Seminar Topos Theory Lecture 2 Notes

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We will cover the following subjects:

* recap on sites and sheaves,
* associated sheaves of set-valued presheaves on sites,
* (co-)limits of sheaves,
* associated sheaves of algebra-valued presheaves.

Recap

Recall that, in a (small) category $\mathcal{C}$, a *sink* R on an object C is a set of arrows $R = \{f_i : C_i \to C\}$ into C. If D is some object, $R(D)$ denotes the set of arrows in R with domain D. We can view the hom-presheaf $\text{よ}_C = \mathrm{Hom}(-, C)$ as a sink into C, and it is the largest sink. A sink is called a *sieve* if it is closed under precomposition (i.e. if $f \in R(D)$ and $g \in \text{よ}_D(E)$, then $f \circ g \in R(E)$). Alternatively, a sink into C is a *sub-presheaf* of よ_C.

Recall that a *site* is a category $\mathcal{C}$ with a function J that assigns to each object C of $\mathcal{C}$ a collection $J(C)$ of sieves on C, called *covering sieves*. This function J is called a *Grothendieck topology*, and it is required to satisfy the following for all objects C of $\mathcal{C}$:

* (S1: existence of trivial covers) $\text{よ}_C \in J(C)$,
* (S2: pullback-stability) if $S \in J(C)$ and $d\colon D \to C$ is a morphism, then $d^*(S) = \{e\colon E \to D : d \circ e \in S\} \in J(D)$,
* (S3: locality) if $S \in J(C)$, and T is a sieve on C such that for every $d \in S(D)$ we have $d^*(T) \in J(D)$, then $T \in J(C)$.

The axiom S2 can alternatively be rephrased as saying that J is a *subfunctor* of the functor that assigns to each object C the collection of sieves on C (and assigns to morphisms the corresponding pullback operation). As a fun fact (which may or may not come back later in the course), this sieve functor is exactly the *subobject classifier* Ω of the category $\mathrm{PSh}(\mathcal{C})$, so a Grothendieck topology can be viewed as a specific kind of subobject of Ω.

A sink R on C is called a *(J-)cover* if the sieve generated by R is a covering sieve on C. A *basis* for a Grothendieck topology is a function K that assigns to every object a collection of *basic covers*, satisfying similar axioms to the ones above. A sieve on C is then declared to be covering if it contains a basic cover.

On any category $\mathcal{C}$, we have two extreme examples of Grothendieck topologies: the trivial topology, which has $J(C) = \{ \text{よ}_C \}$, and the maximal topology, which has $J(C) = \{\text{all sieves on } C\}$.

As a classical example, if $\mathrm{Open}(X)$ is the lattice of open subsets of a topological space X, then taking as covers the open covers (note that in a preordered set, any morphisms are uniquely determined by their domain and codomain) gives us a Grothendieck topology on $\mathrm{Open}(X)$. This is called the *canonical topology* on $\mathrm{Open}(X)$ (though note that we can define the canonical topology more generally on any category). More generally (ignoring non-sober spaces), this construction works for any *locale*, i.e. the objects are the elements of a complete Heyting algebra.

Generally speaking, we'd like covering sieves S on C to express a notion of 'joint surjectivity', where the 'images' of the morphisms of S 'touch' 'every part' of C. There's various ways of expressing this. Let us say that two morphisms $d, d' \in \text{よ}_C$ are *dense-disjoint* if there exist no morphisms e, e' such that $d \circ e = d' \circ e'$. We say that a sink R on C is *dense-covering* if there exists no morphism $f \in \text{よ}_C$ that is dense-disjoint from every $g \in R$. Then the dense-covering sieves form a Grothendieck topology J on $\mathcal{C}$, called the *dense topology.* It's a straightforward exercise to show that we can alternatively describe the dense topology as

$$J(C) = \Big\{ \text{sieves } S \text{ on } C : \text{for any } f \in \text{よ}_C(D) \text{ there exists } g \in \text{よ}_D(E) \text{ such that } f \circ g \in S(E) \Big\}.$$

If $\mathcal{C}$ satisfies the *right Ore condition*, meaning that any diagram $D \xrightarrow{f} C \xleftarrow{g} E$ can be completed to a commutative square, then the dense topology coincides with the *atomic topology*, whose covers are all the non-empty sinks.

Associated sheaves

If $(\mathcal{C}, J)$ is a site, then a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is said to be a *sheaf*, in short, if for every object C of $\mathcal{C}$ and every covering sieve $S \in J(C)$, any *matching family* $(s_f \in F(\mathrm{dom}\, f) : f \in S)$ has a unique collation $s \in F(C)$.

'Matching' means that if $f \in S(D)$ and $g \in \text{よ}_D$, then $s_{f \circ g} = F(g)(s_f)$. This is equivalent to the statement that if $f, f' \in S$ and g, g' are such that $f \circ g = f' \circ g'$, then $F(g)(s_f) = F(g')(s_{f'})$. We say that s is a 'collation' of this family if $s_f = F(f)(s)$ for all $f \in S$.

Another way to say that a family $(s_f : f \in S)$ is matching is to say that it defines a natural transformation $S \to F$, where we view S as a sub-presheaf of よ_C. That such a family has a unique collation means it extends uniquely to a natural transformation $\text{よ}_C \to F$ (and the collation is the image of id_C under this transformation), which by the Yoneda lemma corresponds naturally to an element of $F(C)$. This shows that F is a sheaf if and only if for all C and $S \in J(C)$ we have that the natural restriction map

$$F(C) \cong \mathrm{PSh}(\text{よ}_C, F) \to \mathrm{PSh}(S, F)$$

is a bijection.

A presheaf P is called *separated* if collations of matching families don't necessarily exist, but are unique if they do. We say that P *has collations* if any matching family has a not necessarily unique collation.

As an example of a separated presheaf that's not a sheaf, consider a topological space X, and its associated canonical site $(\mathrm{Open}(X), J)$. Letting P be the presheaf that associates to every open subset U the set of *bounded* real-valued continuous maps on U (with the usual restriction maps), then this presheaf is separated, but for non-compact U it will not necessarily have all collations.

As an example of a presheaf with collations that's not separated, let $X = \{0, 1\}$ be a discrete two-point space. We define $P\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$ as $P(X) = \mathbb{R}^3$, $P(\{0\}) = P(\{1\}) = \mathbb{R}$, and $P(\varnothing) = \{*\}$. We define the restriction maps $\pi_1\colon P(X) \to P(\{0\})$ and $\pi_3\colon P(X) \to P(\{1\})$ as the projections onto the first and third coordinates, respectively. See the diagram:

$$
\begin{tikzcd}
 & \{0,1\} & \\
\{0\} \arrow[ur] & & \{1\} \arrow[ul] \\
 & \varnothing \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{R}^3 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
\mathbb{R} \arrow[dr] & & \mathbb{R} \arrow[dl] \\
 & \{*\} &
\end{tikzcd}
$$

Then a matching family for the cover $\{\{0\},\{1\}\}$ of X is a pair of real numbers $(a,b) \in \mathbb{R}^2$, which has the collation $(a,0,b) \in P(X)$, but it is certainly not unique.

Given a presheaf P, we'd like to give a sheaf $P^\ddagger$ that 'best approximates' P. Specifically, we want $P^\ddagger$ to be such that if F is a sheaf and $\phi\colon P \to F$ a morphism of presheaves, then ϕ factors uniquely through an 'extension' $\overline{\phi}\colon P^\ddagger \to F$. More precisely, we'd like a natural bijection

$$\mathrm{Sh}(P^\ddagger, F) \cong \mathrm{PSh}(P, F),$$

In other words, we want to find a *left adjoint* functor to the inclusion functor $i\colon \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{PSh}(\mathcal{C})$. Said in other words still, we would like to show that $\mathrm{Sh}(\mathcal{C}, J)$ is a *reflective subcategory* of $\mathrm{PSh}(\mathcal{C})$. This functor $(-)^\ddagger\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, J)$ will be called the *associated sheaf functor* (or 'sheafification functor', if you prefer). In particular we would need

$$\mathrm{Sh}(P^\ddagger, P^\ddagger) \cong \mathrm{PSh}(P, P^\ddagger),$$

so we have a canonical morphism $\eta\colon P \to P^\ddagger$, whose unique extension is the identity $\mathrm{id}\colon P^\ddagger \to P^\ddagger$.

So $P^\ddagger$ is a sheaf, and its sections 'contain' the sections of P through η. So in particular, $P^\ddagger$ needs to contain collations of matching families of sections of P. Taking the path of least resistance, a good first try is to *define* $P^\ddagger(C)$ as the set

$$M(C) = \Big\{ (s_f \in P(\operatorname{dom} f) : f \in S) : S \in J(C) \Big\}$$

of matching families of P-sections. Note that we can make M into a presheaf on $\mathcal{C}$ by letting $M(g)$ be the map that sends a family $(s_f : f \in S)$ to $(s_{g \circ h} : h \in g^*(S))$.

However, this clearly cannot work, as restricting a matching family to a smaller covering sieve should give you the same collation (by uniqueness of collations). Define an equivalence relation $\sim$ on $M(C)$ by $(s_f)_{f \in S} \sim (t_g)_{g \in T}$ if there exists a covering sieve $R \subseteq S \cap T$ such that $(s_f : f \in R) = (t_g : g \in R)$. Then

$$P^+(C) = M(C)/{\sim} = \operatorname*{colim}_{S \in J(C)} \Big\{ \text{matching } S\text{-families} \Big\}$$

is a pretty good (though not quite perfect) guess! The restriction maps of M are well-defined w.r.t. $\sim$, so P^+ is a presheaf. Let's try to prove that P^+ is a sheaf.

Let $S \in J(C)$, and let $\big([s_g : g \in T_f]_f : f \in S\big)$ be a matching family of classes of matching families. To try to make a single matching family on C which collates these families, fix representatives $[s_g : g \in T_f]$ for the classes and let $R = \{ f \circ g : f \in S,\ g \in T_f \}$. This is a sieve on C, and it follows from axiom S3 that it is a covering sieve. For $f \in S$ and $g \in T_f$, define $t_{f \circ g} = s_g \in P(\operatorname{dom} g)$.

To see that this is well-defined, we need that if $f, f' \in S$ and $g \in T_f$, $g' \in T_{f'}$ are such that $f \circ g = f' \circ g'$, then $t_{f \circ g} = t_{f' \circ g'}$, so $s_g = s_{g'}$. Now note that $f \circ g \in S$, so by the matching condition we have

$$
\begin{aligned}
[u_\ell : \ell \in T_{f \circ g}]_{f \circ g} &= P^+(g)([s_h : h \in T_f]_f) = [s_{g \circ k} : k \in g^*(T_f)]; \\
[u_\ell : \ell \in T_{f' \circ g'}]_{f' \circ g'} &= P^+(g')([s_{h'} : h' \in T_{f'}]_{f'}) = [s_{g' \circ k'} : k' \in (g')^*(T_{f'})].
\end{aligned}
$$

Because $f \circ g = f' \circ g'$, the leftmost classes are equal, so the rightmost classes are equal too. It follows by definition that there exists some covering sieve $R \in J(\operatorname{dom} g)$ such that $Q \subseteq g^*(T_f) \cap (g')^*(T_{f'})$ and $s_{g \circ q} = s_{g' \circ q}$ (i.e. $P(q)(s_g) = P(q)(s_{g'})$) for all $q \in Q$.

Here's where we hit a snag. We'd like to conclude that $s_g = s_{g'}$, but we can only do so if P is separated! If we assume that P is separated, then $(s_g : f \circ g \in R)$ is a well-defined matching family of P-sections, and its $\sim$-class is independent of the choice of representatives $(s_g : g \in T_f)_f$. Moreover, it is a collation of these classes, and it is the unique class collating the given classes of families. In other words, P^+ is a sheaf.

For separated P, we have a canonical morphism $\eta \colon P \to P^+$, mapping a section $s \in P(C)$ to the matching family class $[P(f)(s) : f \in よ_C]$. In fact, if $\phi \colon P \to F$ is a morphism of separated presheaves in which F is a sheaf, then this morphism factors through $\eta \colon P \to P^+$, because the image of a collation of P-sections (s_f) can be taken as the collation of the images $(\phi(s_f))$. Moreover, this extension $\overline{\phi}$ is unique by uniqueness of collations. This shows that $\mathrm{Sh}(\mathcal{C}, J)$ is reflective in the category of separated presheaves $\mathrm{PSh}^{\mathrm{sep}}(\mathcal{C}, J)$.

In the first homework exercise you will work out an example of a non-separated presheaf P for which P^+ is not a sheaf.

To fix this construction, we first quotient the sets $P(C)$ by an equivalence relation. Define $s \approx t$ if there exists a covering sieve $S \in J(C)$ such that $P(f)(s) = P(f)(t)$ for all $f \in S$. Note that $s \approx t$ is a sufficient condition to conclude that $\eta(s) = \eta(t)$ in $P^\ddagger(C)$. Define

$$P^s(C) = P(C)/{\approx}\ .$$

Restriction is consistent with $\approx$ (by axiom S2), so this defines a presheaf. It's not difficult to see that P^s is a separated presheaf, and in fact if $\phi \colon P \to Q$ is a morphism of presheaves where Q is separated, then ϕ factors uniquely through the quotient map $P \to P^s$. The presheaf P^s is called the *separation* of P. This shows that the separated presheaves are reflective in PSh.

Putting these constructions together, for a presheaf P and a sheaf F we have natural bijections

$$\mathrm{PSh}(P, F) \cong \mathrm{PSh}^{\mathrm{sep}}(P^s, F) \cong \mathrm{Sh}((P^s)^+, F).$$

This shows that $(-)^\ddagger = (-)^+ \circ (-)^s$ gives us our desired associated sheaf functor.

As an aside, applying $(-)^+$ to an arbitrary presheaf may not in general give a sheaf, but it does result in a separated presheaf (in a sense, the relation $\approx$ is subsumed by $\sim$).

For convenience later, we give a quick argument as two why $(-)^+ \circ (-)^+$ is also an associated sheaf functor, in the sense that it is left adjoint to the inclusion functor $i \colon \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{PSh}(\mathcal{C})$. Because P^+ is constructed of matching families of sections of P, if F is any sheaf and $\phi \colon P \to F$ is a morphism of presheaves, then ϕ factors uniquely through the canonical morphism $P \to P^+$. The only obstruction to $(-)^+$ being an associated sheaf functor is that P^+ might itself not be a sheaf. Moreover, we can show that P^+ is always a separated presheaf, so $(P^+)^+$ is a sheaf. It follows that the canonical morphism $P \to (P^+)^+$ factors uniquely through $P^\ddagger$, but we also have that the canonical morphism $P \to P^\ddagger$ factors uniquely through P^+, hence through $(P^+)^+$. By uniqueness these morphisms $(P^+)^+ \leftrightarrows P^\ddagger$ are inverse isomorphisms.

This is why I use the $\ddagger$ symbol; it's twice the $+$ symbol. The single plus functor is not a reflection functor, though (it is not idempotent), so I prefer to build $(-)^\ddagger$ out of $(-)^s$ and $(-)^+$.

Consider the trivial topology J. Any presheaf P is a sheaf for this topology, so the $(-)^\ddagger$ is naturally isomorphic to the identity functor. Boring!

For the maximal topology, note that if $\varnothing \in J(C)$ (the empty sieve covers C), then a matching family of sections of a sheaf F indexed by the empty sieve must have a unique collation in $F(C)$, so because any section is a collation of an empty family we must have that $F(C)$ contains exactly one element. It follows that the only sheaves on the maximal topology are (up to isomorphism) the constant sheaves on $\{*\}$. Note that this is the terminal presheaf T. The canonical morphism $P \to P^\ddagger$ is the unique morphism $P \to T$.

If P is a presheaf on a topological space X whose sections over U are functions from U into some fixed set S and the restriction maps are the usual ones, then P is separated. Moreover, the associated sheaf $P^\ddagger$ also has such functions as its sections. More precisely, if P embeds into the sheaf of all functions into S, then so does $P^\ddagger$ by reflection. If P is the presheaf of bounded real-valued continuous maps, then its associated sheaf is the sheaf of maps f into $\mathbb{R}$ such that there exists an open cover of its domain such that f restricted to the sets in this cover is a bounded continuous map. This is the sheaf of real-valued continuous maps!

(Co-)limits of sheaves

We will use $(-)^\ddagger$ to construct limits and colimits in $\mathrm{Sh}(\mathcal{C}, J)$. First, we recall these notions in $\mathrm{PSh}(\mathcal{C})$.

If $D\colon I \to \mathrm{PSh}(\mathcal{C})$ is a diagram of presheaves (I is some small domain category), then we claim that $\lim_i D_i$ and $\operatorname{colim}_i D_i$ exist, and that they may be given by

$$\Big(\lim_{i\in I} D_i\Big)(C) = \lim_{i\in I}\Big(D_i(C)\Big),$$
$$\Big(\operatorname*{colim}_{i\in I} D_i\Big)(C) = \operatorname*{colim}_{i\in I}\Big(D_i(C)\Big),$$

with appropriate restriction maps. To see this, consider a cone of presheaf morphisms $\phi_i\colon D_i \to P$ to some presheaf P. As these are natural transformations, we can take components at objects C of $\mathcal{C}$ to get cones of sets $(\phi_i)_C\colon D_i(C) \to P(C)$. By definition these cones correspond to set functions $f_C\colon \lim_{i\in I} D_i(C) \to P(C)$, which in turn can be sewn together into a natural transformation $f\colon \lim_i \Big(D_i(-)\Big) \to P$. These correspondences are natural and bijective, so $\lim_i \Big(D_i(-)\Big)$ is the limit of the diagram D. For colimits, turn the document upside down and read this paragraph again. This shows that $\mathrm{PSh}(\mathcal{C})$ is (co-)complete.

It is not a terrible chore to work out that if every D_i in the diagram is a sheaf, then so is $\lim_i D_i$. For colimits, there's issues. Consider the example from the previous section, where we had sheaves for the maximal topology on some category $\mathcal{C}$. These were all isomorphic to the terminal presheaf T. However, the coproduct of T with itself as a presheaf sends every object of $\mathcal{C}$ to a two-point set, so this is not a sheaf. To fix this, we use the associated sheaf functor.

Given a diagram $D\colon I \to \mathrm{Sh}(\mathcal{C}, J)$, the we know that the colimit presheaf $\operatorname{colim}_i^{\mathrm{PSh}} D_i$ solves the 'equation'

$$\mathrm{Nat}(D, \mathrm{const}_P) \cong \mathrm{PSh}\left(\operatorname*{colim}_i{}^{\mathrm{PSh}} D_i, P\right)$$

for all presheaves P. That is, cones from D to P (i.e. natural transformations $D \Rightarrow \mathrm{const}_P$) correspond naturally to morphisms $\operatorname{colim}_i^{\mathrm{PSh}} D_i \to P$. As a limit of sheaves, we want some sheaf C to fit in the equation

$$\mathrm{Nat}(D, \mathrm{const}_F) \cong \mathrm{Sh}(C, F)$$

for every sheaf F. Using the associated sheaf functor, we have

$$\mathrm{Nat}(D, \mathrm{const}_F) \cong \mathrm{PSh}\left(\operatorname*{colim}_i{}^{\mathrm{PSh}} D_i, F\right) \cong \mathrm{Sh}\left(\left(\operatorname*{colim}_i{}^{\mathrm{PSh}} D_i\right)^\ddagger, F\right),$$

so taking the sheaf associated to the colimit presheaf gets us the colimit sheaf.

We have constructed arbitrary small (co-)limits in $\mathrm{Sh}(\mathcal{C}, J)$, so sheaf categories are (co-)complete. Moreover, the inclusion functor $i\colon \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{PSh}(\mathcal{C})$ creates (hence, by completeness or by adjointness, preserves) all limits, and (as a left adjoint) the associated sheaf functor $(-)^\ddagger\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, J)$ preserves all colimits.

As a final comment on limits of sheaves, we will show that $(-)^\ddagger$ preserves finite limits. Because $(-)^\ddagger \cong (-)^+ \circ (-)^+$, it suffices to show that $(-)^+$ preserves finite limits. Recall that

$$P^+(C) \cong \operatorname*{colim}_{S\in J(C)} \Big\{\text{matching } S\text{-families}\Big\}.$$

Moreover, the set of matching S-families is naturally isomorphic to $\mathrm{PSh}(S, P)$. It's a general fact that covariant hom-functors preserve limits, so

$$\Big(\lim_{i\in I} P_i\Big)^+(C) \cong \operatorname*{colim}_{S\in J(C)} \mathrm{PSh}\Big(S, \lim_{i\in I} P_i\Big) \cong \operatorname*{colim}_{S\in J(C)}\Big(\lim_{i\in I} \mathrm{PSh}(S, P_i)\Big).$$

However, note that the colimit is taken over the poset of covering sieves on C ordered by reverse inclusion, which is a directed set because the intersection of two covering sieves is a covering sieve. It follows that this is a filtered colimit, which commute with finite limits, so

$$\Big(\lim_{i\in I} P_i\Big)^+(C) \cong \lim_{i\in I}\Big(\operatorname*{colim}_{S\in J(C)} \mathrm{PSh}(S, P_i)\Big) \cong \lim_{i\in I}\Big(P_i^+(C)\Big) \cong \lim_{i\in I}\Big(P_i^+\Big)(C),$$

and all of these bijections are natural, so $(-)^+$ preserves finite limits, so $(-)^\ddagger$ does as well.

This shows that the associated sheaf functor $\mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, J)$ is a left adjoint that preserves finite limits, hence $((-)^\ddagger, i)$ is a geometric morphism of topoi $\mathrm{Sh}(\mathcal{C}, J) \to \mathrm{PSh}(\mathcal{C})$.

Associated sheaves of algebra-valued presheaves

As an application of limits of sheaves, let us consider the category of algebra-valued presheaves $\mathrm{PSh}(\mathcal{C}; \mathcal{A})$, where $\mathcal{A}$ is some category of algebraic structures. We will treat the case where $\mathcal{A} = \mathbf{Ab}$, the category of abelian groups, but the narrative generalizes straightforwardly.

Consider a presheaf of abelian groups $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$. Then for every object C, we have a group structure on the set $P(C)$. This consists of a binary operation $P(C)\times P(C) \to P(C)$, a unit element $\{*\} \to P(C)$, and an inversion map $P(C) \to P(C)$, but let us focus on the binary operation. We know that for $f\colon D \to C$, the restriction map $P(f)\colon P(C) \to P(D)$ is a group homomorphism by definition. This states exactly that the diagram

$$\begin{tikzcd}
P(C) \times P(C) \arrow[r, "P(f)\times P(f)"] \arrow[d, "-\cdot-"'] & P(D) \times P(D) \arrow[d, "-\cdot-"] \\
P(C) \arrow[r, "P(f)"] & P(D)
\end{tikzcd}$$

is commutative. In other words, the group operation is a natural transformation $P \times P \Rightarrow P$ (i.e. a morphism of presheaves). Moreover, that the group operation is commutative is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
P(C) \times P(C) \arrow[rr, "\mathrm{swap}"] \arrow[dr, "-\cdot-"'] & & P(C) \times P(C) \arrow[dl, "-\cdot-"] \\
& P(C) &
\end{tikzcd}$$

so this commutativity extends to a commutative diagram of natural transformations. In this way we can identify (in the sense of an equivalence of categories) presheaves of abelian groups with *abelian group objects* in the category of set-valued presheaves $\mathrm{PSh}(\mathcal{C})$.

A presheaf of abelian groups F is a sheaf of abelian groups precisely when the underlying presheaf of sets is a sheaf of sets. This shows that sheaves of abelian groups are abelian group objects in $\mathrm{Sh}(\mathcal{C}, J)$. How do we construct an associated sheaf of a presheaf of abelian groups?

We can apply a functor to any diagram, and it will preserve commutativity of diagrams by functoriality. Using the associated sheaf functor, we get

$$\begin{tikzcd}
(P \times P)^\ddagger \arrow[d, "(-\cdot-)^\ddagger"'] \\
P^\ddagger
\end{tikzcd}$$

Because $(-)^\ddagger$ preserves finite products, this yields an natural operation $P^\ddagger \times P^\ddagger \to P^\ddagger$. Because any functor preserves commutative diagrams, this operation (together with the unit and inversion morphisms we can similarly construct) yields an abelian group object in $\mathrm{Sh}(\mathcal{C}, J)$.

Moreover, we know that if $\phi\colon P \to F$ is a morphism of presheaves of abelian groups where F is a sheaf, then ϕ factors uniquely through the sheaf of sets $P^\ddagger$. To see that both the morphisms $P \to P^\ddagger$ and $P^\ddagger \to F$ are group object homomorphisms, note the commutativity of

$$\begin{tikzcd}
P \times P \arrow[r] \arrow[d, "- \cdot -"'] \arrow[rr, bend left, "\phi \times \phi"] & P^\ddagger \times P^\ddagger \arrow[r] \arrow[d, "(- \cdot -)^\ddagger"'] & F \times F \arrow[d, "- \cdot -"'] \\
P \arrow[r] \arrow[rr, bend right, "\phi"'] & P^\ddagger \arrow[r] & F
\end{tikzcd}$$

The left square commutes because $(- \cdot -)^\ddagger$ is by construction the unique morphism $P^\ddagger \times P^\ddagger \to P^\ddagger$ through which the composite $P \times P \to P \to P^\ddagger$ factors. To see that the right square commutes, note that there exists a unique morphism $P^\ddagger \times P^\ddagger \to F$ through which $\phi \circ (- \cdot -) \colon P \times P \to F$ factors. Using the fact that $\phi \circ (- \cdot -) = (- \cdot -) \circ (\phi \times \phi)$ and the commutativity of the left square, we can show that both paths through the right square satisfy this equation, so they must be equal.

This shows that the sheaf associated to a presheaf of abelian groups is a sheaf of abelian groups in a canonical way, so we have constructed associated sheaves of abelian groups. This same construction works for any algebraic structure that can be expressed in terms of commutative diagrams and natural morphisms between finite products (including terminal objects, recall the unit map $\{*\} \to P(C)$).

Homework Exercises (9 points)

Exercise 1: Sheaf conditions on representable presheaves (6 points)

On a small category $\mathcal{C}$, consider an object C and the hom-presheaf よ_C.

(a) (2 points)

Show that よ_C is separated if and only if every covering sieve $S \in J(C)$ is *jointly epimorphic*, meaning that if $h, h' \colon C \to D$ are any two morphisms such that $h \circ f =' \circ f$ for all $f \in S$, then $h = h'$.

(b) (4 points)

If S is any sieve on C, then we can view S as a full subcategory of the comma category $(\mathcal{C} \downarrow C)$, whose objects are $\mathcal{C}$-morphisms $f \colon D \to C$ and whose arrows are $\mathcal{C}$-morphisms g that fit into commutative triangles

$$\begin{tikzcd}
D \arrow[rr, "g"] \arrow[dr, "f"'] & & D' \arrow[dl, "f'"] \\
& C &
\end{tikzcd}$$

with composition being that of $\mathcal{C}$. We have a forgetful functor $(\mathcal{C} \downarrow C) \to \mathcal{C}$, so any sieve S yields a diagram $D_S \colon S \to \mathcal{C}$ as the composite

$$S \longrightarrow (\mathcal{C} \downarrow C) \longrightarrow \mathcal{C}$$

Show that よ_C is a sheaf if and only if for every $S \in J(C)$ we have that C is the colimit of D_S, and the elements $f \in S$ are the colimit injections.

If $(\mathcal{C}, J)$ is a site such that every representable presheaf is a sheaf (or, equivalently by this exercise, such that every covering sieve consists of colimit injections), then J is said to be *subcanonical.* Any small category has a largest subcanonical topology, called the *canonical topology.* You don't need to show this for the exercise, but think about how this compares to the canonical topology on $\mathrm{Open}(X)$ for a topological space X.

(c) (bonus: 2 points)

Find an example of a site $(\mathcal{C}, J)$ such that for some object C of $\mathcal{C}$, the presheaf $(\text{よ}_C)^+$ is not a sheaf. (*Hint: my own example has only five objects, maybe there's even smaller examples!*)

Exercise 2: sheaves of modules (3 points)

Fix a commutative unital ring R. Describe the structure of an R-module as a set equipped with a collection set functions satisfying various commutative diagrams. Conclude that a sheaf of R-modules is an R-module object in the category of sheaves.