Contents

1 Tournament Formats

1.1 Definitions

Definition 1.1.1: Gameplay Function

A *gameplay function* g on a list of teams $\mathcal{T} = [t_1, ..., t_n]$ is a nondeterministic function $g : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ with the following properties:

- $\mathbb{P}[g(t_i, t_j) = t_i] + \mathbb{P}[g(t_i, t_j) = t_j] = 1$.
- $\mathbb{P}[g(t_i, t_j) = t_i] = \mathbb{P}[g(t_j, t_i) = t_j]$.

A gameplay function represents a process in which two teams compete in a game, with one of them emerging as the winner. This model simplifies away effects like home-field advantage or teams improving over the course of a tournament: a gameplay function is fully described by a single probability for each pair of teams in the list.

Definition 1.1.2: Playing, Winning, and Losing

When g is queried on input (t_i, t_j) we say that t_i and t_j *played a game*. We say that the team that got outputted by g *won*, and the team that did not *lost*.

The information in a gameplay function can be encoded into a *matchup table*.

Definition 1.1.3: Matchup Table

The *matchup table* implied by a gameplay function g on a list of teams $\mathcal{T}$ of length n is a n-by-n matrix $\mathbf{M}$ such that $\mathbf{M}_{ij} = \mathbb{P}[t_i \text{ beats } t_j]$.

For example, let $\mathcal{T}$ =[Favorites, Rock, Paper, Scissors, Conceders], and g be such that the Conceders concede every game they play, the Favorites are 70% favorites against Rock, Paper, and Scissors, and Rock, Paper, and Scissors match up with each other as their name implies. Then the matchup table would look like so:

Figure 1.1.4: The Matchup Table for $(\mathcal{T}, g)$

	Favorites	Rock	Paper	Scissors	Conceders
Favorites	0.5	0.7	0.7	0.7	1.0
Rock	0.3	0.5	0.0	1.0	1.0
Paper	0.3	1.0	0.5	0.0	1.0
Scissors	0.3	0.0	1.0	0.5	1.0
Conceders	0.0	0.0	0.0	0.0	0.5

Theorem 1.1.5

If $\mathbf{M}$ is the matchup table for $(\mathcal{T}, g)$, then $\mathbf{M} + \mathbf{M}^T$ is the matrix of all ones.

Proof. $(\mathbf{M} + \mathbf{M}^T)_{ij} = \mathbf{M}_{ij} + \mathbf{M}_{ji} = \mathbb{P}[t_i \text{ beats } t_j] + \mathbb{P}[t_j \text{ beats } t_i] = 1.$ $\square$

Definition 1.1.6: Tournament Format

A *tournament format* is an algorithm that takes as input a list of teams $\mathcal{T}$ and a gameplay function g and outputs a champion $t \in \mathcal{T}$.

We use a gameplay function rather than a matchup table in the definition of a tournament format because a tournament format cannot simply look at the matchup table itself in order to decide which teams are best. Instead, formats query the gameplay function (have teams play games) in order to gather information about the teams. That said, matchup tables will often be useful in our *analysis* of tournament formats.

We also introduce some shorthand to help make notation more concise.

Definition 1.1.7: $\mathbb{W}_{\mathcal{A}}(t, \mathcal{T})$

$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T})$ is the probability that team $t \in \mathcal{T}$ wins tournament format $\mathcal{A}$ when it is run on the list of teams $\mathcal{T}$.

(This chapter will be fleshed out but I'm including the important definitions here for the sake of the next chapter.)

2 Brackets

2.1 Bracket Signatures

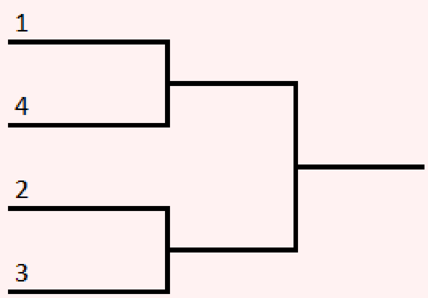
Definition 2.1.1: Bracket

A *bracket* is a tournament format in which:

- Teams don't play any games after their first loss,
- Games are played until only one team has no losses, and that team is crowned champion, and
- The matchups between game winners are determined in advance of the outcomes of any games.

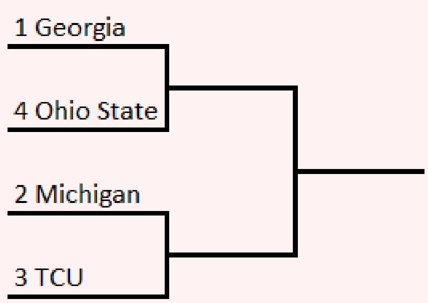
We can draw a bracket as a tree-like structure in the following way.

Figure 2.1.2: The 2023 College Football Playoff



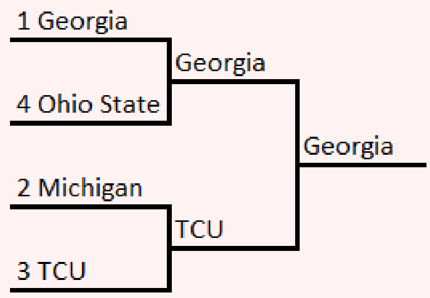
The numbers 1, 2, 3, and 4 indicate where t_1, t_2, t_3, and t_4 in $\mathcal{T}$ are placed to start. In the actual 2023 College Football Playoff, the list of teams $\mathcal{T}$ was Georgia, Michigan, TCU, and Ohio State, in that order, so the bracket was filled in like so.

Figure 2.1.3: The 2023 CFP After Team Placement



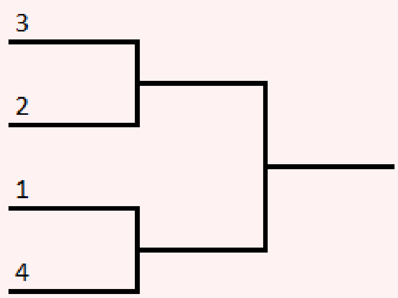
As games are played, we write the name of the winning teams on the corresponding lines. This bracket tells us that Georgia played Ohio State, and Michigan played TCU. Georgia and TCU won their respective games, and then Georgia beat TCU, winning the tournament.

Figure 2.1.4: The 2023 CFP After Completion



Rearranging the way the bracket is pictured, if it doesn't affect any of the matchups, does not create a new bracket. For example, Figure 2.1.5 is just another way to draw the same 2023 CFP Bracket.

Figure 2.1.5: Alternative Drawing of the 2023 CFP



One key piece of bracket vocabulary is the *round*.

Definition 2.1.6: Round

A *round* is a set of games such that the winners of each of those games have the same number of games remaining to win the tournament.

For example, the 2023 CFP has two rounds. The first round included the games Georgia vs Ohio State and Michigan vs TCU, and the second round was just a single game: Georgia vs TCU.

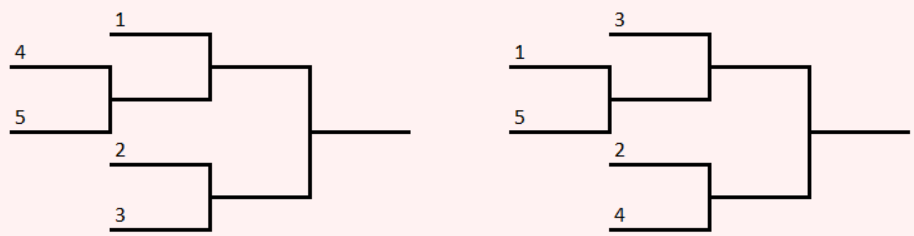
Another important concept is the *shape* of a bracket.

Definition 2.1.7: Shape

The *shape* of a bracket is the tree that underlies it.

For example, the following two brackets have the same shape.

Figure 2.1.8: Two Brackets with the Same Shape



Definition 2.1.9: Bye

A team has a *bye* in round r if it plays no games in round r or before.

One way to describe the shape of a bracket its signature.

Definition 2.1.10: Bracket Signature

The *signature* $[[\mathbf{a_0}; \ldots; \mathbf{a_r}]]$ of an r-round bracket $\mathcal{A}$ is list such that a_i is the number of teams with i byes.

The signature of a bracket is defined only by its shape: the two brackets in Figure 2.1.8 have the same shape, so they also have the same signature.

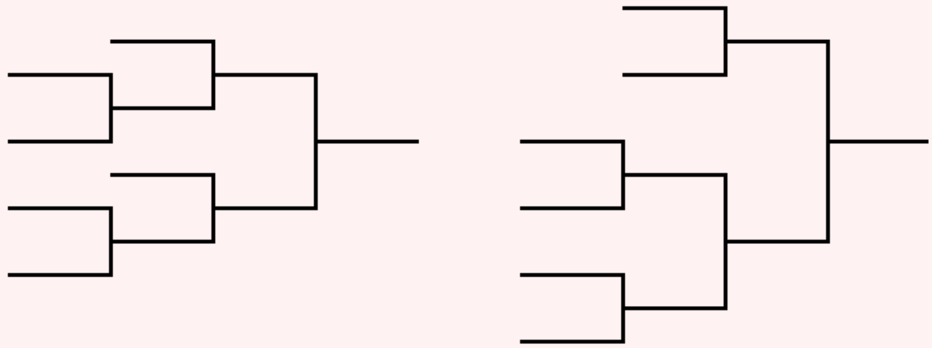
The signatures of the brackets discussed in this section are shown in Figure 2.1.11. It's worth verifying the signatures we've seen so far and trying to draw brackets with the signatures we haven't yet before moving on.

Figure 2.1.11: The Signatures of Some Brackets

Bracket	Signature
2023 College Football Playoff	$[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$
The brackets in Figure 2.1.8	$[[\mathbf{2}; \mathbf{3}; \mathbf{0}; \mathbf{0}]]$
The brackets in Figure 2.1.12	$[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$
2023 WCC Men's Basketball Tournament	$[[\mathbf{4}; \mathbf{2}; \mathbf{2}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$

Two brackets with the same shape must have the same signature, but the converse is not true: two brackets with different shapes can have the same signature. For example, both bracket shapes depicted in Figure 2.1.12 have the signature $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$.

Figure 2.1.12: Two Shapes with the Signature $[[4;2;0;0]]$



Despite this, bracket signatures are a useful way to talk about the shape of a bracket. Communicating a bracket's signature is a lot easier than communicating its shape, and much of the important information (such as how many games each team must win in order to win the tournament) is contained in the signature.

Bracket signatures have one more important property.

Theorem 2.1.13

Let $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ be a list of natural numbers. Then $\mathcal{A}$ is a bracket signature if and only if

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = 1.$$

Proof. Let $\mathcal{A}$ be the signature for some bracket. Assume that every game in the bracket was a coin flip, and consider each team's probability of winning the tournament. A team that has i byes must win $r - i$ games to win the tournament, and so will do so with probability $\left(\frac{1}{2}\right)^{r-i}$. For each $i \in \{0, ..., r\}$, there are a_i teams with i byes, so (because any two teams winning are mutually exclusive)

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i}$$

is the probability that one of the teams wins, which is 1.

We prove the other direction by induction on r. If $r = 0$, then the only list with the desired property is $[[\mathbf{1}]]$, which is the signature for the unique one-team bracket. For any other r, first note that a_0 must be even: if it were odd, then

$$\begin{aligned}
\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} &= \frac{1}{2^r} \cdot \sum_{i=0}^{r} a_i \cdot 2^i \\
&= \frac{1}{2^r} \cdot \left(a_0 + 2\sum_{i=1}^{r} a_i \cdot 2^{i-1}\right) \\
&= k/2^r && \text{for some odd } k \\
&\neq 1.
\end{aligned}$$

Now, consider the signature $\mathcal{B} = [[\mathbf{a_1 + a_0/2}; \mathbf{a_2}; \ldots; \mathbf{a_r}]]$. By induction, there exists a bracket with signature $\mathcal{B}$. But if we take that bracket and replace $a_0/2$ of the teams with no byes with a game whose winner gets placed on that line, we get a new bracket with signature $\mathcal{A}$. $\square$

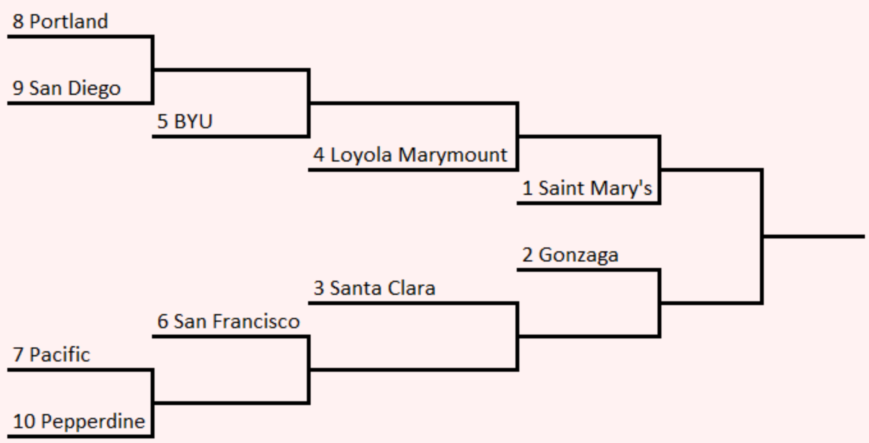
In the next few sections, we will use the language and properties of bracket signatures to describe the brackets that we work with. For now though, let's return to the 2023 College Football Playoff. The bracket used in the 2023 CFP has a special property that not all brackets have: it is *balanced*.

Definition 2.1.14: Balanced Bracket

A *balanced bracket* is a bracket in which none of the teams have byes.

The 2023 West Coast Conference Men's Basketball Tournament, on the other hand, is unbalanced.

Figure 2.1.15: The 2023 WCC Men's Basketball Tournament



Saint Mary's and Gonzaga each have three byes and so only need to win two games to win the tournament, while Portland, San Diego, Pacific, and Pepperdine need to win five. Unsurprisingly, this format conveys a massive advantage to Saint Mary's and Gonzaga, but this was intentional: those two teams were being rewarded for doing the best during the regular season.

In many cases, however, it is undesirable to grant advantages to certain teams over others. One might hope, for any n, to be able to construct a balanced bracket for n teams, but unfortunately this is rarely possible.

Theorem 2.1.16

There exists an n-team balanced bracket if and only if n is a power of two.

Proof. A bracket is balanced if no teams have byes, which is true exactly when its signature is of the form $\mathcal{A} = [[\mathbf{n}; \mathbf{0}; \ldots; \mathbf{0}]]$ where n is the number of teams in the bracket. If n is a power of two, then by Theorem 2.1.13 $\mathcal{A}$ is indeed a bracket signature and so points to a balanced bracket for n teams. If n is not a power of two, however, then Theorem 2.1.13 tells us that $\mathcal{A}$ is not a bracket signature, and so no balanced brackets exist for n teams. $\square$

Given this, brackets are not a great option when we want to avoid giving some teams advantages over others unless we have a power of two teams. They are a fantastic tool, however, if doling out advantages is the goal, perhaps after some teams did better during the regular season and ought to be rewarded with an easier path in the bracket.

2.2 Proper Brackets

Definition 2.2.1: Seeding

The *seeding* of an n-team bracket is the arrangement of the numbers 1 through n in the bracket.

Together, the shape and seeding fully specify a bracket.

Definition 2.2.2: i-seed

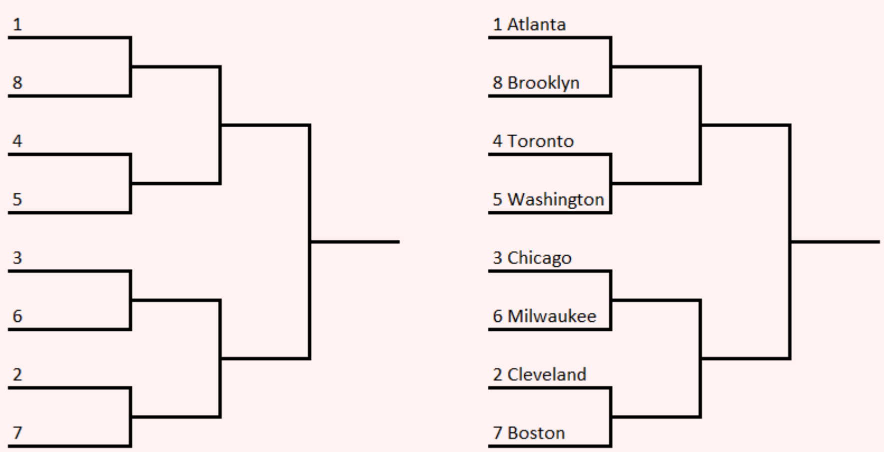
In a list of teams $\mathcal{T} = [t_1, ..., t_n]$, we refer to t_i as the i-seed.

Definition 2.2.3: Higher and Lower Seeds

Somewhat confusingly, convention is that smaller numbers are the *higher seeds*, and greater numbers are the *lower seeds*.

Seeding is typically used to reward better and more deserving teams. As an example, on the left is the eight-team bracket used in the 2015 NBA Eastern Conference Playoffs. At the end of the regular season, the top eight teams in the Eastern Conference were ranked and placed into the bracket as shown on the right.

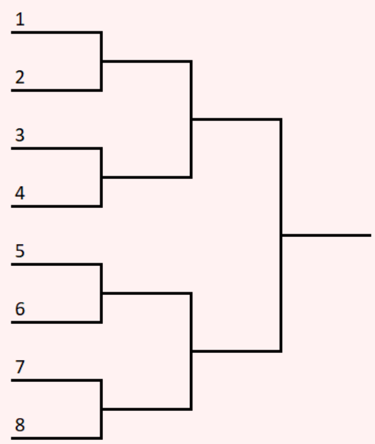
Figure 2.2.4: 2015 NBA Eastern Conference Playoffs



Despite this bracket being balanced, the higher seeds are still at advantage: they have an easier set of opponents. Compare 1-seed Atlanta, whose first two rounds are versus 8-seed Brooklyn and then (most likely) 4-seed Toronto, versus 7-seed Boston, whose first two rounds are versus 2-seed Cleveland and then (most likely) 3-seed Chicago. Atlanta's schedule is far easier: despite them having the same number of games to win as Boston, Atlanta is expected to play lower seeds in each round than Boston will.

Thus, we've identified two ways in which brackets can convey an advantage onto certain teams: by giving them more byes, and by giving them easier (expected) opponents. Not every seeding of a bracket does this: for example, consider the following alternative seeding for the 2015 NBA Eastern Conference Playoffs.

Figure 2.2.5: An Alternative Seeding of the 2015 NBA Eastern Conference Playoffs



This seeding does a very poor job of rewarding the higher-seeded teams: the 1- and 2-seeds are matched up in the first round, while the easiest road is given to the 7-seed, who plays the 8-seed in the first round and then (most likely) the 5-seed in the second. Since the whole point of seeding is to give the higher-seeded teams an advantage, we introduce the concept of a *proper seeding*.

Definition 2.2.6: Chalk

We say a tournament *went chalk* if the higher-seeded team won every game during the tournament.

Definition 2.2.7: Proper Seeding

A *proper seeding* of a bracket is one such that if the bracket goes chalk, in every round it is better to be a higher-seeded team than a lower-seeded one, where:

(1) It is better to have a bye than to play a game.

(2) It is better to play a lower seed than to play a higher seed.

Definition 2.2.8: Proper Bracket

A *proper bracket* is a bracket that has been properly seeded.

It is clear that the actual 2015 NBA Eastern Conference Playoffs was properly seeded, while our alternative seeding was not.

We now quickly derive a few lemmas about proper brackets.

Lemma 2.2.9

In a proper bracket, if m teams have a bye in a given round, those teams must be seeds 1 through m.

Proof. If they did not, the seeding would be in violation of condition (1). $\square$

Lemma 2.2.10

If a proper bracket goes chalk, then after each round the m teams remaining will be the top m seeds.

Proof. We will prove the contrapositive. Assume that for some $i < j$, after some round, t_i has been eliminated but t_j is still alive. Let k be the seed of the team that t_i lost to. Because the bracket went chalk, $k < i$. Now consider what t_j did in that round. If they had a bye, then the bracket violates condition (1). Assume instead they played t_ℓ. They beat t_ℓ, so $j < \ell$, giving,

$$k < i < j < \ell.$$

In the round that t_i was eliminated, t_i played t_k, while t_j played t_ℓ, violating condition (2). Thus, the bracket is not proper. $\square$

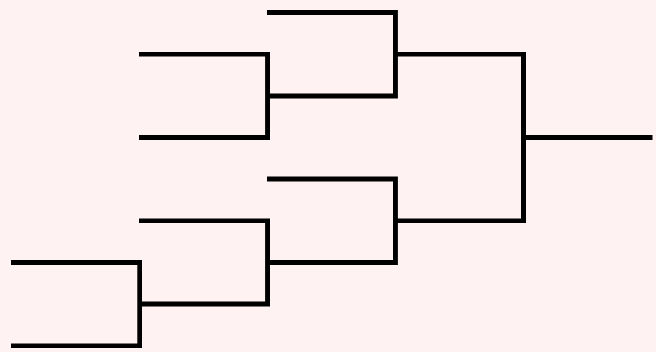
Lemma 2.2.11

In a proper bracket, if m teams have a bye and k games are being played in a given round, then if the bracket goes chalk those matchups will be seed $m + i$ vs seed $(m + 2k + 1) - i$ for $i \in \{1, ..., k\}$.

Proof. In the given round, there are $m + 2k$ teams remaining. Theorem 2.2.10 tells us that (if the bracket goes chalk) those teams must be seeds 1 through $m + 2k$. Theorem 2.2.9 tells us that seeds 1 through m must have a bye, so the teams playing must be seeds $m + 1$ through $m + 2k$. Then condition (2) tells us that the matchups must be exactly $m + i$ vs seed $(m + 2k + 1) - i$ for $i \in \{1, ..., k\}$. $\square$

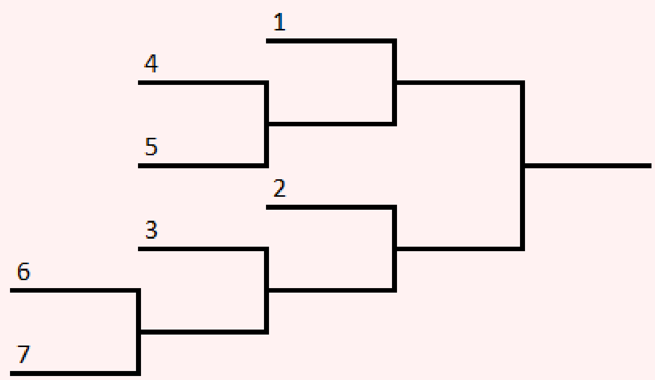
We can use Lemmas 2.2.9 through 2.2.11 to properly seed various bracket shapes. For example, consider the following seven-team shape.

Figure 2.2.12: A Seven-Team Bracket Shape



Lemma 2.2.9 tells us that the first-round matchup must be between the 6-seed and the 7-seed. Lemma 2.2.11 tells us that if the bracket goes chalk, the second-round matchups must be 3v6 and 4v5, so the 3-seed play the winner of the first-round matchup. Finally, we can apply Lemma 2.2.11 again to the semifinals to find that the 1-seed should play the winner of the 4v5 matchup, while the 2-seed should play the winner of the 3v(6v7) matchup. In total, our proper seeding looks like so.

Figure 2.2.13: A Seven-Team Bracket, Properly Seeded

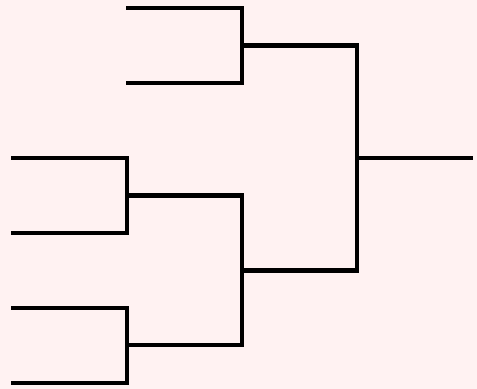


We can also quickly simulate the bracket going chalk to verify Lemma 2.2.10.

Lemmas 2.2.9 through 2.2.11 are quite powerful. It is not a coincidence that we managed to specify exactly what a proper seeding of the above bracket must look like with no room for variation: soon we will prove that the proper seeding for a particular bracket shape is unique.

But not every shape admits even this one proper seeding. Consider the following six-team shape.

Figure 2.2.14: A Six-Team Bracket Shape



This shape admits no proper seedings. Lemma 2.2.9 requires that the two teams getting byes be the 1- and 2-seed, but this violates Lemma 2.2.11 which requires that in the second round the 1- and 2-seeds do not play each other. So how can we think about which shapes admit proper seedings?

Theorem 2.2.15: The Fundamental Theorem of Brackets

There is exactly one proper bracket with each bracket signature.

Proof. Let $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ be an r-round bracket signature. We proceed by induction on r. If $r = 0$, then the only possible bracket signature is $[[\mathbf{1}]]$, and it points to the unique one-team bracket, which is indeed proper.

For any other r, the first-round matchups of a proper bracket with signature $\mathcal{A}$ are defined by Lemma 2.2.11. Then if those matchups go chalk, we are left with a proper bracket of signature $[[\mathbf{a_0/2 + a_1}; \mathbf{a_2}; ...; \mathbf{a_r}]]$, which induction tells us exists admits exactly one proper bracket.

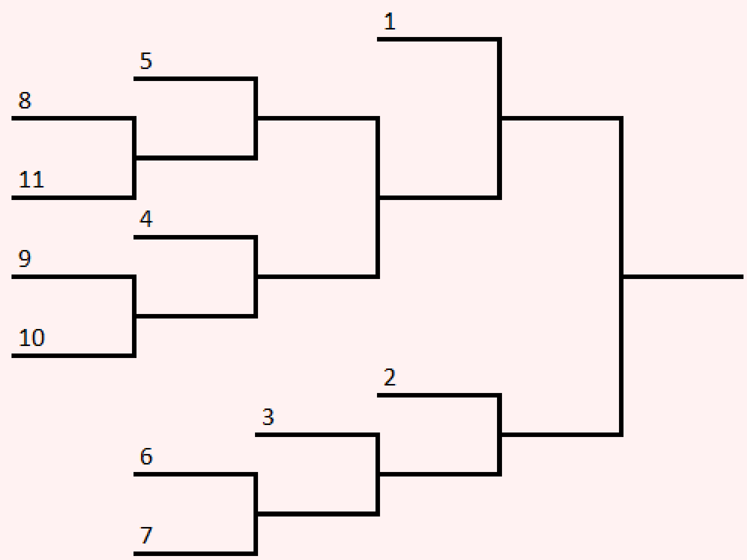
Thus both the first-round matchups and the rest of the bracket are determined, and by combining them we get a proper bracket with signature $\mathcal{A}$, so there is exactly one proper bracket with signature $\mathcal{A}$. $\square$

The fundamental theorem of brackets means that we can refer to the proper bracket $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ in a well-defined way, as long as

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = 1.$$

In practice, virtually every sports league that uses a traditional bracket uses a proper one: while different leagues take very different approaches to how many byes to give teams (compare the 2023 West Coast Conference Men's Basketball Tournament with the 2015 NBA Eastern Conference Playoffs), they are almost all proper. This makes bracket signatures a convenient labeling system for the set of brackets that we might reasonably encounter. They also are a powerful tool for specifying new brackets: if you are interested in (say) an eleven-team bracket where four teams get no byes, four teams get one bye, one team gets two byes and two teams get three byes, we can describe the proper bracket with those specs as $[[\mathbf{4};\mathbf{4};\mathbf{1};\mathbf{2};\mathbf{0};\mathbf{0}]]$ and use Lemmas 2.2.9 through 2.2.11 to draw it with ease.

Figure 2.2.16: The Proper Bracket of Signature $[[\mathbf{4};\mathbf{4};\mathbf{1};\mathbf{2};\mathbf{0};\mathbf{0}]]$

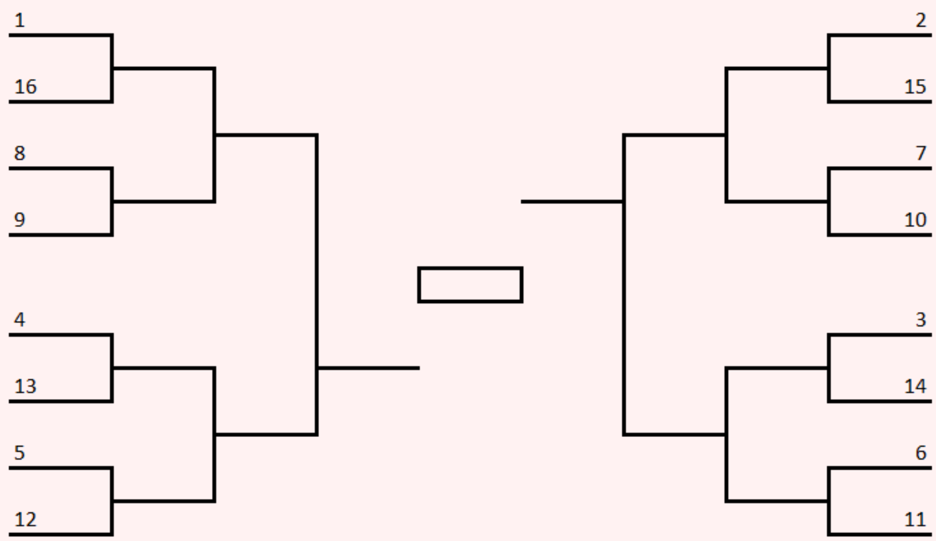


Due to these properties, we will almost exclusively discuss proper bracket from here on out: unless stated otherwise, assume all brackets are proper.

2.3 Ordered Brackets

Consider the proper bracket $[[\mathbf{16}; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$, which was used in the 2021 NCAA Men's Basketball Tournament South Region, and is shown below. (Sometimes brackets are drawn in the manner below, with teams starting on both sides and the winner of each side playing in the championship game.)

Figure 2.3.1: 2021 NCAA Men's Basketball Tournament South Region



The definition of a proper seeding ensures that as long as the bracket goes chalk (that is, higher seeds always beat lower seeds), it will always be better to be a higher seed than a lower seed. But what if it doesn't go chalk?

One counter-intuitive fact about the NCAA Basketball Tournament is that it is probably better to be a 10-seed than a 9-seed. (This doesn't violate the proper seeding property because 9-seeds have an easier first-round matchup than 10-seeds, and for further rounds, proper seedings only care about what happens if the bracket goes chalk, which would eliminate both the 9-seed and 10-seed in the first round.) Why? Let's look at whom each seed-line matchups against in the first two rounds.

Figure 2.3.2: NCAA Basketball Tournament 9- and 10-seed Schedules

Seed	First Round	Second Round
9	8	1
10	7	2

The 9-seed has an easier first-round matchup, while the 10-seed has an easier second-round matchup. However, this isn't quite symmetrical. Because the teams are probably drawn from a roughly normal distribution, the expected difference in skill between the 1-

and 2-seeds is far greater than the expected difference between the 7- and 8-seeds, implying that the 10-seed does in fact have an easier route than the 9-seed.

Nate Silver [7] investigated this matter in full, finding that in the NCAA Basketball Tournament, seed-lines 10 through 15 give teams better odds of winning the region than seed-lines 8 and 9. Of course this does not mean that the 11-seed (say) has a better chance of winning a given region than the 8-seed does, as the 8-seed is a much better team than the 11-seed. But it does mean that the 8-seed would love to swap places with the 11-seed, and that doing so would increase their odds to win the region.

This is not a great state of affairs: the whole point of seeding is confer an advantage to higher-seeded teams, and the proper bracket $[[\mathbf{16}; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$ is failing to do that. Not to mention that giving lower-seeded teams an easier route than higher-seeded ones can incentivize teams to lose during the regular season in order to try to get a lower but more advantageous seed.

To fix this, we need a stronger notion of what makes a bracket effective than properness. The issue with proper seedings is the false assumption that higher-seeded teams will always beat lower-seeded teams. A more nuanced assumption, initially proposed by H.A. David [3], might look like this.

Definition 2.3.3: Strongly Stochastically Transitive

A list of teams $\mathcal{T}$ is *strongly stochastically transitive* if for each i, j, k such that $j < k$,

$$\mathbb{P}[t_i \text{ beats } t_j] \leq \mathbb{P}[t_i \text{ beats } t_k].$$

A list of teams being strongly stochastically transitive (SST) captures the intuition that each team ought to do better against lower-seeded teams than against higher-seeded teams. A few quick implications of this definition are stated below.

Corollary 2.3.4

(1) If $\mathcal{T}$ is SST, then for each $i < j$,

$$\mathbb{P}[t_i \text{ beats } t_j] \geq 0.5.$$

(2) If $\mathcal{T}$ is SST, then for each i, j, k, ℓ such that $i < j$ and $k < \ell$,

$$\mathbb{P}[t_i \text{ beats } t_\ell] \geq \mathbb{P}[t_j \text{ beats } t_k].$$

(3) If $\mathcal{T}$ is SST, then the matchup table $\mathbf{M}$ is monotonically increasing along each row and monotonically decreasing along each column.

Note that not every set of teams can be seeded to be SST. Consider, for example, the game of rock-paper-scissors. Rock beats paper which beats scissors which beats rock, so no ordering of these "teams" will be SST. For our purposes, however, SST will work well enough.

Our new, nuanced alternative a proper bracket is an *ordered bracket*, first defined by Chen and Hwang [2] (though we use the name proposed by Edwards [4]).

Definition 2.3.5: Ordered

A tournament format $\mathcal{A}$ is *ordered* if, for any SST list of teams $\mathcal{T}$, if $i < j$, then $\mathbb{W}_{\mathcal{A}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{T})$

In an informal sense, a bracket being ordered is the strongest thing we can want without knowing more about why the tournament is being played. Depending on the situation, we might be interested in a format that almost always declares the most-skilled team as the winner, or in a format that gives each team roughly the same chance of winning, or anywhere in between. But certainly, better teams should win more, which is what the ordered bracket condition requires.

In particular, a bracket being ordered is a stronger claim than it being proper.

Theorem 2.3.6

Every ordered bracket is proper.

Proof. Let $\mathcal{A}$ be an ordered n-team bracket with r rounds.

Let $\mathcal{T}$ be SST with matchup table $\mathbf{M}$ where $\mathbf{M}_{ij} = 0.5$. A team that plays their first game in the ith round will win the tournament with probability $(0.5)^{r-i}$, so teams that get more byes will have a higher probability to win the tournament than teams with fewer byes. This implies that higher-seeded teams must have more byes than lower-seeded teams, so in each round, the teams with byes must be the highest-seeded teams that are still alive. Thus, condition (1) is met.

We show that condition (2) is met by proving the stronger condition from Lemma 2.2.11: if m teams have a bye and k games are being played in round s, then if the bracket goes chalk, those matchups will be t_{m+i} vs $t_{(m+2k+1)-i}$ for $i \in \{1, ..., k\}$. We show this by strong induction on s and on i.

Assume that this is true for every round up until s and for all $i < j$ for some j. Let $\ell = (m + 2k + 1) - j$. We want to show that if the the bracket goes chalk, t_{m+j} will face off against seed t_ℓ in the given round. Consider the following SST matchup table: every game is a coin flip, except for games involving a team seeded ℓ or lower, in which case the higher seed always wins. Then, each team seeded between $\ell - 1$ and $m + j$ will win the tournament with probability $(\frac{1}{2})^{r-s}$, other than the team slated to play t_ℓ in round s who wins with probability $(\frac{1}{2})^{r-i-1}$. In order for $\mathcal{B}$ to be ordered, that team must be t_{m+j}.

Thus $\mathcal{A}$ satisfies both conditions, and so is a proper bracket. □

With Theorem 2.3.6, we can use the language of bracket signatures to describe ordered brackets without worrying that two ordered brackets might share a signature. Now we examine three particularly important examples of ordered brackets.

We begin with the unique one-team bracket.

Figure 2.3.7: The One-Team Bracket $[[\mathbf{1}]]$

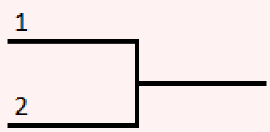
Theorem 2.3.8

The one-team bracket $[[\mathbf{1}]]$ is ordered.

Proof. Since there is only team, the ordered bracket condition is vacuously true. □

Next we look at the unique two-team bracket.

Figure 2.3.9: The Two-Team Bracket $[[\mathbf{2};\mathbf{0}]]$



Theorem 2.3.10

The two-team bracket $[[\mathbf{2};\mathbf{0}]]$ is ordered.

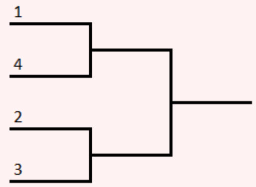
Proof. Let $\mathcal{A} = [[\mathbf{2};\mathbf{0}]]$. Then,

$$\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) = \mathbb{P}[t_1 \text{ beats } t_2] \geq 0.5 \geq \mathbb{P}[t_2 \text{ beats } t_1] = \mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T})$$

so $\mathcal{A}$ is ordered. □

And thirdly, we show that the balanced four-team bracket is ordered, first proved by Horen and Riezman [5].

Figure 2.3.11: The Four-Team Bracket $[[4; 0; 0]]$



Theorem 2.3.12

The four-team bracket $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$ is ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$ and let $p_{ij} = \mathbb{P}[t_i \text{ beats } t_j]$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) &= p_{14} \cdot (p_{23}p_{12} + p_{32}p_{13}) \\
&= p_{14}p_{23}p_{12} + p_{14}p_{32}p_{13} \\
&\geq p_{14}p_{23}p_{21} + p_{24}p_{41}p_{23} \\
&= p_{23} \cdot (p_{14}p_{21} + p_{41}p_{24}) \\
&= \mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T})
\end{aligned}$$

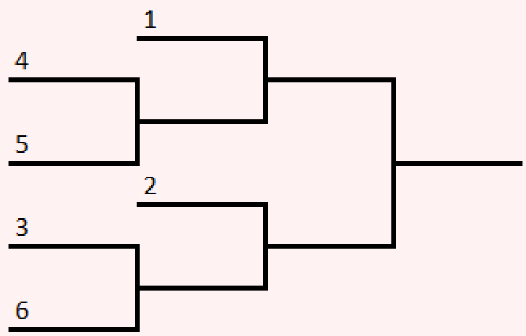
$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T}) &= p_{23} \cdot (p_{14}p_{21} + p_{41}p_{24}) \\
&\geq p_{32} \cdot (p_{14}p_{31} + p_{41}p_{34}) \\
&= \mathbb{W}_{\mathcal{A}}(t_3, \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_3, \mathcal{T}) &= p_{32} \cdot (p_{14}p_{31} + p_{41}p_{34}) \\
&= p_{32}p_{14}p_{31} + p_{32}p_{41}p_{34} \\
&\geq p_{42}p_{23}p_{41} + p_{32}p_{41}p_{43} \\
&= p_{41} \cdot (p_{23}p_{42} + p_{32}p_{43}) \\
&= \mathbb{W}_{\mathcal{A}}(t_4, \mathcal{T})
\end{aligned}$$

Thus $\mathcal{A}$ is ordered. □

However, not every proper bracket is ordered. One particularly important example of a non-ordered proper bracket is $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$

Figure 2.3.13: The Six-Team Bracket $[[4; 2; 0; 0]]$



Theorem 2.3.14

The six-team bracket $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$, and let $\mathcal{T}$ have the following matchup table.

	t_1	t_2	t_3	t_4	t_5	t_6
t_1	0.5	0.5	0.5	0.5	0.5	1
t_2	0.5	0.5	0.5	0.5	0.5	1
t_3	0.5	0.5	0.5	0.5	0.5	0.5
t_4	0.5	0.5	0.5	0.5	0.5	0.5
t_5	0.5	0.5	0.5	0.5	0.5	0.5
t_6	0	0	0.5	0.5	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) = 0.5 \cdot 0.5 = 0.25,$$

but

$$\mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T}) = (0.5 \cdot 0.5 + 0.5 \cdot 1) \cdot 0.5 = 0.375.$$

Thus $\mathcal{A}$ is not ordered. $\square$

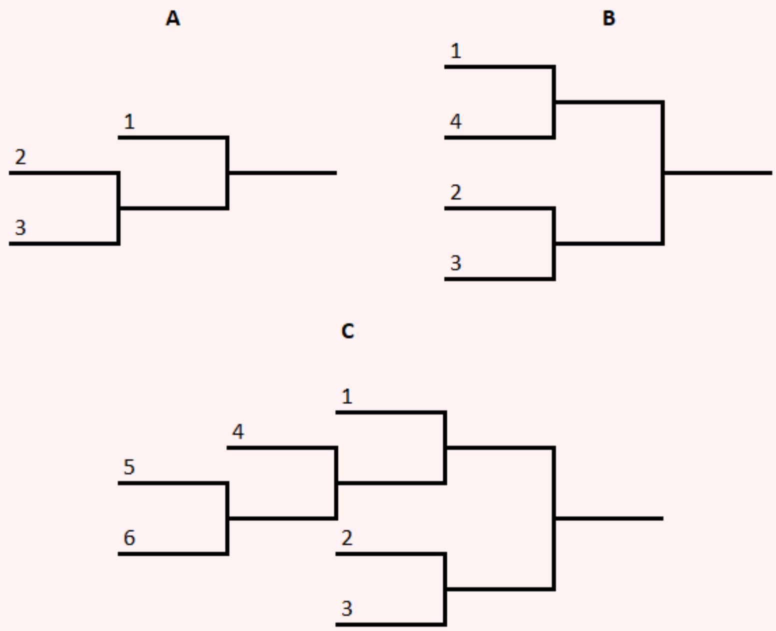
In the next section, we move on from describing particular ordered and non-ordered brackets in favor of a more general result.

2.4 Edwards's Theorem

We now attempt to completely classify the set of ordered brackets. Edwards [4] originally accomplished this without access to the machinery of bracket signatures or proper brackets: we present a quicker proof that makes use of the fundamental theorem of brackets and develop two nice lemmas along the way.

We begin with the stapling lemma, which allows us to combine two smaller ordered brackets into a larger ordered one by having the winner of one of the brackets be treated as the lowest seed in the other. This is depicted in Figure 2.4.1.

Figure 2.4.1: Setup of the Stapling Lemma with $\mathcal{A} = [[\mathbf{2};\mathbf{1};\mathbf{0}]]$, $\mathcal{B} = [[\mathbf{4};\mathbf{0};\mathbf{0}]]$, and $\mathcal{C} = [[\mathbf{2};\mathbf{1};\mathbf{3};\mathbf{0};\mathbf{0}]]$



Lemma 2.4.2: The Stapling Lemma

If $\mathcal{A} = [[\mathbf{a_0};...;\mathbf{a_r}]]$ and $\mathcal{B} = [[\mathbf{b_0};...;\mathbf{b_s}]]$ are ordered brackets, then $\mathcal{C} = [[\mathbf{a_0};...;\mathbf{a_r}+\mathbf{b_0}-\mathbf{1};...;\mathbf{b_s}]]$ is an ordered bracket as well.

Proof. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be as specified. Let $\mathcal{T}$ be an SST list of teams $n+m-1$ teams, and let $\mathcal{R}, \mathcal{S} \subset \mathcal{T}$ be the lowest n and the highest $m-1$ seeds of $\mathcal{T}$ respectively. We divide proving that $\mathcal{C}$ is ordered into proving three sub-statements:

1. For $i < j < m$, $\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})$
2. $\mathbb{W}_{\mathcal{C}}(t_{m-1}, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_m, \mathcal{T})$
3. For $m \leq i < j$, $\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})$

Together, these show that $\mathcal{C}$ is ordered.

We begin with the first sub-statement. Let $i < j < m$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) &= \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_k\}) \\
&\geq \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_j, \mathcal{S} \cup \{t_k\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})
\end{aligned}$$

The first and last equalities follow from the structure of $\mathcal{C}$, and the inequality follows from $\mathcal{B}$ being ordered.

Now the second sub-statement.

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_{m-1}, \mathcal{T}) &= \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_{m-1}, \mathcal{S} \cup \{t_k\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_m, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_{m-1}, \mathcal{S} \cup \{t_m\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_m, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_m, \mathcal{S} \cup \{t_m\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_m, \mathcal{T})
\end{aligned}$$

The equalities follow from the structure of $\mathcal{C}$, the first inequality follows from probabilities being non-negative, and the second inequality follows from $\mathcal{B}$ being ordered.

Finally, we show the third sub-statement. Let $m \leq i < j$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}}(t_i, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_i\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_i\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_j, \mathcal{S} \cup \{t_j\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})
\end{aligned}$$

The equalities follow from the structure of $\mathcal{C}$, the first inequality from $\mathcal{A}$ being ordered, and the second inequality from the teams being SST.

We have shown all three sub-statements, and so $\mathcal{C}$ is ordered. $\square$

Now, if we begin with the set of brackets $\{[[\mathbf{1}]], [[\mathbf{2}; \mathbf{0}]], [[\mathbf{4}; \mathbf{0}; \mathbf{0}]]\}$ and then repeatedly apply the stapling lemma, we can construct a set of brackets that we know are ordered. In other words,

Corollary 2.4.3

Any bracket signature formed by the following process is ordered:

1. Start with the list $[[\mathbf{0}]]$ (note that this not yet a bracket signature).
2. As many times as desired, prepend the list with $[[\mathbf{1}]]$ or $[[\mathbf{3};\mathbf{0}]]$.
3. Then, add 1 to the first element in the list, turning it into a bracket signature.

Corollary 2.4.3 uses the tools that we have developed so far to identify a set of ordered brackets. Somewhat surprisingly, this set is complete: any bracket not reachable using the process in Corollary 2.4.3 is not ordered. To prove this we first need to show the containment lemma.

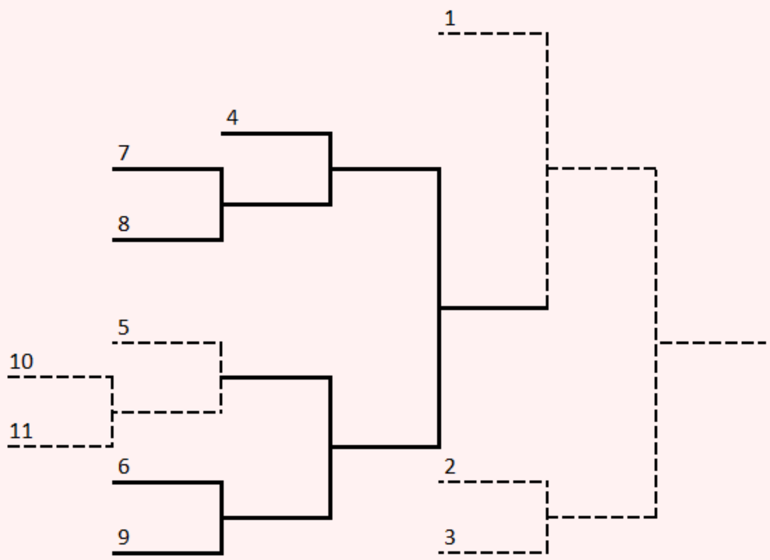
Definition 2.4.4: Containment

Let $\mathcal{A}$ and $\mathcal{B}$ be bracket signatures. We say $\mathcal{A}$ *contains* $\mathcal{B}$ if there exists some i such that

- At least as many games are played in the $(i+1)$th round of $\mathcal{A}$ as in the first round of $\mathcal{B}$, and
- For $j > 1$, there are exactly as many games played in the $(i+j)$th round of $\mathcal{A}$ as in the jth round of $\mathcal{B}$.

Intuitively, $\mathcal{A}$ containing $\mathcal{B}$ means that if $\mathcal{A}$ went chalk, and games within each round were played in order of largest seed-gap to smallest seed-gap, then at some point, there would be a bracket of shape $\mathcal{B}$ used to determine to identify the last team in the rest of bracket $\mathcal{A}$. Figure 2.4.5 shows $\mathcal{A} = [[\mathbf{2};\mathbf{5};\mathbf{1};\mathbf{0};\mathbf{3};\mathbf{0};\mathbf{0}]]$ containing $\mathcal{B} = [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$. After the 10v11 game and the 5v(10v11) game, there is a bracket of shape $\mathcal{B}$ (the solid lines) that must be played to determine the last team in the rest of the bracket

Figure 2.4.5: Setup of the Containment Lemma with $\mathcal{A} = [[2; 5; 1; 0; 3; 0; 0]]$ and $\mathcal{B} = [[4; 2; 0; 0]]$.



Lemma 2.4.6: The Containment Lemma

If $\mathcal{A}$ contains $\mathcal{B}$, and $\mathcal{B}$ is not ordered, then neither is $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be a bracket signature with r rounds and n teams, and let $\mathcal{B}$ have s round and m teams, such that $\mathcal{A}$ contains $\mathcal{B}$ and $\mathcal{B}$ is not ordered. Let k be the number of teams in $\mathcal{A}$ that get at least $s+i$ byes (where i is from the definition of contains).

$\mathcal{B}$ is not ordered, so let $\mathbf{M}$ be a matchup table that violates the orderedness condition, where none of the win probabilities are 0. (If we have an $\mathbf{M}$ that includes 0s, we can replace them with ϵ. For small enough ϵ, $\mathbf{M}$ will still violate the condition.) Let p be the minimum probability in $\mathbf{M}$. Let $\mathbf{P}$ be a matchup table in which the lower-seeded team wins with probability p, and let $\mathbf{Z}$ be a matchup table in which the lower-seeded team wins with probability 0.

Now, consider the following block matchup table on $\mathcal{T}$, a list of n teams.

	t_1 - t_k	t_{k+1} - t_{k+m}	t_{k+m+1} - t_n
t_1 - t_k	$\mathbf{P}$	$\mathbf{P}$	$\mathbf{Z}$
t_{k+1} - t_{k+m}	$\mathbf{P}$	$\mathbf{M}$	$\mathbf{Z}$
t_{k+m+1} - t_n	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$

Let $\mathcal{S} \subset \mathcal{T}$ be the sublist of teams seeded between $k+1$ and $k+m$. Then, for $t_j \in \mathcal{S}$,

$$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T}) = \mathbb{W}_{\mathcal{B}}(t, \mathcal{S}) \cdot p^{r-s-i},$$

since t_j wins any games it might have to play in rounds i or before automatically, any games after $s+i$ with probability p, and any games in between according to $\mathbf{M}$.

However, $\mathbf{M}$ (and thus $\mathcal{S}$) violates the orderedness condition for $\mathcal{B}$, and so $\mathcal{T}$ does for $\mathcal{A}$. □

With the containment lemma shown, we can proceed to the main theorem.

Theorem 2.4.7: Edwards's Theorem

The only ordered brackets are those described by Corollary 2.4.3.

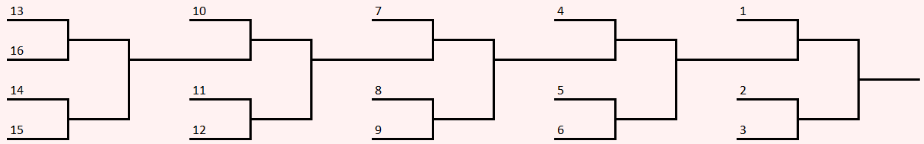
Proof. Let $\mathcal{A}$ be a proper bracket not described by Corollary 2.4.3. The corollary describes all proper brackets in which each round either has only game, or has two games but is immediately followed by a round with only one game. Thus $\mathcal{A}$ must include at least two successive rounds with two or more games each.

The final round in such a chain will be followed by a round with a single game, and so the final round must have only two games. Thus, $\mathcal{A}$ includes a sequence of three rounds, the first of which has at least two games, the second of which has exactly two games, and the third of which has one game.

Therefore, $\mathcal{A}$ contains $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$. But we know that $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$ is not ordered, and so by the containment lemma, neither is $\mathcal{A}$. □

Edwards's Theorem is both exciting and disappointing. On one hand, it means that we can fully describe the set of ordered brackets, making it easy to check whether a given bracket is ordered or not. On the other hand, it means that in an ordered bracket at most three teams can be introduced each round, so the length of the shortest ordered bracket on n teams grows linearly with n (rather than logarithmically as is the case for the shortest proper bracket). If we want a bracket on many teams to be ordered, we risk forcing lower-seeded teams to play a large number of games, and we only permit the top-seeded teams to play a few. For example, the shortest ordered bracket that could've been used in the 2021 NCAA Basketball South Region is $[[\mathbf{4}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{0}]]$, which is played over a whopping ten rounds.

Figure 2.4.8: The Shortest Sixteen-Team Ordered Bracket



Because of this, few leagues use ordered brackets, and those who do usually have so few teams that every proper bracket is ordered (the 2023 College Football Playoffs, for example). Even the Korean Baseball Organization League, which uses a somewhat unconventional $[[\mathbf{2}; \mathbf{1}; \mathbf{1}; \mathbf{1}; \mathbf{0}]]$, only sends five teams to the playoffs, and again every five-team proper bracket is ordered. If the KBO League ever expanded to the six-team bracket $[[\mathbf{2}; \mathbf{1}; \mathbf{1}; \mathbf{1}; \mathbf{1}; \mathbf{0}]]$, we would have a case of an ordered bracket being used when a proper non-ordered bracket exists on the same number of teams.

2.5 Reseeded Brackets

Edwards's Theorem naturally raises the question: is there some bracket-like tournament format, one where undefeated teams face off until only one reamins, that expands the space of signatures that are ordered? *Reseeded* brackets are our first attempt at an answer.

Ultimately, the reason that proper brackets are not, in general, ordered, is that lower-seeded teams are treated, if they win, as the team that they beat for the rest of the format. Consider again the proper bracket analyzed by Silver: $[[\mathbf{16};\mathbf{0};\mathbf{0};\mathbf{0};\mathbf{0}]]$. If an 11-seed wins in the first round, they take on the schedule of a 6-seed for the rest of the tournament, while if the 9-seed wins, they take on the schedule of an 8-seed. Given that a 6-seed has an easier schedule than an 8-seed, it's not hard to see why it might be preferable to be an 11-seed rather than a 9-seed.

Reseeding (poorly named) fixes this by resorting the match-ups every round: if an 11-seed keeps winning, they will have to play teams according to their seed, rather than getting an effective upgrade to 6-seed status.

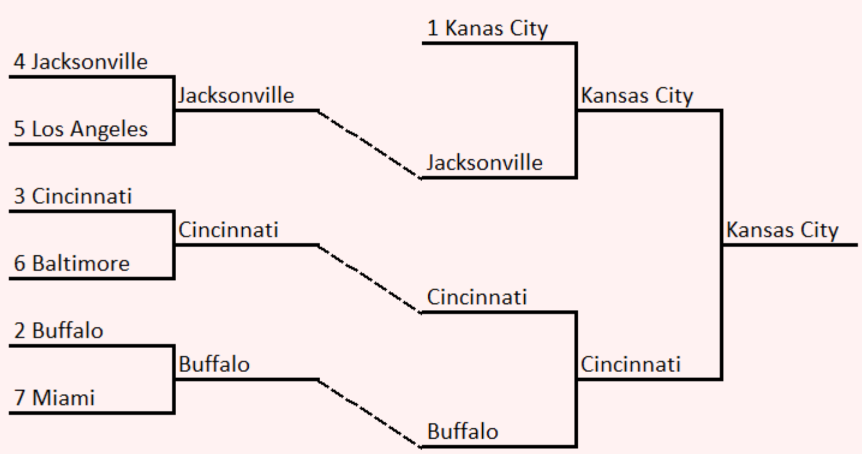
Definition 2.5.1: Reseeded Brackets

In a *reseeded* bracket, after each round, match up the highest-seeded team with the lowest-seeded team, second-highest vs second-lowest, etc.

Note that by Definition 2.1.1, a reseeded bracket is not a bracket at all, as matchups between teams that have not yet lost are not determined in advance of the outcomes of any games. However, because reseeded brackets act so similarly to traditional brackets, and because colloquially they are referred to as brackets, we opt to continue using the word "bracket" to describe them.

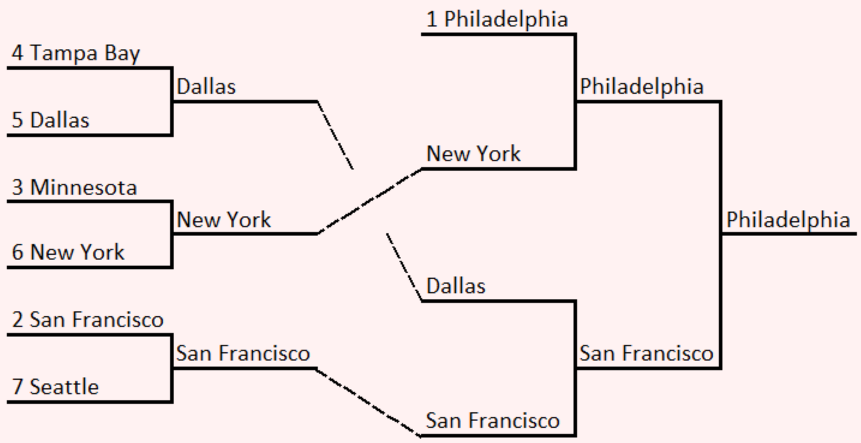
In 2023, both National Football League conferences use a reseeded bracket with signature $[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$. (The superscript R indicates this is reseeded bracket.) If the first round of the bracket goes chalk, then it looks just like a normal bracket.

Figure 2.5.2: 2023 National Football League AFC Playoffs



The dotted lines are drawn after the first round of games has been played: if there are some first-round upsets, then the bracket is rearranged to ensure that it is still better to be a higher seed rather than a lower seed.

Figure 2.5.3: 2023 National Football League NFC Playoffs



In the NFC, 6-seed New York upset 3-seed Minnesota. Had a conventional bracket been used, the semifinal matchups would have been 1-seed vs 5-seed and 2-seed vs 6-seed: the 2-seed would have had an easier draw than the 1-seed, while the 6-seed would have an easier draw than the 5-seed. Reseeding fixes this by matching 6-seed New York with top-seed Philadelphia, and 2-seed San Francisco with 5-seed Dallas.

Reseeding is a powerful technique. For one, the fundamental theorem still applies to reseeded brackets, allowing us to refer to reseeded brackets by their signatures as well.

Theorem 2.5.4

There is exactly one proper reseeded bracket with each bracket signature.

Proof. The definition of properness ensures that there is only one way byes can be distributed such that a reseeded bracket can be proper. Additionally, because reseeded brackets have no additional parameters beyond which seeds get how many byes, there is no more than one reseeded bracket with each signature that could be proper. Finally, that bracket is indeed proper: if the bracket goes to chalk, the matchups will be the exact same as a traditional bracket, which by the fundamental theorem is a proper set of matchups. □

But what about orderedness? It's intuitive to think that all proper reseeded are ordered: it feels like almost by definition, the higher-seeded teams have an easier path than the lower-seeded ones. Hwang [6] conjectured a weaker version of this.

Conjecture 2.5.5

All balanced proper reseeded brackets are ordered.

Unfortunately, neither the stronger claim that all proper reseeded brackets are ordered, nor Hwang's weaker conjecture are true. Our classification of the ordered reseeded brackets takes the same route as our proof of Edwards's Theorem did: we first examine the orderedness of certain important brackets, and then we use the stapling and containment lemmas to specify the complete set of ordered reseeded brackets.

The proofs of the stapling and containment lemmas for reseeded brackets, as well as the fact that all ordered reseeded brackets are proper, are so similar to the corresponding proofs for traditional brackets that we just state them without proof.

Theorem 2.5.6

All ordered reseeded brackets are proper.

Lemma 2.5.7: The Stapling Lemma for Reseeding

If $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]^R$ and $\mathcal{B} = [[\mathbf{b_0}; ...; \mathbf{b_s}]]^R$ are ordered reseeded brackets, then $\mathcal{C} = [[\mathbf{a_0}; ...; \mathbf{a_r} + \mathbf{b_0} - \mathbf{1}; ...; \mathbf{b_s}]]^R$ is an ordered reseeded bracket as well.

Lemma 2.5.8: The Containment Lemma for Reseeding

If $\mathcal{A}$ and $\mathcal{B}$ are reseeded brackets, $\mathcal{A}$ contains $\mathcal{B}$, and $\mathcal{B}$ is not ordered, then neither is $\mathcal{A}$.

We now examine particular brackets.

Theorem 2.5.9

$[[\mathbf{1}]]^R$, $[[\mathbf{2}; \mathbf{0}]]^R$, and $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]^R$ are ordered.

Proof. Since no reseeding is done in a bracket of two or fewer rounds, and since the traditional brackets of these signatures are ordered, so are the reseeded brackets. □

Our primary example of a reseeded bracket that is ordered despite the traditional bracket of the same signature not being ordered is $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]^R$.

Theorem 2.5.10

$[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]^R$ is ordered.

Proof. This can be shown by computing the probability of each team winning the format and then applying the SST conditions to establish the inequalities, as we did in Theorem 2.3.12. In the interest of brevity, however, we instead give an intuitive argument.

$\mathbb{W}_A(t_1, \mathcal{T}) \geq \mathbb{W}_A(t_2, \mathcal{T})$ because from those two teams perspectives, this format is just $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]^R$. $\mathbb{W}_A(t_2, \mathcal{T}) \geq \mathbb{W}_A(t_3, \mathcal{T})$ because t_2 has better odds if t_3 wins in the first round and they meet in the semifinals, and certainly has better odds if t_3 loses in the first round. $\mathbb{W}_A(t_4, \mathcal{T}) \geq \mathbb{W}_A(t_5, \mathcal{T})$ because t_4 is at least as likely to win the first-round matchup, and then their paths would be identical.

$\mathbb{W}_A(t_3, \mathcal{T}) \geq \mathbb{W}_A(t_4, \mathcal{T})$ holds because if both teams win the first round then t_3 has better odds in the remaining $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]^R$ bracket. Meanwhile if only one does, then t_3 will be joined by t_5 while t_4 will be joined by t_6, and so t_3 is more likely to dodge playing t_1 in the finals. The same argument applies to show that $\mathbb{W}_A(t_5, \mathcal{T}) \geq \mathbb{W}_A(t_6, \mathcal{T})$ as well. $\square$

Unfortunately, that is where the power of reseeding to convert non-ordered signatures into ordered ones ends. The following two signatures are not ordered.

Theorem 2.5.11

$[[\mathbf{6}; \mathbf{1}; \mathbf{0}; \mathbf{0}]]^R$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{6}; \mathbf{1}; \mathbf{0}; \mathbf{0}]]^R$, and let $\mathcal{T}$ have the following matchup table.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	0.5	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$
t_2	p	0.5	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$
t_3	p	p	0.5	0.5	0.5	$1-p$	$1-p$
t_4	p	p	0.5	0.5	0.5	0.5	0.5
t_5	p	p	0.5	0.5	0.5	0.5	0.5
t_6	p	p	p	0.5	0.5	0.5	0.5
t_7	p	p	p	0.5	0.5	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_6, \mathcal{T}) = O(p^3),$$

but

$$\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) = 0.25p^2 + O(p^3).$$

Thus, for small enough p, $\mathbb{W}_{\mathcal{A}}(t_6, \mathcal{T}) < \mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T})$, so $\mathcal{A}$ is not ordered. $\square$

Theorem 2.5.12

$[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$, and let $\mathcal{T}$ have the following matchup table.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
t_1	0.5	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$
t_2	p^2	0.5	0.5	0.5	$1-p$	$1-p$	$1-p^2$	$1-p^2$
t_3	p^2	0.5	0.5	0.5	$1-p$	$1-p$	$1-p$	$1-p$
t_4	p^2	0.5	0.5	0.5	0.5	$1-p$	$1-p$	$1-p$
t_5	p^2	p	p	0.5	0.5	$1-p$	$1-p$	$1-p$
t_6	p^2	p	p	p	p	0.5	$1-p$	$1-p$
t_7	p^2	p^2	p	p	p	p	0.5	0.5
t_8	p^2	p^2	p	p	p	p	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) = 0.25p^5 + O(p^6)$$

but

$$\mathbb{W}_{\mathcal{A}}(t_8, \mathcal{T}) = 0.5p^5 + O(p^6).$$

Thus, for small enough p, $\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) < \mathbb{W}_{\mathcal{A}}(t_8, \mathcal{T})$, so $\mathcal{A}$ is not ordered. $\square$

Recapping,

Figure 2.5.13: Which Proper Reseeded Brackets are Ordered

Ordered	Not Ordered
$[[\mathbf{1}]]^R$	$[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$
$[[\mathbf{2};\mathbf{0}]]^R$	$[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$
$[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$	
$[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$	

Finally, we apply the stapling and containment lemmas to complete the theorem.

Theorem 2.5.14

The ordered reseeded brackets are exactly those corresponding to signatures that can be generated in the following way.

1. Start with the list $[[\mathbf{0}]]^R$ (note that this not yet a bracket signature).

2. As many times as desired, prepend the list with $[[\mathbf{1}]]$, $[[\mathbf{3};\mathbf{0}]]$, or $[[\mathbf{3};\mathbf{2};\mathbf{0}]]$.

3. Then, add 1 to the first element in the list, turning it into a bracket signature.

Proof. The stapling lemma, combined with the fact that $[[\mathbf{1}]]^R$, $[[\mathbf{2};\mathbf{0}]]^R$, $[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$, and $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ are ordered, ensure that any reseeded brackets generated by the above procedure is indeed ordered. Left is to use the containment lemma to ensure that these are the only ones.

Let $\mathcal{A}$ be a bracket signature that cannot be generated by the procedure. Then, either there is a round in which three or more games are to be played, or there is a round in which exactly two games are played and the next two rounds each have exactly two games played as well.

Let i be the latest such round. If round i is the first of three rounds with two games each, then round $i+3$ must have only one game played (otherwise i would not be the latest such round). But then $\mathcal{A}$ contains $[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$, and so is not ordered.

If round i has three or more games, then round $i+1$ must contain exactly two games (any less and not every winner would have a game, any more and i would not be the latest such round.) Then, if round $i+2$ has one game, then $\mathcal{A}$ contains $[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$, and if it has two, then $\mathcal{A}$ contains $[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$. In either case, $\mathcal{A}$ is not ordered.

Thus, the ordered reseeded brackets are exactly those generated by the procedure. □

So, the space of ordered reseeded brackets is slightly larger than the space of ordered traditional brackets, although perhaps this is not quite as much of an expansion as we would've liked or expected. Despite this, reseeded brackets definitely *feel* more ordered than traditional brackets of the same signature, even if neither is ordered in the definitional sense.

Conjecture 2.5.15

There is some reasonable restriction on a set of teams that is stronger than SST under which all reseeded brackets ordered.

In the meantime, reseeding remains an important tool in our tournament design toolkit. But it is not without its drawbacks, as discussed by Baumann, Matheson, and Howe [1].

In a reseeded bracket, teams and spectators alike don't know who they will play or where their next game will be until the entire previous round is complete. This can be an especially big issue if parts of the bracket are being played in different locations on short turnarounds: in the NCAA Basketball Tournament, the first two rounds are played over a weekend at various pre-determined locations. It would cause problems if teams had to pack up and travel across the country because they got reseeded and their opponent and thus location changed.

In addition, part of what makes the NCAA Basketball Tournament (affectionately known as “March Madness”) such a fun spectator experience is the fact that these matchups are known ahead of time. In “bracket pools,” groups of fans each fill out their own brackets, predicting who will win each game and getting points based on how many they get right. If it wasn’t clear where in the bracket the winner of a given game was supposed to go, this experience would be diminished.

Finally, reseeding gives the top seed(s) an even greater advantage than they already have: instead of playing against merely the *expected* lowest-seeded team(s) each round, they would get to play against the *actual* lowest-seeded team(s). In March Madness, “Cinderella Stories,” that is, deep runs by low seeds, would become much less common.

In many ways, the NFL conference playoffs are a perfect place to use a reseeded bracket: games are played once a week, giving plenty of time for travel; only seven teams make the playoffs in each, so a huge March Madness-style bracket challenge is unlikely; as a professional league, the focus is far more on having the best team win and protecting Cinderella Stories isn’t as important; and because the bracket is only three rounds long, reseeding is only required once. Somewhat ironically, the NFL conference playoffs used to use the format $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ which is ordered, but have since allowed a seventh team from each conference into the playoffs and changed to the non-ordered $[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$.

Other leagues with similar structures might consider adopting forms of reseeding to protect their incentives and competitive balance (looking at you, Major League Baseball), but in many cases, the traditional bracket structure is too appealing to adopt a reseeded one.

3 References

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