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of speculation concerning the nature of the dark matter. For a more advanced and extensive treatment of the problem see the review by V. Trimble, "Existence and nature of dark matter in the universe," Annu. Rev. Astron. Astrophys. **25**, 425–472 (1987). The article also appears in the reprint collection edited by E. W. Kolb and M. S. Turner *loc. cit.* 67–114. See, also, the recent report by M. Spiro, "Detection of Dark Matter," in *Physical Cosmology*, edited by A. Blanchard *et al.* (Editions Frontieres, Gif-sur-Yvette Cedex, France, 1991), pp. 237–247.

A polytropic model of the Sun

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We show that the presently accepted standard model of the Sun exhibits polytropic power-law behavior $P=K\rho^{\gamma}$ over certain regions of the Sun's interior. We then develop a three-polytype model that gives a good representation of this standard solar model. The model is easily computable.

I. INTRODUCTION

Several years ago, Clayton[1] presented an interesting phenomenological model of the Sun, based on taking the pressure gradient in the form

$$\frac{dP}{dR}=-\left(\frac{4\pi}{3}\right)G\rho_c^2R\exp\left[-\left(\frac{R}{a}\right)^2\right]. \tag{1}$$

Here, R is the radial distance from the center of the Sun. Clayton's motivation was to obtain a model that was analytically solvable, but which at the same time gave a reasonable representation of some of the main characteristics of the Sun. The form (1) provides such a model. Only some basic calculus needs to be used to derive expressions for the pressure, mass, density, and temperature as functions of the radial distance. This is clearly a very instructive model for students in both physics and astronomy.

In the present paper, we develop a more advanced model of the Sun. It is based on the idea of polytropes. The theory of polytropes is a familiar one in astrophysics, covered in most introductory texts.[2,3] The classic book by Chandrasekhar[4] contains a more detailed discussion. By a polytrope, we mean a gas which obeys the property that the local pressure P and density ρ are related through a power law:

$$P=K\rho^{\gamma}. \tag{2}$$

It is customary to write $\gamma=1+(1/N)$, where N is called the polytropic index. Specific cases often examined in class correspond to $\gamma=1$ (an isothermal gas sphere), $\gamma=4/3$ (Eddington's standard model with purely radiative heat transport), and $\gamma=5/3$ (a gas sphere with adiabatic convective heat transport). Unlike Clayton's model, the model presented here cannot be solved analytically, but requires the use of a computer to numerically integrate a second-order differential equation. A routine for doing this is often available in many of the computer packages accessible to students.

A model based on polytropes gives a surprisingly good representation of the Sun, and thus makes the effort to obtain such a model particularly worthwhile. It could lead to interesting research projects. The reason for this model's success can be seen from Fig. 1. There, the pressure obtained in the sophisticated standard solar model (SSM) of Bahcall and Ulrich[5,6] is plotted on a log–log scale against the density. Three regions clearly emerge. In each region, the SSM output is approximated rather well by a straight line, indicating polytropic behavior.

Of these three regions, the outermost one (low pressure, low density) represents the convective zone where heat transport is achieved by adiabatic convection. The slope corresponds to $\gamma=5/3$. The inner regions constitute the radiative zone, with slopes γ about 1.26 and 1.05, respectively. It may seem surprising that the radiative zone has two regions. However, this is not inconceivable, considering that the dominant physical processes within the radiative zone change significantly with radial distance: The central region is where most of the Sun's energy is produced by nuclear fusion; further out, where little energy is produced, the outward flow of this energy is impeded by increasing opacity.

Our model is based on the above characteristics of the SSM, the goal being to obtain a representation of the Sun in terms of three polytropes. The lines in Fig. 1 indicate how well our final result approximates the SSM.

In the following, for completeness, we shall start with a brief description of a single polytrope (this is what is usually covered in standard coursework[2,3]). Next we examine a two-polytrope model. This may be regarded as a first approximation to a Sun that includes both a convective and a radiative zone. It provides useful experience with matching physical quantities at the interface. Finally, we build a three-polytrope version which gives a good approximation to the major features of the SSM.

II. ONE POLYTROPE

Most textbooks[2,3] deal with the case of a single polytrope. The basic equations are those for hydrostatic equilibrium $dP(R)/dR=-GM(R)\rho/R^2$ and for mass continuity $dM(R)/dR=4\pi R^2\rho(R)$. $M(R)$ is the mass of solar

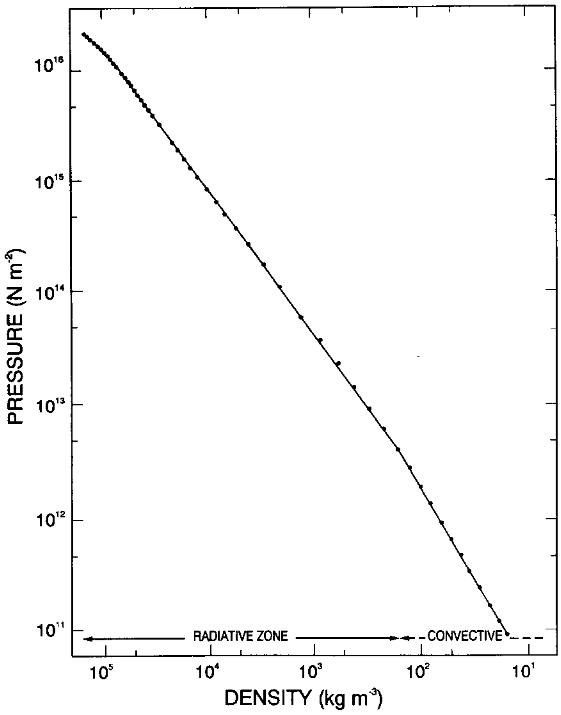

Fig. 1. Plot of the standard solar model SSM values (Ref. 5) of pressure P and density ρ. The center of the Sun is on the left, the convective zone on the right. Three distinct regions are apparent. Each region has a polytropic power-law behavior $P=K\rho^{\gamma}$, with different values of K and γ. The lines correspond to our phenomenological three-polytrope model, given by (15)–(17) in the text.

matter within a radius R. Eliminating $M(R)$ between these two equations yields a differential equation in P and ρ. The additional assumption of polytropic behavior $P=K\rho^{\gamma}$ yields a second-order differential equation in ρ.

It is customary at this stage to change to dimensionless quantities ξ, θ, where

$$R=\alpha\xi, \quad \rho=\lambda\theta^{N}, \tag{3}$$

with $\alpha=[(N+1)K/4\pi G]^{1/2}\lambda^{(1-N)/2N}$. The resulting equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^{N}. \tag{4}$$

This second-order differential equation is called the Lane–Emden equation, and is the crux of all polytropic models. Numerical integration can be done either using a standard package. if one is available, or by writing one's own computer routine. We used a standard Runge–Kutta method to integrate (4). The mass $M(R)$ is related to θ and ξ by

$$M(R)=4\pi\alpha^3\lambda\xi^2(-\theta'), \tag{5}$$

where $\theta'=d\theta/d\xi$.

The solutions of the Lane–Emden equation fall into two categories: those that are finite, and those that are singular at the origin $\xi=0$. However, for a single polytrope, we need consider only the former, since the density at the center is finite. A number of these (all normalized to have $\theta=1$ at $\xi=0$) are shown in Fig. 2(a) for different values of polytropic index N. The corresponding sun is taken to extend from $\xi=0$ (where the slope is 0) out to the first zero of θ at $\xi=\xi_0$ (where the slope is θ_0').

It is also useful to calculate two other quantities U, V defined to be

$$U=\frac{\xi\theta^{N}}{-\theta'}=\frac{d(\ln M)}{d(\ln R)}=\frac{4\pi R^3\rho}{M}, \tag{6}$$

$$V=\frac{(N+1)\xi(-\theta')}{\theta}=\frac{-d(\ln P)}{d(\ln R)}=\frac{GM\rho}{PR}. \tag{7}$$

These are particularly convenient when imposing continuity across interfaces in multipolytrope models. The U, V curves that correspond to the solutions in Fig. 2(a) are shown in Fig. 2(b).

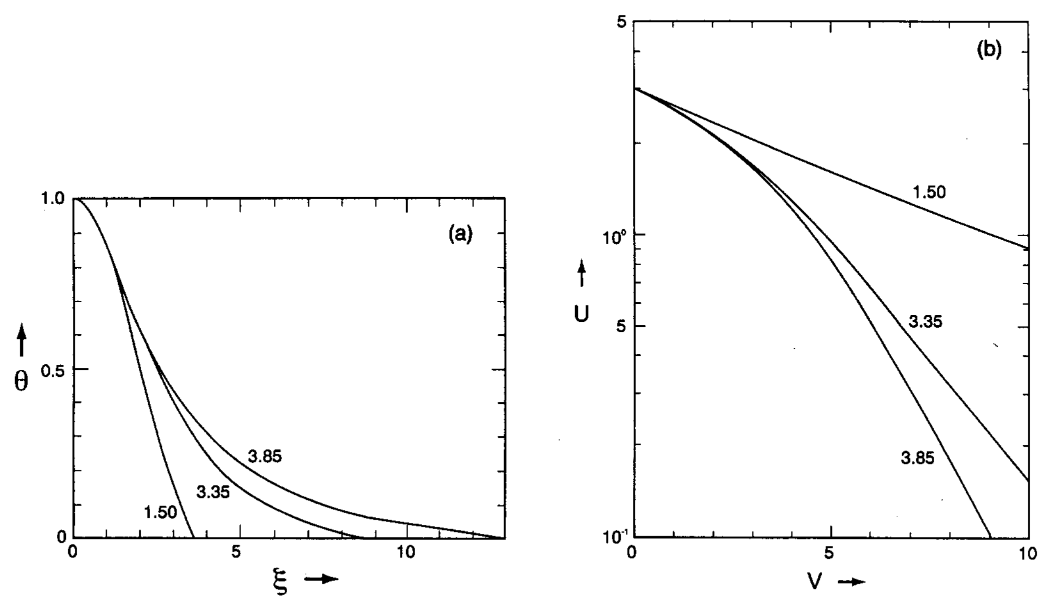

Fig. 2. (a) Solutions of the Lane–Emden Eq. (4) that are finite at the origin $\xi=0$ (E solutions). The polytropic index $N=3.85$, 3.35, and 1.50. The corresponding sun would extend from $\xi=0$ out to the first zero at $\xi=\xi_0$. (b) The corresponding curves in the U, V plane. U and V are the quantities defined in (6), (7).

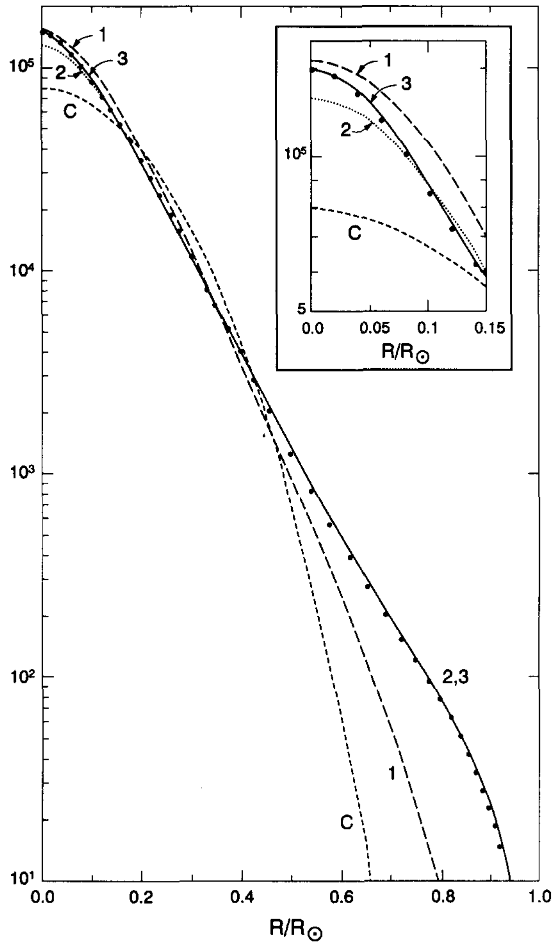

Fig. 3. Density profiles $\rho(R)$ as a function of the radial distance $R/R_{\odot}$. The dots are the SSM values. The label C corresponds to the analytical model of Clayton (Ref. 1), while 1, 2, and 3 correspond to our one-, two-, and three-polytrope models given in the text. The insert shows the radial dependence for small $R/R_{\odot}$.

As a specific example, let us work through the case with $N=3.35$ ($\gamma=1.2985$) shown in Fig. 2. The first step is to integrate the corresponding Lane–Emden equation. This yields $\xi_0=8.568$ and slope $\theta_0'=-0.0262$. These values, together with (3), (5) and the physical constraints $R_{\odot}=6.96\times10^8$ m, $M_{\odot}=1.99\times10^{30}$ kg, allow us to evaluate the unknowns K and λ. The parametric form of the polytrope is then

$$\rho=\rho_c\theta^N,\quad P=K\rho^\gamma=P_c\theta^{N+1},\tag{8}$$

with $\rho_c=\lambda=1.53\times10^5$ kg m^{-3}, $P_c=K\lambda^\gamma=3.00\times10^{16}$ nt m^{-2}. ρ_c and P_c correspond to the central density and pressure, to be compared with the SSM values[5,6] of 1.48×10^5 kg m^{-3} and 2.29×10^{16} nt m^{-2}, respectively.

The computer output tells us how θ varies with ξ. From this, (3) and (8), one can immediately deduce the density and pressure profiles $\rho(R)$, $P(R)$. In Fig. 3, we show the resulting density profile $\rho(R)$ from this one-polytrope model. Characteristic of such models, the density falls off too rapidly for large radial distances. Nonetheless it comes closer to the SSM (indicated by dots in Fig. 3) than Clayton's analytical model.

One can also determine the temperature profile $T(R)$ if we know the equation of state. In most cases, it is sufficient to take the ideal gas equation

$$P=(k/\mu m_a)\rho T,\tag{9}$$

where k is the Boltzmann constant, m_a the mass of an atomic mass unit, and μ the mean molecular weight. This gives the temperature as

$$T=T_c\theta,\tag{10}$$

with $T_c=(\mu m_a/k)K\lambda^{1/N}$. We see however that, to proceed, we must know the mean molecular weight. Following Clayton,[1] we could take μ to be a constant and aim to describe an initial sun with uniform composition. However, in general, μ is not a constant—in the SSM, it varies from 0.854 at the helium rich center to 0.613 at the surface. In all that follows, we shall use the mean molecular weight as given by Bahcall and Ulrich.[5,6] In this case, we obtain a central temperature of $T_c=20.2\times10^6$ K, compared with the SSM value of 15.6×10^6 K.

By choosing different values of N, one can easily obtain other one-polytrope models. This provides useful experience. But none of these models provides a really good approximation to the Sun. The real Sun is more complicated, with convective and radiative zones. As a next step therefore, we consider two-polytrope models.

III. TWO POLYTROPES

We now develop a model sun with two polytropic regions, representing convective and radiative zones. We use ξ, θ as the variables in the Lane–Emden equation for the convective zone with index N_1, and η, ϕ as the variables for the radiative zone with index N_2. The parametric forms for the convective and radiative zone polytropes are

$$\rho=\lambda_1\theta^{N_1},\quad P=K_1\rho^{\gamma_1},\tag{11}$$

and

$$\rho=\lambda_2\phi^{N_2},\quad P=K_2\rho^{\gamma_2},\tag{12}$$

respectively. The main challenge is to learn how to fit these two polytropes together. Since it is the physical quantities P, ρ, and M that are continuous across the interface (not θ or ϕ), the variables U and V given by (6) and (7) prove to be very useful.

We shall again illustrate the method by taking a specific example.

Let us start by considering the convective zone. We take $\gamma_1=5/3$ ($N_1=1.50$) corresponding to adiabatic heat transport; thus $\xi_0=3.6538$ and $\theta_0'=-0.2033$. However, since this polytrope is not being used anywhere in the vicinity of $\xi=0$, it is possible to consider all of the solutions of the Lane–Emden equations for $N_1=1.50$. These may be generated by starting at ξ_0 with arbitrary starting slope and integrating inwards. As mentioned above, we used a standard Runge–Kutta method; $\theta_0=0$, $\xi_0=3.6538$ and the starting slope θ_0' provide the initial values which allow us to start the integration process inwards. Choosing the starting slope as $\theta_0'=-0.2033$ yields the solution which is finite at the origin. (This is a good check on one's computer program; the solution should pass through $\theta=1$ at $\xi=0$.) Solutions with starting slopes less negative than θ_0' are of particular interest (in the literature, they are referred to as M-solutions) since these are the ones which intersect the polytrope that represents the radiative zone. Three such solutions, translated into the U, V variables, are shown in Fig. 4. Solutions with starting slopes more neg-

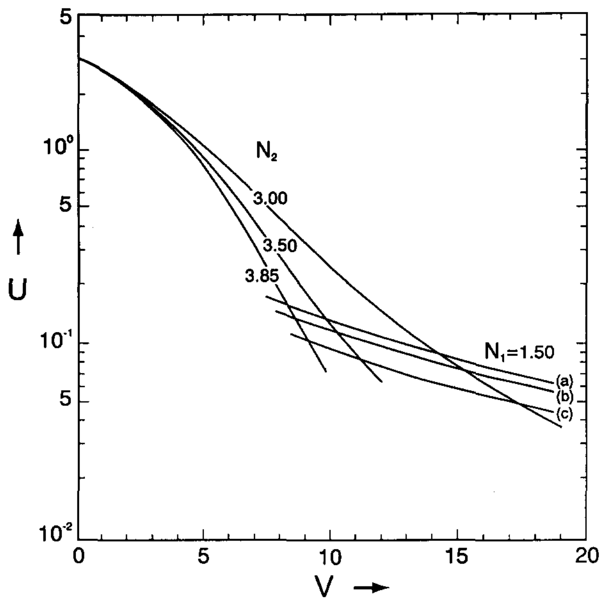

Fig. 4. Solutions of the Lane–Emden Eq. (4). The center of the Sun occurs at $U=3.0$, $V=0.0$. We draw E solutions for polytropic index $N=3.85$, 3.50, and 3.00 to represent the radiative zone. The $N_1=1.50$ solutions represent the convective zone; they are M solutions with starting slopes given by (a) -0.0040, (b) -0.0032, and (c) -0.0020.

ative than θ_0' (the F solutions) do not intersect the radiative polytrope and so need not be considered here.

For specificity, let us choose the solution with a starting slope of -0.0032 at ξ_0. As we shall see below, this gives a good approximation to the SSM. Using (3), (5), and the values of $R_\odot$, $M_\odot$, we can immediately deduce K_1, λ_1. The resulting polytrope representing the convective zone is then given by

$$\rho=5.363\times10^5\theta^{N_1},\quad P=9.888\times10^8\rho^{\gamma_1}, \tag{13}$$

with $\gamma_1=1+1/N_1=5/3$, and $\xi_i\leqslant\xi\leqslant\xi_0$ (ξ_i being the value of ξ at the bottom of the convective zone).

For the radiative zone, since we have a finite density at the center, we must choose a solution which is finite at the origin (these are called E solutions). Three such solutions are shown in Fig. 4, for $N_2=3.85$, 3.50, and 3.00. Once we have selected N_2 the remaining polytropic parameters K_2, λ_2 can be deduced by imposing continuity at the interface with the convective zone.

To see how this happens, let us take $N_2=3.85$. Using Fig. 4, we can obtain the values U_i, V_i corresponding to the point of intersection with the polytrope representing the convective zone. From the computer output, we can read off the corresponding values of ξ_i, θ_i, and η_i, ϕ_i. From (13), the former pair determine ρ_i, P_i; using the latter pair in (12) then yields K_2 and λ_2. Thus we find

$$\rho=1.299\times10^5\phi^{N_2},\quad P=8.087\times10^9\rho^{\gamma_2}, \tag{14}$$

with $\gamma_2=1+1/N_2=1.260$.

The above procedure determines the two matching polytropes. There are several steps involved. They were as follows:

(i) Take $N_1=1.50$ for adiabatic convection.
(ii) Choose an M solution. Using $R_\odot$, $M_\odot$, this determines K_1, λ_1.
(iii) Choose an N_2. It has to be the E solution.
(iv) Read off U_i, V_i.
(v) Find the corresponding (ξ_i, θ_i), (η_i, ϕ_i).
(vi) Deduce K_2, λ_2.

Only two choices need to be made, namely at steps (ii) and (iii). The rest is automatic, but requires patience. By taking different choices, we can generate a variety of two-polytrope models of the Sun.

Knowing $\theta(\xi)$ and $\phi(\eta)$, we can deduce the density and pressure profiles $\rho(R)$, $P(R)$. The above model yields a central density of $\rho_c=1.299\times10^5$ kg m^{-3} and a central pressure of $P_c=2.237\times10^{16}$ nt m^{-2}. The density profile $\rho(R)$ is shown in Fig. 3. As can be seen, this two-polytrope model provides a good approximation to the SSM throughout most of the Sun, a noticeable deviation between the two occurring only in the Sun's most central region.

The temperature profile $T(R)$ follows from the ideal gas equation (throughout the Sun, the radiative pressure $(1/3)aT^4$ is at least three orders of magnitude smaller than the thermal pressure, so we neglect it) and an assumption about the mean molecular weight $\mu(R)$. With the SSM value $\mu_c=0.854$, we obtain $T_c=17.8\times10^6$ K.

Two-polytrope models provide a greatly improved approximation to the SSM. It should be satisfying to students to generate models of the Sun which specifically incorporate two regions, reflecting the two zones that occur in the real Sun. However, as indicated in Fig. 1, the SSM calls for three matching polytropes, which is the case we consider next.

IV. THREE POLYTROPES

As can be seen from the SSM points in Fig. 1, a better approximation can be obtained if we use two polytropes to represent the radiative zone. This complicates the model of course, but in practice the steps we take to knit the three polytropes together are similar to those described in the previous section. Again, matching at the interfaces pins down the values of some of the parameters. Our representation will be such that the core region will correspond to an E solution of the Lane–Emden equation with polytropic index N_3, the outer region of the radiative zone to an M solution with polytropic index N_2, and the convective zone to an M solution with polytropic index N_1. M solutions in the outer two regions are called for in order to achieve intersections of all three polytropes.

There are several ways of fitting three polytropes together, depending on what initial assumptions are made. One could, for example, fix the convective and innermost regions; the continuity conditions then determine the parameters of the intermediate polytrope. An alternative is to start with the convective zone and work inwards towards the center. We shall follow the latter procedure in what we describe in the specific example below.

But first, we make an improvement on our convective zone. Previously, we took a convective zone with $N_1=1.50$, and followed an M solution inwards from ξ_0. Since $\theta=0$ at ξ_0, all such models starting at ξ_0 have surface pressure, density, and temperature equal to zero. It is possible to do better than this, namely obtain a model of the convective zone where these quantities are nonzero at the surface. As in the SSM, we shall take the outer edge of the convective zone to have a temperature $T_s=5770$ K and a mean molecular weight $\mu_s=0.613$. The ratio P_s/ρ_s then follows from the ideal gas equation. To completely determine the polytrope for the convective zone, we need one additional piece of information. For simplicity, we shall take it to be K_1, obtained from a fit to the SSM in the convective zone (Fig. 1); this gives $K_1=9.83\times10^8$. This immediately yields ρ_s and P_s; also since $R_\odot$, $M_\odot$ are known, we can deduce U_s, V_s from (6), (7). However, there is only *one* M solution of the $N_1=1.50$ polytrope which passes through this point (U_s, V_s): it is the solution with starting slope of $-0.003\,25$. The computer output for this polytrope yields (ξ_s, θ_s), and, hence, λ_1. The resulting parametric form for this polytrope is

$$\rho=5.268\times10^5\theta^{N_1},\quad P=9.847\times10^8\rho^{\gamma_1} \tag{15}$$

with $\gamma_1=1+1/N_1=5/3$, and $\xi_i\leqslant\xi\leqslant\xi_s$. This is close to the form (13), which may be used if one is willing to use a model with zero values of ρ, P, and T at the edge of the convective zone.

Proceeding inwards, we now fix the depth of the convective zone. The graph shown in Fig. 1 corresponds to taking this depth at $R_i=0.714\,R_\odot$. This pinpoints ξ_i, θ_i, U_i, V_i, ρ_i, P_i, and M_i. We now know the intersection point (U_i, V_i) in the U, V plane through which the polytrope with index N_2 for the intermediate region must pass. A good approximation to this intermediate region can be obtained (see Fig. 1) if we take $\gamma_2=1.264$. With this value of γ_2, only one M solution of this polytrope passes through (U_i, V_i). Having found it and, hence, the corresponding values η_i, ϕ_i, we can deduce from continuity of ρ and M at this interface the values of K_2 and λ_2. We find

$$\rho=1.196\times10^5\phi^{N_2},\quad P=7.810\times10^9\rho^{\gamma_2} \tag{16}$$

with $\gamma_2=1+1/N_2=1.264$.

The final step is to determine the polytrope for the innermost region representing the core. This is easier to determine since, whatever value of the polytropic index N_3 we select, the appropriate solution to consider is the E solution. For the graph in Fig. 1, we took $\gamma_3=1.05$. By going through the same procedure as before, namely finding the intersection of this E solution in the U, V plane with the now-known polytrope (16) for the intermediate region and imposing continuity there, we can determine K_3 and λ_3. The resulting parametric form for the core polytrope is

$$\rho=1.483\times10^5\psi^{N_3},\quad P=8.798\times10^{10}\rho^{\gamma_3}, \tag{17}$$

where ζ, ψ are the Lane–Emden variables in this region, and $\gamma_3=1+1/N_3=1.05$.

This completes the determination of the three polytropes which provide the approximation to the SSM shown in Fig. 1. Again, for convenience, let us summarize the major steps involved.

(i) Take T_s, μ_s, $R_\odot$, $M_\odot$, and $N_1=1.50$. This determines the point (U_s, V_s) which serves to anchor the whole three-polytrope representation.

(ii) Choose K_1. The value of λ_1, and hence the parametric form of the convective zone polytrope, can now be deduced.

(iii) Choose the depth of the convective zone. This locates the position of the interface at (U_i, V_i).

(iv) Choose N_2. The values of K_2 and λ_2, and hence the parametric form for the polytrope of this intermediate region, can now be deduced.

(v) Choose N_3. Continuity at the interface with the intermediate polytrope fixes K_3 and λ_3, and hence the polytrope for the core region.

In all, four choices have to be made. The straight lines in Fig. 1 correspond to the representations (15)–(17) above. Again, pressure, density, and mass profiles as a function of R can be immediately deduced, the density profile being shown in Fig. 3. It clearly approximates the SSM well. The temperature profile follows from the ideal gas equation, or if one wants to be more precise in the core region, the modified equation of state

$$P=(k/\mu m_a)\rho T(1+D). \tag{18}$$

The quantity $D(R)$ arises from corrections due to the effect of electron degeneracy in the core, and can be calculated.[5,6] It is at most a 2% effect.

V. REMARKS

In this paper, we have shown how to obtain a three-polytrope model of the Sun. As preliminary steps, we first built one- and two-polytrope models. As illustrated in Fig. 1, the three-polytrope model given in (15)–(17) provides a remarkably good approximation to the SSM.

This study can lead to interesting research projects for students. For example, by combining this model with nuclear reaction rates,[3] one can calculate[7] the production rates of neutrinos in the Sun. Also by choosing values of the parameters slightly different from the ones given here, one can generate a whole family of model suns, each of which may be considered as a small perturbation of the SSM. It is then possible to study the sensitivity of quantities such as the neutrino production rate or the helioseismological eigenfrequencies of acoustic waves in the Sun on the temperature and pressure profiles of the Sun.

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