HOMEWORK FOR THE COURSE FOUNDATIONS OF MATHEMATICS 2025-2026

LÉONARD GUETTA

1. Homework 1 [Deadline: November 20, 2025, 23:59]

Homework longer than 3 pages long will not be accepted!

(1) Let $f\colon X \to Y$ be a injective function.
  (a) (2 pts) Prove without the axiom of choice that if X is non-empty, then there exists a surjection $g\colon Y \to X$ such that $g(f(x)) = x$ for every $x \in X$.
  (b) (1 pt) What happens when $X = \emptyset$? When can you define g as above?

(2) (3 pts) Prove that
$$|\mathbb{N}^{\mathbb{N}}| = 2^{|\mathbb{N}|}.$$

(3) (4 pts) Let X be an infinite set. Prove that
$$|X| = |X| + 1.$$
Hint: Proposition 1.2.2 of the textbook.

2. Homework 2 [Deadline: November 27, 2025, 23:59]

The purpose of this homework is to prove the existence of nonprincipal ultrafilters.

Let X be a set. A *filter* on X is a subset of $\mathcal{P}(X)$ satisfying the following conditions:

(F1) $X \in \mathcal{F}$,
(F2) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$,
(F3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(NB: Definition 2.5.6 in the book also requires that $\emptyset \notin \mathcal{F}$, but we do not impose this condition.) Note that since filters are subsets of the powerset $\mathcal{P}(X)$, we can compare filters via the inclusion relation on $\mathcal{P}(X)$.

A filter $\mathcal{F}$ is called proper if it is not equal to $\mathcal{P}(X)$, or equivalently if $\emptyset \notin \mathcal{F}$. A filter is called an *ultrafiler* if it is a maximal proper filter for the inclusion relation.

(1) (1pt) Let A be a subset of X. Show that the set $\{B \subseteq X | A \subseteq B\}$ is a filter. This is the *filter generated by* A. Such filters are also called *principal*.

(2) (1pt) Let $\mathcal{C}$ be the set $\{B \subset X \mid X - B \text{ is finite}\}$. Show that $\mathcal{C}$ is a filter. This filter is called the *cofinite filter*.
(3) (3pt) Let $\mathcal{F}$ be a proper filter on X. Use Zorn's lemma to prove that there exists an ultrafilter on X that contains $\mathcal{F}$.
(4) (4pt) Let $\mathcal{F}$ be a proper filter on X and suppose that A is a subset of X satisfying the following condition: For all $B \in \mathcal{F}$, the intersection $A \cap B$ is nonempty. Show that there exists a proper filter $\mathcal{F}'$ such that $A \in \mathcal{F}'$ and $\mathcal{F} \subseteq \mathcal{F}'$.
(5) (1pt) Let $\mathcal{U}$ be an ultrafilter and A a subset of X. Show that $A \in \mathcal{U}$ or $X - A \in \mathcal{U}$.
(6) (1pt) Show that a principal ultrafilter on X is generated by $\{x\}$ for some $x \in X$.
(7) (1pt) Assume that X is infinite. Show that the cofinite filter $\mathcal{C}$ is proper and not contained in any principal ultrafilter. Conclude that there exist ultrafilters that are not principal.

3. Homework 3 [Deadline: December 04, 2025, 23:59]

Homework longer than 4 pages long will not be accepted!

Let us say that a function $f\colon (P, \leq_P) \to (Q, \leq_Q)$ between posets is

- *monotone* if
$$x \leq_P y \text{ implies } f(x) \leq_Q f(y)$$
- *strictly monotone* if
$$x <_P y \text{ implies } f(x) <_Q f(y),$$
- *continuous* if it is monotone and if it preserves least upper bounds. This last condition means that for any subset $A \subseteq P$ which admits a least upper bound x, then $f(x)$ is a least upper bound of $f(A)$.

(1) (4pts) Show that if L is a well-ordered set and $f\colon L \to L$ is strictly monotone, then for any $x \in L$, we have
$$x \leq f(x).$$
(2) (1pt) If you replace "strictly monotone" by "monotone" in the previous question, is the assertion still true? Justify your answer.
(3) (5pts) Let L be a nonempty well-ordered set. Show that any function $f\colon L \to L$ strictly monotone and continuous admits a fixed point, i.e. there exists $x \in L$ such that
$$x = f(x).$$

4. Homework 4 [Deadline: December 11, 2025, 23:59]

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(This exercise was inspired by Exercise 87 of the textbook.)

Recall that a **simple undirected graph** consists of a set of vertices and for each *unordered* pair $\{x, y\}$ of vertices, a set of edges (such an edge is said to be "between x and y"), such that:

- for any vertices x and y, there is at most one edge between x and y,
- for any vertex x, there is no edge between x and x.

For an integer $k \geq 1$, a k-colouring of a graph is a function f from the set of vertices to the set $\{1, \ldots, k\}$ such that if there is an edge between two vertices x and y, then $f(x) \neq f(y)$.

(1) (2pts) Show that there is a language $\mathcal{L}$ with no constants, no function symbols and exactly one relation symbol, and a theory T in that language such that a model of T is exactly a simple undirected graph.
(2) (5pts) Using the compactness theorem, show that if a graph G is such that every finite subgraph of G admits a k-coloring, then G admits a k-coloring.

5. Homework 5 [Deadline: January 15, 2026, 23:59]

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Demonstrate by constructing proof trees:

(1) (5pts) $(\phi \to \exists x \psi) \vdash \exists x(\phi \to \psi)$ (here it is assumed that the variable x does not occur in ϕ).
(2) (5pts) $\vdash (\phi \to \psi) \vee (\psi \to \phi)$.

6. Homework 5 [Deadline: January 15, 2026, 23:59]

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(This is exercise 129 from the textbook.)

Let L be a language, T an L-theory and L' an extension of L.

(1) (4pts) Show that the poset of all L'-theories which are conservative extension of T, ordered by inclusion, satisfies the hypothesis of Zorn's Lemma.
(2) (6 pts) By Zorn's Lemma, there is a maximal L'-theory U which is conservative over T. Show that for every L'-sentence $\psi \notin U$, there are an L-sentence ϕ and an L'-sentence $\gamma \in U$ such that $\gamma \wedge \psi \vDash \phi$ and $T \nvDash \phi$.