Open Quantum System

Lectures:

Last time:

- Daniel Manzano, A short introduction to the Lindblad master equation (*all*)
- Breuer and Petruccione, The Theory of Open Quantum Systems (*ch. 3 - 4.3*)
- Daniel A. Lidar, Notes on the Theory of Open Quantum Systems (*up to ch. 12*)

Today:

- Buča, B., Tindall, J. & Jaksch, D. Non-stationary coherent quantum many-body dynamics through dissipation
- Victor V. Albert & Liang Jiang, Symmetries and conserved quantities in Lindblad master equations
- Cameron Booker, Berislav Buča, Dieter Jaksch, Non-stationarity and Dissipative Time Crystals: Spectral Properties and Finite-Size Effects
- Victor V. Albert. Lindbladians with multiple steady states: theory and applications. A Dissertation Presented to the Faculty of the Graduate School of Yale University

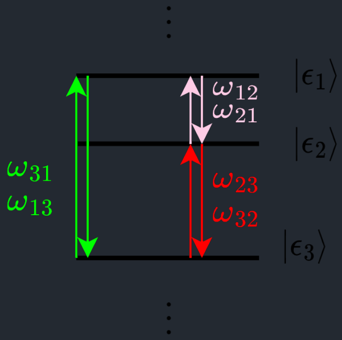
November 27, 2023

Recalling notations

Assumption $\mathfrak{a}$:

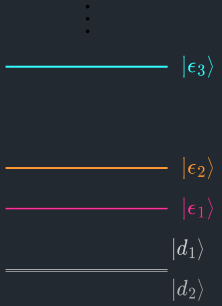
$$\forall \omega \ \exists! \{m, n\} : e_m - e_n = \omega$$

This implied that $\mathscr{H}_S = \mathrm{span}(|m\rangle_\omega \langle n|)$



Assumption $\mathfrak{b}$:

$\mathfrak{a}$ + 1 degeneracy of order 1



Conditions on A - assumption $\mathfrak{a}$

- Last time derivation -

$$\left[|m\rangle_\omega \langle m| \, L_\nu^{(\dagger)} \, |n\rangle_\omega \langle n| \, , A \right] = 0 \quad \text{with } A = \sum_{kl} a_{kl} \, |k\rangle\langle l|$$

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- Relations from last time:

$$\left.\begin{aligned} & a_{nm}\langle m|L_\nu|n\rangle = 0\,, \quad a_{mn}\langle n|L_\nu^\dagger|m\rangle = 0 \\ & \begin{cases} a_{nn}\langle m|L_\nu|n\rangle = a_{mm}\langle m|L_\nu|n\rangle \\ a_{nn}\langle n|L_\nu^\dagger|m\rangle = a_{mm}\langle n|L_\nu^\dagger|m\rangle \end{cases} \end{aligned}\right\} \quad \forall\{m, n, \nu\} \tag{1}$$

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Question - does this imply that $a_{mn} = 0$?

Since from def. of L_ν - $H_{\text{int}} = \sum_\nu S_\nu \otimes B_\nu$, with $L_\nu \overset{\text{u}}{\sim} S_\nu$?

Conditions on A

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- Thus, condition on a_{kl} depends on whether L_ν couples $|m\rangle$, $|n\rangle$ $\quad \forall\nu$?

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Conclusion 1

$$\text{If } \forall\, |m\rangle \neq |n\rangle, \;\; \exists\nu \;\text{ such that } \langle m| \, L_\nu \, |n\rangle \neq 0 \quad \Longrightarrow \quad A = \lambda\mathbb{I}$$

Intuitively, $A = \lambda\mathbb{I}$ if H_{int} does not couple all $|k\rangle\langle l|$. I.e. if the H_{int} is not in the form $H_{\text{int}} = \sum_{m<n\in\sigma(H_S)} |m\rangle\langle n| \otimes B_{m,n}$.

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Conclusion 2

$$\text{If } \exists m \neq n \ \text{ such that } \forall\nu,\ \langle m| L_\nu |n\rangle = 0 \quad \implies \quad A \neq \lambda\mathbb{I}$$

Conditions on A - assumption $\mathfrak{b}$

Obtained relation from last time for assumption $\mathfrak{b}$:

$$a_{aa} \langle b| L_\nu |a\rangle = a_{bb} \langle b| L_\nu |a\rangle$$
$$a_{aa} \langle a| L_\nu |b\rangle = a_{bb} \langle a| L_\nu |b\rangle$$
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Then, for some non-degenerate $|g\rangle$

$$a_{ag} \langle g| L_\nu |a\rangle = a_{ag} \langle g| L_\nu |b\rangle = 0$$
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And additional relations

$$a_{aa} \langle g| L_\nu |a\rangle + a_{ba} \langle g| L_\nu |b\rangle = a_{gg} \langle g| L_\nu |a\rangle$$
$$\ldots$$

Conclusions on A under $\mathfrak{b}$

One can write the form of A

$$
\left(\begin{array}{cc|ccc}
\multicolumn{2}{c|}{\mathbf{A}_{\mathbf{0}}^{*}} & \ddots & * & \ddots \\
 & & * & \ddots & * \\
\hline
\ddots & * & & & \\
* & \ddots & & \lambda\mathbf{I} & \\
\ddots & * & & &
\end{array}\right)
$$

If same conditions on L_ν ($L_\nu \equiv \mathrm{span}(|g\rangle\langle a,b|)\ \forall\,|g\rangle$), then it is block diagonal with $\mathbf{A}_{\mathbf{0}}^{*}$ and $\lambda\mathbf{I}$.

Examples

- Spontaneous emission

$$H_{SB} = g \sum_{\nu} L_{\nu} \otimes B_{\nu} = g\big(|e\rangle\langle g| \otimes a + |g\rangle\langle e| \otimes a^{\dagger}\big)$$

$$L_1 = |e\rangle\langle g| \quad \text{and} \quad L_2 = |g\rangle\langle e|$$

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- Spin Chain Coupled to Magnetic Fluctuations -
  - Heizenberg XY model $H_S = J \sum_i (s_i^x s_{i+1}^x + s_i^y s_{i+1}^y + \Delta s_i^z s_{i+1}^z)$
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  - Collection of bosonic modes $H_B = \sum_i \omega_i b_i^{\dagger} b_i$
  - Interaction $H_{SB} \equiv \sum_{ik} g_{ik} S_i^{+} \otimes b_k + g_{ik}^{*} S_i^{-} \otimes b_k^{\dagger}$
  - Jump operators $S^{+} \equiv |0\rangle\langle 1|$ and $S^{-} \equiv |1\rangle\langle 0|$ for every i.

Symmetries & Conserved quantities

Steady-state:

- Conserved quantities J generate symmetries U
- If $\mathscr{L}$ has no purely imaginary eigenvalues, let $\{M_\nu\}$ of $\mathrm{L}_{ss} \subseteq \mathrm{L}$ such that

$$\rho_{ss} = \lim_{t\to\infty} e^{\mathscr{L}t} = \sum_\nu \rho_\nu M_\nu \quad \text{with} \qquad \rho_\nu = \mathrm{Tr}\left[J_\nu^\dagger M_\nu\right]$$

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Oscillating-coherence

- Oscillating coherence state ρ_∞, such that $\mathscr{L}\rho_\infty = i\lambda\rho_\infty$ or $\lim_{t\to\infty} e^{\mathscr{L}t}\rho_\infty = e^{i\lambda t}\rho_\infty$
- In that case, the conserved quantities $\widehat{J}$ also oscillate:

$$e^{\mathscr{L}^\dagger t}\widehat{J_{\mu,\nu}} = e^{-i\lambda_{\mu,\nu}t}\widehat{J_{\mu,\nu}}$$

Symmetries & Conserved quantities

Theorem

The steady-states, steady-state coherences and oscillating coherences form a basis of the $\mathbb{L}$ *space.*

Symmetries & Conserved quantities

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The steady-states, steady-state coherences and oscillating coherences form a basis of the L *space.*

- For an oscillating coherence, one can write using a basis $\{M_\nu\}$ of the L_{ss}, of dimension D and the basis $\{O_{\mu,\nu}\}$ for the oscillating coherences,

$$\rho_\infty = \underbrace{\sum_{\mu=1}^{D} \rho_\mu M_\mu}_{\substack{\text{steady-state } \& \\ \text{steady-state coherence}}} + \underbrace{\sum_{\substack{\mu,\nu \\ \mu\neq\nu}} \rho_{\mu,\nu}(t)\, O_{\mu,\nu}}_{\text{oscillating coherence}} \quad (2)$$

$$\equiv |\rho_\infty\rangle\rangle = \sum_{\mu=1}^{D} \rho_\mu |M_\mu\rangle\rangle + \sum_{\mu\neq\nu} \rho_{\mu,\nu}(t)\, |O_{\mu,\nu}\rangle\rangle \quad (3)$$

Symmetries & Conserved quantities

$$\rho_\infty = \sum_{\mu=1}^{D} \rho_\mu M_\mu + \sum_{\mu\neq\nu} \rho_{\mu,\nu}(t)\ O_{\mu,\nu}$$

- With $\rho_{\mu,\nu}(t)$ defined via

$$\rho_{\mu,\nu}(t) = \langle\langle\, \widehat{J}_{\mu,\nu} | \rho_\infty \rangle\rangle = \mathrm{Tr}\left[\widehat{J}^\dagger_{\mu,\nu}\ \rho_\infty\right] \tag{4}$$

$$\rho_{\mu,\nu}(t) = e^{i\lambda_{\mu,\nu} t} \langle\langle\, \widehat{J}_{\mu,\nu} | \rho_{in} \rangle\rangle = e^{i\lambda_{\mu,\nu} t}\mathrm{Tr}\left[\widehat{J}^\dagger_{\mu,\nu}\ \rho_{in}\right] \tag{5}$$

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- The oscillating coherence via operator A:

$$\mathscr{L}(\rho_{ss}) \quad \xrightarrow[\exists A]{[H,A]\rho_{ss}=\lambda A\rho_{ss}\ \&\ \cdots} \quad \mathscr{L}(A\rho_{ss}) = i\lambda A\rho_{ss} \tag{6}$$

$$\implies \rho_\infty = A\rho_{ss} \tag{7}$$

Symmetries & Conserved quantities

- Applying $\mathscr{L}$ to $\rho_\infty = \sum_{\mu=1}^{D} \rho_\mu M_\mu + \sum_{\mu\neq\nu} \rho_{\mu,\nu}(t)\; O_{\mu,\nu}$

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$$\mathscr{L}(\rho_\infty) = \sum_{\mu=1}^{D} \rho_\mu \mathscr{L}(M_\mu) + \sum_{\mu\neq\nu} \rho_{\mu,\nu}(t)\; \mathscr{L}(O_{\mu,\nu})$$

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$$i\lambda\rho_\infty = \sum_{\mu\neq\nu} \rho_{\mu,\nu}(t) i\lambda_{\mu,\nu} O_{\mu,\nu}$$

$$A\rho_{ss} = \sum_{\mu\neq\nu} \frac{\lambda_{\mu,\nu}}{\lambda} \rho_{\mu,\nu}(t) O_{\mu,\nu}$$

Symmetries & Conserved quantities

- It is possible to write ρ_∞ and ρ_{ss} in a more formal form (B. Baumgartner), (V.V. Albert, L. Jiang)
- For example, a block diagonal form

$$\rho_{ss} = \bigoplus_{\iota} \Big[\sum_{\mu,\nu}^{n_\iota} \rho^{(\iota)}_{\mu,\nu} \, |\mu\rangle_\iota \langle \nu| \otimes T^{(\iota)} \Big]$$

$$\rho_\infty(t) = \bigoplus_{\iota} \Big[\sum_{\mu,\nu}^{n_\iota} \rho^{(\iota)}_{\mu,\nu} e^{i\lambda^{(\iota)}_{\mu,\nu} t} \, |\mu\rangle_\iota \langle \nu| \otimes T^{(\iota)} \Big]$$