Simplicial cheat-sheet

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1 Simplicial and co-simplicial objects

1.1. For any non-negative integer n, we denote by $[n]$ the ordered set

$$\{0 < 1 < .. < n\}$$

(for $n = 0$, this is just a point). The category Δ is the category whose objects are all the $[n]$ for $n \geq 0$ and whose morphisms are non-decreasing functions between these ordered sets. For $n \geq 0$ and $0 \leq i \leq n$, we denote by $\delta^i \colon [n-1] \to [n]$ the only non-decreasing injective function such that i is not in its image. Explicitly, we have

$$\delta^i(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \geq i. \end{cases}$$

The δ^i's are referred to as the co-face maps. Similarly, for $n \geq 0$ and $0 \leq i \leq n$, we denote by $\sigma^i \colon [n+1] \to [n]$ the only non-decreasing surjective function such that hits i twice. Explicitly, we have

$$\sigma^i(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i. \end{cases}$$

The σ^i's are referred to as the co-degeneracy maps. As is easily checked, we have the following so-called *co-simplicial identities*

$$\begin{aligned} \delta^j\delta^i &= \delta^i\delta^{j-1} \text{ if } i < j \\ \sigma^j\delta^i &= \delta^i\sigma^{j-1} \text{ if } i < j \\ id &= \sigma^j\delta^j = \sigma^j\delta^{j-1} \\ \delta^j\sigma^i &= \delta^{i-1}\sigma^j \text{ if } i > j+1 \\ \sigma^j\sigma^i &= \sigma^i\sigma^{j+1} \text{ if } i \leq j. \end{aligned}$$

As one can prove, Δ is in fact the category freely generated by the morphisms ∂^i and σ^i and subject to the co-simplicial identities.

Definition 1.2. A simplicial object X in a category $\mathcal{C}$ is a functor

$$X \colon \Delta^{\mathrm{op}} \to \mathcal{C}.$$

A morphism $X \to X'$ of simplicial objects is simply a natural transformation. We denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$. Dually, a co-simplicial object Y in $\mathcal{C}$ is a functor

$$Y \colon \Delta \to \mathcal{C},$$

and morphisms of co-simplicial objects are natural transformations. We denote by $\cos\mathcal{C}$ the category of simplicial objects.[1]

Remark 1.3. In practise, a co-simplicial object $Y\colon \Delta \to \mathcal{C}$ consists of the data of

- a sequence $(Y^n)_{n\geq 0}$ of objects of $\mathcal{C}$
- morphisms $\partial^i\colon Y^n \to Y^{n+1}$ for all $n \geq 0$ and $0 \leq i \leq n$,
- morphisms $s^i\colon Y^{n+1} \to Y^n$ for all $n \geq 0$ and $0 \leq i \leq n$,

satisfying the co-simplicial identities. Dually, a simplicial object $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ consists of the data of

- a sequence $(X_n)_{n\geq 0}$ of objects of $\mathcal{C}$,
- morphisms $\partial_i\colon X_n \to X_{n+1}$ for all $n \geq 0$ and $0 \leq i \leq n$,
- morphisms $s_i\colon X_{n+1} \to X_n$ for all $n \geq 0$ and $0 \leq i \leq n$,

satisfying the simplicial identities (i.e. the dual of the co-simplicial identities).

Remark 1.4. Note that we have

$$\cos\mathcal{C} = (s(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}.$$

Example 1.5. The case of $\mathcal{C} = \mathsf{Set}$ is fundamental, the objects of sSet are referred to as *simplicial sets*. Note that the Yoneda embedding defines a co-simplicial object in sSet

$$y_\Delta\colon \Delta \to \mathsf{sSet}.$$

Usually, we use the notation

$$\Delta[n] := y_\Delta[n].$$

Example 1.6. Since we can see any ordered set as a category and any non-decreasing function as a functor, we have a canonical functor

$$i\colon \Delta \to \mathsf{Cat},$$

where Cat is the category of small categories. This defines a co-simplicial object in Cat. Observe that this functor is fully faithful, and is often identified with a full subcategory inclusion.

Example 1.7. For any $n \geq 0$, let Δ^n be the standard topological n-simplex, that is Δ^n is the convex hull in $\mathbb{R}^n$ of the standard basis $(e_0, e_1, \cdots, e_n)$ of $\mathbb{R}^n$. One defines co-face maps as

$$\begin{aligned} \delta^i\colon \Delta^{n-1} &\to \Delta^n \\ \sum_{j=0}^{n-1} t_j e_j &\mapsto \left(\sum_{j=0}^{i-1} t_j e_j + \sum_{j=i+1}^{n+1} t_{j-1} e_j \right), \end{aligned}$$

[1]The notation $\cos\mathcal{C}$ is not standard.

and co-degeneracy maps as

$$\begin{aligned}
&\sigma^i \colon \Delta^{n+1} \to \Delta^n \\
&\sum_{j=0}^{n+1} t_j e_j \mapsto \left(\sum_{j=0}^{i-1} t_j e_j + (t_i + t_{i+1}) e_i + \sum_{j=i+1}^{n} t_{j+1} e_j \right).
\end{aligned}$$

One checks (exercise) that the co-simplicial identities are satisfied and thus defines a co-simplicial object in the category Top of topological spaces

$$\begin{aligned}
\Delta &\to \mathsf{Top} \\
[n] &\mapsto \Delta^n.
\end{aligned}$$

As it turns out, co-simplicial objects allow to define functors into the category of simplicial sets.

1.8. Let $i \colon \Delta \to \mathcal{C}$ a co-simplicial object in a category $\mathcal{C}$. We denote by i^* the functor from $\mathcal{C}$ to the category of simplicial sets sSet (i.e. simplicial objects in Set) defined as

$$\begin{aligned}
i^* \colon \mathcal{C} &\to \mathsf{sSet} \\
X &\mapsto ([n] \mapsto \mathrm{Hom}_{\mathcal{C}}(i[n], X))
\end{aligned}$$

If $\mathcal{C}$ is co-complete (i.e. has all colimits), then i^* has a left adjoint $i_!$ defined by

$$\begin{aligned}
i_! \colon \mathsf{sSet} &\to \mathcal{C} \\
X &\mapsto \operatorname*{colim}_{([n], \Delta[n] \to X)} i[n]
\end{aligned}$$

Example 1.9. If we consider $i \colon \Delta \to \mathsf{Cat}$, then the functor i^* is often denoted as N. For a small category C, $N(C)$ is referred to as the *nerve of* C.

Example 1.10. Consider the co-simplicial object $i \colon \Delta \to \mathsf{Top}$, $[n] \mapsto \Delta^n$. Then the functor i^* is often denoted as Sing. For a topological space X, Sing$X$ is referred to as the *singular complex of* X.

2 Simplicial objects and chain complexes

2.1. Let $\mathcal{A}$ be an abelian category. Given a simplicial objet $X \colon \Delta^{\mathrm{op}} \to \mathcal{A}$, we define a chain complex $C_\bullet X$ as

$$C_k X := X_k$$

and the boundary map is given by

$$\begin{aligned}
\partial \colon C_k X &\to C_{k-1} X \\
x &\mapsto \sum_{i=0}^{n} (-1)^i \partial_i(x).
\end{aligned}$$

One indeed checks using the simplicial identities that we have $\partial \circ \partial = 0$. This construction canonically defines a functor

$$\begin{aligned}
C \colon s\mathcal{A} &\to \mathsf{Ch}_{\geq 0}(\mathcal{A}) \\
X &\mapsto C_\bullet X.
\end{aligned}$$

2.2. In the definition of the functor $C_\bullet \colon s\,\mathcal{A} \to \mathsf{Ch}_{\geq 0}(s\,\mathcal{A})$, we haven't make use of the degeneracy maps. We can actually use them to refine the functor $C_\bullet$ as follows. For $X \colon \Delta^{\mathrm{op}} \to \mathcal{A}$, we say that a n-simplex $x \in X_n$, with $n \geq 1$, is *degenerate* if it is in the image of one of the $s_i \colon X_{n-1} \to X_n$. We denote by $D_n X$ the subobject of X_n generated by degenerated simplices, and we define $N_n X$ as

$$N_n X := C_n X / D_n X.$$

One can check that the differential $\partial \colon C_n X \to C_{n-1} X$ induces a morphism

$$\partial \colon N_n X \to N_{n-1} X.$$

The chain complex $N_\bullet X$ is called the *normalized chain complex* associated to X. This construction extends naturally to a functor

$$N_\bullet \colon s\,\mathcal{A} \to \mathsf{Ch}_{\geq 0}(s\,\mathcal{A}).$$

By construction, we have a natural transformation

$$p \colon C_\bullet \Rightarrow N_\bullet.$$

Proposition 1. *Let $\mathcal{A}$ be an abelian category. For every object X of $s\,\mathcal{A}$, the morphism of chain complexes*

$$p_X \colon C_\bullet X \to N_\bullet X$$

is a quasi-isomorphism.

Proof. (Sketch) By construction, we have a short exact sequence of chain complexes

$$0 \to D_\bullet X \to C_\bullet X \to N_\bullet X \to 0.$$

Using simplicial combinatorics, one can show that the homology of $D_\bullet X$ is trivial. The result follows them from the long exact sequence on homology. □

The main interest of the normalized chain complex functor comes from the following theorem, known as the Dold–Kan correspondance.

Theorem 2.3. *(Dold–Kan correspondance) Let $s\,\mathcal{A}$ be an abelian category. The functor*

$$N_\bullet \colon s\,\mathcal{A} \to \mathsf{Ch}_{\geq 0}\mathcal{A}$$

is an equivalence of categories.

Remark 2.4. Since the opposite category of an abelian category is an abelian category, all the previous results can be dualized. In particular, if $s\,\mathcal{A}$ is an abelian category, we have an equivalence of categories between co-simplicial objects in $\mathcal{A}$ and non-negatively graded cochain complexes in $\mathcal{A}$

$$\mathrm{co}s\,\mathcal{A} \simeq \mathrm{co}\mathsf{Ch}^{\geq 0}(\mathcal{A}).$$

2.5. In light of the Dold–Kan correspondance, we see that looking at simplicial (resp. co-simplicial) objects in an abelian category is the same thing as looking at non-negatively graded chain complexes (resp. cochain complexes) in $\mathcal{A}$. The point is that the notion of simplicial and co-simplicial objects in a category makes sense even if the category is not pre-additive. In other words, looking at (co)-simplicial objects in a (nice) category is a way to do "non-abelian homological algebra". Of course, we are particularly interested in the case that the ambient category is a topos $\mathcal{E}$, hence giving rise to the notion of (co)simplicial sheaves.