

Lecture 2: Projective varieties – Exercamples

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Exercexample 1. • Which result in linear algebra is the fiber dimension theorem generalizing?

- Find an example where equality does not always hold for all $y \in Y$.
- What is the dimension of the variety of $m \times n$ -matrices of rank $\leq r$?

Exercexample 2. Show that any lines $L_1, L_2 \subseteq \mathbb{P}^2$ intersect ($L_i = \mathbb{V}(\ell_i)$, $0 \neq \ell_i \in S_1$).

Exercexample 3. • $I \subseteq S$ is homogeneous iff $f \in I$ implies $f_j \in I$ for its graded components $f = \sum_{j=0}^{\deg f} f_j$

- $I(X)$ (homogeneous vanishing ideal) = $I(\hat{X})$ (ideal of cone)

Exercexample 4. Let $f_1, \dots, f_n \in S$ be homogeneous.

- Show that $X = \mathbb{V}_{\mathbb{P}^n}(f_1, \dots, f_n) \neq \emptyset$.
- Show that if X is finite, then $\dim \mathbb{V}(f_1, \dots, f_k) = n - k$ for $k = 1, \dots, n$.
- Show that this is generally *fails* in \mathbb{A}^3 .

Exercexample 5. • Let $\phi: X \rightarrow Y$ be a morphism of affine varieties. If X is irreducible and ϕ is surjective, then Y is irreducible.

- Show that the set of matrices $X \subseteq \mathbb{C}^{m \times n}$ of rank $\leq r$ is an irreducible variety
- Compute the dimension of X .

Exercexample 6. • Let $X = \{(t, t^2, t^3) \mid t \in \mathbb{C}\} \subseteq \mathbb{A}^3$. Show that $X = \mathbb{V}(x_1^2 - x_2, x_1^3 - x_3)$. We will see later that these equations generate $I = I(X) \subseteq R$.

- Describe $\overline{X} \subseteq \mathbb{P}^3$. Can you find generators of $I(\overline{X}) = I^h$ up to radical?
- Compute the number of intersection points of X with a general plane $H \subseteq \mathbb{P}^3$.

Exercexample 7. • Compute the Hilbert function and degree of a hypersurface $\mathbb{V}(f) \subseteq \mathbb{P}^n$, $f \in S_d$.

- Show that the Hilbert function of a set of points $\{p_1, \dots, p_r\} \subseteq \mathbb{P}^n$ is eventually constant r .

Exercexample 8. • Compute the Hilbert function of a hypersurface $\mathbb{V}(f)$, $f \in S_d$

- Compute the Hilbert function for a set of 3 points in \mathbb{P}^2

Exercexample 9. • What is the Hilbert polynomial of \mathbb{P}^n ? Of a hypersurface?

- Show that if $X \cap Y = \emptyset$, then $P_{X \cup Y} = P_X + P_Y$, but equality may not hold for Hilbert functions.

Exercexample 10. Explore the theorem for the BKK-general system

$$\mathcal{F} = \{-1 + x - y + xy, 2 + x + \frac{1}{2}y - xy\}.$$

What are its roots in \mathbb{A}^2 ? In \mathbb{P}^2 ?