Lecture 3: Multiplicity and reducedness

Algebraic Geometry Tools for Polynomial Systems in Engineering

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Today: Multiplicities and how to avoid them

0. Leftovers from the previous lectures
    - Intersection multiplicity
1. How to show a given set of equations defines a variety in a reduced way?
    - What does this even mean?
    - Smoothness
    - Affine bundles
    - Generic smoothness
    - Unmixedness theorem
2. A glimpse at Thom–Porteous?

Multiplicity

Definition (Local ring, intersection multiplicity)

The *local ring* at $p \in \mathbb{A}^n$ is $\mathcal{O}_{\mathbb{A}^n,p} = \left\{ \frac{f}{g} \in \mathbb{C}(x_1,\dots,x_n) \;\middle|\; g(p) \neq 0 \right\}$. The *multiplicity* of an ideal $I = \langle f_1,\dots,f_s\rangle \subseteq R$ at $p \in \mathbb{V}(I)$ is

$$\mathrm{mult}_p(I) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^n,p}/\langle f_1,\dots,f_s\rangle_{\mathcal{O}_{\mathbb{A}^n,p}} \overset{p=\mathbf{0}}{=} \dim_{\mathbb{C}} \mathbb{C}[\![\underline{x}]\!]/I\mathbb{C}[\![\underline{x}]\!].$$ *power series*

- $0 < \mathrm{mult}_p(I) < \infty$ if and only if p is isolated in $\mathbb{V}(I)$ (a component)
- $\mathrm{mult}_p(I) = 1$ iff $\langle f_1,\dots,f_s\rangle_{\mathcal{O}_{\mathbb{A}^n,p}} = \langle x_1 - p_1,\dots,x_n - p_n\rangle_{\mathcal{O}_{\mathbb{A}^n,p}}$ if and only if the Jacobi matrix of $f_1,\dots,f_s$ has maximal rank at p

Exercample

- Show that this agrees with your experience from multiplicity of univariate polynomials
- Let $p = \mathbf{0} \in \mathbb{A}^3$. Compute $\mathrm{mult}_{\mathbf{0}}(x^3, y^3, z^3)$ and $\mathrm{mult}_{\mathbf{0}}(\text{all mon's of deg. } 3)$
- Let $f = y^2 - x^2(x+1)$ and $g = y$. Compute $\mathbb{V}(f,g)$ and the intersection multiplicities

Bézout with multiplicities

- The intersection multiplicity of $p \in \mathbb{P}^n$ at $I \subseteq S$ is defined by dehomogenizing to an affine chart $p \in U_i \subseteq \mathbb{P}^n$

Exercample

If $\mathbb{V}(I) \subseteq \mathbb{P}^n$ is a finite set $p_1, \ldots, p_r$, then $\mathrm{hf}_{S/I}(t)$ eventually stabilizes at the value $m = \sum_{i=1}^r \mathrm{mult}_{p_i}(I)$

Theorem (Projective Bézout with multiplicities)

If $f_1, \ldots, f_n \in S$ are homogeneous of degree d_i such that $X = \mathbb{V}(f_1, \ldots, f_n)$ is a finite set, then

$$\sum_{p \in X} \mathrm{mult}_p(I) = d_1 \cdots d_n.$$

- If you found "Bézout-many" roots (with multiplicities), then you found them all!

We don't like multiplicities!

- Usually interested in situations where all multiplicities are 1. But how to prove it?
- Useful refinement: "defines multiplicity-free on an open set"

Definition (Defining a variety in a reduced way)

$\mathcal{F} \subseteq R$ define $X = \mathbb{V}(I) \subseteq \mathbb{A}^n$ *in a reduced way at* $p \in \mathbb{A}^n$ if $\mathcal{F} \cdot \mathcal{O}_{\mathbb{A}^n,p} = I(X) \cdot \mathcal{O}_{\mathbb{A}^n,p}$.
$\mathcal{F}$ defines X in an open set $U \subseteq \mathbb{A}^n$ if $\mathcal{F}$ defines X locally around every $p \in U$.

Exercample

Consider $\mathcal{F} = \{xy, y^2\} \subseteq \mathbb{C}[x, y]$. What is $X = \mathbb{V}(\mathcal{F})$ and $I(X)$? Where does $\mathcal{F}$ define X in a reduced way?

- Tools to show that equations define variety in a reduced way:

Smoothness, affine bundle, generic fibres, unmixedness theorem, reduced degeneration, ...

How to be radical i: Smoothness

Definition (Smoothness)

Let $\langle f_1, \dots, f_s \rangle$ be the vanishing ideal of $X \subseteq \mathbb{A}^n$, then the smooth locus of X is

$$X^{\mathsf{sm}} := \left\{ p \in X \;\middle|\; \operatorname{rank} \left[\frac{\partial f_i}{\partial x_j}(p) \right]_{i,j} \geq n - \dim_p X \right\}, \qquad X^{\mathsf{sing}} = X \setminus X^{\mathsf{sm}}$$

- The smooth locus of X is a dense open subset
- If $X \subseteq Y$, then $X^{\mathsf{sing}} \subseteq Y^{\mathsf{sing}}$. Points on multiple components of X are always singular
- Useful: If $\mathbb{V}(f_1, \dots, f_m) = X$ and *these* equations satisfy the property from the definition at $p \in X$, then $f_1, \dots, f_s$ define X in a reduced way at p

Exercample

- Show the first statement for $X = \mathbb{V}(f)$
- Find the singular locus of $\mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2$ and $\mathbb{V}(xyz + xyw + xwz + yzw) \subseteq \mathbb{P}^3$

How to be radical ii: Affine bundles

- Let $\mathcal{F} = \mathcal{F}'(y) \cup \mathcal{F}''(x,y) \subseteq \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_m]$
- Let $X = \mathbb{V}(\mathcal{F}) \subseteq \mathbb{A}^{n+m}$, $Y = \mathbb{V}(\mathcal{F}') \subseteq \mathbb{A}^m$
- Assume that the equations $\mathcal{F}''$ are of degree ≤ 1 in x and that for all $b \in Y$, the affine-linear system $\mathcal{F}''(x,b)$ has constant rank r
- Then $X \to Y$ is an affine bundle of rank r, $X =$ "base space", $Y =$ "total space"

Lemma (Affine bundles transfer niceness from base to total space)

If Y is irreducible/reduced/smooth/of dimension d, them X is irreducible/reduced/smooth/of dimension $d + (n-r)$.

Exercample

Let $\mathcal{F} = \{x_1y_2 - x_2y_1, y_1^2 + y_2^2 - 1\} \subseteq \mathbb{C}[x_1,x_2,y_1,y_2]$. Show that $\mathbb{V}(\mathcal{F})$ is a smooth irreducible variety of dimension 2. What do points $x, y \in (\mathbb{R}^2 \times \mathbb{R}^2) \cap X$ "represent"?

How to be radical iii: Generic fibers

Definition (Scheme-theoretic fiber)

Let $\phi\colon X \to Y \subseteq \mathbb{A}^n$ be a morphism and $y \in Y$. The *scheme-theoretic fiber* $f^{-1}(y)$ in $X \subseteq \mathbb{A}^m$ is defined by $I(X) \cup \{\, \phi_1(x) - y_1, \ldots, \phi_n(x) - y_n \,\}$.

Theorem (Generic smoothness)

Let $\phi\colon X \to Y$ be a dominant morphism of irreducible varieties. There exists a dense open $U \subseteq Y$ such that for all $y \in U$ the scheme-theoretic fiber $f^{-1}(y) \subseteq X$ is a smooth subvariety, in particular defined in a reduced way.

If $\dim X = \dim Y$, then $\phi^{-1}(y)$ is a set of the same number of points for all $y \in U$.

Exercample

- Use this to argue that the degree of a variety is well-defined.
- Show that if $f \in \mathbb{C}[x,t]$ is a polynomial with $f(x,t)$ having a multiple root in x for almost all t, then f is *not* irreducible.

How to be radical iv: Unmixedness theorem

- The notion of a *Cohen–Macaulay ideal/ring* is a bit technical, so we will black-box it

Theorem (Hochster–Eagon)

If $f_1, \ldots, f_m \in R$ *are such that* $\emptyset \neq \mathbb{V}(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$ *has dimension* $\leq n - m$*, then the ideal* $I = \langle f_1, \ldots, f_m \rangle$ *is Cohen–Macaulay.*

Far more generally, if $M \in R^{e \times f}$ *is a matrix such that* $X = \{\, x \in \mathbb{A}^n \mid \operatorname{rank} M(x) \leq r \,\}$ *has dimension* $\leq n - (e-r)(f-r)$*, then* $I = \langle (r+1)\text{-minors of } M \rangle$ *is Cohen–Macaulay.*

Theorem (Unmixedness theorem)

If I *is a Cohen–Macaulay ideal such that* $X = \mathbb{V}(I)$ *is an irreducible variety and for some* $p \in X$*,* I *defines* X *in a reduced way at* p*, then* I *defines* X *in a reduced way everywhere.*

Exercample

The ideal of 2-minors of $\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$ defines an irreducible variety in a reduced way.

How to be radical v: From incidence to minors

- Let $M(a) \in R^{k\times m}$ be a matrix polynomial
- Consider the projection $p\colon \mathcal{K} \to D_{m-1}(M)$ from the *Kernel incidence* to the *degeneracy locus*

$$\mathcal{K} := \{\, (a,[v]) \in \mathbb{A}^n \times \mathbb{P}^{m-1} \mid M(a)v = 0 \,\} \longrightarrow D_{m-1}(M) = \{\, a \mid \operatorname{rank} M(a) < m \,\}$$

- $\mathcal{K}$ is defined by k equations $\mathcal{F}$ in $\mathbb{A}^n \times \mathbb{P}^{m-1}$ (i-th row of $M(a)\cdot v = 0$)

Lemma

Assume the following:

- *For all $a \in \mathbb{A}^n$ we have $\operatorname{rank} M(a) \in \{m-1, m\}$*
- *$\mathcal{F}$ defines $\mathcal{K}$ in a reduced way*

Then the ideal of minors of M define $D_{m-1}(M)$ in a reduced way.

Extended example: The variety of low-rank Hankel matrices

- Let $y = (y_0, \ldots, y_d) \in \mathbb{C}[y_0, \ldots, y_d]$ be a sequence. The r*-th Hankel matrix* of y is

$$H_r(y) = \begin{bmatrix} y_0 & y_1 & \cdots & y_r \\ y_1 & y_2 & \cdots & y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{d-r} & y_{d-r+1} & \cdots & y_d \end{bmatrix} \in S^{(d-r+1)\times(r+1)}$$

- The variety of rank r Hankel matrices is

$$\mathcal{X}_{d,r} = \left\{ y \in \mathbb{A}^{d+1} \;\middle|\; \operatorname{rank} H_r(y) \le r \right\}$$

- $y \in \mathcal{X}_{d,r}$ correspond to sequences satisfying a linear recurrence relation of order r

Exercample

$\mathcal{X}_{d,r}$ is an irreducible variety of dimension $2r$. Its defining ideal is generated by the $(r+1)$-minors of $H_r(y)$. We have $\mathcal{X}_{d,r}^{\mathrm{sing}} = \mathcal{X}_{d,r-1}$.

Next time: Thom–Porteous

- Let $M \in S^{f \times e}$ be a matrix such that each column consists of homogeneous polynomials of degree $d_1, \ldots, d_e$
- Consider the rational function

$$\Psi(T) = \frac{1}{(1 - d_1 T) \cdots (1 - d_e T)}$$

- Let $\{\Psi\}^k$ be the k-th coefficient in the series expansion $\Psi = \sum_{k \geq 0} \{\Psi\}^k T^k$

Theorem (Giambelli–Thom–Porteous, corank 1)

If $X = D_{e-1}(M)$ is a variety of dimension $n - (e - f + 1) \geq 0$, then $\deg X = \{\Psi\}^{f-e+1}$.

Exercample

- How does this theorem extend Bézout's theorem?
- What is the degree of $\mathbb{P}(\mathcal{X}_{d,r}) \subseteq \mathbb{P}^d$?