Contents

1 Tournament Formats

1.1 Definitions

Definition 1.1.1: Gameplay Function

A *gameplay function* g on a list of teams $\mathcal{T} = [t_1, ..., t_n]$ is a nondeterministic function $g : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ with the following properties:

- $\mathbb{P}[g(t_i, t_j) = t_i] + \mathbb{P}[g(t_i, t_j) = t_j] = 1$.
- $\mathbb{P}[g(t_i, t_j) = t_i] = \mathbb{P}[g(t_j, t_i) = t_j]$.

A gameplay function represents a process in which two teams compete in a game, with one of them emerging as the winner. This model simplifies away effects like home-field advantage or teams improving over the course of a tournament: a gameplay function is fully described by a single probability for each pair of teams in the list.

Definition 1.1.2: Playing, Winning, and Losing

When g is queried on input (t_i, t_j) we say that t_i and t_j *played a game*. We say that the team that got outputted by g *won*, and the team that did not *lost*.

The information in a gameplay function can be encoded into a *matchup table*.

Definition 1.1.3: Matchup Table

The *matchup table* implied by a gameplay function g on a list of teams $\mathcal{T}$ of length n is a n-by-n matrix $\mathcal{M}$ such that $\mathcal{M}_{ij} = \mathbb{P}[t_i \text{ beats } t_j]$.

For example, let $\mathcal{T}$ =[Favorites, Rock, Paper, Scissors, Conceders], and g be such that the Conceders concede every game they play, the Favorites are 70% favorites against Rock, Paper, and Scissors, and Rock, Paper, and Scissors match up with each other as their name implies. Then the matchup table would look like so:

Figure 1.1.4: The Matchup Table for $(\mathcal{T}, g)$

	Favorites	Rock	Paper	Scissors	Conceders
Favorites	0.5	0.7	0.7	0.7	1.0
Rock	0.3	0.5	0.0	1.0	1.0
Paper	0.3	1.0	0.5	0.0	1.0
Scissors	0.3	0.0	1.0	0.5	1.0
Conceders	0.0	0.0	0.0	0.0	0.5

Theorem 1.1.5

If $\mathcal{M}$ is the matchup table for $(\mathcal{T}, g)$, then $\mathcal{M} + \mathcal{M}^T$ is the matrix of all ones.

Proof. $(\mathcal{M} + \mathcal{M}^T)_{ij} = \mathcal{M}_{ij} + \mathcal{M}_{ji} = \mathbb{P}[t_i \text{ beats } t_j] + \mathbb{P}[t_j \text{ beats } t_i] = 1$. □

Definition 1.1.6: Tournament Format

A *tournament format* is an algorithm that takes as input a list of teams $\mathcal{T}$ and a gameplay function g and outputs a champion $t \in \mathcal{T}$.

We use a gameplay function rather than a matchup table in the definition of a tournament format because a tournament format cannot simply look at the matchup table itself in order to decide which teams are best. Instead, formats query the gameplay function (have teams play games) in order to gather information about the teams. That said, matchup tables will often be useful in our *analysis* of tournament formats.

We also introduce some shorthand to help make notation more concise.

Definition 1.1.7: $\mathbb{W}_{\mathcal{A}}(t, \mathcal{T})$

$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T})$ is the probability that team $t \in \mathcal{T}$ wins tournament format $\mathcal{A}$ when it is run on the list of teams $\mathcal{T}$.

Finally, we will focus our study on the subset of tournament formats that fulfill the *network condition*.

Definition 1.1.8: Network Condition

A tournament format fulfills the *network condition* if after t_i plays t_j, the rest of the format is identical no matter which team won, except for t_i and t_j are swapped.

(This chapter will be fleshed out but I'm including the important definitions here for the sake of the next chapter.)

2 Brackets

2.1 Bracket Signatures

Definition 2.1.1: Elimination Format

An *elimination* tournament format is one in which

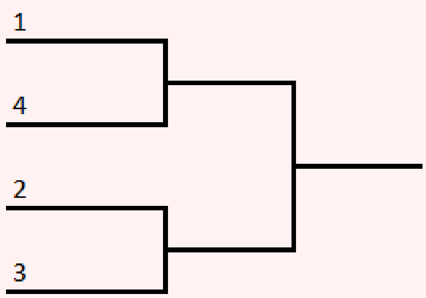
- Teams don't play any games after their first loss, and
- Games are played until only one team has no losses, and that team is crowned champion.

Definition 2.1.2: Bracket

A *bracket* is a elimination tournament format that upholds the network condition.

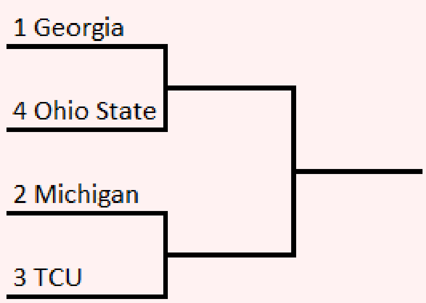
We can draw a bracket as a tree-like structure in the following way.

Figure 2.1.3: The 2023 College Football Playoff



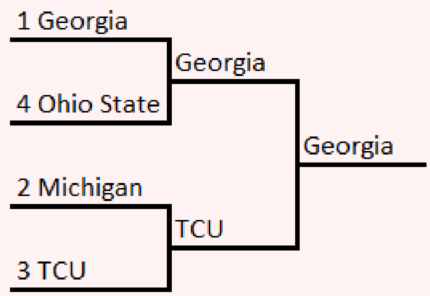
The numbers 1, 2, 3, and 4 indicate where t_1, t_2, t_3, and t_4 in $\mathcal{T}$ are placed to start. In the actual 2023 College Football Playoff, the list of teams $\mathcal{T}$ was Georgia, Michigan, TCU, and Ohio State, in that order, so the bracket was filled in like so.

Figure 2.1.4: The 2023 CFP After Team Placement



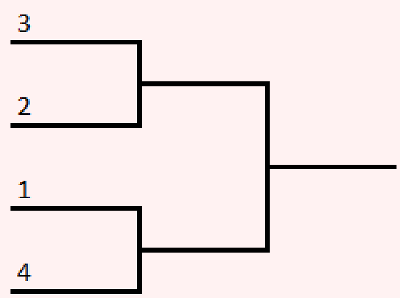
As games are played, we write the name of the winning teams on the corresponding lines. This bracket tells us that Georgia played Ohio State, and Michigan played TCU. Georgia and TCU won their respective games, and then Georgia beat TCU, winning the tournament.

Figure 2.1.5: The 2023 CFP After Completion



Rearranging the way the bracket is pictured, if it doesn't affect any of the matchups, does not create a new bracket. For example, Figure 2.1.6 is just another way to draw the same 2023 CFP Bracket.

Figure 2.1.6: Alternative Drawing of the 2023 CFP



One key piece of bracket vocabulary is the *round*.

Definition 2.1.7: Round

A *round* is a set of games such that the winners of each of those games have the same number of games remaining to win the tournament.

For example, the 2023 CFP has two rounds. The first round included the games Georgia vs Ohio State and Michigan vs TCU, and the second round was just a single game: Georgia vs TCU.

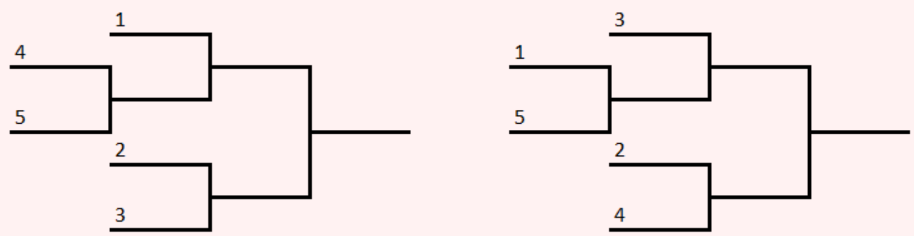
Another important concept is the *shape* of a bracket.

Definition 2.1.8: Shape

The *shape* of a bracket is the tree that underlies it.

For example, the following two brackets have the same shape.

Figure 2.1.9: Two Brackets with the Same Shape



Definition 2.1.10: Bye

A team has a *bye* in round r if it plays no games in round r or before.

One way to describe the shape of a bracket its signature.

Definition 2.1.11: Bracket Signature

The *signature* $[[\mathbf{a_0}; \ldots; \mathbf{a_r}]]$ of an r-round bracket $\mathcal{A}$ is list such that a_i is the number of teams with i byes.

The signature of a bracket is defined only by its shape: the two brackets in Figure 2.1.9 have the same shape, so they also have the same signature.

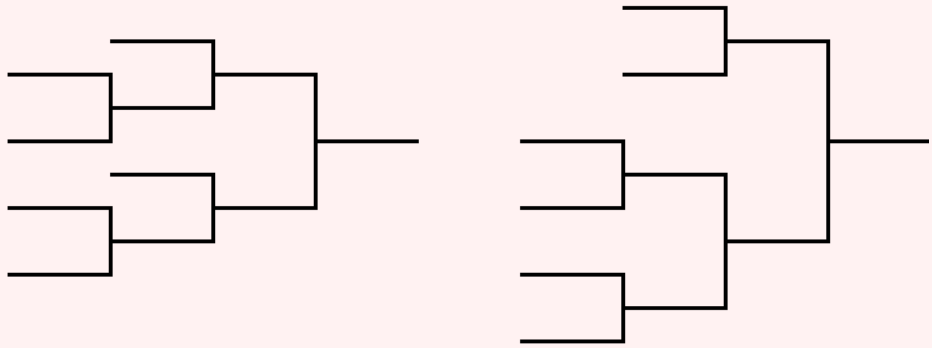
The signatures of the brackets discussed in this section are shown in Figure 2.1.12. It's worth verifying the signatures we've seen so far and trying to draw brackets with the signatures we haven't yet before moving on.

Figure 2.1.12: The Signatures of Some Brackets

Bracket	Signature
2023 College Football Playoff	$[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$
The brackets in Figure 2.1.9	$[[\mathbf{2}; \mathbf{3}; \mathbf{0}; \mathbf{0}]]$
The brackets in Figure 2.1.13	$[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$
2023 WCC Men's Basketball Tournament	$[[\mathbf{4}; \mathbf{2}; \mathbf{2}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$

Two brackets with the same shape must have the same signature, but the converse is not true: two brackets with different shapes can have the same signature. For example, both bracket shapes depicted in Figure 2.1.13 have the signature $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$.

Figure 2.1.13: Two Shapes with the Signature $[[4; 2; 0; 0]]$



Despite this, bracket signatures are a useful way to talk about the shape of a bracket. Communicating a bracket's signature is a lot easier than communicating its shape, and much of the important information (such as how many games each team must win in order to win the tournament) is contained in the signature.

Bracket signatures have one more important property.

Theorem 2.1.14

Let $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ be a list of natural numbers. Then $\mathcal{A}$ is a bracket signature if and only if

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = 1.$$

Proof. Let $\mathcal{A}$ be the signature for some bracket. Assume that every game in the bracket was a coin flip, and consider each team's probability of winning the tournament. A team that has i byes must win $r - i$ games to win the tournament, and so will do so with probability $\left(\frac{1}{2}\right)^{r-i}$. For each $i \in \{0, ..., r\}$, there are a_i teams with i byes, so (because any two teams winning are mutually exclusive)

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i}$$

is the probability that one of the teams wins, which is 1.

We prove the other direction by induction on r. If $r = 0$, then the only list with the desired property is $[[\mathbf{1}]]$, which is the signature for the unique one-team bracket. For any other r, first note that a_0 must be even: if it were odd, then

$$\begin{aligned}
\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} &= \frac{1}{2^r} \cdot \sum_{i=0}^{r} a_i \cdot 2^i \\
&= \frac{1}{2^r} \cdot \left(a_0 + 2\sum_{i=1}^{r} a_i \cdot 2^{i-1}\right) \\
&= k/2^r && \text{for some odd } k \\
&\neq 1.
\end{aligned}$$

Now, consider the signature $\mathcal{B} = [[\mathbf{a_1 + a_0/2; a_2; ...; a_r}]]$. By induction, there exists a bracket with signature $\mathcal{B}$. But if we take that bracket and replace $a_0/2$ of the teams with no byes with a game whose winner gets placed on that line, we get a new bracket with signature $\mathcal{A}$. □

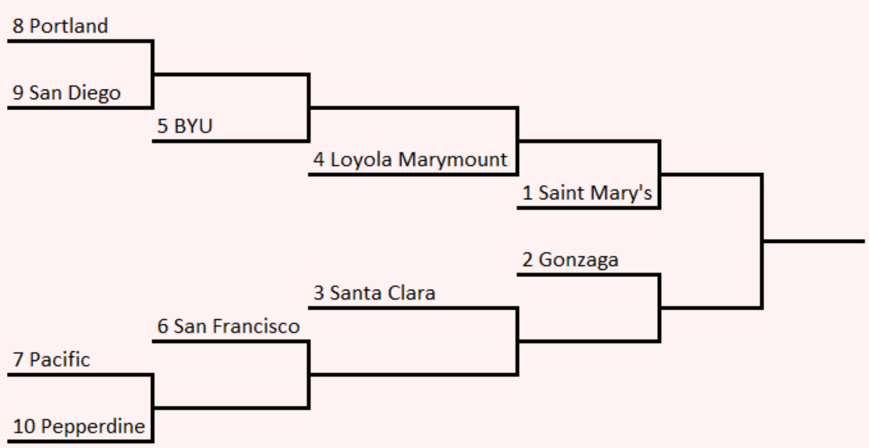
In the next few sections, we will use the language and properties of bracket signatures to describe the brackets that we work with. For now though, let's return to the 2023 College Football Playoff. The bracket used in the 2023 CFP has a special property that not all brackets have: it is *balanced*.

Definition 2.1.15: Balanced Bracket

A *balanced bracket* is a bracket in which none of the teams have byes.

The 2023 West Coast Conference Men's Basketball Tournament, on the other hand, is unbalanced.

Figure 2.1.16: The 2023 WCC Men's Basketball Tournament



Saint Mary's and Gonzaga each have three byes and so only need to win two games to win the tournament, while Portland, San Diego, Pacific, and Pepperdine need to win five. Unsurprisingly, this format conveys a massive advantage to Saint Mary's and Gonzaga, but this was intentional: those two teams were being rewarded for doing the best during the regular season.

In many cases, however, it is undesirable to grant advantages to certain teams over others. One might hope, for any n, to be able to construct a balanced bracket for n teams, but unfortunately this is rarely possible.

Theorem 2.1.17

There exists an n-team balanced bracket if and only if n is a power of two.

Proof. A bracket is balanced if no teams have byes, which is true exactly when its signature is of the form $\mathcal{A} = [[\mathbf{n}; \mathbf{0}; ...; \mathbf{0}]]$ where n is the number of teams in the bracket. If n is a power of two, then by Theorem 2.1.14 $\mathcal{A}$ is indeed a bracket signature and so points to a balanced bracket for n teams. If n is not a power of two, however, then Theorem 2.1.14 tells us that $\mathcal{A}$ is not a bracket signature, and so no balanced brackets exist for n teams. $\square$

Given this, brackets are not a great option when we want to avoid giving some teams advantages over others unless we have a power of two teams. They are a fantastic tool, however, if doling out advantages is the goal, perhaps after some teams did better during the regular season and ought to be rewarded with an easier path in the bracket.

2.2 Proper Brackets

Definition 2.2.1: Seeding

The *seeding* of an n-team bracket is the arrangement of the numbers 1 through n in the bracket.

Together, the shape and seeding fully specify a bracket.

Definition 2.2.2: i-seed

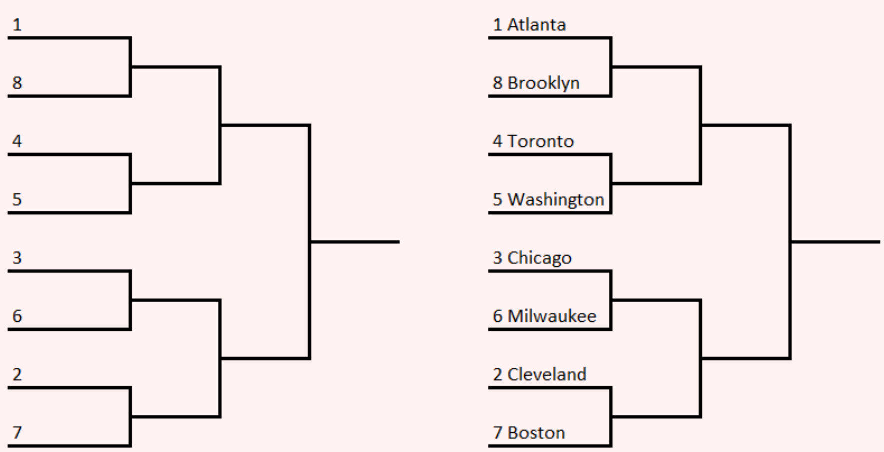
In a list of teams $\mathcal{T} = [t_1, ..., t_n]$, we refer to t_i as the i-seed.

Definition 2.2.3: Higher and Lower Seeds

Somewhat confusingly, convention is that smaller numbers are the *higher seeds*, and greater numbers are the *lower seeds*.

Seeding is typically used to reward better and more deserving teams. As an example, on the left is the eight-team bracket used in the 2015 NBA Eastern Conference Playoffs. At the end of the regular season, the top eight teams in the Eastern Conference were ranked and placed into the bracket as shown on the right.

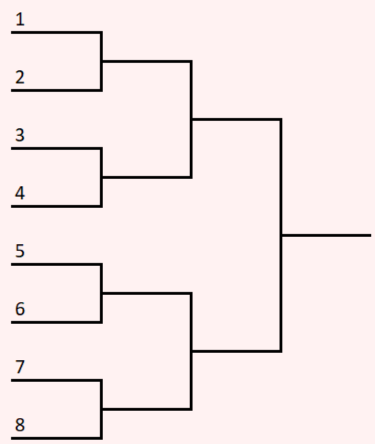
Figure 2.2.4: 2015 NBA Eastern Conference Playoffs



Despite this bracket being balanced, the higher seeds are still at advantage: they have an easier set of opponents. Compare 1-seed Atlanta, whose first two rounds are versus 8-seed Brooklyn and then (most likely) 4-seed Toronto, versus 7-seed Boston, whose first two rounds are versus 2-seed Cleveland and then (most likely) 3-seed Chicago. Atlanta's schedule is far easier: despite them having the same number of games to win as Boston, Atlanta is expected to play lower seeds in each round than Boston will.

Thus, we've identified two ways in which brackets can convey an advantage onto certain teams: by giving them more byes, and by giving them easier (expected) opponents. Not every seeding of a bracket does this: for example, consider the following alternative seeding for the 2015 NBA Eastern Conference Playoffs.

Figure 2.2.5: An Alternative Seeding of the 2015 NBA Eastern Conference Playoffs



This seeding does a very poor job of rewarding the higher-seeded teams: the 1- and 2-seeds are matched up in the first round, while the easiest road is given to the 7-seed, who plays the 8-seed in the first round and then (most likely) the 5-seed in the second. Since the whole point of seeding is to give the higher-seeded teams an advantage, we introduce the concept of a *proper seeding*.

Definition 2.2.6: Chalk

We say a tournament *went chalk* if the higher-seeded team won every game during the tournament.

Definition 2.2.7: Proper Seeding

A *proper seeding* of a bracket is one such that if the bracket goes chalk, in every round it is better to be a higher-seeded team than a lower-seeded one, where:

(1) It is better to have a bye than to play a game.

(2) It is better to play a lower seed than to play a higher seed.

Definition 2.2.8: Proper Bracket

A *proper bracket* is a bracket that has been properly seeded.

It is clear that the actual 2015 NBA Eastern Conference Playoffs was properly seeded, while our alternative seeding was not.

We now quickly derive a few lemmas about proper brackets.

Lemma 2.2.9

In a proper bracket, if m teams have a bye in a given round, those teams must be seeds 1 through m.

Proof. If they did not, the seeding would be in violation of condition (1). □

Lemma 2.2.10

If a proper bracket goes chalk, then, after each round, the m teams remaining will be the top m seeds.

Proof. We will prove the contrapositive. Assume that for some $i < j$, after some round, t_i has been eliminated but t_j is still alive. Let k be the seed of the team that t_i lost to. Because the bracket went chalk, $k < i$. Now consider what t_j did in that round. If they had a bye, then the bracket violates condition (1). Assume instead they played t_ℓ. They beat t_ℓ, so $j < \ell$, giving,

$$k < i < j < \ell.$$

In the round that t_i was eliminated, t_i played t_k, while t_j played t_ℓ, violating condition (2). Thus, the bracket is not proper. □

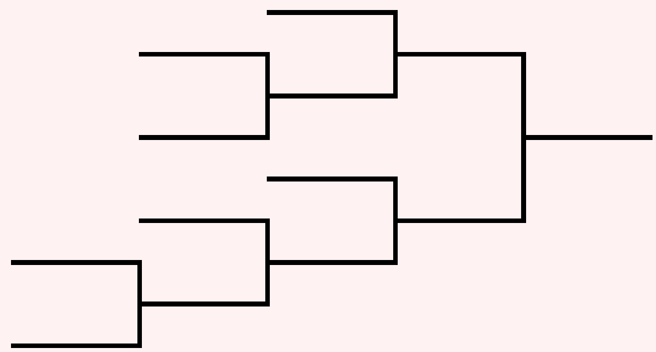
Lemma 2.2.11

In a proper bracket, if m teams have a bye and k games are being played in a given round, then if the bracket goes chalk those matchups will be seed $m + i$ vs seed $(m + 2k + 1) - i$ for $i \in \{1, ..., k\}$.

Proof. In the given round, there are $m + 2k$ teams remaining. Theorem 2.2.10 tells us that (if the bracket goes chalk) those teams must be seeds 1 through $m+2k$. Theorem 2.2.9 tells us that seeds 1 through m must have a bye, so the teams playing must be seeds $m + 1$ through $m + 2k$. Then condition (2) tells us that the matchups must be exactly $m + i$ vs seed $(m + 2k + 1) - i$ for $i \in \{1, ..., k\}$. □

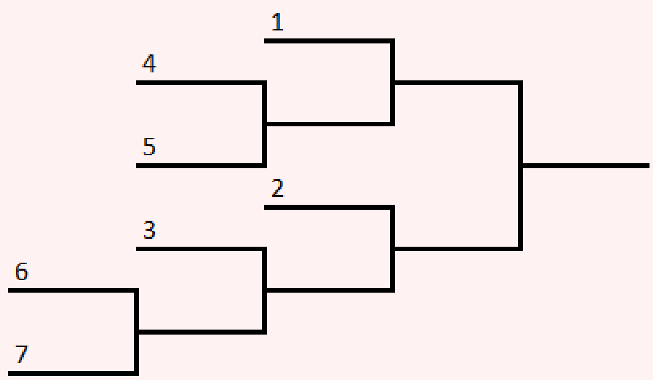
We can use Lemmas 2.2.9 through 2.2.11 to properly seed various bracket shapes. For example, consider the following seven-team shape.

Figure 2.2.12: A Seven-Team Bracket Shape



Lemma 2.2.9 tells us that the first-round matchup must be between the 6-seed and the 7-seed. Lemma 2.2.11 tells us that if the bracket goes chalk, the second-round matchups must be 3v6 and 4v5, so the 3-seed play the winner of the first-round matchup. Finally, we can apply Lemma 2.2.11 again to the semifinals to find that the 1-seed should play the winner of the 4v5 matchup, while the 2-seed should play the winner of the 3v(6v7) matchup. In total, our proper seeding looks like so.

Figure 2.2.13: A Seven-Team Bracket, Properly Seeded

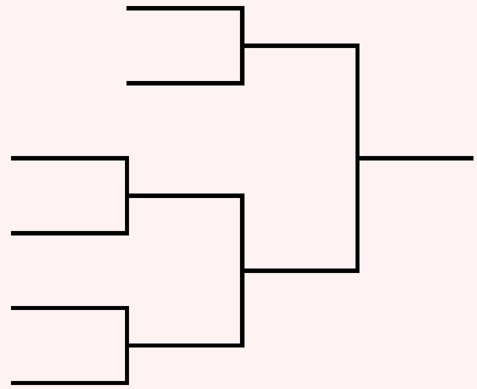


We can also quickly simulate the bracket going chalk to verify Lemma 2.2.10.

Lemmas 2.2.9 through 2.2.11 are quite powerful. It is not a coincidence that we managed to specify exactly what a proper seeding of the above bracket must look like with no room for variation: soon we will prove that the proper seeding for a particular bracket shape is unique.

But not every shape admits even this one proper seeding. Consider the following six-team shape.

Figure 2.2.14: A Six-Team Bracket Shape



This shape admits no proper seedings. Lemma 2.2.9 requires that the two teams getting byes be the 1- and 2-seed, but this violates Lemma 2.2.11 which requires that in the second round the 1- and 2-seeds do not play each other. So how can we think about which shapes admit proper seedings?

Theorem 2.2.15: The Fundamental Theorem of Brackets

There is exactly one proper bracket with each bracket signature.

Proof. Let $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ be an r-round bracket signature. We proceed by induction on r. If $r = 0$, then the only possible bracket signature is $[[\mathbf{1}]]$, and it points to the unique one-team bracket, which is indeed proper.

For any other r, the first-round matchups of a proper bracket with signature $\mathcal{A}$ are defined by Lemma 2.2.11. Then if those matchups go chalk, we are left with a proper bracket of signature $[[\mathbf{a_0/2 + a_1}; \mathbf{a_2}; ...; \mathbf{a_r}]]$, which induction tells us exists admits exactly one proper bracket.

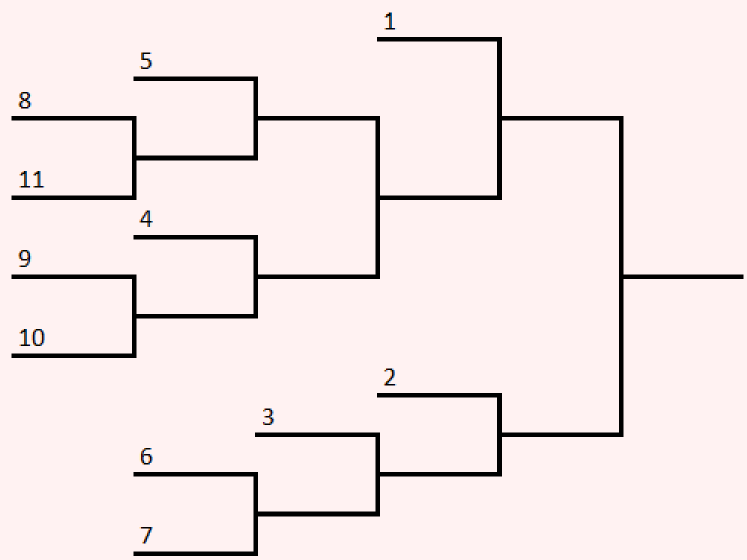
Thus both the first-round matchups and the rest of the bracket are determined, and by combining them we get a proper bracket with signature $\mathcal{A}$, so there is exactly one proper bracket with signature $\mathcal{A}$. $\square$

The fundamental theorem of brackets means that we can refer to the proper bracket $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ in a well-defined way, as long as

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = 1.$$

In practice, virtually every sports league that uses a traditional bracket uses a proper one: while different leagues take very different approaches to how many byes to give teams (compare the 2023 West Coast Conference Men's Basketball Tournament with the 2015 NBA Eastern Conference Playoffs), they are almost all proper. This makes bracket signatures a convenient labeling system for the set of brackets that we might reasonably encounter. They also are a powerful tool for specifying new brackets: if you are interested in (say) an eleven-team bracket where four teams get no byes, four teams get one bye, one team gets two byes and two teams get three byes, we can describe the proper bracket with those specs as $[[\mathbf{4};\mathbf{4};\mathbf{1};\mathbf{2};\mathbf{0};\mathbf{0}]]$ and use Lemmas 2.2.9 through 2.2.11 to draw it with ease.

Figure 2.2.16: The Proper Bracket of Signature $[[\mathbf{4};\mathbf{4};\mathbf{1};\mathbf{2};\mathbf{0};\mathbf{0}]]$

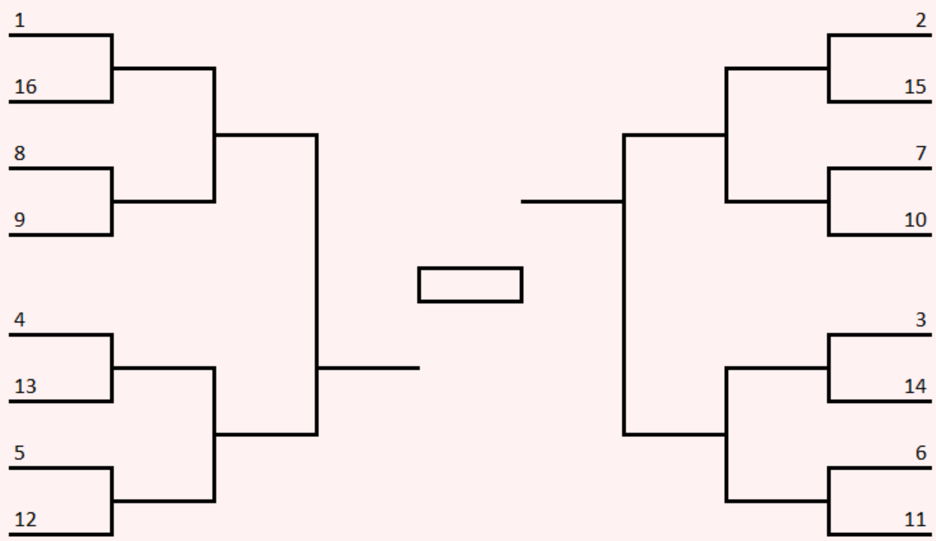


Due to these properties, we will almost exclusively discuss proper bracket from here on out: unless stated otherwise, assume all brackets are proper.

2.3 Ordered Brackets

Consider the proper bracket $[[\mathbf{16};\mathbf{0};\mathbf{0};\mathbf{0};\mathbf{0}]]$, which was used in the 2021 NCAA Men's Basketball Tournament South Region, and is shown below. (Sometimes brackets are drawn in the manner below, with teams starting on both sides and the winner of each side playing in the championship game.)

Figure 2.3.1: 2021 NCAA Men's Basketball Tournament South Region



The definition of a proper seeding ensures that as long as the bracket goes chalk (that is, higher seeds always beat lower seeds), it will always be better to be a higher seed than a lower seed. But what if it doesn't go chalk?

One counter-intuitive fact about the NCAA Basketball Tournament is that it is probably better to be a 10-seed than a 9-seed. (This doesn't violate the proper seeding property because 9-seeds have an easier first-round matchup than 10-seeds, and for further rounds, proper seedings only care about what happens if the bracket goes chalk, which would eliminate both the 9-seed and 10-seed in the first round.) Why? Let's look at whom each seed-line matchups against in the first two rounds.

Figure 2.3.2: NCAA Basketball Tournament 9- and 10-seed Schedules

Seed	First Round	Second Round
9	8	1
10	7	2

The 9-seed has an easier first-round matchup, while the 10-seed has an easier second-round matchup. However, this isn't quite symmetrical. Because the teams are probably drawn from a roughly normal distribution, the expected difference in skill between the 1-

and 2-seeds is far greater than the expected difference between the 7- and 8-seeds, implying that the 10-seed does in fact have an easier route than the 9-seed.

Nate Silver [14] investigated this matter in full, finding that in the NCAA Basketball Tournament, seed-lines 10 through 15 give teams better odds of winning the region than seed-lines 8 and 9. Of course this does not mean that the 11-seed (say) has a better chance of winning a given region than the 8-seed does, as the 8-seed is a much better team than the 11-seed. But it does mean that the 8-seed would love to swap places with the 11-seed, and that doing so would increase their odds to win the region.

This is not a great state of affairs: the whole point of seeding is confer an advantage to higher-seeded teams, and the proper bracket $[[\mathbf{16};\mathbf{0};\mathbf{0};\mathbf{0};\mathbf{0}]]$ is failing to do that. Not to mention that giving lower-seeded teams an easier route than higher-seeded ones can incentivize teams to lose during the regular season in order to try to get a lower but more advantageous seed.

To fix this, we need a stronger notion of what makes a bracket effective than properness. The issue with proper seedings is the false assumption that higher-seeded teams will always beat lower-seeded teams. A more nuanced assumption, initially proposed by H.A. David [7], might look like this.

Definition 2.3.3: Strongly Stochastically Transitive

A list of teams $\mathcal{T}$ is *strongly stochastically transitive* if for each i, j, k such that $j < k$,

$$\mathbb{P}[t_i \text{ beats } t_j] \leq \mathbb{P}[t_i \text{ beats } t_k].$$

A list of teams being strongly stochastically transitive (SST) captures the intuition that each team ought to do better against lower-seeded teams than against higher-seeded teams. A few quick implications of this definition are stated below.

Corollary 2.3.4

(1) If $\mathcal{T}$ is SST, then for each $i < j$,

$$\mathbb{P}[t_i \text{ beats } t_j] \geq 0.5.$$

(2) If $\mathcal{T}$ is SST, then for each i, j, k, ℓ such that $i < j$ and $k < \ell$,

$$\mathbb{P}[t_i \text{ beats } t_\ell] \geq \mathbb{P}[t_j \text{ beats } t_k].$$

(3) If $\mathcal{T}$ is SST, then the matchup table $\mathcal{M}$ is monotonically increasing along each row and monotonically decreasing along each column.

Note that not every set of teams can be seeded to be SST. Consider, for example, the game of rock-paper-scissors. Rock beats paper which beats scissors which beats rock, so no ordering of these "teams" will be SST. For our purposes, however, SST will work well enough.

Our new, nuanced alternative a proper bracket is an *ordered bracket*, first defined by Chen and Hwang [4] (though we use the name proposed by Edwards [8]).

Definition 2.3.5: Ordered

A tournament format $\mathcal{A}$ is *ordered* if, for any SST list of teams $\mathcal{T}$, if $i < j$, then $\mathbb{W}_{\mathcal{A}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{T})$

In an informal sense, a bracket being ordered is the strongest thing we can want without knowing more about why the tournament is being played. Depending on the situation, we might be interested in a format that almost always declares the most-skilled team as the winner, or in a format that gives each team roughly the same chance of winning, or anywhere in between. But certainly, better teams should win more, which is what the ordered bracket condition requires.

In particular, a bracket being ordered is a stronger claim than it being proper.

Theorem 2.3.6

Every ordered bracket is proper.

Proof. Let $\mathcal{A}$ be an ordered n-team bracket with r rounds.

Let $\mathcal{T}$ be SST with matchup table $\mathcal{M}$ where $\mathcal{M}_{ij} = 0.5$. A team that plays their first game in the ith round will win the tournament with probability $(0.5)^{r-i}$, so teams that get more byes will have a higher probability to win the tournament than teams with fewer byes. This implies that higher-seeded teams must have more byes than lower-seeded teams, so in each round, the teams with byes must be the highest-seeded teams that are still alive. Thus, condition (1) is met.

We show that condition (2) is met by proving the stronger condition from Lemma 2.2.11: if m teams have a bye and k games are being played in round s, then if the bracket goes chalk, those matchups will be t_{m+i} vs $t_{(m+2k+1)-i}$ for $i \in \{1, ..., k\}$. We show this by strong induction on s and on i.

Assume that this is true for every round up until s and for all $i < j$ for some j. Let $\ell = (m + 2k + 1) - j$. We want to show that if the the bracket goes chalk, t_{m+j} will face off against seed t_ℓ in the given round. Consider the following SST matchup table: every game is a coin flip, except for games involving a team seeded ℓ or lower, in which case the higher seed always wins. Then, each team seeded between $\ell - 1$ and $m + j$ will win the tournament with probability $(\frac{1}{2})^{r-s}$, other than the team slated to play t_ℓ in round s who wins with probability $(\frac{1}{2})^{r-i-1}$. In order for $\mathcal{B}$ to be ordered, that team must be t_{m+j}.

Thus $\mathcal{A}$ satisfies both conditions, and so is a proper bracket. □

With Theorem 2.3.6, we can use the language of bracket signatures to describe ordered brackets without worrying that two ordered brackets might share a signature. Now we examine three particularly important examples of ordered brackets.

We begin with the unique one-team bracket.

Figure 2.3.7: The One-Team Bracket $[[\mathbf{1}]]$

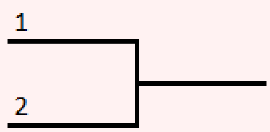
Theorem 2.3.8

The one-team bracket $[[\mathbf{1}]]$ is ordered.

Proof. Since there is only team, the ordered bracket condition is vacuously true. □

Next we look at the unique two-team bracket.

Figure 2.3.9: The Two-Team Bracket $[[\mathbf{2};\mathbf{0}]]$



Theorem 2.3.10

The two-team bracket $[[\mathbf{2};\mathbf{0}]]$ is ordered.

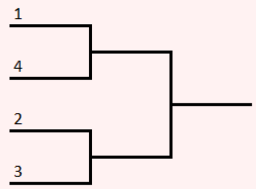
Proof. Let $\mathcal{A} = [[\mathbf{2};\mathbf{0}]]$. Then,

$$\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) = \mathbb{P}[t_1 \text{ beats } t_2] \geq 0.5 \geq \mathbb{P}[t_2 \text{ beats } t_1] = \mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T})$$

so $\mathcal{A}$ is ordered. □

And thirdly, we show that the balanced four-team bracket is ordered, first proved by Horen and Riezman [9].

Figure 2.3.11: The Four-Team Bracket $[[4; 0; 0]]$



Theorem 2.3.12

The four-team bracket $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$ is ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$ and let $p_{ij} = \mathbb{P}[t_i \text{ beats } t_j]$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) &= p_{14} \cdot (p_{23}p_{12} + p_{32}p_{13}) \\
&= p_{14}p_{23}p_{12} + p_{14}p_{32}p_{13} \\
&\geq p_{14}p_{23}p_{21} + p_{24}p_{41}p_{23} \\
&= p_{23} \cdot (p_{14}p_{21} + p_{41}p_{24}) \\
&= \mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T})
\end{aligned}$$

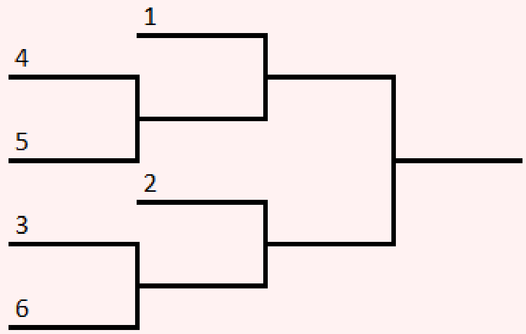
$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T}) &= p_{23} \cdot (p_{14}p_{21} + p_{41}p_{24}) \\
&\geq p_{32} \cdot (p_{14}p_{31} + p_{41}p_{34}) \\
&= \mathbb{W}_{\mathcal{A}}(t_3, \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_3, \mathcal{T}) &= p_{32} \cdot (p_{14}p_{31} + p_{41}p_{34}) \\
&= p_{32}p_{14}p_{31} + p_{32}p_{41}p_{34} \\
&\geq p_{42}p_{23}p_{41} + p_{32}p_{41}p_{43} \\
&= p_{41} \cdot (p_{23}p_{42} + p_{32}p_{43}) \\
&= \mathbb{W}_{\mathcal{A}}(t_4, \mathcal{T})
\end{aligned}$$

Thus $\mathcal{A}$ is ordered. □

However, not every proper bracket is ordered. One particularly important example of a non-ordered proper bracket is $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$

Figure 2.3.13: The Six-Team Bracket $[[4; 2; 0; 0]]$



Theorem 2.3.14

The six-team bracket $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$, and let $\mathcal{T}$ have the following matchup table.

	t_1	t_2	t_3	t_4	t_5	t_6
t_1	0.5	0.5	0.5	0.5	0.5	1
t_2	0.5	0.5	0.5	0.5	0.5	1
t_3	0.5	0.5	0.5	0.5	0.5	0.5
t_4	0.5	0.5	0.5	0.5	0.5	0.5
t_5	0.5	0.5	0.5	0.5	0.5	0.5
t_6	0	0	0.5	0.5	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) = 0.5 \cdot 0.5 = 0.25,$$

but

$$\mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T}) = (0.5 \cdot 0.5 + 0.5 \cdot 1) \cdot 0.5 = 0.375.$$

Thus $\mathcal{A}$ is not ordered. $\square$

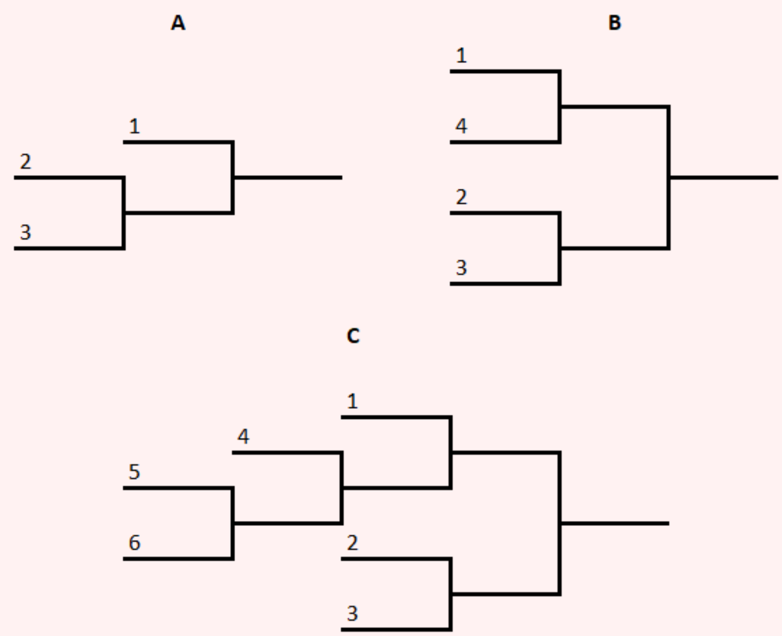
In the next section, we move on from describing particular ordered and non-ordered brackets in favor of a more general result.

2.4 Edwards's Theorem

We now attempt to completely classify the set of ordered brackets. Edwards [8] originally accomplished this without access to the machinery of bracket signatures or proper brackets: we present a quicker proof that makes use of the fundamental theorem of brackets and develop two nice lemmas along the way.

We begin with the stapling lemma, which allows us to combine two smaller ordered brackets into a larger ordered one by having the winner of one of the brackets be treated as the lowest seed in the other. This is depicted in Figure 2.4.1.

Figure 2.4.1: Setup of the Stapling Lemma with $\mathcal{A} = [[\mathbf{2}; \mathbf{1}; \mathbf{0}]]$, $\mathcal{B} = [[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$, and $\mathcal{C} = [[\mathbf{2}; \mathbf{1}; \mathbf{3}; \mathbf{0}; \mathbf{0}]]$



Lemma 2.4.2: The Stapling Lemma

If $\mathcal{A} = [[\mathbf{a_0}; \ldots; \mathbf{a_r}]]$ and $\mathcal{B} = [[\mathbf{b_0}; \ldots; \mathbf{b_s}]]$ are ordered brackets, then $\mathcal{C} = [[\mathbf{a_0}; \ldots; \mathbf{a_r} + \mathbf{b_0} - \mathbf{1}; \ldots; \mathbf{b_s}]]$ is an ordered bracket as well.

Proof. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be as specified. Let $\mathcal{T}$ be an SST list of teams $n+m-1$ teams, and let $\mathcal{R}, \mathcal{S} \subset \mathcal{T}$ be the lowest n and the highest $m-1$ seeds of $\mathcal{T}$ respectively. We divide proving that $\mathcal{C}$ is ordered into proving three sub-statements:

1. For $i < j < m$, $\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})$
2. $\mathbb{W}_{\mathcal{C}}(t_{m-1}, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_m, \mathcal{T})$
3. For $m \leq i < j$, $\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})$

Together, these show that $\mathcal{C}$ is ordered.

We begin with the first sub-statement. Let $i < j < m$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) &= \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_k\}) \\
&\geq \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_j, \mathcal{S} \cup \{t_k\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})
\end{aligned}$$

The first and last equalities follow from the structure of $\mathcal{C}$, and the inequality follows from $\mathcal{B}$ being ordered.

Now the second sub-statement.

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_{m-1}, \mathcal{T}) &= \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_{m-1}, \mathcal{S} \cup \{t_k\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_m, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_{m-1}, \mathcal{S} \cup \{t_m\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_m, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_m, \mathcal{S} \cup \{t_m\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_m, \mathcal{T})
\end{aligned}$$

The equalities follow from the structure of $\mathcal{C}$, the first inequality follows from probabilities being non-negative, and the second inequality follows from $\mathcal{B}$ being ordered.

Finally, we show the third sub-statement. Let $m \leq i < j$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}}(t_i, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_i\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_i\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_j, \mathcal{S} \cup \{t_j\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})
\end{aligned}$$

The equalities follow from the structure of $\mathcal{C}$, the first inequality from $\mathcal{A}$ being ordered, and the second inequality from the teams being SST.

We have shown all three sub-statements, and so $\mathcal{C}$ is ordered. $\square$

Now, if we begin with the set of brackets $\{[[\mathbf{1}]], [[\mathbf{2}; \mathbf{0}]], [[\mathbf{4}; \mathbf{0}; \mathbf{0}]]\}$ and then repeatedly apply the stapling lemma, we can construct a set of brackets that we know are ordered. In other words,

Corollary 2.4.3

Any bracket signature formed by the following process is ordered:

1. Start with the list $[[\mathbf{0}]]$ (note that this not yet a bracket signature).
2. As many times as desired, prepend the list with $[[\mathbf{1}]]$ or $[[\mathbf{3}; \mathbf{0}]]$.
3. Then, add 1 to the first element in the list, turning it into a bracket signature.

Corollary 2.4.3 uses the tools that we have developed so far to identify a set of ordered brackets. Somewhat surprisingly, this set is complete: any bracket not reachable using the process in Corollary 2.4.3 is not ordered. To prove this we first need to show the containment lemma.

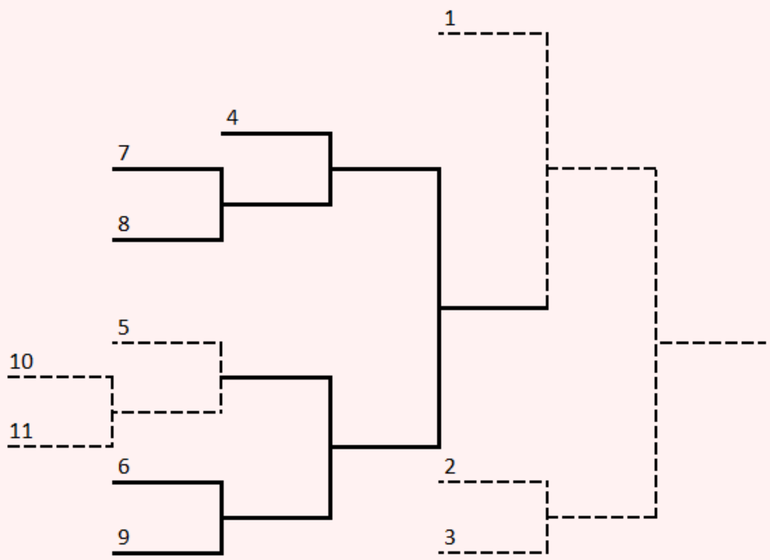
Definition 2.4.4: Containment

Let $\mathcal{A}$ and $\mathcal{B}$ be bracket signatures. We say $\mathcal{A}$ *contains* $\mathcal{B}$ if there exists some i such that

- At least as many games are played in the $(i+1)$th round of $\mathcal{A}$ as in the first round of $\mathcal{B}$, and
- For $j > 1$, there are exactly as many games played in the $(i+j)$th round of $\mathcal{A}$ as in the jth round of $\mathcal{B}$.

Intuitively, $\mathcal{A}$ containing $\mathcal{B}$ means that if $\mathcal{A}$ went chalk, and games within each round were played in order of largest seed-gap to smallest seed-gap, then at some point, there would be a bracket of shape $\mathcal{B}$ used to determine to identify the last team in the rest of bracket $\mathcal{A}$. Figure 2.4.5 shows $\mathcal{A} = [[\mathbf{2}; \mathbf{5}; \mathbf{1}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{0}]]$ containing $\mathcal{B} = [[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$. After the 10v11 game and the 5v(10v11) game, there is a bracket of shape $\mathcal{B}$ (the solid lines) that must be played to determine the last team in the rest of the bracket

Figure 2.4.5: Setup of the Containment Lemma with $\mathcal{A} = [[2; 5; 1; 0; 3; 0; 0]]$ and $\mathcal{B} = [[4; 2; 0; 0]]$.



Lemma 2.4.6: The Containment Lemma

If $\mathcal{A}$ contains $\mathcal{B}$, and $\mathcal{B}$ is not ordered, then neither is $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be a bracket signature with r rounds and n teams, and let $\mathcal{B}$ have s round and m teams, such that $\mathcal{A}$ contains $\mathcal{B}$ and $\mathcal{B}$ is not ordered. Let k be the number of teams in $\mathcal{A}$ that get at least $s+i$ byes (where i is from the definition of contains).

$\mathcal{B}$ is not ordered, so let $\mathcal{M}$ be a matchup table that violates the orderedness condition, where none of the win probabilities are 0. (If we have an $\mathcal{M}$ that includes 0s, we can replace them with ϵ. For small enough ϵ, $\mathcal{M}$ will still violate the condition.) Let p be the minimum probability in $\mathcal{M}$. Let $\mathbf{P}$ be a matchup table in which the lower-seeded team wins with probability p, and let $\mathbf{Z}$ be a matchup table in which the lower-seeded team wins with probability 0.

Now, consider the following block matchup table on $\mathcal{T}$, a list of n teams.

	t_1 - t_k	t_{k+1} - t_{k+m}	t_{k+m+1} - t_n
t_1 - t_k	$\mathbf{P}$	$\mathbf{P}$	$\mathbf{Z}$
t_{k+1} - t_{k+m}	$\mathbf{P}$	$\mathcal{M}$	$\mathbf{Z}$
t_{k+m+1} - t_n	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$

Let $\mathcal{S} \subset \mathcal{T}$ be the sublist of teams seeded between $k+1$ and $k+m$. Then, for $t_j \in \mathcal{S}$,

$$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T}) = \mathbb{W}_{\mathcal{B}}(t, \mathcal{S}) \cdot p^{r-s-i},$$

since t_j wins any games it might have to play in rounds i or before automatically, any games after $s+i$ with probability p, and any games in between according to $\mathcal{M}$.

However, $\mathcal{M}$ (and thus $\mathcal{S}$) violates the orderedness condition for $\mathcal{B}$, and so $\mathcal{T}$ does for $\mathcal{A}$. $\square$

With the containment lemma shown, we can proceed to the main theorem.

Theorem 2.4.7: Edwards's Theorem

The only ordered brackets are those described by Corollary 2.4.3.

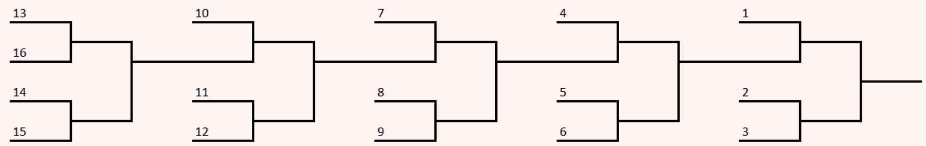
Proof. Let $\mathcal{A}$ be a proper bracket not described by Corollary 2.4.3. The corollary describes all proper brackets in which each round either has only game, or has two games but is immediately followed by a round with only one game. Thus $\mathcal{A}$ must include at least two successive rounds with two or more games each.

The final round in such a chain will be followed by a round with a single game, and so the final round must have only two games. Thus, $\mathcal{A}$ includes a sequence of three rounds, the first of which has at least two games, the second of which has exactly two games, and the third of which has one game.

Therefore, $\mathcal{A}$ contains $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$. But we know that $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$ is not ordered, and so by the containment lemma, neither is $\mathcal{A}$. $\square$

Edwards's Theorem is both exciting and disappointing. On one hand, it means that we can fully describe the set of ordered brackets, making it easy to check whether a given bracket is ordered or not. On the other hand, it means that in an ordered bracket at most three teams can be introduced each round, so the length of the shortest ordered bracket on n teams grows linearly with n (rather than logarithmically as is the case for the shortest proper bracket). If we want a bracket on many teams to be ordered, we risk forcing lower-seeded teams to play a large number of games, and we only permit the top-seeded teams to play a few. For example, the shortest ordered bracket that could've been used in the 2021 NCAA Basketball South Region is $[[\mathbf{4}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{3}; \mathbf{0}; \mathbf{0}]]$, which is played over a whopping ten rounds.

Figure 2.4.8: The Shortest Sixteen-Team Ordered Bracket



Because of this, few leagues use ordered brackets, and those who do usually have so few teams that every proper bracket is ordered (the 2023 College Football Playoffs, for example). Even the Korean Baseball Organization League, which uses a somewhat unconventional $[[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{1};\mathbf{0}]]$, only sends five teams to the playoffs, and again every five-team proper bracket is ordered. If the KBO League ever expanded to the six-team bracket $[[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{1};\mathbf{1};\mathbf{0}]]$, we would have a case of an ordered bracket being used when a proper non-ordered bracket exists on the same number of teams.

2.5 Reseeded Brackets

Edwards's Theorem tells us that the number of rounds required to construct on ordered bracket grows linearly with the number of teams involved. This can be somewhat frustrating, as part of the power of elimination formats is the ability to crown a champion in a number of rounds logorithmic in the number of teams participating.

Recall though that brackets are elimination formats that uphold the network condition in particular. Could there exist an ordered elimination format whose number of rounds grows logorithmically with the number of participating teams if we no longer require the network condition? *Reseeded* brackets are our first attempt at an answer.

Ultimately, the reason that proper brackets are not, in general, ordered, is that lower-seeded teams are treated, if they win, as the team that they beat for the rest of the format. Consider again the proper bracket analyzed by Silver: $[[\mathbf{16}; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$. If an 11-seed wins in the first round, they take on the schedule of a 6-seed for the rest of the tournament, while if the 9-seed wins, they take on the schedule of an 8-seed. Given that a 6-seed has an easier schedule than an 8-seed, it's not hard to see why it might be preferable to be an 11-seed rather than a 9-seed.

Reseeding (poorly named) fixes this by resorting the match-ups every round: if an 11-seed keeps winning, they will have to play teams according to their seed, rather than getting an effective upgrade to 6-seed status.

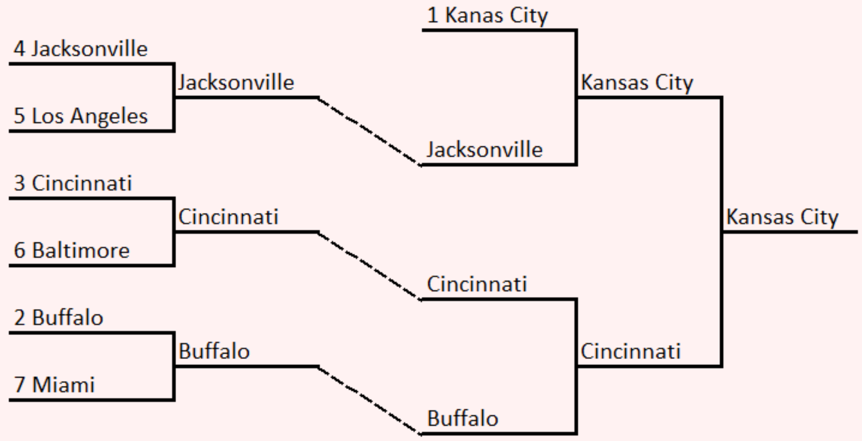
Definition 2.5.1: Reseeded Brackets

In a *reseeded* bracket, after each round, match up the highest-seeded team with the lowest-seeded team, second-highest vs second-lowest, etc.

Note that by Definition 2.1.2, a reseeded bracket is not a bracket at all, as matchups between teams that have not yet lost are not determined in advance of the outcomes of any games. However, because reseeded brackets act so similarly to traditional brackets, and because colloquially they are referred to as brackets, we opt to continue using the word "bracket" to describe them.

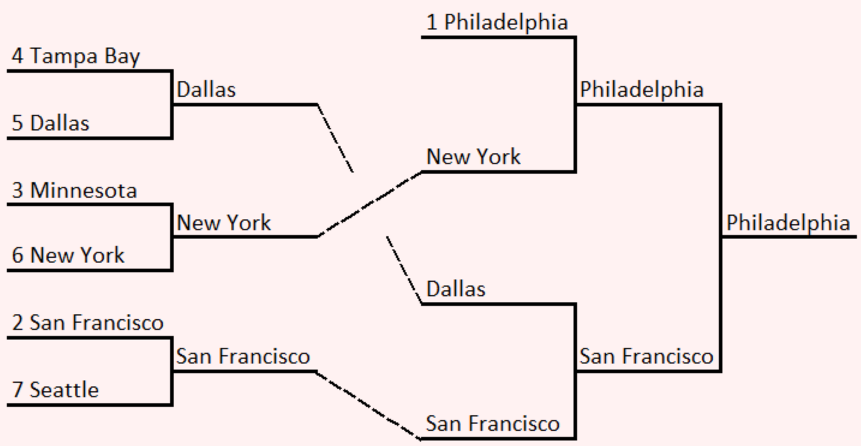
In 2023, both National Football League conferences use a reseeded bracket with signature $[[\mathbf{6}; \mathbf{1}; \mathbf{0}; \mathbf{0}]]^R$. (The superscript R indicates this is reseeded bracket.) If the first round of the bracket goes chalk, then it looks just like a normal bracket.

Figure 2.5.2: 2023 National Football League AFC Playoffs



The dotted lines are drawn after the first round of games has been played: if there are some first-round upsets, then the bracket is rearranged to ensure that it is still better to be a higher seed rather than a lower seed.

Figure 2.5.3: 2023 National Football League NFC Playoffs



In the NFC, 6-seed New York upset 3-seed Minnesota. Had a conventional bracket been used, the semifinal matchups would have been 1-seed vs 5-seed and 2-seed vs 6-seed: the 2-seed would have had an easier draw than the 1-seed, while the 6-seed would have an easier draw than the 5-seed. Reseeding fixes this by matching 6-seed New York with top-seed Philadelphia, and 2-seed San Francisco with 5-seed Dallas.

Reseeding is a powerful technique. For one, the fundamental theorem still applies to reseeded brackets, allowing us to refer to reseeded brackets by their signatures as well.

Theorem 2.5.4

There is exactly one proper reseeded bracket with each bracket signature.

Proof. The definition of properness ensures that there is only one way byes can be distributed such that a reseeded bracket can be proper. Additionally, because reseeded brackets have no additional parameters beyond which seeds get how many byes, there is no more than one reseeded bracket with each signature that could be proper. Finally, that bracket is indeed proper: if the bracket goes to chalk, the matchups will be the exact same as a traditional bracket, which by the fundamental theorem is a proper set of matchups. □

But what about orderedness? It's intuitive to think that all proper reseeded are ordered: it feels like almost by definition, the higher-seeded teams have an easier path than the lower-seeded ones. Hwang [10] conjectured a weaker version of this.

Conjecture 2.5.5

All balanced proper reseeded brackets are ordered.

Unfortunately, neither the stronger claim that all proper reseeded brackets are ordered, nor Hwang's weaker conjecture are true. Our classification of the ordered reseeded brackets takes the same route as our proof of Edwards's Theorem did: we first examine the orderedness of certain important brackets, and then we use the stapling and containment lemmas to specify the complete set of ordered reseeded brackets.

The proofs of the stapling and containment lemmas for reseeded brackets, as well as the fact that all ordered reseeded brackets are proper, are so similar to the corresponding proofs for traditional brackets that we just state them without proof.

Theorem 2.5.6

All ordered reseeded brackets are proper.

Lemma 2.5.7: The Stapling Lemma for Reseeding

If $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]^R$ and $\mathcal{B} = [[\mathbf{b_0}; ...; \mathbf{b_s}]]^R$ are ordered reseeded brackets, then $\mathcal{C} = [[\mathbf{a_0}; ...; \mathbf{a_r} + \mathbf{b_0} - \mathbf{1}; ...; \mathbf{b_s}]]^R$ is an ordered reseeded bracket as well.

Lemma 2.5.8: The Containment Lemma for Reseeding

If $\mathcal{A}$ and $\mathcal{B}$ are reseeded brackets, $\mathcal{A}$ contains $\mathcal{B}$, and $\mathcal{B}$ is not ordered, then neither is $\mathcal{A}$.

We now examine particular brackets.

Theorem 2.5.9

$[[\mathbf{1}]]^R$, $[[\mathbf{2};\mathbf{0}]]^R$, and $[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$ are ordered.

Proof. Since no reseeding is done in a bracket of two or fewer rounds, and since the traditional brackets of these signatures are ordered, so are the reseeded brackets. □

Our primary example of a reseeded bracket that is ordered despite the traditional bracket of the same signature not being ordered is $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$.

Theorem 2.5.10

$[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ is ordered.

Proof. This can be shown by computing the probability of each team winning the format and then applying the SST conditions to establish the inequalities, as we did in Theorem 2.3.12. In the interest of brevity, however, we instead give an intuitive argument.

$\mathbb{W}_A(t_1,\mathcal{T}) \geq \mathbb{W}_A(t_2,\mathcal{T})$ because from those two teams perspectives, this format is just $[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$. $\mathbb{W}_A(t_2,\mathcal{T}) \geq \mathbb{W}_A(t_3,\mathcal{T})$ because t_2 has better odds if t_3 wins in the first round and they meet in the semifinals, and certainly has better odds if t_3 loses in the first round. $\mathbb{W}_A(t_4,\mathcal{T}) \geq \mathbb{W}_A(t_5,\mathcal{T})$ because t_4 is at least as likely to win the first-round matchup, and then their paths would be identical.

$\mathbb{W}_A(t_3,\mathcal{T}) \geq \mathbb{W}_A(t_4,\mathcal{T})$ holds because if both teams win the first round then t_3 has better odds in the remaining $[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$ bracket. Meanwhile if only one does, then t_3 will be joined by t_5 while t_4 will be joined by t_6, and so t_3 is more likely to dodge playing t_1 in the finals. The same argument applies to show that $\mathbb{W}_A(t_5,\mathcal{T}) \geq \mathbb{W}_A(t_6,\mathcal{T})$ as well. □

Unfortunately, that is where the power of reseeding to convert non-ordered signatures into ordered ones ends. The following two signatures are not ordered.

Theorem 2.5.11

$[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$, and let $\mathcal{T}$ have the following matchup table.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	0.5	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$
t_2	p	0.5	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$
t_3	p	p	0.5	0.5	0.5	$1-p$	$1-p$
t_4	p	p	0.5	0.5	0.5	0.5	0.5
t_5	p	p	0.5	0.5	0.5	0.5	0.5
t_6	p	p	p	0.5	0.5	0.5	0.5
t_7	p	p	p	0.5	0.5	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_6, \mathcal{T}) = O(p^3),$$

but

$$\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) = 0.25p^2 + O(p^3).$$

Thus, for small enough p, $\mathbb{W}_{\mathcal{A}}(t_6, \mathcal{T}) < \mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T})$, so $\mathcal{A}$ is not ordered. $\square$

Theorem 2.5.12

$[[\mathbf{4}; \mathbf{2}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]^R$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4}; \mathbf{2}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]^R$, and let $\mathcal{T}$ have the following matchup table.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
t_1	0.5	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$
t_2	p^2	0.5	0.5	0.5	$1-p$	$1-p$	$1-p^2$	$1-p^2$
t_3	p^2	0.5	0.5	0.5	$1-p$	$1-p$	$1-p$	$1-p$
t_4	p^2	0.5	0.5	0.5	0.5	$1-p$	$1-p$	$1-p$
t_5	p^2	p	p	0.5	0.5	$1-p$	$1-p$	$1-p$
t_6	p^2	p	p	p	p	0.5	$1-p$	$1-p$
t_7	p^2	p^2	p	p	p	p	0.5	0.5
t_8	p^2	p^2	p	p	p	p	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) = 0.25p^5 + O(p^6)$$

but

$$\mathbb{W}_{\mathcal{A}}(t_8, \mathcal{T}) = 0.5p^5 + O(p^6).$$

Thus, for small enough p, $\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) < \mathbb{W}_{\mathcal{A}}(t_8, \mathcal{T})$, so $\mathcal{A}$ is not ordered. $\square$

Recapping,

Figure 2.5.13: Which Proper Reseeded Brackets are Ordered

Ordered	Not Ordered
$[[\mathbf{1}]]^R$	$[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$
$[[\mathbf{2};\mathbf{0}]]^R$	$[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$
$[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$	
$[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$	

Finally, we apply the stapling and containment lemmas to complete the theorem.

Theorem 2.5.14

The ordered reseeded brackets are exactly those corresponding to signatures that can be generated in the following way.

1. Start with the list $[[\mathbf{0}]]^R$ (note that this not yet a bracket signature).
2. As many times as desired, prepend the list with $[[\mathbf{1}]]$, $[[\mathbf{3};\mathbf{0}]]$, or $[[\mathbf{3};\mathbf{2};\mathbf{0}]]$.
3. Then, add 1 to the first element in the list, turning it into a bracket signature.

Proof. The stapling lemma, combined with the fact that $[[\mathbf{1}]]^R$, $[[\mathbf{2};\mathbf{0}]]^R$, $[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$, and $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ are ordered, ensure that any reseeded brackets generated by the above procedure is indeed ordered. Left is to use the containment lemma to ensure that these are the only ones.

Let $\mathcal{A}$ be a bracket signature that cannot be generated by the procedure. Then, either there is a round in which three or more games are to be played, or there is a round in which exactly two games are played and the next two rounds each have exactly two games played as well.

Let i be the latest such round. If round i is the first of three rounds with two games each, then round $i+3$ must have only one game played (otherwise i would not be the latest such round). But then $\mathcal{A}$ contains $[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$, and so is not ordered.

If round i has three or more games, then round $i+1$ must contain exactly two games (any less and not every winner would have a game, any more and i would not be the latest such round.) Then, if round $i+2$ has one game, then $\mathcal{A}$ contains $[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$, and if it has two, then $\mathcal{A}$ contains $[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$. In either case, $\mathcal{A}$ is not ordered.

Thus, the ordered reseeded brackets are exactly those generated by the procedure. □

So, the space of ordered reseeded brackets is slightly larger than the space of ordered traditional brackets, although perhaps this is not quite as much of an expansion as we would've liked or expected. Despite this, reseeded brackets definitely *feel* more ordered than traditional brackets of the same signature, even if neither is ordered in the definitional sense.

Conjecture 2.5.15

There is some reasonable restriction on a set of teams that is stronger than SST under which all reseeded brackets ordered.

In the meantime, reseeding remains an important tool in our tournament design toolkit. But it is not without its drawbacks, as discussed by Baumann, Matheson, and Howe [2].

In a reseeded bracket, teams and spectators alike don't know who they will play or where their next game will be until the entire previous round is complete. This can be an especially big issue if parts of the bracket are being played in different locations on short turnarounds: in the NCAA Basketball Tournament, the first two rounds are played over a weekend at various pre-determined locations. It would cause problems if teams had to pack up and travel across the country because they got reseeded and their opponent and thus location changed.

In addition, part of what makes the NCAA Basketball Tournament (affectionately known as "March Madness") such a fun spectator experience is the fact that these matchups are known ahead of time. In "bracket pools," groups of fans each fill out their own brackets, predicting who will win each game and getting points based on how many they get right. If it wasn't clear where in the bracket the winner of a given game was supposed to go, this experience would be diminished.

Finally, reseeding gives the top seed(s) an even greater advantage than they already have: instead of playing against merely the *expected* lowest-seeded team(s) each round, they would get to play against the *actual* lowest-seeded team(s). In March Madness, "Cinderella Stories," that is, deep runs by low seeds, would become much less common.

In many ways, the NFL conference playoffs are a perfect place to use a reseeded bracket: games are played once a week, giving plenty of time for travel; only seven teams make the playoffs in each, so a huge March Madness-style bracket challenge is unlikely; as a professional league, the focus is far more on having the best team win and protecting Cinderella Stories isn't as important; and because the bracket is only three rounds long, reseeding is only required once. Somewhat ironically, the NFL conference playoffs used to use the format $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ which is ordered, but have since allowed a seventh team from each conference into the playoffs and changed to the non-ordered $[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$.

Other leagues with similar structures might consider adopting forms of reseeding to protect their incentives and competitive balance (looking at you, Major League Baseball), but in many cases, the traditional bracket structure is too appealing to adopt a reseeded one.

2.6 Randomization

Given that reseeding doesn't solve the orderedness problem presented by Edwards's Theorem, we turn to a new approach at generating potentially ordered elimination tournaments: randomization.

Definition 2.6.1: Totally Randomized Bracket

A *totally randomized bracket* is a traditional bracket except the teams are randomly placed onto the starting lines instead of being placed according to seed.

This randomization means that the shape of a totally randomized bracket uniquely defines it. There is a no notion of two totally randomized brackets having the same shapes but different seedings, as the teams are randomly placed in the bracket the beginning of the format.

Chung and Hwang conjectured that all totally randomized brackets were ordered [4]. After all, the teams are all being treated identically: how could a better team be at a disadvantage relative to a worse one?

Conjecture 2.6.2

All totally randomized brackets are ordered.

Indeed, Lemma 2.6.3 seems to provide some evidence for the conjecture.

Lemma 2.6.3

Let $\mathcal{A}$ be a totally randomized bracket with signature $[[\mathbf{a_0}; ...; \mathbf{a_r}]]$, let $\mathcal{S}$ be a list of teams, and let $\mathcal{T}$ be the list of teams produced by replacing a given team $s \in \mathcal{S}$ with a team t such that for all other teams u,

$$\mathbb{P}[t \text{ beats } u] \geq \mathbb{P}[s \text{ beats } u].$$

Then,

$$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T}) \geq \mathbb{W}_{\mathcal{A}}(s, \mathcal{S}).$$

Proof. Let X be the set of subsets of $\mathcal{S} \setminus \{s\} = T \setminus \{t\}$, and for each set of teams $Y \in X$, let P_Y be the probability that s or t will have to beat exactly the set of teams Y in order to win the format (noting that this probability is the same for s and t).

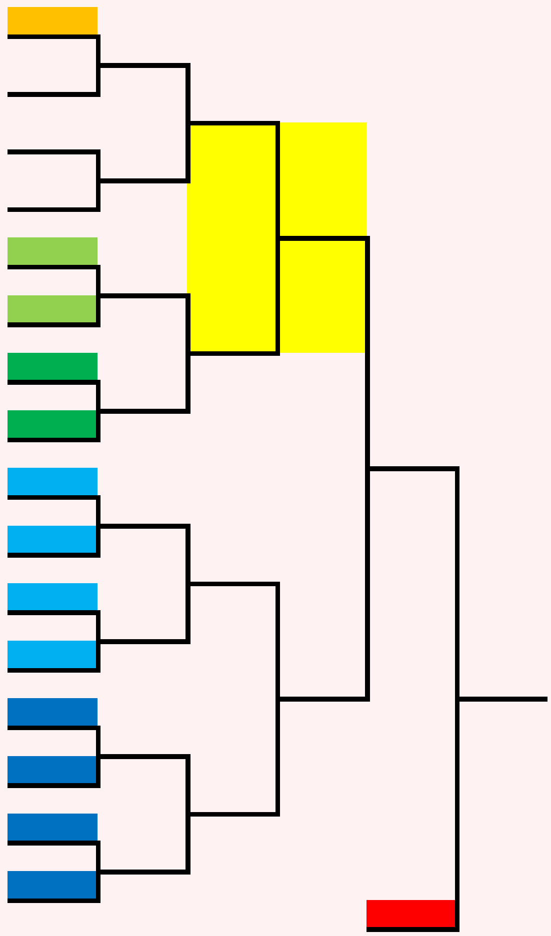
Then,

$$\begin{aligned}\mathbb{W}_{\mathcal{A}}(t,\mathcal{T}) &= \sum_{Y\in X}\left(P_Y\cdot\prod_{u\in Y}\mathbb{P}[t\text{ beats }u]\right)\\ &\geq \sum_{Y\in X}\left(P_Y\cdot\prod_{u\in Y}\mathbb{P}[s\text{ beats }u]\right)\\ &= \mathbb{W}_{\mathcal{A}}(s,\mathcal{S})\end{aligned}$$

□

But the evidence for Conjecture 2.6.2 isn't that strong: Lemma 2.6.3 is true for all traditional brackets as well, even though they, too, are are not always ordered. Indeed, despite the lemma, Conjecture 2.6.2 turns out to be false. This was first shown by Israel with the following counterexample [11].

Figure 2.6.4: The Proper Bracket Shape of Signature $[[16; 0; 0; 0; 1; 0]]$



Theorem 2.6.5

The totally randomized bracket whose shape is the proper bracket shape of signature $[[\mathbf{16}; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{1}; \mathbf{0}]]$ is not ordered.

Proof. Let $\mathcal{A}$ be the format in question, and let $\mathcal{T}$ be the list of seventeen teams containing one copy of each of t_1, t_3, t_4, and t_5, and thirteen copies of t_2 with the following matchup table.

	t_1	t_2	t_3	t_4	t_5
t_1					
t_2	0.5				
t_3	0	$2p$			
t_4	0	p	$2p$		
t_5	0	p	p	0.5	

Let $i \in \{4,5\}$ and let $j = 9 - i$. For t_i to win $\mathcal{A}$ without getting placed on the red starting line, they must win at least four games against teams t_1, t_2, or t_3, which happens with probability $O(p^4)$. Thus we let $\mathcal{B}_i$ be the format identical to $\mathcal{A}$ except we enforce that t_i will be placed on the red starting line and note that

$$\mathbb{W}_{\mathcal{A}}(t_i, \mathcal{T}) = \frac{1}{17}\mathbb{W}_{\mathcal{B}_i}(t_i, \mathcal{T}) + O(p^4).$$

Now t_j reaches the finals of $\mathcal{B}_i$ with probability $O(p^4)$, t_3 reaches the finals of $\mathcal{B}_i$ with probability $O(p^3)$ and so t_i beats them in the finals with probability $O(p^4)$, and of course t_i cannot beat t_1 in the finals. Thus,

$$\mathbb{W}_{\mathcal{B}_i}(t_i, \mathcal{T}) = p \cdot \mathbb{P}[t_2 \text{ reaches the finals of } \mathcal{B}_i] + O(p^4).$$

Since t_3 and t_j reach the finals of $\mathcal{B}_i$ with probability $O(p^3)$ and $O(p^4)$ respectively,

$$\mathbb{W}_{\mathcal{B}_i}(t_i, \mathcal{T}) = p \cdot \mathbb{P}[t_1 \text{ doesn't reach the finals of } \mathcal{B}_i] + O(p^4).$$

Assume now without loss of generality that t_1 gets placed on the orange starting line.

Any difference in $\mathbb{P}[t_1 \text{ doesn't reach the finals of } \mathcal{B}_i]$ between $i \in \{4,5\}$ will have to come as a result of a game involving t_j (as t_j is the only difference in t_1's route to the finals between $\mathcal{B}_4$ and $\mathcal{B}_5$), and because t_4 and t_5 have the same probability of beating every team other than t_3, it will have to be as a result of a game against t_3. However, because neither t_3 nor t_j can beat t_1, in order to play each other in a game whose winner doesn't immediately play t_1, they will have to be placed on two colored starting lines of the same color.

If t_3 and t_j are placed on two of the light blue or dark blue starting lines, then any difference in $\mathbb{P}[t_1 \text{ doesn't reach the finals of } \mathcal{B}_i]$ between $i \in \{4,5\}$ will be induced by t_j winning its first three games, with happens with probability $O(p^3)$.

However, if t_3 and t_j are placed on the two dark green or two light green starting lines, then when $i = 4$, t_1 will play t_2 in the yellow game with probability

$$p_{35}p_{23} + p_{53}p_{25} = ((1-p)(1-2p) + (p)(1-p)) = 1 - 2p + p^2,$$

while when $i = 5$, t_1 will play t_2 in the yellow game with probability

$$p_{34}p_{23} + p_{43}p_{24} = ((1-2p)(1-2p) + (2p)(1-p)) = 1 - 2p + 2p^2.$$

Thus,

$$\begin{aligned} &\mathbb{P}[t_1 \text{ plays } t_2 \text{ in the yellow game of } \mathcal{B}_5] \\ &- \mathbb{P}[t_1 \text{ plays } t_2 \text{ in the yellow game of } \mathcal{B}_4] \\ &= cp^2 + O(p^3) \end{aligned}$$

for some constant c, so

$$\begin{aligned}&\mathbb{P}[t_1 \text{ doesn't reach the finals of } \mathcal{B}_5]\\ &- \mathbb{P}[t_1 \text{ doesn't reach the finals of } \mathcal{B}_4]\\ &= cp^2 + O(p^3)\end{aligned}$$

for some constant c, so

$$\mathbb{W}_{\mathcal{B}_5}(t_5, \mathcal{T}) - \mathbb{W}_{\mathcal{B}_4}(t_4, \mathcal{T}) = cp^3 + O(p^4)$$

for some constant c, so

$$\mathbb{W}_{\mathcal{A}}(t_5, \mathcal{T}) - \mathbb{W}_{\mathcal{A}}(t_4, \mathcal{T}) = cp^3 + O(p^4)$$

for some constant c.

Therefore $\mathcal{A}$ is not ordered. □

Chung and Hwang's conjecture was rescued by Chen and Hwang who restricted the domain of the claim to balanced formats [3].

Theorem 2.6.6

All totally randomized balanced brackets are ordered.

Proof. Let $\mathcal{A}_r$ be the totally randomized balanced bracket on 2^r teams. We induct on r. Clearly the one-team format $\mathcal{A}_0$ is ordered. For any other r, let $\mathcal{T}$ be a list of teams, and let t_i and t_j be teams such that $i < j$.

Let $\mathcal{B}_r$ be the totally randomized balanced bracket on 2^r teams except t_i and t_j are forced to play each other in the first round, and let $\mathcal{C}_r$ be the totally randomized balanced bracket on 2^r teams except t_i and t_j cannot play each other in the first round. Then,

$$\mathbb{W}_{A_r}(t_i, \mathcal{T}) = \left(\frac{1}{2^r - 1}\right) \mathbb{W}_{B_r}(t_i, \mathcal{T}) + \left(\frac{2^r - 2}{2^r - 1}\right) \mathbb{W}_{C_r}(t_i, \mathcal{T})$$

and likewise for t_j.

Because $p_{ij} \geq p_{ji}$, and by Lemma 2.6.3, $\mathbb{W}_{B_r}(t_i, \mathcal{T}) \geq \mathbb{W}_{B_r}(t_j, \mathcal{T})$. Thus left is to show that $\mathbb{W}_{C_r}(t_i, \mathcal{T}) \geq \mathbb{W}_{C_r}(t_j, \mathcal{T})$.

For two other teams t_a and t_b, let M_{ab} be the set of $2^{r-1} - 2$ team subsets of $\mathcal{T} \setminus \{t_i, t_j, t_a, t_b\}$, and for $\mathcal{S} \in M_{ab}$, let $P_{\mathcal{S}}$ be the probability that the teams in $\mathcal{S}$ all win their first-round games and none of them play any of t_i, t_j, t_a, or t_b in the first round. Now,

$$\begin{aligned}
\mathbb{W}_{C_r}(t_i, \mathcal{T}) &= \frac{1}{2} \sum_{t_a, t_b \in \mathcal{T} \setminus \{t_i, t_j\}} \sum_{\mathcal{S} \in M_{ab}} P_{\mathcal{S}} \cdot ((p_{ia}p_{jb} + p_{ib}p_{ja}) \cdot \mathbb{W}_{A_{r-1}}(t_i, \mathcal{S} \cup \{t_i, t_j\}) \\
&+ p_{ia}p_{bj} \cdot \mathbb{W}_{A_{r-1}}(t_i, \mathcal{S} \cup \{t_i, t_b\}) + p_{ib}p_{aj} \cdot \mathbb{W}_{A_{r-1}}(t_i, \mathcal{S} \cup \{t_i, t_a\})) \\
&\geq \frac{1}{2} \sum_{t_a, t_b \in \mathcal{T} \setminus \{t_i, t_j\}} \sum_{\mathcal{S} \in M_{ab}} P_{\mathcal{S}} \cdot ((p_{ia}p_{jb} + p_{ib}p_{ja}) \cdot \mathbb{W}_{A_{r-1}}(t_j, \mathcal{S} \cup \{t_i, t_j\}) \\
&+ p_{ja}p_{bi} \cdot \mathbb{W}_{A_{r-1}}(t_j, \mathcal{S} \cup \{t_j, t_b\}) + p_{jb}p_{ai} \cdot \mathbb{W}_{A_{r-1}}(t_j, \mathcal{S} \cup \{t_j, t_a\})) \\
&= \mathbb{W}_{C_r}(t_i, \mathcal{T})
\end{aligned}$$

The inequality follows by comparing each term to its corresponding term: the $\mathbb{W}_{A_{r-1}}(t_i, \mathcal{S} \cup \{t_i, t_j\})$ inequality is by induction, while the other two terms are by Lemma 2.6.3.

Thus, $\mathcal{A}_r$ is ordered. $\square$

Chen and Hwang also conjectured that Theorem 2.6.6 could be extended to totally randomized nearly balanced brackets as well, but this remains an open question.

Definition 2.6.7: Nearly Balanced

A bracket is *nearly balanced* if no team receives more than one bye.

Conjecture 2.6.8

All totally randomized nearly balanced brackets are ordered.

Conjecture 2.6.8 would be a nice strengthening of Theorem 2.6.6: unlike balanced brackets, there is a nearly balanced bracket for every number of teams. If the conjecture held, then total randomization would be a format that can provide orderedness to arbitrary numbers of teams n without having $O(n)$ rounds, as the traditional and reseeded options do.

Of course, this orderedness does not come without drawbacks. For one, the randomization makes the orderedness feel a bit cheap: once the randomization is complete, before any games have even been played, the orderedness is lost. (Compare to the proper and reseeded ordered brackets, which maintain their orderedness throughout the whole tournament.)

But secondly, total randomness has the undesirable property that it might make for some very lopsided and anti-climatic brackets. It could be that top-two teams, whom everyone wants to see face off in the championship game, are set to play each other in the first round!

To fix this, we define a new class of randomized brackets: *cohort randomized brackets*, first defined by Schwenk [13].

Definition 2.6.9: Cohort Randomized Bracket

The r-round *cohort randomized bracket* on is the traditional balanced bracket on 2^r teams, except, for each i, seeds $2^i + 1$ through 2^{i+1} are shuffled randomly before play.

Thus the 1- and 2-seeds are locked into their places, the 3- and 4-seeds exchange places half the time, seeds 5-8 are randomly shuffled, and as are 9-16, 17-32, etc.

Cohort randomization solves the second issue with total randomness: that good teams might meet earlier than desired.

Theorem 2.6.10

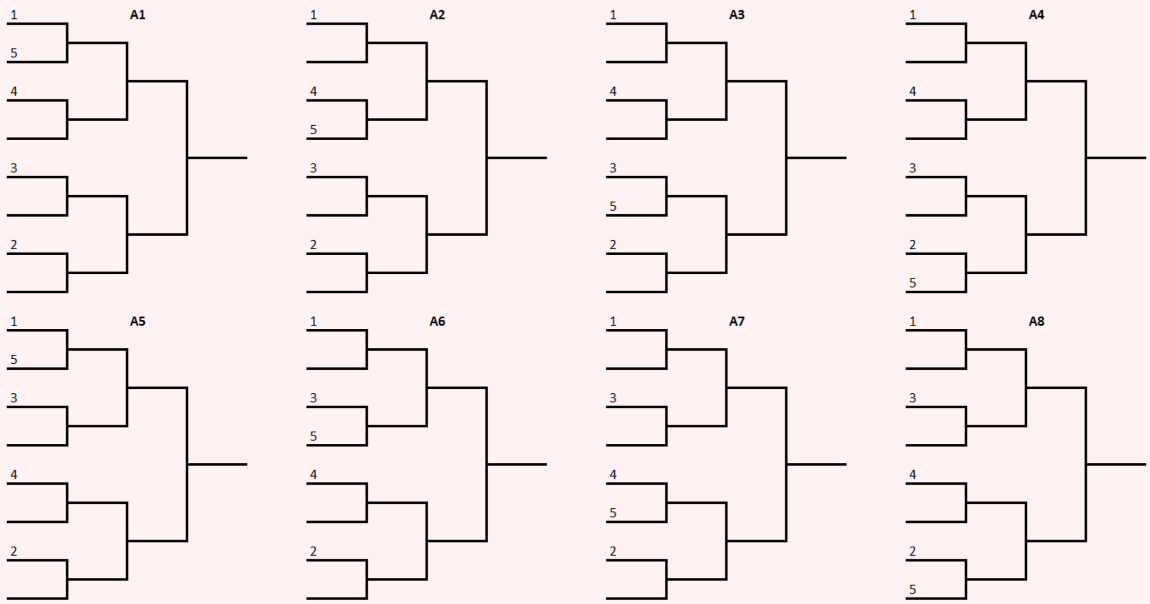
If the r-round cohort randomized bracket goes chalk, after the ith round, the top 2^{r-i} seeds will remain.

Proof. We proceed by induction on r. If $r = 0$, then there are no rounds and so the theorem holds. For any other r, in the first round, the top 2^{r-1} seeds will face the bottom 2^{r-1} seeds, and because the format goes chalk, the bottom half of teams will be eliminated. Thus after the first round, the top 2^{r-1} seeds will remain. The remaining format is just the $r-1$-round cohort randomized bracket, for which the theorem holds by induction. $\square$

It seems as though cohort randomized brackets ought to be ordered: being in a higher cohort seems certainly preferable to being a lower cohort, as you delay confrontation with the other higher-cohorted teams until later, and if two teams are in the same cohort, they are treated identically and thus it seems that the better team would win more.

Unfortunately, like many other formats we've seen thus far, cohort randomized brackets are not (for more than two rounds) ordered.

Figure 2.6.11: Setup of Theorem 2.6.12



Theorem 2.6.12

The eight-team cohort randomized bracket is not ordered.

Proof. Let $\mathcal{A}$ be the eight-team cohort randomized bracket, and let $\mathcal{T}$ have the following matchup table for $0 < p < 0.5$.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
t_1								
t_2	0.5							
t_3	0.5	0.5						
t_4	0.5	0.5	0.5					
t_5	p	0.5	0.5	0.5				
t_6	p	p	p	p	0.5			
t_7	p	p	p	p	0.5	0.5		
t_8	p	p	p	p	0.5	0.5	0.5	

Note that because t_6, t_7, and t_8 each have identical matchups against every other team, permutations of those teams don't affect the probability of any other teams winning the tournament. Thus we consider the eight possible randomizations in Figure 2.6.11 noting that for $i \in \{2, 3\}$,

$$\mathbb{W}_{\mathcal{A}}(t_i, \mathcal{T}) = \frac{1}{8} \sum_{j=1}^{8} \mathbb{W}_{\mathcal{A}_j}(t_i, \mathcal{T}).$$

(The three empty starting lines can be filled with any permutation of $\{t_6, t_7, t_8\}$.)

Some calculation finds

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}_1}(t_2, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}_1}(t_3, \mathcal{T}) \\
\mathbb{W}_{\mathcal{A}_2}(t_2, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}_2}(t_3, \mathcal{T}) \\
\mathbb{W}_{\mathcal{A}_3}(t_2, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}_4}(t_3, \mathcal{T}) \\
\mathbb{W}_{\mathcal{A}_4}(t_2, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}_3}(t_3, \mathcal{T}) \\
\mathbb{W}_{\mathcal{A}_6}(t_2, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}_8}(t_3, \mathcal{T}) \\
\mathbb{W}_{\mathcal{A}_7}(t_2, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}_5}(t_3, \mathcal{T}) \\
\mathbb{W}_{\mathcal{A}_8}(t_2, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}_6}(t_3, \mathcal{T})
\end{aligned}$$

However, letting $q = 1 - p$, $r = \frac{1}{2}q + \frac{1}{4}$, and $s = pq + \frac{1}{2}q$

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}_5}(t_2, \mathcal{T}) &= qs\left(q\frac{1}{2} + p(pr + qs)\right) \\
&< qs\left(q\frac{1}{2} + p\left(\frac{1}{2}r + \frac{1}{2}s\right)\right) \qquad \text{because } r < s \text{ and } p < \frac{1}{2} < q \\
&= \mathbb{W}_{\mathcal{A}_7}(t_3, \mathcal{T})
\end{aligned}$$

Thus,

$$\mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T}) < \mathbb{W}_{\mathcal{A}}(t_3, \mathcal{T})$$

so $\mathcal{A}$ is not ordered. □

If cohort randomized brackets don't solve the orderedness problem, why would we use them over traditional proper brackets? Cohort randomization is most famously found on the ATP Tour, a collection of tournaments played by professional tennis players that all use almost identical formats: large balanced brackets. Additionally, the seeding for these tournaments is set by the ATP rankings, which tend to be slow to update. As a result, if every ATP Tour tournament used the proper seeding, the 6-seed and 27-seed would play each other in the first round at every tournament until one of them moved up or moved down. These rematches were deemed undesirable and so this randomization procedure was introduced: The 1-seed's quarterfinals matchup (if everything goes chalk) is now randomly drawn from the 5- through 8-seeds, instead of always being the 8-seed.

But Theorem 2.6.12 tells us that they are not ordered. Unfortunately, this chapter ends an unsatisfying note, with two key questions in the field remaining open.

Open Question 2.6.13

For a given signatures $\mathcal{A}$, does there exists an ordered elimination format with signature $\mathcal{A}$?

Of course, if $\mathcal{A}$ is a balanced signature, then we know the answer is yes: a totally randomized bracket will suffice. Thus the second open question,

Open Question 2.6.14

For each r, does there exists a balanced ordered elimination format such that if the format goes chalk, after the ith round, the top 2^{r-i} seeds will remain?

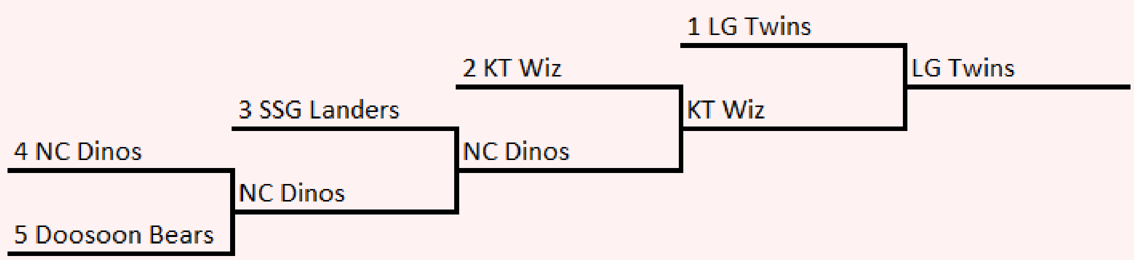
3 Multibrackets

3.1 Consolation Brackets

In the pervious chapter, we discussed brackets and similar formats (reseeded and cohort randomized brackets) paying attention only to which team is declared champion. Edwards's Theorem and its analogies, for example, makes claims only about which teams are most likely to win the tournament: all participants that don't are grouped together as losers. Real tournaments, however, do not always operate in this way: for example, explicitly or not, the team that lost the championship game is often considered to have earned the second-place finish.

Third-place is often harder to determine. If a team is given a bye all the way to the finals, and thus there is only one semifinal, then the loser of that semifinal can be unambiguously granted third. The 2023 Korean Baseball Organization League Playoffs have this property: they use a bracket of signature $[[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{1};\mathbf{0}]]$, and so could easily assign third-place to the NC Dinos, who lost in the sole semifinal. (The LG Twins won the format, and finals losers KT Wiz came in second.)

Figure 3.1.1: 2023 KBO League Playoffs



But in most brackets (those brackets whose signature's penultimate digit is a zero), assigning third-place is trickier: there are two teams who lost in the semifinal and have an equal claim to the place. There are a number of strategies that a league might use in the face of this ambiguity.

The first option is to just not assign a third-place at all. In the wise words of Will Ferrell from Talladega Nights, "If you ain't first, you're last." Who cares who came in third: you didn't win, you didn't even come in second, so you lost. This approach is taken by all four major American professional sports leagues (the NFL, NBA, NHL, and MLB).

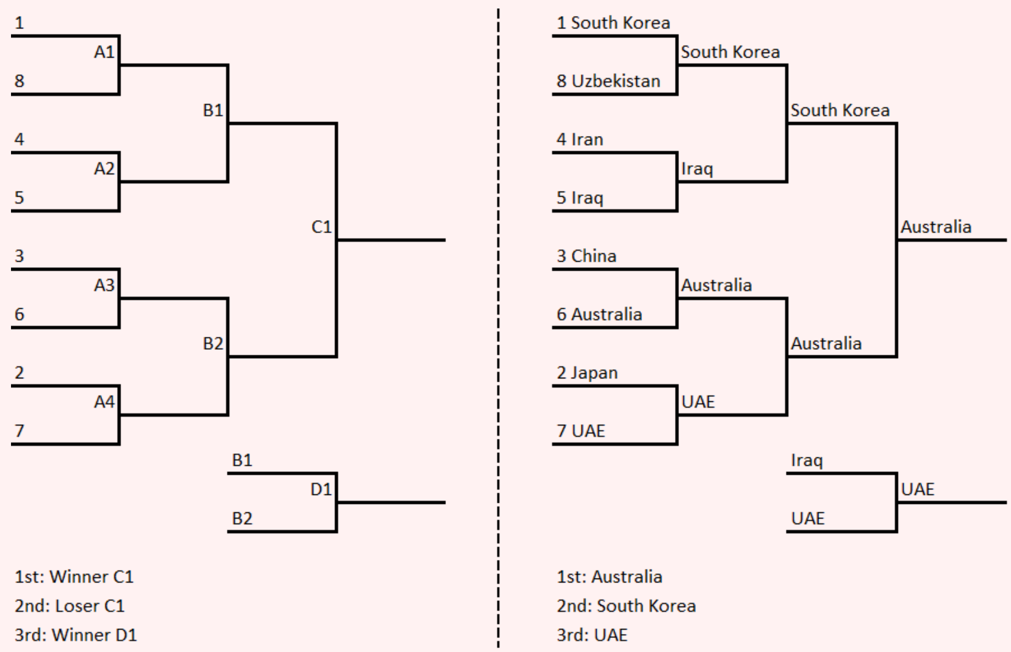
The second option is to declare the two semifinal losers co-third-place finishers. In many ways, this is the same as the first option, but with a single sentence added to the end of a press-release indicating that the teams in question each finished third. (This option also has the unsatisfying property that four teams will be able to claim a top-three finish. This can be easily fixed, however, by just granting both teams fourth-place instead.)

The third option is to use some (relatively) arbitrary tiebreaker to select the third-place team. A few potential such tiebreakers are: whichever team was seeded higher, whichever team lost to the tournament champion (as opposed to the tournament runner-up), or if the teams played each other during the "regular season" portion of a tournament, whichever team won that game.

None of these are particularly satisfying. While they may do alright when giving out third-place isn't super important, if we really want to assign third in a fair and equitable way, say because there is a bronze medal or spot in a future tournament up grabs, these options will not do.

Instead, the best thing to do is play a game: The 2015 Asian Football Confederation Asian Cup did exactly that.

Figure 3.1.2: The 2015 AFC Asian Cup



In the 2015 AFC Asian Cup, after the main bracket is complete, with the winner of the final game (Australia) being crowned champion and the loser (South Korea) coming in second, the two semifinal losers (Iraq and the UAE) are matched up in the third-place game.

A quick note about Figure 3.1.2: each game in the figure is labeled. In the primary bracket, first-round games are **A1** through **A4**, while the semifinals are **B1** and **B2**, and the finals is game **C1**. The third-place game is labeled **D1**: even though it could be played concurrently to the championship game, it is part of a different bracket and so we label it as a different round.

We indicate that the third-place game is to be played between the losers of games **B1** and **B2** by labeling the starting lines in the third-place game with those games. This is not ambiguous because the winners of those games always continue on in the original bracket, so such labels only refer to the losers.

The third-place game, which can also be viewed as a two-team bracket of signature $[[\mathbf{2}; \mathbf{0}]]$, in an example of a *consolation bracket*.

Definition 3.1.3: Consolation Bracket

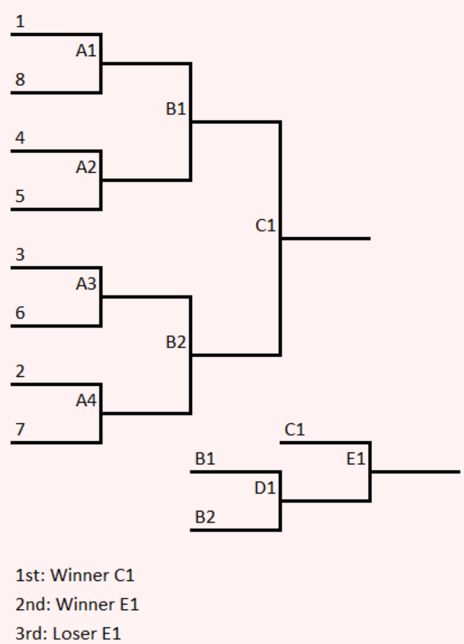
A *consolation bracket* is a bracket in which teams that did not win the tournament compete for an n-place finish for some n.

The third-place game, as used by the 2015 AFC Asian Cup, is a common and well-liked consolation bracket used for selecting the top-three teams after a bracket. But it is far from the only way that the AFC could have doled out gold, silver, and bronze.

In fact, it's not clear the loser of **C1**, who comes in second place, is really more deserving than the winner of **D1**, who comes in third. The UAE might argue: South Korea and we both finished with two wins and one loss – a first-round win, a win against Iraq, and a loss against Australia. The only reason that South Korea came in second and we came in third was because South Korea lucked out by having Australia on the other half of the bracket as them. That's not fair!

If the AFC took this complaint seriously, they could modify their format to add a game **E1** for second-place to be played between the loser of **C1** and the winner of **D1**, with the loser coming in third.

Figure 3.1.4: The 2015 AFC Asian Cup Alternative



If the AFC used the format in Figure 3.1.4 in 2015, then South Korea and the UAE would have played each other for second place after all of the other games were completed. In some sense, this is a more equitable format than the one used in reality: we have the same data about the UAE and South Korea and so we ought to let them play for second-place instead of having decided almost randomly.

However, swapping formats doesn't come without costs. For one thing, South Korea and the UAE would've had to play a fourth game: if the AFC had only three days to put on the tournament and teams can play at most one game a day, then the format in Figure 3.1.4 isn't feasible.

Another concern: what if Iraq had beaten the UAE when they played in game **D1**? Then the two teams with a claim to second-place would have been South Korea and Iraq, except South Korea already beat Iraq! One option is to say "tough luck, later games being more important than earlier ones is a staple in sports". But another is to designate game **E1** as *contingent*.

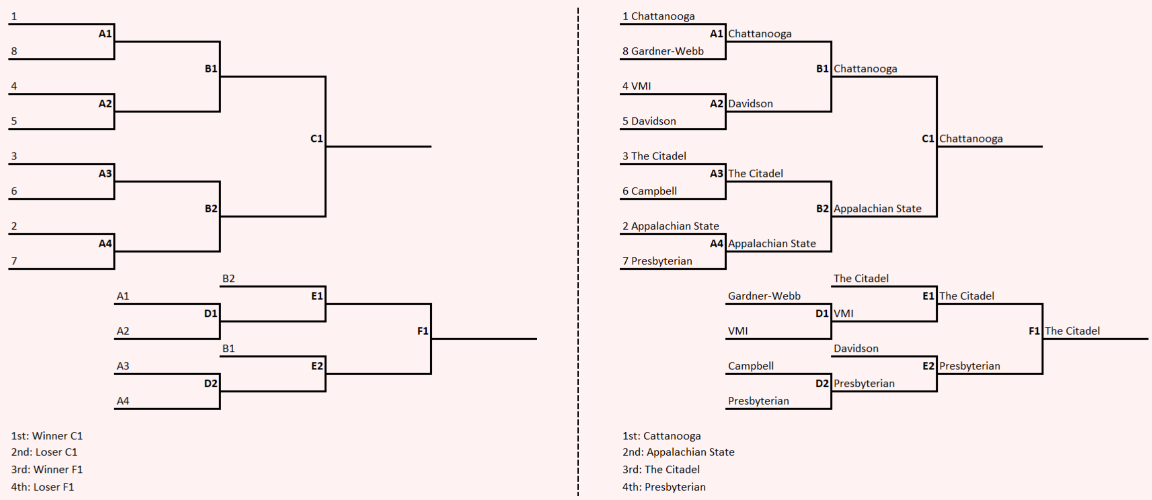
Definition 3.1.5: Contingent Game

We say a game in a tournament format is *contingent* if under certain circumstances (most commonly if the teams have already played earlier in the tournament,) the game is skipped and the result of a previous game is used.

Ultimately, whether game **E1** should be in included or not depends on the purpose of the tournament. If there is a huge difference between the prizes for coming in second and third, for instance, if the top two finishing teams in the Asian Cup qualified for the World Cup, then **E1** is quite important. If, on the other hand, this is a self-contained format played purely for bragging rights, **E1** could probably be left out. In reality, the 2015 AFC Asian Cup qualified only its winner to another tournament (the 2017 Confederations Cup), and gave medals to its top three, so game **E1**, which distinguishes between second- and third-place, was probably unnecessary.

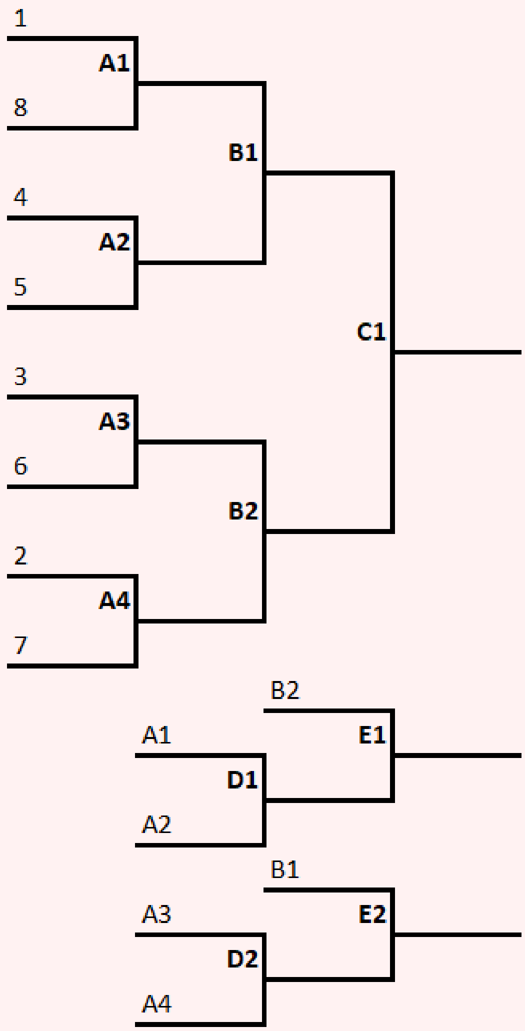
Consider if instead of just the champion, the top four teams from the Asian Cup advanced to the Confederations Cup. One could imagine an easy extension of the format used presently, in which the loser of game **D1** is awarded fourth-place, to determine the four teams that qualify. However, this format would be quite poor: whether or not a team qualifies for the Confederations Cup would be solely determined by the result of their first-game and so the **B**, **C**, and **D**-round games might as well not even be played. A better format for selecting the top-four would allow first-round losers to win their way back onto the podium, as was employed by the 2023 Southern Conference Wrestling Championships.

Figure 3.1.6: 2023 Southern Conference Wrestling Championships



The format in Figure 3.1.6 is a dramatic improvement for selecting a top-four over that in Figure 3.1.2. In the 2023 Southern Conference Wrestling Championships, teams finish in the top-four if and only if they win two games before they lose two, which is a nice property to have. The one downside is it takes a fourth round: if there is not enough time for a fourth round, or if there is safety risk to teams playing four matches in a row, the format isn't feasible. Though if we only care about the top-four, and not the specifics of which team came in third or in fourth, we could drop game **F1**, ensuring that each team plays at most three games.

Figure 3.1.7: 2023 Southern Conference Wrestling Championships Alternative



1st: Winner C1

2nd: Loser C1

4th: Winner E1, Winner E2

(As discussed earlier, we opt to both the **E**-round winners in fourth, to ensure that no more than m teams can claim a top-m finish for any m.)

The four formats with consolation brackets presented thus far are examples of *multi-brackets*.

Definition 3.1.8: Multibracket

A *multibracket* is a collection of one or more brackets coupled with a specification of which winners and losers of which games receive which places. Starting lines in multibrackets can be marked a seed, or by a game, indicating that loser of the specified game should be placed there.

Since which game each team plays in next (and which place each team ends up in) can be derived only from which game that team played in most recently and whether they won or lost that game, this definition is equivalent to saying that the format upholds the network condition.

Definition 3.1.9: Network Condition

A tournament format upholds the *network condition* if after t_i plays t_j, the rest of the format is identical no matter which team won, except for t_i and t_j are swapped.

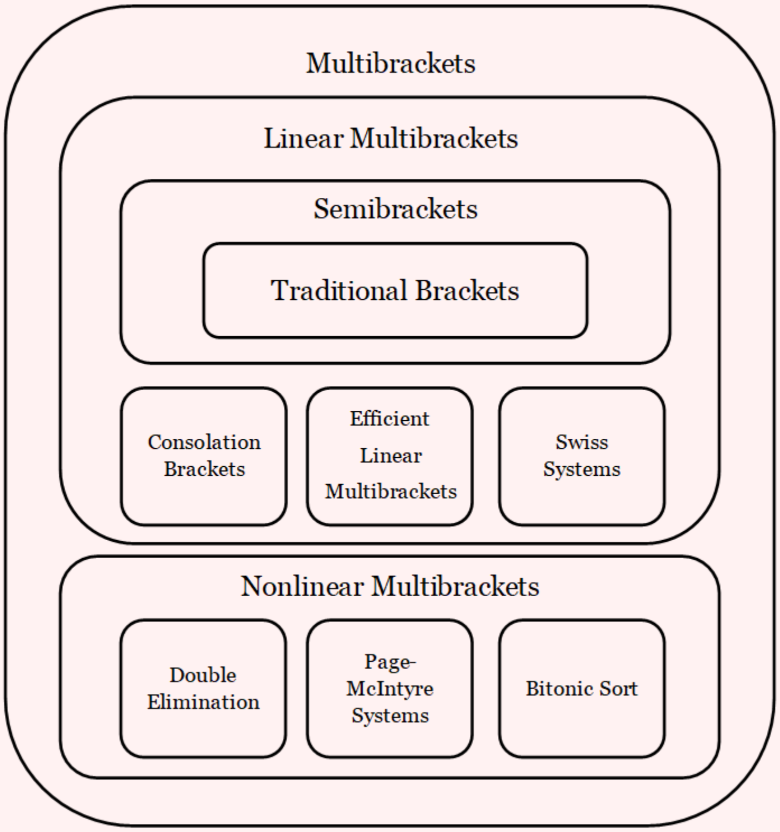
Definition 3.1.10: Multibracket

A *multibracket* is a tournament format that upholds the network condition.

(Note that this means multibrackets with contingent games are technically not multibrackets at all. However, they are close enough to being multibrackets and are important enough tools for tournament design that we include them in our discussion, in the same way that in the last chapter we discussed reseeded brackets even though they are technically not brackets.)

We will see in the coming sections that many formats used in a variety of settings are actually just examples of multibrackets. Figure 3.1.11 gives an outline of what the space of multibrackets looks like: we will spent the rest of the chapter examining the various categories in more detail.

Figure 3.1.11: The Space of Multibrackets



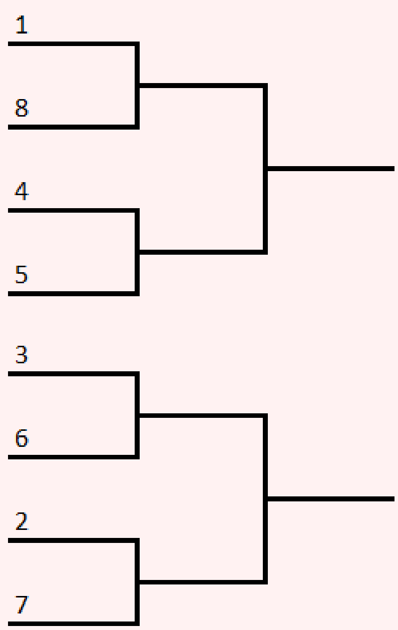
3.2 Semibrackets

In this section, we move on from consoloation brackets to focus on semibrackets, which as indicated by Figure 3.1.11 is a generalization of the traditional bracket. We will then use the notion of a semibracket to define linear multibrackets, which we will study for a few sections before addressing nonlinear multibrackets at the end of the chapter.

Consider now the following tournament design problem: we are tasked with designing an eight-team tournament to select the top two teams who will go on to compete in the national tournament. However, there's only enough time for two rounds: perhaps due to field space or team fatigue, each team can only play two games. What design should we use?

The most natural answer to this question is to use an traditional eight-team bracket, but leave the championship game unplayed. This format is displayed in the figure below.

Figure 3.2.1: $[[8; 0; 0; 0]]$ with no Championship Game



The format in Figure 3.2.1 does exactly what we need. The championship game being left unplayed is not a bug but a feature: each team plays a maximum of two games, and the two teams that advance to the national tournament are clear.

While it would be reasonable to describe the format in Figure 3.2.1 as two brackets that run side-by-side, it would be nice to be able to describe it as a single format: a bracket in which the championship game is left unplayed.

Definition 3.2.2: Semibracket

A *semibracket* is a tournament format in which:

- Teams don't play any games after their first loss.

- The matchups between game winners are determined in advance of the outcomes of any games.

All teams that finish a semibracket with no losses are declared co-champions.

Thus semibrackets are a generalization of brackets: a bracket is a semibracket in which only one team is left undefeated and declared champion.

Figure 3.2.3 describes which properties various bracket-like formats require.

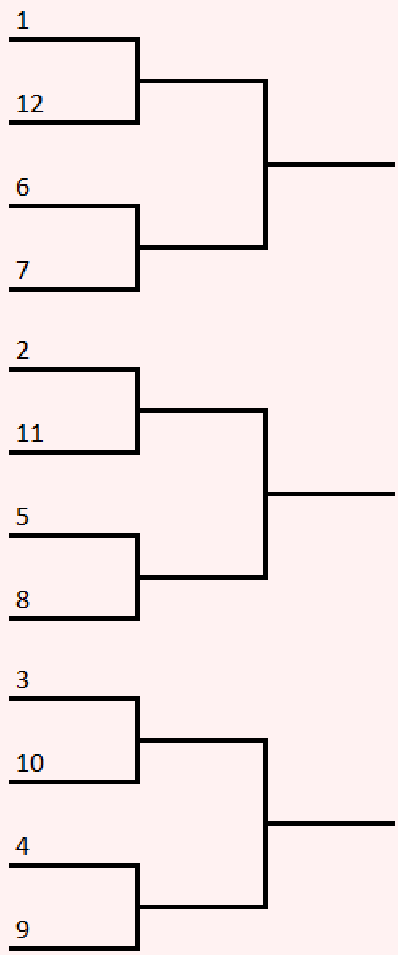
Figure 3.2.3: Properties of Networked Formats

Format	No games after first loss	Only one team finishes undefeated	Network condition
Traditional Bracket	✔	✔	✔
Semibracket	✔	✘	✔
Multibracket	✘	✘	✔

The format in Figure 3.2.1 is not a particularly exciting example of a semibracket: it is just a traditional bracket minus one game. Are there any examples of semibrackets that are not traditional brackets with some rounds left uncompleted?

Indeed there are. Let's modify the original problem so that we need to pick a top three teams out of twelve. Again, no team can play more then two games. The natural choice is shown below in Figure 3.2.4.

Figure 3.2.4: A More Exciting Semibracket



There is no potential for the format in Figure 3.2.4 to be completed into a traditional bracket, the next round would include three teams: an odd number. But as a semibracket, this is still a viable format, one that nicely solves the tournament design problem that we were given.

Definition 3.2.5: Rank of a Semibracket

The *rank* of a semibracket is how many co-champions it crowns. If the semibracket $\mathcal{A}$ has rank m, we say $\text{Rank}(\mathcal{A}) = m$ or that $\mathcal{A}$ *ranks* m *teams.*

Traditional brackets are exactly the semibrackets that rank one team. The formats in Figures 3.2.1 and 3.2.4 rank two and three teams, respectively.

We can adapt the concept of a bracket signature to semibrackets.

Definition 3.2.6: Semibracket Signature

The *signature* $[[\mathbf{a_0}; ...; \mathbf{a_r}]]_m$ of an r-round semibracket $\mathcal{A}$ is a list such that a_i is the number of teams with i byes and $m = \mathrm{Rank}(\mathcal{A})$. (In the case where $m = \mathrm{Rank}(\mathcal{A}) = 1$, it can be omitted.)

Thus the signature of traditional brackets are the same as when they are viewed as semibrackets that rank one team. The signatures of the formats in Figures 3.2.1 and 3.2.4 are $[[\mathbf{8}; \mathbf{0}; \mathbf{0}]]_2$ and $[[\mathbf{12}; \mathbf{0}; \mathbf{0}]]_3$, respectively.

In analogy with traditional bracket signature's Theorem 2.1.14, we have Theorem 3.2.7.

Theorem 3.2.7

Let $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]_m$ be a list of natural numbers. Then $\mathcal{A}$ is a semibracket signature if and only if

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = m.$$

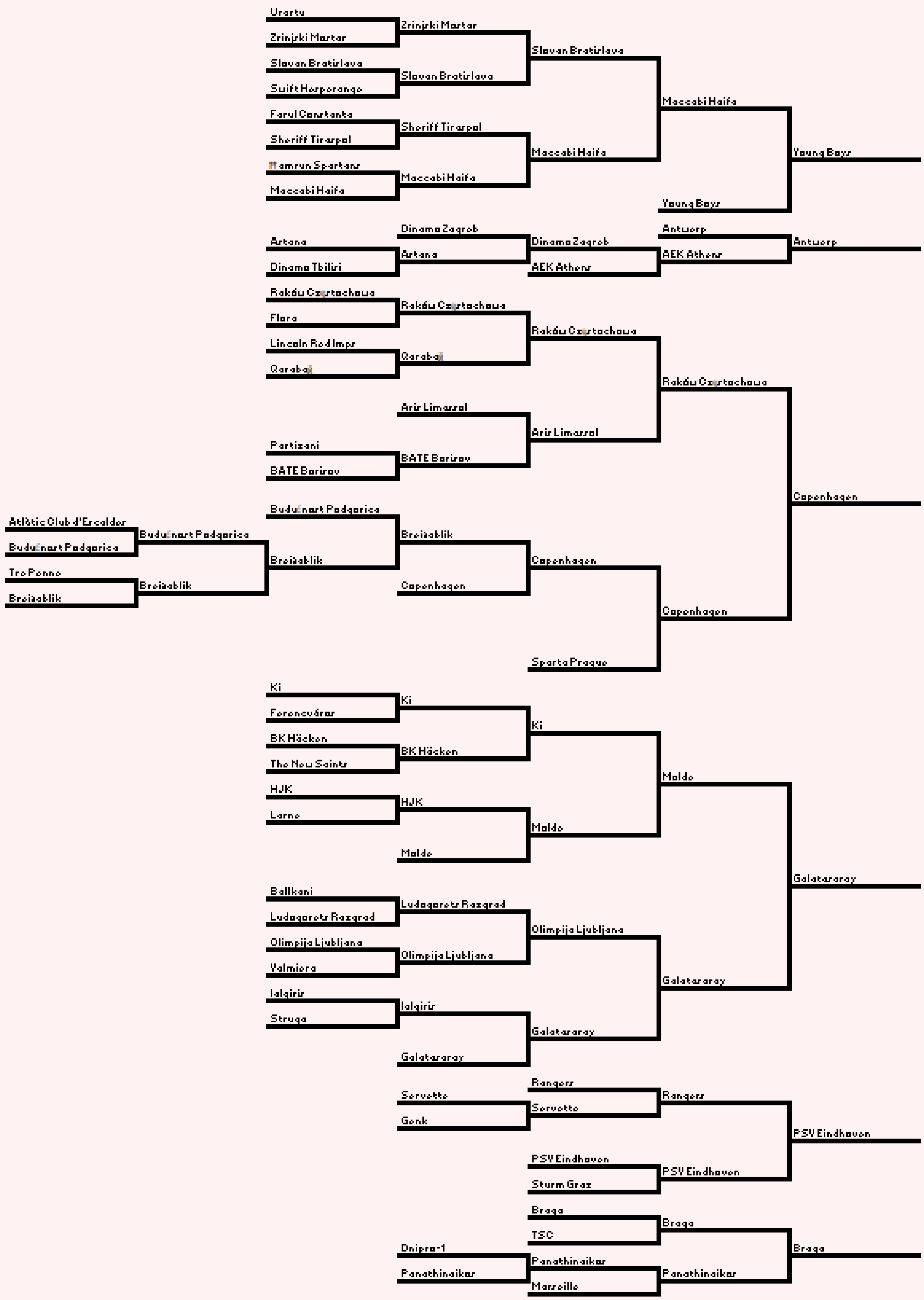
The proof is almost identical to that of Theorem 2.1.14 so we leave it out for brevity. Likewise, properness can be defined in the same way for semibracket, and the fundamental theorem still applies. (Again with a nearly identical proof that is left out for brevity.)

Theorem 3.2.8

Each semibracket signature admits exactly one proper semibracket.

Semibrackets are used in practice in situations where the excitement of a single elimination tournament is desired, but multiple winners are needed. The 2023-2024 Union of European Football Associations Champions League Qualifying Phase, for example, used a (non-proper) semibracket of signature $[[\mathbf{4}; \mathbf{0}; \mathbf{29}; \mathbf{9}; \mathbf{8}; \mathbf{2}; \mathbf{0}]]_6$ to determine the final six teams that would get to compete in the Group Stage.

Figure 3.2.9: The 2023-2024 UEFA Champions League Qualifying Phase



Finally, we give a few descriptors to describe certain semibracket shapes.

Definition 3.2.10: Trivial Semibracket

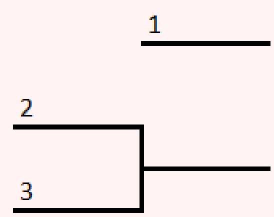
We say a semibracket is *trivial* if every team is declared co-champion without playing any games. Equivalently, a semibracket is trivial its signature is of the form $[[\mathbf{m}]]_m$.

Definition 3.2.11: Competitive Semibracket

We say a semibracket is *competitive* if no teams are declared co-champion without winning at least one game. Equivalently, a semibracket is competitive its signature ends in a 0.

Clearly the two categories are mutually exclusive. Restricting briefly to the domain of traditional brackets, the two categories are also collectively exhaustive: there is no traditional bracket that is neither competitive nor trivial. (In fact, the only trivial traditional bracket is $[[\mathbf{1}]]$, every other traditional bracket is competitive.) However, this dichotomy does not apply to semibrackets: there are semibrackets that are neither trivial nor competitive. The simplest example is $[[\mathbf{2};\mathbf{1}]]_2$, where the 1-seed is automatically one co-champion (so it's not competitive), but the 2- and 3-seeds play to be the other co-champion (so it's not trivial).

Figure 3.2.12: $[[\mathbf{2};\mathbf{1}]]_2$



These two properties of semibrackets will sometimes be useful in defining and proving theorems about certain types of multibrackets down the line. In the next section, we will use semibrackets to construct a particularly nice kind of multibrackets: *linear multibrackets.*

3.3 Linear Multibrackets

In the past two sections, we have looked at semibrackets, as well as formats with a consolation bracket, as examples of multibrackets. Let's back up a bit from specific examples, however, and ask what information we can learn about arbitrary multibrackets. One potential question to ask is if the fundamental theorem of brackets, which held for traditional brackets and semibrackets, holds for multibrackets as well. But before we can do that, we need to define what a multibracket signature and proper multibracket seeding might look like.

This is trickier than it seems: for arbitrary multibrackets, there isn't a natural generalization of signatures and properness. But there is a subset of multibrackets for which these notions generalize, allowing us to examine the fundamental theorem as it applies to this subset. These multibrackets are called *linear multibrackets*.

Definition 3.3.1: Linear Multibracket

A *linear multibracket* is a multibracket that can be arranged into a sequence of semibrackets such that

(a) If a team loses in a given semibracket but is not eliminated, they are sent to a later semibracket, and

(b) Each team that wins the nth semibracket finishes in mth place, where m is the sum of the ranks of the first n semibrackets.

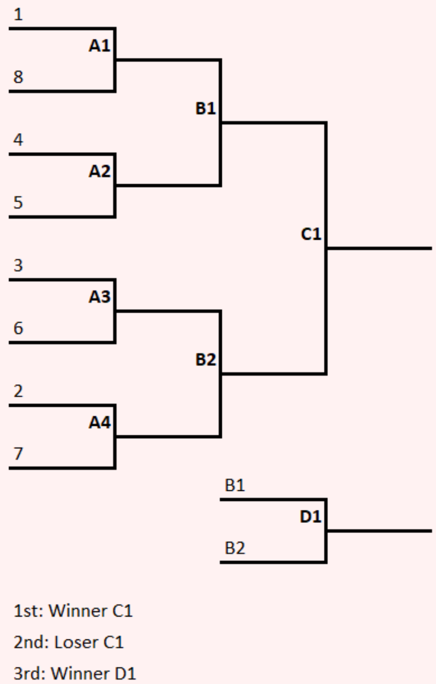
A linear multibracket can then be easily imbued with a signature derived from the signatures of the semibrackets in the sequence.

Definition 3.3.2: Linear Multibracket Signatures

If the sequence of semibrackets that constitute a linear multibracket have signature $\mathcal{A}_1, ..., \mathcal{A}_k$ then the linear multibracket has signature $\mathcal{A}_1 \to ... \to \mathcal{A}_k$.

Let's confirm that the multibrackets from the previous section are indeed linear. First, the 2015 AFC Asian Cup.

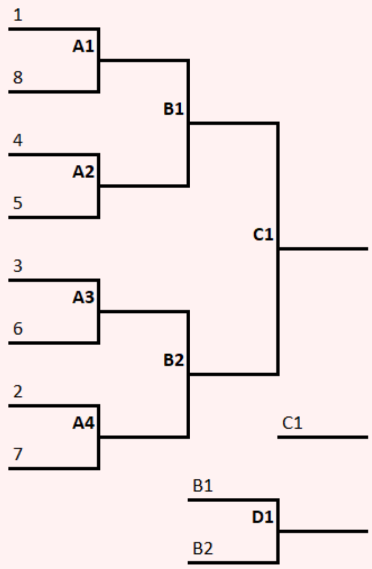
Figure 3.3.3: The 2015 AFC Asian Cup



Looking at Figure 3.3.3, it can be tempting to say that the 2015 AFC Asian Cup is a linear multibracket with signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{2};\mathbf{0}]]$. But this is not quite right: The format with this signature would give second place to the winner of **D1** (as the winner of the second bracket), while outright eliminating the loser of **C1** (as a team that did not win any bracket). But in fact, we want to give second place to the loser of **C1**, and then third place to the winner of the consolation bracket with signature $[[\mathbf{2};\mathbf{0}]]$. We can do this by adding a second bracket with signature $[[\mathbf{1}]]$ while sliding the bracket with signature $[[\mathbf{2};\mathbf{0}]]$ to third.

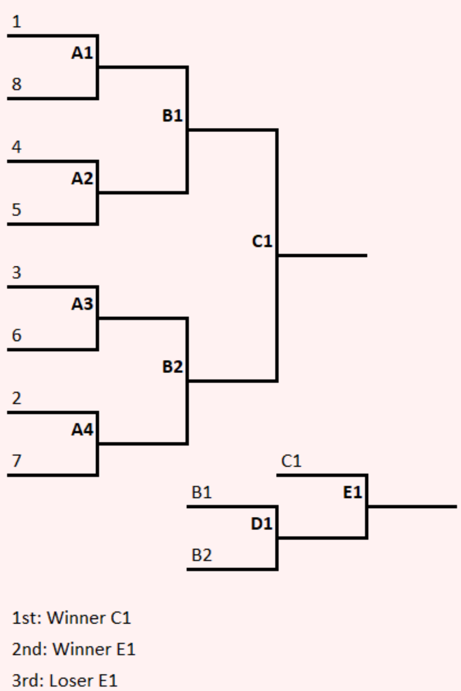
Thus in total, the 2015 Asian Cup is a linear multibracket with signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]]$. To make clear that the middle one-team bracket is included, we include it in the figure. This also allows us to drop the labeling of which teams finish in which place, as they are guaranteed by the linearity.

Figure 3.3.4: $[[8; 0; 0; 0]] \to [[1]] \to [[2; 0]]$



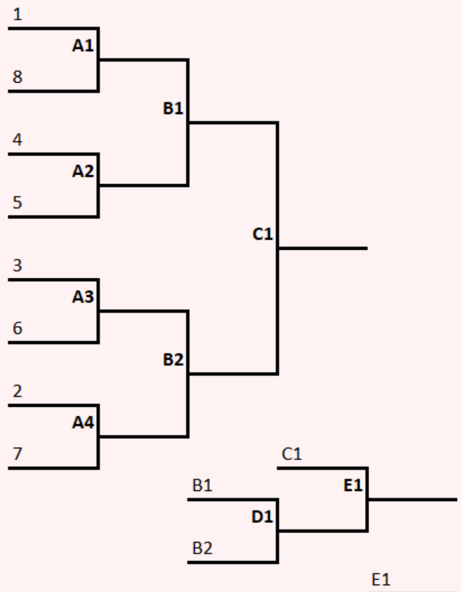
Next, let's examine our alternative to the 2015 AFC Asian Cup.

Figure 3.3.5: The 2015 AFC Asian Cup Alternative



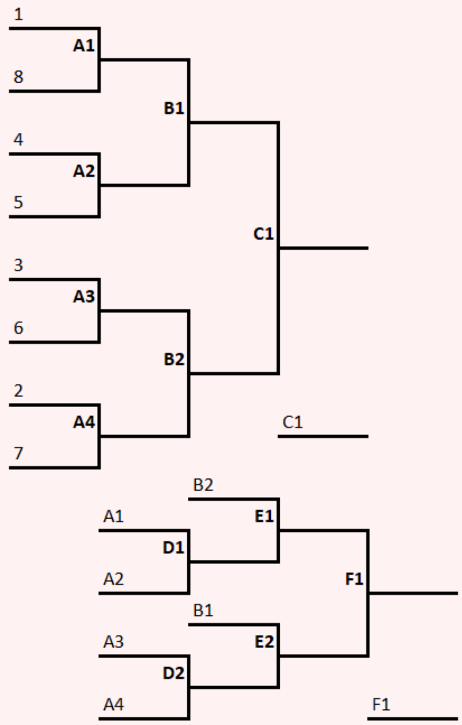
Again, a quick look indicates a signature of $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{2}; \mathbf{1}; \mathbf{0}]]$. And while this signature would correctly assign a first- and second-place, it doesn't assign third-place. Instead, we need a signature of $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{2}; \mathbf{1}; \mathbf{0}]] \to [[\mathbf{1}]]$.

Figure 3.3.6: $[[8; 0; 0; 0]] \to [[\mathbf{2}; \mathbf{1}; \mathbf{0}]] \to [[\mathbf{1}]]$



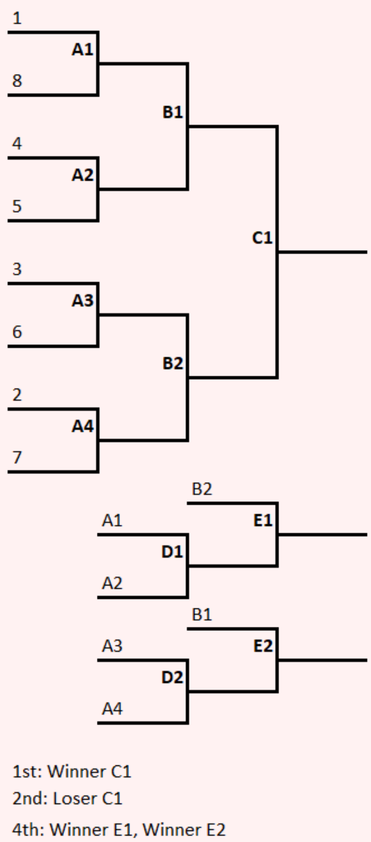
A similar analysis finds that the signature of the 2023 Wrestling Championships is $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]]$.

Figure 3.3.7: $[[8; 0; 0; 0]] \to [[1]] \to [[4; 2; 0; 0]] \to [[1]]$.



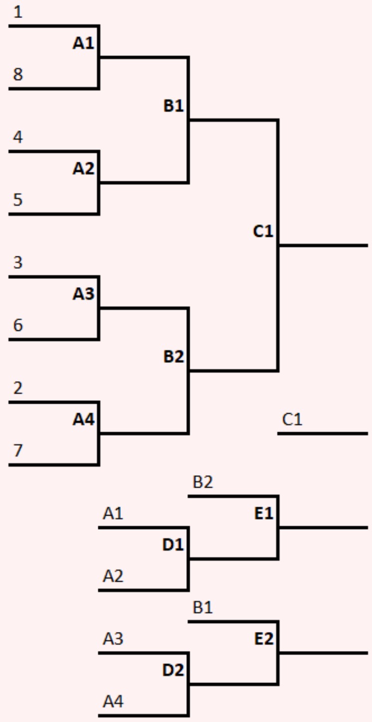
In all three examples so far, every semibracket has had rank one (that is, been a traditional bracket). However our final example, the 2023 Southern Conference Wrestling Championships Alternative, requires a semibracket of rank greater than one. (Recall the motivation for the complex multibracket: we want to identify the top-four team while not eliminating any team from contention after just a single loss.)

Figure 3.3.8: The 2023 Southern Conference Wrestling Championships Alternative



An attempt to give a signature of this format without the use of non-traditional semibrackets might be $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \rightarrow [[\mathbf{1}]] \rightarrow [[\mathbf{2};\mathbf{1};\mathbf{0}]] \rightarrow [[\mathbf{2};\mathbf{1};\mathbf{0}]]$. Unfortunately, this isn't quite the same format: it assigns third place to the winner of **E1** and fourth to the loser of **E2**. We want to treat both winners identically: this is the exact problem that semibrackets were developed to solve. Using semibrackets, we can see that the signature should be $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \rightarrow [[\mathbf{1}]] \rightarrow [[\mathbf{4};\mathbf{2};\mathbf{0}]]_2$.

Figure 3.3.9: $[[8;0;0;0]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2};\mathbf{0}]]_2$



This format is differentiated from the (admittedly a bit strange) format in which the winner of game **E1** comes in third and the winner of game **E2** comes in fourth by the lettering of the games: the fact that games **E1** and **E2** are both **E**-round games means they must come from the same semibracket. If games **D2** and **E2** were instead **F1** and **G1** respectively, then we would indeed have a linear multibracket of signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{1};\mathbf{0}]] \to [[\mathbf{2};\mathbf{1};\mathbf{0}]]$.

Now that we have defined linear multibrackets and developed a notion of signature, we can turn to the other half of the fundamental theorem: properness.

3.4 Properness

In traditional brackets, defining properness is relatively straightforward: the teams are totally ordered by seed, and a bracket is proper if, assuming it goes chalk, in every round is better to be a higher seed than a lower one. Properness in the setting of linear multibrackets is a little trickier, as many teams competing in lower semibracket are not there by virtue of being a particular seed, but instead because they lost an earlier game. To account for this, we define a new concept first, *labels*, before defining an ordering on these labels and then properness using this ordering.

One quick note before we proceed: in previous sections, we never formally defined how we name certain games in a linear multibracket. To define properness it will be important to use a specific convention: each game is given a name consisting of a letter and then a number. Every game in a given round of a semibracket must have the same letter, and the letters must be distributed in a way that satisfies two requirements. First, that in a given semibracket later rounds have later letters in the alphabet than earlier rounds. And second, that every round of a lower semibracket has a later letter in the alphabet than every round of an upper semibracket. Finally, numbers are distributed such that no two games have the same name.

With that established, we define a label.

Definition 3.4.1: Label

A *label* in a multibracket is either

(a) a seed, or

(b) the name of a game.

Definition 3.4.2: Label Used by a Semibracket

We say the label $\mathbf{L}$ is *used by the semibracket* $\mathcal{A}_i$ in a linear multibracket if $\mathbf{L}$ is placed on one of the starting lines of $\mathcal{A}_i$.

Each label in a linear multibracket is used by at most one of its semibrackets.

Definition 3.4.3: Labels Available to a Semibracket

Let $\mathcal{A} = \mathcal{A}_1 \to \ldots \to \mathcal{A}_k$ be a linear multibracket. We say a label $\mathbf{L}$ is *available to the semibracket* $\mathcal{A}_i$ if $\mathbf{L}$ is either

(a) a seed, or

(b) the name of a game in semibracket $\mathcal{A}_j$ for $j < i$,

and $\mathbf{L}$ is not on a starting line of any semibracket $\mathcal{A}_j$ for $j < i$.

Linearity guarantees that every label used by a given semibracket is available to it.

Traditional brackets (and thus the primary semibracket of a linear multibracket) only have the seeds available to them, and so defining a total order on the labels available to a linear bracket is easy: higher seeds are better and more deserving than lower seeds. We want to develop an analogous ordering for the labels available to a later semibracket: rather than a total ordering, we divide the labels into tiers.

Definition 3.4.4: Linear Multibracket Tiers

Let $\mathcal{A} = \mathcal{A}_1 \to ... \to \mathcal{A}_k$ be a linear multibracket, and let **L1** and **L2** be two labels available to the semibracket $\mathcal{A}_i$. We say a label **L1** is of a higher tier than label **L2** if

(a) **L1** and **L2** are both seeds and **L1** is a higher seed, or

(b) **L1** is the name of the game and **L2** is a seed, or

(c) **L1** and **L2** are both the names of a game, but **L1** is later in the alphabet than **L2** is.

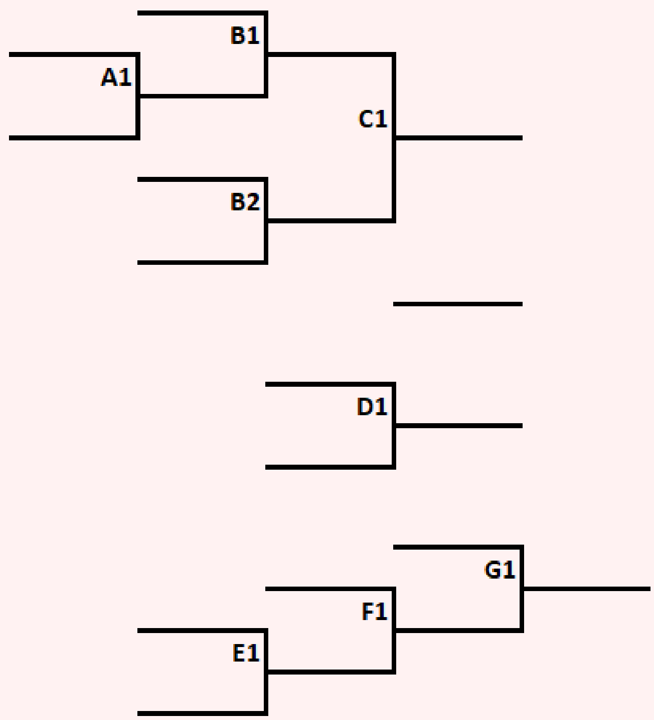
If **L1** and **L2** are names of games in the same round, then they are of the same tier.

While condition (a) is intuitive, conditions (b) and (c) might be a bit more confusing: why should the name of a game be a higher tier than a particular seed, and and why should the names of games with later letters of the alphabet be of a higher tier than the names of a games with an earlier letter of the alphabet.

To answer these questions, consider the seven-team linear multibracket shape of signature

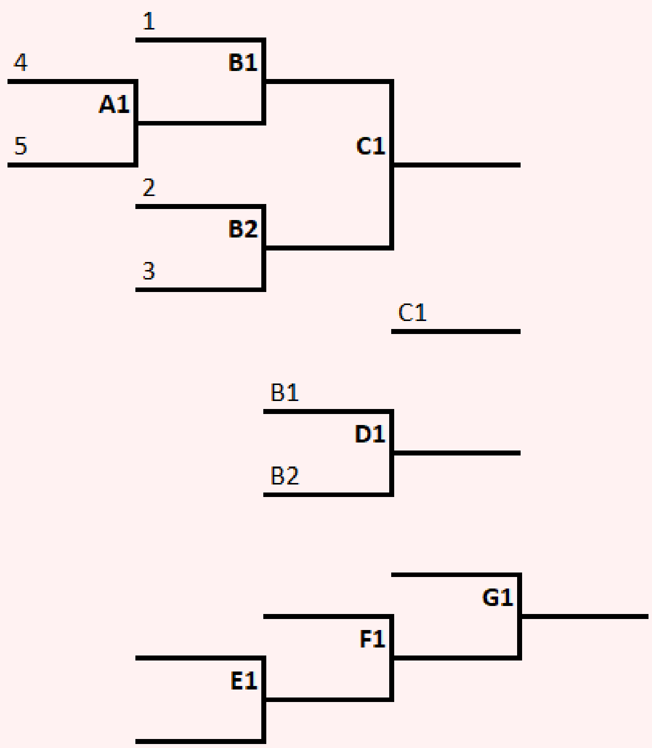
$$[[\mathbf{2};\mathbf{3};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{0}]].$$

Figure 3.4.5: $[[2;3;0;0]] \to [[1]] \to [[2;0]] \to [[2;1;1;0]]$



Before fully defining properness for linear multibrackets, let's try to fill out the starting lines in Figure 3.4.5 just based on what feels intuitively proper. The primary bracket is easy enough: we use the proper seeding of $[[\mathbf{2};\mathbf{3};\mathbf{0};\mathbf{0}]]$ to fill it out. The second and third brackets are also pretty clear: the second bracket just assigns second place, and so ought to contain only the championship game loser, while the third bracket is a third-place game and so should be played between the two semifinal losers. Filling this all in, we are left with the following linear multibracket.

Figure 3.4.6: $[[2;3;0;0]] \to [[1]] \to [[2;0]] \to [[2;1;1;0]]$

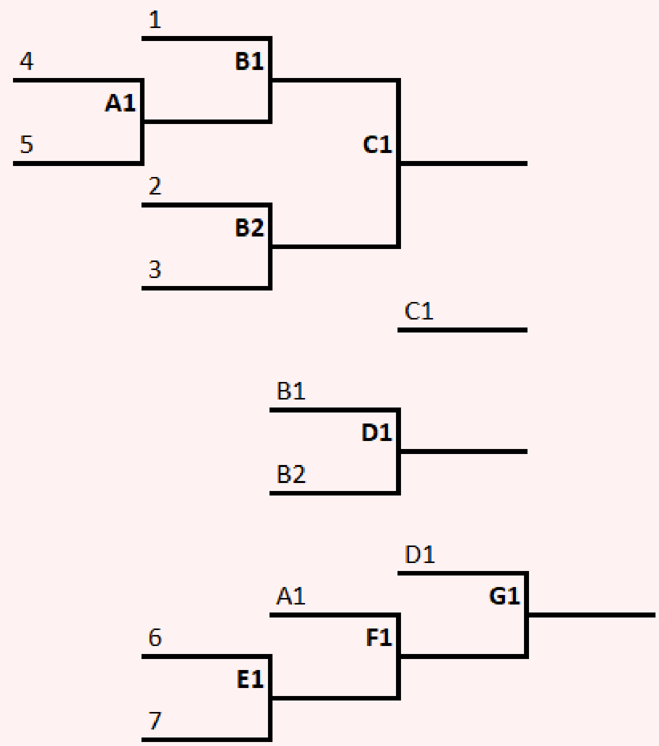


The only remaining choice is how to fill out the final semibracket. There are four labels available: **D1**, **A1**, 6 and 7. Which label should go where?

The central idea behind properness is that, if a format goes to chalk, you should never prefer to be a lower seed than a higher seed, or prefer to lose than to win. With this in mind, the proper seeding becomes clear. The loser of **D1** should get the double bye to the finals of the fourth bracket: if they didn't, then the 4- and 5-seeds might prefer to lose game **A1** rather than risk losing in games **B1** and **D1** and getting a worse starting line in the fourth-place bracket. Similarly, the loser of **A1** should get the single bye: if they didn't, then a team interested in a top-four finish might prefer to be the 6- or 7-seed to get a better spot in the fourth-place bracket, rather than the 4- or 5-seed and risk losing game **A1** and having to win three more games to claim fourth in the format.

Thus, the proper seeding of $[[\mathbf{2};\mathbf{3};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{0}]]$ is displayed below.

Figure 3.4.7: $[[2; 3; 0; 0]] \to [[1]] \to [[2; 0]] \to [[2; 1; 1; 0]]$



This example justifies why conditions (b) and (c) are what they are. We now formalize the analysis we just conducted to define properness on linear multibrackets.

Definition 3.4.8: Label Representing a Team

We say the label $\mathbf{L}$ represents the team t if either

(a) $\mathbf{L}$ is a seed and t is the $\mathbf{L}$-seed, or

(b) $\mathbf{L}$ is the name of a game and t lost in game $\mathbf{L}$.

Definition 3.4.9: Proper Linear Multibracket

A linear multibracket is proper, if, assuming teams representing higher-tiered labels always beat teams representing lower-tiered ones, then in every round of every semibracket it is better to be a team representing a higher-tiered label than a lower-tiered label, where:

(a) It is better to have already won a semibracket than to have not.

(b) It is better to be competing in a semibracket than to be available to a semibracket but not competing.

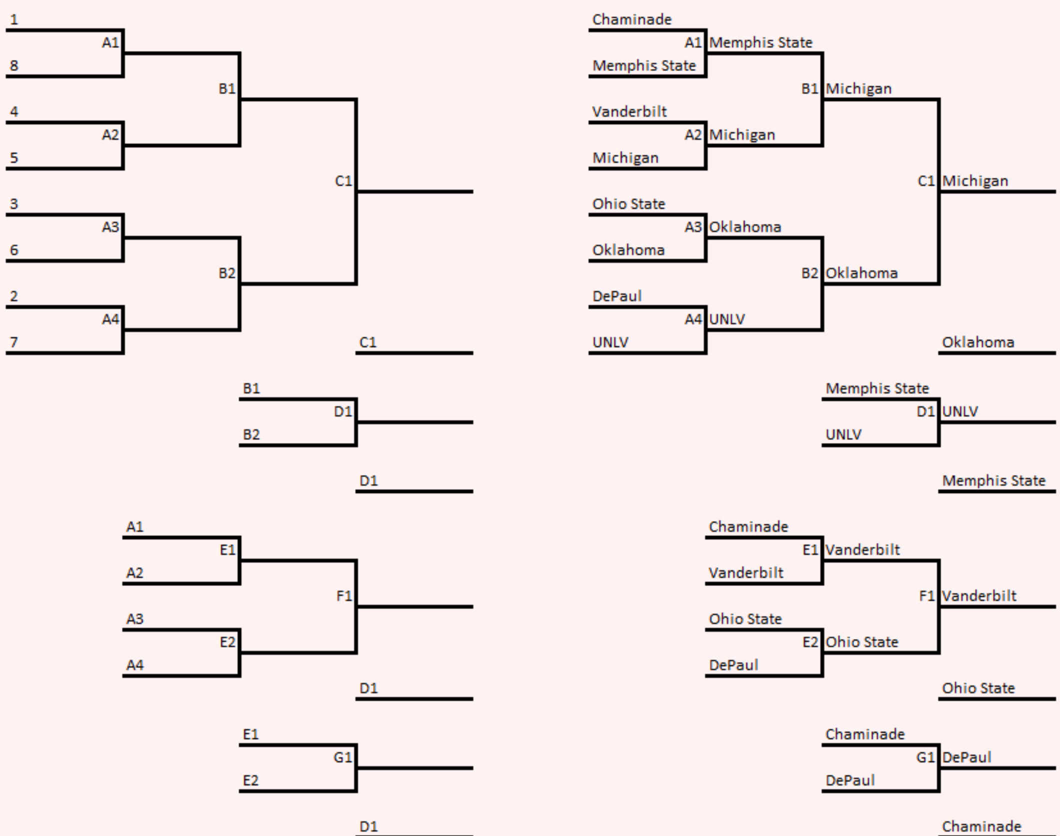
(c) It is better to have a bye than be playing a game.

(d) It is better to be playing the team representing a lower-tiered label than the team representing a higher-tiered one.

With signatures and properness defined, we can address the question posed last section: does the fundamental theorem apply to linear multibrackets? There are two ways to answer this question. The first is a cheap hack that shows the answer is no, and the second is a more thorough analysis that also shows the answer is no.

We begin with the cheap hack. Consider the 1988 Men's College Basketball Maui Invitational, which was a multibracket of signature $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{4}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]]$.

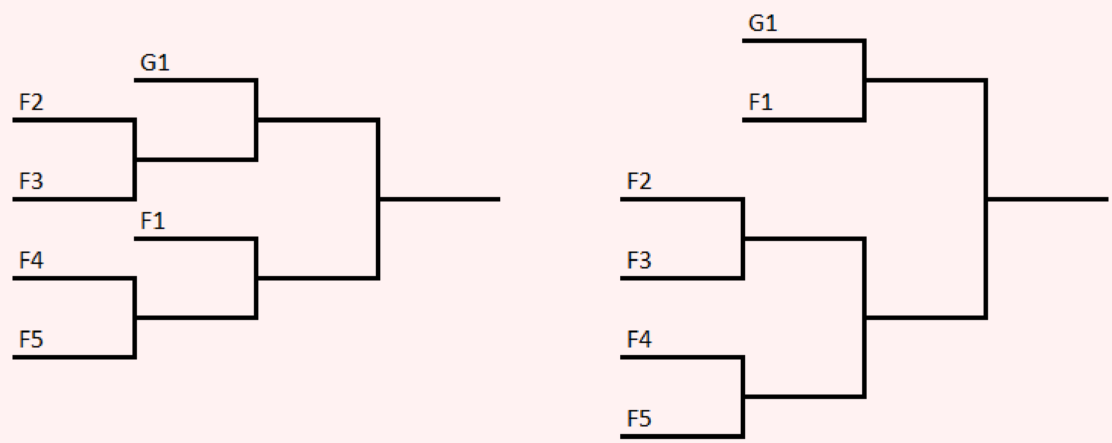
Figure 3.4.10: The 1998 Men's College Basketball Maui Invitational



The cheap idea that proves the fundamental theorem doesn't apply to linear multibrackets is that we can swap (say) **A2** and **A3** and the resulting multibracket has the same signature and is still proper. Why is this cheap? Because its easily patched over: it would still be a meaningful and important result for the fundamental theorem to be true up to rearranging labels in the same tier.

Unfortunately, this too is not the case. Consider a linear multibracket containing at some point a bracket of signature $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$, in which the six teams that are set to play in the bracket (by properness) are one team of a higher tier (which we will name **G1**), and five teams of a lower tier (which we will name **F1**, **F2**, **F3**, **F4**, and **F5**.) What might a proper instantiation of $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$ look like? In fact there are two.

Figure 3.4.11: Two Proper Instantiations of $[[4;2;0;0]]$ in a Linear Multi-bracket



Because linear multibracket properness doesn't require that teams in the same tier be treated equally, only that teams in higher tiers be treated better, both options in Figure 3.4.11 are proper. This issue is not fixable by adjusting the wording of the fundamental theorem. The two brackets are more than just a shuffling of same-tiered teams away from each other: they are of a different shape! Thus the fundamental theorem doesn't hold for linear multibrackets: we are left only with the existence half.

Theorem 3.4.12

There is at least one proper linear multibracket with each linear bracket signature.

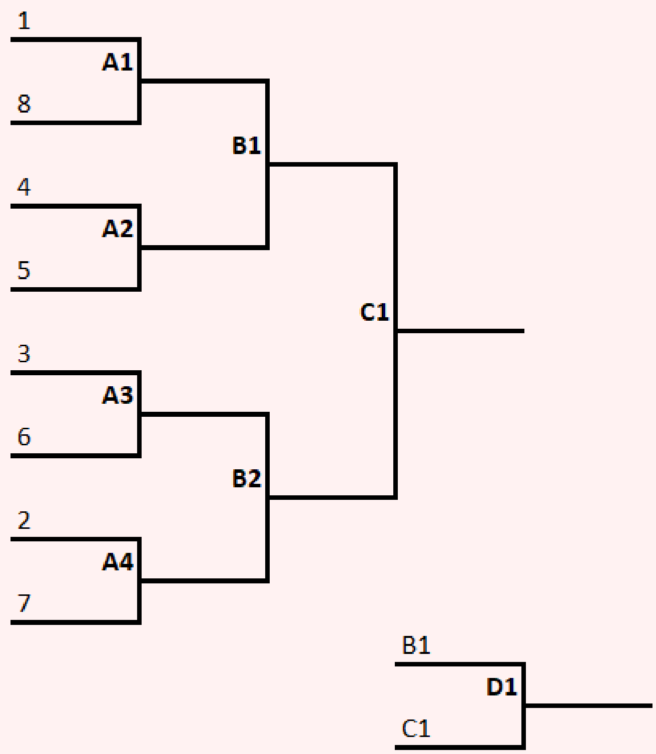
Proof. Let $\mathcal{A} = \mathcal{A}_1 \to \ldots \to \mathcal{A}_k$. We proceed by induction on k. For $k = 1$, $\mathcal{A} = \mathcal{A}_1$, so the proper semibracket of signature $\mathcal{A}_1$ suffices. For larger k, begin with the proper linear multibracket of signature $\mathcal{A}_1 \to \ldots \to \mathcal{A}_{k-1}$, and then add a semibracket to the end whose shape is the shape of the proper semibracket of signature $\mathcal{A}_k$, and whose seeding is derived by replacing the 1-seed with the highest-tiered remaining label, and then the 2-seed with the highest-tiered remaining label, etc. $\square$

It is reasonable to insist, however, not only that teams of higher tier should be treated better than teams of lower tiers, but also that teams of the same tier should be treated equally. What it means for two teams to be treated equally turns out to be a somewhat nuanced question that is treated in the next section.

3.5 Respectfulness

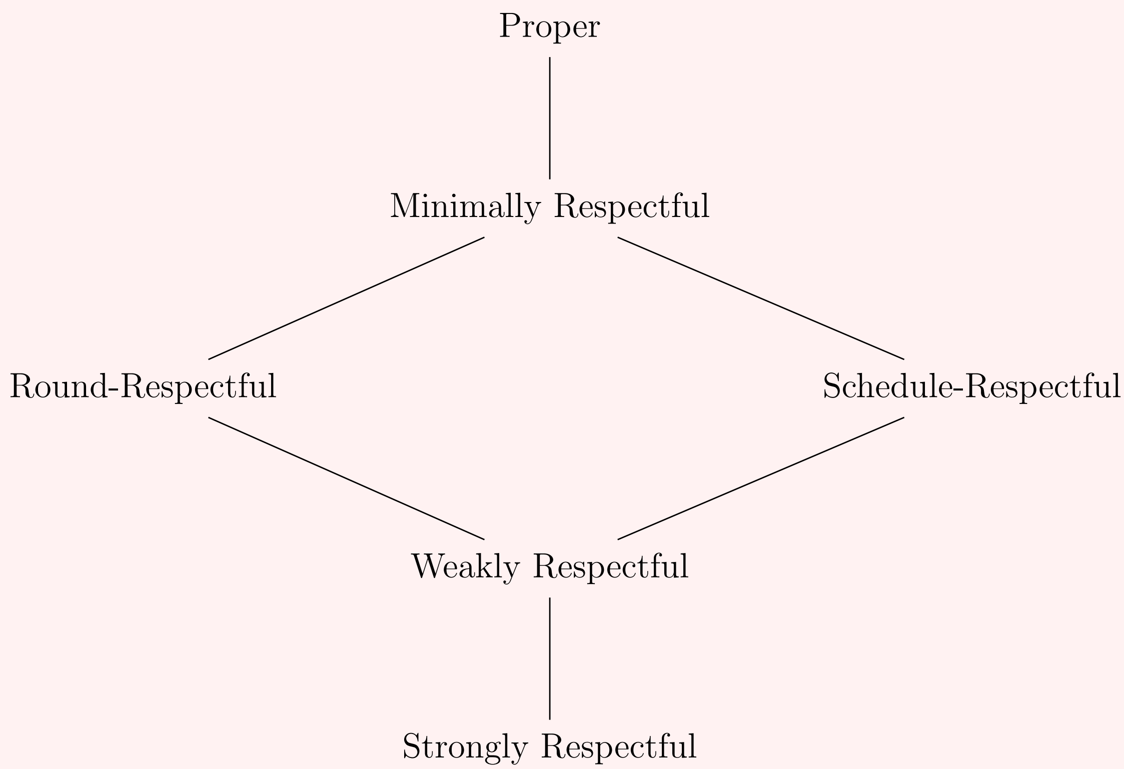
Properness as defined in the previous section has a weakness: it doesn't require that teams in the *same* tier be treated similarly, only that teams in higher tiers be treated better than teams in lower tiers. To illustrate this, consider the following proper linear multibracket of signature $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{2}; \mathbf{0}]]$.

Figure 3.5.1: $[[8; 0; 0; 0]] \to [[2; 0]]$



This linear multibracket is proper: the primary bracket is simply the proper seeding of $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$, and the teams that lost in the semifinals are given at least as good of spots as the teams that lost in the first rounds. But it still feels wrong. As we discussed at the end of the last chapter, properness only guarantees that teams of a higher tier are treated better than teams of a lower tier. But we also have an intuition that teams of the same tier ought to be treated the same. This intuition has a name: *respectfulness*. Unlike properness, respectfulness comes in a few different levels, the weakest of which is *minimal respectfulness*.

Figure 3.5.2: Respectfulness Properties (Weakest at the Top, Strongest at the Bottom)

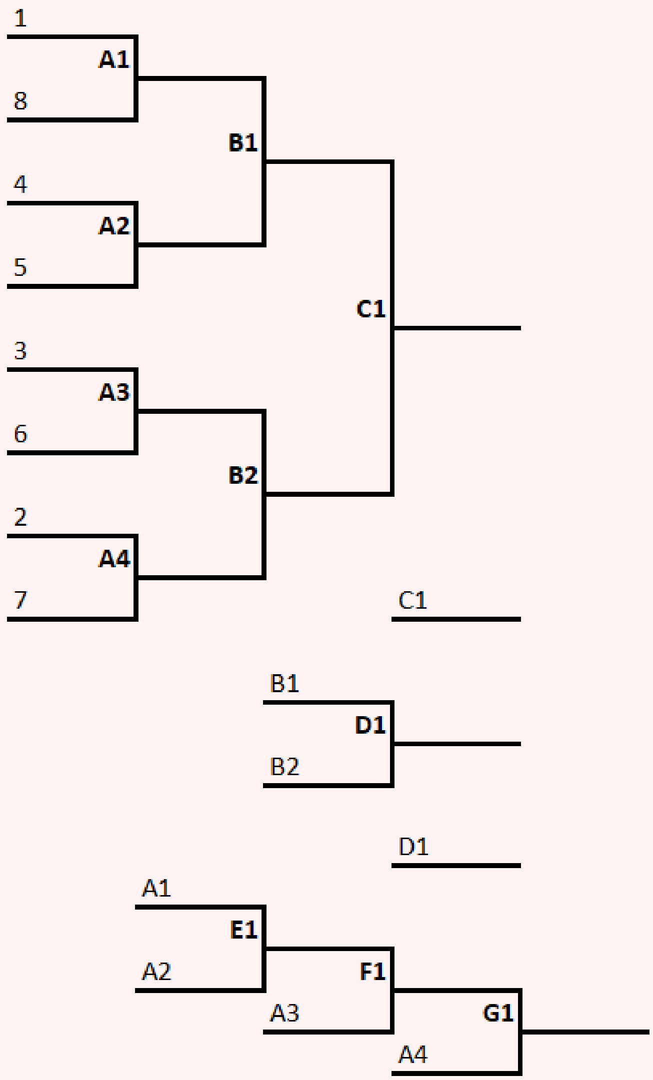


Definition 3.5.3: Minimally Respectful Linear Multibracket

A proper linear multibracket is *minimally respectful* if, for every round of every semibracket, all the losers of that round fall into the same other semibracket (or are all eliminated).

But minimal respectfulness as the name implies, is just the minimum. Consider, the following minimally respectful linear multibracket of signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{0}]]$.

Figure 3.5.4: $[[8;0;0;0]] \to [[1]] \to [[2;0]] \to [[1]] \to [[2;1;1;0]]$



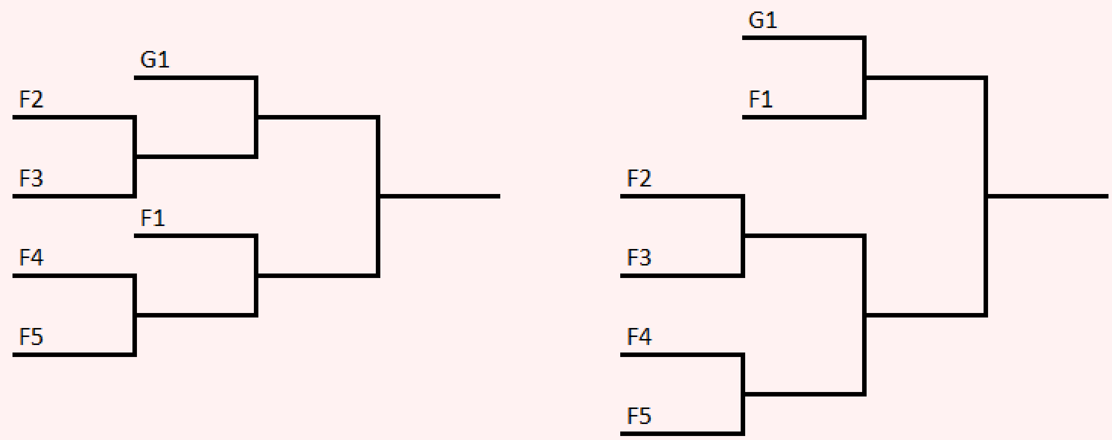
This linear multibracket is minimally respectful: every team that lost in the first round of the primary bracket falls in into the same semibracket. But still, teams that lost in the same round are not being treated the same: some first-round losers are getting more byes than others. It is not *round-respectful*.

Definition 3.5.5: Round-Respectful Linear Multibracket

A minimally respectful linear multibracket is *round-respectful* if, for every round of every semibracket, all the losers of that round fall into the same round of the same other semibracket (or are all eliminated).

The next level of respectfulness to examine is called schedule-respectfulness, and to understand it, we recall our example from last chapter that showed the fundamental theorem doesn't apply to linear multibrackets.

Figure 3.5.6: $[[4; 2; 0; 0]]$



Both of these options for a bracket of signature $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$ are proper, and in fact, both are minimally respectful while neither is round-respectful. But are they equally good? The right bracket seems a little bit more fair (that is, respectful). In the left bracket, **F1** lucks out, getting both a first-round bye and dodging the highest-tiered **G1** until the final game of the bracket. In the left bracket, however, the advantages are distributed: **F1** gets a first-round bye, but has the toughest second-round matchup, while the other **F** round losers each have to play an extra game but are on the opposite side of the bracket as **G1**. We say the right bracket, but not the left, is *schedule-respectful*

Definition 3.5.7: Schedule-Respectful Linear Multibracket

A *schedule-respectful* linear multibracket is a minimally respectful linear multibracket in which [[RIGOR IZE THIS DEFINITION]]

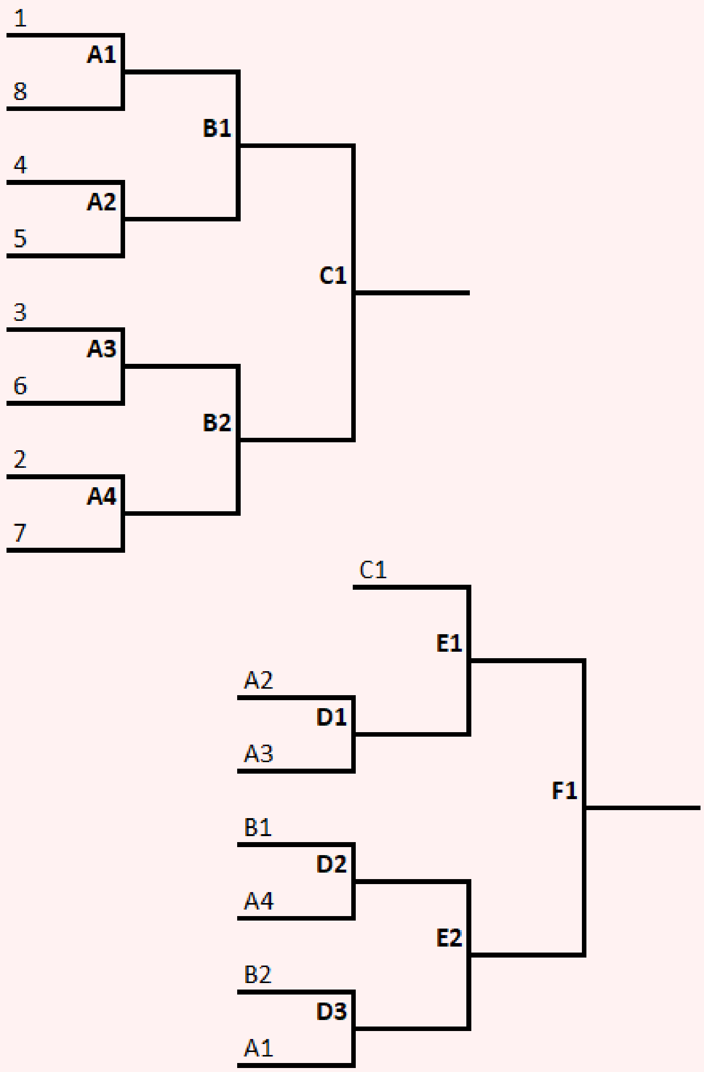
A given minimally respectful linear multibracket can be any of round-respectful, schedule-respectful, neither, or both. If a linear multibracket is both round- and schedule-respectful, we say it is *weakly respectful*.

Definition 3.5.8: Weakly Respectful Linear Multibracket

A linear multibracket is *weakly respectful* if it is both round-respectful and schedule-respectful.

The linear multibracket in Figure 3.5.9 is weakly respectful.

Figure 3.5.9: $[[8;0;0;0]] \to [[6;1;0;0]]$



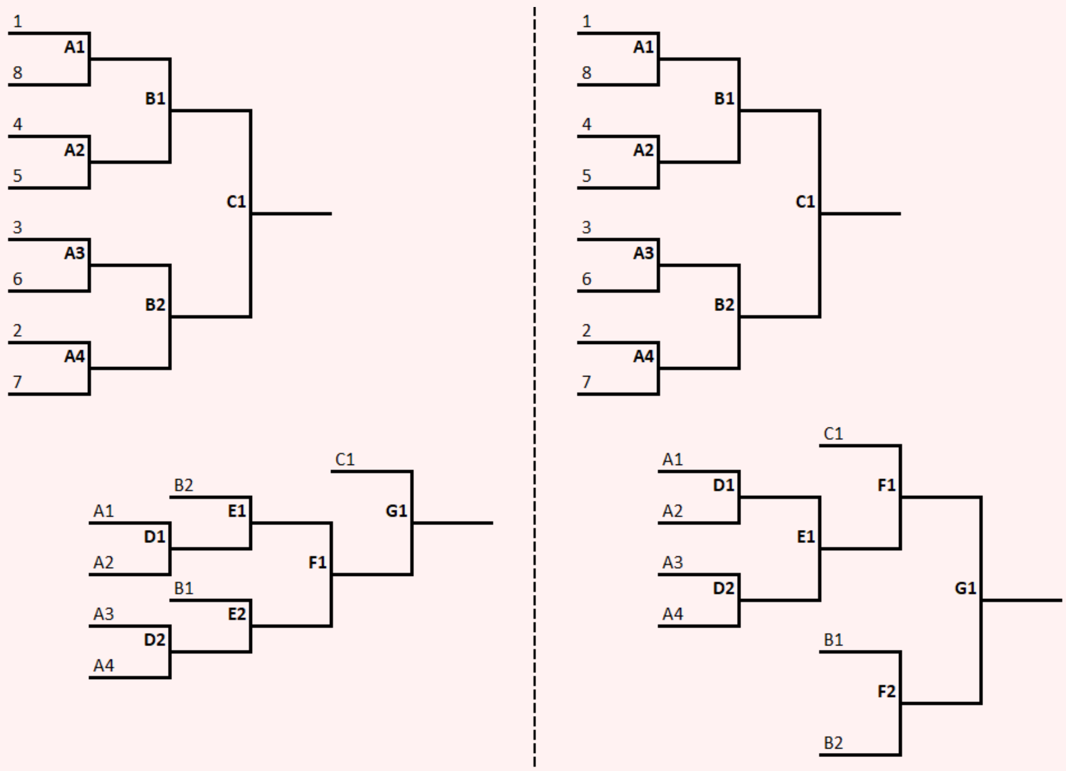
Finally, we can simply require that teams that lost in the same round of the a given semibracket be given symmetric spots in the semibracket they fall into.

Definition 3.5.10: Strongly Respectful Linear Multibracket

A *strongly respectful* linear multibracket is a minimally respectful one in which teams that lost in the same round of the primary bracket are given the same path in the linear multibracket.

Strong respectfulness is the gold standard of respectfulness in linear multibrackets. The linear multibracket in FIgure 3.5.9 is not strongly respectful, but both linear multibrackets displayed below are.

Figure 3.5.11: Two Strongly Respectful Linear Multibrackets

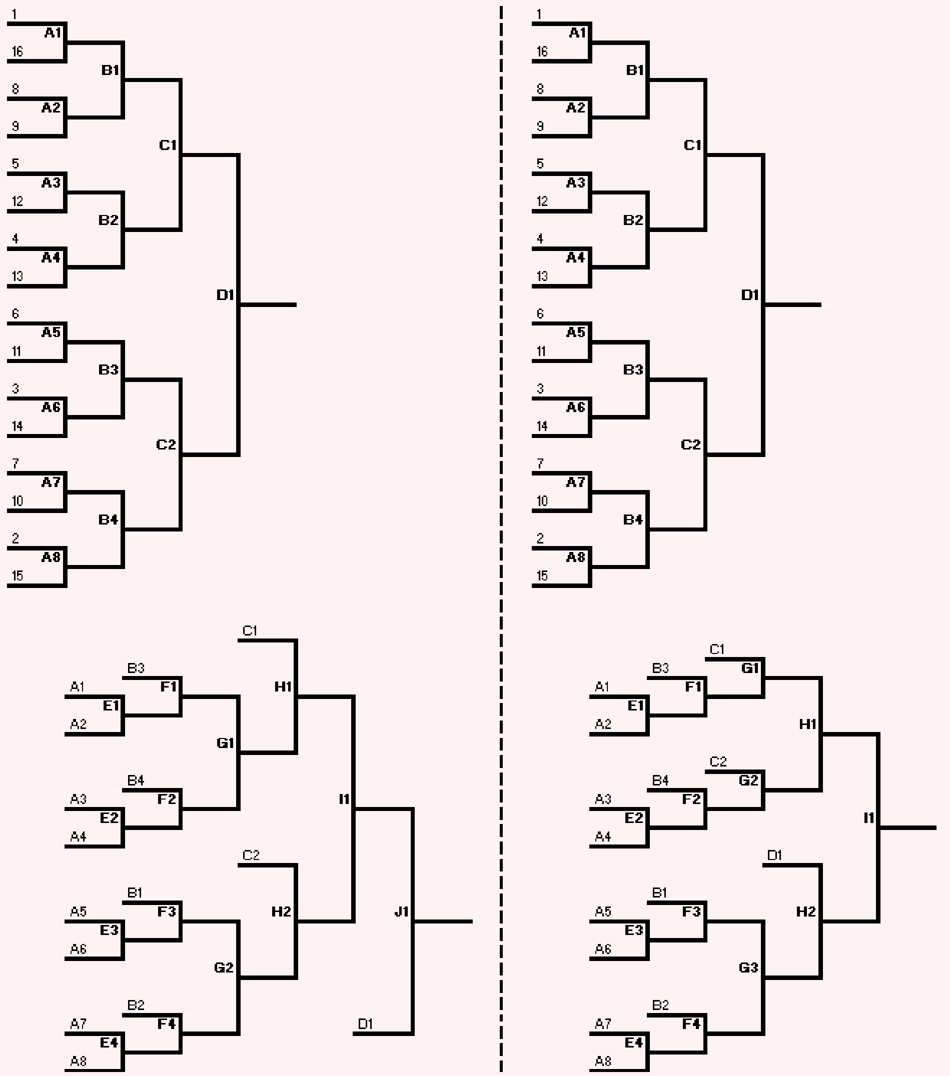


It's pretty rare to find linear multibrackets that aren't strongly respectful in use in real life, and for good reason: it is the maximum level of fairness. But despite this, there are compelling reasons to use less respectful formats in certain cases.

One reason is if there simply aren't more respectful ones available. Imagine you are tasked with designing a second-place bracket for a linear bracket whose primary bracket is $[[\mathbf{6}; \mathbf{1}; \mathbf{0}; \mathbf{0}]]$, and it's important for every team to get a chance at second-place even if they lost in the first round of the primary bracket. There is no strongly respectful bracket that meets these criteria: in fact the best we can do is minimally respectful.

In other cases, weakly respectful formats can allow for faster formats than their strongly respectful counterparts. Compare the following two linear multibrackets, for example.

Figure 3.5.12: $[[\mathbf{16};\mathbf{0};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{8};\mathbf{4};\mathbf{0};\mathbf{2};\mathbf{0};\mathbf{1};\mathbf{0}]]$ and $[[\mathbf{16};\mathbf{0};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{8};\mathbf{4};\mathbf{2};\mathbf{1};\mathbf{0};\mathbf{0}]]$



Both formats begin with a primary bracket of signature $[[\mathbf{16};\mathbf{0};\mathbf{0};\mathbf{0};\mathbf{0}]]$ and then look to select a second-place finisher without eliminating any teams from second-place contention without a second loss. The format on the left is strongly respectful, but a team could be forced to play seven games over the course of the tournament. The format on the right, on the other hand, doesn't ask any team to play more than six games, but it is only weakly respectful. Despite this, Dabney [5] found that under certain assumptions, the right format is actually more accurate at selecting its top two teams.

Overall, various tournament design problems might call for different levels of respectfulness: having access to the range of descriptors in Figure 3.5.2 equips one to answer the various tournament design problems presented to them.

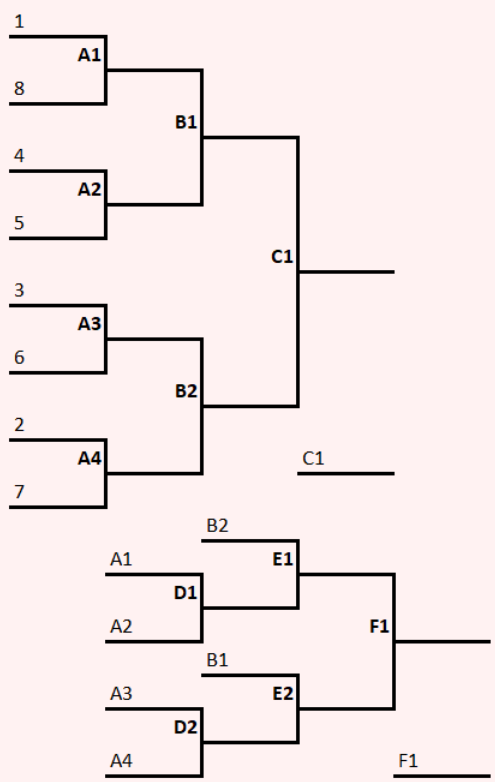
3.6 Efficient Linear Multibrackets

In the past few sections, we have looked at multibrackets (and in particular linear multibrackets) as a solution to the tournament design question of how to crown a champion as well as give out certain consolation places.

We now consider a slightly different tournament design problem: we no longer care about which teams finish in first or any other specific place, only about which teams finish in the top-m for a particular m. This is a problem commonly faced at regional tournaments in which the top-m teams qualify for a national tournament: the ranking of the teams within the region aren't relevant, only which teams are above and below the cutoff.

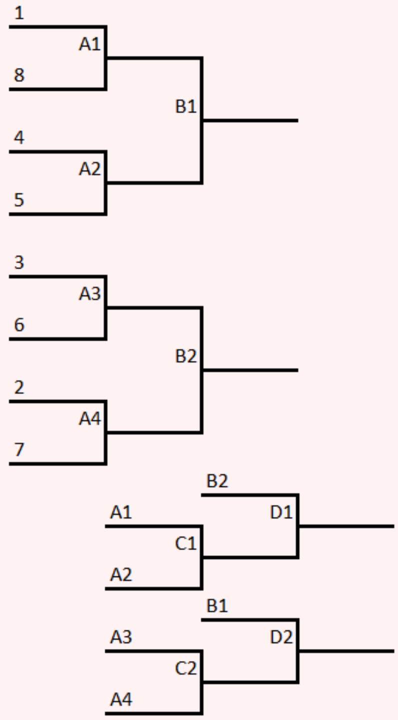
Recall the format used in the 2023 Southern Conference Wrestling Championships.

Figure 3.6.1: 2023 Southern Conference Wrestling Championships



If we were only interested in the top four teams, rather than the rank of the team within those top four slots, games **C1** and **F1** become unnecessary: no matter what the results of those games are, the top four teams are the same. A more efficient format would leave those games unplayed, resulting in the following format.

Figure 3.6.2: An Efficient Format for Selecting a Top Four



Instead of being composed of four traditional brackets, the format in Figure 3.6.2 is composed of two semibrackets each of which have rank two: one with the **A** and **B** round games, and one with the **C** and **D** round games. And, as desired, there no games played between two teams such that both the winner and loser of each of those games are guaranteed to finish in the top four.

This format has signature $[[\mathbf{8};\mathbf{0};\mathbf{0}]]_2 \to [[\mathbf{4};\mathbf{2}]]_2$, and we say that it is *weakly efficient*.

Definition 3.6.3: Weakly Efficient

A linear multibracket is *weakly efficient* if there are no games played within it such that both the winner and loser of that game are guaranteed to be ranked by the format.

Identifying whether a proper linear multibracket is weakly efficient can be done just by looking at its signature.

Theorem 3.6.4

A proper linear multibracket with signature $\mathcal{A} = \mathcal{A}_1 \to ... \to \mathcal{A}_k$ is weakly efficient if and only if there is some integer j with $1 \leq j \leq k$ such that every semibracket $\mathcal{A}_i$ with $i < j$ is trivial and every semibracket $\mathcal{A}_i$ with $i > j$ is competitive.

Proof. Assume first that some j exists. Let $\mathbf{G}$ be a game. Because all semibrackets $\mathcal{A}_i$ with $i < j$ are trivial, $\mathbf{G}$ must be in a semibracket $\mathcal{A}_i$ for $i \geq j$, so the loser of $\mathbf{G}$ is either eliminated outright, or falls into a semibracket $\mathcal{A}_i$ for $i > j$, which is competitive, in which case they will play another game. If they continue losing, they will continue falling into competitive semibrackets, until they are eliminated outright and do not get ranked. Thus if the loser of $\mathbb{P}[$ beats $]$loses the rest of their games they will not get ranked, and so $\mathcal{A}$ is weakly efficient.

Assume now that no such j exists. Thus there must be some i such that $\mathcal{A}_i$ is nontrivial and $\mathcal{A}_{i+1}$ is noncompetitive. $\mathcal{A}_{i+1}$ is noncompetitive, so at least one team must win $\mathcal{A}_{i+1}$ without playing a game. Because $\mathcal{A}$ is proper, this team must be a championship game loser of $\mathcal{A}_i$, if such a game exists. $\mathcal{A}_i$ is nontrivial so such a game does indeed exist: let $\mathbf{G}$ be that game. Then the winner of $\mathbf{G}$ wins $\mathcal{A}_i$, and the loser of $\mathbf{G}$ wins $\mathcal{A}_{i+1}$, so $\mathcal{A}$ is not weakly efficient. □

The USA Ultimate Manual of Championship Series Tournament Formats [15], which is used to determine the formats to be used at the various sectional and regional tournaments in the sport of ultimate frisbee, contains a host of weakly efficient linear multibrackets for selecting the top m teams out of a list of n for various values of m and n, after a "regular season" portion of the tournament has been played to establish seeds.

A couple examples are Figure 3.6.5, which selects a top six out of seven, and 3.6.6, which selects a top five out of sixteen in at most five games per team. (In reality, sometimes additional games are played to determine placements within the top-m, but we display only the weakly efficient part of the format here.)

Figure 3.6.5: $[[\mathbf{1}]] \to [[\mathbf{1}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{0}]]_2 \to [[\mathbf{2};\mathbf{0}]]$

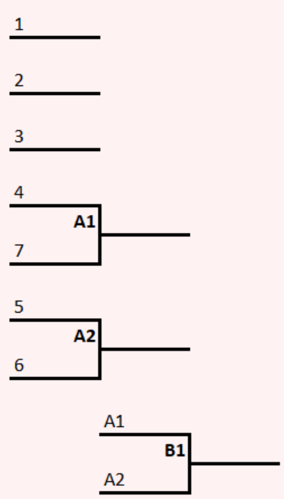
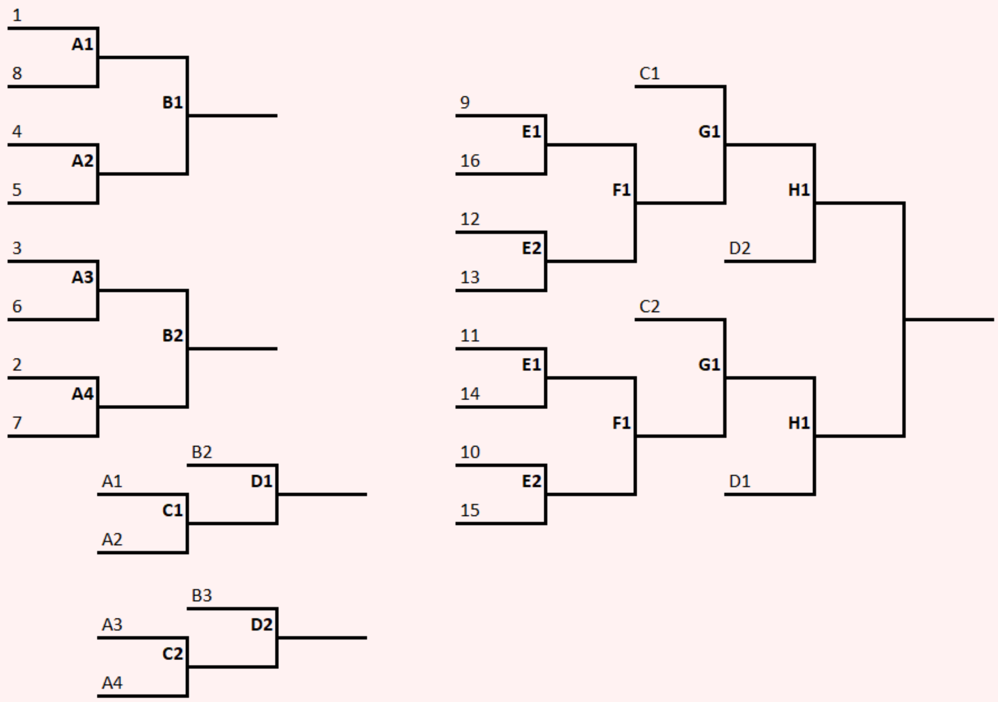


Figure 3.6.6: $[[8;0;0]]_2 \to [[4;2;0]]_2 \to [[8;0;2;2;0;0]]$



We note two things about the notion of weak efficiency presented above. First, Theorem 3.6.4 implies that a weakly efficient multibracket can begin with a long string of trivial semibrackets before the nontrivial ones begin. While this is sufficient for avoiding playing unnecessary games, it does not completely remove unnecessary semibrackets: the set of leading trivial semibrackets

$$[[\mathbf{m_1}]]_{m_1} \to \dots \to [[\mathbf{m_j}]]_{m_j}$$

of a weakly efficient multibracket can be combined into a single trivial semibracket

$$[[\mathbf{m_1} + \dots + \mathbf{m_j}]]_{(m_1+\dots+m_j)}$$

without affecting which teams end up ranked. Applying this to the format in Figure 3.6.5 yields a signature of

$$[[\mathbf{3}]]_3 \to [[\mathbf{4};\mathbf{0}]]_2 \to [[\mathbf{2};\mathbf{0}]].$$

In fact, if there is at least one game played in a weakly efficient multibracket, trivial semibrackets can removed entirely, converting a multibracket of signature

$$[[\mathbf{m_1}]]_{m_1} \to [[\mathbf{a_1};\dots;\mathbf{a_r}]]_{m_2} \to \dots \to \mathcal{A}_k$$

into one of signature

$$[[\mathbf{a_1};\dots;\mathbf{a_r} + \mathbf{m_1}]]_{m_1+m_2} \to \dots \to \mathcal{A}_k.$$

Applying this to the format in Figure 3.6.5 yields a signature of

$$[[\mathbf{4};\mathbf{3}]]_5 \to [[\mathbf{2};\mathbf{0}]].$$

To patch this, we strengthen the notion of weak efficiency into just *efficiency*.

Definition 3.6.7: Efficient

We say a proper linear multibracket is *efficient* if one of three conditions hold:

(a) It is a single trivial semibracket.

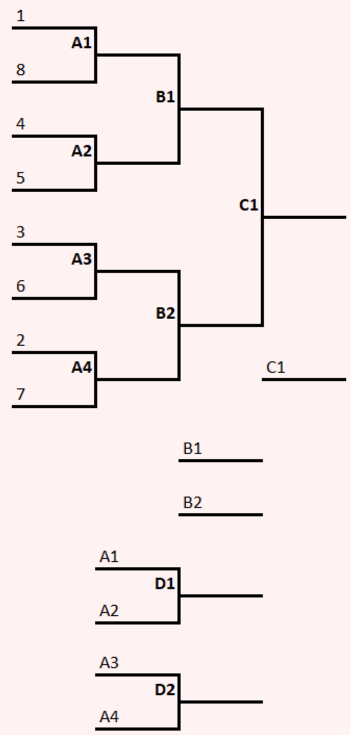
(b) It is a sequence of competitive semibrackets.

(c) It is a single nontrivial noncompetitive semibracket followed by a sequence of competitive semibrackets.

Theorem 3.6.4 says that in each of these three cases no games are played between teams guaranteed to be ranked, and the process detailed above can reduce any weakly efficient signature into a signature that takes one of those three forms.

The second thing to note is that efficiency makes a lot of sense if we are only interested in the top-m teams (where m is the sum of the ranks of the semibrackets in our format) and not in the rankings of the teams within them. But sometimes we might be interested in the intermediate rankings as well. For example, let's say we want to design an eight-team tournament format in which the top team receives the grand prize, second-place receives a second-place prize, while the third- through sixth-place each get equivalent consolation prizes, and seventh and eighth each get nothing. While not efficient (or even weakly efficient), the following format assigns the desired places without playing any games between teams that are guaranteed to receive the same prize.

Figure 3.6.8: $[[8;0;0;0]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2}]]_4$



To account for this, we introduce the notion of a *prize structure*.

Definition 3.6.9: Prize Structure

A *prize structure* $\mathcal{P}$ is a sequence $(\mathbf{p_1}, ..., \mathbf{p_m})$ indicating that the top p_1 teams in a format receive some prize, the next p_2 receive some smaller prize, etc. Any teams finishing in place $1 + \sum_{i=1}^{m} p_i$ or worse receive no prize.

Then,

Definition 3.6.10: Efficient with Respect to a Prize Struture

We say a proper linear multibracket $\mathcal{A} = \mathcal{A}_1 \to ... \to \mathcal{A}_k$ is *efficient with respect to a prize structure* $\mathcal{P} = (\mathbf{p_1}, ..., \mathbf{p_m})$ if

- $\mathcal{A}_j$ being noncompetitive implies that for some $\ell < m$,

$$\sum_{i=1}^{j-1} \text{Rank}(\mathcal{A}_j) = \sum_{i=1}^{\ell} p_i,$$

and

- $\mathcal{A}_j$ being trivial implies that for some $\ell < m$,

$$\sum_{i=1}^{j} \mathrm{Rank}(\mathcal{A}_j) = \sum_{i=1}^{\ell} p_i.$$

(The first condition ensures that the format is weakly efficient with respect to the prize structure, and the second condition ensures that there are no trivial semibrackets that could be combined with another semibracket as per the process detailed before Definition 3.6.7.)

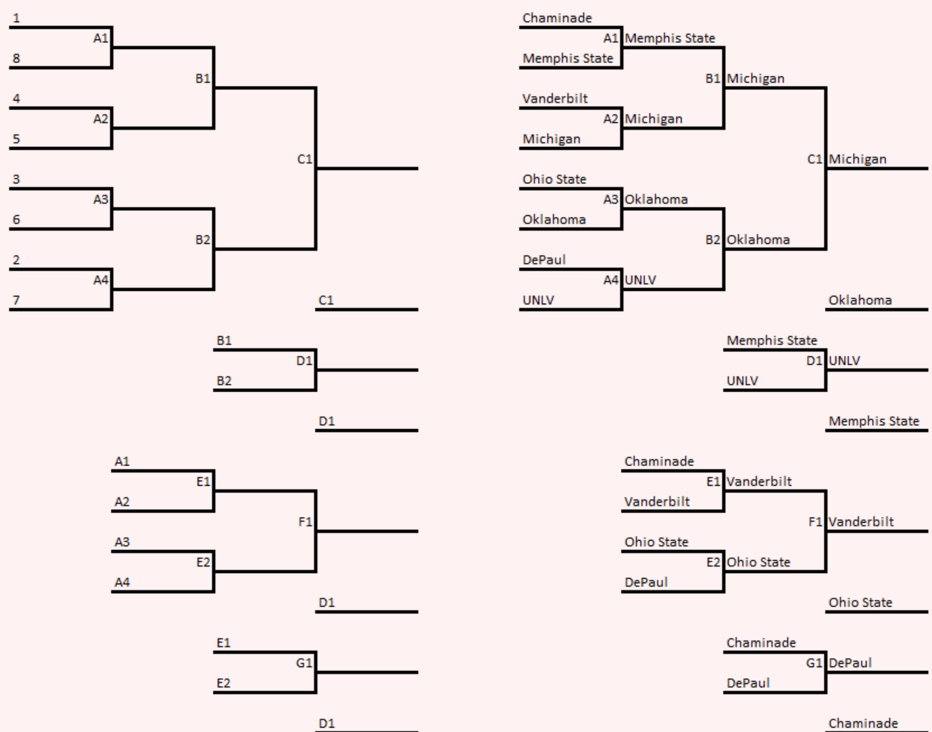
So the proper linear multibracket $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2}]]_4$ is efficient with respect to the prize structure $(\mathbf{1},\mathbf{1},\mathbf{4})$. A linear multibracket being efficient is the same as it being efficient with respect the prize structure $(\mathbf{m})$, where m is the sum of the ranks of its semibrackets.

Efficient multibrackets are great tournament designs for tournaments whose primary goal is to select the top m teams to move on to the next stage of the competitions, as discussed in the beginning of this section. They do so excitingly, with each spot in the top-m being awarded as the winner of a particular game; efficiently, with no games being played between teams who will receive the same prize; and fairly, as the multibracket rules ensure that winning is always better than losing. It is not surprising that many sports with regional tournaments that qualify teams for a national one use such formats.

3.7 Swiss Systems

Consider the 1998 Men's College Basketball Maui Invitational, which used a proper linear multibracket of signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]]$.

Figure 3.7.1: The 1998 Men's College Basketball Maui Invitational



The format used in the Maui Invitational has several nice properties. First, it is strongly respectful. Secondly, unlike some of the efficient linear multibrackets we saw last section, every team starts in the primary bracket, ensuring that if a team wins every game they will win the tournament. Third, the sum of the ranks of its semibrackets sum to the number of participating teams, ensuring that every team ends up ranked. Fourth, every bracket is either trivial or competitive, ensuring that the final rankings are as granular as possible.

Fifth, every team plays the same number of games, in this case, three. And sixth, every game is between teams with the same record, hopefully leading to evenly matched and exciting games. Linear multibracket with all six of these properties are called *swiss formats*, named because of their first recorded use at a chess tournament in Zürich, Switzerland in 1895.

Definition 3.7.2: Swiss Formats

A *swiss format* is a minimally respectful linear multibracket with the following five properties.

- Every team starts in the primary semibracket.
- Every team wins a semibracket.
- Every semibracket is either trivial or competitive.
- Every team plays the same number of games.
- Every game is between two teams with the same record.

Note that even though the Maui Invitational was strongly respectful, we only require minimal respectfulness: this allows for a substantially larger space of swiss formats while still maintaining the important properties of the class of formats.

Definition 3.7.3: r-Round Swiss

We say a swiss format in which each team plays r games is an r-round swiss format.

Thus the 1998 Men's College Basketball Maui Invitational was an 8-team 3-round swiss format.

Often times, we will interested in just the signature of a swiss format, rather than the specific details of the entire format.

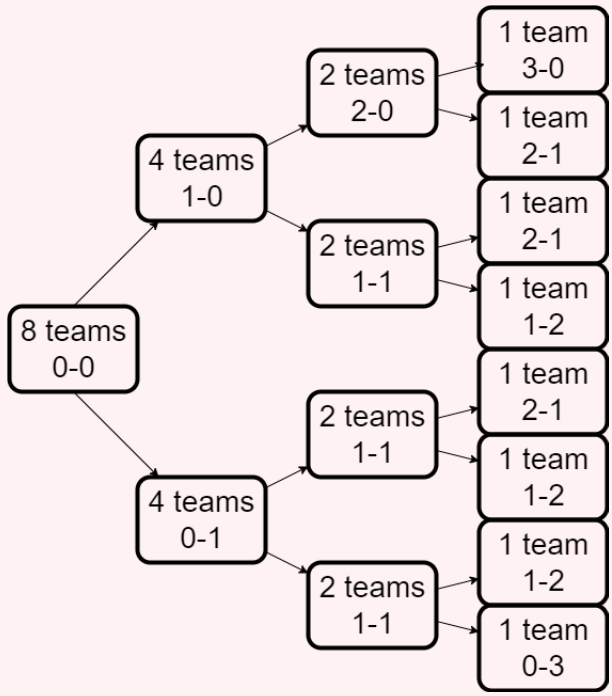
Definition 3.7.4: Swiss Signature

A *swiss signature* is a linear multibracket signature that admits a swiss format.

Thus $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]]$ is a swiss signature.

Without a close inspection, it can be difficult to see from the diagram, and certainly from the signature, that the Maui Invitational meets all the requirements of a swiss system: the last one in particular is tricky to confirm. But there is another diagram that we can use to depict swiss signatures in much more intuitive way: flowcharts. Figure 3.7.5 depicts the flow chart for the 1998 Men's College Basketball Maui Invitational.

Figure 3.7.5: The Maui Invitational Flowchart



The signature used in the Maui Invitational is a particular example of a family of swiss signature known as the the *standard swiss signatures*, which we abbreviate by $\mathcal{S}_r$ for some r.

Definition 3.7.6: Standard Swiss Signature ($\mathcal{S}_r$)

$\mathcal{S}_r$, or the *standard r-round swiss signature*, is the multibracket signature defined recursively by

$$\mathcal{S}_0 = [[\mathbf{1}]],$$

and

$$\mathcal{S}_r = [[\mathbf{2^r}; ...; \mathbf{0}]] \to \mathcal{S}_0 \to \mathcal{S}_1 \to ... \to \mathcal{S}_i \to ... \to \mathcal{S}_{r-1}.$$

Thus we have

$$\begin{aligned}
\mathcal{S}_0 &= [[\mathbf{1}]] \\
\mathcal{S}_1 &= [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]] \\
\mathcal{S}_2 &= [[\mathbf{4}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]] \\
\mathcal{S}_3 &= [[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]]
\end{aligned}$$

Intuitively, you can think of the $\mathcal{S}_r$ as a 2^r team tournament, where, after the first round of games, the winners and losers each go off and play through separate instances of $\mathcal{S}_{r-1}$.

Figures 3.7.7 and 3.7.8 display $\mathcal{S}_0, \mathcal{S}_1$, and $\mathcal{S}_2$ as a linear multibracket and as a flowchart, while the 1998 Men's College Basketball Maui Invitational was an instance of the standard swiss signature $\mathcal{S}_3$.

Figure 3.7.7: $\mathcal{S}_0, \mathcal{S}_1$, **and** $\mathcal{S}_2$

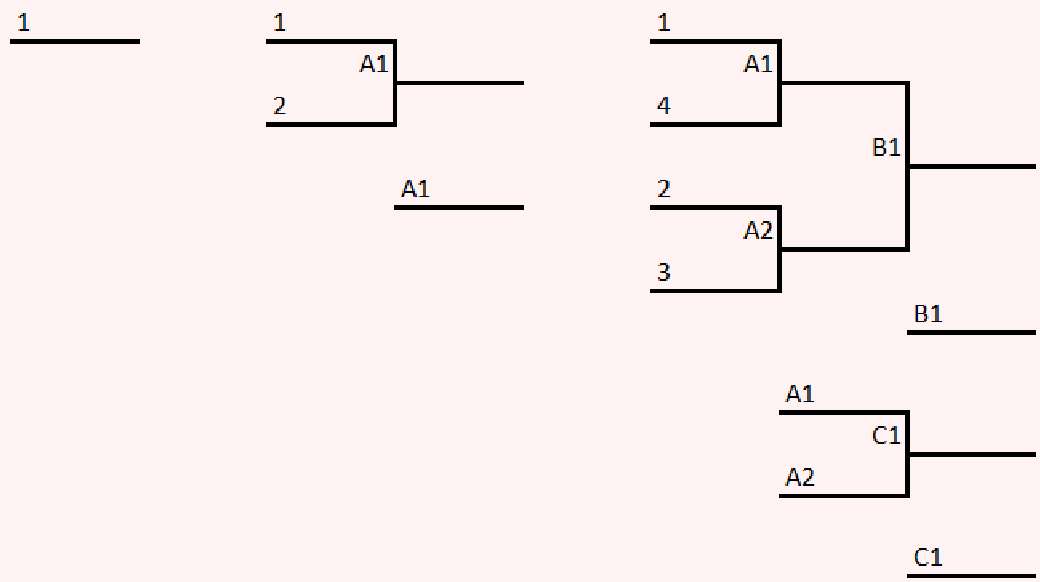
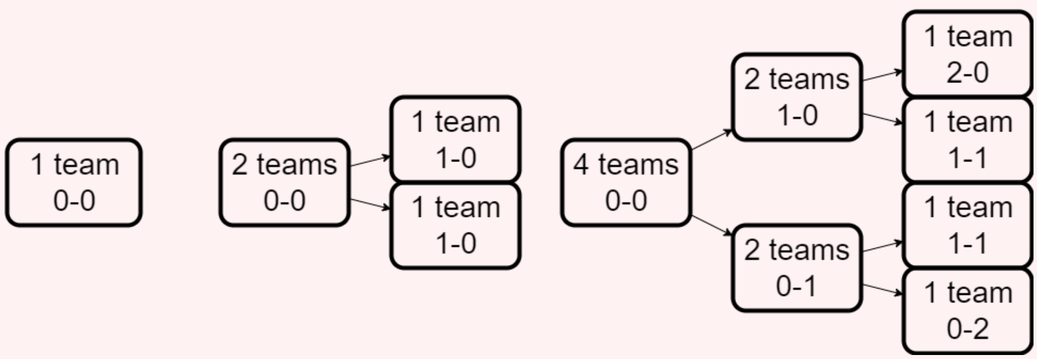


Figure 3.7.8: $\mathcal{S}_0, \mathcal{S}_1$, **and** $\mathcal{S}_2$



The standard swiss signatures are particularly nice: in addition to the other swiss format requirements, the primary semibracket of a standard swiss format has rank one, so a single champion is crowned. Not every swiss format has this property: consider, for example, the following 8-team swiss format.

Figure 3.7.9: $[[8;0;0]]_2 \to [[2]]_2 \to [[4;0]]_2 \to [[2]]_2$

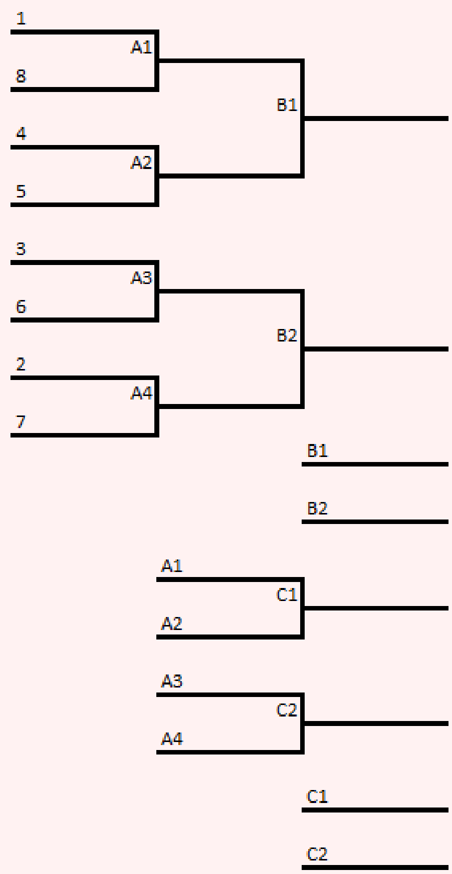
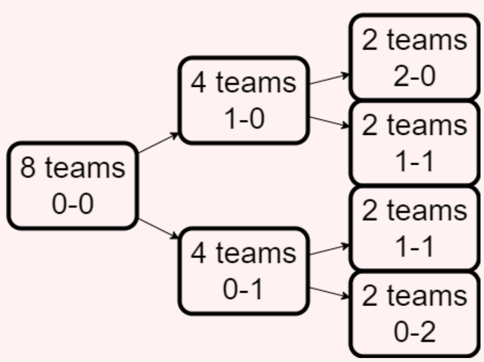


Figure 3.7.10: $[[8;0;0]]_2 \to [[2]]_2 \to [[4;0]]_2 \to [[2]]_2$



The format in Figure 3.7.9 is a swiss format, but it doesn't crown an individual champion, as two teams end the format undefeated. The swiss signature $[[\mathbf{8};\mathbf{0};\mathbf{0}]]_2 \to [[\mathbf{2}]]_2 \to [[\mathbf{4};\mathbf{0}]]_2 \to [[\mathbf{2}]]_2$ is not *compact*.

Definition 3.7.11: Compact

We say a swiss signature is *compact* if its primary semibracket has rank one.

We can think of a non-compact swiss signature as multiple copies of another, smaller swiss signature running in parallel. For example, the format in Figures 3.7.9 and 3.7.10 can be viewed as two independent instances of the format $\mathcal{S}_2$ being played side by side. We use this to introduce mx notation.

Definition 3.7.12: $m\text{x}\mathcal{A}$

If $m \in \mathbb{N}$ and $\mathcal{A}$ is a multibracket signature, then $m\text{x}\mathcal{A}$ is the multibracket signature formed by multiplying every number in every signature in $\mathcal{A}$ by m.

So $[[\mathbf{8};\mathbf{0};\mathbf{0}]]_2 \to [[\mathbf{2}]]_2 \to [[\mathbf{4};\mathbf{0}]]_2 \to [[\mathbf{2}]]_2 = 2\text{x}\mathcal{S}_2$.

With the standard swiss signature and mx notation defined, we are ready for Figure 3.7.13, which details the various swiss signatures for 1-, 2-, 4-, and 8-teams.

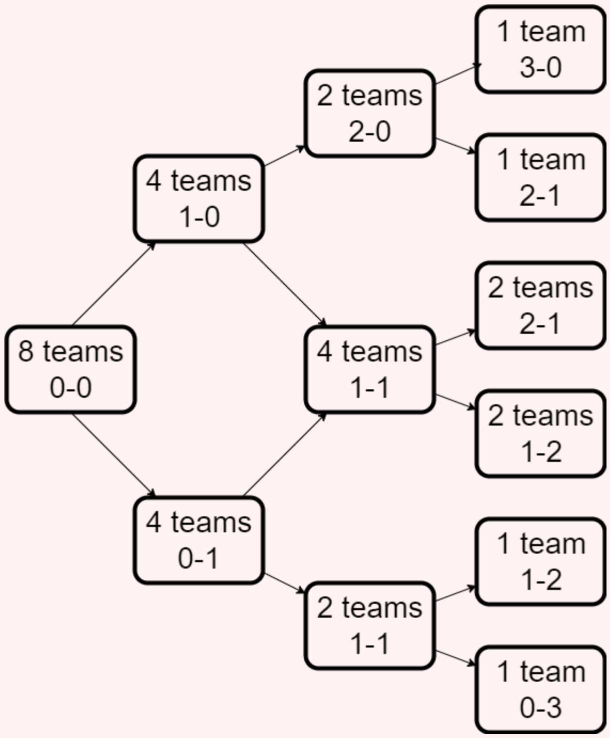
Figure 3.7.13: The 1-, 2-, 4-, and 8-team Swiss Signatures

	1 Team	2 Teams	4 Teams	8 Teams
0 Rounds	$\mathcal{S}_0$	$2\text{x}\mathcal{S}_0$	$4\text{x}\mathcal{S}_0$	$8\text{x}\mathcal{S}_0$
1 Round		$\mathcal{S}_1$	$2\text{x}\mathcal{S}_1$	$4\text{x}\mathcal{S}_1$
2 Rounds			$\mathcal{S}_2$	$2\text{x}\mathcal{S}_2$
3 Rounds				$\mathcal{S}_3, \mathcal{T}_3$

The swiss signatures on the diagonal of Figure 3.7.13 are the compact ones. While standard swiss signature and mx notation are sufficient for explaining almost every signature in Figure 3.7.13, there is a second 8-team 3-round swiss signature, $\mathcal{T}_3$, that we have yet to define. It's worth attempting to construct $\mathcal{T}_3$ before reading on.

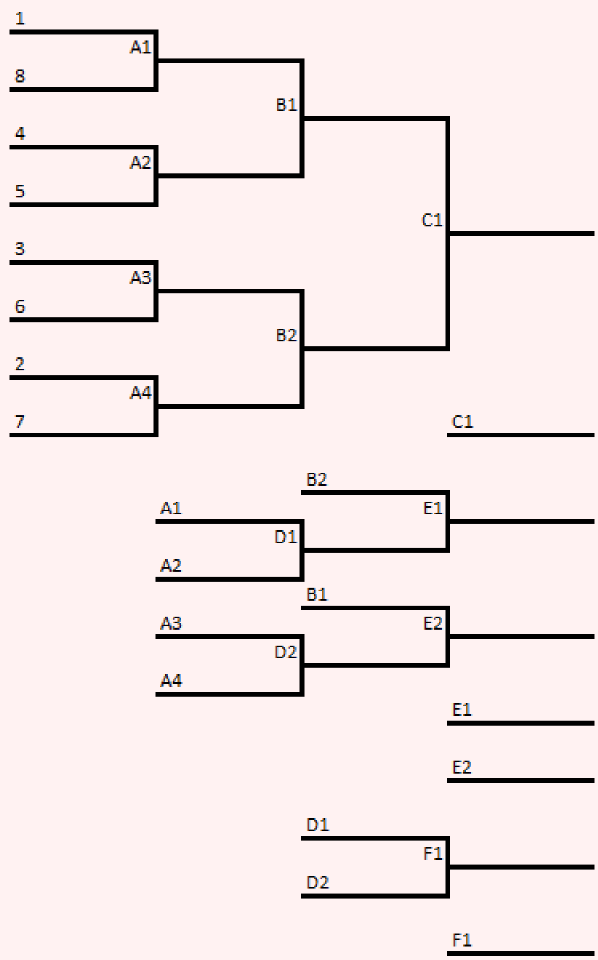
The key insight is to realize that teams with the same record in vertically adjacent cells of the flowchart can actually play against each other without violating any of the swiss format requirements, merging the cells. Thus the flow chart for $\mathcal{T}_3$ looks like so.

Figure 3.7.14: $\mathcal{T}_3$



We can use the flowchart to reconstruct the bracket and signature.

Figure 3.7.15: $\mathcal{T}_3$



$\mathcal{T}_3 = [[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4}; \mathbf{2}; \mathbf{0}]]_2 \to [[\mathbf{2}]]_2 \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]]$. $\mathcal{T}_3$ is also very similar to the format used in Figure 3.1.6 by the 2023 Southern Conference Wrestling Championships: both use a primary eight-team balanced bracket and let their first-round losers fight their way back for a top-half finish.

Theorem 3.7.16

$\mathcal{S}_3$ and $\mathcal{T}_3$ are the only compact 3-round swiss signatures.

Proof. Any compact 3-round swiss signature must begin with $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]]$. Now let $\mathcal{A}$ be the semibracket that first-round primary brackets losers fall into. $\mathcal{A}$ must have two rounds, and the first-round primary bracket losers must all get no byes (otherwise they would not play the requisite three games). Thus $\mathcal{A} = [[\mathbf{4}; \mathbf{a_1}; \mathbf{0}]]_{(a_1/2+1)}$ for some a_1. As neither of the two semifinal winners can fall into $\mathcal{A}$, $a_1 \leq 2$. Additionally, if $a_1 = 1$, $\mathcal{A}$ would not be a signature. Thus, $a_1 = 0$ or 2.

If $a_1 = 0$, then in between the first two brackets and $\mathcal{A}$, we must have two more brackets for the second-round losers of the primary bracket: $[[\mathbf{2};\mathbf{0}]]$ and $[[\mathbf{1}]]$. Then $\mathcal{A}$ must be followed by $[[\mathbf{1}]]$ for the loser of its championship game, and then $[[\mathbf{2};\mathbf{0}]]$ and $[[\mathbf{1}]]$ so that the last two teams get a third game. This is the swiss signature $\mathcal{S}_3$.

If $a_1 = 2$, then the losers of the two championship games of $\mathcal{A}$ have already played all three of their games and so need to fall into the bracket $[[\mathbf{2}]]$. Then we need $[[\mathbf{2};\mathbf{0}]]$ and $[[\mathbf{1}]]$ so that the last two teams get a third game. This is the swiss signature $\mathcal{T}_3$. $\square$

A similar style of proof for other numbers of teams and rounds can be used to determine that there are no other signatures missing from Figure 3.7.13.

Figure 3.7.13 tells us that there are five 8-team swiss signatures. How would a tournament designer decide which 3-round signature to use? Well, it depends on what the prize structure of the format is. If the goal is to identify a top-three, then signature $\mathcal{S}_3$ is preferable: signature $\mathcal{T}_3$ doesn't even recognize a third-place, instead assigning fourth-place to two teams. But if the goal is to identify a top-four, signature $\mathcal{T}_3$ is preferable: the team that comes in fourth in signature $\mathcal{S}_3$ actually finishes with only one win, while the team that comes in fifth finishes with two. While it is still reasonable to grant the one-win team fourth-place – they had a more difficult slate of opponents – this is a somewhat messy situation that is solved by just using signature $\mathcal{T}_3$.

(McGarry and Schutz [12] considered outright swapping the positions of the fourth- and fifth-place teams at the conclusion of $\mathcal{S}_3$, but this format is not proper and provides some incentive for losing in the first round in order to get an easier path to a top-half finish. Simply using $\mathcal{T}_3$ when identifying the top-four teams is preferable.)

For similar reasons, both formats are good for selecting a top-one or top-seven, and $\mathcal{S}_3$ but not $\mathcal{T}_3$ is good for selecting a top-five. Finally, it might seem that $\mathcal{S}_3$ and $\mathcal{T}_3$ are good formats for selecting a top-two or top-six: in both cases, the top two and top six teams are clearly defined, and there are no teams with better records that don't make the cut. However, notice that if we use $\mathcal{S}_3$ or $\mathcal{T}_3$ to select a top-two, the final round of games are meaningless: the two teams that finish in the top-two are the two teams that win their first two games, irrespective of how the third round of games went. Better than using either format $\mathcal{S}_3$ or $\mathcal{T}_3$ would be to use the non-compact 2x$\mathcal{S}_2$, shortening the format down to two rounds without losing any important games.

There are eight compact 4-round signatures: $\mathcal{S}_4$ and seven others. The count of compact r-round signatures for general r, however, is still an open question.

Conjecture 3.7.17

Let s_r be the number of compact r-round swiss signatures. Then s_r is given by:

$$s_0 = s_1 = 1$$
$$s_r = s_{r-1} \cdot \sum_{i=1}^{r-1} s_i$$

Overall, swiss formats are very useful and practical tournament designs: they give each team the same number of games, they ensure that games are being played between teams that have the same record and thus, hopefully, similar skill levels, and, for many values of m, they efficiently identify a top-m in a fair and satisfying way.

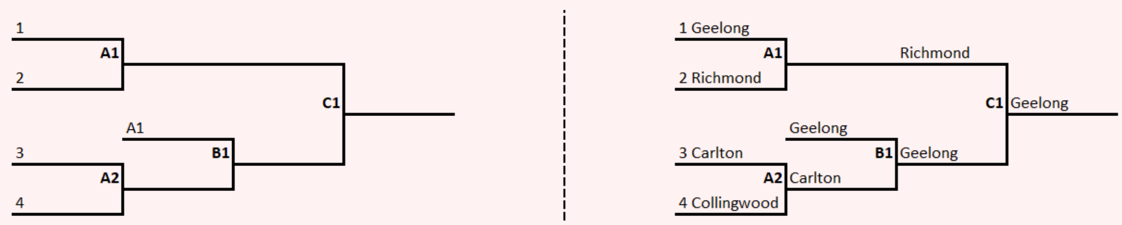
Further, swiss or near-swiss formats are great when the number of teams is exceedingly large. Even if not every requirement in Definition 3.7.2 is met, or the number of teams isn't a power of two, or or the signature is not compact, or there is a round at the end that doesn't affect placement for important places, formats that are swiss in spirit tend to do a great job of gathering a lot of meaningful data about a large number of teams in a small number of rounds. For this reason, they are often used in large tournaments for board or cards games, such as chess or Magic: The Gathering.

3.8 Nonlinear Multibrackets

For the four sections we have focused our study on *linear* multibrackets with the property that when a team loses in a given semibracket they drop into a different, lower semibracket. But many leagues use nonlinear multibrackets as well, and so while our tools of signature and properness are less equipped to study them, we look at what the space looks like.

An simple example of a nonlinear multibracket is the 1931 Victorian Football League Playoffs, sometimes called the Page-McIntyre system.

Figure 3.8.1: 1931 Victorian Football League Playoffs



Nonlinear multibrackets are a bit strange: while the winner of game **A1** goes directly to the final, the loser falls into the semifinal of the *same bracket*. This poses problems for both attempts to define a signature as well as a notion of properness.

Beginning with signature, the shape of the bracket is a bit strange: the winner of game **A1** gets a bye *after winning a game*, something that never happens in a traditional bracket. Attempts to give this bracket a signature might lead to $[[\mathbf{4};\mathbf{1};\mathbf{0};\mathbf{0}]]$ or even $[[\mathbf{4};\mathbf{0};\mathbf{0};\mathbf{0}]]$, neither of which are actually bracket signatures (they both violate Theorem 2.1.14). The issue here is that game **A1** is actually a semifinal, and so "should" (if it didn't deliver its loser to the other semifinal) live in the second round, producing a signature of $[[\mathbf{2};\mathbf{3};\mathbf{0};\mathbf{0}]]$. But then of course this format is quite different from traditional brackets with that same signature. Bracket signatures on nonlinear multibrackets are in general not well-defined.

To make matters worse, the first round appears to have an "improper" set of matchups: the games are 1v2 and 3v4 rather than then "proper" 1v4 and 2v3. However, properness is a much trickier concept for nonlinear multibrackets. While the 1- and 2-seeds to have tougher first round matchups than the 3- and 4-seeds, this is compensated by them getting an extra life: if they lose, they play the winner of the 3v4 matchup, while the 3v4 loser is just eliminated, so no team would prefer to be seeded lower than they are. One could imagine developing this intuition of extra lives into a formal notion of properness, but we leave that question untreated.

Open Question 3.8.2

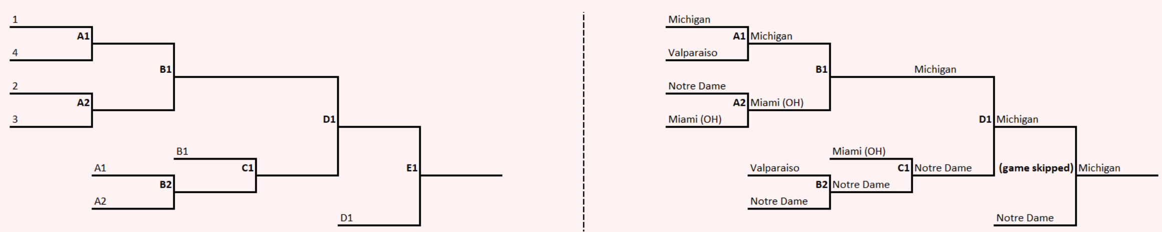
What would a notion of signature and properness look like for nonlinear multibrackets?

One final thing of note is that **C1** can be a rematch of game **A1**. In fact, this pretty likely: if the bracket goes chalk, the 1- and 2-seed will find themselves replaying the game they played just two rounds ago. In Figure 3.8.1, the bracket did not go chalk, but game **C1** was still a rematch. This can be pretty unsatisfying: indeed, in the 1931 Victorian Football League Playoffs, Geelong and Richmond each beat each other once, but Geelong won the game that mattered and so was declared champion.

Again as discussed in the previous section, one option would be to be to make game **C1** contingent on it not being a rematch: if it is a rematch, then the game is skipped and whichever team won the previous game is declared champion. While this solution is effective for the second-place game in our alternative AFL Asian Cup format (Figure 3.1.4), it doesn't work here. Making the game contingent would mean that the loser of **A1** is actually eliminated upon their loss: even if they win **B1**, they wouldn't have the ability to play in the championship game.

A better solution might be a *double-elimination tournament*, as employed by the 2016 NCAA Softball Ann Arbor Regional.

Figure 3.8.3: The 2016 NCAA Softball Ann Arbor Regional



Definition 3.8.4: Double-Elimination Tournament

A *double-elimination* tournament is a multibracket (plus one contingent game) consisting of a *winners' bracket*, where every team starts, a *losers'* bracket, that every winners' bracket loser falls into, and a *grand finals*, in which the winner of the winners' bracket plays the winner of the losers' bracket for the championship, with the losers' bracket winner needing to win twice, while the winners' bracket winner only needs to win once.

A double-elimination tournament guarantees that the winner will finish undefeated or with only one loss, while every other team finishes with two.

The 2016 NCAA Softball Ann Arbor Regional is an example of a double-elimination format: the winners' bracket consists of games **A1**, **A2**, and **B1**; the losers' bracket consists of games **B2** and **C1**; and the grand finals is game **D1** and then, if necessary, **E1**. Michigan finished undefeated while Valparaiso, Notre Dame, and Miami (OH) each finished with two losses.

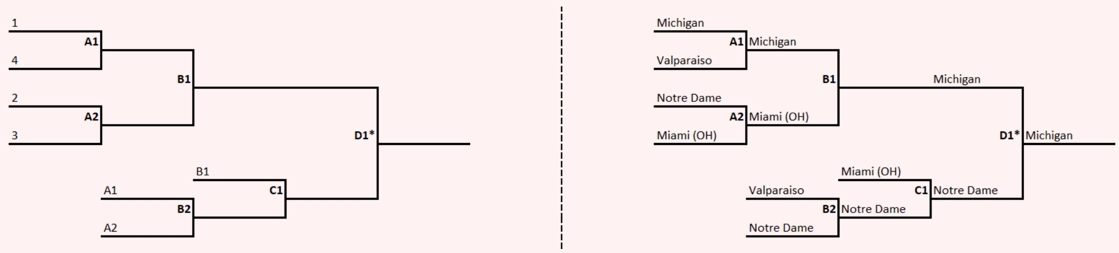
Because double-elimination tournaments are so common, and all use a contingent game that is played only if the lower team wins (**E1** in the case of Figure 3.8.3), that contingent game has a name.

Definition 3.8.5: Recharge Game

A *recharge game* is a contingent game in a multibracket that is a rematch of a previous game and played only if the lower team won the first game.

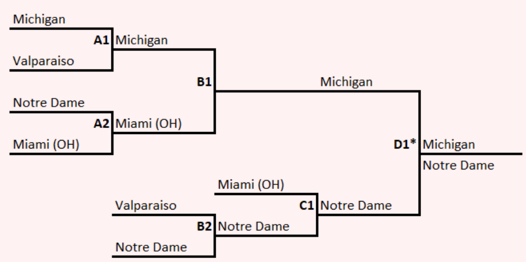
Recharge games are so common that we introduce a special notation: if the name of a game has a star after it, then that game is followed by a recharge game (if necessary). This allows us to condense the format in the Figure 3.8.3 a little bit, as displayed in Figure 3.8.6.

Figure 3.8.6: The 2016 NCAA Softball Ann Arbor Regional



The only issue with this notation is that, if the recharge game was triggered but won by the upper team, there is no natural place to denote that the recharge game was played. We adopt the convention of writing the the lower team *under* the line that the winner of the recharge game is placed over in this case. This is depicted in Figure 3.8.7.

Figure 3.8.7: Figure 3.8.6 if Notre Dame Beat Michigan Once



While the recharge game is necessary to ensure that the format is a truly a double-elimination tournament, as well as preventing the problem in the Page-McIntyre System where the champion and runner-up each finish with one-loss, it's not all upside. For one thing, Dabney [6] found some evidence that a tournament with no recharge game actually does a better job of crowing the best team as champion then the truer double-elimination with the recharge game included. Additionally, formats with recharge games tend to be less exciting, as they risk not playing a true championship game (a game in which either team wins the format if they win that game).

In any case, whether the recharge game is used or not, double-elimination tournaments are a powerful tool in a tournament designer's arsenal, as they are in some sense more accurate than their single elimination counterparts. We prove this fact for a simplified case where the winners' and losers' bracket are relatively nice, and where there is a single best team that is favored against every other team with a constant probability $1/2 < p < 1$.

Theorem 3.8.8

Let n be a positive integer, p be a probability such that $1/2 < p < 1$, and $\mathcal{T}$ be a list of 2^n teams with a team $t \in \mathcal{T}$ such that for every other team s,

$$\mathbb{P}[t \text{ beats } s] = p.$$

Let $\mathcal{A}$ be the balanced bracket on 2^n teams, let $\mathcal{B}$ be a bracket on $2^n - 1$ teams such that the linear multibracket $\mathcal{A} \to \mathcal{B}$ is round-respectful, and let $\mathcal{C}$ be the double-elimination format with winners' bracket $\mathcal{A}$ and losers' bracket $\mathcal{B}$. Then,

$$\mathbb{W}_{\mathcal{C}}(t, \mathcal{T}) \geq \mathbb{W}_{\mathcal{A}}(t, \mathcal{T})$$

with equality only when $n = 1$ and there is no recharge round.

Proof. To win $\mathcal{A}$, t simply has to win n games. Thus

$$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T}) = p^n.$$

Now consider $\mathcal{C}$. Let r be the number of rounds in $\mathcal{B}$, let r_i be the round of $\mathcal{B}$ that teams that lose in the ith round of $\mathcal{A}$ fall into, and let $c_i = r - r_i + 1$, so teams that lose in the ith round of $\mathcal{A}$ need to win c_i games in $\mathcal{B}$ in order to make the grand finals.

Since there are 2^{n-i} i-round losers, by Theorem 2.1.14,

$$\sum_{i=1}^{n} 2^{n-i} \cdot \left(\frac{1}{2}\right)^{c_i} = 1,$$

so,

$$\sum_{i=1}^{n} \left(\frac{1}{2}\right)^{c_i+i-1} = \left(\frac{1}{2}\right)^{n-1}. \qquad (*)$$

Letting $q = 1 - p$, note that t wins the winners' bracket with probability p^n, and the losers' bracket with probability

$$\sum_{i=1}^{n} p^{i-1} \cdot q \cdot p^{c_i} = q \cdot \sum_{i=1}^{n} p^{c_i+i-1} \geq q \cdot p^{n-1},$$

with the inequality coming by equation $(*)$ because $p > \frac{1}{2}$, and with equality only when $n = 1$.

Now, if there is a recharge round, then

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t,\mathcal{T}) &= \mathbb{W}_{\mathcal{A}}(t,\mathcal{T})\cdot(p+qp)+\mathbb{W}_{\mathcal{B}}(t,\mathcal{T})\cdot p^2 \\
&\geq p^n(p+qp)+(q\cdot p^{n-1})\cdot p^2 && \text{with equality only when } n=1 \\
&= p^n(p+2qp) \\
&> p^n \\
&= \mathbb{W}_{\mathcal{A}}(t,\mathcal{T}).
\end{aligned}$$

If there is no recharge round, then

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t,\mathcal{T}) &= \mathbb{W}_{\mathcal{A}}(t,\mathcal{T})\cdot p+\mathbb{W}_{\mathcal{B}}(t,\mathcal{T})\cdot p \\
&\geq p^n(p)+(q\cdot p^{n-1})\cdot p && \text{with equality only when } n=1 \\
&= p^n \\
&= \mathbb{W}_{\mathcal{A}}(t,\mathcal{T}).
\end{aligned}$$

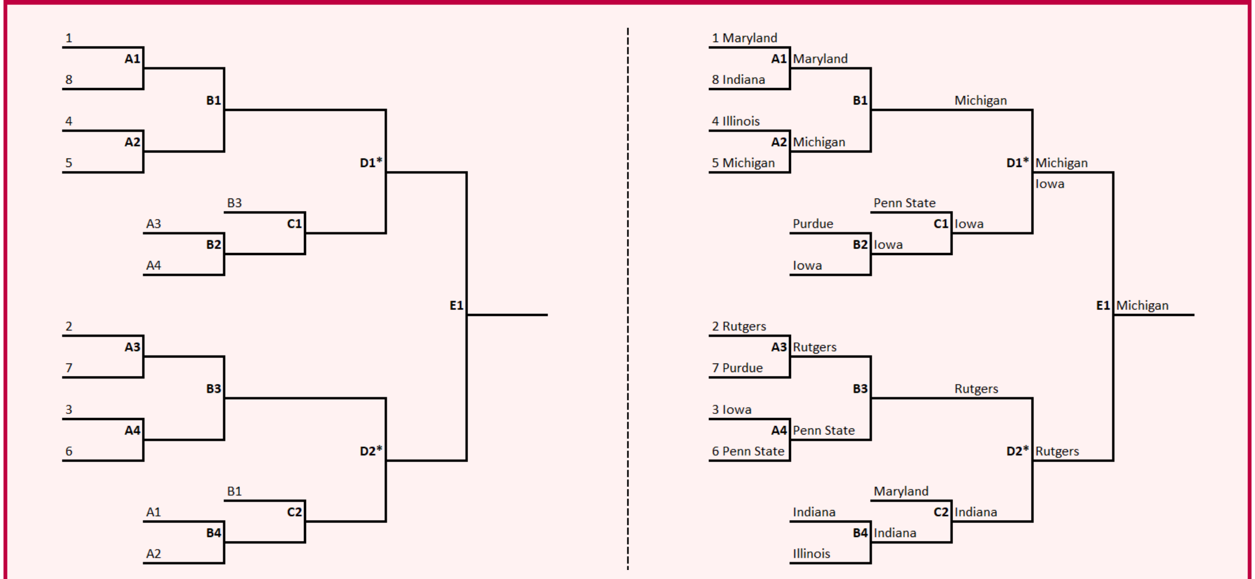
Thus,

$$\mathbb{W}_{\mathcal{C}}(t,\mathcal{T}) \geq \mathbb{W}_{\mathcal{A}}(t,\mathcal{T})$$

with equality only when $n=1$ and there is no recharge round. $\square$

We conclude our discussion of nonlinear multibrackets with a few more interesting examples. The first is the 2022 Big Ten Baseball Tournament.

Figure 3.8.9: The 2022 Big Ten Baseball Tournament



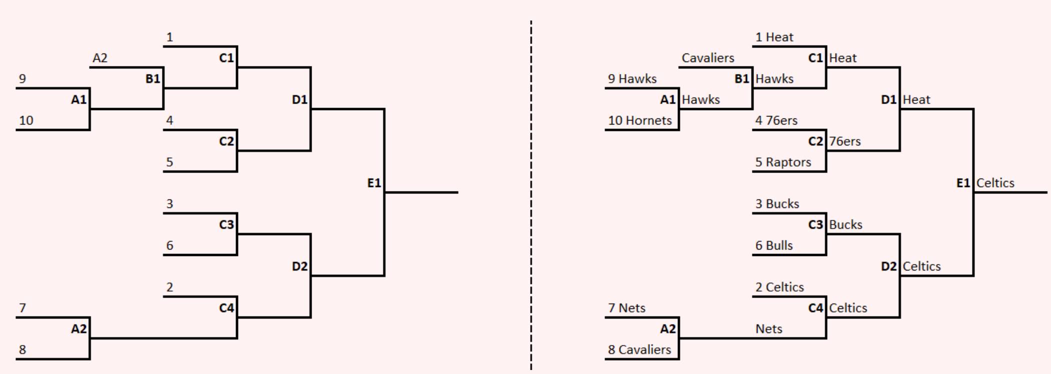
The 2022 Big Ten Baseball Tournament wanted to balance two effects: first, that double-elimination formats lead to more accurate results, but second, that championship games are exciting and double-elimination games risk not including one. 2022 Big Ten Baseball Tournament innovates to solve the latter issue by including recharge games in the *semifinals*, and then having the championship game be single winner-take-all game.

Note that this format does not fully solve all the problems it is attempting to tackle: for one thing, it is not a true double-elimination, as Rutgers gets eliminated with only a single loss. That said, Michigan is unambiguously the most deserving winner: every team other than Michigan and Rutgers lost once, and Michigan defeated Rutgers in their one matchup.

However, this property was not guaranteed: had Penn State beaten Iowa in game **C1**, Michigan twice in game **D1** and the recharge game, and then Rutgers in the final, we would be back to the issue with the Page-McIntyre System. Penn State and Rutgers would have each finished with only one-loss to the other team, with the champion being determined somewhat arbitrarily by who won the most recent game. This illustrates an important point: the desire for an unambiguous champion and the desire for an unambiguous championship *game* are fundamentally in conflict in the world of multibrackets.

Another interesting nonlinear multibracket of note is the NBA Playoffs. You may recall from Figure 2.2.4 that in 2004, the NBA Eastern Conference Playoffs used a simple bracket of signature $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$ to determine its champion (the Western Conference did the same, and then the two conference champions played each other in the NBA finals). However, in 2020, after a much of the NBA regular season was cut short due to Covid, there was a feeling that the regular season wasn't as accurate a measure as it usually is. So the playoffs were expanded slightly: if the 8th and 9th place teams were close enough in record, the playoff for that conference expanded to $[[\mathbf{2}; \mathbf{7}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$, allowing both teams in. After the success of that system, the playoffs were expanded further starting in 2021 to the following nonlinear multibracket.

Figure 3.8.10: 2022 NBA Eastern Conference Playoffs



The first two rounds of the new NBA playoffs are similar in structure to the Page-McIntyre system: two lower-seeded teams play each other and two higher-seeded teams play each other, and then the winner of the first game plays the loser of the second. But because the two qualifying teams get dumped into a larger eight-team bracket, rather than facing off immediately, the issues of the original Page-McIntyre system are avoided.

A final nice example of nonlinear multibrackets is bitonic sort. Bitonic sort was developed by Batcher [1] as a networked sorting algorithm with low delay (the sorting-theory equivalent to a low number of rounds). As every sorting algorithm can be transformed into a tournament format, and every networked sorting algorithm can be transformed into a multibracket, we can construct an nonlinear multibracket that executes Batcher's bitonic sort.

Definition 3.8.11: Bitonic Sort

The *bitonic sort* on 2^r teams proceed by diving the teams into two groups of 2^{r-1} teams, recursively running the bitonic sort on 2^{r-1} teams on each group, and then running the standard swiss format $\mathcal{S}_r$ on the full group of 2^r teams, with one of the groups getting the odd seeds in $\mathcal{S}_r$ and the other group getting the even seeds.

The 8-team bitonic sort is displayed in Figure 3.8.12. The **A**-, **B**-, and **C**- round games facilitate the running of two parallel instantiations of the 4-team bitonic sort, while the **D**-, **E**-, and **F**$-$round games carry out $\mathcal{S}_3$.

Figure 3.8.12: 8-Team Bitonic Sort

We leave it to the reader to verify that bitonic sort is in fact a sorting algorithm: that is, if the matchup table is SST with all win probabilities being 0 or 1 (even if the teams are not seeded in the correct order initially!), bitonic sort will correctly sort the teams. Impressively, the 8-team bitonic sort does this in only six rounds: no team needs to play every other team in order to complete the sort.

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5 References

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