Open Quantum System

Lectures:

Last time:

- Daniel Manzano, A short introduction to the Lindblad master equation (*all*)
- Breuer and Petruccione, The Theory of Open Quantum Systems (*ch. 3 - 4.3*)
- Daniel A. Lidar, Notes on the Theory of Open Quantum Systems (*up to ch. 12*)

Today:

- Breuer and Petruccione, The Theory of Open Quantum Systems (*ch. 3 - 4.3*)
- Daniel A. Lidar, Notes on the Theory of Open Quantum Systems (*up to ch. 12*)
- B.Kraus, H.P.Buchler, S. Diehl, A. Kantian, A. Micheli, & P. Zoller, Preparation of Entangled States by Quantum Markov Processes
- Buča, B., Tindall, J. & Jaksch, D. Non-stationary coherent quantum many-body dynamics through dissipation
- Victor V. Albert & Liang Jiang, Symmetries and conserved quantities in Lindblad master equations
- Cameron Booker, Berislav Buča, Dieter Jaksch, Non-stationarity and Dissipative Time Crystals: Spectral Properties and Finite-Size Effects

November 12, 2023

Recalling

- Derivation of the Lindblad equation via different approximations:
  - Von Neumann evolution equation, $H_{\text{int}} = \sum_k S_k \otimes B_k \rightarrow$ Weak coupling, Born, Markov & Rotating wave $\rightarrow$ Redfield equation
    Note: no stationarity of the system was assumed!
  - Yielding (Schrodinger pic.):

$$\begin{aligned} \frac{d}{dt}\rho(t) = [H + H_{\text{LS}}, \rho(t)] + \\ + \sum_{k,l,\omega} \gamma_{k,l}(\omega) \left[S_l(\omega)\rho(t)S_k^\dagger(\omega) - \frac{1}{2}\left\{ S_k^\dagger(\omega)S_l(\omega), \rho(t) \right\} \right] \end{aligned} \tag{1}$$

  - With $\gamma_{k,l}$, $\pi_{k,l}$ - defined via $\Gamma_{k,l}(\omega)$'s real and imaginary parts.

$$\begin{aligned} H_{\text{LS}} = \sum_{\omega,k,l} \pi_{k,l}(\omega) S_k^\dagger(\omega) S_l(\omega) \\ \Gamma_{k,l}(\omega) = \int_0^\infty e^{i\omega s} \text{Tr}_B \left[B_k^\dagger(t) B_l(t-s) \rho_B(0) \right] \end{aligned} \tag{2}$$

- With the operators $S_k(\omega)$ defined via

$$S_k(\omega) = \sum_{\epsilon' - \epsilon = \omega} \Pi_\epsilon S_k \Pi_{\epsilon'} \equiv \sum_{\epsilon' - \epsilon = \omega} |\epsilon\rangle\langle\epsilon| \, S_k \, |\epsilon'\rangle\langle\epsilon'| \tag{3}$$

- Yielding a time evolution (Dirac's pic) in the form

$$H_{\text{int}}(t) = \sum_{\alpha,\omega} e^{-i\omega t} S_\alpha(\omega) \otimes B_\alpha(t) \ , \quad B_\alpha(t) = e^{iH_B t} B_\alpha e^{-iH_B t} \tag{4}$$

-

$$\frac{d}{dt}\rho(t) = [H + H_{\text{LS}}, \rho(t)] + \sum_{k,l,\omega} \gamma_{k,l}(\omega) \left[S_l(\omega)\rho(t)S_k^\dagger(\omega) - \frac{1}{2}\{S_k^\dagger(\omega)S_l(\omega), \rho(t)\} \right]$$

which we can diagonalize over (k, l), by defining $\gamma(\omega) = U\Sigma(\omega)U^\dagger$, with $\Sigma(\omega) = \text{Diag}(\varsigma_k(\omega))$ and the transformation of the jump operators - $L_k(\omega) = \sum_l U_{lk} S_l(\omega)$. Yielding

Lindblad

$\frac{d}{dt}\rho(t) = [H + H_{\text{LS}}, \rho(t)] + \sum_\omega \sum_k \varsigma_k(\omega) \left[L_k(\omega)\rho(t)L_k^\dagger(\omega) - \frac{1}{2}\{L_k^\dagger(\omega)L_k(\omega), \rho(t)\} \right]$

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$$\varphi \in \mathrm{DS} : L_k |\varphi\rangle = 0 \qquad \& \qquad L_k \rho L_k^\dagger \neq 0$$

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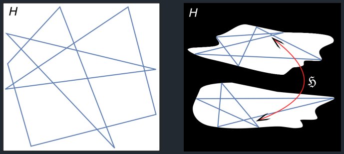
- Dark Hamiltonian - driver of the non stationary effects

(non)-Stationarity

- Question - how to determine the steady states and non-steady states?



(a) Ergodicity

(b) Oscillating coherences via $\mathfrak{H}$

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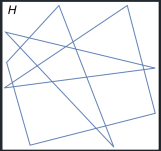
- Before that, one defines:
  - Symmetry-preserving dissipation
  - Dark Hamiltonian $\mathfrak{H}$ - not necessarily hermitian



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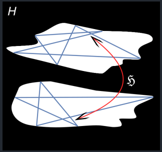

(b) Oscillating coherences via $\mathfrak{H}$

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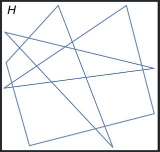

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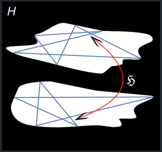

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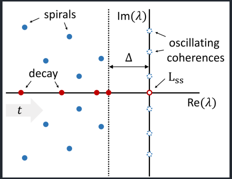

(c) Eigenvalues of $\mathscr{L}$

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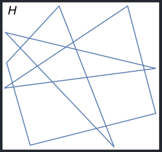
  - Decays
  - Oscillating coherences
  - Spirals



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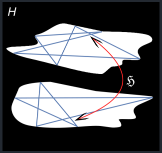

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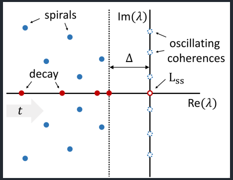

(c) Eigenvalues of $\mathscr{L}$

(non)-Stationarity

Theorem

Let $|\rho_\infty\rangle\rangle$ *- steady state, s.t.* $\mathscr{L}|\rho_\infty\rangle\rangle = 0$*. If* $\exists$ *A s.t.*

$$[H, A]\rho_\infty = \lambda A\rho_\infty \qquad [L_k, A]\rho_\infty = [L_k^\dagger, A]\rho_\infty = 0$$

Then,

$$\mathscr{L}(A|\rho_\infty\rangle\rangle) = i\lambda A|\rho_\infty\rangle\rangle \qquad \textit{i.e. } A|\rho_\infty\rangle\rangle \in EigSp(\mathscr{L})$$

with λ *- purely imaginary (See oscillating coherences).*

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Intuitively, given $|\rho_\infty\rangle\rangle$, the *"symmetry" A* will *"induce"* the non-stationary states.

(non)-Stationarity

One defines

- Super-operator $\hat{\hat{A}}$: $\quad \hat{\hat{A}}|\rho\rangle\rangle \equiv \hat{A}|\rho\rangle\rangle$, with operator $\hat{A}$ with defined properties.

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- The state $|\rho_\infty\rangle\rangle$ as before, $\mathscr{L}|\rho_\infty\rangle\rangle = 0$
- The eigen-operator relation for super-operators

$$[\mathscr{L}, \hat{\hat{A}}] = i\lambda\hat{\hat{A}}$$

Corollary

If there exists an operator $\hat{A}$, such that

$$[H, \hat{A}] = \lambda \hat{A} \qquad \text{and} \qquad [L_k, \hat{A}] = [L_k^\dagger, \hat{A}] = 0 \quad \forall k$$

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Then, one can define

$$|\rho_{nm}\rangle\rangle = \left(\hat{A}\right)^n |\rho_\infty\rangle\rangle \left(\hat{A}^\dagger\right)^m$$

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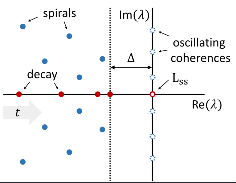

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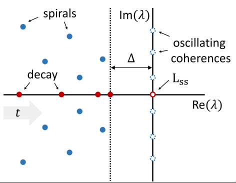

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- Properties of eigenstates $|\rho\rangle\rangle \leftrightarrow \sigma(\mathscr{L})$ derived independently.
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- Symmetry-preserving dissipations & dark Hamiltonian

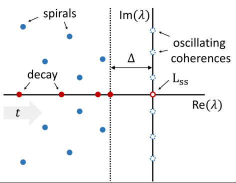

Figure 2

Implications?

Last times:

$$\mathscr{L}\rho = \dot{\rho} = -i[H + H_{\mathrm{LS}}, \rho] + \sum_{\omega,k} \varsigma_k(\omega)\Big(L_k(\omega)\rho L_k^\dagger(\omega) - \tfrac{1}{2}\{L_k^\dagger(\omega)L_k(\omega), \rho\} \Big)$$

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With the diagonalized elements $L_k(\omega) = \sum_l U_{lk} S_l(\omega)$ and

$$S_l(\omega) = \sum_{\omega=\epsilon'-\epsilon} \Pi_\epsilon S_l \Pi_{\epsilon'} \equiv \sum_{\omega=\epsilon'-\epsilon} |\epsilon\rangle\langle\epsilon| \, S_l \, |\epsilon'\rangle\langle\epsilon'|$$

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In the literature in question -

$$\mathscr{L}\rho = -i[H, \rho] + \sum_\mu \Big(2\tilde{L}_\mu \rho \tilde{L}_\mu^\dagger - \{\tilde{L}_\mu^\dagger \tilde{L}_\mu, \rho\}\Big)$$

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One may try to suggest to go from the 2nd to the 1st:

$$H \mapsto H + H_{\mathrm{Ls}} \quad \mu \mapsto (\omega, k) \qquad \Longrightarrow \qquad \tilde{L}_{\mu=(\omega,k)} \mapsto \sqrt{\tfrac{\varsigma_k(\omega)}{2}} L_k(\omega)$$

Implications?

Using the definitions:

$$L_\mu \mapsto \sqrt{\frac{S_k(\omega)}{2}} L_k(\omega) = \sqrt{\frac{S_k(\omega)}{2}} \sum_{\omega=\epsilon'-\epsilon} |\epsilon\rangle\langle\epsilon| \, L_k \, |\epsilon'\rangle\langle\epsilon'|$$

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For an operator A such that $[H, A] = \lambda A$ and $[L_\mu, A] = [L_\mu^\dagger, A] = 0$,

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How about the condition on H? That is, $[H, A] = \lambda A$. One suggested:

$$[H, A] \mapsto [H + H_{\mathrm{Ls}}] \quad \Longrightarrow \quad [H + H_{\mathrm{Ls}}] = \lambda A$$

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How to interpret A?

- Symmetry generator? (angular momentum, energy)
- Ladders?

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How to interpret A?

- Symmetry generator? (angular momentum, energy)
- Ladders?
- Example: $[L_z, L_\pm] = \pm\hbar L_\pm$, $\qquad [H, a^\dagger] = \hbar\omega a^\dagger$

Implications?

Let us recall again:

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- Can we expand more?
- Is L_k Hermitian? How are A and L_k related?
- Remember the definition of L_k - unitary transform of S_k, defined via $H_{\text{int}} = \sum_k S_k \otimes B_k$. Thus $H_{\text{int}} = H_{\text{int}}^\dagger$, S_k - not necessarily hermitian.