CSC 411 Lecture 08: Generative Models for Classification

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Today

- Classification – Bayes classifier
- Estimate input probability densities from data
- Naive Bayes

Classification

- Given inputs **x** and classes y we can do classification in several ways. How?

(features)

$\mathbf{x}$

e.g:
- height
- weight
- color

(class label)

y

elefant

bear

Discriminative Classifiers

- **Discriminative** classifiers try to either:
  - learn mappings directly from the space of inputs $\mathcal{X}$ to class labels $\{0, 1, 2, \ldots, K\}$

(features) $\quad\mathbf{x} \longrightarrow y\quad$ (class label)

e.g:
- height
- weight
- color

elefant

bear

For example:

Decision trees

Discriminative Classifiers

- **Discriminative** classifiers try to either:
  - or try to learn $p(y|\mathbf{x})$ directly

(features)

$\mathbf{x}$ $\longrightarrow$ $p(y|\mathbf{x})$

(class probability)

e.g:
- height
- weight
- color

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For example:

- Logistic Regression

Generative Classifiers

How about this approach: build a model of "what data for a class looks like"

- **Generative** classifiers try to model $p(\mathbf{x}, y)$. If we know $p(y)$ we can easily compute $p(\mathbf{x}|y)$.
- Classification via Bayes rule (thus also called Bayes classifiers)

(prob. of features given label) (class label)

$$p(\mathbf{x}|y) \longleftarrow y$$

e.g:
- height
- weight
- color

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Tries to model:
- How does data look like for a class?

Classification (How?)

Generative vs Discriminative

Two approaches to classification:

- Discriminative classifiers estimate parameters of decision boundary/class separator directly from labeled examples. Tries to solve: How do I separate the classes?
  - learn $p(y|\mathbf{x})$ directly (logistic regression models)
  - learn mappings from inputs to classes (least-squares, decision trees)
- Generative approach: model the distribution of inputs characteristic of the class (Bayes classifier). Tries to solve: What does each class "look" like?
  - Build a model of $p(\mathbf{x}|y)$
  - Apply Bayes Rule

Bayes Classifier

- Aim to classify text into spam/not-spam (yes C=1; no C=0)
- Use bag-of-words features, get binary vector $\mathbf{x}$ for each patient
- Given features $\mathbf{x} = [x_1, x_2, \cdots, x_d]^T$ we want to compute class probabilities using Bayes Rule:

$$p(C|\mathbf{x}) = \frac{p(\mathbf{x}|C)p(C)}{p(\mathbf{x})}$$

- More formally

$$\text{posterior} = \frac{\text{Class likelihood} \times \text{prior}}{\text{Evidence}}$$

- How can we compute $p(\mathbf{x})$ for the two class case? (Do we need to?)

$$p(\mathbf{x}) = p(\mathbf{x}|C=0)p(C=0) + p(\mathbf{x}|C=1)p(C=1)$$

- To compute $p(C|\mathbf{x})$ we need: $p(\mathbf{x}|C)$ and $p(C)$

Classification: Simple Example

- Let's start with a simple (but slightly redundant) example.
- Imagine that we have some biased coins and we observe a single outcome from one of these coins.
- We have $P(x|C) = Ber(\theta_C) = \theta_C^x \cdot (1 - \theta_C)^{1-x}$
- Notice that we have different parameters for each c oin
- How can I fit the distribution to my data?
- Simple approach - maximum likelihood

MLE for Bernoulli

- Assumption: data points are independent and identically distributed (i.i.d)

$$p(\mathcal{D}_C|C) = \prod_{n=1}^{N} p(x^{(n)}|C) = \prod_{n=1}^{N} \theta_C^{x^{(n)}} \cdot (1-\theta_C)^{1-x^{(n)}} = \theta_C^{N_C} \cdot (1-\theta_C)^{N-N_C}$$

- We define $N_C = \sum_{i=1}^{N} x^{(n)}$ the number of ones (heads) seen.
- N and N_C are called sufficient statistics - hold all the information we need to compute $P(\mathcal{D}_C|C)$
- We can minimize the negative log-likelihood (NLL)

$$\ell_{log-loss} = -\log(p(x^{(1)}, \cdots, x^{(N)}|C)) = -N_C \log(\theta_C) - (N - N_C)\log(1-\theta_C)$$

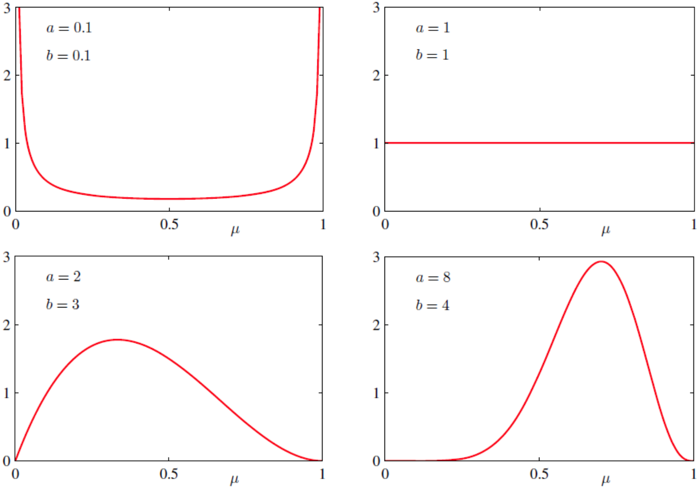
$$\frac{\partial \ell_{log-loss}}{\partial \theta_C} = -\frac{N_C}{\theta_C} + \frac{N-N_C}{1-\theta_C} = 0 \Rightarrow \theta_C = \frac{N_C}{N}$$

Beta-Binomial

- MLE solution $\theta_C = \frac{N_C}{N}$. What if $N_C = 0$?
- Example: Some rare word unseen in a training corpus.
- In that case $P(x|C) = 0$ no matter what other information we have!
- Solution: A prior over θ.
- Simple (conjugate) prior: Beta distribution
  - $Beta(\theta|a, b) \propto \theta^{a-1}(1-\theta)^{b-1}$

Beta Distribution

- Examples of $Beta(\theta|a, b) \propto \theta^{a-1}(1-\theta)^{b-1}$:



[Image credit: Bishop]

Beta-Binomial

- Likelihood $p(\mathcal{D}_C|\theta_C) = \theta_C^{N_C} \cdot (1-\theta_C)^{N-N_C}$
- Prior $P(\theta_C) = Beta(\theta_C|a,b) \propto \theta_C^{a-1}(1-\theta_C)^{b-1}$

$$p(\theta_C|\mathcal{D}_C) = \frac{p(\mathcal{D}_C|\theta_C)P(\theta_C)}{p(\mathcal{D}_C)} \propto \theta_C^{N_C} \cdot (1-\theta_C)^{N-N_C}\theta_C^{a-1}(1-\theta_C)^{b-1}$$

$$= \theta_C^{N_C+a-1} \cdot (1-\theta_C)^{N-N_C+b-1}$$

- We have $P(\theta_C|\mathcal{D}_C) = Beta(N_C + a, N - N_C + b)$
- MAP estimation $\theta_{C,map} = \frac{N_C+a-1}{N+a+b-2}$ (show!)

Beta-Binomial *

- Can we do better then the using the MAP estimator? A more Bayesian approach.
- We have $P(\theta_C|\mathcal{D}_C) = Beta(N_c + a, N - N_C + b)$, what is $P(x = 1|\mathcal{D}_C)$?

$$P(x = 1|\mathcal{D}_C) = \int_0^1 P(x = 1|\theta_C)P(\theta_C|\mathcal{D}_C)$$

$$= \int_0^1 \theta_C P(\theta_C|\mathcal{D}_C) = \mathbb{E}[\theta_C|\mathcal{D}_C]$$

- Beta(a,b) has a closed form mean $\frac{a}{a+b}$ (a bit of work to show) so $\theta_C = P(x = 1|\mathcal{D}_C) = \frac{N_C+a}{N+a+b}$
- Equivalent to pseudo-counts, adding a fictitious positive examples and b negative ones.

Moving beyond coins

- In the real world we tend to have a vector of observations $\mathbf{x} = [x_1, .., x_d]$.
- Modelling $p(\mathbf{x}, y)$ in this case is much more complex.

$$p(x_1, \cdots, x_d, y) = p(x_1 | x_2, \cdots, x_d, y) \cdots p(x_{d-1} | x_d, y) p(x_d, y)$$

- We need to make some assumptions!
- The Naive-Bayes Model is born from a particularly strong assumption.

Naive-Bayes for Bernoulli variables

- Make the (naive) assumption - dimensions $\mathbf{x} = [x_1, .., x_d]$ are independent given the class y.

$$P(\mathbf{x}|y = C, \theta_C) = \prod_{j=1}^{d} p(x_j|y = C, \theta_{jC}) = \prod_{j=1}^{d} \theta_{jC}^{x_j}(1 - \theta_{jC})^{(1-x_j)} =$$

$$\exp\left(\sum_{j=1}^{d} x_j \log(\theta_{jC}/(1 - \theta_{jC})) + \sum_{j=1}^{d} \log(1 - \theta_{jC})\right) = \exp(\mathbf{w}_C^T\mathbf{x} + w_{0C})$$

- Define $w_{Cj} = \log(\theta_{jC}/(1 - \theta_{jC}))$, $w_{0C} = \sum_{j=1}^{d} \log(1 - \theta_{jC})$

Naive-Bayes for Bernoulli variables

- How do we classify?

$$P(y = C|\mathbf{x}) \propto P(y = C)P(\mathbf{x}|y = C) = \exp(\mathbf{w}_C^T\mathbf{x} + b_C)$$

- $w_{Cj} = \log(\theta_{jC}/(1 - \theta_{jC}))$, $b_C = w_{0C} + \log(P(y = C))$
- Linear classifier! Model is similar to logistic regression, but different optimization.
  - No gradients - just need to count! Really fast to train.
  - Doesn't take into account correlation between features.

Example: 20newsgroups

Table: Top word per topic

Topic	Naive Bayes	Logistic regression
'alt.atheism',	don	enlightening
'comp.graphics',	thanks	needed
'comp.os.ms-windows.misc',	windows	windows
'comp.sys.ibm.pc.hardware',	thanks	disappointing
'comp.sys.mac.hardware',	mac	mac
'comp.windows.x',	window	xtvaappinitialize
'misc.forsale',	sale	semd
'rec.autos',	car	car
'rec.motorcycles',	bike	bike
'rec.sport.baseball',	year	950k
'rec.sport.hockey',	team	hockey
'sci.crypt',	key	encryption
'sci.electronics',	use	cci
'sci.med',	don	melittin
'sci.space',	space	launch
'soc.religion.christian',	god	satan
'talk.politics.guns',	people	gun
'talk.politics.mideast',	people	kidding
'talk.politics.misc',	people	paranoia
'talk.religion.misc'	people	compuserve

Beyond Bernoulli

- We focused on binary features, x_i, but Naive bayes is more general.
- Discrete features - multinomial.
- Continuous features - Gaussian (or any other).
- No problem to mix (unlike logistic regression)!

NB recap

- Learning parameters:
  - Estimate $P(y = C)$, e.g. $P(y = C) = \frac{\#\, class\ C}{\#\, data\ points}$
  - For each class C and feature x_i estimate the distribution $p(x_i|y = C)$
- At test time:
  - For each class compute $S_C = \log(P(Y = C)) + \sum_{i=1}^{d} \log(p(x_i|y = C))$
  - Classify according to $max_C S_C$
- Probabilities: $P(y = C|\mathbf{x}) = \frac{\exp(S_C)}{\sum_{i=1}^{L} \exp(S_i)}$

NB recap

- Pros:
  - Really fast to train (single pass through data!).
  - Fast to test.
  - Less over-fitting, sometimes better then logistic on small data sets
  - Easy to add/remove classes
  - Can handle partial data.
- Cons:
  - When naive i.i.d assumption doesn't hold (almost always) - can perform much worse.