

Lecture 1: Affine varieties – Exercamples

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Exercample 1. • Show that $\mathbb{V}(\mathcal{F}) = \mathbb{V}(\langle \mathcal{F} \rangle_R)$, where $\langle \mathcal{F} \rangle_R := \{ \sum_i g_i f_i \mid g_i \in R, f_i \in \mathcal{F} \}$
• If $X, Y \subseteq \mathbb{A}^n$ are varieties, then so are $X \cap Y$ and $X \cup Y$.

Exercample 2. Show that for all \mathcal{F}, X

- $\mathbb{V}(\dots)$ and $I(\dots)$ are *antitone*: $\mathcal{F} \subseteq \mathcal{G} \implies \mathbb{V}(\mathcal{F}) \supseteq \mathbb{V}(\mathcal{G})$
- $\mathcal{F} \subseteq I(\mathbb{V}(\mathcal{F}))$ and $I(\mathbb{V}(I(X))) = I(X)$
- $X \subseteq \mathbb{V}(I(X))$ and $\mathbb{V}(I(\mathbb{V}(\mathcal{F}))) = \mathbb{V}(\mathcal{F})$
- What is $I(p)$, $p = (p_1, \dots, p_n) \in \mathbb{A}^n$?

Exercample 3. • Every open set of X is a union of $D_X(f) = \{x \in X \mid f(x) \neq 0\}$, $f \in R$

- Y is dense in X if and only if every non-empty open set of X meets Y
- X is irreducible iff every non-empty open set is dense
- $X = \bigcup_{i=1}^c X_i$ uniquely for X_i closed irreducible, $X_i \not\subseteq X_j$ for $i \neq j$ (“components”)

Exercample 4. \sqrt{I} is an ideal. $I(X)$ is a radical ideal. Maximal \implies prime \implies radical.

Exercample 5. Let $I \subseteq R$ be an ideal such that $\mathbb{V}(I) = \{\mathbf{0}\}$. Show that $\ell := \dim_{\mathbb{C}} R/I < \infty$. For $g \in R$, show that the linear map $m_g: R/I \rightarrow R/I$ has characteristic polynomial t^ℓ .

Exercample 6. Describe the vanishing ideal of the following variety of skew-symmetric matrices:

$$\left\{ A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{bmatrix} \mid \text{rank } A < 4 \right\} \subseteq \mathbb{A}^6.$$

Exercample 7. • Morphisms $X \rightarrow \mathbb{A}^1 = \mathbb{C}$ are in bijection with $R/I(X)$

- Show that a morphism $X \rightarrow Y$ is *continuous*: Preimages of open/closed sets from Y are open/closed in X
- Describe the image of $\phi: \mathbb{A}^2 \rightarrow \mathbb{A}^2, (x, y) \mapsto (x, xy)$

Exercample 8. • If $X, Y \subseteq \mathbb{A}^n$ are such that $X \cap Y = \emptyset$, show that $\mathcal{O}(X \cup Y) = \mathcal{O}(X) \times \mathcal{O}(Y)$. (Hint: Chinese remainder theorem).

- What is the coordinate ring of a set of points?
- Show that a morphism $\phi: X \rightarrow Y$ induces a \mathbb{C} -algebra homomorphism $\phi^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. Is this correspondence reversible?
- Show that $X = \mathbb{V}(xy - 1)$ is not *isomorphic* to \mathbb{A}^1 (no mutually-inverse isomorphisms).

Exercample 9. • Let $X \subseteq \mathbb{A}^n$ be a variety, and $X = X_1 \cup \dots \cup X_r$ for subvarieties X_i . Show that $\dim X = \max\{\dim X_1, \dots, \dim X_r\}$

- If $X \rightarrow Y$ is dominant, show that $\dim X \geq \dim Y$
- Show $\dim \mathbb{A}^1 = 1$. Can you argue $\dim \mathbb{A}^2 = 2$? $\dim \mathbb{A}^3 = 3$?

Exercample 10. • Which result in linear algebra is the fiber dimension theorem generalizing?

- Find an example where equality does not always hold.
- What is the dimension of the variety of $m \times n$ -matrices of rank $\leq r$.

Exercample 11. • Show the first statement for $X = \mathbb{V}(f)$

- Find the smooth locus of $\mathbb{V}(y^2 - x^3)$

Exercample 12. Let $\mathcal{F} = \{x_1y_2 - x_2y_1, y_1^2 + y_2^2 - 1\} \subseteq \mathbb{C}[x_1, x_2, y_1, y_2]$. Show that $\mathbb{V}(\mathcal{F})$ is a smooth irreducible variety of dimension 2. What do points $x, y \in (\mathbb{R}^2 \times \mathbb{R}^2) \cap X$ “represent”?