms getting "closer and closer" together can be characterized by *Cauchy sequence*. [Note: We have studied this concept in the special case of $\mathbb{R}$ in MATH2241. Here we consider a more general one.]

A sequence $\{x_n\}$ in X is said to be a **Cauchy sequence** if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for any $n, m \geq N$.

2.2.4 As a *convergent* sequence also has the feature that the terms get "closer and closer" together, intuitively it seems that it should also be a Cauchy sequence. This is indeed the case.

Proposition 2.2.a. Every convergent sequence is Cauchy.

Proof. Consider any convergent sequence $\{x_n\} \to a$ in X. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, a) < \frac{\varepsilon}{2} \quad \text{for any } n \geq N.$$

Thus, for any $n, m \geq N$, by triangle inequality (M3),

$$d(x_n, x_m) \leq d(x_n, a) + d(x_m, a) < \varepsilon \quad \text{for any } n, m \geq N.$$

Thus, $\{x_n\}$ is Cauchy. □

2.2.5 Intuitively, a Cauchy sequence "stabilizes" as the index gets larger. Hence, naturally we would expect that the *distances* between terms respectively taken from two Cauchy sequences "stabilize" also. This intuitive idea is proven below.

Proposition 2.2.b. Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences. Then, the sequence $\{d(x_n, y_n)\}$ converges in $\mathbb{R}$.

Proof. Since the sequence $\{d(x_n, y_n)\}$ is in $\mathbb{R}$, it suffices to prove that the sequence $\{d(x_n, y_n)\}$ is Cauchy.

Fix any $\varepsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} d(x_n, x_m) &< \frac{\varepsilon}{2} \quad \text{for any } n, m \geq N_1 \\ d(y_n, y_m) &< \frac{\varepsilon}{2} \quad \text{for any } n, m \geq N_2. \end{aligned}$$

We then choose $N = \max\{N_1, N_2\}$. Then, for any $n, m \geq N$, we have

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &= |d(x_n, y_n) - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)| \\ &\leq |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \quad \text{(triangle inequality)} \\ &\leq |d(y_n, y_m)| + |d(x_n, x_m)| \quad \text{(M3)} \\ &\leq d(y_n, y_m) + d(x_n, x_m) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This shows that $\{d(x_n, y_n)\}$ is Cauchy, as desired. □

2.2.6 In our previous $\sqrt{2}$ example, the rational sequence there can be shown to be Cauchy but not convergent. So a Cauchy sequence is not necessarily convergent. Indeed, the convergence of Cauchy sequence allows us to characterize the notion of "completeness". Intuitively, divergence of a Cauchy sequence indicates a "missing point". To be "complete", there should be no "missing point", thus every Cauchy sequence should be convergent.

A metric space (X, d) is said to be **complete** if *every* Cauchy sequence in X converges in X. A subset S of X is said to be **complete** if, when considered as a metric space itself, (S, d) is complete.

2.2.7 Next, we shall establish an important result that the Euclidean space $\mathbb{R}^k$ is complete for any $k \in \mathbb{N}$.

Theorem 2.2.c. The metric space $\mathbb{R}^k$ is complete for any $k \in \mathbb{N}$.

Proof. We shall base our proof on the fact that $\mathbb{R}$ is complete. Consider any Cauchy sequence $\{(x_n^{(1)}, \ldots, x_n^{(k)})\}$ in $\mathbb{R}^k$. Note that for every $i = 1, \ldots, k$, $\{x_n^{(i)}\}$ is a Cauchy sequence in $\mathbb{R}$. Then by the completeness of $\mathbb{R}$, we must have $\{x_n^{(i)}\} \to a^{(i)}$ for some $a^{(i)} \in \mathbb{R}$. This then implies that

$$\{(x_n^{(1)}, \ldots, x_n^{(k)})\} \to (a^{(1)}, \ldots, a^{(k)}) \in \mathbb{R}^k.$$

□

2.2.8 The next result connects *compactness* with *completeness*. It turns out that compactness, being a rather strong condition, does implies completeness.

Theorem 2.2.d. Every compact subset S of a metric space X is complete.

Proof. Let $\{x_n\}$ be any Cauchy sequence in S.

<u>Case 1</u>: $\{x_n : n \in \mathbb{N}\}$ is a finite set.

Then there exists $N \in \mathbb{N}$ such that $x_n = x \in S$ for any $n \geq N$. This shows $\lim_{n\to\infty} x_n = x \in S$. [Note: For this case, we do not even need compactness!]

<u>Case 2</u>: $\{x_n : n \in \mathbb{N}\}$ is an infinite set.

By Theorem 1.8.b, X has the Boltzano-Weierstrass property. Hence, as an infinite subset of X, the set $\{x_n : n \in \mathbb{N}\}$ has an accumulation point a in $\{x_n : n \in \mathbb{N}\}$. Fix any $\varepsilon > 0$. First, since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for any $n, m \geq N$. Also, since a is an accumulation point of $\{x_n : n \in \mathbb{N}\}$, the intersection $B(a, \varepsilon/2) \cap \{x_n : n \in \mathbb{N}\}$ is infinite. Thus, there exists $m \geq N$ such that $x_m \in B(a, \varepsilon/2)$.

Now, applying triangle inequality, we have

$$d(x_n, a) \leq d(x_n, x_m) + d(x_m, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for any $n \geq N$. This shows $\{x_n\} \to a$, as desired. □

2.2.9 The following result also concerns about the completeness of some subset of a metric space. But here we need to impose a strong requirement that the original metric space should be complete.

Proposition 2.2.e. Every closed subset S of a complete metric space X is complete.

Proof. Fix any Cauchy sequence $\{x_n\}$ in S. Since the metric for S is induced by that for X, $\{x_n\}$ is still Cauchy when considered as a sequence in X.

Because X is complete, $\{x_n\}$ converges to some $x \in X$. Since $\{x_n\}$ is in S, we have $x \in \overline{S}$ by Proposition 2.1.c.

As S is closed in X, $\overline{S} = S$ by Proposition 1.6.e, thus $x \in S$. Hence $\{x_n\}$ is convergent in S. □

Construction of Completion (Optional)

2.2.10 Given a metric space (X, d) which may not be complete, we can construct the *completion* of (X, d), denoted by $(\widetilde{X}, \widetilde{d})$. The **completion** of (X, d) is the smallest complete metric space containing X, i.e., for any complete metric space (Y, d_Y) with $Y \supseteq X$, the set $\widetilde{X}$ is a subset of Y (with possibly some "embedding"). [Note: The metric d_Y is the same as d except that its domain is changed to $Y \times Y$.]

Example: The completion of $\mathbb{Q}$ equipped with standard Euclidean metric is $\mathbb{R}$ with the same metric.

2.2.11 Here we will demonstrate how to actually construct such completion. As suggested previously, every Cauchy sequence in X may be seen as corresponding to a "point", to which the terms are "getting closer". The intuitive idea is then to include all the Cauchy sequences in X in the completion, so that all the possible corresponding "points" are included — There is not "missing point"!

2.2.12 However, it is not hard to notice that doing in this way would result in many redundancies. It appears that many Cauchy sequences actually correspond to the same "point". For example, both the Cauchy sequences $\{0.9, 0.99, 0.999, 0.9999, \dots\}$ and $\{1.1, 1.01, 1.001, 1.0001, \dots\}$ appear to correspond to the same "point": 1. But we want the completion to be "smallest", so all those redundancies should be removed.

2.2.13 This leads us to consider the idea of *equivalence classes*, to "represent" each "point" by using only one element, namely the *equivalence class* containing all the "equivalent" Cauchy sequences corresponding to the same "point".

Hence, we would like to define an equivalence relation $\sim$ on the set of all Cauchy sequences in X to capture our intuitive idea of "corresponding to the same point".

2.2.14 To define such equivalence relation, we first introduce the following function. For any two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X, define the function $\widetilde{\delta} : X \times X \to \mathbb{R}$ by

$$\widetilde{\delta}(\{x_n\}, \{y_n\}) = \lim_{n \to \infty} d(x_n, y_n).$$

Note that the limit $\lim_{n \to \infty} d(x_n, y_n)$ always exists by Proposition 2.2.b, so the function $\widetilde{\delta}$ is well-defined.

2.2.15 Next, we define $\{x_n\}$ and $\{y_n\}$ as *equivalent*, written as $\{x_n\} \sim \{y_n\}$, if $\widetilde{\delta}(\{x_n\}, \{y_n\}) = 0$.

[Intuition 💡: When the distance between two Cauchy sequences are zero, the terms in each Cauchy sequence should "get closer" to the same "point".]

2.2.16 We now let

$$\widetilde{X} = \{\text{all equivalence classes of Cauchy sequences in } X\}.$$

Note that $\widetilde{\delta}(\{x_n\}, \{y_n\})$ is invariant after replacing either of the sequences by a Cauchy sequence equivalent to it.

Proof. WLOG, we only prove that $\widetilde{\delta}(\{x_n\}, \{y_n\}) = \widetilde{\delta}(\{x_n\}, \{z_n\})$, where $\{z_n\}$ be any Cauchy sequence equivalent to $\{y_n\}$, i.e., $\{z_n\} \sim \{y_n\}$.

Using (M3) of the underlying metric d, we have

$$\widetilde{\delta}(\{x_n\}, \{y_n\}) \leq \widetilde{\delta}(\{x_n\}, \{z_n\}) + \underbrace{\widetilde{\delta}(\{z_n\}, \{y_n\})}_{0} = \widetilde{\delta}(\{x_n\}, \{z_n\}).$$

On the other hand, using (M3) of the metric d again, we get

$$\widetilde{\delta}(\{x_n\}, \{z_n\}) \leq \widetilde{\delta}(\{x_n\}, \{y_n\}) + \underbrace{\widetilde{\delta}(\{y_n\}, \{z_n\})}_{0} = \widetilde{\delta}(\{x_n\}, \{y_n\}).$$

This shows that $\widetilde{\delta}(\{x_n\}, \{y_n\}) = \widetilde{\delta}(\{x_n\}, \{z_n\})$. □

With this property, the following function $\widetilde{d} : \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ is well-defined:

$$\widetilde{d}([s_1], [s_2]) = \widetilde{\delta}(s_1, s_2)$$

where $s_1 = \{x_n\}$ and $s_2 = \{y_n\}$ are any elements taken from the equivalence classes $[s_1]$ and $[s_2]$ in $\widetilde{X}$ respectively.

2.2.17 We can then verify that $\widetilde{d}$ is indeed a metric as follows.

Proof. (M1): Due to the nonnegativity of the underlying metric d, we must have $d([s_1], [s_2]) = \widetilde{\delta}(s_1, s_2) \geq 0$. Next, observe that $\widetilde{\delta}(s_1, s_2) = 0$ iff $s_1 = s_2$, due to the underlying metric d. Since equivalence classes are either equal or disjoint, it follows that $d([s_1], [s_2]) = 0$ iff $[s_1] = [s_2]$.

(M2): It follows from (M2) of the underlying metric d: $d(x_n, y_n) = d(y_n, x_n)$ for any $n \in \mathbb{N}$.

(M3): It follows from (M3) of the underlying metric d. □

2.2.18 Now we define an *embedding* $\iota : X \to \widetilde{X}$ by $\iota(x) = [\{x, x, x, \dots\}]$, i.e., the equivalence class containing the sequence $\{x_n\}$ with $x_n = x$ for any $n \in \mathbb{N}$.

The embedding ι is an *isometry* or *distance-preserving map*, i.e., $d(x, y) = \widetilde{d}(\iota(x), \iota(y))$ for any $x, y \in X$. Then, by identifying every $x \in X$ with the corresponding embedding $\iota(x) \in \widetilde{X}$ ("treating them as the same"), we may say that X is a "subset" of $\widetilde{X}$ (with embedding).

2.2.19 Next, we will show that $(\widetilde{X}, \widetilde{d})$ is indeed a *complete* metric space. We want to show that every Cauchy sequence of equivalence classes in $\widetilde{X}$ converges to a certain equivalence class in $\widetilde{X}$.

Fix any Cauchy sequence $\{[s_n]\}_{n=1}^{\infty}$ of equivalence classes in $\widetilde{X}$, where s_n is a Cauchy sequence $\{x_m^{(n)}\}_{m=1}^{\infty}$ in X for any $n \in \mathbb{N}$.

To prove the convergence of $\{[s_n]\}_{n=1}^{\infty}$, we will use a special kind of diagonal argument. First fix any $k \in \mathbb{N}$. Since $s_k = \{x_m^{(k)}\}_{m=1}^{\infty}$ is Cauchy in X, there exists $N_k \in \mathbb{N}$ such that

$$d\Big(x_{m_1}^{(k)}, x_{m_2}^{(k)}\Big) < \frac{1}{k}$$

for any $m_1, m_2 \geq N_k$. We denote the N_kth term in the sequence s_k, namely $x_{N_k}^{(k)}$, by y_k. Then particularly we would have

$$d\Big(x_{m_1}^{(k)}, y_k\Big) = d\Big(x_{m_1}^{(k)}, x_{N_k}^{(k)}\Big) < \frac{1}{k} \tag{3}$$

for any $m_1 \geq N_k$.

We claim that the sequence $\{[s_n]\}$ converges to the equivalence class containing the sequence $\{y_k\}_{k=1}^{\infty} = \{x_{N_k}^{(k)}\}_{k=1}^{\infty}$, which is in $\widetilde{X}$.

Proof. Firstly, we shall prove that $\{y_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X, which shows that the equivalence class containing $\{y_k\}_{k=1}^{\infty}$ is indeed in $\widetilde{X}$. Fix any $\varepsilon > 0$. Then, we choose a sufficiently large $N \in \mathbb{N}$ such that:

(a) $1/N < \varepsilon/3$, and

(b) Considering any $m, n \geq N$, there should exist a sufficiently large $j \in \mathbb{N}$ such that $d(x_j^{(n)}, x_j^{(m)}) < \varepsilon/3$ since $\{[s_i]\}_{i=1}^{\infty}$ is Cauchy, which implies that $\widetilde{\delta}(\{x_j^{(n)}\}, \{x_j^{(m)}\}) = \lim_{j\to\infty} d(x_j^{(n)}, x_j^{(m)})$ can be arbitrarily small, as long as m, n are large enough.

Then, for any $m, n \geq N$ with a sufficiently large $j \in \mathbb{N}$, that exceeds both N_m and N_n, and makes $d(x_j^{(n)}, x_j^{(m)}) < \varepsilon/3$, we have

$$d(y_n, y_m) = d(x_{N_n}^{(n)}, x_{N_m}^{(m)}) \leq d(x_{N_n}^{(n)}, x_j^{(n)}) + d(x_j^{(n)}, x_j^{(m)}) + d(x_j^{(m)}, x_{N_m}^{(m)}) < \frac{1}{n} + \frac{\varepsilon}{3} + \frac{1}{m} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

establishing the Cauchyness of $\{y_k\}_{k=1}^{\infty}$ in X.

Next, we want to show that $\{[s_n]\} \to [\{y_k\}_{k=1}^{\infty}]$. In other words, we need to show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\widetilde{d}([s_n], [\{y_k\}_{k=1}^{\infty}]) = \widetilde{\delta}(s_n, \{y_k\}_{k=1}^{\infty}) = \widetilde{\delta}(\{x_k^{(n)}\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty}) = \lim_{k\to\infty} d(x_k^{(n)}, y_k) = \lim_{k\to\infty} d(x_k^{(n)}, x_{N_k}^{(k)}) < \varepsilon$$

for any $n \geq N$.

We first fix any $\varepsilon > 0$, and then choose a sufficiently large $N \in \mathbb{N}$ such that (i) $1/N < \varepsilon/2$ and (ii) for any $m, n \geq N$, we have $d(y_n, y_m) = d(x_{N_n}^{(n)}, x_{N_m}^{(m)}) < \varepsilon/2$. Then, for any $n \geq N$, we have

$$\begin{aligned}
\widetilde{d}([s_n], [\{y_k\}_{k=1}^{\infty}]) &= \lim_{k\to\infty} d(x_k^{(n)}, x_{N_k}^{(k)}) \\
&\leq \lim_{k\to\infty} d(x_k^{(n)}, x_{N_n}^{(n)}) + \lim_{k\to\infty} d(x_{N_n}^{(n)}, x_{N_k}^{(k)}) \\
&< \frac{1}{n} + \frac{\varepsilon}{2} && \text{(using (3) for the first term; (ii) for the second term)} \\
&\leq \frac{1}{N} + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

completing the proof. □

2.2.20 Finally, we shall show that $(\widetilde{X}, \widetilde{d})$ is the *smallest* complete metric space containing X. Consider another complete metric space (Y, d_Y) containing X also, where d_Y is the same as d except that the domain becomes $Y \times Y$.

Note that for every class $[s] \in \widetilde{X}$, the Cauchy sequence s in X is also a Cauchy sequence in Y, thus converges to some $y \in Y$ by the completeness of Y. From this, define a function $\rho : \widetilde{X} \to Y$ by $\rho([s]) = y$.

We claim that the function ρ is an isometry, and so the set $\widetilde{X}$ can be regarded as a subset of Y with "embedding".

Proof. Consider any two classes $[s_1], [s_2] \in \widetilde{X}$. Suppose the corresponding Cauchy sequences s_1 and s_2 converge to y_1 and y_2 (both in Y) respectively.

Since the definitions of d_Y and d coincide on $X \times X$, we may extend the definition of $\widetilde{\delta}$ through replacing the underlying metric d by d_Y, and changing its domain to $Y \times Y$. Correspondingly, the domain of metric $\widetilde{d}$ can be extended to $\widetilde{Y} \times \widetilde{Y}$ where $\widetilde{Y}$ is the set of all equivalence classes of Cauchy sequences in Y.

Now, to show that
$$\widetilde{d}([s_1],[s_2]) = d_Y(\rho([s_1]),\rho([s_2])) = d_Y(y_1,y_2),$$
we first pick the Cauchy sequences $\{y_1,y_1,\dots\} \in [s_1]$ and $\{y_2,y_2,\dots\} \in [s_2]$.[7] Then, we have
$$\widetilde{d}([s_1],[s_2]) = \widetilde{\delta}(\{y_1,y_1,\dots\},\{y_2,y_2,\dots\}) = d_Y(y_1,y_2),$$
as desired. □

2.3 Continuous Functions

2.3.1 As we have mentioned at the very beginning, in MATH3401 we are studying continuous functions between metric spaces. We have analyzed metric spaces in Section 1. It is now time to study *continuous functions*, another central concept in MATH3401.

2.3.2 The concept of continuous functions is utilized for studying the "equivalence problem" in metric space topology, namely determining whether two metric spaces are "equivalent" in some sense. More specifically, we would like to investigate whether two metric spaces have "similar shape". We will introduce a notion called *homeomorphism*, which is related to the "equivalence" of metric spaces. Simply speaking, homeomorphism is a bijective continuous function whose inverse is also continuous.

Throughout Section 2.3, we shall use the notations (X, d_X) and (Y, d_Y) to denote arbitrary metric spaces. Let us first generalize the ε-δ definition of *limit of function* studied in MATH2241 to the context of metric spaces.

2.3.3 Let S be a subset of X, and $f : S \to Y$ be a function. Suppose that $p \in X$ is an accumulation point of S. Then, we write $\lim_{x\to p} f(x) = b$ or $f(x) \to b$ as $x \to p$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d_Y(f(x), b) < \varepsilon$$
for any $x \in S \setminus \{p\}$ with $d_X(x,p) < \delta$, or in other words,
$$f(x) \in B_Y(b,\varepsilon)$$
for any $x \in B_X(p,\delta) \cap S \setminus \{p\}$, or more compactly:
$$f\big((B_X(p,\delta) \cap S \setminus \{p\})\big) \subseteq B_Y(b,\varepsilon).$$

[Note: We require $p \in X$ to be an accumulation point of S to ensure that $B_X(p,\delta) \cap S \setminus \{p\}$ is nonempty for any $\delta > 0$ ➔ excluding "boring" cases.]

2.3.4 We can relate the notion of limit of function between metric spaces with that for real functions (studied in MATH2241) as follows.
$$\lim_{x\to p} f(x) = b \iff \lim_{x\to p} d_Y(f(x), b) = 0$$

Proof. Note that
$$\begin{aligned}
&\lim_{x\to p} f(x) = b \\
\iff &\text{for any } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that } |d_Y(f(x),b) - 0| = d_Y(f(x),b) < \varepsilon \\
&\quad \text{for any } x \in B_X(p,\delta) \cap S \setminus \{p\} \\
\iff &\lim_{x\to p} d_Y(f(x), b) = 0.
\end{aligned}$$
□

[7] We have $\widetilde{\delta}(\{y_1,y_1,\dots\}, s_1) = 0$ since $s_1 \to y_1$; similar for another sequence.

2.3.5 Like MATH2241, we have the following sequential criterion for limits of functions.

Proposition 2.3.a. Let $p \in X$ be an accumulation point of $S \subseteq X$, $b \in Y$, and $f : S \to Y$ be any function. Then, $\lim_{x \to p} f(x) = b$ iff $\lim_{n \to \infty} f(x_n) = b$ for every sequence $\{x_n\}$ in $S \setminus \{p\}$ which converges to p.

Proof. "$\Rightarrow$": Assume that $\lim_{x \to p} f(x) = b$. This means for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), b) < \varepsilon$ for any $x \in B_X(p, \delta) \cap S \setminus \{p\}$.

Next, consider any sequence $\{x_n\}$ in $S \setminus \{p\}$ which converges to p. For the positive $\delta > 0$, by convergence there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $d_X(x_n, p) < \delta$, which implies that

$$x_n \in B_X(p, \delta) \cap S \setminus \{p\},$$

thus $d_Y(f(x_n), b) < \varepsilon$. This proves $\lim_{n \to \infty} f(x_n) = b$.

"$\Leftarrow$": We prove by contrapositive. Assume that $\lim_{x \to p} f(x) \neq b$. Then there exists $\varepsilon_0 > 0$ such that for any $n \geq N$, there exists $x_n \in B(p, 1/n) \cap S \setminus \{p\}$ with $d(f(x_n), b) \geq \varepsilon_0$.

By construction we have $0 < d(x_n, p) < 1/n$ for any $n \in \mathbb{N}$. Hence, by sandwich theorem, we have $\{d(x_n, p)\} \to 0$, which means that $\{x_n\} \to p$. On the other hand, we have $d(f(x_n), b) \geq \varepsilon_0$ for any $n \in \mathbb{N}$, for some $\varepsilon_0 > 0$. This suggests that $\lim_{n \to \infty} f(x_n) \neq b$. ☐

2.3.6 Next, we will consider a result applicable when the codomain is $\mathbb{R}^n$, related to the Euclidean norm $\|\cdot\|$.

Proposition 2.3.b. Let S be a subset of X, $p \in X$ be an accumulation point of S, and $f : S \to \mathbb{R}^n$ be any function. Then,

$$\lim_{x \to p} f(x) = b \implies \lim_{x \to p} \|f(x)\| = \|b\|,$$

where $\|y\| = d_E(y, \mathbf{0})$ for any $y \in \mathbb{R}^n$ ($\mathbf{0}$ is the zero vector in $\mathbb{R}^n$ and d_E is the Euclidean metric).

Proof. Assume that $\lim_{x \to p} f(x) = b$. Then, $\lim_{x \to p} d(f(x), b) = \lim_{x \to p} \|f(x) - b\| = 0$. By reverse triangle inequality, we have

$$0 \leq \big| \|f(x)\| - \|b\| \big| \leq \|f(x) - b\|.$$

Hence, using sandwich theorem, we get

$$\lim_{x \to p} \big| \|f(x)\| - \|b\| \big| = 0,$$

implying that $\lim_{x \to p} \|f(x)\| = \|b\|$. ☐

2.3.7 Now, we discuss the concept of continuity. Like the limit of function, the definition here is also generalized from the one in MATH2241. Let S be a subset of X. A function $f : S \to Y$ is **continuous** at $p \in S$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for any $x \in S$ with $d_X(x, p) < \delta$, or more compactly, $f(B_X(p, \delta) \cap S) \subseteq B_Y(f(p), \varepsilon)$. The function f is said to be **continuous on S** if it is continuous at every $p \in S$. Sometimes we simply call f as **continuous** when f is continuous on its domain.

2.3.8 Every point $p \in S$ must be an adherent point of S, thus can be classified into (i) accumulation point of S and (ii) isolated point of S. We have the following results about continuity for (i) and (ii) respectively.

(a) If $p \in S$ is an isolated point of S, then f must be continuous at p.
Proof. Assume that $p \in S$ is an isolated point of S. Then there exists $\delta_0 > 0$ such that $B_X(p, \delta_0) \cap S = \{p\}$. Now, for any $\varepsilon > 0$, choose $\delta = \delta_0$, and then we have

$$d_Y(f(x), f(p)) = d_Y(f(p), f(p)) = 0 < \varepsilon$$

for any $x \in B_X(p, \delta) \cap S = \{p\}$. ☐

(b) If $p \in S$ is an accumulation point of S, then f is continuous at p iff $\lim_{x\to p} f(x) = f(p)$.

Proof. Assume that $p \in S$ is an accumulation point of S.

"$\Rightarrow$": For any $\varepsilon > 0$, the continuity of f suggests that there exists $\delta > 0$ such that $f(B_X(p,\delta)\cap S) \subseteq B_Y(f(p),\varepsilon)$, which implies $f(B_X(p,\delta) \cap S \setminus \{p\}) \subseteq B_Y(f(p),\varepsilon)$, thus $\lim_{x\to p} f(x) = f(p)$.

"$\Leftarrow$": For any $\varepsilon > 0$, $\lim_{x\to p} f(x) = f(p)$ suggests that there exists $\delta > 0$ such that $f(B_X(p,\delta) \cap S \setminus \{p\}) \subseteq B_Y(f(p),\varepsilon)$. Note that $f(p)$ always belongs to $B_Y(f(p),\varepsilon)$. Thus, we can conclude that

$$f(B_X(p,\delta)) \subseteq B_Y(f(p),\varepsilon),$$

meaning that f is continuous at p. $\square$

2.3.9 Without specifying whether $p \in S$ is an isolated point or accumulation point of S, we can have the following criterion for continuity based on limit of sequence.

Proposition 2.3.c. Let S be a subset of X. A function $f : S \to Y$ is continuous at $p \in S$ iff $\lim_{n\to\infty} f(x_n) = f(p)$ whenever $\{x_n\}$ is a sequence in S converging to p.

Proof. Case 1: $p \in X$ is an isolated point of X.

In this case, f is always continuous at p. On the other hand, consider any sequence $\{x_n\}$ in S with $\{x_n\} \to p$. Since p is an isolated point of X, there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, $x_n = p$ (which implies that $f(x_n) = f(p)$). Thus, $\lim_{n\to\infty} f(x_n) = f(p)$.

Case 2: $p \in X$ is an accumulation point of X.

Note that

$$\begin{aligned}
&f \text{ is continuous at } p \\
\iff& \lim_{x\to p} f(x) = f(p) && \text{([2.3.8]b)} \\
\iff& \lim_{n\to\infty} f(x_n) = f(p) \text{ for any sequence } \{x_n\} \text{ in } S\setminus\{p\} \text{ converging to } p && \text{(Proposition 2.3.a)} \\
\iff& \lim_{n\to\infty} f(x_n) = f(p) \text{ for any sequence } \{x_n\} \text{ in } S \text{ converging to } p.
\end{aligned}$$

To prove the last equivalence, consider:

- "$\Leftarrow$": Immediate.
- "$\Rightarrow$": Assume that $\lim_{n\to\infty} f(x_n) = f(p)$ for any sequence $\{x_n\}$ in $S \setminus \{p\}$ converging to p.

  Now, consider any sequence $\{x_n\}$ in S converging to p. If it only contains finitely many terms different from p, then it is immediate that $\lim_{n\to\infty} f(x_n) = f(p)$. So henceforth we suppose that it has infinitely many terms different from p.

  For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

  $$d_X(x_n, p) < \varepsilon$$

  for any $n \geq N$. After excluding all terms equal to p from the sequence $\{x_n\}$, we can get another sequence $\{y_n\}$ in $S \setminus \{p\}$. Note that $\{y_n\}$ also converges to p since it is just a subsequence of $\{x_n\}$. By assumption, we then have

  $$\lim_{n\to\infty} f(y_n) = f(p).$$

  Now, we add the excluded terms that are equal to p back to the sequence $\{y_n\}$ to reassemble $\{x_n\}$. Observe that we still have

  $$\lim_{n\to\infty} f(x_n) = f(p),$$

  by considering the definition.

$\square$

2.3.10 Next we consider composition of continuous functions. Here we generalize the corresponding result in MATH2241 to make it applicable for general metric spaces.

Proposition 2.3.d. If $f : X \to Y$ is continuous at $p \in X$ and $g : Y \to Z$ is continuous at $f(p) \in Y$, then the composition $g \circ f : X \to Z$ is continuous at p.

Proof. Fix any $\varepsilon > 0$. Due to the continuity of g, there exists $\delta_1 > 0$ such that

$$y \in B_Y(f(p), \delta_1) \implies g(y) \in B_Z(g \circ f(p), \varepsilon).$$

Using this δ_1 to serve as the role of "ε", due to the continuity of f, there exists $\delta > 0$ such that

$$x \in B_X(p, \delta) \implies f(x) \in B_Y(f(p), \delta_1).$$

Combining the two implications gives

$$x \in B_X(p, \delta) \implies g \circ f(x) = g(f(x)) \in B_Z(g \circ f(p), \varepsilon),$$

meaning that $g \circ f$ is continuous at p. □

2.3.11 Some further properties about "combining" continuous functions in different ways are as follows.

Proposition 2.3.e.

(a) If $f, g : X \to \mathbb{C}$ are continuous at p, then so are $f + g$, $f - g$, fg, and f/g. For f/g, we need to require that $g(p) \neq 0$.

(b) If $f, g : X \to \mathbb{C}^n$ are continuous at p, then so are $f + g$, λf, $f \cdot g$, and $\|f\|$, where $\lambda \in \mathbb{C}$ and "$\cdot$" denotes (pointwise) dot/inner product.

(c) If $f_i : X \to \mathbb{C}$ is continuous at p for any $i = 1, \ldots, n$, then so is the vector-valued function $f = (f_1, \ldots, f_n) : X \to \mathbb{C}^n$. The converse also holds.

Proof. Omitted. □

Remarks:

- Since the identity function from $\mathbb{C}$ to $\mathbb{C}$ is continuous, we can obtain many continuous functions based on Proposition 2.3.e.
- We consider the set $\mathbb{C}$ of complex numbers in this result since it is a "large" set equipped with some natural arithmetic operations ("+", "−", "·", "/").

2.4 Relationship Between Continuity and Topological Concepts

2.4.1 Next, we will discuss the relationships between continuity and some topological concepts discussed in Section 1. They give us some useful criteria and necessary/sufficient conditions for continuity.

2.4.2 The first one is about an important criterion of continuity using the concept of openness and closedness.

Theorem 2.4.a.

(a) A function $f : X \to Y$ is continuous on X iff the preimage $f^{-1}(T)$ of any open set $T \subseteq Y$ is open in X.

(b) A function $f : X \to Y$ is continuous on X iff the preimage $f^{-1}(T)$ of any closed set $T \subseteq Y$ is closed in X.

[Note: The two instances of "open" ("closed") carry different meanings. One of them is "in Y", while another is "in X".]

Proof.

(a) "$\Rightarrow$": Assume that f is continuous on X and consider any open set $T \subseteq Y$. For any $x_0 \in f^{-1}(T)$, we have $f(x_0) = y_0 \in T$. By the openness of T, there exists $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq T$.
Since f is continuous at x_0, there exists $\delta > 0$ such that

$$f(B_X(x_0, \delta)) \subseteq B_Y(f(y_0), \varepsilon),$$

which implies that

$$B_X(x_0, \delta) \subseteq f^{-1}(B_Y(y_0, \varepsilon)) \subseteq f^{-1}(T),$$

hence $f^{-1}(T)$ is open in X.
"$\Leftarrow$": Assume that the preimage $f^{-1}(T)$ of any open set $T \subseteq Y$ is open in X. Fix any $\varepsilon > 0$. Then we want to find $\delta > 0$ such that

$$f(B_X(p, \delta)) \subseteq B_Y(f(p), \varepsilon).$$

Since the open ball $B_Y(f(p), \varepsilon)$ is open in Y, by assumption, the preimage $f^{-1}(B_Y(f(p), \varepsilon))$ is open in X. Now, since p lies in the preimage $f^{-1}(B_Y(f(p), \varepsilon))$, by openness there exists $\delta > 0$ such that

$$B(p, \delta) \subseteq f^{-1}(B_Y(f(p), \varepsilon)),$$

which implies that

$$f(B(p, \delta)) \subseteq B(f(p), \varepsilon),$$

as desired.

(b) It follows from (a) and the fact that $f^{-1}(Y \setminus T) = X \setminus f^{-1}(T)$:

$$\begin{aligned} & f \text{ is continuous on } X \\ \iff & f^{-1}(S) \text{ is open in } X \text{ for any open } S \subseteq Y && \text{(a)} \\ \iff & f^{-1}(Y \setminus T) \text{ is open in } X \text{ for any closed } T \subseteq Y \\ \iff & X \setminus f^{-1}(T) \text{ is open in } X \text{ for any closed } T \subseteq Y \\ \iff & f^{-1}(T) \text{ is closed in } X \text{ for any closed } T \subseteq Y. \end{aligned}$$

□

[⚠ Warning: The criterion here is about *preimage*, but not *image*. In general, the image of an open (closed) set under a continuous function may not be open (closed).]

2.4.3 A function that maps open sets to open sets is said to be *open*; a function that maps closed sets to closed sets is said to be *closed*. More explicitly, a function $f : X \to Y$ is **open** (**closed**) if the image $f(S)$ is open (closed) in Y for any open (closed) subset S of X.

Open mappings may not be closed and closed mappings may not be open in general.

2.4.4 Counterexamples about open/closed/continuous mappings: [Note: The metric spaces involved below are all equipped with Euclidean metrics (with suitable domains).]

(a) *Continuous but not open mapping:* the constant zero function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for any $x \in \mathbb{R}$.
Proof. It is clearly continuous but the open set $(0, 1) \subseteq \mathbb{R}$ is mapped to $\{0\}$ which is *not* open in $\mathbb{R}$. □

(b) *Continuous but not closed mapping:* the identity function $f : X = (0, 1) \to \mathbb{R}$ defined by $f(x) = x$ for any $x \in (0, 1)$.
Proof. It is clearly continuous. But, $(0, 1)$ is *closed* in $X = (0, 1)$, while $f((0, 1)) = (0, 1)$ is *not closed* in $\mathbb{R}$. □

(c) *Continuous mapping that is neither open nor closed:* function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2/(1 + x^2)$ for any $x \in \mathbb{R}$.
Proof. It is clearly continuous. The set $\mathbb{R}$ which is both open and closed in $\mathbb{R}$, but $f(\mathbb{R}) = [0, 1)$ which is *neither* open *nor* closed in $\mathbb{R}$. So it is neither open nor closed mapping. □

(d) *Open but not continuous mapping:* function $f : \mathbb{R} \to \{0, 1\}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ 1 & \text{if } x < 0. \end{cases}$$

Proof. It is clearly an open mapping since $\varnothing$, $\{0\}$, $\{1\}$, and $\{0, 1\}$ are all open in $\{0, 1\}$. But it is not continuous at 0. $\square$

(e) *Closed but not continuous mapping:* function $f : \mathbb{R} \to \{0, 1\}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ 1 & \text{if } x < 0. \end{cases}$$

Proof. It is clearly a closed mapping since $\varnothing$, $\{0\}$, $\{1\}$, and $\{0, 1\}$ are all closed in $\{0, 1\}$. But it is not continuous at 0. $\square$

(f) *Bijective open but not continuous mapping:* function $f : [0, \infty) \to \{-1\} \cup (0, \infty)$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x = 0, \\ x & \text{if } x > 0. \end{cases}$$

Proof. It is clearly bijective. To show that it is an open mapping, consider the following. Fix any open subset S of $[0, \infty)$.

<u>Case 1</u>: $0 \in S$.

Then, the image $f(S) = \{-1\} \cup (S \setminus \{0\})$. Note that $\{-1\}$ and $S \setminus \{0\}$ are both open in $\{-1\} \cup (0, \infty)$, so do their union.

<u>Case 2</u>: $0 \notin S$.

Then the image $f(S) = S \subseteq (0, \infty)$, and we can see that S is open in $\{-1\} \cup (0, \infty)$.

However, we can clearly see that f is not continuous at 0. $\square$

(g) *Open but not closed mapping:* function $f : X = (0, 1) \to \mathbb{R}$ defined by $f(x) = x$ for any $x \in (0, 1)$.

Proof. It is an open mapping since every every set $S \subseteq (0, 1)$ which is open in $(0, 1)$ is also open in $\mathbb{R}$. On the other hand, $(0, 1)$ is closed in $(0, 1)$ but $f((0, 1)) = (0, 1)$ is not closed in $\mathbb{R}$. Thus it is not a closed mapping. $\square$

(h) *Closed but not open mapping:* the constant zero function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for any $x \in \mathbb{R}$.

Proof. It is a closed mapping since $\varnothing$ and $\{0\}$ are both closed in $\mathbb{R}$. However, the open set $(0, 1) \subseteq \mathbb{R}$ is mapped to $\{0\}$ which is not open in $\mathbb{R}$. Thus it is not an open mapping. $\square$

2.4.5 After that, we are going to discuss some results/counterexamples about *preservation* of various kinds of points discussed in Section 1 under a continuous function $f : X \to Y$.

(a) *not preserving interior points:* Take $X = Y = \mathbb{R}$, where X and Y are equipped with discrete and Euclidean metrics respectively. Define f by $f(x) = x$ and take $S = [0, 1]$.

Note that $[0, 1]^\circ = [0, 1]$ under discrete metric while $[0, 1]^\circ = (0, 1)$ under Euclidean metric. Thus,

$$f(S^\circ) = f([0, 1]) = [0, 1] \not\subseteq (0, 1) = f(S)^\circ.$$

(b) *preserving adherent points:* Fix any $S \subseteq X$. Since $S \subseteq f^{-1}(f(S)) \subseteq f^{-1}(\overline{f(S)})$, we have

$$\overline{S} \subseteq \overline{f^{-1}(\overline{f(S)})} = f^{-1}(\overline{f(S)}).$$

The last equality holds since $\overline{f(S)}$ is closed in Y and thus the preimage $f^{-1}(\overline{f(S)})$ is closed in X. Hence, we have

$$f(\overline{S}) \subseteq f(f^{-1}(\overline{f(S)})) \subseteq \overline{f(S)}.$$

(c) i. *preserving accumulation points under injectivity:* Consider any $S \subseteq X$ and fix any $y \in f(S')$. Then $y = f(x)$ for some $x \in S'$. As $x \in S'$, there is an infinite sequence of distinct points $\{s_n\}$ in S such that $\{s_n\} \to x$.
Due to the injectivity of f, $\{f(s_n)\}$ is still an infinite sequence of distinct points in $f(S)$. Also, since f is continuous, $\{f(s_n)\} \to f(x)$. Thus $y \in f(x) \in f(S)'$. This means $f(S') \subseteq f(S)'$.

ii. *not preserving accumulation points in general:* Take $X = \mathbb{R}$ and $Y = \{0\}$ (both with Euclidean metric). Define f by $f(x) = 0$ and take $S = [0, 1]$. Then,

$$f(S') = f([0,1]) = \{0\} \not\subseteq \varnothing = \{0\}' = f(S)'$$

(d) *not preserving isolated points:* Take $X = Y = \mathbb{R}$, where X and Y are equipped with discrete and Euclidean metrics respectively (same setting as the one for interior points). Define f by $f(x) = x$ and take $S = X = \mathbb{R}$.
Note that $\mathbb{R}' = \varnothing$ under discrete metric while $\mathbb{R}' = \mathbb{R}$ under Euclidean metric. Thus,

$$f(S \setminus S') = f(\mathbb{R} \setminus \varnothing) = f(\mathbb{R}) = \mathbb{R} \not\subseteq \varnothing = \mathbb{R} \setminus \mathbb{R} = f(S) \setminus f(S)'.$$

(e) i. *preserving boundary points under injectivity:* Consider any $S \subseteq X$ and fix any $y \in f(\partial S)$. Then $y = f(x)$ for some $x \in \partial S$. As $x \in \partial S = \overline{S} \cap \overline{X \setminus S}$, there are sequences $\{s_n\}$ in S and $\{t_n\}$ in $X \setminus S$ such that $\{s_n\} \to x$ and $\{t_n\} \to x$.
Note that $\{f(s_n)\}$ is in $f(S)$. Also, due to the injectivity of f, $\{f(t_n)\}$ is in $Y \setminus f(S)$. Since f is continuous, $\{f(s_n)\} \to f(x) = y$ and $\{f(t_n)\} \to f(x) = y$. Hence $y \in \overline{f(S)} \cap \overline{Y \setminus f(S)} = \partial f(S)$. This means $f(\partial S) \subseteq \partial f(S)$.

ii. *not preserving boundary points in general:* Take $X = \mathbb{R}$ and $Y = \{0\}$. Define f by $f(x) = 0$ and take $S = [0, 1]$ (same as the one for accumulation points). Then,

$$f(\partial S) = f(\{0, 1\}) = \{0\} \not\subseteq \varnothing = \partial\{0\} = \partial f(S).$$

To summarize, we have the following for continuous functions:

type of points	preserve in general?	preserve assuming injectivity?
interior	✗	✗
adherent	✓	✓
accumulation	✗	✓
isolated	✗	✗
boundary	✗	✓

2.4.6 A remarkable property of continuous function is that it preserves *compactness*.

Theorem 2.4.b. Let $f : X \to Y$ be a continuous function and $C \subseteq X$ be a compact set. Then, the image $f(C)$ is compact in Y.

Proof. Take any open cover $\{U_\lambda : \lambda \in \Lambda\}$ of $f(C)$:

$$\bigcup_{\lambda \in \Lambda} U_\lambda \supseteq f(C).$$

This implies that

$$f^{-1}\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) \supseteq C.$$

Due to the continuity of f, the preimage $f^{-1}(U_\lambda)$ is open in X for any $\lambda \in \Lambda$. Hence, $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ serves as an open cover of C. By the compactness of C, there exists a finite sub-cover $\{f^{-1}(U_i) : i = 1, \dots, n\}$ of C:

$$f^{-1}\left(\bigcup_{i=1}^{n} U_i\right) = \bigcup_{i=1}^{n} f^{-1}(U_i) \supseteq C,$$

which implies that

$$\bigcup_{i=1}^{n} U_i \supseteq f(C),$$

so we have found a finite sub-cover of $f(C)$: $\{U_i : i = 1, \ldots, n\}$. □

2.4.7 Using Theorem 2.4.b, we can generalize the *boundedness theorem* in MATH2241 as follows.

Corollary 2.4.c. If $f : X \to Y$ is continuous and $C \subseteq X$ is compact, then f is bounded on C, i.e., the image $f(C)$ is bounded.

Proof. By Theorem 2.4.b, we know that $f(C)$ is compact, which implies that it is bounded by Theorem 1.8.a. □

2.4.8 Next, we generalize the *extreme value theorem* in MATH2241.

Theorem 2.4.d. If $f : X \to \mathbb{R}$ is continuous on a compact set $C \subseteq X$, then f attains its maximum and minimum in C, i.e., there exist $p, q \in C$ such that

$$f(p) = \max f(C) \quad \text{and} \quad f(q) = \min f(C)$$

Proof. By Theorem 2.4.b, the image $f(C)$ is compact in $\mathbb{R}$. Thus, it is also closed and bounded in $\mathbb{R}$ by Theorem 1.8.a. By the completeness axiom of $\mathbb{R}$, the supremum $M = \sup f(C)$ exists in $\mathbb{R}$. Now fix any $r > 0$. Firstly, since M is the supremum of $f(C)$, there exists $y \in f(C)$ such that $M - r < y$. Next, as M is an upper bound of $f(C)$, we have $y \leq M < M + r$. It follows that $B(M, r) \cap f(C) = (M - r, M + r) \cap f(C) \supseteq \{y\} \neq \varnothing$. This means that M is an adherent point of $f(C)$. Hence, we have $M \in \overline{f(C)} = f(C)$, where the equality holds due to the closedness of $f(C)$. This means that $\max f(C) = M = f(p)$ for some $p \in C$. Similarly, we can show that $f(q) = \min f(C)$ for some $q \in C$.

□

2.4.9 Now we consider the continuity of inverse function. The following result suggests a sufficient condition for the inverse of a continuous function to be also continuous.

Proposition 2.4.e. Let $f : X \to Y$ be a function. Suppose that X is compact, and f is both bijective and continuous. Then, the inverse $f^{-1} : Y \to X$ is also continuous.

Proof. First of all, due to the bijectivity of f, the inverse f^{-1} exists. Now fix any closed set $C \subseteq X$. By the compactness of X and Theorem 1.8.c, C is a compact set. Hence, by the continuity of f and Theorem 2.4.b, $f(C)$ is also compact, thus closed in Y.

Note that the preimage of C under f^{-1} is

$$(f^{-1})^{-1}(C) = \{y \in Y : f^{-1}(y) \in C\} = \{y \in Y : f(f^{-1}(y)) \in f(C)\} = \{y \in Y : y \in f(C)\} = f(C).$$

So, the preimage $(f^{-1})^{-1}(C)$ is closed in Y. By Theorem 2.4.a, we conclude that f^{-1} is continuous. □

2.4.10 For Proposition 2.4.e to hold, it is important that X is compact. Without the compactness, we can construct a counterexample as follows. Let $X = Y = [0, 1]$. Equip X with the discrete metric and Y with the standard Euclidean metric. Note that X is *not* compact since $\mathcal{F} = \{\{x\} : x \in X\}$ is an open cover of X without any finite subcover. Here, since the discrete metric is used, $\{x\}$ is an open set in X.

Now consider the function $f : X \to Y$ defined by $f(x) = x$ for any $x \in X$. It is both bijective and continuous. Its inverse is $f^{-1} : Y \to X$, defined by $f^{-1}(y) = y$ for any $y \in Y$. Note that f^{-1} is *not* continuous. To see this, take $\{0\} \subseteq X$. It is open in X under the discrete metric, but the preimage $f^{-1}(\{0\}) = \{0\}$ is *not* open in Y under the Euclidean metric.

2.5 Homeomorphisms

2.5.1 Now, we introduce an important notion that has already been mentioned at the beginning of Section 2.3: *homeomorphism*. A bijective continuous function $f : X \to Y$ whose inverse f^{-1} is also continuous is called a **homeomorphism** or a **topological mapping** of X onto Y. Two metric spaces X and Y are said to be **homeomorphic**, denoted by $X \cong Y$, if there is a homeomorphism of X onto Y.

[Note: $\cong$ is an equivalence relation. To see this, note that:

(a) The identity function from X to X is a homeomorphism.

(b) If $f : X \to Y$ is a homeomorphism, then so is $f^{-1} : Y \to X$.

(c) If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then so is the composition $g \circ f : X \to Z$.

]

2.5.2 The idea of homeomorphic metric spaces can be intuitively understood as follows. Let $f : X \to Y$ be a continuous function, and imagine that X and Y are elastic geometric objects. Intuitively, the continuous function f "moves" the points in X such that the points "close together" originally are still "close together" in Y: there is not "abrupt change". In other words, there can be bending, twisting, stretching, dilating, contracting, etc. on X, but there cannot be *tearing* on X, since it can cause points that were originally close together to become far apart. In short, f "continuously deforms" the object X to the object Y. So, two metric spaces are homeomorphic if one can be "continuously deformed" to another, and back.

2.5.3 In the following, we will prove a result that gives several criteria for homeomorphism. Before proving that, we consider the following lemma.

Lemma 2.5.a. Let $f : X \to Y$ be a bijective function. Then the following are equivalent.

(a) f is continuous.

(b) f^{-1} is open.

(c) f^{-1} is closed.

Proof. For clarity, here we denote the preimage and image of a set S under a function g by $g^{\leftarrow}(S)$ and $g^{\rightarrow}(S)$ respectively.

Note that the preimage of T under f is

$$\begin{aligned} f^{\leftarrow}(T) &= \{x \in X : f(x) \in T\} \\ &= \{x \in X : f^{-1}(f(x)) \in f^{-1\rightarrow}(T)\} \\ &= \{x \in X : x \in f^{-1\rightarrow}(T)\} \\ &= f^{-1\rightarrow}(T), \end{aligned}$$

the image of T under f^{-1}.

Then, by Theorem 2.4.a, we know that:

- f is continuous iff $f^{-1\rightarrow}(T) = f^{\leftarrow}(T)$ is open in X for any open set $T \subseteq Y$, i.e., f^{-1} is open.
- f is continuous iff $f^{-1\rightarrow}(T) = f^{\leftarrow}(T)$ is closed in X for any closed set $T \subseteq Y$, i.e., f^{-1} is closed.

Hence the result follows. □

Theorem 2.5.b. Let $f : X \to Y$ be a bijective function. Then the following are equivalent.

(a) f is a homeomorphism, i.e., f and f^{-1} are both continuous.

(b) f is continuous and open.

(c) f is continuous and closed.

Proof. Note that f^{-1} is bijective. By Lemma 2.5.a, the following are equivalent:

- f^{-1} is continuous.
- $f = (f^{-1})^{-1}$ is open.
- $f = (f^{-1})^{-1}$ is closed.

Then the result follows. □

2.5.4 A property of a set which remains invariant under homeomorphisms/topological mappings is called a **topological property** or a **topological invariant**.

Examples of topological properties:

- *openness/closedness:* By Theorem 2.5.b, a homeomorphism is a mapping that is both open and closed, thus preserving both openness and closedness.
- *compactness:* By Theorem 2.4.b, compactness is preserved by continuous function, hence also by homeomorphism.

Non-example of topological property:

- *distance between two points:* Take $X = (0, 1)$ and $Y = (0, 2)$, both equipped with the standard Euclidean metric d. Then, the function $f : X \to Y$ defined by $f(x) = 2x$ can be shown to be a homeomorphism. However, we have

$$d(0.25, 0.75) = |0.75 - 0.25| \neq |1.5 - 0.5| = d(0.5, 1.5) = d(f(0.25), f(0.75)).$$

2.5.5 We see that a homeomorphism may not preserve the distance. In view of this, we define a notion related to the preservation of distance. A function $f : X \to Y$ is called an **isometry**[8] if it preserves the metric or distance, i.e.,

$$d_Y(f(p), f(q)) = d_X(p, q)$$

for any $p, q \in X$, where d_X and d_Y denote the metrics for X and Y respectively. Two metric spaces X and Y are said to be **isometric** if there is an isometry of X onto Y.

2.5.6 The concepts of isometry and homeomorphism are related by the following result.

Proposition 2.5.c. Let $f : X \to Y$ be a function. If f is a surjective isometry, then it is a homeomorphism.

Proof. Firstly, we prove the injectivity of f. Suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Then, we have

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = 0,$$

which implies that $x_1 = x_2$, establishing the injectivity. Together with the surjectivity of f, we know that f is bijective, thus f^{-1} exists.

Next, we will prove that f is continuous. Fix any $x \in X$ and any $\varepsilon > 0$. Consider any $z \in B_X(x, \varepsilon)$. Then, we have

$$d_Y(f(z), f(x)) = d_X(z, x) < \varepsilon,$$

thus $f(z) \in B_Y(f(x), \varepsilon)$. Since z is arbitrary, we have

$$f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon),$$

by choosing $\delta = \varepsilon$. This means that f is continuous.

Finally, we will show that f^{-1} is continuous. Note that

$$d_Y(y_1, y_2) = d_Y(f(f^{-1}(y_1)), f(f^{-1}(y_2))) = d_X(f^{-1}(y_1), f^{-1}(y_2))$$

[8] *iso:* "equal"; *metry:* "measure"

for any $y_1, y_2 \in Y$, so $f^{-1} : Y \to X$ is also an isometry. Thus, by replacing "f" by "f^{-1}" in the argument above, we have, for any $y \in Y$ and any $\varepsilon > 0$,

$$f^{-1}(B_Y(y, \delta)) \subseteq B_X(f^{-1}(y), \varepsilon)$$

where $\delta = \varepsilon$, establishing the continuity of f^{-1}. $\square$

3 Connectedness

3.1 Connectedness

3.1.1 We have generalized the *extreme value theorem* in Theorem 2.4.d. Then the next natural thing to do is to generalize the *intermediate value theorem* we learn in MATH2241. It turns out that the generalization uses the notion of *connectedness*.

3.1.2 It is somehow complicated to directly define the concept of connectedness. So, we shall first define the notion of <u>dis</u>connectedness, and then define connectedness as the state of being *not* disconnected.

3.1.3 A metric space X is said to be **disconnected** if $X = A \sqcup B$ for some nonempty and disjoint sets A and B which are open in X. A metric space X is said to be **connected** if it is not disconnected. A subset S of X is said to be **connected** (in X) if it is connected when considered as a metric space itself under the induced metric.

Example: $X = [0,1] \cup [2,3]$ (equipped with standard Euclidean metric) is disconnected, by noting that $[0,1]$ and $[2,3]$ are nonempty disjoint open subsets of X whose union is X.

3.1.4 The following gives several criteria for being disconnected.

Proposition 3.1.a. Let X be a metric space. The following are equivalent.

(a) X is disconnected, i.e., $X = A \sqcup B$ for some nonempty and disjoint sets A and B which are *open* in X.

(b) $X = A \sqcup B$ for some nonempty and disjoint sets A and B which are *closed* in X.

(c) There exists a proper nonempty closed and open subset of X.

Proof. <u>(a) $\implies$ (b)</u>: Assume that $X = A \sqcup B$ for some nonempty and disjoint sets A and B which are *open* in X. Note that as complements of each other, $B = X \setminus A$ and $A = X \setminus B$ are also *closed* in X.

<u>(b) $\implies$ (c)</u>: Assume that $X = A \sqcup B$ for some nonempty and disjoint sets A and B which are *closed* in X. Then $A = X \setminus B$ is open in X. Since B is nonempty, $A = X \setminus B$ is a *proper* subset of X. It then follows that A is a proper nonempty closed and open subset of X.

<u>(c) $\implies$ (a)</u>: Assume that there exists a proper nonempty closed and open subset of X. Define $B = X \setminus A$. Since A is a *proper* subset of X, the set B is nonempty. Furthermore, since A is closed in X, $B = X \setminus A$ is open in X. Also, we can see that $A \cap B = \varnothing$ and $X = A \sqcup B$. □

3.1.5 To prove that a metric space is <u>dis</u>connected, we can just find an example of A and B satisfying the specified conditions. On the other hand, it is much more difficult to prove that a metric space is connected, since we need to show that there *do not exist* any such sets A and B. We need to consider all possible choices!

To simplify the arguments, the use of a continuous function which takes only two possible values is very helpful.

3.1.6 Any continuous function from X to $\{0,1\}$ equipped with discrete metric is called a **2-valued function** on X. The following result suggests how 2-valued function can help us to prove connectedness.

Theorem 3.1.b. A metric space X is connected iff the only possible 2-valued functions on X are constant functions.

Proof. "$\Leftarrow$": We prove by contrapositive. Assume that X is disconnected. Then we can write $X = A \sqcup B$ for some nonempty and disjoint sets A and B which are open in X. Then define a function $f : X \to \{0,1\}$, where $\{0,1\}$ is equipped with discrete metric, by

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

Note that every subset of $\{0,1\}$ is open. Then we can check that $f^{-1}(\varnothing) = \varnothing$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = B$, and $f^{-1}(\{0,1\}) = A \sqcup B = X$ are all open in X. Thus, f is continuous, hence is a non-constant 2-valued function.

"$\Rightarrow$": Assume that X is connected. Let $f : X \to \{0,1\}$ be a 2-valued function. Assume to the contrary that f is non-constant. Then the preimages $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are both nonempty. Also, due to the continuity of f, the preimages are both open in X. Note also that $f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = \varnothing$. Thus, we can write

$$X = f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}),$$

which implies that X is disconnected, contradiction. $\square$

3.1.7 Like compactness, *connectedness* is also preserved by continuous functions by the following result.

Proposition 3.1.c. Let $f : X \to Y$ be a continuous function and $S \subseteq X$ be a connected set in X. Then, the image $f(S)$ is connected in Y.

Proof. Let $g : f(S) \to \{0,1\}$ be any 2-valued function on $f(S)$. Then, due to the continuity of both f and g, the composition $g \circ f : X \to \{0,1\}$ is also continuous, so is the restriction $g \circ f|_S : S \to \{0,1\}$. But then it just means that $g \circ f|_S : S \to \{0,1\}$ is a 2-valued function on S.

Due to the connectedness of S, $g \circ f|_S$ must be a constant function. It then forces g to be a constant function as well, hence $f(S)$ is connected. $\square$

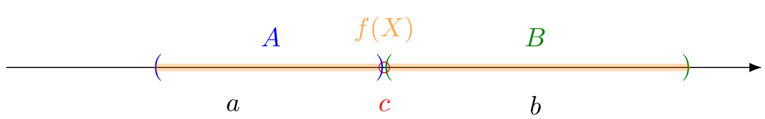
Particularly, this implies that connectedness is a topological property, i.e., preserved by homeomorphisms.

3.1.8 We can then generalize the intermediate value theorem we learn in MATH2241.

Theorem 3.1.d (Intermediate value theorem)**.** Let $f : X \to \mathbb{R}$ be a continuous function on a connected metric space X. If f takes on two different values $a, b \in X$, then f takes on every real number $c \in [a,b]$, i.e., for any $c \in [a,b]$, there exists $x \in X$ such that $f(x) = c$.

Proof. Since f is continuous and X is connected, by Proposition 3.1.c we know that $f(X)$ is connected in $\mathbb{R}$. Now assume to the contrary that there exist $c \in [a,b]$ such that $f(x) \neq c$ for any $x \in X$. Indeed, we have $c \in (a,b)$ since f takes on a and b by assumption.

Then, define $A = (-\infty, c) \cap f(X)$ and $B = (c, \infty) \cap f(X)$.



Note that we have $f(X) = A \sqcup B$ where A and B are nonempty disjoint open subsets of $f(X)$, thus $f(X)$ is disconnected, contradiction. $\square$

3.1.9 The next result concerns the connectedness of union of connected sets. It turns out that, similar to openness, arbitrary union of connected sets is connected, *provided that* the connected sets involved have nonempty intersection.

Proposition 3.1.e. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a collection of connected subsets of X with $\bigcap_{\alpha\in\Lambda} U_\alpha \neq \varnothing$. Then, the union $\bigcup_{\alpha\in\Lambda} U_\alpha$ is connected.

Proof. Fix any $t \in \bigcap_{\alpha\in\Lambda} U_\alpha$ and let $f : \bigcup_{\alpha\in\Lambda} \to \{0,1\}$ be a 2-valued function on $\bigcup_{\alpha\in\Lambda}$. Then for any $\beta \in \Lambda$, the restriction $f|_{U_\beta} : U_\beta \to \{0,1\}$ is a 2-valued function on U_β. Since U_β is connected, $f|_{U_\beta}$ must be a constant function by Theorem 3.1.b. Particularly, we must have $f(x) = f(t)$ for any $x \in U_\beta$, since t is always an element of U_β.

Since this holds for any $\beta \in \Lambda$, we conclude that $f(x) = f(t)$ for any $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$, thus f is a constant function. So, by Theorem 3.1.b, $\bigcup_{\alpha\in\Lambda} U_\alpha$ is connected. $\square$

3.1.10 Recall from Theorem 1.5.h that every nonempty open subset of $\mathbb{R}$ can be expressed as the union of countably many disjoint open intervals in $\mathbb{R}$ uniquely. Using the notion of connectedness, we can obtain a corresponding result in $\mathbb{R}^n$. In the case for proving Theorem 1.5.h, we have utilized the notion of *maximal open interval*. Here, we need an analogous notion. A maximal connected set in a metric space is called a **connected component** of the metric space.

Here, maximality of a connected set S in a metric space X means that for any connected set T in X with $T \supseteq S$, we have $T = S$. There is not a connected set in X that is a *proper superset* of S. In particular, this implies a connected component must be nonempty, since any singleton is a connected set in X, which is a proper superset of an empty set.

3.1.11 Like the case for $\mathbb{R}$, such maximal connected set is indeed a certain union. To be more precise, we have the following result.

Proposition 3.1.f. Let X be a metric space, $x \in X$, and U_x be the union of all connected sets in X that contain x. Then, U_x is the unique connected component of X that contains x.

Proof. First we show that U_x is a connected component of X. Note that U_x is nonempty since at least $\{x\}$ is a connected set containing x. Thus, by Proposition 3.1.e, U_x is connected. Since every connected set in X that contains x must be a subset of the union U_x, the union U_x is indeed a maximal connected set in X, thus a connected component of X.

Next we prove the uniqueness. Suppose that V_x is also a connected component of X that contains x. Then in particular, V_x is a connected set in X that contains x, thus $V_x \subseteq U_x$. But then by the maximality of V_x, we must have $U_x = V_x$, establishing the uniqueness. $\square$

Consider any connected component U of X and pick any $x \in U$. By Proposition 3.1.f, we know that the union of all connected sets in X that contain x, i.e., U_x, is the unique connected component of X that contains x. This then means that U is just given by this unique connected component U_x.

3.1.12 Observe that connected components of X are either identical or disjoint.

Proof. The proof is similar to the one for the corresponding property in Lemma 1.5.g. Here again we will prove that given any two connected components U_x and U_y, if $U_x \cap U_y \neq \varnothing$, then $U_x = U_y$. Here the notations U_x and U_y carry the meaning suggested in Proposition 3.1.f.

Firstly, since U_x and U_y are connected with nonempty intersection, $U_x \cup U_y$ is connected also, and contains x. By the maximality of U_x and U_y respectively, we have $U_x \cup U_y \subseteq U_x$ and $U_x \cup U_y \subseteq U_y$. This implies that $U_x \cup U_y = U_x$ and $U_x \cup U_y = U_y$ as another subset inclusion always holds. Thus, $U_x = U_y$. $\square$

3.1.13 By going through all elements in any subset S of X and applying Proposition 3.1.f for each of them, we can obtain a unique collection of disjoint connected components of S whose union is S, which contains all distinct connected components of S. In other words, S can always be uniquely decomposed into a disjoint union of connected components of S.

3.1.14 Next we will introduce some properties of connected components. Before that, we consider the following lemmas.

Lemma 3.1.g. If $S \subseteq X$ is connected, then every $T \subseteq X$ with $S \subseteq T \subseteq \overline{S}$ is connected.

Proof. Let $f : T \to \{0, 1\}$ be any 2-valued function on T. Then the restriction of f on $S \subseteq T$, $f|_S$, is a 2-valued function on S. Since S is connected, $f|_S$ is constant by Theorem 3.1.b. WLOG, suppose that $f|_S \equiv 0$. For any $x \in T \subseteq \overline{S}$, there exists a sequence $\{x_n\}$ in S such that $\{x_n\} \to x$, by Proposition 2.1.c. Then by the continuity of f, we have $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$, by Proposition 2.3.c. $\square$

Lemma 3.1.h. A set $I \subseteq \mathbb{R}$ is an interval iff it is connected.

Proof. "$\Leftarrow$": We prove by contrapositive. Assume that I is not an interval. Then, there exists $c \in (a,b)$ with $a, b \in I$ and $c \notin I$. Define $A = (-\infty, c) \cap I$ and $B = (c, \infty) \cap I$, which are nonempty disjoint open subsets of I. By writing $I = A \sqcup B$, we know that I is disconnected.

"$\Rightarrow$": Let $I \subseteq \mathbb{R}$ be an interval. If I is a singleton, there is nothing to prove. So henceforth assume that I is a nondegenerate interval, i.e., one that has more than one element.

Let $f : I \to \{0, 1\}$ be any 2-valued function on I. Assume to the contrary that there are distinct points $a, b \in I$ such that $f(a) \neq f(b)$. WLOG, assume $a < b$, $f(a) = 0$ and $f(b) = 1$. Since I is an interval, it contains every value between a and b also, i.e., $[a, b] \subseteq I$.

Let $t = \inf\{x \in [a, b] : f(x) = 1\}$. Then, $t \in (a, b] \subseteq I$ and $f(x) = 0$ for any $x < t$ in I. By the continuity of f, we have $f(t) = \lim_{x \to t^-} f(x) = 0$.

On the other hand, since infimum is a lower bound, for any $n \in \mathbb{N}$, there exists $x_n \in [t + t + \frac{1}{n}) \cap I$ such that $f(x_n) = 1$. Observe that $\{x_n\}$ is a sequence in I converging to t, by sandwich theorem. Thus, by Proposition 2.3.c, we have $f(t) = \lim_{n \to \infty} f(x_n) = 1$, contradiction. □

3.1.15 With the help of Lemmas 3.1.g and 3.1.h, we can prove the following result about some properties of connected components.

Proposition 3.1.i. Let X be a metric space.

(a) Every connected component of X is closed in X.

(b) Connected components of X may not be open in X.

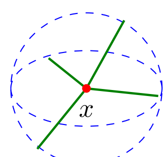
(c) Every connected component of an open subset $S \subseteq \mathbb{R}^n$ (as a metric subspace) is open in $\mathbb{R}^n$.

Proof.

(a) Let S be any connected component of X. By Lemma 3.1.g, $\overline{S} \supseteq S$ is also connected. By the maximality of S, we then have $\overline{S} = S$, meaning that S is closed in X by Proposition 1.6.e.

(b) Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Since the only connected set containing 0 is $\{0\}$, the singleton $\{0\}$ is a connected component of X. But it is not open in X.

(c) Let $T \subseteq S$ be any connected component of S. Then for any $x \in T \subseteq S$, due to the openness of S, there exists $r > 0$ such that the open ball $B_{\mathbb{R}^n}(x, r) = B(x, r) \subseteq S \subseteq \mathbb{R}^n$.



open ball $B(x, r)$

Note that every radius of the open ball $B(x, r)$ (green line segments in the picture above) is homeomorphic to the interval $[0, r) \subseteq \mathbb{R}$, which is connected by Lemma 3.1.h. Since connectedness is a topological property, every radius is connected. Furthermore, the intersection of all the radii is $\{x\}$, thus nonempty. Noting that the open ball $B(x, r)$ is indeed the union of all the radii (with nonempty intersection), we conclude by Proposition 3.1.e that the open ball $B(x, r)$ is connected.

Since T contains x, it must be the union of all connected sets containing x by Proposition 3.1.f. Hence, we must have $B(x, r) \subseteq T$. Thus, T is open in $\mathbb{R}^n$.

□

3.1.16 To close Section 3.1, we will prove a result analogous to Theorem 1.5.h but for $\mathbb{R}^n$ below.

Theorem 3.1.j. Every nonempty open subset S of $\mathbb{R}^n$ can be uniquely expressed as a *countable* disjoint union of nonempty open connected sets in $\mathbb{R}^n$. Furthermore, these open connected sets are all the connected components of S.

Proof. From [3.1.13], we know that S can be expressed as a disjoint union of the connected components of S. So to prove the existence of such expression, it suffices to prove that (i) each connected component is open in $\mathbb{R}^n$, and (ii) the disjoint union is countable. Since $S \subseteq \mathbb{R}^n$, (i) follows from Proposition 3.1.i.

Next, due to Lindelöf's theorem (Theorem 1.5.f), we can assume WLOG that such collection of connected components is countable, proving (ii).

It then remains to prove the uniqueness. Suppose we can write $S = \sqcup_{n=1}^{\infty} C_n$ where each C_n is a nonempty open connected subset of S. Consider any C_n and pick any $x \in C_n$. Due to the connectedness of C_n, we know that $C_n \subseteq U_x$ where U_x denotes the connected component containing x, i.e., the union of all connected sets containing x. Note that we can write $C_n = S \setminus \sqcup_{m \neq n} C_m$ where $\sqcup_{m \neq n} C_m$ is open in S (as a union of open sets), thus C_n is closed in S as well.

Since $C_n \subseteq U_x$, this then implies that $C_n = C_n \cap U_x$ is both closed and open in U_x, by [1.5.5]. We claim that $C_n = U_x$. If this was not the case, i.e., C_n is a proper subset of U_x, then C_n would be a proper nonempty closed and open subset of U_x, which implies that U_x is disconnected by Proposition 3.1.a, contradiction. From this, we can deduce that the collection $\{C_n\}$ is indeed just the collection of all the connected components of S. This proves the uniqueness. □

3.2 Path-connectedness

3.2.1 In Section 3.2, we will discuss another notion of "connectedness", which is called path-connectedness. Recall that we have defined connectedness as a state of being not disconnected. This is somewhat indirect and may be unintuitive. In contrast, the concept of path-connectedness is more direct and carries a clear geometric intuition. It turns out that path-connectedness implies connectedness, so the former provides us an "intuitive" way to prove connectedness.

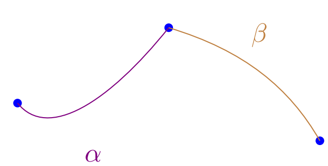
3.2.2 Before defining path-connectedness, we first define what a path is. A **path** in a metric space X is a continuous function $\gamma : [0,1] \to X$. For convenience, sometimes we call the image of γ ($\gamma([0,1])$) as path also.

In the latter sense of path, every path in X would be a connected set in X. This is because $[0,1]$ is connected (as an interval; see Lemma 3.1.h) and continuous function preserves connectedness (Proposition 3.1.c).

3.2.3 Two paths can be "joined" to form another path. If $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$ are paths with $\alpha(0) = p$, $\alpha(1) = \beta(0) = q$ ("ending point" of α = "starting point" of β), and $\beta(1) = r$, then the function $\gamma : [0,1] \to X$ defined by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le 1/2, \\ \beta(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

is a path in X joining p to r through α and β.



This is called the **product** of the paths α and β, denoted by $\beta \cdot \alpha$. [⚠ Warning: This does not denote the usual product of two functions!]

3.2.4 A metric space X is **path-connected** if any two points p, q in X can be joined by a path in X, i.e., there exists a path γ in X such that $\gamma(0) = p$ and $\gamma(1) = q$. We can also define path-connectedness for a subset S of X, by considering it as a metric subspace.

3.2.5 We first prove that path-connectedness implies connectedness.

Theorem 3.2.a. Every path-connected set X is connected.

Proof. Consider any 2-valued function $f : X \to \{0, 1\}$ and a specific point $a \in X$.

Fix any $x \in X$. By the path-connectedness, there exists a path γ in X such that $\gamma(0) = a$ and $\gamma(1) = x$. As the image $\gamma([0, 1])$ (also referred as path) is connected, the restriction $f|_{\gamma([0,1])}$ is constant, thus $f(x) = f(a) =$ constant. Hence f is a constant function. $\square$

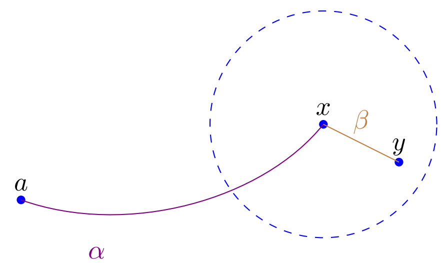
3.2.6 How about another implication? It does not hold in general, but it does hold true under the following special case.

Theorem 3.2.b. Every open connected set S in $\mathbb{R}^n$ is path-connected.

Proof. Fix $a \in S$. Let $A = \{x \in S : x \text{ and } a \text{ are joined by a path in } S\}$ and $B = S \setminus A$. Note that $A \neq \varnothing$ as $a \in A$. By construction, A is path-connected. Also, we observe that $A \sqcup B = S$.

It then suffices to show that A and B are both open in S, which forces $B = \varnothing$ (thus $S = A$ is path-connected) by the connectedness of S.

For any $x \in A \subseteq S$, there is a path α in S joining a to x. By the openness of S, there exists $r_x > 0$ such that $B_{\mathbb{R}^n}(x, r_x) \subseteq S$. It is not hard to show that the open ball $B_{\mathbb{R}^n}(x, r_x)$ is path-connected (by considering the line segment joining two points). So for any $y \in B_{\mathbb{R}^n}(x, r_x)$, there is a path β in S joining x to y.

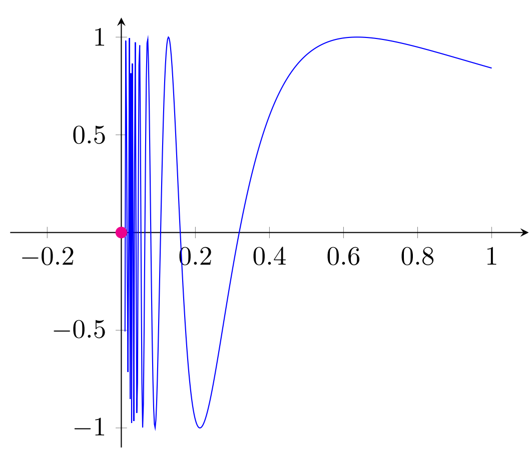


Then the product $\beta \cdot \alpha$ is a path in S joining a to y, thus $y \in A$. As $y \in B_{\mathbb{R}^n}(x, r_x)$ is arbitrary, we conclude that $B_{\mathbb{R}^n}(x, r_x) \subseteq A$, thus A is open in $\mathbb{R}^n$. Since $A = A \cap S$, by [1.5.5], A is also open in S.

Next, for any $z \in B \subseteq S$, there exists $r_z > 0$ such that $B_{\mathbb{R}^n}(z, r_z) \subseteq S$. Fix any $w \in B_{\mathbb{R}^n}(z, r_z) \subseteq S$, and there is a path γ in S joining w to z.

If we had $w \in S \setminus B = A$, then there would be a path δ in S joining a to w, hence the product $\gamma \cdot \delta$ would be a path in S joining a to z. This means $z \in A$, contradiction. Thus we must have $w \in B$. As $w \in B_{\mathbb{R}^n}(z, r_z)$ is arbitrary, we conclude that $B_{\mathbb{R}^n}(z, r_z) \subseteq B$, and thus, similarly, B is open in S. $\square$

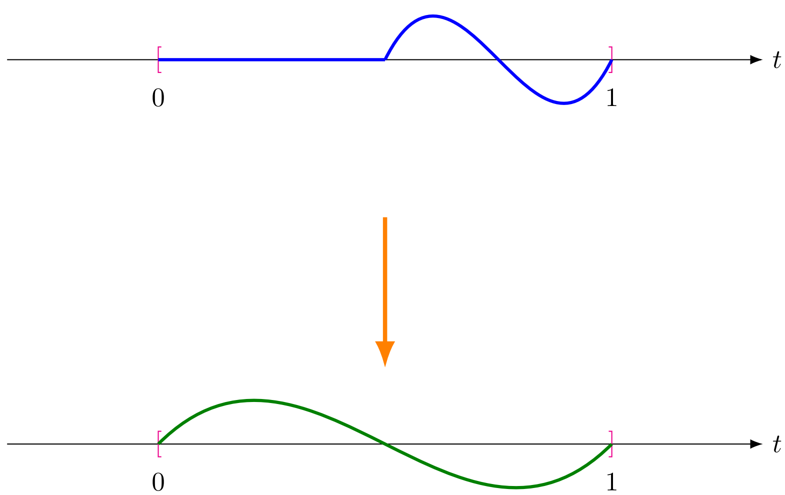
3.2.7 To close Section 3.2, we present an example which is connected but not path-connected. Define $A = \left\{(x, \sin \frac{1}{x}) \in \mathbb{R}^2 : x \in (0, 1]\right\}$, $B = \{(0, 0)\}$, and $X = A \cup B$. [Note: X is known as the **topologist's sine curve**.]



Then, X is connected but not path-connected.

Proof. Connected: Since $(0,1]$ is connected and the function $g(x) = (x, \sin 1/x)$ is continuous (as each component is continuous), A is connected. Now consider any 2-valued function f on X. The restriction $f|_A \equiv C$ for some constant C, due to the connectedness of A. Since $(0,0) \in \overline{A}$ and f is continuous, $f((0,0)) = C$ also (by Propositions 2.1.c and 2.3.c), implying that f is constant.

Not path-connected: Assume to the contrary that X is path-connected. Then there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = (0,0)$ and $\gamma(1) = (1/\pi, 0)$. Note that the path can "stay" at $(0,0)$ for some time, but will eventually "leave" it since $\gamma(1) \neq (0,0)$. So, by removing the period of "staying" at $(0,0)$ and scaling the remaining part accordingly, we can assume WLOG that $\inf\{t \in [0,1] : \gamma(t) \neq (0,0)\} = 0$, i.e., the path "immediately" leaves $(0,0)$ after the "initial time point".

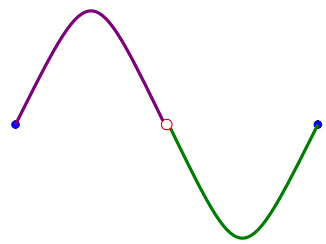


By the continuity of γ, there exists $\delta_0 > 0$ such that for any $t < \delta_0$, $\|\gamma(t) - (0,0)\| < \frac{1}{2}$.

By the assumption about the infimum, there exists a sequence $\{t_n\} \to 0$ such that $\gamma(t_n) \neq (0,0)$ for any $n \in \mathbb{N}$. Then, choose a $0 < t_{n_1} < \delta_0$ and choose $N_1 \in \mathbb{N}$ such that $N_1\pi < \frac{1}{t_{n_1}} < (N_1+1)\pi$. Next, choose $0 < t_{n_2} < \dfrac{1}{4N_1\pi} < \dfrac{1}{(N_1+1)\pi} < t_{n_1} < \delta_0$ such that $\gamma(t_{n_2}) \neq (0,0)$.

Note that these choices of t_{n_1} and t_{n_2} are sufficiently far apart that the path γ reaches at least one "peak" between t_{n_1} and t_{n_2}, i.e., there exists $s \in (t_{n_1}, t_{n_2})$ such that $\gamma(s) = (x, 1)$ for some $x \in (0,1]$.

Note that $\gamma(t_{n_1}), \gamma(t_{n_2}) \in \gamma([0,\delta_0])$. Since $\gamma([0,\delta_0])$ is connected, we conclude that every point p "between" $\gamma(t_{n_1})$ and $\gamma(t_{n_2})$, i.e., p with $p = \gamma(u)$ where $u \in (t_{n_1}, t_{n_2})$, belongs to $\gamma([0,\delta_0])$ as well, for otherwise, one can construct two nonempty disjoint open subsets of $\gamma([0,\delta_0])$ whose union is $\gamma([0,\delta_0])$, through splitting based on the "missing point".



Particularly, it means that $(x,1)=\gamma(s)\in\gamma([0,\delta_0])$. But $\|\gamma(s)-(0,0)\|=\|(x,1)-(0,0)\|>1>1/2$, contradiction. $\square$

4 Uniform Continuity and Uniform Convergence

4.0.1 In the rest of this notes, we will mainly investigate *sequences of functions*. Of course they can be studied using the notions introduced in Section 2, but here we will also discuss some extra concepts that are exclusive to them to make the investigation more fruitful.

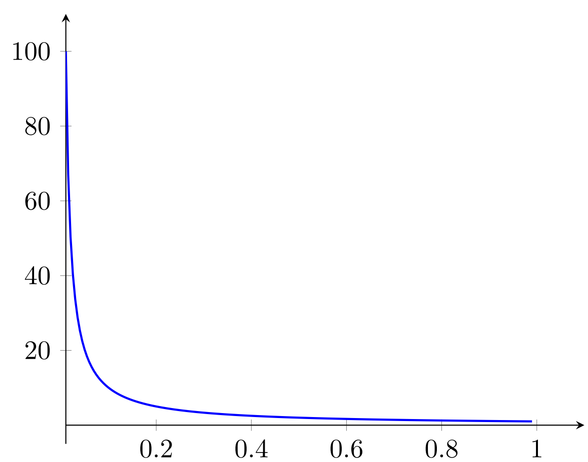
We shall start with the concept of uniform continuity.

4.1 Uniform Continuity

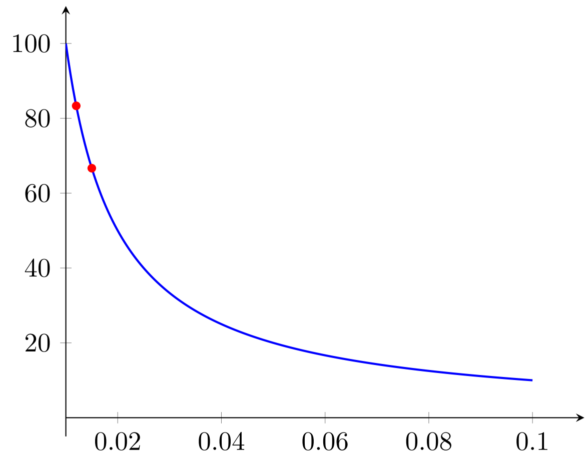
4.1.1 As suggested by the name "uniform continuity", one may naturally expect that it should have some relationship with the continuity we study in Section 2.3. Recall that a function $f : S \to Y$ is continuous at $p \in S$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ for any $x \in S$ with $d_X(x, p) < \delta$. Note that the δ here possibly depends on both ε and p, so explicitly we can write $\delta = \delta(\varepsilon, p)$. When a function is continuous on S, it means that with the flexibility of adjusting the "δ" for different points, we are able to choose a suitable $\delta = \delta(\varepsilon, p)$ for every point p.

On the other hand, for the function to be *uniformly continuous* on S, we need to choose a suitable $\delta = \delta(\varepsilon)$ that works for every point p, <u>without</u> the flexibility of adjusting the "δ" for different points. The choice needs to be in an "uniform manner" and be independent from the point p in question.

4.1.2 For example, consider the function $f : X \to \mathbb{R}$ defined by $f(x) = 1/x$ where $X = (0, 1)$.



By Proposition 2.3.e, we know that f is continuous on X. But is it *uniformly continuous* on X? It turns out that this is not the case. Intuitively, the reason is that regardless of what the "uniform" $\delta = \delta(\varepsilon)$ is chosen to be, we are able to find two inputs $x_1, x_2 \in X$ that are located at the very left and are sufficiently close together such that $|x_1 - x_2| < \delta$ but $|f(x_1), f(x_2)| = |1/x_1 - 1/x_2|$ is large, due to the rapid spike of the function when the input is near 0.



4.1.3 To formalize the intuitive reason, we first define uniform continuity. A function $f : X \to Y$ is **uniformly continuous** on a subset S of X if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_Y(f(x_1, x_2)) < \varepsilon$$

for any $x_1, x_2 \in S$ with $d_X(x_1, x_2) < \delta$, or more compactly,

$$f(B_X(x, \delta) \cap S) \subseteq B_Y(f(x), \varepsilon)$$

for every $x \in S$.

From the definition, we can observe that uniform continuity on S implies continuity on S, because the "uniform" $\delta = \delta(\varepsilon)$ here can already serve as a suitable $\delta = \delta(\varepsilon, p)$ for satisfying the condition of being continuous at every point $p \in S$.

4.1.4 Going back to the example in [4.1.2], we can formalize the argument as follows. First set $\varepsilon = 1/2$, and fix any $\delta > 0$. Note that there exists a sufficiently large $N \in \mathbb{N}$ such that $\frac{1}{2N} < \delta$. We then choose $x_1 = \frac{1}{N}$ and $x_2 = \frac{1}{2N}$. Note that $|x_1 - x_2| = \frac{1}{2N} < \delta$, but $|f(x_1) - f(x_2)| = N > \varepsilon$. This shows no "uniform" δ exists, and thus f cannot possibly be uniformly continuous on X.

4.1.5 From the example in [4.1.2], we know that continuity on S does not imply uniform continuity on S. However, with an extra assumption about the *compactness*, we can actually show this implication, as suggested by the Heine-Cantor theorem.

Theorem 4.1.a (Heine-Cantor theorem)**.** If $f : X \to Y$ is continuous on a compact subset C of X, then f is uniformly continuous on C.

Proof. Fix any $\varepsilon > 0$. By continuity of f on C, for any $c \in C$, there exists $\delta_c > 0$ such that $d_Y(f(x), f(c)) < \varepsilon/2$ for any $x \in B_X(c, \delta_c) \cap C$.

Note that $\{B_X(c, \delta_c/2\}_{c \in C}$ is an open cover of C, so by compactness of C, there exists a finite subcover $\{B_X(c_i, \delta_{c_i}/2\}_{i=1}^n \supseteq C$.

Let $\delta = \min\{\delta_1/2, \ldots, \delta_n/2\} > 0$. Fix any $x, y \in C$ with $d(x, y) < \delta$. Then $x \in B(c_j, \delta_{c_j}/2) \cap C$ for some $j = 1, \ldots, n$, thus $d_X(y, c_j) \leq d_X(y, x) + d_X(x, c_j) < \delta + \delta_{c_j}/2 \leq \delta_{c_j}$.

This means $x, y \in B_X(c_j, \delta_{c_j}) \cap C$, thus by the continuity of f,

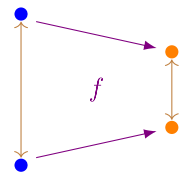
$$d(f(x), f(y)) \leq d(f(x), f(c_j)) + d(f(c_j), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

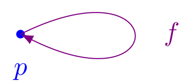
4.1.6 Next, we will consider a special kind of uniform continuous function: *contraction*. Let $f : X \to X$ be a function (same metric d is equipped to both the domain and codomain). The function f is a **contraction** of X if there exists $\alpha < 1$ (called the **contraction constant**) such that

$$d(f(x), f(y)) \le \alpha d(x, y)$$

for any $x, y \in X$.



A contraction is a distance-decreasing map and brings the points "closer". It turns out that it is important for studying the concept of *fixed point*, which is in turn helpful for dealing with the existence and uniqueness problem of solutions to various differential and integral equations (see Theorem 4.2.g). Here, a point $p \in X$ is called a **fixed point** of f if $f(p) = p$.



To see that a contraction is uniformly continuous, first fix any $\varepsilon > 0$. By choosing $\delta = \varepsilon/\alpha$, we have

$$d(f(x), f(y)) \le \alpha d(x, y) < \alpha\delta < \varepsilon$$

for any $x, y \in S$ with $d(x, y) < \delta$.

4.1.7 A remarkable result concerning fixed point is the *Banach's fixed point theorem*.

Theorem 4.1.b (Banach's fixed point theorem)**.** Every contraction of a complete metric space has a unique fixed point.

Proof. Let $f : X \to X$ be a contraction. Fix any $a \in X$. Define a sequence $\{p_n\}$ in X by iteratively applying f: $p_0 = a$ and $p_{n+1} = f(p_n)$ for any $n \in \mathbb{N}_0$. (More explicitly, we have $p_n = \overbrace{(f \circ \cdots \circ f)}^{n \text{ times}}(a)$ for any $n \in \mathbb{N}$.)

We claim that $\{p_n\}$ is Cauchy. To see this, note that for any $n \in \mathbb{N}$, due to the contraction property we have $d(p_{n+1}, p_n) = d(f(p_n), f(p_{n-1})) \le \alpha d(p_n, p_{n-1})$ for some $\alpha < 1$. Then by induction, for any $n \in \mathbb{N}$ we have $d(p_{n+1}, p_n) \le c\alpha^n$ where the constant c is $d(p_1, p_0)$. So, when we fix any $\varepsilon > 0$, by choosing sufficiently large $N \in \mathbb{N}$ such that $c\alpha^N/(1-\alpha) < \varepsilon$, we have for any $n > m \ge N$,

$$\begin{aligned}
d(p_m, p_n) &= d(p_m, p_{m+1}) + d(p_{m+1}, p_{m+2}) + \cdots + d(p_{n-1}, p_n) \\
&\le c(\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) \\
&< c\alpha^m/(1-\alpha) \\
&\le c\alpha^N/(1-\alpha) \\
&< \varepsilon.
\end{aligned}$$

Then, by the completeness of the metric space, $\{p_n\}$ converges to some point $p \in X$. By the continuity of the contraction f, we have

$$f(p) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} p_{n+1} = p,$$

thus p is a fixed point. This establishes the existence part.

For uniqueness, suppose $p, q \in X$ are fixed points of f. Then,

$$d(p,q) = d(f(p), f(q)) \leq \alpha d(p,q).$$

As $\alpha < 1$, this forces $d(p,q) = 0$, or $p = q$. $\square$

4.2 Uniform Convergence of Sequences of Functions

4.2.1 After discussing uniform continuity and some related results, we now change our focus to *uniform convergence*, which is about the notion of convergence for a sequence of functions. Unless otherwise specified, throughout we restrict our attention to real-valued functions defined on a nonempty subset S of a metric space X.

4.2.2 It turns out that there are two kinds of convergence for a sequence of functions: (i) *pointwise* convergence and (ii) *uniform* convergence. The former is about convergence in the values taken by the functions. To be more precise, let $\{f_n\}$ be a sequence of functions on S and f be a function on S. If $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in S$, then the sequence $\{f_n\}$ **converges pointwisely** to f on S, written as "$\{f_n\} \to f$ pointwisely on S". If $\{f_n\} \to f$ pointwisely on S for some function f on S, we say that $\{f_n\}$ **converges pointwisely**/is **pointwisely convergent**.

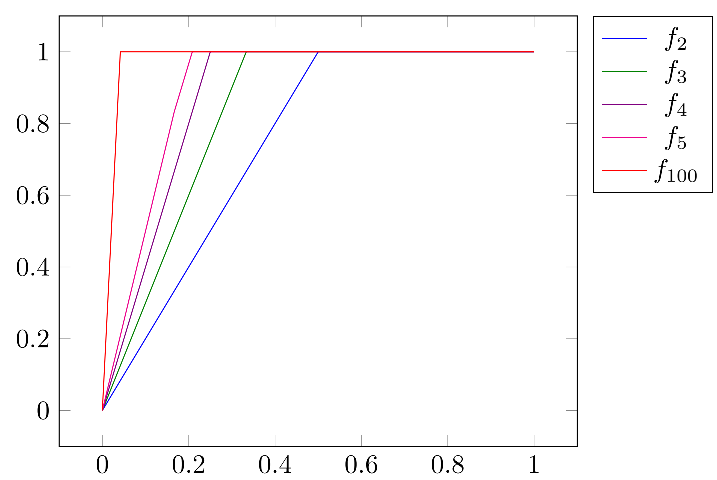
The pointwise convergence holds as long as, for every $x \in S$, the sequence $\{f_n(x)\}$ "eventually" converges to $f(x)$. It does not tell us how the *speeds* of convergence for different x compare. It is possible that the convergence happens "very quickly" for some x, while "very slowly" for some other x.

4.2.3 Uniform convergence concerns also the convergence speeds for different x. Intuitively, it requires that the pointwise convergence for different x should take place at a similar "speed". Convergence occurs at a "uniform" rate. To illustrate this, consider the following sequence of functions that converges pointwisely but not uniformly.

Let $S = [0,1] \subseteq \mathbb{R}$ and define $f_n : S \to \mathbb{R}$ by

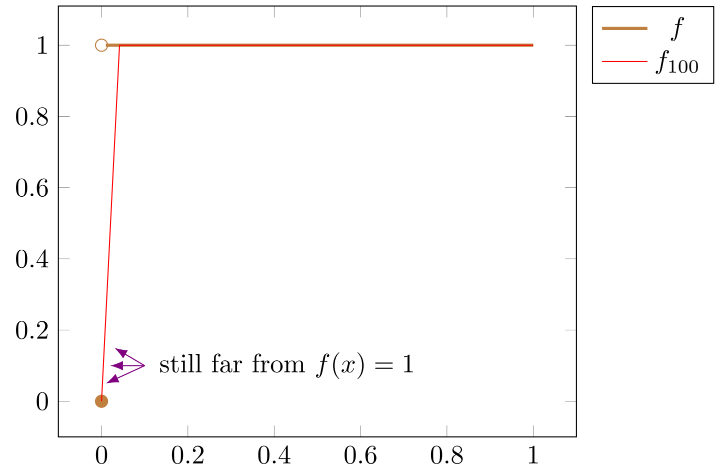
$$f_n(x) = \begin{cases} nx & \text{if } x \in [0, 1/n], \\ 1 & \text{if } x \in [1/n, 1], \end{cases}$$

for any $n \in \mathbb{N}$.



It can be shown that

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



So, $\{f_n\}$ converges pointwisely to f on S. However, the convergence turns out to be not uniform. For example, take $N = 100$ and consider $f_N = f_{100}$. For any $x \in [1/100, 1]$, we have $f_{100}(x) = f(x)$, so for these x the pointwise convergence has already been "completed". On the other hand, for any $0 < x < 1/100$, $f_{100}(x)$ still has a certain distance from the limit $f(x) = 1$, and particularly, $f_{100}(x)$ is still very close to 0 for very small x. In this sense, the convergence is not "uniform". Note that similar phenomenon occurs for other N also.

4.2.4 Let us now define the notion of uniform convergence. A sequence of functions $\{f_n\}$ on S **converges uniformly** on S to a function f defined on S, written as "$\{f_n\} \to f$ uniformly on S", if for any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon$$

for any $n \geq N$ and any $x \in S$. If $\{f_n\} \to f$ uniformly on S for some function f on S, we say that $\{f_n\}$ **converges pointwisely**/is **uniformly convergent**.

This means that for any point $x \in S$, $\{f_n(x)\} \to f(x)$ at a "uniform" rate. More specifically, we can always find a sufficiently large N that "works" for every $x \in S$, i.e., from the Nth function onward $(f_N, f_{N+1}, \ldots)$, the value taken at every $x \in S$ is very close to the corresponding limit $f(x)$ (within a threshold ε). There is no point $x \in S$ at which it is still "far from convergence".

It is not hard to see that uniform convergence implies pointwise convergence (to the same limit), just like how uniform continuity implies continuity.

4.2.5 Here we introduce several criteria for uniform convergence.

Proposition 4.2.a. Let $\{f_n\}$ be a sequence of functions on S. Then the following are equivalent.

(a) $\{f_n\} \to f$ uniformly on S.

(b) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup\{|f_n(x) - f(x)| : x \in S\} < \varepsilon$$

for any $n \geq N$.

(c) $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in S\} = 0$.

Proof. (b) $\iff$ (c): It follows from the definition of limit of a sequence of real numbers, since $\{\sup\{|f_n(x) - f(x)| : x \in S\}\}_{n=1}^{\infty}$ is just a sequence of real numbers.

(a) $\implies$ (b): Suppose $\{f_n\} \to f$ uniformly on S. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon/2$$

for any $n \geq N$ and any $x \in S$. Thus, $\sup\{|f_n(x) - f(x)| : x \in S\} \leq \varepsilon/2 < \varepsilon$ for any $n \geq N$.

(b) $\implies$ (a): For any $\varepsilon > 0$, we have $|f_n(x) - f(x)| \leq \sup\{|f_n(x) - f(x)| : x \in S\} < \varepsilon$ for any $n \geq N$ and any $x \in S$. The uniform convergence then follows. $\square$

4.2.6 Intuitively, if a sequence of functions converges uniformly, the behaviour of the functions can be "controlled" in a uniform manner, and there would not be abrupt changes in the behaviour in some "parts" of the input. Consequently, some common properties can be "passed" to the limiting function.

The first property that can be "passed" is continuity at a point.

Proposition 4.2.b. Suppose that $\{f_n\} \to f$ uniformly on S, and f_n is continuous at $c \in S$ for every $n \in \mathbb{N}$. Then f is also continuous at c.

Proof. Fix any $\varepsilon > 0$, from the uniform convergence, there exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \varepsilon/3$ for any $x \in S$.

By the continuity of f_N at c, there exists $\delta = \delta(\varepsilon, c) > 0$ such that $|f_N(x) - f_N(c)| < \varepsilon/3$ for any $x \in S$ with $d(x, c) < \delta$.

Thus, for any $x \in S$ with $d(x, c) < \delta$, we have

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

[Note: Note that $\{f_n\} \to f$ uniformly on S implies that $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in S$. Hence, symbolically, we can write

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{n\to\infty} f_n(c) = f(c) = \lim_{x\to c} f(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x),$$

which means that "$\lim_{n\to\infty}$" commutes with "$\lim_{x\to c}$" under uniform convergence.]

4.2.7 Next, we show that uniform continuity can be "passed" to the limiting function as well.

Proposition 4.2.c. Suppose that $\{f_n\} \to f$ uniformly on S, and f_n is uniformly continuous on S for every $n \in \mathbb{N}$. Then f is also uniformly continuous on S.

Proof. (Similar to the proof for Proposition 4.2.b) Fix any $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \varepsilon/3$ for any $x \in S$.

By the uniform continuity of f_N, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f_N(x) - f_N(y)| < \varepsilon/3$ for any $x, y \in S$ with $d(x, y) < \delta$.

Hence, for any $x, y \in S$ with $d(x, y) < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

4.2.8 In Section 2.1, we have introduced the concept of convergence in a metric space. By considering a metric space of functions equipped with some metric, we can also talk about convergence for a sequence in functions. How is the uniform convergence related to the convergence in this metric space sense? The following theorem suggests the relationship.

Theorem 4.2.d. Let $\{f_n\}$ be a sequence in the metric space $(C_{\text{bd}}(S), d)$ where $C_{\text{bd}}(S)$ denotes the set of all bounded continuous real-valued functions on S and

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in S\}$$

(the L^∞ norm introduced in [1.2.5]). Then, $\{f_n\} \to f$ uniformly on X iff $\{d(f_n, f)\} \to 0$, i.e., $\{f_n\} \to f$ in the metric space $(C_{\text{bd}}(S), d)$.

Proof. It follows directly from Proposition 4.2.a. □

Remarks:

- In view of this result, the metric d here is sometimes called the **uniform metric**.
- Under the special case where S is compact, we have $C_{\text{bd}}(S) = C(S)$, since all continuous real-valued functions on S must be bounded by Corollary 2.4.c.

4.2.9 Recall from Section 2.2 that convergence implies Cauchyness, and the converse holds only when the underlying metric space is complete. But it turns out that for a sequence of functions on S, uniform convergence is *equivalent* to "uniform Cauchyness", regardless of the completeness of the underlying metric space.

A sequence of functions $\{f_n\}$ is **uniformly Cauchy** on S if for any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon$$

for any $n, m \geq N$ and any $x \in S$.

The following theorem establishes the equivalence between uniform convergence and uniform Cauchyness.

Theorem 4.2.e (Cauchy's criterion for uniform convergence of sequences of functions)**.** Let $\{f_n\}$ be a sequence of functions on S. Then, $\{f_n\} \to f$ uniformly on S for some function f iff $\{f_n\}$ is uniformly Cauchy on S.

Proof. "$\Rightarrow$": Assume $\{f_n\} \to f$ uniformly on S. Fix any $\varepsilon > 0$, and there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon/2$ for any $n \geq N$ and any $x \in S$. By triangle inequality, for any $m, n \geq N$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

"$\Leftarrow$": Under the uniform Cauchyness, for any $x \in S$, $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$, thus converges to $f(x) \triangleq \lim_{m\to\infty} f_m(x)$.

Fix any $\varepsilon > 0$. Due to the uniform Cauchyness, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon/2$ for any $m, n \geq N$ and $x \in S$. Therefore, for any $n \geq N$, we have

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon,$$

establishing the uniform convergence. □

[Note: Although we do not require completeness for the metric space X, of which S is a subset for this result, it is important that the functions are real-valued so that the codomain is $\mathbb{R}$, which is complete.]

4.2.10 As a corollary, we can show that $(C_{\text{bd}}(S), d)$, where d is the uniform metric, is a complete metric space.

Corollary 4.2.f. Let $(C_{\text{bd}}(S), d)$ be the metric space as defined in Theorem 4.2.d. Then $(C_{\text{bd}}(S), d)$ is complete.

Proof. Fix any Cauchy sequence $\{f_n\}$ in $C_{\text{bd}}(S)$. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

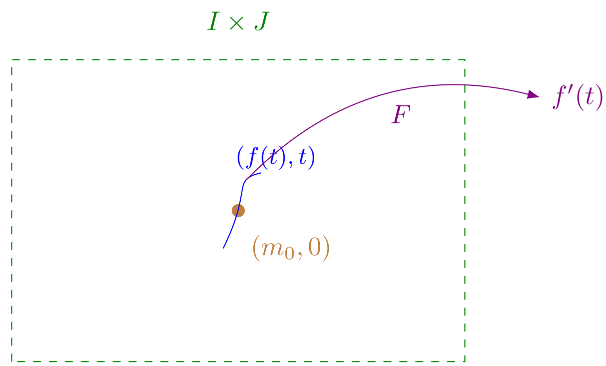
$$|f_n(x) - f_m(x)| \leq d(f_n, f_m) < \varepsilon$$

for any $m, n \geq N$ and $x \in S$. This shows $\{f_n\}$ is uniformly Cauchy on S, thus by Theorem 4.2.e, $\{f_n\} \to f$ uniformly for some function f. By Proposition 4.2.b, f is a continuous function on S. Next, due to the uniform convergence, there exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < 1$ for any $x \in S$. Since f_N is bounded, $|f_N(x)| \leq M$ for some $M > 0$. Hence by triangle inequality, $|f(x)| = |f_N(x)| + |f(x) - f_N(x)| < M + 1$ for any $x \in S$, meaning that f is bounded. Thus, $f \in C_{\text{bd}}(S)$.

By Theorem 4.2.d, we know that $\{d(f_n, f)\} \to 0$, thus $\{f_n\} \to f \in C_{\text{bd}}(S)$ by [2.1.3]. This shows $\{f_n\}$ is convergent in $C_{\text{bd}}(S)$. □

4.2.11 Before going further into the discussion about uniform convergence, we shall apply the concepts we learn here to prove a result concerning the existence and uniqueness of solutions to ODEs, which illustrates a "practical" application of the abstract concepts here.

Theorem 4.2.g (Picard-Lindelöf theorem)**.** Let I and J be open intervals in $\mathbb{R}$ containing a constant $m_0 \in \mathbb{R}$ and 0 respectively. Let $F : I \times J \to \mathbb{R}$ be a continuously differentiable[9] function. Then there exists $\delta > 0$ such that, over $(-\delta, \delta)$, there is a unique function f such that $f(0) = m_0$ and $f'(t) = F(f(t), t)$.[10]



Proof. First note that

$$\begin{aligned}
& f(0) = m_0 \quad \text{and} \quad f'(s) = F(f(s), s) \\
\iff\ & f(t) = m_0 + \int_0^t F(f(s), s)\, \mathrm{d}s \\
\iff\ & f \text{ is a fixed point of } \Lambda \text{ where } \Lambda(g) = m_0 + \int_0^t F(g(s), s)\, \mathrm{d}s\,.
\end{aligned}$$

We then reduced the problem to finding a unique fixed point, for which the *Banach's fixed point theorem* would be helpful.

Up to passing I and J to subintervals such that $I \times J$ still contains $(m_0, 0)$, we can assume that $M = \sup_{(x,y)\in I\times J} |F(x, y)| < \infty$ and $M' = \sup_{(x,y)\in I\times J} |\partial_1 F| < \infty$ (where $\partial_1 F$ denotes the partial derivative of F with respect to its first argument). Choose $\delta = \min\left\{\frac{1}{M}, \frac{1}{2M'}\right\}$ (for reasons that will become transparent very soon).

Consider the metric space $C([-\delta, \delta])$ (the set of all real-valued continuous functions on $[-\delta, \delta]$) with the uniform metric d, and consider the closed ball $\overline{B}(m_0, 1) \subseteq C([-\delta, \delta])$ where "m_0" here denotes the constant function on $[-\delta, \delta]$ that always takes the value of m_0. We define Λ on the closed ball $\overline{B}(m_0, 1)$.

For any $g \in \overline{B}(m_0, 1) \subseteq C([-\delta, \delta])$ and any $t \in [-\delta, \delta]$,

$$|\Lambda(g)(t) - m_0| = \left| \int_0^t F(g(s), s)\, \mathrm{d}s \right| \leq |t| M \leq \delta M \leq 1.$$

This shows $\Lambda(g) \in \overline{B}(m_0, 1)$, so we can define Λ as a function from $\overline{B}(m_0, 1)$ to $\overline{B}(m_0, 1)$.

Next, we want to show that Λ is a contraction of $\overline{B}(m_0, 1)$. Consider

$$\begin{aligned}
d(\Lambda(g), \Lambda(h)) &= \sup_{t\in[-\delta,\delta]} |\Lambda(g)(t) - \Lambda(h)(t)| \\
&= \sup_{t\in[-\delta,\delta]} \left| \int_0^t [F(g(s), s) - F(h(s), s)]\, \mathrm{d}s \right| \\
&\leq \sup_{t\in[-\delta,\delta]} \int_0^t |F(g(s), s) - F(h(s), s)|\, \mathrm{d}s\,.
\end{aligned}$$

[9] It means that (i) F is differentiable and (ii) the two partial derivatives of F are continuous.

[10] This means f "locally solves" the initial value problem specified here near 0, uniquely.

For the integrand, applying mean value theorem on the first argument of F (with the second argument fixed), there exists $v \in (g(s), h(s))$ such that $\partial_1 F(v, s)(g(s) - h(s)) \le M'[g(s) - h(s)]$.

Thus, we can write

$$\begin{aligned}
\sup_{t\in[-\delta,\delta]} \int_0^t |F(g(s), s) - F(h(s), s)| \,\mathrm{d}s &\le \sup_{t\in[-\delta,\delta]} \int_0^t M' \underbrace{|g(s) - h(s)|}_{d(g,h)} \,\mathrm{d}s \\
&\le \sup_{t\in[-\delta,\delta]} \underbrace{M'\delta d(g, h)}_{\text{free of } t} \\
&= M'\delta d(g, h) \\
&\le \frac{1}{2} d(g, h),
\end{aligned}$$

so Λ is a contraction of $\overline{B}(m_0, 1)$.

As $[-\delta, \delta]$ is compact in $\mathbb{R}$, every continuous real-valued function on $[-\delta, \delta]$ is bounded, by Corollary 2.4.c. Thus $C([-\delta, \delta]) = C_{\mathrm{bd}}([-\delta, \delta])$. Then by Corollary 4.2.f, $(C([-\delta, \delta]), d)$ is complete. Since $\overline{B}(m_0, 1)$ is a closed subset of $C([-\delta, \delta])$, it follows by Proposition 2.2.e that the metric subspace $(\overline{B}(m_0, 1), d)$ (with the induced uniform metric d) is complete also.

It then follows by Banach's fixed point theorem (Theorem 4.1.b) that Λ has a unique fixed point, namely the function f. □

4.3 Uniform Convergence of Series of Functions

4.3.1 Now we go back to the discussion about uniform convergence. After considering *sequences* of functions, we consider *series* of functions. A familiar example of series of functions is the exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} f_k(x),$$

where $f_k(x) = x^k/k!$ for any $k \in \mathbb{N}_0$. Each term in the infinite sum can be regarded as a function in x. How to "make sense of" this kind of infinite sum?

4.3.2 Like how we handle series of real numbers in MATH2241, we start with *partial sums*. For every $k \in \mathbb{N}$, let $f_k : S \to \mathbb{R}$ be a function. For any $n \in \mathbb{N}$, define the **nth partial sum** of the series $\sum_{k=1}^{\infty} f_k$ as the function $s_n : S \to \mathbb{R}$ given by

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \quad \text{for any } x \in S.$$

4.3.3 After that, we can define *uniform convergence* for series, utilizing the concept of uniform convergence of sequences of functions. The infinite series $\sum_{k=1}^{\infty} f_k$ **converges uniformly** on S if there exists a function $f : S \to \mathbb{R}$ such that $\{s_n\} \to f$ uniformly on S. In this case, we write "$\sum_{k=1}^{\infty} f_k(x) = f(x)$ uniformly on S".

Analogously, if there is a function $f : S \to \mathbb{R}$ such that $\{s_n\} \to f$ *pointwisely* on S, then we say that the infinite series $\sum_{k=1}^{\infty} f_k$ **converges pointwisely** on S, written as "$\sum_{k=1}^{\infty} f_k(x) = f(x)$ pointwisely on S".

Again, if $\sum_{k=1}^{\infty} f_k(x) = f(x)$ uniformly (pointwisely resp.) on S for some function $f : S \to \mathbb{R}$, we say that $\{f_n\}$ **converges uniformly** (**converges pointwisely** resp.)/is **uniformly convergent** (**pointwisely convergent** resp.).

It turns out that in the case of exponential function, the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$[11] converges uniformly on any finite interval in $\mathbb{R}$. The limiting function in the uniform convergence sense can then be *defined* as the exponential function. This gives rise to one approach of defining exponential function.

[11] We may change the index to make the sum starts at $k = 1$ to match with the discussion above.

4.3.4 For uniform convergence of infinite series of functions, we also have a Cauchy's criterion.

Theorem 4.3.a (Cauchy's criterion for uniform convergence of series of functions)**.** The infinite series $\sum_{k=1}^{\infty} f_k$ converges uniformly on S iff for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| < \varepsilon$$

for any $m > n \geq N$ and $x \in S$.

Proof. Consider

$$\begin{aligned}
&\sum_{k=1}^{\infty} f_k \text{ converges uniformly on } S \\
\iff &\{s_n\} \to f \text{ uniformly on } S \text{ for some function } f : S \to \mathbb{R} \\
\iff &\{s_n\} \text{ is uniformly Cauchy on } S \qquad \text{(Theorem 4.2.e)} \\
\iff &\text{for any } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } |s_m(x) - s_n(x)| < \varepsilon \text{ for any } m > n \geq N \text{ and } x \in S \\
\iff &\text{for any } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } \left| \sum_{k=n+1}^{m} f_k(x) \right| < \varepsilon \text{ for any } m > n \geq N \text{ and } x \in S.
\end{aligned}$$

□

4.3.5 It turns out that like uniform convergence of sequences of functions, the property of continuity at a point can be "passed" to the limiting function, under the uniform convergence of series of functions.

Proposition 4.3.b. Suppose that f_k is continuous at $c \in S$ for every $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} f_k(x) = f(x)$ uniformly on S. Then f is also continuous at c.

Proof. By assumption, the sequence of partial sums $\{s_n\} \to f$ uniformly on S. Since f_k is continuous at $c \in S$ for each $k \in \mathbb{N}$, the same is true for every partial sum s_n.

Thus, applying Proposition 4.2.b on $\{s_n\}$ suggests that f is continuous at c. □

4.3.6 Apart from the Cauchy's criterion (Theorem 4.3.a), the following result is also an useful test of uniform convergence of series of functions.

Theorem 4.3.c (Weierstrass M-test)**.** Let $\sum_{k=1}^{\infty} f_k$ be a series of functions satisfying

(a) $|f_k(x)| \leq M_k$ for any $k \in \mathbb{N}$ and $x \in X$, and

(b) $\sum_{k=1}^{\infty} M_k$ converges.

Then, $\sum_{k=1}^{\infty} f_k$ converges uniformly on X.

Proof. Fix any $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} M_k$ converges, there exists $N \in \mathbb{N}$ such that $\left| \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{N} M_k \right| = \sum_{k=N+1}^{\infty} M_k < \varepsilon$.[12] Thus, for any $m > n \geq N$ and $x \in S$,

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k \leq \sum_{k=N+1}^{\infty} M_k < \varepsilon,$$

and then the uniform convergence follows from Theorem 4.3.a. □

[12] Note that $M_k \geq 0$ for any $k \in \mathbb{N}$ due to (a).

4.4 Commutativity of Limit with Integration/Differentiation

4.4.1 Next, we are going to investigate questions concerning the validity of "switching orders of limit and integration/differentiation":

Q1 Is it true that $\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$?

Q2 Is it true that $\lim_{n\to\infty} f_n'(x) = (\lim_{n\to\infty} f_n(x))'$?

[Note: Here "$\lim_{n\to\infty}$" is the usual limit for real numbers, and note that the function f defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is the *pointwise* limit of $\{f_n\}$.]

4.4.2 Unfortunately, in general the answers to both Q1 and Q2 are negative, as the following examples illustrate.

- *Negative example for Q1:* For any $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = n^2(1-x)x^n$ for any $x \in [0,1]$. Then, note that $\{f_n\} \to f \equiv 0$ pointwisely on $[0,1]$.
  Now for any $n \in \mathbb{N}$, compute
  $$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{n^2}{(n+1)(n+2)},$$
  and thus
  $$\lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{n^2}{(n+1)(n+2)} = \lim_{n\to\infty}\frac{1}{1+3/n+2/n^2} = 1 \neq 0 = \int_0^1 f(x)\,\mathrm{d}x\,.$$
- *Negative example for Q2:* For any $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{1}{\sqrt{n}}\sin nx$ for any $x \in \mathbb{R}$. Note that for any $x \in \mathbb{R}$, we have
  $$|f_n(x)| = \left|\frac{1}{\sqrt{n}}\sin nx\right| \le \left|\frac{1}{\sqrt{n}}\right|.$$
  Hence, by sandwich theorem, $\lim_{n\to\infty} f_n(x) = 0$, and thus $\{f_n\} \to f \equiv 0$ pointwisely on $\mathbb{R}$.
  Clearly we have $f'(x) = 0$ for any $x \in \mathbb{R}$. But for any $x \in \mathbb{R}$, we have $f_n'(x) = \sqrt{n}\cos nx$, and hence $\lim_{n\to\infty} f_n'(x)$ does not exist.

4.4.3 Nevertheless, the answers can become affirmative with some additional conditions imposed. For Q1, the answer is affirmative under uniform convergence.

Proposition 4.4.a. Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$ which converges uniformly to f on $[a,b]$. Then,
$$\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x\,.$$

Proof. The second equality follows from the uniform convergence, so it suffices to prove the first.

Firstly, by Proposition 4.2.b, the uniform limit f is continuous. Since continuity implies (Riemann) integrability, all f_n's and f are integrable over $[a,b]$.

Now, using the assumption that $\{f_n\} \to f$ uniformly on $[a,b]$, we have for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that
$$|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}$$
for any $n \ge N$ and $x \in [a,b]$. It then follows that
$$\left|\int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x\right| = \left|\int_a^b f_n(x) - f(x)\,\mathrm{d}x\right| \le \int_a^b |f_n(x) - f(x)|\,\mathrm{d}x \le \frac{\varepsilon}{2} < \varepsilon.$$

□

4.4.4 Although the uniform convergence ensures the commutativity of limit with integration, this condition is not a necessary condition.

Example: For any $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for any $x \in [0,1]$. Then, $\{f_n\} \to f$ pointwisely on $[0,1]$ where

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

Then, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{n+1} = 0 = \int_0^1 f(x)\,\mathrm{d}x\,.$$

Although the limit is commutative with integration here, the convergence $\{f_n\} \to f$ is not uniform.

4.4.5 Now, we consider Q2. Again the answer becomes affirmative with some additional conditions.

Proposition 4.4.b. Let $\{f_n\}$ be a sequence of differentiable functions on a bounded open interval $S = (a,b)$ where $a < b$. Suppose that

(i) $\{f_n'\} \to g$ uniformly on S for some function g, and

(ii) there exists $x_0 \in S$ such that $\{f_n(x_0)\}$ converges.

Then,

(a) $\{f_n\} \to f$ uniformly on S for some function f, and

(b) f is differentiable, with $(\lim_{n\to\infty} f_n(x))' = f'(x) = g(x) = \lim_{n\to\infty} f_n'(x)$ for any $x \in S$.

Proof. Fix any $c \in (a,b)$. For any $n \in \mathbb{N}$, we define $g_n : S \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} \frac{f_n(x)-f_n(c)}{x-c} & \text{if } x \neq c, \\ f_n'(c) & \text{if } x = c. \end{cases}$$

By construction, g_n is continuous at c.

By mean value theorem, for any $x \neq c$,

$$\begin{aligned} |g_n(x) - g_m(x)| &= \left| \frac{[f_n(x) - f_m(x)] - [f_n(c) - f_m(c)]}{x-c} \right| \\ &= \left| \frac{(f_n - f_m)(x) - (f_n - f_m)(c)}{x-c} \right| \\ &= |(f_n - f_m)'(t)| \\ &= |f_n'(t) - f_m'(t)| \end{aligned}$$

for some t between x and c.

Since $\{f_n'\}$ converges uniformly on S, it is uniformly Cauchy on S. Thus, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$,

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

for any $x \in S$. It then follows that for any $m, n \geq N$,

$$|g_n(x) - g_m(x)| = \begin{cases} |f_n'(t) - f_m'(t)| & \text{if } x \neq c, \\ |f_n'(c) - f_m'(c)| & \text{if } x = c \end{cases} < \varepsilon$$

for any $x \in S$, hence $\{g_n\}$ is uniformly Cauchy.

(a) We take $c = x_0$ for each g_n defined above. For any $m, n \in \mathbb{N}$, we have

$$\begin{aligned} f_n(x) - f_m(x) &= [f_n(x_0) + g_n(x_0)(x - x_0)] - [f_m(x_0) + g_m(x_0)(x - x_0)] \\ &= f_n(x_0) - f_m(x_0) + (x - x_0)(g_n(x) - g_m(x)) \end{aligned}$$

for any $x \in S$. Since $\{f_n(x_0)\}$ converges, there exists $N_1 \in \mathbb{N}$ such that

$$|f_n(x_0) - f_m(x_0)| < \varepsilon/2$$

for any $n, m \geq N_1$. On the other hand, since $\{g_n\}$ is uniformly Cauchy on S, there exists $N_2 \in \mathbb{N}$ such that

$$|g_n(x) - g_m(x)| < \frac{\varepsilon}{2(b-a)}$$

for any $m, n \geq N_2$ and any $x \in S$.

It follows that for any $m, n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x_0) - f_m(x_0)| + |x - x_0||g_n(x) - g_m(x)| \\ &< \frac{\varepsilon}{2} + (b-a) \cdot \frac{\varepsilon}{2(b-a)} \\ &= \varepsilon, \end{aligned}$$

for any $x \in S$. Thus, $\{f_n\}$ is uniformly Cauchy, hence converges uniformly to some function f by Theorem 4.2.e.

(b) For any $c \in S$, we have

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \lim_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} \\ &= \lim_{n \to \infty} \lim_{x \to c} \frac{f_n(x) - f_n(c)}{x - c} && \text{(by Proposition 4.2.b)} \\ &= \lim_{n \to \infty} \lim_{x \to c} g_n(x) \\ &= \lim_{n \to \infty} g_n(c) && (g_n \text{ is continuous}) \\ &= \lim_{n \to \infty} f'(c) \\ &= g(c). \end{aligned}$$

□

4.4.6 Note that in Proposition 4.4.b, apart from requiring the uniform convergence of $\{f_n'\}$, we also need an additional condition that $\{f_n(x_0)\}$ converges for some $x_0 \in S$. To illustrate the necessity of this additional condition, consider the following example.

Let $S = (0, 1)$ and for any $n \in \mathbb{N}$, define $f_n(x) = \ln nx$ for any $x \in S$. Note that for each $n \in \mathbb{N}$, f_n is differentiable and $f_n'(x) = 1/x$ for any $x \in S$. Thus, we have $\{f_n'\} \to g$ uniformly on S, where $g(x) = 1/x$ for any $x \in S$.

However, for any $x \in S$, $\{f_n(x)\}$ diverges, so the condition (ii) fails. So Proposition 4.4.b does not apply. Also observe that since $\{f_n\}$ does not even converge pointwisely, it does not converge uniformly.

4.4.7 We also have a series version of Proposition 4.4.b:

Corollary 4.4.c. Let $\{f_n\}$ be a sequence of differentiable functions on a bounded open interval $S \subseteq \mathbb{R}$. Suppose that

(i) $\sum_{n=1}^{\infty} f_n'(x) = g(x)$ uniformly on S, and

(ii) there exists $x_0 \in S$ such that $\sum_{n=1}^{\infty} f_n(x_0)$ converges.

Then,

(a) $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on S, and
(b) f is differentiable, with $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ for any $x \in S$.

Proof. For any $n \in \mathbb{N}$, define the partial sums $F_n : S \to \mathbb{R}$ by $F_n(x) = \sum_{k=1}^{n} f_k(x)$ for any $x \in S$. Since each f_k is differentiable, each F_n is differentiable.

Also, we have

$$\{F_n'\} = \left\{ \left(\sum_{k=1}^{n} f_k \right)' \right\} = \left\{ \sum_{k=1}^{n} f_k' \right\} \overset{\text{(i)}}{\to} g$$

uniformly on S. Furthermore, we have

$$\{F_n(x_0)\} = \left\{ \sum_{k=1}^{n} f_k(x_0) \right\} \overset{\text{(ii)}}{\to} \sum_{k=1}^{\infty} f_k(x_0) \in \mathbb{R}.$$

So, the conditions (i) and (ii) in Proposition 4.4.b are satisfied, and hence there exists a differentiable function $f : S \to \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} f_n = \lim_{n\to\infty} F_n = f$$

uniformly on S, with

$$f'(x) = g(x) = \sum_{n=1}^{\infty} f_n'(x)$$

for any $x \in S$. □

4.5 Arzelà-Ascoli Theorem

4.5.1 To close Section 4, we will prove an important theorem in analysis known as *Arzelà-Ascoli theorem*. To motivate the discussion, we consider its application to partial differential equations (PDEs). By Picard-Lindelöf theorem (Theorem 4.2.g), we can analyze the existence and uniqueness of solutions to ODEs. But the situation is much more complicated for PDEs.

To assess the qualitative features of PDEs like existence and uniqueness of solutions, a possible approach is to first try to find some "approximated" solution f_n for each $n \in \mathbb{N}$.

Next, we investigate the converging behaviour of $\{f_n\}$. Sometimes we can find a convergent subsequence of $\{f_n\}$ converging to some function f. If we are "lucky" enough, the function f could be the exact solution to the PDE. Hence, we are interested in knowing whether a given sequence of functions $\{f_n\}$ has a convergent subsequence, and Arzelà-Ascoli theorem suggests a necessary and sufficient condition for it to happen, under some assumptions.

4.5.2 Before stating Arzelà-Ascoli theorem, we introduce/recall some terminologies and results related to different kinds of *compactness*. Let X be a metric space. A set $S \subseteq X$

(a) (recall) is *compact* if every open cover of S has a finite subcover;

(b) (recall) has the *Boltzano-Weierstrass property* or is *limit point compact* if every infinite subset of S has an accumulation point in S;

(c) is **sequential compact** if every sequence in S has a subsequence which is convergent in S;

(d) is **relatively sequential compact** if every sequence in S has a subsequence which is convergent in X;

(e) is **relatively compact** if $\overline{S}$ is compact.

Theorem 4.5.a. We have (a) $\iff$ (b) $\iff$ (c) $\implies$ (d) $\iff$ (e).

4.5.3 Next, we introduce some more terminologies and results for preparation of the proof of Arzelà-Ascoli theorem. Let X be a metric space.

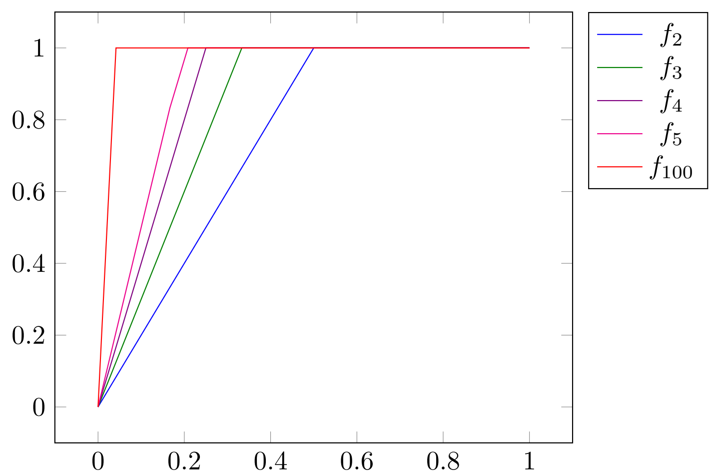
- A set $S \subseteq X$ is **dense** if $\overline{S} = X$.
- A set $\mathcal{F} \subseteq C(X)$ is **pointwise bounded** if for any $x \in X$, $\sup\{|f(x)| : f \in \mathcal{F}\} < \infty$. [Note: $C(X)$ denotes the set of all continuous real-valued functions on X.]
- A set $\mathcal{F} \subseteq C(X)$ is **equicontinuous at $x_0 \in X$** if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that for any $x \in B(x_0, \delta)$, $|f(x) - f(x_0)| < \varepsilon$ for any $f \in \mathcal{F}$. [Intuition 💡: When $\mathcal{F}$ is equicontinuous at x_0, all functions in the family $\mathcal{F}$ vary over a given neighbourhood of x_0 at an "equal" rate.]

  The set $\mathcal{F}$ is **equicontinuous (on X)** if $\mathcal{F}$ is equicontinuous at every point in X.

4.5.4 Example of being *not* equicontinuous: Let $S = [0, 1] \subseteq \mathbb{R}$ and define $f_n : S \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} nx & \text{if } x \in [0, 1/n], \\ 1 & \text{if } x \in [1/n, 1], \end{cases}$$

for any $n \in \mathbb{N}$.



Then, $\{f_n\}$ is *not* equicontinuous at 0. [Intuition 💡: The function f_n changes at a "faster" rate around 0 as n gets large, so the "rate of change around 0" is not equal for all functions in $\{f_n\}$.]

Proof. For any $\delta > 0$, there exists $N > 2/\delta$, which means $\delta/2 > 1/N$. Now consider $x_N = \min\{\delta/2, 1\} \in B(x_0, \delta)$. We have $f_N(x_N) = 1$ by construction. Next, take $\varepsilon = 1/2$, and then $|f_N(x_N) - f_N(0)| = 1 > \varepsilon$. ☐

4.5.5 A metric space X is **separable** if it contains a countable dense subset. Example: $X = [0, 1]$ is separable since $S = \mathbb{Q} \cap [0, 1] \subseteq X$ is countable and dense.

Lemma 4.5.b. Any compact metric space X is separable.

Proof. Since X is compact, for each $n \in \mathbb{N}$, the open cover $\{B(x, 1/n) : x \in X\}$ of X has a finite subcover $\{B(x_i^{(n)}, 1/n)\}_{i=1}^{m_n}$. Let $S_n = \{x_i^{(n)}\}_{i=1}^{m_n}$ denote the set containing all "centres" in the finite subcover.

Then, the union $S = \bigcup_{n=1}^{\infty} S_n$ is countable and dense.

<u>Countable</u>: Since each S_n is finite, one can use a similar argument as the proof of countability of $\mathbb{Q}$.

<u>Dense</u>: Fix any $x \in X$. For any $r > 0$, there exists $N \in \mathbb{N}$ such that $1/N < r$. Since $\{B(x_i^{(N)}, 1/N)\}_{i=1}^{m_N}$ covers X, there exists $j \in \{1, \ldots, m_N\}$ such that $x \in B(x_j^{(N)}, 1/N)$, which in turn means that $x_j^{(N)} \in B(x, 1/N) \subseteq B(x, r)$. Thus, $B(x, r) \cap S \neq \varnothing$, and so $x \in \overline{S}$. This shows $X \subseteq \overline{S}$, and the reverse subset inclusion is clear since every adherent point of S must belong to X by definition. ☐

4.5.6 Now we will prove Arzelà-Ascoli theorem.

Theorem 4.5.c (Arzelà-Ascoli theorem)**.** Let X be a compact metric space, and denote by $C(X)$ the set of all real-valued continuous functions on X. Equip $C(X)$ with the uniform metric d. Then, the family $\mathcal{F} \subseteq C(X)$ is relatively compact iff it is equicontinuous and pointwise bounded.

Proof. "$\Rightarrow$":

$\mathcal{F}$ is relatively compact $\implies$ $\mathcal{F}$ is pointwise bounded:

We prove by contrapositive. Suppose $\mathcal{F}$ is not pointwise bounded. Then for some $x_0 \in X$, $\sup_{f\in\mathcal{F}} |f(x_0)| = \infty$, which means that there exists $f_n \in \mathcal{F}$ such that $|f_n(x_0)| > n$ for any $n \in \mathbb{N}$.

This implies that $\{f_n(x_0)\}$ has no convergent subsequence in $\mathbb{R}$, and thus the sequence $\{f_n\}$ in $\mathcal{F}$ has no subsequence which is convergent in $C(X)$. Hence, $\mathcal{F}$ is not relatively compact by Theorem 4.5.a.

$\mathcal{F}$ is relatively compact $\implies$ $\mathcal{F}$ is equicontinuous:

We again prove by contrapositive. Suppose $\mathcal{F}$ is not equicontinuous at some $x_0 \in X$. Then there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$, there exists $f_n \in \mathcal{F}$ such that $|f_n(x_n) - f_n(x_0)| > \varepsilon_0$ for some $x_n \in B(x_0, 1/n)$.

Now assume to the contrary that $\mathcal{F}$ is relatively compact. Then the sequence $\{f_n\}$ in $\mathcal{F}$ would have a convergent subsequence $\{f_{n_k}\} \to f$ for some function $f \in C(X)$ (under the uniform metric d).

Since f is continuous, there exists $\delta > 0$ such that for any $x \in B(x_0, \delta)$, $|f(x) - f(x_0)| < \varepsilon_0/3$. Also, by Theorem 4.2.d, we have $\{f_{n_k}\} \to f$ uniformly on X, so there exists sufficiently large K such that $|f_{n_K}(x) - f(x)| < \varepsilon_0/3$ for any $x \in X$, and that $1/n_K < \delta$.

We consider $x_{n_K} \in B(x_0, 1/n_K) \subseteq B(x_0, \delta)$ in particular. We have

$$\begin{aligned}|f_{n_K}(x_0) - f_{n_K}(x_{n_K})| &\leq |f_{n_K}(x_0) - f(x_0)| + |f(x_0) - f(x_{n_K})| + |f(x_{n_K}) - f_{n_K}(x_{n_K})| \\ &< \varepsilon_0/3 + \varepsilon_0/3 + \varepsilon_0/3 \\ &= \varepsilon_0,\end{aligned}$$

contradiction.

"$\Leftarrow$": Suppose $\mathcal{F}$ is equicontinuous and pointwise bounded. By Lemma 4.5.b, since X is compact, there is a countable dense set $A = \{a_i : i \in \mathbb{N}\} \subseteq X$.

Now, fix any sequence $\{f_n\}$ in $\mathcal{F}$. Due to the pointwise boundedness, the real-valued sequence $\{f_n(a_1)\}$ is bounded. Then, by Boltzano-Weierstrass theorem (for real-valued sequences), $\{f_n(a_1)\}$ has a convergent subsequence (in $\mathbb{R}$), so there exists $J_1 = \{f_{1,n}\} \subseteq \{f_n\}$ such that $\{f_{1,n}(a_1)\}$ converges.

Applying a similar argument on the sequence $J_1 = \{f_{1,n}\}$ in $\mathcal{F}$ with a_1 replaced by a_2, we know that there exists $J_2 = \{f_{2,n}\} \subseteq \{f_{1,n}\}$ such that $\{f_{2,n}(a_2)\}$ converges. Also, since $\{f_{2,n}\} \subseteq \{f_{1,n}\}$, we have $\{f_{2,n}(a_1)\}$ converges as well.

Continuing this process ad infinitum, for any $k \in \mathbb{N}$, there exists $J_k = \{f_{k,n}\}_{n=1}^{\infty}$ such that $\{f_{k,n}(a_i)\}$ converges for any $i = 1, \dots, k$.

After that, line up the functions in $J_1, J_2, \dots$ as an infinite array in the following way:

$$\begin{array}{cccccc} f_{1,1} & f_{1,2} & f_{1,3} & \cdots & \text{(convergent at } a_1) \\ f_{2,1} & f_{2,2} & f_{2,3} & \cdots & \text{(convergent at } a_1, a_2) \\ f_{3,1} & f_{3,2} & f_{3,3} & \cdots & \text{(convergent at } a_1, a_2, a_3) \\ \vdots & \vdots & \vdots & \ddots & \end{array}$$

Define $g_n = f_{n,n}$ for any $n \in \mathbb{N}$. Then, by construction, $\{g_n\}$ converges at every point $a_i \in A$, since $\{g_n\} \setminus \{g_1, \dots, g_k\} \subseteq J_k$ for any $k \in \mathbb{N}$ and removing first finitely many terms in the sequence does not affect its converging behaviour.

Now, it suffices to show that $\{g_n\}$ is uniformly Cauchy, hence converges uniformly to some function $g \in C(X)$. This means that $\{g_n\} \to g \in C(X)$ in the metric space $(C(X), d)$ by Theorem 4.2.d, and thus $\{g_n\}$ serves as a subsequence of $\{f_n\}$ which is convergent in $C(X)$, so $\mathcal{F}$ is relatively compact by Theorem 4.5.a.

Since $\mathcal{F}$ is equicontinuous, for any $\varepsilon > 0$ and $x \in X$, there exists $\delta_x = \delta(\varepsilon, x)$ such that for any $y \in B(x, \delta_x)$, $|f(y) - f(x)| < \varepsilon/10$ for any $f \in \mathcal{F}$.

As $\{B(x, \delta_x)\}_{x \in X}$ is an open cover of X, it has a finite subcover $\{B(x_i, \delta_i)\}_{i=1}^{M}$. Take $\delta = \min\{\delta_1, \ldots, \delta_M\} > 0$. Since A is dense, for any $i = 1, \ldots, M$, there exists $b_i \in A$ such that $d(x_i, b_i) < \delta$.

Note that we only have finitely many b_i's, namely $b_1, \ldots, b_M$. So, there exists sufficiently large $N \in \mathbb{N}$ such that $\{g_N\}$ converges, hence Cauchy, at all $b_1, \ldots, b_M \in A$. Thus, for each $i = 1, \ldots, M$, there exists $N^{(i)} \in \mathbb{N}$ such that $|g_n(b_i) - g_m(b_i)| < \varepsilon/10$ for any $m, n \geq N^{(i)}$.

Then, choose $N^* = \max\{N^{(1)}, \ldots, N^{(M)}\}$, and fix any $m, n \geq N^*$. We have $|g_n(b_i) - g_m(b_i)| < \varepsilon/10$ for any $i = 1, \ldots, M$.

Now, for any $x \in X$, it must belong to an open ball in the finite subcover, i.e., $x \in B(x_i, \delta_i)$ for some $i = 1, \ldots, M$. Hence,

$$\begin{aligned} |g_n(x) - g_n(b_i)| &\leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_n(b_i)| \\ &< \varepsilon/10 + \varepsilon/10 \\ &= \varepsilon/5. \end{aligned}$$

Similarly, we can show that $|g_m(x) - g_m(b_i)| < \varepsilon/5$.

Finally, by triangle inequality, for any $x \in X$ we have

$$\begin{aligned} |g_n(x) - g_m(x)| &\leq |g_n(x) - g_n(b_i)| + |g_n(b_i) - g_m(b_i)| + |g_m(b_i) - g_m(x)| \\ &< \varepsilon/5 + \varepsilon/10 + \varepsilon/5 \\ &= \varepsilon/2 \\ &< \varepsilon, \end{aligned}$$

hence $\{g_n\}$ is uniformly Cauchy. $\square$

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Concepts and Terminologies

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