The Foundations: Logic and Proofs

Chapter 1, Part II: Predicate Logic 술어논리

With Question/Answer Animations

Section Summary[1]

Predicates 술어

Variables 변수

Quantifiers 한정기호

- Universal Quantifier
- Existential Quantifier

Negating Quantifiers 한정기호의 부정

- De Morgan’s Laws for Quantifiers

Translating English to Logic

Introducing Predicate Logic 술어 논리

Predicate logic uses the following new features:

- Variables 변수: *x*, *y*, *z*
- Predicates 술어: *P*(*x*), *M*(*x*)
- Quantifiers 한정기호 (뒤에서 자세히 설명):

Propositional functions 명제 함수 are a generalization of propositions 명제의 일반화.

- 명제함수는 변수와 술어를 가짐, e.g., *P*(*x*)
- 변수는 정의역의 원소로 바꿀 수 있다.

Propositional Functions 명제함수

Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later). 명제함수는 명제변수에 정의역의 하나의 값을 넣으면 명제가 된다 (진리값을 가짐).

The statement *P(x)* is said to be the value of the propositional function *P* at *x*.

문장 P(x) 는 명제함수 P 의 x 에서의 값

For example, let *P(x)* denote "*x* > 0" and the domain 정의역 be the integers 정수. Then:

P(-3) is false.

P(0) is false.

P(3) is true.

Often the domain is denoted by *U*. 종종 정의역은 *U 로 표기.*

So in this example *U* is the integers.

Examples of Propositional Functions

Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

R(2,-1,5)

Solution: F

R(3,4,7)

Solution: T

R(x, 3, z)

Solution: Not a Proposition

Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

Q(2,-1,3)

Solution: T

Q(3,4,7)

Solution: F

Q(x, 3, z)

Solution: Not a Proposition

Compound Expressions 복합수식

Connectives from propositional logic carry over to predicate logic.

If *P(x)* denotes "$x > 0$," find these truth values:

$P(3) \vee P(-1)$ **Solution**: T

$P(3) \wedge P(-1)$ **Solution**: F

$P(3) \rightarrow P(-1)$ **Solution**: F

$P(3) \rightarrow \neg P(-1)$ **Solution**: T

Expressions with variables are not propositions and therefore do not have truth values. For example,

$P(3) \wedge P(y)$

$P(x) \rightarrow P(y)$

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers 한정기호

Charles Peirce (1839-1914)

We need *quantifiers* to express the meaning of English words including *all* and *some* : 영어의 ‘모두’, ‘어떤’ 을 표현하기 위해 필요

- “All men are Mortal.” 모든 사람은 죽는다
- “Some cats do not have fur.” 어떤 고양이는 털이 없다

The two most important quantifiers are:

- *Universal Quantifier 전칭 한정기호,* **“**For all,” symbol: ∀
- *Existential Quantifier 존재 한정기호*, “There exists,” symbol: ∃

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

$\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

$\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

$\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.

2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.

3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Existential Quantifier 존재 한정기호

$\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Properties of Quantifiers

$\exists x\, P(x)$ and $\forall x\, P(x)$ 의 진리값은 명제함수$P(x)$ 와 정의역 U에 의해 결정됨

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers 한정기호의 우선순위

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators. 한정기호는 다른 논리 연산자들보다 우선순위가 높다.

For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$

$\forall x\, (P(x) \vee Q(x))$ means something different.

Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating from English to Logic[1]

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java." 이 수업의 모든 학생은 Java 수업을 들었다.

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(*x*) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Translating from English to Logic 2

Example 2: Translate the following sentence into predicate logic: “Some student in this class has taken a course in Java.” 이 수업을 듣는 어떤 학생은 Java 수업을 들었다.

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class 이고 J(*x*) denoting “x has taken a course in Java” , translate as

$$\exists x \; J(x)$$

Solution 2: But if *U* is all people 이고 S(x) denoting “x is a student in this class”, then translate as

$\exists x \; (S(x) \wedge J(x))$

$\exists x \; (S(x) \rightarrow J(x))$ is not correct. What does it mean?

한정기호를 논리합과 논리곱으로 표현해 보면

If the domain is finite 유한하면, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

If *U* consists of the integers 1,2, and 3:

$$\forall x P\left(x\right) \equiv P\left(1\right) \wedge P\left(2\right) \wedge P\left(3\right)$$

$$\exists x P\left(x\right) \equiv P\left(1\right) \wedge P\left(2\right) \wedge P\left(3\right)$$

Even if the domains are infinite 무한하면, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions[1]

한정기호있는 수식의 부정

Consider $\forall x\, J(x)$

"Every student in your class has taken a course in Java."

Here $J(x)$ is "x has taken a course in Java" and

the domain is students in your class.

Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg \forall x\, J(x)$ and $\exists x \neg J(x)$ are equivalent

Negating Quantified Expressions 2

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg\exists xP(x)$	$\forall x\neg P(x)$	For every *x*, *P*(*x*) is false.	There is *x* for which *P*(*x*) is true.
$\neg\forall xP(x)$	$\exists x\neg P(x)$	There is an *x* for which *P* (*x*) is false.	*P*(*x*) is true for every *x*.

The reasoning in the table shows that:

$$\neg\forall xP(x) \equiv \exists x\neg P(x)$$

$$\neg\exists xP(x) \equiv \forall x\neg P(x)$$

These are important. You will use these.

Nested Quantifiers

Section 1.4

Section Summary 2

Nested Quantifiers

Order of Quantifiers

Translating from Nested Quantifiers into English

Translating Mathematical Statements into Statements involving Nested Quantifiers.

Translated English Sentences into Logical Expressions.

Negating Nested Quantifiers.

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "Every real number has an inverse" is

$$\forall x\, \exists y(x + y = 0)$$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x\, \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

Nested Loops

- To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
- To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
  - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\, \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers 1

Example 1: Let *U* be the real numbers,

Define *P(x,y) : x* · *y* = 0

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$

   Answer:False

2. $\forall x \exists y P(x, y)$

   Answer:True

3. $\exists x \forall y P(x, y)$

   Answer:True

4. $\exists x \exists y P(x, y)$

   Answer:True

Questions on Order of Quantifiers 2

Example 2: Let *U* be the real numbers,

Define *P(x,y)* : *x* / *y* = 1

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$

   Answer:False

2. $\forall x \exists y P(x, y)$

   Answer:False

3. $\exists x \forall y P(x, y)$

   Answer:False

4. $\exists x \exists y P(x, y)$

   Answer:True

Quantifications of Two Variables

Statement	**When True?**	**When False**
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \forall y P(x, y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x\left(C\left(X\right)\vee \exists y\left(C\left(y\right)\wedge F\left(X,y\right)\right)\right)$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x\forall y\forall z\left(\left(F\left(x,y\right)\wedge F\left(x,z\right)\wedge\left(y\neq z\right)\right)\rightarrow\neg F\left(y,z\right)\right)$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. Introduce the variables *x* and *y*, and specify the domain, to obtain:
   "For all positive integers *x* and *y*, *x* + *y* is positive."
3. The result is:

$$\forall x \forall y \left(\left(x > 0 \right) \wedge \left(y > 0 \right) \right) \to \left(x + y > 0 \right)$$

where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."

2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.

3. Then the statement can be expressed as:

$$\exists w \forall a \exists f \left(P\left(w, f\right) \wedge Q\left(f, a\right)\right)$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: “Brothers are siblings.”

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: “Siblinghood is symmetric.”

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: “Everybody loves somebody.”

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: “There is someone who is loved by everyone.”

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: “There is someone who loves someone.”

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: “Everyone loves himself”

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:a

1. $\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$
2. $\forall w \neg \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f \left(\neg P(w, f) \vee \neg Q(f, a) \right)$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"