Nested Quantifiers
중첩된 한정기호

Section 1.4

Nested Quantifiers 중첩된 한정기호

Example: “Every real number 실수 has an inverse 역” is

$\forall x\, \exists y(x + y = 0)$

where the domains 정의역 of x and y are the real numbers 실수.

We can also think of nested propositional functions:

$\forall x\, \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Order of Quantifiers 한정기호의 순서

(순서가 중요!!!)

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers 실수. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\ \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers[1]

Example 1: Let *U* be the real numbers 정의역이 실수,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$

   Answer:False

2. $\forall x \exists y P(x, y)$

   Answer:True

3. $\exists x \forall y P(x, y)$

   Answer:True

4. $\exists x \exists y P(x, y)$

   Answer:True

Questions on Order of Quantifiers 2

Example 2: Let *U* be the real numbers,

Define *P(x,y)* : *x / y* = 1

What is the truth value of the following:

1. $\forall x \forall y P(x, y)$

   Answer: False

2. $\forall x \exists y P(x, y)$

   Answer: False

3. $\exists x \forall y P(x, y)$

   Answer: False

4. $\exists x \exists y P(x, y)$

   Answer: True

Quantifications of Two Variables

Statement	**When True?**	**When False**
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x, y* for which *P*(*x,y*) is false.
$\forall x \exists y P(x, y)$	For every *x* there is a *y* for which *P*(*x,y*) is true.	There is an x such that *P*(*x,y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x,y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x, y* for which *P*(*x,y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. 한정기호와 정의역을 구체적으로 적어보면:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. Introduce the variables *x* and *y*, and specify the domain, to obtain:
   "For all positive integers *x* and *y*, *x* + *y* is positive."
3. The result is:

$$\forall x \forall y \big((x > 0) \land (y > 0) \big) \rightarrow (x + y > 0)$$

여기서 x, y 두 변수의 정의역은 모든 정수

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 3: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 4: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

Example 5: "Everyone loves himself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: 아래 논리 수식을 부정하시오.

$$\exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$$

Solution:

$$\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:a

1. $\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$
2. $\forall w \neg \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f \left(\neg P(w, f) \vee \neg Q(f, a) \right)$ by De Morgan's for $\wedge$.