1. Beta - Binomial conjugation:

proof: For convenience, we take coin tossing as an example.

Assume that we toss a coin N times, with m times resulting in heads.

Previously, there have been a heads a times and tails b times.

Denote the probability of tossing heads by θ.

Then $$p(\theta \mid event) = \frac{p(event \mid \theta) \cdot p(\theta)}{p(event)}$$

$$(*) \quad = \frac{\cancel{\binom{N}{m}} \cdot \theta^{m} \cdot (1-\theta)^{N-m} \cdot \theta^{a-1} \cdot (1-\theta)^{b-1} \cdot \cancel{\frac{\Gamma(a+b)}{\Gamma(a) \cdot \Gamma(b)}}}{\int_0^1 \cancel{\binom{N}{m}} \cdot \theta^{m} \cdot (1-\theta)^{N-m} \cdot \theta^{a-1} \cdot (1-\theta)^{b-1} \cdot \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}} \, d\theta}$$

$\because \int_0^1 B(\theta \mid m+a, N-m+b)\, d\theta$

$$= \int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} \cdot \frac{\Gamma(N+a+b)}{\Gamma(m+a)\Gamma(N-m+b)} = 1$$

$$\therefore \int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} = \left(\frac{\Gamma(N+a+b)}{\Gamma(m+a)\Gamma(N-m+b)}\right)^{-1}$$

Thus (*) becomes $\theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} \cdot \frac{\Gamma(N+a+b)}{\Gamma(m+a)\Gamma(N-m+b)}$

$$= B(\theta \mid m+a, N-m+b) \quad \#$$

2. Gamma-Poisson conjugation:

Gamma distribution: $f(x \mid a, \beta) = \frac{\beta^a}{\Gamma(a)} \cdot x^{a-1} \cdot e^{-\beta x}$

proof: By Bayes Rule, we have $P(\lambda|x) = \dfrac{P(x|\lambda)\cdot P(\lambda)}{P(x)}$

$$\propto P(x|\lambda)\cdot P(\lambda)$$

$\hookrightarrow$ x : event

Assume there are n data points,

$$P(x|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{(x_i)!} = \frac{\lambda^{\Sigma x_i}\cdot e^{-n\lambda}}{\Pi(x_i)!} \propto \lambda^{\Sigma x_i}\cdot e^{-n\lambda}$$

$P(\lambda) \propto \lambda^{\alpha-1} e^{-\beta x}$ (prior)

Thus, $P(\lambda|x) \propto \lambda^{\Sigma x_i}\cdot e^{-n\lambda}\cdot \lambda^{\alpha-1}\cdot e^{-\beta x}$

$$= \lambda^{n\bar{x}}\cdot \lambda^{\alpha-1}\cdot e^{-(\beta+n)\lambda}$$

$$= \lambda^{(n\bar{x}+\alpha-1)}\cdot e^{-(\beta+n)\lambda} = \text{gamma}(n\bar{x}+\alpha, \beta+n)$$ #