# 14 Equity and Commodity

## 14.1 Equity

We next show how to extend the IR/FX setup to include multiple equity processes in order to cover equity derivatives. We make the simplest possible choice here and model each equity with *Geometric Brownian Motion* driven by the short rate-dividend yield differential, in analogy to the FX model of the previous section. The starting point is the equity process $S_k$, expressed in units of a foreign (nominal) bank account $Q^{B_i}$, where $i$ denotes the currency of equity $S_k$:

$$dS_k(t)/S_k(t) = [n_i(t) - q_k(t)]dt + \sigma_k^S(t)\, dW_{S_k}^{B_i}(t) \tag{14.1}$$

As a first step, our aim is to derive the dynamics of $S_k$ under the LGM numeraire $Q^{N_i}$ by finding the drift adjustment $\phi_{S_k}^{N_i}(t)$ such that

$$dW_{S_k}^{N_i}(t) = dW_{S_k}^{B_i}(t) - \phi_{S_k}^{N_i}(t)\, dt \tag{14.2}$$

Consider the equity asset $\hat{S}_k$ (i.e. the sum of the equity price process $S_k$ and the dividend process $q_k\, S_k$) expressed in units of foreign LGM numeraire. This is a tradable asset and as such is a martingale under the foreign LGM measure when expressed in units of the foreign LGM numeraire. Therefore the SDE given by

$$\begin{aligned} d\left(\frac{\hat{S}_k(t)}{N_i(t)}\right) = \frac{\hat{S}_k(t)}{N_i(t)} \Big( \Big[ & n_i(t) + \phi_{S_k}^{N_i}(t)\, \sigma_k^S(t) - n_i(t) + \rho_{il}^{zs}\, H_i^z(t)\, \alpha_i^z(t)\, \sigma_k^S(t) \Big] dt \\ & + \sigma_k^S(t)\, dW_{S_k}^{N_i}(t) - H_i^z(t)\, \alpha_i^z(t)\, dW_{z_i}^{N_{i_j}}(t) \Big) \end{aligned}$$

must have zero drift. Solving for $\phi_{S_k}^{N_i}(t)$ yields

$$\phi_{S_k}^{N_i}(t) = \rho_{il}^{zs}\, H_i^z(t)\, \alpha_i^z(t) \tag{14.3}$$

Next we derive the dynamics of $S_k$ under the domestic LGM measure $Q^{N_0}$ by finding the drift adjustment $\phi^{N_0}_{S_k}(t)$ such that

$$dW^{N_0}_{S_k}(t) = dW^{N_i}_{S_k}(t) - \phi^{N_0}_{S_k}(t)\,dt. \tag{14.4}$$

Firstly, inserting (14.2) and (14.3) into (14.4) yields

$$dW^{N_0}_{S_k}(t) = dW^{B_i}_{S_k}(t) - (\rho^{zs}_{il}\,H^z_i(t)\,\alpha^z_i(t) + \phi^{N_0}_{S_k}(t))dt. \tag{14.5}$$

Now, the equity asset $\hat{S}_k$, converted into *domestic* currency and expressed in units of *domestic* numeraire must be a martingale under the *domestic* LGM numeraire. Therefore the SDE given by

$$\begin{aligned}
\frac{d\left(\frac{\hat{S}_k(t)\,x_i(t)}{N_0(t)}\right)}{\frac{\hat{S}_k(t)\,x_i(t)}{N_0(t)}} &= \Big[n_i(t) + \phi^{N_0}_{S_k}(t)\,\sigma^S_k(t) + \rho^{zs}_{il}\,H^z_i(t)\,\alpha^z_i(t)\,\sigma^S_k(t) \\
&\quad + n_0(t) - n_i(t) + \rho^{zx}_{0i}\,H^z_0(t)\,\alpha^z_0(t)\,\sigma^x_i(t) \\
&\quad - n_0(t) + \rho^{xs}_{il}\,\sigma^x_i(t)\,\sigma^S_k(t) - \rho^{zs}_{0l}\,H^z_0(t)\,\alpha^z_0(t)\,\sigma^S_k(t) \\
&\quad - \rho^{zx}_{0i}\,H^z_0(t)\,\alpha^z_0(t)\,\sigma^x_i(t)\Big]dt \\
&\quad + \sigma^S_k(t)\,dW^{N_0}_{S_k}(t) + \sigma^x_i(t)\,dW^{N_0}_{x_i}(t) \\
&\quad - H^z_0(t)\,\alpha^z_0(t)\,dW^{N_0}_{z_0}(t) \\
&= \Big[\phi^{N_0}_{S_k}(t)\,\sigma^S_k(t) + \rho^{zs}_{il}\,H^z_i(t)\,\alpha^z_i(t)\,\sigma^S_k(t) \\
&\quad + \rho^{xs}_{il}\,\sigma^x_i(t)\,\sigma^S_k(t) - \rho^{zs}_{0l}\,H^z_0(t)\,\alpha^z_0(t)\,\sigma^S_k(t)\Big]dt \\
&\quad + \sigma^S_k(t)\,dW^{N_0}_{S_k}(t) + \sigma^x_i(t)\,dW^{N_0}_{x_i}(t) - H^z_0(t)\,\alpha^z_0(t)\,dW^{N_0}_{z_0}(t)
\end{aligned}$$

must be driftless. Solving for $\phi^{N_0}_{S_k}(t)$ yields

$$\phi^{N_0}_{S_k}(t) = \rho^{zs}_{0l}\,H^z_0(t)\,\alpha^z_0(t) - \rho^{zs}_{il}\,H^z_i(t)\,\alpha^z_i(t) - \rho^{xs}_{il}\,\sigma^x_i(t) \tag{14.6}$$

and inserting (14.6) and (14.5) into (14.1) gives the dynamics of $S_k$ under the domestic LGM measure:

$$\begin{aligned}
dS_k(t)/S_k(t) &= [n_i(t) - q_k(t) + \rho^{zs}_{0l}\,H^z_0(t)\,\alpha^z_0(t)\,\sigma^S_k(t) - \epsilon_i\,\rho^{xs}_{il}\,\sigma^x_i(t)\,\sigma^S_k(t)]dt \\
&\quad + \sigma^S_k(t)\,dW^{N_0}_{S_k}(t)
\end{aligned} \tag{14.7}$$

This equity evolution model can be calibrated to the "risk-free" yield curve, the dividend yield term structure for the given equity and a selection of European equity options (such as a series of at-the-money options – using a procedure that is similar to the calibration of FX options in Section 12. The exposure evolution of a basic Equity Forward is therefore also (qualitatively) similar to that of an FX Forward which we have seen already in Section 12.5.

## 14.2 Commodity

In this section we explore simple modelling approaches to cover commodity derivatives. The simplest possibility is again to model spot commodity prices using *Geometric Brownian Motion (GBM)* which we have seen applied to precious metals (gold, silver, platinum, palladium, including copper) with drift determined by appropriate *lease rate curves*. For other commodities, modelling the futures/forward price instead seems to be the more common approach. Particularly popular in this context is the *Gabillon model* [67]. See also [7] where Andersen develops a framework for construction of Markov models for commodity derivatives which also covers a Markov diffusion model corresponding to Gabillon's approach.

**The two-factor model and its calibration**

Gabillon used the following model for the futures price $F(t,T)$ observed at time time $t$ with delivery in $T$:

$$\frac{dF(t,T)}{F(t,T)} = g(T)\left(\sigma_S\, e^{-\kappa(T-t)}\, dW_t^S + \sigma_L \left(1 - e^{-\kappa(T-t)}\right) dW_t^L\right). \tag{14.8}$$

The futures price curve is driven by two factors representing shocks to the short and far long end of the curve, respectively. For $t \to T$, close to futures expiry, only the short term $\sigma_S\, dW_t^S$ is left, while for very long terms $T - t \to \infty$, long before futures expiry, only $\sigma_L\, dW_t^L$ drives $F(t,T)$, that is in fact a mix of both terms drives $F(t,T)$ for any finite time to expiry $T - t$. Parameter $\kappa$ controls the "decay" of the short term factor and the "switching over" to the long term factor. The time-dependent parameter $g(T)$ that we added here to Gabillon's original model, which has constant parameters only, allows extra flexibility in calibrating the model. Overall, the dynamics are log-normal with instantaneous volatility $\sigma_{tT}$

$$\sigma_{tT}^2 = g^2(T)\left[\sigma_L^2 + (\sigma_S^2 + \sigma_L^2 - 2\,\rho\,\sigma_s\sigma_L)e^{-2\kappa(T-t)} - 2\,(\sigma_L^2 - \rho\,\sigma_S\sigma_L)\,e^{-\kappa(T-t)}\right]$$

and Black volatility $\Sigma_{tT}$

$$\Sigma_{tT}^2 = \frac{1}{T-t}\int_t^T \sigma_{sT}^2\, ds. \tag{14.9}$$

The model is typically calibrated to the futures price curve as initial values $F(0,T)$. Parameter $g(T)$ can be used to achieve a perfect match with at-the-money futures options. This leaves the constant parameters $(\sigma_S, \sigma_L, \kappa, \rho)$ to be determined by either matching selected exotics (such as Asian options or commodity swaptions) or by fitting empirical covariances of futures prices with different expiries. We will explore the latter possibility now.

Historical commodity price data is available for long periods. For example, the UBS Constant Maturity Commodity Index (CMCI) family covers 28 commodity futures contracts representing the energy, precious metals, industrial metals, agricultural and livestock sectors [140]. It provides daily futures price fixings back to 1997 for expiries from three months up to three years (e.g. 3M, 6M, 1Y, 2Y, 3Y) from each date in the time series. This allows constructing an empirical covariance matrix for each sector by computing

$$V_{ij}^* = \frac{1}{\tau}\sum_{k=1}^{m} \ln\frac{F(t_k, t_k+\tau_i)}{F(t_{k-1}, t_{k-1}+\tau_i)}\,\ln\frac{F(t_k, t_k+\tau_j)}{F(t_{k-1}, t_{k-1}+\tau_j)}, \qquad \tau = t_m - t_0,$$

where indices $i, j = 1, \ldots, 5$ label the available contract expiries. On the other hand one can compute the corresponding model-implied covariance matrix (setting $g(T) = 1$)

$$\begin{aligned}
V_{ij} &= \frac{1}{\tau}\int_t^{t+\tau} \frac{dF(s, s+\tau_i)\, dF(s, s+\tau_j)}{F(s, s+\tau_i)\, F(s, s+\tau_j)} \\
&= \frac{1}{\tau}\int_t^{t+\tau} \left(\sigma_S\, e^{-\kappa\,\tau_i}\, dW_s^S + \sigma_L\left(1 - e^{-\kappa\,\tau_i}\right) dW_s^L\right) \\
&\quad \times \left(\sigma_S\, e^{-\kappa\,\tau_j}\, dW_s^S + \sigma_L\left(1 - e^{-\kappa\,\tau_j}\right) dW_s^L\right) ds \\
&= \sigma_S^2\, e^{-\kappa(\tau_i+\tau_j)} + \sigma_L^2\left(1 - e^{-\kappa\,\tau_i}\right)\left(1 - e^{-\kappa\,\tau_j}\right) \\
&\quad + \rho\,\sigma_S\sigma_L\left(e^{-\kappa\tau_i} + e^{-\kappa\tau_j} - 2\, e^{-\kappa(\tau_i+\tau_j)}\right)
\end{aligned}$$

The model parameters $(\sigma_S, \sigma_L, \kappa, \rho)$ can then be chosen such that the square deviation $\sum_{i,j}(V_{ij} - V_{ij}^*)^2$ between empirical and model-implied covariances is minimized.

To tailor the model then to a particular commodity in the sector, one can adjust $g(T)$ for each available futures option contract expiry $T$ so that model Black volatility $\Sigma_T$ (which contains $g(T)$ as a multiplicative factor) matches the market quoted volatilities of futures options for this commodity.

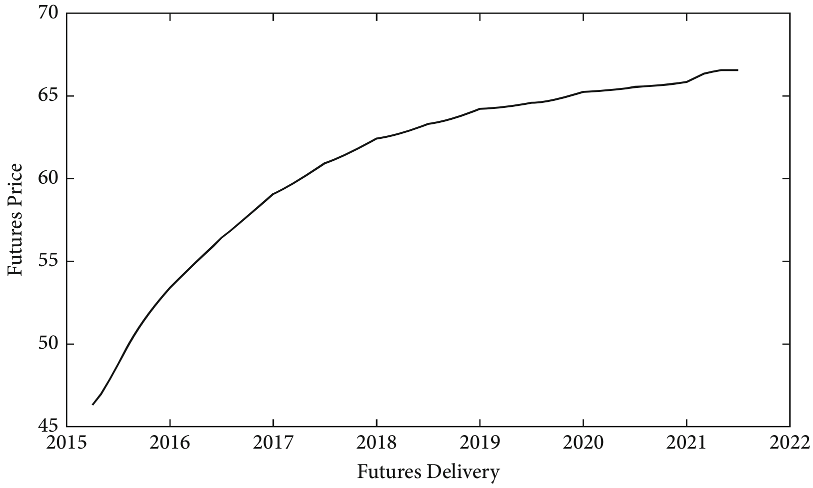


**Figure 14.1** Crude Oil WTI futures prices as of 22 January 2015

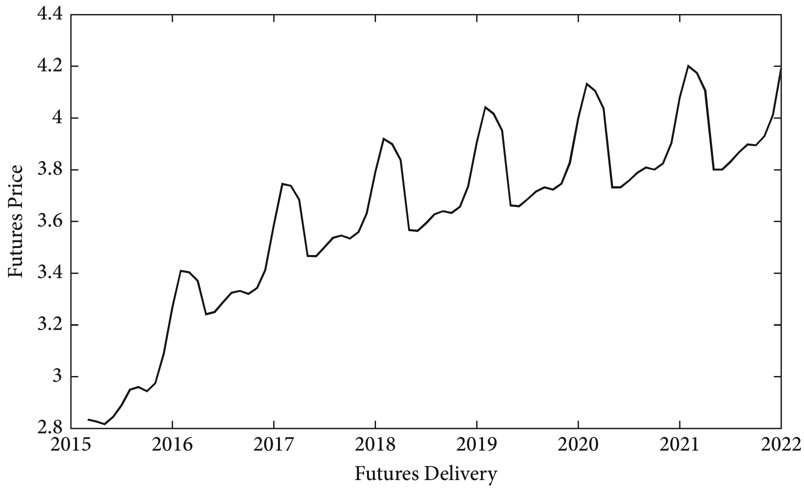


**Figure 14.2** Natural Gas futures prices as of 22 January 2015

Figures 14.1 and 14.2 show typical commodity futures curves with less (oil) or more (gas) pronounced seasonality impact. Futures options implied volatilities are quoted by multiplicative strike (0.2–0.4 times ATM strike) and by option expiry (monthly). The underlying future contract is assumed to mature at expiry, that is $T_1 = T_2$.

**Propagation**

In order to propagate a particular futures price $F(t,T)$, it is convenient to express the futures curve dynamics in terms of an artificial 2-d spot price process

$$S_t = A(t)\exp\left(B(t)\left(X_1(t)+X_2(t)\right)\right), \qquad F(t,T) = \mathbb{E}_t[S(T)]$$

where $X_i(t)$ follow Ornstein-Uhlenbeck processes starting at $X_i(0)=0$

$$\begin{aligned} dX_1(t) &= -\kappa\, X_1(t)\, dt + \sigma_1(t)\, dW_1(t) \\ dX_2(t) &= \sigma_2(t)\, dW_2(t) \\ dW_1(t)\, dW_2(t) &= \rho_{12}\, dt \end{aligned} \tag{14.10}$$

in the risk-neutral measure of the commodity currency with

$$\begin{aligned} \sigma_1^2(t) &= \alpha^2(t)\,\left(\sigma_S^2 + \sigma_L^2 - 2\,\rho\,\sigma_S\sigma_L\right) \\ \sigma_2^2(t) &= \alpha^2(t)\,\sigma_L^2 \\ \rho_{12} &= \frac{\sigma_S\,\rho - \sigma_L}{\sigma_1(t)/\alpha(t)} \end{aligned}$$

The conditional futures curve in terms of variables $X_{1,2}(t)$ then reads

$$\begin{aligned} F(t,T) = F(0,T)\,\exp\Big[g(T)\,\Big(X_1(t)\,e^{-\kappa\,(T-t)} + X_2(t)\Big) \\ - \frac{1}{2}\,g^2(T)\,(V(0,T) - V(t,T))\Big] \end{aligned}$$

with the variance of log-returns of futures prices

$$\begin{aligned} V(t,T) &= \int_t^T \sigma_2^2\, ds + e^{-2\kappa T}\int_t^T \sigma_1^2\, e^{2\kappa s}\, ds + 2\,\rho_{12}\, e^{-\kappa T}\int_t^T \sigma_1\,\sigma_2\, e^{\kappa s}\, ds \\ &= \sigma_2^2\,(T-t) + \sigma_1^2\,\frac{1-e^{-2\kappa(T-t)}}{2\kappa} + 2\,\rho_{12}\,\sigma_1\,\sigma_2\,\frac{1-e^{-\kappa(T-t)}}{\kappa} \end{aligned}$$

This matches with the Black variance $(T-t)\cdot\Sigma_{tT}^2$ in (14.9) for $g(T)=1$, ensures $F(t,T)$ is a martingale, and satisfies the initial condition $F(0,T)$. The Ornstein-Uhlenbeck processes are propagated as usual using

$$\begin{aligned} X_1(t) - X_1(s) &= -X_1(s)\,\Big(1 - e^{-\kappa(t-s)}\Big) + \int_s^t \sigma_1(\tau)\, e^{-\kappa(t-\tau)}\, dW_1(\tau) \\ X_2(t) - X_2(s) &= \int_s^t \sigma_2(\tau)\, dW_2(\tau) \end{aligned}$$