Reformulate GAN Loss

$$\min_{\theta} \max_{\varphi} \mathbb{E}_{x\sim p^*}\left[\log(D_\varphi(x))\right] + \underbrace{\mathbb{E}_{z\sim P_0}\left[\log\left(1-D_\varphi(G_\theta(z))\right)\right]}_{\mathbb{E}_{x\sim P_G}\left[\log\left(1-D_\varphi(x)\right)\right]}$$

$D_\varphi \to D$

$G_\theta \to G$

P_G ↳ generated distribution of $G_\theta(z)$, $z\sim P_0$

$$\min_{G} \max_{D} \underbrace{\mathbb{E}_{x\sim p^*}\left(\log D(x)\right) + \mathbb{E}_{x\sim P_G}\left(\log(1-D(x))\right)}_{L(D,G)}$$

① What is the optimal discriminator D given a fixed G.

Fix G, max over D.

$$L(D,G) = \int \log(D(x))\cdot p^*(x)\,dx + \int \log(1-D(x))\cdot P_G(x)\,dx$$

$$= \int \underbrace{\log(D(x))\cdot p^*(x) + \log(1-D(x))\cdot P_G}_{\text{max over } D \text{ for each } x.}\,dx$$

[let $\ell(u;a,b) = a\log u + b\log(1-u)$

want to max ℓ over $u\in\mathbb{R}$.

$$\frac{\partial \ell}{\partial u} = \frac{a}{u} - \frac{b}{1-u} = 0 \Rightarrow a(1-u) = bu$$

$$\Rightarrow u = \frac{a}{a+b}$$]

Let $u = D(x)$, $a = p^*(x)$, $b = P_G(x)$

Thus, D is maximized at

$$D(x) = D^*(x) = \frac{p^*(x)}{p^*(x)+P_G(x)}.$$

② Fix $D = D^*$, train G.

$$\min_G \max_D L(G,D) = \min_G L(G,D^*)$$

$$L(G,D^*) = \int \log(D^*(x))\,p^*(x) + \log(1-D^*(x))\,p_G(x)\,dx$$

$$= \int \log\left(\frac{p^*(x)}{p^*(x)+p_G(x)}\right) p^*(x)\,dx + \int \log\left(\frac{p_G(x)}{p^*(x)+p_G(x)}\right) p_G(x)\,dx$$

Let $q_G(x) = \frac{1}{2}\left(p^*(x) + p_G(x)\right)$

new distribution

$$= \int p^*(x) \cdot \log\left(\frac{p^*(x)}{q_G(x)}\right) dx + \int p_G(x) \cdot \log\left(\frac{p_G(x)}{q_G(x)}\right) dx$$

$$\underline{-2\log 2}$$

$\int p^*(x) \log 2\, dx = \log 2$

$$= D_{KL}(p^* \| q_G) + D_{KL}(p_G \| q_G) - 2\log 2$$

Example (Non-convergence of GD for min-max problems)

Let $L(\theta, \varphi) = \theta\varphi$, $\theta, \varphi \in \mathbb{R}$

$$\min_\theta \left[\max_\varphi L(\theta,\varphi)\right]$$

① Solution: if $\theta \neq 0$, $\max_\varphi L(\theta,\varphi) = +\infty$

if $\theta = 0$, $\max_\varphi L(\theta,\varphi) = 0$

Hence, $\{\theta = 0, \varphi \text{ arbitrary}\}$ are the solutions

(2) Gradient descent/ascent:

$$\begin{cases} \varphi_{k+1} = \varphi_k + \varepsilon \nabla_\varphi L = \varphi_k + \varepsilon \theta_k \\ \theta_{k+1} = \theta_k - \varepsilon \nabla_\theta L = \theta_k - \varepsilon \varphi_k \end{cases}$$

Let $r_k^2 = \theta_k^2 + \varphi_k^2$

$$\Rightarrow \quad r_{k+1}^2 = \theta_{k+1}^2 + \varphi_{k+1}^2$$

$$= (\theta_k - \varepsilon \varphi_k)^2 + (\varphi_k - \varepsilon \theta_k)^2$$

$$= (1+\varepsilon^2)\underbrace{(\theta_k^2 + \varphi_k^2)}_{r_k^2}$$