APM466 A2

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Questions - 100 points

1. *(40 points)* Suppose that company X has four states of solvency: good, bad, crisis, and default. Suppose also that the one year transition (between solvency states) probability matrix is given by:

$$P = \begin{array}{c|cccc} \text{state} & \text{good} & \text{bad} & \text{crisis} & \text{default} \\ \hline \text{good} & 8/10 & 1/10 & 1/10 & 0 \\ \text{bad} & 1/10 & 5/10 & 2/10 & 2/10 \\ \text{crisis} & 1/10 & 3/10 & 3/10 & 3/10 \\ \text{default} & 0 & 0 & 0 & 1 \end{array}$$

For the following questions, feel free to use a computer to aid your calculations. For part a)&b), you must state your final answer with a small explanation (explicit calculations discouraged in your report). For part c), a formal proof is not needed, just a 1 or 2 sentence explanation.

(a) *(10 points)* What is the two year transition probability matrix?

Answer: The two year transition probability matrix is:

$$P^2 = \begin{pmatrix} 0.66 & 0.16 & 0.13 & 0.05 \\ 0.15 & 0.32 & 0.17 & 0.36 \\ 0.14 & 0.25 & 0.16 & 0.45 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

because: we want the two year transition probability matrix, P is given for t = 1, so we want to compute P to the exponent t = 2, which is just multiplying P by itself.

(b) *(10 points)* What is the probability that if company X is currently in a "crisis" solvency state, they will default within the next month?

Answer: If company X is currently in a "crisis" solvency state, the probability that they will default within the next month is $\simeq 0.0376$, because: we raise P to the exponent 1/12 to convert the annual matrix to monthly and get the 1 month time period for t, then we isolate the value for the 3rd row and 4th column to obtain the transition probability from crisis to default.

Using the formula:

$$A^{1/12} = PD^{1/12}P^{-1}$$

and the Matlab commands

```
P = [0.8,0.1,0.1,0;0.1,0.5,0.2,0.2;0.1,0.3,0.3,0.3;0,0,0,1]
[V,D]=eig(P);
A1=V*D^(1/12)*inv(V);

%which produces:

A1 =

    0.9802    0.0088    0.0142   -0.0032
    0.0106    0.9287    0.0444    0.0163
    0.0124    0.0675    0.8826    0.0376
         0         0         0    1.0000
```

(c) *(10 points)* What is $\lim_{t\to\infty} P^t$?

Using the formula:

$$A^n = PD^nP^{-1} \Rightarrow \lim_{n\to\infty} A^n = \lim_{n\to\infty} PD^nP^{-1} = P \lim_{n\to\infty} D^nP^{-1} \tag{1}$$

To diagonalize the 4x4 matrix P, I write the following code in Matlab:

```
format long
P = [0.8,0.1,0.1,0;0.1,0.5,0.2,0.2;0.1,0.3,0.3,0.3;0,0,0,1]
[V,D]=eig(P);
A=V*D*inv(V);

%which produces:

V =

0.8489    0.5340   -0.0626    0.5000
0.3981   -0.6632   -0.4659    0.5000
0.3476   -0.5244    0.8826    0.5000
     0         0         0    0.5000



D =

0.8878         0         0         0
     0    0.5776         0         0
     0         0    0.1345         0
     0         0         0    1.0000
```

Where V would be our P matrix in equation 1 above, and D is our new D matrix in equation 1.

$$\lim_{t\to\infty} D^t = \begin{pmatrix} \lim_{t\to\infty} 0.8878^t & 0 & 0 & 0 \\ 0 & \lim_{t\to\infty} 0.5776^t & 0 & 0 \\ 0 & 0 & \lim_{t\to\infty} 0.1345^t & 0 \\ 0 & 0 & 0 & \lim_{t\to\infty} 1^t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Since for x^t where $x \in (0,1)$, as t increases the denominator in x increases faster than the numerator and hence $x \to 0$

We then have:

$$\lim_{t\to\infty} P^t = V \lim_{t\to\infty} D^t V^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0.999999999999993 \\ 0 & 0 & 0 & 0.999999999999998 \\ 0 & 0 & 0 & 0.999999999999998 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Which was obtained using the Matlab commands:

```
F = [0,0,0,0;0,0,0,0;0,0,0,0;0,0,0,1]
V*F*inv(V)
```

(d) *(10 points)* If $t \in \mathbb{N}$, $(t < \infty)$, given that the company X has not yet defaulted, is it guaranteed (/with probability 1) that company X will default within t years?
(Hint: Either use induction or show that $\exists t < \infty$ for which $P^t_{ij} = 0\ \forall j \neq 4, P^t_{ij} = 1$ if $j = 4$.)

Answer: No, because: using the first equality in equation (1) in part c), we see that we will raise our matrix D, which is the eigenvalues of P to the exponent t. The eigenvalues of P are non zero, these values raised to an exponent are products of each other, which means we are asking if there exists a finite t for which x^t where x $\in (0, 1) = 0$:

We know that taking a product of non-zero numbers is still non zero. Using that fact we can say that our matrix D to the finite exponent t will have non zero entries for i,j=1,2,3. Then we multiply this matrix by V and V inverse which are both non zero matrices, which will produce products of non-zero numbers which we know cannot be zero.

If we run the computation in Matlab for a random t (100) we see:

```
B=V*D^(100)*inv(V); %B will represent our A^100
B

%which produces:

B =


   0.000004900688766   0.000002590045705   0.000001714875722   0.999990794389800
   0.000002298322378   0.000001214678240   0.000000804241492   0.999995682757888
   0.000002006599050   0.000001060500574   0.000000702160075   0.999996230740299
           0                   0                   0           1.000000000000000
```

Using our answer from c, we know that as we increase t, we approach the matrix in equation (3). That is, for any finite t we choose, like 100, we can choose t+1 that will produce a P_{ij} closer to 0 for i,j=1,2,3 and so on for t+2, t+3...

```
B1 = V*D^(101)*inv(V); % t+1

B1 =

   0.000004351043155   0.000002299554446   0.000001522540734   0.999991826861658
   0.000002040549875   0.000001078443805   0.000000714040333   0.999996166965983
   0.000001781545305   0.000000941558215   0.000000623408043   0.999996653488436
           0                   0                   0           1.000000000000000

```

```
BN = V*D^(1000)*inv(V); % t=1000

BN =

   0.000000000000000   0.000000000000000   0.000000000000000   0.999999999999993
   0.000000000000000   0.000000000000000   0.000000000000000   0.999999999999998
   0.000000000000000   0.000000000000000   0.000000000000000   0.999999999999998
           0                   0                   0           1.000000000000000

%the numbers for Pij; i,j=1,2,3 are extremely small, but still not zero evidenced by the 4th column
%having entries not equal to 1 for i=1,2,3
```

I.e. if we call δ the distance between each P_{ij} and 0 for i,j =1,2,3, then for any finite t we choose, there exists another t with $\delta/N, N \in \mathbb{N}$ that we can choose suggesting that there is no finite t that we can choose for P^t to guarantee with probability 1 that company X will default in that time period, because we will approach 0 for P_{ij} i,j = 1,2,3 but never reach this limit.

2. *(40 points)* Assume that Germany's bonds are risk-free and Italy's bonds are risk-prone, and that each country issues zero coupon bonds with a face value of 1. We denote a German bond with an outstanding term of i years simply by its current price P_i^G, and an Italian bond with outstanding term of i years also simply by P_i^I. Finally, assume everything henceforth is priced using continuous discounting, and zero recovery under default.

   (a) *(10 points)* Given $\{P_1^G, \dots, P_n^G\}$ and $\{P_1^I, \dots, P_n^I\}$, derive a closed form formula for the credit spread, h_i, at time $i \in \{1, \dots, n\}$ for Italy in terms of i, P_i^G, and P_i^I.

   Answer: Since

$$r_i^I = r_i^G + h_i \tag{4}$$

   where r_i^G is risk free and h_i captures the risk. We also have:

$$r(T) = -\frac{log(P/N)}{T} \tag{5}$$

   Plugging (5) into (4) for each country we get:

$$\frac{ln(P_i^I/N)}{i} = \frac{ln(P_i^G/N)}{i} + h_i \tag{6}$$

   Solving for h_i

$$h_i = \frac{ln(P_i^G/N)}{i} - \frac{ln(P_i^I/N)}{i} = \frac{1}{i}(ln(\frac{p_i^G}{p_i^I})) \tag{7}$$

   (b) *(10 points)* Under a two state markov chain model (solvency and default), write Italy's ith-year probability transition matrix, P^i, in terms of just i and h_i.

   Answer: Since

$$h_i = -\frac{ln(q_i)}{t_i} \Rightarrow q_i = e^{-h_i t_i} \Rightarrow q_i = -\frac{P_i^G}{P_i^I} \tag{8}$$

   Therefore we have:

$$P^i = \begin{pmatrix} q_i & 1-q_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-h_i i} & 1-e^{-h_i i} \\ 0 & 1 \end{pmatrix}$$

(c) *(10 points)* If the Italian government issues a one-off asset, A, that pays $C_i, i = 1, \ldots, n$, at time i, find the price of this asset in terms of $\{1, \ldots, n\}$, $\{h_1, \ldots, h_n\}$, $\{P_1^G, \ldots, P_n^G\}$, and $\{C_1, \ldots, C_n\}$.

Answer: Since

$$A = \sum_{i=1}^{n} C_i e^{-(r_i^G + h_i)i}$$

plugging in $q_i = e^{-h_i i}$, and N=1 we see that:

$$A = \sum_{i=1}^{n} C_i e^{-(r_i^G)i} e^{-h_i i} = \sum_{i=1}^{n} C_i e^{ln(P_i^G/N) - h_i i} = \sum_{i=1}^{n} C_i (P_i^G/N) e^{-h_i i} = \sum_{i=1}^{n} C_i (P_i^G) e^{-h_i i}$$

(d) *(10 points)* First find $\partial_{h_i} A$, then use this to say what would happen to the price of A given Italy's probability of default (by any time $i \geq 1$) increases.

Answer: Taking the derivative in h_i, we see that:

$$\partial_{h_i} A = -iC_i P_i^G e^{-h_i i} = -iC_i P_i^G q_i = -iC_i e^{-r_i i} q_i$$

And thus since the partial derivative of A with respect to h_i is negative, it is decreasing. We see that if Italy's probability of default increases (q_i decreases), the partial derivative will be less negative, hence the decrease of A's price will be less, but it will still decrease.

3. *(20 points)* List 4 simplifications (I.e., assumptions that might not be true in real life) that are made under Merton's Credit Risk Model.

Max 1 sentence per assumption.

(a) *Assumption 1:* All options are European and are exercised only at the time of expiration, obviously, American type options exist in the real world as well.

(b) *Assumption 2:* No dividends are paid out, this is not true as investors will eventually want to obtain a payout from their investment.

(c) *Assumption 3:* Underlying stocks' volatility and risk-free rates are constant, not true, there can be time periods of low volatility and time periods with a lot of uncertainty with high volatility in the stock market.

(d) *Assumption 4:* Returns on underlying stocks are regularly distributed. False, the distribution of stock returns is more like a multimodal distribution, with different peaks in frequency around a percent return during bear markets and peaks in frequency around percent returns during a bull market.