# HKU STAT3910 Study Notes

Chiu Ka Long (Leo)*

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## Contents



*email: leockl@connect.hku.hk; personal website: https://leochiukl.github.io

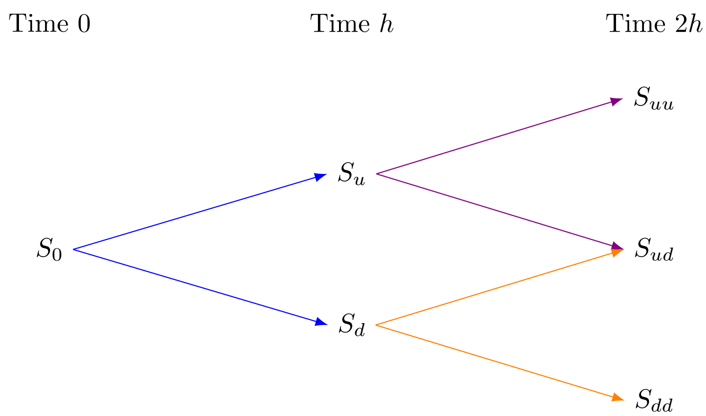
# 1 Further Topics on Binomial Option Pricing Model

1.0.1 We have learnt the basics of binomial option pricing model in STAT3905. Here, we will investigate it in more details and study some further topics about it.

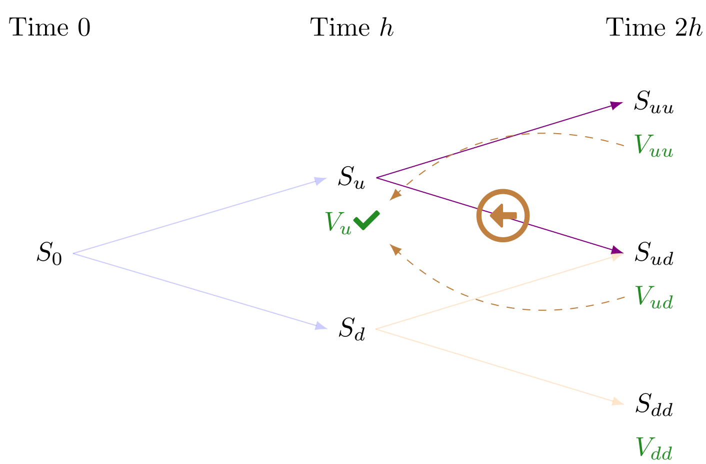
## 1.1 Dynamic Hedging

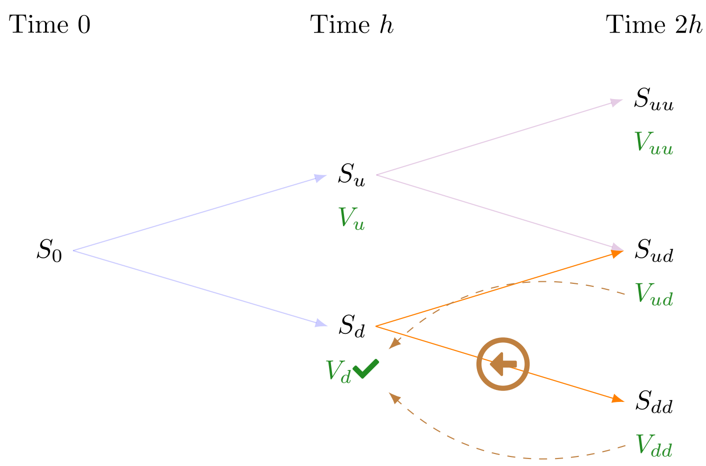
1.1.1 In STAT3905, the main tool for pricing an option in a multi-period binomial tree is *risk-neutral pricing*. Even in the backward induction process, each step we are utilizing one-period risk-neutral pricing. Can we use the approach of *pricing by replication*, just like the one-period binomial tree?

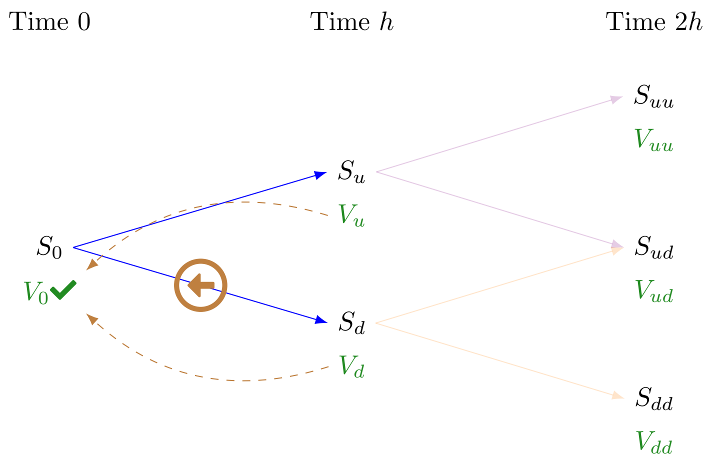
1.1.2 It turns out that constructing a replicating portfolio in the multi-period binomial tree setting is more tricky, since such replicating portfolio is not "static" but "dynamic" — we need to adjust its components depending on the market movements at the intermediate nodes. This way of construction is known as **dynamic hedging**.




1.1.3 Recall the backward induction process, which looks like the following.

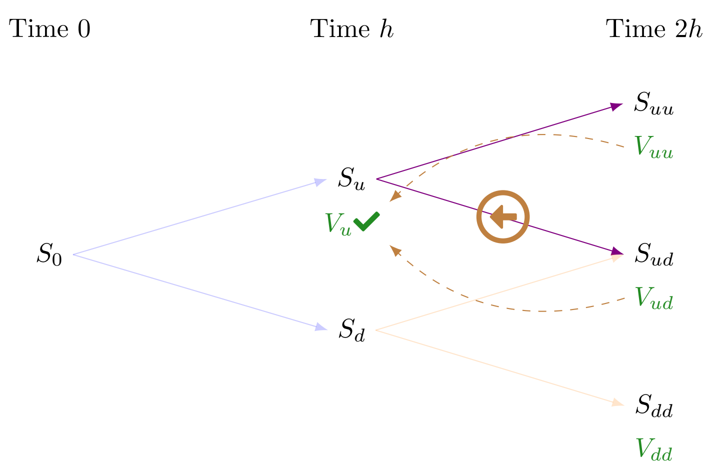





1.1.4 Here, in each step of the backward induction, we shall use the replicating portfolio approach instead of using the risk-neutral pricing formula. Through this, we can know how to construct a *dynamic* replicating portfolio, and thus also know how to *exploit arbitrage opportunities*, which is the main application of this dynamic hedging process.

1.1.5 Let us start at the $u$ node:



We can use the standard formulas for finding out the replicating portfolio *at $u$ node*:

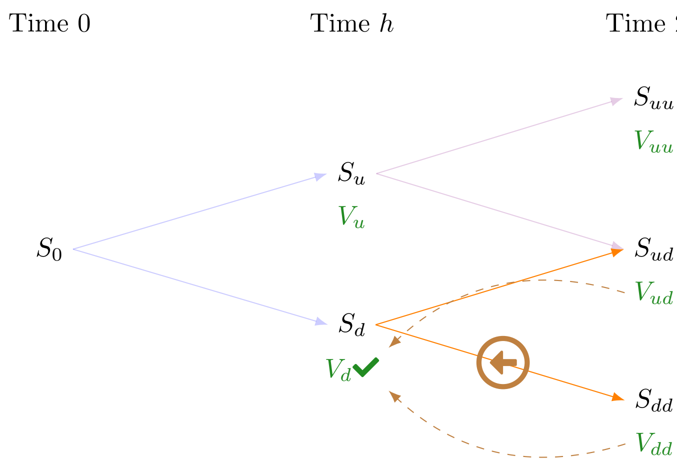
$$\Delta_u = e^{-\delta h} \cdot \frac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}},$$
$$B_u = e^{-rh} \cdot \frac{uV_{ud} - dV_{uu}}{u - d}.$$

From this we know that the replicating portfolio at $u$ node should be comprised of $\Delta_u$ shares of stock and $B_u$ in risk-free bond. Then of course we can carry out *pricing by replication* and conclude that

$$V_u = \Delta_u S_u + B_u.$$

Here, we add a subscript $u$ to signify that this replicating portfolio is for $u$ node.

At the same time point $h$, if we are at $d$ node instead of $u$ node, the components of replicating portfolio would be different.




In this case, applying the standard formulas again, we would have

$$\Delta_d = e^{-\delta h} \cdot \frac{V_{ud} - V_{dd}}{S_{ud} - S_{dd}},$$
$$B_d = e^{-rh} \cdot \frac{uV_{dd} - dV_{ud}}{u - d}.$$

As the input values in the formulas differ, the resulting replicating portfolio is generally different from that for $u$ node. Nonetheless, we can again use pricing by replication to calculate $V_d$:

$$V_d = \Delta_d S_d + B_d.$$

Having the values of $V_u$ and $V_d$, we can go to the initial node (time 0) and construct a replicating portfolio there, using the standard formulas again:

$$\Delta_0 = e^{-\delta h} \cdot \frac{V_u - V_d}{S_u - S_d},$$
$$B_0 = e^{-rh} \cdot \frac{uV_d - dV_u}{u - d}.$$

Of course one can then calculate the time-0 value $V_0$. But more importantly, this dynamic hedging approach gives us *three* replicating portfolios to be constructed:

$$(\Delta_0, B_0), (\Delta_u, B_u), (\Delta_d, B_d).$$

How should we actually replicate the values/payoffs of the derivative using these three pairs of values?

1.1.6 To actually perform the replication, we start with the replicating portfolio at the initial node: $(\Delta_0, B_0)$. By construction, when we have $\Delta_0$ shares of stock and $B_0$ in risk-free bond at time 0, the portfolio value would match with the value of the derivative for both $u$ and $d$ nodes, i.e., the portfolio value would be $V_u$ at $u$ node and $V_d$ at $d$ node. There is nothing special for this period.

However, when we are at time $h$, the above argument suggests that when we are at $u$ or $d$ node, the replicating portfolio should be $(\Delta_u, B_u)$ or $(\Delta_d, B_d)$ respectively. Generally, *adjustments* are needed to *rebalance* our existing portfolio (resulting from the time-0 replicating portfolio) into one of these, depending on which node we are at.

More explicitly, the portfolio we have at time $h$ before rebalancing is given by $(\Delta_0 e^{\delta h}, Be^{rh})$ due to the reinvestment of dividends and accumulation of interest.

- At $u$ node, we need to change $(\Delta_0 e^{\delta h}, Be^{rh})$ to $(\Delta_u, B_u)$.
- At $d$ node, we need to change $(\Delta_0 e^{\delta h}, Be^{rh})$ to $(\Delta_d, B_d)$.

To change the components, we need to buy/sell suitable shares of stocks and borrow/lend suitable amount of money. A natural question then arises: Do these transactions incur any cost/yield any gain?

1.1.7 It turns out that these transactions must be costless, overall speaking. By construction of the initial replicating portfolio, we must have:

$$\begin{cases} \Delta_0 e^{\delta h} S_u + Be^{rh} = V_u, \\ \Delta_0 e^{\delta h} S_d + Be^{rh} = V_d. \end{cases}$$

But on the other hand, pricing by replication at $u$ and $d$ nodes suggests that

$$\begin{cases} \Delta_u S_u + B_u = V_u, \\ \Delta_d S_d + B_d = V_d. \end{cases}$$

Combining these two systems, this just means

$$\begin{cases} \Delta_0 e^{\delta h} S_u + Be^{rh} = \Delta_u S_u + B_u, \\ \Delta_0 e^{\delta h} S_d + Be^{rh} = \Delta_d S_d + B_d, \end{cases}$$

i.e., the total portfolio value *remains unchanged* after the rebalancing! This explains why those transactions must be overall costless. Due to this feature, sometimes the replicating portfolio here is said to be **self-financing** as it can *finance itself*.

1.1.8 After performing the rebalancing at time $h$, we can just hold the new replicating portfolio until time $2h$, to match with the payoff of the derivative at time $2h$.

To summarize, in dynamic hedging, generally we need to perform rebalancing at intermediate nodes, so that we can replicate the payoff of derivative at *every node*. After obtaining the "dynamic" replicating portfolio, we can use the usual buy-low-sell-high approach to exploit arbitrage opportunity (if exists). [Note: In this case, we need to be dynamic in the buy-low-sell-high approach. We may need to perform rebalancing at some intermediate time.]

## 1.2 Asian Options

1.2.1 In STAT3905, we have only considered option which is *path-independent* in the sense that its payoff depends only the underlying asset price at the time of expiration (for European option) or time of exercise (for American option). Here, using the binomial option pricing model, we will analyze *path-dependent* options whose payoff depends also on intermediate underlying asset prices ("path"). We will consider two examples of path-dependent options: *Asian option* (in Section 1.2) and *barrier option* (in Section 1.3).

1.2.2 **Asian option** is a path-dependent option whose payoff depends on a suitably defined *average price* of the underlying asset over the life of the option.

To be more specific, we start with a plain vanilla European call with strike price $K$.[1] Its payoff at time $T$ (maturity) is

$$(S_T - K)_+.$$

After replacing either $S_T$ or $K$ by the "average price", we can incorporate path dependency in the payoff calculation.

Here we have two ways to compute the "average price" based on the "path" of the underlying asset prices. Given $n = T/h$ asset prices $S_h, S_{2h}, \ldots, S_{nh}$ in $h$-year interval, we have:

(a) **arithmetic average**:

$$A_T = \frac{1}{n}\sum_{i=1}^{n} S_{ih}.$$

(b) **geometric average**:

$$G_T = \left(\prod_{i=1}^{n} S_{ih}\right)^{1/n}.$$

The arithmetic average is more familiar to us and is the "usual" way of computing average. However, it turns out that using geometric average makes the option pricing more mathematically tractable, especially in the continuous-time *Black-Scholes model* (in Section 2).

1.2.3 After fixing a method for computing average, say geometric average, we can use $G_T$ to replace either $S_T$ or $K$, and then the resulting payoff would respectively be

$$(G_T - K)_+ \quad \text{or} \quad (S_T - G_T)_+.$$

We get an Asian option in either case.

In the former case, we call the resulting Asian option as *geometric average price Asian call option*:

- geometric average: the way of computing average
- price: the asset price $S_T$ gets replaced
- call: we start with a call option

There are 3 "dimensions".

By the same logic, in the latter case, we call the resulting Asian option as *geometric average strike Asian call option* (strike ➔ strike price $K$ gets replaced).

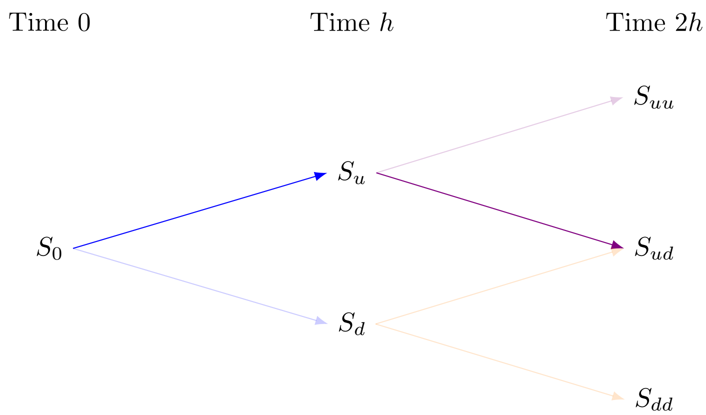
1.2.4 In a similar manner, we can get the payoff formulas for the other 6 kinds of Asian options (there are in total $2 \times 2 \times 2 = 8$ kinds of Asian options). The payoff formulas for all 8 kinds of Asian option are summarized below.

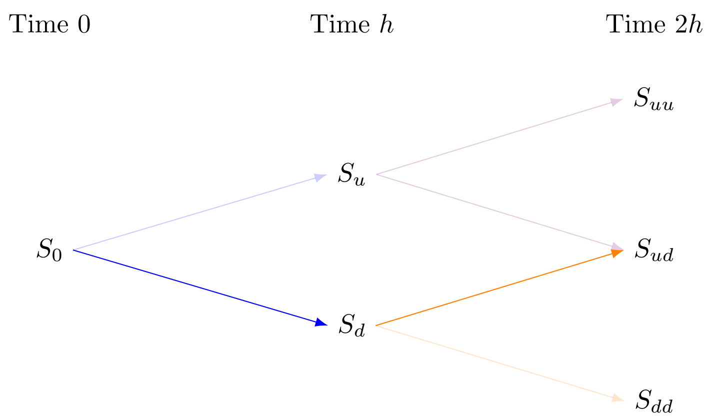
(1) arithmetic average price call: $(A_T - K)_+$
(2) arithmetic average price put: $(K - A_T)_+$
(3) arithmetic average strike call: $(S_T - A_T)_+$
(4) arithmetic average strike put: $(A_T - S_T)_+$
(5) geometric average price call: $(G_T - K)_+$
(6) geometric average price put: $(K - G_T)_+$
(7) geometric average strike call: $(S_T - G_T)_+$
(8) geometric average strike put: $(G_T - S_T)_+$

[1] Here we focus on *European* Asian option rather than *American* Asian option.

1.2.5 After knowing the payoff structure of an Asian option, we start discussing how to *price* an Asian option using the binomial option pricing model. It turns out that we can construct replicating portfolios in a similar manner as before and obtain an analogous *risk-neutral pricing formula*. But there is some twist: Due to the *path-dependent* nature, different asset price paths that result in the *same* terminal asset price may lead to *different* payoffs for an Asian option!

For example, the payoffs of an Asian option can be different for the following two paths, although they both lead to the same terminal asset price $S_{ud}$. This is because the *intermediate* asset price can differ, which influences the average calculation.




1.2.6 Therefore, although the risk-neutral pricing formula is most similar to before, we need to consider all the $2^n$ asset price paths instead of just the $n+1$ terminal asset prices, for an $n$-period binomial tree. We need to separately calculate the payoff for *each* path, and multiply it by the corresponding risk-neutral probability for travelling *in this specific path*.

For example, the risk-neutral pricing formula in a two-period binomial tree takes the following form:

$$\begin{aligned} V_0 &= e^{-rT} \sum_{\text{all } 2^2=4 \text{ paths}} \text{RN probability for travelling in that path} \times \text{payoff for that path} \\ &= e^{-rT}[(p^*)^2 V_{uu} + p^*(1-p^*)V_{ud} + (1-p^*)p^* V_{du} + (1-p^*)^2 V_{dd}]. \end{aligned}$$

Remarks:

- We need to consider the $ud$ and $du$ paths separately as $V_{ud} \neq V_{du}$ in general.
- The RN probability for travelling in the $ud$ path, i.e., "up" and then "down" *in this order* (not just one "up" and one "down"), is $p^*(1-p^*)$, not $2p^*(1-p^*)$! This is similar for the $du$ path.

## 1.3 Barrier Options

1.3.1 A **barrier option** is another path-dependent option whose payoff depends on whether the underlying asset price *reaches* or *hits* a specified level, called the **barrier**, over the life of the option.

1.3.2 Like Asian option, a barrier option also has three dimensions:

(1) *"up" or "down":* It refers to whether the asset price needs to go *up* ↑ or *down* ↓ from time 0 to hit the barrier $B$. In other words, it is an **up option** (**down option**) if the barrier $B$ is higher (lower) than the current asset price $S_0$.

(2) *"in" or "out":* The option can be either *knocked-in* or *knocked-out*. When the barrier is hit,

- **knocked-in**: a plain vanilla European option *comes into existence*;
- **knocked-out**: the original plain vanilla European option *ceases to exist* and becomes worthless 🗑.

[Note: The underlying plain vanilla option is supposed to have a fixed expiration time, independent from the "hitting time".]

(3) *"call" or "put":* It refers to whether the underlying plain vanilla option is a call or put option.

1.3.3 Comparing a barrier option and an otherwise identical plain vanilla option, we can observe that the barrier option *never pays more* than the plain vanilla option. Hence, the barrier option must be no more expensive than the plain vanilla one. This makes barrier option as a more "economic" alternative to the plain vanilla one.

1.3.4 To be more precise, we can express payoff formula of a barrier option mathematically as follows. Consider a $T$-year $K$-strike underlying European call/put. Let $M_T = \max_{t\in[0,T]} S_t$ and $m_T = \min_{t\in[0,T]} S_t$ be the maximum and minimum asset prices in the path respectively.

Then, the time-$T$ payoffs of the 8 kinds of barrier option are given by:

(1) up-and-in call: $(S_T - K)_+ \mathbf{1}_{\{M_T \geq B\}}$ [Note: $M_T \geq B$ means that $S_t \geq B$ for some time $t \in [0,T]$ → asset price has gone high enough to hit the barrier → "knocked-in".]

(2) up-and-in put: $(K - S_T)_+ \mathbf{1}_{\{M_T \geq B\}}$

(3) up-and-out call: $(S_T - K)_+ \mathbf{1}_{\{M_T < B\}}$ [Note: $M_T < B$ means that $S_t < B$ for *any* time $t \in [0,T]$ → asset price has *never* gone high enough to hit the barrier → never "knocked-out"]

(4) up-and-out put: $(K - S_T)_+ \mathbf{1}_{\{M_T < B\}}$

(5) down-and-in call: $(S_T - K)_+ \mathbf{1}_{\{m_T \leq B\}}$ [Note: $m_T \leq B$ means that $S_t \leq B$ for some time $t \in [0,T]$ → asset price has gone low enough to hit the barrier → "knocked-in".]

(6) down-and-in put: $(K - S_T)_+ \mathbf{1}_{\{m_T \leq B\}}$

(7) down-and-out call: $(S_T - K)_+ \mathbf{1}_{\{m_T > B\}}$ [Note: $m_T > B$ means that $S_t > B$ for *any* time $t \in [0,T]$ → asset price has *never* gone low enough to hit the barrier → never "knocked-out"]

(8) down-and-out put: $(K - S_T)_+ \mathbf{1}_{\{m_T > B\}}$

1.3.5 A remarkable relationship between otherwise identical knocked-in and knocked-out barrier options is known as *barrier options parity*:

$$\text{knocked-in option price} + \text{knocked-out option price} = \text{plain vanilla option price.}$$

*Proof.* We can just check the payoff formulas for all 4 types of combinations of otherwise identical knocked-in and knocked-out options (up/down & call/put). Here we only check the combination of up-and-in call and up-and-out call. Others can be checked similarly.

In this case, according to [1.3.4], the payoff of the combination of up-and-in call and up-and-out call is

$$(S_T - K)_+ \mathbf{1}_{\{M_T \geq B\}} + (S_T - K)_+ \mathbf{1}_{\{M_T < B\}} = (S_T - K)_+,$$

which equals the payoff of the plain vanilla call. The parity then follows by *law of one price*. □

# 2 Pricing in Black-Scholes Model

2.0.1 In STAT3905, we have studied Black-Scholes model. It will be studied again in STAT3910, but in more details. We will consider a more general form of Black-Scholes model involving the notion of *Brownian motion*, something we have learnt in STAT3903. But it turns out that this is still not the "real" form. We will only cover it in full mathematical details in STAT3911.

## 2.1 Model Formulation

2.1.1 In the **Black-Scholes model** (or **Black-Scholes framework**), we are assumed to be in a perfect market having the following two assets:

- a risky stock which pays dividend continuously at a dividend yield $\delta$, where its time-$t$ price is

$$S_t = S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma W_t\right],$$

  for some nonnegative parameters $\alpha$ and $\sigma$, and $\{W_t\}$ is a standard Brownian motion or Wiener process; [Note: The stock prices form another stochastic process $\{S_t\}$, which is called **geometric Brownian motion**.]
- a risk-free zero-coupon bond with an annual continuously compounded risk-free rate $r$.

Recall from STAT3903 that a *Wiener process* $\{W_t\}$ $(t \geq 0)$ is defined by the following properties:

(a) (starting at 0) $W_0 = 0$.

(b) (continuity) $W_t$ is continuous function of time $t$.

(c) (independent increments) Fix any $n$ time points with $0 < t_1 < t_2 < \cdots < t_n$. Then the increments

$$W_{t_1} = W_{t_1} - \underbrace{W_0}_{0}, \quad W_{t_2} - W_{t_1}, \quad \ldots \quad , \quad W_{t_n} - W_{t_{n-1}}$$

over non-overlapping time intervals are independent random variables.

(d) (stationary increments) For any $s \geq 0$ and any $t \geq 0$, $W_{s+t} - W_s \sim \mathrm{N}(0, t)$, so the distribution depends only on the time length $t$ and is free of $s$.

2.1.2 In this more general characterization of Black-Scholes model, the parameters $\alpha$ and $\sigma$ still carry the same meanings as before.

Interpretations of $\alpha$ and $\sigma$:

- $\alpha$ is the *continuously compounded expected rate of return on the stock*:

$$\begin{aligned}\mathbb{E}[S_t] &= \exp\left[\ln S_0 + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t\right]\mathbb{E}\left[e^{\sigma W_t}\right] \\ &= \exp\left[\ln S_0 + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t\right]\exp\left(\frac{\sigma^2 t}{2}\right) \\ &= S_0 e^{(\alpha-\delta)t}.\end{aligned}$$

[Note: By setting $s = 0$ in the stationary increment property, we know $W_t = W_t - W_0 \sim \mathrm{N}(0, t)$. Now, recalling the formula of *moment generating function* for normal distribution, we have

$$\mathbb{E}\left[e^{\sigma W_t}\right] = \exp\left(\frac{\sigma^2 t}{2}\right).$$

]

This means

$$\underbrace{S_0}_{\text{beginning value}} e^{\alpha t} = \underbrace{e^{\delta t}\mathbb{E}[S_t]}_{\text{expected ending value}}.$$

- $\sigma$ is the *volatility of the stock*:

$$\mathrm{Var}\left(\ln\frac{S_t}{S_0}\right) = \mathrm{Var}\left(\text{non-random constant} + \sigma W_t\right) = \mathrm{Var}\left(\sigma W_t\right) = \sigma^2 t.$$

2.1.3 Now, we shall show that the model studied in STAT3905 is indeed a special case of the one we study here. First fix any time interval $[s, s+t]$. Then note that

$$\ln\frac{S_{s+t}}{S_s} = \ln\left(\frac{S_{s+t}}{S_0}\cdot\frac{S_0}{S_s}\right) = \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma(W_{s+t} - W_s).$$

By the stationary increments property, we have $W_{s+t} - W_s \sim \mathrm{N}\left(0, t\right)$, so we indeed have

$$\ln\frac{S_{s+t}}{S_s} \sim \boxed{\mathrm{N}\left(\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)}.$$

We can also express this in terms of log-normal distribution:

$$\frac{S_{s+t}}{S_s} \sim \boxed{\mathrm{LN}\left(\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)}.$$

In particular, the distribution is free of $s$ and only depends on $t$, due to the stationary increments property of Weiner process. We have *stationary stock price ratios* in the Black-Scholes model.

2.1.4 To reduce this to the case for STAT3905, we can set $s = 0$, which yields

$$\frac{S_t}{S_0} \sim \mathrm{LN}\left(\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right),$$

or

$$S_t \sim \mathrm{LN}\left(\ln S_0 + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right),$$

which is the same as the one for STAT3905, from the perspective of marginal distribution.

2.1.5 But the advantage of using the more general Black-Scholes model here is that we can also describe the relationship between different stock prices — we can model the "joint" behaviour of stock prices. Consider $n$ time points with $0 < t_1 < t_2 < \cdots < t_n$. Then the stock price ratios

$$\frac{S_{t_1}}{S_0},\quad \frac{S_{t_2}}{S_{t_1}},\quad \ldots\quad,\quad \frac{S_{t_n}}{S_{t_{n-1}}}$$

are functions of the independent increments $W_{t_1} - W_0$, $W_{t_2} - W_{t_1}$, ..., $W_{t_n} - W_{t_{n-1}}$. Thus, these stock price ratios over non-overlapping time intervals are independent also.

## 2.2 Probabilistic Quantities Under Black-Scholes Model

2.2.1 Since the marginal distribution of the more general Black-Scholes model here is just the lognormal distribution we have studied in STAT3905, the formulas for probabilistic quantities introduced in STAT3905 are also applicable here:

(a) *(true) exercise probability:* (call) $\mathbb{P}(S_T > K) = \boxed{\Phi(\widehat{d_2})}$; (put) $\mathbb{P}(S_T < K) = \boxed{\Phi(-\widehat{d_2})}$, where

$$\widehat{d_2} = \frac{\ln(S_0/K) + (\alpha - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

(b) *mean and variance of $S_t$:* $\mathbb{E}[S_t] = \boxed{S_0 e^{(\alpha-\delta)t}}$ and $\mathrm{Var}\left(S_t\right) = \boxed{(\mathbb{E}[S_t])^2(e^{\sigma^2 t} - 1)}$.

2.2.2 Here we introduce several more probabilistic quantities. The first one is *quantiles*. For the case of stock price $S_t$ here, the $p$th quantile, or $100p$th percentile, of $S_t$ is just the value of $c$ that makes $\mathbb{P}(S_t \leq c) = p$. We shall denote it by $\pi_p(S_t)$.

[Note: In the financial or risk management context, the quantiles are also known as *value at risk* (VaR).]

2.2.3 The formula for $\pi_p(S_t)$ is given by

$$\pi_p(S_t) = \boxed{S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t} \cdot \pi_p(Z)\right]}$$

where $\pi_p(Z)$ is the $100p$th percentile of a *standard normal* random variable $Z$.

*Proof.* We shall use without proof the following general result: If $g$ is a *strictly increasing* continuous function applied to a random variable $X$, then $\pi_p(g(X)) = g(\pi_p(X))$ — We can "interchange" $\pi_p$ and $g$.

First, note that for a Weiner process $\{W_t\}_{t\geq 0}$, for any $t \geq 0$, we can write $W_t \overset{d}{=} \sqrt{t}Z$ where $Z$ is a standard normal random variable, since $W_t \sim \mathrm{N}(0, t)$. Thus, we have

$$S_t \overset{d}{=} S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right].$$

With a fixed $t \geq 0$, we define the function $g$ by

$$g(x) = S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}x\right],$$

which is continuous and strictly increasing. Then, applying the general result, we get

$$\pi_p(S_t) = \pi_p(g(Z)) = g(\pi_p(Z)) = S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t} \cdot \pi_p(Z)\right].$$

□

2.2.4 Next we will consider quantities involving multiple stock prices at different time points. For these quantities, knowledge of *marginal* distributions of stock prices is not enough. We need to know their *joint* distribution. This is a place where the more general characterization of Black-Scholes model here is useful.

2.2.5 Examples:

(a) *Probability involving multiple stock prices:* We write

$$\begin{aligned}\mathbb{P}(S_1 < S_2 < S_3) &= \mathbb{P}(S_1 < S_2 \text{ and } S_2 < S_3) \\ &= \mathbb{P}\left(\frac{S_2}{S_1} > 1 \text{ and } \frac{S_3}{S_2} > 1\right) \\ &= \mathbb{P}\left(\frac{S_2}{S_1} > 1\right)\mathbb{P}\left(\frac{S_3}{S_2} > 1\right) \qquad \text{(independent price ratios).}\end{aligned}$$

Then we can carry out the remaining computations based on the property

$$\frac{S_{s+t}}{S_s} \sim \mathrm{LN}\left(\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right).$$

(b) *Geometric average of stock prices:* We write

$$\begin{aligned}G_3 &= (S_1 S_2 S_3)^{1/3} \\ &= S_0 \exp\left[\left(\alpha - \delta - \frac{1}{2}\sigma^2\right)\left(\frac{1+2+3}{3}\right) + \sigma\left(\frac{W_1 + W_2 + W_3}{3}\right)\right] \\ &= S_0 \exp[a + b(W_1 + W_2 + W_3)]\end{aligned}$$

where $a$ and $b$ are some constants. Then, we write

$$W_1 + W_2 + W_3 = 3W_1 + 2(W_2 - W_1) + (W_3 - W_2).$$

[Note: The idea for obtaining this expression is as follows.
To apply the independent increment property of Weiner process, we first write

$$W_1 + W_2 + W_3 = aW_1 + b(W_2 - W_1) + c(W_3 - W_2)$$

and we would like to find the values of $a$, $b$, and $c$. Comparing the coefficient of $W_3$, we conclude that $c = 1$. Next, comparing the coefficient of $W_2$ gives $b = 2$. Finally, comparing the coefficient of $W_1$ gives $a = 3$.]

Since $W_1, W_2 - W_1, W_3 - W_2 \overset{\text{iid}}{\sim} \mathrm{N}(0,1)$, we have

$$W_1 + W_2 + W_3 \sim \mathrm{N}\left(0, (1^2 + 2^2 + 3^2)(1)\right) \equiv \mathrm{N}(0, 14).$$

From this, we can calculate probabilistic quantities regarding $G_3$.

## 2.3 Risk-Neutral Pricing

2.3.1 After discussing the formulation of the Black-Scholes model, we would like to *price* options under the Black-Scholes model, just like the case for binomial option pricing model. Similarly, we will utilize *risk-neutral pricing*. Entering into the risk-neutral world provides us a convenient way to price options.

2.3.2 The stock price formula in [2.1.1] describes the behaviour of stock prices in the *real* world.

$$S_t = S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma W_t\right], \quad t \geq 0.$$

- The parameter $\alpha$ describes the continuously compounded expected rate of return on the stock in the *real* world. To compensate for the risk inherent to the stock, the return rate $\alpha$ is usually higher than the risk-free rate $r$.
- The stochastic process $\{W_t\}$ is a Weiner process in the *real* world.

2.3.3 Next, we want to enter into the *risk-neutral* world. Before that, we do some preparatory work. First define another stochastic process $\{\widetilde{W}_t\}$ by

$$\widetilde{W}_t = W_t + \frac{\alpha - r}{\sigma}t$$

for any $t \geq 0$. Then, we can write the stock price formula as:

$$\begin{aligned}
S_t &= S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \\
&= S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t + \sigma\left(\widetilde{W}_t - \frac{\alpha - r}{\sigma}t\right)\right] \\
&= S_0 \exp\left[\left(\alpha - \delta - \frac{\sigma^2}{2}\right)t - (\alpha - r)t + \sigma\widetilde{W}_t\right] \\
&= \boxed{S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)t + \sigma\widetilde{W}_t\right]}.
\end{aligned}$$

The final expression is very similar to the formula in [2.1.1]. The only changes are that $\alpha$ becomes $r$, and $W_t$ becomes $\widetilde{W}_t$.

2.3.4 It turns out that the stochastic process $\{\widetilde{W}_t\}$ is a Weiner process in the *risk-neutral* world (not real world ⚠). In the risk-neutral world, the *risk-neutral probability measure* $\mathbb{Q}$ is used for calculating probabilities, instead of the real-world probability measure $\mathbb{P}$. To justify that such risk-neutral world we "imagine" here actually *exists*[2], many mathematical technicalities are involved, and they are to be discussed in STAT3911. Here, we shall focus on the application of risk-neutral pricing rather than its proof.

2.3.5 Consequently, in the risk-neutral world, the stock price process $\{S_t\}$ is still a geometric Brownian motion like the real world, with just one change: The continuously compounded expected rate of return changes from $\alpha$ to the risk-free rate $r$. So, the risky stock also earns risk-free rate in this world, confirming that we are actually in the *risk-neutral* world.

Particularly, the distributions of stock price ratios and individual stock prices in the risk-neutral world, where the probability measure $\mathbb{Q}$ is used, are given by

$$\frac{S_{s+t}}{S_s} \overset{\text{RN}}{\sim} \text{LN}\left(\left(r-\delta-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$$

and

$$S_t \overset{\text{RN}}{\sim} \text{LN}\left(\ln S_0 + \left(r-\delta-\frac{\sigma^2}{2}\right)t, \sigma^2 t\right),$$

for any $s \geq 0$ and $t \geq 0$.

2.3.6 Since every asset earns the risk-free rate $r$ in the risk-neutral world, we can calculate the price of a European option/derivative[3] using the risk-neutral pricing formula, just like the case for binomial option pricing model: The current option price $V_0$ is given by

$$V_0 = \boxed{e^{-rT}\mathbb{E}_{\mathbb{Q}}[\text{terminal payoff}]}.$$

Although the form of this formula is the same as that in the binomial tree model, due to the great difference between the binomial tree model and Black-Scholes model, the computation of $\mathbb{E}_{\mathbb{Q}}[\text{terminal payoff}]$ would be significantly different.

2.3.7 To illustrate the usage of risk-neutral pricing formula under Black-Scholes model, we consider an example about pricing a *power derivative* on the stock 🍎 (under Black-Scholes model), whose payoff is $V_T = S_T^a$ at the expiration time $T$, where $a$ is a fixed real number.

Applying the risk-neutral pricing formula in [2.3.6], the current price of the power derivative is

$$\begin{aligned}
V_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[S_T^a] \\
&= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left[S_0 \exp\left(\left(r-\delta-\frac{1}{2}\sigma^2\right)T + \sigma\widetilde{W}_T\right)\right]^a\right] \\
&= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[S_0^a \exp\left(a\left(r-\delta-\frac{1}{2}\sigma^2\right)T + a\sigma\widetilde{W}_T\right)\right] \\
&= e^{-rT}S_0^a \exp\left[a\left(r-\delta-\frac{1}{2}\sigma^2\right)T\right]M_{\widetilde{W}_T}(a\sigma) \qquad (M_{\widetilde{W}_T} \text{ denotes the mgf of } \widetilde{W}_T\text{, under the risk-neutral world}).
\end{aligned}$$

Since $\widetilde{W}_T \overset{\text{RN}}{\sim} \text{N}(0, T)$, we have

$$M_{\widetilde{W}_T}(a\sigma) = \exp\left(\frac{T(a\sigma)^2}{2}\right).$$

It follows that

$$V_0 = e^{-rT}S_0^a \exp\left[a\left(r-\delta-\frac{1}{2}\sigma^2\right)T\right]\exp\left(\frac{T(a\sigma)^2}{2}\right) = \boxed{S_0^a \exp\left\{[(a-1)r - a\delta]T + \frac{1}{2}a(a-1)\sigma^2 T\right\}}.$$

Special cases:

[2]We can "imagine" many crazy things that do not exist at all...

[3]Pricing of American option/derivative under the Black-Scholes model is beyond the scope.

- $a = 0$: The time-$T$ payoff of the power derivative is just $V_T = S_T^0 = 1$ always, so the power derivative is essentially a risk-free bond with \$1 payable at time $T$. Using the formula derived here, the current price is

$$V_0 = S_0^0 \exp\left\{[(0-1)r - 0\delta]T + \frac{1}{2}0(0-1)\sigma^2 T\right\} = e^{-rT},$$

as expected.

- $a = 1$: Then time-$T$ payoff of the derivative is $V_T = S_T$, the time-$T$ stock price. In other words, the derivative delivers one unit of stock 🍎 at time $T$, and thus it can be treated as a prepaid forward on the stock 🍎. Using the formula above, we get

$$V_0 = S_0^1 \exp\left\{[(1-1)r - 1\delta]T + \frac{1}{2}1(1-1)\sigma^2 T\right\} = S_0 e^{-\delta T},$$

which is precisely the current price of the $T$-year prepaid forward on the stock.

## 2.4 The Black-Scholes Formula for European Call and Put

2.4.1 By applying the risk-neutral pricing formula in [2.3.6] to price European call and put, we can obtain the famous *Black-Scholes formula*.

2.4.2 Consider a $K$-strike $T$-year European call on a stock 🍎 with dividends payable continuously at the dividend yield $\delta$ under the Black-Scholes model. Since the time-$T$ payoff of the call is $(S_T - K)_+$, using the risk-neutral pricing formula in [2.3.6] suggests that the current price of the call is

$$C_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+].$$

After some tedious algebra (see STAT3905 for more details), we get

$$C_0 = \boxed{S_0 e^{-\delta T}\Phi(d_1) - K e^{-rT}\Phi(d_2)}$$

where $d_1 = \dfrac{\ln(S_0/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T} = \dfrac{\ln(S_0/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, and $\Phi(\cdot)$ denotes the standard normal cdf.

2.4.3 We can express the formula in [2.4.2] using a generic Black-Scholes type pricing function defined by:

$$\mathrm{BS}\,(s_1, \delta_1; s_2, \delta_2; \sigma, T) = s_1 e^{-\delta_1 T}\Phi(d_1) - s_2 e^{-\delta_2 T}\Phi(d_2)$$

where $d_1 = \dfrac{\ln(s_1/s_2) + (\delta_2 - \delta_1 + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. The notations carry the following meanings:

- $s_1$ is the current price of the asset gained if the option is exercised, and $\delta_1$ is the "dividend yield" (continuously compounded "growth rate") of the asset.
- $s_2$ is the current price of the asset lost if the option is exercised, and $\delta_2$ is the "dividend yield" of the asset.
- $\sigma$ is the annualized volatility of the asset whose prices follow a geometric Brownian motion.
- $T$ is the time to maturity of the option.

By plugging in suitable arguments into the generic "BS" function as suggested by the meanings above, it can be shown that $\mathrm{BS}\,(s_1, \delta_1; s_2, \delta_2; \sigma, T)$ gives the correct time-0 option price. (See Section 2.6 for more details.)

[Note: The time-$t$ option price $(0 < t < T)$ is given by $\mathrm{BS}\,(s_1, \delta_1; s_2, \delta_2; \sigma, T - t)$, by noting that the time to maturity becomes $T - t$, at time $t$. Furthermore, $s_1$ and $s_2$ become the time-$t$ prices.]

2.4.4 Example: For the case of the call option, when it is exercised, the stock 🍎 is gained and the "pile of cash" 💵 that worth the strike price is lost. So:

- $s_1$ should be the current price of the stock 🍎 ($S_0$), and $\delta_1$ should be the dividend yield of 🍎 ($\delta$).
- $s_2$ should be the current price of the "pile of cash" 💵 ($K$), and $\delta_2$ should be the "dividend yield" of 💵: the continuously compounded "growth rate" of 💵, namely the risk-free rate $r$.
- $\sigma$ is the annualized volatility of the stock 🍎.

Hence, we have

$$C_0 = \boxed{\text{BS}\,(S_0, \delta; K, r; \sigma, T)},$$

which coincides with the formula in [2.4.2].

2.4.5 For a $K$-strike $T$-year European put on 🍎, we can again apply the generic Black-Scholes pricing function to get its time-0 price:

$$P_0 = \boxed{\text{BS}\,(K, r; S_0, \delta; \sigma, T)} = Ke^{-rT}\Phi(d_1^P) - S_0e^{-\delta T}\Phi(d_2^P).$$

Here we add a superscript "$P$" to each of $d_1$ and $d_2$ to emphasize that they are for the European put, and are *different* from the $d_1^C$ and $d_2^C$ for the European call ⚠ (we also add the superscript $C$ here for clarity). More specifically, they are given by $d_1^P = \dfrac{\ln(K/S_0) + (\delta - r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2^P = d_1^P - \sigma\sqrt{T}$.

In STAT3905, the formula for the time-0 put price is given by

$$P_0 = Ke^{-rT}\Phi(-d_2^C) - S_0e^{-\delta T}\Phi(-d_1^C),$$

which looks different from the expression above. However, we can show that $-d_2^C = d_1^P$ and $-d_1^C = d_2^P$, so these two expressions are actually equivalent.

*Proof.* Note that

$$d_1^P = \frac{\ln(K/S_0) + (\delta - r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -\frac{\ln(S_0/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -d_2^C,$$

and

$$d_2^P = \frac{\ln(K/S_0) + (\delta - r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -\frac{\ln(S_0/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = -d_1^C.$$

□

2.4.6 The Black-Scholes formula is not just limited to European call/put on stocks paying dividends continuously. We can also use it to price European option on some other kind of asset.

2.4.7 The first kind of asset we consider here is *currency* €. We will use the Black-Scholes formula to price European option on currency €, which are also known as **currency option**. The key idea is to treat currency as a "special stock". For a currency option, there are generally *two* currencies involved:

- **underlying currency**: the currency on which the option is written; the underlying asset for the option
- **benchmark currency**: the currency on which value is measured/denominated

Suppose that the underlying currency is euro € and the benchmark currency is dollar $. Let $X_t$ denotes the time-$t$ dollar/euro exchange rate, i.e., 1 euro can be exchanged for $X_t$ dollars at time $t$, vice versa. Symbolically, €1 = $\$X_t$.

2.4.8 To price a **\$**-denominated $T$-year $\$K$-strike European option on €1 under the Black-Scholes model, we assume that the exchange rate process $\{X_t\}_{t\geq 0}$ is a geometric Brownian motion with constant volatility $\sigma$ and "dividend yield" $r_{\text{underlying}} = r_{€}$ (risk-free rate for the underlying currency: euro). Also, the "risk-free rate $r$" in this case is the risk-free rate for the *benchmark* currency (dollar), denoted by $r_{\text{benchmark}} = r_{\$}$. Thus, considering the risk-neutral world, we can write

$$X_t = X_0 \exp\left[\left(r_{\text{benchmark}} - r_{\text{underlying}} - \frac{\sigma^2}{2}\right)t + \sigma\widetilde{W}_t\right]$$

for any $t \geq 0$. From this, we can then associate the notations here with the ones for stock option as follows:

| Currency option | Stock option |
|---|---|
| $X_t$ | $S_t$ |
| $r_{\text{benchmark}}$ | $r$ |
| $r_{\text{underlying}}$ | $\delta$ |
| $\sigma$ (for exchange rates) | $\sigma$ (for stock prices) |

2.4.9 Hence, the current prices of **\$**-denominated $T$-year $\$K$-strike European call and put on €1 are respectively

$$C_0 = \boxed{\text{BS}\left(X_0, r_{\text{underlying}}; K, r_{\text{benchmark}}; \sigma, T\right)}$$

and

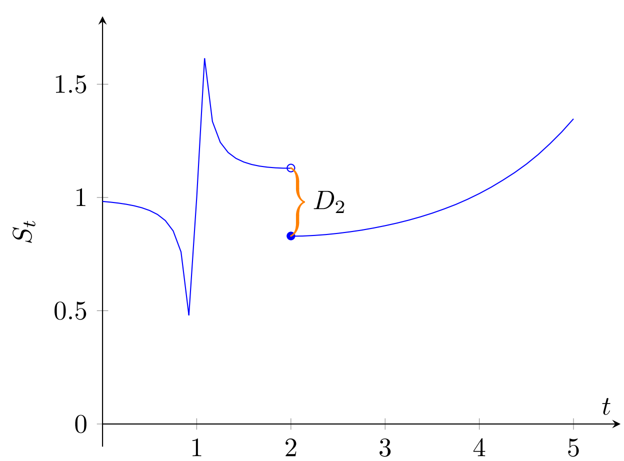
$$P_0 = \boxed{\text{BS}\left(K, r_{\text{benchmark}}; X_0, r_{\text{underlying}}; \sigma, T\right)}.$$

2.4.10 The second kind of asset to be considered is *stock paying discrete dividends*. The underlying stocks we have dealt with so far pay continuous proportional dividends at a dividend yield $\delta$. What if the stock pays discrete dividends instead? Here we shall consider the case where both the amount and timing of the discrete dividend payments are known.

2.4.11 Consider a stock that pays dividend $D_{t_i}$ at time $t_i$, for every $i = 1, \ldots, n$, where $0 < t_1 < \cdots < t_n < T$. The time $T$ is the time to maturity of a European option on such stock.

The following shows an example of stock price path for a stock with discrete dividend payable at time 2:

Stock price path




Due to the no-arbitrage principle, the stock price jumps downward immediately at each dividend payment time, by the amount of the dividend at that time. This makes the stock price path *discontinuous*.

If the stock price process was a geometric Brownian motion, the continuity of the sample path for the underlying Weiner process would imply the continuity of the stock price path as well. Hence, this suggests that the stock price process $\{S_t\}_{t\geq 0}$ cannot possibly be a geometric Brownian motion. So how should we apply the Black-Scholes model to price a $T$-year $K$-strike option written on such stock?

2.4.12 The key idea is to consider the *prepaid forward price* instead. From STAT3905, the time-$t$ price of a prepaid forward for one share of stock to be delivered at time $T$ $(\geq t)$ is

$$F_{t,T}^P(S) = S_t - \mathrm{PV}_t(\text{dividends}) = S_t - \sum_{i:t_i>t} D_{t_i} e^{-r(t_i-t)}.$$

Although the stock price $S_{t_j}$ has a downward jump of size $D_{t_j}$ at each dividend payment time $t_j$, the corresponding present value $\mathrm{PV}_{t_j}(\text{dividends})$ drops by the same amount. This ensures that the prepaid forward price $F_{t,T}^P(S)$ is a continuous function of time $t$. Therefore, we can assume that the prepaid forward price process $\{F_{t,T}^P(S)\}_{t\in[0,T]}$ is a geometric Brownian motion, and then we can proceed the option pricing, utilizing this assumption.

2.4.13 Since there is no dividend payable after time $T$, the time-$T$ prepaid forward price is the same as the time-$T$ stock price, i.e., $F_{T,T}^P(S) = S_T$. Then, by the risk-neutral pricing formula in [2.3.6], the current price of the European call on the stock is

$$C_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[(F_{T,T}^P(S) - K)_+\right].$$

From this, we can see that such European call can also be viewed as the otherwise identical European call on the *prepaid forward* on the stock, also maturing at time $T$, for option pricing purpose.

2.4.14 With the assumption that $\{F_{t,T}^P(S)\}_{t\in[0,T]}$ is a geometric Brownian motion, we can use the generic Black-Scholes pricing function to price such European call, with the following arguments:

- $s_1 = F_{0,T}^P(S)$: the time-0 price of the asset gained by exercising the call, namely the prepaid forward maturing at time $T$.
- $\delta = 0$: the prepaid forward itself does not pay any dividend.
- $\sigma$: the volatility of the *prepaid forward*. That is, we have

$$\mathrm{Var}\left(\ln F_{t,T}^P(S)\right) = \sigma^2 t$$

  for any $0 \leq t \leq T$.

The current price of the European call is thus

$$C_0 = \boxed{\mathrm{BS}\left(F_{0,T}^P(S), 0; K, r; \sigma, T\right)}.$$

Similarly, the current price of the otherwise identical European *put* is

$$P_0 = \boxed{\mathrm{BS}\left(K, r; F_{0,T}^P(S), 0; \sigma, T\right)}.$$

## 2.5 Pricing Complex Options under Black Scholes Model: Treating the Underlying Asset as a Special Stock

2.5.1 By "tweaking" the generic Black-Scholes pricing function suitably, we are able to price some options with complex payoff structure in the Black-Scholes model also. One approach is to treat the underlying asset as a "special stock", somewhat like what we did for currency options, but in a more complicated manner. We shall consider two examples in Section 2.5: (European) power option and (geometric average price) Asian option.

2.5.2 The first example here is (European) *power options*. For any fixed $a \in \mathbb{R}$, an European **power call option** (**power put option**) has a payoff $(S_T^a - K)_+$ ($(K - S_T^a)_+$) at the expiration time $T$. When $a = 1$, it reduces to the plain vanilla European call/put option.

2.5.3 The treatments for power call option and power put option are analogous, so we focus on a power call option here. By [2.3.6], the current price of the option is

$$C_0^{\text{power}} = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T^a - K)_+].$$

To actually compute this option price, we try to "match" the expression with something we are more familiar with. By [2.4.2], we know that

$$\begin{aligned} e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+] &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)T + \sigma\widetilde{W}_T\right] - K\right)_+\right] \\ &= \mathrm{BS}\left(S_0, \delta; K, r; \sigma, T\right) \\ &= S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2). \end{aligned}$$

So the idea is to express $S_T^a$ in the form of $S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)T + \sigma\widetilde{W}_T\right]$ (matching with the expression in [2.3.3]), with possibly different inputs $S_0, r, \delta, \sigma$, so that we can apply this familiar formula.

2.5.4 After some algebra, we can write

$$S_T^a = S_0^a \exp\left[a\left(r - \delta - \frac{\sigma^2}{2}\right)T + a\sigma\widetilde{W}_T\right] = S_0^a \exp\left[\left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)T + \sigma^*\widetilde{W}_T\right]$$

where $\sigma^* = a\sigma$ and $\delta^* = r - a(r - \delta) - \frac{1}{2}a(a-1)\sigma^2$. Practically, instead of memorizing the formulas of $\sigma^*$ and $\delta^*$ here, we can solve the following equation

$$a\left(r - \delta - \frac{\sigma^2}{2}\right)T + a\sigma\widetilde{W}_T = \left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)T + \sigma^*\widetilde{W}_T$$

for the unknowns $\sigma^*$ and $\delta^*$ by comparing coefficients, i.e., solving the following system:

$$\begin{cases} a\left(r - \delta - \dfrac{\sigma^2}{2}\right)T & = \left(r - \delta^* - \dfrac{(\sigma^*)^2}{2}\right)T, \\ a\sigma & = \sigma^*. \end{cases}$$

With the expression in terms of $\sigma^*$ and $\delta^*$, we can price the power option as follows:

$$C_0^{\text{power}} = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S_0^a \exp\left[\left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)T + \sigma^*\widetilde{W}_T\right] - K\right)_+\right] = \mathrm{BS}\left(S_0^a, \delta^*; K, r; \sigma^*, T\right).$$

2.5.5 Next, we consider a geometric average price call option with payoff $(G_T - K)_+$ where $G_T = (\prod_{i=1}^n S_{ih})^{1/n}$ with $T = nh$. The approach for pricing such option is the same, but the actual procedure would be more complicated.

Example: Consider a 3-year geometric average price call with payoff $(G_3 - K)_+$ with $G_3 = (S_1 \times S_2 \times S_3)^{1/3}$. Like the usual way to deal with the geometric average as discussed before, we first write

$$\begin{aligned} G_3 &= \left(S_0^3 e^{\left(r-\delta-\frac{\sigma^2}{2}\right)(1)+\sigma\widetilde{W}_1} e^{\left(r-\delta-\frac{\sigma^2}{2}\right)(2)+\sigma\widetilde{W}_2} e^{\left(r-\delta-\frac{\sigma^2}{2}\right)(3)+\sigma\widetilde{W}_3}\right)^{1/3} \\ &= S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)\frac{1+2+3}{3} + \frac{\sigma}{3}(\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3)\right] \\ &= S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)(2) + \frac{\sigma}{3}(\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3)\right]. \end{aligned}$$

Now, note that $\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3 = 3\widetilde{W}_1 + 2(\widetilde{W}_2 - \widetilde{W}_1) + (\widetilde{W}_3 - \widetilde{W}_2)$. By the independent increment property, we have $\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3 \overset{\text{RN}}{\sim} \mathrm{N}\left(0, 1^2 + 2^2 + 3^2\right) \equiv \mathrm{N}\left(0, 14\right)$.
Then, the key idea is to note that, in the risk-neutral world,

$$\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3 \overset{d}{=} \sqrt{\frac{14}{3}}\widetilde{W}_3$$

since both follow the $\mathrm{N}\left(0, 14\right)$ distribution. In view of this, our next step is to find $\delta^*$ and $\sigma^*$ such that

$$\begin{aligned} S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)(2) + \frac{\sigma}{3}(\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3)\right] &\overset{d}{=} S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)(2) + \frac{\sigma}{3}\sqrt{\frac{14}{3}}\widetilde{W}_3\right] \\ &= S_0 \exp\left[\left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)(3) + \sigma^*\widetilde{W}_3\right]. \end{aligned}$$

Practically, we need to solve the following equation

$$\left(r - \delta - \frac{\sigma^2}{2}\right)(2) + \frac{\sigma}{3}\sqrt{\frac{14}{3}}\widetilde{W}_3 = \left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)(3) + \sigma^*\widetilde{W}_3$$

for the unknowns $\sigma^*$ and $\delta^*$, by comparing coefficients, i.e., solving the following system:

$$\begin{cases} \left(r - \delta - \dfrac{\sigma^2}{2}\right)(2) &= \left(r - \delta^* - \dfrac{(\sigma^*)^2}{2}\right)(3), \\ \dfrac{\sigma}{3}\sqrt{\dfrac{14}{3}} &= \sigma^*. \end{cases}$$

[Note: It can be verified that we have

$$\delta^* = \frac{1}{2}\left(\frac{2}{3}r + \frac{4}{3}\delta + \frac{4}{27}\sigma^2\right) \quad \text{and} \quad \sigma^* = \sqrt{\frac{14}{27}}\sigma.$$

]

Finally, with the expression in terms of $\sigma^*$ and $\delta^*$, we can price the geometric average price call option as follows:

$$\begin{aligned} C_0^{\text{geo}-\text{price}} &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[(G_3 - K)_+] \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S_0 \exp\left[\left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)(3) + \sigma^*\widetilde{W}_3\right] - K\right)_+\right] \\ &= \mathrm{BS}\left(S_0, \delta^*; K, r; \sigma^*, T\right). \end{aligned}$$

2.5.6 Some natural follow-up questions on [2.5.5] would be the following.

Q1 How about geometric average *strike* call option with payoff $(S_T - G_T)_+$ where $G_T = (\prod_{i=1}^n S_{ih})^{1/n}$ with $T = nh$? Can we use a similar approach to price it?
*Answer.* Although such call option can be priced, the approach used is somewhat different. The approach in [2.5.5] does not work since both $S_T$ and $G_T$ are random variables. It turns out that we need to use the concept of *exchange option* (to be introduced in Section 2.6) for pricing it, so we defer the discussion on how to price such call option to [2.6.12].

Q2 How about *arithmetic* average price/strike options? Can we use a similar approach to price them?
*Answer.* It turns out that it is much more challenging to deal with arithmetic average options under Black-Scholes model since the arithmetic average $A_T = \sum_{i=1}^n S_{ih}/n$ (with $T = nh$) does *not* follow a lognormal distribution anymore. [⚠ Warning: Sum of lognormal random variables is not lognormally distributed! Instead, it follows a very complex distribution and is much less mathematically tractable. ] In fact, no closed-form formula for pricing arithmetic average options under Black-Scholes model is currently known! Consequently, one usually needs to resort to numerical methods for pricing arithmetic average options approximately, such as Monte Carlo simulation.

## 2.6 Exchange Options

2.6.1 *Exchange option* is a complex option with particular importance. It unifies the concepts of "call" and "put" options together, and they can be viewed just as special cases of "exchanging" something:

- *call:* buying an asset at a fixed strike price $\leftrightarrow$ "exchanging" the strike price ("pile of cash") for the asset
- *put:* selling an asset at a fixed strike price $\leftrightarrow$ "exchanging" the asset for the strike price ("pile of cash")

Generally, an exchange option can incorporate exchange of *any two* assets. Both assets can be risky, unlike the plain vanilla call/put options!

2.6.2 Setting: We are in a perfect market with two assets. For $i = 1, 2$:

- $S_t^{(i)}$ denotes the time-$t$ price of asset $i$.
- The "dividend yield" of asset $i$ is $\delta^{(i)}$.

Given two such assets, we consider a $T$-year European exchange option which gives us the right to exchange asset 2 for asset 1, or more precisely, give up one unit of asset 2 in return for one unit of asset 1, at time $T$. The time-$T$ payoff of the exchange option is then $(S_T^{(1)} - S_T^{(2)})_+$.

2.6.3 To price such exchange option, since two risky assets are involved, we need a more general assumption than the usual assumption in the Black-Scholes model. We shall assume that the process of the *ratio* of the two asset prices $\left\{\dfrac{S_t^{(1)}}{S_t^{(2)}}\right\}$ is a geometric Brownian motion with volatility parameter $\sigma$. Here, $\sigma$ is the volatility of the ratio of asset prices, so $\operatorname{Var}\left(\ln \dfrac{S_t^{(1)}}{S_t^{(2)}}\right) = \sigma^2 t$.

2.6.4 With this general assumption, we can obtain the following pricing formula in terms of "BS" function.

**Theorem 2.6.a.** Under the assumption in [2.6.3], the time-0 price of the $T$-year European exchange option with payoff $(S_T^{(1)} - S_T^{(2)})_+$ is

$$V_0 = \mathrm{BS}\left(S_0^{(1)}, \delta^{(1)}; S_0^{(2)}, \delta^{(2)}; \sigma, T\right) = S_0^{(1)} e^{-\delta^{(1)}T}\Phi(d_1) - S_0^{(2)} e^{-\delta^{(2)}T}\Phi(d_2)$$

where

$$d_1 = \frac{\ln\left(S_0^{(1)}/S_0^{(2)}\right) + (\delta^{(2)} - \delta^{(1)} + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

*Proof.* Take asset 2 as the "benchmark" asset (just like the benchmark currency). Using cash as benchmark, the time-$T$ payoff of the exchange option is $\$(S_T^{(1)} - S_T^{(2)})_+$. On the other hand, using asset 2 as benchmark, the time-$T$ payoff is

$$\frac{\$(S_T^{(1)} - S_T^{(2)})_+}{\$S_T^{(2)}} = \left(\frac{S_T^{(1)}}{S_T^{(2)}} - 1\right)_+ \text{ units of asset 2,}$$

which is just the time-$T$ payoff of a 1-strike plain vanilla European call option on asset 1, with all values and prices expressed in the units of asset 2. (Note that the time-$T$ price of asset 1 is $S_T^{(1)}/S_T^{(2)}$ units of asset 2.) Pricing this plain vanilla call using the Black-Scholes call price formula ([2.4.2]), the time-0 price is

$$\mathrm{BS}\left(\frac{S_0^{(1)}}{S_0^{(2)}}, \delta^{(1)}; 1, \delta^{(2)}; \sigma, T\right) = \frac{S_0^{(1)}}{S_0^{(2)}} e^{-\delta^{(1)}T}\Phi(d_1) - (1)e^{-\delta^{(2)}T}\Phi(d_2) \text{ units of asset 2,}$$

where

$$d_1 = \frac{\ln\left(\frac{S_0^{(1)}}{S_0^{(2)}}/1\right) + (\delta^{(2)} - \delta^{(1)} + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

(same as the ones given in the theorem). To convert this price to the price in the units of cash, we can just multiply it by $S_0^{(2)}$ to get:

$$V_0 = S_0^{(2)}\left[\frac{S_0^{(1)}}{S_0^{(2)}}e^{-\delta^{(1)}T}\Phi(d_1) - (1)e^{-\delta^{(2)}T}\Phi(d_2)\right] = S_0^{(1)}e^{-\delta^{(1)}T}\Phi(d_1) - S_0^{(2)}e^{-\delta^{(2)}T}\Phi(d_2).$$

□

2.6.5 Note that the assumption in [2.6.3] is only about the *ratio* of asset prices, but not the individual asset prices. It does not assume that the two asset prices are geometric Brownian motions individually.

If we make a stronger assumption that $\{S_t^{(1)}\}$ and $\{S_t^{(2)}\}$ form a *multivariate* geometric Brownian motion[4], then the two stock price processes would be geometric Brownian motions individually, say with volatilities $\sigma_1$ and $\sigma_2$ respectively. Here, we shall also assume that the correlation coefficient $\operatorname{Corr}\left(\ln\frac{S_t^{(1)}}{S_0^{(1)}}, \ln\frac{S_t^{(2)}}{S_0^{(2)}}\right) = \operatorname{Corr}\left(\ln S_t^{(1)}, \ln S_t^{(2)}\right)$ is a constant $\rho$ always. Then, the volatility of the resulting geometric Brownian motion for the ratio of prices can be obtained by

$$\sigma = \boxed{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}.$$

[Warning: We have $-2\rho\sigma_1\sigma_2$ rather than $+2\rho\sigma_1\sigma_2$ in the expression.]

This is sometimes known as the **blended volatility** as it "blends" the individual volatilities $\sigma_1$ and $\sigma_2$ together.

*Proof.* Note that

$$\begin{aligned}
\sigma^2 &= \operatorname{Var}\left(\ln\frac{S_1^{(1)}}{S_1^{(2)}}\right) \\
&= \operatorname{Var}\left(\ln S_1^{(1)} - \ln S_1^{(2)}\right) \\
&= \operatorname{Var}\left(\ln S_1^{(1)}\right) + \operatorname{Var}\left(\ln S_1^{(2)}\right) - 2\operatorname{Cov}\left(\ln S_1^{(1)}, \ln S_1^{(2)}\right) \\
&= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.
\end{aligned}$$

□

2.6.6 So far we have only considered "one-to-one" exchange option where one unit of asset is exchanged for one unit of another asset. In general, exchange option can also incorporate exchanging unequal numbers of assets. Consider a European exchange option which gives us the right to obtain $c_1$ units of asset 1 by giving up $c_2$ units of asset 2 at time $T$. The time-$T$ payoff would then become $(c_1S_T^{(1)} - c_2S_t^{(2)})_+$. What would be the impacts on the pricing?

2.6.7 It turns out that we only need to update two inputs in the "BS" function to get the time-0 price:

$$V_0 = \boxed{\operatorname{BS}\left(c_1S_0^{(1)}, \delta^{(1)}; c_2S_0^{(2)}, \delta^{(2)}; \sigma, T\right)}.$$

*Proof.* We can see this by viewing such exchange option as a special kind of "one-to-one" exchange option, where one "bundle" ("bundle 2") containing $c_2$ units of asset 2 is to be exchanged for another "bundle" ("bundle 1") containing $c_1$ units of asset 1.

[4] See STAT3911 for more details about this.

The current prices of bundles 1 and 2 are $c_1 S_0^{(1)}$ and $c_2 S_0^{(2)}$ respectively, hence the corresponding inputs in the "BS" function. The "dividend yields" of the bundles remain the same as the underlying asset as the number of assets available do not affect the growth rate. Next, consider

$$\mathrm{Var}\left(\ln \frac{c_1 S_1^{(1)}}{c_2 S_1^{(2)}}\right) = \mathrm{Var}\left(\ln \frac{c_1}{c_2} + \ln \frac{S_1^{(1)}}{S_1^{(2)}}\right) = \mathrm{Var}\left(\ln \frac{S_1^{(1)}}{S_1^{(2)}}\right) = \sigma^2,$$

so the volatility remains the same as well. $\square$

2.6.8 As an application, we can use the concepts of exchange options to price some other exotic derivatives: maximum and minimum claims. The idea is to express rewrite the payoff such that some terms match with the ones for exchange options, thereby "decomposing" the claims.

2.6.9 Consider first a $T$-year maximum contingent claim whose time-$T$ payoff is $\max\left\{S_T^{(1)}, S_T^{(2)}\right\}$, the maximum of the two asset prices. Writing

$$\max\{S_T^{(1)}, S_T^{(2)}\} = S_T^{(1)} + \left(S_T^{(2)} - S_T^{(1)}\right)_+ = S_T^{(2)} + \left(S_T^{(1)} - S_T^{(2)}\right)_+,$$

we see that the maximum claim can be decomposed into: (i) a $T$-year prepaid forward on <u>asset 1</u> + a time-$T$ exchange option for <u>exchanging asset 1 for asset 2</u>; or (ii) a $T$-year prepaid forward on <u>asset 2</u> + a time-$T$ exchange option for <u>exchanging asset 2 for asset 1</u>. These decompositions induce the following formulas for the time-0 price of maximum claim:

(i) $V_0^{\max} = \boxed{S_0^{(1)} e^{-\delta^{(1)}T} + \mathrm{BS}\left(S_0^{(2)}, \delta^{(2)}; S_0^{(1)}, \delta^{(1)}; \sigma, T\right)}$.

(ii) $V_0^{\max} = \boxed{S_0^{(2)} e^{-\delta^{(2)}T} + \mathrm{BS}\left(S_0^{(1)}, \delta^{(1)}; S_0^{(2)}, \delta^{(2)}; \sigma, T\right)}$.

[⚠ Warning: Be careful about the orders of the inputs in the "BS" functions!]

2.6.10 Next, consider a $T$-year minimum contingent claim whose time-$T$ payoff is $\min\left\{S_T^{(1)}, S_T^{(2)}\right\}$, the minimum of the two asset prices. We can utilize our knowledge on how to price the maximum claim for pricing this minimum claim. First write

$$\min\left\{S_T^{(1)}, S_T^{(2)}\right\} = S_T^{(1)} + S_T^{(2)} - \max\left\{S_T^{(1)}, S_T^{(2)}\right\}$$

(which should be somewhat familiar to students who have taken STAT3909...). With this decomposition, the time-0 price of the minimum claim can be obtained by

$$V_0^{\min} = \boxed{S_0^{(1)} e^{-\delta^{(1)}T} + S_0^{(2)} e^{-\delta^{(2)}T} - V_0^{\max}},$$

where $V_0^{\max}$ is the time-0 price of the corresponding maximum claim.

2.6.11 Alternatively, we can use a more direct approach as follows. Write instead (i)

$$\begin{aligned}\min\left\{S_T^{(1)}, S_T^{(2)}\right\} &= S_T^{(1)} + \min\left\{S_T^{(2)} - S_T^{(1)}, 0\right\} \\ &= S_T^{(1)} - \max\left\{S_T^{(1)} - S_T^{(2)}, 0\right\} \\ &= S_T^{(1)} - \left(S_T^{(1)} - S_T^{(2)}\right)_+,\end{aligned}$$

or (ii)

$$\begin{aligned}\min\left\{S_T^{(1)}, S_T^{(2)}\right\} &= S_T^{(2)} + \min\left\{S_T^{(1)} - S_T^{(2)}, 0\right\} \\ &= S_T^{(2)} - \max\left\{S_T^{(2)} - S_T^{(1)}, 0\right\} \\ &= S_T^{(2)} - \left(S_T^{(2)} - S_T^{(1)}\right)_+.\end{aligned}$$

Consequently, the time-0 price of the minimum claim is given by

(i) $V_0^{\min} = \boxed{S_0^{(1)}e^{-\delta^{(1)}T} - \text{BS}\left(S_0^{(1)}, \delta^{(1)}; S_0^{(2)}, \delta^{(2)}; \sigma, T\right)}$, or

(ii) $V_0^{\min} = \boxed{S_0^{(2)}e^{-\delta^{(2)}T} - \text{BS}\left(S_0^{(2)}, \delta^{(2)}; S_0^{(1)}, \delta^{(1)}; \sigma, T\right)}$.

[⚠ Warning: Again, be careful about the orders of the inputs in the "BS" functions!]

2.6.12 After learning the concept of exchange option, we are able to price the geometric average strike call option as mentioned in Q1, though the procedure would be even more complicated than the one in [2.5.5]!

For illustration, let us use a 3-year geometric average *strike* call with payoff $(S_3 - G_3)_+$ with $G_3 = (S_1 \times S_2 \times S_3)^{1/3}$ as an example. Like what we did in [2.5.5], we are going to handle the geometric average $G_3$ by treating it as the time-3 price of a "special stock". More specifically, we are going to treat this geometric average strike call as a 3-year exchange option granting us the right to exchange one unit of "special stock" ("asset 2" with time-3 price: $G_3$) for one unit of the normal stock ("asset 1" with time-3 price: $S_3$).

By [2.3.6], the geometric average strike call price is then

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}\big[(S_T - G_T)_+\big]$$

with $T = 3$. Here, the key idea is to "match" $S_T$ and $G_T$ with the expressions of $S_T^{(1)}$ and $S_T^{(2)}$ respectively, under the (stronger) assumption specified in [2.6.5], i.e., $\{S_t^{(1)}\}$ and $\{S_t^{(2)}\}$ are both geometric Brownian motions individually with constant correlation coefficient $\rho = \text{Corr}\left(\ln S_t^{(1)}, \ln S_t^{(2)}\right)$ always. Then by [2.3.3] we can write:

$$S_T^{(1)} = S_0^{(1)} \exp\left[\left(r - \delta^{(1)} - \frac{\sigma_1^2}{2}\right)T + \sigma_1\widetilde{W}_T^{(1)}\right]$$

and

$$S_T^{(2)} = S_0^{(2)} \exp\left[\left(r - \delta^{(2)} - \frac{\sigma_2^2}{2}\right)T + \sigma_2\widetilde{W}_T^{(2)}\right]$$

[Note: Generally, the "$\widetilde{W}_T$" for the prices of assets 1 and 2 could be different, so we add superscripts to differentiate them.]

Our goal is then to match the expressions of $S_T$ and $G_T$ with the ones of $S_T^{(1)}$ and $S_T^{(2)}$.

2.6.13 By [2.3.3], we have

$$S_3 = S_0 \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)(3) + \sigma\widetilde{W}_3\right].$$

By [2.5.5], we have

$$G_3 \overset{d}{=} S_0 \exp\left[\left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)(3) + \sigma^*\widetilde{W}_3\right]$$

with

$$\delta^* = \frac{1}{2}\left(\frac{2}{3}r + \frac{4}{3}\delta + \frac{4}{27}\sigma^2\right) \quad \text{and} \quad \sigma^* = \sqrt{\frac{14}{27}}\sigma.$$

Hence, we can match these expressions with the ones of $S_3^{(1)}$ and $S_3^{(2)}$ by setting:

- $S_0^{(1)} = S_0^{(2)} = S_0$;
- $\delta^{(1)} = \delta$ and $\delta^{(2)} = \delta^*$;
- $\sigma_1 = \sigma$ and $\sigma_2 = \sigma^*$.

The last piece of information that is missing for pricing the geometric average strike call is the value of the blended volatility $\sigma^{\text{blended}} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$. To find it, we need to deduce what the constant correlation coefficient $\rho$ is.

2.6.14 Since the correlation coefficient $\rho$ is constant by assumption, we can deduce its value by finding the time-3 correlation $\operatorname{Corr}\left(\ln S_3^{(1)}, \ln S_3^{(2)}\right) = \operatorname{Corr}(\ln S_3, \ln G_3)$. [Note: The correlation is the same in both real and risk-neutral worlds, so we will work in the risk-neutral world below.]

Consider first the covariance

$$
\begin{aligned}
\operatorname{Cov}(\ln S_3, \ln G_3) &= \operatorname{Cov}\left(\ln S_0 + \left(r - \delta - \frac{\sigma^2}{2}\right)(3) + \sigma\widetilde{W}_3, \ln S_0 + \left(r - \delta - \frac{\sigma^2}{2}\right)(2) + \frac{\sigma}{3}(\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3)\right) \\
&= \operatorname{Cov}\left(\sigma\widetilde{W}_3, \frac{\sigma}{3}(\widetilde{W}_1 + \widetilde{W}_2 + \widetilde{W}_3)\right) \qquad \text{(drop all non-random terms)} \\
&= \frac{\sigma^2}{3}\left[\operatorname{Cov}\left(\widetilde{W}_3, \widetilde{W}_1\right) + \operatorname{Cov}\left(\widetilde{W}_3, \widetilde{W}_2\right) + \operatorname{Cov}\left(\widetilde{W}_3, \widetilde{W}_3\right)\right] \\
&= \frac{\sigma^2}{3}(1 + 2 + 3) \qquad (1) \\
&= 2\sigma^2.
\end{aligned}
$$

[⚠ Warning: Substituting $G_3$ by the expression $S_0 \exp\left[\left(r - \delta^* - \frac{(\sigma^*)^2}{2}\right)(3) + \sigma^*\widetilde{W}_3\right]$ in the covariance could yield an incorrect result since this expression is only equal *in distribution* to $G_3$, and such equality in distribution does not guarantee that the covariance would remain unchanged after the substitution (unlike *marginal* probabilistic quantities such as their individual expectations). Thus, to obtain the correct covariance, we should use *actual* expressions of $S_3$ and $G_3$.]

To get (1), we note that for any $t = 1, 2$,

$$
\begin{aligned}
\operatorname{Cov}\left(\widetilde{W}_3, \widetilde{W}_t\right) &= \mathbb{E}\left[\widetilde{W}_3\widetilde{W}_t\right] - \underbrace{\mathbb{E}\left[\widetilde{W}_3\right]\mathbb{E}\left[\widetilde{W}_t\right]}_{0} \\
&= \mathbb{E}\left[\widetilde{W}_t(\widetilde{W}_3 - \widetilde{W}_t)\right] + \underbrace{\mathbb{E}\left[\widetilde{W}_t^2\right]}_{=\operatorname{Var}\left(\widetilde{W}_t\right)=t} \\
&= \underbrace{\mathbb{E}\left[\widetilde{W}_3\right]}_{0}\mathbb{E}\left[\widetilde{W}_3 - \widetilde{W}_t\right] + t \qquad \text{(independent increments)} \\
&= t,
\end{aligned}
$$

and of course $\operatorname{Cov}\left(\widetilde{W}_3, \widetilde{W}_3\right) = \operatorname{Var}\left(\widetilde{W}_3\right) = 3$.

Hence, the correlation coefficient is

$$
\rho = \operatorname{Corr}(\ln S_3, \ln G_3) = \frac{\operatorname{Cov}(\ln S_3, \ln G_3)}{\sqrt{\operatorname{Var}(\ln S_3)\operatorname{Var}(\ln G_3)}} = \frac{2\sigma^2}{\sigma\sqrt{3} \times \sigma^*\sqrt{3}} = \frac{2\sigma^2}{3\sqrt{\frac{14}{27}}\sigma^2} = \frac{2}{3}\sqrt{\frac{27}{14}}.
$$

Knowing the value of $\rho$, we can then obtain the blended volatility:

$$
\begin{aligned}
\sigma^{\text{blended}} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} &= \sqrt{\sigma^2 + (\sigma^*)^2 - \frac{4}{3}\sqrt{\frac{27}{14}}\sigma\sigma^*} \\
&= \sqrt{\sigma^2 + \frac{14}{27}\sigma^2 - \frac{4}{3}\sqrt{\frac{27}{14}}\sqrt{\frac{14}{27}}\sigma^2} \\
&= \sigma\sqrt{\frac{5}{27}}.
\end{aligned}
$$

2.6.15 After obtaining the value of blended volatility, we are finally (!) able to price the geometric average strike call option:

$$C_0^{\text{geo-strike}} = \text{BS}\left(S_0, \delta; S_0, \delta^*; \sigma^{\text{blended}}, T\right) = \text{BS}\left(S_0, \delta; S_0, \frac{1}{2}\left(\frac{2}{3}r + \frac{4}{3}\delta + \frac{4}{27}\sigma^2\right); \sigma\sqrt{\frac{5}{27}}, 3\right).$$

# 3 Hedging in Black-Scholes Model

3.0.1 After discussing how to utilize the Black-Scholes model to price different options in Section 2, we shall shift our focus to the usage of Black-Scholes model in hedging and risk management in Section 3.

## 3.1 Option Greeks

3.1.1 The Black-Scholes pricing function ("BS" function) introduced in Section 2 gives us the price of an option at a single time point (usually time 0). But for the purpose of risk management, this is not enough and we would also like to know how the option prices *vary* over time, due to the changes in input parameters as time elapses.

To investigate the risks arising from the variations in option price due to the changes in various parameters, a typical approach is *sensitivity analysis*, which can be implemented using option Greeks.

3.1.2 Mathematically, **option Greeks** are partial derivatives of an option price with respect to a single input of interest. (We investigate the effect from the parameters one at a time!) They are usually denoted by Greek letters, hence the name "option Greeks".

Practically, option Greeks can be interpreted as measures of magnitude of changes in option prices in response to (small) changes in one input parameter, holding other parameters fixed. This is an important interpretation that makes option Greeks an useful tool for risk management.

3.1.3 For each option Greek discussed here (delta $\Delta$, gamma $\Gamma$, theta $\Theta$), our exploration shall be guided by the following three questions of interest for risk managers:

Q1 How to compute the option Greek?

Q2 How do the option Greeks of European calls and puts (that are otherwise the same) compare?

Q3 **(Important!)** How does the option Greek behave *qualitatively*?

Here we shall focus on European calls and puts in the Black-Scholes model.

## 3.2 Delta

3.2.1 The **delta** of an option is the partial derivative of the option price with respect to the current price of the underlying asset. Letting time 0 be the current time, the delta is denoted by $

$$\Delta_0 = \frac{\partial V_0}{\partial S_0}$$

where $V_0$ and $S_0$ are the time-0 option and underlying asset prices respectively.

Delta can be interpreted as a measure of the sensitivity of the option price in response to the changes in the current price of the underlying asset. For example, if delta equals 1.2, then an unit increase the underlying asset price would *approximately* increase the option price by 1.2. (The approximation works better if the changes in the underlying asset prices are smaller.)

3.2.2 Now let us answer Q1 by deriving a closed-form formula for option delta. First we consider European call. Partially differentiating the Black-Scholes call price $C_0$ with respect to the current stock price $S_0$ gives

$$\Delta_0^C = \frac{\partial C_0}{\partial S_0} = \boxed{e^{-\delta T}\Phi(d_1)}.$$

*Proof.* [⚠ Warning: It may be tempting to just write

$$\Delta_0^C = \frac{\partial}{\partial S_0}[S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)] = \frac{\partial}{\partial S_0} S_0 \underbrace{e^{-\delta T}\Phi(d_1)}_{\text{"coefficient"}} = e^{-\delta T}\Phi(d_1)$$

as the "proof", but this is indeed mathematically problematic as $d_1$ and $d_2$ are also functions of $S_0$! (It is just a coincidence for the result to be the same as the correct one.) Nonetheless, this "proof" can serve as a mnemonic for the formula.]

Note first that

$$\frac{\partial d_2}{\partial S_0} = \frac{\partial}{\partial S_0}(d_1 - \underbrace{\sigma\sqrt{T}}_{\text{independent from } S_0}) = \frac{\partial d_1}{\partial S_0}.$$

Hence, using product rule and chain rule, we have

$$\Delta_0^C = e^{-\delta T}\Phi(d_1)+S_0e^{-\delta T}\phi(d_1)\frac{\partial d_1}{\partial S_0}-Ke^{-rT}\phi(d_2)\frac{\partial d_2}{\partial S_0} = e^{-\delta T}\Phi(d_1)+[S_0e^{-\delta T}\phi(d_1)-Ke^{-rT}\phi(d_2)]\frac{\partial d_1}{\partial S_0}$$

where $\phi$ denotes the standard normal pdf.

It then suffices to show that

$$S_0e^{-\delta T}\phi(d_1) - Ke^{-rT}\phi(d_2) = 0,$$

which follows from the somewhat lengthy algebra below:

$$\begin{aligned}
\frac{\phi(d_1)}{\phi(d_2)} &= \exp\left(-\frac{(d_1^2-d_2^2)}{2}\right) \\
&= \exp\left(-\frac{(d_1-d_2)(d_1+d_2)}{2}\right) \\
&= \exp\left(-\frac{(\sigma\sqrt{T})(2d_1-\sigma\sqrt{T})}{2}\right) \\
&= \exp\left[-\big(\ln(S_0/K)+(r-\delta+\sigma^2/2)T\big)+(\sigma^2/2)T\right] \qquad \left(d_1 = \frac{\ln(S_0/K)+(r-\delta+\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&= \exp\left[-\ln\frac{S_0}{K}-(r-\delta)T\right] \\
&= \frac{Ke^{-rT}}{S_0e^{-\delta T}}.
\end{aligned}$$

□

3.2.3 Next, we shall answer [3.4.3] by utilizing the put-call parity (and also deriving a closed-form formula for put delta). Consider European call and put that are otherwise the same: same strike price $K$ and time to maturity $T$, with current prices $C_0$ and $P_0$ respectively.

By put-call parity,

$$C_0 - P_0 = S_0e^{-\delta T} - Ke^{-rT}.$$

Partially differentiating both sides with respect to $S_0$ gives

$$\Delta_0^C - \Delta_0^P = e^{-\delta T}.$$

This suggests the relationship between call and put deltas:

$$\boxed{\Delta_0^P = \Delta_0^C - e^{-\delta T}};$$

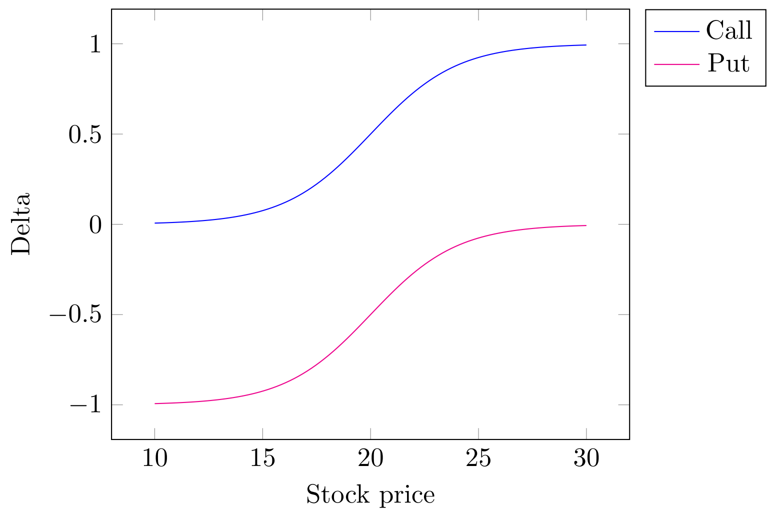
they just differ by a constant! Using this, we can also obtain a closed-form formula for put delta:

$$\Delta_0^P = e^{-\delta T}\Phi(d_1) - e^{-\delta T} = -e^{-\delta T}(1-\Phi(d_1)) = \boxed{-e^{-\delta T}\Phi(-d_1)},$$

where $d_1$ here is the one for European *call* (computed based on $C_0 = \mathrm{BS}(S_0,\delta;K,r;\sigma,T)$).

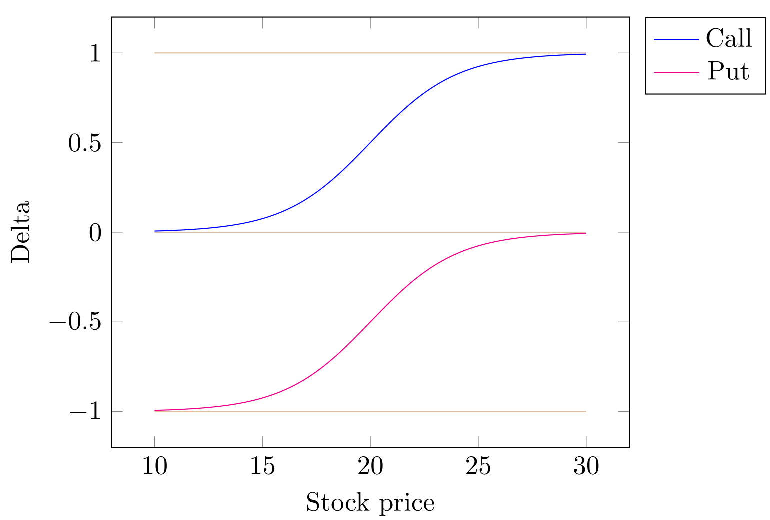
3.2.4 Last but definitely not least, we shall answer Q3 and describe some qualitative properties of delta through some intuitive reasoning (which is a bit informal, but can be justified mathematically under Black-Scholes model). We will also illustrate these properties using graphs, which give us some idea about what the "shape" looks like.

3.2.5 *Property 1: Deltas of call and put that are otherwise identical are parallel.* This is due to the constant difference $e^{-\delta T}$ between call and put delta observed when answering [3.4.3]. Hence, call and put delta share many similar features.




3.2.6 *Property 2: Call and put deltas satisfy the bounds:* $\boxed{0 < \Delta_0^C < 1, \quad -1 < \Delta_0^P < 0}$.

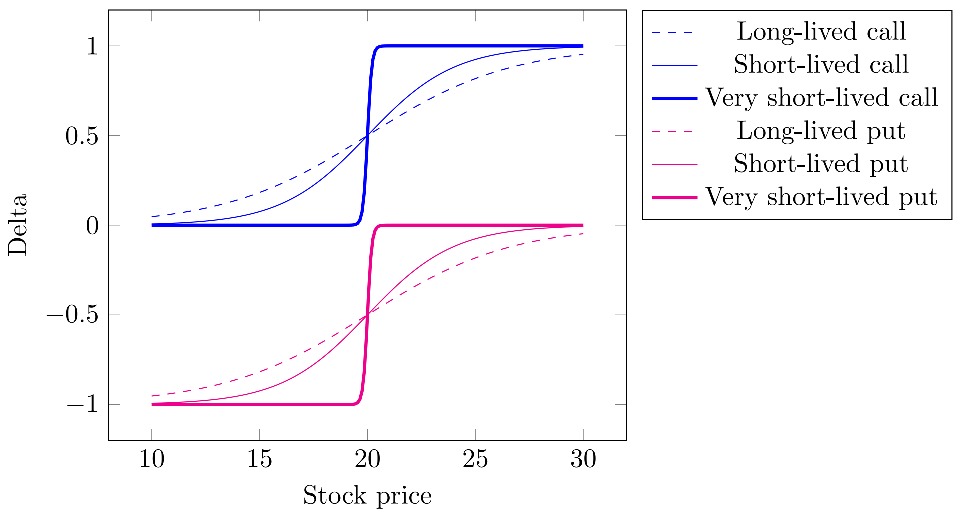
- *(sign of call and put deltas)* As the stock price ↑, the call (put), which gives a right to acquire (lose) the now more valuable stock, also becomes more (less) valuable, which leads to a higher (lower) fair price **$**. Hence, the call (put) delta is positive (negative).
- *(not too sensitive)* The call (put) delta is always less than 1 (greater than −1), since even if the stock price ↑ by \$1, it is irrational to pay \$1 more (less) for just a *right* to buy (sell) the stock at a fixed strike price. (If the option turns out to expire worthless, then this ↑ in stock price means nothing!)



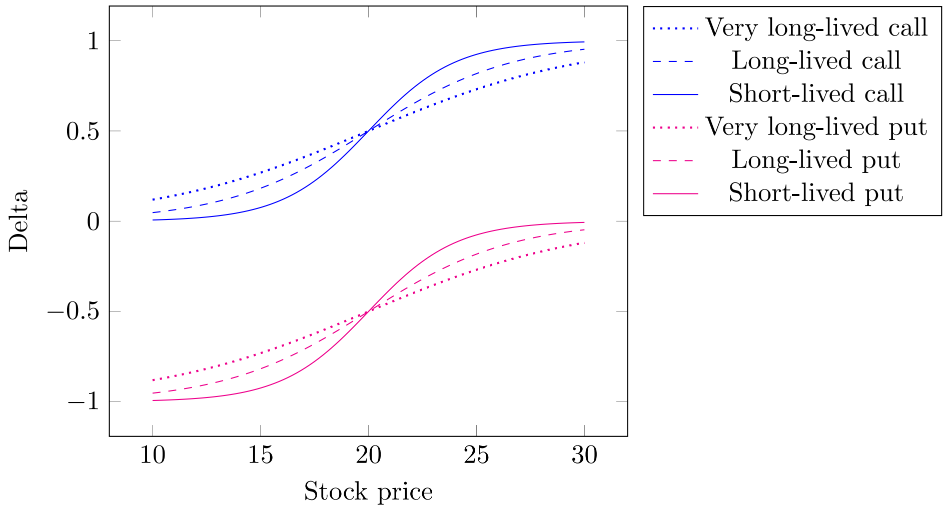
3.2.7 *Property 3: Delta is an increasing function of $S_0$, with the most rapid increase occurring near strike price especially for short-lived options.*

- *(increasing in $S_0$)* As $S_0$ 🡅, the call (put) is more (less) likely to expire in-the-money, thus being more (less) sensitive to stock price changes. Hence, the call (put) delta becomes more positive (less negative).
- *(steep increase near strike price, especially for short-lived options)* As $S_0$ increases past the strike price $K$, the call (put) is much more (less) likely to expire in-the-money, especially for short-lived ones as there is not much room for further movements in stock prices for them.
- *(delta of deep in-the-money option is close to 1 in absolute value)* We can observe that $\Delta_0^C \approx 1$ for very high $S_0$ and $\Delta_0^P \approx -1$ for very low $S_0$, meaning that a deep in-the-money call/put is extremely sensitive to stock price changes — its price moves in almost the same magnitude as the stock price!

  This is because a deep in-the-money call/put would almost surely expire in the money, so an unit increase in $S_0$ would lead to almost an unit increase (decrease) in the call (put) price.
- *(delta of deep out-of-the-money option is close to 0)* We can observe that $\Delta_0^C \approx 0$ for very low $S_0$ and $\Delta_0^P \approx 0$ for very high $S_0$, meaning that a deep out-of-the-money has almost no sensitivity to stock price changes. This is because a deep out-of-the-money call/put would almost surely expire worthless, so changes in $S_0$ would hardly lead to any change in the call/put price (which would be very close to 0 as the call/put is almost a "trash" 🗑).




3.2.8 *Property 4: Delta gets larger in absolute value as $T$ increases for deep out-of-the-money (in-the-money) option (with $S_0$ fixed).* As a deep out-of-the-money (in-the-money) option has a longer time to live, there is a larger room for further movements in stock prices, thus it has a higher chance to eventually expire in-the-money (out-of-the-money), hence being more (less) sensitive to the stock price changes.




### 3.3 Gamma

3.3.1 From the graphs for the delta in Section 3.2, we observe that delta can vary drastically as $S_0$ changes, so we would like to also study the sensitivity of delta in response to stock price changes (sensitivity of sensitivity measures!). This can be done by using gamma.

The **gamma** of an option is the partial derivative of the option delta with respect to the current price of the underlying asset, i.e., the second partial derivative of the option price with respect to the current asset price:

$$\Gamma_0 = \frac{\partial \Delta_0}{\partial S_0} = \frac{\partial^2 V_0}{\partial S_0^2}.$$

Gamma can be interpreted as (i) a measure of the sensitivity of the delta in response to asset price changes, or (ii) a measure of the convexity of the option price as a function of $S_0$. For the purpose of understanding the qualitative properties, the interpretation (i) is usually more helpful.

3.3.2 To answer Q1 for option gamma, we derive a closed-form formula for the gamma of a European call. By [3.2.2] we have $\Delta_0^C = e^{-\delta T}\Phi(d_1)$. Then, using chain rule, partially differentiating both sides with respect to $S_0$ gives

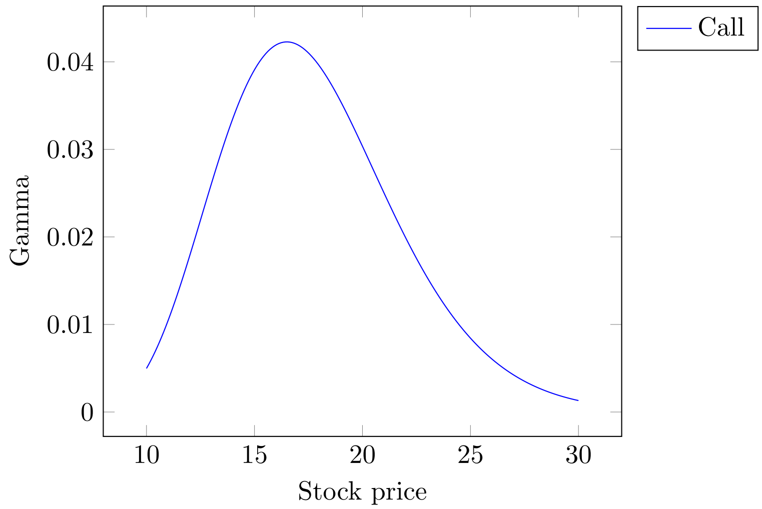
$$\Gamma_0^C = \frac{\partial}{\partial S_0}[e^{-\delta T}\Phi(d_1)] = e^{-\delta T}\phi(d_1)\frac{\partial d_1}{\partial S_0} = \boxed{\frac{e^{-\delta T}\phi(d_1)}{S_0\sigma\sqrt{T}}}.$$

3.3.3 Next, we will answer [3.4.3], by partially differentiating the relation $\Delta_0^P = \Delta_0^C - e^{-\delta T}$ from [3.2.3], with respect to $S_0$:

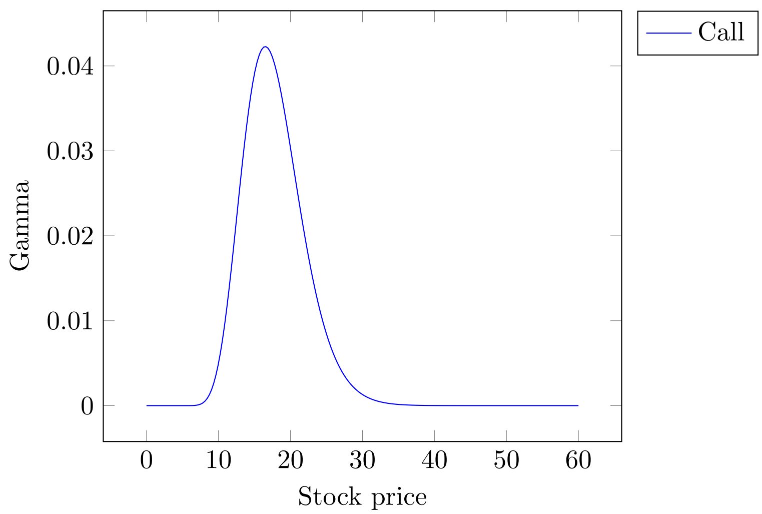
$$\frac{\partial \Delta_0^P}{\partial S_0} = \frac{\partial \Delta_0^C}{\partial S_0} - \underbrace{\frac{\partial e^{-\delta T}}{\partial S_0}}_{0} \implies \Gamma_0^P = \Gamma_0^C,$$

meaning that the gamma of call and put that are otherwise identical are always the same. Consequently, when answering Q3 in the following, it suffices to focus on call gamma, without loss of generality.

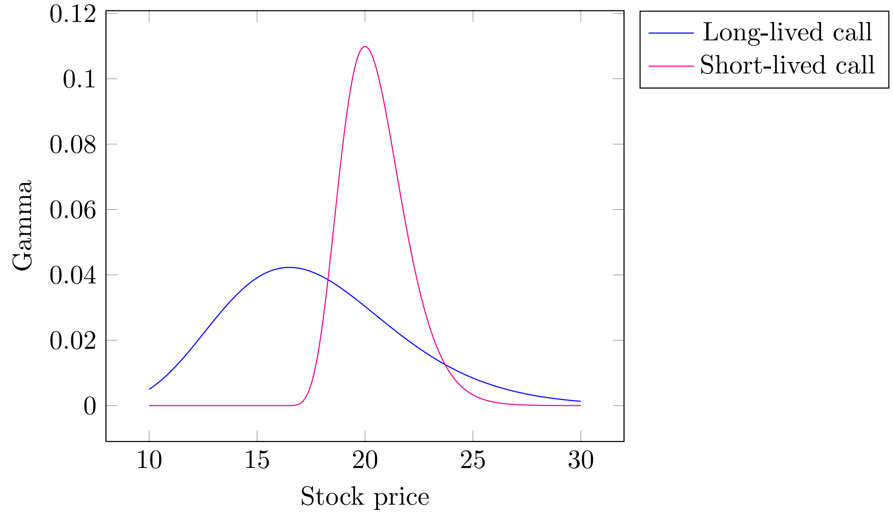
3.3.4 *Property 1: Gamma is always positive.* This is because the delta is a strictly increasing function of $S_0$, as we have seen in Section 3.2.




3.3.5 *Property 2: Gamma is almost zero for deep in-the-money/out-of-the-money option.* When the call is deep in-the-money (out-of-the-money), the delta is extremely close to 1 (0) and hardly response to any change in stock price.




3.3.6 *Property 3: Gamma is very large when the stock price is near the strike price, especially for short-lived options.* This follows from the property of delta that it rises most rapidly near the strike price especially for short-lived options, thus the rate of change of delta is very large around that region, hence a larger gamma.




## 3.4 Theta

3.4.1 The next and final Greek to be discussed here is related to the impact of *passage of time* on the option price. As passage of time is involved, we need to analyze the option prices dynamically instead of just focusing on the time-0 price.

Consider a $T$-year European option. The time-$t$ price of the option ($0 \le t \le T$), denoted by $V_t$, is based on the time-$t$ underlying asset price $S_t$ and the remaining time to expiration $T-t$.

The (time-$t$) **theta** is the partial derivative of the time-$t$ option price with respect to the current time $t$ (here we treat time $t$ as "now"):

$$\Theta_t = \frac{\partial V_t}{\partial t}.$$

The theta $\Theta_t$ can be interpreted as the approximated change in the option price $V_t$ when the remaining time to expiration $T-t$ *decreases* by 1 year (i.e., 1 year has "passed"). Treating 1 day to be 1/365 years, the approximated change after the "passage" of 1 day can be obtained by $\Theta_t \times (1/365)$.

3.4.2 We first answer Q1 for theta. It turns out that theta has a rather complicated closed-form formula. Consider a European call and its time-$t$ Black-Scholes call price

$$C_t = \mathrm{BS}\left(S_t, \delta; K, r; \sigma, T-t\right) = S_t e^{-\delta(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where $d_1 = \dfrac{\ln(S_t/K) + (r-\delta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$. Partially differentiating the call price $C_t$ with respect to the current time $t$ while holding all other things fixed[5] gives

$$\begin{aligned}
\Theta_t^C &= \delta S_t e^{-\delta(T-t)}\Phi(d_1) + S_t e^{-\delta(T-t)}\phi(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\phi(d_2)\frac{\partial d_2}{\partial t} \\
&= \delta S_t e^{-\delta(T-t)}\Phi(d_1) - rKe^{-r(T-t)}\Phi(d_2) + S_t e^{-\delta(T-t)}\phi(d_1)\frac{\partial d_1}{\partial t} - Ke^{-r(T-t)}\phi(d_2)\left(\frac{\partial d_1}{\partial t} - \frac{\sigma}{2\sqrt{T-t}}\right) \\
&= \delta S_t e^{-\delta(T-t)}\Phi(d_1) - rKe^{-r(T-t)}\Phi(d_2) + \underbrace{\left[S_t e^{-\delta(T-t)}\phi(d_1) - Ke^{-r(T-t)}\phi(d_2)\right]}_{0}\frac{\partial d_1}{\partial t} \\
&\quad - Ke^{-r(T-t)}\phi(d_2)\frac{\sigma}{2\sqrt{T-t}} \\
&= \boxed{\delta S_t e^{-\delta(T-t)}\Phi(d_1) - rKe^{-r(T-t)}\Phi(d_2) - \frac{Ke^{-r(T-t)}\phi(d_2)\sigma}{2\sqrt{T-t}}}.
\end{aligned}$$

[5] In particular, we treat $S_t$ as a fixed value that does not change with time $t$. In other words, after the time changes from $t$ to some other time $t^*$, we still use the time-$t$ price $S_t$ (the price at the original time point).

where $S_te^{-\delta(T-t)}\phi(d_1)-Ke^{-r(T-t)}\phi(d_2)=0$ can be shown by using a similar argument as in the proof for the formula for call delta.

3.4.3 After that, we shall answer for theta. Again we will utilize the put-call parity. Partially differentiating both sides of the put-call parity $C_t-P_t=S_te^{-\delta(T-t)}-Ke^{-r(T-t)}$ with respect to $t$ gives

$$\Theta_t^C-\Theta_t^P=\delta S_te^{-\delta(T-t)}-rKe^{-r(T-t)}.$$

Rearranging this gives

$$\begin{aligned}\Theta_t^P&=\boxed{\Theta_t^C-\delta S_te^{-\delta(T-t)}+rKe^{-r(T-t)}}\\&=\delta S_te^{-\delta(T-t)}\Phi(d_1)-rKe^{-r(T-t)}\Phi(d_2)-\frac{Ke^{-r(T-t)}\phi(d_2)\sigma}{2\sqrt{T-t}}-\delta S_te^{-\delta(T-t)}+rKe^{-r(T-t)}\\&=\boxed{-\delta S_te^{-\delta(T-t)}\Phi(-d_1)+rKe^{-r(T-t)}\Phi(-d_2)-\frac{Ke^{-r(T-t)}\phi(d_2)\sigma}{2\sqrt{T-t}}},\end{aligned}$$

where $d_1$ and $d_2$ are based on the time-$t$ call price $C_t$.

3.4.4 Lastly, we will deal with the somewhat hard-to-answer question: Q3 for theta. As hinted by the complicated formulas for call and put thetas, one can expect that the qualitative properties of theta are rather complex and the explanations would be quite intricate.

The complex behaviour of theta can be explained through analyzing the following opposing forces:

F1 $t\uparrow\implies$ *less uncertainty ahead* $\implies V_t\downarrow$.

The option price increases as there is more uncertainty ahead, due to the *asymmetric* option payoffs. For a call (put) option:

- If the stock price falls below (rises above) the strike price, the payoff would be zero, no matter how far the stock price is away from the strike price.
- On the other hand, if the stock price rises above (falls below) the strike price, for every unit increase in the difference, the call payoff would increase by the same amount.

Hence, as the stock price fluctuates in a larger magnitude, the option holder would benefit more from the upside and would *not* lose more from the downside, making the option more valuable.

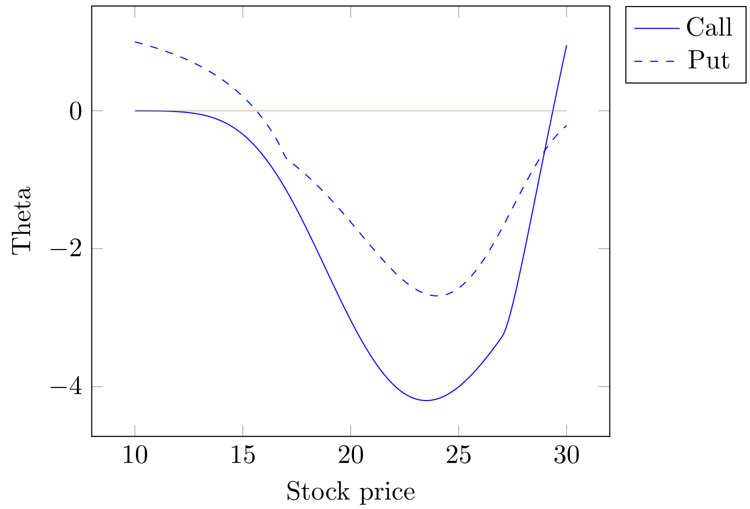
F2
- *call:* $t\uparrow\implies$ *receive underlying asset sooner* $\implies V_t\uparrow$.
- *put:* $t\uparrow\implies$ *receive strike price sooner* $\implies V_t\uparrow$.

As the remaining time to expiration reduces, the holder of a call (put), if exercised, will receive the underlying stock (strike price) and earn dividends (interest) sooner, making the option more valuable.

F3
- *call:* $t\uparrow\implies$ *lose strike price sooner* $\implies V_t\downarrow$.
- *put:* $t\uparrow\implies$ *lose underlying asset sooner* $\implies V_t\downarrow$.

As the remaining time to expiration reduces, the holder of a call (put), if exercised, will lose the strike price (underlying asset) and lose interest (dividends) sooner, making the option less valuable.

3.4.5 *Property 1: Theta is mostly negative.* In view of this, we say that most options suffer from **time decay**. This is because the force F1 is dominating for most options, hence the option price drops as the remaining time to maturity decreases (all else equal).

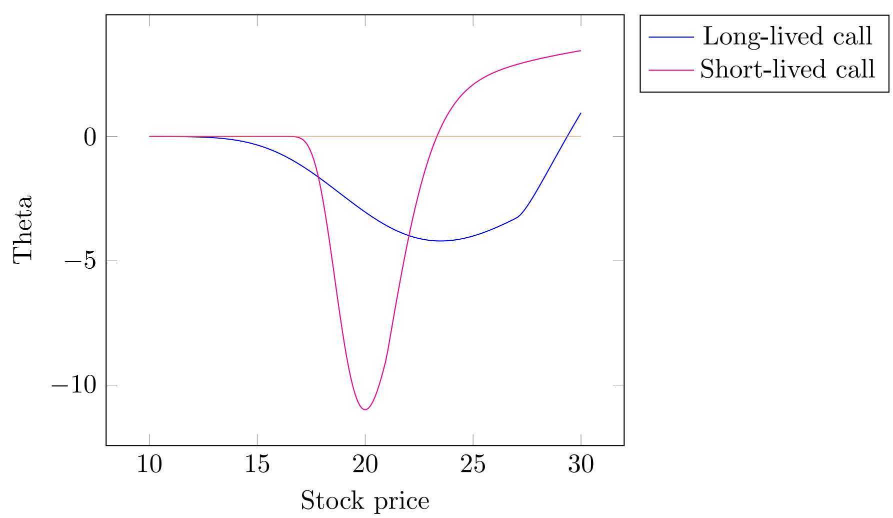


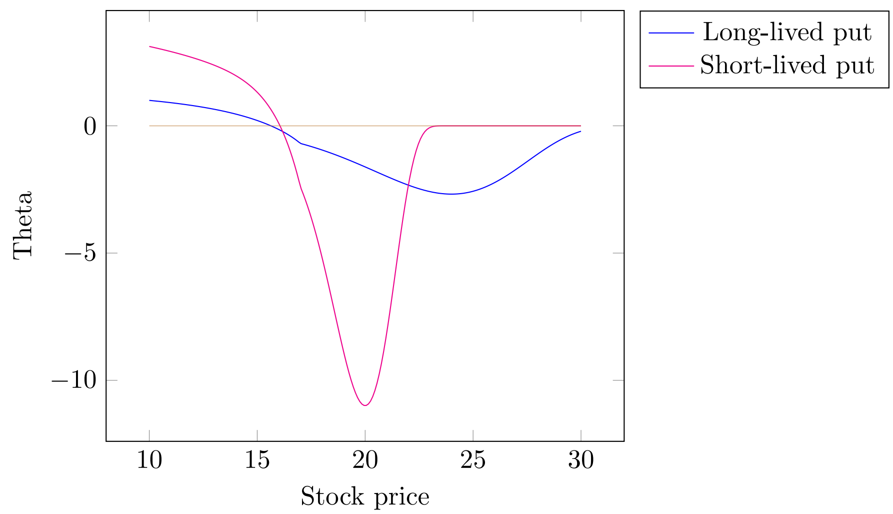

Yet, in some rare cases, theta can be positive. For this to happen, the force F2 needs to outweigh the forces F1 and F3 combined. Examples:

- *deep in-the-money call on a dividend-paying stock:* The time-$T$ payoff of such deep-in-the-money call (with very small strike price $K$) is extremely close to $S_T$, so the call is extremely similar to a prepaid forward on the stock. Hence, its time-$t$ price is $C_t \approx S_t e^{-\delta(T-t)}$, which is a strictly increasing function of $t$ when $\delta > 0$. ($S_t$ is held constant and does not vary with $t$.)
- *deep in-the-money put:* The time-$T$ payoff of such deep-in-the-money put (with very large strike price $K$) is extremely close to $K$, so the put is extremely similar to a zero-coupon bond of $K$ maturing at time $T$. Hence, its time-$t$ price is $P_t \approx Ke^{-r(T-t)}$, which is also a strictly increasing function of $t$.

3.4.6 *Property 2: Theta is close to zero for deep out-of-the-money options, especially for short-lived ones.* This is because deep out-of-the-money option has a very high chance to expire worthless, thus does not response much to changes in the remaining time to maturity.

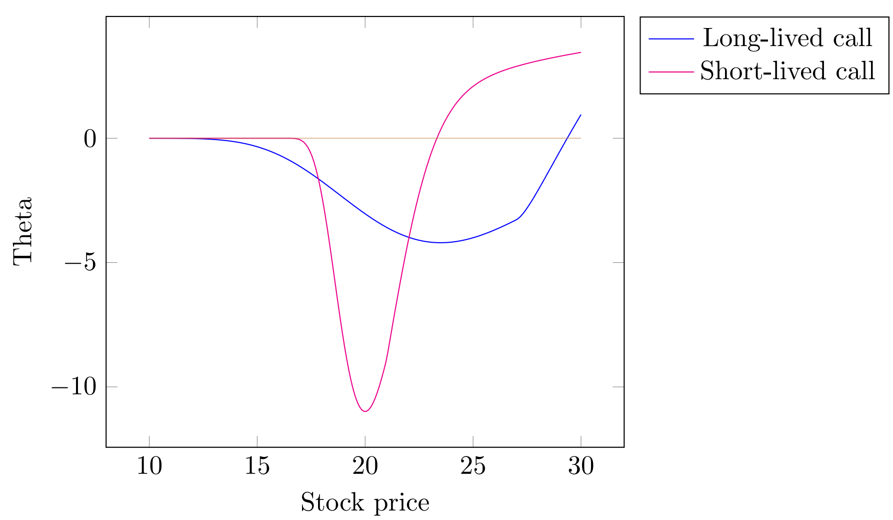
For short-lived options in particular, as there is only a very short time left for the stock price to change, the option would almost surely expire worthless, hence there is virtually no sensitivity to changes in the remaining time to maturity.

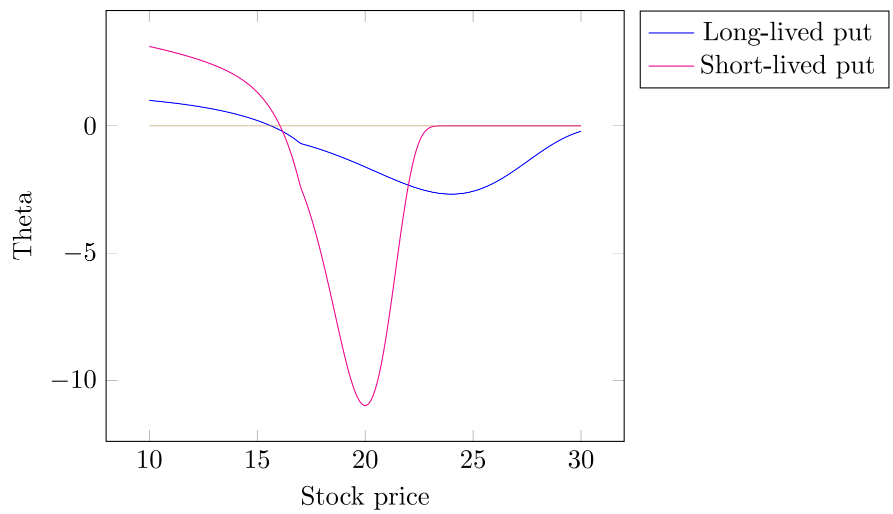





3.4.7 *Property 3: Theta is very negative for at-the-money options, especially for the short-lived ones.* For at-the-money option, changes in stock price often "passes through" the strike price, hence such option gains a lot from the asymmetry of option payoff. Thus, as the remaining time to maturity drops, the less uncertainty ahead as suggested in the force F1 reduces a huge amount of the option price.

This effect is more significant for short-lived at-the-money options since a tiny drop in the remaining time to maturity already leads to a relatively large decrease in the time to maturity.






## 3.5 Delta-Hedging

3.5.1 After discussing three option Greeks: delta ($\Delta$), gamma ($\Gamma$), theta ($\Theta$), we shall discuss how they are useful for designing and implementing hedging strategies (under Black-Scholes model). Firstly, we will discuss a simple hedging strategy known as *delta hedging*.

3.5.2 For illustration purpose, in Section 3, we shall consider a setting where hedging is critical and we will analyze how different hedging strategies can protect us. We shall consider the perspective from a **market-maker**, who is an intermediary or trader that is ready to buy (sell) derivatives from customers who wish to sell (buy), i.e., a counterparty to the customers.

For example, assume that a market-maker has *sold* a European call option on a nondividend-paying stock and has therefore incurred a delta of $-\Delta_0^C$ in his portfolio.

[Note: In general, the greek of a *portfolio* is also a partial derivative with respect to the same input of interest, but the "option price" is replaced by "portfolio value". For example, in this case, the market-maker's portfolio consists of $-1$ unit of call, hence the portfolio delta is $\dfrac{\partial(-C_0)}{\partial S_0} = -\dfrac{\partial C_0}{\partial S_0} = -\Delta_0^C$.
]

3.5.3 Since the market-maker's delta is negative[6], he is exposed to *upside* stock price risk, i.e., the risk that the stock price rises in the future. To be more specific, we consider the **overnight profit** (i.e., the profit on the next day, or at time 1/365) of the market-maker.

The overnight profit can be computed by

$$-C_{1/365} + C_0 e^{r/365},$$

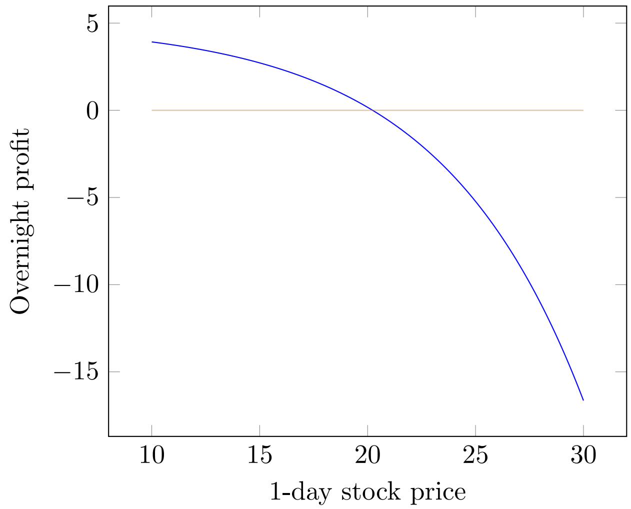
which is the payoff at time 1/365 of the following actions performed at time 0 that result in zero net cash flow at that time:

(1) sell a European call (time-0 CF: $+C_0$)

(2) lend $C_0$ (time-0 CF: $-C_0$)

Then, at time 1/365, the action (1) results in a payoff of $-C_{1/365}$ (from closing out the short call position) and the action (2) results in a payoff of $+C_0 e^{r/365}$.

[6]Recall that call delta is always positive.




3.5.4 We can observe that as $S_{1/365}$ rises, the overnight profit drops *indefinitely*! There is no limit on how much the market-maker can lose, meaning that he can potentially *bankrupt*! In view of this, there is a desperate need to hedge against this dangerous position ⚠.

[Note: Sometimes the act of selling a call without owning the underlying asset is called selling a **naked call** since it is as dangerous as being "naked"...]

3.5.5 Market-makers earn profit by buying at the bid price (from customers who wish to sell immediately) and selling at the ask price (from customers who wish to buy immediately) to capture by the bid-ask spread, instead of speculating on the asset prices in the market.

Thus, it is of great interest for market-makers to protect themselves against the risk arising from their actions, by making the profit graph "flatter" (having less potential loss at the cost of having less potential profit from price movements as well).

3.5.6 A simple way of protection against the risk from selling/writing a call is *delta-hedging*. To form **delta-hedging**, we perform the following actions at time 0, in the case of selling call:

(1) sell a European call (time-0 CF: $+C_0$)

(2) buy $\Delta_0^C$ shares of stock to *delta-hedge* the market-maker's position (time-0 CF: $-\Delta_0^C S_0$)

(3) borrow $Ke^{-rT}\Phi(d_2)$ at risk-free interest rate, or investing $-Ke^{-rT}\Phi(d_2)$ in a risk-free bond (time-0 CF: $+Ke^{-rT}\Phi(d_2)$)

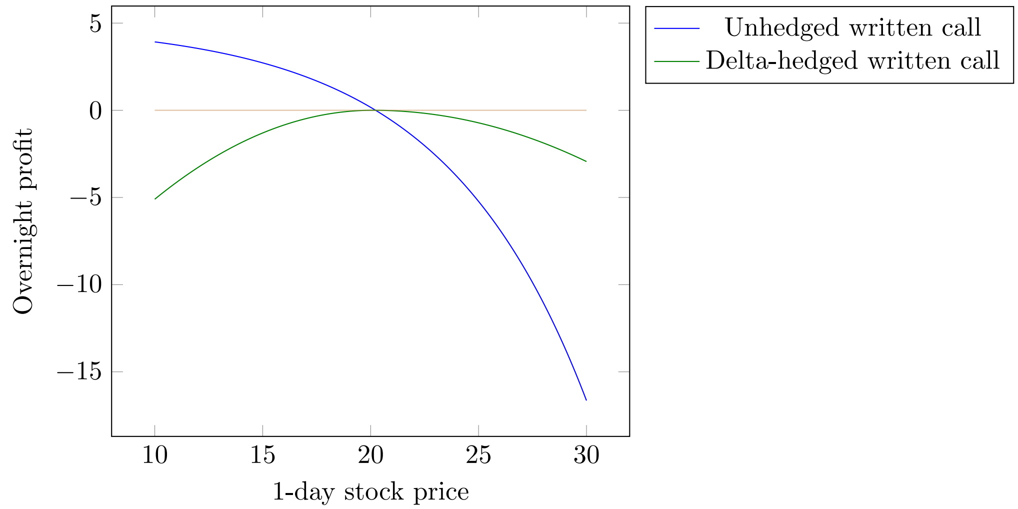
Note that the time-0 net cash flow of these actions is

$$+C_0 - \Delta_0^C S_0 + Ke^{-rT}\Phi(d_2) = S_0 \underbrace{e^{-\delta T}\Phi(d_1)}_{\Delta_0^C} - Ke^{-rT}\Phi(d_2) - \Delta_0^C S_0 + Ke^{-rT}\Phi(d_2) = 0.$$

After these actions, the delta of market-maker's portfolio is exactly zero at time 0, and the portfolio is said to be **delta-neutral**.

3.5.7 After that, the time-$t$ profit of a delta-hedged written call is given by

$$\underbrace{-C_t}_{(1)} \underbrace{+\Delta_0^C S_t}_{(2)} \underbrace{-B_0 e^{rt}}_{(3)}.$$




We can observe that after delta-hedging, the market-maker's profit declines at a more controllable rate as the stock price increases, meaning that the upside stock price risk drops a lot. In exchange for this protection, the delta-hedged written call has a negative and decreasing profit when the stock price decreases, unlike the unhedged written call. Nonetheless, in terms of potential loss, the delta-hedged written call would be safer.

## 3.6 Hedging Multiple Option Greeks

3.6.1 Delta-hedging is by no means perfect. In fact, even the market maker delta hedges his short call position, there would still be concern on the upside stock price risk since it is not completely *eliminated*. The potential loss is still unlimited, and the market-maker still has a potential to bankrupt (although with a much lower chance).

3.6.2 This issue arises since delta-hedging can only form a *local* protection against the stock price risk. When the 1-day stock price is near the time-0 price, the overnight profit can be ensured to be very close to zero by the design of delta-hedging.

However, as there is a larger stock price movement and the stock price is quite far away from the time-0 price, the protection would start to fall apart and fail to work. This is because the delta varies as the stock price changes. More specifically, since the market-maker's portfolio has a negative gamma, the portfolio delta drops as the stock price rises. Particularly, the market-maker's delta becomes negative again when the stock price rises from the time-0 price, and thus his portfolio is no longer delta-neutral.

3.6.3 Since the root cause for the issue faced by delta-hedging is the negative gamma, a natural way to tackle this is to not just neutralize delta, but also neutralize gamma. This motivates us to hedge multiple Greeks (both delta and gamma in this context).

However, since the gamma of the underlying stock is $\dfrac{\partial^2 S_0}{\partial S_0^2} = 0$, buying/selling stocks is not helpful for neutralizing the gamma. We need to also trade other securities in order to neutralize it.

3.6.4 Consider the general case where we want to hedge a given option position with respect to $m$ option Greeks by using $m$ securities (in addition the given option). For each $i = 1, \dots, m$, we let $x_i$ be the number of units to buy for the $i$th security. [Note: When $x_i < 0$, it means that we *sell* $-x_i$ units of the $i$th security.]

To find the appropriate number of units to buy for each security that can neutralize all the $m$ Greeks, we need to solve the following system of $m$ linear equations (one for each Greek) for $x_1, \ldots, x_m$:

$$\begin{cases} \text{Greek 1 of whole portfolio} = 0 \\ \text{Greek 2 of whole portfolio} = 0 \\ \qquad\qquad \vdots \\ \text{Greek } m \text{ of whole portfolio} = 0, \end{cases}$$

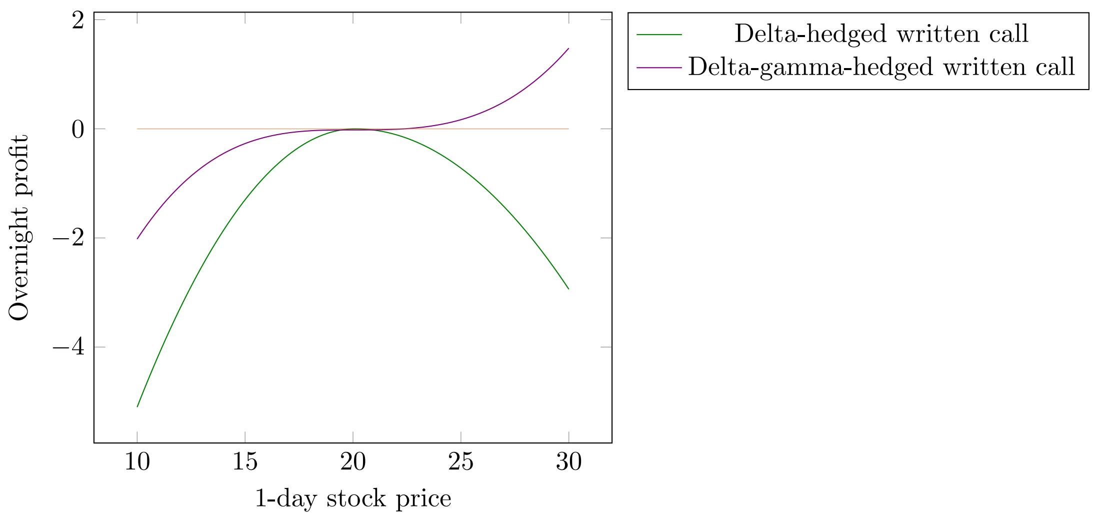
where the LHS of each equation is a linear combination of $x_1, \ldots, x_m$, depending on the values of Greeks for each security.

3.6.5 In our case here, we want to hedge both delta and gamma. Suppose that the call sold by the market-maker is of 20-strike. Then, we may utilize a 25-strike call on the same underlying asset to perform the delta-gamma-hedging. Let $x_1$ and $x_2$ be the number of units to buy for the stock and the 25-strike call. Then, the system we need to solve is

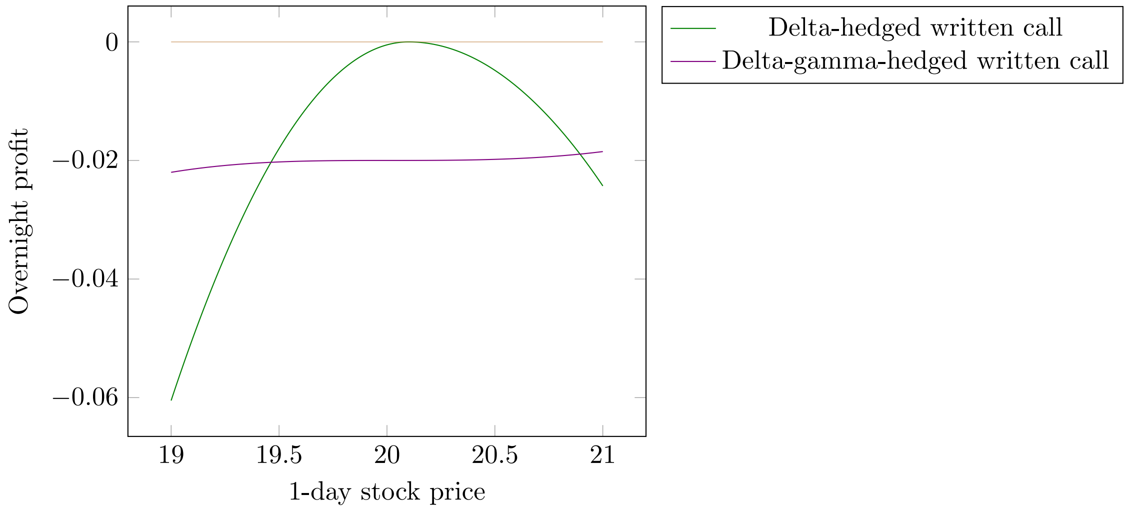
$$\begin{cases} x_1(1) + x_2\Delta_0^{C,K=25} - \Delta_0^{C,K=20} & = 0 \\ x_1(0) + x_2\Gamma_0^{C,K=25} - \Gamma_0^{C,K=20} & = 0 \end{cases}$$

where the superscripts of the Greeks signify the call strike prices.

After performing the delta-gamma-hedging, the overnight profit would look something like below.




To inspect the graph more clearly, we zoom in the graph for the part where the stock price is near 20:




We can observe that delta-gamma-hedging has a much better performance than delta-hedging. Not only the upside stock price risk is virtually nonexistent, the downside stock price risk is also reduced (as the overnight profit when the stock price drops becomes less negative). In exchange for this improvement, when the stock price does not move much after 1 day (which is quite likely in such a short time span), the overnight profit becomes more negative. Still, delta-gamma-hedging is a much more prudent and effective hedging strategy than mere delta-hedging.

## 3.7 Dynamic Hedging

3.7.1 Throughout Section 3.7, we shall assume that the underlying stock does not pay dividends.

Apart from hedging multiple option Greeks, another possible approach to improve the effectiveness of delta-hedging is to perform *dynamic hedging*, somewhat like what we do in Section 1.1.

3.7.2 Like Section 1.1, here we also need to construct replicating portfolios dynamically, but we are doing that in the Black-Scholes model rather than the binomial tree model. [Note: In this context, another name for replicating portfolio is **hedge portfolio**.] Recall from Section 3.5 that to perform delta-hedging of a written call, we need to do the following actions at time 0 (in addition to selling the call):

- *(stock part)* buying $\Delta_0^C = \Phi(d_1)$ shares of the stock
- *(bond part)* investing $B_0 = -Ke^{-rT}\Phi(d_2)$ in a risk-free bond

Note that the number of shares in the stock part is the "coefficient" of $S_0$ in the Black-Scholes pricing formula, and the value for the bond part is precisely the other term in that formula. [Note: This is the same in the case of delta-hedging of a written *put*.]

Hence, these actions form a portfolio with time-0 price

$$\Delta_0^C S_0 + B_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) = C_0,$$

which replicates the value of the call at time 0. However, as time passes, the call value would change and hence the old portfolio $(\Delta_0^C, B_0)$ is generally unable to replicate it. We need to adjust the stock and bond parts in the portfolio, just like Section 1.1.

3.7.3 But under the Black-Scholes model, the call price can *vary continuously*, unlike the case in the binomial tree model. Consequently, as one may expect, the adjustment needs to take place in a continuous manner to successfully complete the replication. It turns out that such continuous rebalancing requires no cost (the portfolio is self-financing). More details will be discussed in STAT3911.

3.7.4 In practice, it is impossible to rebalance continuously. One can only do so at discrete time points. So here we shall investigate the effect of rebalancing at some regular intervals, like monthly, weekly, daily, etc.

Since we are only doing rebalancing at discrete time points rather than in a continuous manner, our dynamic replicating portfolio will neither be able complete the replication *perfectly* nor be self-financing. The value of our portfolio will only approximate the option payoff at maturity, and in each rebalancing nonzero cash flow may arise due to the mismatch between the values of the hedge portfolio *needed* (the new one) and the replicating portfolio *available* (the old one "brought forward"). (In contrast, if the rebalancing is done continuously, then these two values coincide.)

3.7.5 More specifically, suppose the current time is $t$ and we have set up the hedge portfolio $(\Delta_t, B_t)$ for an European call/put, and the next rebalancing time is $t+h$.

- *value needed at time $t+h$:* The proper hedge portfolio needed at time $t+h$ becomes $(\Delta_{t+h}, B_{t+h})$, and the total cost of setting such portfolio up is the time-$t+h$ option price:

$$V_{t+h} = \boxed{\Delta_{t+h}S_{t+h} + B_{t+h}}.$$

- *value available at time $t+h$:* The value of the old hedge portfolio created at time $t$ *brought forward* to time $t+h$ is

$$V_{t+h}^{\text{bf}} = \boxed{\Delta_t S_{t+h} + B_t e^{rh}}.$$

  Here, "brought forward" means that we hold the old hedge portfolio from time $t$ to $t+h$. When the time reaches $t+h$, the new stock price is $S_{t+h}$ (while the number of shares owned remains the same due to the absence of dividends) and the value of risk-free investment grows to $B_t e^{rh}$ (at the risk-free interest rate $r$).

3.7.6 In general, $V_{t+h}$ and $V_{t+h}^{\text{bf}}$ are different, and the difference $V_{t+h} - V_{t+h}^{\text{bf}}$ is the additional cash flow ("net cash flow") required to be injected, for rebalancing at time $t+h$ (this is also known as **rebalancing cost**):

$$\text{Net cash flow}_{t+h} = V_{t+h} - V_{t+h}^{\text{bf}}.$$

If Net cash flow$_{t+h} > 0$, then extra fund needs to be injected to update the hedge portfolio ☹. On the other hand, if Net cash flow$_{t+h} < 0$, then some amount of money can be extracted during the rebalancing ☺.

3.7.7 To compute all the net cash flows involved in the time points $0, h, 2h, \ldots, T$, we can use the following table and the formulas above repetitively:

| Time $t$ | $S_t$ | Stock part | Bond Part | Total | Old hedge brought forward $V_t^{\text{bf}}$ | Net cash flow |
|---|---|---|---|---|---|---|
| $0$ | $S_0$ | $\Delta_0$ | $B_0$ | $V_0$ | $0$ | $V_0$ |
| $h$ | $S_h$ | $\Delta_h$ | $B_h$ | $V_h$ | $\Delta_0 S_h + B_0 e^{rh}$ | $V_h - V_h^{\text{bf}} =?$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $kh$ | $S_{kh}$ | $\Delta_{kh}$ | $B_{kh}$ | $V_{kh}$ | $\Delta_{(k-1)h}S_{kh} + B_{(k-1)h}e^{rh}$ | $V_{kh} - V_{kh}^{\text{bf}} =?$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $T$ | $S_T$ | $\Delta_T$ | $B_T$ | $V_T$ | $\Delta_{T-h}S_T + B_{T-h}e^{rh}$ | $V_T - V_T^{\text{bf}} =?$ |

[Note: $V_T$ equals the payoff the option at maturity.]

# 4 Interest Rate Derivatives

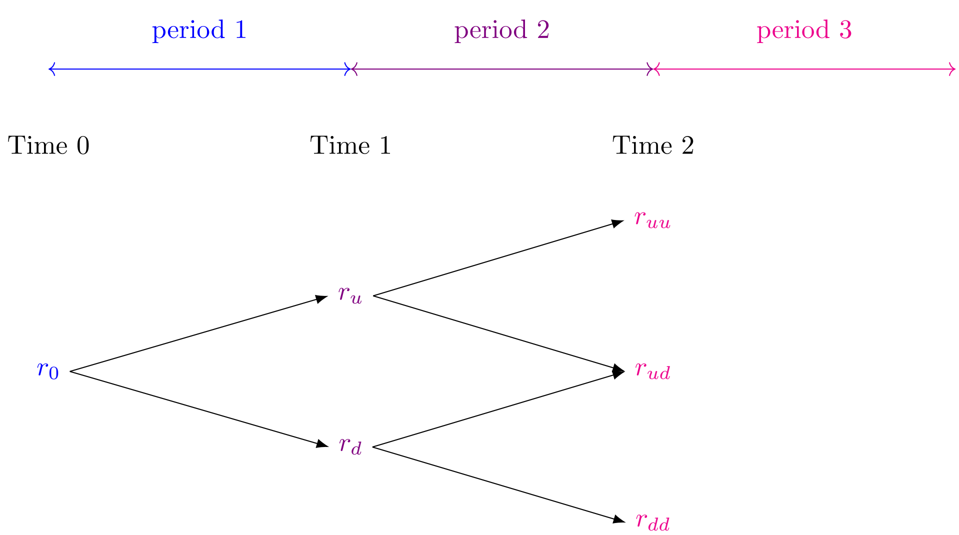
4.0.1 In Section 4, we will apply the methodologies from the binomial tree model and Black-Scholes model to price **interest rate derivatives**, which are derivatives whose payoffs depend on interest rates, e.g., zero-coupon bonds (simplest one!), options on zero-coupon bonds, and interest rate caps and floors.

## 4.1 The Binomial Tree Approach

4.1.1 To apply the binomial tree model for pricing interest rate derivatives, as one may expect, we are going to assume that (risk-free) *interest rates* (not stock prices this time) evolve in a binomial fashion. [Note: The interest rates involved can be continuously compounded, effective annual, or in some other compounding frequency. Be careful about the specification of interest rate!]

4.1.2 More specifically, at the end of each period, the interest rate over the period can change to *two* possible values: one corresponds to the "up" move and another corresponds to the "down" move. The new interest rate is then applied over the next period.

Example: The following is a *3-period* ( ) binomial interest rate tree where the length of each period is 1 year.




- The interest rate over period 1 is $r_0$ (non-random).
- The interest rate over period 2 is $r_u$ or $r_d$ (random).
- The interest rate over period 3 is $r_{uu}$, $r_{ud}$, or $r_{dd}$ (random).

Apart from this special feature that the value at each node applies for one period (but not just for a single time point), another special feature for binomial interest rate tree is that the *risk-neutral probability* of an up move $p^*$ is *not* directly computed based on the tree parameters, but is rather an *additional* tree parameter to be specified. [ Warning: We do not have $p^* = \frac{e^{(r-\delta)h}-d}{u-d}$ in a binomial tree *interest rate* tree! ] [Note: Roughly speaking, this situation arises since unlike a stock, interest rate is not tradable directly, so we are unable to just obtain the risk-neutral probability simply by a replicating portfolio argument (like what we did for binomial stock price tree).]

4.1.3 Given a binomial interest rate tree together with the additional parameter $p^*$, we can price interest rate derivatives using the *risk-neutral pricing* method, similar to what we did before for stock options, but with some subtleties .

Here, the interest rates are used for *two* purposes:

(1) defining the payoff of the interest rate derivative

(2) discounting the payoff from the time of payment to time 0

Notably, interest rate derivatives are always *path-dependent* in the sense that the discounted payoffs for different paths would be different in general, since at least the discounting involved would be different (even if the payoffs are the same).

4.1.4 By risk-neutral pricing[7], the time-0 price of an interest rate derivative is

$$V_0 = \boxed{\sum_{\text{all paths}} \text{discount factor} \times \text{RN probability} \times \text{payoff}}.$$

Note that "discount factor" here generally differs for different interest rate paths, and thus needs to be placed inside the summation (unlike the case for binomial stock price trees).

Here, we will discuss three examples of interest rate derivatives for illustrate the usage of such risk-neutral pricing formula:

(1) zero-coupon bonds

(2) options on zero-coupon bonds

(3) interest rate caps and floors

4.1.5 Let us start with the simplest interest rate derivative: (risk-free) zero-coupon bonds (ZCBs). We have already learnt what a zero-coupon bond is in STAT2902: It (only) pays a fixed amount, which is known as the face value, at the maturity date of the bond. In the case where the face value equals 1, by risk-neutral pricing, the time-0 bond price is

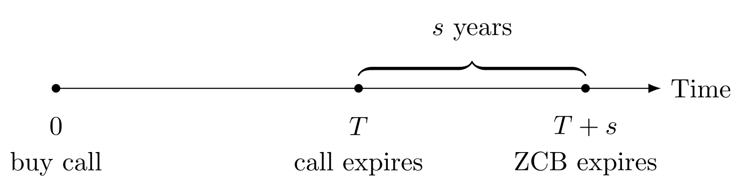
$$B_0 = \boxed{\sum_{\text{all paths}} \text{discount factor} \times \text{RN probability}}.$$

since payoff $\equiv$ face value $= 1$.

[Note: In general, when the face value of the zero-coupon bond is $F$, its time-0 price is just given by $F \times B_0$ (as we have payoff $\equiv F$ instead).]

In STAT2902, we have learnt the pricing of zero-coupon bonds in a simpler setting where the interest rate is a known constant. Here we are in a slightly more complex setting where the interest rates evolve according to a binomial tree, and hence we need to do the discounting on a *path-by-path* basis.

4.1.6 After warming up with pricing zero-coupon bonds, we now consider some more complicated interest rate derivatives. First we discuss the pricing of options on zero-coupon bonds. Consider a $K$-strike $T$ year European call option on a $s$-year zero-coupon bond paying 1 (i.e., the zero-coupon bond matures at time $T + s$, not time $s$ ).




The time-$T$ payoff of such call option is $(P(T, T + s) - K)_+$ where $P(T, T + s)$ is the time-$T$ price of the underlying ZCB that matures at time $T + s$. It is a random variable, viewing from time 0. So, plugging this into the risk-neutral pricing formula, the time-0 price of the call is

$$C_0 = \boxed{\sum_{\text{all paths}} \text{discount factor} \times \text{RN probability} \times (P(T, T + s) - K)_+}.$$

[7] We shall not discuss the justification of risk-neutral pricing in this context as it would involve technical concepts to be discussed in STAT3911.

For the otherwise identical *put* option, its time-0 price is

$$P_0 = \boxed{\sum_{\text{all paths}} \text{discount factor} \times \text{RN probability} \times (K - P(T, T+s))_+}.$$

4.1.7 Lastly, we consider interest rate caps and floors. They are formed by some building blocks known as interest rate *caplets* and *floorlets*, which are similar to call and put options on the interest rate, respectively.

4.1.8 An **interest rate caplet** is characterized by 3 elements:

(1) *a fixed single time period:* we are focusing on interest rates over this period

(2) *cap rate:* a fixed interest over the period, serving as a "cap" on the actual interest payment

(3) *notional amount:* the amount you borrowed "notionally", which is used for the "notional" calculations of interest payments

For illustration purpose, let us consider a concrete example about interest rate caplet. Suppose that you have borrowed 100 at a floating rate with annual interest rate payments (made at the end of each year), i.e., the interest rate charged "floats" according to the market interest rate. Assume that the effective interest rate over this year is 8%, which is supposed to be known and fixed at the beginning of this year. Then, at the end of this year, you will need to pay an interest of $100 \times 8\% = 8$.

Note that here you are exposed to the upside interest rate risk. If the realized interest rate over this year were abnormally high (say 1000%), you would need to pay a huge amount of interest ($100 \times 1000\% = 1000$ ⚠). To protect against this risk, an interest rate caplet that "caps" the actual interest payment can be purchased.

Suppose that you have purchased a 5% (effective annual) interest rate caplet on a 100 loan over this year. Knowing that the interest rate over this year is 8% at the beginning, which exceeds the cap rate 5%, there would be a caplet payment made to you *at the beginning* of this year, of amount

$$\frac{100 \times (8\% - 5\%)}{1 + 8\%}.$$

This payment amount can insure against the interest rate risk, since if you invest this amount at the 8% (risk-free) interest rate over this year, you can receive $100 \times (8\% - 5\%)$ at the end of the year. Hence, your net payment at the end of the year will just be

$$\underbrace{100 \times 8\%}_{\text{original interest payment}} - \underbrace{100 \times (8\% - 5\%)}_{\text{return from investment}} = 100 \times 5\% = 5,$$

so it is like the interest payment is capped at the amount required under 5% interest rate.

On the other hand, if the effective interest rate over this year is *lower* than 5%, then there would not be any caplet payment.

So, in general, the caplet payment, made at the beginning of the period, is given by

$$\text{notional amount} \times \frac{(\text{interest rate} - \text{cap rate})_+}{1 + \text{interest rate}}$$

(assuming the interest rates are all effective annual).

4.1.9 After understanding interest rate caplet, an **interest rate floorlet** works analogously, and is again characterized by 3 elements:

(1) *a fixed single time period:* we are focusing on interest rates over this period

(2) *floor rate:* a fixed interest over the period, serving as a "floor" on the actual interest payment

(3) *notional amount:* the amount you lent "notionally", which is used for the "notional" calculations of interest payments

An interest rate floorlet is used for another purpose: hedging against the interest rate risk for the *lender*. Continuing from the example above, now suppose that you are the lender of the loan rather than the borrower.

This time, you are exposed to the downside interest rate risk. If the realized interest rate over this period were abnormally low (say 0.001%), you would only receive a tiny amount of interest ($100 \times 0.001\% = 0.001$ ⚠). To protect against this risk, you can purchase an interest rate floor.

Suppose that you have purchased a 5% (effective annual) interest rate caplet on a 100 loan over this year. Assuming that the interest rate over this year is known to be 2% at the beginning, which is below the cap rate 5%, there would be a floor payment made to you *at the beginning* of this year, of amount

$$\frac{100 \times (5\% - 2\%)}{1 + 2\%}.$$

Similarly, after investing this amount at 2% interest rate over this year, your net amount to be received at the end of the year will be

$$100 \times 2\% + 100 \times (5\% - 2\%) = 100 \times 5\% = 5.$$

In general, the floor payment, made at the beginning of the period, is given by

$$\text{notional amount} \times \frac{(\text{cap rate} - \text{interest rate})_+}{1 + \text{interest rate}}$$

(assuming the interest rates are all effective annual).

4.1.10 Each interest rate caplet/floorlet is only over a single time period. To hedge against interest rate risk over multiple time periods, we can purchase multiple interest rate caplets/floorlets, and a collection of interest rate caplets (floorlets) over different time periods is known as a **interest rate cap** (**interest rate floor**).

Since an interest rate cap (floor) comprises of multiple interest rate caplets (floorlets), there would be multiple potential caplet (floorlet) payments made at different time points also. Hence, to price an interest rate cap (floor) in a binomial interest rate tree, it is not enough to just look at the payoffs at terminal nodes — we also need to consider the earlier nodes (somewhat like American options...), as payments may arise at those earlier nodes as well!

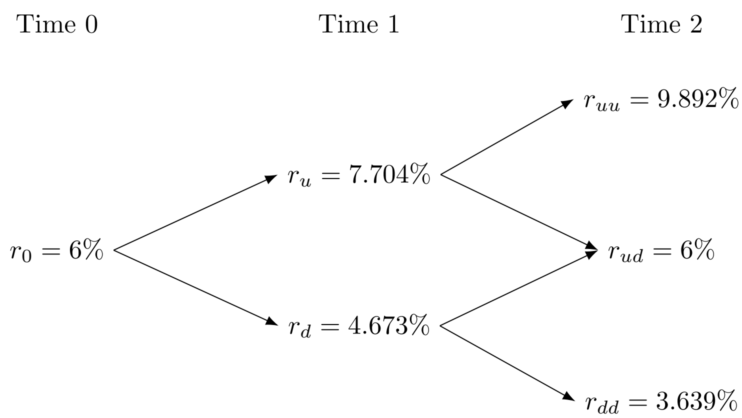
So, by risk-neutral pricing, the time-0 price of an interest rate cap is the sum of the prices of all the caplets in the composition:

$$V_0 = \boxed{\sum_{\text{all paths to nonzero caplet payment}} \text{discount factor} \times \text{RN probability} \times \text{caplet payment}}.$$

[⚠ Warning: It is possible that a single caplet payment is triggered by *multiple* paths! For example, in case there is a caplet payment at the $ud$ node, then there would be 2 paths reaching the $ud$ node, where the discounting is done differently.]

We replace "caplet" by "floorlet" in the formula for an interest rate floor.

4.1.11 Example: Consider the following binomial interest rate tree with $p^* = 1/2$ (where all interest rates are effective annual rates):




Suppose that we want to price a 7.5% interest rate cap on a 100 *3-year* loan.

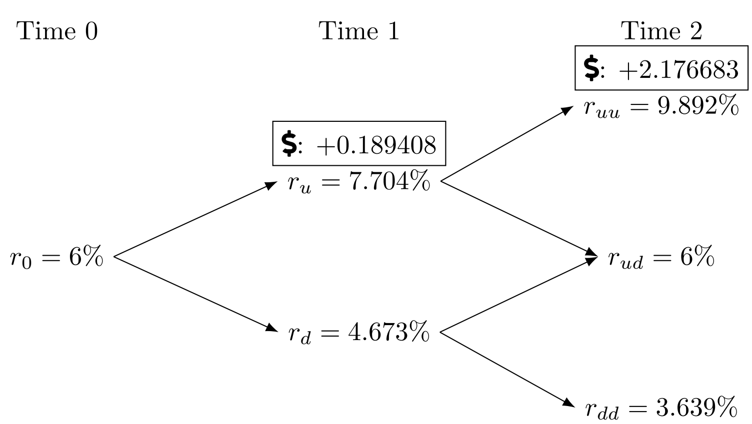
The first step is to understand the composition of such interest rate cap. Here, it consists of 3 interest rate caplets:

(a) 7.5% interest rate caplet on a 100 loan over first year (time 0 to time 1) ➔ potential caplet payment at time 0 (initial node)
(b) 7.5% interest rate caplet on a 100 loan over second year (time 1 to time 2) ➔ potential caplet payment at time 1 ($u$ or $d$ node)
(c) 7.5% interest rate caplet on a 100 loan over third year (time 2 to time 3) ➔ potential caplet payment at time 2 ($uu$, $ud$ or $du$ node)

Next, since the caplet payment is nonzero only when the realized effective interest rate exceeds 7.5%, there would only be nonzero caplet payments at the $u$ node and $uu$ node.

Then, we calculate the caplet payments at $u$ and $uu$ nodes:

- *$u$ node:*
$$\frac{100 \times (7.704\% - 7.5\%)}{1 + 7.704\%} = 0.189408.$$
[Note: The realized interest rate $r_u = 7.704\%$ is applied for the year starting at time 1, i.e., the second year, and the caplet payment here is made at the beginning of second year, i.e., time 1.]
- *$uu$ node:*
$$\frac{100 \times (9.892\% - 7.5\%)}{1 + 9.892\%} = 2.176683.$$




Finally, by risk-neutral pricing, the time-0 price of the interest rate cap is

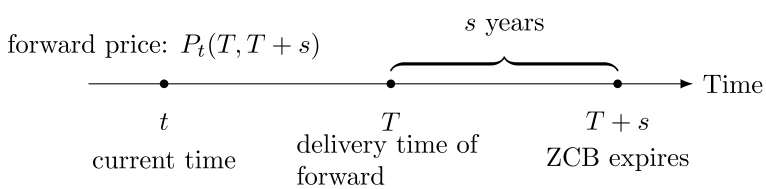
$$V_0 = (1 + r_0)^{-1} \times p^* \times 0.189408 + (1 + r_0)^{-1}(1 + r_u)^{-1} \times (p^*)^2 \times 2.176683 = 0.5660.$$

## 4.2 The Black-Scholes Approach

4.2.1 After discussing the pricing of interest rate derivatives using the binomial tree model in Section 4.1, we will discuss how to perform the pricing under the Black-Scholes model. Here, we need to assume a certain price process follows geometric Brownian motion. But what price process should be assumed to follow a geometric Brownian motion?

4.2.2 It turns out that we are assuming that the *bond forward price* (i.e., the delivery/forward price of a forward contract on a zero-coupon bond) follows a geometric Brownian motion. Before explaining the rationale behind this assumption, we will have some discussions about the bond forward price.

Throughout, we shall use the (seemingly complex, but useful) notation $P_t(T, T+s)$ to denote the time-$t$ forward price (i.e., the delivery price negotiated at time $t$) of a forward on a ZCB paying 1 at time $T+s$, to be delivered at time $T$.




More specifically, after entering into such forward at time $t$, we are obligated to pay the forward price $P_t(T, T+s)$ to get the ZCB paying 1 at time $T+s$, at time $T$.

4.2.3 In the special case where $t = T$, the time-$t$ "forward price" $P_t(T, T+s) = P_t(t, T+s)$ is nothing but just the time-$t$ spot price of the ZCB paying 1 at time $T+s$, since under the absence of arbitrage opportunities, the only possible price negotiated is the time-$t$ spot price.

4.2.4 Under the no-arbitrage principle, we have the following "ratio formula" for computing bond forward price:

$$\boxed{P_t(T, T+s) = \frac{P_t(t, T+s)}{P_t(t, T)}}$$

for any $t \in [0, T]$. Rearranging the equation gives $P_t(t, T+s) = P_t(t, T) \times P_t(T, T+s)$, which looks like the well-known formula ${}_{t+u}p_x = {}_tp_x \times {}_up_{x+t}$ we learn in STAT3901!

*Proof.* The main idea is to construct two strategies to get the ZCB paying 1 at time $T+s$ (corresponding to $P_t(t, T+s) = P_t(t, T) \times P_t(T, T+s)$) and use the law of one price to equate their time-$t$ values. The two strategies are as follows:

(1) at time $t$ (current time): buying the ZCB paying 1 at time $T+s$, at the time-$t$ spot price

(2) at time $t$ (current time): buying a ZCB paying $P_t(T, T+s)$ at time $T$, and entering into the forward on ZCB (then at time $T$, the payment from the ZCB will be just enough for paying the delivery price)

The time-$t$ value of (1) is $P_t(t, T+s)$, and the time-$t$ value of (2) is $P_t(t, T) \times P_t(T, T+s)$. □

4.2.5 Now, we consider a $T$-year $K$-strike European call option on an $s$-year ZCB paying 1 (i.e., ZCB expires at time $T+s$). The time-$T$ payoff this call is

$$(P_T(T, T+s) - K)_+ = \left(P_T(T, T+s) - K \times \underbrace{P_T(T, T)}_{1}\right)_+ \triangleq (S_T^{(1)} - S_T^{(2)})+$$

After writing the payoff in such a way, it becomes clear that the call on ZCB can indeed be viewed as an *exchange option*. More specifically, we let:

- asset 1 (underlying asset): a ZCB of 1 maturing at time $T+s$ (time-$T$ price: $S_T^{(1)} = P_T(T, T+s)$, which is random)
- asset 2 (strike asset): a ZCB of $K$ maturing at time $T$ (time-$T$ price: $S_T^{(2)} = K \times P_T(T,T)$, which is nonrandom)

Then, the call option can be seen as an exchange option which gives us the right to exchange asset 2 for asset 1.

Remarks:

- The time-$t$ (spot) prices of assets 1 and 2 are $S_t^{(1)} = P_t(t, T+s)$ and $S_t^{(2)} = K \times P_t(t,T)$.
- Both assets 1 and 2 here are nondividend-paying, since ZCB does not pay any dividends.

4.2.6 This point of view motivates us to use the pricing assumption as suggested in [2.6.3] for pricing options on ZCB, namely assuming that the ratio of the two asset prices $\left\{\dfrac{S_t^{(1)}}{S_t^{(2)}}\right\}_{t\in[0,T]} = \left\{\dfrac{P_t(t,T+s)}{K\times P_t(t,T)}\right\}_{t\in[0,T]}$ is a geometric Brownian motion (GBM) with volatility parameter $\sigma$.

The expression $\dfrac{P_t(t,T+s)}{K\times P_t(t,T)}$ appears to be quite familiar. Indeed, it is just $P_t(T,T+s)/K$ by [4.2.4]. Since $\{P_t(T,T+s)/K\}$ follows a GBM iff $\{P_t(T,T+s)\}$ follows a GBM, and both GBMs would have the same volatility parameter $\sigma$[8], the assumption above is just equivalent to the assumption that the *bond forward price process* $\{P_t(T,T+s)\}_{t\in[0,T]}$ follows a geometric Brownian motion with volatility $\sigma$, where $\sigma$ is the volatility of the bond forward:

$$\mathrm{Var}\left(\ln P_t(T,T+s)\right) = \sigma^2 t,$$

for any $t \in [0,T]$. Usually the pricing assumption is phrased in this way (i.e., in terms of bond forward price).

4.2.7 With this pricing assumption, we can apply the formulas derived in Section 2.6 to compute the time-0 prices of European options on ZCBs:

- *$T$-year $K$-strike call on ZCB:*

$$\begin{aligned} C_0 &= \mathrm{BS}\left(S_0^{(1)}, 0; S_0^{(2)}, 0; \sigma, T\right) \\ &= \boxed{\mathrm{BS}\left(P_0(0,T+s), 0; K\times P_0(0,T), 0; \sigma, T\right)} \\ &= P_0(0,T+s)\Phi(d_1) - K\times P_0(0,T)\Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\ln[P_0(0,T+s)/(K\times P_0(0,T))] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

- *$T$-year $K$-strike put on ZCB:*

$$\begin{aligned} P_0 &= \mathrm{BS}\left(S_0^{(2)}, 0; S_0^{(1)}, 0; \sigma, T\right) \\ &= \boxed{\mathrm{BS}\left(K\times P_0(0,T), 0; P_0(0,T+s), 0; \sigma, T\right)} \\ &= K\times P_0(0,T)\Phi(d_1) - P_0(0,T+s)\Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\ln[(K\times P_0(0,T))/P_0(0,T+s)] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

[8] This can be seen by noting that $\mathrm{Var}\left(\ln(P_t(T,T+s)/K)\right) = \mathrm{Var}\left(\ln P_t(T,T+s) - \ln K\right) = \mathrm{Var}\left(\ln P_t(T,T+s)\right)$.

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## Concepts and Terminologies



## Results

### Section 1

- [1.3.4]: payoff formulas for barrier options
- [1.3.5]: barrier options parity

### Section 2

- [2.1.3]: distribution of stock price ratios in Black-Scholes model
- [2.1.5]: stock price ratios over disjoint time intervals are independent in Black-Scholes model
- [2.2.1]a: true exercise probability in Black-Scholes model
- [2.2.1]b: mean and variance of stock price in Black-Scholes model
- [2.2.3]: quantile of stock price in Black-Scholes model
- [2.3.3]: risk-neutral expression of $S_t$
- [2.3.6]: risk-neutral pricing formula in Black-Scholes model
- [2.4.3]: generic Black-Scholes pricing function
- [2.4.14]: Black-Scholes pricing formula for stocks paying discrete dividends

- Theorem 2.6.a: pricing formula for "one-to-one" exchange option
- [2.6.5]: formula for blended volatility
- [2.6.7]: pricing formula for "$c_1$-to-$c_2$" exchange option
- [2.6.9]: pricing formulas for maximum claim
- [2.6.10]: relationship between prices of minimum and maximum claims
- [2.6.11]: direct pricing formulas for minimum claim

### Section 3

- [3.2.2]: call delta formula
- [3.2.3]: relationship between call and put deltas
- [3.3.2]: call gamma formula
- [3.3.3]: relationship between call and put gammas
- [3.4.2]: call theta formula
- [3.4.3]: relationship between call and put thetas
- [3.5.6]: actions needed for delta-hedging
- [3.5.7]: profit formula for delta-hedging
- [3.6.4]: method for hedging multiple greeks
- [3.7.5]: formulas for values needed and available of hedge portfolios in dynamic hedging
- [3.7.6]: net cash flow formula for dynamic hedging

### Section 4

- [4.1.4]: general risk-neutral pricing formula for interest rate derivatives
- [4.1.5]: risk-neutral pricing formula for zero-coupon bonds
- [4.1.6]: risk-neutral pricing formula for European calls/puts on zero-coupon bonds
- [4.1.10]: risk-neutral pricing formula for interest rate caps/floors
- [4.2.4]: ratio formula for bond forward price
- [4.2.7]: Black-Scholes pricing formula for European calls/puts on zero-coupon bonds