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Answers (Please provide a full mathematical proof - no marks for unsupported or unclear answers. Answers should be well structured and thought out/planned before typing so as to mimic the standards for a proof one would find in an upper-year mathematics textbook.)

1. **(b)** because:

Proof. Credit Spread is defined as $h_i = \frac{-ln(q_i)}{t_i}$, where q_i is probability of solvency at time t, therefore if h_i is increasing with all else equal q_i must decrease.

Note that value of bond is defines as:

$$V = \sum_{i=1}^{n} P_i e^{-r_i t_i} q_i \tag{1}$$

Hence if q_i is decreasing, from (1) we can see that V will decrease → Price of bond decreases as credit spread increases. □

2. **(a)** because:

Proof. *NOTE: proof needs full derivation*

Define $X_t = log(S_t)$, $\quad S_t = e^{x_t}$, $\quad dS_t = \mu S_t d_t + \sigma^2 S_t dW_t$

With our answer (a), we should find that dlog$(S_t) = (\mu - \frac{1}{2}\sigma^2)d_t + \sigma dW_t$

Using Ito's lemma:

$$f(s) = log(s), \quad \partial_s f = \frac{1}{s}, \quad \partial_s^2 f = \partial_s \frac{1}{s} = -\frac{1}{s^2} \tag{2}$$

$$dX_t = df(S_t) = \partial_s f(S_t) dS_t + \frac{1}{2}\partial_s^2 f(S_t)(dS_t)^2 = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} dS_t^2 \tag{3}$$

Using our answer a),

$$(dS_t)^2 = \sigma^2 S_t^2 d_t + 2_t^2 \mu dW_t d_t + \mu^2 S_t^2 d_t^2 \tag{4}$$

Note that when $d_t \to 0$, it does so faster than $dW_t \to 0$, since $dW_t^2 = O(d_t)$, so we can simplify $(dS_t)^2 = \sigma^2 S_t^2 d_t$

Plugging into equation 3 for dS_t^2 and dS_t:

$$\frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} dS_t^2 = \frac{\mu S_t d_t + \sigma^2 S_t dW_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 d_t = (\mu - \frac{1}{2}\sigma^2)d_t + \sigma dW_t \tag{5}$$

$$\Rightarrow dX_t = (\mu - \frac{1}{2}\sigma^2)d_t + \sigma dW_t \tag{6}$$

Which is what we wanted.

□

3. **(b)** because:

Proof. $V = B + S$ where B and S are bond and stock prices, respectively. If S = 0, V = B. We can price S with the Black-Scholes methodology obtaining:

$$S = VN(d_1) + -Ke^{-rt}N(d_2) \Rightarrow V = \frac{Ke^{-rt}N(d_2)}{N(d_1)} \tag{7}$$

Since the value of the bond is given by B = V - S, we know that by Black-Scholes:

$$B = Ke^{-rt}N(d_2) + N[1 - N(d_1)] \Rightarrow V = Ke^{-rt}N(d_2) + N[1 - N(d_1)] \neq \frac{Ke^{-rt}N(d_2)}{N(d_1)} \tag{8}$$

Contradiction □

4. **(a)** because:

Proof. Given that $r = 0, e^{-rt} = 1$

We first find the expected value in year one of the bond:

$$E[X_1] = q(R)(coupon) + (1 - q)(coupon) = 0.5(0.5)(10) + (0.5)(10) = 7.5 \tag{9}$$

Next we find the expected value in year 2 of the bond defining Di as default in year i and $\tilde{\mathrm{D}}$ as no default and sum the payoffs in each case to find expected value:

$$E[X_2] = Pay(D1) + Pay(D2) + Pay(\tilde{D}) \tag{10}$$

= 0.5(5 + 55) + 0.25(10 + 55) + 0.25(10 + 110) = 30 + 16.25 + 30 = 76.25 □

5. **(a)** because:

Proof. Using the expected credit loss formula:

$$ECL = \mathbb{E}\left[\mathbb{1} \cdot CE \cdot LGD\right] \tag{11}$$

Where $R = 1 - LGD$, D is our probability of defaulting and hence can be represented as the expectation of our indicator function, CE is our total credit exposure and the expectation of this can be represented as our total investment E as that would be our maximum possible exposure in the event of mass default.

$$\Rightarrow ECL = \mathbb{E}\left[\mathbb{1}\right] \cdot \mathbb{E}\left[CE\right] \cdot \mathbb{E}\left[1 - R\right] = D \cdot E \cdot (1 - R) \tag{12}$$

□

6. **(b)**because:

Proof. We will calculate: Var = D(Loss After Recover - Expected Loss) + (1-D)(0 - Expected Loss). Since all bonds are perfectly correlated the variance for each will be the same, and hence the Loss After Recover is N(90/N) = 90 and Expected Loss is N(18/N) = 18

$$\sigma^2 = 0.2(90 - 18)^2 + (1 - 0.2)(0 - 18)^2 = 1296 = 36^2 \Rightarrow \sigma = 36 \tag{13}$$

□

7. **(b)** because:

Proof. Now bonds are uncorrelated, we need to calculate variance with the original loss values:

$$\sigma_i^2 = 0.2(\frac{90-18}{N})^2 + (1-0.2)(\frac{0-18}{N})^2 = \frac{1296}{N^2} \tag{14}$$

This is for an individual bond, so we multiply by N for the portfolio and then calculate standard deviation:

$$\sigma = \sqrt{N \cdot \frac{1296}{N^2}} = \frac{36}{\sqrt{N}} \tag{15}$$

□

8. **(e)** because:

Proof. Using:

$$Var(\sum_i X_i) = \sum_i \mathbb{E}\left[X_i X_j\right] - \mu_i \mu_j = \sum_i \sigma_i^2 + \sum_{i \neq j} \sigma_{i,j} = \sum_i \sigma_i^2 + \sum_{i \neq j} \rho \sigma_i \sigma_j \tag{16}$$

We get:

$$\sigma(\sum_i X_i) = \sqrt{\sum_i \frac{1296}{N^2} + \sum_{i \neq j} \rho \frac{36^2}{N^2}} = 36\sqrt{\frac{N}{N^2} + (N)(N-1)\rho \frac{1}{N^2}} \tag{17}$$

Note that $\sum_{i \neq j} \sigma_i \sigma_j = (N)(N-1)$ because in the summation we have N for i bonds and N-1 for j bonds which is one less than i.

From here we get the answer (e)

$$\sigma(\sum_i X_i) = 36\sqrt{\frac{N}{N^2} + \frac{(N^2-N)\rho}{N^2}} \tag{18}$$

□