Q1 $x, y \in H \Rightarrow b^T x \le c,\ b^T y \le c$

$b^T[\lambda x + (1-\lambda)y]$, $\lambda \in [0,1]$

$= b^T(\lambda x) + b^T(1-\lambda)y$

$= \lambda b^T x + (1-\lambda) b^T y$

$\le \lambda \cdot c + (1-\lambda)\cdot c$

$\le c$

$\therefore \lambda x + (1-\lambda) y \in H$

$\therefore$ H is a convex set.

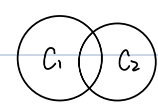
Q2. (a) $x, y \in C_1 \cap C_2$.

$$\begin{cases} \lambda x + (1-\lambda) y \in C_1 \\ \lambda x + (1-\lambda) y \in C_2 \end{cases} \Rightarrow \lambda x + (1-\lambda) y \in C_1 \cap C_2$$

$\therefore C_1 \cap C_2$ is convex.

(b) $C_1 = \{x \in R^2 \mid \|x\|_2 \le 1\}$

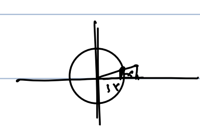
$C_2 = \{x \in R^2 \mid (x_1 - 1)^2 + x_2^2 \le 1\}$



C_1, C_2 are both convex, but $C_1 \cup C_2$ is not convex.

Q3. (a) $\Pi_C(x) = \min_{y \in C} \frac{1}{2}\|y - x\|^2$, $C = \{y \in R \mid \|y\|_2 \le r\}$

(b) $(3,1)^T \notin C$



By solving graphically:

$\sqrt{y^2 + (3y)^2} = 2 \Rightarrow y = \sqrt{\frac{2}{5}}$

$\therefore \Pi_C = \left(3\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}\right)^T$

Q4. (a) gradient: $\begin{cases} 400x_1^3 - 400x_1x_2 + 2x_1 - 2 & ① \\ 200(x_2 - x_1^2) & ② \end{cases}$

hessian:

$$\begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

(b) $x_1 = 1,\ x_2 = 1 \Rightarrow \begin{cases} ① = 0 \\ ② = 0 \end{cases}$

$\therefore x^* = (1;1)$ is a local minimizer.

(c) $x^* = (1;1) \Rightarrow$ Hessian: $\begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$

let $y \in \mathbb{R}^2$ and $y \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y^T \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix} y$

$\Rightarrow (802y_1 - 400y_2,\ -400y_1 + 200y_2)\,y$

$\Rightarrow 802y_1^2 - 400y_1y_2 - 400y_1y_2 + 200y_2^2$

$\Rightarrow 802y_1^2 - 800y_1y_2 + 200y_2^2$.

$\Rightarrow 2y_1^2 + 200(4y_1^2 - 4y_1y_2 + y_2^2)$

$\Rightarrow 2y_1^2 + 200(2y_1 - y_2)^2 > 0$. (as y_1, y_2 cannot both be 0)

$\therefore$ Hessian is positive-definite

$\therefore x^* = (1,1)^T$ is an unique local minimizer.

Q5 $\begin{cases} 2(x_1 + x_2^2) \\ 2(x_1 + x_2^2)\cdot 2x_2 \end{cases}$ $\begin{bmatrix} 2 & 4x_2 \\ 4x_2 & 12x_2^2 \end{bmatrix}$

(a) $x^{(0)} = (1,0)^T \Rightarrow -\nabla f(\cdot) = (-2,0)^T$

(b) $-\nabla f(\cdot) = (-2,0)^T$ set $y = (-1,0)^T$

then $-\nabla f(\cdot)^T y = 2 > 0.$

$\therefore (-1,0)^T$ is a descent direction.

(c) $\min_{\alpha>0} \phi(\alpha) = f(x^0 + \alpha^0 p^0)$

$= [(x_1^0 - 2\alpha) + x_2^{0\,2}]^2$

$= (1-2\alpha)^2$

$\phi'(\alpha) = 4(2\alpha - 1) = 0 \Rightarrow \alpha = \frac{1}{2}$ $\quad \binom{2}{0}(-2,0)$

$\therefore x^{(1)} = (1,0)^T - \frac{1}{2}(2,0)^T = (0,0)^T$ $\quad (2,0)\binom{-2}{0}$

(d) ① $\alpha = 1$ $f(x^0 + \alpha p^0) = 1 > f(x^0) + 10^{-4} \times 1 \times (-4) = 1 - 4\times 10^{-4}$

② $\alpha = 0.9$ $f(x^0 + \alpha p^0) = 0.64 \le f(x^0) + 10^{-4} \times 0.9 \times -4 = 1 - 3.6\times 10^{-4}$

$\therefore x^{(1)} = (-0.8, 0)^T$

(e) $\nabla f(x^{(k)}) + Hf(x^{(k)})\, p = 0$

$\Rightarrow (2,0)^T + \begin{bmatrix} 2, & 0 \\ 0, & 0 \end{bmatrix} p = 0$

$\Rightarrow (2,0)^T + (2p_1, p_2)^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow p = (-1,0)^T$

(f) $Hf(x^{(k)})^{-1} = \begin{bmatrix} \frac{1}{2}, & 0 \\ 0, & \frac{1}{2} \end{bmatrix}$

$x^{(1)} = (1,0)^T - \begin{bmatrix} \frac{1}{2}, & 0 \\ 0, & \frac{1}{2} \end{bmatrix}(2,0)^T = (1,0)^T - (1,0)^T = (0,0)^T$

26 (a) $dom(f) = \{x \in \mathbb{R} \mid f(x) < +\infty\} \Rightarrow x \in (-\infty, +\infty)$

$f^*(y) = \sup\{\langle x, y\rangle - f(x)\}$

$= \sup(xy - \frac{1}{2}x^2 - 4x)$

$= \sup(-\frac{1}{2}x^2 + (y-4)x)$

let $\phi(x) = -\frac{1}{2}x^2 + (y-4)x$

$\phi'(x) = -x + y - 4 = 0 \Rightarrow x = y - 4$

$\therefore \phi(x) \leq -\frac{1}{2}(y-4)^2 + (y-4)^2$

$\leq \frac{1}{2}(y-4)^2$

$\therefore f^*(y) = \frac{1}{2}(y-4)^2$.

(b) $dom(f) = \{x \in \mathbb{R}^n \mid f(x) < +\infty\} \Rightarrow \{x \mid x_i > 0, i = 1, 2, \cdots, n\}$.

$f^*(y) = \sup\{\langle y, x\rangle - f(x)\}$

$= \sup\{y^T x + \sum \log x_i\}$

① $y_i \geq 0 \Rightarrow f^*(y) = \sum \log x_i + y_i x_i \Rightarrow +\infty$.

② $y_i < 0 \Rightarrow f^*(y) = \sum(\log x_i + y_i x_i)$

let $\phi(x_i) = \log x_i + y_i x_i$

$\phi'(x_i) = \frac{1}{x_i} + y_i \Rightarrow x_i = -\frac{1}{y_i}$

$\therefore f^*(y) = \sum(\log -\frac{1}{y_i} - 1)$

$\therefore f^*(y) = \begin{cases} +\infty & \text{if } y_i \geq 0 \\ \sum(\log -\frac{1}{y_i} - 1) & \text{if } y_i < 0 \end{cases}$

Q7

(a) $f^*(y) = \sup \{ \langle y, x \rangle - f(x) \}$

$= \sup \{ y^T x - \lambda \|x\|_2 \}$

$y^T x - \lambda \|x\|_2 \le \|x\|_2 \|y\|_2 - \lambda \|x\|_2$

① $\|y\|_2 \le \lambda \Rightarrow \|x\|_2 \|y\|_2 - \lambda \|x\|_2 \le 0$

$\Rightarrow f^*(y) = 0.$

② $\|y\|_2 > \lambda \Rightarrow$ construct $\bar{x} = (my_1, my_2, \cdots, my_n)$ for a $m > 0$.

$\Rightarrow y^T x - \lambda \|x\|_2 = m \|y\|_2^2 - \lambda m \|y\|_2$

$= m (\|y\|_2^2 - \lambda \|y\|_2)$

$\ge m \|y\|_2 (\|y\|_2 - \lambda) \to \infty$ as $m \Rightarrow \infty$.

$\therefore f^*(y) = \delta_C(y)$, $C = \{ y \in R^n \mid \|y\|_2 \le \lambda \}$.

(b) $P_{f^*}(\cdot) = \underset{y}{\operatorname{argmin}} \{ \delta_C(y) + \frac{1}{2} \|y - x\|^2 \} \Rightarrow \underset{y \in C}{\operatorname{argmin}} \frac{1}{2} \|y - x\|^2 \Rightarrow \Pi_C(x)$

(c) $x = P_f(x) + P_{f^*}(x)$

$= P_f(x) + \Pi_C(x)$

$\therefore P_f(x) = x - \Pi_C(x)$

(d) $C = \{ y \in R^n \mid \|y\|_2 \le 1 \}$.

$\therefore P_{f^*}(x) = \min_{y \in C} \frac{1}{2} \|y - x\|^2 = (\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})^T$

$P_f(x) = (1, 1)^T - (\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})^T = (1 - \sqrt{\frac{1}{2}}, 1 - \sqrt{\frac{1}{2}})^T$

Q8

(a) $f(x) = \frac{1}{2} \| Ax - b \|_2^2$

$g(x) = \delta_C (\|x\|_\infty)$, $C \in [0, r]$

$\therefore \min_x \frac{1}{2} \|Ax - b\|_2^2 + \delta_C(\|x\|_\infty)$, $C \in [0, r]$

(b) $\beta^{k+1} = \underset{\beta}{\arg\min} \; -(\beta^k), < \nabla f(\beta^k), \beta - \beta^k > + g(\beta) + \frac{1}{2\partial_k} \|\beta - \beta^{(k)}\|^2 \}$

$= \underset{\beta}{\arg\min} \left\{ \frac{1}{2\partial_k} \|\beta - (\beta^k - \partial_k \nabla f(\beta^k)\|^2 + g(\beta) \right\}$

$= \Pi_D (\beta^k - \partial_k A^T(Ax - b))$, $D \in \{ \beta \in R^n \mid \|\beta\|_\infty \le r \}$.

(c) $\nabla f(x) = Ax - b = (-2, -2)^T$

$x^1 = \Pi_D \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right]$

$= \Pi_D [(4, 4)^T]$, $D \in \{ x \in R^n \mid \|x\|_\infty \le 1 \}$

$= (1, 1)^T$

Q9.

(a) $\min_{y,z} \ \|y\|_1 + \delta_C(z)$, s.t. $y - z = 0$, $C = \{z \in R^n \mid \|z-b\|_2 \le r\}$

(b) $L_\sigma(y, z, x) = f(y) + \delta_C(z) + \langle x, y-z \rangle + \frac{\sigma}{2}\|y-z\|^2$

$= f(y) + \delta_C(z) + \frac{\sigma}{2}\|y - z + \sigma^{-1}x\|^2 - \frac{1}{2\sigma}\|x\|^2.$

subproblem $-y$:

$\arg\min_y \{\frac{1}{\sigma}f(y) + \frac{1}{2}\|y - (z - \sigma^{-1}x)\|^2\}$

$= P_{\frac{1}{\sigma}\|y\|_1}(z - \sigma^{-1}x)$

subproblem $-z$:

$\Pi_C(y + \sigma^{-1}x)$, $C = \{z \in R^n \mid \|z-b\|_2 \le r\}$.

$\therefore$

$y^{k+1} = P_{\frac{1}{\sigma}\|y\|_1}(z^k - \sigma^{-1}x^k)$

$z^{k+1} = \Pi_C(y^{k+1} + \sigma^{-1}x^k)$

$x^{k+1} = x^k + \tau\sigma(y^{k+1} - z^{k+1})$

(c) $y^1 = \arg\min_y \{\|y\|_1 + \frac{1}{2}\|y\|^2\} = (0,0)^T$

$z^1 = \Pi_C(y^1 + x^0) = (2 - \frac{2}{5}\sqrt{5}, 1 - \frac{1}{5}\sqrt{5})^T$

$x^1 = (\frac{2}{5}\sqrt{5} - 2, \frac{1}{5}\sqrt{5} - 1)^T$

Q10.

① $x_1^{(1)} = \underset{x_1}{\arg\min}\ x_1^4 - 0 = 0.$

$x_2^{(1)} = \underset{x_2}{\arg\min}\ 5x_2^2 - 10x_2 = 1$

$\therefore x^{(1)} = (0, 1)^T$

② $x_1^{(2)} = \underset{x_1}{\arg\min}\ x_1^4 - 4x_1 = 1$

$x_2^{(2)} = \underset{x_2}{\arg\min}\ 5x_2^2 - 14x_2 = \frac{7}{5}$

$\therefore x_2^{(2)} = (1, \frac{7}{5})^T$

Q11

(a) $\min\ 2x_{11} + x_{21} + 3x_{12} + 4x_{22}$

$$\text{s.t.} \begin{cases} x_{11} + x_{12} = 0.4 \\ x_{21} + x_{22} = 0.6 \\ x_{11} + x_{21} = 0.2 \\ x_{12} + x_{22} = 0.8 \\ x_{11}, x_{12}, x_{21}, x_{22} \geq 0. \end{cases}$$

(b) $f(x) = 2x_{11} + x_{21} + 3x_{12} + 4x_{22}.$

$$\text{s.t.} \begin{cases} g_1(x) = x_{11} + x_{12} - 0.4 = 0 \\ g_2(x) = x_{21} + x_{22} - 0.6 = 0 \\ g_3(x) = x_{11} + x_{21} - 0.2 = 0 \\ g_4(x) = x_{12} + x_{22} - 0.8 = 0. \\ h_1(x) = -x_{11} \leq 0,\ h_2(x) = -x_{12} \leq 0,\ h_3(x) = -x_{21} \leq 0,\ h_4(x) = -x_{22} \leq 0. \end{cases}$$

let $x_{11} = 0.2,\ x_{12} = 0.2,\ x_{21} = 0,\ x_{22} = 0.6,$

then $g_i(x) = 0,\ h_i(x) \leq 0\ \ \forall i,$

$\Rightarrow$ slater's condition holds.

$$L(x, u, v) = 2x_{11} + x_{21} + 3x_{12} + 4x_{22} + \sum_{i=1}^{4} v_i g_i(x) + \sum_{j=1}^{4} u_j h_j(x)$$

$$\frac{\partial}{\partial x} L(x^*, u^*, v^*) = \begin{cases} 2 + v_1^* + v_3^* - u_1^* = 0. & (x_{11}) \quad \text{KKT condition.} \\ 1 + v_2^* + v_3^* - u_3^* = 0 & (x_{21}) \\ 3 + v_1^* + v_4^* - u_2^* = 0 & (x_{12}) \\ 4 + v_2^* + v_4^* - u_4^* = 0 & (x_{22}) \end{cases}$$

$g_i(x^*) = 0$, $h_j(x^*) \le 0$, $u_j^* \ge 0$, $u_j^* h_j(x^*) = 0$. $\forall i \in [4]$, $j \in [4]$.

(c) ① $x_{11} = 0.2$, $x_{12} = 0.2$ $x_{21} = 0$ $x_{22} = 0.6$.

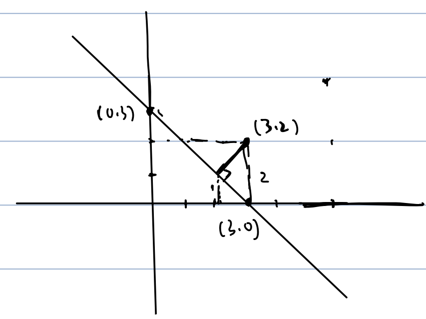
② $x_{11} = 0.1$, $x_{12} = 0.3$ $x_{21} = 0.1$ $x_{22} = 0.5$

(d) in ① cost is $2 \times 0.2 + 0 + 3 \times 0.2 + 4 \times 0.6 = 3.4$

in ② cost is $2 \times 0.1 + 0.1 + 3 \times 0.3 + 4 \times 0.5 = 3.2$

∴ second is better.

Q12.



(a) $\min\limits_{x_1, x_2} \quad (x_1 - 3)^2 + (x_2 - 2)^2$

s.t. $x_1 + x_2 \le 3$.

$x_1^* = 2$, $x_2^* = 1$

(b) $L(x, u) = (x_1 - 3)^2 + (x_2 - 2)^2 + u_1(x_1 + x_2 - 3)$

let $x_1 = x_2 = 1$, then $h(x) = x_1 + x_2 - 3 = -1 \le 0$.

∴ slater's condition holds.

KKT condition:

$$\frac{\partial}{\partial x} L(x^*, u^*) = \begin{cases} 2x_1^* + u_1^* - 6 = 0. \\ 2x_2^* + u_1^* - 4 = 0. \end{cases}$$

$x_1^* + x_2^* - 3 \le 0$, $u_1^* \ge 0$, $u_1^*(x_1^* + x_2^* - 3) = 0$.

(c) $\theta(u_1) = \inf\limits_{x_1, x_2} \quad (x_1 - 3)^2 + (x_2 - 2)^2 + u_1(x_1 + x_2 - 3)$

$= -3u_1 + \inf\limits_{x_1} (x_1 - 3)^2 + u_1 x_1 + \inf\limits_{x_2} (x_2 - 2)^2 + u_1 x_2$

$= -3u_1 + 3u_1 - \frac{1}{4}u_1^2 + 2u_1 - \frac{1}{4}u_1^2$

$= -\frac{1}{2}u_1^2 + 2u_1$

$\therefore$ dual is $\max \ -\frac{1}{2}u_1^2 + 2u_1$

s.t. $u_1 \geq 0$.

(d) $\theta'(u_1) = -u_1 + 2 = 0 \Rightarrow u_1 = 2$

$\therefore y^* = 2$.

(e) $x_1^* = 2$, $x_2^* = 1$, $y^* = 2$

from (b):
$$\begin{cases} 2\times 2 + 2 - 6 = 0. \\ 2\times 1 + 2 - 4 = 0. \\ 2 + 1 - 3 \leq 0. \\ 2 \geq 0 \\ 2\cdot(2+1-3) = 0 \end{cases}$$

$\therefore$ KKT condition is satisfied.