HOMEWORK FOR THE COURSE FOUNDATIONS OF MATHEMATICS 2025-2026

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1. Homework 1 [Deadline: November 20, 2025, 23:59]

Homework longer than 3 pages long will not be accepted!

(1) Let $f\colon X \to Y$ be a injective function.
   (a) (2 pts) Prove without the axiom of choice that if X is non-empty, then there exists a surjection $g\colon Y \to X$ such that $g(f(x)) = x$ for every $x \in X$.
   (b) (1 pt) What happens when $X = \emptyset$? When can you define g as above?

(2) (3 pts) Prove that
$$|\mathbb{N}^{\mathbb{N}}| = 2^{|\mathbb{N}|}.$$

(3) (4 pts) Let X be an infinite set. Prove that
$$|X| = |X| + 1.$$
Hint: Proposition 1.2.2 of the textbook.

2. Homework 2 [Deadline: November 27, 2025, 23:59]

The purpose of this homework is to prove the existence of nonprincipal ultrafilters.

Let X be a set. A *filter* on X is a subset of $\mathcal{P}(X)$ satisfying the following conditions:

(F1) $X \in \mathcal{F}$,
(F2) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$,
(F3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(NB: Definition 2.5.6 in the book also requires that $\emptyset \notin \mathcal{F}$, but we do not impose this condition.) Note that since filters are subsets of the powerset $\mathcal{P}(X)$, we can compare filters via the inclusion relation on $\mathcal{P}(X)$.

A filter $\mathcal{F}$ is called proper if it is not equal to $\mathcal{P}(X)$, or equivalently if $\emptyset \notin \mathcal{F}$. A filter is called an *ultrafiler* if it is a maximal proper filter for the inclusion relation.

(1) (1pt) Let A be a subset of X. Show that the set $\{B \subseteq X | A \subseteq B\}$ is a filter. This is the *filter generated by* A. Such filters are also called *principal*.

(2) (1pt) Let $\mathcal{C}$ be the set $\{B \subset X \mid X - B \text{ is finite}\}$. Show that $\mathcal{C}$ is a filter. This filter is called the *cofinite filter*.

(3) (3pt) Let $\mathcal{F}$ be a proper filter on X. Use Zorn's lemma to prove that there exists an ultrafilter on X that contains $\mathcal{F}$.

(4) (4pt) Let $\mathcal{F}$ be a proper filter on X and suppose that A is a subset of X satisfying the following condition: For all $B \in \mathcal{F}$, the intersection $A \cap B$ is nonempty. Show that there exists a proper filter $\mathcal{F}'$ such that $A \in \mathcal{F}'$ and $\mathcal{F} \subseteq \mathcal{F}'$.

(5) (1pt) Let $\mathcal{U}$ be an ultrafilter and A a subset of X. Show that $A \in \mathcal{U}$ or $X - A \in \mathcal{U}$.

(6) (1pt) Show that a principal ultrafilter on X is generated by $\{x\}$ for some $x \in X$.

(7) (1pt) Assume that X is infinite. Show that the cofinite filter $\mathcal{C}$ is proper and not contained in any principal ultrafilter. Conclude that there exist ultrafilters that are not principal.