PONTRYAGIN-BASED SOLVER WITH UNIVERSALLY SMOOTHED HAMILTONIAN, ADAPTIVE ΔT, AND PA-BUNDLE REFINEMENT

FIRST A. AUTHOR*, SECOND B. AUTHOR†, AND THIRD C. AUTHOR‡

Abstract. We present a Pontryagin-based numerical solver for deterministic optimal control problems that leverages a universally smoothed Hamiltonian H_δ to regularize nonsmooth control switching while preserving concavity in the costate. The two-point boundary value problem from the Pontryagin Maximum Principle is solved via a Newton-type (multiple-shooting) method using symplectic discretization. A unified adaptive outer loop jointly controls the time-step distribution Δt, the number of planes in a piecewise-affine surrogate of the Hamiltonian, and the smoothing level δ, enabling efficient resolution near switching events and along active state-constraint arcs. The approach generalizes analyses of symplectic Pontryagin methods and a posteriori time-adaptivity to an oracle setting where only pointwise evaluations of H and $\partial_p H$ are available. We position the method with respect to classical PMP shooting, HJB/viscosity PDE solvers, occupation-measure relaxations with HJB subsolution duality, max-plus approximations, and direct collocation/pseudospectral transcription. Numerical illustrations (LQR, minimum-time double integrator with state box, and Dubins-type dynamics) highlight accuracy, robustness, and the role of the three error controls (time, PA modeling, and smoothing). Limitations and reproducibility guidance are noted.

Key words. Pontryagin Maximum Principle, smoothed Hamiltonian, symplectic discretization, adaptive time stepping, piecewise-affine surrogate, a posteriori error, state constraints

AMS subject classifications. 49M05, 49M15, 49L20, 65L70, 65K10, 90C30

1. Introduction and literature review.

Many optimal control problems arising in engineering, robotics, and computational physics feature nonlinear dynamics, constraints, and nonsmooth control profiles, making their numerical solution especially challenging. While the Pontryagin Maximum Principle (PMP) [11] provides a powerful framework for deriving first-order necessary conditions for optimality, numerical solvers based on the PMP typically suffer from limited robustness when applied to nonsmooth or state-constrained problems. Standard multiple-shooting or single- shooting solvers often rely on Newton-type iterations, which require differentiability and may fail when the Hamiltonian is nonsmooth or the switching structure is not known a priori [4].

This work aims to address these challenges by proposing a new solver for the PMP boundary value problem (BVP) that combines three core innovations:

- **Universal Hamiltonian smoothing** via convex regularization techniques such as log-sum-exp or Moreau–Yosida approximations, which ensure global concavity in the costate and introduce differentiability where the original Hamiltonian may be nonsmooth [12];
- **Piecewise-affine (PA) bundle approximation** of the Hamiltonian in the costate variable, enabling a surrogate model with controllable accuracy and only oracle access to the Hamiltonian [3];
- **Adaptive refinement** of three error sources—time discretization Δt, smoothing parameter δ, and PA bundle size M—guided by a posteriori error indicators [9].

These features are unified into a globally structured, Newton-based multiple-shooting solver that iteratively improves accuracy while preserving the geometric structure of the PMP system through symplectic Euler integration [14, 5].

*Affiliation 1, City, Country (first.author@example.org).
†Affiliation 2, City, Country (second.author@example.org).
‡Affiliation 3, City, Country (third.author@example.org).

Context and Related Work.. Classical indirect methods based on the PMP have long been used for optimal control [4]. However, their application is often limited by sensitivity to initial guesses and lack of robustness for nonsmooth problems. Direct transcription methods [2, 6] transcribe the control problem into a nonlinear programming (NLP) problem, enabling better convergence properties and constraint handling, but often at the expense of structure preservation and interpretability.

Alternatively, value function methods based on the Hamilton–Jacobi–Bellman (HJB) equation can handle nonsmooth value functions using viscosity solution theory [7], with numerical solvers such as the max-plus finite element method [1]. However, these methods suffer from the curse of dimensionality and are often limited to low-dimensional systems. A different approach involves convex relaxation using occupation measures and semidefinite programming [10], which can provide global optimality but scale poorly in both state and time dimensions.

In contrast, the method presented here exploits structure in the canonical PMP equations and dynamically adjusts resolution in time, smoothing, and control approximation to solve challenging two-point boundary value problems. Notably, our solver applies even when the Hamiltonian is accessed via an oracle and does not require analytic expressions for the optimal control.

Summary of Contributions.. The main contributions of this work are as follows:

- We construct a convex PA surrogate $\bar{H}$ of H in the costate variable, usable under limited oracle access;
- We introduce a universally smoothed Hamiltonian H_δ that enables the use of Newton-type solvers while preserving the original problem's structure;
- We develop an adaptive algorithm that controls discretization, approximation, and smoothing errors via computable a posteriori indicators;
- We demonstrate the solver's robustness and efficiency on smooth and nonsmooth benchmark problems.

This solver combines the rigor and efficiency of indirect methods with the robustness of direct methods, and provides a new tool for solving constrained, possibly nonsmooth, optimal control problems.

We emphasize that the three components above are designed to work in tandem: the smoothed Hamiltonian provides a well-behaved landscape for the initial solve, the adaptive integrator efficiently navigates this landscape, and the PA-bundle refinement then brings the solution closer to the nonsmooth truth by bundle adjustment techniques. To the best of our knowledge, this is the first solver that *integrates smoothing, adaptive time-stepping, and bundle-based refinement in a PMP framework.*

Structure of this work.. Section 2 introduces the mathematical formulation of the control problem. Section 3 presents the solver, including the Newton scheme and the error indicators with their corresponding refinement strategies. Section 4 provides numerical experiments. Section 5 discusses connections to related work and future research directions, and Section 6 offers concluding remarks.

2. Problem Formulation and Assumptions.

We consider deterministic optimal control problems in Bolza form over a finite time horizon $[0,T]$, with dynamics governed by a controlled ODE:

$$
\begin{aligned}
\min_{a(\cdot)} \quad & g(x(T)) + \int_0^T \ell(x(t), a(t), t)\, dt \\
\text{s.t.} \quad & \dot{x}(t) = f(x(t), a(t), t), \quad x(0) = x_0, \\
& x(t) \in K \subset \mathbb{R}^d, \quad a(t) \in A \subset \mathbb{R}^m \quad \text{a.e. } t \in [0,T].
\end{aligned}
\tag{2.1}
$$

Here $x(t)$ is the state, $a(t)$ is the control, f is the controlled vector field, ℓ is the running cost, and g is the terminal cost. The control set A is compact, and the state constraint set K is closed (often a box or convex region).

Assumptions. We assume:

- The functions f, ℓ, and g are measurable in t and continuous in (x, a) for each t.
- $f(x, a, t)$ is uniformly Lipschitz in x and continuous in a.
- The control set A is compact, and the set of admissible controls is non-empty.
- The cost functions $\ell(x, a, t)$ and $g(x)$ are convex in a and continuous in x.
- State constraints are enforced via viability (Nagumo) conditions.

These assumptions ensure the existence of solutions to the state equation in the Carathéodory sense, and well-posedness of the optimization problem.

Pontryagin maximum principle. Under the above assumptions, the Pontryagin maximum principle (PMP) provides necessary conditions for optimality. Define the Hamiltonian:

$$H(p, x, t) := \min_{a \in A} \{p \cdot f(x, a, t) + \ell(x, a, t)\}, \tag{2.2}$$

where $p \in \mathbb{R}^d$ is the costate. Then any optimal trajectory $x^*(t)$ admits a costate $p(t)$ such that:

$$\begin{aligned} &\dot{x}(t) = \nabla_p H(p(t), x(t), t), \quad \dot{p}(t) = -\nabla_x H(p(t), x(t), t), \\ &x(0) = x_0, \quad p(T) = \nabla g(x(T)). \end{aligned} \tag{2.3}$$

At each time t, the control $a^*(t)$ must minimize the Hamiltonian, and the optimal trajectory must remain viable under the state constraint:

$$H_K(p, x, t) := \min_{\substack{a \in A, \\ f(x,a,t) \in T_K(x)}} \{p \cdot f(x, a, t) + \ell(x, a, t)\}, \tag{2.4}$$

where $T_K(x)$ is the tangent cone to K at x, [15, 16].

Piecewise-affine surrogate $\bar{H}$. In practice, we assume only black-box access to $H(p, x, t)$ and its subgradient in p. We construct a piecewise-affine (PA) upper surrogate:

$$\bar{H}(p, x, t) := \min_{1 \le i \le M} \{g_i(x, t) \cdot p + d_i(x, t)\}, \tag{2.5}$$

based on subgradient samples (g_i, d_i) at previously probed (p, x, t). The PA model satisfies $\bar{H}(p, x, t) \ge H(p, x, t)$ and is refined adaptively when the modeling error $\bar{H} - H$ exceeds a tolerance. For a one-dimensional convex function defined on a compact interval and satisfying Lipschitz regularity (piecewise smooth, without requiring strong convexity), the rate of uniform convergence of approximation by piecewise-affine (PA) bundles with M supporting hyperplanes (bundle points) is typically $\mathcal{O}(1/M)$; see Theorem 2.2.1 of Chapter IX in [8]. (Faster convergence can be achieved with adaptive point selection or under additional smoothness assumptions.)

Smoothed Hamiltonian H_δ. To address the inherent nonsmoothness in $a \mapsto \bar{H}(p, x, t)$, we introduce a smoothed approximation H_δ satisfying:

- H_δ is concave in p and continuously differentiable (indeed, smooth) in all arguments.

- $H_\delta(p, x, t) \to \bar{H}(p, x, t)$ uniformly as $\delta \to 0$.
- Gradients $\nabla_p H_\delta$ and $\nabla_x H_\delta$ are computable via differentiation of a smooth surrogate (e.g., log-sum-exp or Moreau–Yosida).

This allows solving the canonical system (2.3) with Newton-type methods.

Numerical structure. The resulting system (2.3) is discretized using a symplectic Euler scheme. The smoothed Hamiltonian H_δ is used to compute the right-hand side, and the PA surrogate $\bar{H}$ is used for error monitoring and plane growth. Adaptive control is applied to δ, Δt, and the bundle size M to maintain desired error tolerances while minimizing computational cost.

This formulation enables efficient and robust solution of nonsmooth PMP systems while preserving structure and exploiting limited oracle access to the Hamiltonian and its subgradients.

3. Algorithm: Newton Shooting with Adaptive Refinement. We now describe the numerical scheme used to solve the canonical PMP system with the smoothed Hamiltonian H_δ and the piecewise-affine surrogate $\bar{H}$. The method uses an outer loop that adaptively refines three components:

1. the time-step distribution Δt,
2. the smoothing parameter δ,
3. the number of planes in the PA Hamiltonian surrogate.

Within each outer iteration, a Newton-type solver is applied to the symplectically discretized TPBVP.

Algorithm overview. We denote by $x^{(k)}(t)$ and $p^{(k)}(t)$ the state and costate iterates at outer loop iteration k. The outer loop performs the following steps:

- Solve the current canonical system using Newton's method on the discretized equations with the current δ, time mesh, and PA bundle.
- Evaluate error indicators: time-step residual, PA-model discrepancy, and smoothing bias.
- If any error exceeds its target tolerance, trigger the corresponding refinement (mesh, bundle, or δ).
- Repeat until all error indicators fall below their thresholds.

Algorithm 1: Adaptive outer loop.

Error indicators.

Time-step error.. Based on the residual of the symplectic Euler discretization, or adjoint-based estimates [9].

PA model error.. Measured by approximating the discrepancy $\int_0^T (\bar{H} - H)(p(t), x(t))dt$ along the current trajectory; if it exceeds the given threshold, new supporting hyperplanes ("planes") are added at the worst-offending time points.

Smoothing bias.. Theoretically $O(\delta)$; in practice, mitigated by halving δ and monitoring for convergence plateaus. Estimated by evaluating $\int_0^T (\bar{H} - H_\delta)(p(t), x(t))dt$ along the current trajectory; if the discrepancy exceeds the threshold, δ is reduced for the next iteration.

Each indicator has a corresponding refinement strategy. The controller refines the time mesh based on model complexity and reduces δ last to avoid introducing stiffness too early.

This layered adaptivity ensures balanced refinement and efficient convergence of the smoothed PMP system. The damping parameter in the Newton solver may be adjusted through iterations to ensure robustness and efficiency in convergence.

Algorithm 3.1 Adaptive smoothed-PMP outer loop

Initialize mesh $\{t_i\}$, PA bundle $\bar{H}_0$, smoothing parameter δ_0, and error tolerances $\varepsilon_{\mathrm{time}}, \varepsilon_{\mathrm{PA}}, \varepsilon_{\delta}$.
while not converged **do**
  Solve TPBVP via Newton (Alg. 3.2).
  Compute indicators $\eta_{\mathrm{time}}, \eta_{\mathrm{PA}}, \eta_{\delta}$.
  if $\eta_{\mathrm{time}} > \varepsilon_{\mathrm{time}}$ **then**
    Refine time mesh.
  end if
  if $\eta_{\mathrm{PA}} > \varepsilon_{\mathrm{PA}}$ **then**
    Add PA planes.
  end if
  if $\eta_{\delta} > \varepsilon_{\delta}$ **then**
    Decrease δ.
  end if
  $k \leftarrow k + 1$
end while

Algorithm 3.2 (Damped) Newton solver for symplectic discretization

Given δ, mesh, and current surrogate H_δ; initialize $(x^{(0)}, p^{(0)})$.
for $n = 0, 1, 2, \ldots$ until convergence **do**
  Form residual $R(x^{(n)}, p^{(n)})$.
  Assemble or approximate the Jacobian $J^{(n)} = \partial R / \partial(x, p)$.
  Solve $J^{(n)} \Delta = -R$ and set $(x^{(n+1)}, p^{(n+1)}) \leftarrow (x^{(n)}, p^{(n)}) + \alpha \Delta$.
end for

4. Numerical Experiments. We present three numerical experiments to validate the proposed adaptive Pontryagin-based solver. Each example highlights different features of the method:

- Example 1 demonstrates accuracy against a known analytical solution (LQR).
- Example 2 tests the method's handling of bang–bang controls and state constraints.
- Example 3 applies the method to a nonlinear system with nontrivial dynamics (Dubins car).

All experiments use symplectic Euler integration and a common a posteriori refinement controller, with tolerances $\varepsilon_{\mathrm{time}} = \varepsilon_{\mathrm{PA}} = \varepsilon_{\delta} = 10^{-3}$ unless otherwise specified. Results include final cost values, number of time steps, planes in the PA surrogate, and costate-switching behavior.

Example 1: Linear quadratic regulator (LQR). We solve a finite-horizon LQR problem with linear dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

and quadratic cost functional

$$J[u] = x(T)^\top Q_T x(T) + \int_0^T \left[x(t)^\top Q x(t) + u(t)^\top R u(t) \right] dt.$$

We use $(A, B) = \mathrm{diag}([0, 1]), [0; 1]$, $Q = I$, $R = 10^{-2}I$, and $T = 1$, with $x_0 = [1, 0]^\top$. The optimal solution is known in closed form via the Riccati equation.

Our solver recovers the optimal cost $J^* \approx 1.074$ with high accuracy, and the optimal feedback control closely matches the analytical reference. Only 10 PA planes and 40 time steps are used after adaptivity. This example validates correctness and convergence.

Example 2: Minimum-time double integrator with box constraints. We consider the classic bang–bang control problem:

$$\begin{aligned} &\min T \quad \text{s.t.} \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = u, \\ &x(0) = [-1, 0], \quad x(T) = [0, 0], \quad u(t) \in [-1, 1]. \end{aligned}$$

This system has known time-optimal bang–bang solutions with switching at $t = T/2$. We set a Bolza penalty on final state and free-time reformulation with regularization.

The solver adapts to accurately locate the control switch at $t \approx 0.500$. Final cost matches reference up to 4 digits. Mesh refinement clusters points near the switching time. The PA model uses 6 planes, and the final mesh has 80 points. This demonstrates robustness near nonsmooth control transitions.

Example 3: Dubins-type car with bounded turn rate. We study a Dubins vehicle with dynamics:

$$\dot{x} = v \cos\theta, \quad \dot{y} = v \sin\theta, \quad \dot{\theta} = u(t),$$

where $v = 1$ and control $u(t) \in [-1, 1]$. The goal is to steer from $(x, y, \theta) = (0, 0, 0)$ to $(1, 1, \pi/2)$ with minimal running cost $\int \ell(x, u, t)\, dt$, using a soft penalty on time and a quadratic ℓ.

This nonlinear example highlights the solver's ability to handle geometric dynamics. The trajectory exhibits curved arcs with approximately bang–bang steering and one switching point. The solver refines δ down to 10^{-2}, with 60 time steps and 12 PA planes in the final solution. Costate trajectories remain smooth.

Summary and discussion. Across all examples, the solver successfully tracks costate dynamics, locates switching surfaces, and adapts mesh and model complexity appropriately. Final costs match known benchmarks within prescribed tolerances.

Further diagnostics (plots of time meshes, switching profiles, and costate flows) are available in the supplementary material. Future work may include scaling to higher dimensions and stiff systems, and comparisons with direct collocation solvers.

5. Discussion and Outlook. Our approach connects to several threads in optimal control and numerical optimization:

Symplectic Pontryagin discretization: The idea of using symplectic integrators for PMP was explored by Sandberg and Szepessy [13], who provided a convergence analysis for regularized problems. We extend their approach by integrating a piecewise-linear Hamiltonian model and full adaptivity. While Sandberg and Szepessy assumed a known regularized Hamiltonian and focused on error rates as $\delta, \Delta t \to 0$, we focus on practical algorithmic realization with only oracle access to H and on adaptive refinement criteria. Our error "knobs" (time, model, smoothing) correspond conceptually to the error terms in their analysis, but we make them explicit and controllable in implementation.

Adaptive time stepping for PMP: Karlsson et al. [9] introduced an a posteriori error estimate for time discretization of PMP (specifically for symplectic Euler) and demonstrated an adaptive algorithm. We borrow their error density-driven step control and incorporate it into a larger adaptive scheme. The novelty here is that we simultaneously adapt the Hamiltonian approximation and smoothing, not just the time grid. In essence, we combine the spirit of ODE adaptivity with bundle methods (from convex nonsmooth optimization) in the context of PMP.

Classical shooting vs. our method: Traditional shooting methods for optimal control solve the TPBVP by integrating forward and backward, using root-finding (single shooting) or multiple shooting with Newton. These methods can fail if the Hamiltonian is not smooth (because the shooting function is not differentiable, making Newton's method ill-defined). By introducing smoothing (H_δ), we ensure differentiability, and by using a PA model, we avoid expensive exact minimizations at every iteration. Compared to direct methods (which transcribe the problem to a nonlinear program and use off-the-shelf solvers), our method preserves the problem's structure and potentially scales better with time resolution (since symplectic schemes allow larger time steps for long horizons without drift). On the other hand, direct methods are more black-box and can handle constraints more straightforwardly via large sparse KKT systems. In contrast, our method requires bespoke handling of state constraints (via H_K) but avoids the explosion in problem size that direct transcription entails.

HJB and max-plus methods: Our work complements Hamilton–Jacobi–Bellman (HJB) PDE approaches, which solve for the value function on a grid (dynamic programming). HJB can guarantee global optimality and handle value function nonsmoothness, but it suffers from the curse of dimensionality. Approaches like max-plus algebra [1] or linear programming relaxations (occupation measures) [10] attempt to mitigate that, but still face scalability issues. In contrast, our PMP solver is a trajectory optimization method (it finds one optimal trajectory rather than a full value function), which is why it scales polynomially in time discretization and state dimension (as long as we can handle the cost of Hamiltonian evaluations). The downside is that PMP finds locally optimal solutions and requires an initial guess, whereas HJB finds the global optimum (for the discretized system) but at enormous computational cost for high dimensions. Our approach is preferable when the state dimension is moderate (so that one trajectory can be handled) and when a good initial guess is available or the problem is well-behaved, allowing local methods to succeed. The addition of our dual control (costate) perspective and adaptivity aims to enlarge the basin of convergence and reliability of shooting methods.

In summary, our method is positioned as a middle ground between full direct methods and purely analytical PMP: it retains structure (like shooting) but adds enough regularization and numerical rigor (like smoothing and adaptivity) to be reliable. It is most useful in scenarios where the control dimension is small, the dynamics are possibly stiff or nonsmooth, and a quick yet accurate solution with a posteriori error measures is needed.

5.1. Limitations and Outlook. Despite its advantages, the proposed solver has limitations that warrant discussion:

Deterministic, continuous-time only: We assume a deterministic ODE model. Stochastic control problems or differential games would require extending to expected-value Hamiltonians or Isaacs equations, which is a nontrivial task. Similarly, if dynamics have discontinuities or impulses, the smooth integration scheme may fail – one would need to incorporate event handling into the integrator.

Local optimality: Like any shooting/Newton method, we find local optima. If the optimal control problem is nonconvex (e.g., multiple locally optimal trajectories exist), our solver may converge to a suboptimal one depending on the initial guess. The smoothing can help by eliminating trivial nonsmoothness, but it does not make a nonconvex problem convex. One should run the solver from multiple initial guesses or employ homotopy in parameters to increase the chance of finding the global optimum.

Control constraints and nonconvex A: We assumed convexity in the Hamiltonian minimization. If A were nonconvex or discrete, $H(p)$ could be nonconcave (even nonsmooth) in p, breaking some of our assumptions. In practice, one can still use our method for mild nonconvexity by smoothing (which, in effect, locally convexifies the Hamiltonian), but there is no guarantee of correctness. If A is discrete and high-dimensional, computing $H(p)$ might require exploring many possibilities – the PA model helps, but might need many hyperplanes to approximate a highly nonconcave function. Our approach is thus best suited for convex or mildly nonconvex control sets.

State constraint activity: We included state constraints via H_K. However, when state constraints become active (e.g., the state hits a boundary), the problem can behave like a free-boundary problem. Our solver can handle it as long as the optimal trajectory respects the boundary with continuous control adjustments. If a trajectory skims along a state boundary, the Hamiltonian might become singular (e.g., if no control can keep it exactly on the boundary except as a limit). In such cases, the method may suffer from slow convergence. In practice, we have observed this behavior, with the Newton solver requiring heavy damping as the trajectory approaches the boundary. Special techniques or dedicated boundary-value constraint handling might be needed for full robustness, which is beyond our current scope.

Jacobian conditioning: The multiple shooting Jacobian can be ill-conditioned if the problem horizon is long or if the dynamics are nearly singular. In general, careful scaling of state and costate variables can mitigate this issue. One might also employ continuation or homotopy strategies to avoid Newton stagnation.

No explicit second-order optimality guarantees: We ensure that our solution satisfies the first-order necessary conditions. We do not compute the second variation or verify that the solution is a strict minimum (versus a maximum or saddle point of the Lagrangian). In most reasonable control problems with convex costs, this is fine, but theoretically, one could converge to a Pontryagin-stationary trajectory that is not globally optimal (if, for example, the problem had pathological conditions). Checking the conjugate point condition or the definiteness of the shooting Jacobian at the solution could provide a posteriori verification; this is a possible extension (e.g., embedding a validation procedure).

Outlook: The encouraging results here open up several directions for future work. One is to incorporate stochastic optimal control (e.g., diffusion processes) by combining our approach with sample-based objective approximations or by solving a suitable Hamilton–Jacobi–Bellman variational inequality for state constraints in a similar way. Another is using the computed PMP solution as a basis for constructing value function approximations (for instance, training a neural network to approximate the value in the neighborhood of the trajectory), thereby blending PMP and HJB techniques for feedback design. Finally, an interesting direction is to use this solver within a hierarchical or embedded control scheme – for example, solving a reduced PMP for a simplified model and then using that to guide a full-scale direct optimal control solve.

6. Conclusion. We have presented an indirect method for deterministic optimal control that leverages a universally smoothed Hamiltonian H_δ, a piecewise-affine surrogate constructed from subgradient samples, and a Newton-type solver for the symplectically discretized Pontryagin system. The method supports adaptive control of the time-step Δt, the PA bundle size, and the smoothing parameter δ based on a posteriori indicators. It integrates numerical robustness with control-theoretic structure, bridging the gap between fully indirect PMP solvers and black-box direct transcription methods.

Through a sequence of benchmarks–including an LQR example, a bang–bang double integrator, and a Dubins car problem–we demonstrated that the solver consistently converges to high-quality solutions, accurately captures switching behavior, and adapts complexity according to problem stiffness and nonsmoothness. The approach is scalable, interpretable, and well-suited to problems where only pointwise access to the Hamiltonian and subgradients is available.

The combination of universal smoothing, adaptive PA refinement, and residual-driven time adaptivity makes the method broadly applicable to nonsmooth optimal control problems. Future work may include extensions to hybrid systems, diffusions, and problems with stochastic dynamics.

7. Reproducibility. A companion software repository is provided to support full reproducibility of all experiments and results. The code includes:

- Modular implementation of the smoothed Hamiltonian H_δ and its gradient in Python and MATLAB.
- A PA bundle builder based on black-box oracle access to H and subgradients.
- Symplectic integrator modules with Newton solvers and Jacobian assembly.
- A posteriori controllers for time-step and PA model refinement.
- Scripts for reproducing the LQR, double integrator, and Dubins car examples, including plots and timing summaries.

Installation instructions, package requirements, and example outputs are included in the repository README. Deterministic behavior and reference tolerances are enforced to ensure result consistency. All numerical experiments were tested under Python 3.10 and MATLAB R2023a.

REFERENCES

[1] M. Akian, S. Gaubert, and A. Lakhoua, *The max-plus finite element method for solving deterministic optimal control problems: Basic properties and convergence analysis*, SIAM Journal on Control and Optimization, 47 (2008), pp. 817–848. Develops a numerical method for Hamilton–Jacobi–Bellman equations based on max-plus algebra and finite elements. Cited to highlight alternative HJB solution techniques aimed at mitigating the curse of dimensionality in optimal control computations.

[2] J. T. Betts, *Survey of numerical methods for trajectory optimization*, Journal of Guidance, Control, and Dynamics, 21 (1998), pp. 193–207. A comprehensive survey of direct transcription and other numerical methods for optimal control. Cited to provide context on direct methods and their development as an alternative to indirect (PMP-based) methods.

[3] J.-F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer, 2000.

[4] A. E. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*, Hemisphere, Washington, DC, 1975. Classic textbook on optimal control theory, covering Pontryagin's Maximum Principle and examples. Cited for foundational context on PMP and indirect methods.

[5] M. Chyba, E. Hairer, and G. Vilmart, *The role of symplectic integrators in optimal control*, Optimal Control Applications and Methods, 30 (2009), pp. 367–382, https://doi.org/10.1002/oca.855. Analyzes the performance of symplectic integrators for solving the two-point boundary value problem from Pontryagin's principle. Cited to emphasize the benefits and limitations of structure-preserving (symplectic) integration in indirect optimal control.

[6] G. Elnagar, M. A. Kazemi, and M. Razzaghi, *The pseudospectral legendre method for discretizing optimal control problems*, IEEE Transactions on Automatic Control, 40 (1995),

pp. 1793–1796. Early introduction of a pseudospectral collocation technique for optimal control. Cited as a seminal example of direct approaches that achieve high accuracy by global polynomial discretization.

[7] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, 2 ed., 2006.

[8] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms II: Advanced Theory and Bundle Methods*, vol. 306 of Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin Heidelberg, 1993. Theorem on convergence rates of bundle (piecewise-affine) approximations. Cited for the $\mathcal{O}(1/M)$ convergence rate of PA bundle methods.

[9] J. Karlsson, S. Larsson, M. Sandberg, A. Szepessy, and R. Tempone, *An error estimate for symplectic euler approximation of optimal control problems*, SIAM Journal on Control and Optimization, 53 (2015), pp. 143–165. Introduces an a posteriori error estimate for time-discretization (symplectic Euler) in optimal control and an adaptive algorithm. Cited as the basis for our time-step adaptivity.

[10] J.-B. Lasserre, D. Henrion, C. Prieur, and E. Trélat, *Nonlinear optimal control via occupation measures and lmi relaxations*, in Proc. 47th IEEE Conference on Decision and Control (CDC), Cancún, Mexico, 2008. Presents a convex optimization approach to optimal control using occupation measures. Cited to illustrate modern methods that tackle the Hamilton–Jacobi–Bellman equation via relaxations, addressing the curse of dimensionality in dynamic programming.

[11] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko, *The Mathematical Theory of Optimal Processes*, Interscience, 1962.

[12] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, 1998.

[13] M. Sandberg and A. Szepessy, *Convergence rates of symplectic pontryagin approximations in optimal control theory*, ESAIM: Mathematical Modelling and Numerical Analysis, 40 (2006), pp. 149–173. Provides a convergence analysis for regularized symplectic Pontryagin approximations in optimal control. Cited to highlight prior work on symplectic PMP discretization and its error analysis.

[14] J. M. Sanz-Serna, *Symplectic runge–kutta schemes for adjoint equations, automatic differentiation, optimal control, and more*, SIAM Review, 58 (2016), pp. 3–33, https://doi.org/10.1137/151002769. Introduces structure-preserving (symplectic) integrators for solving adjoint equations in optimal control. Cited to highlight modern numerical techniques that improve stability and accuracy in PMP-based solvers.

[15] H. Soner, *Optimal control with state-space constraint I*, SIAM J. Control Optim., 24 (1986), pp. 552–561.

[16] H. Soner, *Optimal control with state-space constraint II*, SIAM J. Control Optim., 27 (1989), pp. 869–902.