Open Quantum System

Lectures:

Last time:

- Daniel Manzano, A short introduction to the Lindblad master equation (*all*)
- Breuer and Petruccione, The Theory of Open Quantum Systems (*ch. 3 - 4.3*)
- Daniel A. Lidar, Notes on the Theory of Open Quantum Systems (*up to ch. 12*)

Today:

- Buča, B., Tindall, J. & Jaksch, D. Non-stationary coherent quantum many-body dynamics through dissipation
- Victor V. Albert & Liang Jiang, Symmetries and conserved quantities in Lindblad master equations
- Cameron Booker, Berislav Buča, Dieter Jaksch, Non-stationarity and Dissipative Time Crystals: Spectral Properties and Finite-Size Effects
- Victor V. Albert. Lindbladians with multiple steady states: theory and applications. A Dissertation Presented to the Faculty of the Graduate School of Yale University

November 20, 2023

Dynamical symmetries

Diagonal Lindblad

$$\mathscr{L}\rho = -i[H_S + H_{LS}, \rho] + \sum_\mu \left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\} \right) \quad \text{with } L_\mu \equiv L_\nu(\omega)$$

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$$\forall\ \mu \equiv (\nu, \omega)\,, \quad \Big[\underbrace{L_\mu}_{L_\nu(\omega)}, A\Big] = 0 \qquad \text{with } \ L_\nu(\omega) := \sum_{\omega=\epsilon'-\epsilon} |\epsilon\rangle\langle\epsilon|\, L_\nu\, |\epsilon'\rangle\langle\epsilon'|$$

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Remembering

$$H_{\text{int}} \equiv \sum_\alpha S_\alpha \otimes B_\alpha \quad \text{and} \quad L_\nu = \sum_i u_{i,\nu} S_i$$

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Question: what can one say on A and/or L_ν?

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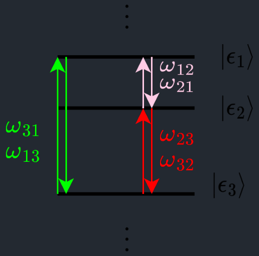
Assumption $\textcircled{\mathfrak{a}}$ - for every ω, a unique pair (ϵ, ϵ') exists.
Note - assuption **not** the same as non-degeneracy



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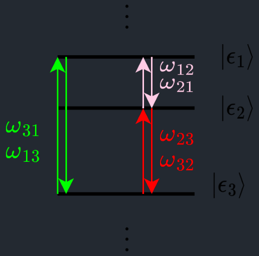
$$\left[|\epsilon\rangle\langle\epsilon| L_\nu |\epsilon'\rangle\langle\epsilon'|, A\right] = 0 \qquad \forall\,(\nu, (\epsilon', \epsilon)) \tag{3}$$

$$\left[|\epsilon'\rangle\langle\epsilon'| L_\nu^\dagger |\epsilon\rangle\langle\epsilon|, A\right] = 0 \qquad \forall\,(\nu, (\epsilon', \epsilon)) \tag{4}$$



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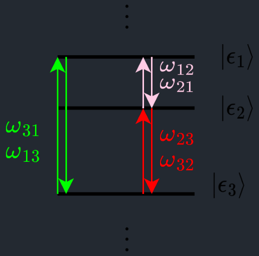

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The states $\{|\epsilon\rangle\}$ - the chosen orthonormal eigenkets of the Hamiltonian H_S.
Clearly, the ket-bra's span the Liouville space -

$$\mathfrak{D}(\mathscr{H}) = \mathrm{span}(\{|\epsilon\rangle\langle\epsilon'|\}) \quad \forall\,(\nu, (\epsilon', \epsilon))$$

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Using the completeness relation,

$$L_\nu = \sum_{\varepsilon,\varepsilon'} |\varepsilon\rangle\langle\varepsilon| \, L_\nu \, |\varepsilon'\rangle\langle\varepsilon'|$$

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$$\begin{aligned} L_\nu A - A L_\nu &= \sum_{\varepsilon,\varepsilon'} |\varepsilon\rangle\langle\varepsilon| L_\nu |\varepsilon'\rangle\langle\varepsilon'| A - \sum_{\varepsilon,\varepsilon'} A |\varepsilon\rangle\langle\varepsilon| L_\nu |\varepsilon'\rangle\langle\varepsilon'| \\ &= \sum_{\varepsilon} \sum_{\varepsilon'} \underbrace{\Big(|\varepsilon\rangle\langle\varepsilon| L_\nu |\varepsilon'\rangle\langle\varepsilon'| A - A |\varepsilon\rangle\langle\varepsilon| L_\nu |\varepsilon'\rangle\langle\varepsilon'| \Big)}_{0 \text{ for all } (\nu,(\varepsilon,\varepsilon'))} = 0 \end{aligned} \tag{5}$$

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The same process can be repeated for $L_\nu^\dagger$.
Meaning that if such dynamical symmetry exists,

$$[L_\nu, A] = [L_\nu^\dagger, A] = 0$$

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Then, the commutator relation will be given by

$$\left[|m\rangle\langle m| L_\nu |n\rangle\langle n| , \sum_{kl} a_{kl} |k\rangle\langle l| \right] = 0 \tag{6}$$

$$\sum_{kl} a_{kl} |m\rangle\langle m| L_\nu |n\rangle\langle n| |k\rangle\langle l| - \sum_{kl} a_{kl} |k\rangle\langle l| |m\rangle\langle m| L_\nu |n\rangle\langle n| \tag{7}$$

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$$\sum_{kl} a_{kl} \left|m\right\rangle\!\left\langle m\right| L_\nu \left|n\right\rangle\!\left\langle n\right| \left|k\right\rangle\!\left\langle l\right| - \sum_{kl} a_{kl} \left|k\right\rangle\!\left\langle l\right| \left|m\right\rangle\!\left\langle m\right| L_\nu \left|n\right\rangle\!\left\langle n\right| =$$

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- Meaning that $a_{nm} = 0 \implies$ contradiction

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Using the same expansion, and commutator relations -

$$d_\epsilon \left|\epsilon\middle\rangle\middle\langle\epsilon\right| L_\nu \left|\epsilon'\middle\rangle\middle\langle\epsilon'\right| - d_{\epsilon'} \left|\epsilon\middle\rangle\middle\langle\epsilon\right| L_\nu \left|\epsilon'\middle\rangle\middle\langle\epsilon'\right| = 0$$

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The newly introduced $\rho_{mn} := A^n \rho_\infty (A^\dagger)^m$ thus do not give any additional imformation. The condition $\mathscr{L}\rho_{mn} = -i(m-n)\lambda\rho_{mn}$ therefore is redundant.

Assumption: continuity

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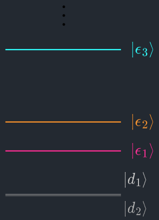
- Continious spectrum:

$$S_\alpha(\omega) = \iint_{\epsilon,\epsilon'\in\sigma(H_S)} |\epsilon\rangle\langle\epsilon| \, S_\alpha \, |\epsilon'\rangle\langle\epsilon'| \, \delta\big(\omega - (\epsilon - \epsilon')\big) d\epsilon \, d\epsilon' \tag{10}$$

Degeneracies?

- In addition to the assumption $\mathfrak{a}$, one can add degeneracies to the ground level $|d_1\rangle$, $|d_2\rangle$.



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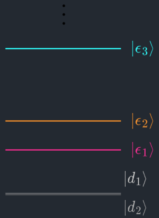

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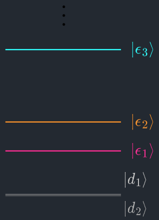

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$$\left[|d_1\rangle\langle d_1| \, L_\nu \, |d_2\rangle\langle d_2| + |d_2\rangle\langle d_2| \, L_\nu \, |d_1\rangle\langle d_1| \, , \underbrace{A}_{\equiv a_{ij}} \right] = 0$$

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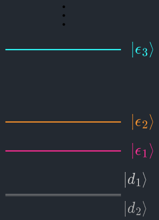

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Using same procedures as before, one can obtain some relations as for example

$$a_{d_1,d_2} \langle d_2| L_\nu |d_1\rangle - a_{d_2,d_1} \langle d_1| L_\nu |d_2\rangle = 0$$
$$a_{d_1,d_1} \langle d_2| L_\nu |d_1\rangle - a_{d_2,d_2} \langle d_2| L_\nu |d_1\rangle = 0$$

Degeneracies

This suggest the form of A

$$\left(\begin{array}{c|ccc} \mathbf{A}_0^* & \ddots & * & \ddots \\ & * & \ddots & * \\ \hline \ddots \quad * & & & \\ * \quad \ddots & & \lambda\mathbf{I} & \\ \ddots \quad * & & & \end{array}\right)$$

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In fact, if $\mathscr{L}$ does **not** have purely imaginary eigenvalues (no oscillating coherences), then for $\rho_{ss} \in \mathrm{L}_{ss}$ of dim $= D$

$$\rho_{ss} = \lim_{t\to\infty} e^{\mathscr{L}t} \rho_{in} = \sum_{\mu}^{D} \langle J_\mu | \rho_{in} \rangle M_\mu \tag{11}$$

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Q

How are those oscillating coherence quantities related to each other and to the steady-state quantities?