

**HOMEWORK FOR THE COURSE FOUNDATIONS OF
MATHEMATICS
2025-2026**

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1. HOMEWORK 1 [DEADLINE: NOVEMBER 20, 2025, 23:59]

Homework longer than 3 pages long will not be accepted!

- (1) Let $f: X \rightarrow Y$ be an injective function.
 - (a) (2 pts) Prove without the axiom of choice that if X is non-empty, then there exists a surjection $g: Y \rightarrow X$ such that $g(f(x)) = x$ for every $x \in X$.
 - (b) (1 pt) What happens when $X = \emptyset$? When can you define g as above?
- (2) (3 pts) Prove that

$$|\mathbb{N}^{\mathbb{N}}| = 2^{|\mathbb{N}|}.$$

- (3) (4 pts) Let X be an infinite set. Prove that

$$|X| = |X| + 1.$$

Hint: Proposition 1.2.2 of the textbook.

2. HOMEWORK 2 [DEADLINE: NOVEMBER 27, 2025, 23:59]

The purpose of this homework is to prove the existence of nonprincipal ultrafilters.

Let X be a set. A *filter* on X is a subset of $\mathcal{P}(X)$ satisfying the following conditions:

- (F1) $X \in \mathcal{F}$,
- (F2) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$,
- (F3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(NB: Definition 2.5.6 in the book also requires that $\emptyset \notin \mathcal{F}$, but we do not impose this condition.) Note that since filters are subsets of the powerset $\mathcal{P}(X)$, we can compare filters via the inclusion relation on $\mathcal{P}(X)$.

A filter \mathcal{F} is called *proper* if it is not equal to $\mathcal{P}(X)$, or equivalently if $\emptyset \notin \mathcal{F}$. A filter is called an *ultrafilter* if it is a maximal proper filter for the inclusion relation.

- (1) (1pt) Let A be a subset of X . Show that the set $\{B \subseteq X \mid A \subseteq B\}$ is a filter. This is the *filter generated by A* . Such filters are also called *principal*.

- (2) (1pt) Let \mathcal{C} be the set $\{B \subset X \mid X - B \text{ is finite}\}$. Show that \mathcal{C} is a filter. This filter is called the *cofinite filter*.
- (3) (3pt) Let \mathcal{F} be a proper filter on X . Use Zorn's lemma to prove that there exists an ultrafilter on X that contains \mathcal{F} .
- (4) (4pt) Let \mathcal{F} be a proper filter on X and suppose that A is a subset of X satisfying the following condition: For all $B \in \mathcal{F}$, the intersection $A \cap B$ is nonempty. Show that there exists a proper filter \mathcal{F}' such that $A \in \mathcal{F}'$ and $\mathcal{F} \subseteq \mathcal{F}'$.
- (5) (1pt) Let \mathcal{U} be an ultrafilter and A a subset of X . Show that $A \in \mathcal{U}$ or $X - A \in \mathcal{U}$.
- (6) (1pt) Show that a principal ultrafilter on X is generated by $\{x\}$ for some $x \in X$.
- (7) (1pt) Assume that X is infinite. Show that the cofinite filter \mathcal{C} is proper and not contained in any principal ultrafilter. Conclude that there exist ultrafilters that are not principal.

3. HOMEWORK 3 [DEADLINE: DECEMBER 04, 2025, 23:59]

Homework longer than 4 pages long will not be accepted!

Let us say that a function $f: (P, \leq_P) \rightarrow (Q, \leq_Q)$ between posets is

- *monotone* if

$$x \leq_P y \text{ implies } f(x) \leq_Q f(y)$$

- *strictly monotone* if

$$x <_P y \text{ implies } f(x) <_Q f(y),$$

- *continuous* if it is monotone and if it preserves least upper bounds. This last condition means that for any subset $A \subseteq P$ which admits a least upper bound x , then $f(x)$ is a least upper bound of $f(A)$.

- (1) (4pts) Show that if L is a well-ordered set and $f: L \rightarrow L$ is strictly monotone, then for any $x \in L$, we have

$$x \leq f(x).$$

- (2) (1pt) If you replace “strictly monotone” by “monotone” in the previous question, is the assertion still true? Justify your answer.
- (3) (5pts) Let L be a nonempty well-ordered set. Show that any function $f: L \rightarrow L$ strictly monotone and continuous admits a fixed point, i.e. there exists $x \in L$ such that

$$x = f(x).$$

4. HOMEWORK 4 [DEADLINE: DECEMBER 11, 2025, 23:59]

Homework longer than 4 pages long will not be accepted!

(This exercise was inspired by Exercise 87 of the textbook.)

Recall that a **simple undirected graph** consists of a set of vertices and for each *unordered* pair $\{x, y\}$ of vertices, a set of edges (such an edge is said to be “between x and y ”), such that:

- for any vertices x and y , there is at most one edge between x and y ,
- for any vertex x , there is no edge between x and x .

For an integer $k \geq 1$, a k -colouring of a graph is a function f from the set of vertices to the set $\{1, \dots, k\}$ such that if there is an edge between two vertices x and y , then $f(x) \neq f(y)$.

- (1) (2pts) Show that there is a language \mathcal{L} with no constants, no function symbols and exactly one relation symbol, and a theory T in that language such that a model of T is exactly a simple undirected graph.
- (2) (5pts) Using the compactness theorem, show that if a graph G is such that every finite subgraph of G admits a k -coloring, then G admits a k -coloring.

5. HOMEWORK 5 [DEADLINE: JANUARY 15, 2026, 23:59]

Homework longer than 3 pages long will not be accepted!

Demonstrate by constructing proof trees:

- (1) (5pts) $(\phi \rightarrow \exists x\psi) \vdash \exists x(\phi \rightarrow \psi)$ (here it is assumed that the variable x does not occur in ϕ).
- (2) (5pts) $\vdash (\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$.

6. HOMEWORK 5 [DEADLINE: JANUARY 25, 2026, 23:59]

Homework longer than 3 pages long will not be accepted!

(This is exercise 129 from the textbook.)

Let L be a language, T an L -theory and L' an extension of L .

- (1) (4pts) Show that the poset of all L' -theories which are conservative extension of T , ordered by inclusion, satisfies the hypothesis of Zorn’s Lemma.
- (2) (6 pts) By Zorn’s Lemma, there is a maximal L' -theory U which is conservative over T . Show that for every L' -sentence $\psi \notin U$, there are an L -sentence ϕ and an L' -sentence $\gamma \in U$ such that $\gamma \wedge \psi \models \phi$ and $T \not\models \phi$.