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1 Tournament Formats

1.1 Definitions

Definition 1.1.1: Gameplay Function

A *gameplay function* g on a list of teams $\mathcal{T} = \{t_1, ..., t_n\}$ is a nondeterministic function $g : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ with the following properties:

- $\mathbb{P}[g(t_i, t_j) = t_i] + \mathbb{P}[g(t_i, t_j) = t_j] = 1$.
- $\mathbb{P}[g(t_i, t_j) = t_i] = \mathbb{P}[g(t_j, t_i) = t_j]$.

A gameplay function represents a process in which two teams compete in a game, with one of them emerging as the winner. This model simplifies away effects like home-field advantage or teams improving over the course of a tournament: a gameplay function is fully described by a single probability for each pair of teams in the list.

Definition 1.1.2: Playing, Winning, and Losing

When g is queried on input (t_i, t_j) we say that t_i and t_j *played a game*. We say that the team that got outputted by g *won*, and the team that did not *lost*.

The information in a gameplay function can be encoded into a *matchup table*.

Definition 1.1.3: Matchup Table

The *matchup table* implied by a gameplay function g on a list of teams $\mathcal{T}$ of length n is a n-by-n matrix $\mathbf{M}$ such that $\mathbf{M}_{ij} = \mathbb{P}[t_i \text{ beats } t_j]$.

For example, let $\mathcal{T} = \{\text{Favorites, Rock, Paper, Scissors, Conceders}\}$, and g be such that the Conceders concede every game they play, the Favorites are 70% favorites against Rock, Paper, and Scissors, and Rock, Paper, and Scissors matchup with each other as their name implies. Then the matchup table would look like so:

Figure 1.1.4: The Matchup Table for $(\mathcal{T}, g)$

	Favorites	Rock	Paper	Scissors	Conceders
Favorites	0.5	0.7	0.7	0.7	1.0
Rock	0.3	0.5	0.0	1.0	1.0
Paper	0.3	1.0	0.5	0.0	1.0
Scissors	0.3	0.0	1.0	0.5	1.0
Conceders	0.0	0.0	0.0	0.0	0.5

Theorem 1.1.5

If $\mathbf{M}$ is the matchup table for $(\mathcal{T}, g)$, then $\mathbf{M} + \mathbf{M}^T$ is the matrix of all ones.

Proof. $(\mathbf{M} + \mathbf{M}^T)_{ij} = \mathbf{M}_{ij} + \mathbf{M}_{ji} = \mathbb{P}[t_i \text{ beats } t_j] + \mathbb{P}[t_j \text{ beats } t_i] = 1.$ $\square$

Definition 1.1.6: Tournament Format

A *tournament format* is an algorithm that takes as input a list of teams $\mathcal{T}$ and a gameplay function g and outputs a champion $t \in \mathcal{T}$.

We use a gameplay function rather than a matchup table in the definition of a tournament format because a tournament format cannot simply look at the matchup table itself in order to decide which teams are best. Instead, formats query the gameplay function (have teams play games) in order to gather information about the teams. That said, matchup tables will often be useful in our *analysis* of tournament formats.

We also introduce some shorthand to help make notation more concise.

Definition 1.1.7: $\mathbb{W}_{\mathcal{A}}(t, \mathcal{T})$

$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T})$ is the probability that team $t \in \mathcal{T}$ wins tournament format $\mathcal{A}$ when it is run on the list of teams $\mathcal{T}$.

(This chapter will be fleshed out but I'm including the important definitions here for the sake of the next chapter.)

2 Brackets

2.1 Brackets and Rounds

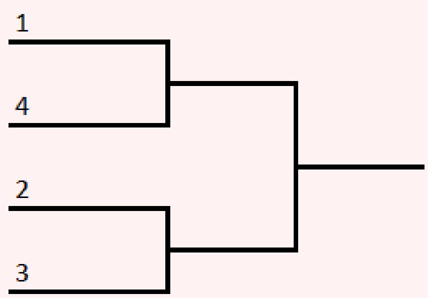
Definition 2.1.1: Bracket

A *bracket* is a tournament format in which:

- Teams don't play any games after their first loss,
- Games are played until only one team has no losses, and that team is crowned champion, and
- The matchups between teams that have not yet lost are determined based on the ordering of the teams in $\mathcal{T}$ in advance of the outcomes of any games.

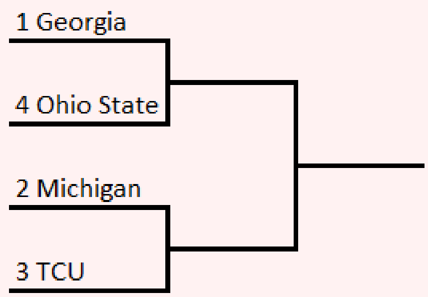
We can draw a bracket as a tree-like structure in the following way:

Figure 2.1.2: The 2023 College Football Playoff



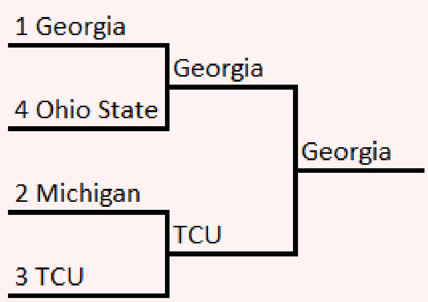
The numbers 1, 2, 3, and 4 indicate where t_1, t_2, t_3, and t_4 in $\mathcal{T}$ are placed to start. In the actual 2023 College Football Playoff, the list of teams $\mathcal{T}$ was Georgia, Michigan, TCU, and Ohio State, in that order, so the bracket was filled in like so:

Figure 2.1.3: The 2023 CFP After Team Placement



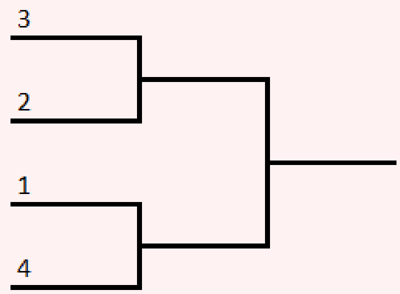
As games are played, we write the name of the winning teams on the corresponding lines. This bracket tells us that Georgia played Ohio State, and Michigan played TCU. Georgia and TCU won their respective games, and then Georgia beat TCU, winning the tournament.

Figure 2.1.4: The 2023 CFP After Completion



Rearranging the way the bracket is pictured, if it doesn't affect any of the matchups, does not create a new bracket. For example, Figure 2.1.5 is just another way to draw the same 2023 CFP Bracket.

Figure 2.1.5: Alternative Drawing of the 2023 CFP



One key piece of bracket vocabulary is the *round*.

Definition 2.1.6: Round

A *round* is a set of games such that the winners of each of those games have the same number of games remaining to win the tournament.

For example, the 2023 CFP has two rounds. The first round included the games Georgia vs Ohio State and Michigan vs TCU, and the second round was just a single game: Georgia vs TCU.

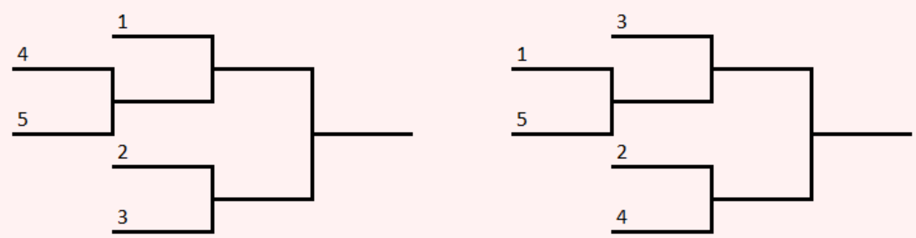
2.2 Bracket Signatures

Definition 2.2.1: Shape

The *shape* of a bracket is the tree that underlies it.

For example, the following two brackets have the same shape:

Figure 2.2.2: Two Brackets with the Same Shape



Definition 2.2.3: Bye

A team has a *bye* in round r if it plays no games in round r or before.

One way to describe the shape of a bracket its signature.

Definition 2.2.4: Bracket Signature

The *signature* $[[\mathbf{a_0}; ...; \mathbf{a_r}]]$ of an r-team bracket $\mathcal{A}$ is list such that a_i is the number of teams with i byes.

The signature of a bracket is defined only by its shape: the two brackets in Figure 2.2.2 have the same shape, so they also have the same signature.

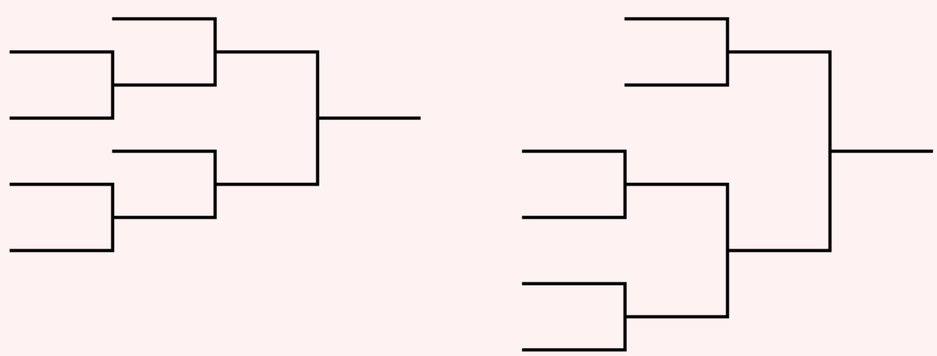
The signatures of the brackets discussed in this section are shown in Figure 2.2.5. It's worth verifying the signatures we've seen so far and trying to draw brackets with the signatures we haven't yet before moving on.

Figure 2.2.5: The Signatures of Some Brackets

Bracket	Signature
2023 College Football Playoff	$[[\mathbf{4};\mathbf{0};\mathbf{0}]]$
The brackets in Figure 2.2.2	$[[\mathbf{2};\mathbf{3};\mathbf{0};\mathbf{0}]]$
The brackets in Figure 2.2.6	$[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$
2023 WCC Men's Basketball Tournament	$[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]$

Two brackets with the same shape must have the same signature, but the converse is not true: two brackets with different shapes can have the same signature. For example, both bracket shapes depicted in Figure 2.2.6 have the signature $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$.

Figure 2.2.6: Two Shapes with the Signature $[[4;2;0;0]]$



Despite this, bracket signatures are a useful way to talk about the shape of a bracket. Communicating a bracket's signature is a lot easier than communicating its shape, and much of the important information (such as how many games each team must win in order to win the tournament) is contained in the signature.

Bracket signatures have one more important property.

Theorem 2.2.7

Let $\mathcal{A} = [[\mathbf{a_0};\ldots;\mathbf{a_r}]]$ be a list of natural numbers. Then $\mathcal{A}$ is a bracket signature if and only if

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = 1.$$

Proof. Let $\mathcal{A}$ be the signature for some bracket. Assume that every game in the bracket was a coin flip, and consider each team's probability of winning the tournament. A team that has i byes must win $r-i$ games to win the tournament, and so will do so with probability $\left(\frac{1}{2}\right)^{r-i}$. For each $i \in \{0, ..., r\}$, there are a_i teams with i byes, so (because any two teams winning are mutually exclusive)

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i}$$

is the probability that one of the teams wins, which is 1.

We prove the other direction by induction on r. If $r = 0$, then the only list with the desired property is $[[\mathbf{1}]]$, which is the signature for the unique one-team bracket. For any other r, first note that a_0 must be even: if it were odd, then

$$\begin{aligned}
\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} &= \frac{1}{2^r} \cdot \sum_{i=0}^{r} a_i \cdot 2^i \\
&= \frac{1}{2^r} \cdot \left(a_0 + 2\sum_{i=1}^{r} a_i \cdot 2^{i-1}\right) \\
&= k/2^r & \text{for some odd } k \\
&\neq 1.
\end{aligned}$$

Now, consider the signature $\mathcal{B} = [[\mathbf{a_1 + a_0/2; a_2; ...; a_r}]]$. By induction, there exists a bracket with signature $\mathcal{B}$. But if we take that bracket and replace $a_0/2$ of the teams with no byes with a game whose winner gets placed on that line, we get a new bracket with signature $\mathcal{A}$. $\square$

The operation of transforming a bracket signature $\mathcal{A} = [[\mathbf{a_0; ...; a_r}]]$ into a bracket signature with one fewer round $\mathcal{B} = [[\mathbf{a_1 + a_0/2; a_2; ...; a_r}]]$ that we used at the end of the proof of Theorem 2.2.7 will become somewhat frequent, as we often induct on the number of rounds in a bracket, so it has a name:

Definition 2.2.8: The Successor Signature

If $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$, then the successor signature

$$\mathfrak{S}(\mathcal{A}) = [[\mathbf{a_1} + \mathbf{a_0}/\mathbf{2}; \mathbf{a_2}; ...; \mathbf{a_r}]].$$

(The successor signature of zero-round signatures is undefined.)

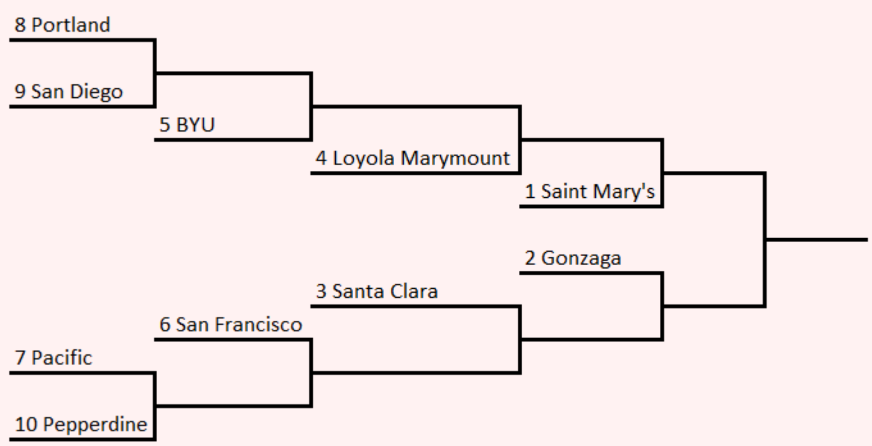
In the next few sections, we will use the language and properties of bracket signatures to describe the brackets that we work with. For now though, let's return to the 2023 College Football Playoff. The bracket used in the 2023 CFP has a special property that not all brackets have: it is *balanced*.

Definition 2.2.9: Balanced Bracket

A *balanced bracket* is a bracket in which none of the teams have byes.

The 2023 West Coast Conference Men's Basketball Tournament, on the other hand, is unbalanced:

Figure 2.2.10: The 2023 WCC Men's Basketball Tournament



Saint Mary's and Gonzaga each have three byes and so only need to win two games to win the tournament, while Portland, San Diego, Pacific, and Pepperdine need to win five. Unsurprisingly, this format conveys a massive advantage to Saint Mary's and Gonzaga, but this was intentional: those two teams were being rewarded for doing the best during the regular season.

In many cases, however, it is undesirable to grant advantages to certain teams over others. One might hope, for any n, to be able to construct a balanced bracket for n teams, but unfortunately this is rarely possible.

Theorem 2.2.11

There exists an n-team balanced bracket if and only if n is a power of two.

Proof. A bracket is balanced if no teams have byes, which is true exactly when its signature is of the form $\mathcal{A} = [[\mathbf{n}; \mathbf{0}; \ldots; \mathbf{0}]]$ where n is the number of teams in the bracket. If n is a power of two, then by Theorem 2.2.7 $\mathcal{A}$ is indeed a bracket signature and so points to a balanced bracket for n teams. If n is not a power of two, however, then Theorem 2.2.7 tells us that $\mathcal{A}$ is not a bracket signature, and so no balanced brackets exist for n teams. □

Given this, brackets are not a great option when we want to avoid giving some teams advantages over others unless we have a power of two teams. They are a fantastic tool, however, if doling out advantages is the goal, perhaps after some teams did better during the regular season and ought to be rewarded with an easier path in the bracket.

2.3 Proper Brackets

Definition 2.3.1: Seeding

The *seeding* of an n-team bracket is the arrangement of the numbers 1 through n in the bracket.

Together, the shape and seeding fully specify a bracket.

Definition 2.3.2: i-seed

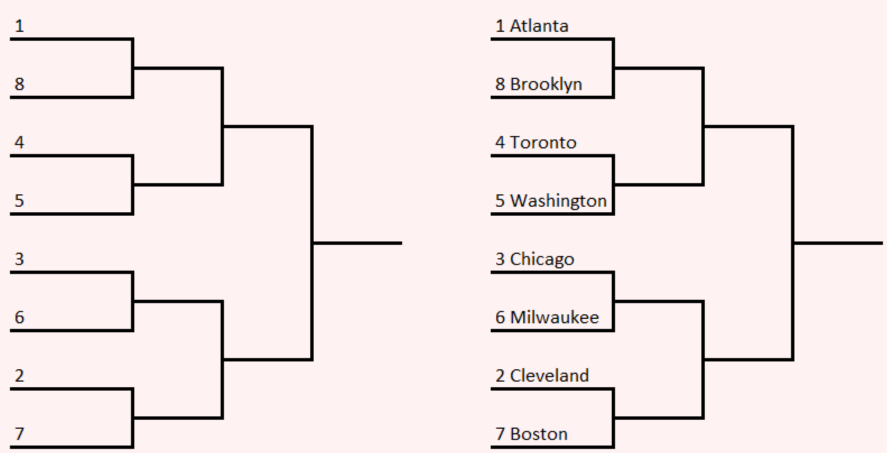
In a list of teams $\mathcal{T} = \{t_1, ..., t_n\}$, we refer to t_i as the i-seed.

Definition 2.3.3: Higher and Lower Seeds

Somewhat confusingly, convention is that smaller numbers are the *higher seeds*, and greater numbers are the *lower seeds*.

Seeding is typically used to reward better and more deserving teams. As an example, on the left is the eight-team bracket used in the 2015 NBA Eastern Conference Playoffs. At the end of the regular season, the top eight teams in the Eastern Conference were ranked and placed into the bracket as shown on the right.

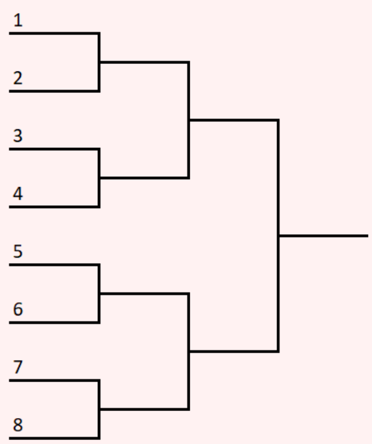
Figure 2.3.4: 2015 NBA Eastern Conference Playoffs



Despite this bracket being balanced, the higher seeds are still at advantage: they have an easier set of opponents. Compare 1-seed Atlanta, whose first two rounds are versus 8-seed Brooklyn and then (most likely) 4-seed Toronto, versus 7-seed Boston, whose first two rounds are versus 2-seed Cleveland and then (most likely) 3-seed Chicago. Atlanta's schedule is far easier: despite them having the same number of games to win as Boston, Atlanta is expected to play lower seeds in each round than Boston will.

Thus, we've identified two ways in which brackets can convey an advantage onto certain teams: by giving them more byes, and by giving them easier (expected) opponents. Not every seeding of a bracket does this: for example, consider the following alternative seeding for the 2015 NBA Eastern Conference Playoffs.

Figure 2.3.5: An Alternative Seeding of the 2015 NBA Eastern Conference Playoffs



This seeding does a very poor job of rewarding the higher-seeded teams: the 1- and 2-seeds are matched up in the first round, while the easiest road is given to the 7-seed, who plays the 8-seed in the first round and then (most likely) the 5-seed in the second. Since the whole point of seeding is to give the higher-seeded teams an advantage, we introduce the concept of a *proper seeding*.

Definition 2.3.6: Chalk

We say a tournament *went chalk* if the higher-seeded team won every game during the tournament.

Definition 2.3.7: Proper Seeding

A *proper seeding* of a bracket is one such that if the bracket goes chalk, in every round it is better to be a higher-seeded team than a lower-seeded one, where:

(1) It is better to have a bye than to play a game.

(2) It is better to play a lower seed than to play a higher seed.

Definition 2.3.8: Proper Bracket

A *proper bracket* is a bracket that has been properly seeded.

It is clear that the actual 2015 NBA Eastern Conference Playoffs was properly seeded, while our alternative seeding was not.

A few quick lemmas about proper brackets:

Lemma 2.3.9

In a proper bracket, if m teams have a bye in a given round, those teams must be seeds 1 through m.

Proof. If they did not, the seeding would be in violation of condition (1). $\square$

Lemma 2.3.10

If a proper bracket goes chalk, then after each round the m teams remaining will be the top m seeds.

Proof. We will prove the contrapositive. Assume that for some $i < j$, after some round, t_i has been eliminated but t_j is still alive. Let k be the seed of the team that t_i lost to. Because the bracket went chalk,

$k < i$. Now consider what t_j did in that round. If they had a bye, then the bracket violates condition (1). Assume instead they played t_ℓ. They beat t_ℓ, so $j < \ell$, giving,

$$k < i < j < \ell.$$

In the round that t_i was eliminated, t_i played t_k, while t_j played t_ℓ, violating condition (2). Thus, the bracket is not proper. □

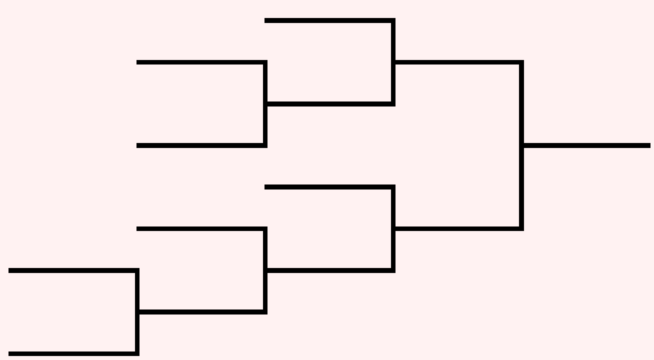
Lemma 2.3.11

In a proper bracket, if m teams have a bye and k games are being played in a given round, then if the bracket goes chalk those matchups will be seed $m + i$ vs seed $(m + 2k + 1) - i$ for $i \in \{1, ..., k\}$.

Proof. In the given round, there are $m+2k$ teams remaining. Theorem 2.3.10 tells us that (if the bracket goes chalk) those teams must be seeds 1 through $m+2k$. Theorem 2.3.9 tells us that seeds 1 through m must have a bye, so the teams playing must be seeds $m+1$ through $m+2k$. Then condition (2) tells us that the matchups must be exactly $m + i$ vs seed $(m + 2k + 1) - i$ for $i \in \{1, ..., k\}$. □

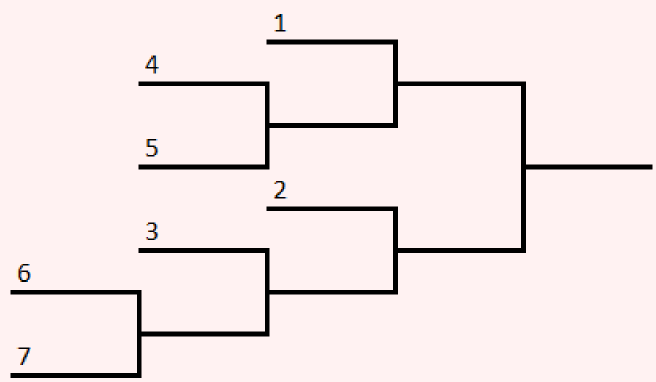
We can use Lemmas 2.3.9 through 2.3.11 to properly seed various bracket shapes. For example, consider the following seven-team shape:

Figure 2.3.12: A Seven-Team Bracket Shape



Lemma 2.3.9 tells us that the first-round matchup must be between the 6-seed and the 7-seed. Lemma 2.3.11 tells us that if the bracket goes chalk, the second-round matchups must be 3v6 and 4v5, so the 3-seed play the winner of the first-round matchup. Finally, we can apply Lemma 2.3.11 again to the semifinals to find that the 1-seed should play the winner of the 4v5 matchup, while the 2-seed should play the winner of the 3v(6v7) matchup. In total, our proper seeding looks like:

Figure 2.3.13: A Seven-Team Bracket, Properly Seeded

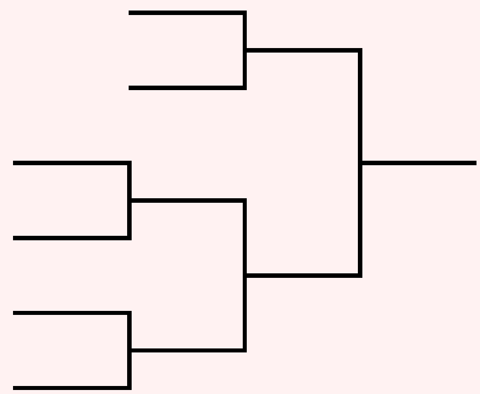


We can also quickly simulate the bracket going chalk to verify Lemma 2.3.10.

Lemmas 2.3.9 through 2.3.11 are quite powerful. It is not a coincidence that we managed to specify exactly what a proper seeding of the above bracket must look like with no room for variation: soon we will prove that the proper seeding for a particular bracket shape is unique.

But not every shape admits even this one proper seeding. Consider the following six-team shape:

Figure 2.3.14: A Six-Team Bracket Shape



This shape admits no proper seedings. Lemma 2.3.9 requires that the two teams getting byes be the 1- and 2-seed, but this violates Lemma 2.3.11 which requires that in the second round the 1- and 2-seeds do not play each other. So how can we think about which shapes admit proper seedings?

Theorem 2.3.15: The Fundamental Theorem of Brackets

There is exactly one proper bracket with each bracket signature.

Proof. Let $\mathcal{A}$ be an r-round bracket signature. We proceed by induction on r. If $r = 0$, then the only possible bracket signature is $[[\mathbf{1}]]$, and it points to the unique one-team bracket, which is indeed proper.

For any other r, the first-round matchups of a proper bracket with signature $\mathcal{A}$ are defined by Theorem 2.3.11. Then if those matchups go chalk, we are left with a proper bracket of signature $\mathfrak{S}(\mathcal{A})$, which induction tells us exists admits exactly one proper bracket.

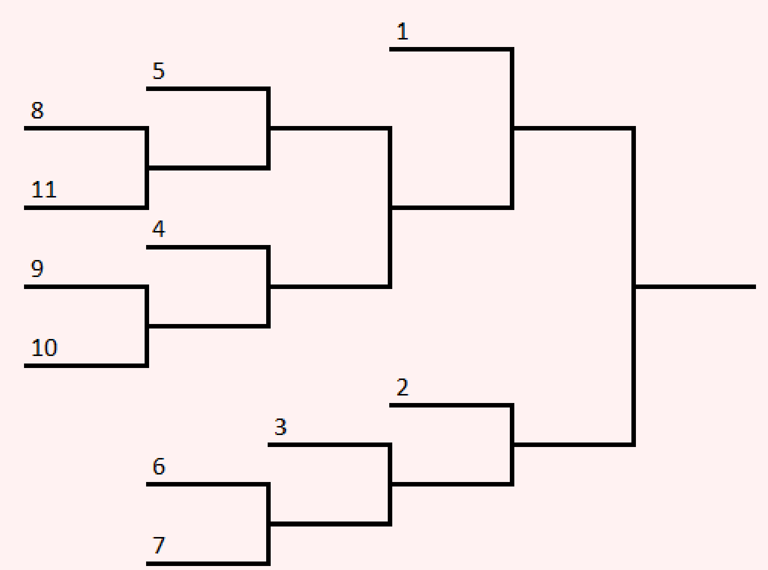
Thus both the first-round matchups and the rest of the bracket are determined, and by combining them we get a proper bracket with signature $\mathcal{A}$, so there is exactly one proper bracket with signature $\mathcal{A}$. $\square$

The fundamental theorem of brackets means that we can refer to the proper bracket $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ in a well-defined way, as long as

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = 1.$$

In practice, virtually every sports league that uses a traditional bracket uses a proper one: while different leagues take very different approaches to how many byes to give teams (compare the 2023 West Coast Conference Men's Basketball Tournament with the 2015 NBA Eastern Conference Playoffs), they are almost all proper. This makes bracket signatures a convenient labeling system for the set of brackets that we might reasonably encounter. They also are a powerful tool for specifying new brackets: if you are interested in (say) an eleven-team bracket where four teams get no byes, four teams get one bye, one team gets two byes and two teams get three byes, we can describe the proper bracket with those specs as $[[\mathbf{4}; \mathbf{4}; \mathbf{1}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]$ and use Lemmas 2.3.9 through 2.3.11 to draw it with ease:

Figure 2.3.16: The Proper Bracket of Signature $[[4; 4; 1; 2; 0; 0]]$

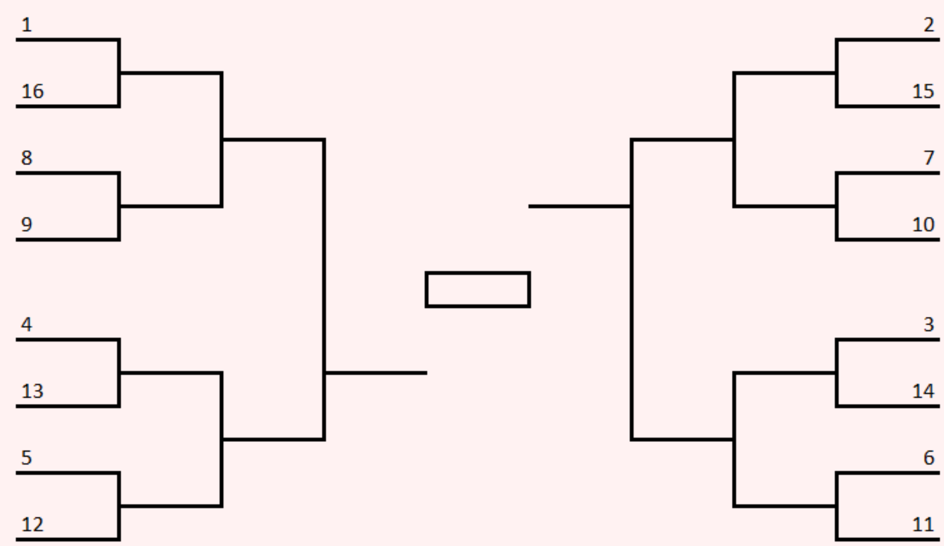


Due to these properties, we will almost exclusively discuss proper bracket from here on out: unless stated otherwise, assume all brackets are proper.

2.4 Ordered Brackets

Consider the proper bracket $[[\mathbf{16}; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$, which was used in the 2022 NCAA Women's Basketball Tournament Wichita Region, and is shown below. (Sometimes brackets are drawn in the manner below, with teams starting on both sides and the winner of each side playing in the championship game.)

Figure 2.4.1: 2022 NCAA Women's Basketball Tournament Wichita Region



The definition of a proper seeding ensures that as long as the bracket goes chalk (that is, higher seeds always beat lower seeds), it will always be better to be a higher seed than a lower seed. But what if it doesn't go chalk?

One counter-intuitive fact about the NCAA Basketball Tournament is that it is probably better to be a 10-seed than a 9-seed. (This doesn't violate the proper seeding property because 9-seeds have an easier first-round matchup than 10-seeds, and for further rounds, proper seedings only care about what happens if the bracket goes chalk, which would eliminate both the 9-seed and 10-seed in the first round.) Why? Let's look at whom each seed-line matchups against in the first two rounds:

Figure 2.4.2: NCAA Basketball Tournament 9- and 10-seed Schedules

Seed	First Round	Second Round
9	8	1
10	7	2

The 9-seed has an easier first-round matchup, while the 10-seed has an easier second-round matchup. However, this isn't quite symmetrical. Because the teams are probably drawn from a roughly normal distribution, the expected difference in skill between the 1- and 2-seeds is far greater than the expected difference between the 7- and 8-seeds, implying that the 10-seed does in fact have an easier route than the 9-seed.

Nate Silver [7] investigated this matter in full, finding that in the NCAA Basketball Tournament, seed-lines 10 through 15 give teams better odds of winning the region than seed-lines 8 and 9. Of course this does not mean that the 11-seed (say) has a better chance of winning a given region than the 8-seed does, as the 8-seed is a much better team than the 11-seed. But it does mean that the 8-seed would love to swap places with the 11-seed, and that doing so would increase their odds to win the region.

This is not a great state of affairs: the whole point of seeding is confer an advantage to higher-seeded teams, and the proper bracket $[[\mathbf{16};\mathbf{0};\mathbf{0};\mathbf{0};\mathbf{0}]]$ is failing to do that. Not to mention that giving lower-seeded teams an easier route than higher-seeded ones can incentivize teams to lose during the regular season in order to try to get a lower but more advantageous seed.

To fix this, we need a stronger notion of what makes a bracket effective than properness. The issue with proper seedings is the false assumption that higher-seeded teams will always beat lower-seeded teams. A more nuanced assumption, initially proposed by H.A. David [2], might look like this:

Definition 2.4.3: Strongly Stochastically Transitive

A list of teams $\mathcal{T}$ is *strongly stochastically transitive* if for each i, j, k such that $j < k$,

$$\mathbb{P}[t_i \text{ beats } t_j] \leq \mathbb{P}[t_i \text{ beats } t_k].$$

A list of teams being strongly stochastically transitive (SST) captures the intuition that each team ought to do better against lower-seeded teams than against higher-seeded teams. A few quick implications of this definition are:

Corollary 2.4.4

If $\mathcal{T}$ is SST, then for each $i < j$, $\mathbb{P}[t_i \text{ beats } t_j] \geq 0.5$.

Corollary 2.4.5

If $\mathcal{T}$ is SST, then for each i, j, k, ℓ such that $i < j$ and $k < \ell$,

$$\mathbb{P}[t_i \text{ beats } t_\ell] \geq \mathbb{P}[t_j \text{ beats } t_k].$$

Corollary 2.4.6

If $\mathcal{T}$ is SST, then the matchup table $\mathbf{M}$ is monotonically increasing along each row and monotonically decreasing along each column.

Note that not every set of teams can be seeded to be SST. Consider, for example, the game of rock-paper-scissors. Rock beats paper which beats scissors which beats rock, so no ordering of these "teams" will be SST. For our purposes, however, SST will work well enough.

Our new, nuanced alternative a proper bracket is an *ordered bracket*, first defined by Chen and Hwang [1].

Definition 2.4.7: Ordered

A tournament format $\mathcal{A}$ is *ordered* if, for any SST list of teams $\mathcal{T}$, if $i < j$, then $\mathbb{W}_{\mathcal{A}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{T})$

In an informal sense, a bracket being ordered is the strongest thing we can want without knowing more about why the tournament is being played. Depending on the situation, we might be interested in a format that almost always declares the most-skilled team as the winner, or in a format that gives each team roughly the same chance of winning, or anywhere in between. But certainly, better teams should win more, which is what the ordered bracket condition requires.

In particular, a bracket being ordered is a stronger claim than it being proper.

Theorem 2.4.8

Every ordered bracket is proper.

Proof. Let $\mathcal{A}$ be an ordered n-team bracket with r rounds.

Let $\mathcal{T}$ be SST with matchup table $\mathbf{M}$ where $\mathbf{M}_{ij} = 0.5$. A team that plays their first game in the ith round will win the tournament with probability $(0.5)^{r-i}$, so teams that get more byes will have a higher probability to win the tournament than teams with fewer byes. This implies that higher-seeded teams must have more byes than lower-seeded teams, so in each round, the teams with byes must be the highest-seeded teams that are still alive. Thus, condition (1) is met.

We show that condition (2) is met by proving the stronger condition from Lemma 2.3.11: if m teams have a bye and k games are being played in round s, then if the bracket goes chalk, those matchups will be t_{m+i} vs $t_{(m+2k+1)-i}$ for $i \in \{1, ..., k\}$. We show this by strong induction on s and on i.

Assume that this is true for every round up until s and for all $i < j$ for some j. Let $\ell = (m + 2k + 1) - j$. We want to show that if the the bracket goes chalk, t_{m+j} will face off against seed t_ℓ in the given round. Consider the following SST matchup table: every game is a coin flip, except for games involving a team seeded ℓ or lower, in which case the higher seed always wins. Then, each team seeded between $\ell - 1$ and $m + j$ will win the tournament with probability $(\frac{1}{2})^{r-s}$, other than the team slated to play t_ℓ in round s who wins with probability $(\frac{1}{2})^{r-i-1}$. In order for $\mathcal{B}$ to be ordered, that team must be t_{m+j}.

Thus $\mathcal{A}$ satisfies both conditions, and so is a proper bracket. □

With Theorem 2.4.8, we can use the language of bracket signatures to describe ordered brackets without worrying that two ordered brackets might share a signature. Now we examine three particularly important examples of ordered brackets.

We begin with the unique one-team bracket.

Figure 2.4.9: The One-Team Bracket $[[\mathbf{1}]]$

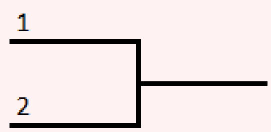
Theorem 2.4.10

The one-team bracket $[[\mathbf{1}]]$ is ordered.

Proof. Since there is only team, the ordered bracket condition is vacuously true. $\square$

Next we look at the unique two-team bracket.

Figure 2.4.11: The Two-Team Bracket $[[\mathbf{2}; \mathbf{0}]]$



Theorem 2.4.12

The two-team bracket $[[\mathbf{2}; \mathbf{0}]]$ is ordered.

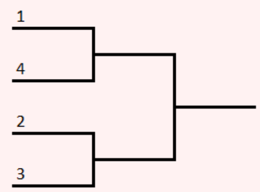
Proof. Let $\mathcal{A} = [[\mathbf{2}; \mathbf{0}]]$. Then,

$$\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) = \mathbb{P}[t_1 \text{ beats } t_2] \geq 0.5 \geq \mathbb{P}[t_2 \text{ beats } t_1] = \mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T})$$

so $\mathcal{A}$ is ordered. $\square$

And thirdly, we show that the balanced four-team bracket is ordered, first proved by Horen and Riezman [4].

Figure 2.4.13: The Four-Team Bracket $[[4;0;0]]$



Theorem 2.4.14

The four-team bracket $[[\mathbf{4};\mathbf{0};\mathbf{0}]]$ is ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4};\mathbf{0};\mathbf{0}]]$ and let $p_{ij} = \mathbb{P}[t_i \text{ beats } t_j]$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) &= p_{14} \cdot (p_{23}p_{12} + p_{32}p_{13}) \\
&= p_{14}p_{23}p_{12} + p_{14}p_{32}p_{13} \\
&\geq p_{14}p_{23}p_{21} + p_{24}p_{41}p_{23} \\
&= p_{23} \cdot (p_{14}p_{21} + p_{41}p_{24}) \\
&= \mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T})
\end{aligned}$$

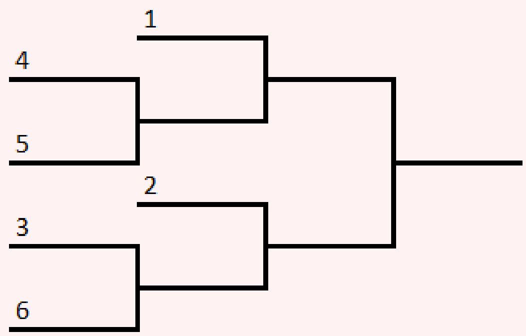
$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T}) &= p_{23} \cdot (p_{14}p_{21} + p_{41}p_{24}) \\
&\geq p_{32} \cdot (p_{14}p_{31} + p_{41}p_{34}) \\
&= \mathbb{W}_{\mathcal{A}}(t_3, \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
\mathbb{W}_{\mathcal{A}}(t_3, \mathcal{T}) &= p_{32} \cdot (p_{14}p_{31} + p_{41}p_{34}) \\
&= p_{32}p_{14}p_{31} + p_{32}p_{41}p_{34} \\
&\geq p_{42}p_{23}p_{41} + p_{32}p_{41}p_{43} \\
&= p_{41} \cdot (p_{23}p_{42} + p_{32}p_{43}) \\
&= \mathbb{W}_{\mathcal{A}}(t_4, \mathcal{T})
\end{aligned}$$

Thus $\mathcal{A}$ is ordered. □

However, not every proper bracket is ordered. One particularly important example of a non-ordered proper bracket is $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$

Figure 2.4.15: The Six-Team Bracket $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$



Theorem 2.4.16

The six-team bracket $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$, and let $\mathcal{T}$ have the following matchup table:

	t_1	t_2	t_3	t_4	t_5	t_6
t_1	0.5	0.5	0.5	0.5	0.5	1
t_2	0.5	0.5	0.5	0.5	0.5	1
t_3	0.5	0.5	0.5	0.5	0.5	0.5
t_4	0.5	0.5	0.5	0.5	0.5	0.5
t_5	0.5	0.5	0.5	0.5	0.5	0.5
t_6	0	0	0.5	0.5	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_1, \mathcal{T}) = 0.5 \cdot 0.5 = 0.25,$$

but

$$\mathbb{W}_{\mathcal{A}}(t_2, \mathcal{T}) = (0.5 \cdot 0.5 + 0.5 \cdot 1) \cdot 0.5 = 0.375.$$

Thus $\mathcal{A}$ is not ordered. $\square$

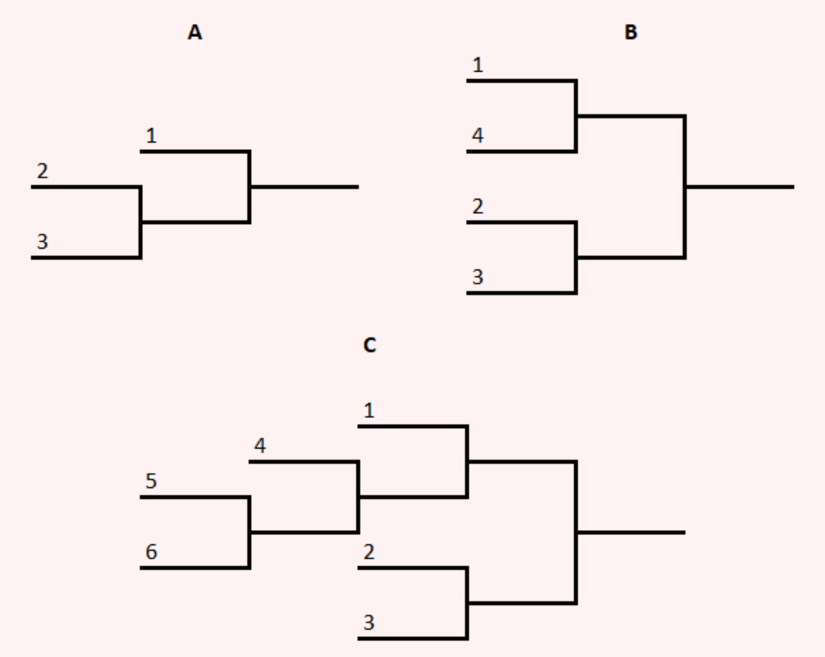
In the next section, we move on from describing particular ordered and non-ordered brackets in favor of a more general result.

2.5 Edwards's Theorem

We now attempt to completely classify the set of ordered brackets. Edwards [3] originally accomplished this without access to the machinery of bracket signatures or proper brackets: we present a quicker proof that makes use of the fundamental theorem of brackets and develop two nice lemmas along the way.

We begin with the stapling lemma, which allows us to combine two smaller ordered brackets into a larger ordered one by having the winner of one of the brackets be treated as the lowest seed in the other. This is depicted in Figure 2.5.1.

Figure 2.5.1: Setup of the Stapling Lemma with $\mathcal{A} = [[\mathbf{2}; \mathbf{1}; \mathbf{0}]]$, $\mathcal{B} = [[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$, and $\mathcal{C} = [[\mathbf{2}; \mathbf{1}; \mathbf{3}; \mathbf{0}; \mathbf{0}]]$



Lemma 2.5.2: The Stapling Lemma

If $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ and $\mathcal{B} = [[\mathbf{b_0}; ...; \mathbf{b_s}]]$ are ordered brackets, then $\mathcal{C} = [[\mathbf{a_0}; ...; \mathbf{a_r} + \mathbf{b_0} - \mathbf{1}; ...; \mathbf{b_s}]]$ is an ordered bracket as well.

Proof. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be as specified. Let $\mathcal{T}$ be an SST list of teams $n+m-1$ teams, and let $\mathcal{R}, \mathcal{S} \subset \mathcal{T}$ be the lowest n and the highest $m-1$ seeds of $\mathcal{T}$ respectively. We divide proving that $\mathcal{C}$ is ordered into proving three sub-statements:

1. For $i < j < m$, $\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})$
2. $\mathbb{W}_{\mathcal{C}}(t_{m-1}, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_m, \mathcal{T})$
3. For $m \leq i < j$, $\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) \geq \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})$

Together, these show that $\mathcal{C}$ is ordered.

We begin with the first sub-statement. Let $i < j < m$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) &= \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_k\}) \\
&\geq \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_j, \mathcal{S} \cup \{t_k\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})
\end{aligned}$$

The first and last equalities follow from the structure of $\mathcal{C}$, and the inequality follows from $\mathcal{B}$ being ordered.

Now the second sub-statement.

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_{m-1}, \mathcal{T}) &= \sum_{k=m}^{n+m-1} \mathbb{W}_{\mathcal{A}}(t_k, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_{m-1}, \mathcal{S} \cup \{t_k\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_m, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_{m-1}, \mathcal{S} \cup \{t_m\}) \\
&\geq \mathbb{W}_{\mathcal{A}}(t_m, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_m, \mathcal{S} \cup \{t_m\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_m, \mathcal{T})
\end{aligned}$$

The equalities follow from the structure of $\mathcal{C}$, the first inequality follows from probabilities being non-negative, and the second inequality follows from $\mathcal{B}$ being ordered.

Finally, we show the third sub-statement. Let $m \le i < j$. Then,

$$\begin{aligned}
\mathbb{W}_{\mathcal{C}}(t_i, \mathcal{T}) &= \mathbb{W}_{\mathcal{A}}(t_i, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_i\}) \\
&\ge \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_i, \mathcal{S} \cup \{t_i\}) \\
&\ge \mathbb{W}_{\mathcal{A}}(t_j, \mathcal{R}) \cdot \mathbb{W}_{\mathcal{B}}(t_j, \mathcal{S} \cup \{t_j\}) \\
&= \mathbb{W}_{\mathcal{C}}(t_j, \mathcal{T})
\end{aligned}$$

The equalities follow from the structure of $\mathcal{C}$, the first inequality from $\mathcal{A}$ being ordered, and the second inequality from the teams being SST.

We have shown all three sub-statements, and so $\mathcal{C}$ is ordered. □

We can now construct the set of brackets that we have thus far shown are ordered by starting with $\{[[\mathbf{1}]], [[\mathbf{2}; \mathbf{0}]], [[\mathbf{4}; \mathbf{0}; \mathbf{0}]]\}$ and then repeatedly applying the stapling lemma on the set to expand it. In other words,

Corollary 2.5.3

Any bracket signature formed by the following process is ordered:

1. Start with the list $[[\mathbf{0}]]$ (note that this not yet a bracket signature).
2. As many times as desired, prepend the list with $[[\mathbf{1}]]$ or $[[\mathbf{3}; \mathbf{0}]]$.
3. Then, add 1 to the first element in the list, turning it into a bracket signature.

Corollary 2.5.3 uses the tools that we have developed so far to demark a set of brackets as ordered. Somewhat surprisingly, this set is complete: any bracket not reachable using the process in Corollary 2.5.3 is not ordered. To prove this we first need to show the containment lemma.

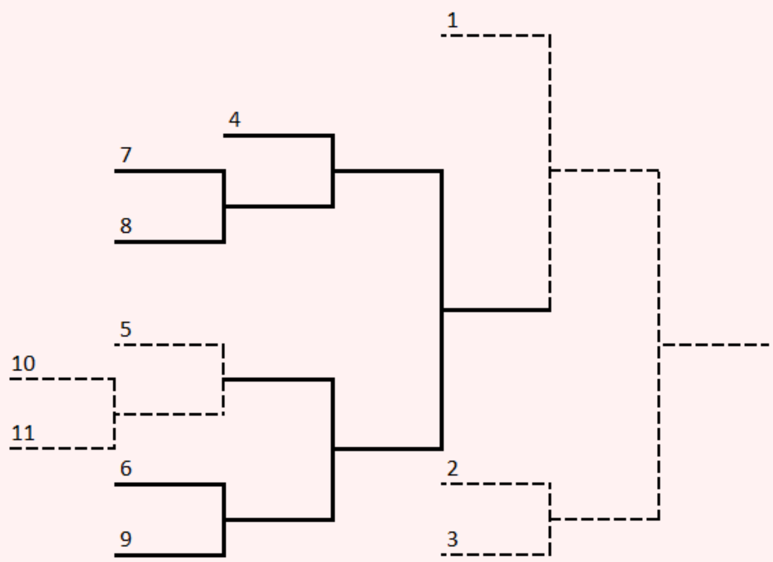
Definition 2.5.4: Containment

Let $\mathcal{A}$ and $\mathcal{B}$ be bracket signatures. We say $\mathcal{A}$ *contains* $\mathcal{B}$ *in round* i if

- At least as many games are played in the $(i+1)$th round of $\mathcal{A}$ as in the first round of $\mathcal{B}$, and

- For $j > 1$, there are exactly as many games played in the $(i+j)$th round of $\mathcal{A}$ as in the jth round of $\mathcal{B}$.

Figure 2.5.5: Setup of the Containment Lemma with $\mathcal{A} = [[\mathbf{2};\mathbf{5};\mathbf{1};\mathbf{0};\mathbf{3};\mathbf{0};\mathbf{0}]]$ and $\mathcal{B} = [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$.



Lemma 2.5.6: The Containment Lemma

If $\mathcal{A}$ contains $\mathcal{B}$, and $\mathcal{B}$ is not ordered, then neither is $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be a bracket signature with r rounds and n teams, and let $\mathcal{B}$ have s round and m teams, such that $\mathcal{A}$ contains $\mathcal{B}$ in round i, and $\mathcal{B}$ is not ordered. Let k be the number of teams in $\mathcal{A}$ that get at least $s+i$ byes.

$\mathcal{B}$ is not ordered, so let $\mathbf{M}$ be a matchup table that violates the orderedness condition, where none of the win probabilities are 0. (If we have an $\mathbf{M}$ that includes 0s, we can replace them with ϵ. For small enough ϵ, $\mathbf{M}$ will still violate the condition.) Let p be the minimum probability in $\mathbf{M}$. Let $\mathbf{P}$ be a matchup table in which the lower-seeded team wins with probability p, and let $\mathbf{Z}$ be a matchup table in which the lower-seeded team wins with probability 0.

Now, consider the following block matchup table on $\mathcal{T}$, a list of n teams:

	t_1 - t_k	t_{k+1} - t_{k+m}	t_{k+m+1} - t_n
t_1 - t_k	**P**	**P**	**Z**
t_{k+1} - t_{k+m}	**P**	**M**	**Z**
t_{k+m+1} - t_n	**Z**	**Z**	**Z**

Let $\mathcal{S} \subset \mathcal{T}$ be the sublist of teams seeded between $k+1$ and $k+m$. Then, for $t_j \in \mathcal{S}$,

$$\mathbb{W}_{\mathcal{A}}(t, \mathcal{T}) = \mathbb{W}_{\mathcal{B}}(t, \mathcal{S}) \cdot p^{r-s-i},$$

since t_j wins any games it might have to play in rounds i or before automatically, any games after $s+i$ with probability p, and any games in between according to **M**.

However, **M** (and thus $\mathcal{S}$) violates the orderedness condition for $\mathcal{B}$, and so $\mathcal{T}$ does for $\mathcal{A}$. □

With the containment lemma shown, we can proceed to the main theorem:

Theorem 2.5.7: Edwards's Theorem

The only ordered brackets are those described by Corollary 2.5.3.

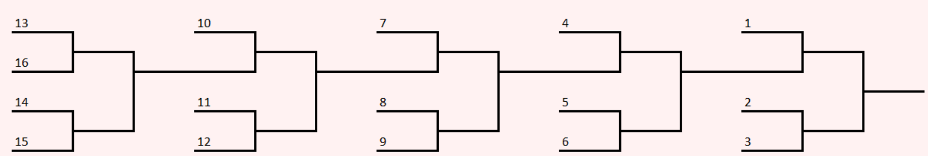
Proof. Let $\mathcal{A}$ be a proper bracket not described by Corollary 2.5.3. The corollary describes all proper brackets in which each round either has only game, or has two games but is immediately followed by a round with only one game. Thus $\mathcal{A}$ must include at least two successive rounds with two or more games each.

The final round in such a chain will be followed by a round with a single game, and so the final round must have only two games. Thus, $\mathcal{A}$ includes a sequence of three rounds, the first of which has at least two games, the second of which has exactly two games, and the third of which has one game.

Therefore, $\mathcal{A}$ contains $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$. But we know that $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$ is not ordered, and so by the containment lemma, neither is $\mathcal{A}$. $\square$

Edwards's Theorem is both exciting and disappointing. On one hand, it means that we can fully describe the set of ordered brackets, making it easy to check whether a given bracket is ordered or not. On the other hand, it means that in an ordered bracket at most three teams can be introduced each round, so the length of the shortest ordered bracket on n teams grows linearly with n (rather than logarithmically as is the case for proper brackets). If we want a bracket on many teams to be ordered, we risk forcing lower-seeded teams to play large numbers of games, and we only permit the top-seeded teams to play a few. For example, the shortest ordered bracket that could've been used in the 2022 NCAA Women's Basketball Wichita Region is $[[\mathbf{4};\mathbf{0};\mathbf{3};\mathbf{0};\mathbf{3};\mathbf{0};\mathbf{3};\mathbf{0};\mathbf{3};\mathbf{0};\mathbf{0}]]$, which is played over a whopping ten rounds.

Figure 2.5.8: The Shortest Sixteen-Team Ordered Bracket



Because of this, few leagues use ordered brackets, and those who do usually have so few teams that every proper bracket is ordered (the 2023 College Football Playoffs, for example). Even the Korean Baseball Organization League, which uses a somewhat unconventional $[[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{1};\mathbf{0}]]$, only sends five teams to the playoffs, and again every five-team proper bracket is ordered. If the KBO League ever expanded to the six-team bracket $[[\mathbf{2};\mathbf{1};\mathbf{1};\mathbf{1};\mathbf{1};\mathbf{0}]]$, we would have a case of an ordered bracket being used when a proper non-ordered bracket exists on the same number of teams.

Open Question 2.5.9

Which brackets are ordered if $\mathcal{T}$ is not SST?

Edward's Theorem assumes that the inputted list of teams are SST, and in particular, are in the correct order such that better teams are seeded higher. In real life, however, seeding is not that simple: if we knew who the best team was going into the tournament, we wouldn't need to conduct the tournament at all. Instead, seeding is often a guess as to the strengths of various teams based on the information known at the time, and in some cases, such as the 2023 Major League Baseball Playoffs, certain higher-seeded are generally agreed to be weaker than certain lower-seeded teams. Which brackets remain ordered under weaker assumptions than SST is still open.

3 Alternative Seedings

3.1 Reseeding

Edwards's Theorem naturally raises the question: is there some tournament format that mantains the basic idea of the bracket, but expands the space of signatures that are ordered. *Reseeded* brackets are our first attempt at an answer.

Ultimately, the reason that proper brackets are not, in general, ordered, is that lower-seeded teams are treated, if they win, as the team that they beat for the rest of the format. Consider again the proper bracket analyzed by Silver: $[[\mathbf{16}; \mathbf{0}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$. If an 11-seed wins in the first round, they take on the schedule of a 6-seed for the rest of the tournament, while if the 9-seed wins, they take on the schedule of an 8-seed. Given that a 6-seed has an easier schedule than an 8-seed, it's not hard to see why it might be preferable to be an 11-seed rather than a 9-seed.

Reseeding (poorly named) fixes this by resorting the match-ups every round: if an 11-seed keeps winning, they will have to play teams according to their seed, rather than getting an effective upgrade to 6-seed status.

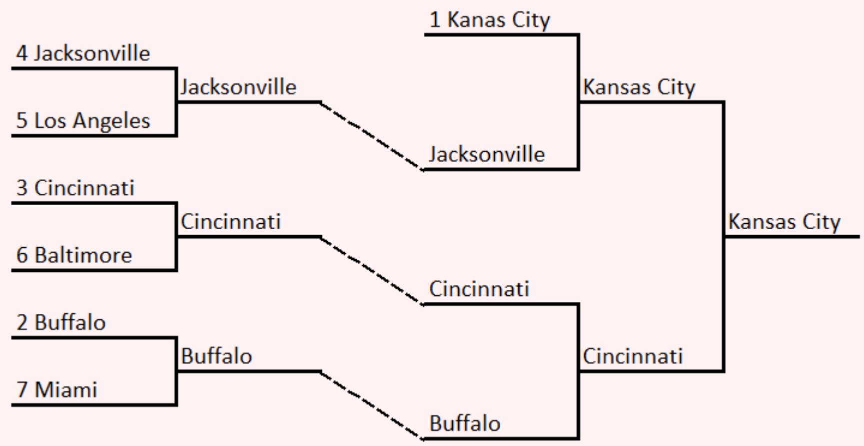
Definition 3.1.1: Reseeded Brackets

In a *reseeded* bracket, after each round, match up the highest-seeded team with the lowest-seeded team, second-highest vs second-lowest, etc.

Note that by Definition 2.1.1, a reseeded bracket is not a bracket at all, as matchups between teams that have not yet lost are not determined in advance of the outcomes of any games. However, because reseeded brackets act so similarly to traditional brackets, and because colloquially they are referred to as brackets, we opt to continue using the word "bracket" to describe them.

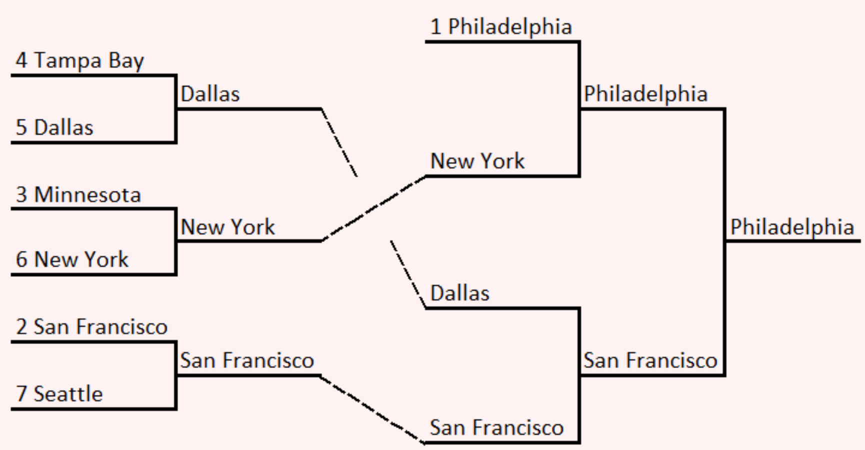
Both National Football League conferences use a reseeded bracket with signature $[[\mathbf{6}; \mathbf{1}; \mathbf{0}; \mathbf{0}]]^R$. (The superscript R indicates this is reseeded bracket.) If the first round of the bracket goes chalk, then it looks just like a normal bracket:

Figure 3.1.2: 2023 National Football League AFC Playoffs



The dotted lines are drawn after the first round of games has been played: if there are some first-round upsets, then the bracket is rearranged to ensure that it is still better to be a higher seed rather than a lower seed.

Figure 3.1.3: 2023 National Football League NFC Playoffs



In the NFC, 6-seed New York upset 3-seed Minnesota. Had a conventional bracket been used, the semifinal matchups would have been 1-seed vs 5-seed and 2-seed vs 6-seed: the 2-seed would have had an easier draw than the 1-seed, while the 6-seed would have an easier draw than the 5-seed. Reseeding fixes this by matching 6-seed New York is with top-seed Philadelphia, and

2-seed San Francisco with 5-seed Dallas.

Reseeding is a powerful technique. For one, the fundamental theorem still applies to reseeded brackets, allowing us to refer to reseeded brackets by their signatures as well.

Theorem 3.1.4

There is exactly one proper reseeded bracket with each bracket signature.

Proof. The definition of properness ensures that there is only one way byes can be distributed such that a reseeded bracket can be proper. Additionally, because reseeded brackets have no additional parameters beyond which seeds get how many byes, there is no more than one reseeded bracket with each signature that could be proper. Finally, that bracket is indeed proper: if the bracket goes to chalk, the matchups will be the exact same as a traditional bracket, which by the fundamental theorem is a proper set of matchups. □

But what about orderedness? It's intuitive to think that all proper reseeded are ordered: it feels like almost by definition, the higher-seeded teams have an easier path than the lower-seeded ones. Hwang [5] conjectured exactly this.

Conjecture 3.1.5

All proper reseeded brackets are ordered.

Unfortunately, this is not true. Our classification of the ordered reseeded brackets takes the same route as our proof of Edwards's Theorem did: we first examine the orderedness of certain important brackets, and then we use the stapling and containment lemmas to specify the complete set of ordered reseeded brackets.

Note that the proofs of the stapling and containment lemmas for reseeded brackets, as well as the fact that all ordered reseeded brackets are proper, are so similar to the corresponding proofs for traditional brackets that we just state them without proof.

Theorem 3.1.6

All ordered reseeded brackets are proper.

Lemma 3.1.7: The Stapling Lemma for Reseeding

If $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]^R$ and $\mathcal{B} = [[\mathbf{b_0}; ...; \mathbf{b_s}]]^R$ are ordered reseeded brackets, then $\mathcal{C} = [[\mathbf{a_0}; ...; \mathbf{a_r} + \mathbf{b_0} - \mathbf{1}; ...; \mathbf{b_s}]]^R$ is an ordered reseeded bracket as well.

Lemma 3.1.8: The Containment Lemma for Reseeding

If $\mathcal{A}$ and $\mathcal{B}$ are reseeded brackets, $\mathcal{A}$ contains $\mathcal{B}$, and $\mathcal{B}$ is not ordered, then neither is $\mathcal{A}$.

We now examine particular brackets.

Theorem 3.1.9

$[[\mathbf{1}]]^R$, $[[\mathbf{2}; \mathbf{0}]]^R$, and $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]^R$ are ordered.

Proof. Since no reseeding is done in a bracket of two or fewer rounds, and since the traditional brackets of these signatures are ordered, so are the reseeded brackets. □

Our primary example of a reseeded bracket that is ordered despite the traditional bracket of the same signature not being ordered is $[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]^R$.

Theorem 3.1.10

$[[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]]^R$ is ordered.

Proof. This can be shown by computing the probability of each team winning the format and then applying the SST conditions to establish the inequalities, as we did in Theorem 2.4.14. In the interest of brevity, however, we instead give an intuitive argument.

$\mathbb{W}_A(t_1, \mathcal{T}) \geq \mathbb{W}_A(t_2, \mathcal{T})$ because from those two teams perspectives, this format is just $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]^R$. $\mathbb{W}_A(t_2, \mathcal{T}) \geq \mathbb{W}_A(t_3, \mathcal{T})$ because t_2

has better odds if t_3 wins in the first round and they meet in the semifinals, and certainly has better odds if t_3 loses in the first round. $\mathbb{W}_A(t_4, \mathcal{T}) \geq \mathbb{W}_A(t_5, \mathcal{T})$ because t_4 is at least as likely to win the first-round matchup, and then their paths would be identical.

$\mathbb{W}_A(t_3, \mathcal{T}) \geq \mathbb{W}_A(t_4, \mathcal{T})$ holds because if both teams win the first round then t_3 has better odds in the remaining $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]^R$ bracket. Meanwhile if only one does, then t_3 will be joined by t_5 while t_4 will be joined by t_6, and so t_3 is more likely to dodge playing t_1 in the finals. The same argument applies to show that $\mathbb{W}_A(t_5, \mathcal{T}) \geq \mathbb{W}_A(t_6, \mathcal{T})$ as well. $\square$

Unfortunately, that is where the power of reseeding to convert non-ordered signatures into ordered ones ends. The following two signatures are not ordered:

Theorem 3.1.11

$[[\mathbf{6}; \mathbf{1}; \mathbf{0}; \mathbf{0}]]^R$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{6}; \mathbf{1}; \mathbf{0}; \mathbf{0}]]^R$, and let $\mathcal{T}$ have the following matchup table:

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	0.5	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$
t_2	p	0.5	$1-p$	$1-p$	$1-p$	$1-p$	$1-p$
t_3	p	p	0.5	0.5	0.5	$1-p$	$1-p$
t_4	p	p	0.5	0.5	0.5	0.5	0.5
t_5	p	p	0.5	0.5	0.5	0.5	0.5
t_6	p	p	p	0.5	0.5	0.5	0.5
t_7	p	p	p	0.5	0.5	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_6, \mathcal{T}) = O(p^3),$$

but

$$\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) = 0.25p^2 + O(p^3).$$

Thus, for small enough p, $\mathbb{W}_{\mathcal{A}}(t_6, \mathcal{T}) < \mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T})$, so $\mathcal{A}$ is not ordered. $\square$

Theorem 3.1.12

$[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ is not ordered.

Proof. Let $\mathcal{A} = [[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$, and let $\mathcal{T}$ have the following matchup table:

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
t_1	0.5	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$	$1-p^2$
t_2	p^2	0.5	0.5	0.5	$1-p$	$1-p$	$1-p^2$	$1-p^2$
t_3	p^2	0.5	0.5	0.5	$1-p$	$1-p$	$1-p$	$1-p$
t_4	p^2	0.5	0.5	0.5	0.5	$1-p$	$1-p$	$1-p$
t_5	p^2	p	p	0.5	0.5	$1-p$	$1-p$	$1-p$
t_6	p^2	p	p	p	p	0.5	$1-p$	$1-p$
t_7	p^2	p^2	p	p	p	p	0.5	0.5
t_8	p^2	p^2	p	p	p	p	0.5	0.5

Then

$$\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) = 0.25p^5 + O(p^6)$$

but

$$\mathbb{W}_{\mathcal{A}}(t_8, \mathcal{T}) = 0.5p^5 + O(p^6).$$

Thus, for small enough p, $\mathbb{W}_{\mathcal{A}}(t_7, \mathcal{T}) < \mathbb{W}_{\mathcal{A}}(t_8, \mathcal{T})$, so $\mathcal{A}$ is not ordered. □

Recapping,

Figure 3.1.13: Which Proper Reseeded Brackets are Ordered

Ordered	Not Ordered
$[[\mathbf{1}]]^R$	$[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$
$[[\mathbf{2};\mathbf{0}]]^R$	$[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$
$[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$	
$[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$	

Finally, we apply the stapling and containment lemmas to complete the theorem.

Theorem 3.1.14

The ordered reseeded brackets are exactly those corresponding to signatures that can be generated in the following way:

1. Start with the list $[[\mathbf{0}]]^R$ (note that this not yet a bracket signature).
2. As many times as desired, prepend the list with $[[\mathbf{1}]]$, $[[\mathbf{3};\mathbf{0}]]$, or $[[\mathbf{3};\mathbf{2};\mathbf{0}]]$.
3. Then, add 1 to the first element in the list, turning it into a bracket signature.

Proof. The stapling lemma, combined with the fact that $[[\mathbf{1}]]^R$, $[[\mathbf{2};\mathbf{0}]]^R$, $[[\mathbf{4};\mathbf{0};\mathbf{0}]]^R$, and $[[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$ are ordered, ensure that any reseeded brackets generated by the above procedure is indeed ordered. Left is to use the containment lemma to ensure that these are the only ones.

Let $\mathcal{A}$ be a bracket signature that cannot be generated by the procedure. Then, either there is a round in which three or more games are to be played, or there is a round in which two games and the next two rounds each have two games played as well.

Let i be the latest such round. If round i is the first of three rounds with two games each, then round $i+3$ must have only one game played (otherwise i would not be the latest such round). But then $\mathcal{A}$ contains $[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$, and so is not ordered.

If round i has three or more games, then round $i+1$ must contain exactly two games (any less and not every winner would have a game, any more and i would not be the latest such round.) Then, if round $i+2$ has one game, then $\mathcal{A}$ contains $[[\mathbf{6};\mathbf{1};\mathbf{0};\mathbf{0}]]^R$, and if it has two, then $\mathcal{A}$ contains $[[\mathbf{4};\mathbf{2};\mathbf{2};\mathbf{0};\mathbf{0}]]^R$. In either case, $\mathcal{A}$ is not ordered.

Thus, the ordered reseeded brackets are exactly those generated by the procedure. $\square$

So, the space of ordered reseeded brackets is slightly larger than the space of ordered traditional brackets, although perhaps this is not quite as much of an expansion as we would've liked or expected. This intuitive notion, however, that reseeded brackets are "more" ordered than their traditional counterparts of the same signature, even if neither is ordered in the definitional sense, remains as an open question.

Conjecture 3.1.15

There is some reasonable restriction on a set of teams that is stronger than SST that makes all reseeded brackets ordered.

In the meantime, however, reseeding is an important tool in our tournament design toolkit. But it is not without its drawbacks.

In a reseeded bracket, teams and spectators alike don't know who they will play or where their next game will be until the entire previous round is complete. This can be an especially big issue if parts of the bracket are being played in different locations on short turnarounds: in the 2022 NCAA Women's Basketball Tournament, the first two rounds are played over a weekend on the various college campuses of the highest-seeded teams. It would cause problems if teams had to pack up and travel across the country because they got reseeded and their opponent changed.

In addition, part of what makes the NCAA Basketball Tournament (affectionately known as "March Madness") such a fun spectator experience is the fact that these matchups are known ahead of time. In "bracket pools," groups of fans each fill out their own brackets, predicting who will win each game and getting points based on how many they get right. If it wasn't clear where in the bracket the winner of a given game was supposed to go, this experience would be diminished.

Finally, reseeding gives the top seed(s) an even greater advantage than they already have: instead of playing against merely the *expected* lowest-seeded team(s) each round, they would get to play against the *actual* lowest-seeded team(s). In March Madness, "Cinderella Stories," that is, deep runs by low seeds, would become much less common.

In many ways, the NFL conference playoffs are a perfect place to use a reseeded bracket: games are played once a week, giving plenty of time for travel; only seven teams make the playoffs in each, so a huge March Madness-style bracket challenge is unlikely; as a professional league, the focus is far more on having the best team win and protecting Cinderella Stories isn't as important; and because the bracket is only three rounds long, reseeding is only required once.

Other leagues with similar structures might consider adopting forms of reseeding to protect their incentives and competitive balance (looking at you, Major League Baseball), but in many cases, the traditional bracket structure is too appealing to adopt a reseeded one.

In the coming sections, we will develop the framework of *tiered seeding* which will be used in our next attempt to generate ordered brackets of arbitrary signatures: *cohort randomized seeding*.

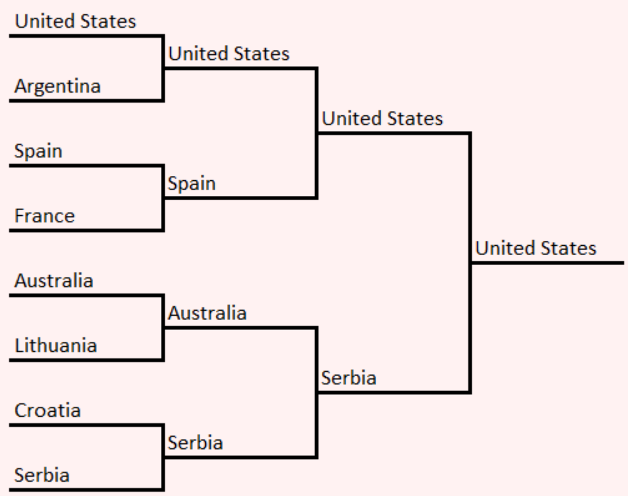
3.2 Tiered Seeding

Consider the 2016 Olympic Basketball Tournament. Twelve teams qualified for the Olympics, and they were divided into two groups of six teams each. Each group conducted a mini-tournament, ranking the teams in each group from first through sixth (the specifics of the mini-tournament are not relevant). Then, the bottom two teams in each group were eliminated, with the remaining eight teams (four from each group) entering the bracket $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]]$. The entire format as it played out is displayed in Figure 3.2.1.

Figure 3.2.1: The 2016 Olympic Basketball Tournament

Group A	
Rank	Team
1	United States
2	Australia
3	France
4	Serbia
5	Venezuela
6	China

Group B	
Rank	Team
1	Croatia
2	Spain
3	Lithuania
4	Argentica
5	Brazil
6	Nigeria



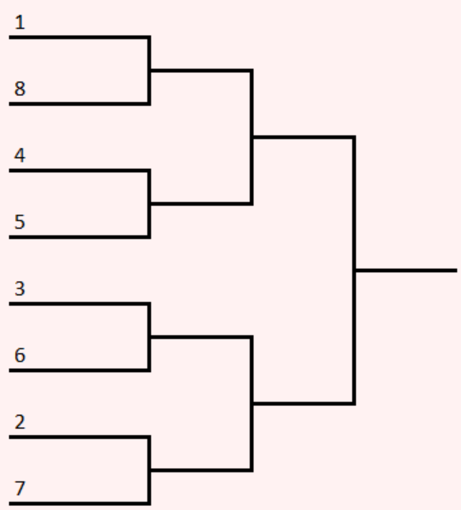
The seeding going into the bracket portion of the 2016 Olympic Basketball Tournament is a little different than the seedings that we have discussed so far. Rather than the ranking of the teams being a complete ordering, it is a partial one: teams are grouped into tiers, and the tiers are ranked. Two teams, one from each group, occupy each tier.

Definition 3.2.2: Tiered Seeding

A *tiered seeding* is a partial ordering on the teams entering a tournament.

This is as opposed to traditional seedings, which are a complete ordering (although we can view a traditional seeding as a special example of a tiered seeding where each tier has a single team). When filling out a bracket using a tiered seeding, we continually assign the top remaining seeds to the teams in the top remaining tier. Recall the proper bracket $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$:

Figure 3.2.3: $[[8; 0; 0; 0]]$



The United States and Croatia, as the two teams in the top tier, are given seeds one and two. The two tier-two teams, Australia and Spain, get seeds three and four, and so on. The actual algorithm used for assigning the seeds to the teams within each tier can be arbitrary: in the particular case of the 2016 Olympic Basketball Tournament, teams from Group A were given the odd seeds and teams from Group B the evens.

We can describe a tiered seeding with a list of integers indicating how many teams are in each tier. The eight teams that advanced to bracket at the Olympics were divided into four pools of two teams each, so we write $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$. A quick notational note: we list tier sizes in reverse order, with the size of the lowest tier coming first, and the size of the top tier coming last. This is done to keep it consistent with bracket signatures, in which the lower-seeded teams are listed earlier, and higher-seeded teams that get more byes are listed later.

The tiered seeding $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$ interacts very nicely with the proper bracket $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$: there is no advantage for a team being assigned a particular seed within their tier.

Definition 3.2.4: Strongly Respecful

A bracket *strongly respects* a tiered seeding if, as long as teams' win probabilities are defined only by what tier they are in (that is, for all t_i, t_j that share a tier and t_k, t_ℓ that share a tier, $\mathbb{P}[t_i \text{ beats } t_k] = \mathbb{P}[t_j \text{ beats } t_\ell]$), then teams in the same tier have the same probability of winning the tournament.

Sometimes, it is not possible to generate a bracket that strongly respects a tiered seeding (for example, the tiered seeding $(\mathbf{3}, \mathbf{1})$), so we also introduce the concept of a bracket weakly respecting a tiered seeding.

Definition 3.2.5: Weakly Respectful

A bracket *weakly respects* a tiered seeding if each team in a tier is given the same number of byes.

Although no signature strongly respects the tiered seeding $(\mathbf{3}, \mathbf{1})$, the bracket $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]]$ is preferable to $[[\mathbf{2}; \mathbf{1}; \mathbf{1}; \mathbf{0}]]$ because at least it weakly respects it. The names of the two conditions come from strong respectfulness being a stronger condition than weak respectfulness.

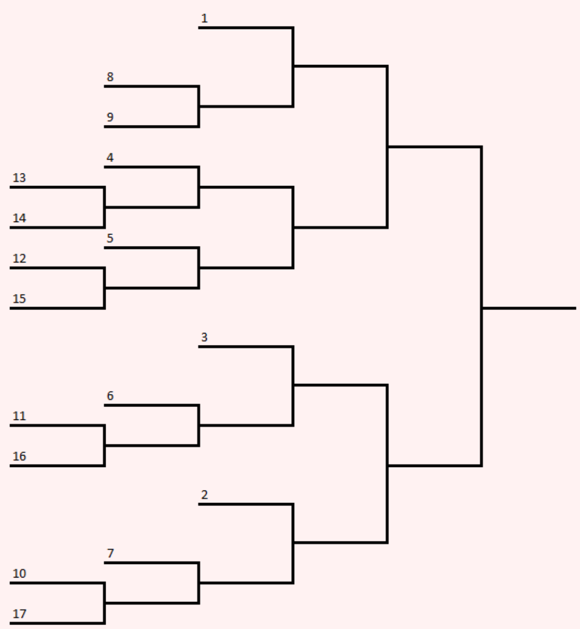
Theorem 3.2.6

If a bracket strongly respects a tiered seeding then it weakly respects it as well.

Proof. If a bracket strongly respects a tiered seeding, then all teams within the same tier must have the same probability of winning the tournament if every game is a coin flip. If indeed every game is coinflip, two teams have the same chance of winning the tournament only if they have the same number of byes, so the bracket must weakly respect the tiered seeding as well. □

How can we tell whether an arbitrary bracket respects an arbitrary tiered seeding? Weak respectfulness is somewhat straightforward to check: we can simply matchup the signature with the tiered seeding to see if we ever have to split a tier across two different levels in the signature. For example, consider the bracket signature $\mathcal{A} = [[\mathbf{8}; \mathbf{6}; \mathbf{3}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$ and the tiered seeding $(\mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{1})$. The four teams in the second-highest tier are distributed over two levels: two the them (seeds 2 and 3) get two byes and two of them (seeds 4 and 5) get a single bye, so $[[\mathbf{8}; \mathbf{6}; \mathbf{3}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$ does not weakly respect $(\mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{1})$.

Figure 3.2.7: $[[8; 6; 3; 0; 0; 0]]$



If a bracket signature *does* weakly respect a tiered seeding, we can combine the information of the bracket signature and the tiered seeding into a single list of lists called the *tiered signature*.

Definition 3.2.8: Tiered Signature

If a bracket signature $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ weakly respects a tiered signature $\mathcal{B}$, then the *tiered signature* of the signature-seeding pair $(\mathcal{A}, \mathcal{B})$ is a list $\mathcal{C} = [[\mathcal{C}_\mathbf{0}; ...; \mathcal{C}_\mathbf{r}]]$ where $\mathcal{C}_i$ is the sublist of $\mathcal{B}$ corresponding to the a_i teams that get i byes.

The bracket $\mathcal{A} = [[\mathbf{8}; \mathbf{6}; \mathbf{3}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$ weakly respects the tiered seeding $\mathcal{B} = (\mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{1})$, and the associated tiered signature of this pair is

$$\mathcal{C} = [[(\mathbf{4}, \mathbf{4}); (\mathbf{4}, \mathbf{2}); (\mathbf{2}, \mathbf{1}); (); (); ()]].$$

The somewhat trivial tiered signature of the 2016 Olympic Basketball Tournament is

$$[[(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}); (); (); ()]].$$

Note that we can easily extract both the bracket signature and the tiered seeding from the tiered signature. For the former, sum each sublist, and for the latter, concatenate the sublists into a single list. Sometimes, we will refer a tiered signature as being strongly respectful as a shorthand for saying that the associated tiered seeding respects the associate bracket signature.

Checking for strong respectfulness seems to be much trickier than weak respectfulness. Somehow, we need to be able to verify that for any distribution of win probabilities, (as long as teams within the same tier have the same matchup table,) teams within the same tier have the same probability of winning the tournament. Luckily, there is a simple algorithm for doing just that, which we will explore in the next section.

3.3 The Palandromic Algorithm

In this section, we present an algorithm for verifying whether a bracket signature strongly respects a tiered seeding. We will first intuitively describe what the algorithm is doing and then we will formally specify it, before running the algorithm on a few examples and then finally proving its correctness.

The idea behind the algorithm is to ensure that in each round, teams of the same tier are being assigned opponents of the same tier. This is done by keeping track of the tiers of the teams that will be playing in each round, and ensuring that the round-specific tiered signatures are palandromic. Formally,

Definition 3.3.1: The Palandromic Algorithm for Tiered Signatures

Let $\mathcal{A}$ be a bracket signature and $\mathcal{B}$ be a tiered seeding. First, check if $\mathcal{A}$ weakly respects $\mathcal{B}$. If it doesn't, then it certainly doesn't strongly respect it. If it does, then let $\mathcal{C}$ be the tiered signature of $(\mathcal{A}, \mathcal{B})$.

We define $\mathcal{F}$, a recursive operator that maps a tiered signature to either **true** or **false**. Then, if $\mathcal{F}(\mathcal{C})$ is true, $\mathcal{A}$ strongly respects $\mathcal{B}$, otherwise it does not.

The operator $\mathcal{F}$ is defined in the following way on $\mathcal{C} = [[\mathcal{C}_\mathbf{0}; ...; \mathcal{C}_\mathbf{r}]]$.

- If $r = 0$, then $\mathcal{F}(\mathcal{C})$ is **true**.
- Otherwise, if $\mathcal{C}_0$ is not palandromic, then $\mathcal{F}(\mathcal{C})$ is **false**.
- Otherwise, let $\mathcal{D}_0$ be the right half of $\mathcal{C}_0$ concatenated with $\mathcal{C}_1$, and $\mathcal{D} = \mathcal{F}([[\mathcal{D}_\mathbf{0}; \mathcal{C}_\mathbf{2}; ...; \mathcal{C}_\mathbf{r}]])$. Then, $\mathcal{F}(\mathcal{C}) = \mathcal{F}(\mathcal{D})$.

(For the last step, if $\mathcal{C}_0$ has odd length, then the first element of $\mathcal{D}_0$ is half of the middle element of $\mathcal{C}_0$. The middle element of $\mathcal{C}_0$ will always be even because it is palandromic and its sum must be even.)

Let's go over a few examples. Consider the bracket signature $\mathcal{A} = [[\mathbf{8}; \mathbf{6}; \mathbf{3}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$ along with the tiered seeding $\mathcal{B} = (\mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{1})$. As we verified earlier, $\mathcal{A}$ weakly respects $\mathcal{B}$, so we can apply the palandromic algorithm to check if it is strongly respectful.

$$\begin{aligned}\mathcal{F}(\mathcal{C}) &= \mathcal{F}([[(\mathbf{4},\mathbf{4});(\mathbf{4},\mathbf{2});(\mathbf{2},\mathbf{1});();();()]]) \\ &= \mathcal{F}([[(\mathbf{4},\mathbf{4},\mathbf{2});(\mathbf{2},\mathbf{1});();();()]]) \\ &= \textbf{false} \text{ (because } (\mathbf{4},\mathbf{4},\mathbf{2}) \text{ is not palandromic.)}\end{aligned}$$

Figure 3.3.2: $[[(\mathbf{4},\mathbf{4});(\mathbf{4},\mathbf{2});(\mathbf{2},\mathbf{1});();();()]]$

We can verify this result intuitively with the help of the bracket $\mathcal{A}$. In the second round, for example, two of the Tier 4 teams play each other, while two of them play the winner of a Tier 5 vs Tier 6 matchup. If the Tier 5 and 6 teams are much worse than the rest of the teams, it is not hard to imagine that the two Tier 4 teams who have to play each other are at a severe disadvantage.

Let's instead consider the bracket signature $\mathcal{A} = [[\mathbf{8};\mathbf{0};\mathbf{6};\mathbf{0};\mathbf{0};\mathbf{1};\mathbf{0}]]$ along with the tiered seeding $\mathcal{B} = (\mathbf{8},\mathbf{4},\mathbf{2},\mathbf{1})$. $\mathcal{A}$ weakly respects $\mathcal{B}$ with tiered signature

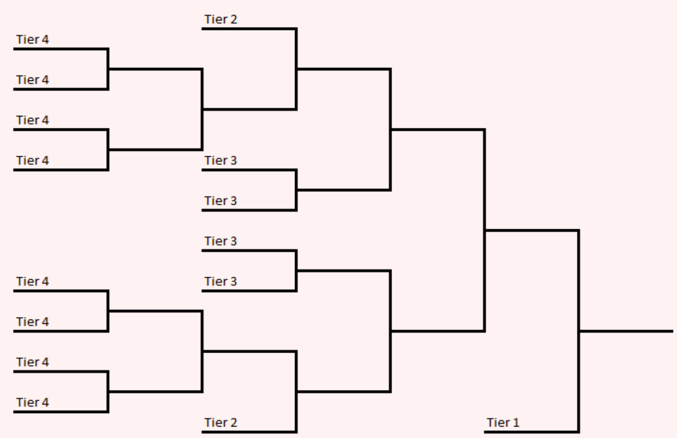
$$\mathcal{C} = [[(\mathbf{8});();(\mathbf{4},\mathbf{2});();();(\mathbf{1});()]]$$

Applying the palandromic algorithm,

$$\begin{aligned}
\mathcal{F}(\mathcal{C}) &= \mathcal{F}([[(\mathbf{8}); (); (\mathbf{4}, \mathbf{2}); (); (); (\mathbf{1}); ()]]) \\
&= \mathcal{F}([[(\mathbf{4}); (\mathbf{4}, \mathbf{2}); (); (); (\mathbf{1}); ()]]) \\
&= \mathcal{F}([[(\mathbf{2}, \mathbf{4}, \mathbf{2}); (); (); (\mathbf{1}); ()]]) \\
&= \mathcal{F}([[(\mathbf{2}, \mathbf{2}); (); (\mathbf{1}); ()]]) \\
&= \mathcal{F}([[(\mathbf{2}); (\mathbf{1}); ()]]) \\
&= \mathcal{F}([[(\mathbf{1}, \mathbf{1}); ()]]) \\
&= \mathcal{F}([[(\mathbf{1})]]) \\
&= \textbf{true}
\end{aligned}$$

So $\mathcal{A}$ does strongly respect $\mathcal{B}$. This can also be seen intuitively by looking at the bracket: teams in each tier have the same exact path throughout the tournament.

Figure 3.3.3: $[[(\mathbf{8}); (); (\mathbf{4}, \mathbf{2}); (); (); (\mathbf{1}); ()]]$



Finally, we leave as an exercise to the reader to use the palandromic algorithm to show that the 2016 Olympic Basketball Tournament was strongly respectful.

Hopefully, these three examples have given a sense as to why the palandromic algorithm accurately ascertains whether a bracket signature strongly respects a tiered seeding. We will prove it by induction.

Theorem 3.3.4

The palandromic algorithm correctly ascertains whether a bracket signature strongly respects a tiered seeding.

Proof. Let $\mathcal{A}$ be a bracket signature with r rounds and $\mathcal{B}$ be a tiered seeding. If $\mathcal{A}$ doesn't weakly respect $\mathcal{B}$, then the palandromic algorithm will correctly say that it doesn't strongly respect $\mathcal{B}$ either. Assume then that $\mathcal{A}$ does weakly respect $\mathcal{B}$, where $\mathcal{C} = [[\mathcal{C}_{\mathbf{0}}; ...; \mathcal{C}_{\mathbf{r}}]]$. is the tiered signature of the pair $(\mathcal{A}, \mathcal{B})$.

We proceed by induction on r. If $r = 0$, then $\mathcal{A} = [[\mathbf{1}]]$, $\mathcal{B} = (\mathbf{1})$, and $\mathcal{C} = [[(\mathbf{1})]]$. The palandromic algorithm will correctly claim that $\mathcal{A}$ strongly respects $\mathcal{B}$ without any recursive calls.

For any other r, we will show that the palandromic algorithm returns **false** if and only if $\mathcal{A}$ does not strongly respect $\mathcal{B}$.

Assume first that A does not strongly respect B. Then, for some tier, either teams in that tier are not all equally likely to make it out of the first round, or they are not all equally likely to win the bracket, conditional on having made it out of the first round. In the former case, this would be caused by teams in the same tier having first-round matchups in different tiers, meaning C_0 would not be palandromic, and so the palandromic algorithm would fail on its first iteration. In the latter case, this would imply that $\mathcal{D} = [[\mathcal{D}_{\mathbf{0}}; \mathcal{C}_{\mathbf{2}}; ...; \mathcal{C}_{\mathbf{r}}]]$ is not a strongly respectful tiered signature, (where $\mathcal{D}_0$ is the right half of $\mathcal{C}_0$ concatenated with $\mathcal{C}_1$), so by induction, $\mathcal{F}(\mathcal{C}) = \mathcal{F}(\mathcal{D}) =$ **false**. In either case, the palandromic algorithm correctly identifies that A does not strongly respect B.

Now, assume that the palandromic algorithm returns **false**. If it did so in the first iteration, then that means that there are two tiers T_0, T_1 for which some but not all teams in T_0 are matched up in the first-round against teams in T_1. Consider a list of teams such that teams in T_1 always lose, and all other games are coin-flips. Then, the teams in T_0 matched up against T_1 teams in the first-round will win the tournament with probability $(0.5)^{r-1}$, while the teams that are not will win with probability 0.5^r, so $\mathcal{A}$ does not strongly respect $\mathcal{B}$.

Meanwhile, if the palandromic algorithm failed at a later iteration, then by induction, $\mathcal{D} = [[\mathcal{D}_\mathbf{0}; \mathcal{C}_\mathbf{2}; ...; \mathcal{C}_\mathbf{r}]]$ is not a strongly respectful tiered signature, (where $\mathcal{D}_0$ is the right half of $\mathcal{C}_0$ concatenated with $\mathcal{C}_1$). However, if we consider a set of teams such that all of the first-round matchups in $\mathcal{C}$ are guaranteed wins for the higher tier, then a team's probability of winning the entire bracket (as long as they are in a tier that will win in the first-round) is the same as their probability of winning $\mathcal{D}$. Because $\mathcal{D}$ is not a strongly respectful tiered signature, some teams in the same tier have different tournament-win probabilities, so $\mathcal{C}$ is also not strongly respectful. Thus, $\mathcal{A}$ does not strongly respect $\mathcal{B}$.

So by induction, the palandromic algorithm claims that a bracket signature strongly respects a tiered seeding if and only if it truly does so. $\square$

With the palandromic algorithm in our back pocket, we can now quickly identify the relation between a given bracket signature and tiered seeding: whether it is strongly, weakly, properly, or not at all respectful. The concept of tiered seedings will show up in a few different places down the line: tiers are a powerful and generalizable tool for understanding tournament formats from Wimbledon to the NCAA Softball Tournament to the World Cup, as we shall investigate in the coming sections.

3.4 Cohort Randomized Seeding

4 Multibrackets

4.1 Simple Multibrackets

While the bracket is a very powerful and important tournament design, it has two intimately related shortcomings. First, brackets lead teams to play wildly different numbers of games: in the bracket $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$, for example, two teams will play three games, two teams will play two, and four teams will play only one.

And second, brackets tend to do a very poor job of ranking teams beyond just selecting a winner. Again considering the bracket $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$: the first-place finisher by definition the team that remains undefeated, and we can easily grant the loser of the championship game second place, but the two semifinals losers both might have a claim to the third place, and sorting through the fifth- through eight-place finishes is even trickier.

These problems are reflections of each other: the reason that ranking the lower-placing teams is so hard is because they play so few games. It's easier to differentiate between first and second because both teams have played three games, but differentiating between the four teams that have played only a single game is nigh impossible.

In some cases, these problems do not cause concern: perhaps we are only interested in crowning a champion and don't care about exactly who came in third, or maybe this bracket is being played at the conclusion of a long season and so teams playing variable numbers of games is not a big deal. But the interconnected nature of the two problems lets us solve them together, leveraging the extra games that lower-ranked teams have left in order to rank them.

To do this, we consider a class of formats called *simple multibrackets*.

Definition 4.1.1: Simple Multibracket

A *simple multibracket* is a sequence of brackets in which all teams begin in the first bracket, and then the losers of certain games in the upper brackets fall into the lower brackets rather than being eliminated outright, and teams place based on which bracket they won.

Definition 4.1.2: Primary Bracket

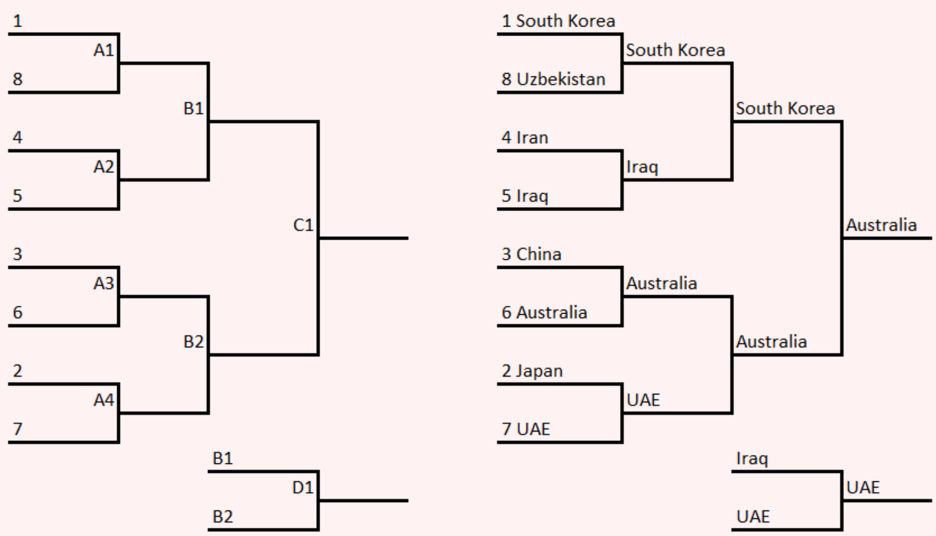
The first bracket in a multibracket is called the *primary bracket.*

Definition 4.1.3: Order of a Simple Multibracket

The *order* of a simple multibracket is the number of brackets it contains.

Simple multibrackets of order one are just traditional brackets. The simplest example of a simple multibracket with order greater than one is the third-place game, in which the losers of the two semifinal games play each other for third. The 2015 Asian Football Confederation Asian Cup, whose bracket is of signature $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]]$, employs a third-place game.

Figure 4.1.4: 2015 AFL Asian Cup



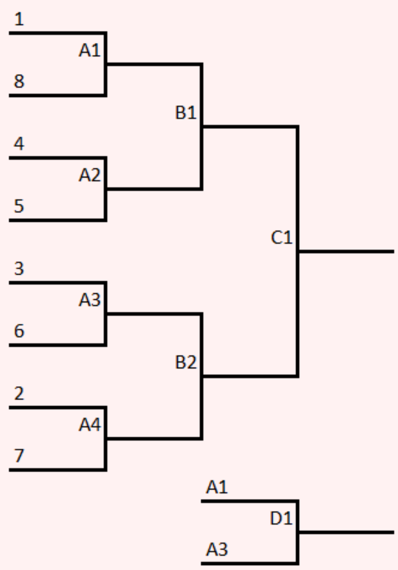
Each game in this figure is labeled. In the primary bracket, first-round games are **A1** through **A4**, while the semifinals are **B1** and **B2**, and the finals is game **C1**. The labeling system works as follows: each round in the first bracket is given a letter, and then each round in the second bracket, etc. Then games are assigned a number from the top of the bracket to the bottom within each round.

Thus, the third-place game is labeled **D1**: even though it could be played concurrently to the championship game, it is part of a different bracket and so we label it as a different round.

We indicate that the third-place game is to be played in between the losers of games **B1** and **B2** by labeling the starting lines in the third-place game with those games. This is not ambiguous because the winners of those games always continue on in the original bracket, so such labels only refer to the losers.

In theory, the Asian Cup could have had any two teams play for third.

Figure 4.1.5: 2015 AFL Asian Cup Alternative Format



In this alternative, the third-place game is played between the losers of **A1** and **A3** instead of between the two semifinal losers. This is probably a bad idea: the losers of **B1** and **B2** each made it further in the main bracket than the losers of **A1** and **A3** did, and so they ought to be the teams playing the third-place game.

Definition 4.1.6: Weakly Proper Simple Multibracket

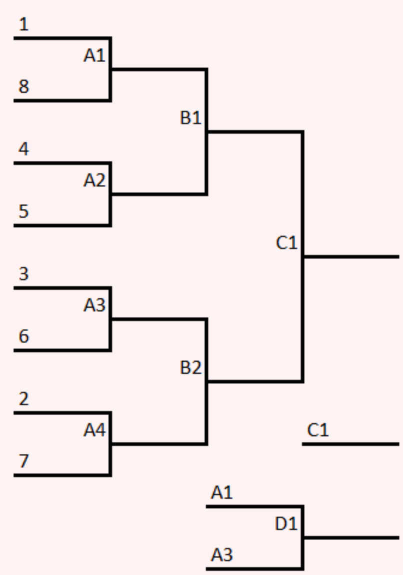
A simple multibracket is *weakly proper* if each m-team bracket $\mathcal{A}_i$ includes the m teams that lost most recently but have not yet been placed, (where rounds later in the alphabet are more recent,) and teams that lose in the same round are placed in the same bracket.

Since the losers of **B1** and **B2** lost in a later round than the losers of **A1** through **A4**, if the Asian Cup wants to be weakly proper, its third-place game must be played between the losers of **B1** and **B2**.

Note that as a simple multibracket of order two, the Asian Cup is actually not quite weakly proper: remember that a team's final place is based on which bracket it wins. Thus, for the Asian Cup to have a top-three, it needs to have order three. Otherwise, the winner of game **D1** would come in second place, leaving the loser of **C1** out in the cold and violating the properness condition.

In reality, there is an implied one-team bracket in between the primary bracket and the third-place game. Rigorously, the Asian Cup simple multibracket is weakly proper with order three and looks like this:

Figure 4.1.7: 2015 AFL Asian Cup, Rigorously



However, for clarity, we will omit zero-round brackets when drawing multibracket figures. Much like the case with traditional brackets, we will almost always focus our study on weakly proper multibrackets: so in a figure, if a zero-round bracket would be necessary for a multibracket to be weakly proper, that bracket is implied.

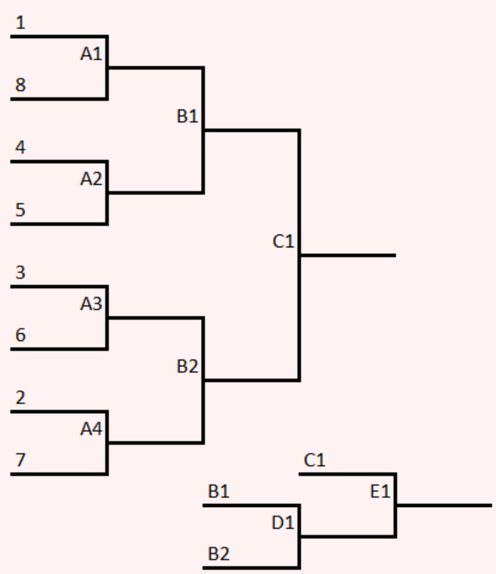
Definition 4.1.8: Multibracket Signature

The *signature* of a multibracket is the list of signatures of its brackets.

So the 2015 AFL Asian Cup has signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]]$. But the simple multibracket with this signature is far from the only simple multibracket that the AFL could have used to give out gold, silver, and bronze. In fact, it's not clear the loser of **C1**, who comes in second place, is really more deserving than the winner of **D1**, who comes in third. One could imagine the UAE arguing: we and South Korea both finished with two wins and one loss – a first-round win, a win against Iraq, and a loss against Australia. The only reason that South Korea came in second and we came in third was because South Korea lucked out by having Australia on the other half of the bracket as them. That's not fair!

If the AFL took this complaint seriously, they could modify their format to have signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{2};\mathbf{1};\mathbf{0}]] \to [[\mathbf{1}]]$. (Again, the bracket of signature $[[\mathbf{1}]]$ is implied and left out of the figure.)

Figure 4.1.9: $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{2};\mathbf{1};\mathbf{0}]] \to [[\mathbf{1}]]$



If the AFL used the format in Figure 4.1.9 in 2015, then South Korea and the UAE would have played each other for second place after all of the other games were completed. In some sense, this is a more equitable format than the one used in reality: we have the same data about the UAE and South Korea and so we ought to let them play for second place instead of having decided almost randomly.

However, swapping formats doesn't come without costs. For one thing, South Korea and the UAE would've had to play a fourth game: if the AFL had only three days to put on the tournament and teams can play at most one game a day, then the format in Figure 4.1.9 isn't feasible.

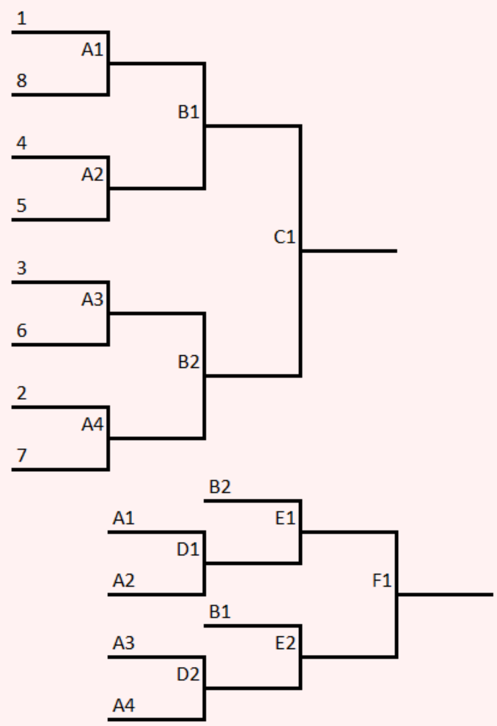
Another concern: what if Iraq had beaten the UAE when they played in game **D1**? Then the two teams with a claim to second place would have been South Korea and Iraq, except South Korea already beat Iraq! In this world, South Korea being given second place without having to win a rematch with Iraq seems more equitable than giving Iraq a second chance to win. To address this, one could imagine a format in which game **E1** is played only if it is not a rematch, although this would no longer be a multibracket and is a bit out of scope.

Ultimately, whether including game **E1** is worth it depends on the goal of the format. If there is a huge difference between the prizes for coming in second and third, for example, if the top-two finishing teams in the Asian Cup qualified for the World Cup, then **E1** is quite important. If, on the other hand, this is self-contained format played purely for bragging rights, **E1** could probably be left out. In reality, the 2015 AFL Asian Cup qualified only its winner to another tournament (the 2017 Confederations Cup), and gave medals to its top three, so game **E1**, which distinguishes between second and third place, is probably unnecessary.

Let's imagine, however, that instead of just the champion, the top four teams from the Asian Cup advanced to the Confederations Cup. In this case, the format used in 2015 would be quite poor, as teams finish in the top four based only on the result of their first-round game: the rest of the games don't even have to be played. (Formally, the simple multibracket $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]]$ has order three and so doesn't even assign a fourth place, but it could easily be extended to the following simple multibracket of order four $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]]$, which has the property mentioned above.)

A better format for selecting the top four teams might look like this:

Figure 4.1.10: $[[8;0;0;0]] \rightarrow [[1]] \rightarrow [[4;2;0;0]] \rightarrow [[1]]$



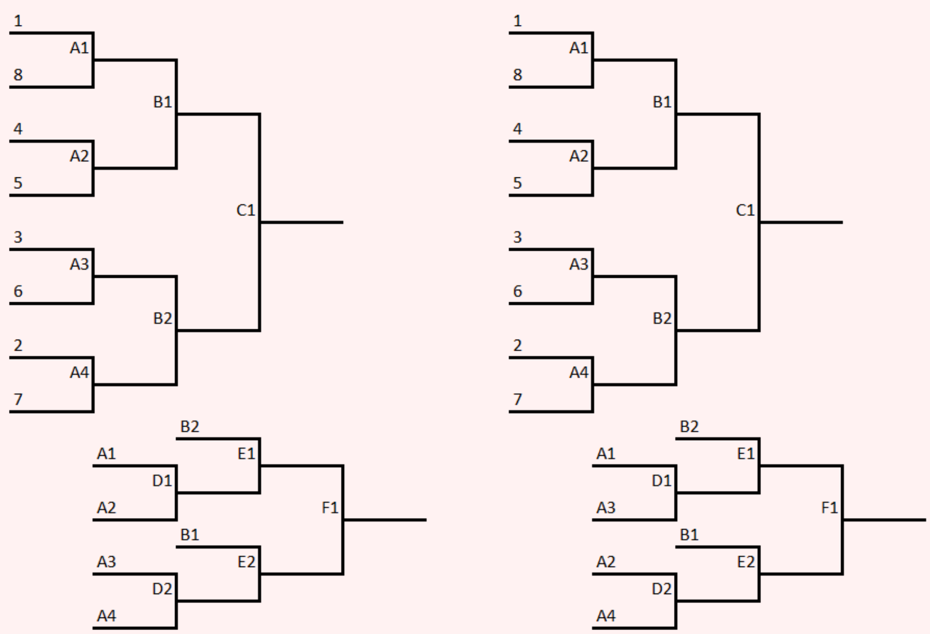
The weakly proper simple multibracket in Figure 4.1.10 selects a top-four without having the selection be completely determined by the first-round games. In fact, $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \rightarrow [[\mathbf{1}]] \rightarrow [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]] \rightarrow [[\mathbf{1}]]$ has the attractive property that a team will finish in the top four if and only if it wins two of its first three games.

Simple multibrackets are a powerful generalization of brackets: they keep the excitement of every game mattering and the power of sorting through many teams in relatively few games, but allow for teams to continue playing and fighting for lower places even after losing their first games.

4.2 Multibracket Considerations

At the end of the previous section, we arrived at the weakly proper simple multibracket in Figure 4.1.10 as a great option for selecting a first-, second-, third-, and fourth-place team out of eight competing teams. And while we presented it as the natural instantiation of of the signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]]$, it is far from the only one. Below, we display the format from Figure 4.1.10 on the left, along with a different weakly proper format of the same signature on the right.

Figure 4.2.1: Two Simple Multibrackets with Signature $[[8;0;0;0]] \to [[1]] \to [[4;2;0;0]] \to [[1]]$



The primary brackets are identical, and both formats are weakly respectful: the difference comes in where the **A**-round loses are placed. Why do we prefer the format on the left? Rematches.

In any tournament format, rematches are far from ideal. From an information theoretical perspective, a rematch is less informative then a new matchup: we already have some data on how those two team compare. From a competitive perspective, they are unsatisfying: without the ability to play a third "rubber" match, if each team wins one game, we are left in a disappointing state of uncertainty. And in a multibracket of signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]]$ they can be particularly frustrating to the teams themselves: to end in the top-four, teams must win two games before they lose two. If a team loses two games to the same team, they might reasonably feel like they didn't get a fair shake.

So the choice between the two simple multibrackets in Figure 4.2.1 comes down to which bracket will lead to fewer rematches. A quick analysis shows that this is done by the format on the left, where only **F1** could be a rematch, as opposed to the format on the right, where any of **E1**, **E1**, or **F1** could be.

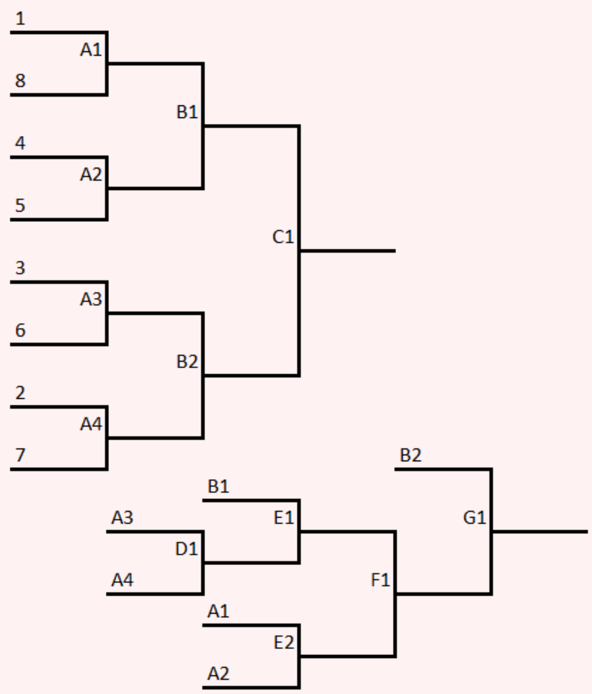
In fact, the simple multibracket on the left is the weakly proper bracket of signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]]$ with the fewest rematch opportunities, and so we consider it the canonical multibracket with that signature. In the general case, however, finding this multibracket is an open question.

Open Question 4.2.2

Given a multibracket signature $\mathcal{A}$, which instantiation of it has the fewest number of expected rematches?

Not only is the bracket on the left not the only instantiation of its signature (though we established it is the one that best avoids rematches), its signature is not the only one of order four. Another option might be $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{3};\mathbf{0};\mathbf{1};\mathbf{0}]] \to [[\mathbf{1}]]$.

Figure 4.2.3: $[[8; 0; 0; 0]] \to [[1]] \to [[2; 3; 0; 1; 0]] \to [[1]]$



Like the simple multibrackets in Figure 4.2.1, the simple multibracket in Figure 4.2.3 is weakly proper: for both (or technically, all four) of its brackets, teams that lost more recently are given more byes than teams that lost earlier. However, it still feels unfair in some sense: the losers of **B1** and **B2** lost in the same round and so they ought to be on the same foot. Instead, the loser of **B1** has to win two games just to get in the same spot that the loser of **B2** starts in. We can use the already-established language of tiered seeding to express this.

Definition 4.2.4: Weakly Respectful Multibrackets

A multibracket is *weakly respectful* if each of its brackets weakly respects the tiering that groups teams by the letter of the round that they lost, with teams that lost more recently getting at least as many byes.

Definition 4.2.5: Strongly Respectful Multibrackets

A multibracket is *strongly respectful* if each of its brackets strongly respects the tiering that groups teams by the letter of the round that they lost.

Thus, the multibrackets of signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]]$ are weakly and strongly respectful, while ones of signature $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{3};\mathbf{0};\mathbf{1};\mathbf{0}]] \to [[\mathbf{1}]]$ are not respectful at all. As a general rule, we prefer multibrackets that are more respectful over ones that are less so, although there can be other factors that might convince us to use less respectful multibrackets, and under certain circumstances no strongly or weakly respectful brackets will be available.

One natural question to ask is if there's a property called that fills in the final square in the grid.

Figure 4.2.6: Strongly Respectful?

Weakly Respectful	Strongly Respectful
Weakly Proper	???

Recall that respectfulness involves treating teams that lost in the same round similarly, while properness involves treating teams that lost more recently better. Additionally, the weak properties focus just on which bracket or how many byes teams are getting, while the strong properties try to be more complete. Thus,

Definition 4.2.7: Strongly Proper Multibrackets

A multibracket is *strongly proper* if

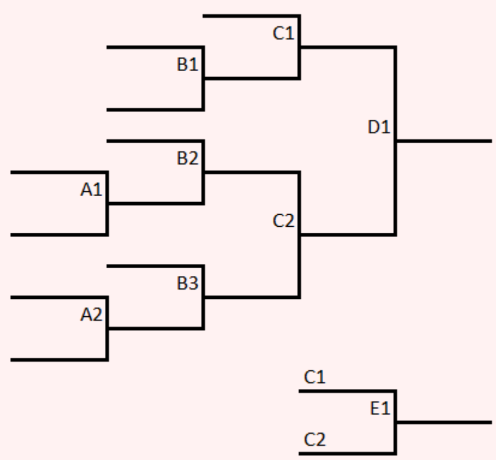
- It is weakly proper,
- Each of its brackets are proper,
- Teams that lost more recently are given the slots of higher seeds

than teams that lost less recently.

In the traditional bracket world, the idea that proper brackets were preferable to non-proper ones was pretty well accepted, and when combined with the fundamental theorem, meant that there was a clear best bracket for each bracket signature. While strong properness seems like a natural extension of the accepted notion of properness in traditional brackets, it is not obvious that it is quite as desirable in the context of multibrackets. Let's examine two examples to illustrate this point.

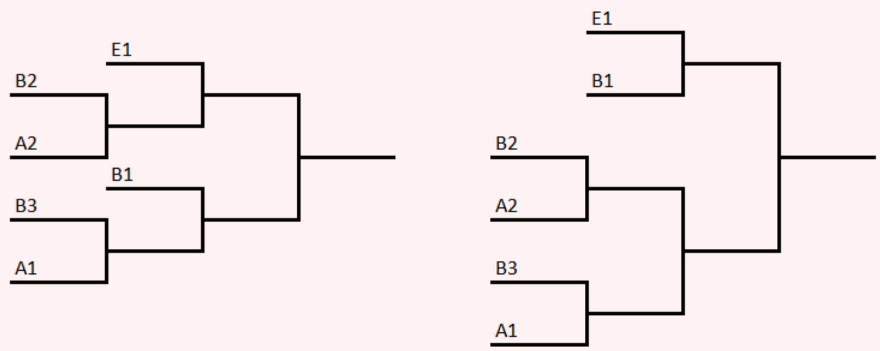
First, consider a simple multibracket with signature $[[\mathbf{4};\mathbf{4};\mathbf{1};\mathbf{0};\mathbf{0}]] \rightarrow [[\mathbf{1}]] \rightarrow [[\mathbf{2};\mathbf{0}]] \rightarrow [[\mathbf{4};\mathbf{2};\mathbf{0};\mathbf{0}]]$. The first three brackets are shown in Figure 4.2.8 (with the second bracket implied by weak properness left off of the figure).

Figure 4.2.8: $[[4;4;1;0;\mathbf{0}]] \rightarrow [[\mathbf{1}]] \rightarrow [[\mathbf{2};\mathbf{0}]]$



All of the teams that didn't win one the first three brackets will play in the third bracket, but what should that look like? Below, a strongly proper bracket is on the left, but a potentially better option is on the right.

Figure 4.2.9: Two options for $[[4; 2; 0; 0]]$



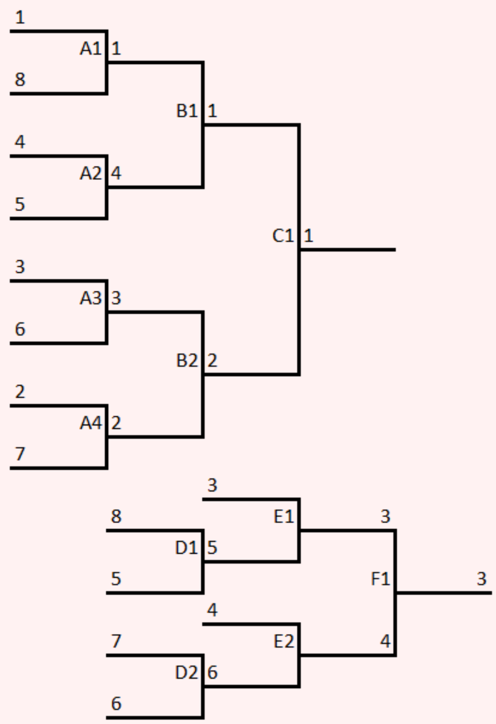
In the strongly proper bracket, there is clear variation between the difficulty of the paths of the round-**B** losers: **B1** both gets a first-round bye and avoids the likely best team **E1** until the finals, while **B2** gets neither of those luxuries. This is fine in the traditional bracket setting when we want to treat the 2-seed better than the 4-seed, but in a multibracket this is far from ideal. The alternative bracket on the right distributes the advantages more evenly by having the round-**B** loser that receives the bye face **E1** in the semifinals.

There are certainly still reasons to support the strongly proper bracket over the alternative, but it's not so cut and dry. One mitigating factor is strong respectfulness: if a multibracket is strongly respectful, then each team that lost in the same round is treated identically, and so strong properness will not give out these lopsided advantages.

For a second example of strongly proper multibrackets not being quite ideal, recall the strongly proper format in Figure 4.1.10 that we began the section with: $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]]$.

The notion of properness in the traditional setting is that, in each round, if the bracket goes to chalk, in each round it is better to be a higher seed than a lower seed. But what happens when $[[\mathbf{8}; \mathbf{0}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4}; \mathbf{2}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]]$ goes to chalk?

Figure 4.2.10: $[[8;0;0;0]] \to [[1]] \to [[4;2;0;0]] \to [[1]]$ **Chalk**



In round **E**, the matchups are not proper: the 4-seed has an easier opponent than the 3-seed. We could fix this by moving around where each seed starts in the primary bracket, but then the primary bracket would no longer be proper. (We could also fix this by swapping where the losers of **B1** and **B2** go, but this would massively increase the rate of rematches.)

Overall, strong properness is simply not as nice of a condition in the world of multibrackets as properness was for traditional brackets. While it is still often used, especially when there is a desire to drastically shrink the space of formats that need analyzing such as in monte-carlo simulations of formats, it leaves a lot to be desired.

Open Question 4.2.11

Is there a better alternative property to strong properness?

We conclude the section with an important lemma about weakly proper multibrackets.

Lemma 4.2.12

In a weakly proper multibracket $\mathcal{A}$, if the loser of game $\mathbf{G}$ goes to bracket $\mathcal{A}_j$, then the winner of game $\mathbf{G}$ will either:

1. Win a bracket $\mathcal{A}_i$ for $i \leq j$, or

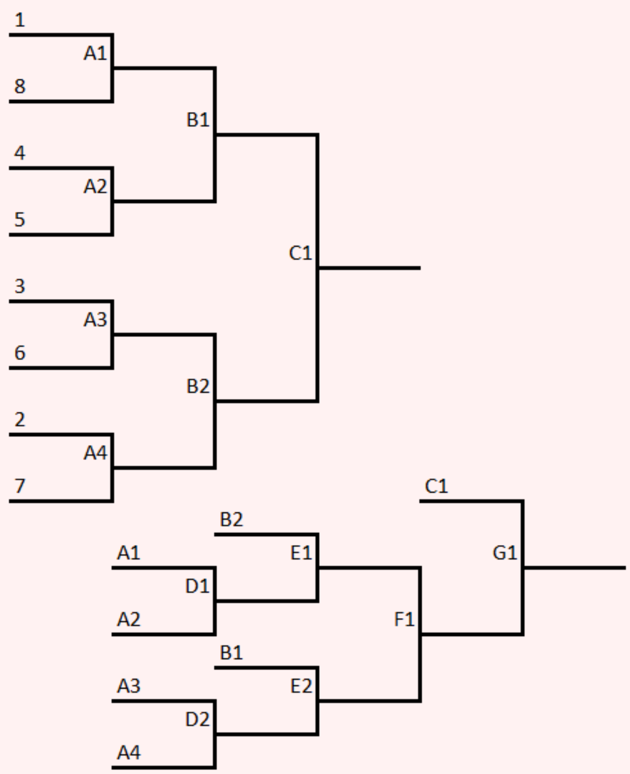
2. Lose a game in $\mathcal{A}_j$.

Proof. Let $\mathcal{A}$ be a weakly proper multibracket $\mathcal{A}$, and $\mathbf{G}$ be a game in $\mathcal{A}$ such that the loser of $\mathbf{G}$ goes to bracket $\mathcal{A}_j$. Let t be the team that won $\mathbf{G}$. Assume that t does not win any bracket $\mathcal{A}_i$ for $i \leq j$. Thus, t must have lost at least one game after playing $\mathbf{G}$. Upon losing this game, t will have lost more recently than the loser of game $\mathbf{G}$, and so must fall into bracket $\mathcal{A}_i$ for $i \leq j$. If they fall into $\mathcal{A}_i$ for $i < j$, then again they must lose and again fall into $\mathcal{A}_i$ for $i \leq j$. At some point, then, t must fall into $\mathcal{A}_j$. And since t does not win bracket $\mathcal{A}_j$, they must lose in $\mathcal{A}_j$ as well. $\square$

4.3 Semibrackets

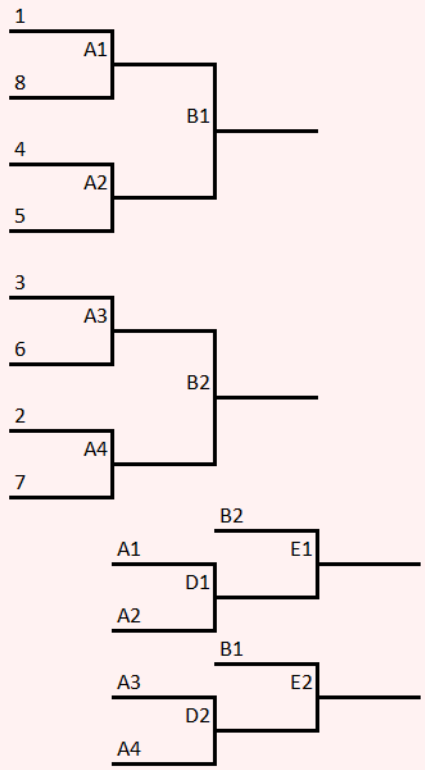
One common application of multibrackets is when an n-team tournament is needed to select the top m teams to move on to the next stage of the competitions (perhaps this is the regional tournament, and the top m teams from this region qualify for nationals.) For a tournament such as this, a natural option would be an m-team multibracket of order n. This is exactly what the USA Ultimate Manual of Championship Series Tournament Formats [8] calls for in many circumstances. One such example is the following eight-team order four simple multibracket:

Figure 4.3.1: $[[8;0;0;0]] \to [[4;2;0;1;0]] \to [[1]] \to [[1]]$



One observation about this format is that a few of the games played seem unnecessary. In particular, games **F1** and **G1** are downright silly: after round **E**, the four teams that advance to nationals are already set: the winner and loser of **C1**, and the winners of the two round **E** games. Perhaps more subtly, game **C1** is also unnecessary: both the winner and loser will advance. A more efficient alternative might look like so.

Figure 4.3.2: A More Efficient Alternative



The format in Figure 4.3.2 isn't an example of anything we've defined thus far. It's almost a simple multibracket but not quite: for one thing, some teams start in the "second bracket." To formally describe what is going on in Figure 4.3.2, we introduce the notion of semibracket.

Definition 4.3.3: Semibracket

A *semibracket* is a tournament format in which:

- Teams don't play any games after their first loss,
- The matchups between teams that have not yet lost are determined based on the ordering of the teams in $\mathcal{T}$ in advance of the

outcomes of any games.

All teams that finish a semibracket with no losses are declared co-champions.

Definition 4.3.4: Order of a Semibracket

The *order* of a semibracket is the number of co-champions it produces.

This is the same definition of a bracket but without the requirement that games be played until only one team is without losses. Note that semibrackets of order one are just traditional brackets. Figure 4.3.2 can be viewed then as being composed of two semibrackets of order two rather than of four brackets.

Many of the important properties of traditional brackets are true of semibrackets as well, up to and including the fundamental theorem. We state the key ones here without proof for brevity, though the proofs are analogous to those presented for traditional brackets.

Definition 4.3.5: Semibracket Signature

The *signature* $[[\mathbf{a_0}; ...; \mathbf{a_r}]]$ of an r-team semibracket $\mathcal{A}$ is a list of natural numbers, such that a_i is the number of teams with i byes.

Theorem 4.3.6

Let $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ be a list of natural numbers. Then $\mathcal{A}$ is a semibracket signature of order z if and only if

$$\sum_{i=0}^{r} a_i \cdot \left(\frac{1}{2}\right)^{r-i} = z.$$

Note that the semibracket signature summation sums to the order of the semibracket. As traditional brackets are just semibrackets of order one, this is consistent with Theorem 2.2.7.

Definition 4.3.7: Proper Semibracket

A *proper semibracket* is a semibracket that has been properly seeded.

Theorem 4.3.8

There is exactly one proper semibracket with each semibracket signature.

One strange feature of semibrackets is that a semibracket can declare arbitrarily many teams champion without any of them having to win a game.

Definition 4.3.9: Trivial

We say semibrackets with signature $[[\mathbf{n}]]$ for some n are *trivial.*

While the only trivial traditional bracket is the one-team bracket $[[\mathbf{1}]]$, there is a trivial semibracket of each order. There are even semibrackets in which some but not all teams are declared champion without playing any games.

Definition 4.3.10: Semitrivial

We say a semibracket $\mathcal{A} = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ is *semitrivial* if $r \geq 1$ and $a_r \neq 0$.

For example, the semibracket $[[\mathbf{4}; \mathbf{3}]]$ is semitrivial, but not trivial. There are no semitrivial traditional brackets.

We can combine semibrackets into larger formats in the same way that we combined traditional brackets into simple multibrackets.

Definition 4.3.11: Multibracket

A *multibracket* is a sequence of semibrackets in which the losers of certain games in the upper semibrackets fall into the lower semibrackets rather than being eliminated outright, and teams place based on which semibracket they won.

Definition 4.3.12: Order of a Multibracket

The *order* of a multibracket is the sum of the orders of the semibrackets it consists of.

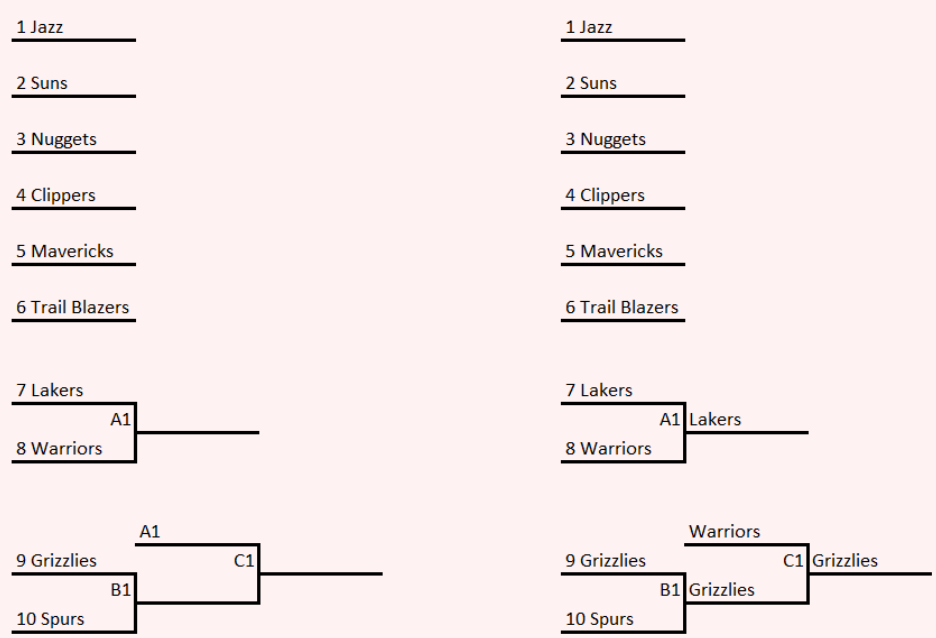
Definition 4.3.13: Weakly Proper Multibracket

A multibracket is *weakly proper* if each m-team bracket $\mathcal{A}_i$ includes either the m teams that lost most recently but have not yet been placed, or all of the teams that have lost and not yet been placed, (where rounds later in the alphabet are more recent,) and teams that lose in the same round are placed in the same bracket.

Thus, saying that a multibracket is *simple* means that all of its semi-brackets are of order one, and that all teams start in the primary bracket: in generalized multibrackets, some teams (in particular lower-seeded teams) might start in lower brackets.

An example of a non-simple multibracket that took advantage of both properties is the 2021 NBA Western Conference Play-in Tournament, which was a ten-team multibracket with order eight and following signature: $[[\mathbf{6}]] \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{2}; \mathbf{1}; \mathbf{0}]]$. The play-in tournament was used to whittle the top ten teams in the conference down to eight teams who would qualify for the playoffs.

Figure 4.3.14: 2021 NBA Western Conference Play-in



The top six seeds get slotted into the primary semibracket where they don't have to play any games in order to advance to playoffs. Seeds 7 and 8 each must win one of their next two games to qualify, while seeds 9 and 10 must win both of their next two.

While it is reasonable for a multibracket to have a trivial semibracket as one of its brackets, (for example, if we want to grant the loser of the championship game of the primary bracket second-place,) semitrivial brackets are much harder to justify. Any multibracket $\mathcal{A} = \mathcal{A}_1 \to ... \to \mathcal{A}_k$ in which $\mathcal{A}_i = [[\mathbf{a_0}; ...; \mathbf{a_r}]]$ is semitrivial can be reconstructed as

$$\mathcal{A}' = \mathcal{A}_1 \to ... \to \mathcal{A}_{i-1} \to [[\mathbf{a_r}]] \to [[\mathbf{a_0}; ...; \mathbf{a_{r-1}}; \mathbf{0}]] \to \mathcal{A}_{i+1} \to ... \to \mathcal{A}_k.$$

$\mathcal{A}'$ plays all of the same games as $\mathcal{A}$, but with more granularity in the final rankings.

Because weak properness and a lack of semitrivial semibrackets are such important and universal criteria, we combine them into the notion of a *standard* multibracket.

Definition 4.3.15: Standard

A multibracket is *standard* if it is weakly proper and contains no semitrivial semibrackets.

It is now clear that both the 2021 NBA Western Conference Play-in format as well as the format in Figure 4.3.2 are standard multibrackets. Both of these formats share an additional property that the original format in Figure 4.3.1 does not: they are *efficient*.

Definition 4.3.16: Efficient

A multibracket is *efficient* if there are no games played within it such that both the winner and loser of that game are guaranteed to win a semibracket.

Identifying whether a standard multibracket is efficient can be done by looking at its signature.

Theorem 4.3.17

A standard multibracket $\mathcal{A} = \mathcal{A}_1 \to ... \to \mathcal{A}_k$ is efficient if and only if there is some j such that all brackets $\mathcal{A}_i$ for $i < j$ are trivial and all brackets $\mathcal{A}_i$ for $i \geq j$ are not.

Proof. Let $\mathcal{A}$ be a standard multibracket.

Assume no such j exists, and let i be the first trivial bracket that follows a nontrivial one. Thus there is at least one game $\mathbf{G}$ such that the loser drops into $\mathcal{A}_i$. Because $\mathcal{A}_i$ is trivial, the loser of $\mathbf{G}$ wins $\mathcal{A}_i$. Applying Lemma 4.2.12, we see that the winner of $\mathbf{G}$ will either win a semibracket as well, or lose in $\mathcal{A}_i$. But $\mathcal{A}_i$ is trivial, so they must win a semibracket. Thus, $\mathcal{A}$ is not efficient.

Now assume that such a j exists. We will show by inducting on the semibrackets in $\mathcal{A}$ in reverse that none of the semibrackets contain a game that violates the efficiency condition. Firstly, $\mathcal{A}_k$ upholds the condition because any team that loses a game in $\mathcal{A}_k$ doesn't fall into another semibracket, much less have a chance to win one.

Now we must show that if all of the semibrackets from $\mathcal{A}_{i+1}$ to $\mathcal{A}_k$ uphold the condition, then $\mathcal{A}_i$ does as well. If $i < j$, then $\mathcal{A}_i$ is trivial so there are no games to violate the condition with. Otherwise, let $\mathbf{G}$ be a game in $\mathcal{A}_i$. If the loser of $\mathbf{G}$ does not fall into another semibracket, then we are done. If they do, then because that bracket is not trivial, they will play another game. However, by induction, the loser of this game is not guaranteed to win a semibracket. Thus neither is the loser of $\mathbf{G}$.

So by induction, if such a j exists, then $\mathcal{A}$ is efficient. Thus we have proved the theorem. $\square$

Efficient multibrackets are great tournament designs for tournaments whose primary goal is to select the top m teams to move on to the next stage of the competitions, as discussed in the beginning of this section. They do so excitingly, with each spot in the top m being awarded as the winner of a particular game; efficiently, with no games being played between teams who have each already clinched spots; and fairly (as long as the format is as proper and respectful as desired.)

4.4 Swiss Systems

Many tournaments, particularly those in which there are many teams each looking to play a similar number of games against teams of similar skill, use a set of formats referred to as *swiss systems.* In particular, swiss systems or near-variants are commonly used in board game tournaments, such as chess or Magic: The Gathering.

The idea behind a swiss system is to play a fixed number of rounds, and in each round have each matchup be between teams with the same record. This gives every team a bunch of games, while ensuring that teams are paired with teams that are probably similarly skilled. We can formally describe a swiss system in the language of multibrackets.

Definition 4.4.1: Swiss System

A *swiss system* is a standard multibracket signature in which

1. Each matchup is between teams of the same record,
2. All teams play the same number of games,
3. All teams start in the primary semibracket, and
4. The order of the multibracket is equal to the number of teams participating.

The first two requirements come from the intuitive notion of a swiss systems, while the last two requirements are technical detail that ensures we don't double count systems: for example once where teams that finish with no wins drop into a trivial last semibracket, and one where they don't drop into any semibracket at all.

Definition 4.4.2: r-Round Swiss

We say a swiss system is an *r-round swiss* if each team plays r games.

We begin our analysis by noting a key structural fact about Swiss systems.

Theorem 4.4.3

All r-round swiss systems are on $m \cdot 2^r$ teams for some m. Further, m divides the order of each its semibrackets.

Proof. In order for a multibracket to be swiss system, its primary semibracket must be balanced. (Otherwise, teams that get byes will play fewer games than teams that don't.) Additionally, since each winner of the primary semibracket will play all of their games in that bracket, it must be exactly r rounds long. A balanced semibracket that is r rounds long has signature $[[\mathbf{m \cdot 2^r}; \mathbf{0}; ...; \mathbf{0}]]$ for some m. Thus, since every team starts in the primary semibracket, there must be $m \cdot 2^r$ teams participating for some m.

To prove the second half of the theorem, we will show by induction on s that after each team has played s games, the number of teams in each semibracket is divisible by $m \cdot 2^{r-s}$. Then the case of $r = s$ shows that the number of teams that win each semibracket is divisible by m, and so m divides the order of each semibracket.

For $s = 0$, all $m \cdot 2^r$ teams are in the primary semibracket and no teams are in any of the others, so the statement holds. Now assume the statement holds for $s - 1$. Let t_i be the number of teams in the ith semibracket after each team has played $s - 1$ games. By induction $m \cdot 2^{r-s+1}$ divides t_i for all i. After each team plays their sth game, the ith semibracket contains $t_i/2$ teams that just won, and $t_i/2$ teams just lost and are dropped into another semibracket. Thus, each semibracket now has $\sum_{i \in S} t_i/2$ teams in it, for some set S. However, each $t_i/2$ is divisible by $m \cdot 2^{r-s}$, so the inductive case holds.

Therefore by induction, m divides the order of each its semibrackets.

$\square$

Theorem 4.4.3 indicates that r-round swiss systems on $m \cdot 2^r$ teams are, in a sense, actually m different simultaneous and identical swiss tournaments each operating on 2^r teams. Because of this, it is useful to study just the swiss systems that operate on 2^r teams, as this will give us strong insights into the full space of swiss systems.

Definition 4.4.4: Compact

We say a swiss system is *compact* if its primary semibracket has order one.

Theorem 4.4.3 guarantees that r-round compact swiss systems are on exactly 2^r teams.

One useful fact about compact swiss system is that it's easy to count how many teams will have each record.

Theorem 4.4.5

After each team in an r-round compact swiss system has played s games, for each i, $2^{r-s} \cdot \binom{s}{i}$ teams will have i wins.

Proof. We show this by induction on s. For $s = 0$, no games have yet been played, so all $2^r = 2^{r-0} \cdot \binom{0}{0}$ teams have no wins.

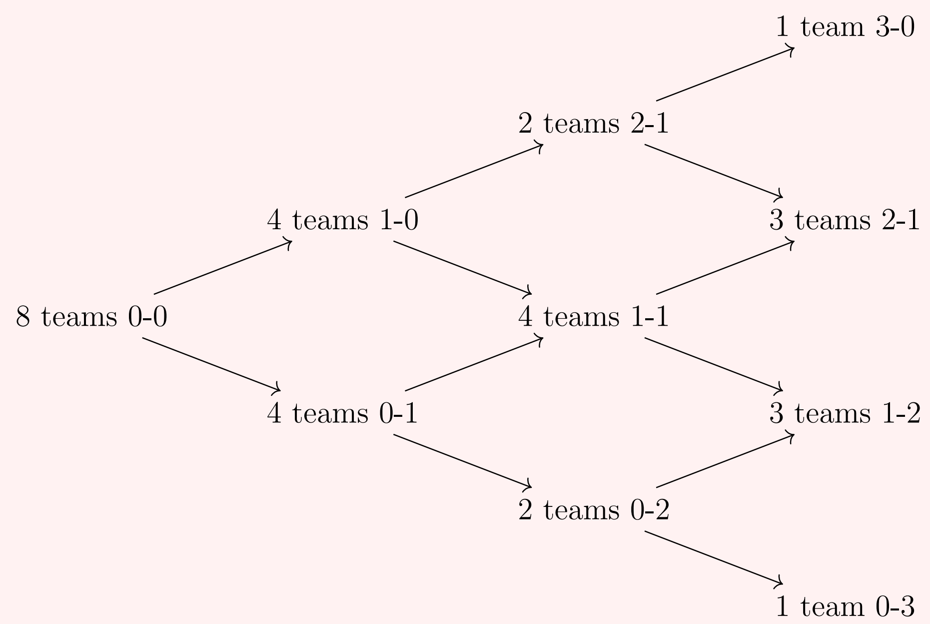
Assume the theorem holds for $s - 1$ and fix i. A team that after s games has i wins will, after $s-1$ games, have had either i or $i-1$ wins. In fact, half of the teams with i wins after $s - 1$ games will have lost and still have i wins, and half of the teams that had $i - 1$ wins will have won and now have i wins. Thus, the number of teams that have i wins after s games is

$$\begin{aligned}
&\frac{1}{2}\left(2^{r-s+1} \cdot \binom{s-1}{i} + 2^{r-s+1} \cdot \binom{s-1}{i-1}\right) \\
&= 2^{r-s}\left(\binom{s-1}{i} + \binom{s-1}{i-1}\right) \\
&= 2^{r-s} \cdot \binom{s}{i}.
\end{aligned}$$

□

Figure 4.4.6 visualizes how Theorem 4.4.5 applies to compact 3-round swiss systems.

Figure 4.4.6: Theorem 4.4.5 with $r = 3$.



We now enumerate the compact r-round swiss systems for various r.

Theorem 4.4.7

There are unique compact 0-, 1-, and 2-round swiss systems.

Proof. Certainly the compact 0-round and 1-round swiss systems are unique: the former is the unique one-team tournament, and the latter is the unique two-team multibracket in which each team plays one game. Their signatures are $[[\mathbf{1}]]$ and $[[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]]$ respectively.

In any compact 2-round swiss system, Theorem 4.4.5 says that after the first round, two teams will have zero wins and two teams will have one win. The two teams with zero wins must play each other, and as must the two teams with one win, so the compact 2-round swiss system is unique with signature $[[\mathbf{4}; \mathbf{0}; \mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]]$. $\square$

Theorem 4.4.8

There are two compact 3-round swiss systems:

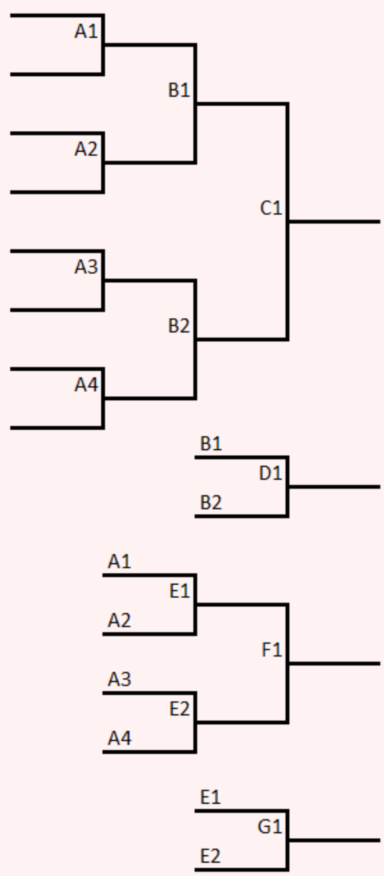
$$\begin{aligned}\mathcal{A} = {} & [[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]] \to \\ & [[\mathbf{4};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]] \\ & \text{and} \\ \mathcal{B} = {} & [[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]] \to [[\mathbf{1}]] \to [[\mathbf{4};\mathbf{2};\mathbf{0}]] \to \\ & [[\mathbf{2}]] \to [[\mathbf{2};\mathbf{0}]] \to [[\mathbf{1}]]\end{aligned}$$

Proof. Any compact 3-round swiss system must have primary bracket $[[\mathbf{8};\mathbf{0};\mathbf{0};\mathbf{0}]]$ and secondary bracket $[[\mathbf{1}]]$. Now let $\mathcal{C}$ be the semibracket that first-round primary brackets losers fall into. $\mathcal{C}$ must have two rounds, and the first-round primary bracket losers must all get no byes (otherwise they may not play the requisit three games). Thus $\mathcal{C} = [[\mathbf{4};\mathbf{c_1};\mathbf{0}]]$ for some c_1. Becuase swiss systems are standard, $c_1 = 0$ or 2.

If $c_1 = 0$, then in between the first two brackets and $\mathcal{C}$, we must have two more brackets for the second-round losers of the primary bracket: $[[\mathbf{2};\mathbf{0}]]$ and $[[\mathbf{1}]]$. Then $\mathcal{C}$ must be followed by $[[\mathbf{1}]]$ for the loser of its championship game, and then $[[\mathbf{2};\mathbf{0}]]$ and $[[\mathbf{1}]]$ so that the last two teams get a third game. In total, this leads to the swiss system $\mathcal{A}$.

If $c_1 = 2$, then the losers of the two championship games of $\mathcal{C}$ have already played all three of their games and so need to fall into the bracket $[[\mathbf{2}]]$. Then we need $[[\mathbf{2};\mathbf{0}]]$ and $[[\mathbf{1}]]$ so that the last two teams get a third game. In total, this leads to the swiss system $\mathcal{B}$. $\square$

Figure 4.4.9: The Two Compact 3-Round Swiss Systems



How would a tournament designer decide which compact 3-round system to use? Well, it depends on what the prize structure of the format is. If the goal is to identify a top-three, then system $\mathcal{A}$ is preferable: after all, system $\mathcal{B}$ has the two teams that win its third semibracket tie for third place. But if the goal is to identify a top-four, system $\mathcal{B}$ is preferable: the team that comes in fourth in system $\mathcal{A}$ actually finishes with only one win, while the team the comes in fifth finishes with two. While it is still reasonable to grant the one-win team fourth-place – they had a more difficult slate of oponnents – this is a somewhat messy situation that is solved by just using system $\mathcal{B}$.

(McGarry and Schutz [6] considered outright swapping the positions of the fourth- and fifth-place teams at the conclusion of $\mathcal{A}$, but this provides some incentive for losing in the first round in order to get an easier path to a top-half finish. Simply using $\mathcal{B}$ when identifying the top-four teams is a much preferable solution.)

For similar reasons, both formats are good for selecting a top-one or top-seven, and $\mathcal{A}$ but not $\mathcal{B}$ is good for selecting a top-five. Finally, it might seem that $\mathcal{A}$ and $\mathcal{B}$ are good formats for selecting a top-two or top-six: in both cases the top two and top six teams are clearly defined, and there are no teams with better records that don't make the cut. However, notice that if we use $\mathcal{A}$ or $\mathcal{B}$ to select a top-two, the final round of games are meaningless: the two teams that finish in the top-two are the two teams that win their first two games, irrespective or how game three went. Better than using either format $\mathcal{A}$ or $\mathcal{B}$ would be to use a non-compact 2-round swiss for eight teams, so that a third, meaningless round is avoided.

We formalize the notion of a swiss sytem being good at selecting a top-m in Definition 4.4.10 and summerize what we have deduced so far in Figure 4.4.11.

Definition 4.4.10: Supporting a Top-m

We say a swiss system $\mathcal{A} = \mathcal{A}_1 \to \dots \to \mathcal{A}_k$ *supports a top-m* if

- For some j, $\sum_{i=1}^{j} \text{Order}(\mathcal{A}_i) = m$.
- No team that wins $\mathcal{A}_i$ for $i > j$ finishes with a better record then a team that wins $\mathcal{A}_i$ for $i \leq j$.
- $\mathcal{A}_j$ is not trivial.

Figure 4.4.11: Which Compact Systems Support Top-ms

	Top-m							
Compact Swiss System	1	2	3	4	5	6	7	8
Unique 1-round System	✔	✘						
Unique 2-round System	✔	✘	✔	✘				
$\mathcal{A}$	✔	✘	✔	✘	✔	✘	✔	✘
$\mathcal{B}$	✔	✘	✘	✔	✘	✘	✔	✘

Note that even if no *compact* swiss system on n teams supports a top-m, we can still sometimes use a non-compact swiss system to identify a top-m. For example, there is no eight-team compact system that supports a top-2, but we can still use a 2-round swiss identify a top two teams out of eight.

We state without proof that there are eight compact 4-round swiss systems, and in Figure 4.4.12 we indicate which of these compact 4−round systems support various top-ms.

Figure 4.4.12: Support of Compact 4-Round Swiss Systems

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathcal{A}_1$	✔	✘	✔	✘	✘	✘	✘	✘	✘	✘	✘	✘	✔	✘	✔	✘
$\mathcal{A}_2$	✔	✘	✔	✘	✘	✘	✘	✘	✘	✘	✘	✔	✘	✘	✔	✘
$\mathcal{A}_3$	✔	✘	✘	✔	✘	✘	✘	✘	✘	✘	✘	✘	✔	✘	✔	✘
$\mathcal{A}_4$	✔	✘	✘	✔	✘	✘	✘	✘	✘	✘	✘	✔	✘	✘	✔	✘
$\mathcal{A}_5$	✔	✘	✔	✘	✘	✔	✘	✘	✘	✔	✘	✘	✔	✘	✔	✘
$\mathcal{A}_6$	✔	✘	✔	✘	✘	✔	✘	✘	✘	✘	✔	✘	✘	✘	✔	✘
$\mathcal{A}_7$	✔	✘	✘	✘	✔	✘	✘	✘	✘	✔	✘	✘	✔	✘	✔	✘
$\mathcal{A}_8$	✔	✘	✘	✘	✔	✘	✘	✘	✘	✘	✔	✘	✘	✘	✔	✘

Thus there is a compact 4-round swiss system that supports a top-m for each $m \in \{1, 3, 4, 5, 6, 10, 11, 12, 13, 15\}$, but not for $m \in \{2, 7, 8, 9, 14, 16\}$. The general question, however, is still open.

Open Question 4.4.13

For which r and m is there a compact r-round swiss system that supports a top-m?

We make progress on Open Question 4.4.13 with the following theorem.

Theorem 4.4.14

For $r \geq 3$, there is a compact r-round swiss system that supports a top-m for

$$m = 1, 3, 4, (2^r - 4), (2^r - 3), \text{ or } (2^r - 1),$$

and no such system for

$$m = 2, (2^r - 2), \text{ or } 2^r.$$

Proof. We prove the first half of the theorem inductively. If $r = 3$, then the system $\mathcal{A}$ from Theorem 4.4.8 supports each of $1, 3, (2^r - 3)$, and $(2^r - 1)$. For any other r, let $\mathcal{A}$ be the compact $(r-1)$-round system that supports those four top-ms. Now consider the compact r-round system $\mathcal{C}$ in which, after the first round of games, the winners and losers each play out $\mathcal{A}$ on their own. $\mathcal{C}$ is a compact r-round swiss system that supports a top-$1, 3, (2^r - 3)$, and $(2^r - 1)$. The same inductive argument on $\mathcal{B}$ generates compact r-round swiss systems that support a top $1, 4, (2^r - 4)$, and $(2^r - 1)$.

For the second half of the theorem, we note that any compact system must begin with $[[\mathbf{2^r}; \mathbf{0}; ...; \mathbf{0}]] \to [[\mathbf{1}]] \to ...$, so a system cannot support a top-2. Similarly, any compact system must end with $... \to [[\mathbf{2}; \mathbf{0}]] \to [[\mathbf{1}]]$ so that the two teams with no wins can compete for $(2^r - 1)$th place. The team in $(2^r - 2)$th place, then, must have won the third-to-last semibracket, which must be trivial (otherwise the team they just beat would have nowhere to go). Finally, a compact system cannot support a top 2^r, because the (2^r)th place team wins the final semibracket, which is also trivial. $\square$

Overall, swiss systems very useful and practical tournament designs: they give each team the same number of games, they ensure that games are being played between teams that have the same record and thus, hopefully, similar skill levels, and, for many values of m, they efficiently identify a top-m in a fair and satisfying way.

Further, near-swiss systems are great when the number of teams is exceedingly large. Even if the number of teams is not a power of two, or the system is not compact, or it is being used to identify a top-m that it doesn't technically support, formats that are swiss in spirit tend to do a great job of gathering a lot of meaningful data about a large number of teams in a minimal number of rounds.

4.5 Multiple Elimination

5 Round Robins

5.1 Intro

Definition 5.1.1: Round-Robin

A *round robin* is a tournament format in which each team plays each other team once, and then teams are ranked according to how many games they won.

Round robins, or close variants, are used in many leagues across many sports, especially during the regular season or qualifying rounds. For example, the 2014 Ivy League Football Regular Season was structured as round robin. At the conclusion of a round robin, a league table can be used to display the results and rank the teams.

Figure 5.1.2: 2014 Ivy League Football Regular Season

Rank	Team	Games	Wins	Losses
1	Harvard	7	7	0
2	Dartmouth	7	6	1
3	Yale	7	5	2
4	Princeton	7	4	3
5	Brown	7	3	4
6	Penn	7	2	5
7	Cornell	7	1	6
8	Columbia	7	0	7

At the end of an n-team round robin, each team has played each other team once, for a total of $n-1$ games. There are n possible records a team could have after playing $n-1$ games, so it is possible for each team to end the tournament with a different record: the 2014 Ivy League Football Regular Season has this property.

However, this is far from guaranteed: consider the 2019 Big 12 Football Regular Season (strangely enough, in 2019 the Big 12 had only ten teams).

Figure 5.1.3: 2019 Big 12 Football Regular Season

Rank	Team	Games	Wins	Losses
1	Oklahoma	9	8	1
2	Baylor	9	8	1
3	Texas	9	5	4
4	Oklahoma State	9	5	4
5	Kansas State	9	5	4
6	Iowa State	9	5	4
7	West Virginia	9	3	6
8	TCU	9	3	6
9	Texas Tech	9	2	7
10	Kansas	9	1	8

Nearly every team, including the two leaders Oklahoma and Baylor, ended the season tied with at least one other team.

6 References

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