DSA5103 Assignment 4

Instructions Due on April 21, 11:59pm. The final exam will take the same format as this assignment.

1. (5 marks) Given $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, show that the half-space $H = \{x \in \mathbb{R}^n \mid b^T x \le c\}$ is a convex set.

2. (5 marks) Given two convex sets C_1 and C_2 in $\mathbb{R}^n$,

   (a) Show that $C = C_1 \cap C_2$ is also convex ($\cap$ denotes the intersection of two sets).

   (b) Give an example where $C_1 \cup C_2$ is not convex but C_1 and C_2 are convex ($\cup$ denotes the union of two sets).

3. (5 marks) Let $C = \{x \in \mathbb{R}^n \mid \|x\|_2 \le r\}$.

   (a) Give the formula for the projection onto C: $\Pi_C(x)$.

   (b) When $r = 2$ and $x = (3, 1)^T$, compute $\Pi_C(x)$.

4. (5 marks) Consider the function

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2,\ x = (x_1; x_2) \in \mathbb{R}^2.$$

   (a) Compute the gradient and Hessian of the function.

   (b) Show that $x^* = (1, 1)^T$ is a local minimizer of the function. [Hint: consider the sufficient condition]

   (c) Show that $x^* = (1, 1)^T$ is the only local minimizer of the function. [Hint: consider the necessary condition]

5. (10 marks) Consider the function

$$f(x) = (x_1 + x_2^2)^2,\ x = (x_1; x_2) \in \mathbb{R}^2.$$

   (a) At $x^{(0)} = (1, 0)^T$, compute the steepest descent direction.

   (b) At $x^{(0)} = (1, 0)^T$, construct another descent direction (different from the steepest descent direction).

(c) Start from $x^{(0)} = (1,0)^T$, apply steepest descent method with exact line search and determine the next iterate $x^{(1)}$.

(d) Start from $x^{(0)} = (1,0)^T$, apply steepest descent method with backtracking line search and determine the next iterate $x^{(1)}$. In particular, set $\bar{\alpha} = 1, \rho = 0.9, c_1 = 10^{-4}$ and do backtracking line search until the Armijo condition holds

$$f(x^{(k)} + \alpha p^{(k)}) \leq f(x^{(k)}) + c_1 \alpha \nabla f(x^{(k)})^T p^{(k)}$$

(e) At $x^{(0)} = (1,0)^T$, compute the Newton direction.

(f) Start from $x^{(0)} = (1,0)^T$, apply pure Newton's method (step size = 1) and determine the next iterate $x^{(1)}$.

6. (10 marks) (a) Let $f(x) = \frac{1}{2}x^2 + 4x$, $x \in \mathbb{R}$. Write down the effective domain of f. Compute its conjugate function f^*.

   (b) Let $f(x) = -\sum_{i=1}^{n} \log x_i$, $x \in \mathbb{R}^n$. Write down the effective domain of f. Compute its conjugate function f^*. ($\log x = \ln x$, $(\log x)' = \frac{1}{x}$.)

7. (10 marks) Let $f(x) = \lambda \|x\|_2$, $x \in \mathbb{R}^n$.

   (a) Compute its conjugate function f^*.

   (b) Give the formula for $P_{f^*}(\cdot)$ (the proximal mapping of f^*).

   (c) Give the formula for $P_f(\cdot)$ (the proximal mapping of f). [Hint: Moreau decomposition]

   (d) Set $\lambda = 1$, $b = (1,1)^T$, compute $P_{f^*}(b)$ and $P_f(b)$.

8. (10 marks) Consider the problem

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2}\|Ax - b\|_2^2 \quad \text{s.t.} \quad \|x\|_\infty \leq r$$

   where $r > 0$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$ are given, $\|x\|_\infty = \max_i |x_i|$.

   (a) Transform the above problem into the structure

$$\min_x \quad f(x) + g(x)$$

   where f is differentiable and g is non-differentiable.

   (b) Give the PG (proximal gradient) iterations.

   (c) Set $r = 1$, $\alpha = 1$(step length), $m = n = 2$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Initialize $x^{(0)} = (0,0)^T$. Compute the first iteration $x^{(1)}$ of PG.

9. (10 marks) Consider the problem

$$\min_{x\in\mathbb{R}^n} \quad \|x\|_1 \quad \text{s.t.} \quad \|x-b\|_2 \le r$$

where $r > 0$, and $b \in \mathbb{R}^n$ are given, $\|\cdot\|_1, \|\cdot\|_2$ denote the ℓ_1, ℓ_2 norms.

(a) Transform the above problem into the 2-block separable structure

$$\min_{y,z} \quad f(y) + g(z) \quad \text{s.t.} \quad Ay + Bz = c$$

that ADMM can handle.

(b) Give the ADMM iterations.

(c) Set $r = 1$, $\sigma = 1$, $\tau = 1$, $m = n = 2$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Initialize every variable by zero. Compute the first iteration of ADMM.

10. (10 marks) Apply coordinate descent method for

$$\min_{x=(x_1;x_2)\in\mathbb{R}^2} \quad f(x_1, x_2) = x_1^4 - 4x_1x_2 + 5x_2^2 - 10x_2$$

with initial point $x^{(0)} = (0; 0)$. Find $x^{(1)}$ and $x^{(2)}$.

11. (10 marks) Two supplies $a_1 = 0.4$ and $a_2 = 0.6$ of goods must be transport to meet two demands $b_1 = 0.2$ and $b_2 = 0.8$ of customers. The cost of transporting one unit of ith good to jth customer is c_{ij} ($c_{11} = 2$, $c_{21} = 1$, $c_{12} = 3$, $c_{22} = 4$). Determine the optimal transportation plan (x_{ij} unit of ith good transported to jth customer) to minimize the cost.

(a) Form the above problem into an optimization problem.

(b) Give the KKT conditions.

(c) Construct two feasible points (i.e., two feasible transportation plans).

(d) For the two feasible transportation plans in (c), which one is better (i.e., with less cost)?

12. (10 marks) Consider the problem

$$\begin{aligned} \max_{x_1,x_2} \quad & 6x_1 + 4x_2 - 13 - x_1^2 - x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 \le 3 \end{aligned}$$

(a) Convert the problem into an equivalent minimization problem. Solve it graphically and give the solution x_1^*, x_2^*.

(b) Give the KKT conditions.

(c) Derive the Lagrange dual problem.

(d) Solve the Lagrange dual problem and give the dual solution y^*.

(e) Verify that x_1^*, x_2^*, y^* satisfy the KKT conditions.