

**HOMEWORK FOR THE COURSE FOUNDATIONS OF  
MATHEMATICS**  
**2025-2026**

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1. HOMEWORK 1 [DEADLINE: NOVEMBER 20, 2025, 23:59]

*Homework longer than 3 pages long will not be accepted!*

- (1) Let  $f: X \rightarrow Y$  be a injective function.
  - (a) (2 pts) Prove without the axiom of choice that if  $X$  is non-empty, then there exists a surjection  $g: Y \rightarrow X$  such that  $g(f(x)) = x$  for every  $x \in X$ .
  - (b) (1 pt) What happens when  $X = \emptyset$ ? When can you define  $g$  as above?
- (2) (3 pts) Prove that

$$|\mathbb{N}^{\mathbb{N}}| = 2^{|\mathbb{N}|}.$$

- (3) (4 pts) Let  $X$  be an infinite set. Prove that

$$|X| = |X| + 1.$$

Hint: Proposition 1.2.2 of the textbook.

2. HOMEWORK 2 [DEADLINE: NOVEMBER 27, 2025, 23:59]

The purpose of this homework is to prove the existence of nonprincipal ultrafilters.

Let  $X$  be a set. A *filter* on  $X$  is a subset of  $\mathcal{P}(X)$  satisfying the following conditions:

- (F1)  $X \in \mathcal{F}$ ,
- (F2) if  $A \in \mathcal{F}$  and  $A \subseteq B$ , then  $B \in \mathcal{F}$ ,
- (F3) if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ .

(NB: Definition 2.5.6 in the book also requires that  $\emptyset \notin \mathcal{F}$ , but we do not impose this condition.) Note that since filters are subsets of the powerset  $\mathcal{P}(X)$ , we can compare filters via the inclusion relation on  $\mathcal{P}(X)$ .

A filter  $\mathcal{F}$  is called proper if it is not equal to  $\mathcal{P}(X)$ , or equivalently if  $\emptyset \notin \mathcal{F}$ . A filter is called an *ultrafilter* if it is a maximal proper filter for the inclusion relation.

- (1) (1pt) Let  $A$  be a subset of  $X$ . Show that the set  $\{B \subseteq X | A \subseteq B\}$  is a filter. This is the *filter generated by A*. Such filters are also called *principal*.

- (2) (1pt) Let  $\mathcal{C}$  be the set  $\{B \subset X \mid X - B \text{ is finite}\}$ . Show that  $\mathcal{C}$  is a filter. This filter is called the *cofinite filter*.
- (3) (3pt) Let  $\mathcal{F}$  be a proper filter on  $X$ . Use Zorn's lemma to prove that there exists an ultrafilter on  $X$  that contains  $\mathcal{F}$ .
- (4) (4pt) Let  $\mathcal{F}$  be a proper filter on  $X$  and suppose that  $A$  is a subset of  $X$  satisfying the following condition: For all  $B \in \mathcal{F}$ , the intersection  $A \cap B$  is nonempty. Show that there exists a proper filter  $\mathcal{F}'$  such that  $A \in \mathcal{F}'$  and  $\mathcal{F} \subseteq \mathcal{F}'$ .
- (5) (1pt) Let  $\mathcal{U}$  be an ultrafilter and  $A$  a subset of  $X$ . Show that  $A \in \mathcal{U}$  or  $X - A \in \mathcal{U}$ .
- (6) (1pt) Show that a principal ultrafilter on  $X$  is generated by  $\{x\}$  for some  $x \in X$ .
- (7) (1pt) Assume that  $X$  is infinite. Show that the cofinite filter  $\mathcal{C}$  is proper and not contained in any principal ultrafilter. Conclude that there exist ultrafilters that are not principal.

3. HOMEWORK 3 [DEADLINE: DECEMBER 04, 2025, 23:59]

*Homework longer than 4 pages long will not be accepted!*

Let us say that a function  $f: (P, \leq_P) \rightarrow (Q, \leq_Q)$  between posets is

- *monotone* if

$$x \leq_P y \text{ implies } f(x) \leq_Q f(y)$$

- *strictly monotone* if

$$x <_P y \text{ implies } f(x) <_Q f(y),$$

- *continuous* if it is monotone and if it preserves least upper bounds. This last condition means that for any subset  $A \subseteq P$  which admits a least upper bound  $x$ , then  $f(x)$  is a least upper bound of  $f(A)$ .

- (1) (4pts) Show that if  $L$  is a well-ordered set and  $f: L \rightarrow L$  is strictly monotone, then for any  $x \in L$ , we have

$$x \leq f(x).$$

- (2) (1pt) If you replace “strictly monotone” by “monotone” in the previous question, is the assertion still true? Justify your answer.
- (3) (5pts) Let  $L$  be a nonempty well-ordered set. Show that any function  $f: L \rightarrow L$  strictly monotone and continuous admits a fixed point, i.e. there exists  $x \in L$  such that

$$x = f(x).$$

## 4. HOMEWORK 4 [DEADLINE: DECEMBER 11, 2025, 23:59]

*Homework longer than 4 pages long will not be accepted!*

*(This exercise was inspired by Exercise 87 of the textbook.)*

Recall that a **simple undirected graph** consists of a set of vertices and for each *unordered* pair  $\{x, y\}$  of vertices, a set of edges (such an edge is said to be ‘between  $x$  and  $y$ ’), such that:

- for any vertices  $x$  and  $y$ , there is at most one edge between  $x$  and  $y$ ,
- for any vertex  $x$ , there is no edge between  $x$  and  $x$ .

For an integer  $k \geq 1$ , a  $k$ -colouring of a graph is a function  $f$  from the set of vertices to the set  $\{1, \dots, k\}$  such that if there is an edge between two vertices  $x$  and  $y$ , then  $f(x) \neq f(y)$ .

- (1) (2pts) Show that there is a language  $\mathcal{L}$  with no constants, no function symbols and exactly one relation symbol, and a theory  $T$  in that language such that a model of  $T$  is exactly a simple undirected graph.
- (2) (5pts) Using the compactness theorem, show that if a graph  $G$  is such that every finite subgraph of  $G$  admits a  $k$ -coloring, then  $G$  admits a  $k$ -coloring.

## 5. HOMEWORK 5 [DEADLINE: JANUARY 15, 2026, 23:59]

*Homework longer than 3 pages long will not be accepted!*

Demonstrate by constructing proof trees:

- (1) (5pts)  $(\phi \rightarrow \exists x \psi) \vdash \exists x (\phi \rightarrow \psi)$  (here it is assumed that the variable  $x$  does not occur in  $\phi$ ).
- (2) (5pts)  $\vdash (\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$ .

## 6. HOMEWORK 5 [DEADLINE: JANUARY 25, 2026, 23:59]

*Homework longer than 3 pages long will not be accepted!*

*(This is exercise 129 from the textbook.)*

Let  $L$  be a language,  $T$  an  $L$ -theory and  $L'$  an extension of  $L$ .

- (1) (4pts) Show that the poset of all  $L'$ -theories which are conservative extension of  $T$ , ordered by inclusion, satisfies the hypothesis of Zorn’s Lemma.
- (2) (6 pts) By Zorn’s Lemma, there is a maximal  $L'$ -theory  $U$  which is conservative over  $T$ . Show that for every  $L'$ -sentence  $\psi \notin U$ , there are an  $L$ -sentence  $\phi$  and an  $L'$ -sentence  $\gamma \in U$  such that  $\gamma \wedge \psi \models \phi$  and  $T \not\models \phi$ .